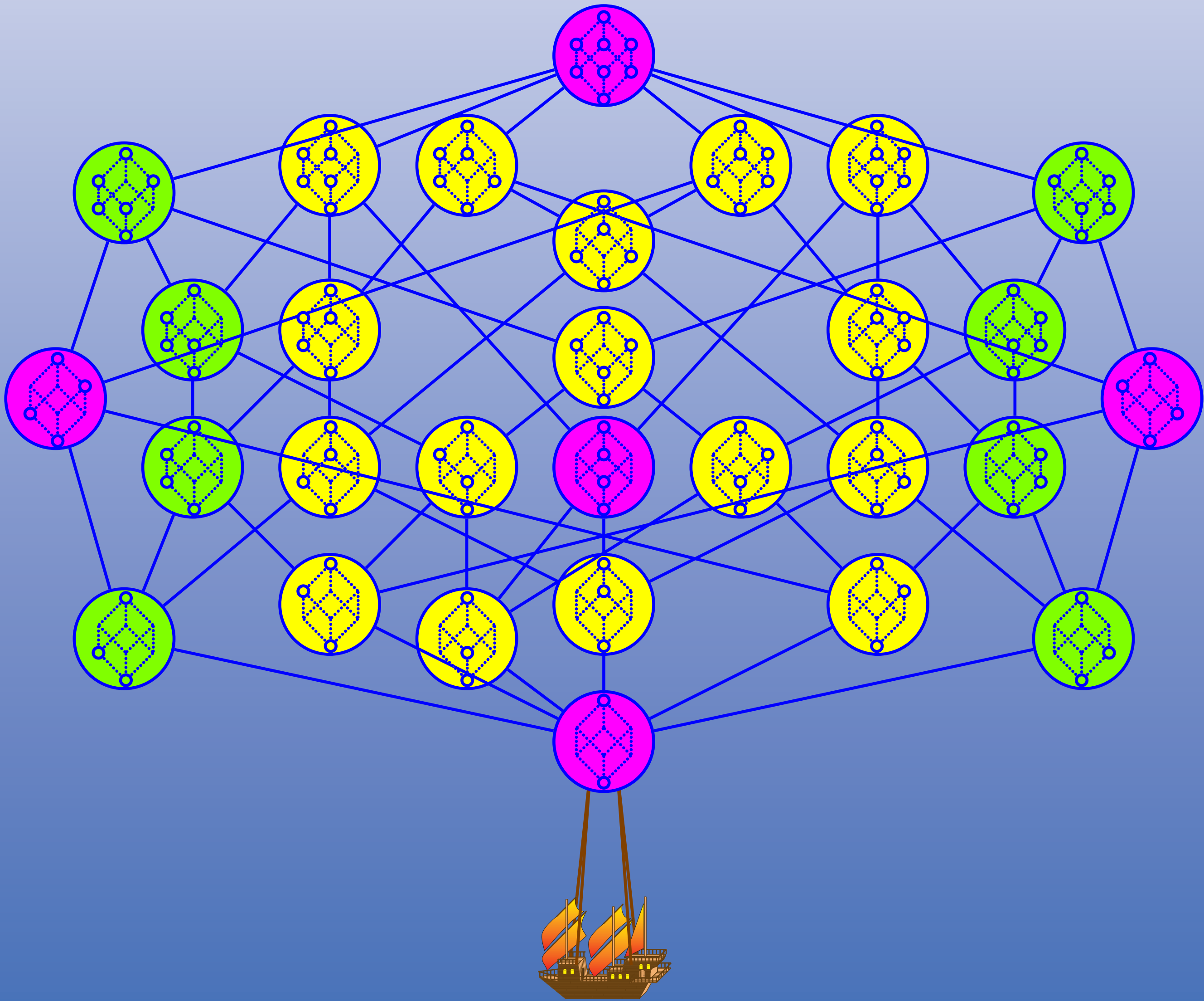
Structure and Analysis of Mathematical Spaces



Daniel J. Greenhoe

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2

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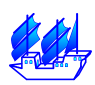
The ship on the cover is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]



[1] Paine (2000) page 63 ⟨Golden Hind⟩

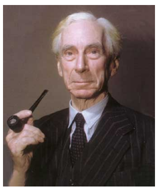
“Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?”

“Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.”

Alfred Edward Housman, English poet (1859–1936) [2]

“The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.”

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



“As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.”

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

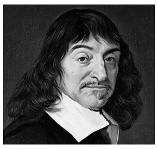
[2] quote: Housman (1936) page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961) page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967) page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.❞



❝Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.❞

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \dots$

…continued on next page…

[5]quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), `http://en.wikipedia.org/wiki/Descartes`, public domain

[6]quote: Cajori (1993) ⟨paragraph 540⟩, image: `http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg`, public domain

symbol	description	
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$
$\mathbb{Z}^{-}$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_{\mathsf{o}}$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_{\mathsf{e}}$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^{+}$	positive real numbers	$(0, \infty)$
$\mathbb{R}^{-}$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^{*}$	extended real numbers	$\mathbb{R}^{*} \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \bigtriangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\not\subseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^{c}	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	

...continued on next page...

symbol	description	
0	"false" condition	
$\neg$	logical NOT operation	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\le$	reflexive ordering relation	"less than or equal to"
$\ge$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lVert\lvert\cdot\rvert\rVert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathscr{T}(X)$	set of topologies on a set X	
$\mathscr{R}(X)$	set of rings of sets on a set X	
$\mathscr{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathscr{S}_{\mathsf{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathscr{I}_{\mathsf{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathscr{B}_{\mathsf{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathscr{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		

...continued on next page...

symbol	description
$\tilde{\mathbf{F}}$	*Fourier Transform* operator
$\hat{\mathbf{F}}$	*Fourier Series* operator
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator
$\mathbf{Z}$	*Z-Transform* operator
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\check{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

Part I

Spaces of Analysis

CHAPTER 1

TOPOLOGICAL SPACES

"Nevertheless I should not pass over in silence the fact that today the feeling among mathematicians is beginning to spread that the fertility of these abstracting methods is approaching exhaustion. The case is this: that all these nice general concepts do not fall into our laps by themselves. But definite concrete problems were first conquered in their undivided complexity, singlehanded by brute force, so to speak. Only afterwards the axiomaticians came along and stated: Instead of breaking the door with all your might and bruising your hands, you should have constructed such and such a key of skill, and by it you would have been able to open the door quite smoothly. But they can construct the key only because they are able, after the breaking in was successful, to study the lock from within and without. Before you can generalize, formalize, and axiomatize, there must be a mathematical substance."

Hermann Weyl (1885–1955), German mathematician, theoretical physicist, and philosopher [1]

1.1 Set structure

1.1.1 Open sets

Definition 1.1. [2] *Let Γ be a set with an arbitrary (possibly uncountable) number of elements. Let 2^X be the* POWER SET *of a set X.*

[1] quote: Weyl (1935a) ⟨memorial address for Emmy Noether (1882–1935)⟩
Weyl (1935c) ⟨in a book of collected works of Hermann Weyl⟩
Weyl (1935b) pages 140–141 ⟨in a book by Auguste Dick about Emmy Noether⟩
image: http://www.hs.uni-hamburg.de/DE/GNT/hh/biogr/weyl.htm

[2] Munkres (2000) page 76, Riesz (1909), Hausdorff (1914), Tietze (1923), Hausdorff (1937) page 258

DEF *A family of sets* $\boldsymbol{T} \subseteq 2^X$ *is a* ***topology*** *on a set* X *if*

1. $\varnothing \in \boldsymbol{T}$ (*$\varnothing$ is in $\boldsymbol{T}$*) *and*
2. $X \in \boldsymbol{T}$ (*X is in $\boldsymbol{T}$*) *and*
3. $U, V \in \boldsymbol{T} \implies U \cap V \in \boldsymbol{T}$ (*the intersection of a finite number of open sets is open*) *and*
4. $\{U_\gamma | \gamma \in \Gamma\} \subseteq \boldsymbol{T} \implies \bigcup_{\gamma\in\Gamma} U_\gamma \in \boldsymbol{T}$ (*the union of an arbitrary number of open sets is open*).

A ***topological space*** *is the pair* $(X, \boldsymbol{T})$. *An* ***open set*** *is any member of* $\boldsymbol{T}$.
A ***closed set*** *is any set D such that D^c is open.*
The set of topologies on a set X is denoted $\mathscr{T}(X)$. *That is,*

$$\mathscr{T}(X) \triangleq \{\boldsymbol{T} \subseteq 2^X | \boldsymbol{T} \text{ is a topology}\}$$

Example 1.1. [3] Let $\mathscr{T}(X)$ be the set of topologies on a set X and 2^X the *power set* (Definition A.1 page 259) on X.

EX

$\{\varnothing, X\}$	is a *topology* in	$\mathscr{T}(X)$	(*indiscrete topology* or *trivial topology*)
2^X	is a *topology* in	$\mathscr{T}(X)$	(*discrete topology*)

Example 1.2 (finite complement topology). [4] Let $\mathscr{T}(X)$ be the set of topologies on a set X and 2^X the *power set* (Definition A.1 page 259) on X.

EX

$$\underbrace{\left\{A \in 2^X | A^c \text{ is finite or } A^c = X\right\} \in \mathscr{T}(X)}_{\text{is a topology on } X}$$

For examples of topologies on the real line, see the following:
Adams and Franzosa (2008) page 31 ⟨"six topologies on the real line"⟩, Salzmann et al. (2007) pages 64–70 ⟨Weird topologies on the real line⟩, Murdeshwar (1990) page 53 ⟨"often used topologies on the real line"⟩, Joshi (1983) pages 85–91 ⟨§4.2 Examples of Topological Spaces⟩

1.1.2 Order structure of a topology

Theorem 1.1. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ *be a* LATTICE.

THM $\boldsymbol{T}$ *is a* TOPOLOGY $\implies$ $(\boldsymbol{T}, \cup, \cap; \subseteq)$ *is a* DISTRIBUTIVE LATTICE

✎PROOF:

1. By Proposition A.10 (page 276), $(\boldsymbol{S}, \subseteq)$ is an *ordered set*.
2. By Proposition A.11 (page 277), $\cup$ is *least upper bound* operation on $(\boldsymbol{S}, \subseteq)$. and $\cap$ is *greatest lower bound* operation on $(\boldsymbol{S}, \subseteq)$.
3. Therefore, by Definition C.3 (page 301), $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is a lattice.
4. By Theorem C.3 (page 302), $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is *idempotent*, *commutative*, *associative*, and *absorptive*.
5. Proof that $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is *distributive*:

[3] Munkres (2000) page 77, Kubrusly (2011) page 107 ⟨Example 3.J⟩, Steen and Seebach (1978) pages 42–43 ⟨II.4⟩, DiBenedetto (2002) page 18
[4] Munkres (2000) page 77

(a) Proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:

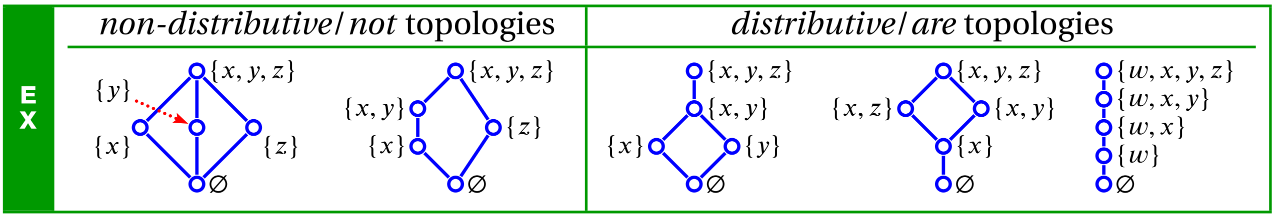
$$
\begin{aligned}
& A \cap (B \cup C) \\
&= \{x \in X | x \in A \wedge x \in (B \cup C)\} && \text{by definition of } \cap \text{ (Definition A.5 page 260)} \\
&= \{x \in X | x \in A \wedge x \in \{x \in X | x \in B \vee x \in C\}\} && \text{by definition of } \cup \text{ (Definition A.5 page 260)} \\
&= \{x \in X | x \in A \wedge (x \in B \vee x \in C)\} \\
&= \{x \in X | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\
&= \{x \in X | x \in A \wedge x \in B\} \cup \{x \in X | x \in A \wedge x \in C\} && \text{by definition of } \cup \text{ (Definition A.5 page 260)} \\
&= (A \cap B) \cup (A \cap C) && \text{by definition of } \cap \text{ (Definition A.5 page 260)}
\end{aligned}
$$

(b) Proof that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:
This follows from the fact that $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is a lattice (item (3) page 4), that $\cap$ distributes over $\cup$ (item (5) page 4).

✏

Remark 1.1. Note that in set structures that are *not* closed under the *set union* operation $\cup$ (Definition A.5 page 260), the set union operation $\cup$ is in general *not* equivalent to the *order join* operation $\vee$ with respect to the *set inclusion* relation $\subseteq$ (Definition A.13 page 276). This is illustrated in the next example.

Example 1.3. There are five unlabeled lattices on a five element set (Proposition C.2 page 307). Of these five, three are *distributive*. The following illustrates that the distributive lattices are isomorphic to topologies, while the non-distributive lattices are not.



✎PROOF:

1. The first two lattices are non-distributive by *Birkhoff distributivity criterion.*

   (a) This lattice is not a topology because, for example,
   $\{x\} \vee \{y\} = \{x, y, z\} \neq \{x, y\} = \{x\} \cup \{y\}$.
   That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.

   (b) This lattice is not a topology because, for example,
   $\{x\} \vee \{y\} = \{y\} \neq \{x, y\} = \{x\} \cup \{y\}$

2. The last three lattices are distributive by *Birkhoff distributivity criterion.*

   (a) This lattice is the topology $\boldsymbol{T}_{13}$ of Example 1.6 (page 7). On the set $\{x, y, z\}$, there are a total of three topologies that have this order structure (Example 1.6 page 7):
   $$
   \begin{array}{lcl}
   \boldsymbol{T}_{13} &=& \{\ \varnothing,\ \{x\},\ \{y\},\ \{x,y\},\ \{x,y,z\}\ \} \\
   \boldsymbol{T}_{25} &=& \{\ \varnothing,\ \{x\},\ \{z\},\ \{x,z\},\ \{x,y,z\}\ \} \\
   \boldsymbol{T}_{46} &=& \{\ \varnothing,\ \{y\},\ \{z\},\ \{y,z\},\ \{x,y,z\}\ \}
   \end{array}
   $$

   (b) This lattice is the topology $\boldsymbol{T}_{31}$ of Example 1.6 (page 7). On the set $\{x, y, z\}$, there are a total of three topologies that have this order structure (Example 1.6 page 7):
   $$
   \begin{array}{lcl}
   \boldsymbol{T}_{31} &=& \{\ \varnothing,\ \{x\},\ \{x,y\},\ \{x,z\},\ \{x,y,z\}\ \} \\
   \boldsymbol{T}_{52} &=& \{\ \varnothing,\ \{y\},\ \{x,y\},\ \{y,z\},\ \{x,y,z\}\ \} \\
   \boldsymbol{T}_{64} &=& \{\ \varnothing,\ \{z\},\ \{x,z\},\ \{y,z\},\ \{x,y,z\}\ \}
   \end{array}
   $$

(c) This lattice is a topology by Definition 1.1 (page 3).

✏

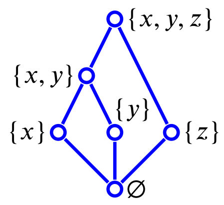
Example 1.4. The set structure

$$S \triangleq \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,y,z\}\}$$

ordered by the set inclusion relation $\subseteq$ is illustrated by the Hasse diagram to the right. Note that

$$\{x\} \vee \{z\} = \{x,y,z\} \neq \{x,z\} = \{x\} \cup \{z\}.$$

That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.



1.1.3 Number of topologies

Theorem 1.2. [5]

THM

The ***number of topologies*** t_n *on a finite set* X_n *with* n *elements is*

n	0	1	2	3	4	5	6	7	8
t_n	1	1	4	29	355	6942	209,527	9,535,241	642,779,354

n	9	10
t_n	63,260,289,423	8,977,053,873,043

Proposition 1.1. [6] *Let* t_n *be the number of topologies on a finite set with* n *elements.*

PRP

$$\lim_{n\to\infty} \frac{t_n}{2^{\frac{n^2}{4}}} = \infty \qquad \textit{(lower bound)}$$

$$\lim_{n\to\infty} \frac{t_n}{2^{\left(\frac{1}{2}+\epsilon\right)n^2}} = 0 \quad \forall\epsilon>0 \qquad \textit{(upper bound)}$$

$$t_n > n t_{n-1} \qquad \textit{(rate of growth)}$$

1.1.4 Closed sets

Theorem 1.3. [7] *Let* (X, T) *be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3). Let* T^* *be the set of all* CLOSED *sets in* (X, T)*. Let* Γ *be a set with an arbitrary (possibly uncountable) number of elements.*

THM

1. $\varnothing \in T^*$ *($\varnothing$ is* CLOSED *in* (X,T)*)*
2. $X \in T^*$ *(X is* CLOSED *in* (X,T)*)*
3. $A, B \in T^* \implies A \cup B \in T^*$ *(the union of a finite number of closed sets is* CLOSED*)*
4. $\{D_\gamma | \gamma \in \Gamma\} \subseteq T^* \implies \bigcap_{\gamma\in\Gamma} D_\gamma \in T^*$ *(the intersection of an arbitrary number of closed sets is* CLOSED*)*

---

[5] Sloane (2014) ⟨http://oeis.org/A000798⟩, Brown and Watson (1996) page 31, Comtet (1974) page 229, Comtet (1966), Chatterji (1967) page 7, Evans et al. (1967), Krishnamurthy (1966) page 157,

[6] Chatterji (1967) pages 6–7, Kleitman and Rothschild (1970)

[7] Aliprantis and Burkinshaw (1998) page 35, Hausdorff (1937) page 258

✎PROOF:

$$
\begin{array}{rcll}
\varnothing \text{ is open} & & & \text{by Definition 1.1 page 3} \\
& \implies & \varnothing^c \text{ is closed} & \text{by Definition 1.1 page 3} \\
& \implies & X \text{ is closed} & \text{because } \varnothing^c = X \\
X \text{ is open} & & & \text{by Definition 1.1 page 3} \\
& \implies & X^c \text{ is closed} & \text{by Definition 1.1 page 3} \\
& \iff & \varnothing \text{ is closed} & \\
A, B \text{ are closed} & \implies & A^c, B^c \text{ are open} & \text{by Definition 1.1 page 3} \\
& \implies & A^c \cap B^c \text{ is open} & \text{by Definition 1.1 page 3} \\
& \implies & \left(A^c \cap B^c\right)^c \text{ is closed} & \text{by Definition 1.1 page 3} \\
& \implies & A \cup B \text{ is closed} & \text{by } \textit{Demorgan's law} \text{ (Theorem A.7 page 274)} \\
\left(\!\left(A_\gamma\right)\!\right)_{\gamma\in\Gamma} \text{ are closed} & \implies & \left(\!\left({A^c}_\gamma\right)\!\right)_{\gamma\in\Gamma} \text{ are open} & \text{by Definition 1.1 page 3} \\
& \implies & \bigcup_{\gamma\in\Gamma} {A^c}_\gamma \text{ is open} & \text{by Definition 1.1 page 3} \\
& \implies & \left(\bigcup_{\gamma\in\Gamma} {A^c}_\gamma\right)^c \text{ is closed} & \text{by Definition 1.1 page 3} \\
& \implies & \bigcap_{\gamma\in\Gamma} A_\gamma \text{ is closed} & \text{by } \textit{Demorgan's law} \text{ (Theorem A.7 page 274)}
\end{array}
$$

✏

Example 1.5. There are four topologies on the set $X \triangleq \{x, y\}$:

EX topologies on $\{x, y\}$	corresponding closed sets
$T_0 = \{\varnothing, X\}$	$\{\varnothing, X\}$
$T_1 = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y\}, X\}$
$T_2 = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x\}, X\}$
$T_3 = \{\varnothing, \{x\}, \{y\}, X\}$	$\{\varnothing, \{x\}, \{y\}, X\}$

The topologies (X, T_1) and (X, T_2), as well as their corresponding closed set topological spaces, are all *Serpiński spaces*.

Example 1.6. There are a total of 29 *topologies* (Definition 1.1 page 3) on the set $X \triangleq \{x, y, z\}$:

topologies on $\{x, y, z\}$	corresponding closed sets
$T_{00} = \{\varnothing, X\}$	$\{\varnothing, X\}$
$T_{01} = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y, z\}, X\}$
$T_{02} = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x, z\}\ X\}$
$T_{04} = \{\varnothing, \{z\}, X\}$	$\{\varnothing, \{x, y\}, X\}$
$T_{10} = \{\varnothing, \{x, y\}, X\}$	$\{\varnothing, \{z\}, X\}$
$T_{20} = \{\varnothing, \{x, z\}, X\}$	$\{\varnothing, \{y\}, X\}$
$T_{40} = \{\varnothing, \{y, z\}, X\}$	$\{\varnothing, \{x\}, X\}$
$T_{11} = \{\varnothing, \{x\}, \{x, y\}, X\}$	$\{\varnothing, \{z\}, \{y, z\}, X\}$
$T_{21} = \{\varnothing, \{x\}, \{x, z\}, X\}$	$\{\varnothing, \{y\}\ \{y, z\}, X\}$
$T_{41} = \{\varnothing, \{x\}, \{y, z\}, X\}$	$\{\varnothing, \{x\}, \{y, z\}, X\}$
$T_{12} = \{\varnothing, \{y\}, \{x, y\}, X\}$	$\{\varnothing, \{z\}, \{x, z\}\ X\}$
$T_{22} = \{\varnothing, \{y\}, \{x, z\}, X\}$	$\{\varnothing, \{y\}, \{x, z\}, X\}$
$T_{42} = \{\varnothing, \{y\}, \{y, z\}, X\}$	$\{\varnothing, \{x\}, \{x, z\}, X\}$
$T_{14} = \{\varnothing, \{z\}, \{x, y\}, X\}$	$\{\varnothing, \{z\}, \{x, y\}, X\}$
$T_{24} = \{\varnothing, \{z\}, \{x, z\}, X\}$	$\{\varnothing, \{y\}, \{x, y\}, X\}$
$T_{44} = \{\varnothing, \{z\}, \{y, z\}, X\}$	$\{\varnothing, \{x\}, \{x, y\}, X\}$
$T_{31} = \{\varnothing, \{x\}, \{x, y\}, \{x, z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{y, z\}, X\}$
$T_{52} = \{\varnothing, \{y\}, \{x, y\}, \{y, z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x, z\}, X\}$
$T_{64} = \{\varnothing, \{z\}, \{x, z\}, \{y, z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x, y\}, X\}$

$T_{13} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, X\}$	$\{\emptyset, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{25} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, X\}$	$\{\emptyset, \{y\}, \{x,y\}, \{y,z\}, X\}$
$T_{46} = \{\emptyset, \{y\}, \{z\}, \{y,z\}, X\}$	$\{\emptyset, \{x\}, \{x,y\}, \{x,z\}, X\}$
$T_{33} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	$\{\emptyset, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{53} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{\emptyset, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{35} = \{\emptyset, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	$\{\emptyset, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$
$T_{65} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\emptyset, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$
$T_{56} = \{\emptyset, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	$\{\emptyset, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$
$T_{66} = \{\emptyset, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\emptyset, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$
$T_{77} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	$\{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$

1.1.5 Bases for topologies

Definition 1.2. [8] *Let (X, T) be a* TOPOLOGICAL SPACE (Definition 1.1 page 3).

DEF

*A set $B \subseteq 2^X$ is a **base** for a topology (X, T) if*

1. $B \subseteq T$ *and*
2. $\forall U \in T, \exists \{B_\gamma \in B\}$ such that $U = \bigcup \{B_\gamma \in B\}$

*An element $A \in B$ is called a **basic open set**.*

Theorem 1.4. [9] *Let (X, T) be a* TOPOLOGICAL SPACE (Definition 1.1 page 3).

THM

$$\{B \text{ is a BASE for } T\} \iff \left\{\begin{array}{l}\text{For every } x \in X \text{ and for every OPEN SET } U \text{ containing } x, \\ \text{there exists } B \in B \text{ such that } x \in B \subseteq U.\end{array}\right\}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned} x \in U \in T &\implies \exists\{B_\gamma \in B\} \text{ such that } U = \bigcup\{B_\gamma \in B\} && \text{by "}B\text{ is a } base\text{" hypothesis} \\ &\implies \exists B_\gamma \text{ such that } x \in B_\gamma \subseteq U && \text{because } B_\gamma \subseteq \bigcup\{B_\gamma \in B\} \end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned} U \in T &\implies \forall x \in U \exists\{B_\gamma \in B\} \text{ such that } U = \bigcup\{B_\gamma \in B \mid x \in B_\gamma\} && \text{by right hypothesis} \\ &\implies \{B \text{ is a } base \text{ for } T\} && \text{by definition of } base\text{: Definition 1.2} \end{aligned}$$

✏

Theorem 1.5. [10] *Let (X, T) be a* TOPOLOGICAL SPACE (Definition 1.1 page 3) *and $B \subseteq 2^X$.*

THM

$$B \text{ is a base for } (X, T) \iff \left\{\begin{array}{lllll} 1. & x \in X & \implies & \exists B \in B \text{ such that } x \in B & \text{and} \\ 2. & B_1, B_2 \in B & \implies & B_1 \cap B_2 \in B & \end{array}\right.$$

Example 1.7. Consider the 29 topologies on the set $\{x, y, z\}$ (Example 1.6 page 7).

[8] Joshi (1983) page 92 ⟨(3.1) Definition⟩, Davis (2005) page 46 ⟨Definition 4.15⟩
[9] Joshi (1983) pages 92–93 ⟨(3.2) Proposition⟩, Davis (2005) page 46
[10] Bollobás (1999) page 19

This family of sets	is a *base* for these topologies on $\{x, y, z\}$:
$\{\{x\}, \{y, z\}\}$	T_{00}, T_{01}, T_{40}, and T_{41}.
$\{\{y\}, \{x, z\}\}$	T_{00}, T_{02}, T_{20}, and T_{22}.
$\{\{z\}, \{x, y\}\}$	T_{00}, T_{04}, T_{10}, and T_{14}.
$\{\{x\}, \{x, y\}, \{x, z\}\}$	T_{00}, T_{11}, T_{21}, and T_{31}.
$\{\{y\}, \{x, y\}, \{y, z\}\}$	T_{00}, T_{12}, T_{42}, and T_{52}.
$\{\{z\}, \{x, z\}, \{y, z\}\}$	T_{00}, T_{24}, T_{44}, and T_{64}.
$\{\{x\}, \{y\}, \{x, y, z\}\}$	T_{00}, T_{01}, T_{02}, T_{10}, T_{11}, T_{12}, and T_{13}.
$\{\{x\}, \{z\}, \{x, y, z\}\}$	T_{00}, T_{01}, T_{04}, T_{20}, T_{21}, T_{24}, and T_{25}.
$\{\{y\}, \{z\}, \{x, y, z\}\}$	T_{00}, T_{02}, T_{04}, T_{40}, T_{42}, T_{44}, and T_{46}.
$\{\{x\}, \{y\}, \{x, z\}\}$	T_{00}, T_{01}, T_{02}, T_{10}, T_{11}, T_{12}, T_{13}, T_{20}, T_{21}, T_{22}, T_{31}, and T_{33}.
$\{\{x\}, \{y\}, \{y, z\}\}$	T_{00}, T_{01}, T_{02}, T_{10}, T_{11}, T_{12}, T_{13}, T_{40}, T_{41}, T_{42}, T_{52}, and T_{53}.
$\{\{x\}, \{z\}, \{x, y\}\}$	T_{00}, T_{01}, T_{04}, T_{10}, T_{11}, T_{14}, T_{20}, T_{21}, T_{24}, T_{25}, T_{31}, and T_{35}.
$\{\{x\}, \{z\}, \{y, z\}\}$	T_{00}, T_{01}, T_{04}, T_{20}, T_{21}, T_{24}, T_{25}, T_{40}, T_{41}, T_{44}, T_{64}, and T_{65}.
$\{\{y\}, \{z\}, \{x, z\}\}$	T_{00}, T_{02}, T_{04}, T_{20}, T_{22}, T_{24}, T_{40}, T_{42}, T_{44}, T_{46}, T_{64}, and T_{66}.
$\{\{y\}, \{z\}, \{x, z\}\}$	T_{00}, T_{02}, T_{04}, T_{10}, T_{12}, T_{14}, T_{40}, T_{42}, T_{44}, T_{46}, T_{52}, and T_{56}.
$\{\{x\}, \{y\}, \{z\}\}$	all 29 of the topologies.

Example 1.8. [11] Let (X, d) be a *metric space*.

EX The set $\boldsymbol{B} \triangleq \{\mathsf{B}(x, r) \,|\, x \in X,\ r \in \mathbb{N}\}$ (the set of all open balls in (X, d)) is a *base* for a topology on (X, d).

Example 1.9 (the standard topology on the real line). [12]

EX The set $\boldsymbol{B} \triangleq \{(a : b) | a, b \in \mathbb{R},\ a < b\}$ is a *base* for the metric space $(\mathbb{R}, |b - a|)$ (the *usual metric space* on $\mathbb{R}$).

Example 1.10. [13]

EX The set $\boldsymbol{B} \triangleq \{(a : b) | a, b \in \mathbb{Q},\ a < b\}$ is a *base* for the metric space $(\mathbb{R}, |b - a|)$ (the *usual metric space* on $\mathbb{R}$).

The possible advantage of this base over the base of Example 1.9 is that this base is *countable*.

Example 1.11 (lower limit topology/the Sorgenfrey line topology). [14]

EX The set $\boldsymbol{B} \triangleq \{[a : b) | a, b \in \mathbb{R},\ a < b\}$ is a *base* for the metric space $(\mathbb{R}, |b - a|)$ (the *usual metric space* on $\mathbb{R}$).

Under this topology, the *cumulative distribution functions* of probability theory are *continuous*.

Counterexample 1.1. [15] Definition 1.1 (page 3) states that the intersection of a *finite* number of open sets is also open. But under this definition, in general it is *not* true that the intersection of an infinite number of open sets is open. Take for example the *standard topology on the real line* (Example 1.9 page 9):

1. Let $\left(A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)\right)_{n\in\mathbb{N}}$ be a sequence of real intervals. That is
$$\left((-1, 1), \left(-\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{3}, \frac{1}{3}\right), \left(-\frac{1}{4}, \frac{1}{4}\right), \left(-\frac{1}{5}, \frac{1}{5}\right), \ldots\right)$$
2. Then $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$; that is, $\bigcap_{n\in\mathbb{N}} A_n$ is a set with just one value (0).

[11] Davis (2005) page 46 ⟨Example 4.16⟩
[12] Munkres (2000) page 81, Davis (2005) page 46 ⟨Example 4.16⟩
[13] Davis (2005) page 46 ⟨Example 4.16⟩
[14] Munkres (2000) pages 81–82, Davis (2005) page 48 ⟨Example 4.21⟩
[15] Rosenlicht (1968) page 40

3. A single value is *not* an open set because any ball with radius greater than 0 is not in the set (Lemma 3.3 page 38).

4. Therefore, $\bigcap_{n\in\mathbb{N}} A_n$ is not open.

1.1.6 Order structure of the topologies on a set

In general for a given set X, there is not just one possible topology. Rather, for any sizeable set X, there are myriads of topologies. Some of these topologies are subsets of other topologies; in such a case, Definition 1.3 (page 10) states that we say that the subset topology is *coarser* than the other and that the other superset topology is *finer*. And not only does any individual topology generate a lattice, but as demonstrated by Theorem 1.6 (page 10), all the topologies taken together also form a lattice. Examples of lattices of topologies are provided by the following:

- Example 1.12 (page 10): lattice of the 4 topologies of a 2 element set X.
- Example 1.13 (page 10): lattice of the 29 topologies of a 3 element set X.

Definition 1.3. [16] *Let $(X, \boldsymbol{S})$ and $(X, \boldsymbol{T})$ be two* TOPOLOGICAL SPACES *(Definition 1.1 page 3) on a set X.*

DEF

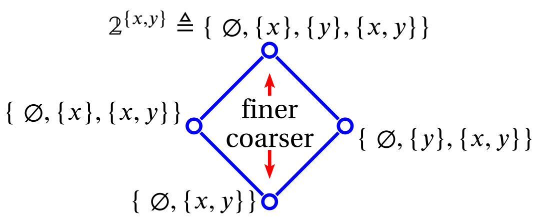
$\boldsymbol{S}$ is **coarser** *than $\boldsymbol{T}$ and $\boldsymbol{T}$ is* **finer** *than $\boldsymbol{S}$ if*
$\boldsymbol{S} \subseteq \boldsymbol{T}$.
$\boldsymbol{S}$ is **strictly coarser** *than $\boldsymbol{T}$ and $\boldsymbol{T}$ is* **strictly finer** *than $\boldsymbol{S}$ if*
$\boldsymbol{S} \subsetneq \boldsymbol{T}$.

Theorem 1.6 (Lattice of topologies). [17]

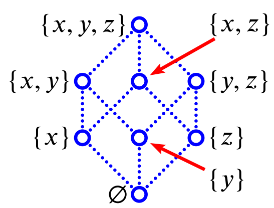
THM

$$\underbrace{\mathscr{T}(X) \triangleq \{\boldsymbol{T}_1, \boldsymbol{T}_2, \dots | \boldsymbol{T}_n \text{ is a topology on } X\}}_{\text{the set of topologies on } X} \implies (\mathscr{T}(X), \cup, \cap; \subseteq) \text{ is a lattice.}$$

Example 1.12. [18] Example 1.5 (page 7) lists the four topologies on the set $X \triangleq \{x, y\}$. The lattice of these topologies $(\{\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{T}_4\}, \cup, \cap; \subseteq)$ is illustrated by the figure below and to the right. Note that there are only four valid topologies out of a total sixteen possible families of sets: $(2^{|2^X|} = 2^{2^{|X|}} = 2^{2^2} = 2^4 = 16)$. Half of the sixteen families are not valid topologies because they do not contain $\varnothing$ and half of the remaining are not valid because they do not contain X. This leaves $16 \times \frac{1}{2} \times \frac{1}{2} = 4$ topologies.



Example 1.13. [19] Let a given topology in $\mathscr{T}(\{x, y, z\})$ be represented by a Hasse diagram as illustrated to the right, where a circle present means the indicated set is in the topology, and a circle absent means the indicated set is not in the topology. Example 1.6 (page 7) lists the 29 topologies $\mathscr{T}(\{x, y, z\})$. The lattice of these 29 topologies $(\mathscr{T}(\{x, y, z\}), \cup, \cap; \subseteq)$ is illustrated in Figure 1.1 (page 11) and Figure 1.2 (page 12). The five topologies $\boldsymbol{T}_1, \boldsymbol{T}_{41}, \boldsymbol{T}_{22}, \boldsymbol{T}_{14}$, and $\boldsymbol{T}_{77}$ are also *algebras of sets* (Definition A.9 page 266); these five sets are shaded in Figure 1.1 and represented as solid dots in Figure 1.2.



[16] Munkres (2000) page 77
[17] Larson and Andima (1975), Vaidyanathaswamy (1960) page 131, Birkhoff (1936a), Stone (1936b), Wallman (1938)
[18] Isham (1999) page 44, Isham (1989) page 1515
[19] Isham (1999) page 44, Isham (1989) page 1516, Steiner (1966) page 386

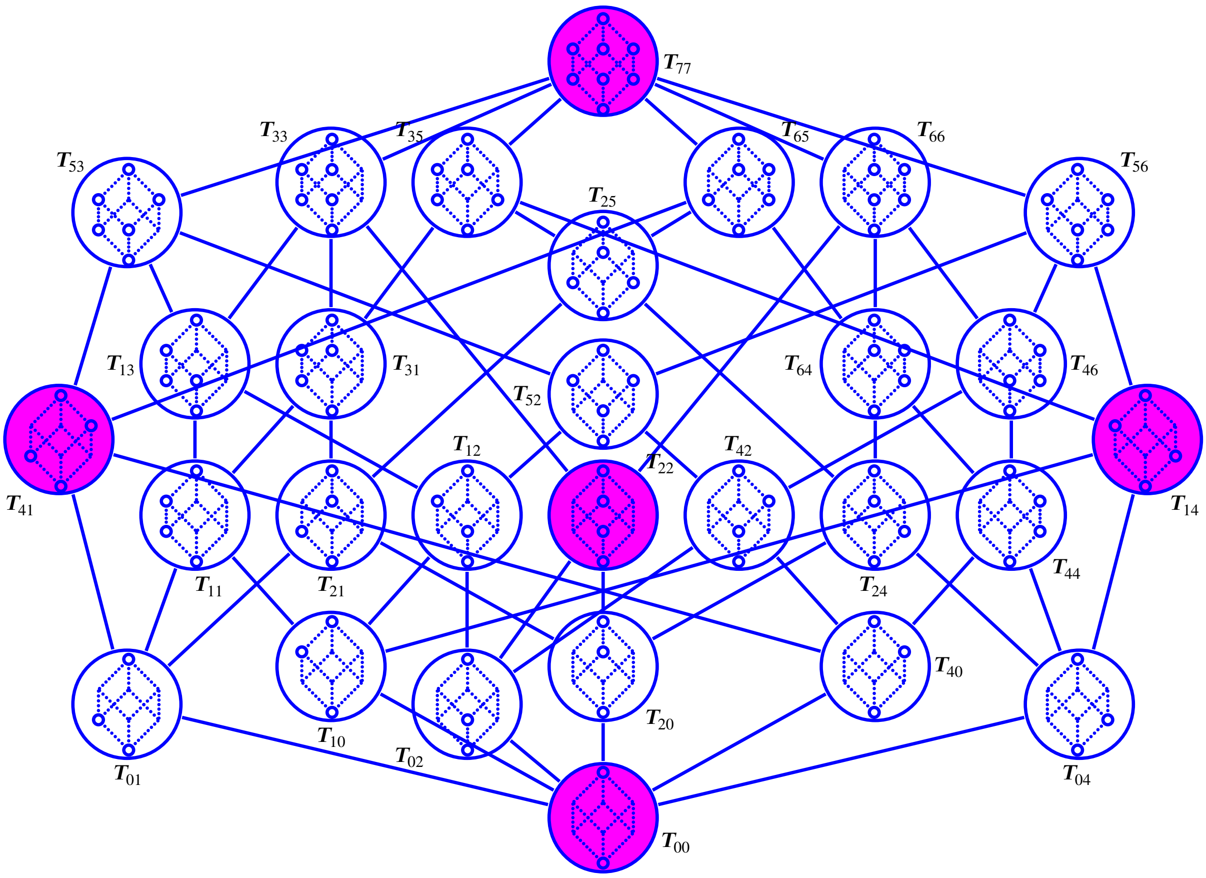


Figure 1.1: Lattice of topologies on $X \triangleq \{x, y, z\}$ (see Example 1.13 page 10)

Theorem 1.7. [20] *Let $\mathscr{T}(X)$ be the lattice of topologies on a set X with $|X|$ elements.*

THM
$$\begin{array}{lclcl} |X| & \le & 2 & \Longrightarrow & \mathscr{T}(X) \text{ is DISTRIBUTIVE} \\ |X| & \ge & 3 & \Longrightarrow & \mathscr{T}(X) \text{ is NOT MODULAR (and not distributive)} \end{array}$$

Theorem 1.8. [21] *Let $\mathscr{T}(X)$ be the lattice of topologies on a set X.*

THM
$$\mathscr{T}(X) \text{ is SELF-DUAL} \quad \Longleftrightarrow \quad |X| \le 3$$

Theorem 1.9. [22]

THM *Every lattice of topologies is complemented.*

Theorem 1.10. [23]

THM *Every topology except the discrete and indiscrete topology in the lattice of topologies on a set X has at least $|X| - 1$ complements.*

Let $\hat{\Sigma}(X)$ be the set of all topologies on X except for the discrete and indiscrete topologies on X.

Example 1.14. Example 1.6 (page 7) lists the 29 topologies on a set $X \triangleq \{x, y, z\}$. By Theorem 1.10 (page 11), with the exception of T_{00} (the indiscrete topology) and T_{77} (the discrete topology), each

[20] Steiner (1966) page 384
[21] Steiner (1966) page 385
[22] van Rooij (1968), Steiner (1966) page 397, Gaifman (1961), Hartmanis (1958)
[23] Hartmanis (1958), Schnare (1968) page 56, Watson (1994), Brown and Watson (1996) page 32

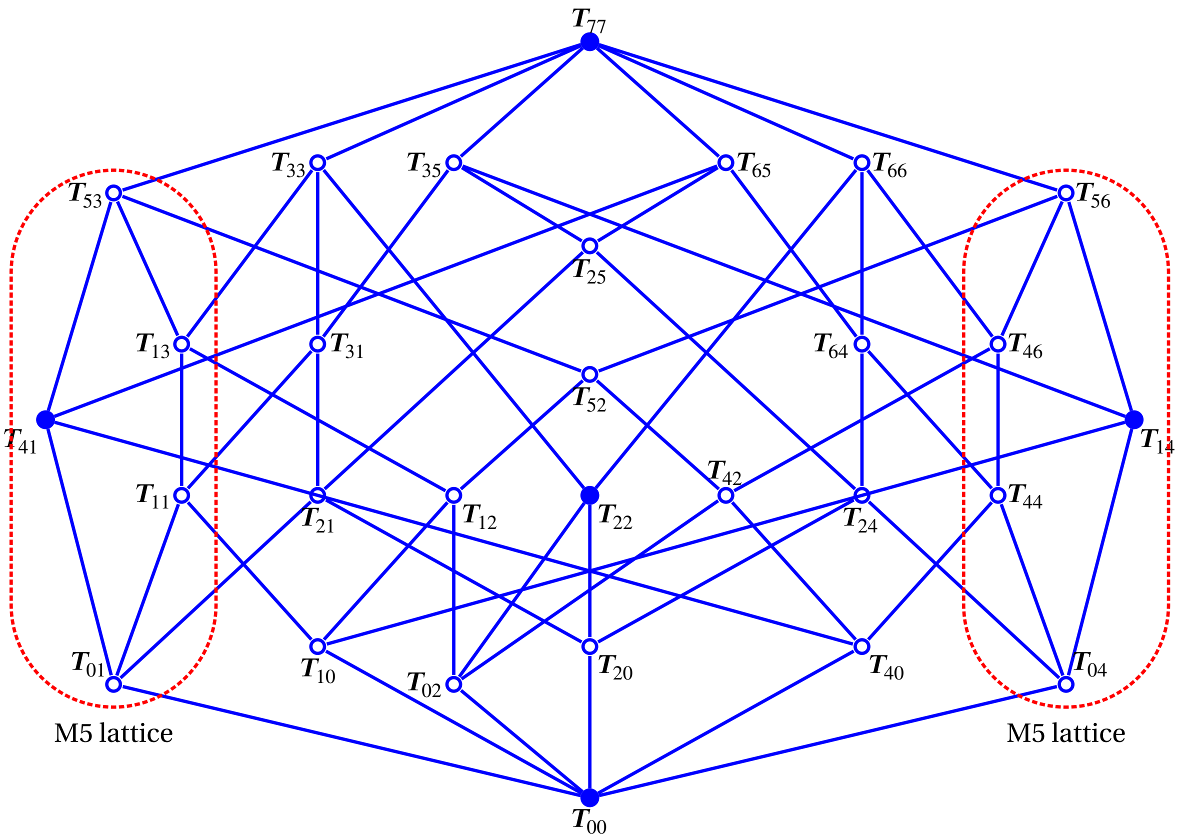


Figure 1.2: Lattice of topologies of $X \triangleq \{x, y, z\}$ (see Example 1.13 page 10)

of those topologies has exactly $|X| - 1 = 3 - 1 = 2$ complements. Listed below are the 29 topologies on $\{x, y, z\}$ along with their respective complements.

topologies on $\{x, y, z\}$	1st complement	2nd compl.
$T_{00} = \{\emptyset, X\}$	T_{77}	
$T_{01} = \{\emptyset, \{x\}, X\}$	T_{56}	T_{66}
$T_{02} = \{\emptyset, \{y\}, X\}$	T_{65}	T_{35}
$T_{04} = \{\emptyset, \{z\}, X\}$	T_{53}	T_{33}
$T_{10} = \{\emptyset, \{x,y\}, X\}$	T_{65}	T_{66}
$T_{20} = \{\emptyset, \{x,z\}, X\}$	T_{53}	T_{56}
$T_{40} = \{\emptyset, \{y,z\}, X\}$	T_{33}	T_{35}
$T_{11} = \{\emptyset, \{x\}, \{x,y\}, X\}$	T_{64}	T_{46}
$T_{21} = \{\emptyset, \{x\}, \{x,z\}, X\}$	T_{52}	T_{46}
$T_{41} = \{\emptyset, \{x\}, \{y,z\}, X\}$	T_{22}	T_{14}
$T_{12} = \{\emptyset, \{y\}, \{x,y\}, X\}$	T_{64}	T_{25}
$T_{22} = \{\emptyset, \{y\}, \{x,z\}, X\}$	T_{41}	T_{14}
$T_{42} = \{\emptyset, \{y\}, \{y,z\}, X\}$	T_{31}	T_{25}
$T_{14} = \{\emptyset, \{z\}, \{x,y\}, X\}$	T_{41}	T_{22}
$T_{24} = \{\emptyset, \{z\}, \{x,z\}, X\}$	T_{52}	T_{13}
$T_{44} = \{\emptyset, \{z\}, \{y,z\}, X\}$	T_{31}	T_{13}
$T_{31} = \{\emptyset, \{x\}, \{x,y\}, \{x,z\}, X\}$	T_{42}	T_{44}
$T_{52} = \{\emptyset, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{21}	T_{24}
$T_{64} = \{\emptyset, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{11}	T_{12}
$T_{13} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, X\}$	T_{24}	T_{44}
$T_{25} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, X\}$	T_{12}	T_{42}
$T_{46} = \{\emptyset, \{y\}, \{z\}, \{y,z\}, X\}$	T_{11}	T_{21}
$T_{33} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{04}	T_{40}
$T_{53} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	T_{04}	T_{20}
$T_{35} = \{\emptyset, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	T_{02}	T_{40}
$T_{65} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{02}	T_{10}
$T_{56} = \{\emptyset, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	T_{01}	T_{20}
$T_{66} = \{\emptyset, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{01}	T_{10}
$T_{77} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	T_{00}	

Theorem 1.11. [24]

THM: $\mathscr{T}(X)$ *is a topology of sets* $\implies$ $\begin{cases} \mathscr{T}(X) \text{ is atomic.} \\ \mathscr{T}(X) \text{ is anti-atomic.} \end{cases}$

Theorem 1.12. [25] *Let* $\mathscr{T}(X)$ *be the lattice of topologies on a set* X *and let* $n \triangleq |X|$.

THM:

$\mathscr{T}(X)$ *contains* $2^n - 2$ *atoms*	*for finite* X.
$\mathscr{T}(X)$ *contains* $2^{|X|}$ *atoms*	*for infinite* X.
$\mathscr{T}(X)$ *contains* $n(n-1)$ *anti-atoms*	*for finite* X.
$\mathscr{T}(X)$ *contains* $2^{2^{|X|}}$ *anti-atoms*	*for infinite* X.

[24] Larson and Andima (1975) page 179, Frölich (1964), Vaidyanathaswamy (1960), Vaidyanathaswamy (1947)

[25] Larson and Andima (1975) page 179, Frölich (1964)

1.2 Derived Sets

1.2.1 Definitions

Several useful set structures can be derived from the simple concept of the open set (next definition).

Definition 1.4. [26] *Let (X, T) be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3). Let 2^X be the* POWER SET *of X.*

DEF

*The set A^- is the **closure** of $A \in 2^X$ if $A^- \triangleq \bigcap \{D \in 2^X | A \subseteq D \text{ and } D \text{ is CLOSED}\}$.*
*The set A° is the **interior** of $A \in 2^X$ if $A^\circ \triangleq \bigcup \{U \in 2^X | U \subseteq A \text{ and } U \text{ is OPEN}\}$.*
*A point x is a **closure point** of A if $x \in A^-$.*
*A point x is an **interior point** of A if $x \in A^\circ$.*
*A point x is an **accumulation point** of A if $x \in (A \setminus \{x\})^-$*
*A point x in A^- is a **point of adherence** in A or is **adherent** to A if $x \in A^-$.*

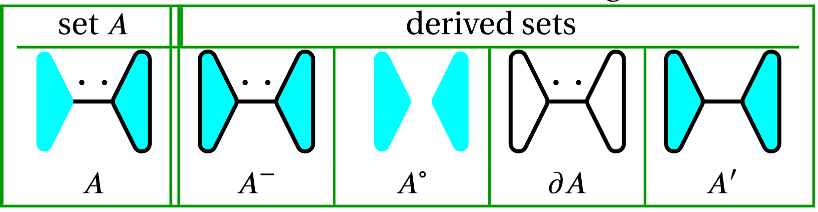
Definition 1.5. [27] *Let (X, T) be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3). Let 2^X be the* POWER SET *of X.*

DEF

*The set ∂A is the **boundary** of $A \in 2^X$ if $\partial A \triangleq A^- \cap \left(A^c\right)^-$.*
*The set A^e is the **exterior** of $A \in 2^X$ if $A^e \triangleq \left(A^c\right)^\circ$.*
*A point x in X is a **boundary point** of A if $x \in \partial A$.*
*A point x in X is an **exterior point** of A if $x \in A^e$.*
*A point x in A^- is a **point of adherence** in A or is **adherent** to A if $x \in A^-$.*
*The set A' is the **derived set** of $A \in 2^X$ if*
$$A' \triangleq \{x \in X | x \text{ is an accumlation point of } A\}.$$

Example 1.15. [28] Let A be the set illustrated as follows in a topogical space (X, T). The sets defined in Definition 1.4 page 14 are illustrated to the right of A.

EX



1.2.2 Resulting properties

Proposition 1.2. [29] *Let (X, T) be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3).*

PRP

$$\left\{ \begin{array}{l} x \text{ is an ACCUMULATION POINT} \\ \text{of a set } A \text{ in } (X, T) \end{array} \right\} \iff \left\{ \begin{array}{l} \text{Every open set containing } x \text{ also contains} \\ \text{another point } y \in A,\ y \neq x). \end{array} \right\}$$

Proposition 1.3. [30] *Let (X, T) be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3). Let A^- be the* CLOSURE, *A°*

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[26] Gemignani (1972) pages 55–56 ⟨Definition 3.5.7⟩, McCarty (1967) page 90, Munkres (2000) page 95 ⟨§Closure and Interior of a Set⟩, Thron (1966) pages 21–22 ⟨definition 4.8, defintion 4.9⟩, Kelley (1955) page 42, Kubrusly (2001) pages 115–116

[27] Gemignani (1972) pages 55–56 ⟨Definition 3.5.7⟩, McCarty (1967) page 90, Munkres (2000) page 95 ⟨§Closure and Interior of a Set⟩, Thron (1966) pages 21–22 ⟨definition 4.8, defintion 4.9⟩, Kelley (1955) page 42, Kubrusly (2001) pages 115–116, Murdeshwar (1990) page 48 ⟨exterior A^e⟩, Joshi (1983) page 110 ⟨exterior A^e⟩

[28] McCarty (1967) page 90

[29] Kubrusly (2001) pages 115–116 ⟨Proposition 3.26⟩, Murdeshwar (1990) page 48 ⟨1.24 Exercises (19)⟩

[30] McCarty (1967) page 90 ⟨IV.1 THEOREM⟩

the INTERIOR*, and* ∂A *the* BOUNDARY *of a set* A*. Let* 2^X *be the* POWER SET *of* X*.*

PRP

1. A^- *is* CLOSED $\forall A \in 2^X$.
2. A° *is* OPEN $\forall A \in 2^X$.
3. ∂A *is* CLOSED $\forall A \in 2^X$.

✎PROOF:

$$\begin{aligned}
A^- &\triangleq \bigcap \left\{D \in 2^X | A \subseteq D \text{ and } D \text{ is closed}\right\} && \text{by Definition 1.4 page 14}\\
&\implies A^- \text{ is } \textit{closed} && \text{by Theorem 1.3 page 6}\\
A^\circ &\triangleq \bigcap \left\{U \in 2^X | U \subseteq A \text{ and } U \text{ is open}\right\} && \text{by Definition 1.4 page 14}\\
&\implies A^\circ \text{ is } \textit{open} && \text{by Definition 1.1 page 3}\\
\partial A &\triangleq A^- \cap \left(A^c\right)^- && \text{by Definition 1.4 page 14}\\
&\implies \partial A \text{ is } \textit{closed} && \text{by (1) and Theorem 1.3 page 6}
\end{aligned}$$

✏

Lemma 1.1. [31] *Let* A^- *be the* CLOSURE*,* A° *the* INTERIOR*, and* ∂A *the* BOUNDARY *of a set* A *in a topological space* (X, T)*. Let* 2^X *be the* POWER SET *of* X*.*

LEM

1. $A^\circ \subseteq A \subseteq A^-$ $\forall A \in 2^X$.
2. $A' \subseteq A^-$ $\forall A \in 2^X$.
3. $A = A^\circ \iff A$ *is* OPEN $\forall A \in 2^X$.
4. $A = A^- \iff A$ *is* CLOSED $\forall A \in 2^X$.
5. $\left\{\begin{array}{ll} D \in 2^X \text{ is CLOSED} & \textit{and} \\ A \subseteq D & \end{array}\right\} \implies A^- \subseteq D$ $\left(\begin{array}{l} A^- \textit{ is the smallest CLOSED set} \\ \textit{containing } A \end{array}\right)$ $\forall A \in 2^X$.
6. $\left\{\begin{array}{ll} U \in 2^X \text{ is OPEN} & \textit{and} \\ U \subseteq A & \end{array}\right\} \implies U \subseteq A^\circ$ $\left(\begin{array}{l} A^\circ \textit{ is the largest OPEN set} \\ \textit{contained in } A \end{array}\right)$ $\forall A \in 2^X$.

✎PROOF:

1. Proof that $A^\circ \subseteq A \subseteq A^-$:

$$\begin{aligned}
A^\circ &\triangleq \bigcup \left\{U \in 2^X | U \subseteq A \text{ and } U \text{ is open}\right\} && \text{by Definition 1.4 page 14}\\
&\subseteq A\\
A &\subseteq \bigcap \left\{D \in 2^X | A \subseteq D \text{ and } D \text{ is closed}\right\}\\
&\triangleq A^- && \text{by Definition 1.4 page 14}
\end{aligned}$$

2. Proof that $A' \subseteq A^-$:

$$\begin{aligned}
A' &\triangleq \{x \in X | x \in (A \setminus \{x\})^-\} && \text{by definition of } A'\text{: Definition 1.5 page 14}\\
&\subseteq \{x \in X | x \in A^-\} && \text{by Theorem 1.16 page 18}\\
&= A^-
\end{aligned}$$

3. Proof that $A = A^\circ \implies A$ is *open*: by Proposition 1.3 page 14.

4. Proof that $A = A^\circ \impliedby A$ is *open*:

$$\begin{aligned}
A^\circ &\triangleq \bigcup \left\{U \in 2^X | U \subseteq A \text{ and } U \text{ is open}\right\} && \text{by Definition 1.4 page 14}\\
&= A && \text{by “}A \text{ is } \textit{open}\text{” hypothesis}
\end{aligned}$$

[31] McCarty (1967) pages 90–91 ⟨IV.1 THEOREM⟩, Aliprantis and Burkinshaw (1998) PAGE 59

5. Proof that $A = A^- \implies A$ is *closed*: by Proposition 1.3 page 14.

6. Proof that $A = A^- \impliedby A$ is *closed*:

$$\begin{aligned} A^- &\triangleq \bigcap \left\{ D \in 2^X | A \subseteq D \text{ and } D \text{ is closed} \right\} && \text{by Definition 1.4 page 14} \\ &= A && \text{by "} A \text{ is } \textit{closed} \text{" hypothesis} \end{aligned}$$

7. Proof that $\mathit{cls}\,A$ is the smallest *closed* set containing A:

$$\begin{aligned} A^- &\triangleq \bigcap \left\{ B \in 2^X | B \text{ is } \textit{closed} \text{ and } A \subseteq B \right\} && \text{by definition of } A^-\text{: Definition 1.4 page 14} \\ &\subseteq D && \text{because } D \text{ is } \textit{closed} \text{ by hypothesis} \end{aligned}$$

8. Proof that $\mathit{int}\,A$ is the largest *open* set contained in A:

$$\begin{aligned} U &\subseteq \bigcup \left\{ V \in 2^X | V \text{ is } \textit{closed} \text{ and } V \subseteq A \right\} && \text{by definition of } \bigcap \\ &\triangleq A^\circ && \text{by definition of } A^\circ\text{: Definition 1.4 page 14} \end{aligned}$$

✏

Theorem 1.13 (Kuratowski closure properties). [32] *Let* (X, T) *be a* TOPOLOGICAL SPACE.

THM						
1.	$\varnothing^-$	$=$	$\varnothing$		(NORMALIZED)	*and*
2.	A	$\subseteq$	A^-	$\forall A \in 2^X$	(EXTENSIVE)	*and*
3.	$(A^-)^-$	$=$	A^-	$\forall A \in 2^X$	(IDEMPOTENT)	*and*
4.	$(A \cup B)^-$	$=$	$A^- \cup B^-$	$\forall A, B \in 2^X$	(ADDITIVE).	

✎PROOF:

1. Proof that $\varnothing^- = \varnothing$:

$$\begin{aligned} &\varnothing \text{ is } \textit{closed} && \text{by Theorem 1.3 page 6} \\ &\implies \varnothing^- = \varnothing && \text{by Lemma 1.1 page 15} \end{aligned}$$

2. Proof that $A \subseteq A^-$: by Lemma 1.1 page 15

3. Proof that $(A^-)^- = A^-$:

$$\begin{aligned} (A^-)^- &\triangleq \left(\bigcap \left\{ D \in 2^X | (A^-) \subseteq D \text{ and } D \text{ is closed} \right\} \right) && \text{by Definition 1.4} \\ &= A^- && \text{because } (A^-) \text{ is closed by Proposition 1.3} \end{aligned}$$

4. Proof that $A^- \cup B^- = (A \cup B)^-$:

$$\begin{aligned} A^- \cup B^- &= (A^- \cup B^-)^- && \text{by Theorem 1.3 (page 6) } A^- \cup B^- \text{ is } \textit{closed} \\ & && \text{and by Lemma 1.1 (page 15)} \\ &\supseteq (A \cup B)^- && \text{by Lemma 1.1 page 15} \end{aligned}$$

$$\begin{aligned} A^- \cup B^- &\subseteq \left[\underbrace{(A \cup B)^-}_{A \subseteq (A \cup B)^-} \right]^- \cup \left[\underbrace{(A \cup B)^-}_{B \subseteq (A \cup B)^-} \right]^- && \text{because } A \subseteq (A \cup B)^- \text{ and } B \subseteq (A \cup B)^- \\ &= (A \cup B)^- \cup (A \cup B)^- && \text{by item (3)} \\ &\triangleq (A \cup B)^- && \text{by Theorem A.5 page 273} \end{aligned}$$

[32] Kelley (1955) page 43 ⟨1.8 THEOREM⟩, Davis (2005) PAGE 45, Thron (1966) PAGES 21–22, Hausdorff (1937) PAGE 258, Kuratowski (1922) PAGE 182, Riesz (1906)

Theorem 1.14. *Let* (X,T) *be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3)*.

1.	$\varnothing^\circ$	$=$	$\varnothing$		(NORMALIZED)	and
2.	A°	$\subseteq$	A	$\forall A\in 2^X$	(EXTENSIVE)	and
3.	$(A^\circ)^\circ$	$=$	A°	$\forall A\in 2^X$	(IDEMPOTENT)	and
4.	$(A\cup B)^\circ$	$=$	$A^\circ \cap B^\circ$	$\forall A,B\in 2^X$	(ADDITIVE).	

✎PROOF:

1. Proof that $\varnothing^\circ = \varnothing$:

$$\begin{aligned} &\varnothing \text{ is } \textit{open} && \text{by Definition 1.1 page 3}\\ &\implies \varnothing^\circ = \varnothing && \text{by Lemma 1.1 page 15}\end{aligned}$$

2. Proof that $A \supseteq A^\circ$: by Lemma 1.1 page 15.

3. Proof that $(A^\circ)^\circ = A^\circ$:

$$\begin{aligned} (A^\circ)^\circ &\triangleq \bigcup \{U\in 2^X | U \subseteq A^\circ \text{ and } U \text{ is open}\} && \text{by Definition 1.4 page 14}\\ &= A^\circ && \text{by Proposition 1.3 page 14}\end{aligned}$$

4. Proof that $A^\circ \cap B^\circ = (A\cap B)^\circ$:

$$\begin{aligned} A^\circ \cap B^\circ &= (A^\circ \cap B^\circ)^\circ && \text{by Definition 1.1 (page 3) and Lemma 1.1 (page 15)}\\ &\subseteq (A\cap B)^\circ && \text{by Lemma 1.1 page 15}\end{aligned}$$

$$\begin{aligned} A^\circ \cap B^\circ &\supseteq \Bigg[\underbrace{(A\cap B)^\circ}_{A\supseteq (A\cap B)^\circ}\Bigg]^\circ \cap \Bigg[\underbrace{(A\cap B)^\circ}_{B\supseteq (A\cap B)^\circ}\Bigg]^\circ && \text{because } A\supseteq (A\cap B)^\circ \text{ and } B \supseteq (A\cap B)^\circ\\ &= (A\cap B)^\circ \cap (A\cap B)^\circ && \text{by item (3)}\\ &\triangleq (A\cap B)^\circ && \text{by Theorem A.5 page 273}\end{aligned}$$

Theorem 1.15. [33] *Let* (X,T) *be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3)*.

1.	$\varnothing'$	$=$	$\varnothing$		(NORMALIZED)	and
2.	A'	$\subseteq$	A^-	$\forall A\in 2^X$	(EXTENSIVE)	and
3.	$(A\cup B)'$	$=$	$A'\cup B'$	$\forall A,B\in 2^X$	(ADDITIVE).	

✎PROOF:

1. Proof that $\varnothing' = \varnothing$:

$$\begin{aligned} \varnothing' &= \{x\in X | x \in (x\in \varnothing\setminus\{x\})^-\} && \text{by definition of } A'\text{: Definition 1.5 page 14}\\ &= \{x \in X | x\in (x\in\varnothing)^-\} && \text{by definition of } \varnothing\end{aligned}$$

[33] Mukherjee (2005) page 32 ⟨2.2.14 Theorem⟩, Murdeshwar (1990) page 48 ⟨1.24 Exercises (19)⟩

2. Proof that $A' \subseteq A^-$: by Lemma 1.1 page 15.

3. Proof that $A' \cup B' = (A \cup B)'$:

$$\begin{aligned}
A' \cup B' &\triangleq \{x \in X | x \in (x \in A \backslash \{x\})^-\} \cup \{x \in X | x \in (x \in B \backslash \{x\})^-\} && \text{by Definition 1.5 page 14}\\
&= \{x \in X | x \in (x \in A \backslash \{x\})^- \text{ or } x \in (x \in B \backslash \{x\})^-\} && \text{by definition of } \cup\\
&= \{x \in X | x \in (x \in A \backslash \{x\})^- \cup (x \in B \backslash \{x\})^-\} && \text{by definition of } \cup\\
&= \left\{x \in X | x \in \left[x \in (A \cup B) \backslash \{x\}\right]^-\right\} && \text{by Theorem 1.13 page 16}\\
&\triangleq (A \cup B)' && \text{by Definition 1.5 page 14}
\end{aligned}$$

✏

Theorem 1.16. [34] *Let A^- be the* CLOSURE *and A° the* INTERIOR (Definition 1.4 page 14) *of a set A on the topological space (X, T).*

THM
$$A \subseteq B \implies \left\{\begin{array}{lllll} 1. & A^- & \subseteq & B^- & (\text{ISOTONE}) \text{ and}\\ 2. & A^\circ & \subseteq & B^\circ & (\text{ISOTONE}) \text{ and}\\ 3. & A' & \subseteq & B' & (\text{ISOTONE}) \end{array}\right\} \quad \forall A, B \in 2^X$$

✎PROOF:

$$\begin{aligned}
A^- &\subseteq A^- \cup B^-\\
&= (A \cup B)^- && \text{by Theorem 1.13 page 16 } (\textit{additivity})\\
&= B^- && \text{by } A \subseteq B \text{ hypothesis}
\end{aligned}$$

$$\begin{aligned}
A^\circ &\triangleq \bigcup \{U \in 2^X | U \subseteq A \text{ and } U \text{ is open}\} && \text{by Definition 1.4 page 14}\\
&\subseteq \bigcup \{U \in 2^X | U \subseteq B \text{ and } U \text{ is open}\} && \text{by } A \subseteq B \text{ hypothesis}\\
&\triangleq B^\circ && \text{by Definition 1.4 page 14}
\end{aligned}$$

$$\begin{aligned}
A' &\triangleq \{x \in X | x \in (A \backslash \{x\})^-\} && \text{by Definition 1.5 page 14}\\
&\subseteq \{x \in X | x \in (B \backslash \{x\})^-\} && \text{by } A \subseteq B \text{ hypothesis}\\
&\triangleq B' && \text{by Definition 1.5 page 14}
\end{aligned}$$

✏

Theorem 1.17. [35] *Let (X, T) be a* TOPOLOGICAL SPACE (Definition 1.1 page 3).

THM

1.	A^{-c}	$=$	$A^{c\circ}$	$\forall A \in 2^X$	*(the complement of the closure is the interior of the complement)*
2.	$A^{\circ c}$	$=$	A^{c-}	$\forall A \in 2^X$	*(the complement of the interior is the closure of the complement)*
3.	A^-	$=$	$A^{c\circ c}$	$\forall A \in 2^X$	*(the complement of the interior of the complement is the closure)*
4.	A°	$=$	A^{c-c}	$\forall A \in 2^X$	*(the complement of the closure of the complement is the interior)*
5.	∂A	$=$	$\partial\left(A^c\right)$	$\forall A \in 2^X$	*(the boundary of the complement is the boundary)*

✎PROOF:

[34] McCarty (1967) page 90 ⟨IV.1 THEOREM⟩, Davis (2005) PAGE 45, Thron (1966) PAGE 42 ⟨THEOREM 8.1⟩, Kubrusly (2001) PAGE 116, Kuratowski (1922) PAGE 183
[35] Murdeshwar (1990) page 43 ⟨Theorem 1.16⟩, McCarty (1967) page 90 ⟨1 THEOREM⟩, Aliprantis and Burkinshaw (1998) PAGES 59–60

1. Proof that $A^\circ = \left(\left(A^c\right)^-\right)^c$:

$$\begin{aligned}
\left(\left(A^c\right)^-\right)^c &= \left(\bigcap\left\{D\in 2^X | A^c\subseteq D \text{ and } D \text{ is } \textit{closed}\right\}\right)^c && \text{by Definition 1.4}\\
&= \bigcup\left\{D^c\in 2^X | A^c\subseteq D \text{ and } D \text{ is } \textit{closed}\right\} && \text{by } \textit{de Morgan's law} \text{ (Theorem A.6 page 274)}\\
&= \bigcup\left\{D^c\in 2^X | D^c\subseteq A \text{ and } D^c \text{ is } \textit{open}\right\} && \text{by Definition 1.1 page 3}\\
&\triangleq A^\circ && \text{by Definition 1.4 page 14}
\end{aligned}$$

2. Proof that $A^- = \left(\left(A^c\right)^\circ\right)^c$:

$$\begin{aligned}
\left(\left(A^c\right)^\circ\right)^c &= \left(\bigcup\left\{U\in 2^X | U\subseteq A^c \text{ and } U \text{ is } \textit{open}\right\}\right)^c && \text{by Definition 1.4}\\
&= \bigcap\left\{U^c\in 2^X | U\subseteq A^c \text{ and } U \text{ is } \textit{open}\right\} && \text{by } \textit{de Morgan's law} \text{ (Theorem A.6 page 274)}\\
&= \bigcup\left\{U^c\in 2^X | A\subseteq U^c \text{ and } U^c \text{ is } \textit{closed}\right\} && \text{by Definition 1.1 page 3}\\
&\triangleq A^- && \text{by Definition 1.4 page 14}
\end{aligned}$$

3. Proof that $(A^\circ)^c = \left(A^c\right)^-$:

$$\begin{aligned}
(A^\circ)^c &= \left(\left(\left(A^c\right)^-\right)^c\right)^c && \text{by item (1)}\\
&= \left(A^c\right)^- && \text{by Theorem A.6 page 274}
\end{aligned}$$

4. Proof that $(A^-)^c = \left(A^c\right)^\circ$:

$$\begin{aligned}
(A^-)^c &= \left(\left(\left(A^c\right)^\circ\right)^c\right)^c && \text{by item (2)}\\
&= \left(A^c\right)^\circ && \text{by Theorem A.6 page 274}
\end{aligned}$$

5. Proof that $\partial A = \partial\left(A^c\right)$:

$$\begin{aligned}
\partial A &= A^- \cap \left(A^c\right)^- && \text{by Definition 1.5 page 14}\\
&= \left(A^c\right)^- \cap A^- && \text{by Theorem A.5 page 273}\\
&= \left(A^c\right)^- \cap \left(\left(A^c\right)^c\right)^- && \text{by Theorem A.6 page 274}\\
&= \partial\left(A^c\right) && \text{by Definition 1.5 page 14}
\end{aligned}$$

✏

Theorem 1.18. [36] *Let (X, T) be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3).*

THM

$$\begin{array}{llllllll}
1. & A^- &=& A^\circ\cup\partial A &=& A\cup\partial A &=& A\cup A' \quad \forall A\in 2^X\\
2. & \partial A &=& A^-\setminus A^\circ & & & & \forall A\in 2^X\\
3. & (A\setminus A^\circ)^\circ &=& \varnothing & & & & \forall A\in 2^X
\end{array}$$

✎PROOF:

1. Proof that $A^- = A\cup\partial A$:

(a) lemma: $A\cup\left(A^c\right)^- = X$

$$\begin{aligned}
A\cup\left(A^c\right)^- &\supseteq A\cup A^c && \text{by Lemma 1.1 page 15}\\
&= X && \text{by Theorem A.6 page 274}\\
A\cup\left(A^c\right)^- &\subseteq X
\end{aligned}$$

[36] Aliprantis and Burkinshaw (1998) pages 59–60, McCarty (1967) page 90 ⟨1 THEOREM⟩, KUBRUSLY (2001) PAGE 116

(b) Proof that $A^- = A \cup \partial A$:

$$\begin{aligned}
A \cup \partial A &= A \cup \left[A^- \cap \left(A^c\right)^-\right] && \text{by Definition 1.5 page 14}\\
&= [A \cup A^-] \cap \left[A \cup \left(A^c\right)^-\right] && \text{by Theorem A.5 page 273}\\
&= [A \cup A^-] \cap X && \text{by item (1a)}\\
&= A^- \cap X && \text{by Lemma 1.1 page 15}\\
&= A^- && \text{by Theorem A.6 page 274}
\end{aligned}$$

2. Proof that $A^- = A^\circ \cup \partial A$:

$$\begin{aligned}
A^\circ \cup \partial A &= A^\circ \cup \left[A^- \cap \left(A^c\right)^-\right] && \text{by definition of } \partial A \text{ (Definition 1.5 page 14)}\\
&= [A^\circ \cup A^-] \cap \left[A^\circ \cup \left(A^c\right)^-\right] && \text{by Theorem A.5 page 273}\\
&= [A^\circ \cup A^-] \cap \left[A^\circ \cup (A^\circ)^c\right] && \text{by Theorem 1.17 page 18}\\
&= [A^\circ \cup A^-] \cap X && \text{by Theorem 1.17 page 18}\\
&= [A^\circ \cup A^-] && \text{by Theorem A.6 page 274}\\
&= A^- && \text{by Lemma 1.1 page 15}
\end{aligned}$$

3. Proof that $A^- = A \cup A'$:

(a) Proof that $A \cup A' \subseteq A^-$:

$$\begin{aligned}
A \cup A' &\triangleq A \cup \{x \in X | x \in (A \setminus \{x\})^-\} && \text{by definition of } A'\text{: Definition 1.5 page 14}\\
&\subseteq A \cup \{x \in X | x \in A^-\} && \text{by Theorem 1.16 page 18}\\
&= A \cup A^- &&\\
&\subseteq A^- \cup A^- && \text{by Lemma 1.1 page 15}\\
&= A^- && \text{by Theorem A.5 page 273}
\end{aligned}$$

(b) Proof that $A^- \supseteq A \cup A'$:

$$\begin{aligned}
x \notin A \cup A' &\implies x \notin A \cup \{x \in X | x \in (A \setminus \{x\})^-\} && \text{by definition of } A'\text{: Definition 1.5 page 14}\\
&\implies x \notin \{x \in X | x \in (A \setminus \{x\})^-\} &&\\
&\implies x \notin \{x \in X | x \in (A)^-\} &&\\
&\iff x \notin A^- &&\\
&\implies A^- \subseteq A \cup A' &&
\end{aligned}$$

4. Proof that $\partial A = A^- \setminus A^\circ$:

$$\begin{aligned}
A^- \setminus A^\circ &= A^- \cap (A^\circ)^c && \text{by Theorem A.1 page 261}\\
&= A^- \cap \left[((A^c)^-)^c\right]^c && \text{by Theorem 1.17}\\
&= A^- \cap (A^c)^- && \text{by } \textit{idempotent} \text{ property (Theorem A.6 page 274)}\\
&= \partial A && \text{by Definition 1.5 page 14}
\end{aligned}$$

5. Proof that $(A \setminus A^\circ)^\circ = \emptyset$:

$$\begin{aligned}
(A \setminus A^\circ)^\circ &= \left[A \cap (A^\circ)^c\right]^\circ && \text{by Theorem A.1 page 261}\\
&= \left[\left[\left(A \cap (A^\circ)^c\right)^c\right]^-\right]^c && \text{by Theorem 1.17}\\
&= \left[\left(A^c \cup A^\circ\right)^-\right]^c && \text{by } \textit{idempotent} \text{ property (Theorem A.6 page 274)}\\
&= \left[\left(A^c\right)^- \cup (A^\circ)^-\right]^c && \text{by Theorem 1.13 page 16}\\
&= \left(\left(A^c\right)^-\right)^c \cap ((A^\circ)^-)^c && \text{by } \textit{de Morgan's law} \text{ (Theorem A.6 page 274)}\\
&= A^\circ \cap ((A^\circ)^-)^c && \text{by Theorem 1.17}\\
&= \emptyset && \text{because } A^\circ \subseteq (A^\circ)^- \text{ by Lemma 1.1 page 15}
\end{aligned}$$

Proposition 1.4. [37] *Let (X, T) be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3). Let A° be the* INTERIOR, *∂A be the* BOUNDARY, *and A^e be the* EXTERIOR *of a set A.*

PRP
$$X = \underbrace{A^\circ \cup \partial A \cup A^e}_{\text{partition of } X} \qquad \forall A \in 2^X$$

✎PROOF:

$$\begin{aligned}
A^\circ \cup \partial A \cup A^e &= A^\circ \cup \left(A^- \cap A^{c-}\right) \cup A^{c\circ} && \text{by Definition 1.5 page 14}\\
&= \left[(A^\circ \cup A^-) \cap \left(A^\circ \cup A^{c-}\right)\right] \cup A^{c\circ} && \text{by Theorem A.5 page 273}\\
&= \left[(A^-) \cap \left(A^\circ \cup A^{c-}\right)\right] \cup A^{c\circ} && \text{because } A^\circ \subseteq A^-\text{: Theorem 1.19 page 21}\\
&= \left[A^- \cap \left(A^\circ \cup A^{\circ c}\right)\right] \cup A^{c\circ} && \text{by Theorem 1.17 page 18}\\
&= [A^- \cap X] \cup A^{c\circ} && \text{by Theorem A.6 page 274}\\
&= A^- \cup A^{c\circ} && \text{by Theorem A.6 page 274}\\
&= A^- \cup A^{-c} && \text{by Theorem 1.17 page 18}\\
&= X && \text{by Theorem A.6 page 274}
\end{aligned}$$

Theorem 1.19. [38] *Let A^- be the* CLOSURE, *A° the* INTERIOR, *∂A the* BOUNDARY, *and A' the* DRIVED SET *of a set A in a topological space (X, T). Let 2^X be the* POWER SET *of X.*

THM

1. $A^\circ \subseteq A \subseteq A^- \qquad \forall A \in 2^X.$
2. $A' \subseteq A^- \qquad \forall A \in 2^X.$
3. $A = A^\circ \iff A \text{ is OPEN} \iff A \cap \partial A = \varnothing \qquad \forall A \in 2^X.$
4. $A = A^- \iff A \text{ is CLOSED} \iff A \cap \partial A = \varnothing \qquad \forall A \in 2^X.$
5. $A = A^- \iff A \text{ is CLOSED} \iff A' \subseteq A \qquad \forall A \in 2^X.$

✎PROOF:

1. Proof that $A^\circ \subseteq A \subseteq A^-$: by Lemma 1.1 page 15
2. Proof that $A' \subseteq A^-$: by Lemma 1.1 page 15
3. Proof that $A = A^\circ \iff A$ is *open*: by Lemma 1.1 page 15
4. Proof that $A = A^- \iff A$ is *closed*: by Lemma 1.1 page 15
5. Proof that A is *open* $\implies A \cap \partial A = \varnothing$:

$$\begin{aligned}
A \cap \partial A &\triangleq A \cap \left(A^- \cap A^{c-}\right) && \text{by Definition 1.5 page 14}\\
&= A^\circ \cap \left(A^- \cap A^{c-}\right) && \text{by “}A\text{ is \textit{open}” hypothesis}\\
&= (A^\circ \cap A^-) \cap \left(A^\circ \cap A^{c-}\right) && \text{by Theorem A.5 page 273}\\
&= A^\circ \cap \left(A^\circ \cap A^{c-}\right) && \text{by Lemma 1.1}\\
&= A^\circ \cap A^{c-} && \text{by Theorem A.6}\\
&= A^\circ \cap A^{\circ c} && \text{by Theorem 1.17}\\
&= \varnothing && \text{by Theorem A.6}
\end{aligned}$$

[37] Haaser and Sullivan (1991) page 43
[38] Aliprantis and Burkinshaw (1998) page 59, McCarty (1967) page 90 ⟨IV.1 THEOREM⟩, KUBRUSLY (2001) PAGE 116

6. Proof that A is *open* $\Longleftarrow$ $A \cap \partial A = \varnothing$:

$$\begin{aligned}
\varnothing &= A \cap \partial A \\
&\triangleq A \cap \left(A^- \cap A^{c-}\right) && \text{by Definition 1.5 page 14} \\
&= A \cap A^{c-} && \text{by Lemma 1.1 page 15} \\
&= A \cap A^{\circ c} && \text{by Theorem 1.17 page 18} \\
\Longrightarrow \quad A &= A^{\circ} && \text{by Theorem A.6 page 274}
\end{aligned}$$

7. Proof that A is *closed* $\Longrightarrow$ $\partial A \subseteq A$:

$$\begin{aligned}
\partial A &\triangleq A^- \cap A^{c-} && \text{by Definition 1.5 page 14} \\
&= A \cap A^{c-} && \text{by "}A\text{ is } \textit{closed}\text{" hypothesis and Lemma 1.1 page 15} \\
&\subseteq A
\end{aligned}$$

8. Proof that A is *closed* $\Longleftarrow$ $\partial A \subseteq A$:

$$\begin{aligned}
A^- &= A \cup \partial A && \text{by Theorem 1.18 page 19} \\
&= A && \text{by "}\partial A \subseteq A\text{ hypothesis} \\
&\Longrightarrow A \text{ is } \textit{closed} && \text{by Theorem 1.19 page 21}
\end{aligned}$$

9. Proof that $A = A^-$ $\Longrightarrow$ $A' \subseteq A$:

$$\begin{aligned}
A' &\subseteq A^- && \text{by Lemma 1.1 page 15} \\
&= A && \text{by } A = A^- \text{ hypothesis}
\end{aligned}$$

10. Proof that $A = A^-$ $\Longleftarrow$ $A' \subseteq A$:

$$\begin{aligned}
A^- &= A \cup A' && \text{by Theorem 1.18 page 19} \\
&\subseteq A \cup A && \text{by } A' \subseteq A \text{ hypothesis} \\
&\subseteq A && \text{by Theorem A.5 page 273}
\end{aligned}$$

✏

A weakened form of the closure properties of Theorem 1.13 (page 16) can be used to define a topology (next theorem).

Theorem 1.20 (Kuratowski closure axioms). [39] *Let* f *be a set function on* 2^X.

THM

$$\left\{\begin{array}{lrcllll}
1. & \mathsf{f}(\varnothing) & = & \varnothing & & (\text{NORMALIZED}) & \text{and} \\
2. & A & \subseteq & \mathsf{f}(A) & \forall A \in 2^X & (\text{EXTENSIVE}) & \text{and} \\
3. & \mathsf{f}(\mathsf{f}(A)) & \subseteq & \mathsf{f}(A) & \forall A \in 2^X & & \text{and} \\
4. & \mathsf{f}(A \cup B) & = & \mathsf{f}(A) \cup \mathsf{f}(B) & \forall A, B \in 2^X & (\text{ADDITIVE}). &
\end{array}\right\}
\Longrightarrow
\left\{\begin{array}{l}
(X, \boldsymbol{T}(\mathsf{f})) \text{ is a topological space where} \\
\boldsymbol{T}(\mathsf{f}) \triangleq \left\{A \in 2^X \,|\, \mathsf{f}(A^c) = A^c\right\}
\end{array}\right\}$$

Lemma 1.2. [40] *Let* $(X, \boldsymbol{T})$ *be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3)*.

LEM

$$\underbrace{\{x \in A^-\}}_{x \text{ is ADHERENT to } A} \iff \underbrace{\{A \cap U \neq \varnothing \quad \forall x \in U \in \boldsymbol{T}\}}_{\text{every open set containing } x \text{ meets } A}$$

[39] Thron (1966) pages 42–43, Murdeshwar (1990) pages 45–46
[40] Kubrusly (2001) page 115 ⟨Proposition 3.25⟩

✎PROOF:

1. Proof that $x \in A^- \implies A \cap U \neq \varnothing$:

$$\begin{aligned}
\{x \in U \text{ and } A \cap U = \varnothing\} &\implies \{x \notin U^c \text{ and } A \subseteq U^c\} \\
&\implies A^- \subseteq U^c && \text{by Lemma 1.1 page 15} \\
&\implies x \notin A^- && \text{because } x \in U \iff x \notin U^c \\
&\implies A \cap U \neq \varnothing && \text{because "} x \notin A^- \text{" contradicts "} x \in A^- \text{" hypothesis}
\end{aligned}$$

2. Proof that $x \in A^- \impliedby A \cap U \neq \varnothing$:

$$\begin{aligned}
x \notin A^- &\implies x \in \underbrace{A^{-c}}_{open} && \text{by definition of } A^-\text{: Definition 1.4 page 14} \\
&\implies \varnothing \neq A^- \cap \underbrace{A^{-c}}_{\textit{open} \text{ set containing } x} && \text{by right hypothesis} \\
&\qquad = \left(A^{-c}\right)^c \cap A^{-c} \\
&\qquad = \varnothing && \text{(contradiction)} \\
&\implies x \in A^-
\end{aligned}$$

✏

1.3 Supported topological properties

Definition 1.6. [41] *Let* (X, T) *be a* TOPOLOGICAL SPACE (Definition 1.1 page 3).

DEF: *A set* Y *is* ***dense*** *in* X *if* $Y^- = X$.

Definition 1.7. [42] *Let* (X, T) *be a* TOPOLOGICAL SPACE (Definition 1.1 page 3).

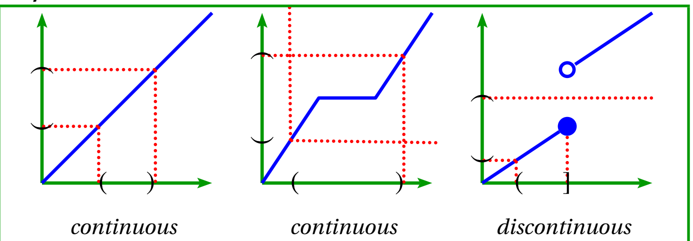
DEF: *The set* X *is* ***separable*** *if it contains a* COUNTABLE DENSE *subset.*

Definition 1.8. [43] *Let* (X, T_x) *and* (Y, T_y) *be topological spaces. Let* f *be a function in* Y^X.

DEF: *A function* $\mathsf{f} \in Y^X$ *is* ***continuous*** *if for every* OPEN SET $U \in T_y$, $\mathsf{f}^{-1}(U)$ *is also* OPEN.
A function is ***discontinuous*** *if it is not* CONTINUOUS.
The ***set of all continuous functions*** $\mathscr{C}(X, Y)$ *in the function space* Y^X *is*
$$\mathscr{C}(X, Y) \triangleq \{\mathsf{f} \in Y^X | \mathsf{f} \text{ is CONTINUOUS in } X\}.$$

Example 1.16.

EX:



[41] Murdeshwar (1990) page 248 ⟨2.21 Theorem and Definition⟩, Joshi (1983) page 133 ⟨(5.1.6) Definition⟩
[42] Murdeshwar (1990) page 248 ⟨16.1 Definition⟩, Joshi (1983) page 133 ⟨(6.1.3) Definition⟩
[43] Davis (2005) page 34

Definition 1.8 (previous definition) defines continuity using open sets. Continuity can alternatively be defined using closed sets or closure (next theorem).

Theorem 1.21. [44] *Let* $(X, \mathbf{T})$ *and* $(Y, \mathbf{S})$ *be topological spaces. Let* f *be a function in* Y^X.

THM

The following are equivalent:

1. f *is* CONTINUOUS $\iff$
2. B *is closed in* $(Y, \mathbf{S}) \implies \mathsf{f}^{-1}(B)$ *is closed in* $(X, \mathbf{T})$ $\quad \forall B \in 2^Y \iff$
3. $\mathsf{f}(A^-) \subseteq \mathsf{f}(A)^-$ $\quad \forall A \in 2^X \iff$
4. $\mathsf{f}^{-1}(B)^- \subseteq \mathsf{f}^{-1}(B^-)$ $\quad \forall B \in 2^Y$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned} B \text{ is } closed &\iff B^c \text{ is } open && \text{by definition of a } closed\ set \text{ (Definition 1.1 page 3)} \\ &\implies \mathsf{f}^{-1}(B^c) \text{ is } open && \text{by (1) and Definition 1.8 page 23} \\ &\iff \left[\mathsf{f}^{-1}(B)\right]^c \text{ is } open && \text{because } \mathsf{f}^{-1}(B^c) = \left[\mathsf{f}^{-1}(B)\right]^c \\ &\iff \mathsf{f}^{-1}(B) \text{ is } closed && \text{by definition of a } closed\ set \text{ (Definition 1.1 page 3)} \end{aligned}$$

2. Proof that (2) $\implies$ (3):

(a) lemma: Proof that $\mathsf{f}^{-1}[\mathsf{f}(A)^-]$ is *closed*:

$$\begin{aligned} &\mathsf{f}(A)^- \text{ is } closed && \text{by Proposition 1.3 page 14} \\ &\implies \mathsf{f}^{-1}[\mathsf{f}(A)^-] \text{ is } closed && \text{by (2)} \end{aligned}$$

(b) lemma: Proof that $A \subseteq \mathsf{f}^{-1}[\mathsf{f}(A)^-]$:

$$\begin{aligned} A &\subseteq \mathsf{f}^{-1}[\mathsf{f}(A)] && \text{by result from function theory} \\ &\subseteq \mathsf{f}^{-1}[\mathsf{f}(A)^-] && \text{by Lemma 1.1 page 15} \end{aligned}$$

(c) Proof that (2) $\implies$ (3):

$$\begin{aligned} \mathsf{f}(A^-) &\subseteq \mathsf{f}\left(\mathsf{f}^{-1}[\mathsf{f}(A)^-]^-\right) && \text{by item (2b)} \\ &= \mathsf{f}\left(\mathsf{f}^{-1}[\mathsf{f}(A)^-]\right) && \text{by item (2b)} \\ &\subseteq \mathsf{f}(A)^- && \text{by result from function theory} \end{aligned}$$

3. Proof that (3) $\implies$ (4):

$$\begin{aligned} \mathsf{f}^{-1}(B)^- &\subseteq \mathsf{f}^{-1}\mathsf{f}\left[\mathsf{f}^{-1}(B)^-\right] && \text{by result from function theory} \\ &\subseteq \mathsf{f}^{-1}\left(\left[\mathsf{f}\mathsf{f}^{-1}(B)\right]^-\right) && \text{by result from function theory} \\ &\subseteq \mathsf{f}^{-1}(B^-) && \text{by result from function theory} \end{aligned}$$

[44] McCarty (1967) pages 91–92 ⟨IV.2 THEOREM⟩

4. Proof that (4) $\Longrightarrow$ (1):

$$\begin{aligned}
U \text{ is } \textit{open} &\Longrightarrow U^c \text{ is } \textit{closed} && \text{by Definition 1.1 page 3}\\
&\Longrightarrow \mathsf{f}^{-1}(U^c) = \mathsf{f}^{-1}(U^{c-}) && \text{by Lemma 1.1 page 15}\\
&\qquad\qquad\;\; \supseteq \mathsf{f}^{-1}(U^c)^- && \text{by (4)}\\
&\qquad\qquad\;\; \supseteq \mathsf{f}^{-1}(U^c) && \text{by Lemma 1.1 page 15}\\
&\Longrightarrow \mathsf{f}^{-1}(U^c) = \mathsf{f}^{-1}(U^c)^-\\
&\Longleftrightarrow \mathsf{f}^{-1}(U^c) \text{ is } \textit{closed} && \text{by Lemma 1.1 page 15}\\
&\Longleftrightarrow \left[\mathsf{f}^{-1}(U^c)\right]^c \text{ is } \textit{open} && \text{by Definition 1.1 page 3}\\
&\Longleftrightarrow \mathsf{f}^{-1}(U) \text{ is } \textit{open} && \text{because } \mathsf{f}^{-1}(U) = \left[\mathsf{f}^{-1}(U^c)\right]^c\\
&\Longrightarrow \mathsf{f} \text{ is } \textit{continuous} && \text{by Definition 1.8 page 23}
\end{aligned}$$

✏

1.4 Neighborhoods

Definition 1.9. [45] *Let* $(X, \boldsymbol{T})$ *be a* TOPOLOGICAL SPACE (Definition 1.1 page 3). *Let* A° *be the* INTERIOR *of a set* A *(Definition 1.4 page 14).*

DEF

A set $N_x \in 2^X$ *is a* ***neighborhood*** *of an element* $x \in X$ *if* $x \in N_x{}^\circ$.

A set N_x *is an* ***open neighborhood*** *of an element* $x \in X$ *if* N_x *is a neighborhood of* x *and* $N_x \in \boldsymbol{T}$.

Proposition 1.5. [46] *Let* $(X, \boldsymbol{T})$ *be a* TOPOLOGICAL SPACE (Definition 1.1 page 3).

PRP

$$\left\{\begin{array}{l}\text{A set } N_x \text{ is a } \textbf{neighborhood}\\ \text{of an element } x \in X\end{array}\right\} \Longleftrightarrow \left\{\exists U \in \boldsymbol{T} \quad \text{such that} \quad x \in U \subseteq N_x\right\}$$

Example 1.17. Example 1.6 (page 7) lists the 29 topologies on a set $X \triangleq \{x, y, z\}$. These topologies are listed next along with their open and closed neighborhoods of the element $x \in X$:

topologies on $\{x, y, z\}$	open nbhds. of x	not open nbhds.
$\boldsymbol{T}_{00} = \{\varnothing, X\}$	X	
$\boldsymbol{T}_{01} = \{\varnothing, \{x\}, X\}$	$\{x\}$, X	$\{x, y\}$, $\{x, z\}$
$\boldsymbol{T}_{02} = \{\varnothing, \{y\}, X\}$	X	$\{x, y\}$
$\boldsymbol{T}_{04} = \{\varnothing, \{z\}, X\}$	X	$\{x, z\}$
$\boldsymbol{T}_{10} = \{\varnothing, \{x, y\}, X\}$	$\{x, y\}$, X	
$\boldsymbol{T}_{20} = \{\varnothing, \{x, z\}, X\}$	$\{x, z\}$, X	
$\boldsymbol{T}_{40} = \{\varnothing, \{y, z\}, X\}$	X	
$\boldsymbol{T}_{11} = \{\varnothing, \{x\}, \{x, y\}, X\}$	$\{x\}$, $\{x, y\}$, X	$\{x, z\}$
$\boldsymbol{T}_{21} = \{\varnothing, \{x\}, \{x, z\}, X\}$	$\{x\}$, $\{x, z\}$, X	$\{x, y\}$
$\boldsymbol{T}_{41} = \{\varnothing, \{x\}, \{y, z\}, X\}$	$\{x\}$, X	$\{x, y\}$, $\{x, z\}$
$\boldsymbol{T}_{12} = \{\varnothing, \{y\}, \{x, y\}, X\}$	$\{x, y\}$, X	
$\boldsymbol{T}_{22} = \{\varnothing, \{y\}, \{x, z\}, X\}$	$\{x, z\}$, X	$\{x, y\}$
$\boldsymbol{T}_{42} = \{\varnothing, \{y\}, \{y, z\}, X\}$	X	$\{x, y\}$
$\boldsymbol{T}_{14} = \{\varnothing, \{z\}, \{x, y\}, X\}$	$\{x, y\}$, X	$\{x, z\}$
$\boldsymbol{T}_{24} = \{\varnothing, \{z\}, \{x, z\}, X\}$	$\{x, z\}$, X	
$\boldsymbol{T}_{44} = \{\varnothing, \{z\}, \{y, z\}, X\}$	X	$\{x, z\}$
$\boldsymbol{T}_{31} = \{\varnothing, \{x\}, \{x, y\}, \{x, z\}, X\}$	$\{x\}$, $\{x, y\}$, $\{x, z\}$, X	

[45] Murdeshwar (1990) page 88 ⟨3.1 Definition⟩, Davis (2005) page 43 ⟨Definition 4.7⟩
[46] Murdeshwar (1990) page 88 ⟨3.1 Definition⟩,

$T_{52} = \{\varnothing, \{y\}, \{x,y\}, \{x,z\}, X\}$	$\{x,y\}, \{x,z\}, X$	
$T_{64} = \{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{x,z\}, X$	
$T_{13} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$	$\{x\}, \{x,y\}, X$	$\{x,z\}$
$T_{25} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$	$\{x\}, \{x,z\}, X$	$\{x,y\}$
$T_{46} = \{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$	X	$\{x,y\}, \{x,z\}$
$T_{33} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	$\{x\}, \{x,y\}, \{x,z\}, X$	
$T_{53} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{x\}, \{x,y\}, X$	$\{x,z\}$
$T_{35} = \{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	$\{x\}, \{x,y\}, X$	$\{x,z\}$
$T_{65} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{x\}, \{x,z\}, X$	$\{x,y\}$
$T_{56} = \{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	$\{x,y\}, X$	$\{x,z\}$
$T_{66} = \{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{x,z\}, X$	$\{x,y\}$
$T_{77} = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	$\{x\}, \{x,y\}, \{x,z\}, X$	

Definition 1.10. [47] *Let (X, T) be a* TOPOLOGICAL SPACE (Definition 1.1 page 3).

DEF

*A set $Y \subseteq X$ is **disconnected** if there exists $A, B \subseteq X$ such that*

1. $A \cup B = Y$ *and*
2. $A \cap B = \varnothing$.

*In this case, Y is said to be **disconnected** by the sets A and B, and the pair A, B is a **separation** of Y.*
*If a set is not disconnected, then it is **connected**.*

Definition 1.11. [48]

DEF

A TOPOLOGICAL SPACE *is a **Hausdorff space** if*

$$\forall x, y \in X,\ \exists N \in \mathbf{N}x \text{ and } M \in \mathbf{N}y \quad \text{such that} \quad N \cap M = \varnothing.$$

Definition 1.12. [49] *Let (X, T) be a topological space and A, B, $\{A_i\}$, $\{B_i\}$, $\{M_i\} \subseteq X$.*

DEF

*A sequence $(A_i)_{i \in I}$ is a **cover** of a set A in the topological space (X, T) if*

$$A \subseteq \bigcup_{i \in I} A_i.$$

*A sequence $(B_i)_{i \in J}$ is a **subcover** of set A with respect to a cover $(A_i)_{i \in I}$ if*

$$\{B_i\}_{i \in J} \subsetneq \{A_i\}_{i \in I}.$$

*A sequence $(M_i)_{i \in K}$ is a **minimal cover** of A if $(M_i)_{i \in K}$ is a cover and $(M_i)_{i \in K \setminus \{n\}}$ is not a cover.*
*A cover $(A_i)_{i \in I}$ is a **proper cover** of A if A is not a member.*
*A cover $(A_i)_{i \in I}$ is a **open cover** of A if it consists entirely of open sets.*

Definition 1.13. [50]

DEF

*A set $A \subseteq X$ is **compact** in the topological space (X, T) if any open cover of A has a finite subcover.*

[47] Munkres (2000) page 148 ⟨§Connected Spaces⟩, Dieudonné (1969) page 67, Carothers (2000) page 78
[48] Aliprantis and Burkinshaw (1998) page 60, Hausdorff (1914)
[49] Aliprantis and Burkinshaw (1998) page 48
[50] Aliprantis and Burkinshaw (1998) page 62

CHAPTER 2

DISTANCE SPACES

A *distance space* (Definition 2.1 page 27) can be defined as a *metric space* (Definition 3.1 page 33) without the *triangle inequality* constraint. Much of the material in this chapter about *distance spaces* is standard in *metric spaces*. However, this chapter revisits what may commonly be associated with metric spaces to demonstrate "how far we can go", and can't go, without the *triangle inequality*.

2.1 Fundamental structure of distance spaces

Definition 2.1. [1]

DEF

A function d *in the set* $\mathbb{R}^{X \times X}$ *is a* ***distance*** *if*

1. $\mathsf{d}(x,y) \geq 0 \quad \forall x,y \in X$ *(NON-NEGATIVE) and*
2. $\mathsf{d}(x,y) = 0 \iff x = y \quad \forall x,y \in X$ *(NONDEGENERATE) and*
3. $\mathsf{d}(x,y) = \mathsf{d}(y,x) \quad \forall x,y \in X$ *(SYMMETRIC)*

The pair (X, d) *is a* ***distance space*** *if* d *is a* DISTANCE *on a set* X.
A DISTANCE *is also called a* ***dissimilarity***.

In a *metric space* (Definition 3.1 page 33), it is sometimes useful to know the maximum distance between any two points in the set. This maximum distance is called the *diameter* of the set (next definition).

Remark 2.1. The *diameter* is an example of a broader class of functions called *set functions*.[2] Some *distance spaces* (Definition 2.1 page 27) and all *metric spaces* (Definition 3.1 page 33) induce *topological spaces* (Definition 1.1 page 3). However the *set function set diameter* and the related property of *boundedness* (Definition 2.3 page 28) are fundamentally *distance space* concepts, not topological ones.[3]

Definition 2.2. [4] *Let* (X, d) *be a* DISTANCE SPACE *and* 2^X *be the* POWER SET *of* X *(Definition A.1 page 259).*

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[1] Menger (1928) page 76 ⟨"Abstand *a b* definiert ist…" (distance from *a* to *b* is defined as…")⟩, Wilson (1931) page 361 ⟨§1., "distance", "semi-metric space"⟩, Blumenthal (1938) page 38, Blumenthal (1953) page 7 ⟨"DEFINITION 5.1. A distance space is called semimetric provided…"⟩, Galvin and Shore (1984) page 67 ⟨"distance function"⟩, Laos (1998) page 118 ⟨"distance space"⟩, Khamsi and Kirk (2001) page 13 ⟨"semimetric space"⟩, Bessenyei and Pales (2014) page 2 ⟨"semimetric space"⟩, Deza and Deza (2014) page 3 ⟨"**distance** (or **dissimilarity**)"⟩

[2] Pap (1995) page 7, Hahn and Rosenthal (1948), Choquet (1954)

[3] in *metric space*: Munkres (2000) page 121

[4] in *metric space*: Hausdorff (1937) page 166, Copson (1968) page 23, Michel and Herget (1993) page 267, Molchanov (2005) page 389

DEF: *The* **diameter** $\operatorname{diam} A$ *of a set* $A \in 2^X$ *is* $\operatorname{diam} A \triangleq \begin{cases} 0 & \text{for } A = \varnothing \\ \sup\{\mathsf{d}(x,y) \,|\, x, y \in A\} & \text{otherwise} \end{cases}$

Definition 2.3. [5] *Let* (X, d) *be a* DISTANCE SPACE. *Let* 2^X *be the* POWER SET *of* X.

DEF: *A set* A *is* **bounded** *in* (X, d) *if* $A \in 2^X$ *and* $\operatorname{diam} A < \infty$.

Remark 2.2. Let $(\!(x_n)\!)_{n\in\mathbb{Z}}$ be a *sequence* in a *distance space* (X, d). The *distance space* (X, d) does not necessarily have all the nice properties that a *metric space* (Definition 3.1 page 33) has. In particular, note the following:

REM:

1.	d is a *distance* in (X, d)	$\not\Longrightarrow$	d is *continuous* in (X, d)	(Example 9.5 page 139).
2.	B is an *open ball* in (X, d)	$\not\Longrightarrow$	B is *open* in (X, d)	(Example 9.4 page 138).
3.	$\boldsymbol{B}$ is the set of all *open balls* in (X, d)	$\not\Longrightarrow$	$\boldsymbol{B}$ is a *base* for a topology on X	(Example 9.4 page 138).[6]
4.	$(\!(x_n)\!)$ is *convergent* in (X, d)	$\not\Longrightarrow$	limit is *unique*	(Example 9.3 page 137).
5.	$(\!(x_n)\!)$ is *convergent* in (X, d)	$\not\Longrightarrow$	$(\!(x_n)\!)$ is *Cauchy* in (X, d)	(Example 9.4 page 138).

2.2 Open sets in distance spaces

Definition 2.4. [7] *Let* (X, d) *be a* DISTANCE SPACE *(Definition 2.1 page 27). Let* $\mathbb{R}^+$ *be the* SET OF POSITIVE REAL NUMBERS.

DEF:
An **open ball** *centered at* x *with radius* r *is the set* $\mathsf{B}(x, r) \triangleq \{y \in X \,|\, \mathsf{d}(x,y) < r\}$.
A **closed ball** *centered at* x *with radius* r *is the set* $\overline{\mathsf{B}}(x, r) \triangleq \{y \in X \,|\, \mathsf{d}(x,y) \le r\}$.

Definition 2.5. *Let* (X, d) *be a* DISTANCE SPACE. *Let* $X \setminus A$ *be the* SET DIFFERENCE *of* X *and a set* A.

DEF:
A set U *is* **open** *in* (X, d) *if* $U \in 2^X$ *and for every* x *in* U *there exists* $r \in \mathbb{R}^+$ *such that* $\mathsf{B}(x, r) \subseteq U$.
A set U *is an* **open set** *in* (X, d) *if* U *is* OPEN *in* (X, d). *A set* D *is* **closed** *in* (X, d) *if* $(X \setminus D)$ *is* OPEN.
A set D *is a* **closed set** *in* (X, d) *if* D *is* CLOSED *in* (X, d).

Theorem 2.1. [8] *Let* (X, d) *be a* DISTANCE SPACE. *Let* N *be any (finite) positive integer. Let* Γ *be a* SET *possibly with an uncountable number of elements.*

THM:
1. X *is* OPEN.
2. $\varnothing$ *is* OPEN.
3. *each element in* $\left\{U_n \,\middle|\, n=1,2,\ldots,N\right\}$ *is* OPEN $\Longrightarrow \bigcap_{n=1}^{N} U_n$ *is* OPEN.
4. *each element in* $\left\{U_\gamma \in 2^X \,\middle|\, \gamma \in \Gamma\right\}$ *is* OPEN $\Longrightarrow \bigcup_{\gamma\in\Gamma} U_\gamma$ *is* OPEN.

✎PROOF:

1. Proof that X is *open* in (X, d):

[5] in *metric space*: Thron (1966) page 154 ⟨definition 19.5⟩, Bruckner et al. (1997) page 356
[6] Heath (1961) page 810 ⟨THEOREM⟩, Galvin and Shore (1984) page 71 ⟨2.3 LEMMA⟩
[7] in *metric space*: Aliprantis and Burkinshaw (1998) page 35
[8] in *metric space*: Dieudonné (1969) pages 33–34, Rosenlicht (1968) page 39

(a) By definition of *open set* (Definition 2.5 page 28), X is *open* $\iff \forall x \in X \quad \exists r \quad \text{such that} \quad \mathsf{B}(x, r) \subseteq X$.
(b) By definition of *open ball* (Definition 2.4 page 28), it is always true that $\mathsf{B}(x, r) \subseteq X$ in (X, d).
(c) Therefore, X is *open* in (X, d).

2. Proof that $\varnothing$ is *open* in (X, d):
   (a) By definition of *open set* (Definition 2.5 page 28), $\varnothing$ is *open* $\iff \forall x \in X \quad \exists r \quad \text{such that} \quad \mathsf{B}(x, r) \subseteq \varnothing$.
   (b) By definition of *empty set* $\varnothing$, this is always true because no x is in $\varnothing$.
   (c) Therefore, $\varnothing$ is *open* in (X, d).

3. Proof that $\bigcup U_\gamma$ is *open* in (X, d):
   (a) By definition of *open set* (Definition 2.5 page 28), $\bigcup U_\gamma$ is *open* $\iff \forall x \in \bigcup U_\gamma \quad \exists r \quad \text{such that} \quad \mathsf{B}(x, r) \subseteq \bigcup U_\gamma$.
   (b) If $x \in \bigcup U_\gamma$, then there is at least one $U \in \bigcup U_\gamma$ that contains x.
   (c) By the left hypothesis in (4), that set U is open and so for that x, $\exists r \quad \text{such that} \quad \mathsf{B}(x, r) \subseteq U \subseteq \bigcup U_\gamma$.
   (d) Therefore, $\bigcup U_\gamma$ is *open* in (X, d).

4. Proof that U_1 and U_2 are *open* $\implies U_1 \cap U_2$ is *open*:
   (a) By definition of *open set* (Definition 2.5 page 28), $U_1 \cap U_2$ is *open* $\iff \forall x \in U_1 \cap U_2 \quad \exists r \quad \text{such that} \quad \mathsf{B}(x, r) \subseteq U_1 \cap U_2$.
   (b) By the left hypothesis above, U_1 and U_2 are *open*; and by the definition of *open sets* (Definition 2.5 page 28), there exists r_1 and r_2 such that $\mathsf{B}(x, r_1) \subseteq U_1$ and $\mathsf{B}(x, r_2) \subseteq U_2$.
   (c) Let $r \triangleq \min\{r_1, r_2\}$. Then $\mathsf{B}(x, r) \subseteq U_1$ and $\mathsf{B}(x, r) \subseteq U_2$.
   (d) By definition of *set intersection* $\cap$ then, $\mathsf{B}(x, r) \subseteq U_1 \cap U_2$.
   (e) By definition of *open set* (Definition 2.5 page 28), $U_1 \cap U_2$ is *open*.

5. Proof that $\bigcap_{n=1}^{N} U_n$ is *open* (by induction):
   (a) Proof for $N = 1$ case: $\bigcap_{n=1}^{N} U_n = \bigcap_{n=1}^{1} U_n = U_1$ is *open* by hypothesis.
   (b) Proof that N case $\implies N + 1$ case:

$$\begin{aligned}
\bigcap_{n=1}^{N+1} U_n &= \left(\bigcap_{n=1}^{N} U_n\right) \cap U_{N+1} && \text{by property of } \bigcap \\
&\implies \textit{open} && \text{by "}N\text{ case" hypothesis and (4) lemma page 29}
\end{aligned}$$

✏

Corollary 2.1. *Let* (X, d) *be a* DISTANCE SPACE.

COR: *The set* $T \triangleq \{U \in 2^X | U \text{ is OPEN in } (X, \mathsf{d})\}$ *is a* TOPOLOGY *on* X, *and* (X, T) *is a* TOPOLOGOGICAL SPACE.

✎PROOF: This follows directly from the definition of an *open* set (Definition 2.5 page 28), Theorem 2.1 (page 28), and the definition of *topology* (Definition 1.1 page 3). ✏

Of course it is possible to define a very large number of topologies even on a finite set with just a handful of elements;[9] and it is possible to define an infinite number of topologies even on a *linearly*

[9]For a finite set X with n elements, there are 29 topologies on X if $n = 3$, 6942 topologies on X if $n = 5$, and and 8,977,053,873,043 (almost 9 trillion) topologies on X if $n = 10$. References: Sloane (2014) ⟨http://oeis.org/A000798⟩, Brown and Watson (1996) page 31, Comtet (1974) page 229, Comtet (1966), Chatterji (1967) page 7, Evans et al. (1967), Krishnamurthy (1966) page 157

ordered infinite set like the *real line* $(\mathbb{R}, \leq)$.[10] Be that as it may, Definition 2.6 (next definition) defines a single but convenient *topological space* in terms of a *distance space*. Note that every *metric space* conveniently and naturally induces a *topological space* because the *open balls* of the metric space form a *base* for the *topology*. This is not the case for all distance spaces. But if the open balls of a *distance space* are all *open*, then those open balls induce a topology (next theorem).[11]

Definition 2.6. *Let* (X, d) *be a* DISTANCE SPACE.

DEF *The set* $\boldsymbol{T} \triangleq \{U \in 2^X | U \text{ is OPEN in } (X, \mathsf{d})\}$ *is the* ***topology induced by*** (X, d) ***on*** X. *The pair* $(X, \boldsymbol{T})$ *is called the* ***topological space induced by*** (X, d).

For any *distance space* (X, d), no matter how strange, there is guaranteed to be at least one *topological space induced by* (X, d)—and that is the *indiscrete topological space* (Example 1.1 page 4) because for any distance space (X, d), $\varnothing$ and X are *open sets* in (X, d) (Theorem 2.1 page 28).

Theorem 2.2. *Let* $\boldsymbol{B}$ *be the set of all* OPEN BALLS *in a* DISTANCE SPACE (X, d).

THM $\{\text{every OPEN BALL in } \boldsymbol{B} \text{ is OPEN}\} \iff \{\boldsymbol{B} \text{ is a BASE for a TOPOLOGY}\}$

✎PROOF:

$\boxed{\text{every } \textit{open ball} \text{ in } \boldsymbol{B} \text{ is } \textit{open}}$

$\implies$ for every x in $B_y \in \boldsymbol{B}$ there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq B_y$ — by definition of *open* (Definition 2.5 page 28)

$\implies \left\{\begin{array}{l}\text{for every } x \in X \text{ and for every } B_y \in \boldsymbol{B} \text{ containing } x, \\ \text{there exists } B_x \in \boldsymbol{B} \text{ such that } \quad x \in B_x \subseteq B_y.\end{array}\right\}$ — because $\forall (x, r) \in X \times \mathbb{R}^+, \mathsf{B}(x, r) \subseteq X$

$\implies \boxed{\boldsymbol{B} \text{ is a } \textit{base} \text{ for } \boldsymbol{T}}$ — by Theorem 1.4 page 8

$\implies \left\{\begin{array}{l}\text{for every } x \in X \text{ and for every } U \subseteq \boldsymbol{T} \text{ containing } x, \\ \text{there exists } B_x \in \boldsymbol{B} \text{ such that } \quad x \in B_x \subseteq U.\end{array}\right\}$ — by Theorem 1.4 page 8

$\implies \left\{\begin{array}{l}\text{for every } x \in X \text{ and for every } B_y \in \boldsymbol{B} \subseteq \boldsymbol{T} \text{ containing } x, \\ \text{there exists } B_x \in \boldsymbol{B} \text{ such that } \quad x \in B_x \subseteq B_y.\end{array}\right\}$ — by definition of *base* (Definition 1.2 page 8)

$\implies \left\{\begin{array}{l}\text{for every } x \in B_y \in \boldsymbol{B} \subseteq \boldsymbol{T}, \\ \text{there exists } B_x \in \boldsymbol{B} \text{ such that } \quad x \in B_x \subseteq B_y.\end{array}\right\}$

$\implies \boxed{\text{every } \textit{open ball} \text{ in } \boldsymbol{B} \text{ is } \textit{open}}$ — by definition of *open* (Definition 2.5 page 28)

⇒

[10] For examples of topologies on the real line, see the following: Adams and Franzosa (2008) page 31 ⟨"six topologies on the real line"⟩, Salzmann et al. (2007) pages 64–70 ⟨Weird topologies on the real line⟩, Murdeshwar (1990) page 53 ⟨"often used topologies on the real line"⟩, Joshi (1983) pages 85–91 ⟨§4.2 Examples of Topological Spaces⟩

[11] *metric space*: Definition 3.1 page 33; *open ball*: Definition 2.4 page 28; *base*: Definition 1.2 page 8; *topology*: Definition 1.1 page 3; not all open balls are open in a distance space: Example 9.3 (page 137) and Example 9.4 (page 138);

2.3 More about distance spaces

More material concerning distance spaces is available within this text:

- Sequences in distance spaces Section 9.3 (page 134)
- Distance Spaces with Power Triangle Inequalities CHAPTER 13 (page 185)

CHAPTER 3

METRIC SPACES

3.1 Algebraic structure

> ❝ *The Epicureans are wont to ridicule this theorem, saying it is evident even to an ass and needs no proof; it is as much the mark of an ignorant man, they say, to require persuasion of evident truths as to believe what is obscure without question. ...That the present theorem is known to an ass they make out from the observation that, if straw is placed at one extremity of the sides, an ass in quest of provender will make his way along the one side and not by way of the two others.* ❞
>
> Proclus Lycaeus (412 – 485 AD), Greek philosopher, commenting on the Epicureans opinion regarding the *triangle inequality* property. [1]

A *metric space* is simply a set together with a "*distance*" function, which is called the *metric* of the *metric space* (Definition 3.1 page 33) (next definition). With a metric on a set, we can measure the distance between points in the set.

Definition 3.1. [2] *Let X be a set and $\mathbb{R}^{\vdash}$ the set of non-negative real numbers.*

DEF

*A function $\mathsf{d} \in \mathbb{R}^{\vdash X \times X}$ is a **metric** on X if*

1.	$\mathsf{d}(x,y) \geq 0$	$\forall x,y \in X$	(NON-NEGATIVE)	*and*
2.	$\mathsf{d}(x,y) = 0 \iff x = y$	$\forall x,y \in X$	(NONDEGENERATE)	*and*
3.	$\mathsf{d}(x,y) = \mathsf{d}(y,x)$	$\forall x,y \in X$	(SYMMETRIC)	*and*
4.	$\mathsf{d}(x,y) \leq \mathsf{d}(x,z) + \mathsf{d}(z,y)$	$\forall x,y,z \in X$	(SUBADDITIVE/TRIANGLE INEQUALITY).[3]	

*A **metric space** is the pair (X, d).*

Actually, it is possible to significantly simplify the definition of a metric to an equivalent statement requiring only half as many conditions. These equivalent conditions (a "*characterization*") are stated in Theorem 3.1 (next).

[1] Lycaeus (circa 450) page 251

[2] Dieudonné (1969) page 28, Copson (1968) page 21, Hausdorff (1937) page 109, Fréchet (1928), Fréchet (1906) page 30

[3] Euclid (circa 300BC) ⟨Book I Proposition 20⟩

Theorem 3.1 (metric characterization). [4] *Let* d *be a function in* $(\mathbb{R}^{\vdash})^{X\times X}$.

THM			
$\mathsf{d}(x,y)$ *is a metric*	$\iff$	$\left\{\begin{array}{lrclll} \textit{1.} & \mathsf{d}(x,y) & = & 0 \iff x=y & \forall x,y\in X & \textit{and} \\ \textit{2.} & \mathsf{d}(x,y) & \le & \mathsf{d}(z,x)+\mathsf{d}(z,y) & \forall x,y,z\in X & \end{array}\right.$	

✎PROOF:

1. Proof that $[\mathsf{d}(x,y)$ is a metric$] \implies [(1)$ and $(2)]$:

   1a. Proof that $\mathsf{d}(x,y)=0 \iff x=y$: by left hypothesis 2 ($\mathsf{d}(x,y)$ is *nondegenerate*)

   1b. Proof that $\mathsf{d}(x,y)\le \mathsf{d}(z,x)+\mathsf{d}(z,y)$:

$$\begin{aligned}
\mathsf{d}(x,y) &\le \mathsf{d}(x,z)+\mathsf{d}(z,y) && \text{by right hypothesis 4 (triangle inequality)}\\
&= \mathsf{d}(z,x)+\mathsf{d}(z,y) && \text{by right hypothesis 3 (commutative)}
\end{aligned}$$

2. Proof that $[\mathsf{d}(x,y)$ is a metric$] \impliedby [(1)$ and $(2)]$:

   2a. Proof that $\mathsf{d}(x,y)\ge 0$:

$$\begin{aligned}
0 &= \frac{1}{2}\cdot 0 \\
&= \frac{1}{2}\,\mathsf{d}(y,y) && \text{by right hypothesis 1}\\
&= \frac{1}{2}\,\mathsf{d}(y,z)\Big|_{z=y} \\
&\le \frac{1}{2}\,[\mathsf{d}(x,y)+\mathsf{d}(x,z)]_{z=y} && \text{by right hypothesis 2}\\
&= \frac{1}{2}\,[\mathsf{d}(x,y)+\mathsf{d}(x,y)] \\
&= \mathsf{d}(x,y)
\end{aligned}$$

   2b. Proof that $\mathsf{d}(x,y)=0 \iff x=y$: by right hypothesis 1

   2c. Proof that $\mathsf{d}(x,y)=\mathsf{d}(y,x)$:

$$\begin{aligned}
\mathsf{d}(x,y)|_{z=y} &\le [\mathsf{d}(z,x)+\mathsf{d}(z,y)]_{z=y} && \text{by right hypothesis 2}\\
&= \mathsf{d}(y,x)+\cancelto{0}{\mathsf{d}(y,y)} \\
&= \mathsf{d}(y,x) && \text{by right hypothesis 1}\\
\mathsf{d}(y,x)|_{z=x} &\le [\mathsf{d}(z,y)+\mathsf{d}(z,x)]_{z=x} && \text{by right hypothesis 2}\\
&= \mathsf{d}(x,y)+\cancelto{0}{\mathsf{d}(x,x)} \\
&= \mathsf{d}(x,y) && \text{by right hypothesis 1}
\end{aligned}$$

   2d. Proof that $\mathsf{d}(x,y)\le \mathsf{d}(x,z)+\mathsf{d}(z,y)$:

$$\begin{aligned}
\mathsf{d}(x,y) &\le \mathsf{d}(z,x)+\mathsf{d}(z,y) && \text{by right hypothesis 2}\\
&= \mathsf{d}(x,z)+\mathsf{d}(z,y) && \text{by result 2c}
\end{aligned}$$

✏

The *triangle inequality* property stated in the definition of metrics (Definition 3.1 page 33) axiomatically endows a metric with an upper bound. Lemma 3.1 (next) demonstrates that there is a complementary lower bound similar in form to the triangle-inequality upper bound.

[4] Busemann (1955a) page 3, Michel and Herget (1993) page 264, Giles (1987) page 18

Lemma 3.1. [5] *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33). Let* $|\cdot|$ *be the* ABSOLUTE VALUE *function (Definition F.4 page 342).*

LEM

$$\begin{array}{lrcll} 1. & |\mathsf{d}(x,p) - \mathsf{d}(p,y)| & \le & \mathsf{d}(x,y) & \forall x,y,p \in X \\ 2. & \mathsf{d}(x,p) - \mathsf{d}(p,y) & \le & \mathsf{d}(x,y) & \forall x,y,p \in X \end{array}$$

✎PROOF:

1. Proof that $|\mathsf{d}(x,p) - \mathsf{d}(p,y)| \le \mathsf{d}(x,y)$:

$$\begin{aligned} |\mathsf{d}(x,p) - \mathsf{d}(p,y)| &\le |\mathsf{d}(x,y) + \mathsf{d}(y,p) - \mathsf{d}(p,y)| && \text{by } \textit{subadditive} \text{ property (Definition 3.1 page 33)} \\ &= |\mathsf{d}(x,y) + \mathsf{d}(p,y) - \mathsf{d}(p,y)| && \text{by } \textit{symmetry} \text{ property of metrics (Definition 3.1 page 33)} \\ &= |\mathsf{d}(x,y) + 0| \\ &= \mathsf{d}(x,y) && \text{by } \textit{non-negative} \text{ property of metrics (Definition 3.1 page 33)} \end{aligned}$$

2. Proof that $\mathsf{d}(x,p) \ge \mathsf{d}(p,y) \implies \mathsf{d}(x,p) - \mathsf{d}(p,y) \le \mathsf{d}(x,y)$:

$$\begin{aligned} \mathsf{d}(x,p) - \mathsf{d}(p,y) &= |\mathsf{d}(x,p) - \mathsf{d}(p,y)| && \text{by left hypothesis and definition of } |\cdot| \\ &\le \mathsf{d}(x,y) && \text{by item (1)} \end{aligned}$$

3. Proof that $\mathsf{d}(x,p) \le \mathsf{d}(p,y) \implies \mathsf{d}(x,p) - \mathsf{d}(p,y) \le \mathsf{d}(x,y)$:

$$\begin{aligned} |\mathsf{d}(x,p) - \mathsf{d}(p,y)| &\le 0 && \text{by left hypothesis} \\ &\le \mathsf{d}(x,y) && \text{by } \textit{non-negative} \text{ property of metrics (Definition 3.1 page 33)} \end{aligned}$$

✏

The *triangle inequality* property stated in the definition of metrics (Definition 3.1 page 33) can be extended from two to any finite number of metrics (next).

Proposition 3.1. [6] *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33) and* $(\!| x_n \in X |\!)_1^N$ *an* N-TUPLE *(Definition 9.1 page 131) on* X.

PRP

$$\mathsf{d}(x_1, x_N) \le \sum_{n=1}^{N-1} \mathsf{d}(x_n, x_{n+1}) \qquad \forall N \in \mathbb{N} \setminus 1$$

✎PROOF: Proof by induction:

Proof that the $\{N = 2 \text{ case}\}$ is true:

$$\mathsf{d}(x_1, x_2) \le \sum_{n=1}^{2-1} \mathsf{d}(x_n, x_{n+1})$$

Proof for that the $\{N \text{ case}\} \implies \{N+1 \text{ case}\}$:

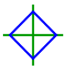
$$\begin{aligned} \mathsf{d}(x_1, x_{N+1}) &\le \mathsf{d}(x_1, x_N) + \mathsf{d}(x_N, x_{N+1}) && \text{by } \textit{subadditive} \text{ property (Definition 3.1 page 33)} \\ &\le \left(\sum_{n=1}^{N-1} \mathsf{d}(x_N, x_{N+1}) \right) + \mathsf{d}(x_N, x_{N+1}) && \text{by } \{N \text{ case}\} \text{ hypothesis} \\ &= \sum_{n=1}^{N} \mathsf{d}(x_n, x_{n+1}) \end{aligned}$$

✏

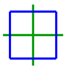
[5] Dieudonné (1969) page 28, Michel and Herget (1993) page 266
[6] Dieudonné (1969) page 28, Rosenlicht (1968) page 37

3.2 Open and closed balls

Open balls will often "appear" different in different metric spaces. Some examples include the following (Example 3.1 page 42):

 *taxi-cab metric*

 *Euclidean metric*

 *sup metric*

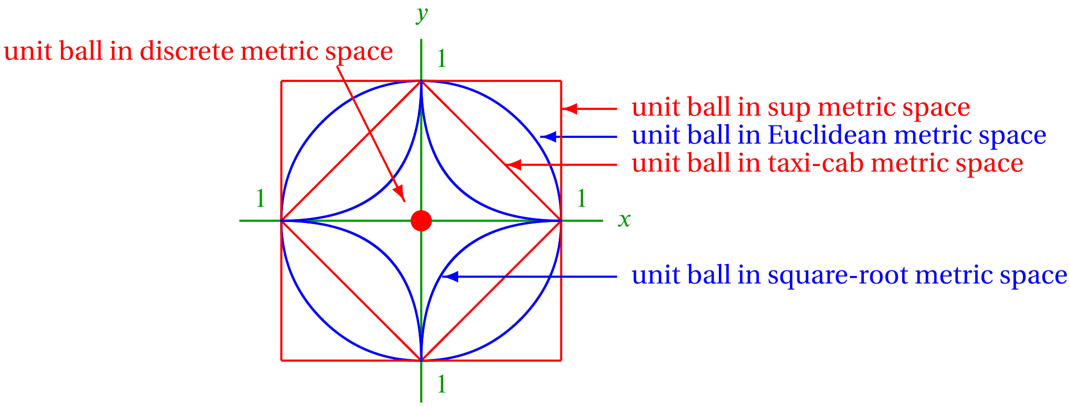


Figure 3.1: Balls on the set $\mathbb{R}^2$ using assorted metrics

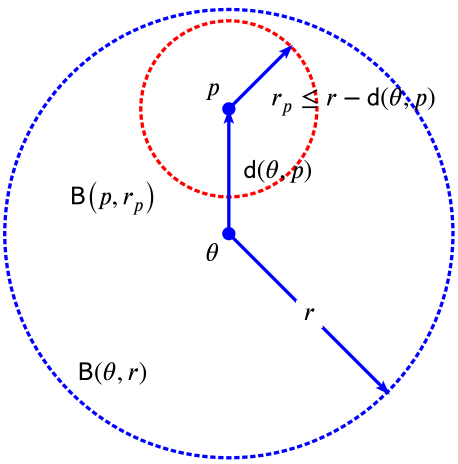


Figure 3.2: Every point in an open ball is contained in an open ball that is contained in the original open ball (Lemma 3.2 page 36)

Lemma 3.2 (next) demonstrates that every point in an open ball is contained in an open ball that is contained in the original open ball (see Figure 3.2 page 36 for an illustration).

Lemma 3.2. [7] *Let* B *be an* OPEN BALL *(Definition 2.4 page 28) in a* METRIC SPACE (X, d).

LEM

$$p \in \mathsf{B}(x, r) \iff \exists r_p \quad \text{such that} \quad \mathsf{B}(p, r_p) \subseteq \mathsf{B}(x, r)$$

[7] Rosenlicht (1968) pages 40–41, Aliprantis and Burkinshaw (1998) page 35

✎PROOF:

1. lemma: Proof that $p \in \mathsf{B}(x,r) \implies \exists r_p \in \mathbb{R}^+ \text{ such that } r_p < r - \mathsf{d}(\theta,p)$:

$$\begin{aligned}
p \in \mathsf{B}(x,r) &\iff \mathsf{d}(\theta,p) < r && \text{by definition of \textit{open ball} (Definition 2.4 page 28)}\\
&\iff 0 < r - \mathsf{d}(\theta,p) && \text{by property of real numbers}\\
&\implies \exists r_p \in \mathbb{R}^+ \text{ such that } 0 < r_p < r - \mathsf{d}(\theta,p) && \text{by property of real numbers}
\end{aligned}$$

2. Proof for ($\implies$) case:

$$\begin{aligned}
\mathsf{B}(p,r_p) &\triangleq \left\{x \in X | \mathsf{d}(p,x) < r_p \in \mathbb{R}^+\right\} && \text{by definition of \textit{open ball} (Definition 2.4 page 28)}\\
&\subseteq \{x \in X | \mathsf{d}(p,x) < r - \mathsf{d}(\theta,p)\} && \text{by left hypothesis and item (1)}\\
&= \{x \in X | \mathsf{d}(p,x) + \mathsf{d}(\theta,p) < r\} && \text{by property of real numbers}\\
&= \{x \in X | \mathsf{d}(\theta,p) + \mathsf{d}(p,x) < r\} && \text{by \textit{symmetry} of metrics (Definition 3.1 page 33)}\\
&\subseteq \{x \in X | \mathsf{d}(\theta,x) < r\} && \text{by \textit{subadditive} property (Definition 3.1 page 33),}\\
& && \quad \mathsf{d}(\theta,x) \le \mathsf{d}(\theta,p) + \mathsf{d}(p,x)
\end{aligned}$$

3. Proof for ($\impliedby$) case:

$$\begin{aligned}
p &= \{x \in X | \mathsf{d}(p,x) = 0\} && \text{by \textit{nondegenerate} property of metrics: Definition 3.1 page 33}\\
&\subseteq \left\{x \in X | \mathsf{d}(p,x) < r_p \in \mathbb{R}^+\right\} && \text{because } 0 < r_p\\
&\triangleq \mathsf{B}(p,r_p) && \text{by definition of \textit{open ball} (Definition 2.4 page 28)}\\
&\subseteq \mathsf{B}(x,r) && \text{by right hypothesis}
\end{aligned}$$

✏

3.3 Topological structure

3.3.1 Topologies induced by metrics

Theorem 3.2 (page 37) shows that in a *metric space* (Definition 3.1 page 33) (X,d), the metric d always induces a topology $\boldsymbol{T}$ on X. The set X together with topology $\boldsymbol{T}$ is a *topological space*. More specifically, the set of *open balls* in a metric space form a *base* for a *topological space*. Therefore, *every metric space* (Definition 3.1 page 33) *is* a topological space, and everything that is true of a topological space is also true for all *metric spaces*.

Theorem 3.2. *Let (X,d) be a* METRIC SPACE *(Definition 3.1 page 33).*

THM: *The set of all* OPEN BALLS *in (X,d) is a* BASE *for the topological space $(X,\boldsymbol{T})$ where*
$$\boldsymbol{T} \triangleq \left\{U \in 2^X | U \text{ is the union of balls in } (X,\mathsf{d})\right\}.$$

✎PROOF:

1. The set of all *open balls* in (X,d) is a *base* for $(X,\boldsymbol{T})$ by Lemma 3.2 (page 36) and Theorem 1.4 (page 8).
2. $\boldsymbol{T}$ is a topology on X by Definition 1.2 (page 8).

✏

3.3.2 Open and closed sets

Corollary 3.1 (next) identifies four fundamental properties of open sets in metric spaces. These properties are the same as those defining a topology (Definition 1.1 page 3).

Corollary 3.1. *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33).*

COR

1.				X	*is* OPEN.
2.				$\varnothing$	*is* OPEN.
3.	$\left\{U_n \,\middle|\, n=1,2,\ldots,N\right\}$	*are* OPEN	$\Longrightarrow$	$\displaystyle\bigcap_{n=1}^{N} U_n$	*is* OPEN.
4.	$\left\{U_\gamma \in 2^X \,\middle|\, \gamma \in \mathbb{R}\right\}$	*are* OPEN	$\Longrightarrow$	$\displaystyle\bigcup_{\gamma\in\Gamma} U_\gamma$	*is* OPEN.

✎PROOF:

1. The *metric space* (X, d) is a *topological space* by Theorem 3.2 (page 37).
2. The four properties are true for any topological space by Theorem 1.14 (page 17).

✏

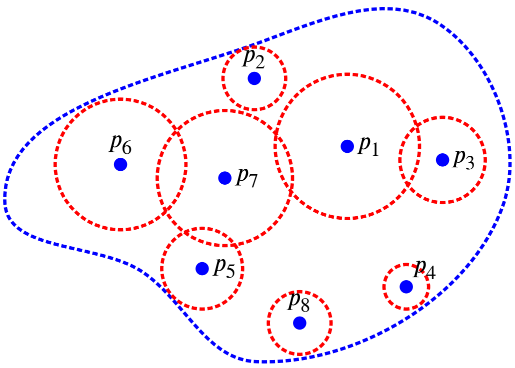


Figure 3.3: Every point in an open set is contained in an open ball that is contained in the original open set (Lemma 3.3 page 38)

Lemma 3.3 (next) demonstrates that every point in an open set is contained in an open ball that is contained in the original open set (see also Figure 3.3 page 38).

Lemma 3.3. *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33).*

LEM

$$\left\{U \in 2^X \text{ is open in } (X, \mathsf{d})\right\} \iff \left\{\begin{array}{ll}\forall x \in U,\ \exists r \in \mathbb{R}^+ & \text{such that} \\ \mathsf{B}(x, r) \subseteq U & \end{array}\right\}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned} U &= \bigcup\left\{\mathsf{B}(x_\gamma, r_\gamma) \,|\, \mathsf{B}(x_\gamma, r_\gamma) \subseteq U\right\} && \text{by left hypothesis and Theorem 3.2 page 37} \\ &\supseteq \mathsf{B}(x, r) && \text{because } x \text{ must be in one of those balls in } U \end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
U &= \bigcup \{x \in X | x \in U\} && \text{by definition of union operation } \bigcup \\
&= \bigcup \left\{\mathsf{B}(x,r) \,|\, x \in U \text{ and } \mathsf{B}(x,r) \subseteq U\right\} && \text{by right hypothesis} \\
&\implies U \text{ is open} && \text{by Theorem 3.2 page 37 and Corollary 3.1 page 38}
\end{aligned}$$

✏

Corollary 3.2. *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33). Let* N *be a finite* WHOLE NUMBER.

COR

1.				X	*is* CLOSED.
2.				$\varnothing$	*is* CLOSED.
3.	$\left\{D_\gamma \in 2^X \mid \gamma \in \mathbb{R}\right\}$	*are* CLOSED	$\implies$	$\bigcap_{\gamma\in\mathbb{R}} D_\gamma$	*is* CLOSED.
4.	$\left\{D_n \in 2^X \mid n = 1, 2, \ldots, N\right\}$	*are* CLOSED	$\implies$	$\bigcup_{n=1}^{N} D_n$	*is* CLOSED.

✎PROOF:

1. (X, d) is a *topological space* by Theorem 3.2 page 37.
2. The four properties are true of all topological spaces by Theorem 1.3 page 6.

✏

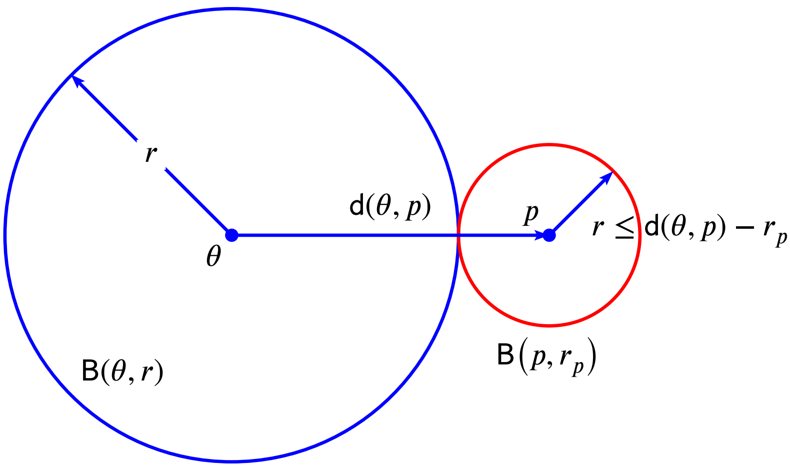


Figure 3.4: Every closed ball is a closed set (Proposition 3.2 page 39)

Proposition 3.2. [8] *Let* B *be an* OPEN BALL *and* $\overline{\mathsf{B}}(\cdot,\cdot)$ *a* CLOSED BALL *(Definition 2.4 page 28) in a metric space* (X, d).

PRP

Every OPEN BALL	$\mathsf{B}(x,r)$	*in* (X, d)	*is* OPEN	$\forall x\in X$ *and* $\forall r\in\mathbb{R}^+$.
Every CLOSED BALL	$\overline{\mathsf{B}}(x,r)$	*in* (X, d)	*is* CLOSED	$\forall x\in X$ *and* $\forall r\in\mathbb{R}^+$.

✎PROOF:

[8] Rosenlicht (1968) pages 40–41, Aliprantis and Burkinshaw (1998) page 35

1. Proof that every open ball is open:

$$\begin{aligned} &\text{The union of any set of open balls is open} && \text{by Theorem 3.2 page 37}\\ &\implies \text{the union of a set of just one open ball is open}\\ &\implies \text{every open ball is open.}\end{aligned}$$

2. lemma: $p \in \left(\overline{\mathsf{B}}(x, r)\right)^{\mathsf{c}} \implies r \le \mathsf{d}(\theta, p) - r_p$:

3. Proof that every closed ball is closed (see Figure 3.4 page 39 for illustration):

$$\begin{aligned}
\left(\overline{\mathsf{B}}(x,r)\right)^{\mathsf{c}} &\triangleq \{x \in X \mid \mathsf{d}(\theta,x) \le r\}^{\mathsf{c}} && \text{by definition of } \textit{closed ball} \text{ (Definition 2.4 page 28)}\\
&= \{x \in X \mid \mathsf{d}(\theta,x) > r\} && \text{by definition of set complement}\\
&\supseteq \left\{x \in X \mid \mathsf{d}(\theta,x) > \mathsf{d}(\theta,p) - r_p\right\} && \text{by item (2)}\\
&= \left\{x \in X \mid \mathsf{d}(\theta,x) - \mathsf{d}(\theta,p) > -r_p\right\} && \text{by property of real numbers}\\
&= \left\{x \in X \mid \mathsf{d}(\theta,p) - \mathsf{d}(\theta,x) < r_p\right\} && \text{by property of real numbers}\\
&= \left\{x \in X \mid \mathsf{d}(p,\theta) - \mathsf{d}(\theta,x) < r_p\right\} && \text{by } \textit{symmetric} \text{ property of metrics (Definition 3.1 page 33)}\\
&\supseteq \left\{x \in X \mid \mathsf{d}(p,x) < r_p\right\} && \text{by Lemma 3.1 page 35}\\
&\triangleq \mathsf{B}(p, r_p) && \text{by definition of } \textit{open ball} \text{ (Definition 2.4 page 28)}\\
&\iff \left(\overline{\mathsf{B}}(\theta,r)\right)^{\mathsf{c}} \text{ is } \textit{open} && \text{by Lemma 3.3 page 38}\\
&\iff \overline{\mathsf{B}}(\theta,r) \text{ is } \textit{closed} && \text{by definition of } \textit{closed set} \text{ (Definition 1.1 page 3)}
\end{aligned}$$

✏

In a metric space, all finite sets are *closed* (Proposition 3.3, next). This is *not* in general true for a topological space (Counterexample 3.1 page 41).

Proposition 3.3. [9] *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33).*

PRP

1.	$\{x \in X\}$	*(a single element set)*	*is* CLOSED *in* (X, d).
2.	$\{x_1, x_2, \ldots, x_n \in X\}$	*(set with finite number of elements)*	*is* CLOSED *in* (X, d).
3.	$\{x \in X \mid \mathsf{d}(\theta, x) = r,\ \theta \in X\}$	*(ring centered at θ with radius r)*	*is* CLOSED *in* (X, d).

✎PROOF:

1. Proof that any single element set is closed:
   (a) Let $\left(\!\left(\overline{\mathsf{B}}(x_n, r_n)\right)\!\right)_{n\in\mathbb{Z}}$ be a sequence of all the closed balls containing a.
   (b) Then $\{a\} = \bigcap_{n\in\mathbb{Z}} \overline{\mathsf{B}}(x_n, r_n)$.
   (c) By Proposition 3.2 page 39, every closed ball $\overline{\mathsf{B}}(\cdot,\cdot)$ is a closed set.
   (d) By Corollary 3.2 page 39, the infinite intersection of closed sets is also closed. So, $\bigcap_{n\in\mathbb{Z}} \overline{\mathsf{B}}(x_n, r_n)$ is closed.
   (e) Therefore, $\{x\}$ is closed.
2. Proof that any finite element set is closed:
   (a) By the previous result, any single element set $\{x\}$ is closed.

[9] Rosenlicht (1968) page 42

(b) By Corollary 3.2 (page 39), the finite union of closed sets is also closed.

(c) Therefore,

$$\{x_1, x_2, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\}$$

is closed.

3. Proof that any ring is closed:

(a) By Proposition 3.2 (page 39), the closed ball $\overline{\mathsf{B}}(\theta, r)$ is a closed set.

(b) By Proposition 3.2 (page 39), the open ball $\mathsf{B}(\theta, r)$ is an open set and by Definition 1.1 (page 3), its complement $(\mathsf{B}(\theta, r))^c$ is a closed set.

(c) By Corollary 3.2 (page 39), the intersection of the two closed sets $\overline{\mathsf{B}}(\theta, r) \cap \left((\mathsf{B}(\theta, r))^c\right)$ is also a closed set.

(d) Therefore, the ring is a closed set because

$$\underbrace{\{x \in X | \mathsf{d}(\theta, x) = r\}}_{\text{ring}} = \underbrace{\overline{\mathsf{B}}(\theta, r) \cap \left((\mathsf{B}(\theta, r))^c\right)}_{\text{intersection of two closed sets}}.$$

✎

Counterexample 3.1. Unlike *metric spaces* (Proposition 3.3 page 40), a finite set in a *topological space* (Definition 1.1 page 3) $(X, \boldsymbol{T})$ is *not* in general *closed* (Definition 1.4 page 14).

CNT The finite set $\{x\}$ is *not* closed in the topological space (a *Serpiński space*)

$$\Bigg(\underbrace{\{x, y\}}_{X}, \underbrace{\{\varnothing, \{x\}, \{x, y\}\}}_{\boldsymbol{T}}\Bigg).$$

✎Proof:

1. A set is *closed* if it is the complement of an open set (Definition 1.1 page 3).
2. The set $\{x\}$ is *not* the complement of any open set in the topology.
3. Therefore, $\{x\}$ is not closed.

✎

3.3.3 Equivalence and Order on metric spaces

Definition 3.2. [10] *Let* (X, d_1) *be a* METRIC SPACE *(Definition 3.1 page 33) that induces the* TOPOLOGY *(Definition 1.1 page 3)* $(X, \boldsymbol{T}_1)$ *and* (X, d_2) *be a* METRIC SPACE *that induces the* TOPOLOGY $(X, \boldsymbol{T}_2)$.

DEF d_1 *and* d_2 *are* ***equivalent*** *if*
$\boldsymbol{T}_1 = \boldsymbol{T}_2$.

[10] Davis (2005) page 20

Theorem 3.3. [11] *Let* $\{B_1(x,y)\}$ *be* OPEN BALLS (*Definition 2.4 page 28*) *on a* METRIC SPACE (X, d_1) *that induces the* TOPOLOGY (X, T_1) *and* $\{B_2(x,y)\}$ *be* OPEN BALLS *on a* METRIC SPACE (X, d_2) *that induces the* TOPOLOGY (X, T_2).

THM

$$\left.\begin{array}{ll} 1. & \exists\alpha > 0 \text{ such that } \mathsf{d}_1(x,y) \le \alpha \mathsf{d}_2(x,y) \\ & \forall x, y \in X \quad \textit{and} \\ 2. & U \textit{ is open in } (X, \mathsf{d}_1) \end{array}\right\} \implies \left\{\begin{array}{ll} 1. & U \textit{ is open in } (X, \mathsf{d}_2) \quad \textit{and} \\ 2. & T_2 \subseteq T_1 \end{array}\right.$$

✎PROOF:

1. Proof that U is open in (X, d_2):

$$\begin{array}{lll} & U \text{ is open in } (X, \mathsf{d}_1) & \text{by left hypothesis 2.} \\ \implies & \forall x \in U,\ \exists r > 0 \text{ such that } B_d(x,r) \subseteq X & \text{by Lemma 3.3 page 38} \\ \implies & \forall x \in U,\ \exists r > 0 \text{ such that } \{y \in X | \mathsf{d}_1(x,y) < r\} \subseteq X & \text{by Definition 2.4 page 28} \\ \implies & \forall x \in U,\ \exists r > 0 \text{ such that } \{y \in X | \mathsf{d}_1(x,y) < \alpha r\} \subseteq X & \\ \implies & \forall x \in U,\ \exists r > 0 \text{ such that } \{y \in X | \mathsf{d}_2(x,y) < r\} \subseteq X & \text{by left hypothesis 1.} \\ \implies & \forall x \in U,\ \exists r > 0 \text{ such that } B_2(x,y) \subseteq X & \text{by Definition 2.4 page 28} \\ \implies & U \text{ is open in } (X, \mathsf{d}_2) & \text{by Lemma 3.3 page 38} \end{array}$$

2. Proof that $T_2 \subseteq T_1$:
Because U is open in $(X, \mathsf{d}_1) \implies U$ is open in (X, d_2), (see above), then $T_2 \subseteq T_1$.

✏

Example 3.1. [12] Let R be a *commutative ring* and $|\cdot| \in R^R$ be the *absolute value* (Definition F.4 page 342) on R.

EX

The following *metric spaces* are all *equivalent* for any $n \in \mathbb{N}$:

1. $\left(R^n,\ \mathsf{d}_1(\boldsymbol{x},\boldsymbol{y}) \triangleq \sum_{i=1}^{n} |x_i - y_i|\right)$ (l_1-*metric* or *taxi-cab metric*)
2. $\left(R^n,\ \mathsf{d}_2(\boldsymbol{x},\boldsymbol{y}) \triangleq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}\right)$ (l_2-*metric* or *Euclidean metric*)
3. $\left(R^n,\ \mathsf{d}_\infty(\boldsymbol{x},\boldsymbol{y}) \triangleq \max\{|x_i - y_i|\,|\, i = 1, 2, \ldots n\}\right)$ (l_∞-*metric* or *sup metric*)

✎PROOF:

1. Proof that (R^n, d_1) and (R^n, d_2) are equivalent:

Let

$$\boldsymbol{z}^{(1)} \triangleq \begin{bmatrix} y_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \boldsymbol{z}^{(2)} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} \quad \boldsymbol{z}^{(3)} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ x_4 \\ x_5 \\ \vdots \\ x_n \end{bmatrix} \quad \boldsymbol{z}^{(k)} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \\ x_{k+1}, x_{k+2}, \vdots \\ x_n \end{bmatrix}$$

[11] Davis (2005) page 20
[12] Davis (2005) pages 20–21

$$\begin{aligned}
\mathsf{d}_2(\boldsymbol{x},\boldsymbol{y}) &\le \mathsf{d}_2(\boldsymbol{x},\boldsymbol{z}^{(1)}) + \mathsf{d}(\boldsymbol{z}^{(1)},\boldsymbol{y}) && \text{by definition of } \textit{metric} \text{ (Definition 3.1 page 33)}\\
&\le \mathsf{d}_2(\boldsymbol{x},\boldsymbol{z}^{(1)}) + \mathsf{d}(\boldsymbol{z}^{(1)},\boldsymbol{z}^{(2)}) + \mathsf{d}(\boldsymbol{z}^{(2)},\boldsymbol{y}) && \text{by definition of } \textit{metric} \text{ (Definition 3.1 page 33)}\\
&\le \mathsf{d}_2(\boldsymbol{x},\boldsymbol{z}^{(1)}) + \mathsf{d}(\boldsymbol{z}^{(1)},\boldsymbol{z}^{(2)}) + \mathsf{d}(\boldsymbol{z}^{(2)},\boldsymbol{z}^{(3)}) + \mathsf{d}(\boldsymbol{z}^{(3)},\boldsymbol{y}) && \text{by definition of } \textit{metric} \text{ (Definition 3.1 page 33)}\\
&\ \vdots\\
&\le \mathsf{d}_2(\boldsymbol{x},\boldsymbol{z}^{(1)}) + \mathsf{d}(\boldsymbol{z}^{(1)},\boldsymbol{z}^{(2)}) + \mathsf{d}(\boldsymbol{z}^{(2)},\boldsymbol{z}^{(3)}) + \cdots\\
&\qquad + \mathsf{d}(\boldsymbol{z}^{(n-2)},\boldsymbol{z}^{(n-1)}) + \mathsf{d}(\boldsymbol{z}^{(n-1)},\boldsymbol{y}) && \text{by definition of } \textit{metric} \text{ (Definition 3.1 page 33)}\\
&= \sqrt{\sum_{i=1}^{n} |x_i - z_i^{(1)}|^2} + \sqrt{\sum_{i=1}^{n} |z_i^{(1)} - z_i^{(2)}|^2} + \cdots\\
&\qquad + \sqrt{\sum_{i=1}^{n} |z_i^{(n-2)} - z_i^{(n-1)}|^2} + \sqrt{\sum_{i=1}^{n} |z_i^{(n-1)} - y_i|^2}\\
&= \sqrt{|x_1-y_1|^2} + \sqrt{|x_2-y_2|^2} + \sqrt{|x_3-y_3|^2} + \cdots\\
&\qquad + \sqrt{|x_{n-1}-y_{n-1}|^2} + \sqrt{|x_n-y_n|^2}\\
&= |x_1-y_1| + |x_2-y_2| + |x_3-y_3| + \cdots + |x_n-y_n|\\
&= \sum_{i=1}^{n} |x_i - y_i|\\
&= \mathsf{d}_1(\boldsymbol{x},\boldsymbol{y}) && \text{by definition of } \textit{metric} \text{ (Definition 3.1 page 33)}
\end{aligned}$$

By Theorem 3.3 (page 42), $\mathsf{d}_2(\boldsymbol{x},\boldsymbol{y}) \le \mathsf{d}_1(\boldsymbol{x},\boldsymbol{y})$ implies that (R^n, d_1) and (R^n, d_2) are equivalent.

2. Proof that (R^n, d_1) and (R^n, d_∞) are equivalent:

$$\begin{aligned}
\mathsf{d}_1(\boldsymbol{x},\boldsymbol{y}) &= \sum_{i=1}^{n} |x_i - y_i| && \text{by definition of } \textit{metric} \text{ (Definition 3.1 page 33)}\\
&\le n \cdot \max\left\{|x_i - y_i| \,|\, i = 1, 2, \ldots, N\right\}\\
&= n\,\mathsf{d}_\infty(\boldsymbol{x},\boldsymbol{y}) && \text{by definition of } \textit{metric} \text{ (Definition 3.1 page 33)}
\end{aligned}$$

By Theorem 3.3 (page 42), $\mathsf{d}_1(\boldsymbol{x},\boldsymbol{y}) \le n\mathsf{d}_\infty(\boldsymbol{x},\boldsymbol{y})$ implies that $(\mathbb{C}^n, \mathsf{d}_1)$ and $(\mathbb{C}^n, \mathsf{d}_\infty)$ are equivalent.

✎

3.3.4 Metrics induced by topologies

There are many topological spaces that are induced by metric spaces, and others that are not. A topology that is induced by a metric is called *metrizable* (next definition).

Definition 3.3. *Let* (X, T) *be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3).*

DEF	
D E F	*A* TOPOLOGY *is **metrizable** if it is induced by a* METRIC.

Example 3.2. [13] Let $\mathscr{T}(X)$ be the set of *topologies* (Definition 1.1 page 3) on a set X and 2^X the *power set* (Definition A.1 page 259) on X.

EX			
E X	$\{\varnothing, X\}$	is called the *indiscrete topology*.	It is *not* metrizable.
	2^X	is called the *discrete topology*.	It *is* metrizable.

[13] Munkres (2000) page 77, Kubrusly (2011) page 107 ⟨Example 3.J⟩, Steen and Seebach (1978) pages 42–43 ⟨II.4⟩, DiBenedetto (2002) page 18

Example 3.3. [14]

EX

The **Sierpiński space** $(X, \mathbf{T})$ is a topological space with topology $\mathbf{T} \triangleq \{\varnothing, \{x\}, \{x, y\}\}$ on the set $X \triangleq \{x, y\}$.
It is *not* metrizable.
The Sierpiński space is also called the **two-point connected space**.

3.4 Additional properties

3.4.1 Separable metric spaces

Definition 1.7 page 23 gives the definition of a separable space.

Theorem 3.4. [15] *Let* (Y, d) *be a subspace of a* METRIC SPACE (X, d).

THM

(X, d) *is* SEPARABLE $\implies$ (Y, d) *is* SEPARABLE *(Definition 1.7 page 23)*

3.4.2 Compact metric spaces

Definition 3.4 (Borel-Lebesgue axiom). [16] *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33). Let* I *be an* INFINITE INDEXING SET *and* $J \subsetneq I$ *be a* FINITE INDEXING SET. *Let* $(\!(U_n)\!)_{n\in I}$ *be a* SEQUENCE *(Definition 9.1 page 131) of* OPEN SETS *(Definition 1.1 page 3).*

DEF

A set A *is **compact** if*

$$\underbrace{A \subseteq \bigcup_{n\in I} U_n}_{A\text{ is covered by an infinite union of open sets}} \implies \underbrace{A \subseteq \bigcup_{n\in J} U_n}_{A\text{ is covered by a finite union of open sets}}$$

Proposition 3.4. [17] *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33). Let* Y *be a subset of* X.

PRP

$$\left\{\begin{array}{lll} 1. & (X, \mathsf{d}) \text{ is COMPACT} & \text{and} \\ 2. & Y \text{ is CLOSED in } (X, \mathsf{d}) & (Y = Y^-) \end{array}\right\} \implies \{(Y, \mathsf{d}) \text{ is COMPACT}\}$$

✎PROOF:

1. Because (Y, d) is a metric space, there exists a sequence of open sets $(\!(A_n)\!)_{i\in I}$, where I is an infinite indexing set, such that
$$Y \subseteq \bigcup_{i\in I} A_i.$$
2. By left hypothesis 2, Y is closed, which means its complement Y^c is open.
3. Combining the above two statements, we have
$$X = Y \cup Y^c \subseteq \left(\bigcup_{i\in I} A_i\right) \cup Y^c.$$

[14] Joshi (1983) page 90 ⟨example 9⟩, Davis (2005) page 42 ⟨Example 4.4.4⟩
[15] Runda (2005) page 32 ⟨Theorem 2.2.17⟩
[16] Dieudonné (1969) pages 57–58, Rosenlicht (1968) page 54
[17] Dieudonné (1969) page 62, Rosenlicht (1968) page 54

4. By left hypothesis 1, X is compact and therefore can be covered by a *finite* number of open sets. Let J be a finite indexing set such that

$$X \subseteq \left(\bigcup_{i\in J} A_i\right) \cup Y^c.$$

5. By left hypothesis 2, $Y \subseteq X$. Therefore

$$Y \subseteq X \subseteq \left(\bigcup_{i\in J} A_i\right) \cup Y^c.$$

6. And so, Y is covered by a finite number of open sets ($(A_i)_{i\in J}$ and Y^c), and Y is therefore *compact*.

✏

Proposition 3.5 (Nested set property). [18] *Let (X, d) be a* METRIC SPACE *and $(Y_n)_{n\in\mathbb{Z}}$ a* SEQUENCE *(Definition 9.1 page 131) of sets.*

THM

$$\left.\begin{array}{lllll} 1. & (X,\mathsf{d}) \text{ is COMPACT} & & & \text{and} \\ 2. & Y_n \subseteq X & \forall n\in\mathbb{Z} & (Y_n \text{ are subsets of } X) & \text{and} \\ 3. & Y_n \neq \varnothing & \forall n\in\mathbb{Z} & (Y_n \text{ are non-empty}) & \text{and} \\ 4. & Y_n \supseteq Y_{n+1} & \forall n\in\mathbb{Z} & (\text{nested subsets}) & \text{and} \\ 5. & Y_n = Y_n^- & \forall n\in\mathbb{Z} & (Y \text{ is closed}). & \end{array}\right\} \implies \underbrace{\left|\bigcap_{n\in\mathbb{Z}} Y_n\right| \geq 1}_{\text{all subsets have at least one common element}}$$

✎PROOF: Proof is by contradiction.

1. Note that $Y_n \supseteq Y_{n+1} \iff {Y^c}_n \subseteq {Y^c}_{n+1}$

2. Suppose that the statement is false; that is, $\left|\bigcap_{n\in\mathbb{Z}} Y_n\right| = 0$.

$$\begin{aligned} \left|\bigcap_{n\in\mathbb{Z}} Y_n\right| = 0 &\iff \bigcap_{n\in\mathbb{Z}} Y_n = \varnothing \\ &\iff \bigcup_{n\in\mathbb{Z}} {Y^c}_n = X && \text{by } \textit{de Morgan's law} \text{ (Theorem A.7 page 274)} \\ &\implies \bigcup_{n=1}^{N} {Y^c}_n = X \text{ for some finite } N && \text{by compactness hypothesis} \\ &\implies {Y^c}_N = X && \text{because } {Y^c}_n \subseteq {Y^c}_{n+1} \\ &\iff Y_N = \varnothing \end{aligned}$$

3. But this is a *contradiction*, because by left hypothesis 3, $Y_n \neq \varnothing$.

4. Therefore, $\left|\bigcap_{n\in\mathbb{Z}} Y_n\right| \geq 1$.

✏

[18] Rosenlicht (1968) page 55

3.4.3 Orthogonality on metric linear spaces

Definition 3.5. [19] *Let $(\mathbf{V}, \mathsf{d})$ be a* METRIC LINEAR SPACE. *Let $[x_1 : x_2]$ and $[y_1 : y_2]$ be* LINE SEGMENTS *in the linear space $\mathbf{V}$ that intersect at a point $p \in [x_1 : x_2]$.*

DEF
*The line segments $[x_1 : x_2]$ and $[y_1 : y_2]$ are **orthogonal** in the metric linear space $(\mathbf{V}, \mathsf{d})$ if*

$$\underbrace{\mathsf{d}(y_1, p) \le \mathsf{d}(y_2, q) \qquad \forall q \in [x_1 : x_2]}_{p \text{ is the closest point in } [x_1 : x_2] \text{ to } y_1}$$

3.5 Metric transforms

If we know that one or more functions are metrics, then we can use them to generate other metrics. This is demonstrated by the following:

- Theorem 3.5 (page 46): generate a metric using an isometry.
- Theorem 3.6 (page 46): generate a metric using a monotone function.
- Theorem 3.8 (page 48): generate a metric using a *metric preserving function.*
- Theorem 3.9 (page 51): generate a metric from a linear combination of metrics.
- Theorem 3.10 (page 52): generate an N-dimensional metric from weighted 1-dimensional metrics.

3.5.1 Metric transforms on the domains of metrics

Definition 3.6. [20] *Let (X, d) and (Y, p) be* METRIC SPACES *(Definition 3.1 page 33).*

DEF
*The function $\mathsf{f} \in Y^X$ is an **isometry** on $(Y, \mathsf{p})^{(X, \mathsf{d})}$ if*

$$\mathsf{d}(x, y) = \mathsf{p}(\mathsf{f}(x), \mathsf{f}(y)) \qquad \forall x, y \in X$$

*The spaces (X, d) and (Y, p) are **isometric** if there exists an isometry on $(Y, \mathsf{p})^{(X, \mathsf{d})}$.*

Theorem 3.5. [21] *Let (X, d) and (Y, p) be* METRIC SPACES. *Let f be a function in Y^X and f^{-1} its inverse in X^Y.*

THM
$$\left\{\mathsf{f} \text{ is an } \textbf{isometry} \text{ on } (Y, \mathsf{p})^{(X, \mathsf{d})}\right\} \iff \left\{\mathsf{f}^{-1} \text{ is an } \textbf{isometry} \text{ on } (X, \mathsf{d})^{(Y, \mathsf{p})}\right\}$$

If a function p is a *metric* and a function g is *injective*, then the function $\mathsf{d}(x, y) \triangleq \mathsf{p}(\mathsf{g}(x), \mathsf{g}(y))$ is also a *metric* (next theorem). For an example of this with $\mathsf{p}(x, y) \triangleq |x - y|$ and $\mathsf{g} \triangleq \arctan(x)$, see Example 3.26 (page 65).

Theorem 3.6 (Pullback metric/g-transform metric). [22] *Let X and Y be sets. Let g be a function in Y^X.*

THM
$$\left\{\begin{array}{ll} 1. & \mathsf{p} \text{ is a metric on } Y \quad \text{and} \\ 2. & \mathsf{g} \text{ is } \text{INJECTIVE} \end{array}\right\} \implies \left\{\begin{array}{c} \mathsf{d}(x, y) = \mathsf{p}(\mathsf{g}(x), \mathsf{g}(y)) \quad \forall x, y \in X \\ \text{is a metric on } Y \end{array}\right\}$$

✎PROOF:

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[19] Birkhoff (1935) page 169

[20] Thron (1966) page 153 ⟨definition 19.4⟩, Giles (1987) page 124 ⟨Definition 6.22⟩, Khamsi and Kirk (2001) page 15 ⟨Definition 2.4⟩, Kubrusly (2001) page 110

[21] Thron (1966) page 153 ⟨theorem 19.5⟩

[22] Deza and Deza (2009) page 81

1. Proof that $x = y \implies \mathsf{d}(x,y) = 0$:

$$\begin{aligned}
\mathsf{d}(x,y) &\triangleq \mathsf{p}(\phi(x),\phi(y)) && \text{by definition of } \mathsf{d}\\
&= \mathsf{p}(\phi(x),\phi(x)) && \text{by } x = y \text{ hypothesis}\\
&= 0 && \text{by \textit{nondegenerate} property of metric } \mathsf{p} \text{ (Definition 3.1 page 33)}\\
&= 0
\end{aligned}$$

2. Proof that $x = y \impliedby \mathsf{d}(x,y) = 0$:

$$\begin{aligned}
0 &= \mathsf{d}(x,y) && \text{by right hypothesis}\\
&\triangleq \mathsf{p}(\phi(x),\phi(y)) && \text{by definition of } \mathsf{d}\\
\implies \mathsf{p}(\phi(x),\phi(y)) &= 0 \text{ for } n = 1,2,\ldots,N && \text{because } \mathsf{p} \text{ is non-negative}\\
\implies x &= y && \text{by left hypothesis 2}
\end{aligned}$$

3. Proof that $\mathsf{d}(x,y) \le \mathsf{d}(z,x) + \mathsf{d}(z,y)$:

$$\begin{aligned}
\mathsf{d}(x,y) &\triangleq \mathsf{p}(\phi(x),\phi(y)) && \text{by definition of } \mathsf{d}\\
&\le \big(\mathsf{p}(\phi(x),\phi(z)) + \mathsf{d}(\phi(z),\phi(y))\big) && \text{by \textit{subadditive} property of } \mathsf{p} \text{ (Definition 3.1 page 33)}\\
&= \mathsf{p}(\phi(z),\phi(x)) + \mathsf{p}(\phi(z),\phi(y)) && \text{by \textit{symmetry} property of metric } \mathsf{p} \text{ (Definition 3.1 page 33)}\\
&\triangleq \mathsf{d}(z,x) + \mathsf{d}(z,y) && \text{by definition of } \mathsf{d}
\end{aligned}$$

⇨

3.5.2 Metric preserving functions

Definition 3.7. [23] *Let $\mathbb{M}$ be the set of all* METRIC SPACES *on a set X.*

DEF

A FUNCTION $\phi \in \mathbb{R}^{\vdash \mathbb{R}^{\vdash}}$ *is **metric preserving** if*

$\mathsf{d}(x,y) \triangleq \phi \circ \mathsf{p}(x,y)$ *is a metric on X for all $(X,\mathsf{p}) \in \mathbb{M}$*

Theorem 3.7 (next theorem) presents some necessary conditions for a function ϕ to be *metric preserving*. Theorem 3.8 (page 48) presents some sufficient conditions. But first some conditions that are *not* necessary:

1. It is *not* necessary for ϕ to be *continuous* (see Example 3.8 page 50).
2. It is *not* necessary for ϕ to be *nondecreasing* (see Example 3.10 page 50).
3. It is *not* necessary for ϕ to be *monotonic* (see Example 3.11 page 51).

Theorem 3.7 (necessary conditions). [24] *Let $\mathscr{R}\mathsf{f}$ be the* RANGE *of a function* f.

THM

$$\left\{ \begin{array}{l}\phi \text{ is a metric}\\ \text{preserving function}\end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & \phi^{-1}(0) = \{0\} & \text{and}\\ 2. & \mathscr{R}\phi \subseteq \mathbb{R}^{\vdash} & \text{and}\\ 3. & \phi(x+y) \le \phi(x) + \phi(y) & (\phi \text{ is SUBADDITIVE}) \end{array} \right\}$$

✎PROOF:

1. Proof that ϕ is a *metric preserving function* $\implies \phi^{-1}(0) = \{0\}$:

   (a) Suppose that the statement is not true and $\phi^{-1}(0) = \{0, a\}$.

[23] Vallin (1999) page 849 ⟨Definition 1.1⟩, Corazza (1999) page 309, Deza and Deza (2009) page 80

[24] Corazza (1999) page 310 ⟨Proposition 2.1⟩, Deza and Deza (2009) page 80

(b) Then $\phi(a) = 0$ and for some x, y such that $x \neq y$ and $\mathsf{d}(x, y) = a$ we have

$$\begin{aligned}\phi \circ \mathsf{d}(x, y) &= \phi(a)\\ &= 0\\ &\implies \phi \circ \mathsf{d} \text{ is } \textit{not} \text{ a metric}\\ &\implies \phi \text{ is } \textit{not} \text{ a metric preserving function}\end{aligned}$$

(c) But this contradicts the original hypothesis, and so it must be that $\phi^{-1}(0) = \{0\}$.

2. Proof that $\mathscr{R}\phi \subseteq \mathbb{R}^{\vdash}$:

$$\begin{aligned}\mathscr{R}\phi \circ \mathsf{d} &\subseteq \mathscr{R}\mathsf{d}\\ &\subseteq \mathbb{R}^{\vdash}\end{aligned}$$

3. Proof that ϕ is a metric preserving function $\implies$ ϕ is *subadditive*:
   (a) For ϕ to be a *metric preserving function*, by definition it must work with *all metric spaces*.
   (b) So to develop necessary conditions, we can pick any metric space we want (because it is necessary that ϕ preserves it as a metric space).
   (c) For this proof we choose the metric space $(\mathbb{R}, \mathsf{d})$ where $\mathsf{d}(x, y) \triangleq |x - y|$ for all $x, y \in \mathbb{R}^{\vdash}$:

$$\begin{aligned}\phi(x) + \phi(y) &= \phi(|(x + y) - x|) + \phi(|x - 0|) && \text{by definition of } |\cdot|\\ &= (\phi \circ \mathsf{d})(x + y, x) + (\phi \circ \mathsf{d})(x, 0) && \text{by definition of } \mathsf{d}\\ &\geq (\phi \circ \mathsf{d})(x + y, 0) && \text{by left hypothesis and Definition 3.1 page 33}\\ &= \phi(|(x + y) - 0|) && \text{by definition of } \mathsf{d}\\ &= \phi(x + y) && \text{because } x, y \in \mathbb{R}^{\vdash}\end{aligned}$$

✏

Theorem 3.8 (sufficient conditions). [25] *Let ϕ be a function in $\mathbb{R}^{\mathbb{R}}$.*

THM

$$\left.\begin{array}{llll}1. & x \geq y \implies \phi(x) \geq \phi(y) \quad \forall x,y \in \mathbb{R}^{\vdash} & \text{(NONDECREASING)} & \textit{and}\\ 2. & \phi(0) = 0 & & \textit{and}\\ 3. & \phi(x + y) \leq \phi(x) + \phi(y) \quad \forall x,y \in \mathbb{R}^{\vdash} & \text{(SUBADDITIVE).} & \end{array}\right\} \implies \begin{array}{l}\phi \textit{ is a } \text{METRIC}\\ \text{PRESERVING FUNCTION}\end{array}$$

✎PROOF:

1. Proof that $\phi \circ \mathsf{d}(x, y) = 0 \implies x = y$:

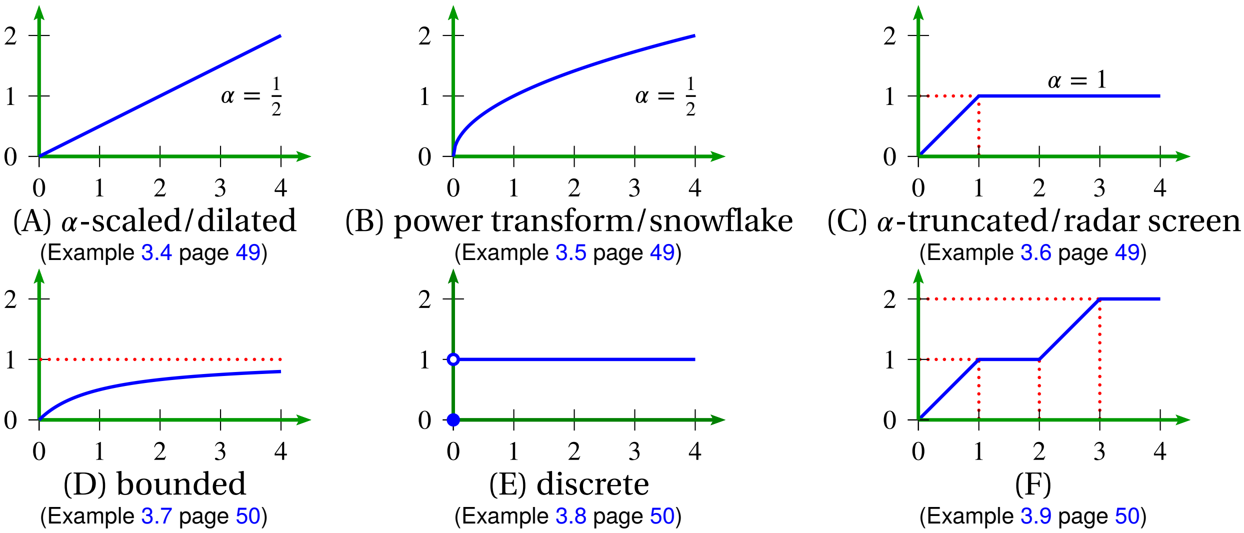
$$\begin{aligned}\phi \circ \mathsf{d}(x, y) = 0 &\implies \mathsf{d}(x, y) = 0 && \text{by } \phi \text{ hypothesis 2}\\ &\implies x = y && \text{by } \textit{nondegenerate} \text{ property page 33}\end{aligned}$$

2. Proof that $\phi \circ \mathsf{d}(x, y) = 0 \impliedby x = y$:

$$\begin{aligned}\phi \circ \mathsf{d}(x, y) &= \phi \circ \mathsf{d}(x, x) && \text{by } x = y \text{ hypothesis}\\ &= \phi(0) && \text{by } \textit{nondegenerate} \text{ property page 33}\\ &= 0 && \text{by } \phi \text{ hypothesis 2}\end{aligned}$$

3. Proof that $\phi \circ \mathsf{d}(x, y) \leq \phi \circ \mathsf{d}(z, x) + \phi \circ \mathsf{d}(z, y)$:

$$\begin{aligned}\phi \circ \mathsf{d}(x, y) &\leq \phi(\mathsf{d}(x, z) + \mathsf{d}(z, y)) && \text{by } \phi \text{ hypothesis 1 and triangle inequality page 33}\\ &\leq \phi(\mathsf{d}(z, x) + \mathsf{d}(z, y)) && \text{by symmetric property of } \mathsf{d} \text{ page 33}\\ &\leq \phi \circ \mathsf{d}(z, x) + \phi \circ \mathsf{d}(z, y) && \text{by } \phi \text{ hypothesis 3}\end{aligned}$$



(A) α-scaled/dilated (Example 3.4 page 49)
(B) power transform/snowflake (Example 3.5 page 49)
(C) α-truncated/radar screen (Example 3.6 page 49)
(D) bounded (Example 3.7 page 50)
(E) discrete (Example 3.8 page 50)
(F) (Example 3.9 page 50)

Figure 3.5: metric preserving functions

✏

Example 3.4 (α-scaled metric/dilated metric). [26] Let (X, d) be a *metric space* (Definition 3.1 page 33).

EX	$\phi(x) \triangleq \alpha x,\ \alpha \in \mathbb{R}^+$, is *metric preserving*	$\left\{\begin{array}{l}\mathsf{p}(x,y) \triangleq \alpha\mathsf{d}(x,y) \text{ is a metric on } X \\ \text{(see Figure 3.5 page 49 (A))}\end{array}\right\}$

✎PROOF:

1. Note that ϕ satisfies the conditions of Theorem 3.8 (page 48).
2. Therefore, by Theorem 3.8 (page 48), $\mathsf{d}(x,y)$ is a metric on X.

✏

Example 3.5 (power transform metric/snowflake transform metric). [27] Let (X, d) be a *metric space* (Definition 3.1 page 33).

EX	$\phi(x) \triangleq x^\alpha,\ \alpha \in (0:1]$, is *metric preserving*	$\left(\begin{array}{l}\mathsf{p}(x,y) \triangleq [\mathsf{d}(x,y)]^\alpha \text{, is a metric on } X \\ \text{(see Figure 3.5 page 49 (B))}\end{array}\right)$

✎PROOF:

1. Note that ϕ satisfies the conditions of Theorem 3.8 (page 48) for $0 < \alpha \leq 1$.
2. Therefore, by Theorem 3.8 (page 48), $\mathsf{d}(x,y)$ is a metric on X.

✏

Example 3.6 (α-truncated metric/radar screen metric). [28] Let (X, d) be a *metric space* (Definition 3.1 page 33).

EX	$\phi(x) \triangleq \min\{\alpha,\ x\},\ \alpha \in \mathbb{R}^+$, is *metric preserving*	$\left(\begin{array}{l}\mathsf{p}(x,y) \triangleq \min\{\alpha,\ \mathsf{d}(x,y)\} \\ \text{is a metric on } X \\ \text{(see Figure 3.5 page 49 (C))}\end{array}\right)$

[25] Corazza (1999) ⟨Proposition 2.3⟩, Deza and Deza (2009) page 80, Kelley (1955) page 131 ⟨Problem C⟩
[26] Deza and Deza (2006) page 44
[27] Deza and Deza (2009) page 81, Deza and Deza (2006) page 45
[28] Giles (1987) page 33, Deza and Deza (2006) pages 242–243

✎PROOF:

1. Note that ϕ satisfies the conditions of Theorem 3.8 (page 48).
2. $\mathsf{d}(x,y) \triangleq \min\{\alpha, \mathsf{p}(x,y)\} = \phi \circ \mathsf{p}(x,y)$
3. Therefore, by Theorem 3.8 (page 48), $\mathsf{d}(x,y)$ is a metric.

✏

Example 3.7 (bounded metric). [29] Let (X, d) be a *metric space* (Definition 3.1 page 33).

EX $\phi(x) \triangleq \dfrac{x}{1+x}$ is *metric preserving* (see Figure 3.5 page 49 (D)) $\left(\mathsf{p}(x,y) \triangleq \dfrac{\mathsf{d}(x,y)}{1+\mathsf{d}(x,y)} \text{ is also a metric on } X\right)$

✎PROOF:

1. Note that ϕ satisfies the conditions of Theorem 3.8 (page 48).
2. $\mathsf{d}(x,y) \triangleq \dfrac{\mathsf{p}(x,y)}{1+\mathsf{p}(x,y)} = \phi \circ \mathsf{p}(x,y)$
3. Therefore, by Theorem 3.8 (page 48), $\mathsf{d}(x,y)$ is a metric.

✏

Example 3.8. [30] Let ϕ be a function in $\mathbb{R}^{\mathbb{R}}$.

EX $\phi(x) \triangleq \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x > 0 \end{cases}$ is a *discontinuous metric preserving function* (see Figure 3.5 page 49 (E)).

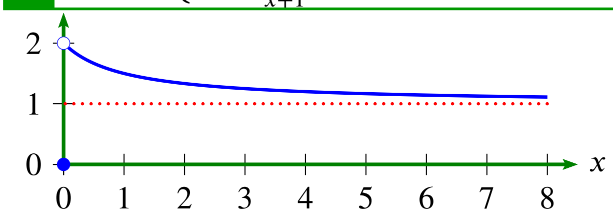
✎PROOF: This result follows directly from Theorem 3.8 page 48. ✏

Example 3.9. Let ϕ be a function in $\mathbb{R}^{\mathbb{R}}$.

EX $\phi(x) \triangleq \begin{cases} x & \text{for } 0 \le x < 1, & 1 & \text{for } 1 \le x \le 2, \\ x-1 & \text{for } 2 < x < 3, & 2 & \text{for } x \ge 3 \end{cases}$ is a *metric preserving function* (see Figure 3.5 page 49 (F)).

Example 3.10. Let ϕ be a function in $\mathbb{R}^{\mathbb{R}}$.

EX $\phi(x) \triangleq \begin{cases} 0 & \text{for } x = 0 \\ 1 + \frac{1}{x+1} & \text{for } x > 0 \end{cases}$ is a metric preserving function



✎PROOF:

1. Note that $\phi \circ \mathsf{d}(x,x) = 0 \iff x = 0$.

[29] Vallin (1999) page 849, Aliprantis and Burkinshaw (1998) page 39
[30] Corazza (1999) page 311

2. Lemma: $\frac{1}{a+b} \le \frac{1}{a} + \frac{1}{b}$ for $a, b \in \mathbb{R}^+$:

$$\begin{aligned}\frac{1}{a+b} &\le \frac{1}{a}\\ &\le \frac{1}{a} + \frac{1}{b}\end{aligned}$$

3. Proof that $\phi \circ \mathsf{d}$ is *subadditive*:

$$\begin{aligned}
\phi \circ \mathsf{d}(x,y) &= 1 + \frac{1}{1+\mathsf{d}(x,y)} && \text{by definition of } \phi\\
&\le 1 + \frac{1}{1+\mathsf{d}(x,z)+\mathsf{d}(z,y)} && \text{by \textit{subadditive} property of metric } \mathsf{d}\\
&\le 1 + \frac{1}{1+\mathsf{d}(x,z)+1+\mathsf{d}(z,y)}\\
&\le 1 + \frac{1}{1+\mathsf{d}(x,z)} + \frac{1}{1+\mathsf{d}(z,y)} && \text{by Example 2}\\
&\le 1 + \frac{1}{1+\mathsf{d}(x,z)} + 1 + \frac{1}{1+\mathsf{d}(z,y)}\\
&= \phi \circ \mathsf{d}(x,z) + \phi \circ \mathsf{d}(z,y) && \text{by definition of } \phi
\end{aligned}$$

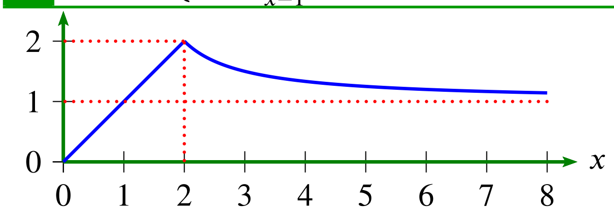
4. Therefore, by Theorem 3.1 (page 34), $\phi \circ \mathsf{d}(x,y)$ is a *metric* and ϕ is a *metric preserving function*.

✎

Example 3.11. [31] Let ϕ be a function in $\mathbb{R}^{\mathbb{R}}$.

EX

$$\phi(x) \triangleq \begin{cases} x & \text{for } x \le 2\\ 1 + \frac{1}{x-1} & \text{for } x > 2\end{cases} \qquad \text{is a \textit{nonmonotonic} metric preserving function}$$



3.5.3 Product metrics

Theorem 3.9 (Fréchet product metric). [32] *Let X be a set.*

THM

$$\left\{\begin{array}{lll} 1. & (\!|\mathsf{p}_n|\!) \text{ are METRICS on } X & \text{and}\\ 2. & \alpha_n \ge 0 \quad \forall n = 1,2,\ldots,N & \text{and}\\ 3. & \max\left\{\alpha_n \middle| n=1,2,\ldots,N\right\} > 0 & \end{array}\right\} \implies \left\{\begin{array}{c} \mathsf{d}(x,y) = \sum_{n=1}^{N} \alpha_n \mathsf{p}_n(x,y)\\ \text{is a METRIC on } X\end{array}\right\}$$

✎PROOF:

[31] Corazza (1999) page 309, Doboš (1998) page 25 ⟨Example 1⟩, Jůza (1956)
[32] Deza and Deza (2006) page 47, Deza and Deza (2009) page 84, Steen and Seebach (1978) pages 64–65 ⟨Example 37.7⟩, Isham (1999) page 10

1. Proof that $x = y \implies \mathsf{d}(x,y) = 0$:

$$\begin{aligned}
\mathsf{d}(x,y) &= \sum_{n=1}^{N} \alpha_n\, \mathsf{p}_n(x,y) && \text{by definition of } \mathsf{d}\\
&= \sum_{n=1}^{N} \alpha_n\, \mathsf{p}_n(x,x) && \text{by left hypothesis}\\
&= \sum_{n=1}^{N} 0 && \text{by } \textit{nondegenerate} \text{ property of metrics (Definition 3.1 page 33)}\\
&= 0
\end{aligned}$$

2. Proof that $x = y \impliedby \mathsf{d}(x,y) = 0$:

$$\begin{aligned}
0 &= \mathsf{d}(x,y) && \text{by right hypothesis}\\
&= \sum_{n=1}^{N} \alpha_n\, \mathsf{p}_n(x,y) && \text{by definition of } \mathsf{d}\\
\implies \mathsf{p}_n(x,y) &= 0 \qquad \forall x,y \in \boldsymbol{X} && \text{by metric properties page 33}\\
\implies x &= y \qquad \forall x,y \in \boldsymbol{X} && \text{by non-degenerate property of metrics page 33}
\end{aligned}$$

3. Proof that $\mathsf{d}(x,y) \le \mathsf{d}(z,x) + \mathsf{d}(z,y)$:

$$\begin{aligned}
\mathsf{d}(x,y) &= \sum_{n=1}^{N} \alpha_n\, \mathsf{p}_n(x,y) && \text{by definition of } \mathsf{d}\\
&\le \sum_{n=1}^{N} \alpha_n \left[\mathsf{p}_n(x,z) + \mathsf{p}_n(z,y)\right] && \text{by } \textit{subadditive} \text{ property (Definition 3.1 page 33)}\\
&= \sum_{n=1}^{N} \alpha_n \left[\mathsf{p}_n(z,x) + \mathsf{p}_n(z,y)\right] && \text{by } \textit{symmetry} \text{ property (Definition 3.1 page 33)}\\
&= \sum_{n=1}^{N} \alpha_n\, \mathsf{p}_n(z,x) + \sum_{n=1}^{N} \alpha_n\, \mathsf{p}_n(z,y)\\
&= \mathsf{d}(z,x) + \mathsf{d}(z,y) && \text{by definition of } \mathsf{d}
\end{aligned}$$

✏

Theorem 3.10 (Power mean metrics)**.** *Let X be a set. Let $(\!(x_n \in X)\!)_1^N$ and $(\!(y_n \in X)\!)_1^N$ be N-tuples on X.*

THM

$$\left.\begin{array}{ll}
1. & \mathsf{p} \textit{ is a } \text{METRIC} \textit{ on } X \quad \textit{and}\\
2. & \displaystyle\sum_{n=1}^{N} \lambda_n = 1
\end{array}\right\} \implies \left\{\begin{array}{l}
\mathsf{d}\big((\!(x_n)\!), (\!(y_n)\!)\big) \triangleq \left(\displaystyle\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(x_n, y_n)\right)^{\frac{1}{r}},\\
\qquad r \in [1:\infty], \quad \textit{is a } \text{METRIC} \textit{ on } X
\end{array}\right.$$

Moreover, if $r = \infty$, then $\mathsf{d}\big((\!(x_n)\!), (\!(y_n)\!)\big) = \max\limits_{n=1,\dots,N} \mathsf{p}(x_n, y_n)$.

✎PROOF:

1. Proof that $\lparen x_n \rparen = \lparen y_n \rparen \implies \mathsf{d}(\lparen x_n \rparen, \lparen y_n \rparen) = 0$ for $r \in [1 : \infty)$:

$$\begin{aligned}
\mathsf{d}(\lparen x_n \rparen, \lparen y_n \rparen) &\triangleq \left(\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(x_n, y_n)\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{d} \\
&= \left(\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(x_n, x_n)\right)^{\frac{1}{r}} && \text{by } \lparen x_n \rparen = \lparen y_n \rparen \text{ hypothesis} \\
&= \left(\sum_{n=1}^{N} 0\right)^{\frac{1}{r}} && \text{because } \mathsf{p} \text{ is } \textit{nondegenerate} \\
&= 0
\end{aligned}$$

2. Proof that $\lparen x_n \rparen = \lparen y_n \rparen \impliedby \mathsf{d}(\lparen x_n \rparen, \lparen y_n \rparen) = 0$ for $r \in [1 : \infty)$:

$$\begin{aligned}
0 &= \mathsf{d}(\lparen x_n \rparen, \lparen y_n \rparen) && \text{by } \mathsf{d}(\lparen x_n \rparen, \lparen y_n \rparen) = 0 \text{ hypothesis} \\
&\triangleq \left(\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(x_n, y_n)\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{d} \\
\implies \left(\mathsf{p}(x_n, y_n)\right)^{\frac{1}{r}} &= 0 \text{ for } n = 1, 2, \ldots, N && \text{because } \mathsf{p} \text{ is } \textit{non-negative} \\
\implies \lparen x_n \rparen &= \lparen y_n \rparen && \text{because } \mathsf{p} \text{ is } \textit{nondegenerate}
\end{aligned}$$

3. Proof that d satisfies the triangle inequality property for $r = 1$:

$$\begin{aligned}
\mathsf{d}(\lparen x_n \rparen, \lparen y_n \rparen) &\triangleq \left(\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(x_n, y_n)\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{d} \\
&= \sum_{n=1}^{N} \lambda_n \mathsf{p}(x_n, y_n) && \text{by } r = 1 \text{ hypothesis} \\
&\le \sum_{n=1}^{N} \lambda_n \left[\mathsf{p}(z_n, x_n) + \mathsf{p}(z_n, y_n)\right] && \text{by } \textit{triangle inequality} \\
&= \sum_{n=1}^{N} \lambda_n \mathsf{p}(z_n, x_n) + \sum_{n=1}^{N} \lambda_n \mathsf{p}(z_n, y_n) \\
&= \left(\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(z_n, x_n)\right)^{\frac{1}{r}} + \left(\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(z_n, y_n)\right)^{\frac{1}{r}} && \text{by } r = 1 \text{ hypothesis} \\
&\triangleq \mathsf{d}(\lparen z_n \rparen, \lparen x_n \rparen) + \mathsf{d}(\lparen z_n \rparen, \lparen y_n \rparen) && \text{by definition of } \mathsf{d}
\end{aligned}$$

4. Proof that d satisfies the triangle inequality property for $r \in (1 : \infty)$:

$$\begin{aligned}
&\mathsf{d}(\lparen x_n \rparen, \lparen y_n \rparen) \\
&\triangleq \left(\sum_{n=1}^{N} \lambda_n \mathsf{p}^r(x_n, y_n)\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{d} \\
&\le \left(\sum_{n=1}^{N} \lambda_n \left[\mathsf{p}(z_n, x_n) + \mathsf{p}(z_n, y_n)\right]^r\right)^{\frac{1}{r}} && \text{by } \textit{subadditive} \text{ property (Definition 3.1 page 33)} \\
&= \left(\sum_{n=1}^{N} \left[\lambda_n^{\frac{1}{r}} \mathsf{p}(z_n, x_n) + \lambda_n^{\frac{1}{r}} \mathsf{p}(z_n, y_n)\right]^r\right)^{\frac{1}{r}} && \text{by } \textit{subadditive} \text{ property (Definition 3.1 page 33)}
\end{aligned}$$

$$\leq \left(\sum_{n=1}^{N}\left[\lambda_n^{\frac{1}{r}}\mathsf{p}(z_n,x_n)\right]^r\right)^{\frac{1}{r}} + \left(\sum_{n=1}^{N}\left[\lambda_n^{\frac{1}{r}}\mathsf{p}(z_n,y_n)\right]^r\right)^{\frac{1}{r}} \quad \text{by } \textit{Minkowski's inequality} \text{ (Theorem 11.5 page 169)}$$

$$\leq \left(\sum_{n=1}^{N}\lambda_n\mathsf{p}^r(z_n,x_n)\right)^{\frac{1}{r}} + \left(\sum_{n=1}^{N}\lambda_n\mathsf{p}^r(z_n,y_n)\right)^{\frac{1}{r}}$$

$$\triangleq \mathsf{d}\left(\langle\!|z_n|\!\rangle, \langle\!|x_n|\!\rangle\right) + \mathsf{d}\left(\langle\!|z_n|\!\rangle, \langle\!|y_n|\!\rangle\right) \quad \text{by definition of } \mathsf{d}$$

5. Proof for the $r = \infty$ case:

   (a) Proof that $\mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) = \max\langle\!|x_n|\!\rangle$: by Theorem 11.3 page 164

   (b) Proof that $\langle\!|x_n|\!\rangle = \langle\!|y_n|\!\rangle \implies \mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) = 0$:

$$\begin{aligned}
\mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) &\triangleq \max\left\{\mathsf{p}\left(x_n,y_n\right) | n = 1,2,\ldots,N\right\} && \text{by definition of } \mathsf{d}\\
&= \max\left\{\mathsf{p}\left(x_n,x_n\right) | n = 1,2,\ldots,N\right\} && \text{by } \langle\!|x_n|\!\rangle = \langle\!|y_n|\!\rangle \text{ hypothesis}\\
&= 0 && \text{because } \mathsf{p} \text{ is } \textit{nondegenerate}
\end{aligned}$$

   (c) Proof that $\langle\!|x_n|\!\rangle = \langle\!|y_n|\!\rangle \impliedby \mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) = 0$:

$$\begin{aligned}
0 &= \mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) && \text{by } \mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) = 0 \text{ hypothesis}\\
&\triangleq \max\left\{\mathsf{p}\left(x_n,y_n\right) | n = 1,2,\ldots,N\right\} && \text{by definition of } \mathsf{d}\\
\implies \mathsf{p}\left(x_n,y_n\right) &= 0 \text{ for } n = 1,2,\ldots,N\\
\implies \langle\!|x_n|\!\rangle &= \langle\!|y_n|\!\rangle && \text{because } \mathsf{p} \text{ is } \textit{nondegenerate}
\end{aligned}$$

   (d) Proof that d satisfies the triangle inequality property:

$$\begin{aligned}
&\mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right)\\
&\triangleq \max\left\{\mathsf{p}\left(x_n,y_n\right) | n = 1,2,\ldots,N\right\} && \text{by definition of } \mathsf{d}\\
&\leq \max\left\{\mathsf{p}\left(x_n,z_n\right) + \mathsf{p}\left(z_n,y_n\right) | n = 1,2,\ldots,N\right\} && \text{by } \textit{subadditive} \text{ property}\\
&\leq \max\left\{\mathsf{p}\left(x_n,z_n\right) | n = 1,2,\ldots,N\right\} + \max\left\{\mathsf{p}\left(z_n,y_n\right) | n = 1,2,\ldots,N\right\} && \text{by } \textit{non-negative} \text{ property}\\
&= \max\left\{\mathsf{p}\left(z_n,x_n\right) | n = 1,2,\ldots,N\right\} + \max\left\{\mathsf{p}\left(z_n,y_n\right) | n = 1,2,\ldots,N\right\} && \text{by } \textit{symmetry} \text{ property}\\
&\triangleq \mathsf{d}(\langle\!|z_n|\!\rangle, \langle\!|x_n|\!\rangle) + \mathsf{d}(\langle\!|z_n|\!\rangle, \langle\!|y_n|\!\rangle) && \text{by definition of } \mathsf{d}
\end{aligned}$$

⇨

Example 3.12 (Generalized Taxi-Cab Metric). Let X be a set. Let $\langle\!|x_n \in X|\!\rangle_1^N$ and $\langle\!|y_n \in X|\!\rangle_1^N$ be N-tuples on X.

EX
$$\{\mathsf{p} \text{ is a metric on } X\} \implies \mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) \triangleq \sum_{n=1}^{N}\mathsf{p}\left(x_n,y_n\right) \quad \forall x_n,y_n \in X \quad \text{is a metric on } X$$

✎PROOF:

$$\mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) = \sum_{n=1}^{N}\mathsf{p}\left(x_n,y_n\right)$$

$$= \underbrace{(N^r)\underbrace{\left(\sum_{n=1}^{N}\frac{1}{N}\mathsf{p}^r\left(x_n,y_n\right)\right)^{\frac{1}{r}}}_{\text{metric by Theorem 3.10 page 52}}}_{\text{metric by Theorem 3.8 page 48 (see also Example 3.4 page 49)}} \quad \text{where } r \triangleq 1$$

$$\implies \mathsf{d}\left(\langle\!|x_n|\!\rangle, \langle\!|y_n|\!\rangle\right) \text{ is a metric}$$

In the French railway system, a large number of railway lines go through Paris. This means that often the distance from city x to city y is $\mathsf{d}(x,p)+\mathsf{d}(p,y)$ where p represents Paris. This situation gives motivation for the *French Railroad Metric* (next).

Proposition 3.6. [33] *Let X be a set and $\mathsf{p} \in \mathbb{R}^{X\times X}$ be a function.*

PRP

If p is a metric, then the following functions are also metrics:

1. $$\mathsf{d}_f(x,y;z) = \left\{\begin{array}{lll} 0 & \text{for} & x=y \\ \mathsf{p}(x,z)+\mathsf{p}(z,y) & \text{for} & x\neq y \end{array}\right\} \quad \text{(French railway metric)}$$
2. $$\mathsf{d}_{\mathsf{p}}(x,y) = \left\{\begin{array}{lll} 0 & \text{for} & x=y \\ \mathsf{p}(0,x)+\mathsf{p}(0,y) & \text{for} & x\neq y \end{array}\right\} \quad \text{(post office metric)}$$

✎PROOF:

1. Proof the $\mathsf{d}_f(x,y;z)$ is a metric:

Proof that $x=y \implies \mathsf{d}_f(x,y;z)=0$:

$$\begin{aligned} \mathsf{d}_f(x,y;z) &= \mathsf{d}_f(x,x) && \text{by left hypothesis} \\ &= 0 && \text{by definition of } \mathsf{d}_f \end{aligned}$$

Proof that $x=y \impliedby \mathsf{d}_f(x,y;z)=0$:

$$\begin{aligned} 0 &= \mathsf{d}_f(x,y;z) && \text{by right hypothesis} \\ &\triangleq \begin{cases} 0 & \text{for } x=y \\ \mathsf{p}(x,z)+\mathsf{p}(z,y) & \text{for } x\neq y \end{cases} && \text{by definition of } \mathsf{d}_f \\ &\geq \begin{cases} 0 & \text{for } x=y \\ \mathsf{d}_f(x,y;z) & \text{for } x\neq y \end{cases} && \text{by Definition 3.1} \\ &\geq 0 && \text{by Definition 3.1} \\ \implies \mathsf{d}_f(x,y;z) &= 0 \qquad \forall x,y\in \boldsymbol{X} \\ \implies x &= y && \text{by Definition 3.1} \end{aligned}$$

Proof that $\mathsf{d}_f(x,y;z) \leq \mathsf{d}_f(u,x)+\mathsf{d}_f(u,y)$:

$$\begin{aligned} \mathsf{d}_f(x,y;z) &\triangleq \begin{cases} 0 & \text{for } x=y \\ \mathsf{p}(x,z)+\mathsf{p}(z,y) & \text{for } x\neq y \end{cases} && \text{by definition of } \mathsf{d}_f \\ &\leq \begin{cases} 0 & \text{for } x=y \\ \mathsf{p}(u,z)+\mathsf{p}(z,x) & \text{for } x\neq y \end{cases} \\ &\quad + \begin{cases} 0 & \text{for } x=y \\ \mathsf{p}(u,z)+\mathsf{p}(z,y) & \text{for } x\neq y \end{cases} && \text{by Definition 3.1} \\ &= \mathsf{d}_f(u,x)+\mathsf{d}_f(u,y) && \text{by definition of } \mathsf{d}_f \end{aligned}$$

2. Proof for Post Office Metric: this is a special case of the French Railroad metric (with $z=0$).

[33] Giles (1987) pages 17,34, Runde (2005) page 25

3.6 Examples

Example 3.13. [34] Let $|\cdot| \in \mathbb{R}^{\vdash R}$ be an *absolute value* (Definition F.4 page 342) function on a *ring* (Definition F.2 page 341) R.

The function $\mathsf{d}(x,y) \triangleq |x-y|$ is a *metric*. This metric is called the **usual metric**. It is defined on any ring such as the ring of *real numbers*, *rational numbers*, *complex numbers*, etc.

✎PROOF: Proof by use of Theorem 3.1 (page 34) ...

Proof that $x = y \implies \mathsf{d}(x,y) = 0$:

$$\begin{aligned}
\mathsf{d}(x,y) &\triangleq |x-y| && \text{by definition of d}\\
&= |x-x| && \text{by left hypothesis}\\
&= 0
\end{aligned}$$

Proof that $x = y \impliedby \mathsf{d}(x,y) = 0$:

$$\begin{aligned}
0 &= \mathsf{d}(x,y) && \text{by right hypothesis}\\
&= |x-y| && \text{by definition of d}\\
\implies x &= y && \text{by property of } |\cdot|
\end{aligned}$$

Proof that $\mathsf{d}(x,y) = \mathsf{d}(z,x) + \mathsf{d}(z,y)$:

$$\begin{aligned}
\mathsf{d}(x,y) &= |x-y| && \text{by definition of d}\\
&= |x-z+z-y|\\
&\le |x-z| + |z-y| && \text{by } \textit{subadditive} \text{ property of } |\cdot| \text{ (Definition F.4 page 342)}\\
&= |z-x| + |z-y|\\
&= \mathsf{d}(z,x) + \mathsf{d}(z,y) && \text{by definition of d (Definition 3.1 page 33)}
\end{aligned}$$

✎

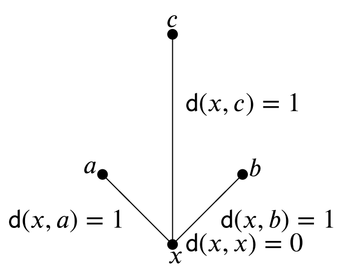
Example 3.14 (The discrete metric). [35] Let X be a set and $\mathsf{d} \in \mathbb{R}^{X \times X}$.

$$\mathsf{d}(x,y) \triangleq \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases} \quad \text{is a metric.}$$

d is *not* generated by a norm.

$$\mathsf{B}(0,1) = \{0\}$$

$$\operatorname{diam} \mathsf{B}(0,1) = 0$$



This metric is called the *discrete metric*. It is unusual among metrics because so little is required of the set X. In particular, X does not need to be equipped with any order structure (does not need to be a partially or totally ordered set). The diameter of (X, d) is 1.

✎PROOF:

1. Proof that $\mathsf{d}(x,y)$ is a metric (using Theorem 3.1 page 34):

---

[34] Davis (2005) page 16

[35] Busemann (1955a) page 4 ⟨COMMENTS ON THE AXIOMS⟩, Giles (1987) page 13, Copson (1968) page 24, Khamsi and Kirk (2001) page 19 ⟨Example 2.1⟩

Proof that $x = y \implies \mathsf{d}(x, y) = 0$:

$$\begin{aligned}
\mathsf{d}(x, y) &\triangleq \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases} && \text{by definition of } \mathsf{d} \\
&= 0 && \text{by } x = y \text{ hypothesis}
\end{aligned}$$

Proof that $x = y \impliedby \mathsf{d}(x, y) = 0$:

$$\begin{aligned}
0 &= \mathsf{d}(x, y) && \text{by } \mathsf{d}(x, y) = 0 \text{ hypothesis} \\
&\triangleq \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases} && \text{by definition of } \mathsf{d} \\
\implies x &= y
\end{aligned}$$

Proof that $\mathsf{d}(x, y) \leq \mathsf{d}(z, x) + \mathsf{d}(z, y)$:

$$\begin{aligned}
\mathsf{d}(x, y) &\triangleq \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases} && \text{by definition of } \mathsf{d} \\
&\leq \begin{cases} 1 & \text{for } z \neq x \\ 0 & \text{for } z = x \end{cases} + \begin{cases} 1 & \text{for } z \neq y \\ 0 & \text{for } z = y \end{cases} \\
&= \mathsf{d}(z, x) + \mathsf{d}(z, y) && \text{by definition of } \mathsf{d}
\end{aligned}$$

2. Proof that d is not generated by a norm:

$$\begin{aligned}
\|\alpha x\| &= \mathsf{d}(\alpha x, 0) && \text{for some function } \|\cdot\| \\
&= \begin{cases} 1 & \text{for } \alpha x \neq 0 \\ 0 & \text{for } \alpha x = 0 \end{cases} && \text{by definition of } \mathsf{d} \\
&= \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \\
&= \mathsf{d}(x, 0) && \text{by definition of } \mathsf{d} \\
&\neq |\alpha| \mathsf{d}(x, 0) \\
&= |\alpha| \, \|x\|
\end{aligned}$$

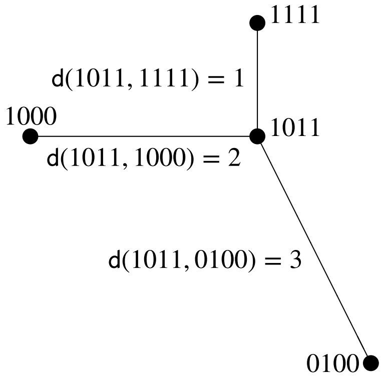
3. Proof that $\mathsf{B}(0, 1) = \{0\}$:

$$\begin{aligned}
\mathsf{B}(0, 1) &= \{x \in \boldsymbol{X} | \mathsf{d}(0, x) < 1\} && \text{by definition of open ball } \mathsf{B} \text{ page 28} \\
&= \{0\}
\end{aligned}$$

4. Proof that $\operatorname{diam} \mathsf{B}(0, 1) = 0$:

$$\begin{aligned}
\operatorname{diam} \mathsf{B}(0, 1) &= \operatorname{diam} \{0\} && \text{by previous result} \\
&= \sup \{\mathsf{d}(x, y) \, | x, y \in \{0\}\} && \text{by definition of diam page 27} \\
&= \sup \{\mathsf{d}(0, 0)\} \\
&= \sup \{0\} && \text{by \textit{non-degenerate} propery of } \mathsf{d} \text{ (Definition 3.1 page 33)} \\
&= 0
\end{aligned}$$

3.6.1 Metrics on finite sets



Example 3.15 (Hamming distance). Let $\boldsymbol{x} \triangleq (\!(x_1, x_2, \ldots, x_n)\!)$, $\boldsymbol{y} \triangleq (\!(y_1, y_2, \ldots, y_n)\!)$, and $x_i, y_i \in \{0, 1\}$.

EX

The *Hamming distance* between $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \sum_{i=1}^{n} \mathsf{p}\left(x_i, y_i\right)$$

where

$$\mathsf{p}(x, y) \triangleq \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases}.$$

The function d is a metric.

✎PROOF: Example 3.14 (page 56) already showed that p is a metric. And because of this and by Proposition 3.6 (page 55), d is also a metric. ✑

Example 3.16 (lattice metric). [36] Let $L = (X, \circledless, \vee, \wedge)$ be a lattice.

EX

Let $\|x\| : X \to \mathbb{R}$ be a function that satisfies the conditions

1. $x \circledless y \implies \|x\| \leq \|y\| \qquad \forall x,y \in X \qquad$ (*monotonic*)
2. $\|x \vee y\| + \|x \wedge y\| = \|x\| + \|y\| \qquad \forall x,y \in X$

Then $\mathsf{d}(x, y) \triangleq \|x \vee y\| - \|x \wedge y\|$ is a **metric** on L.

✎PROOF:

1. Proof that $\mathsf{d}(x, y) \geq 0$:

$$\begin{aligned} \mathsf{d}(x, y) &= \|x \vee y\| - \|x \wedge y\| && \text{by definition of } \mathsf{d}(x, y) \\ &\geq 0 && \text{by condition 1 and because } x \vee y \geq x \wedge y \end{aligned}$$

2. Proof that $\mathsf{d}(x, y) = 0 \implies x = y$:

$$\begin{aligned} \mathsf{d}(x, y) = 0 &\implies \|x \vee y\| = \|x \wedge y\| && \text{by definition of } \mathsf{d}(x, y) \\ &\implies x \vee y = x \wedge y && \text{by definition of } \|x\| \text{ condition 1} \\ &\implies x = y && \text{by definition of } \vee \text{ and } \wedge \end{aligned}$$

3. Proof that $\mathsf{d}(x, y) = 0 \impliedby x = y$:

$$\begin{aligned} \mathsf{d}(x, y) &= \|x \vee y\| - \|x \wedge y\| && \text{by definition of } \mathsf{d}(x, y) \\ &= \|x \vee x\| - \|x \wedge x\| && \text{by right hypothesis} \\ &= \|x\| - \|x\| && \text{by \textit{idempotent} property (Theorem C.3 page 302)} \\ &= 0 \end{aligned}$$

4. Proof that $\mathsf{d}(x, y) = \mathsf{d}(y, x)$:

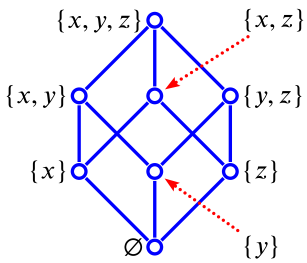
$$\begin{aligned} \mathsf{d}(x, y) &= \|x \vee y\| - \|x \wedge y\| && \text{by definition of } \mathsf{d}(x, y) \\ &= \|y \vee x\| - \|y \wedge x\| && \text{by \textit{commutative} property (Theorem C.3 page 302)} \\ &= \mathsf{d}(y, x) && \text{by definition of } \mathsf{d}(x, y) \end{aligned}$$

[36] Blumenthal (1970) page 25

5. Proof that $\mathsf{d}(x,y) \le \mathsf{d}(x,z) + \mathsf{d}(z,y)$:

$$\begin{aligned}
&\mathsf{d}(x,z) + \mathsf{d}(z,y) \\
&= \underbrace{(\|x \vee z\| - \|x \wedge z\|)}_{\mathsf{d}(x,y)} + \underbrace{(\|z \vee y\| - \|z \wedge y\|)}_{\mathsf{d}(z,y)} && \text{by definition of } \mathsf{d}(x,y) \\
&= (\|x \vee z\| + \|z \vee y\|) - (\|x \wedge z\| + \|z \wedge y\|) \\
&= (\|(x \vee z) \vee (z \vee y)\| + \|(x \vee z) \wedge (z \vee y)\|) \\
&\quad - (\|(x \wedge z) \vee (z \wedge y)\| + \|(x \wedge z) \wedge (z \wedge y)\|) && \text{by definition of } \|x\| \\
&= (\|(x \vee y) \vee z\| + \|(x \vee z) \wedge (z \vee y)\|) \\
&\quad - (\|(x \wedge z) \vee (z \wedge y)\| + \|(x \wedge y) \wedge z\|) && \text{by Theorem C.3 page 302} \\
&\ge (\|(x \vee y) \vee z\| + \|(x \wedge y) \vee z\|) \\
&\quad - (\|(x \vee y) \wedge z\| + \|(x \wedge y) \wedge z\|) && \text{by \textit{distributive inequality} (Theorem C.6 page 305)} \\
&\ge (\|(x \vee y) \vee z\| + \|(x \vee y) \wedge z\|) \\
&\quad - (\|(x \wedge y) \vee z\| + \|(x \wedge y) \wedge z\|) && \text{by \textit{minimax inequality} (Theorem C.5 page 304)} \\
&= (\|x \vee y\| + \|z\|) - (\|x \wedge y\| + \|z\|) && \text{by definition of } \|\cdot\| \text{ page 87} \\
&= \|x \vee y\| - \|x \wedge y\| \\
&= \mathsf{d}(x,y) && \text{by definition of } \mathsf{d} \text{ page 33}
\end{aligned}$$

✎



Example 3.17 (metric on powerset lattice). Let X be a set, 2^X the power set of X and $|A|$ the order of a set A (the number of elements in A). The tupple $(2^X, \cup, \cap; \subseteq)$ is a *lattice*.[37] A metric $\mathsf{d}(A,B) : 2^X \to \mathbb{R}$ can be defined as

EX $\quad \mathsf{d}(A,B) \triangleq \|A \cup B\| - \|A \cap B\| \quad$ where $\quad \|A\| \triangleq |A| \qquad \forall A,B \in 2^X$

The *Hasse diagram* for $(2^{\{x,y,z\}}, \subseteq)$ is illustrated in the figure to the left.

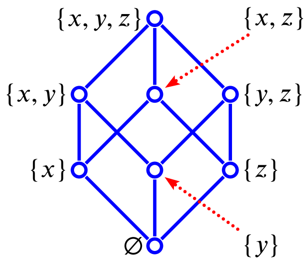
✎PROOF: Proof that $\|A\|$ satisfies the conditions of a lattice norm:

1. $A \subseteq B \implies |A| \le |B| \implies \|A\| \le \|B\|$

2. $\|A \cup B\| + \|A \cap B\| = |A \cup B| + |A \cap B| = |A| + |B| = \|A\| + \|B\|$

✎

Example 3.18 (Symmetric difference metric / Fréchet-Nikodym-Aronszayn distance). [38] Let X be a set, 2^X the power set of X, $A \triangle B$ the symmetric difference of sets $A, B \subseteq X$, and $|A|$ the order of a set A (the number of elements in A).[39]



EX $\quad$ The function $\quad \left\{\mathsf{d}(A,B) \triangleq |A \triangle B| \qquad \forall A,B \in 2^X\right\} \quad$ is a metric.

The tupple $(2^X, \subseteq, \cup, \cap)$ is a *lattice*. The *Hasse diagram* for $(2^{\{x,y,z\}}, \subseteq)$ is illustrated in the figure to the left. Notice that the distance (the metric) $\mathsf{d}(A,B)$ between any two sets A and B is just the shortest number of nodes that one must travel to get from A to B.

[37] $2^{\{x,y,z\}}$ *lattice example:* Example C.2 page 306

[39] Deza and Deza (2006) page 25

✎PROOF: The distance between any two sets is simply the number of elements that are different between the two sets. Therefore, this example is essentially the same as Example 3.15 (page 58) (Hamming distance example). ✏

3.6.2 Metrics on infinite sets

Example 3.19. [40] Let $\mathsf{d} : X \to \mathbb{R}$, $\mathsf{x} : X \to Y$, and $\mathsf{y} : X \to Y$ be functions on a set X. Then (X, d) is a metric space if d is defined as

EX $\mathsf{d}(\mathsf{x},\mathsf{y}) = \sup_{t \in A} |\mathsf{x}(t) - \mathsf{y}(t)|$

✎PROOF:

1. Proof that $\mathsf{d}(\mathsf{x},\mathsf{y}) = 0 \implies \mathsf{x} = \mathsf{y}$:

$$\begin{aligned}
0 &= \mathsf{d}(\mathsf{x},\mathsf{y}) && \text{by left hypothesis}\\
&= \sup_{t\in A} |\mathsf{x}(t) - \mathsf{y}(t)| && \text{by definition of } \mathsf{d}\\
\implies & \mathsf{x} = \mathsf{y}
\end{aligned}$$

2. Proof that $\mathsf{d}(\mathsf{x},\mathsf{y}) = 0 \impliedby \mathsf{x} = \mathsf{y}$:

$$\begin{aligned}
\mathsf{d}(\mathsf{x},\mathsf{y}) &= \sup_{t\in A} |\mathsf{x}(t) - \mathsf{y}(t)| && \text{by definition of } \mathsf{d}\\
&= \sup_{t\in A} |\mathsf{x}(t) - \mathsf{x}(t)| && \text{by right hypothesis}\\
&= 0
\end{aligned}$$

3. Proof that $\mathsf{d}(\mathsf{x},\mathsf{y}) \le \mathsf{d}(\mathsf{x},\mathsf{z}) + \mathsf{d}(\mathsf{y},\mathsf{z})$:

$$\begin{aligned}
\mathsf{d}(\mathsf{x},\mathsf{y}) &= \sup_{t\in A} |\mathsf{x}(t) - \mathsf{y}(t)| && \text{by definition of } \mathsf{d}\\
&= \sup_{t\in A} |\mathsf{x}(t) - \mathsf{z}(t) + \mathsf{z}(t) - \mathsf{y}(t)|\\
&\le \sup_{t\in A} |\mathsf{x}(t) - \mathsf{z}(t)| + \sup_{t\in A} |\mathsf{z}(t) - \mathsf{y}(t)|\\
&= \sup_{t\in A} |\mathsf{x}(t) - \mathsf{z}(t)| + \sup_{t\in A} |\mathsf{y}(t) - \mathsf{z}(t)|\\
&= \mathsf{d}(\mathsf{x},\mathsf{z}) + \mathsf{d}(\mathsf{y},\mathsf{z}) && \text{by definition of } \mathsf{d}
\end{aligned}$$

✏

Example 3.20 (p-adic metric). [41] For any rational number $x \in \mathbb{Q}$, there exists

1. the sequence of all prime numbers $(\!(p_i)\!)_{i\in\mathbb{N}} = (\!(1, 2, 3, 5, \ldots)\!)$,
2. a sequence of numerator exponents $(\!(n_i \in \mathbb{W})\!)_{i\in\mathbb{N}}$, and ,
3. a sequence of denominator exponents $(\!(m_i \in \mathbb{W})\!)_{i\in\mathbb{N}}$

such that $$x = \frac{\prod_{i\in\mathbb{N}} p_i^{n_i}}{\prod_{i\in\mathbb{N}} p_i^{m_i}}.$$

[40] Dieudonné (1969) page 29

[41] Dieudonné (1969) page 30

EX Then for any prime number p, the pair $(\mathbb{Q}, \mathsf{d}(x,y;p))$ is a metric space where

$$\mathsf{d}(x,y;p) = \begin{cases} 0 & \text{for } x = y \\ \dfrac{1}{p^{\theta(x-y;p)}} & \text{for } x \neq y \end{cases} \qquad \forall x,y \in \mathbb{Q}$$

where the function θ is defined as

$$\theta(x;p) = n_i - m_i \qquad \text{where the value of index } i \text{ is such that } p = p_i.$$

3.6.3 Metrics on n-tuples

Example 3.21. Let $(\!(x_n)\!)_1^N$ and $(\!(y_n)\!)_1^N$ be n-tuples over a set X. Let ϕ be a function in $\mathbb{R}^X$ on X.

EX

$$\left\{\begin{array}{ll} 1. & \phi \text{ is } \textit{convex} \\ 2. & \phi \text{ is } \textit{strictly monotonic} \\ 3. & \phi(0) = 0 \\ 4. & \log \circ \phi \circ \exp \text{ is } \textit{convex} \\ 5. & \phi(-x) = \phi(x)\ (\textit{even}) \end{array}\right\} \implies \left\{\begin{array}{l} \mathsf{d}\left((\!(x_n)\!), (\!(y_n)\!)\right) \triangleq \phi^{-1}\left(\displaystyle\sum_{n=1}^{N} \lambda_n \phi(x_n - y_n)\right) \\ \text{is a } \textit{metric}. \end{array}\right\}$$

In the special case that $\phi(x) \triangleq |x|$ such that

$$\mathsf{d}\left((\!(x_n)\!)_1^N, (\!(y_n)\!)_1^N\right) \triangleq \left(\sum_{n=1}^{N} \lambda_n |x_n - y_n|^r\right)^{\frac{1}{r}},$$

$\|x\| \triangleq \mathsf{d}(x,x)$ is a *norm*.

✎PROOF:

1. Proof that $(\!(x_n)\!)_1^N = (\!(y_n)\!)_1^N \implies \mathsf{d}\left((\!(x_n)\!)_1^N, (\!(y_n)\!)_1^N\right) = 0$: by definition of d.

2. Proof that $(\!(x_n)\!)_1^N = (\!(y_n)\!)_1^N \impliedby \mathsf{d}\left((\!(x_n)\!)_1^N, (\!(y_n)\!)_1^N\right) = 0$: by *strictly monotonic* property.

3. Proof that $\mathsf{d}\left((\!(x_n)\!)_1^N, (\!(y_n)\!)_1^N\right) \le \mathsf{d}\left((\!(z_n)\!)_1^N, (\!(x_n)\!)_1^N\right) \mathsf{d}\left((\!(z_n)\!)_1^N, (\!(y_n)\!)_1^N\right)$:

$$\begin{aligned} \mathsf{d}\left((\!(x_n)\!), (\!(y_n)\!)\right) &= \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n - y_n)\right) && \text{by definition of } \mathsf{d} \\ &= \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n - z_n + z_n - y_n)\right) \\ &\le \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n - z_n)\right) + \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(z_n - y_n)\right) && \text{by Theorem 11.2 page 163} \\ &= \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(z_n - x_n)\right) + \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(z_n - y_n)\right) && \text{by } \textit{even} \text{ property} \\ &= \mathsf{d}\left((\!(z_n)\!), (\!(x_n)\!)\right) + \mathsf{d}\left((\!(z_n)\!), (\!(y_n)\!)\right) && \text{by definition of } \mathsf{d} \end{aligned}$$

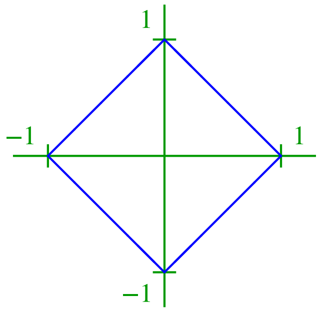
4. Therefore by Theorem 3.1 (page 34), d is a *metric*.

5. $\|\cdot\|$ is a *norm* by Proposition 9.8 (page 148).

⇨

Example 3.22 (Taxi-cab metric). [42]

EX
- $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \sum_{i=1}^{n} |x_i - y_i|$ *is a metric.*
- d *is generated by a norm.*
- $\mathsf{B}(0,1)$ in $(\mathbb{R}^n, \mathsf{d})$ *is convex.*
- $\operatorname{diam} \mathsf{B}(\boldsymbol{x}, r) = 2r$



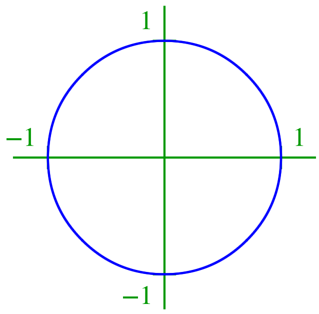
✎PROOF:

1. Proof that d is a metric:
   (a) By Example 3.13 (page 56), $\mathsf{p}(x, y) = |x - y|$ is a metric.
   (b) By the definition of d, $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \sum_{i=1}^{n} |x_i - y_i|$
   (c) And so d is a *Fréchet product metric* and is a *metric* by Theorem 3.9 (page 51).
2. Proof d is generated by a norm:
   (a) d is generated by a norm if and only if $\|\boldsymbol{x}\| \triangleq \sum_{i=1}^{n} |x_i|$ is a norm.
   (b) Proof that $\|\boldsymbol{x}\| \triangleq \sum_{i=1}^{n} |x_i|$ is a norm is given by Example 6.2 (page 89).
3. Proof that the ball is convex:
   By Theorem 6.4 (page 91), all metrics generated by a norm are convex.

✏

Example 3.23 (Euclidean metric). [43]

EX
- $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$ *is a metric.*
- d *is generated by a norm.*
- $\mathsf{B}(0,1)$ in $(\mathbb{R}^2, \mathsf{d})$ *is convex.*
- $\operatorname{diam} \mathsf{B}(\boldsymbol{x}, r) = 2r$



✎PROOF:

1. Proof that d is a metric:
   (a) By Example 3.13 (page 56), $\mathsf{p}(x, y) = |x - y|$ is a metric.
   (b) By the definition of d, $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$

[42] Deza and Deza (2006) page 240
Dieudonné (1969) page 29
[43] Dieudonné (1969) page 29

(c) And so d is a *Fréchet product metric* and is a *metric* by Theorem 3.9 (page 51).

2. Proof d is generated by a norm:

   (a) d is generated by a norm if and only if $\|\boldsymbol{x}\| \triangleq \sqrt{\sum_{i=1}^{n}|x_i|^2}$ is a norm.

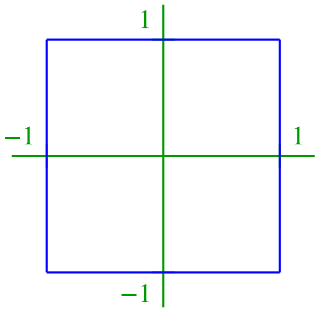
   (b) Proof that $\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^{n}|x_i|^2}$ is a norm is given by Example 6.2 (page 89).

3. Proof that the ball is convex:
   By Theorem 6.4 (page 91), all metrics generated by a norm are convex.

✏

Example 3.24 (Sup metric).

EX
- $\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \triangleq \max\left\{|x_i - y_i| \,|\, i = 1,2,\ldots,n\right\}$ *is a metric.*
- d *is generated by a norm.*
- $\mathsf{B}(0,1)$ in $(\mathbb{R}^n, \mathsf{d})$ *is convex.*
- $\operatorname{diam}\mathsf{B}(\boldsymbol{x},r) = 2\sqrt{2}r$



✎PROOF:

1. Proof that d is a metric:

   (a) By Example 3.13 (page 56), $\mathsf{p}(x,y) = |x-y|$ is a metric.

   (b) By the definition of d, $\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \triangleq \max\left\{\mathsf{p}(x_i,y_i) \,|\, i = 1,2,\ldots n\right\}$

   (c) And so d is a *Fréchet product metric* and is a *metric* by Theorem 3.9 (page 51).

2. Proof d is generated by a norm:

   (a) d is generated by a norm if and only if $\|\boldsymbol{x}\| \triangleq \max\left\{\left|x_i\right| \,|\, i = 1,2,\ldots n\right\}$ is a norm.

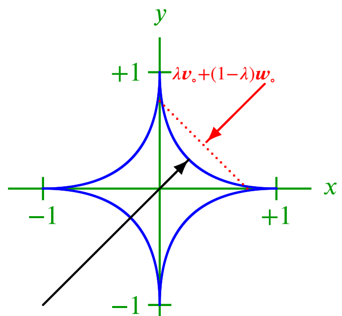
   (b) Proof that $\|\boldsymbol{x}\| \triangleq \max\left\{\left|x_i\right| \,|\, i = 1,2,\ldots n\right\}$ is a norm is given by Example 6.2 (page 89).

3. Proof that the ball is convex:
   By Theorem 6.4 (page 91), all metrics generated by a norm are convex.

✏

Example 3.25 (Parabolic metric). [44]

[44] Norfolk (1991) page 2
http://groups.google.com/group/sci.math/msg/c0eb7e19631c31ea



$x^2+y^2-2xy-2x-2y+1=0$
(parabolic equation)

Let X be a set and $\boldsymbol{x} \triangleq (\!| x_k \in X |\!)_1^N$ and $\boldsymbol{y} \triangleq (\!| y_k \in X |\!)_1^N$ be tuples on X.

EX

1. $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \sum_{i=1}^{n} \sqrt{|x_i - y_i|}$ *is a metric.*
2. d is *not generated by a norm.*
3. $\mathsf{B}(0,1)$ in $(\mathbb{R}^n, \mathsf{d})$ is *not convex.*

✎PROOF:

1. Proof that d is a metric:

Proof that $\boldsymbol{x} = \boldsymbol{y} \implies \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) = 0$:

$$
\begin{aligned}
\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) &= \mathsf{d}(\boldsymbol{x}, \boldsymbol{x}) && \text{by left hypothesis} \\
&= \sum_{i=1}^{n} \sqrt{|x_i - x_i|} && \text{by definition of } \mathsf{d} \\
&= 0
\end{aligned}
$$

Proof that $\boldsymbol{x} = \boldsymbol{y} \impliedby \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) = 0$:

$$
\begin{aligned}
0 &= \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) && \text{by right hypothesis} \\
&= \sum_{i=1}^{n} \sqrt{|x_i - y_i|} && \text{by definition of } \mathsf{d} \\
\implies & x_1 = x_2 \text{ and } y_1 = y_2 && \text{because } |\cdot| \text{ is positive} \\
\implies & \boldsymbol{x} = \boldsymbol{y} && \text{by definitions of } \boldsymbol{v} \text{ and } \boldsymbol{w}
\end{aligned}
$$

Proof that $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \le \mathsf{d}(\boldsymbol{x}, \boldsymbol{z}) + \mathsf{d}(\boldsymbol{z}, \boldsymbol{y})$:

$$
\begin{aligned}
\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) &= \sum_{i=1}^{n} \sqrt{|x_i - y_i|} && \text{by definition of } \mathsf{d} \\
&\le \sum_{i=1}^{n} \sqrt{|x_i - z_i| + |z_i - y_i|} && \text{by triangle inequality property of usual metric } |\cdot| \\
&= \sum_{i=1}^{n} \sqrt{2} \sqrt{\frac{1}{2}|x_i - z_i| + \frac{1}{2}|z_i - y_i|} \\
&= \sum_{i=1}^{n} \sqrt{2} \left(\frac{1}{2}\sqrt{|x_i - z_i|} + \frac{1}{2}\sqrt{|z_i - y_i|} \right) && \text{by Jensen's inequality page 154} \\
&= \frac{\sqrt{2}}{2} \sum_{i=1}^{n} \left(\sqrt{|x_i - z_i|} + \sqrt{|z_i - y_i|} \right) \\
&\le \sum_{i=1}^{n} \sqrt{|z_i - x_i|} + \sum_{i=1}^{n} \sqrt{|z_i - y_i|} \\
&= \mathsf{d}(\boldsymbol{z}, \boldsymbol{x}) + \mathsf{d}(\boldsymbol{z}, \boldsymbol{y})
\end{aligned}
$$

2. Proof d is not generated by a norm:

$$
\begin{aligned}
\|\alpha(\boldsymbol{v}-\boldsymbol{w})\| &= \|\alpha\boldsymbol{v}-\alpha\boldsymbol{w}\| \\
&= \mathsf{d}(\alpha\boldsymbol{v},\alpha\boldsymbol{w}) && \text{for some function } \|\cdot\| \\
&= \sqrt{|\alpha x_1-\alpha x_2|}+\sqrt{|\alpha y_1-\alpha y_2|} && \text{by definition of } \mathsf{d} \\
&= \sqrt{|\alpha||x_1-x_2|}+\sqrt{|\alpha||y_1-y_2|} \\
&= \sqrt{|\alpha|}\left(\sqrt{|x_1-x_2|}+\sqrt{|y_1-y_2|}\right) \\
&= \sqrt{|\alpha|}\mathsf{d}(\boldsymbol{v},\boldsymbol{w}) && \text{by definition of } \mathsf{d} \\
&= \sqrt{|\alpha|}\,\|\boldsymbol{v}-\boldsymbol{w}\| && \text{by definition of function } \|\cdot\| \\
&\neq |\alpha|\,\|\boldsymbol{v}-\boldsymbol{w}\| \\
\implies & \|\cdot\| \text{ is not a norm.} && \text{by homogeneous property of norms page 87}
\end{aligned}
$$

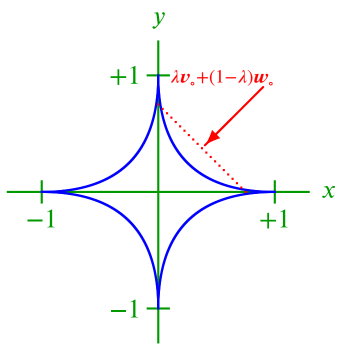
3. Proof that the ball is not convex: Let $\boldsymbol{v} \triangleq \left(\frac{3}{4}, 0\right)$ and $\boldsymbol{w} \triangleq \left(0, \frac{3}{4}\right)$.

$$
\begin{aligned}
\mathsf{d}\left(\mathbb{0}, \tfrac{1}{2}\boldsymbol{v}+(1-\tfrac{1}{2})\boldsymbol{w}\right) &= \mathsf{d}\left(\mathbb{0}, \tfrac{1}{2}\boldsymbol{v}+\tfrac{1}{2}\boldsymbol{w}\right) && \text{let } \lambda=\tfrac{1}{2} \\
&= \mathsf{d}\left((0,0), \tfrac{1}{2}\left(\tfrac{3}{4},0\right)+\tfrac{1}{2}\left(0,\tfrac{3}{4}\right)\right) && \text{by definition of } \boldsymbol{v} \text{ and } \boldsymbol{w} \\
&= \mathsf{d}\left((0,0), \left(\tfrac{3}{8},0\right)+\left(0,\tfrac{3}{8}\right)\right) \\
&= \mathsf{d}\left((0,0), \left(\tfrac{3}{8},\tfrac{3}{8}\right)\right) \\
&= \sqrt{\left|0-\tfrac{3}{8}\right|}+\sqrt{\left|0-\tfrac{3}{8}\right|} && \text{by definition of } \mathsf{d} \\
&= 2\sqrt{\tfrac{3}{8}} \\
&= \tfrac{2}{2}\sqrt{\tfrac{3}{2}} \\
&> 1
\end{aligned}
$$

✏

Example 3.26 (Inverse tangent metric). [45]

Let X be a set and $\boldsymbol{x} \triangleq (\!| x_k \in X |\!)_1^N$ and $\boldsymbol{y} \triangleq (\!| y_k \in X |\!)_1^N$ be sequences on X.



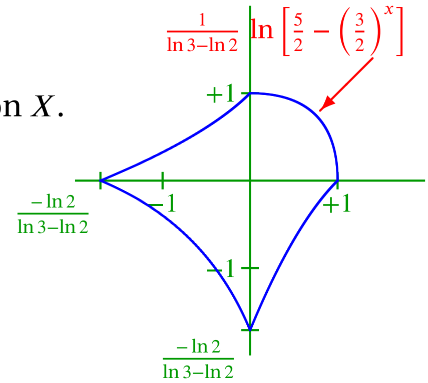
EX $\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \triangleq \sum_{i=1}^{n} |\arctan x_i - \arctan y_i|$ *is a* METRIC.

✎PROOF:

[45] Copson (1968) page 25
Khamsi and Kirk (2001) page 14

1. The function $\mathsf{d}(x, y) \triangleq |x - y|$ is a *metric* (the *usual metric*, Example 3.13 page 56).
2. The function $\mathsf{g}(x) \triangleq \arctan(x)$ is *injective* in $\mathbb{R}^{\mathbb{R}}$.
3. Therefore, d is a *Pullback metric* (or g-*transform metric*), and by Theorem 3.6 (page 46), d is a *metric*.

Example 3.27 (Exponential metric).



Let X be a set and $\boldsymbol{x} \triangleq (\!(x_k \in X)\!)_1^N$ and $\boldsymbol{y} \triangleq (\!(y_k \in X)\!)_1^N$ be sequences on X.

EX

1. $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq 2 \sum_{i=1}^{n} \left| \left(\frac{3}{2}\right)^{x_i} - \left(\frac{3}{2}\right)^{y_i} \right|$ *is a metric.*
2. d is *not generated by a norm.*
3. $\mathsf{B}(\theta, 1)$ in $(\mathbb{R}^n, \mathsf{d})$ is *not convex.*

✎Proof:

1. Proof that d is a metric:
   (a) By Example 3.13 (page 56), $\mathsf{p}(x, y) \triangleq |x - y|$ is a metric (the *usual metric*).
   (b) The function $\mathsf{f}(x) \triangleq 2\left(\left(\frac{3}{2}\right)^x - 1\right)$ is strictly increasing in x. Proof:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} 2\left(\left(\frac{3}{2}\right)^x - 1\right) \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{2}\right)^x \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln\frac{3}{2}}\right)^x \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln\frac{3}{2}} \\
&= 2\left(\ln\frac{3}{2}\right) e^{x\ln\frac{3}{2}} \\
&= 2\left(\ln\frac{3}{2}\right)\left(e^{\ln\frac{3}{2}}\right)^x \\
&= 2\left(\ln\frac{3}{2}\right)\left(\frac{3}{2}\right)^x \\
&> 0 \qquad \forall x \in \mathbb{R}
\end{aligned}
$$

   (c) Therefore, by Theorem 3.6 (page 46), d is a metric.

2. Proof that d is not generated by a norm:

$$
\begin{aligned}
\|\alpha(\boldsymbol{x}-\boldsymbol{y})\| &= \|\alpha\boldsymbol{x}-\alpha\boldsymbol{y}\| \\
&= \mathsf{d}(\alpha\boldsymbol{x},\alpha\boldsymbol{y}) && \text{for some function } \|\cdot\| \\
&= 2\sum_{i=1}^{n}\left|\left(\frac{3}{2}\right)^{\alpha x_i}-\left(\frac{3}{2}\right)^{\alpha y_i}\right| && \text{by definition of } \mathsf{d} \\
&\neq 2\sum_{i=1}^{n}\left|\alpha\left(\frac{3}{2}\right)^{x_i}-\alpha\left(\frac{3}{2}\right)^{y_i}\right| \\
&= |\alpha|2\sum_{i=1}^{n}\left|\left(\frac{3}{2}\right)^{x_i}-\left(\frac{3}{2}\right)^{y_i}\right| \\
&= |\alpha|\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) && \text{by definition of } \mathsf{d} \\
&= |\alpha|\ \|\boldsymbol{x}-\boldsymbol{y}\|
\end{aligned}
$$

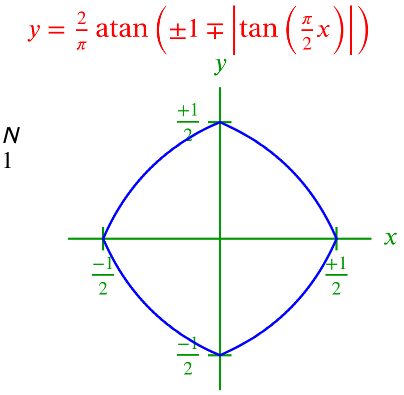
3. Proof that the ball is not convex:

(a) The function $\mathsf{p}(\theta,\boldsymbol{x}) \triangleq 2\sum_{i=1}^{n}\left|\left(\frac{3}{2}\right)^{\theta_i}-\left(\frac{3}{2}\right)^{x_i}\right|$ is not in general convex. Proof:

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i^2}\mathsf{p}(\mathbb{0},\boldsymbol{x}) &= \frac{\partial^2}{\partial x_i^2}2\sum_{i=1}^{n}\left|\left(\frac{3}{2}\right)^{0}-\left(\frac{3}{2}\right)^{x_i}\right| \\
&= \frac{\partial^2}{\partial x_i^2}2\left|1-\left(\frac{3}{2}\right)^{x_i}\right| \\
&= 2\frac{\partial^2}{\partial x_i^2}\left(1-\left(\frac{3}{2}\right)^{x_i}\right) && \text{for } x_i<0 \\
&= -2\frac{\partial}{\partial x_i}\left(\ln\frac{3}{2}\right)\left(\frac{3}{2}\right)^{x_i} && \text{for } x_i<0 \\
&= -2\left(\ln\frac{3}{2}\right)^{2}\left(\frac{3}{2}\right)^{x_i} && \text{for } x_i<0 \\
&< 0 && \text{for } x_i<0 \\
\implies &\ \mathsf{d} \text{ is not convex}
\end{aligned}
$$

(b) Therefore by Theorem 5.2 (page 85), the ball is not convex.

Example 3.28 (Tangential metric).



Let $X = \{x \in \mathbb{R} | x(-1 : 1)\}$ be a set and $\boldsymbol{x} \triangleq (\!|x_i \in X|\!)_1^N$ and $\boldsymbol{y} \triangleq (\!|y_i \in X|\!)_1^N$ be sequences on X.

EX

1. $\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \triangleq \sum_{i=1}^{n}\left|\tan\left(\frac{\pi}{2}x_i\right)-\tan\left(\frac{\pi}{2}y_i\right)\right|$ *is a metric.*
2. d is *not generated by a norm.*
3. $\mathsf{B}(\theta,1)$ in $\left(\mathbb{R}^n,\mathsf{d}\right)$ is *convex.*

✎PROOF:

1. Proof that d is a metric:
   (a) By Example 3.13 (page 56), $\mathsf{p}(x, y) \triangleq |x - y|$ is a metric (the *usual metric*).
   (b) The function $\mathsf{f}(x) \triangleq \tan\left(\frac{\pi}{2}x\right)$ is strictly increasing in x. Proof:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\tan\left(\frac{\pi}{2}x\right)\\
&= \frac{\pi}{2}\sec^2\left(\frac{\pi}{2}x\right)\\
&> 0 \qquad \forall x(-1:1)
\end{aligned}$$

   (c) Therefore, by Theorem 3.6 (page 46), d is a metric.

2. Proof that d is not generated by a norm:

$$\begin{aligned}
\|\alpha(\boldsymbol{x}-\boldsymbol{y})\| &= \|\alpha\boldsymbol{x}-\alpha\boldsymbol{y}\|\\
&= \mathsf{d}(\alpha\boldsymbol{x},\alpha\boldsymbol{y}) && \text{for some function } \|\cdot\|\\
&= \sum_{i=1}^{n}\left|\tan\left(\frac{\pi}{2}\alpha x_i\right)-\tan\left(\frac{\pi}{2}\alpha y_i\right)\right| && \text{by definition of } \mathsf{d}\\
&\neq \sum_{i=1}^{n}\left|\alpha\tan\left(\frac{\pi}{2}x_i\right)-\alpha\tan\left(\frac{\pi}{2}y_i\right)\right|\\
&= |\alpha|\sum_{i=1}^{n}\left|\tan\left(\frac{\pi}{2}x_i\right)-\tan\left(\frac{\pi}{2}y_i\right)\right|\\
&= |\alpha|\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) && \text{by definition of } \mathsf{d}\\
&= |\alpha|\ \|\boldsymbol{x}-\boldsymbol{y}\|
\end{aligned}$$

3. Proof that the ball is convex:
   (a) The function $\mathsf{p}(\theta, \boldsymbol{x}) \triangleq \sum_{i=1}^{n}\left|\tan\left(\frac{\pi}{2}\theta_i\right)-\tan\left(\frac{\pi}{2}x_i\right)\right|$ is convex. Proof:

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2}\mathsf{d}(\mathbb{0},\boldsymbol{x}) &= \frac{\partial^2}{\partial x_i^2}\sum_{i=1}^{n}\left|\tan(0)-\tan\left(\frac{\pi}{2}x_i\right)\right|\\
&= \frac{\partial^2}{\partial x_i^2}\left|\tan(0)-\tan\left(\frac{\pi}{2}x_i\right)\right|\\
&= \begin{cases}\frac{\partial^2}{\partial x_i^2}\tan\left(\frac{\pi}{2}x_i\right) & \text{for } x_i \geq 0\\ \frac{\partial^2}{\partial x_i^2}-\tan\left(\frac{\pi}{2}x_i\right) & \text{for } x_i < 0\end{cases}\\
&= \begin{cases}\frac{\partial}{\partial x_i}\frac{\pi}{2}\sec^2\left(\frac{\pi}{2}x_i\right) & \text{for } x_i \geq 0\\ \frac{\partial}{\partial x_i}-\frac{\pi}{2}\sec^2\left(\frac{\pi}{2}x_i\right) & \text{for } x_i < 0\end{cases}\\
&= \begin{cases}\frac{\pi}{2}\frac{\pi}{2}2\sec^2\left(\frac{\pi}{2}x_i\right)\tan\left(\frac{\pi}{2}x_i\right) & \text{for } x_i \geq 0\\ -\frac{\pi}{2}\frac{\pi}{2}2\sec^2\left(\frac{\pi}{2}x_i\right)\tan\left(\frac{\pi}{2}x_i\right) & \text{for } x_i < 0\end{cases}\\
&\geq 0
\end{aligned}$$

   (b) Therefore by Theorem 5.2 (page 85), the ball is convex.

✏

Example 3.29. [46] Let $\mathsf{d}(x,y) = |x-y|^2$ where $|\cdot|$ is the absolute value on $\mathbb{R}$.

- Balls in $(\mathbb{R}, \mathsf{d})$ are *convex* because they are simple intervals.
- But yet d is *not generated by a norm* because

$$\mathsf{d}(ax, ay) = |ax - ay|^2 = |a(x-y)|^2 = |a|^2|x-y|^2 \neq |a||x-y|^2.$$

Example 3.30. [47] Let $\|\cdot\|_2$ be the l_2 norm. Consider the *post office metric*

$$\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \begin{cases} \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2 & \text{for } \boldsymbol{x} \neq \boldsymbol{y} \\ 0 & \text{for } \boldsymbol{x} = \boldsymbol{y} \end{cases}$$

① The post office metric is *not generated by a norm.*
② The ball generated by the post office metric is in general *not convex.*

✎PROOF:

1. Proof that d is not a norm:

$$\begin{aligned}
\|\mathbb{0}\| &= \|\boldsymbol{x} - \boldsymbol{x}\| \\
&= \mathsf{d}(\boldsymbol{x}, \boldsymbol{x}) && \text{by assumption that } \mathsf{d} \text{ can be generated by a norm } \|\cdot\| \\
&= \|\boldsymbol{x}\|_2 + \|\boldsymbol{x}\|_2 && \text{by definition of the post office metric} \\
&= 2\,\|\boldsymbol{x}\|_2 \\
&\geq 0 && \text{by positive property of } \|\cdot\| \text{ page 87}
\end{aligned}$$

This implies $\|\cdot\|$ is not a norm because it fails the *non-degenerate* property of norms ($\|\mathbb{0}\| = 0$—see Definition 6.1 page 87) and therefore d is not generated by a norm.

2. Proof that the ball generated by d is not convex:
Consider the ball with radius 1 and center $\frac{3}{4}$ generated by the post office metric.

   (a) $\frac{3}{4}$ is in the ball because $\mathsf{d}\left(\frac{3}{4}, \frac{3}{4}\right) = 0 \leq 0$

   (b) $\frac{1}{8}$ is in the ball because $\mathsf{d}\left(\frac{3}{4}, \frac{1}{8}\right) = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \leq 1$

   (c) *But* $\frac{1}{2}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{1}{8} = \frac{7}{16}$ which is *not* in the ball because $\mathsf{d}\left(\frac{7}{16}, \frac{3}{4}\right) = \frac{7}{16} + \frac{3}{4} = \frac{19}{16} > 1.$

✏

Example 3.31 (The bounded metric). [48] Let X be a set and $\mathsf{d} : X^2 \to \mathbb{R}^{\vdash}$.

EX

- $\mathsf{d}(x,y) \triangleq \dfrac{\mathsf{p}(x,y)}{1+\mathsf{p}(x,y)}$ is a metric.
- d is *not* generated by a norm.
- $\mathsf{B}(0,1) = X$
- $\operatorname{diam} \mathsf{B}(0,1) = \operatorname{diam} X$

✎PROOF:

1. Proof that $\mathsf{d}(x,y)$ is a metric (using Theorem 3.1 page 34): Proposition 3.6 (page 55).

[46] http://groups.google.com/group/sci.math/browse_thread/thread/da44b8a80e97d40f/a977cecea243ad0a
[47] Giles (1987) page 17
http://groups.google.com/group/sci.math/msg/38bb848a9c6d5c29
[48] Copson (1968) page 22

2. Proof that d is not generated by a norm:

$$\begin{aligned}
\|\alpha x\| &= \mathsf{d}(\alpha x, 0) && \text{for some function } \|\cdot\| \\
&= \frac{\mathsf{p}(\alpha x, 0)}{1 + \mathsf{p}(\alpha x, 0)} \\
&= \frac{|\alpha|\mathsf{p}(x, 0)}{1 + |\alpha|\mathsf{p}(x, 0)} && \text{assuming p is homogeneous} \\
&= |\alpha|\left[\frac{\mathsf{p}(x, 0)}{1 + |\alpha|\mathsf{p}(x, 0)}\right] \\
&\neq |\alpha|\left[\frac{\mathsf{p}(x, 0)}{1 + \mathsf{p}(x, 0)}\right] \\
&= |\alpha|\mathsf{d}(x, 0) \\
&= |\alpha|\ \|x\|
\end{aligned}$$

3. Proof that $\mathsf{B}(0, 1) = \{0\}$:

$$\begin{aligned}
\mathsf{B}(0, 1) &= \{x \in \boldsymbol{X} | \mathsf{d}(0, x) < 1\} && \text{by definition of open ball B page 28} \\
&= \left\{x \in \boldsymbol{X} | \frac{\mathsf{p}(x, y)}{1 + \mathsf{p}(x, y)} < 1\right\} && \text{by definition d} \\
&= \{x \in \boldsymbol{X}\} \\
&= X
\end{aligned}$$

4. Proof that $\operatorname{diam} \mathsf{B}(0, 1) = \operatorname{diam} X$:

$$\operatorname{diam} \mathsf{B}(0, 1) = \operatorname{diam} X \qquad \text{by previous result}$$

✏

3.7 Literature

LITERATURE SURVEY:

- general reference books about *metric spaces*:
  - Copson (1968)
  - Giles (1987)
- more sophisticated references:
  - Blumenthal (1970)
  - Busemann (1955a)
    - Busemann (1955b)
- *metric spaces* and *convexity*:
  - Khamsi and Kirk (2001)
- "*length spaces*":
  - Burago et al. (2001)
- spaces of *metric spaces*:
  - Burago et al. (2001)
- Very large collections of metric examples:
  - Deza and Deza (2006) ⟨ISBN:0444520872⟩
  - Deza and Deza (2009) ⟨ISBN:3642002331⟩

CHAPTER 4

LINEAR SPACES

❝*The geometric calculus, in general, consists in a system of operations on geometric entities, and their consequences, analogous to those that algebra has on the numbers. It permits the expression in formulas of the results of geometric constructions, the representation with equations of propositions of geometry, and the substitution of a transformation of equations for a verbal argument.*❞
Giuseppe Peano (1858–1932), Italian mathematician, credited with being one of the first to introduce the concept of the *linear space* (*vector space*).[1]

4.1 Definition and basic results

A *metric space* (Definition 3.1 page 33) is a *set* together with nothing else save a *metric* that gives the space a *topology* (Definition 1.1 page 3). A *linear space* (next definition) in general has no topology but does have some additional *algebraic* structure (Appendix F page 341) that is useful in generalizing a number of mathematical concepts. If one wishes to have both algebraic structure and a topology, then this can be accomplished by appending a *topology* to a *linear space* giving a *topological linear space* (Definition 5.1 page 83), a *metric* giving a *metric linear space* (Definition 3.1 page 33), an *inner product* giving an *inner product space* (Definition 7.1 page 99), or a *norm* giving a *normed linear space* (Definition 6.1 page 87).

Definition 4.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD *(Definition F.5 page 342). Let* X *be a set, let* $+$ *be an* OPERATOR *(Definition 15.1 page 205) in* X^{X^2}*, and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

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[1] quote: Peano (1888b) page ix
image http://en.wikipedia.org/wiki/File:Giuseppe_Peano.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991) page 41, Halmos (1948) pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

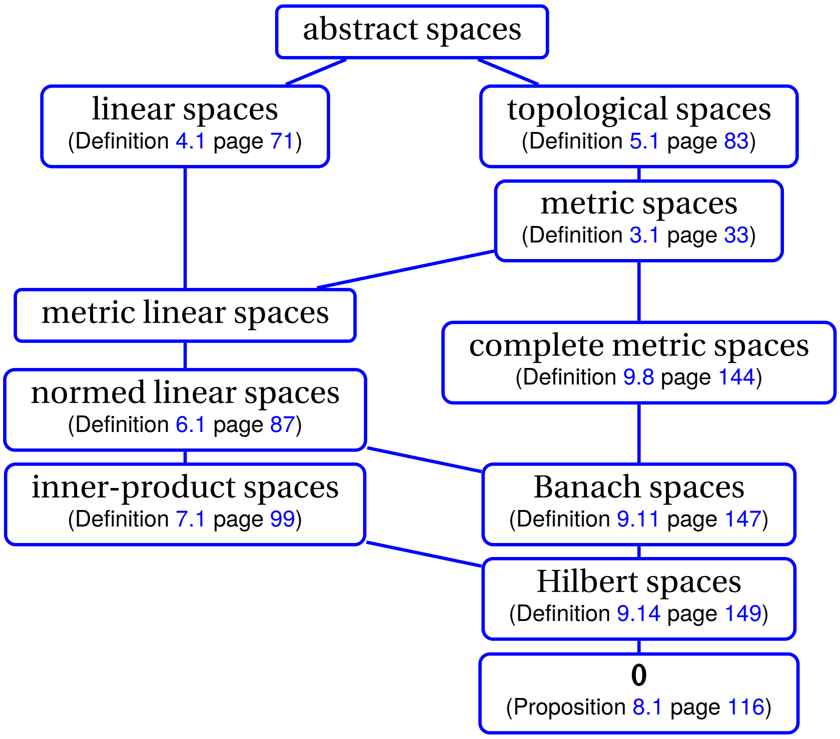

Figure 4.1: Lattice of mathematical spaces

DEF

The structure $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1.	$\exists \mathbb{0} \in X$	such that	$\boldsymbol{x} + \mathbb{0}$	$=$	$\boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	(+ IDENTITY)
2.	$\exists \boldsymbol{y} \in X$	such that	$\boldsymbol{x} + \boldsymbol{y}$	$=$	$\mathbb{0}$	$\forall \boldsymbol{x} \in X$	(+ INVERSE)
3.			$(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$	$=$	$\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$	$\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$	(+ *is* ASSOCIATIVE)
4.			$\boldsymbol{x} + \boldsymbol{y}$	$=$	$\boldsymbol{y} + \boldsymbol{x}$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(+ *is* COMMUTATIVE)
5.			$1 \cdot \boldsymbol{x}$	$=$	$\boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	($\cdot$ IDENTITY)
6.			$\alpha \cdot (\beta \cdot \boldsymbol{x})$	$=$	$(\alpha \cdot \beta) \cdot \boldsymbol{x}$	$\forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$	($\cdot$ ASSOCIATES *with* $\cdot$)
7.			$\alpha \cdot (\boldsymbol{x} + \boldsymbol{y})$	$=$	$(\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y})$	$\forall \alpha \in S$ *and* $\boldsymbol{x}, \boldsymbol{y} \in X$	($\cdot$ DISTRIBUTES *over* +)
8.			$(\alpha + \beta) \cdot \boldsymbol{x}$	$=$	$(\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x})$	$\forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$	($\cdot$ PSEUDO-DISTRIBUTES *over* +)

(Conditions 1–4 are marked $*$.)

The set X *is called the* ***underlying set***. *The elements of* X *are called* ***vectors***. *The elements of* $\mathbb{F}$ *are called* ***scalars***. *A* LINEAR SPACE *is also called a* ***vector space***. *If* $\mathbb{F} \triangleq \mathbb{R}$, *then* $\boldsymbol{\Omega}$ *is a* ***real linear space***. *If* $\mathbb{F} \triangleq \mathbb{C}$, *then* $\boldsymbol{\Omega}$ *is a* ***complex linear space***.

Definition 4.2. *Let* $\boldsymbol{L}_1 \triangleq (X_1, +, \cdot, (\mathbb{F}_1, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{L}_2 \triangleq (X_2, +, \cdot, (\mathbb{F}_2, \dot{+}, \dot{\times}))$.

DEF

$\boldsymbol{\Omega}_2$ *is a* ***linear subspace*** *of* $\boldsymbol{\Omega}_1$ *if*

1. $\boldsymbol{L}_1$ *is a* LINEAR SPACE (*Definition 4.1 page 71*) *and*
2. $\boldsymbol{L}_2$ *is a* LINEAR SPACE (*Definition 4.1 page 71*) *and*
3. $\mathbb{F}_2 \subseteq \mathbb{F}_1$ *and*
4. $X_2 \subseteq X_1$ *and*

Remark 4.1. [3] By the first four conditions ($*$]) listed in Definition 4.1, $(X, +)$ is a **commutative group** (or **abelian group**).

[3] Akhiezer and Glazman (1993) page 1, Haaser and Sullivan (1991) page 41

Often when discussing a linear space, the operator $\cdot$ is simply expressed with juxtaposition (e.g. $\alpha\boldsymbol{x}$ is equivalent to $\alpha \cdot \boldsymbol{x}$). In doing this, there is no risk of ambiguity between scalar-vector multiplication and scalar-scalar multiplication because the operands uniquely identify the precise operator.[4]

Example 4.1 (tuples in $\mathbb{F}^N$). [5] Let $⦇x_n⦈_1^N$ be an *N-tuple* (Definition 9.1 page 131) over a *field* (Definition F.5 page 342) $(\mathbb{F}, +, \cdot, 0, 1)$.

EX

$$\begin{array}{lrcll} \text{Let} & X & \triangleq & \left\{⦇x_n⦈_1^N \,|\, x_n \in \mathbb{F}\right\} & \text{and} \\ & ⦇x_n⦈_1^N + ⦇y_n⦈_1^N & \triangleq & ⦇x_n \dotplus y_n⦈_1^N \qquad \forall ⦇x_n⦈\in X & \text{and} \\ & \alpha \cdot ⦇x_n⦈_1^N & \triangleq & ⦇\alpha \dot{\times} x_n⦈_1^N \qquad \forall ⦇x_n⦈\in X, \alpha\in\mathbb{F}. & \end{array}$$

Then the structure $\left(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times})\right)$ is a *linear space*.

Example 4.2 (real numbers). [6] Let $(\mathbb{R}, +, \cdot, 0, 1)$ be the field of real numbers.

EX

The structure $(\mathbb{R}, +, \cdot, (\mathbb{R}, +, \cdot))$ is a *linear space*.
That is, the field of real numbers forms a linear space over itself.

Example 4.3 (functions). [7] Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a field. Let Y^X be the set of all functions with domain X and range Y.

EX

$$\begin{array}{lrcllll} \text{Let} & [\mathsf{f} + \mathsf{g}](x) & \triangleq & \mathsf{f}(x) + \mathsf{g}(x) & \forall \mathsf{f},\mathsf{g}\in Y^X & \textit{(pointwise addition)} & \text{and} \\ & [\alpha \cdot \mathsf{f}](x) & \triangleq & \alpha \cdot [\mathsf{f}(x)] & \forall \mathsf{f}\in Y^X, \alpha\in\mathbb{F}. & & \end{array}$$

Then the structure $\left(Y^X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times})\right)$ is a *linear space*.

Example 4.4 (functions onto $\mathbb{F}$). [8] Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a field. Let $\mathbb{F}^X$ be the set of all functions with domain X and range $\mathbb{F}$.

EX

$$\begin{array}{lrcllll} \text{Let} & [\mathsf{f} + \mathsf{g}](x) & \triangleq & \mathsf{f}(x) + \mathsf{g}(x) & \forall \mathsf{f},\mathsf{g}\in \mathbb{F}^X & \textit{(pointwise addition)} & \text{and} \\ & [\alpha \cdot \mathsf{f}](x) & \triangleq & \alpha \cdot [\mathsf{f}(x)] & \forall \mathsf{f}\in \mathbb{F}^X, \alpha\in\mathbb{F}. & & \end{array}$$

Then the structure $\left(\mathbb{F}^X, +, \cdot, (\mathbb{F}, +, \cdot)\right)$ is a *linear space*.

Theorem 4.1 (Additive identity properties). [9] *Let $\left(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times})\right)$ be a linear space, 0 the* ADDITIVE IDENTITY ELEMENT *(Definition F.1 page 341) with respect to $\dotplus$, and $\mathbb{0}$ the* ADDITIVE IDENTITY ELEMENT *with respect to $+$.*

THM

$$\begin{array}{llcllll} 1. & 0\boldsymbol{x} & = & \mathbb{0} & & & \forall \boldsymbol{x}\in X \\ 2. & \alpha\mathbb{0} & = & \mathbb{0} & & & \forall \alpha\in\mathbb{F} \\ 3. & \alpha\boldsymbol{x} & = & \mathbb{0} & \Longrightarrow & \alpha = 0 \textit{ or } \boldsymbol{x} = \mathbb{0} & \\ 4. & \boldsymbol{x} + \boldsymbol{x} & = & \boldsymbol{x} & \Longrightarrow & \boldsymbol{x} = \mathbb{0} & \\ 5. & \alpha \neq 0 \textit{ and } \boldsymbol{x} \neq \mathbb{0} & & & \Longrightarrow & \alpha\boldsymbol{x} \neq \mathbb{0} & \end{array}$$

✎PROOF:

[4] *Operator overload* is a technique in which two fundamentally different operators or functions share the same symbol or label. It is inherent in the programming language *C++* and is therein called *operator overload*. In C++, you can define two (or more) operators or functions that share the same symbol or name, but yet are completely different. Two such operators (or functions) are distinguished from each other by the type of their operands. So for example, in C++, you can define an $m \times n$ matrix *type* and use operator overload to define a + operator that operates on this new matrix type. So if variables x and y are of floating point type and A and B are of the matrix type, you can then add either type using the same syntax style:

x+y (add two floating point numbers)
A+B (add two matrices)

Even though both of these operations "look" the same, they are of course fundamentally different.

[5] Kubrusly (2001) page 41 ⟨Example 2D⟩
[6] Kubrusly (2001) page 41 ⟨Example 2D⟩, Hamel (1905)
[7] Kubrusly (2001) page 42 ⟨Example 2F⟩
[8] Kubrusly (2001) page 41 ⟨Example 2E⟩
[9] Berberian (1961) page 6 ⟨Theorem 1⟩, Michel and Herget (1993) page 77

1. Proof that $0\boldsymbol{x} = \mathbb{0}$:

$$\begin{aligned}
0\boldsymbol{x} &= 0\boldsymbol{x} + 0\mathbb{0} && \text{by definition of + additive identity element}\\
&= 0\boldsymbol{x} + 0\boldsymbol{x} + (-0\boldsymbol{x}) && \text{by definition of + additive inverse}\\
&= (0+0)\boldsymbol{x} + (-0\cdot\boldsymbol{x}) && \text{by definition of + additive identity element}\\
&= 0\boldsymbol{x} + (-0\boldsymbol{x}) && \text{by Definition 4.1 property 4}\\
&= \mathbb{0} && \text{by definition of + additive identity element}
\end{aligned}$$

2. Proof that $\alpha\mathbb{0} = \mathbb{0}$:

$$\begin{aligned}
\alpha\mathbb{0} &= \alpha(0\boldsymbol{x}) && \text{by item 1}\\
&= (\alpha 0)\boldsymbol{x} && \text{by Definition 4.1 property 6}\\
&= 0\boldsymbol{x}\\
&= \mathbb{0} && \text{by item 1}
\end{aligned}$$

3. Proof that $\alpha \neq 0$ and $\boldsymbol{x} \neq \mathbb{0} \implies \alpha\boldsymbol{x} \neq \mathbb{0}$: Suppose $\alpha\boldsymbol{x} = \mathbb{0}$. Then

$$\begin{aligned}
\boldsymbol{x} &= \left(\frac{1}{\alpha}\alpha\right)\boldsymbol{x}\\
&= \frac{1}{\alpha}(\alpha\boldsymbol{x})\\
&= \frac{1}{\alpha}\mathbb{0}\\
&= \mathbb{0} && \text{by item 2}\\
&\implies \boldsymbol{x} = \mathbb{0}
\end{aligned}$$

This is a *contradiction* and so $\alpha\boldsymbol{x} \neq \mathbb{0}$.

4. Proof that $\alpha\boldsymbol{x} = \mathbb{0} \implies \alpha = 0$ or $\boldsymbol{x} = \mathbb{0}$: contrapositive argument of item 3

5. Proof that $\boldsymbol{x} + \boldsymbol{x} = \boldsymbol{x} \implies \boldsymbol{x} = \mathbb{0}$:

$$\begin{aligned}
\boldsymbol{x} &= \boldsymbol{x} + \mathbb{0} && \text{by } \textit{additive identity} \text{ property (Definition 4.1 page 71)}\\
&= \boldsymbol{x} + [\boldsymbol{x} + (-\boldsymbol{x})] && \text{by } \textit{additive inverse} \text{ property (Definition 4.1 page 71)}\\
&= [\boldsymbol{x} + \boldsymbol{x}] + (-\boldsymbol{x}) && \text{by } \textit{associative} \text{ property (Definition 4.1 page 71)}\\
&= \boldsymbol{x} + (-\boldsymbol{x}) && \text{by left hypothesis}\\
&= \mathbb{0} && \text{by } \textit{additive inverse} \text{ property (Definition 4.1 page 71)}
\end{aligned}$$

✎

Definition 4.3. [10] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space with vectors* $\boldsymbol{x}, \boldsymbol{y} \in X$. *Let* $-\boldsymbol{y}$ *be the additive inverse of* $\boldsymbol{y}$ *such that* $\boldsymbol{y} + (-\boldsymbol{y}) = \mathbb{0}$.

DEF: *The* ***difference*** *of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is* $\boldsymbol{x} + (-\boldsymbol{y})$ *and is denoted* $\boldsymbol{x} - \boldsymbol{y}$.

Theorem 4.2 (Additive inverse properties). [11] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space,* $\mathbb{0}$ *the* ADDITIVE IDENTITY ELEMENT *with respect to* +, *and* $-\boldsymbol{x}$ *the* ADDITIVE INVERSE *(Definition F.1 page 341) of* $\boldsymbol{x}$ *with respect to* +.

THM:

1.	$\boldsymbol{x} + \boldsymbol{y} = \mathbb{0} \implies \boldsymbol{x} = -\boldsymbol{y}$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$		*(additive inverse is* UNIQUE*)*
2.	$(-\alpha)\boldsymbol{x} = -(\alpha\boldsymbol{x}) = \alpha(-\boldsymbol{x})$	$\forall \boldsymbol{x} \in X, \alpha \in \mathbb{F}$		
3.	$\alpha(\boldsymbol{x} - \boldsymbol{y}) = \alpha\boldsymbol{x} - \alpha\boldsymbol{y}$	$\forall \boldsymbol{x},\boldsymbol{y} \in X, \alpha \in \mathbb{F}$		(DISTRIBUTIVE)
4.	$(\alpha - \beta)\boldsymbol{x} = \alpha\boldsymbol{x} - \beta\boldsymbol{x}$	$\forall \boldsymbol{x} \in X, \alpha,\beta \in \mathbb{F}$		(DISTRIBUTIVE)

[10] Berberian (1961) page 7 ⟨Definition 1⟩

[11] Berberian (1961) page 7 ⟨Corollary 1⟩, Michel and Herget (1993) page 77, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

✎PROOF:

1. Proof that $\mathbf{x}+\mathbf{y}=0 \implies \mathbf{x}=-\mathbf{y}$:

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} - \mathbb{0} \\
&= \mathbf{x} - (\mathbf{x}+\mathbf{y}) && \text{by left hypothesis} \\
&= (\mathbf{x}-\mathbf{x}) - \mathbf{y} \\
&= \mathbb{0} - \mathbf{y} \\
&= -\mathbf{y}
\end{aligned}$$

2. Proof that $(-\alpha)\mathbf{x} = -(\alpha\mathbf{x})$:

$$\begin{aligned}
\mathbb{0} &= 0\mathbf{x} && \text{by Theorem 4.1 page 73} \\
&= (\alpha-\alpha)\mathbf{x} && \text{by field property of } \mathbb{F} \\
&= [\alpha+(-\alpha)]\mathbf{x} && \text{by field property of } \mathbb{F} \\
&= \alpha\mathbf{x} + (-\alpha)\mathbf{x} && \text{by Definition 4.1 page 71} \\
\implies -(\alpha\mathbf{x}) &= (-\alpha)\mathbf{x} && \text{by item (1) page 75}
\end{aligned}$$

3. Proof that $\alpha(-\mathbf{x}) = -(\alpha\mathbf{x})$:

$$\begin{aligned}
\mathbb{0} &= \alpha\mathbb{0} && \text{by Theorem 4.1 page 73} \\
&= \alpha[\mathbf{x}+(-\mathbf{x})] && \text{by definition of additive identity element } -\mathbf{x} \\
&= \alpha\mathbf{x} + \alpha(-\mathbf{x}) && \text{by Definition 4.1 page 71} \\
&= \alpha\mathbf{x} + \alpha(-\mathbf{x}) \\
\implies -(\alpha\mathbf{x}) &= \alpha(-\mathbf{x}) && \text{by item (1) page 75}
\end{aligned}$$

4. Proof that $\alpha(\mathbf{x}-\mathbf{y}) = \alpha\mathbf{x} - \alpha\mathbf{y}$:

$$\begin{aligned}
\alpha(\mathbf{x}-\mathbf{y}) &= \alpha\left[\mathbf{x}+(-\mathbf{y})\right] && \text{by Definition 4.3 page 74} \\
&= \alpha\mathbf{x} + \alpha(-\mathbf{y}) && \text{by Definition 4.1 page 71} \\
&= \alpha\mathbf{x} + (-\alpha\mathbf{y}) && \text{by item (3) page 75} \\
&= \alpha\mathbf{x} - \alpha\mathbf{y} && \text{by Definition 4.3 page 74}
\end{aligned}$$

5. Proof that $(\alpha-\beta)\mathbf{x} = \alpha\mathbf{x} - \beta\mathbf{x}$:

$$\begin{aligned}
(\alpha-\beta)\mathbf{x} &= [\alpha+(-\beta)]\mathbf{x} && \text{by field properties of } \mathbb{F} \\
&= \alpha\mathbf{x} + (-\beta)\mathbf{x} && \text{by Definition 4.1} \\
&= \alpha\mathbf{x} + [-(\beta\mathbf{x})] && \text{by item (2) page 75} \\
&= \alpha\mathbf{x} - (\beta\mathbf{x}) && \text{by Definition 4.3 page 74}
\end{aligned}$$

✏

Theorem 4.3. [12] *Let $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a linear space, $\mathbb{0}$ the additive identity element with respect to $+$, and $-\mathbf{x}$ additive inverse of $\mathbf{x}$ with respect to $+$.*

THM

1. $\alpha\mathbf{x} = \alpha\mathbf{y}$ *and* $\alpha \neq 0 \implies \mathbf{x} = \mathbf{y} \quad \forall \mathbf{x},\mathbf{y}\in X$
2. $\alpha\mathbf{x} = \beta\mathbf{x}$ *and* $\mathbf{x} \neq \mathbb{0} \implies \alpha = \beta \quad \forall \mathbf{x},\mathbf{y}\in X,\ \alpha,\beta\in\mathbb{F}$
3. $\mathbf{z}+\mathbf{x} = \mathbf{z}+\mathbf{y} \implies \mathbf{x} = \mathbf{y} \quad \forall \mathbf{x},\mathbf{y},\mathbf{z}\in X$

[12] Michel and Herget (1993) page 77, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

✎PROOF:

1. Proof that $\alpha x = \alpha y$ and $\alpha \neq 0 \implies x = y$:

$$\begin{aligned}
\mathbb{0} &= \frac{1}{\alpha}(\mathbb{0}) && \text{by left hypothesis } (\alpha \neq 0)\\
&= \frac{1}{\alpha}\big(\alpha x - \alpha y\big) && \text{by left hypothesis } (\alpha x = \alpha y)\\
&= \frac{1}{\alpha}\alpha\big(x - y\big) && \text{by Definition 4.1 page 71}\\
&= x - y
\end{aligned}$$

2. Proof that $\alpha x = \beta x$ and $x \neq \mathbb{0} \implies \alpha = \beta$:

$$\begin{aligned}
\mathbb{0} &= \alpha x + (-\alpha x) && \text{by definition of additive inverse}\\
&= \beta x + (-\alpha x) && \text{by left hypothesis}\\
&= \beta x + (-\alpha) x && \text{by Theorem 4.2 page 74}\\
&= [\beta + (-\alpha)] x && \text{by Definition 4.1 page 71}\\
\implies \beta - \alpha &= 0 && \text{by Theorem 4.1 page 73}\\
\implies \alpha &= \beta && \text{by field properties of } \mathbb{F}
\end{aligned}$$

3. Proof that $z + x = z + y \implies x = y$:

$$\begin{aligned}
\mathbb{0} &= (z + x) - (z + y)\\
&= (x + z) - (z + y) && \text{by Definition 4.1 property 1}\\
&= (x + z) + [(-1)z + (-1)y] && \text{by Definition 4.1 property 3}\\
&= (x + z) + (-z - y) && \text{by previous result 2.}\\
&= x + (z - z) - y\\
&= x - y
\end{aligned}$$

✏

4.2 Order on Linear Spaces

Definition 4.4. [13] *Let $\Omega \triangleq \big(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times})\big)$ be a real linear space.*

DEF

*The pair $(\Omega, \leq)$ is an **ordered linear space** if*

1. $x \leq y \implies x + z \leq y + z \quad \forall z \in X$ *and*
2. $x \leq y \implies \alpha x \leq \alpha y \quad \forall \alpha \in \mathbb{F}$

*A vector x is **positive** if $\mathbb{0} \leq x$.*
*The **positive cone** X^+ of $(X, \leq)$ is the set $X^+ \triangleq \{x \in X | \mathbb{0} \leq x\}$.*

Definition 4.5. [14] *Let $(X, \leq)$ be an ordered linear space.*

DEF

*The tuple $L \triangleq (X, \vee, \wedge; \leq)$ is a **Riesz space** if L is a **lattice**.*
*A RIESZ SPACE is also called a **vector lattice**.*

Theorem 4.4. [15] *Let $(X, \vee, \wedge; \leq)$ be a Riesz space* (Definition 4.5 page 76).

THM

$$\begin{array}{lll|lll|l}
x \vee y &=& -\big[(-x) \wedge (-y)\big] & x \wedge y &=& -\big[(-x) \vee (-y)\big] & \forall x,y \in X\\
x + (y \vee z) &=& (x+y) \vee (x+z) & x + (y \wedge z) &=& (x+y) \wedge (x+z) & \forall x,y,z \in X\\
\alpha(x \vee y) &=& (\alpha x) \vee (\alpha y) & \alpha(x \wedge y) &=& (\alpha x) \wedge (\alpha y) & \forall x,y \in X, \alpha \geq 0\\
x + y &=& (x \wedge y) + (x \vee y) & & & & \forall x,y \in X, \alpha \in \mathbb{F}
\end{array}$$

[13] Aliprantis and Burkinshaw (2006) pages 1–2
[14] Aliprantis and Burkinshaw (2006) page 2
[15] Aliprantis and Burkinshaw (2006) page 3 ⟨Theorem 1.2⟩

✎PROOF:

1. Proof that $\boldsymbol{x} \vee \boldsymbol{y} = -\left[(-\boldsymbol{x}) \wedge (-\boldsymbol{y})\right]$:

$$
\begin{array}{|rcl|rcl|}
\hline
(-\boldsymbol{x}) \wedge (-\boldsymbol{y}) & \le & -\boldsymbol{x} & (-\boldsymbol{x}) \wedge (-\boldsymbol{y}) & \le & -\boldsymbol{y} \\
\boldsymbol{x} & \le & -\left[(-\boldsymbol{x}) \wedge (-\boldsymbol{y})\right] & \boldsymbol{y} & \le & -\left[(-\boldsymbol{x}) \wedge (-\boldsymbol{y})\right] \\
\boldsymbol{x} \vee \boldsymbol{y} & \le & -\left[(-\boldsymbol{x}) \wedge (-\boldsymbol{y})\right] & & & \\
\hline
\boldsymbol{x} & \le & \boldsymbol{x} \vee \boldsymbol{y} & \boldsymbol{y} & \le & \boldsymbol{x} \vee \boldsymbol{y} \\
-(\boldsymbol{x} \vee \boldsymbol{y}) & \le & -\boldsymbol{x} & -(\boldsymbol{x} \vee \boldsymbol{y}) & \le & -\boldsymbol{y} \\
-(\boldsymbol{x} \vee \boldsymbol{y}) & \le & (-\boldsymbol{x}) \wedge (-\boldsymbol{y}) & & & \\
-\left[(-\boldsymbol{x}) \wedge (-\boldsymbol{y})\right] & \le & \boldsymbol{x} \vee \boldsymbol{y} & & & \\
\hline
\end{array}
$$

2. Proof that $\boldsymbol{x} \wedge \boldsymbol{y} = -\left[(-\boldsymbol{x}) \vee (-\boldsymbol{y})\right]$:

$\boldsymbol{x} \vee \boldsymbol{y}$	$=$	$-\left[(-\boldsymbol{x}) \wedge (-\boldsymbol{y})\right]$	by item (1)
$(-\boldsymbol{x}) \vee (-\boldsymbol{y})$	$=$	$-\left[(-(-\boldsymbol{x})) \wedge (-(-\boldsymbol{y}))\right]$	replace $\boldsymbol{x}$ with $-\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{y}$
$(-\boldsymbol{x}) \vee (-\boldsymbol{y})$	$=$	$-\left[\boldsymbol{x} \wedge \boldsymbol{y}\right]$	$-(-\boldsymbol{x}) = \boldsymbol{x}$
$-\left[\boldsymbol{x} \wedge \boldsymbol{y}\right]$	$=$	$(-\boldsymbol{x}) \vee (-\boldsymbol{y})$	by symmetry of $=$ relation
$\boldsymbol{x} \wedge \boldsymbol{y}$	$=$	$-\left[(-\boldsymbol{x}) \vee (-\boldsymbol{y})\right]$	multipy both sides by -1

3. Proof that $\boldsymbol{x} + (\boldsymbol{y} \vee \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) \vee (\boldsymbol{x} + \boldsymbol{z})$:

$$
\begin{array}{|rcl|rcl|}
\hline
\boldsymbol{x} + \boldsymbol{y} & \le & \boldsymbol{x} + (\boldsymbol{y} \vee \boldsymbol{z}) & \boldsymbol{x} + \boldsymbol{z} & \le & \boldsymbol{x} + (\boldsymbol{y} \vee \boldsymbol{z}) \\
(\boldsymbol{x} + \boldsymbol{y}) \vee (\boldsymbol{x} + \boldsymbol{z}) & \le & \boldsymbol{x} + (\boldsymbol{y} \vee \boldsymbol{z}) & & & \\
\hline
\boldsymbol{y} & = & -\boldsymbol{x} + (\boldsymbol{x} + \boldsymbol{y}) & \boldsymbol{z} & = & -\boldsymbol{x} + (\boldsymbol{x} + \boldsymbol{z}) \\
& \le & -\boldsymbol{x} + \left[(\boldsymbol{x} + \boldsymbol{y}) \vee (\boldsymbol{x} + \boldsymbol{z})\right] & & \le & -\boldsymbol{x} + \left[(\boldsymbol{x} + \boldsymbol{y}) \vee (\boldsymbol{x} + \boldsymbol{z})\right] \\
\boldsymbol{y} \vee \boldsymbol{z} & \le & -\boldsymbol{x} + \left[(\boldsymbol{x} + \boldsymbol{y}) \vee (\boldsymbol{x} + \boldsymbol{z})\right] & & & \\
\boldsymbol{x} + (\boldsymbol{y} \vee \boldsymbol{z}) & \le & (\boldsymbol{x} + \boldsymbol{y}) \vee (\boldsymbol{x} + \boldsymbol{z}) & & & \\
\hline
\end{array}
$$

4. Proof that $\boldsymbol{x} + (\boldsymbol{y} \wedge \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) \wedge (\boldsymbol{x} + \boldsymbol{z})$:

$$
\begin{array}{|rcl|rcl|}
\hline
\boldsymbol{x} + \boldsymbol{y} & \ge & \boldsymbol{x} + (\boldsymbol{y} \wedge \boldsymbol{z}) & \boldsymbol{x} + \boldsymbol{z} & \ge & \boldsymbol{x} + (\boldsymbol{y} \wedge \boldsymbol{z}) \\
(\boldsymbol{x} + \boldsymbol{y}) \wedge (\boldsymbol{x} + \boldsymbol{z}) & \ge & \boldsymbol{x} + (\boldsymbol{y} \wedge \boldsymbol{z}) & & & \\
\hline
\boldsymbol{y} & = & -\boldsymbol{x} + (\boldsymbol{x} + \boldsymbol{y}) & \boldsymbol{z} & = & -\boldsymbol{x} + (\boldsymbol{x} + \boldsymbol{z}) \\
& \ge & -\boldsymbol{x} + \left[(\boldsymbol{x} + \boldsymbol{y}) \wedge (\boldsymbol{x} + \boldsymbol{z})\right] & & \ge & -\boldsymbol{x} + \left[(\boldsymbol{x} + \boldsymbol{y}) \wedge (\boldsymbol{x} + \boldsymbol{z})\right] \\
\boldsymbol{y} \wedge \boldsymbol{z} & \ge & -\boldsymbol{x} + \left[(\boldsymbol{x} + \boldsymbol{y}) \wedge (\boldsymbol{x} + \boldsymbol{z})\right] & & & \\
\boldsymbol{x} + (\boldsymbol{y} \wedge \boldsymbol{z}) & \ge & (\boldsymbol{x} + \boldsymbol{y}) \wedge (\boldsymbol{x} + \boldsymbol{z}) & & & \\
\hline
\end{array}
$$

5. Proof that $\alpha(\boldsymbol{x} \vee \boldsymbol{y}) = (\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y})$ for $\alpha \ge 0$:

$$
\begin{array}{|rcl|rcl|l|}
\hline
\boldsymbol{x} & \le & \boldsymbol{x} \vee \boldsymbol{y} & \boldsymbol{y} & \le & \boldsymbol{x} \vee \boldsymbol{y} & \\
\alpha\boldsymbol{x} & \le & \alpha(\boldsymbol{x} \vee \boldsymbol{y}) & \alpha\boldsymbol{y} & \le & \alpha(\boldsymbol{x} \vee \boldsymbol{y}) & \text{by Definition 4.4 page 76} \\
(\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y}) & \le & \alpha(\boldsymbol{x} \vee \boldsymbol{y}) & & & & \\
\hline
\alpha\boldsymbol{x} & \le & (\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y}) & \alpha\boldsymbol{y} & \le & (\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y}) & \\
\boldsymbol{x} & \le & \alpha^{-1}(\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y}) & \boldsymbol{y} & \le & \alpha^{-1}(\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y}) & \\
\boldsymbol{x} \vee \boldsymbol{y} & \le & \alpha^{-1}(\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y}) & & & & \\
\alpha(\boldsymbol{x} \vee \boldsymbol{y}) & \le & (\alpha\boldsymbol{x}) \vee (\alpha\boldsymbol{y}) & & & & \\
\hline
\end{array}
$$

6. Proof that $\alpha(\boldsymbol{x} \wedge \boldsymbol{y}) = (\alpha\boldsymbol{x}) \wedge (\alpha\boldsymbol{y})$ for $\alpha \ge 0$:

$$
\begin{array}{|rcl|rcl|l|}
\hline
\boldsymbol{x} & \ge & \boldsymbol{x} \wedge \boldsymbol{y} & \boldsymbol{y} & \ge & \boldsymbol{x} \wedge \boldsymbol{y} & \\
\alpha\boldsymbol{x} & \ge & \alpha(\boldsymbol{x} \wedge \boldsymbol{y}) & \alpha\boldsymbol{y} & \ge & \alpha(\boldsymbol{x} \wedge \boldsymbol{y}) & \text{by Definition 4.4 page 76} \\
(\alpha\boldsymbol{x}) \wedge (\alpha\boldsymbol{y}) & \ge & \alpha(\boldsymbol{x} \wedge \boldsymbol{y}) & & & & \\
\hline
\end{array}
$$

$$\begin{array}{rcl} \alpha x & \geq & (\alpha x) \wedge (\alpha y) \\ x & \geq & \alpha^{-1}(\alpha x) \wedge (\alpha y) \\ x \wedge y & \geq & \alpha^{-1}(\alpha x) \wedge (\alpha y) \\ \alpha(x \wedge y) & \geq & (\alpha x) \wedge (\alpha y) \end{array}$$	$$\begin{array}{rcl} \alpha y & \geq & (\alpha x) \wedge (\alpha y) \\ y & \geq & \alpha^{-1}(\alpha x) \wedge (\alpha y) \end{array}$$	

7. Proof that $x + y = (x \wedge y) + (x \vee y)$:

$$\begin{array}{rcl} x & \leq & x \vee y \\ x + y & \leq & (x \vee y) + y \\ x + y - (x \vee y) & \leq & y \\ x + y - (x \vee y) & \leq & x \wedge y \\ x + y & \leq & (x \vee y) + (x \wedge y) \end{array}$$	$$\begin{array}{rcl} y & \leq & x \vee y \\ x + vy & \leq & x + (x \vee y) \\ x + vy - (x \vee y) & \leq & x \end{array}$$
$$\begin{array}{rcl} x \wedge y & \leq & x \\ 0 & \leq & x - (x \wedge y) \\ y & \leq & y + x - (x \wedge y) \\ y & \leq & x + y - (x \wedge y) \\ x \vee y & \leq & x + y - (x \wedge y) \\ (x \wedge y) + (x \vee y) & \leq & x + y \end{array}$$	$$\begin{array}{rcl} x \wedge y & \leq & y \\ 0 & \leq & y - (x \wedge y) \\ x & \leq & x + y - (x \wedge y) \\ x & \leq & x + y - (x \wedge y) \end{array}$$

✏

Definition 4.6. [16] *Let* $(X, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition 4.5 page 76).

DEF

x^+	*is defined as* $x^+ \triangleq x \vee \mathbb{0}$	*and is called the* ***positive part*** *of* x.
x^-	*is defined as* $x^- \triangleq (-x) \vee \mathbb{0}$	*and is called the* ***negative part*** *of* x.
$\lvert x \rvert$	*is defined as* $\lvert x \rvert \triangleq x \vee (-x)$	*and is called the* ***absolute value*** *of* x.

Theorem 4.5. [17] *Let* $(X, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition 4.5 page 76).

THM

$$\left.\begin{array}{rcl} y - z & = & x \quad \text{and} \\ y \wedge z & = & \mathbb{0} \end{array}\right\} \iff \left\{\begin{array}{rcl} y & = & x^+ \quad \text{and} \\ z & = & x^- \end{array}\right.$$

✎PROOF:

1. Proof that left hypothesis $\implies$ right hypothesis:

$$\begin{aligned}
x^+ &= x \vee \mathbb{0} && \text{by definition of } x^+ \text{ Definition 4.6 page 78} \\
&= (y - z) \vee \mathbb{0} && \text{by left hypothesis} \\
&= (y - z) \vee (z - z) && \\
&= (y \vee z) - z && \text{by Theorem 4.4 page 76} \\
&= \left[y + z - (y \wedge z)\right] - z && \text{by Theorem 4.4 page 76} \\
&= y - (y \wedge z) && \\
&= y - \mathbb{0} && \text{by left hypothesis} \\
&= y && \\
x^- &= (-x) \vee \mathbb{0} && \text{by definition of } x^- \text{ Definition 4.6 page 78} \\
&= (z - y) \vee \mathbb{0} && \text{by left hypothesis} \\
&= (z - y) \vee (y - y) && \\
&= (z \vee y) - y && \text{by Theorem 4.4 page 76}
\end{aligned}$$

[16] Aliprantis and Burkinshaw (2006) page 4, Istrățescu (1987) page 129
[17] Aliprantis and Burkinshaw (2006) page 4 ⟨Theorem 1.3⟩

$$\begin{aligned}
&= [z + y - (z \wedge y)] - z && \text{by Theorem 4.4 page 76}\\
&= z - (z \wedge y)\\
&= z - \mathbb{0} && \text{by left hypothesis}\\
&= z
\end{aligned}$$

2. Proof that left hypothesis $\Longleftarrow$ right hypothesis:

$$\begin{aligned}
y - z = x^+ - x^- & && \text{by right hypothesis}\\
&= [x \vee \mathbb{0}] - [(-x) \vee \mathbb{0}] && \text{by Definition 4.6 page 78}\\
&= (x \vee \mathbb{0}) + (x \wedge \mathbb{0}) && \text{by Theorem 4.4 page 76}\\
&= x && \text{by Theorem 4.4 page 76}\\
y \wedge z = x^+ \wedge x^- & && \text{by right hypothesis}\\
&= [x^- + (x^+ - x^-)] \wedge [x^- + \mathbb{0}]\\
& x^- + [(x^+ - x^-) \wedge \mathbb{0}] && \text{by Theorem 4.4 page 76}\\
& x^- + [(y - z) \wedge \mathbb{0}] && \text{by right hypothesis}\\
& x^- + (x \wedge \mathbb{0}) && \text{by previous result}\\
& x^- - [-x \vee \mathbb{0}] && \text{by Theorem 4.4 page 76}\\
& x^- - x^- && \text{by definition of } x^- \text{ (Definition 4.6 page 78)}\\
& \mathbb{0}
\end{aligned}$$

✑

Theorem 4.6. [18] *Let* $(X, \vee, \wedge; \leq)$ *be a* RIESZ SPACE *(Definition 4.5 page 76). Let* x^+ *the* POSITIVE PART *of* $x \in X$, x^- *the* NEGATIVE PART *of* $x \in X$, *and* $|x|$ *the* ABSOLUTE VALUE *(Definition 4.6 page 78) of* $x \in X$.

THM

$$\begin{aligned}
|x| &= x^+ + x^- && \forall x \in X\\
x &= (x - y)^+ + (x \wedge y) && \forall x \in X
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
|x| &= x \vee (-x) && \text{by definition of } |x| \text{ (Definition 4.6 page 78)}\\
&= (2x - x) \vee (\mathbb{0} - x)\\
&= (-x + 2x) \vee (-x + \mathbb{0}) && \text{by \textit{commutative} property (Definition 4.1 page 71)}\\
&= (-x) + (2x \vee \mathbb{0}) && \text{by Theorem 4.4 page 76}\\
&= (2x \vee 2\mathbb{0}) - x && \text{by the \textit{commutative} property (Definition 4.1 page 71)}\\
&= 2(x \vee \mathbb{0}) - x && \text{by Theorem 4.4 page 76}\\
&= 2x^+ - x && \text{by definition of } x^+ \text{ (Definition 4.6 page 78)}\\
&= 2x^+ - (x^+ - x^-) && \text{by 1}\\
&= x^+ + x^-\\
x &= x + \mathbb{0} && x + y - y\\
&= (x \vee y) + (x \wedge y) - y && \text{by Theorem 4.4 page 76}\\
&= [(x - y) \vee (y - y)] + (x \wedge y) && \text{by Theorem 4.4 page 76}\\
&= [(x - y) \vee \mathbb{0}] + (x \wedge y)\\
&= (x - y)^+ + (x \wedge y) && \text{by definition of } x^+ \text{ (Definition 4.6 page 78)}
\end{aligned}$$

✑

[18] Aliprantis and Burkinshaw (2006) page 4

Theorem 4.7. [19] *Let* $(X, \vee, \wedge; \le)$ *be a* RIESZ SPACE (Definition 4.5 page 76). *Let* x^+ *the* POSITIVE PART *of* $x \in X$, x^- *the* NEGATIVE PART *of* $x \in X$, *and* $|x|$ *the* ABSOLUTE VALUE (Definition 4.6 page 78) *of* $x \in X$.

THM

1.	$x \vee y$	$=$	$\frac{1}{2}(x + y + \lvert x - y\rvert)$	$\forall x,y \in X$
2.	$x \wedge y$	$=$	$\frac{1}{2}(x + y - \lvert x - y\rvert)$	$\forall x,y \in X$
3.	$\lvert x - y\rvert$	$=$	$(x \vee y) - (x \wedge y)$	$\forall x,y \in X$
4.	$\lvert x\rvert \vee \lvert y\rvert$	$=$	$\frac{1}{2}(\lvert x + y\rvert + \lvert x - y\rvert)$	$\forall x,y \in X$
5.	$\lvert x\rvert \vee \lvert y\rvert$	$=$	$\frac{1}{2}\lvert\lvert x + y\rvert - \lvert x - y\rvert\rvert$	$\forall x,y \in X$

✎PROOF:

$$\begin{aligned}
(x + y + |x - y|) &= (x + y) + \big[(x - y) \vee (y - x)\big] && \text{by Definition 4.6 page 78}\\
&= \big[(x + y) + (x - y)\big] \vee \big[(x + y) + (y - x)\big] && \text{by Theorem 4.4 page 76}\\
&= (2x) \vee (2y)\\
&= 2(x \vee y) && \text{by Theorem 4.4 page 76}\\
(x + y - |x - y|) &= (x + y) - \big[(x - y) \vee (y - x)\big] && \text{by Definition 4.6 page 78}\\
&= (x + y) - \big[(-(y - x)) \vee (-(x - y))\big]\\
&= (x + y) + \big[(y - x) \wedge (x - y)\big] && \text{by Theorem 4.4 page 76}\\
&= \big[(x + y) + (y - x)\big] \wedge \big[(x + y) + (x - y)\big] && \text{by Theorem 4.4 page 76}\\
&= (2y) \wedge (2x)\\
&= 2(y \wedge x) && \text{by Theorem 4.4 page 76}\\
&= 2(x \wedge y)\\
|x - y| &= \frac{1}{2}(x + y + |x - y|) - \frac{1}{2}(x + y + |x - y|)\\
&= (x \vee y) - (x \wedge y) && \text{by 1 and 2}\\
|x + y| + |x - y| &= \frac{1}{2}(\mathbb{0} + |2x + 2y|) + |x - y|\\
&= \frac{1}{2}\big[(x + y) + (-x - y) + |(x + y) - (-x - y)|\big] + |x - y|\\
&= \big[(x + y) \vee (-x - y)\big] + |x - y| && \text{by 1}\\
&= \big[(x + y) + |x - y|\big] \vee \big[(-x - y) + |x - y|\big] && \text{by Theorem 4.4 page 76}\\
&= 2(x \vee y) \vee 2\big[(-y) + (-x) + |(-y) - (-x)|\big] && \text{by 1}\\
&= 2(x \vee y) \vee 2\big[(-y) \vee (-x)\big] && \text{by 1}\\
&= 2([x \vee (-x)] \vee (y \vee (-y)))\\
&= 2(|x| \vee |y|) && \text{by Definition 4.6 page 78}\\
||x + y| - |x - y|| &= 2(|x + y| \vee |x - y|) - (|x + y| + |x - y|) && \text{by 1}\\
&= (|x + y + x - y| + |x + y - x + y|) - 2(|x| \vee |y|) && \text{by 3}\\
&= 2(|x| + |y|) - 2(|x| \vee |y|)\\
&= 2(|x| \vee |y|) && \text{by Theorem 4.4 page 76}
\end{aligned}$$

✏

Definition 4.7. [20] *Let* $(X, \vee, \wedge; \le)$ *be a* RIESZ SPACE (Definition 4.5 page 76). *Let* x^+ *the* POSITIVE PART *of* $x \in X$, x^- *the* NEGATIVE PART *of* $x \in X$, *and* $|x|$ *the* ABSOLUTE VALUE (Definition 4.6 page 78) *of* $x \in X$.

DEF

x and y are ***disjoint****, denoted by* $x \perp y$, *if*

$$|x| \wedge |y| = \mathbb{0}.$$

Two subsets U *and* V *of* X *are* ***disjoint****, denoted by* $U \perp V$ *if*

$$x \perp y \qquad \forall x \in U \text{ and } y \in V$$

[19] Aliprantis and Burkinshaw (2006) page 5 ⟨Theorem 1.4⟩
[20] Aliprantis and Burkinshaw (2006) page 5

Definition 4.8. [21] *Let* $(\boldsymbol{X}, \vee, \wedge; \le)$ *be a* RIESZ SPACE (Definition 4.5 page 76). *Let* $\boldsymbol{x}^+$ *the* POSITIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}^-$ *the* NEGATIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, *and* $|\boldsymbol{x}|$ *the* ABSOLUTE VALUE (Definition 4.6 page 78) *of* $\boldsymbol{x} \in \boldsymbol{X}$. *Let* $\boldsymbol{Y}$ *be a subset of* $\boldsymbol{X}$.

DEF

$\boldsymbol{Y}^d$ *is the* ***disjoint complement*** *of* $\boldsymbol{Y}$ *if* $\quad \boldsymbol{Y}^d \triangleq \{\boldsymbol{x} \in \boldsymbol{X} | \boldsymbol{x} \perp \boldsymbol{y} \quad \forall \boldsymbol{y} \in \boldsymbol{Y}\}$.

The quantity $\boldsymbol{Y}^{dd}$ *is defined as* $\left(\boldsymbol{Y}^d\right)^d$.

Definition 4.9. [22] *Let* $(\boldsymbol{X}, \vee, \wedge; \le)$ *be a* RIESZ SPACE (Definition 4.5 page 76). *Let* $\boldsymbol{x}^+$ *the* POSITIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}^-$ *the* NEGATIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, *and* $|\boldsymbol{x}|$ *the* ABSOLUTE VALUE (Definition 4.6 page 78) *of* $\boldsymbol{x} \in \boldsymbol{X}$.

DEF

$$\begin{aligned}
|A| &\triangleq \{|\boldsymbol{a}| | \boldsymbol{a} \in A\} \\
A^+ &\triangleq \left\{\boldsymbol{a}^+ | \boldsymbol{a} \in A\right\} \\
A^- &\triangleq \{\boldsymbol{a}^- | \boldsymbol{a} \in A\} \\
A \vee B &\triangleq \left\{\boldsymbol{a} \vee \boldsymbol{b} | \boldsymbol{a} \in A \text{ and } \boldsymbol{b} \in B\right\} \\
A \wedge B &\triangleq \left\{\boldsymbol{a} \wedge \boldsymbol{b} | \boldsymbol{a} \in A \text{ and } \boldsymbol{b} \in B\right\} \\
\boldsymbol{x} \vee A &\triangleq \{\boldsymbol{x} \vee \boldsymbol{a} | \boldsymbol{a} \in A\} \\
\boldsymbol{x} \wedge A &\triangleq \{\boldsymbol{x} \wedge \boldsymbol{a} | \boldsymbol{a} \in A\}
\end{aligned}$$

[21] Aliprantis and Burkinshaw (2006) page 5
[22] Aliprantis and Burkinshaw (2006) page 7

CHAPTER 5

TOPOLOGICAL LINEAR SPACES

5.1 Definitions

A *topologial linear space* (often called a *topological vector space*) is basically a *linear space* (Definition 4.1 page 71) with a *topology* (Definition 1.1 page 3). If the topology is generated by a *metric* (Definition 3.1 page 33), then it is a *metric linear space* (Definition 5.5 page 84). If the topology is generated by a *norm* (Definition 6.1 page 87), then it is a *normed linear space*. If the topology is generated by an *inner product* (Definition 7.1 page 99), then it is an *inner product space*.

Definition 5.1. [1]

DEF *The tuple* $\left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times}),\, \boldsymbol{T}\right)$ *is a* ***topological linear space*** *if*

1. $\left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times})\right)$ *is a* LINEAR SPACE *and*
2. $\boldsymbol{T}$ *is a* TOPOLOGY *on* 2^X *and*
3. $(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{x} + \boldsymbol{y}$ *is* CONTINUOUS *on* $\boldsymbol{X}^{\boldsymbol{X}\times\boldsymbol{X}}$ *(Definition 1.8 page 23) and*
4. $(\alpha, \boldsymbol{x}) \to \alpha\boldsymbol{x}$ *is* CONTINUOUS *on* $\boldsymbol{X}^{\mathbb{F}\times\boldsymbol{X}}$. *(Definition 1.8 page 23)* .

Definition 5.2. *Let* $\left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times}),\, \boldsymbol{T}_x\right)$ *be a* TOPOLOGICAL LINEAR SPACE *with topology* $\boldsymbol{T}_x$. *Let* $\left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times}),\, \boldsymbol{T}_y\right)$ *be a* TOPOLOGICAL LINEAR SPACE *with topology* $\boldsymbol{T}_y$. *Let* Y^X *be the set of all functions (operators) from* X *to* Y.

DEF *The set* $\mathscr{C}(X,Y)$ *is the* ***space of continuous operators*** *from* X *to* Y *and is defined as*
$$\mathscr{C}(X,Y) \triangleq \left\{\mathsf{f} \in Y^X \middle| \mathsf{f} \text{ is continuous with respect to } (\boldsymbol{T}_x, \boldsymbol{T}_y)\right\}$$

Definition 5.3. *Let* $\boldsymbol{Y} \triangleq \left(Y,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times}),\, \boldsymbol{T}\right)$ *be a subspace of a* TOPOLOGICAL LINEAR SPACE $\boldsymbol{X} \triangleq \left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times}),\, \boldsymbol{T}\right)$. *Let* Y^- *be the* CLOSURE *of the set* Y *(Definition 1.4 page 14) in the* TOPOLOGICAL SPACE $(X, \boldsymbol{T})$ *(Definition 1.1 page 3)*.

DEF *The subspace* $\boldsymbol{Y}$ *is* ***closed*** *in* $\boldsymbol{X}$ *if*
$$Y = Y^- .$$

Example 5.1. [2] Let A^- be the *closure* (Definition 1.4 page 14) of a set A in a topological space. Let X be the set of all bounded sequences over $\mathbb{R}$. Let Y be the set of all bounded sequences with a finite

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[1] Schaefer and Wolff (1999) page 12 ⟨1. Vector Space Topologies⟩, Robertson and Robertson (1980) page 3 ⟨3. Topological Vector Spaces⟩

[2] Kolmogorov and Fomin (1975) page 140 ⟨Example 1⟩

number of zeros. Let $\boldsymbol{T}$ be the standard topology on $\mathbb{R}$ generated by the metric $\mathsf{d}(x,y) \triangleq |x-y|$.

EX

$\boldsymbol{X}$ is a *topological linear space*.
$\boldsymbol{Y}$ is a *topological linear space*.
But Y is *not closed* in $(\boldsymbol{X},\boldsymbol{T})$ $(Y \neq Y^-)$, because, for example,
$$\left(\!\!\left(1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{n}, \cdots\right)\!\!\right) \text{ is in } Y,$$
but its *closure point*
$$\left(\!\!\left(1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{n}, \cdots, 0, 0, 0, \cdots\right)\!\!\right) \text{ is } \textit{not} \text{ in } Y \text{ (but is in } X).$$

Example 5.2. [3] Let A^- be the *closure* of a set A in a topological space. Let $C_{[a:b]}$ be the set of all *continuous* functions on the real interval $[a:b]$. Let $P_{[a:b]}$ be the set of all *polynomials* on the real interval $[a:b]$. Let $\boldsymbol{T}$ be the standard topology on $\mathbb{R}$ generated by the metric $\mathsf{d}(x,y) \triangleq |x-y|$.

EX

$\left(C_{[a:b]}, \boldsymbol{T}\right)$ is a *topological linear space*.
$\left(P_{[a:b]}, \boldsymbol{T}\right)$ is a *topological linear space*.
But $P_{[a:b]} \neq \left(P_{[a:b]}\right)^- = C_{[a:b]}$, so $P_{[a:b]}$ is *not closed* in $\left(C_{[a:b]}, \boldsymbol{T}\right)$.

✎PROOF: $\left(P_{[a:b]}\right)^- = C_{[a:b]}$ by *Weierstrass' Approximation Theorem*. ✏

5.2 Dual Spaces

Definition 5.4. [4] *Let $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T}\right)$ be a topological linear space. Let $\mathbb{F}^{\boldsymbol{X}}$ be the set of all functionals from X to $\mathbb{F}$. Let $\mathscr{C}(\boldsymbol{X}, \mathbb{F})$ be the space of continuous functionals from X to $\mathbb{F}$.*

DEF

*The **algebraic dual space** $\boldsymbol{X}^\dagger$ of $\boldsymbol{X}$ is* $\quad \boldsymbol{X}^\dagger \triangleq \mathbb{F}^{\boldsymbol{X}}$.
*The **topological dual space** $\boldsymbol{X}^*$ of $\boldsymbol{X}$ is* $\quad \boldsymbol{X}^* \triangleq \mathscr{C}(\boldsymbol{X}, \mathbb{F})$.
*The space $\boldsymbol{X}$ is the **predual** of $\boldsymbol{X}^*$. A topological dual space is also called a **dual space**, **conjugate space** or **adjoint space**.*

Theorem 5.1. *Let $\boldsymbol{X} = \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ be a linear space with dual space $\boldsymbol{X}^*$.*

THM

$\boldsymbol{X}^*$ is a linear space.

5.3 Metric Linear Spaces

Metric space stucture can be added to a linear space resulting in a *metric linear space* (next definition). One key difference between metric linear spaces and normed linear spaces is that the balls in a *normed linear space* (Definition 6.1 page 87) are always *convex* (Definition 10.6 page 152); this is not true for all metric linear spaces.[5]

Definition 5.5. [6] *Let $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$.*

[3] Kolmogorov and Fomin (1975) page 140 ⟨Example 2⟩
[4] Hunter and Nachtergaele (2001) page 116 ⟨Definition 5.54⟩, Kurdila and Zabarankin (2005) page 76 ⟨Definitions 2.2.3, 2.2.4⟩, Hewitt and Stomberg (1965) page 211 ⟨Definition 14.6⟩
[5] Bruckner et al. (1997), page 478
[6] Maddox (1989) page 90, Bruckner et al. (1997) page 477 ⟨Definition 12.3⟩, Rolewicz (1985) page 1

DEF *The tuple $\boldsymbol{\Omega}$ is a **metric linear space** if*

1. *if* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *is a* LINEAR SPACE *and*
2. d *is a* METRIC *in* $\mathbb{R}^{\boldsymbol{X}}$ *and*
3. $\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \quad \forall x,y,z \in X$ (TRANSLATION INVARIANT) *and*
4. $\alpha_n \to \alpha$ *and* $\boldsymbol{x}_n \to \boldsymbol{x} \implies \alpha_n \boldsymbol{x}_n \to \alpha \boldsymbol{x}$

Theorem 5.2. [7] *Let $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d})$ be a metric linear space.*

THM
$$\underbrace{\mathsf{d}(\boldsymbol{\theta}, \lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}) \le \lambda\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{x}) + (1-\lambda)\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{y})}_{\mathsf{d} \text{ is a CONVEX function}} \implies \left\{\begin{array}{c} \mathsf{B}(\theta, r) \in \boldsymbol{\Omega} \\ \text{is convex} \\ \forall \theta \in X, r \in \mathbb{R}^+ \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{d}(\boldsymbol{\theta}, \lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}) &\le \lambda\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{x}) + (1-\lambda)\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{y}) && \text{by convexity hypothesis} \\
&\le \lambda r + (1-\lambda) r && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
&= r \\
\implies \lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y} &\in \mathsf{B}(\theta, r) && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
\implies \mathsf{B}(\theta, r) &\in (\boldsymbol{X}, \mathsf{d}) \text{ is convex} && \forall \theta \in \boldsymbol{X}
\end{aligned}$$

✏

Theorem 5.3. [8] *Let $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \mathsf{d})$ be a real metric linear space.*

THM
$$\left\{\begin{array}{lllll} 1. & \mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) & \forall x,y,z \in \boldsymbol{X} & \text{(TRANSLATION INVARIANT)} & \textit{and} \\ 2. & \mathsf{d}(\lambda\boldsymbol{x}, \lambda\boldsymbol{y}) = \lambda\,\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) & \forall x,y \in \boldsymbol{X}, \lambda \in [0,1] & \text{(HOMOGENEOUS)} & \end{array}\right\}$$
$$\implies \{\mathsf{B}(\theta, r) \in (\boldsymbol{X}, \mathsf{d}) \quad \textit{is } \text{CONVEX} \quad \forall \theta \in \boldsymbol{X}, r \in \mathbb{R}^+\}$$

✎PROOF:

$$\begin{aligned}
&\mathsf{d}(\boldsymbol{\theta}, \lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}) \\
&\quad= \mathsf{d}(\mathbb{0}, \lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}-\boldsymbol{\theta}) && \text{by translation invariance hypothesis} \\
&\quad= \mathsf{d}(\mathbb{0}, \lambda(\boldsymbol{x}-\boldsymbol{\theta})+(1-\lambda)(\boldsymbol{y}-\boldsymbol{\theta})) \\
&\quad\le \mathsf{d}(\mathbb{0}, \lambda(\boldsymbol{x}-\boldsymbol{\theta})) + \mathsf{d}(\lambda(\boldsymbol{x}-\boldsymbol{\theta}), \lambda(\boldsymbol{x}-\boldsymbol{\theta})+(1-\lambda)(\boldsymbol{y}-\boldsymbol{\theta})) && \text{by } \textit{subadditive} \text{ property (Definition 3.1 page 33)} \\
&\quad= \mathsf{d}(\mathbb{0}, \lambda(\boldsymbol{x}-\boldsymbol{\theta})) + \mathsf{d}(\mathbb{0}, \mathbb{0}+(1-\lambda)(\boldsymbol{y}-\boldsymbol{\theta})) && \text{by translation invariance hypothesis} \\
&\quad= \lambda\mathsf{d}(\mathbb{0}, \boldsymbol{x}-\boldsymbol{\theta}) + (1-\lambda)\mathsf{d}(\mathbb{0}, \boldsymbol{y}-\boldsymbol{\theta}) && \text{by homogeneous hypothesis} \\
&\quad= \lambda\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{x}) + (1-\lambda)\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{y}) && \text{by translation invariance hypothesis} \\
&\quad\le \lambda r + (1-\lambda) r && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
&\quad= r \\
&\implies \lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y} \in \mathsf{B}(\theta, r) && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
&\implies \mathsf{B}(\theta, r) \in (\boldsymbol{X}, \mathsf{d}) \text{ is convex} && \forall \theta \in \boldsymbol{X}
\end{aligned}$$

✏

[7] Norfolk (1991) page 5
[8] Norfolk (1991) pages 5–6, http://groups.google.com/group/sci.math/msg/a6f0a7924027957d

CHAPTER 6

NORMED LINEAR SPACES

6.1 Definition and basic results

Definition 6.1. [1] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition 4.1 page 71) and* $|\cdot| \in \mathbb{R}^{\mathbb{F}}$ *the* ABSOLUTE VALUE *function (Definition F.4 page 342).*

DEF

A functional $\|\cdot\|$ *in* $\mathbb{R}^X$ *is a* ***norm*** *if*

1.	$\|x\| \geq 0$	$\forall x \in X$	(STRICTLY POSITIVE)	*and*
2.	$\|x\| = 0 \iff x = \mathbb{0}$	$\forall x \in X$	(NONDEGENERATE)	*and*
3.	$\|\alpha x\| = \|\alpha\|\,\|x\|$	$\forall x \in X,\ \alpha \in \mathbb{C}$	(HOMOGENEOUS)	*and*
4.	$\|x + y\| \leq \|x\| + \|y\|$	$\forall x, y \in X$	(SUBADDITIVE/TRIANGLE INQUALITY).	

A ***normed linear space*** *is the tuple* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

The definition of the *norm* (Definition 6.1 page 87) requires that any two vectors in a norm space be *subadditive* (they satisfy the *triangle inequality* property) such that $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$. Acutally, in **any** normed linear space, this property holds true for **any** finite number of vectors—not just two—such that $\|x_1 + x_2 + \cdots + x_N\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_N\|$ (next theorem).

Theorem 6.1 (triangle inequality). [2] *Let* $(\!(x_n \in X)\!)_1^N$ *be an* N-TUPLE *(Definition 9.1 page 131) of vectors in a* NORMED LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

THM

$$\left\| \sum_{n=1}^{N} x_n \right\| \leq \sum_{n=1}^{N} \|x_n\| \qquad \forall N \in \mathbb{N},\ x_n \in V$$

✎PROOF: Proof is by induction:

---

[1] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

[2] Michel and Herget (1993) page 344, Euclid (circa 300BC) ⟨Book I Proposition 20⟩

1. Proof for the $N = 1$ case:

$$\begin{aligned}
\left\|\sum_{n=1}^{1} x_n\right\| &= \|x_1\| \\
&= \sum_{n=1}^{1} \|x_1\|
\end{aligned}$$

2. Proof for the $N = 2$ case:

$$\begin{aligned}
\left\|\sum_{n=1}^{2} x_n\right\| &= \left\|\sum_{n=1}^{2} x_n\right\| \\
&= \|x_1 + x_2\| \\
&\le \|x_1\| + \|x_2\| && \text{by Definition 6.1 page 87 (triangle inequality)} \\
&= \sum_{n=1}^{2} \|x_n\|
\end{aligned}$$

3. Proof that [N case] $\Longrightarrow$ [$N+1$ case]:

$$\begin{aligned}
\left\|\sum_{n=1}^{N+1} x_n\right\| &= \left\|\sum_{n=1}^{N} x_n + x_{n+1}\right\| \\
&\le \left\|\sum_{n=1}^{N} x_n\right\| + \|x_{n+1}\| && \text{by Definition 6.1 page 87 (triangle inequality)} \\
&\le \sum_{n=1}^{N} \|x_n\| + \|x_{n+1}\| && \text{by left hypothesis} \\
&= \sum_{n=1}^{N+1} \|x_n\|
\end{aligned}$$

✏

Theorem 6.2 (Reverse Triangle Inequality). [3] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition 6.1 page 87).*

THM
$$\underbrace{\big|\|x\| - \|y\|\big| \le \|x - y\|}_{\text{REVERSE TRIANGLE INEQUALITY}} \quad \le \quad \|x\| + \|y\| \qquad \forall x,y \in X$$

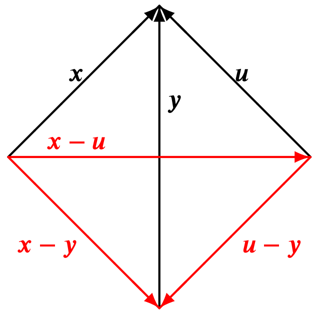
✎PROOF:

$$\begin{aligned}
\big|\|x\| - \|y\|\big| &= \big|\|(x-y) + y\| - \|y\|\big| \\
&\le \big|\|x-y\| + \|y\| - \|y\|\big| && \text{by Definition 6.1 page 87} \\
&= \big|\|x-y\|\big| \\
&= \|x-y\| && \text{by Definition 6.1 page 87}
\end{aligned}$$

$$\begin{aligned}
\|x-y\| &\le \|x - \mathbb{0}\| + \|\mathbb{0} - y\| && \text{by previous result with } u = 0 \\
&= \|x\| + |-1|\,\|y\| && \text{by Definition 6.1 page 87} \\
&= \|x\| + \|y\|
\end{aligned}$$

✏

[3] Aliprantis and Burkinshaw (1998) page 218, Giles (2000) page 2, Banach (1922) page 136



The shortest distance between two vectors is always the difference of the vectors. This is proven in next and illustrated to the left in the Euclidean space $\mathbb{R}^2$ (the plane)

Proposition 6.1. [4] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition 6.1 page 87).*

PRP $$\|x - y\| \;\le\; \|x - u\| + \|u - y\| \qquad \forall x,y,u \in X$$

✎PROOF:

$$\begin{aligned}\|x - y\| &= \|(x - u) + (u - y)\| \\ &\le \|x - u\| + \|u - y\| && \text{by Definition 6.1 page 87}\end{aligned}$$

⇰

Example 6.1 (The usual norm). [5] Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions with domain and range the set of *real numbers* $\mathbb{R}$.

EX The **absolute value** (Definition F.4 page 342) $|\cdot| \in \mathbb{R}^{\mathbb{R}}$ is a *norm.*

Example 6.2 (l_p norms). Let $(\!(x_n)\!)_{n\in\mathbb{Z}}$ be a *sequence* (Definition 9.1 page 131) of real numbers. An uncountably infinite number of norms is provided by the $\ell^p_{\mathbb{F}}$ norms $\|(\!(x_n)\!)\|_p$:

EX $$\|(\!(x_n)\!)\|_p \triangleq \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{1}{p}} \qquad \text{is a norm for } p \in [1 : \infty]$$

6.2 Relationship between metrics and norms

6.2.1 Metrics generated by norms

The concept of *length* is very closely related to the concept of *distance*. Thus it is not surprising that a *norm* (a "length" function) can be used to define a *metric* (a "distance" function) on any *metric linear space* (Definition 5.5 page 84). Another way to say this is that the norm of a normed linear space *induces* a metric on this space. And so every normed linear space also has a metric. And because every normed linear space has a metric, **every normed linear space is also a metric space**. Actually this can be generalized one step further in that every metric space is also a *topological space*. And so **every normed linear space is also a topological space.** In symbols,

$$\text{normed linear space} \quad\Longrightarrow\quad \text{metric space} \quad\Longrightarrow\quad \text{topological space.}$$

[4] Aliprantis and Burkinshaw (1998) page 218
[5] Giles (1987) page 3

Theorem 6.3. [6] *Let* $\mathsf{d} \in \mathbb{R}^{\boldsymbol{X}\times\boldsymbol{X}}$ *be a function on a* REAL *normed linear space* $\big(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \|\cdot\|\big)$. *Let* $\mathsf{B}(\boldsymbol{x}, r) \triangleq \{\boldsymbol{y} \in \boldsymbol{X} \mid \|\boldsymbol{y} - \boldsymbol{x}\| < r\}$ *be the* OPEN BALL (Definition 2.4 page 28) *of radius* r *centered at a point* $\boldsymbol{x}$.

THM $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\|$ *is a metric on* $\boldsymbol{X}$

✎PROOF: The proof follows directly from the definition of a metric (Definition 3.1 page 33) the definition of *norm* (Definition 6.1 page 87). ✏

The previous theorem defined a metric $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y})$ induced by the norm $\|\boldsymbol{x}\|$. The next definition defines this metric formally.

Definition 6.2. [7] *Let* $\big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be a* NORMED LINEAR SPACE (Definition 6.1 page 87).

DEF *The **metric induced by the norm** $\|\cdot\|$ is the function $\mathsf{d} \in \mathbb{R}^{\boldsymbol{X}}$ such that*

$$\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\| \qquad \forall x,y \in X$$

Due to its algebraic structure, every norm is *continuous* (Corollary 6.1 page 90). This makes norm spaces very useful in analysis. For a function f be to *continuous*, for every $\epsilon > 0$ there must exist a $\delta > 0$ such that $|\mathsf{f}(x + \delta) - \mathsf{f}(x)| < \epsilon$. The *Reverse Triangle Inequality* (Theorem 6.2 page 88) shows this to be true when $\mathsf{f}(\cdot) \triangleq \|\cdot\|$.

Corollary 6.1. [8] *Let* $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be a* NORMED LINEAR SPACE (Definition 6.1 page 87).

COR *The norm* $\|\cdot\|$ *is* CONTINUOUS *in* $\boldsymbol{\Omega}$.

✎PROOF: This follows from these concepts:

1. The fact that $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\|$ is a *metric* (Theorem 6.3 page 90).
2. *Continuity* in a metric space.
3. The *Reverse Triangle Inequality* (Theorem 6.2 page 88).

✏

Theorem 6.4 (next) demonstrates that **all open or closed** balls in **any normed linear space** are *convex*. However, the converse is not true—that is, a metric not generated by a norm may still produce a ball that is convex. Here are some examples:

metric name	example		generated by norm	convex ball
Taxi-cab metric	Example 3.22	page 62	✓	✓
Euclidean metric	Example 3.23	page 62	✓	✓
Sup metric	Example 3.24	page 63	✓	✓
Parabolic metric	Example 3.25	page 63		
exponential metric	Example 3.27	page 66		
Tangential metric	Example 3.28	page 67		✓

[6] Michel and Herget (1993) page 344, Banach (1932a) page 53
[7] Giles (2000) page 1 ⟨1.1 Definition⟩
[8] Giles (2000) page 2

Theorem 6.4. [9] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d})$ *be a* METRIC LINEAR SPACE *(Definition 5.5 page 84). Let* B *be the* OPEN BALL *(Definition 2.4 page 28)* $\mathsf{B}(\boldsymbol{p}, r) \triangleq \{\boldsymbol{x} \in \boldsymbol{X} | \mathsf{d}(\boldsymbol{p}, \boldsymbol{x}) < r\}$ *(open ball with respect to metric* d *centered at point* $\boldsymbol{p}$ *and with radius* r*).*

THM

$$\left.\begin{array}{l}\exists \|\cdot\| \in \mathbb{R}^{\boldsymbol{X}} \quad \text{such that} \\ \underbrace{\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{y} - \boldsymbol{x}\|}_{\mathsf{d} \text{ is generated by a norm}}\end{array}\right\} \implies \left\{\begin{array}{ll}1. & \mathsf{B}(\boldsymbol{x}, r) = \boldsymbol{x} + B(0, r) \\ 2. & \mathsf{B}(\mathbb{0}, r) = r\,\mathsf{B}(\mathbb{0}, 1) \\ 3. & \mathsf{B}(\boldsymbol{x}, r) \text{ is CONVEX} \\ 4. & \boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) \iff -\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) \quad \text{(SYMMETRIC)}\end{array}\right.$$

✎PROOF:

1. Proof that $\mathsf{d}(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + v\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y})$ (invariant):

$$\begin{aligned}\mathsf{d}(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + v\boldsymbol{z}) &= \|(\boldsymbol{y} + v\boldsymbol{z}) - (\boldsymbol{x} + \boldsymbol{z})\| && \text{by left hypothesis} \\ &= \|\boldsymbol{y} - \boldsymbol{x}\| \\ &= \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) && \text{by left hypothesis}\end{aligned}$$

2. Proof that $\mathsf{B}(\boldsymbol{x}, r) = \boldsymbol{x} + B(0, r)$:

$$\begin{aligned}\mathsf{B}(\boldsymbol{x}, r) &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) < r\} && \text{by definition of open ball } B \\ &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\boldsymbol{y} - \boldsymbol{y}, \boldsymbol{y} - \boldsymbol{x}) < r\} && \text{by right result 1.} \\ &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{y} - \boldsymbol{x}) < r\} \\ &= \{\boldsymbol{u} + \boldsymbol{x} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\} && \text{let } \boldsymbol{u} \triangleq \boldsymbol{y} - \boldsymbol{x} \\ &= \boldsymbol{x} + \{\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\} \\ &= \boldsymbol{x} + B(0, r) && \text{by definition of open ball } B\end{aligned}$$

3. Proof that $\mathsf{B}(\mathbb{0}, r) = r\,\mathsf{B}(\mathbb{0}, 1)$:

$$\begin{aligned}\mathsf{B}(\mathbb{0}, r) &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{y}) < r\} && \text{by definition of open ball } B \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \frac{1}{r}\mathsf{d}(\mathbb{0}, \boldsymbol{y}) < 1\right\} \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \frac{1}{r}\,\|\boldsymbol{y} - \mathbb{0}\| < 1\right\} && \text{by left hypothesis} \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \left\|\frac{1}{r}\boldsymbol{y} - \frac{1}{r}\mathbb{0}\right\| < 1\right\} && \text{by homogeneous property of } \|\cdot\| \text{ page 87} \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}\left(\frac{1}{r}\mathbb{0}, \frac{1}{r}\boldsymbol{y}\right) < 1\right\} && \text{by left hypothesis} \\ &= \{r\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < 1\} && \text{let } \boldsymbol{u} \triangleq \frac{1}{r}\boldsymbol{y} \\ &= r\ \{\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < 1\} \\ &= r\,B(\mathbb{0}, 1) && \text{by definition of open ball } B\end{aligned}$$

4. Proof that $\mathsf{B}(\boldsymbol{p}, r)$ is convex:
We must prove that for any pair of points $\boldsymbol{x}$ and $\boldsymbol{y}$ in the open ball $\mathsf{B}(\boldsymbol{p}, r)$, any point $\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$ is also in the open ball. That is, the distance from any point $\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$ to the ball's center $\boldsymbol{p}$ must be less

[9] Giles (2000) page 2 ⟨1.2 Remarks⟩, Giles (1987) pages 22–26 ⟨2.4 Theorem, 2.11 Theorem⟩

than r.

$$\begin{aligned}
\mathsf{d}(\boldsymbol{p}, \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= \|\boldsymbol{p} - \lambda\boldsymbol{x} - (1-\lambda)\boldsymbol{y}\| && \text{by left hypothesis} \\
&= \Bigg\|\underbrace{\lambda\boldsymbol{p} + (1-\lambda)\boldsymbol{p}}_{\boldsymbol{p}} - \lambda\boldsymbol{x} - (1-\lambda)\boldsymbol{y}\Bigg\| \\
&= \|\lambda\boldsymbol{p} - \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{p} - (1-\lambda)\boldsymbol{y}\| \\
&\le \|\lambda\boldsymbol{p} - \lambda\boldsymbol{x}\| + \|(1-\lambda)\boldsymbol{p} - (1-\lambda)\boldsymbol{y}\| && \text{by subadditivity property of } \|\cdot\| \text{ page 87} \\
&= |\lambda|\,\|\boldsymbol{p} - \boldsymbol{x}\| + |1-\lambda|\,\|\boldsymbol{p} - \boldsymbol{y}\| && \text{by homogeneous property of } \|\cdot\| \text{ page 87} \\
&= \lambda\,\|\boldsymbol{p} - \boldsymbol{x}\| + (1-\lambda)\,\|\boldsymbol{p} - \boldsymbol{y}\| && \text{because } 0 \le \lambda \le 1 \\
&\le \lambda r + (1-\lambda) r && \text{because } \boldsymbol{x}, \boldsymbol{y} \text{ are in the ball } \mathsf{B}(\boldsymbol{p}, r) \\
&= r
\end{aligned}$$

5. Proof that $\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) \iff -\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r)$ (symmetric):

$$\begin{aligned}
\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) &\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} \,|\, \mathsf{d}(\mathbb{0}, \boldsymbol{y}) < r\} && \text{by definition of open ball } B \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} \,|\, \|\boldsymbol{y} - \mathbb{0}\| < r\} && \text{by left hypothesis} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} \,|\, \|\boldsymbol{y}\| < r\} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} \,|\, \|(-1)(-\boldsymbol{y})\| < r\} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} \,|\, |-1|\,\|-\boldsymbol{y}\| < r\} && \text{by homogeneous property of } \|\cdot\| \text{ page 87} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} \,|\, \|-\boldsymbol{y} - \mathbb{0}\| < r\} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} \,|\, \mathsf{d}(\mathbb{0}, -\boldsymbol{y}) < r\} && \text{by left hypothesis} \\
&\iff \boldsymbol{x} \in \{-\boldsymbol{u} \in \boldsymbol{X} \,|\, \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\} && \text{let } \boldsymbol{u} \triangleq -\boldsymbol{y} \\
&\iff \boldsymbol{x} \in (-\{\boldsymbol{u} \in \boldsymbol{X} \,|\, \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\}) \\
&\iff \boldsymbol{x} \in (-\mathsf{B}(\mathbb{0}, r)) \\
&\iff -\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r)
\end{aligned}$$

⇰

Theorem 6.4 (page 91) demonstrates that if a metric d in a metric space $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$ is generated by a norm, then the ball $B(x, r)$ in that metric linear space is *convex*. However, the converse is not true. That is, it is possible for the balls in a metric space $(\boldsymbol{Y}, \mathsf{p})$ to be *convex*, but yet the metric p not be generated by a norm. Example 3.29 (page 69) gives one such example.

6.2.2 Norms generated by metrics

Every normed linear space is also a metric linear space (Theorem 6.3 page 90). That is, a metric linear space generates a *normed linear space*. However, the converse is not true—not every metric linear space is a *normed linear space*. A characterization of metric linear spaces that *are* normed linear spaces is given by Theorem 6.5 page 94.

Lemma 6.1. [10] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$ *be a* METRIC LINEAR SPACE. *Let* $\|\boldsymbol{x}\| \triangleq \mathsf{d}(\boldsymbol{x}, \mathbb{0})\ \forall \boldsymbol{x} \in X$.

LEM

$$\underbrace{\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X}_{\text{TRANSLATION INVARIANT}} \implies \left\{\begin{array}{lrcll}
1. & \|\boldsymbol{x}\| & = & \|-\boldsymbol{x}\| & \forall \boldsymbol{x} \in X \quad \textit{and} \\
2. & \|\boldsymbol{x}\| & = & 0 \iff \boldsymbol{x} = 0 & \forall \boldsymbol{x} \in X \quad \textit{and} \\
3. & \|\boldsymbol{x}+\boldsymbol{y}\| & \le & \|\boldsymbol{x}\| + \|\boldsymbol{y}\| & \forall \boldsymbol{x},\boldsymbol{y} \in X
\end{array}\right.$$

[10] Oikhberg and Rosenthal (2007) page 599

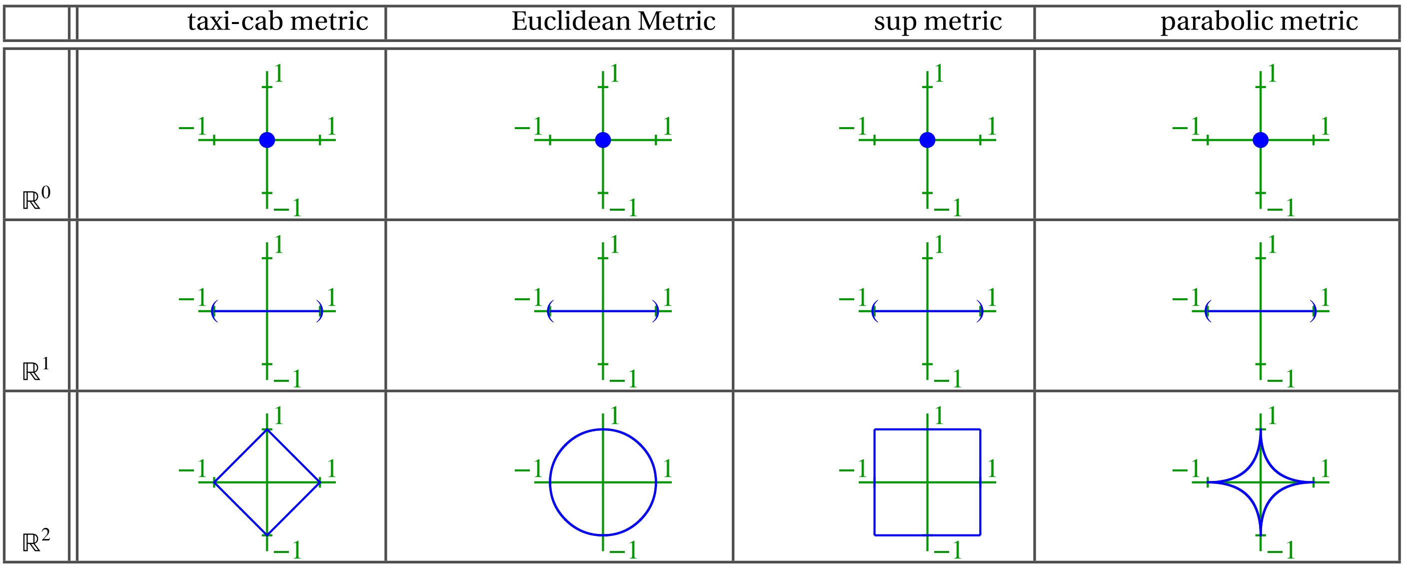


Figure 6.1: Open balls in $(\mathbb{R}^0, \mathsf{d}_n)$, $(\mathbb{R}, \mathsf{d}_n)$, $(\mathbb{R}^2, \mathsf{d}_n)$, and $(\mathbb{R}^3, \mathsf{d}_n)$.

✎PROOF:

1. Proof that $\|\boldsymbol{x}\| = \|-\boldsymbol{x}\|$:

$$\begin{aligned}
\|\boldsymbol{x}\| &= \mathsf{d}(\boldsymbol{x}, \mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\boldsymbol{x} - \boldsymbol{x}, \mathbb{0} - \boldsymbol{x}) && \text{by translation invariance hypothesis} \\
&= \mathsf{d}(\mathbb{0}, -\boldsymbol{x}) \\
&= \|-\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

2a. Proof that $\|\boldsymbol{x}\| = 0 \implies \boldsymbol{x} = 0$:

$$\begin{aligned}
0 &= \|\boldsymbol{x}\| && \text{by left hypothesis} \\
&= \mathsf{d}(\boldsymbol{x}, \mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\boldsymbol{x}, \mathbb{0}) && \text{by definition of } \|\cdot\| \\
&\implies \boldsymbol{x} = \mathbb{0} && \text{by property of metrics page 33}
\end{aligned}$$

2b. Proof that $\|\boldsymbol{x}\| = 0 \impliedby \boldsymbol{x} = 0$:

$$\begin{aligned}
\|\boldsymbol{x}\| &= \mathsf{d}(\boldsymbol{x}, \mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\mathbb{0}, \mathbb{0}) && \text{by right hypothesis} \\
&= 0 && \text{by property of metrics page 33}
\end{aligned}$$

3. Proof that $\|\boldsymbol{x} + \boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$:

$$\begin{aligned}
\|\boldsymbol{x} + \boldsymbol{y}\| &= \mathsf{d}(\boldsymbol{x} + \boldsymbol{y}, \mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\boldsymbol{x} + \boldsymbol{y} - \boldsymbol{y}, \mathbb{0} - \boldsymbol{y}) && \text{by translation invariance hypothesis} \\
&= \mathsf{d}(\boldsymbol{x}, -\boldsymbol{y}) \\
&\le \mathsf{d}(\boldsymbol{x}, \mathbb{0}) + \mathsf{d}(\mathbb{0}, \boldsymbol{y}) && \text{by property of metrics page 33} \\
&= \mathsf{d}(\boldsymbol{x}, \mathbb{0}) + \mathsf{d}(\boldsymbol{y}, \mathbb{0}) && \text{by property of metrics page 33} \\
&= \|\boldsymbol{x}\| + \|\boldsymbol{y}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

✏

Theorem 6.5. [11] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a* LINEAR SPACE. *Let* $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\| \; \forall \boldsymbol{x}, \boldsymbol{y} \in X$.

THM

$$\left.\begin{array}{llll} 1. & \mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X & \text{(TRANSLATION INVARIANT) and} \\ 2. & \mathsf{d}(\alpha\boldsymbol{x}, \alpha\boldsymbol{y}) = |\alpha|\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y} \in X, \alpha \in \mathbb{F} & \text{(HOMOGENEOUS)} \end{array}\right\} \iff \|\cdot\| \textit{ is a } \text{NORM}$$

✎PROOF:

1. Proof of $\Longrightarrow$ assertion:
   (a) Proof that $\|\cdot\|$ is *strictly positive*: This follows directly from the definition of d.
   (b) Proof that $\|\cdot\|$ is *nondegenerate*: This follows directly from Lemma 6.1 (page 92).
   (c) Proof that $\|\cdot\|$ is *homogeneous*: This follows from the second left hypothesis.
   (d) Proof that $\|\cdot\|$ satisfies the *triangle-inequality*: This follows directly from Lemma 6.1 (page 92).
2. Proof of $\Longleftarrow$ assertion:

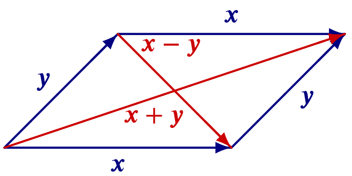
$$\begin{aligned}
\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) &= \|(\boldsymbol{x}+\boldsymbol{z}) - (\boldsymbol{y}+\boldsymbol{z})\| && \text{by definition of } \mathsf{d} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| \\
&= \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) && \text{by definition of } \mathsf{d} \\
\mathsf{d}(\alpha\boldsymbol{x}, \alpha\boldsymbol{y}) &= \|(\alpha\boldsymbol{x}) - (\alpha\boldsymbol{y})\| && \text{by definition of } \mathsf{d} \\
&= \|\alpha(\boldsymbol{x} - \boldsymbol{y})\| \\
&= |\alpha| \, \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \|\cdot\| \text{ page 87} \\
&= |\alpha|\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) && \text{by definition of } \mathsf{d}
\end{aligned}$$

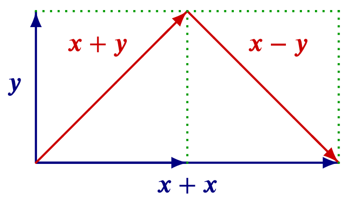
⇰

6.3 Orthogonality on normed linear spaces

Traditionally, *orthogonality* (Definition 7.4 page 111) is a property defined in *inner product spaces* (Definition 7.1 page 99) However, the concept of orthogonality can be extended to *normed linear spaces* (Definition 6.1 page 87). Here are some examples:

① *Isosceles orthogonality*:	Definition 6.3	page 94
② *Pythagorean orthogonality*:	Definition 6.4	page 96
③ *Birkhoff orthogonality*:	Definition 6.5	page 97





Isosceles orthogonality (Definition 6.3 page 94) can be illustrated using a *parallelogram*, as illustrated in the figure to the upper left. In this case, orthogonality implies that the parallelogram is a rectangle, which in turn implies that the lengths of the two diagonals are equal ($\|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}-\boldsymbol{y}\|$). Isosceles orthogonality can also be illustrated with a triangle where the sides are of lengths $\|\boldsymbol{x}+\boldsymbol{y}\|$ and $\|\boldsymbol{x}-\boldsymbol{y}\|$ and base of length $\|\boldsymbol{x}+\boldsymbol{x}\|$. In this case if $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal, then the triangle is *isosceles*. This is illustrated in figure to the lower left. Isosceles orthogonality is formally defined next.

Definition 6.3. [12] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *be a* NORMED LINEAR SPACE *(Definition 6.1 page 87)*.

[11] Bollobás (1999) page 21
[12] James (1945) page 292 ⟨DEFINITION 2.1⟩, Amir (1986) page 24, Dunford and Schwartz (1957) page 93

DEF *Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are* ***orthogonal in the sense of James*** *if*

$$\|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}-\boldsymbol{y}\|.$$

This property is also called ***isosceles orthogonality*** *or* ***James orthogonality***.

Theorem 6.6. *Let $\left(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ be an* INNER PRODUCT SPACE *(Definition 7.1 page 99) with induced norm $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$,* ISOSCELES ORTHOGONALITY *(Definition 6.3 page 94) relation $\obot$, and inner product relation* ORTHOGONALITY *(Definition 7.4 page 111) relation $\perp$.*

THM

$$\underbrace{\boldsymbol{x} \obot \boldsymbol{y}}_{\text{orthogonal in the sense of James}} \iff \underbrace{\boldsymbol{x} \perp \boldsymbol{y}}_{\text{orthogonal in the sense of inner product space}}$$

✎PROOF:

1. Proof that $\boldsymbol{x} \obot \boldsymbol{y} \implies \boldsymbol{x} \perp \boldsymbol{y}$:

$$\begin{aligned}
&4\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= \underbrace{\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2}_{0 \text{ by } \boldsymbol{x} \obot \boldsymbol{y} \text{ hypothesis}} + i\|\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\|\boldsymbol{x}-i\boldsymbol{y}\|^2 && \text{by \textit{polarization identity} (Theorem 7.6 page 106)}\\
&= 0 + i\|\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\|\boldsymbol{x}-i\boldsymbol{y}\|^2 && \text{by } \boldsymbol{x} \obot \boldsymbol{y} \text{ hypothesis}\\
&= i\left[\|\boldsymbol{x}\|^2 + \|i\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid i\boldsymbol{y}\rangle\right] \\
&\quad - i\left[\|\boldsymbol{x}\|^2 + \|-i\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid -i\boldsymbol{y}\rangle\right] && \text{by \textit{Polar Identity} (Lemma 7.1 page 103)}\\
&= i\left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid i\boldsymbol{y}\rangle\right] \\
&\quad - i\left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid -i\boldsymbol{y}\rangle\right] && \text{by Definition 6.1 page 87 and Definition 7.1 page 99}\\
&= 4i\Re\langle \boldsymbol{x} \mid i\boldsymbol{y}\rangle \\
&= 4i\Re\left[i^*\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle\right] \\
&= 0 && \text{because inner product space is \textit{real} } (\mathbb{F}=\mathbb{R})
\end{aligned}$$

2. Proof that $\boldsymbol{x} \obot \boldsymbol{y} \impliedby \boldsymbol{x} \perp \boldsymbol{y}$:

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by \textit{Polar Identity} (Lemma 7.1 page 103)}\\
&= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 0 && \text{by } \boldsymbol{x} \perp \boldsymbol{y} \text{ hypothesis}\\
&= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\Re\cancelto{0 \text{ when } \boldsymbol{x} \perp \boldsymbol{y}}{\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle} && \text{by } \boldsymbol{x} \perp \boldsymbol{y} \text{ hypothesis}\\
&= \|\boldsymbol{x}\|^2 + \|-\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid -\boldsymbol{y}\rangle \\
&= \|\boldsymbol{x}-\boldsymbol{y}\|^2 && \text{by \textit{Polar Identity} (Lemma 7.1 page 103)}
\end{aligned}$$

✏

Theorem 6.7. [13] *Let $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ be a normed linear space and with* ISOSCELES ORTHOGONALITY *(Definition 6.3 page 94) relation $\obot$.*

THM

$$\boldsymbol{x} \obot \boldsymbol{y} \iff \boldsymbol{y} \obot \boldsymbol{x} \iff \alpha\boldsymbol{x} \obot \alpha\boldsymbol{y} \qquad \forall\alpha\in\mathbb{F}$$

[13] Amir (1986) page 24

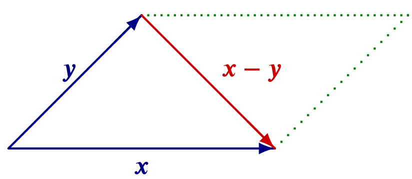
✎PROOF:

$$
\begin{array}{rlll}
\mathbf{x} \textcircled{$\perp$} \mathbf{y} & \implies \|\mathbf{x}+\mathbf{y}\| & = \|\mathbf{x}-\mathbf{y}\| & \text{by Definition 6.3 page 94} \\
& \implies \|\mathbf{x}+\mathbf{y}\| & = |-1|\ \|\mathbf{x}-\mathbf{y}\| & \\
& \implies \|\mathbf{x}+\mathbf{y}\| & = \|-(\mathbf{x}-\mathbf{y})\| & \text{by Definition 6.1 page 87} \\
& \implies \|\mathbf{y}+\mathbf{x}\| & = \|\mathbf{y}-\mathbf{x}\| & \text{by Definition 4.1 page 71} \\
& \implies \mathbf{y} \textcircled{$\perp$} \mathbf{x} & & \text{by Definition 6.3 page 94}
\end{array}
$$

$$
\begin{array}{rlll}
\mathbf{y} \textcircled{$\perp$} \mathbf{x} & \implies \|\mathbf{y}+\mathbf{x}\| & = \|\mathbf{y}-\mathbf{x}\| & \text{by Definition 6.3 page 94} \\
& \implies |\alpha|\ \|\mathbf{y}+\mathbf{x}\| & = |\alpha|\ \|\mathbf{y}-\mathbf{x}\| & \\
& \implies \|\alpha(\mathbf{y}+\mathbf{x})\| & = \|\alpha(\mathbf{y}-\mathbf{x})\| & \text{by Definition 6.1 page 87} \\
& \implies \|\alpha\mathbf{y}+\alpha\mathbf{x}\| & = \|\alpha\mathbf{y}-\alpha\mathbf{x}\| & \\
& \implies \|\alpha\mathbf{x}+\alpha\mathbf{y}\| & = \|-(\alpha\mathbf{x}-\alpha\mathbf{y})\| & \text{by Definition 4.1 page 71} \\
& \implies \|\alpha\mathbf{x}+\alpha\mathbf{y}\| & = |-1|\ \|\alpha\mathbf{x}-\alpha\mathbf{y}\| & \text{by Definition 6.1 page 87} \\
& \implies \|\alpha\mathbf{x}+\alpha\mathbf{y}\| & = \|\alpha\mathbf{x}-\alpha\mathbf{y}\| & \text{by Definition F.4 page 342} \\
& \implies \alpha\mathbf{x} \textcircled{$\perp$} \alpha\mathbf{y} & & \text{by Definition 6.3 page 94}
\end{array}
$$

$$
\begin{array}{rlll}
\alpha\mathbf{x} \textcircled{$\perp$} \alpha\mathbf{y} & \implies \|\alpha\mathbf{x}+\alpha\mathbf{y}\| & = \|\alpha\mathbf{x}-\alpha\mathbf{y}\| & \text{by Definition 6.3 page 94} \\
& \implies \|\alpha(\mathbf{x}+\mathbf{y})\| & = \|\alpha(\mathbf{x}-\mathbf{y})\| & \text{by Definition 4.1 page 71} \\
& \implies |\alpha|\ \|\mathbf{x}+\mathbf{y}\| & = |\alpha|\ \|\mathbf{x}-\mathbf{y}\| & \text{by Definition 6.1 page 87} \\
& \implies \|\mathbf{x}+\mathbf{y}\| & = \|\mathbf{x}-\mathbf{y}\| & \text{by Definition 6.1 page 87} \\
& \implies \mathbf{x} \textcircled{$\perp$} \mathbf{y} & & \text{by Definition 6.3 page 94}
\end{array}
$$

⇨



If a triangle in a plane has two perpendicular sides of lengths a and b and a hypotenuse of length c, then by the *Pythagorean Theorem* (Theorem 7.10 page 112), $a^2 + b^2 = c^2$. This concept of orthogonality can be generalized to normed linear spaces. Two vectors $\mathbf{x}$ and $\mathbf{y}$ (with lengths $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$) are orthogonal when $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x}-\mathbf{y}\|^2$ ($\mathbf{x}-\mathbf{y}$ is a kind of "*hypotenuse*"). This kind of orthogonality is defined next and illustrated in the figure to the left.

Definition 6.4. [14] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition 6.1 page 87).*

DEF
Two vectors $\mathbf{x}$ *and* $\mathbf{y}$ *are* ***orthogonal in the Pythagorean sense*** *if*

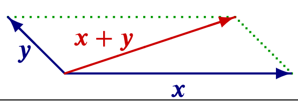
$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

This relationship is also called ***Pythagorean orthogonality***.

Theorem 6.8. [15] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be an* INNER PRODUCT SPACE *(Definition 7.1 page 99) with induced norm* $\|\mathbf{x}\| \triangleq \sqrt{\langle \mathbf{x} \mid \mathbf{x} \rangle}$, PYTHAGOREAN ORTHOGONALITY *(Definition 6.4 page 96) relation* $\textcircled{$\perp$}$*, and inner product relation* ORTHOGONALITY *(Definition 7.4 page 111) relation* $\perp$.

THM

$$\underbrace{\mathbf{x} \textcircled{$\perp$} \mathbf{y}}_{\text{orthogonal in the Pythagorean sense}} \iff \underbrace{\mathbf{x} \perp \mathbf{y}}_{\text{orthogonal in the sense of inner product space}}$$



Besides *isosceles orthogonality* (Definition 6.3 page 94), orthogonality in normed linear spaces can be defined using *Birkhoff orthogonality*, as defined in Definition 6.5 (next) and illustrated to the left.

[14] James (1945) page 292 ⟨DEFINITION 2.2⟩, Amir (1986) page 57, Drljević (1989) page 232
[15] Amir (1986) page 57

Definition 6.5. [16] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE (Definition 6.1 page 87).

DEF

Two vectors $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are* ***orthogonal in the sense of Birkhoff*** *if*

$$\|\boldsymbol{x}\| \le \|\boldsymbol{x} + \alpha \boldsymbol{y}\| \qquad \forall \alpha \in \mathbb{F}.$$

This relationship is also called ***Birkhoff orthogonality***.

Theorem 6.9. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be an* INNER PRODUCT SPACE (Definition 7.1 page 99) *with induced norm* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$, BIRKHOFF ORTHOGONALITY *relation* $\textcircled{\perp}$ (Definition 6.5 page 97), *and inner product relation* ORTHOGONALITY *relation* $\perp$ (Definition 7.4 page 111).

THM

$$\underbrace{\boldsymbol{x} \,\textcircled{\perp}\, \boldsymbol{y}}_{\text{orthogonal in the sense of Birkhoff}} \iff \underbrace{\boldsymbol{x} \perp \boldsymbol{y}}_{\text{orthogonal in the sense of inner product space}}$$

[16] Amir (1986) page 33, Dunford and Schwartz (1957) page 93, James (1947) page 265

CHAPTER 7

INNER PRODUCT SPACES

7.1 Definition and basic results

Definition 7.1. [1] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition 4.1 page 71).

DEF

A FUNCTIONAL (Definition 14.1 page 199) $\langle \triangle \mid \triangledown \rangle \in \mathbb{F}^{X \times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

1.	$\langle \alpha \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$=$	$\alpha \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y} \in X,\ \forall \alpha \in \mathbb{C}$	(HOMOGENEOUS)	*and*
2.	$\langle \boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{u} \rangle$	$=$	$\langle \boldsymbol{x} \mid \boldsymbol{u} \rangle + \langle \boldsymbol{y} \mid \boldsymbol{u} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u} \in X$	(ADDITIVE)	*and*
3.	$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$=$	$\langle \boldsymbol{y} \mid \boldsymbol{x} \rangle^*$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	(CONJUGATE SYMMETRIC).	*and*
4.	$\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle$	$\geq$	0	$\forall \boldsymbol{x} \in X$	(NON-NEGATIVE)	*and*
5.	$\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle$	$=$	$0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	(NON-ISOTROPIC)	

An inner product is also called a ***scalar product***.
The tuple $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *is called an* ***inner product space***.

Theorem 7.1. [2] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a* LINEAR SPACE (Definition 4.1 page 71).

THM

1.	$\langle \boldsymbol{x} \mid \boldsymbol{y} + \boldsymbol{z} \rangle$	$=$	$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \boldsymbol{z} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$		
2.	$\langle \boldsymbol{x} \mid \alpha \boldsymbol{y} \rangle$	$=$	$\alpha^* \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y} \in X,\ \alpha \in \mathbb{F}$		
3.	$\langle \boldsymbol{x} \mid \mathbb{0} \rangle$	$=$	$\langle \mathbb{0} \mid \boldsymbol{x} \rangle = 0$	$\forall \boldsymbol{x} \in X$		
4.	$\langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{z} \rangle$	$=$	$\langle \boldsymbol{x} \mid \boldsymbol{z} \rangle - \langle \boldsymbol{y} \mid \boldsymbol{z} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$		
5.	$\langle \boldsymbol{x} \mid \boldsymbol{y} - \boldsymbol{z} \rangle$	$=$	$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle - \langle \boldsymbol{x} \mid \boldsymbol{z} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$		
6.	$\langle \boldsymbol{x} \mid \boldsymbol{z} \rangle$	$=$	$\langle \boldsymbol{y} \mid \boldsymbol{z} \rangle$	$\forall \boldsymbol{z} \in X \neq \{\mathbb{0}\}$	$\iff$	$\boldsymbol{x} = \boldsymbol{y}$
7.	$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$=$	0	$\forall \boldsymbol{x} \in X$	$\iff$	$\boldsymbol{y} = \mathbb{0}$

✎PROOF:

$\langle \boldsymbol{x} \mid \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{y} + \boldsymbol{z} \mid \boldsymbol{x} \rangle^*$	by *conjugate symmetric* property of $\langle \triangle \mid \triangledown \rangle$	(Definition 7.1 page 99)
$= (\langle \boldsymbol{y} \mid \boldsymbol{x} \rangle + \langle \boldsymbol{z} \mid \boldsymbol{x} \rangle)^*$	by *additive* property of $\langle \triangle \mid \triangledown \rangle$	(Definition 7.1 page 99)
$= \langle \boldsymbol{y} \mid \boldsymbol{x} \rangle^* + \langle \boldsymbol{z} \mid \boldsymbol{x} \rangle^*$	by *distributive* property of $*$	(Definition 17.3 page 252)
$= \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \boldsymbol{z} \rangle$	by *conjugate symmetric* property of $\langle \triangle \mid \triangledown \rangle$	(Definition 7.1 page 99)
$\langle \boldsymbol{x} \mid \alpha \boldsymbol{y} \rangle = \langle \alpha \boldsymbol{y} \mid \boldsymbol{x} \rangle^*$	by *conjugate symmetric* property of $\langle \triangle \mid \triangledown \rangle$	(Definition 7.1 page 99)

[1] Istrăţescu (1987) page 111 ⟨Definition 4.1.1⟩, Bollobás (1999) pages 130–131, Haaser and Sullivan (1991) page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72

[2] Berberian (1961) page 27, Haaser and Sullivan (1991) page 277

$$
\begin{aligned}
&= (\alpha \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle)^* && \text{by \textit{homogeneous} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
&= \alpha^* \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* && \text{by \textit{antiautomorphic} property of } * && \text{(Definition 17.3 page 252)}\\
&= \alpha^* \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by \textit{conjugate symmetric} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
\langle \boldsymbol{x} \,|\, \mathbb{0} \rangle &= \langle \mathbb{0} \,|\, \boldsymbol{x} \rangle^* && \text{by \textit{conjugate symmetric} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
&= \langle 0 \cdot \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* \\
&= (0 \cdot \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle)^* && \text{by \textit{homogeneous} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
&= 0 \\
\langle \mathbb{0} \,|\, \boldsymbol{x} \rangle &= \langle 0 \cdot \boldsymbol{y} \,|\, \boldsymbol{x} \rangle \\
&= (0 \cdot \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle) && \text{by \textit{homogeneous} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
&= 0 \\
\langle \boldsymbol{x} - \boldsymbol{y} \,|\, \boldsymbol{z} \rangle &= \langle \boldsymbol{x} + (-\boldsymbol{y}) \,|\, \boldsymbol{z} \rangle && \text{by definition of } + \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{z} \rangle + \langle -\boldsymbol{y} \,|\, \boldsymbol{z} \rangle && \text{by \textit{additive} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{z} \rangle - \langle \boldsymbol{y} \,|\, \boldsymbol{z} \rangle && \text{by \textit{homogeneous} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
\langle \boldsymbol{x} \,|\, \boldsymbol{y} - \boldsymbol{z} \rangle &= \langle \boldsymbol{y} - \boldsymbol{z} \,|\, \boldsymbol{x} \rangle^* && \text{by \textit{conjugate symmetric} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}\\
&= (\langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle - \langle \boldsymbol{z} \,|\, \boldsymbol{x} \rangle)^* && \text{by 4.}\\
&= \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* - \langle \boldsymbol{z} \,|\, \boldsymbol{x} \rangle^* && \text{by \textit{distributive} property of } * && \text{(Definition 17.3 page 252)}\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle - \langle \boldsymbol{x} \,|\, \boldsymbol{z} \rangle && \text{by \textit{conjugate symmetric} property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition 7.1 page 99)}
\end{aligned}
$$

$$
\begin{aligned}
& \langle \boldsymbol{x} \,|\, \boldsymbol{z} \rangle = \langle \boldsymbol{y} \,|\, \boldsymbol{z} \rangle && \forall \boldsymbol{z} \\
\iff\; & \langle \boldsymbol{x} \,|\, \boldsymbol{z} \rangle - \langle \boldsymbol{y} \,|\, \boldsymbol{z} \rangle = 0 && \forall \boldsymbol{z} && \text{by property of complex numbers}\\
\iff\; & \langle \boldsymbol{x} - \boldsymbol{y} \,|\, \boldsymbol{z} \rangle = 0 && \forall \boldsymbol{z} && \text{by 4.}\\
\iff\; & \boldsymbol{x} - \boldsymbol{y} = \mathbb{0} && \forall \boldsymbol{z} && \text{by \textit{non-isotropic} property of } \langle \triangle \,|\, \triangledown \rangle \text{ (Definition 7.1 page 99)}
\end{aligned}
$$

Proof that $\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle = 0 \implies \boldsymbol{y} = \mathbb{0}$:

1. Suppose $\boldsymbol{y} \neq \mathbb{0}$;
2. Then $\langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle \neq \mathbb{0}$ by the *non-isotropic* property of $\langle \triangle \,|\, \triangledown \rangle$ (Definition 7.1 page 99)
3. But because $\boldsymbol{y} \in X$, the left hypothesis implies that $\langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle = 0$.
4. This is a *contradiction*.
5. Therefore $\boldsymbol{y} \neq \mathbb{0}$ must be incorrect and $\boldsymbol{y} = \mathbb{0}$ must be correct.

Proof that $\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle = 0 \impliedby \boldsymbol{y} = \mathbb{0}$:

$$
\begin{aligned}
\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle &= \langle \boldsymbol{x} \,|\, \mathbb{0} \rangle && \text{by right hypothesis}\\
&= 0 && \text{by Theorem 7.1 page 99}
\end{aligned}
$$

✎

One of the most useful and widely used inequalities in analysis is the *Cauchy-Schwarz Inequality* (sometimes also called the *Cauchy-Bunyakovsky-Schwarz Inequality*). In fact, we will use this inequality shortly to prove that every inner product space *has* a norm and therefore every inner product space *is* a normed linear space.

Theorem 7.2 (Cauchy-Schwarz Inequality). [3] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \triangledown \rangle\right)$ *be an* INNER PRODUCT SPACE *and* $|\cdot| \in \mathbb{R}^{\mathbb{C}}$ *an* ABSOLUTE VALUE *function (Definition F.4 page 342). Let* $\|\cdot\|$ *be a function in* $\mathbb{R}^{\mathbb{F}}$ *such*

[3] Haaser and Sullivan (1991) page 278, Aliprantis and Burkinshaw (1998) page 278, Cauchy (1821) page 455, Bunyakovsky (1859) page 6, Schwarz (1885)

that $\|x\| \triangleq \sqrt{\langle x \,|\, x\rangle}$.[4]

THM

$$
\begin{array}{lclcllll}
|\langle x \,|\, y\rangle|^2 & \leq & \langle x \,|\, x\rangle \; \langle y \,|\, y\rangle & & & & & \forall x,y\in X \\
|\langle x \,|\, y\rangle|^2 & = & \langle x \,|\, x\rangle \; \langle y \,|\, y\rangle & \iff & \exists \alpha \in \mathbb{F} & \text{such that} & y = \alpha x & \forall x,y\in X \\
|\langle x \,|\, y\rangle| & \leq & \|x\| \; \|y\| & & & & & \forall x,y\in X \\
|\langle x \,|\, y\rangle| & = & \|x\| \; \|y\| & \iff & \exists \alpha \in \mathbb{F} & \text{such that} & y = \alpha x & \forall x,y\in X
\end{array}
$$

✎PROOF:

1. Proof that $|\langle x \,|\, y\rangle| \leq \|x\| \, \|y\|$: [5]

   (a) $y = \mathbb{0}$ case:

$$
\begin{aligned}
|\langle x \,|\, y\rangle|^2 &= |\langle x \,|\, \mathbb{0}\rangle|^2 && \text{by } y = \mathbb{0} \text{ hypothesis} \\
&= |\langle \mathbb{0} \,|\, x\rangle|^2 && \text{by Definition 7.1 page 99} \\
&= |\langle 0\mathbb{0} \,|\, x\rangle|^2 && \text{by Definition 4.1 page 71} \\
&= |0 \,\langle \mathbb{0} \,|\, x\rangle|^2 && \text{by Definition 7.1 page 99} \\
&= 0 \\
&= \langle x \,|\, x\rangle \, \langle \mathbb{0} \,|\, \mathbb{0}\rangle \\
&= \langle x \,|\, x\rangle \, \langle y \,|\, y\rangle && \text{by } y = \mathbb{0} \text{ hypothesis}
\end{aligned}
$$

   (b) $y \neq \mathbb{0}$ case: Let $\lambda \triangleq \frac{\langle x \,|\, y\rangle}{\langle y \,|\, y\rangle}$.

$$
\begin{aligned}
0 &\leq \langle x - \lambda y \,|\, x - \lambda y\rangle && \text{by Definition 7.1} \\
&= \langle x \,|\, x - \lambda y\rangle + \langle -\lambda y \,|\, x - \lambda y\rangle && \text{by Definition 7.1} \\
&= \langle x - \lambda y \,|\, x\rangle^* - \lambda \,\langle y \,|\, x - \lambda y\rangle && \text{by Definition 7.1} \\
&= \langle x \,|\, x\rangle^* + \langle -\lambda y \,|\, x\rangle^* - \lambda \,\langle x - \lambda y \,|\, y\rangle^* && \text{by Definition 7.1} \\
&= \langle x \,|\, x\rangle^* - \lambda^* \,\langle y \,|\, x\rangle^* - \lambda \,\langle x \,|\, y\rangle^* - \lambda \,\langle -\lambda y \,|\, y\rangle^* && \text{by Definition 7.1} \\
&= \langle x \,|\, x\rangle - \lambda^* \,\langle x \,|\, y\rangle - \lambda \,\langle x \,|\, y\rangle^* + \lambda\lambda^* \,\langle y \,|\, y\rangle^* && \text{by Definition 7.1} \\
&= \langle x \,|\, x\rangle + \cancelto{0}{\left[\frac{\langle x \,|\, y\rangle}{\langle y \,|\, y\rangle}\lambda^* \,\langle y \,|\, y\rangle - \lambda^* \,\langle x \,|\, y\rangle\right]} - \frac{\langle x \,|\, y\rangle}{\langle y \,|\, y\rangle} \,\langle x \,|\, y\rangle^* && \text{by definition of } \lambda \\
&= \langle x \,|\, x\rangle - \frac{1}{\langle y \,|\, y\rangle} \; |\langle x \,|\, y\rangle|^2 \\
\implies & \; |\langle x \,|\, y\rangle|^2 \leq \langle x \,|\, x\rangle \; \langle y \,|\, y\rangle
\end{aligned}
$$

2. Proof that $|\langle x \,|\, y\rangle|^2 = \langle x \,|\, x\rangle \; \langle y \,|\, y\rangle \iff y = ax$:
   Let $\frac{1}{a} \triangleq \lambda \triangleq \frac{\langle x \,|\, y\rangle}{\langle y \,|\, y\rangle}$. Then...

$$
\begin{aligned}
& & y &= ax \\
&\iff & x &= \lambda y \\
&\iff & x - \lambda y &= \mathbb{0} \\
&\iff & 0 &= \langle x - \lambda y \,|\, x - \lambda y\rangle && \text{by Definition 7.1 page 99} \\
& & &= \langle x \,|\, x - \lambda y\rangle + \langle -\lambda y \,|\, x - \lambda y\rangle && \text{by Definition 7.1 page 99} \\
& & &= \langle x - \lambda y \,|\, x\rangle^* - \lambda \,\langle y \,|\, x - \lambda y\rangle && \text{by Definition 7.1 page 99} \\
& & &\;\;\vdots && \text{(same steps as in 1(b))}
\end{aligned}
$$

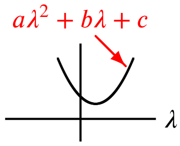
[4] The function $\|\cdot\|$ is a *norm* (Theorem 7.4 page 104) and is called the *norm induced by the inner product* $\langle \triangle \,|\, \nabla\rangle$ (Definition 7.2 page 105).

[5] Haaser and Sullivan (1991), page 278

$$
\begin{aligned}
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle - \frac{1}{\langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle} \; |\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle|^2 \\
\iff \qquad & |\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle|^2 = \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle \; \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle
\end{aligned}
$$

3. Alternate proof for $|\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle| \le \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$: (Note: This is essentially the same proof as used by Schwarz). [6]

   (a) Proof that $\left\{ a\lambda^2 + b\lambda + c \ge 0 \quad \forall \lambda \in \mathbb{R} \right\} \implies \left\{ b^2 \le 4ac \right\}$ (quadratic discriminate inequality):



Let $k \in (0, \infty)$, and $r_1, r_2 \in \mathbb{C}$ be the roots of $a\lambda^2 + b\lambda + c = 0$. Then

$$
\begin{aligned}
0 &\le a\lambda^2 + b\lambda + c && \text{by left hypothesis} \\
&= k(\lambda - r_1)(\lambda - r_2) && \text{by definition of } r_1 \text{ and } r_2 \\
&= k(\lambda^2 - r_1\lambda - r_2\lambda + r_1 r_2) \\
\implies \quad & \lambda^2 - r_1\lambda - r_2\lambda + r_1 r_2 \ge 0 \\
\implies \quad & r_1 = r_2^* && \text{because } r_1 r_2 \ge 0 \text{ for } \lambda = 0
\end{aligned}
$$

The *quadratic equation* places another constraint on r_1 and r_2:

$$
\begin{aligned}
\frac{b^2 + \sqrt{b^2 - 4ac}}{2a} &= r_1 && \text{by quadratic equation} \\
&= r_2^* && \text{by previous result} \\
&= \left(\frac{b^2 - \sqrt{b^2 - 4ac}}{2a} \right)^* && \text{by quadratic equation}
\end{aligned}
$$

The only way for this to be true is if $b^2 \le 4ac$ (the **discriminate** is non-positive).

   (b) Proof that $\langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle \, \lambda^2 + 2|\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle|\lambda + \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle \ge 0 \quad \forall \lambda \in \mathbb{R}$:

$$
\begin{aligned}
0 &\le \langle \boldsymbol{x} + \alpha\boldsymbol{y} \,|\, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle && \text{by Definition 7.1 page 99} \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle + \langle \alpha\boldsymbol{y} \,|\, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle && \text{by Definition 7.1 page 99} \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle + \alpha \, \langle \boldsymbol{y} \,|\, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle && \text{by Definition 7.1 page 99} \\
&= \langle \boldsymbol{x} + \alpha\boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* + \alpha \, \langle \boldsymbol{x} + \alpha\boldsymbol{y} \,|\, \boldsymbol{y} \rangle^* && \text{by Definition 7.1 page 99} \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle^* + \langle \alpha\boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* + \alpha \, \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle^* + \alpha \, \langle \alpha\boldsymbol{y} \,|\, \boldsymbol{y} \rangle^* && \text{by Definition 7.1 page 99} \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle^* + \alpha^* \, \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* + \alpha \, \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle^* + \alpha\alpha^* \, \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle^* && \text{by Definition 7.1 page 99} \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle + \alpha^* \, \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle + \left(\alpha^* \, \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle \right)^* + |\alpha|^2 \, \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle && \text{by Definition 7.1 page 99} \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle + 2\Re \left(\alpha^* \, \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle \right) + |\alpha|^2 \, \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle \\
&\le \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle + 2 \left| \alpha^* \, \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle \right| + |\alpha|^2 \, \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle + 2|\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle||\alpha| + \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle \, |\alpha|^2 \\
&= \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle \, |\alpha|^2 + 2 \, |\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle| \, |\alpha| + \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle \\
&= \underbrace{\langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle}_{a} \lambda^2 + \underbrace{2|\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle|}_{b} \lambda + \underbrace{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}_{c} && \text{because } \lambda \triangleq |\alpha| \in \mathbb{R}
\end{aligned}
$$

[6] Aliprantis and Burkinshaw (1998) page 278, Steele (2004) page 11

(c) The above equation is in the quadratic form used in the lemma of part (a).

$$\left(\underbrace{2|\langle x \mid y\rangle|}_{b}\right)^2 \le 4\underbrace{\langle y \mid y\rangle}_{a}\;\underbrace{\langle x \mid x\rangle}_{c} \qquad \text{by the results of parts (a) and (b)}$$

$$\implies |\langle x \mid y\rangle|^2 \le \langle x \mid x\rangle\,\langle y \mid y\rangle$$

4. Proof that $|\langle x \mid y\rangle| \le \|x\|\,\|y\|$:
This follows directly from the definition $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

5. Proof that $|\langle x \mid y\rangle| = \|x\|\,\|y\| \iff \exists\alpha\in\mathbb{C}$ such that $y = \alpha x$:
This follows directly from the definition $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

⇒

Corollary 7.1. [7] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla\rangle)$ *be an* INNER PRODUCT SPACE.

COR: $\langle x \mid y\rangle$ *is* CONTINUOUS (Definition 1.8 page 23) *in both* x *and* y.

✎PROOF: Let $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

$$\begin{aligned}
|\langle x+\epsilon \mid y\rangle - \langle x \mid y\rangle|^2 &= |\langle x+\epsilon-x \mid y\rangle|^2 && \text{by } \textit{additivity} \text{ of } \langle \triangle \mid \nabla\rangle && \text{(Definition 7.1 page 99)}\\
&= |\langle \epsilon \mid y\rangle|^2\\
&\le \|\epsilon\|^2\,\|y\| && \text{by } \textit{Cauchy-Schwarz Inequality} && \text{(Theorem 7.2 page 100)}
\end{aligned}$$

⇒

7.2 Relationship between norms and inner products

7.2.1 Norms induced by inner products

Lemma 7.1 (Polar Identity). [8] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla\rangle)$ *be an* INNER PRODUCT SPACE (Definition 7.1 page 99). *Let* $\Re z$ *represent the real part of* $z\in\mathbb{C}$. *Let* $\|\cdot\|$ *be a function in* $\mathbb{R}^{\mathbb{F}}$ *such that* $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.[9]

LEM: $$\|x+y\|^2 = \|x\|^2 + 2\mathbf{R_e}\left[\langle x \mid y\rangle\right] + \|y\|^2 \qquad \forall x,y\in X$$

✎PROOF:

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y \mid x+y\rangle && \text{by definition of } \textit{induced norm} && \text{(Theorem 7.4 page 104)}\\
&= \langle x \mid x+y\rangle + \langle y \mid x+y\rangle && \text{by Definition 7.1 page 99}\\
&= \langle x+y \mid x\rangle^* + \langle x+y \mid y\rangle^* && \text{by Definition 7.1 page 99}\\
&= \langle x \mid x\rangle^* + \langle y \mid x\rangle^* + \langle x \mid y\rangle^* + \langle y \mid y\rangle^* && \text{by Definition 7.1 page 99}\\
&= \langle x \mid x\rangle + \langle x \mid y\rangle + \langle x \mid y\rangle^* + \langle y \mid y\rangle && \text{by definition of } \textit{inner product} && \text{(Definition 7.1 page 99)}\\
&= \|x\|^2 + 2\Re\,\langle x \mid y\rangle + \|y\|^2 && \text{by definition of } \textit{induced norm} && \text{(Theorem 7.4 page 104)}
\end{aligned}$$

[7] Bollobás (1999) page 132, Aliprantis and Burkinshaw (1998) page 279 ⟨Lemma 32.4⟩
[8] Conway (1990) page 4, Heil (2011) page 27 ⟨Lemma 1.36(a)⟩
[9] The function $\|\cdot\|$ is a *norm* (Theorem 7.4 page 104) and is called the *norm induced by the inner product* $\langle \triangle \mid \nabla\rangle$ (Definition 7.2 page 105).

✏

Theorem 7.3 (Minkowski's Inequality). [10] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be an* INNER PRODUCT SPACE. *Let* $\|\cdot\|$ *be a function in* $\mathbb{R}^{\mathbb{F}}$ *such that* $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.[11]

THM $\|x + y\| \le \|x\| + \|y\| \qquad \forall x,y \in X$

✎PROOF:

$$\begin{aligned}
\|x+y\|^2 &= \|x\|^2 + 2\Re\langle x \mid y\rangle + \|y\|^2 && \text{by } \textit{Polar Identity} && \text{(Lemma 7.1 page 103)}\\
&\le \|x\|^2 + 2|\langle x \mid y\rangle| + \|y\|^2\\
&\le \|x\|^2 + 2\sqrt{\langle x \mid x\rangle}\sqrt{\langle y \mid y\rangle} + \|y\|^2 && \text{by } \textit{Cauchy-Schwarz Inequality} && \text{(Theorem 7.2 page 100)}\\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

✏

7.2.2 Inner products induced by norms

Theorem 7.4 (induced norm). [12] *Let* $\Omega \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition 7.1 page 99)*.

THM $\|x\| \triangleq \sqrt{\langle x \mid x\rangle} \implies \|\cdot\|$ *is a* NORM

✎PROOF: For a function to be a norm, it must satisfy the four properties listed in Definition 6.1 (page 87).

1. Proof that $\|\cdot\|$ is a norm:
   (a) Proof that $\|x\| > 0$ for $x \ne 0$ (non-negative):
   By Definition 7.1 page 99, all inner products have this property.
   (b) Proof that $\|x\| = 0 \iff x = 0$ (non-isometric):
   By Definition 7.1, all inner products have this property.
   (c) Prove $\|ax\| = |a|\,\|x\|$ (homogeneous):
   $$\|ax\| \triangleq \sqrt{\langle ax \mid ax\rangle} = \sqrt{aa^*\langle x \mid x\rangle} = \sqrt{|a|^2\langle x \mid x\rangle} = |a|\,\|x\|$$
   (d) Proof that $\|x+y\| \le \|x\| + \|y\|$ (subadditive): This is true by *Minkowski's Inequality* (Theorem 11.5 page 169).
2. Proof that every inner product space is a normed linear space:
   Since every inner product induces a norm, so every inner product space has a norm (the norm induced by the inner product) and is therefore a normed linear space.

[10] Aliprantis and Burkinshaw (1998) pages 278–279 ⟨Theorem 32.3⟩, Maligranda (1995), Minkowski (1910) page 115

[11] The function $\|\cdot\|$ is a *norm* (Theorem 7.4 page 104) and is called the *normed induced by the inner product* $\langle \triangle \mid \triangledown \rangle$ (Definition 7.2 page 105).

[12] Aliprantis and Burkinshaw (1998) pages 278–279, Haaser and Sullivan (1991) page 278

✏

Theorem 7.4 (previous theorem) demonstrates that in any inner product space $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$, the function $\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$ is a norm. That is, $\|x\|$ is the *norm induced by the inner product*. This norm is formally defined next.

Definition 7.2. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be an* INNER PRODUCT SPACE *(Definition 7.1 page 99).*

DEF

The ***norm induced by the inner product*** $\langle \triangle \mid \nabla \rangle$ *is defined as*

$$\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$$

Theorem 7.4 (page 104) demonstrates that if a *linear space* (Definition 4.1 page 71) has an *inner product* (Definition 7.1 page 99), then that inner product always induces a *norm* (Definition 6.1 page 87), and the relationship between the two is simply $\|x\| = \sqrt{\langle x \mid x \rangle}$ (Definition 7.2 page 105). But what about the converse? What if a linear space has a norm—can that norm also induce an inner product? The answer in general is "no": Not all norms can induce an inner product. But a less harsh answer is "sometimes": Some norms **can** induce inner products. This leads to some important and interesting questions:

1. How many different inner products can be induced from a single norm? The answer turns out to be **at most** one, but maybe none (Theorem 7.5 page 105).
2. When a norm *can* induce an inner product, what is that (unique) inner product? The inner product expressed in terms of the norm is given by the *Polarization Identity* (Theorem 7.6 page 106).
3. Which norms can induce an inner product and which ones cannot? The answer is that norms that satisfy the *parallelogram law* (Theorem 7.7 page 107) **can** induce an inner product; and the ones that don't, cannot (Theorem 7.7 page 107).

Theorem 7.5. [13] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition 6.1 page 87).*

THM

$$\underbrace{\left.\begin{array}{l} \exists\ \langle \triangle \mid \nabla \rangle \text{ and } (\cdot \mid \cdot) \quad \text{such that} \\ \|x\|^2 = \langle x \mid x \rangle = (x \mid x) \quad \forall x \in X \end{array}\right\}}_{\text{If a norm is induced by two inner products...}} \implies \underbrace{\langle x \mid y \rangle = (x \mid y) \quad \forall x, y \in X}_{\text{...then those two inner products are equivalent.}}$$

✎PROOF:

$$\begin{aligned}
2\langle x \mid y \rangle &= \left[\langle x \mid y \rangle + \langle y \mid x \rangle\right] + \left[\langle x \mid y \rangle - \langle y \mid x \rangle\right] \\
&= \left[\langle x \mid y \rangle + \langle y \mid x \rangle\right] + i\left[-i\langle x \mid y \rangle + i\langle y \mid x \rangle\right] \\
&= \left[\langle x \mid y \rangle + \langle y \mid x \rangle\right] + i\left[-\langle ix \mid y \rangle - \langle y \mid ix \rangle\right] \\
&= \left(\underbrace{\left[\langle x \mid y \rangle + \langle y \mid x \rangle + \langle x \mid x \rangle + \langle y \mid y \rangle\right]}_{\langle x+y \mid x+y \rangle} - \underbrace{\left[\langle x \mid x \rangle + \langle y \mid y \rangle\right]}_{\text{residue}}\right) \\
&\quad - i\left(\underbrace{\left[\langle ix \mid y \rangle + \langle y \mid ix \rangle + \langle ix \mid ix \rangle + \langle y \mid y \rangle\right]}_{\langle ix+y \mid ix+y \rangle} - \underbrace{\left[\langle ix \mid ix \rangle + \langle y \mid y \rangle\right]}_{\text{residue}}\right) \\
&= \left(\left[\langle x+y \mid x+y \rangle\right] - \left[\langle x \mid x \rangle + \langle y \mid y \rangle\right]\right) - i\left(\left[\langle ix+y \mid ix+y \rangle\right] - \left[\langle ix \mid ix \rangle + \langle y \mid y \rangle\right]\right) \\
&= \left(\left[(x+y \mid x+y)\right] - \left[(x \mid x) + (y \mid y)\right]\right) - i\left(\left[(ix+y \mid ix+y)\right] - \left[(ix \mid ix) + (y \mid y)\right]\right)
\end{aligned}$$

[13] Aliprantis and Burkinshaw (1998) page 280, Bollobás (1999) page 132, Jordan and von Neumann (1935) page 721

$$= \left(\underbrace{\left[(x \mid y) + (y \mid x) + (x \mid x) + (y \mid y) \right]}_{(x+y \mid x+y)} - \underbrace{\left[(x \mid x) + (y \mid y) \right]}_{\text{residue}} \right)$$
$$- i \left(\underbrace{\left[(ix \mid y) + (y \mid ix) + (ix \mid ix) + (y \mid y) \right]}_{(ix+y \mid ix+y)} - \underbrace{\left[(ix \mid ix) + (y \mid y) \right]}_{\text{residue}} \right)$$
$$= \left[(x \mid y) + (y \mid x) \right] + i \left[-(ix \mid y) - (y \mid ix) \right]$$
$$= \left[(x \mid y) + (y \mid x) \right] + i \left[-i (x \mid y) + i (y \mid x) \right]$$
$$= \left[(x \mid y) + (y \mid x) \right] + \left[(x \mid y) - (y \mid x) \right]$$
$$= 2 (x \mid y)$$

✏

Theorem 7.6 (Polarization Identities). [14] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}) \right)$ *be a linear space,* $\langle \triangle \mid \nabla \rangle \in \mathbb{F}^{X \times X}$ *a function, and* $\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$.

THM

$\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle \right)$ *is an inner product space* $\implies$

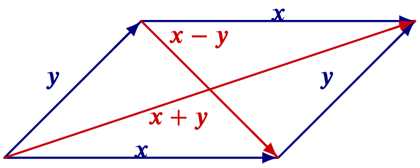
$$\underbrace{4 \langle x \mid y \rangle = \begin{cases} \|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 & \text{for } \mathbb{F} = \mathbb{C} \quad \forall x,y \in X \\ \|x+y\|^2 - \|x-y\|^2 & \text{for } \mathbb{F} = \mathbb{R} \quad \forall x,y \in X \end{cases}}_{\textit{inner product induced by norm}}$$

✎PROOF:

1. These follow directly from properties of *bilinear functionals* (Theorem 14.2 page 201).
2. Alternative proof for $\mathbb{F} = \mathbb{C}$ case:

$$\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2$$
$$= \underbrace{\|x\|^2 + \|y\|^2 + 2\Re \langle x \mid y \rangle}_{\langle x+y \mid x+y \rangle} - \underbrace{\left(\|x\|^2 + \|-y\|^2 + 2\Re \langle x \mid -y \rangle \right)}_{\langle x-y \mid x-y \rangle}$$
$$+ \underbrace{i \left(\|x\|^2 + \|iy\|^2 + 2\Re \langle x \mid iy \rangle \right)}_{i \langle x+iy \mid x+iy \rangle} - \underbrace{i \left(\|x\|^2 + \|-iy\|^2 + 2\Re \langle x \mid -iy \rangle \right)}_{i \langle x-iy \mid x-iy \rangle} \qquad \text{by Lemma 7.1 page 103}$$
$$= \underbrace{\|x\|^2 + \|y\|^2 + 2\Re \langle x \mid y \rangle}_{\langle x+y \mid x+y \rangle} - \underbrace{\left(\|x\|^2 + \|y\|^2 - 2\Re \langle x \mid y \rangle \right)}_{\langle x-y \mid x-y \rangle}$$
$$+ \underbrace{i \left(\|x\|^2 + \|y\|^2 + 2\Re \langle x \mid iy \rangle \right)}_{i \langle x+iy \mid x+iy \rangle} - \underbrace{i \left(\|x\|^2 + \|y\|^2 - 2\Re \langle x \mid iy \rangle \right)}_{i \langle x-iy \mid x-iy \rangle} \qquad \text{by Definition 7.1 page 99}$$
$$= 4\Re \langle x \mid y \rangle + 4i\Re \langle x \mid iy \rangle$$
$$= \underbrace{2 \left(\langle x \mid y \rangle + \langle x \mid y \rangle^* \right)}_{4\Re \langle x \mid y \rangle} + \underbrace{2i \left(\langle x \mid iy \rangle + \langle x \mid iy \rangle^* \right)}_{4i\Re \langle x \mid iy \rangle}$$
$$= 2 \left(\langle x \mid y \rangle + \langle x \mid y \rangle^* \right) + 2i \left(i^* \langle x \mid y \rangle + (i^{**}) \langle x \mid y \rangle^* \right)$$
$$= 2 \left(\langle x \mid y \rangle + \langle x \mid y \rangle^* \right) + 2i \left(-i \langle x \mid y \rangle + i \langle x \mid y \rangle^* \right) \qquad \text{by Definition 7.1 page 99}$$
$$= 2 \langle x \mid y \rangle + 2 \langle x \mid y \rangle^* + 2 \langle x \mid y \rangle - 2 \langle x \mid y \rangle^*$$
$$= 4 \langle x \mid y \rangle$$

[14] Berberian (1961) pages 29–30 ⟨Theorem II.3.3⟩, Istrăţescu (1987) page 110 ⟨Proposition 4.1.5⟩, Bollobás (1999) page 132, Jordan and von Neumann (1935) page 721



In plane geometry ($\mathbb{R}^2$), the *parallelogram law* states that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of its diagonals. This is illustrated in the figure to the left.

Acutally, the parallelogram law can be generalized to *any inner product space* (not just in the plane). And if the parallelogram law happens to hold true in a normed linear space, then that normed linear space is actually an *inner product space*. The parallelogram law and its relation to inner product spaces is stated in the next theorem.

Theorem 7.7 (Parallelogram law). [15] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *and* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.

THM $\boldsymbol{\Omega}$ *is an inner product space* $\iff$ $\underbrace{2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2 = \|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2}_{\text{PARALLELOGRAM LAW / VON NEUMANN-JORDAN CONDITION}}$ $\quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{\Omega}$

✎PROOF:

1. Proof that $[\exists \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ such that $\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle] \implies$ [parallelogram law is true]:

$$
\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 &= \left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle\right]\right] \qquad + \left[\|\boldsymbol{x}\|^2 + \|{-\boldsymbol{y}}\|^2 + 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid -\boldsymbol{y} \rangle\right]\right] \\
&\qquad \text{by Lemma 7.1 page 103} \\
&= \left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle\right]\right] \qquad + \left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle\right]\right] \\
&= 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2
\end{aligned}
$$

2. Proof that $[\exists \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ such that $\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle] \impliedby$ [parallelogram law is true]:
Note that **if** an inner product exists in the norm linear space $(\boldsymbol{\Omega}, \|\cdot\|)$, **then** that norm linear space is actually an inner product space $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$. And if it is an inner product space, then by Theorem 7.6 page 106 that inner product must be given by the **Polarization Identity**
$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \|a\boldsymbol{x}+\boldsymbol{y}\|^2 - \|a\boldsymbol{x}-\boldsymbol{y}\|^2 + i\,\|a\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\,\|a\boldsymbol{x}-i\boldsymbol{y}\|^2$.
Therefore, here we must use the parallelogram law to show that the bilinear function $\mathsf{f}(\boldsymbol{x},\boldsymbol{y}) \triangleq \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ given on the left hand side of the "=" relation is indeed an inner product— that is, that it satisfies the requirements of Definition 7.1 page 99.

   (a) Proof that $\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle \geq 0$ (non-negative):

$$
\begin{aligned}
4\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle &\triangleq \|\boldsymbol{x}+\boldsymbol{x}\|^2 - \cancelto{0}{\|\boldsymbol{x}-\boldsymbol{x}\|^2} + i\,\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - i\,\|\boldsymbol{x}-i\boldsymbol{x}\|^2 && \text{by } \textit{Polarization Identity} \\
&= \|2\boldsymbol{x}\|^2 - \mathbb{0} + i\left(\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - \|\boldsymbol{x}-i\boldsymbol{x}\|^2\right) && \text{by Definition 6.1 page 87} \\
&= |2|^2\,\|\boldsymbol{x}\|^2 + i\left(\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - |i|\,\|\boldsymbol{x}-i\boldsymbol{x}\|^2\right) \\
&= 4\|\boldsymbol{x}\|^2 + i\left(\cancelto{0}{\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - \|i\boldsymbol{x}+\boldsymbol{x}\|^2}\right) && \text{by Definition 6.1 page 87} \\
&= 4\|\boldsymbol{x}\|^2 \\
&\geq 0 && \text{by Definition 6.1 page 87}
\end{aligned}
$$

[15] Amir (1986) page 8, Istrăţescu (1987) page 110, Day (1973) page 151, Halmos (1998a) page 14, Aliprantis and Burkinshaw (1998) pages 280–281 ⟨Theorem 32.6⟩, Riesz (1934) page 36?, Jordan and von Neumann (1935) pages 721–722

(b) Proof that $\langle x \mid x\rangle = 0 \iff x = \mathbb{0}$ (non-isotropic):

$$\begin{aligned}
4\langle x \mid x\rangle &= 4\|x\|^2 && \text{by result of part (a)}\\
&= 0 \quad\iff\quad x = \mathbb{0} && \text{by Definition 6.1 page 87}
\end{aligned}$$

(c) Proof that $\langle x + u \mid y\rangle = \langle x \mid y\rangle + \langle u \mid y\rangle$ (additive):[16]

$$\begin{aligned}
4\langle x + y \mid z\rangle &= 8\left\langle \frac{x+y}{2} \,\middle|\, z\right\rangle && \text{by Definition 7.1 page 99}\\
&= 2\left\|\frac{x+y}{2} + z\right\|^2 - 2\left\|\frac{x+y}{2} - z\right\|^2\\
&\quad + 2i\left\|\frac{x+y}{2} + z\right\|^2 - 2i\left\|\frac{x+y}{2} - iz\right\|^2 && \text{by \textit{Polarization Identity}}\\
&= \left(2\left\|\frac{x+y}{2} + z\right\|^2 + 2\left\|\frac{x-y}{2}\right\|^2\right)\\
&\quad - \left(2\left\|\frac{x+y}{2} - z\right\|^2 + 2\left\|\frac{x-y}{2}\right\|^2\right)\\
&\quad + \left(2i\left\|\frac{x+y}{2} + z\right\|^2 + 2i\left\|\frac{x-y}{2}\right\|^2\right)\\
&\quad - \left(2i\left\|\frac{x+y}{2} - iz\right\|^2 + 2i\left\|\frac{x-y}{2}\right\|^2\right)\\
&= \left(\|x+z\|^2 + \|y+z\|^2\right) - \left(\|x-z\|^2 + \|y-z\|^2\right)\\
&\quad + \left(i\|x+z\|^2 + i\|y+z\|^2\right) - \left(i\|x-iz\|^2 + i\|y-iz\|^2\right) && \text{by parallelogram law}\\
&= \left(\|x+z\|^2 - \|x-z\|^2 + i\|x+z\|^2 - i\|x-iz\|^2\right)\\
&\quad + \left(\|y+z\|^2 - \|y-z\|^2 + i\|y+z\|^2 - i\|y-iz\|^2\right)\\
&= 4\langle x \mid z\rangle + 4\langle y \mid z\rangle && \text{by Polarization Identity}
\end{aligned}$$

(d) Proof that $\langle x \mid y\rangle = \langle y \mid y\rangle^*$ (*conjugate symmetric*):

$$\begin{aligned}
4\langle x \mid y\rangle &\triangleq \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 && \text{by Polarization Identity}\\
&= \|y+x\|^2 - \|y-x\|^2 + i\|i(y-ix)\|^2 - i\|-i(y+ix)\|^2 && \text{by Definition 4.1 page 71}\\
&= \|y+x\|^2 - \|y-x\|^2 + i\|y-ix\|^2 - i\|y+ix\|^2 && \text{by Definition 6.1 page 87}\\
&= \left(\|y+x\|^2 - \|y-x\|^2 - i\|y-ix\|^2 + i\|y+ix\|^2\right)^*\\
&= \left(\|y+x\|^2 - \|y-x\|^2 + i\|y+ix\|^2 - i\|y-ix\|^2\right)^*\\
&\triangleq 4\langle y \mid x\rangle^* && \text{by \textit{Polarization Identity}}
\end{aligned}$$

(e) Proof that $\langle \alpha x \mid y\rangle = \alpha\langle x \mid y\rangle$ (*homogeneous*): [17]

i. Proof that $\langle \alpha x \mid y\rangle$ is linear in α:

$$\begin{aligned}
0 &\le \left|\|\alpha x + y\| - \|\beta x + y\|\right| && \text{by Definition F.4 page 342}\\
&\le \left|\|(\alpha x + y) - (\beta x + y)\|\right| && \text{by Theorem 6.2 page 88}\\
&\le \left|\|(\alpha - \beta)x\|\right|
\end{aligned}$$

This implies that as $a \to \beta$, $\|\alpha x + y\| \to \|\beta x + y\|$, which by definition implies that $\|\alpha x + y\|$ linear in α. And by the parallelogram law, $\langle \alpha x \mid y\rangle$ is also linear in α.

ii. Proof that $\langle nx \mid y\rangle = n\langle x \mid y\rangle$ for $n \in \mathbb{Z}$ (integer case):

[16] Aliprantis and Burkinshaw (1998), page 281
[17] Aliprantis and Burkinshaw (1998), page 138

A. Proof for $n = \pm 1$:

$$\begin{aligned}
\langle n\boldsymbol{x} \,|\, \boldsymbol{y}\rangle &= \langle \pm 1\boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by } n=\pm 1 \text{ hypothesis} \\
&= \pm 1 \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by definition of \textit{inner product}} \quad \text{(Definition 7.1 page 99)} \\
&= n \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by } n=\pm 1 \text{ hypothesis}
\end{aligned}$$

B. Proof for $n = 0$:

$$\begin{aligned}
\langle n\boldsymbol{x} \,|\, \boldsymbol{y}\rangle &= \langle 0\boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by } n=0 \text{ hypothesis} \\
&= \langle \boldsymbol{x}-\boldsymbol{x} \,|\, \boldsymbol{y}\rangle \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle + \langle -1\boldsymbol{x} \,|\, \boldsymbol{y}\rangle \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle - 1\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle - \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle \\
&= 0\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle \\
&= n\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by } n=0 \text{ hypothesis}
\end{aligned}$$

C. Proof for $n = \pm 2$:

$$\begin{aligned}
\langle n\boldsymbol{x} \,|\, \boldsymbol{y}\rangle &= \langle \pm 2\boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by } n=\pm 1 \text{ hypothesis} \\
&= \langle \pm(\boldsymbol{x}+\boldsymbol{x}) \,|\, \boldsymbol{y}\rangle \\
&= \pm\langle \boldsymbol{x}+\boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by definition of \textit{inner product}} \quad \text{(Definition 7.1 page 99)} \\
&= \pm(\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle + \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle) && \text{by additive property} \\
&= \pm 2\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle \\
&= n\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by } n=\pm 2 \text{ hypothesis}
\end{aligned}$$

D. Proof that $[n \text{ case}] \implies [n \pm 1 \text{ case}]$:

$$\begin{aligned}
\langle (n\pm 1)\boldsymbol{x} \,|\, \boldsymbol{y}\rangle &= \langle n\boldsymbol{x} \pm 1\boldsymbol{x} \,|\, \boldsymbol{y}\rangle \\
&= \langle n\boldsymbol{x} \,|\, \boldsymbol{y}\rangle + \langle \pm 1\boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by additive property} \\
&= n\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle \pm 1\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by left hypothesis} \\
&= (n\pm 1)\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle
\end{aligned}$$

iii. Proof that $\langle q\boldsymbol{x} \,|\, \boldsymbol{y}\rangle = q\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle$ for $q \in \mathbb{Q}$ (rational number case):

$$\begin{aligned}
\frac{n}{m}\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle &= \frac{n}{m}\left\langle \frac{m}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle && \text{where } n,m \in \mathbb{Z} \text{ and } m \neq 0 \\
&= \frac{nm}{m}\left\langle \frac{1}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle && \text{by previous result} \\
&= \frac{m}{m}\left\langle \frac{n}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle && \text{by previous result} \\
&= \left\langle \frac{n}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle
\end{aligned}$$

iv. Proof that $\langle r\boldsymbol{x} \,|\, \boldsymbol{y}\rangle = r\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle$ for all $r \in \mathbb{R}$ (real number case):
Because $\mathbb{Q}$ is dense in $\mathbb{R}$ and because $\|\alpha\boldsymbol{x} + \boldsymbol{y}\|$ is continuous in α, so $\langle \alpha\boldsymbol{x} \,|\, \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle$ for all $\alpha \in \mathbb{R}$.

v. Proof that $\langle c\boldsymbol{x} \,|\, \boldsymbol{y}\rangle = c\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle$ for all $c \in \mathbb{C}$ (complex number case):
No proof at this time.

⇒

Remark 7.1. [18] The inner product has already been defined in Definition 7.1 (page 99) as a bilinear function that is *non-negative*, *non-isotropic*, *homogeneous*, *additive*, and *conjugate symmetric*.

[18] Loomis (1953) pages 23–24, Kubrusly (2001) page 317

However, given a normed linear space, we could alternatively define the inner product using the *parallelogram law* (Theorem 7.7 page 107) together with the *Polarization Identity* (Theorem 7.6 page 106). Under this new definition, an inner product *exists* if the parallelogram law is satisfied, and is *specified*, in terms of the norm, by the Polarization Identity.

Of the uncountably infinite number of $\ell^p_{\mathbb{F}}$ norms, only the norm for $p = 2$ induces an inner product (Proposition 7.1, next).

Proposition 7.1. [19] *Let $\left\|(\!(x_n)\!)_{n\in\mathbb{Z}}\right\|_p$ be the $\ell^p_{\mathbb{F}}$ norm* (Definition 9.13 page 148) *of the sequence $(\!(x_n)\!)$ in the space $\ell^p_{\mathbb{F}}$.*

PRP	
	$\left\|(\!(x_n)\!)\right\|_p$ *induces an inner product* $\iff$ $p = 2$

✎PROOF:

1. Proof that $\|\cdot\|_p$ induces an inner product $\Longleftarrow$ $p = 2$ (using the *Parallelogram law* page 107):

$$\begin{aligned}
&\|\boldsymbol{x}+\boldsymbol{y}\|_p^2 + \|\boldsymbol{x}-\boldsymbol{y}\|_p^2 \\
&= \|\boldsymbol{x}+\boldsymbol{y}\|_2^2 + \|\boldsymbol{x}-\boldsymbol{y}\|_2^2 && \text{by right hypothesis} \\
&= \left(\sum_{n\in\mathbb{Z}} |x_n+y_n|^2\right)^{\frac{2}{2}} + \left(\sum_{n\in\mathbb{Z}} |x_n-y_n|^2\right)^{\frac{2}{2}} && \text{by definition of } \|\cdot\|_p \\
&= \sum_{n\in\mathbb{Z}} (x_n+y_n)(x_n+y_n)^* + \sum_{n\in\mathbb{Z}} (x_n-y_n)(x_n-y_n)^* \\
&= \sum_{n\in\mathbb{Z}} \left(|x_n|^2 + |y_n|^2 + 2\Re(x_n y_n)\right) + \sum_{n\in\mathbb{Z}} \left(|x_n|^2 + |y_n|^2 - 2\Re(x_n y_n)\right) \\
&= 2\sum_{n\in\mathbb{Z}} |x_n|^2 + 2\sum_{n\in\mathbb{Z}} |y_n|^2 \\
&= 2\,\|\boldsymbol{x}\|_2^2 + 2\,\|\boldsymbol{y}\|_2^2 && \text{by definition of } \|\cdot\|_p \\
&= 2\,\|\boldsymbol{x}\|_p^2 + 2\,\|\boldsymbol{y}\|_p^2 && \text{by right hypothesis} \\
&\Longrightarrow \|\cdot\|_2 \text{ induces an inner product} && \text{by Theorem 7.7 page 107}
\end{aligned}$$

2. Proof that $\|\cdot\|_p$ induces an inner product $\Longrightarrow$ $p = 2$:

   (a) Let $\boldsymbol{x} \triangleq (1, 0)$ and $\boldsymbol{y} \triangleq (0, 1)$. Then [20]

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|_p^2 + \|\boldsymbol{x}-\boldsymbol{y}\|_p^2 &= \left(\sum_{n\in\mathbb{Z}} |x_n+y_n|^p\right)^{\frac{2}{p}} + \left(\sum_{n\in\mathbb{Z}} |x_n-y_n|^p\right)^{\frac{2}{p}} && \text{by definition of } \|\cdot\|_p \\
&= \left(|1+0|^p + |0+1|^p\right)^{\frac{2}{p}} + \left(|1-0|^p + |0-1|^p\right)^{\frac{2}{p}} && \text{by definitions of } \boldsymbol{x} \text{ and } \boldsymbol{y} \\
&= 2^{\frac{2}{p}} + 2^{\frac{2}{p}} \\
&= 2\cdot 2^{\frac{2}{p}} \\
2\,\|\boldsymbol{x}\|_p^2 + 2\,\|\boldsymbol{y}\|_p^2 &= 2\left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{2}{p}} + 2\left(\sum_{n\in\mathbb{Z}} |y_n|^p\right)^{\frac{2}{p}} && \text{by definition of } \|\cdot\|_p
\end{aligned}$$

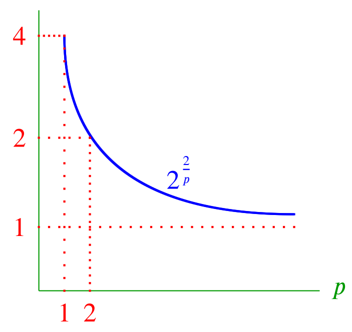
[19] Kubrusly (2001) pages 318–319 ⟨Example 5B⟩
[20] http://groups.google.com/group/sci.math/msg/531b1173f08871e9

$$\begin{aligned}
&= 2\left(|1|^p + |0|^p\right)^{\frac{2}{p}} + 2\left(|1|^p + |0|^p\right)^{\frac{2}{p}} && \text{by definitions of } \boldsymbol{x} \text{ and } \boldsymbol{y}\\
&= 2+2\\
&= 4
\end{aligned}$$

$$\begin{aligned}
2\cdot 2^{\frac{2}{p}} = 4 = \iff\ & 2^{\frac{2}{p}} = 2\\
= \implies\ & p = 2
\end{aligned}$$

(b) Proof that $2^{2/p}$ is monotonic decreasing in p (and so $p = 2$ is the only solution):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}2^{\frac{2}{p}} &= \frac{\mathrm{d}}{\mathrm{d}p}e^{\ln 2^{\frac{2}{p}}}\\
&= \left(e^{\ln 2^{\frac{2}{p}}}\right)\frac{\mathrm{d}}{\mathrm{d}p}\ln 2^{\frac{2}{p}}\\
&= \left(2^{\frac{2}{p}}\right)\frac{\mathrm{d}}{\mathrm{d}p}(2\ln 2)\frac{1}{p}\\
&= \left(2^{\frac{2}{p}}\right)2\ln 2\left(-\frac{1}{p^2}\right)\\
&< 0 \qquad \forall p\in(0,\infty)
\end{aligned}$$



✏

7.3 Orthogonality

Definition 7.3.

DEF: *The **Kronecker delta function** $\bar{\delta}_n$ is defined as* $\bar{\delta}_n \triangleq \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n\neq 0 \end{cases}$ *and* $\forall n\in\mathbb{Z}$

Definition 7.4. *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle \mid \nabla\rangle\right)$ *be an* INNER PRODUCT SPACE (Definition 7.1 page 99).

DEF:
Two vectors $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* X *are **orthogonal** if*
$$\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle = \begin{cases} 0 & \text{for } \boldsymbol{x}\neq\boldsymbol{y} \\ c\in\mathbb{F}\backslash 0 & \text{for } \boldsymbol{x}=\boldsymbol{y}\end{cases}$$
The notation $\boldsymbol{x}\perp\boldsymbol{y}$ *implies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are* ORTHOGONAL.
A set $Y\in 2^X$ *is **orthogonal** if* $\boldsymbol{x}\perp\boldsymbol{y}$ $\forall \boldsymbol{x},\boldsymbol{y}\in Y$.
A set Y *is **orthonomal** if it is* ORTHOGONAL *and* $\langle \boldsymbol{y}\mid\boldsymbol{y}\rangle = 1$ $\forall \boldsymbol{y}\in Y$.
A sequence $(\!(\boldsymbol{x}_n\in X)\!)_{n\in\mathbb{Z}}$ *is **orthogonal** if* $\langle \boldsymbol{x}_n\mid\boldsymbol{x}_m\rangle = c\bar{\delta}_{nm}$ *for some* $c\in\mathbb{R}\backslash 0$.
A sequence $(\!(\boldsymbol{x}_n\in X)\!)_{n\in\mathbb{Z}}$ *is **orthonormal** if* $\langle \boldsymbol{x}_n\mid\boldsymbol{x}_m\rangle = \bar{\delta}_{nm}$.

The definition of the orthogonality relation $\perp$ has several immediate consequences (next theorem):

Theorem 7.8. [21] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle \mid \nabla\rangle\right)$ *be an* INNER PRODUCT SPACE.

THM:

1.	$\boldsymbol{x}\perp\boldsymbol{x} \iff \boldsymbol{x}=\mathbb{0}$	$\forall \boldsymbol{x}\in X$	
2.	$\boldsymbol{x}\perp\boldsymbol{y} \implies \alpha\boldsymbol{x}\perp\boldsymbol{y}$	$\forall\alpha\in\mathbb{R},\boldsymbol{x},\boldsymbol{y}\in X$	(HOMOGENEOUS)
3.	$\boldsymbol{x}\perp\boldsymbol{y} \iff \boldsymbol{y}\perp\boldsymbol{x}$	$\forall\boldsymbol{x},\boldsymbol{y}\in X$	(SYMMETRY)
4.	$\boldsymbol{x}\perp\boldsymbol{y}$ *and* $\boldsymbol{y}\perp\boldsymbol{z} \implies \boldsymbol{x}\perp(\boldsymbol{y}+\boldsymbol{z})$	$\forall\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in X$	(ADDITIVE)
5.	$\exists\beta\in\mathbb{R}$ such that $\boldsymbol{x}\perp(\beta\boldsymbol{x}+\boldsymbol{y})$	$\forall\boldsymbol{x}\in X\backslash\mathbb{0},\boldsymbol{y}\in X$	

[21] James (1945) page 292, Drljević (1989) page 232

Theorem 7.9. *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *be an* INNER PRODUCT SPACE.

THM
$$\left.\begin{array}{lrcll} 1. & \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle &=& 0 & \textit{and} \\ 2. & \boldsymbol{x}+\boldsymbol{y} &=& \mathbb{0} & \end{array}\right\} \iff \left\{\begin{array}{lrcll} 1. & \boldsymbol{x} &=& \mathbb{0} & \textit{and} \\ 2. & \boldsymbol{y} &=& \mathbb{0} & \end{array}\right. \qquad \forall \boldsymbol{x},\boldsymbol{y} \in X$$

✎PROOF:

1. Proof that $\boldsymbol{x} = \boldsymbol{y} = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbb{0} \mid \mathbb{0} \rangle && \text{by non-isotropic property of } \langle \triangle \mid \nabla \rangle \text{ (Definition 7.1 page 99)} \\
&= \langle \boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{x}+\boldsymbol{y} \rangle && \text{by left hypothesis 2} \\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}+\boldsymbol{y} \rangle + \langle \boldsymbol{y} \mid \boldsymbol{x}+\boldsymbol{y} \rangle && \text{by additive property of } \langle \triangle \mid \nabla \rangle \text{ (Definition 7.1 page 99)} \\
&= \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle + \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle^* + \langle \boldsymbol{y} \mid \boldsymbol{y} \rangle && \text{by conjugate symmetric and additive properties of } \langle \triangle \mid \nabla \rangle \\
&= \underbrace{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}_{\geq 0} + 0 + 0 + \underbrace{\langle \boldsymbol{y} \mid \boldsymbol{y} \rangle}_{\geq 0} && \text{by left hypothesis 1} \\
\implies & \boldsymbol{x} = \mathbb{0} \text{ and } \boldsymbol{y} = \mathbb{0} && \text{by non-negative and non-isotropic properties of } \langle \triangle \mid \nabla \rangle
\end{aligned}$$

2. Proof that $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0$:

$$\begin{aligned}
\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle &= \langle \mathbb{0} \mid \mathbb{0} \rangle && \text{by right hypotheses} \\
&= 0 && \text{by non-isotropic property of } \langle \triangle \mid \nabla \rangle \text{ (Definition 7.1 page 99)}
\end{aligned}$$

3. Proof that $\boldsymbol{x}+\boldsymbol{y} = \mathbb{0}$:

$$\begin{aligned}
\boldsymbol{x}+\boldsymbol{y} &= \mathbb{0}+\mathbb{0} && \text{by right hypotheses} \\
&= \mathbb{0}
\end{aligned}$$

⇰

The *triangle inequality for vectors* in a *normed linear space* (Theorem 6.1 page 87) demonstrates that $\left\|\sum_{n=1}^{N} \boldsymbol{x}_n\right\| \leq \sum_{n=1}^{N} \|\boldsymbol{x}_n\|$. The *Pythagorean Theorem* (next) demonstrates that this *inequality* becomes *equality* when the set $\{\boldsymbol{x}_n\}$ is *orthogonal*.

Theorem 7.10 (Pythagorean Theorem). [22] *Let* $\left\{\boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N\right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition 7.1 page 99)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$ *(Definition 7.2 page 105).*

THM
$$\{\boldsymbol{x}_n\} \text{ is ORTHOGONAL} \iff \left\|\sum_{n=1}^{N} \boldsymbol{x}_n\right\|^2 = \sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2 \qquad \forall N \in \mathbb{N}$$

[22] Aliprantis and Burkinshaw (1998) pages 282–283 ⟨Theorem 32.7⟩, Kubrusly (2001) page 324 ⟨Proposition 5.8⟩, Bollobás (1999) pages 132–133 ⟨Theorem 3⟩

✎PROOF: 1. Proof for ($\implies$) case:

$$\begin{aligned}
\left\|\sum_{n=1}^{N} \boldsymbol{x}_n\right\|^2 &= \left\langle \sum_{n=1}^{N} \boldsymbol{x}_n \,\middle|\, \sum_{m=1}^{N} \boldsymbol{x}_m \right\rangle && \text{by def. of } \|\cdot\| && \text{(Definition 6.1 page 87)}\\
&= \sum_{n=1}^{N}\sum_{m=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_m \rangle && \text{by def. of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition 7.1 page 99)}\\
&= \sum_{n=1}^{N}\sum_{m=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_m \rangle \bar{\delta}_{n-m} && \text{by left hypothesis}\\
&= \sum_{n=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_n \rangle && \text{by def. of } \bar{\delta} && \text{(Definition 7.3 page 111)}\\
&= \sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2 && \text{by def. of } \|\cdot\| && \text{(Definition 6.1 page 87)}
\end{aligned}$$

2. Proof for ($\impliedby$) case:

$$\begin{aligned}
&4\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle\\
&= \|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2 + i\,\|\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\,\|\boldsymbol{x}-i\boldsymbol{y}\|^2 && \text{by \textit{polarization identity} (Theorem 7.6 page 106)}\\
&= \left(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2\right) - \left(\|\boldsymbol{x}\|^2 + \|-\boldsymbol{y}\|^2\right) + i\left(\|\boldsymbol{x}\|^2 + \|i\boldsymbol{y}\|^2\right) - i\left(\|\boldsymbol{x}\|^2 + \|-i\boldsymbol{y}\|^2\right) && \text{by right hypothesis}\\
&= \left(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2\right) - \left(\|\boldsymbol{x}\|^2 + |-1|^2\,\|\boldsymbol{y}\|^2\right) + i\left(\|\boldsymbol{x}\|^2 + |i|^2\,\|\boldsymbol{y}\|^2\right) - i\left(\|\boldsymbol{x}\|^2 + |-i|^2\,\|\boldsymbol{y}\|^2\right) && \text{by definition of } \|\cdot\|\\
&= \left(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2\right) - \left(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2\right) + i\left(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2\right) - i\left(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2\right) && \text{by def. of } |\cdot| \text{ (Definition F.4 page 342)}\\
&= 0
\end{aligned}$$

✏

CHAPTER 8

LINEAR SUBSPACES

8.1 Subspaces of a linear space

Linear spaces (Definition 4.1 page 71) can be decomposed into a collection of *linear subspaces* (Definition 8.1 page 116). Often such a collection along with an *order relation* (Definition B.2 page 286) forms a *lattice* (Definition C.3 page 301).

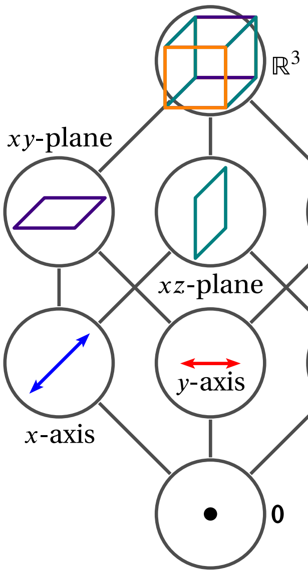
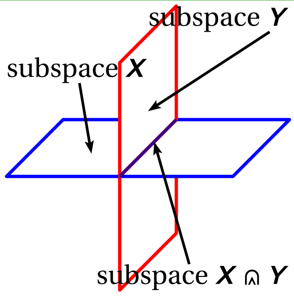


Figure 8.1: lattice of subspaces of $\mathbb{R}^3$ (Example 8.1 page 115)

EX

Example 8.1. The 3-dimensional Euclidean space $\mathbb{R}^3$ contains the 2-dimensional xy-plane and xz-plane subspaces, which in turn both contain the 1-dimensional x-axis subspace. These subspaces are illustrated in the figure to the right and in Figure 8.1 (page 115).



Definition 8.1. [1] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition 4.1 page 71).

DEF

A tupple $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *is a* ***linear subspace*** *of* $\boldsymbol{\Omega}$ *if*

1. $Y \neq \varnothing$ (*Y must contain at least one element*) *and*
2. $Y \subseteq X$ (*Y is a subset of X*) *and*
3. $\boldsymbol{x}, \boldsymbol{y} \in Y \implies \boldsymbol{x} + \boldsymbol{y} \in Y$ (*closed under vector addition*) *and*
4. $\boldsymbol{x} \in Y$ *and* $\alpha \in \mathbb{F} \implies \alpha\boldsymbol{x} \in Y$ (*closed under scalar-vector multiplication*).

A linear subspace is also called a ***linear manifold****.*

Every *linear space* (Definition 4.1 page 71) $\boldsymbol{X}$ has at least two *linear subspaces*—itself and $\mathbf{0}$ (Proposition 8.1 page 116), called the *trivial linear space*. The *linear span* (Definition 8.2 page 117) of every subset of a linear linear space is a subspace (Proposition 8.2 page 117). Every *linear subspace* contains the "zero" vector $\mathbb{0}$, and is *convex* (Definition 10.6 page 152, Proposition 8.3 page 117).

Proposition 8.1. [2] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\mathbf{0} \triangleq (\{\mathbb{0}\}, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

PRP

$$\left\{ \boldsymbol{X} \text{ is a LINEAR SPACE (Definition 4.1 page 71)} \right\} \implies \left\{ \begin{array}{lll} 1. & \mathbf{0} & \text{is a LINEAR SUBSPACE of } \boldsymbol{X} \text{ and} \\ 2. & \boldsymbol{X} & \text{is a LINEAR SUBSPACE of } \boldsymbol{X} \end{array} \right\}$$

✎PROOF: For a structure to be a linear subspace of $\boldsymbol{X}$, it must satisfy the requirements of Definition 8.1 (page 116).

1. Proof that $\{\mathbb{0}\}$ is a linear subspace:

   (a) Note that $\{\mathbb{0}\} \neq \varnothing$.

   (b) Proof that $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbb{0}\} \implies \boldsymbol{x} + \boldsymbol{y} \in \{\mathbb{0}\}$:

   $$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &= \mathbb{0} + \mathbb{0} && \text{by } \boldsymbol{x}, \boldsymbol{y} \in \{\mathbb{0}\} \text{ hypothesis} \\ &= \mathbb{0} \\ &\in \{\mathbb{0}\} \end{aligned}$$

   (c) Proof that $\boldsymbol{x} \in \{\mathbb{0}\}, \alpha \in \mathbb{F} \implies \alpha\boldsymbol{x} \in \{\mathbb{0}\}$:

   $$\begin{aligned} \alpha\boldsymbol{x} &= \alpha\mathbb{0} && \text{by } \boldsymbol{x} \in \{\mathbb{0}\} \text{ hypothesis} \\ &= \mathbb{0} && \text{by definition of } \mathbb{0} \\ &\in \{\mathbb{0}\} \end{aligned}$$

2. Proof that $\boldsymbol{\Omega}$ is a linear subspace of itself:

   (a) Proof that $X \neq \varnothing$:

   $$X \neq \varnothing$$

   (b) Proof that $\boldsymbol{x}, \boldsymbol{y} \in X \implies \boldsymbol{x} + \boldsymbol{y} \in X$:

   $$\boldsymbol{x} + \boldsymbol{y} \in \{\mathbb{0}\} \qquad \text{because } + : X \times X \to X \text{ (} X \text{ is closed under vector addition)}$$

   (c) Proof that $\boldsymbol{x} \in X, \alpha \in \mathbb{F} \implies \alpha\boldsymbol{x} \in X$:

   $$\alpha\boldsymbol{x} \in X \qquad \text{because } \cdot : \mathbb{F} \times X \to X \text{ (} X \text{ is closed under scalar-vector multiplication)}$$

[1] Michel and Herget (1993) page 81 ⟨Definition 3.2.1⟩, Berberian (1961) page 13 ⟨Definition I.5.1⟩, Halmos (1958) page 16
[2] Michel and Herget (1993) pages 81–83, Haaser and Sullivan (1991) page 43

Definition 8.2. [3] *Let $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a linear space and Y be a subset of X.*

DEF

*The **linear span** of Y is defined as* $\operatorname{span} Y \triangleq \left\{ \sum_{\gamma\in\Gamma} \alpha_\gamma \boldsymbol{y}_\gamma \middle| \alpha_\gamma \in \mathbb{F},\ \boldsymbol{y}_\gamma \in Y \right\}$.

*The set Y **spans** a set A if* $A \subseteq \operatorname{span} Y$.

Proposition 8.2. [4] *Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a* LINEAR SPACE (Definition 4.1 page 71). *Let* span *be the* LINEAR SPAN *of a set Y in $\boldsymbol{X}$.*

PRP

$$\left\{ \begin{array}{l} Y \text{ is a SUBSET of the set } X \\ (Y \subseteq X) \end{array} \right\} \implies \left\{ \operatorname{span} Y \quad \text{is a LINEAR SUBSPACE of } \boldsymbol{X}. \right\}$$

Proposition 8.3. [5] *Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a* LINEAR SPACE *and $\mathbb{0}$ the zero vector of $\boldsymbol{X}$.*

PRP

$$\left\{ \boldsymbol{Y} \text{ is a LINEAR SUBSPACE of } \boldsymbol{X} \right\} \implies \left\{ \begin{array}{lll} 1. & \mathbb{0} \in \boldsymbol{Y} & \text{and} \\ 2. & \boldsymbol{Y} \text{ is CONVEX in } \boldsymbol{X} & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\boldsymbol{Y} \text{ is a } \textit{subspace} &\implies \exists(\alpha\boldsymbol{y}) \in \boldsymbol{Y} \quad \forall\alpha\in\mathbb{F} && \text{by Definition 8.1 page 116}\\
&\implies \exists\mathbb{0} \in \boldsymbol{Y} && \text{because } \alpha = 0 \in \mathbb{F}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{Y} \text{ is a linear subspace} &\implies \boldsymbol{x}+\boldsymbol{y} \in Y\ \forall \boldsymbol{x},\boldsymbol{y}\in Y\\
&\implies \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in Y\ \forall \boldsymbol{x},\boldsymbol{y}\in Y\\
&\implies \boldsymbol{Y} \text{ is } \textit{convex}
\end{aligned}$$

Definition 8.3. [6] *Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be* LINEAR SUBSPACES (Definition 8.1 page 116) *of a* LINEAR SPACE (Definition 4.1 page 71) $\boldsymbol{\Omega} \triangleq (\Omega, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF

$$\begin{array}{rcll}
\boldsymbol{X} \hat{+} \boldsymbol{Y} & \triangleq & (\{\boldsymbol{x}+\boldsymbol{y} | \boldsymbol{x} \in X \text{ and } \boldsymbol{y} \in Y\}, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})) & \textit{(Minkowski addition)}\\
\boldsymbol{X} \uplus \boldsymbol{Y} & \triangleq & (X \cup Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})) & \textit{(subspace union)}\\
\boldsymbol{X} \Cap \boldsymbol{Y} & \triangleq & (X \cap Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})) & \textit{(subspace intersection)}
\end{array}$$

Example 8.2. Some examples of operations on subspaces in $\mathbb{R}^3$ are illustrated next:

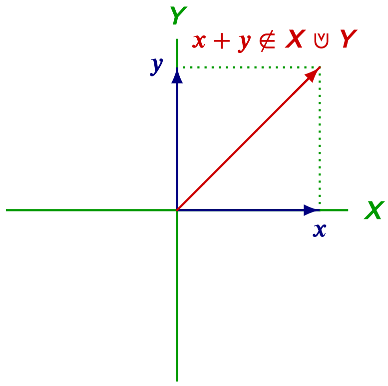
Remark 8.1.

Notice the similarities between the properties of linear subspaces in a linear space (Proposition 8.4 page 118) and the properties of closed sets in a topological space (Theorem 1.3 page 6):

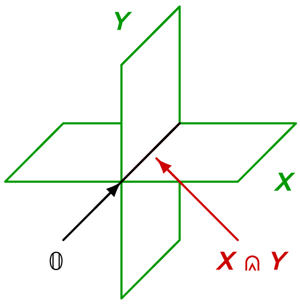
linear subspaces	*closed sets*
$\mathbb{0}$	$\emptyset$
$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$
$\boldsymbol{X} \hat{+} \boldsymbol{Y}$	$X \cup Y$
$\bigcap_{n=1}^{N} \boldsymbol{X}_n$	$\bigcap_{\gamma\in\Gamma} X_\gamma$

[3] Michel and Herget (1993) page 86 ⟨3.3.7 Definition⟩, Kurdila and Zabarankin (2005) page 44, Searcóid (2002) page 71 ⟨Definition 3.2.5—more general definition⟩
[4] Michel and Herget (1993) page 86
[5] Michel and Herget (1993) page 81
[6] Wedderburn (1907) page 79

One key difference is that the union of two linear subspaces is not in general a linear subspace. For example, if $\boldsymbol{x}$ is the vector $[1\ 0]$ in the x direction linear subspace of $\mathbb{R}^2$ and $\boldsymbol{y}$ is the vector $[0\ 1]$ in the y direction linear subspace, then $\boldsymbol{x}+\boldsymbol{y}$ is not in the union of the two linear subspaces (it is not on the x axis or y axis but rather at $(1,1)$).[7]



In general, the set of all linear subspaces of a linear space $\boldsymbol{\Omega}$ is *not* closed under the subspace union ($\uplus$) operation; that is, the union of of two linear subspaces is *not* necessarily a linear subspace. However the set *is* closed under Minkowski sum ($\dotplus$) and subspace intersection ($\Cap$). Proposition 8.4 (next) shows four useful objects are always subspaces. Some of these in Euclidean space $\mathbb{R}^3$ are illustrated to the right.



Proposition 8.4. [8] *Let $\boldsymbol{X}$ be a* LINEAR SPACE *(Definition 4.1 page 71).*

PRP

$$\left\{ \begin{array}{l} \left\{\boldsymbol{X}_n \middle| n=1,2,\ldots,N\right\} \text{ are} \\ \text{LINEAR SUBSPACES of } \boldsymbol{X} \end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_N & \text{is a LINEAR SUBSPACE of } \boldsymbol{X} \\ & \text{and} & \\ 2. & \boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_N & \text{is a LINEAR SUBSPACE of } \boldsymbol{X} \end{array} \right.$$

✎PROOF: For a structure to be a linear subspace of $\boldsymbol{X}$, it must satisfy the requirements of Definition 8.1 (page 116).

1. Proof that $\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_N$ is a *linear subspace* (proof by induction):
   (a) proof for $N=1$ case: by left hypothesis.
   (b) proof for $N=2$ case:
      i. proof that $\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \neq \varnothing$:

$$\begin{aligned} \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 &= \left\{\boldsymbol{v}+\boldsymbol{w} \middle| \boldsymbol{v} \in \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in Y\right\} && \text{by Definition 8.3 page 117} \\ &\supseteq \left\{\boldsymbol{v}+\boldsymbol{w} \middle| \boldsymbol{v} \in \{\mathbb{0}\} \subseteq \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in \{\mathbb{0}\} \subseteq \boldsymbol{X}_2\right\} \\ &= \{\mathbb{0}+\mathbb{0}\} \\ &= \{\mathbb{0}\} \\ &\neq \varnothing \end{aligned}$$

      ii. proof that $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \implies \boldsymbol{x}+\boldsymbol{y} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2$:

$$\begin{aligned} \boldsymbol{x}+\boldsymbol{y} &= (\boldsymbol{v}_1+\boldsymbol{w}_1)+(\boldsymbol{v}_2+\boldsymbol{w}_2) && \text{by } \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \text{ hypothesis} \\ &= \underbrace{(\boldsymbol{v}_1+\boldsymbol{v}_2)}_{\text{in } \boldsymbol{X}_1} + \underbrace{(\boldsymbol{w}_1+\boldsymbol{w}_2)}_{\text{in } \boldsymbol{X}_2 \text{ because } \boldsymbol{X}_2 \text{ is a linear subspace}} && \text{by Definition 4.1 page 71} \\ &\in \left\{\boldsymbol{v}+\boldsymbol{w} \middle| \boldsymbol{v} \in \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in Y\right\} \\ &= \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 && \text{by Definition 8.3 page 117} \end{aligned}$$

[7] Michel and Herget (1993) page 82
[8] Michel and Herget (1993) pages 81–83

iii. proof that $\boldsymbol{v} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2, \alpha \in F \implies \alpha\boldsymbol{v} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2$:

$$\begin{aligned}
\alpha\boldsymbol{x} &= \alpha(\boldsymbol{v}_1 + \boldsymbol{w}_1) && \text{by } \boldsymbol{x} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \text{ hypothesis}\\
&= \underbrace{\alpha\boldsymbol{v}_1}_{\text{in } \boldsymbol{X}_1} + \underbrace{\alpha\boldsymbol{w}_1}_{\text{in } \boldsymbol{X}_2 \text{ because } \boldsymbol{X}_2 \text{ is a linear subspace}} && \text{by Definition 4.1 page 71}\\
&\in \left\{\boldsymbol{v} + \boldsymbol{w} | \boldsymbol{v} \in \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in Y\right\}\\
&= \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 && \text{by Definition 8.3 page 117}
\end{aligned}$$

(c) Proof that [N case] $\implies$ [$N+1$ case]:

$$\begin{aligned}
\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_{N+1} &= \underbrace{\left(\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_N\right)}_{\text{linear subspace by } N \text{ case hypothsis}} \dotplus \boldsymbol{X}_{N+1}\\
&\implies \text{linear subspace by } N = 2 \text{ case (item (1b) page 118)}
\end{aligned}$$

2. Proof that $\boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_N$ is a *linear subspace* (proof by induction):

(a) proof for $N = 1$ case: $\boldsymbol{X}_1$ is a linear subspace by left hypothesis.

(b) Proof for $N = 2$ case:

i. proof that $\boldsymbol{X} \Cap \boldsymbol{Y} \neq \varnothing$:

$$\begin{aligned}
\boldsymbol{X} \Cap \boldsymbol{Y} &= \left\{\boldsymbol{x} \in X | \boldsymbol{x} \in X \text{ and } \boldsymbol{w} \in Y\right\}\\
&\supseteq \left\{\boldsymbol{x} \in X | \boldsymbol{x} \in \{\mathbb{0}\} \subseteq \boldsymbol{X} \text{ and } \boldsymbol{x} \in \{\mathbb{0}\} \subseteq \boldsymbol{Y}\right\}\\
&= \{\mathbb{0} + \mathbb{0}\}\\
&= \{\mathbb{0}\}\\
&\neq \varnothing
\end{aligned}$$

ii. proof that $\boldsymbol{x}, \boldsymbol{y} \in X \Cap \boldsymbol{Y} \implies \boldsymbol{x} + \boldsymbol{y} \in X \Cap \boldsymbol{Y}$:

$$\begin{aligned}
\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X} \Cap \boldsymbol{Y} &\implies \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X} \text{ and } \boldsymbol{x}, \boldsymbol{y} \in Y && \text{by Definition A.5 page 260}\\
&\implies \boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{X} \text{ and } \boldsymbol{x} + \boldsymbol{y} \in Y && \text{because } \boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ are linear subspaces}\\
&\implies \boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{X} \Cap \boldsymbol{Y} && \text{by Definition A.5 page 260}
\end{aligned}$$

iii. proof that $\boldsymbol{v} \in X \Cap \boldsymbol{Y}, \alpha \in F \implies \alpha\boldsymbol{v} \in X \Cap \boldsymbol{Y}$:

$$\begin{aligned}
\boldsymbol{x} \in \boldsymbol{X} \Cap \boldsymbol{Y} &\implies \boldsymbol{x} \in \boldsymbol{X} \text{ and } \boldsymbol{x} \in Y && \text{by Definition A.5 page 260}\\
&\implies \alpha\boldsymbol{x} \in \boldsymbol{X} \text{ and } \alpha\boldsymbol{x} \in Y && \text{because } \boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ are linear subspaces}\\
&\implies \alpha\boldsymbol{x} \in \boldsymbol{X} \Cap \boldsymbol{Y} && \text{by Definition A.5 page 260}
\end{aligned}$$

(c) Proof that [N case] $\implies$ [$N+1$ case]:

$$\begin{aligned}
\boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_{N+1} &= \underbrace{\left(\boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_N\right)}_{\text{linear subspace by } N \text{ case hypothsis}} \Cap \boldsymbol{X}_{N+1}\\
&\implies \text{linear subspace by } N = 2 \text{ case (item (2b) page 119)}
\end{aligned}$$

⇰

Every linear subspace contains the zero vector $\mathbb{0}$ (Proposition 8.3 page 117). But if a pair of linear subspaces of a linear space $\boldsymbol{X}$ *only* have $\mathbb{0}$ in commmon, then any vector in $\boldsymbol{X}$ can be *uniquely* represented by a single vector from each of the two subspaces (next).

Theorem 8.1. [9] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be* LINEAR SUBSPACES *(Definition 8.1 page 116) of a* LINEAR SPACE *(Definition 4.1 page 71)* $\boldsymbol{\Omega} \triangleq (\Omega, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

THM

$$X \cap Y = \{\mathbb{0}\} \iff \left\{ \begin{array}{l} \text{for every } \boldsymbol{u} \in \boldsymbol{X} \hat{+} \boldsymbol{Y} \text{ there exist } \boldsymbol{x} \in X \text{ and } \boldsymbol{y} \in Y \text{ such that} \\ \quad 1. \; \boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y} \qquad \text{and} \\ \quad 2. \; \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are UNIQUE.} \end{array} \right\}$$

✎PROOF:

1. Proof that $X \cap Y = \{\mathbb{0}\} \implies$ *unique* $\boldsymbol{x}, \boldsymbol{y}$:
   Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are not unique, but rather $\boldsymbol{u} = \boldsymbol{x}_1 + \boldsymbol{y}_1 = \boldsymbol{x}_2 + \boldsymbol{y}_2$ where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in Y$.

$$\begin{aligned}
\boldsymbol{u} = \boldsymbol{x}_1 + \boldsymbol{y}_1 = \boldsymbol{x}_2 + \boldsymbol{y}_2 &\implies \underbrace{\boldsymbol{x}_1 - \boldsymbol{x}_2}_{\in X} = \underbrace{\boldsymbol{y}_2 - \boldsymbol{y}_1}_{\in Y} \\
&\implies \boldsymbol{x}_1 - \boldsymbol{x}_2,\ \boldsymbol{y}_2 - \boldsymbol{y}_1 \in X \cap Y \\
&\implies \boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{y}_2 - \boldsymbol{y}_1 = \mathbb{0} && \text{by left hypothesis} \\
&\implies \boldsymbol{x}_1 = \boldsymbol{x}_2 \quad \text{and} \quad \boldsymbol{y}_2 = \boldsymbol{y}_1 \\
&\implies \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are } \textit{unique}
\end{aligned}$$

2. Proof that $X \cap Y = \{\mathbb{0}\} \impliedby$ *unique* $\boldsymbol{x}, \boldsymbol{y}$:

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{x} + \boldsymbol{y} \\
&= \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{y} && \text{for some vector } \boldsymbol{y} \in X \cap Y \\
&= \underbrace{(\boldsymbol{x} + \boldsymbol{y})}_{\in X} + \underbrace{(\boldsymbol{y} - \boldsymbol{y})}_{\in Y} && \text{because } \boldsymbol{x} \in X \text{ and } \boldsymbol{y} \in X \cap Y \ldots \\
&\implies \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are } \textit{not unique} \text{ if } \boldsymbol{y} \neq \mathbb{0} \\
&\implies \boldsymbol{y} = \mathbb{0} && \text{by right hypothesis} \\
&\implies X \cap Y = \{\mathbb{0}\}
\end{aligned}$$

✏

Theorem 8.2. [10] *Let* $\boldsymbol{\Omega}$ *be a linear subspace and* 2^{Ω} *the set of closed linear subspaces of* $\boldsymbol{\Omega}$.

THM

$(2^{\Omega}, \hat{+}, \Cap, \mathbf{0}, \boldsymbol{\Omega}; \subseteq)$ *is a* LATTICE *(Definition C.3 page 301). In particular*

$$\begin{array}{rclrcll}
\boldsymbol{X} \hat{+} \boldsymbol{X} &=& \boldsymbol{X} & \boldsymbol{X} \Cap \boldsymbol{X} &=& \boldsymbol{X} & \forall \boldsymbol{X} \in 2^{\Omega} \\
\boldsymbol{X} \hat{+} \boldsymbol{Y} &=& \boldsymbol{Y} \hat{+} \boldsymbol{X} & \boldsymbol{X} \Cap \boldsymbol{Y} &=& \boldsymbol{Y} \Cap \boldsymbol{X} & \forall \boldsymbol{X}, \boldsymbol{Y} \in 2^{\Omega} \\
(\boldsymbol{X} \hat{+} \boldsymbol{Y}) \hat{+} \boldsymbol{Z} &=& \boldsymbol{X} \hat{+} (\boldsymbol{Y} \hat{+} \boldsymbol{Z}) & (\boldsymbol{X} \Cap \boldsymbol{Y}) \Cap \boldsymbol{Z} &=& \boldsymbol{X} \Cap (\boldsymbol{Y} \Cap \boldsymbol{Z}) & \forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in 2^{\Omega} \\
\boldsymbol{X} \hat{+} (\boldsymbol{X} \Cap \boldsymbol{Y}) &=& \boldsymbol{X} & \boldsymbol{X} \Cap (\boldsymbol{X} \hat{+} \boldsymbol{Y}) &=& \boldsymbol{X} & \forall \boldsymbol{X}, \boldsymbol{Y} \in 2^{\Omega}
\end{array}$$

✎PROOF: These results follow directly from the properties of lattices (Theorem C.3 page 302). ✏

8.2 Subspaces of an inner product space

Definition 8.4. [11] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be an* INNER PRODUCT SPACE *(Definition 7.1 page 99).*

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[9] Michel and Herget (1993) page 83 ⟨Theorem 3.2.12⟩, Kubrusly (2001) page 67 ⟨Theorem 2.14⟩
[10] Iturrioz (1985) pages 56–57
[11] Berberian (1961) page 59 ⟨Definition III.2.1⟩, Michel and Herget (1993) page 382, Kubrusly (2001) page 328

DEF *The* ***orthogonal complement*** $A^\perp$ *in* $\boldsymbol{\Omega}$ *of a set* $A \subseteq X$ *is*
$$A^\perp \triangleq \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A\}.$$
The expression $A^{\perp\perp}$ *is defined as* $\left(A^\perp\right)^\perp$.

Proposition 8.5. [12] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition 7.1 page 99).*

PRP
$$A \subseteq B \implies B^\perp \subseteq A^\perp \quad \forall A, B \in 2^X \quad \text{(ANTITONE)}$$

✎PROOF:

$$\begin{aligned}
B^\perp &\triangleq \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in B\} && \text{by definition of } B^\perp \text{ (Definition 8.4 page 120)}\\
&\subseteq \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A\} && \text{by } A \subseteq B \text{ hypothesis}\\
&= A^\perp && \text{by definition of } A^\perp \text{ (Definition 8.4 page 120)}
\end{aligned}$$

⇒

Every *linear space* $\boldsymbol{X}$ contains $\mathbf{0}$ and $\boldsymbol{X}$ as *linear subspaces* (Proposition 8.1 page 116). If $\boldsymbol{X}$ is also an *inner product space*, then $\mathbf{0}$ and $\boldsymbol{X}$ are *orthogonal complements* of each other (next proposition).

Proposition 8.6. [13] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition 7.1 page 99) and* $\mathbb{0}$ *the* VECTOR ADDITIVE IDENTITY ELEMENT *(Definition 4.1 page 71) in* $\boldsymbol{\Omega}$.

PRP
1. $\{\mathbb{0}\}^\perp = X$
2. $X^\perp = \{\mathbb{0}\}$

✎PROOF:

$$\begin{aligned}
\{\mathbb{0}\}^\perp &= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in \{\mathbb{0}\}\} && \text{by definition of } \perp \text{ (Definition 8.4 page 120)}\\
&= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \mathbb{0} \rangle = 0\}\\
&= X\\
X^\perp &= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in X\} && \text{by definition of } \perp \text{ Definition 8.4 page 120}\\
&= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle = 0\}\\
&= \{\mathbb{0}\}
\end{aligned}$$

⇒

For any set A contained in a linear space $\boldsymbol{X}$, $A^{\perp\perp}$ is a *linear subspace*, and it is the smallest linear subspace containing the set A ($A^{\perp\perp} = \operatorname{span} A$, next theorem). In the case that A is a *linear subspace* rather than just a subset, results simplify significantly (next corollary).

Theorem 8.3. [14] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition 7.1 page 99). Let* $\operatorname{span} A$ *be the span of a set* A *(Definition 8.2 page 117).*

THM
$$\left\{\begin{array}{l} A \text{ is a } \textbf{subset} \text{ of } X\\ (A \subseteq X)\end{array}\right\} \implies \left\{\begin{array}{llll}
1. & A \cap A^\perp = \begin{cases}\{\mathbb{0}\} & \text{if } \mathbb{0} \in A\\ \varnothing & \text{if } \mathbb{0} \notin A\end{cases} & & \text{and}\\
2. & A \subseteq A^{\perp\perp} = \operatorname{span} A & & \text{and}\\
3. & A^\perp = A^{\perp\perp\perp} = A^{\perp^-} = A^{-\perp} = (\operatorname{span} A)^\perp & & \text{and}\\
4. & A^\perp \text{ is a } \textbf{subspace} \text{ of } \boldsymbol{\Omega} & &
\end{array}\right\}$$

[12] Berberian (1961) page 60 ⟨Theorem III.2.2⟩, Kubrusly (2011) page 326
[13] Kubrusly (2011) page 326, Michel and Herget (1993) page 383
[14] Michel and Herget (1993) page 383, Kubrusly (2011) page 326

✎PROOF:

1. Proof that $A \cap A^\perp = \cdots$:

$$\begin{aligned}
A \cap A^\perp &= \{x \in X | x \in A\} \cap \{x \in X | \langle x \mid y\rangle \quad \forall y \in A\} && \text{by definition of } A^\perp\\
&= \left\{x \in X | x \in A \quad \text{and} \quad \langle x \mid y\rangle \quad \forall y \in A\right\}\\
&= \begin{cases} \{\mathbb{0}\} & \text{if} \quad \mathbb{0} \in A \\ \varnothing & \text{if} \quad \mathbb{0} \notin A \end{cases}
\end{aligned}$$

2. Proof that $A \subseteq A^{\perp\perp} = \operatorname{span} A$:

$$\begin{aligned}
x \in A &\implies \{x\}^{\perp\perp} \subseteq A^{\perp\perp}\\
&\implies x \in \{x\}^{\perp\perp} \subseteq A^{\perp\perp}\\
&\implies x \in A^{\perp\perp}
\end{aligned}$$

but

$$x \in A^{\perp\perp} \not\Longrightarrow x \in A$$

Here is an example for the $\not\Longrightarrow$ part using the linear space $\mathbb{R}^3$:

(a) Let $A \triangleq \{i\}$, where i is the unit vector on the x-axis.

(b) Then $A^\perp = \left\{x \in X | x \in \text{yz plane}\right\}$.

(c) Then $A^{\perp\perp} = \{x \in X | x \in \text{x axis}\}$.

(d) Therefore, $A \subsetneq A^{\perp\perp}$

3. Proof for $A^\perp$ equivalent expressions:

(a) Proof that $A^\perp = A^{\perp\perp\perp}$:

$$\begin{aligned}
A^\perp &\subseteq \left(A^\perp\right)^{\perp\perp} && \text{by item (2)}\\
&= \left(A^{\perp\perp}\right)^\perp\\
&= A^{\perp\perp\perp} && \text{by Definition 8.4 page 120}\\
A^{\perp\perp\perp} &= \left(A^{\perp\perp}\right)^\perp && \text{by Definition 8.4 page 120}\\
&\subseteq A^\perp && \text{by item (2) and Proposition 8.5 (page 121)}
\end{aligned}$$

(b) Proof that $A^{\perp\perp\perp} = (\operatorname{span} A)^\perp$: follows directly from item (2) ($A^{\perp\perp} = \operatorname{span} A$).

(c) Proof that $A^\perp = A^{\perp-}$:

i. Let $(\!(x_n)\!)$ be an $A^\perp$-valued sequence that converges to the limit x in X.

ii. The limit point x must be in $A^\perp$ because for all $y \in A$

$$\begin{aligned}
\langle x \mid y\rangle &= \langle \lim x_n \mid y\rangle && \text{by definition of the sequence } (\!(x_n)\!)\\
&= \lim \langle x_n \mid y\rangle\\
&= 0 && \text{because } (\!(x_n)\!) \text{ is } A^\perp\text{-valued}
\end{aligned}$$

iii. Because $\langle x \mid y\rangle = 0 \quad \forall y \in A$, x is in $A^\perp$.

iv. Because $A^\perp$ contains all its limit points, and by the *Closed Set Theorem* (Theorem 9.1 page 132), it must be *closed* ($A^\perp = A^{\perp-}$)

(d) Proof that $A^\perp = A^{-\perp}$:

i. Let $x \in A^\perp$ and $y \in A^-$.
ii. Let $(\!(y_n)\!)$ be an $A^\perp$-valued sequence that converges in X to y.
iii. Thus $A^\perp \perp A^-$ because

$$\begin{aligned}
\langle \boldsymbol{y} \mid \boldsymbol{x} \rangle &= \langle \lim \boldsymbol{y}_n \mid \boldsymbol{x} \rangle && \text{by definition of } (\!(\boldsymbol{y}_n)\!)\\
&= \lim \langle \boldsymbol{y}_n \mid \boldsymbol{x} \rangle \\
&= 0 && \text{because } (\!(\boldsymbol{y}_n)\!) \text{ is } A^\perp\text{-valued}
\end{aligned}$$

iv. Because $A^\perp \perp A^-$, so $A^\perp \subseteq A^{\perp -}$.
v. But $A^{\perp -} \subseteq A^\perp$ because

$$A \subseteq A^- \implies A^{\perp -} \subseteq A^\perp \qquad \text{by } \textit{antitone} \text{ property (Proposition 8.5 page 121)}$$

vi. And so $A^\perp = A^{\perp -}$.

4. Proof that $A^\perp$ is a **subspace** of $\boldsymbol{\Omega}$ (must satisfy the conditions of Definition 8.1 page 116):

(a) Proof that $A^\perp \neq \emptyset$: $A^\perp$ has at least one element, the element $\mathbb{0}$...

$$\begin{aligned}
\langle \mathbb{0} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A & && \text{by definition of } \mathbb{0}\\
\implies \mathbb{0} \in A^\perp & && \text{by definition of } A^\perp \text{ (Definition 8.4 page 120)}
\end{aligned}$$

(b) Proof that $A^\perp \subseteq X$:

$$\begin{aligned}
\boldsymbol{u} \in A^\perp &\implies \boldsymbol{u} \in \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A\} && \text{by definition of } A^\perp \text{ (Definition 8.4 page 120)}\\
&\implies \boldsymbol{u} \in X && \text{by definition of sets}
\end{aligned}$$

(c) Proof that $\boldsymbol{u}, \boldsymbol{v} \in A^\perp \implies (\boldsymbol{u} + \boldsymbol{v}) \in A^\perp$:

$$\begin{aligned}
\boldsymbol{u}, \boldsymbol{v} \in A^\perp &\implies \langle \boldsymbol{u} \mid \boldsymbol{y} \rangle = \langle \boldsymbol{v} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by definition of } A^\perp \text{ (Definition 8.4 page 120)}\\
&\implies \langle \boldsymbol{u} \mid \boldsymbol{y} \rangle + \langle \boldsymbol{v} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A\\
&\implies \langle \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle \text{ (Definition 7.1 page 99)}\\
&\implies \boldsymbol{u} + \boldsymbol{v} \in A^\perp && \text{by definition of } A^\perp \text{ (Definition 8.4 page 120)}
\end{aligned}$$

(d) Proof that $\boldsymbol{v} \in \boldsymbol{\Omega} \implies \alpha \boldsymbol{v} \in A^\perp$:

$$\begin{aligned}
\boldsymbol{v} \in A^\perp &\implies \langle \boldsymbol{v} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by definition of } A^\perp \text{ (Definition 8.4 page 120)}\\
&\implies \alpha \langle \boldsymbol{v} \mid \boldsymbol{y} \rangle = \alpha \cdot 0 \quad \forall \boldsymbol{y} \in A\\
&\implies \langle \alpha \boldsymbol{v} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \nabla \rangle \text{ (Definition 7.1 page 99)}\\
&\implies \alpha \boldsymbol{v} \in A^\perp && \text{by definition of } A^\perp \text{ (Definition 8.4 page 120)}
\end{aligned}$$

✏

Corollary 8.1. *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be* INNER PRODUCT SPACES. *Let* $\operatorname{span} Y$ *be the span of the set* Y (Definition 8.2 page 117).

COR

$$\left\{ \boldsymbol{Y} \text{ is a } \textbf{linear subspace} \text{ of } \boldsymbol{X} \right\} \implies \left\{ \begin{array}{lrcll}
1. & Y \cap Y^\perp & = & \{\mathbb{0}\} & \text{and}\\
2. & Y & = & Y^{\perp\perp} = \operatorname{span} Y & \text{and}\\
3. & Y^\perp & = & Y^{\perp\perp\perp} & \text{and}\\
4. & \multicolumn{4}{l}{Y^\perp \text{ is a } \textbf{subspace} \text{ of } \boldsymbol{X}}
\end{array} \right\}$$

✎PROOF:

1. Proof that $\boldsymbol{Y} \cap Y^{\perp} = \{\mathbb{0}\}$: This follows from Theorem 8.3 (page 121) and the fact that all subspaces contain the zero vector $\mathbb{0}$ (Proposition 8.3 page 117).
2. Proof that $\boldsymbol{Y} = \boldsymbol{Y}^{\perp\perp} = \operatorname{span} \boldsymbol{Y}$: This follows directly from Theorem 8.3 (page 121).
3. Proof that $\boldsymbol{Y}^{\perp} = \boldsymbol{Y}^{\perp\perp\perp}$: This follows directly from Theorem 8.3 (page 121).
4. Proof that $Y^{\perp}$ is a **subspace** of $\boldsymbol{X}$: This follows directly from Theorem 8.3 (page 121).

✏

Theorem 8.4. [15] *Let* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *and* $\boldsymbol{Z} \triangleq \left(Z, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be* LINEAR SUBSPACES *of an* INNER PRODUCT SPACE $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM
$$\boldsymbol{Y} \perp \boldsymbol{Z} \quad \Longrightarrow \quad Y \cap Z = \{\mathbb{0}\}$$

✎PROOF:

$$\begin{aligned}
\boldsymbol{x} \in \boldsymbol{Y} \cap \boldsymbol{Z} &\implies \boldsymbol{x} \in \boldsymbol{Y} \text{ and } \boldsymbol{x} \in \boldsymbol{Z} && \text{by definition of } \cap \\
&\implies \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle = 0 && \text{by hypothesis } \boldsymbol{Y} \perp \boldsymbol{Z} \\
&\implies \boldsymbol{x} = \mathbb{0} && \text{by } \textit{non-isotropic} \text{property of } \langle \triangle \mid \triangledown \rangle \text{ (Definition 7.1 page 99)}
\end{aligned}$$

✏

Theorem 8.5. [16] *Let* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *and* $\boldsymbol{Z} \triangleq \left(Z, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be linear subspaces of an* INNER PRODUCT SPACE $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM
$$\left\{\begin{array}{ll} 1. & \boldsymbol{Y} \perp \boldsymbol{Z} \text{ and} \\ 2. & \boldsymbol{x} \in \boldsymbol{Y} \hat{+} \boldsymbol{Z} \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} 1. & \text{There exists } \boldsymbol{y} \in \boldsymbol{Y} \text{ and } \boldsymbol{z} \in \boldsymbol{Z} \text{ such that } \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z} \text{ and} \\ 2. & \boldsymbol{y} \text{ and } \boldsymbol{z} \text{ are UNIQUE.} \end{array}\right\}$$

✎PROOF:

1. Proof that $\boldsymbol{y}$ and $\boldsymbol{z}$ exist: by definition of Minkowski addition operator $\hat{+}$ (Definition 8.3 page 117).
2. Proof that $\boldsymbol{y}$ and $\boldsymbol{z}$ are *unique*:
   (a) Suppose $\boldsymbol{x} = \boldsymbol{y}_1 + \boldsymbol{z}_1 = \boldsymbol{y}_1 + \boldsymbol{z}_2$ for $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{Y}$ and $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \boldsymbol{Z}$.
   (b) This implies
   $$\begin{aligned}
   \mathbb{0} &= \boldsymbol{x} - \boldsymbol{x} \\
   &= \left(\boldsymbol{y}_1 + \boldsymbol{z}_1\right) - \left(\boldsymbol{y}_1 + \boldsymbol{z}_2\right) \\
   &= \underbrace{\left(\boldsymbol{y}_1 - \boldsymbol{y}_2\right)}_{\text{in } \boldsymbol{Y}} + \underbrace{\left(\boldsymbol{z}_1 - \boldsymbol{z}_2\right)}_{\text{in } \boldsymbol{Z}}
   \end{aligned}$$
   (c) Because $\boldsymbol{y}_1 - \boldsymbol{y}_2 \in \boldsymbol{Y}$, $\boldsymbol{z}_1 - \boldsymbol{z}_2 \in \boldsymbol{Z}$, $(\boldsymbol{y}_1 - \boldsymbol{y}_2) + (\boldsymbol{z}_1 + \boldsymbol{z}_2) = \mathbb{0}$, and $\langle \boldsymbol{y}_1 - \boldsymbol{y}_2 \mid \boldsymbol{z}_1 - \boldsymbol{z}_2 \rangle = 0$, then by Theorem 7.9 (page 112), $\boldsymbol{y}_1 - \boldsymbol{y}_2 = \mathbb{0}$ and $\boldsymbol{z}_1 - \boldsymbol{z}_2 = \mathbb{0}$.
   (d) This implies $\boldsymbol{y}_1 = \boldsymbol{y}_2$ and $\boldsymbol{z}_1 = \boldsymbol{z}_2$.
   (e) This implies $\boldsymbol{y}$ and $\boldsymbol{z}$ are *unique*.

✏

[15] Kubrusly (2001) page 324
[16] Berberian (1961) page 61 ⟨Theorem III.2.3⟩

8.3 Subspaces of a Hilbert Space

Theorem 8.6. [17] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a* HILBERT SPACE *(Definition 9.14 page 149). Let* Y *be a* SUBSET *of* X*, and let* $\mathsf{d}(\boldsymbol{x}, Y) \triangleq \inf_{\boldsymbol{y}\in Y} \|\boldsymbol{x} - \boldsymbol{y}\|$.

THM
$$\left\{\begin{array}{llll} 1. & Y \neq \varnothing & & \text{and} \\ 2. & Y \text{ is CLOSED} & \text{(Definition 1.4 page 14)} & \text{and} \\ 3. & Y \text{ is CONVEX} & \text{(Definition 10.6 page 152)} & \end{array}\right\} \implies \left\{\begin{array}{lll} \multicolumn{3}{l}{\text{There exists } \boldsymbol{p} \in Y \text{ such that}} \\ 1. & \mathsf{d}(\boldsymbol{x}, Y) = \|\boldsymbol{x} - \boldsymbol{p}\| & \text{and} \\ 2. & \boldsymbol{p} \text{ is UNIQUE.} & \end{array}\right\}$$

✎PROOF:

1. Let $\delta \triangleq \inf\{\boldsymbol{x} - \boldsymbol{y} | \boldsymbol{y} \in Y\}$.
2. Let $(\!(\boldsymbol{y}_n)\!)_{n\in\mathbb{Z}}$ be a sequence such that $\|\boldsymbol{x} - \boldsymbol{y}_n\| \to \delta$.
3. Proof that $(\!(\boldsymbol{y}_n)\!)$ is *Cauchy*:

$$\begin{aligned}
&\lim_{m,n\to\infty} \|\boldsymbol{y}_n - \boldsymbol{y}_m\|^2 \\
&= \lim_{m,n\to\infty} \|(\boldsymbol{y}_n - x) + (x - \boldsymbol{y}_m)\|^2 \\
&= \lim_{m,n\to\infty} \left\{ -\|(\boldsymbol{y}_n - x) - (x - \boldsymbol{y}_m)\|^2 + 2\|\boldsymbol{y}_n - x\|^2 + 2\|x - \boldsymbol{y}_m\|^2 \right\} && \text{by } \textit{parallelogram law} \text{ (page 107)} \\
&= \lim_{m,n\to\infty} \left\{ -4 \left\| \underbrace{\left(\frac{1}{2}\boldsymbol{y}_n + \frac{1}{2}\boldsymbol{y}_m\right)}_{\text{in } \boldsymbol{Y} \text{ by } \textit{convexity}} - x \right\|^2 + 2\|\boldsymbol{y}_n - x\|^2 + 2\|x - \boldsymbol{y}_m\|^2 \right\} \\
&\le \lim_{m,n\to\infty} \left\{ -4\delta^2 + 2\|\boldsymbol{y}_n - x\|^2 + 2\|x - \boldsymbol{y}_m\|^2 \right\} && \text{by definition of } \delta \text{ (item (1))} \\
&= -4\delta^2 + \lim_{m,n\to\infty} \left\{ 2\|\boldsymbol{y}_n - x\|^2 \right\} + \lim_{m,n\to\infty} \left\{ 2\|x - \boldsymbol{y}_m\|^2 \right\} \\
&= -4\delta^2 + 2\delta^2 + 2\delta^2 && \text{by definition of } \delta \text{ (item (1))} \\
&= 0
\end{aligned}$$

4. Proof that $\mathsf{d}(\boldsymbol{x}, Y) = \|\boldsymbol{x} - \boldsymbol{y}\|$: because $(\!(\boldsymbol{y}_n)\!)$ is *Cauchy* (item (1)) and by the *closed* hypothesis.
5. Proof that $\boldsymbol{y}$ is *unique*: Because in a metric space, the limit of a convergent sequence is *unique* by Theorem 9.6 page 143.

✏

Theorem 8.7. [18] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a* HILBERT SPACE *(Definition 9.14 page 149). Let* $\mathsf{d}(\boldsymbol{x}, Y) \triangleq \inf_{\boldsymbol{y}\in Y} \|\boldsymbol{x} - \boldsymbol{y}\|$*. Let* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *and* $Y^\perp$ *the* ORTHOGONAL COMPLEMENT *of* Y.

THM
$$\left\{ \boldsymbol{Y} \text{ is a SUBSPACE of } \boldsymbol{H} \right\} \implies \left\{\begin{array}{lll} \multicolumn{3}{l}{\text{There exists } \boldsymbol{p} \in Y \text{ such that}} \\ 1. & \mathsf{d}(\boldsymbol{x}, Y) = \|\boldsymbol{x} - \boldsymbol{p}\| & \text{and} \\ 2. & \boldsymbol{p} \text{ is UNIQUE} & \text{and} \\ 3. & \boldsymbol{x} - \boldsymbol{p} \in Y^\perp. & \end{array}\right\}$$

[17] Kubrusly (2001) page 330 ⟨Theorem 5.13⟩, Aliprantis and Burkinshaw (1998) page 290 ⟨Theorem 33.6⟩, Berberian (1961) page 68 ⟨Theorem III.5.1⟩
[18] Kubrusly (2001) page 330 ⟨Theorem 5.13⟩

Theorem 8.8 (Projection Theorem). [19] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a Hilbert space.*

THM: $\left\{ \boldsymbol{Y} \text{ is a SUBSPACE of } \boldsymbol{H} \right\} \implies \left\{ \boldsymbol{Y} \dotplus \boldsymbol{Y}^{\perp} = \boldsymbol{H} \right\}$

✎PROOF:

$$\begin{aligned}
\boldsymbol{Y} \dotplus \boldsymbol{Y}^{\perp} &= \left[\boldsymbol{Y} \dotplus \boldsymbol{Y}^{\perp}\right]^{\perp\perp} && \text{by Corollary 8.1 page 123} \\
&= \left[\boldsymbol{Y}^{\perp} \Cap \boldsymbol{Y}^{\perp\perp}\right]^{\perp} && \text{by Proposition 8.5 (page 121) and Lemma 16.1 (page 241)} \\
&= \{\mathbb{0}\}^{\perp} && \text{by Corollary 8.1 page 123} \\
&= \boldsymbol{H} && \text{by Proposition 8.6 page 121}
\end{aligned}$$

✏

The inclusion relation $\subseteq$ is an order relation on the set of subspaces of a linear space $\boldsymbol{\Omega}$.

Proposition 8.7. *Let S be the set of subspaces of a linear space $\boldsymbol{\Omega}$. Let $\subseteq$ be the inclusion relation.*

PRP: $(S, \subseteq)$ *is an* ***ordered set***

✎PROOF: $(S, \subseteq)$ is an *ordered set* (Definition B.2 page 286) and because

1. $\boldsymbol{X} \subseteq \boldsymbol{X}$	$\forall \boldsymbol{X} \in S$	(reflexive)	and	preorder
2. $\boldsymbol{X} \subseteq \boldsymbol{Y}$ and $\boldsymbol{Y} \subseteq \boldsymbol{Z} \implies \boldsymbol{X} \subseteq \boldsymbol{Z}$	$\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in S$	(transitive)	and	preorder
3. $\boldsymbol{X} \subseteq \boldsymbol{Y}$ and $\boldsymbol{Y} \subseteq \boldsymbol{X} \implies \boldsymbol{X} = \boldsymbol{Y}$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in S$	(anti-symmetric)		

✏

Theorem 8.9. [20] *Let $\boldsymbol{H}$ be a Hilbert space and $2^{\boldsymbol{H}}$ the set of closed linear subspaces of $\boldsymbol{H}$.*

THM: $\left(2^{\boldsymbol{H}}, \dotplus, \Cap, \mathbf{0}, \boldsymbol{H}; \subseteq\right)$ *is an* ORTHOMODULAR LATTICE *(Definition 16.3 page 247). In particular*

1.	$\boldsymbol{X} \dotplus \boldsymbol{X}^{\perp} = \boldsymbol{H}$	$\forall \boldsymbol{X} \in \boldsymbol{H}$	(COMPLEMENTED)
2.	$\boldsymbol{X} \Cap \boldsymbol{X}^{\perp} = \mathbf{0}$	$\forall \boldsymbol{X} \in \boldsymbol{H}$	(COMPLEMENTED)
3.	$\left(\boldsymbol{X}^{\perp}\right)^{\perp} = \boldsymbol{X}$	$\forall \boldsymbol{X} \in \boldsymbol{H}$	(INVOLUTORY)
4.	$\boldsymbol{X} \le \boldsymbol{Y} \implies \boldsymbol{Y}^{\perp} \le \boldsymbol{X}^{\perp}$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{H}$	(ANTITONE)
5.	$\boldsymbol{X} \le \boldsymbol{Y} \implies \boldsymbol{X} \dotplus \left(\boldsymbol{X}^{\perp} \Cap \boldsymbol{Y}\right) = \boldsymbol{Y}$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in X$	(ORTHOMODULAR IDENTITY)

✎PROOF:

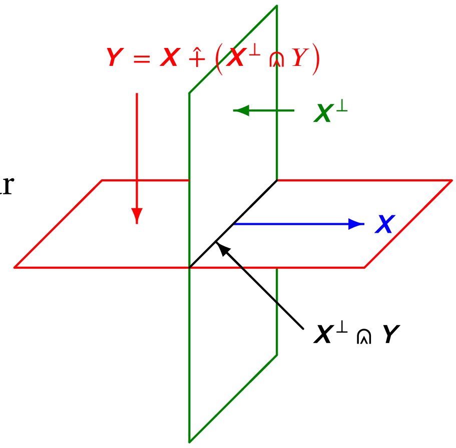
1. Proof for *complemented* (1) property: by *Projection Theorem* (Theorem 8.8 page 126).
2. Proof for *complemented* (2) property: by Corollary 8.1 (page 123).
3. Proof for *involutory* property: by Corollary 8.1 (page 123).
4. Proof for *antitone* property: by Proposition 8.5 (page 121).
5. Proof for *orthomodular identity* property:
6. Proof that lattice is *orthomodular*: by 5 properties and definition of *orthomodular lattice* (Definition 16.3 page 247).

✏

[19] Bachman and Narici (1966) page 172 ⟨Theorem 10.8⟩, Kubrusly (2001) page 339 ⟨Theorem 5.20⟩
[20] Iturrioz (1985) pages 56–57

This concept is illustrated to the right where $\boldsymbol{X}, \boldsymbol{Y} \in 2^{\boldsymbol{H}}$ are linear subspaces of the linear space $\boldsymbol{H}$ and

$$\boldsymbol{X} \subseteq \boldsymbol{Y} \implies \boldsymbol{Y} = \boldsymbol{X} \dotplus \left(\boldsymbol{X}^{\perp} \Cap Y\right).$$



Corollary 8.2. *Let $\boldsymbol{H}$ be a Hilbert space with orthogonality operation $\perp$. Let $\left(2^{\boldsymbol{H}}, \dotplus, \Cap, \mathbf{0}, \boldsymbol{H}; \subseteq\right)$ be the lattice of subspaces of $\boldsymbol{H}$.*

COR

$$\begin{array}{rcll} (\boldsymbol{X} \dotplus \boldsymbol{Y})^{\perp} &=& \boldsymbol{X}^{\perp} \Cap \boldsymbol{Y}^{\perp} & \forall \boldsymbol{X},\boldsymbol{Y} \in 2^{\boldsymbol{H}} \quad \text{(DE MORGAN)} \quad \textit{and} \\ (\boldsymbol{X} \Cap \boldsymbol{Y})^{\perp} &=& \boldsymbol{X}^{\perp} \dotplus \boldsymbol{Y}^{\perp} & \forall \boldsymbol{X},\boldsymbol{Y} \in 2^{\boldsymbol{H}} \quad \text{(DE MORGAN)} \end{array}$$

✎PROOF: By properties of *orthocomplemented lattices* (Theorem 16.1 page 240). ✑

8.4 Subspace Metrics

Definition 8.5 (Hilbert space gap metric). [21] *Let $\boldsymbol{X}$ be a* ***Hilbert space*** *and $\boldsymbol{S}$ the set of subspaces of $\boldsymbol{X}$. Then we define the following metric between subspaces of $\boldsymbol{X}$.*

DEF

$$\begin{array}{rcll} \mathsf{d}(\boldsymbol{V}, \boldsymbol{W}) &\triangleq& \|\mathbf{P} - \mathbf{Q}\| \quad \forall \boldsymbol{V},\boldsymbol{W} \in \boldsymbol{S} & \text{(the distance between subspaces } \boldsymbol{V} \text{ and } \boldsymbol{W} \text{ is the size of the difference of their projection operators)} \\ \textit{where } \boldsymbol{V} &\triangleq& \mathbf{P}\boldsymbol{X} & (\mathbf{P} \text{ is the projection operator that generates the subspace } \boldsymbol{V}) \\ \textit{and } \boldsymbol{W} &\triangleq& \mathbf{Q}\boldsymbol{X} & (\mathbf{Q} \text{ is the projection operator that generates the subspace } \boldsymbol{W}). \end{array}$$

Definition 8.6 (Banach space gap metric). [22] *Let $\boldsymbol{X}$ be a* ***Banach space*** *and $\boldsymbol{S}$ the set of subspaces of $\boldsymbol{X}$. Then we define the following metric between subspaces of $\boldsymbol{X}$.*

DEF

$$\begin{array}{rcll} \mathsf{d}(\boldsymbol{V}, \boldsymbol{W}) &\triangleq& \max\left\{ \sup\limits_{\boldsymbol{v} \in \boldsymbol{V}, \|\boldsymbol{v}\|=1} \mathsf{p}(\boldsymbol{v}, \boldsymbol{W}), \sup\limits_{\boldsymbol{w} \in \boldsymbol{W}, \|\boldsymbol{w}\|=1} \mathsf{p}(\boldsymbol{w}, \boldsymbol{V}) \right\} & \forall \boldsymbol{V},\boldsymbol{W} \in \boldsymbol{S} \\ \textit{where } \mathsf{p}(\boldsymbol{v}, \boldsymbol{W}) &\triangleq& \inf\limits_{\boldsymbol{w} \in \boldsymbol{W}} \|\boldsymbol{v} - \boldsymbol{w}\| & \text{(metric from the point } \boldsymbol{v} \text{ to the subspace } \boldsymbol{W}) \end{array}$$

Definition 8.7 (Schäffer's metric). [23]

DEF

$$\begin{array}{rcl} \mathsf{d}(\boldsymbol{V}, \boldsymbol{W}) &=& \log\left(1 + \max\left\{\mathsf{r}(\boldsymbol{V}, \boldsymbol{W}), \mathsf{r}(\boldsymbol{W}, \boldsymbol{V})\right\}\right) \quad \textit{where} \\ \mathsf{r}(\boldsymbol{V}, \boldsymbol{W}) &\triangleq& \begin{cases} \inf\left\{\|\mathbf{A} - \mathbf{I}\| \mid \mathbf{A}\mathbf{V} = \mathbf{W}\right\} & \textit{if } \mathbf{A} \textit{ and } \mathbf{A}^{-1} \textit{ both exist} \\ 1 & \textit{otherwise} \end{cases} \end{array}$$

8.5 Literature

LITERATURE SURVEY:

[21] Deza and Deza (2006) page 235, Akhiezer and Glazman (1993) page 69, Berkson (1963) page 8, Krein and Krasnoselski (1947)

[22] Akhiezer and Glazman (1993) page 70, Berkson (1963) page 8, Krein et al. (1948)

[23] Massera and Schäffer (1958) pages 562–563, Berkson (1963) pages 7–8

1. Lattice of subspaces
   Birkhoff and Neumann (1936)
   Husimi (1937)
   Sasaki (1954)
   Loomis (1955)
   von Neumann (1960)
   Holland (1970)
   Halmos (1998b)
   Amemiya and Araki (1966)
   Gudder (1979)
   Gudder (2005)

2. Characterizations of lattice of Hilbert subspaces (cf Iturrioz (1985) page 60):
   Kakutani and Mackey (1946) ⟨using Banach spaces⟩
   Piron (1964a) ⟨using pre-Hilbert spaces⟩
      Piron (1964b) ⟨using pre-Hilbert spaces⟩
   Amemiya and Araki (1966) ⟨using pre-Hilbert spaces⟩
   Wilbur (1975) ⟨using locally convex spaces⟩

3. Metrics on subspaces:
   Burago et al. (2001)

Part II

Properties of Spaces

CHAPTER 9

SEQUENCES AND CONVERGENCE

❝It is necessary that the values at which we arrive on increasing continually the number of terms, should approach more and more a fixed limit, and should differ from it only by a quantity which becomes less than any given magnitude: this limit is the value of the series.❞

Joseph Fourier (1768–1830) [1]

9.1 Definitions

Definition 9.1. [2] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the* SET OF INTEGERS.

DEF
*A function f in X^Y is an X-valued **sequence** if $\quad Y = \mathbb{Z}$.*
A sequence may be denoted in the form $(\!(x_n)\!)_{n\in\mathbb{Z}}$ or simply as $(\!(x_n)\!)$.
*A function f in X^Y is an X-valued **n-tuple** if $\quad Y = \{1, 2, \ldots, N\}$.*
An n-tuple may be denoted in the form $\llparenthesis x_n \rrparenthesis_1^N$ or simply as $\llparenthesis x_n \rrparenthesis$.

Definition 9.2. [3] *Let $(\!(x_n)\!)_{n\in\mathbb{Z}}$ and $(\!(y_n)\!)_{n\in\mathbb{Z}}$ be sequences over a field $\mathbb{F}$.*
Let $\llparenthesis x_n \rrparenthesis_1^N$ and $\llparenthesis y_n \rrparenthesis_1^N$ be n-tuples over a field $\mathbb{F}$.

DEF

$$(\!(x_n)\!) + (\!(y_n)\!) \triangleq (\!(x_n + y_n)\!) \qquad \alpha\,(\!(x_n)\!) \triangleq (\!(\alpha x_n)\!) \quad \forall\alpha\in\mathbb{F}$$
$$\llparenthesis x_n \rrparenthesis + \llparenthesis y_n \rrparenthesis \triangleq \llparenthesis x_n + y_n \rrparenthesis \qquad \alpha\,\llparenthesis x_n \rrparenthesis \triangleq \llparenthesis \alpha x_n \rrparenthesis \quad \forall\alpha\in\mathbb{F}$$

[1] quote: Fourier (1878) pages 196–197 ⟨§228⟩
image: http://en.wikipedia.org/wiki/File:Fourier2.jpg, public domain
[2] Bromwich (1908) page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[3] Haaser and Sullivan (1991) page 42 ⟨2·2 Proposition⟩

9.2 Sequences in topological spaces

A *topological space* (Definition 1.1 page 3) provides sufficient structure to support the property of *convergence* (next definition) of a sequence. In a *metric space* (Definition 3.1 page 33), a convergent sequence converges to an *unique* limit (Theorem 9.6 page 143). However in a topological space, a convergent sequence may converge to more than one limit (Example 9.1 page 132).

Definition 9.3. [4] *Let (X, T) be a* TOPOLOGICAL SPACE *(Definition 1.1 page 3).*

DEF

*A sequence $(\!(x_n)\!)_{n\in\mathbb{Z}}$ **converges** to a point x if for each* OPEN SET *(Definition 1.1 page 3) U of x there exists $N \in \mathbb{N}$ such that*

$$x_n \in U \text{ for all } n > N.$$

This condition can be expressed in any of the following forms:

1. *The **limit** of the sequence $(\!(x_n)\!)$ is x.*
2. *The sequence $(\!(x_n)\!)$ is **convergent** with limit x.*
3. $\lim_{n\to\infty} (\!(x_n)\!) = x$.
4. $(\!(x_n)\!) \to x$.

*A sequence that converges is **convergent**. A sequence that does not converge is said to **diverge**, or is **divergent**. An element $x \in A$ is a **limit point** of A if it is the limit of some A-valued sequence $(\!(x_n \in A)\!)$.*

Example 9.1. [5]

EX

Let (X, T_{31}) be a *topological space* where $X \triangleq \{x, y, z\}$ and

$$T_{31} \triangleq \{\emptyset, \{x\}, \{x,y\}, \{x,z\}, \{x,y,z\}\}.$$

In this space, the sequence $(\!(x, x, x, \ldots)\!)$ converges to x. But this sequence also converges to both y and z because x is in every *neighborhood* (Definition 1.9 page 25) of y and x is in every neighborhood of z. That is, the *limit* (Definition 9.3 page 132) of the sequence is *not* unique.

Example 9.2. In contrast to Example 9.1, note that the limit of the sequence $(\!(x, x, x, \ldots)\!)$ *is* unique in a *topological space* with sufficiently high resolution with respect to y and z such as the following:

EX

Define a *topological space* (X, T_{56}) where $X \triangleq \{x, y, z\}$ and

$$T_{56} \triangleq \{\emptyset, \{y\}, \{z\}, \{x,y\}, \{y,z\}, \{x,y,z\}\}.$$

In this space, the sequence $(\!(x, x, x, \ldots)\!)$ converges to x only. The sequence does *not* converge to y or z because there are *open sets* (Definition 1.1 page 3) containing y or z that do not contain x (the open sets $\{y\}$, $\{z\}$, and $\{y, z\}$).

Theorem 9.1 (The Closed Set Theorem). [6] *Let (X, T) be a* TOPOLOGICAL SPACE. *Let A be a subset of X $(A \subseteq X)$. Let A^- be the* CLOSURE *(Definition 1.4 page 14) of A in (X, T).*

THM

$$\underbrace{A \text{ is CLOSED in } (X,T)}_{(A = A^-)} \iff \left\{\begin{array}{l}\text{Every } A\text{-valued sequence } (\!(x_n \in A)\!)_{n\in\mathbb{Z}} \\ \text{that CONVERGES in } (X,T) \\ \text{has its LIMIT in } A\end{array}\right\}$$

✎PROOF:

1. Proof that $\lim (\!(x_n)\!) \in A \implies A = A^-$:

   (a) Proof that $A \subseteq A^-$: by Lemma 1.1 page 15.

[4] Joshi (1983) page 83 ⟨(3.1) Definition⟩, "→" symbol: Leathem (1905) page 13 ⟨section III.11⟩

[5] Munkres (2000) page 98 ⟨Hausdorff Spaces⟩

[6] Kubrusly (2001) page 118 ⟨Theorem 3.30⟩, Haaser and Sullivan (1991) page 75 ⟨6·9 Proposition⟩, Rosenlicht (1968) pages 47–48

(b) Proof that $A^- \subseteq A$:

$$\begin{aligned}
x \in A^- &\implies \text{each open set } U \text{ containing } x \text{ intersects } A && \text{by Lemma 1.2 page 22}\\
&\implies \exists \langle x_n \in A \rangle \text{ that converges to } x && \text{by Definition 9.3 page 132}\\
&\implies x \in A && \text{by left hypothesis}
\end{aligned}$$

2. Proof that $A = A^- \implies \lim \langle x_n \rangle \in A$:

$$\begin{aligned}
\lim \langle x_n \rangle = x &\iff \left\{ \begin{array}{l} \text{for each } \textit{open set } U \text{ containing } x, \text{ there exists } N \\ \text{such that } x_n \in U, \forall n > N. \end{array} \right\} && \text{by Definition 9.3 page 132}\\
&\implies \text{each open set } U \text{ containing } x \text{ intersects } A && \text{because } x_n \text{ in } A\\
&\iff x \in A^- && \text{by Lemma 1.2 page 22}\\
&\iff x \in A && \text{by } A = A^- \text{ hypothesis}
\end{aligned}$$

1. Proof that $\lim \langle x_n \rangle \in A \implies A = A^-$:

2. Proof that $\lim \langle x_n \rangle \in A \impliedby A = A^-$ (proof by contradiction):

$$\begin{aligned}
\lim \langle x_n \rangle \notin A &\implies \lim \langle x_n \rangle \notin A^- && \text{by } A = A^- \text{ hypothesis}\\
&\implies \lim \langle x_n \rangle \in (A^-)^c && \text{by definition of set complement: Definition A.5 page 260}\\
&\implies \exists x_n \quad \text{such that} \quad x_n \in (A^-)^c && \text{because } (A^-)^c \text{ is } \textit{open} \text{ and by Definition 9.3 page 132}\\
&\implies \exists x_n \quad \text{such that} \quad x_n \notin A^- && \text{by definition of set complement: Definition A.5 page 260}\\
&\implies \exists x_n \quad \text{such that} \quad x_n \notin A && \text{by } A = A^- \text{ hypothesis}\\
&\implies \text{contradiction} && \text{by definition of } \langle x_n \in A \rangle\\
&\implies \lim \langle x_n \rangle \in A
\end{aligned}$$

1. Proof that $x \in A \implies A$ is closed (proof by contradiction):

   (a) Suppose that A is not closed.

   (b) Then by Definition 1.1 (page 3), A^c is *not* open.

   (c) If A^c is not open, then there is nothing to prevent x to be located in A^c and yet be so close to A that it is *still* a limit point of $\langle a_n \rangle$. (Note: It is the openness property of A^c that prevents this catastrophe from happening. Without it, as is the case under the "A is not closed supposition", a limit point can get too close to the border and pull in an infinite number of other points from the sequence to the wrong side of the border. That is, the openness property provides a protective buffer on the border that keeps points from being sucked across the border by the limit point.)

   (d) That is, choose an r such that $\mathsf{B}(x, r)\ A^c$.

   (e) Since x is a limit point of the sequence $\langle a_n \rangle$ and by Theorem 9.4 (page 141),

$$\text{for some } 0 < \varepsilon < r, \ \exists N \quad \text{such that} \quad \mathsf{d}\left(a_n, x\right) < \varepsilon < r.$$

   (f) That is, now all the infinite number of points $\langle a_n \rangle_{n>N}$ are inside the ball $\mathsf{B}(x, r)$, and thus *inside* A^c.

   (g) Thus, there are points in $\langle a_n \rangle$ that are in A^c, and not in A where they are by definition supposed to be.

   (h) This is a contradiction, and therefore the original supposition is impossible and A must be closed.

✏

9.3 Sequences in distance spaces

One of the most important applications of *metric space* (Definition 3.1 page 33) analysis, and more generally in *distance space* analysis as well, is the concept of *convergence*. Loosely speaking, a sequence that converges somehow implies that its elements are "getting closer and closer" to some value. In a distance space (of which the metric space is a special case), there are two standard types of sequences often used to describe this:

① *convergent* sequence : The elements of the sequence approach a fixed value x (Definition 9.4 page 134)

② *Cauchy* sequence : The elements of the sequence approach each other (Definition 9.5 page 134)

In a *metric space*, it follows from the *triangle inequality* that the *convergent* condition is "stronger" than the *Cauchy* condition in the sense that all convergent sequences are Cauchy but not all Cauchy sequences are convergent (Theorem 9.5 page 142). This is *not* the case for all *distance spaces* (where the triangle inequality does not hold).

9.3.1 Definitions

Definition 9.4. [7] *Let* $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE *in a* DISTANCE SPACE (X, d).

DEF

The sequence $(\!(x_n)\!)$ ***converges*** *to a* ***limit*** x *if for any* $\varepsilon \in \mathbb{R}^+$, *there exists* $N \in \mathbb{Z}$ *such that for all* $n > N$, $\mathsf{d}(x_n, x) < \varepsilon$.

This condition can be expressed in any of the following forms:

1. *The* ***limit*** *of the sequence* $(\!(x_n)\!)$ *is* x.
2. *The sequence* $(\!(x_n)\!)$ *is* ***convergent*** *with limit* x.
3. $\lim_{n\to\infty} (\!(x_n)\!) = x$.
4. $(\!(x_n)\!) \to x$.

A SEQUENCE *that converges is* ***convergent***.

Definition 9.5. [8] *Let* $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE *in a* DISTANCE SPACE (X, d).

DEF

The sequence $(\!(x_n)\!)$ *is a* ***Cauchy sequence*** *in* (X, d) *if*

for every $\varepsilon \in \mathbb{R}^+$, *there exists* $N \in \mathbb{Z}$ *such that* $\forall n, m > N,\ \mathsf{d}(x_n, x_m) < \varepsilon$ (CAUCHY CONDITION).

A *sequence* is said to be *complete* in a *distance space* (X, d) if every *Cauchy sequence* in (X, d) *converges* to a limit in (X, d) (next definition).

Definition 9.6. [9] *Let* $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE *in a* DISTANCE SPACE (X, d).

DEF

The sequence $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *is* ***complete*** *in* (X, d) *if*

$(\!(x_n)\!)$ *is* CAUCHY *in* (X, d) $\implies$ $(\!(x_n)\!)$ *is* CONVERGENT *in* (X, d).

[7] in *metric space*: Rosenlicht (1968) page 45, Giles (1987) page 37 ⟨3.2 Definition⟩, Khamsi and Kirk (2001) page 13 ⟨Definition 2.1⟩ "→" symbol: Leathem (1905) page 13 ⟨section III.11⟩

[8] in *metric space*: Apostol (1975) page 73 ⟨4.7⟩, Rosenlicht (1968) page 51

[9] in *metric space*: Rosenlicht (1968) page 52

9.3.2 Properties

Proposition 9.1. [10] *Let* $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE *in a* DISTANCE SPACE (X, d).

PRP
$$\left\{ (\!(x_n)\!) \text{ is CAUCHY in } (X,\mathsf{d}) \right\} \implies \left\{ (\!(x_n)\!) \text{ is BOUNDED in } (X,\mathsf{d}) \right\}$$

✎PROOF:

$$\begin{aligned}
(\!(x_n)\!) \text{ is } \textit{Cauchy} &\implies \text{for every } \varepsilon \in \mathbb{R}^+,\ \exists N \in \mathbb{Z} \quad \text{such that} \quad \forall n,m > N,\ \mathsf{d}(x_n, x_m) < \varepsilon && \text{(by Definition 9.5)}\\
&\implies \exists N \in \mathbb{Z} \quad \text{such that} \quad \forall n,m > N,\ \mathsf{d}(x_n, x_m) < 1 && \text{(arbitrarily choose } \varepsilon \triangleq 1)\\
&\implies \exists N \in \mathbb{Z} \quad \text{such that} \quad \forall n,m \in \mathbb{Z},\ \mathsf{d}(x_n, x_{m+1}) < \max\left\{\{1\} \cup \left\{\mathsf{d}(x_p, x_q) \,|\, p,q \not> N\right\}\right\}\\
&\implies (\!(x_n)\!) \text{ is } \textit{bounded} && \text{(by Definition 2.3 page 28)}
\end{aligned}$$

✏

Proposition 9.2. [11] *Let* $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE *in a* DISTANCE SPACE (X, d). *Let* $\mathsf{f} \in \mathbb{Z}^{\mathbb{Z}}$ *be a* STRICTLY MONOTONE *function such that* $\mathsf{f}(n) < \mathsf{f}(n+1)$.

PRP
$$\underbrace{(\!(x_n)\!)_{n\in\mathbb{Z}} \text{ is CAUCHY}}_{\text{sequence is CAUCHY}} \implies \underbrace{\left(\!\left(x_{\mathsf{f}(n)}\right)\!\right)_{n\in\mathbb{Z}} \text{ is CAUCHY}}_{\text{subsequence is also CAUCHY}}$$

✎PROOF:

$$\begin{aligned}
&(\!(x_n)\!)_{n\in\mathbb{Z}} \text{ is } \textit{Cauchy}\\
&\implies \text{for any given } \varepsilon > 0,\ \exists N \quad \text{such that} \quad \forall n,m > N,\ \mathsf{d}(x_n, x_m) < \varepsilon && \text{by Definition 9.5 page 134}\\
&\implies \text{for any given } \varepsilon > 0,\ \exists N' \quad \text{such that} \quad \forall \mathsf{f}(n), \mathsf{f}(m) > N',\ \mathsf{d}\left(x_{\mathsf{f}(n)}, x_{\mathsf{f}(m)}\right) < \varepsilon\\
&\implies \left(\!\left(x_{\mathsf{f}(n)}\right)\!\right)_{n\in\mathbb{Z}} \text{ is } \textit{Cauchy} && \text{by Definition 9.5 page 134}
\end{aligned}$$

✏

Theorem 9.2. [12] *Let* (X, d) *be a* DISTANCE SPACE. *Let* A^- *be the* CLOSURE *(Definition 1.4 page 14) of a* A *in a* TOPOLOGICAL SPACE INDUCED BY (X, d).

THM
$$\left\{\begin{array}{lll} 1. & \text{LIMITS are UNIQUE in } (X,\mathsf{d}) & \text{(Definition 9.4 page 134)} \quad \text{and}\\ 2. & (A,\mathsf{d}) \text{ is COMPLETE in } (X,\mathsf{d}) & \text{(Definition 9.6 page 134)} \end{array}\right\} \implies \underbrace{A \text{ is CLOSED in } (X,\mathsf{d})}_{A = A^-}$$

✎PROOF:

1. Proof that $A \subseteq A^-$: by Lemma 1.1 page 15
2. Proof that $A^- \subseteq A$ (proof that $x \in A^- \implies x \in A$):
   (a) Let x be a point in A^- ($x \in A^-$).
   (b) Define a *sequence* of open balls $\left(\!\left(\mathsf{B}\left(x, \frac{1}{1}\right), \mathsf{B}\left(x, \frac{1}{2}\right), \mathsf{B}\left(x, \frac{1}{3}\right), \ldots\right)\!\right)$.
   (c) Define a *sequence* of points $(\!(x_1,\ x_2,\ x_3,\ \ldots)\!)$ such that $x_n \in \mathsf{B}\left(x_n, \frac{1}{n}\right) \cap A$.

[10] in *metric space*: Giles (1987) page 49 ⟨Theorem 3.30⟩
[11] in *metric space*: Rosenlicht (1968) page 52
[12] in *metric space*: Kubrusly (2001) page 128 ⟨Theorem 3.40⟩, Haaser and Sullivan (1991) page 75 ⟨6·10, 6·11 Propositions⟩, Bryant (1985) page 40 ⟨Theorem 3.6, 3.7⟩, Sutherland (1975) pages 123–124

(d) Then $(\!(x_n)\!)$ is *convergent* in X with limit x by Definition 9.4 page 134

(e) and $(\!(x_n)\!)$ is *Cauchy* in A by Definition 9.5 page 134.

(f) By the hypothesis 2, $(\!(x_n)\!)$ is therefore also *convergent* in A.
Let this limit be y. Note that $y \in A$.

(g) By hypothesis 1, limits are *unique*, so $y = x$.

(h) Because $y \in A$ (item (2f) page 136) and $y = x$ (item (2g) page 136), so $x \in A$.

(i) Therefore, $x \in A^- \implies x \in A$ and $A^- \subseteq A$.

✎

Proposition 9.3. [13] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *be a sequence in a* DISTANCE SPACE (X, d)*. Let* $\mathsf{f} : \mathbb{Z} \to \mathbb{Z}$ *be a strictly increasing function such that* $\mathsf{f}(n) < \mathsf{f}(n+1)$.

PRP

$$\underbrace{(\!(x_n)\!)_{n\in\mathbb{Z}} \to x}_{\textit{sequence converges to limit } x} \implies \underbrace{\left(\!\!\left(x_{\mathsf{f}(n)}\right)\!\!\right)_{n\in\mathbb{Z}} \to x}_{\textit{subsequence converges to the same limit } x}$$

✎PROOF:

$$\begin{aligned}
(\!(x_n)\!)_{n\in\mathbb{Z}} \to x &\implies \forall\varepsilon > 0,\ \exists N \quad \text{such that} \quad \forall n > N,\ \mathsf{d}\big(x_n, x\big) < \varepsilon && \text{by Theorem 9.4 page 141}\\
&\implies \forall\varepsilon > 0,\ \exists \mathsf{f}(N) \quad \text{such that} \quad \forall \mathsf{f}(n) > \mathsf{f}(N),\ \mathsf{d}\big(x_{\mathsf{f}(n)}, x\big) < \varepsilon \\
&\implies \left(\!\!\left(x_{\mathsf{f}(n)}\right)\!\!\right)_{n\in\mathbb{Z}} \to x && \text{by Theorem 9.4 page 141}
\end{aligned}$$

✎

Theorem 9.3 (Cantor intersection theorem). [14] *Let* (X, d) *be a* DISTANCE SPACE *(Definition 2.1 page 27),* $(\!(A_n)\!)_{n\in\mathbb{Z}}$ *a* SEQUENCE *with each* $A_n \in 2^X$*, and* $|A|$ *the number of elements in* A.

THM

$$\left\{\begin{array}{llll}
1. & (X, \mathsf{d}) \text{ is COMPLETE} & & \text{(Definition 9.6 page 134)} \quad \text{and}\\
2. & A_n \text{ is CLOSED} & \forall n\in\mathbb{N} & \text{(Definition 1.1 page 3)} \quad \text{and}\\
3. & \operatorname{diam} A_n \geq \operatorname{diam} A_{n+1} & \forall n\in\mathbb{N} & \text{(Definition 2.2 page 27)} \quad \text{and}\\
4. & \operatorname{diam} (\!(A_n)\!)_{n\in\mathbb{Z}} \to 0 & & \text{(Definition 9.4 page 134)}
\end{array}\right\} \implies \left\{\left|\bigcap_{n\in\mathbb{N}} A_n\right| = 1\right\}$$

✎PROOF:

1. Proof that $\left|\bigcap_{n\in\mathbb{Z}} A_n\right| < 2$:

   (a) Let $A \triangleq \cap A_n$.

   (b) $x \neq y$ and $\{x, y\} \in A \implies \mathsf{d}(x, y) > 0$ and $\{x, y\} \subseteq A_n \forall n$

   (c) $\exists n$ such that $\operatorname{diam} A_n < \mathsf{d}(x, y)$ by left hypothesis 4

   (d) $\implies \exists n$ such that $\sup\left\{\mathsf{d}(x, y) \mid x, y \in A_n\right\} < \mathsf{d}(x, y)$

   (e) This is a contradiction, so $\{x, y\} \notin A$ and $\left|\bigcap A_n\right| < 2$.

2. Proof that $\left|\cap A_n\right| \geq 1$:

   (a) Let $x_n \in A_n$ and $x_m \in A_m$

   (b) $\forall\varepsilon,\ \exists N \in \mathbb{N}$ such that $A_N < \varepsilon$

[13] in *metric space*: Rosenlicht (1968) page 46

[14] in *metric space*: Davis (2005) page 28, Hausdorff (1937) page 150

(c) $\forall m, n > N,\ x_n \in A_n \subseteq A_N$ and $x_m \in A_m \subseteq A_N$
(d) $\mathsf{d}(x_n, x_m) \le \operatorname{diam} A_N < \varepsilon \implies \{x_n\}$ is a Cauchy sequence
(e) Because $\{x_n\}$ is complete, $x_n \to x$.
(f) $\implies x \in (A_n)^- = A_n$
(g) $\implies |A_n| \ge 1$

✎

Definition 9.7. [15] *Let* (X, d) *be a* DISTANCE SPACE. *Let* C *be the set of all* CONVERGENT *sequences in* (X, d).

THM *The* DISTANCE FUNCTION d *is* ***continuous*** *in* (X, d) *if*

$$(\!(x_n)\!), (\!(y_n)\!) \in C \implies \lim_{n\to\infty} (\!(\mathsf{d}(x_n, y_n))\!) = \mathsf{d}\left(\lim_{n\to\infty} (\!(x_n)\!), \lim_{n\to\infty} (\!(y_n)\!)\right).$$

A DISTANCE FUNCTION *is* ***discontinuous*** *if it is not* CONTINUOUS.

Remark 9.1. Rather than defining *continuity* of a *distance function* in terms of the *sequential characterization of continuity* (Definition 9.7 page 137), we could define continuity using an *inverse image characterization of continuity*" (Definition 2.6 page 30). Assuming an equivalent *topological space* is used for both characterizations, the two characterizations are equivalent (Theorem 9.9 page 146). In fact, one could construct an equivalence such as the following:

REM
$$\left\{\begin{array}{l}\mathsf{d} \text{ is } \textit{continuous} \text{ in } \mathbb{R}^{X^2} \\ \text{(Definition 1.8 page 23)} \\ (\textit{inverse image characterization of continuity})\end{array}\right\} \iff \left\{\begin{array}{l}(\!(x_n)\!), (\!(y_n)\!) \in C \implies \\ \lim\limits_{n\to\infty} (\!(\mathsf{d}(x_n, y_n))\!) = \mathsf{d}\left(\lim\limits_{n\to\infty} (\!(x_n)\!), \lim\limits_{n\to\infty} (\!(y_n)\!)\right) \\ \text{(Definition 9.3 page 132)} \\ (\textit{sequential characterization of continuity})\end{array}\right\}$$

Note that just as $(\!(x_n)\!)$ is a sequence in X, so the ordered pair $\left((\!(x_n)\!), (\!(y_n)\!)\right)$ is a sequence in X^2. The remainder follows from Theorem 9.9 (page 146). However, use of the *inverse image characterization* is somewhat troublesome because we would need a topology on X^2, and we don't immediately have one defined and ready to use. In fact, we don't even immediately have a distance space on X^2 defined or even open balls in such a distance space. The result is, for the scope of this chapter, it is arguably not worthwhile constructing the extra structure, but rather this chapter instead uses the *sequential characterization* as a definition (Definition 9.7 page 137).

9.3.3 Examples

Similar distance functions and several of the observations for the examples in this section can be found in Blumenthal (1953) pages 8–13.

In a *metric space*, all *open balls* are *open*, the *open balls* form a *base* for a *topology*, the limits of *convergent* sequences are *unique*, and the *metric function* is *continuous*. In the *distance space* of the next example, none of these properties hold.

Example 9.3. [16] Let (x, y) be an *ordered pair* in $\mathbb{R}^2$. Let $(a : b)$ be an *open interval* and $(a : b]$ a *half-open interval* in $\mathbb{R}$. Let $|x|$ be the *absolute value* of $x \in \mathbb{R}$. The function $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that

$$\mathsf{d}(x, y) \triangleq \left\{\begin{array}{lll} y & \forall (x, y) \in \{4\} \times (0 : 2] & (\textit{vertical half-open interval}) \\ x & \forall (x, y) \in (0 : 2] \times \{4\} & (\textit{horizontal half-open interval}) \\ |x - y| & \text{otherwise} & (\textit{Euclidean}) \end{array}\right\} \text{ is a } \textit{distance} \text{ on } \mathbb{R}.$$

Note some characteristics of the *distance space* $(\mathbb{R}, \mathsf{d})$:

[15] Blumenthal (1953) page 9 ⟨DEFINITION 6.3⟩
[16] A similar distance function d and item (4) page 138 can in essence be found in Blumenthal (1953) page 8.

1. $(\mathbb{R}, \mathsf{d})$ is not a *metric space* because d does not satisfy the *triangle inequality*:
   $$\mathsf{d}(0,4) \triangleq |0-4| = 4 \not\leq 2 = |0-1| + 1 \triangleq \mathsf{d}(0,1) + \mathsf{d}(1,4)$$

2. Not every *open ball* in $(\mathbb{R}, \mathsf{d})$ is *open*.
   For example, the *open ball* $\mathsf{B}(3,2)$ is *not open* because $4 \in \mathsf{B}(3,2)$ *but* for all $0 < \varepsilon < 1$
   $$\mathsf{B}(4,\varepsilon) = (4-\varepsilon : 4+\varepsilon) \cup (0 : \varepsilon) \not\subseteq (1 : 5) = \mathsf{B}(3,2)$$

3. The *open balls* of $(\mathbb{R}, \mathsf{d})$ do not form a *base* for a *topology* on $\mathbb{R}$.
   This follows directly from item (2) and Theorem 2.2 (page 30).

4. In the *distance space* $(\mathbb{R}, \mathsf{d})$, limits are *not unique*;
   For example, the sequence $(\!(1/n)\!)_1^\infty$ converges both to the limit 0 and the limit 4 in $(\mathbb{R}, \mathsf{d})$:
   $$\begin{array}{lllllll}\lim\limits_{n\to\infty} \mathsf{d}(x_n, 0) & \triangleq \lim\limits_{n\to\infty} \mathsf{d}(1/n, 0) & \triangleq \lim\limits_{n\to\infty} |1/n - 0| & = 0 & \implies & (\!(1/n)\!) \to 0 \\ \lim\limits_{n\to\infty} \mathsf{d}(x_n, 4) & \triangleq \lim\limits_{n\to\infty} \mathsf{d}(1/n, 4) & \triangleq \lim\limits_{n\to\infty} (\!(1/n)\!) & = 0 & \implies & (\!(1/n)\!) \to 4\end{array}$$

5. The *topological space* $(X, \boldsymbol{T})$ *induced by* $(\mathbb{R}, \mathsf{d})$ also yields limits of 0 and 4 for the sequence $(\!(1/n)\!)_1^\infty$, just as it does in item (4). This is largely due to the fact that, for small ε, the open balls $\mathsf{B}(0,\varepsilon)$ and $\mathsf{B}(4,\varepsilon)$ are *open*.
   $$\begin{array}{rll}\mathsf{B}(0,\varepsilon) \text{ is } \textit{open} & \implies & \text{for each } U \in \boldsymbol{T} \text{ that contains } 0, \exists N \in \mathbb{N} \text{ such that } 1/n \in U \quad \forall n > N \\ & \iff & (\!(1/n)\!) \to 0 \qquad \text{by definition of } \textit{convergence} \text{ (Definition 9.3 page 132)} \\ \mathsf{B}(4,\varepsilon) \text{ is } \textit{open} & \implies & \text{for each } U \in \boldsymbol{T} \text{ that contains } 4, \exists N \in \mathbb{N} \text{ such that } 1/n \in U \quad \forall n > N \\ & \iff & (\!(1/n)\!) \to 4 \qquad \text{by definition of } \textit{convergence} \text{ (Definition 9.3 page 132)}\end{array}$$

6. The distance function d is *discontinuous* (Definition 9.7 page 137):
   $$\begin{aligned}\lim_{n\to\infty} (\!(\mathsf{d}(1-1/n, 4-1/n))\!) &= \lim_{n\to\infty} (\!(|(1-1/n) - (4-1/n)|)\!) = |1-4| = 3 \neq 4 = \mathsf{d}(0,4) \\ &= \mathsf{d}\left(\lim_{n\to\infty} (\!(1-1/n)\!), \lim_{n\to\infty} (\!(4-1/n)\!)\right)\end{aligned}$$

In a *metric space*, all *convergent* sequences are also *Cauchy*. However, this is not the case for all *distance spaces*, as demonstrated next:

Example 9.4. [17] The function $\mathsf{d}(x,y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that
$$\mathsf{d}(x,y) \triangleq \left\{\begin{array}{lll} |x-y| & \text{for } x = 0 \text{ or } y = 0 \text{ or } x = y & \textit{(Euclidean)} \\ 1 & \text{otherwise} & \textit{(discrete)}\end{array}\right\} \quad \text{is a } \textit{distance} \text{ on } \mathbb{R}.$$
Note some characteristics of the *distance space* $(\mathbb{R}, \mathsf{d})$:

1. (X, d) is not a *metric space* because the *triangle inequality* does not hold:
   $$\mathsf{d}\left(\tfrac{1}{4}, \tfrac{1}{2}\right) = 1 \not\leq \tfrac{3}{4} = \left|\tfrac{1}{4} - 0\right| + \left|0 - \tfrac{1}{2}\right| = \mathsf{d}\left(\tfrac{1}{4}, 0\right) + \mathsf{d}\left(0, \tfrac{1}{2}\right)$$

2. The *open ball* $\mathsf{B}\left(\frac{1}{4}, \frac{1}{2}\right)$ is *not open* because for any $\varepsilon \in \mathbb{R}^+$, no matter how small,
   $$\mathsf{B}(0,\varepsilon) = (-\varepsilon : +\varepsilon) \not\subseteq \left\{0, \tfrac{1}{4}\right\} = \left\{x \in X | \mathsf{d}\left(\tfrac{1}{4}, x\right) < \tfrac{1}{2}\right\} \triangleq \mathsf{B}\left(\tfrac{1}{4}, \tfrac{1}{2}\right)$$

3. Even though not all the *open balls* are *open*, it is still possible to have an *open set* in (X, d). For example, the set $U \triangleq \{1, 2\}$ is *open*:
   $$\begin{array}{lllllllll}\mathsf{B}(1,1) & \triangleq & \{x \in X | \mathsf{d}(1,x) < 1\} & = & \{1\} & \subseteq & \{1,2\} & \triangleq & U \\ \mathsf{B}(2,1) & \triangleq & \{x \in X | \mathsf{d}(2,x) < 1\} & = & \{2\} & \subseteq & \{1,2\} & \triangleq & U\end{array}$$

[17] The distance function d and item (7) page 139 can in essence be found in Blumenthal (1953) page 9

4. By item (2) and Theorem 2.2 (page 30), the *open balls* of $(\mathbb{R}, \mathsf{d})$ do not form a *base* for a *topology* on $\mathbb{R}$.

5. Even though the open balls in $(\mathbb{R}, \mathsf{d})$ do not induce a topology on X, it is still possible to find a set of *open sets* in (X, d) that *is* a topology. For example, the set $\{\varnothing, \{1,2\}, \mathbb{R}\}$ is a topology on $\mathbb{R}$.

6. In $(\mathbb{R}, \mathsf{d})$, limits of *convergent* sequences are *unique*:
$$(\!(x_n)\!) \to x \qquad \Longrightarrow \qquad \lim_{n\to\infty} \mathsf{d}(x_n, x) = \left\{ \begin{array}{rcll} \lim |x_n - 0| & = & 0 & \text{for } x = 0 \quad \text{OR} \\ |x - x| & = & 0 & \text{for constant } (\!(x_n)\!) \text{ for } n > N \quad \text{OR} \\ 1 & \neq & 0 & \text{otherwise} \end{array} \right\}$$
which says that there are only two ways for a sequence to converge: either $x = 0$ or the sequence eventually becomes constant (or both). Any other sequence will *diverge*. Therefore we can say the following:

   (a) If $x = 0$ and the sequence is not constant, then the limit is *unique* and 0.

   (b) If $x = 0$ and the sequence is constant, then the limit is *unique* and 0.

   (c) If $x \neq 0$ and the sequence is constant, then the limit is *unique* and x.

   (d) If $x \neq 0$ and the sequence is not constant, then the sequence diverges and there is no limit.

7. In $(\mathbb{R}, \mathsf{d})$, a *convergent* sequence is not necessarily *Cauchy*. For example,

   (a) the sequence $(\!({}^1\!/\!_n)\!)_{n\in\mathbb{N}}$ is *convergent* with limit 0: $\lim_{n\to\infty} \mathsf{d}({}^1\!/\!_n, 0) = \lim_{n\to\infty} {}^1\!/\!_n = 0$

   (b) However, even though $(\!({}^1\!/\!_n)\!)$ is *convergent*, it is *not Cauchy*: $\lim_{n,m\to\infty} \mathsf{d}({}^1\!/\!_n, {}^1\!/\!_m) = 1 \neq 0$

8. The *distance function* d is *discontinuous* in (X, d):
$$\begin{aligned} &\lim_{n\to\infty} (\!(\mathsf{d}({}^1\!/\!_n, 2 - {}^1\!/\!_n))\!) = 1 \\ &\qquad\qquad \neq 2 = \mathsf{d}(0, 2) = \mathsf{d}\left(\lim_{n\to\infty} (\!({}^1\!/\!_n)\!)\,, \lim_{n\to\infty} (\!(2 - {}^1\!/\!_n)\!) \right) \end{aligned}$$

Example 9.5. [18] The function $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that
$$\mathsf{d}(x, y) \triangleq \left\{ \begin{array}{lll} 2|x - y| & \forall (x, y) \in \{(0,1)\,, (1,0)\} & \text{(dilated Euclidean)} \\ |x - y| & \text{otherwise} & \text{(Euclidean)} \end{array} \right\} \quad \text{is a } \textit{distance} \text{ on } \mathbb{R}.$$
Note some characteristics of the *distance space* $(\mathbb{R}, \mathsf{d})$:

1. $(\mathbb{R}, \mathsf{d})$ is *not* a *metric space* because d does *not* satisfy the *triangle inequality*:
$$\mathsf{d}(0, 1) \triangleq 2|0 - 1| = 2 \not\leq 1 = |0 - {}^1\!/\!_2| + |{}^1\!/\!_2 - 1| \triangleq \mathsf{d}(0, {}^1\!/\!_2) + \mathsf{d}({}^1\!/\!_2, 1)$$

2. The function d is *discontinuous*:
$$\begin{aligned} &\lim_{n\to\infty} (\!(\mathsf{d}(1 - {}^1\!/\!_n, {}^1\!/\!_n))\!) \triangleq \lim_{n\to\infty} (\!(|1 - {}^1\!/\!_n - {}^1\!/\!_n|)\!) = 1 \\ &\qquad \neq 2 = 2|0 - 1| \triangleq \mathsf{d}(0, 1) = \mathsf{d}\left(\lim_{n\to\infty} (\!(1 - {}^1\!/\!_n)\!)\,, \lim_{n\to\infty} (\!({}^1\!/\!_n)\!) \right) \end{aligned}$$

3. In (X, d), *open balls* are *open*:

[18] The distance function d and item (2) page 139 can in essence be found in Blumenthal (1953) page 9

(a) $\mathsf{p}(x,y) \triangleq |x-y|$ is a *metric* and thus all open balls in that do not contain both 0 and 1 are *open*.

(b) By Example 3.4 (page 49), $\mathsf{q}(x,y) \triangleq 2|x-y|$ is also a *metric* and thus all open balls containing 0 and 1 only are *open*.

(c) The only question remaining is with regards to open balls that contain 0, 1 and some other element(s) in $\mathbb{R}$. But even in this case, open balls are still open. For example:
$$\mathsf{B}(-1,2) = (-1:2) = (-1:1) \cup (1:2)$$
Note that both $(-1:1)$ and $(1:2)$ are *open*, and thus by Theorem 2.1 (page 28), $\mathsf{B}(-1,2)$ is *open* as well.

4. By item (3) and Theorem 2.2 (page 30), the *open balls* of $(\mathbb{R},\mathsf{d})$ *do* form a *base* for a *topology* on $\mathbb{R}$.

5. In (X,d), the limits of *convergent* sequences are *unique*. This is demonstrated in Example 13.3 (page 194) using additional structure developed in Chapter 13.

6. In (X,d), *convergent* sequences are *Cauchy*.
This is also demonstrated in Example 13.3 (page 194).

The *distance functions* in Example 9.3 (page 137)–Example 9.5 (page 139) were all *discontinuous*. In the absence of the *triangle inequality* and in light of these examples, one might try replacing the *triangle inequality* with the weaker requirement of *continuity*. However, as demonstrated by the next example, this also leads to an arguably disastrous result.

Example 9.6. [19] The function $\mathsf{d} \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that $\mathsf{d}(x,y) \triangleq (x-y)^2$ is a *distance* on $\mathbb{R}$.
Note some characteristics of the *distance space* $(\mathbb{R},\mathsf{d})$:

1. $(\mathbb{R},\mathsf{d})$ is *not* a *metric space* because the *triangle inequality* does not hold:
$$\mathsf{d}(0,2) \triangleq (0-2)^2 = 4 \not\leq 2 = (0-1)^2 + (1-2)^2 \triangleq \mathsf{d}(0,1) + \mathsf{d}(1,2)$$

2. The *distance function* d is *continuous* in (X,d). This is demonstrated in the more general setting of Chapter 13 in Example 13.4 (page 195).

3. Calculating the length of curves in (X,d) leads to a paradox:[20]

(a) Partition $[0:1]$ into 2^N consecutive line segments connected at the points
$$\left(\!\left(0,\ \tfrac{1}{2^N},\ \tfrac{2}{2^N},\ \tfrac{3}{2^N},\ \ldots,\ \tfrac{2^{N-1}1}{2^N},\ 1\right)\!\right)$$

(b) Then the distance, as measured by d, between any two consecutive points is
$$\mathsf{d}(p_n,p_{n+1}) \triangleq (p_n - p_{n+1})^2 = \left(\frac{1}{2^N}\right)^2 = \frac{1}{2^{2N}}$$

(c) But this leads to the paradox that the total length of $[0:1]$ is 0:
$$\lim_{N\to\infty}\sum_{n=0}^{2^N-1}\frac{1}{2^{2N}} = \lim_{N\to\infty}\frac{2^N}{2^{2N}} = \lim_{N\to\infty}\frac{1}{2^N} = 0$$

[19] Blumenthal (1953) pages 12–13, Laos (1998) pages 118–119
[20] This is the method of "inscribed polygons" for calculating the length of a curve and goes back to Archimedes: Brunschwig et al. (2003) page 26, Walmsley (1920) page 200 ⟨§158⟩,

9.4 Sequences in metric spaces

9.4.1 Cauchy sequences

Lemma 9.1. [21] *Let* $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *be a sequence in the metric space* (X, d).

LEM
$$\left\{\begin{array}{l}(\!(x_n)\!) \text{ is CAUCHY (Definition 9.5 page 134)} \\ \text{in } (X,\mathsf{d})\end{array}\right\} \implies \left\{\begin{array}{l}(\!(x_n)\!) \text{ is BOUNDED} \\ \text{in } (X,\mathsf{d})\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
(\!(x_n)\!) \text{ is } \textit{Cauchy} &\implies \text{for every } \varepsilon \in \mathbb{R}^+,\ \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n,m > N,\ \mathsf{d}(x_n, x_m) < \varepsilon && \text{(Definition 9.5 page 134)} \\
&\implies \exists N \in \mathbb{Z} \quad \text{such that} \quad \forall n,m > N,\ \mathsf{d}(x_n, x_m) < 1 \qquad \text{(arbitrarily choose } \varepsilon \triangleq 1) \\
&\implies \exists N \in \mathbb{Z} \quad \text{such that} \quad \forall n,m \in \mathbb{N},\ \mathsf{d}(x_n, x_{m+1}) < \max\left\{\{1\} \cup \left\{\mathsf{d}(x_p, x_q) \mid p,q \ngtr N\right\}\right\} \\
&\implies (\!(x_n)\!) \text{ is } \textit{bounded} \qquad \text{(by definition of } \textit{bounded})
\end{aligned}$$

✏

Proposition 9.4. [22] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *be a sequence in a metric space* (X, d). *Let* $\mathsf{f} : \mathbb{Z} \to \mathbb{Z}$ *be a strictly increasing function such that* $\mathsf{f}(n) < \mathsf{f}(n+1)$.

PRP
$$\underbrace{(\!(x_n)\!)_{n\in\mathbb{Z}} \text{ is CAUCHY}}_{\textit{Cauchy sequence}} \implies \underbrace{\left(\!\left(x_{\mathsf{f}(n)}\right)\!\right)_{n\in\mathbb{Z}} \text{ is CAUCHY}}_{\textit{subsequence is also Cauchy}}$$

✎PROOF:

$$\begin{aligned}
&(\!(x_n)\!)_{n\in\mathbb{Z}} \text{ is } \textit{Cauchy} \\
&\implies \text{for any given } \varepsilon > 0,\ \exists N \quad \text{such that} \quad \forall n,m > N,\ \mathsf{d}(x_n, x_m) < \varepsilon && \text{by Definition 9.5 page 134} \\
&\implies \text{for any given } \varepsilon > 0,\ \exists N' \quad \text{such that} \quad \forall \mathsf{f}(n), \mathsf{f}(m) > N',\ \mathsf{d}(x_{\mathsf{f}(n)}, x_{\mathsf{f}(m)}) < \varepsilon \\
&\implies \left(\!\left(x_{\mathsf{f}(n)}\right)\!\right)_{n\in\mathbb{Z}} \text{ is } \textit{Cauchy} && \text{by Definition 9.5 page 134}
\end{aligned}$$

✏

9.4.2 Convergence in Metric Space

Theorem 9.4. [23] *Let* (X, T) *be the* TOPOLOGICAL SPACE *induced by a metric space* (X, d). *Let* $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *be a sequence in* (X, d).

THM
$$\underbrace{(\!(x_n)\!) \text{ converges to a limit } x}_{\text{(Definition 9.3 page 132)}} \iff \left\{\begin{array}{l}\text{for any } \varepsilon \in \mathbb{R}^+, \text{ there exists } N \in \mathbb{N} \\ \text{such that for all } n > N, \\ \qquad \mathsf{d}(x_n, x) < \varepsilon\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
(\!(x_n)\!) \to x &\iff x_n \in U \quad \forall U \in N_x,\ n > N && \text{by Definition 9.3 page 132} \\
&\iff \exists \mathsf{B}(x,\varepsilon) \quad \text{such that} \quad x_n \in \mathsf{B}(x,\varepsilon)\, \forall n > N && \text{by Lemma 3.3 page 38} \\
&\iff \mathsf{d}(x_n, x) < \varepsilon && \text{by Definition 2.4 page 28}
\end{aligned}$$

[21] Giles (1987) page 49 ⟨Theorem 3.30⟩
[22] Rosenlicht (1968) page 52
[23] Rosenlicht (1968) page 45, Giles (1987) page 37 ⟨3.2 Definition⟩

A sequence that is *convergent* is always *Cauchy* (next theorem). However, in a metric space, the converse is not true—a sequence that is *convergent* is not in general *Cauchy*. This is in contrast to the special case of a real sequence in the metric space $(\mathbb{R}, |x-y|)$. In this case, all Cauchy sequences are convergent and the Cauchy property is referred to as the *Cauchy condition*.[24]

Theorem 9.5. [25] *Let $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ be a sequence in the metric space (X, d).*

THM
$$\left\{\begin{array}{l}(\!(x_n)\!) \text{ is CONVERGENT} \\ \text{in } (X,\mathsf{d})\end{array}\right\} \implies \left\{\begin{array}{l}(\!(x_n)\!) \text{ is CAUCHY} \\ \text{in } (X,\mathsf{d})\end{array}\right\} \implies \left\{\begin{array}{l}(\!(x_n)\!) \text{ is BOUNDED} \\ \text{in } (X,\mathsf{d})\end{array}\right\}$$

✎PROOF:

1. Proof that *convergent* $\implies$ *Cauchy*:

$$\begin{aligned}\mathsf{d}(x_n, x_m) &\le \mathsf{d}(x_n, x) + \mathsf{d}(x, x_m) && \text{by Definition 3.1 page 33 (triangle inequality)}\\ &< \varepsilon + \varepsilon && \text{by left hypothesis}\\ &= 2\varepsilon\end{aligned}$$

2. Proof that *Cauchy* $\implies$ *bounded*: by Lemma 9.1 (page 141).

Proposition 9.5. [26] *Let $(\!(x_n)\!)_{n\in\mathbb{Z}}$ be a sequence in a metric space (X, d). Let $\mathsf{f} : \mathbb{Z} \to \mathbb{Z}$ be a strictly increasing function such that $\mathsf{f}(n) < \mathsf{f}(n+1)$.*

PRP
$$\left\{\begin{array}{lll}1. & (\!(x_n)\!)_{n\in\mathbb{Z}} \text{ is CAUCHY} & \text{and}\\ 2. & \left(\!\left(x_{\mathsf{f}(n)}\right)\!\right)_{n\in\mathbb{Z}} \text{ is CONVERGENT} & \end{array}\right\} \implies (\!(x_n)\!)_{n\in\mathbb{Z}} \text{ is CONVERGENT.}$$

✎PROOF:

$$\begin{aligned}\mathsf{d}(x_n, x) &= \mathsf{d}(x, x_n)\\ &\le \underbrace{\mathsf{d}(x, x_{\mathsf{f}(n)})}_{<\varepsilon \text{ by left hypothesis 2}} + \underbrace{\mathsf{d}(x_{\mathsf{f}(n)}, x_n)}_{<\varepsilon \text{ by left hypothesis 1}}\\ &< \varepsilon + \varepsilon\\ &= 2\varepsilon\\ \implies & (\!(x_n)\!) \text{ is convergent.}\end{aligned}$$

Proposition 9.6. [27] *Let (X, d) be a METRIC SPACE. Let $(\mathbb{R}, \mathsf{p})$ be a metric space of real numbers with the usual metric $\mathsf{p}(x,y) \triangleq |x-y|$.*

PRP
$$\underbrace{(\!(x_n)\!) \to x \text{ and } (\!(y_n)\!) \to y}_{\text{convergence in } (X,\mathsf{d})} \implies \underbrace{\left(\!\left(\mathsf{d}(x_n, y_n)\right)\!\right) \to \mathsf{d}(x,y)}_{\text{convergence in } (\mathbb{R},\mathsf{p})} \qquad \forall x,y,(\!(x_n)\!)_{n\in\mathbb{Z}},(\!(y_n)\!)_{n\in\mathbb{Z}} \in (X,\mathsf{d})$$

[24] Whittaker (1915) pages 13–15 ⟨2.22⟩
[25] Giles (1987) page 49 ⟨Theorem 3.30⟩, Rosenlicht (1968) page 51, Apostol (1975) pages 72–73 ⟨Theorem 4.6⟩
[26] Rosenlicht (1968) page 52
[27] Berberian (1961) page 37 ⟨Theorem II.4.1⟩

✎PROOF:

$$\begin{aligned}
\mathsf{p}\left(\mathsf{d}(x,y),\mathsf{d}(x_n,y_n)\right) &\triangleq \left|\mathsf{d}(x,y)-\mathsf{d}(x_n,y_n)\right| \\
&\le \left[\mathsf{d}(x,x_n)+\mathsf{d}(x_n,y)\right]-\mathsf{d}(x_n,y_n) && \text{by triangle inequality page 33} \\
&\le \mathsf{d}(x,x_n)+\left[\mathsf{d}(x_n,y_n)+\mathsf{d}(y_n,y)\right]-\mathsf{d}(x_n,y_n) && \text{by triangle inequality page 33} \\
&= \mathsf{d}(x,x_n)+\mathsf{d}(y,y_n) && \text{by definition of metric (Definition 3.1 page 33)} \\
&< \varepsilon+\varepsilon && \text{by left hypothesis} \\
&= 2\varepsilon \\
\implies \mathsf{d}(x_n,y_n) &\to \mathsf{d}(x,y)
\end{aligned}$$

✏

Theorem 9.6 (next) demonstrates that, in a *metric space* (Definition 3.1 page 33), if a sequence *converges* (Definition 9.3 page 132), then the limit it converges to is *unique*—a sequence cannot converge to more than one limit (in a metric space). This is in contrast to the more general topological spaces where a sequence *can* converge to more than one limit (Example 9.1 page 132).

Theorem 9.6 (Uniqueness of limit). [28] *Let* (X,d) *be a* METRIC SPACE. *Let* $x, y, \in X$ *and let* $(\!(x_n)\!)$ *be an* X*-valued sequence.*

THM

$$\underbrace{\left\{(\!(x_n)\!) \to x \text{ and } (\!(x_n)\!) \to y\right\} \quad\implies\quad \{x=y\}}_{\text{the LIMIT of a CONVERGENT sequence is UNIQUE}}$$

✎PROOF:

1. Proof that $\mathsf{d}(x,y) < 2\varepsilon$ for arbitrarily small $\varepsilon > 0$:

$$\begin{aligned}
(\!(x_n)\!) \to x \text{ and } (\!(x_n)\!) \to y &\implies \exists N \text{ such that } \forall n > N,\ \mathsf{d}(x,x_n) < \varepsilon \text{ and } \mathsf{d}(x_n,y) < \varepsilon \\
&\implies \exists N \text{ such that } \forall n > N,\ \mathsf{d}(x,y) = \mathsf{d}(x,x_n)+\mathsf{d}(y_n,y) < 2\varepsilon \\
&\implies \exists N \text{ such that } \forall n > N,\ \mathsf{d}(x,y) < 2\varepsilon \\
&\implies \mathsf{d}(x,y) < 2\varepsilon
\end{aligned}$$

2. Proof that $\mathsf{d}(x,y) = 0$:
   (a) If $\mathsf{d}(x,y) > 0$, then we could choose an arbitrarily small ε such that
   $$\mathsf{d}(x,y) > 2\varepsilon.$$
   (b) But this would contradict the earlier result of $\mathsf{d}(x,y) < 2\varepsilon$.
   (c) Therefore, $\mathsf{d}(x,y) = 0$ (proof by contradiction).
3. Therefore, $x = y$ because by the definition of metrics (Definition 3.1 page 33),
   $$\mathsf{d}(x,y) = 0 \iff x = y.$$

✏

Proposition 9.7. [29] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *be a sequence in a metric space* (X,d). *Let* $\mathsf{f} : \mathbb{Z} \to \mathbb{Z}$ *be a strictly increasing function such that* $\mathsf{f}(n) < \mathsf{f}(n+1)$.

PRP

$$\underbrace{(\!(x_n)\!)_{n\in\mathbb{Z}} \to x}_{\text{sequence converges to limit } x} \quad\implies\quad \underbrace{\left(\!\left(x_{\mathsf{f}(n)}\right)\!\right)_{n\in\mathbb{Z}} \to x}_{\text{subsequence converges to the same limit } x}$$

[28] Rosenlicht (1968) page 46, Thomson et al. (2008) page 32 ⟨Theorem 2.8⟩
[29] Rosenlicht (1968) page 46

✎PROOF:

$$\begin{aligned}
(\!(x_n)\!)_{n\in\mathbb{Z}} \to x &\implies \forall\varepsilon>0,\ \exists N \quad \text{such that} \quad \forall n>N,\ \mathsf{d}(x_n,x)<\varepsilon && \text{by Theorem 9.4 page 141}\\
&\implies \forall\varepsilon>0,\ \exists \mathsf{f}(N) \quad \text{such that} \quad \forall \mathsf{f}(n)>\mathsf{f}(N),\ \mathsf{d}(x_{\mathsf{f}(n)},x)<\varepsilon\\
&\implies (\!(x_{\mathsf{f}(n)})\!)_{n\in\mathbb{Z}} \to x && \text{by Theorem 9.4 page 141}
\end{aligned}$$

✏

9.4.3 Complete metric spaces

Even though a convergent sequence is always Cauchy, not all Cauchy sequences are convergent. That is, the points of a sequence may diverge less and less from each other as $n \to \infty$ (Cauchy sequence), they still may not converge to a single point (which if they did they would be a convergent sequence). However, if all the Cauchy sequences in a *metric space* do converge, and they all converge to points inside the metric space, then that *metric space* is called a *complete metric space*.

Definition 9.8. [30] *Let* (X,d) *be a* METRIC SPACE *(Definition 3.1 page 33).*

DEF

A sequence $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ *is **complete** in* (X,d) *if*

$$\underbrace{(\!(x_n)\!) \text{ is CAUCHY in } (X,\mathsf{d}) \implies (\!(x_n)\!) \text{ is convergent in } (X,\mathsf{d}).}_{\text{every CAUCHY SEQUENCE in } (X,\mathsf{d}) \text{ CONVERGES to a limit in } (X,\mathsf{d})}$$

Theorem 9.7. [31] *Let* (X,d) *be a* METRIC SPACE *(Definition 3.1 page 33). Let A be a subset of X. Let A^- be the* CLOSURE *(Definition 1.4 page 14) of A in* (X,d).

THM

$$\{(A,\mathsf{d}) \text{ is COMPLETE}\} \implies \underbrace{A \text{ is CLOSED in } (X,\mathsf{d})}_{A=A^-}$$

$$\left\{\begin{array}{lll} 1. & (X,\mathsf{d}) \text{ is COMPLETE (Definition 9.6 page 134)} & \text{and} \\ 2. & A \text{ is CLOSED in } (X,\mathsf{d}) & (A=A^-) \end{array}\right\} \implies \{(A,\mathsf{d}) \text{ is COMPLETE}\}$$

✎PROOF:

1. Proof that *complete* $\implies$ *closed*:
   (a) Proof that $A \subseteq A^-$: Lemma 1.1 page 15
   (b) Proof that $A^- \subseteq A$ (proof that $x \in A^- \implies x \in A$): by Theorem 9.2 page 135
2. Proof that *complete* and *closed* $\implies$ *complete*:
   (a) By left hypothesis 2, A is closed in (X,d).
   (b) By Theorem 9.1 (page 132) and because A is closed in (X,d), sequences converge in A.
   (c) Therefore by Definition 9.6 (page 134), (A,d) is complete.

✏

[30] Rosenlicht (1968) page 52
[31] Kubrusly (2001) page 128 ⟨Theorem 3.40⟩, Haaser and Sullivan (1991) page 75 ⟨6·10, 6·11 Propositions⟩, Bryant (1985) page 40 ⟨Theorem 3.6, 3.7⟩, Sutherland (1975) pages 123–124

Corollary 9.1. [32] *Let* (X, d) *be a* METRIC SPACE *(Definition 3.1 page 33). Let* A *be a subset of* X*. Let* A^- *be the* CLOSURE *(Definition 1.4 page 14) of* A *in* (X, d).

COR: $$\{(A, \mathsf{d}) \text{ is COMPLETE}\} \iff \underbrace{A \text{ is CLOSED in } (X, \mathsf{d})}_{A = A^-}$$

✎PROOF: Note that in this corollary, the metric space (X, d) is assumed to be *complete*.

1. Proof that *complete* $\Longrightarrow$ *closed*: by Theorem 9.7 (1).
2. Proof that *complete* $\Longleftarrow$ *closed*: by *complete* hypothesis and Theorem 9.7 (2).

✏

Example 9.7. Let $\mathbb{Q}$ be the set of *rational numbers*.

EX: The metric space $(\mathbb{Q}, \mathsf{d}(x, y) = |x - y|)$ is *not complete.*

✎PROOF: Let $(\!(x_n)\!)_{n\in\mathbb{W}}$ be the sequence of values approximating π truncated to n decimal points:

$$(\!(x_n)\!)_{n\in\mathbb{W}} \triangleq (\!(3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ 3.141592,\ \ldots)\!)$$

This is a Cauchy sequence. However, this sequence (and all sequences converging to an irrational number) does not converge to a rational number ($\mathbb{Q}$) and thus is not in the metric space $(\mathbb{Q}, \mathsf{d})$ and thus $(\mathbb{Q}, \mathsf{d})$ is *not complete.* ✏

Example 9.8 (Cauchy's convergence criterion/Cauchy's criterion). [33] Let $(\!(r_n \in \mathbb{R})\!)_{n\in\mathbb{Z}}$ be a **real** sequence.

EX: The metric space $\left((\!(r_n)\!),\ |r_n - r_m|\right)$ is *complete.*

Theorem 9.8 (Cantor intersection theorem). [34] *Let* (X, d) *be a complete* METRIC SPACE, $(\!(A_n)\!)_{n\in\mathbb{Z}}$ *a sequence with each* $A_n \in 2^X$*, and* $|\,A\,|$ *the number of elements in* A.

THM:
$$\left.\begin{array}{llll} 1. & (X, \mathsf{d}) \text{ is COMPLETE} & & \text{and} \\ 2. & A_n \text{ is CLOSED} & \forall n\in\mathbb{N} & \text{and} \\ 3. & \operatorname{diam} A_{n+1} \le \operatorname{diam} A_n & \forall n\in\mathbb{N} & \text{and} \\ 4. & \operatorname{diam} A_n \to 0 & & \end{array}\right\} \Longrightarrow \left\{\left|\bigcap_{n\in\mathbb{N}} A_n\right| = 1\right\}$$

✎PROOF: By Theorem 9.3 page 136 ✏

9.5 Sequences on normed linear spaces

9.5.1 Convergence in normed linear spaces

Theorem 9.4 (page 141) defines convergence in a general *metric space* (Definition 3.1 page 33). All *normed linear spaces* (Definition 6.1 page 87) are metric spaces, so they inherit this definition with the *metric induced by the norm* (Definition 6.2 page 90).

Definition 9.9. [35] *Let* $\left(X,\ +,\ \cdot,\ (\mathbb{F},\ \dot{+},\ \dot{\times}),\ \|\cdot\|\right)$ *be a normed linear space. Let the metric* d *be defined*

[32] Kubrusly (2001) page 128 ⟨Corollary 3.41⟩
[33] Sohrab (2003) page 54 ⟨Theorem 2.2.5⟩
[34] Davis (2005) page 28, Hausdorff (1937) page 150
[35] Bachman and Narici (1966) page 247, Katznelson (2004) page 67 ⟨section 1.1⟩

as $\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \triangleq \|\boldsymbol{x}-\boldsymbol{y}\|$.

DEF *A sequence* $(\!(\boldsymbol{x}_n \in X)\!)_{n\in\mathbb{Z}}$ ***converges in norm*** *or* ***converges strongly*** *to the* ***limit*** $\boldsymbol{x}$ *if* $(\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{Z}}$ *converges to the limit* $\boldsymbol{x}$ *in the metric space* (X,d). *That is, a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ ***converges strongly*** *in the normed linear space* $\left(X,\,+,\,\cdot,\,(\mathbb{F},\,\dot{+},\,\dot{\times}),\,\|\cdot\|\right)$ *to the* ***limit*** $\boldsymbol{x} \in X$ *if for any* $\varepsilon \in \mathbb{R}^+$ *there exists* $N \in \mathbb{Z}$ *such that*

$$\left\|\boldsymbol{x}_n - \boldsymbol{x}\right\| < \varepsilon \quad \forall n > N.$$

This mode of convergence is called ***strong convergence***.

Definition 9.10. [36] *Let* $\left(X,\,+,\,\cdot,\,(\mathbb{F},\,\dot{+},\,\dot{\times}),\,\|\cdot\|\right)$ *be a normed linear space . Let the metric* d *be defined as* $\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \triangleq \|\boldsymbol{x}-\boldsymbol{y}\|$.

DEF *A sequence* $(\!(\boldsymbol{x}_n \in X)\!)_{n\in\mathbb{Z}}$ ***converges weakly*** *to the* ***limit*** $\boldsymbol{x}$ *if for every functional* $\mathsf{f} \in \mathbb{F}^{\boldsymbol{X}}$, $(\!(\mathsf{f}(\boldsymbol{x}_n))\!)_{n\in\mathbb{Z}}$ *converges to the limit* $\mathsf{f}(\boldsymbol{x})$ *in the metric space* (X,d). *That is, a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ ***converges weakly*** *in the normed linear space* $\left(X,\,+,\,\cdot,\,(\mathbb{F},\,\dot{+},\,\dot{\times}),\,\|\cdot\|\right)$ *to the* ***limit*** $\mathsf{f}(\boldsymbol{x})$ *if for every functional* $\mathsf{f} \in \mathbb{F}^{\boldsymbol{X}}$ *and for any* $\varepsilon \in \mathbb{R}^+$ *there exists* $N \in \mathbb{Z}$ *such that*

$$\left\|\mathsf{f}(\boldsymbol{x}_n) - \mathsf{f}(\boldsymbol{x})\right\| < \varepsilon \quad \forall n > N.$$

This mode of convergence is called ***weak convergence***.

9.5.2 Bounded sequences

9.5.3 Complete normed linear spaces

“*At that time, however, the theory seemed to me to contain for the immediate future nothing but some decades of rather formal and thin work. By this I do not mean to reproach the work of Banach himself but that of the many inferior writers, hungry for easy doctors' theses, who were drawn to it. As I foresaw, it was this class of writers that was first attracted to the theory of Banach spaces.*”

Norbert Wiener (1894–1964), American mathematician [37]

Theorem 9.9. [38] *Let* $(X,\boldsymbol{T})$ *and* $(Y,\boldsymbol{S})$ *be a* TOPOLOGICAL SPACES. *Let* f *be a function in* $(Y,\boldsymbol{S})^{(X,\boldsymbol{T})}$.

THM
$$\underbrace{\left\{\begin{array}{c}\mathsf{f} \textit{ is } \text{CONTINUOUS } \textit{in } (Y,\boldsymbol{S})^{(X,\boldsymbol{T})}\\ \text{(Definition 1.8 page 23)}\end{array}\right\}}_{\text{INVERSE IMAGE CHARACTERIZATION OF CONTINUITY}} \iff \underbrace{\left\{\begin{array}{c}(\!(x_n)\!) \to x \implies \mathsf{f}((\!(x_n)\!)) \to \mathsf{f}(x)\\ \text{(Definition 9.3 page 132)}\end{array}\right\}}_{\text{SEQUENTIAL CHARACTERIZATION OF CONTINUITY}}$$

✎PROOF:

1. Proof for the $\implies$ case (proof by contradiction):

   (a) Let U be an *open set* in $(Y,\boldsymbol{T})$ that contains $\mathsf{f}(x)$ but for which there exists no N such that $\mathsf{f}(x_n) \in U$ for all $n > N$.

   (b) Note that the set $\mathsf{f}^{-1}(U)$ is also *open* by the *continuity* hypothesis.

---

[36] Bachman and Narici (1966) page 231 ⟨Definition 14.1⟩

[37] quote: Wiener (1956) pages 63–64, Werner (2004) page 41
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Wiener_Norbert.html

[38] Ponnusamy (2002) pages 94–96 ⟨“2.59. Proposition.”; in the context of *metric spaces*; includes the “*inverse image characterization of continuity*” and “*sequential characterization of continuity*” terminology; this terminology does not seem to be widely used in the literature in general, but has been adopted for use in this text⟩

(c) If $(\!(x_n)\!) \to x$, then

$$\begin{aligned}
&\mathsf{f}((\!(x_n)\!)) \nrightarrow \mathsf{f}(x) \\
&\implies \text{there exists no } N \text{ such that } \mathsf{f}(x_n) \in U \text{ for all } n > N && \text{by Definition 9.3 (page 132)} \\
&\implies \text{there exists no } M \text{ such that } x_n \in \mathsf{f}^{-1}(U) \text{ for all } n > M && \text{by definition of } \mathsf{f}^{-1} \\
&\implies (\!(x_n)\!) \nrightarrow x \quad \text{by \textit{continuity} hypothesis and def. of \textit{convergence} (Definition 9.3 page 132)} \\
&\implies \text{contradiction of } (\!(x_n)\!) \to x \text{ hypothesis} \\
&\implies \mathsf{f}((\!(x_n)\!)) \to \mathsf{f}(x)
\end{aligned}$$

2. Proof for the $\Longleftarrow$ case (proof by contradiction):

(a) Let D be a *closed* set in $(Y, \boldsymbol{S})$.

(b) Suppose $\mathsf{f}^{-1}(D)$ is *not closed*...

(c) then by the *closed set theorem* (Theorem 9.1 page 132), there must exist a *convergent* sequence $(\!(x_n)\!)$ in $(X, \boldsymbol{T})$, but with limit x *not* in $\mathsf{f}^{-1}(D)$.

(d) Note that $\mathsf{f}(x)$ must be in D. Proof:

i. by definition of D and f, $\mathsf{f}((\!(x_n)\!))$ is in D

ii. by left hypothesis, the sequence $\mathsf{f}((\!(x_n)\!))$ is *convergent* with limit $\mathsf{f}(x)$

iii. by *closed set theorem* (Theorem 9.1 page 132), $\mathsf{f}(x)$ must be in D.

(e) Because $\mathsf{f}(x) \in D$, it must be true that $x \in \mathsf{f}^{-1}(D)$.

(f) But this is a contradiction to item (2c) (page 147), and so item (2b) (page 147) must be wrong, and $\mathsf{f}^{-1}(D)$ must be *closed*.

(g) And so by Theorem 1.21 (page 24), f is *continuous*.

✎

Definition 9.11. [39] *Let* $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be a normed linear space.*

DEF: *The space* NORMED LINEAR SPACE $\boldsymbol{\Omega}$ *is a* ***Banach space*** *if it is* COMPLETE *with respect to the metric* $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{y} - \boldsymbol{x}\|$.

9.5.4 The l_p spaces

Definition 9.12. [40] *Let* $(\!(x_n \in \mathbb{R})\!)_{n\in\mathbb{Z}}$ *be a real sequence.*

DEF: *The space* $\ell^p_{\mathbb{F}}$ *and space* $\ell^\infty_{\mathbb{F}}$ *are defined as*

$$\begin{aligned}
\ell^p_{\mathbb{F}} &\triangleq \left\{ (\!(x_n)\!)_{n\in\mathbb{Z}} \;\middle|\; \sum_{n\in\mathbb{Z}} |x_n|^p < \infty \right\} && \forall 1 \le p < \infty \\
\ell^\infty_{\mathbb{F}} &\triangleq \left\{ (\!(x_n)\!)_{n\in\mathbb{Z}} \;\middle|\; \sup (\!(x_n)\!) < \infty \right\}
\end{aligned}$$

Lemma 9.2. *Let* $(\!(x_n \in \mathbb{F})\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n \in \mathbb{F})\!)_{n\in\mathbb{Z}}$ *be sequences over a field* $\mathbb{F}$.

LEM: $$(\!(x_n)\!), (\!(y_n)\!) \in \ell^p_{\mathbb{F}} \quad\implies\quad \big((\!(x_n)\!) + (\!(y_n)\!)\big) \in \ell^p_{\mathbb{F}} \qquad \forall p \in [1 : \infty]$$

✎PROOF:

[39] Bachman and Narici (1966) page 112 ⟨Definition 8.1⟩, Banach (1932a) page 53 ⟨"espace du type (B)" (space type (B))⟩, Banach (1932b) page 33

[40] Carothers (2000) page 44

1. Proof for $p = 1$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |x_n + y_n|^1 &= \sum_{n\in\mathbb{Z}} |x_n + y_n| \\
&\le \sum_{n\in\mathbb{Z}} \left(|x_n| + |y_n|\right) && \text{by norm properties of } \|\cdot\| \\
&= \sum_{n\in\mathbb{Z}} |x_n|^1 + \sum_{n\in\mathbb{Z}} |y_n|^1 \\
&< \infty && \text{because } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\ell}^1_{\mathbb{F}}
\end{aligned}$$

2. Proof for $1 < p < \infty$: Let $\|\boldsymbol{x}\|_p \triangleq \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{1}{p}}$

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |x_n + y_n|^p &= \Bigg(\underbrace{\left(\sum_{n\in\mathbb{Z}} |x_n + y_n|^p\right)^{\frac{1}{p}}}_{\|\cdot\|_p}\Bigg)^p \\
&= \|\boldsymbol{x} + \boldsymbol{y}\|_p^p && \text{by definition of } \|\cdot\|_p \text{ page 148} \\
&\le \left(\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p\right)^p && \text{by Minkowski's inequality page 169} \\
&\le \infty && \text{by } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\ell}^p_{\mathbb{F}} \text{ hypothesis}
\end{aligned}$$

3. Proof for $p = \infty$:

$$\begin{aligned}
\sup\left\{x_n + y_n | n \in \mathbb{Z}\right\} &\le \sup\left(\!\left(x_n\right)\!\right) + \sup\left(\!\left(y_n\right)\!\right) \\
&\le \infty && \text{by } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\ell}^p_{\mathbb{F}} \text{ hypothesis}
\end{aligned}$$

✏

Definition 9.13. *Let $(\!(x_n)\!)_{n\in\mathbb{Z}}$ be a sequence in the space $\boldsymbol{\ell}^p_{\mathbb{F}}$.*

DEF: *The $\boldsymbol{\ell}^p_{\mathbb{F}}$* ***norm*** *$\left\|(\!(x_n)\!)\right\|_p$ of $(\!(x_n)\!)$ is defined as* $\left\|(\!(x_n)\!)\right\|_p \triangleq \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{1}{p}}$ *for $p \in [1 : \infty]$*

Proposition 9.8. *Let $\left\|(\!(x_n)\!)\right\|_p$ be the $\boldsymbol{\ell}^p_{\mathbb{F}}$ norm of a sequence $(\!(x_n)\!)_{n\in\mathbb{Z}}$ in the space $\boldsymbol{\ell}^p_{\mathbb{F}}$.*

PRP: *$\left\|(\!(x_n)\!)\right\|_p$ is a norm.*

✎PROOF:

Proof that $\|\cdot\|_p \ge 0$:

$$\begin{aligned}
\|\boldsymbol{x}\| &\triangleq \left(\sum_{n\in\mathbb{Z}} |x_n|\right)^{\frac{1}{p}} && \text{by definition of } \|\cdot\| \\
&\ge \sum_{i=1}^{n} 0 \\
&= 0
\end{aligned}$$

Proof that $\|\boldsymbol{x}\| = 0 \implies \boldsymbol{x} = \mathbb{0}$:

$$\begin{aligned}
0 &= \|\boldsymbol{x}\| && \text{by left hypothesis}\\
&= \sum_{i=1}^{n} |x_i| && \text{by definition of } \|\cdot\|\\
&\implies x_i = 0 \quad i = 1, 2, \ldots, n\\
&\implies \boldsymbol{x} = \mathbb{0} && \text{by definition of } \boldsymbol{x}
\end{aligned}$$

Proof that $\|\boldsymbol{x}\| = 0 \impliedby \boldsymbol{x} = \mathbb{0}$:

$$\begin{aligned}
\|\boldsymbol{x}\| &= \sum_{i=1}^{n} |x_i| && \text{by definition of } \|\cdot\|\\
&= \sum_{i=1}^{n} |0| && \text{by right hypothesis}\\
&= 0
\end{aligned}$$

Proof that $\|\alpha\boldsymbol{x}\| = |\alpha|\,\|\boldsymbol{x}\|$:

$$\begin{aligned}
\|\alpha\boldsymbol{x}\| &= \sum_{i=1}^{n} |\alpha x_i| && \text{by definition of } \|\cdot\|\\
&= \sum_{i=1}^{n} |\alpha||x_i|\\
&= |\alpha| \sum_{i=1}^{n} |x_i|\\
&= |\alpha|\,\|\boldsymbol{x}\| && \text{by definition of } \boldsymbol{x}
\end{aligned}$$

Proof that $\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$: by *Minkowski's Inequality* (Theorem 11.5 page 169)

✏

9.6 Complete inner-product spaces

Definition 9.14. [41] *Let* $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \triangledown \rangle\big)$ *be an inner-product space.*[42]

DEF: *The inner-product space* $\boldsymbol{\Omega}$ *is a* ***Hilbert space*** *if it is* COMPLETE *with respect to the metric* $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\| \triangleq \sqrt{\langle \boldsymbol{x} - \boldsymbol{y} \,|\, \boldsymbol{x} - \boldsymbol{y} \rangle}$.

Theorem 9.10 (Complemented-subspace theorem). [43] *Let* $\boldsymbol{B} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be a Banach space.*

THM: $\left\{ \begin{array}{l} \textit{Every closed linear subspace } \boldsymbol{D} \textit{ in } \boldsymbol{B} \\ \textit{has a complement } \boldsymbol{D}^c \textit{ in } \boldsymbol{B} \end{array} \right\} \implies \left\{ \begin{array}{l} \boldsymbol{B} \textit{ is isomorphic} \\ \textit{to a Hilbert space } \boldsymbol{H} \end{array} \right\}$

[41] von Neumann (1929) page 55, "Den abstrakten hilbertschen raum nennen wir $\mathscr{H}$" ("we call the abstract Hilbert space $\mathscr{H}$"), Aliprantis and Burkinshaw (1998) page 288
[42] *complete*: Definition 9.8 page 144
[43] Lindenstrauss and Tzafriri (1971), Day (1973) page 157

9.7 Sequences of functions

For a sequence of real numbers in a metric space, the concept of convergence is well defined and unambiguous (Theorem 9.4 page 141). But for sequences of functions $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle_{n\in\mathbb{Z}}$, on the other hand, there are several different types or "modes" of convergence. Two of the most common modes are *pointwise convergence* (Definition 9.15 page 150) and *uniform convergence* (Definition 9.16 page 150). Both of these are defined in a metric space. In both of these, the value N beyond which the sequence becomes sufficiently "close" to the limit $\mathsf{f}(x)$ depends on a distance parameter ε. The difference between the two modes is that in pointwise convergence, the value N also depends on the value x of the limit $\mathsf{f}(x)$; whereas in uniform convergence, the value N does not depend on x.

Definition 9.15. [44] *Let* $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle_{n\in\mathbb{Z}}$ *be a sequence of functions in a* METRIC SPACE (X, d).

DEF *The sequence* $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle$ ***converges pointwise*** *to a* ***limit*** $\mathsf{f}(x)$ *if for each* $\varepsilon \in \mathbb{R}^+$ *and for each* $x \in X$ *there exists an* $N \in \mathbb{N}$ *(dependent on* x*) such that*

$$\mathsf{d}\big(\mathsf{f}_n(x), \mathsf{f}(x)\big) < \varepsilon.$$

Definition 9.16. [45] *Let* $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle_{n\in\mathbb{Z}}$ *be a sequence of functions in a* METRIC SPACE (X, d).

DEF *The sequence* $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle$ ***converges uniformly*** *to a* ***limit*** $\mathsf{f}(x)$ *if for each* $\varepsilon \in \mathbb{R}^+$ *there exists an* $N \in \mathbb{N}$ *(independent of* x*) such that*

$$\mathsf{d}\big(\mathsf{f}_n(x), \mathsf{f}(x)\big) < \varepsilon \quad \text{for all } x \in X.$$

Theorem 9.11. *Let* $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle_{n\in\mathbb{Z}}$ *be a sequence of functions in a* METRIC SPACE (X, d).

THM $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle$ CONVERGES UNIFORMLY $\implies$ $\langle\!\langle \mathsf{f}_n(x) \rangle\!\rangle$ CONVERGES POINTWISE

✎PROOF: This follows directly from the definition of *uniform convergence* (Definition 9.16 page 150) and the definition of *pointwise convergence* (Definition 9.15 page 150). ✑

[44] Tao (2011) pages 94–96 ⟨section 1.5⟩, Thomson et al. (2008) page 368 ⟨Definition 9.3⟩, Tao (2010) page 117 ⟨Example 1.9.3⟩

[45] Tao (2011) pages 94–96 ⟨section 1.5⟩, Thomson et al. (2008) pages 373–374, Tao (2010) page 117 ⟨Example 1.9.4⟩

CHAPTER 10

INTERVALS AND CONVEXITY

10.1 Intervals

In the real number system, for $a \le b$, the *interval* $[a : b]$ is the set a and b and all the numbers inbetween, as in $[a : b] \triangleq \{x \in \mathbb{R} | a \le x \le b\}$. This concept can be easily generalized:

- In an **ordered set** (Definition B.2 page 286), if two elements x and y are *comparable* and $x \le y$, then we say that x and y and all the elements inbetween, as determined by the ordering relation $\le$, are the interval $[a : b]$ (Definition 10.1 page 151).
- In a **lattice** (Definition C.3 page 301), the concept of the *interval* can be generalized even further. In an arbitrary ordered set, the interval $[x : y]$ of item (10.1) is restricted to the case in which x and y are *comparable* (Definition B.2 page 286). This restriction can be lifted (Definition 10.2 page 151) with the additional structure of upper and lower bounds provided by lattices.
- A **metric space** (Definition 3.1 page 33) in general has no *order relation* $\le$ (Definition B.2 page 286). But intervals can still be defined (Definition 10.4 page 152) in a metric space in terms of the *triangle inequality*.
- A **linear space** (Definition 4.1 page 71) over a real or complex field in general has no *order relation* that compares *vectors* in the space, but the standard order relation $\le$ for real numbers $\mathbb{R}$ can still be used (Definition 10.5 page 152) to define an interval in a linear space.

Definition 10.1 (intervals on ordered sets). [1] *Let $(X, \le)$ be an* ORDERED SET *(Definition B.2 page 286).*

DEF				
	The set $[x : y] \triangleq \{z \in X	x \le z \le y\}$	*is called a* ***closed interval***	*and*
	The set $(x : y] \triangleq \{z \in X	x < z \le y\}$	*is called a* ***half-open interval***	*and*
	The set $[x : y) \triangleq \{z \in X	x \le z < y\}$	*is called a* ***half-open interval***	*and*
	The set $(x : y) \triangleq \{z \in X	x < z < y\}$	*is called an* ***open interval***.	

Definition 10.2 (intervals on lattices). [2] *Let $(X, \vee, \wedge; \le)$ be a* LATTICE *(Definition C.3 page 301).*

DEF		
	The set $[x : y] \triangleq \{z \in X | x \wedge y \le z \le x \vee y\}$	*is called a* ***closed interval***.
	The set $(x : y] \triangleq \{z \in X | x \wedge y < z \le x \vee y\}$	*is called a* ***half-open interval***.
	The set $[x : y) \triangleq \{z \in X | x \wedge y \le z < x \vee y\}$	*is called a* ***half-open interval***.
	The set $(x : y) \triangleq \{z \in X | x \wedge y < z < x \vee y\}$	*is called an* ***open interval***.

[1] Apostol (1975) page 4, Ore (1935) page 409

[2] Duthie (1942) page 2, Ore (1935) page 425 ⟨*quotient structures*⟩

When x and y *are* comparable and $x \le y$, then Definition 10.2 (previous) simplifies to item (10.1) (page 151).

Definition 10.3. [3] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ *be a* LATTICE *with dual* $\mathbf{L}^*$. *Let* $[x : y]$ *be a* CLOSED INTERVAL (Definition 10.2 page 151) *on set* X. *The sublattices* $\mathbf{L}[x : y]$ *and* $\mathbf{L}^*[x : y]$ *are defined as follows:*

DEF
$$\begin{aligned} \mathbf{L}[x : y] &\triangleq \{z \in \mathbf{L} | z \in [x : y]\} && \forall x,y \in X \\ \mathbf{L}^*[x : y] &\triangleq \left\{z \in \mathbf{L}^* | z \in [x : y]\right\} && \forall x,y \in X \end{aligned}$$

Definition 10.4. [4]

DEF
In a METRIC SPACE (X, d) (Definition 3.1 page 33),
the set $[a : b]$ *is the* ***closed interval*** *from* x *to* y *and is defined as*
$$[x : y] \triangleq \{z \in X | \mathsf{d}(x,z) + \mathsf{d}(z,y) = \mathsf{d}(x,y)\}.$$
An element $z \in X$ *is* ***geodesically between*** x *and* y *if* $z \in [x : y]$.

Definition 10.5. [5]

DEF
In a LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ (Definition 4.1 page 71),
$$\begin{aligned} \left[\boldsymbol{x} : \boldsymbol{y}\right] &\triangleq \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} | 0 \le \lambda \le 1\} && \text{is called a \textbf{closed interval}} && \text{and} \\ \left(\boldsymbol{x} : \boldsymbol{y}\right] &\triangleq \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} | 0 < \lambda \le 1\} && \text{is called a \textbf{half-open interval}} && \text{and} \\ \left[\boldsymbol{x} : \boldsymbol{y}\right) &\triangleq \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} | 0 \le \lambda < 1\} && \text{is called a \textbf{half-open interval}} && \text{and} \\ \left(\boldsymbol{x} : \boldsymbol{y}\right) &\triangleq \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} | 0 < \lambda < 1\} && \text{is called an \textbf{open interval}.} \end{aligned}$$

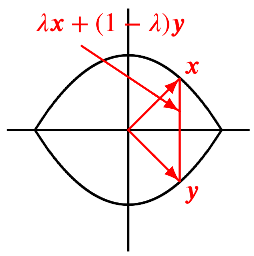
10.2 Convex sets

Using the concept of the *interval* (previous section), we can define the *convex set* (next definition).

Definition 10.6. [6] *Let* X *be a* SET *in an* ORDERED SET $(X, \le)$, *a* LATTICE $(X, \vee, \wedge; \le)$, *a* METRIC SPACE (X, d), *or a* LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

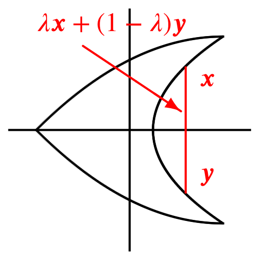
DEF
A subset $D \subseteq X$ *is a* ***convex set*** *in* X *if*
$$x, y \in D \quad \Longrightarrow \quad [x : y] \subseteq D.$$
A set that is ***not*** *convex is* ***concave***.

Example 10.1. Consider the *Euclidean space* $\mathbb{R}^2$ (a special case of a *linear space*).



$\Leftarrow$ $\left\{\text{The figure to the left is a } \textit{convex} \text{ set in } \mathbb{R}^2.\right.$

$\left.\text{The figure to the right is a } \textit{concave} \text{ set in } \mathbb{R}^2.\right\}$ $\Rightarrow$



Example 10.2. In a *metric space* (Definition 3.1 page 33), examples of *convex sets* are convex *balls*. Examples include those balls generated by the following metrics:

- Taxi-cab metric — Example 3.22 — page 62
- Euclidean metric — Example 3.23 — page 62
- Sup metric — Example 3.24 — page 63
- Tangential metric — Example 3.28 — page 67

[3] Maeda and Maeda (1970) page 1
[4] van de Vel (1993) page 8
[5] Barvinok (2002) page 2
[6] Barvinok (2002) page 5

Examples of metrics generating balls which are *not* convex include the following:

- Parabolic metric Example 3.25 page 63
- Exponential metric Example 3.27 page 66

10.3 Convex functions

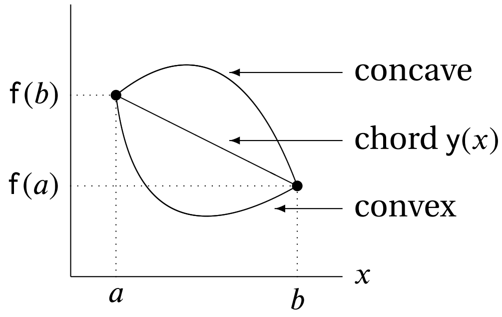


Figure 10.1: Convex and concave functions

Definition 10.7. [7] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition 4.1 page 71) *and* D *a* CONVEX SET (Definition 10.6 page 152) *in* X.

DEF

A function $\mathsf{f} \in F^D$ *is* ***convex*** *if*

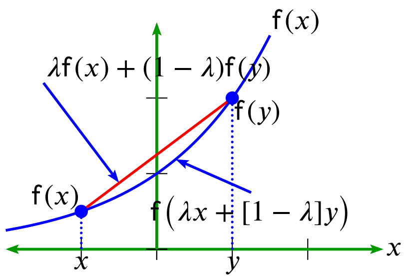
$$\mathsf{f}(\lambda \boldsymbol{x} + [1-\lambda]\boldsymbol{y}) \le \lambda \mathsf{f}(\boldsymbol{x}) + (1-\lambda)\,\mathsf{f}(\boldsymbol{y}) \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in D \text{ and } \forall \lambda \in (0,1)$$

A function $\mathsf{g} \in F^D$ *is* ***strictly convex*** *if*

$$\mathsf{g}(\lambda \boldsymbol{x} + [1-\lambda]\boldsymbol{y}) = \lambda \mathsf{g}(\boldsymbol{x}) + (1-\lambda)\,\mathsf{g}(\boldsymbol{y}) \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in D,\ \boldsymbol{x} \ne \boldsymbol{y},\ \text{and } \forall \lambda \in (0,1)$$

A function $\mathsf{f} \in F^D$ *is* ***concave*** *if* $-\mathsf{f}$ *is* CONVEX.
A function $\mathsf{f} \in F^D$ *is* ***affine*** *if* f *is* CONVEX *and* CONCAVE.

Example 10.3. The function $\mathsf{f}(x) = 2^x$ is a *convex function* (Definition 10.7 page 153), as illustrated to the right.



Definition 10.8. [8] *Let* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition 4.1 page 71).

DEF

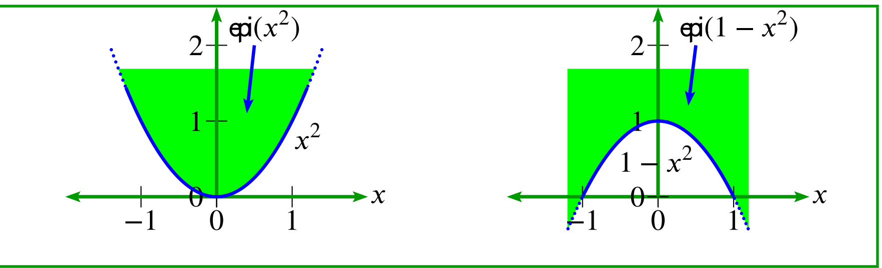
The ***epigraph*** $\mathrm{epi}(\mathsf{f})$ *and* ***hypograph*** $\mathrm{hyp}(\mathsf{f})$ *of a functional* $\mathsf{f} \in \mathbb{R}^X$ *are defined as*

$$\mathrm{epi}(\mathsf{f}) \triangleq \{(x,y) \in X \times \mathbb{R} | y \ge \mathsf{f}(x)\}$$
$$\mathrm{hyp}(\mathsf{f}) \triangleq \{(x,y) \in X \times \mathbb{R} | y \le \mathsf{f}(x)\}$$

Example 10.4.

[7] Simon (2011) page 2, Barvinok (2002) page 2, Bollobás (1999) page 3, Jensen (1906) page 176, Clarkson (1936) ⟨strictly convex⟩
[8] Beer (1993) page 13 ⟨§1.3⟩, Aubin and Frankowska (2009) page 222, Aubin (2011) page 223

EX



Proposition 10.1. [9] *Let* $\left(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times})\right)$ *be a* LINEAR SPACE *(Definition 4.1 page 71). Let* f *be a* FUNCTIONAL *in* $\mathbb{R}^X$.

PRP
$$\left\{\begin{array}{l}\mathsf{f} \textit{ is a}\\ \text{CONVEX FUNCTION}\end{array}\right\} \iff \left\{\begin{array}{l}\text{epi}(\mathsf{f}) \textit{ is a}\\ \text{CONVEX SET}\end{array}\right\}$$

Often a function can be proven to be *convex* or *concave*. *Convex* and *concave* functions are defined in Definition 10.9 (page 154) (next) and illustrated in Figure 10.1 (page 153).

Definition 10.9. *Let*

$$\mathsf{y}(x) = \frac{\mathsf{f}(b) - \mathsf{f}(a)}{b-a}(x-a) + \mathsf{f}(a).$$

A function $\mathsf{f} : \mathbb{R} \to \mathbb{R}$ *is*

DEF

(1).	***convex***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$\le$	$\mathsf{y}(x)$	*for* $x \in (a:b)$
(2).	***concave***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$\ge$	$\mathsf{y}(x)$	*for* $x \in (a:b)$
(3).	***strictly convex***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$<$	$\mathsf{y}(x)$	*for* $x \in (a:b)$
(4).	***strictly concave***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$>$	$\mathsf{y}(x)$	*for* $x \in (a:b)$

Theorem 10.1 (Jensen's Inequality). [10] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a* LINEAR SPACE *(Definition 4.1 page 71), D a subset of X, and* f *a functional in* $\mathbb{F}^D$. *Let* $\sum$ *be the* SUMMATION OPERATOR *(Definition 11.1 page 161).*

THM
$$\left\{\begin{array}{lll}1. & D \text{ is CONVEX} & \textit{and}\\ 2. & \mathsf{f} \text{ is CONVEX} & \textit{and}\\ 3. & \displaystyle\sum_{n=1}^{N} \lambda_n = 1 & \text{(WEIGHTS)}\end{array}\right\} \implies \mathsf{f}\left(\sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n\right) \le \sum_{n=1}^{N} \lambda_n \mathsf{f}(\boldsymbol{x}_n) \qquad \forall \boldsymbol{x}_n \in D,\ N \in \mathbb{N}$$

✎PROOF: Proof is by induction:

1. Proof that statement is true for $N = 1$:

$$\begin{aligned}\mathsf{f}\left(\sum_{n=1}^{N=1} \lambda_n \boldsymbol{x}_n\right) &= \mathsf{f}(\lambda_1 \boldsymbol{x}_1)\\ &\le \mathsf{f}(\lambda_1 \boldsymbol{x}_1)\\ &= \sum_{n=1}^{N=1} \lambda_n \mathsf{f}(\boldsymbol{x}_n)\end{aligned}$$

[9] Udriste (1994) page 63, Kurdila and Zabarankin (2005) page 178 ⟨Proposition 6.1.1⟩, Rockafellar (1970) page 23 ⟨Section 4 Convex Functions⟩, Çinlar and Vanderbei (2013) page 86 ⟨5.4 Theorem⟩
[10] Mitrinović et al. (2010) page 6, Bollobás (1999) page 3, Lay (1982) page 7, Jensen (1906) pages 179–180

2. Proof that statement is true for $N = 2$:

$$\begin{aligned}
\mathrm{f}\left(\sum_{n=1}^{N=2} \lambda_n \boldsymbol{x}_n\right) &= \mathrm{f}(\lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2) \\
&\le \lambda_1 \mathrm{f}(\boldsymbol{x}_1) + \lambda_2 \mathrm{f}(\boldsymbol{x}_2) && \text{by convexity hypothesis} \\
&= \sum_{n=1}^{N=2} \lambda_n \mathrm{f}(\boldsymbol{x}_n)
\end{aligned}$$

3. Proof that if the statement is true for N, then it is also true for $N+1$:

$$\begin{aligned}
\mathrm{f}\left(\sum_{n=1}^{N+1} \lambda_n \boldsymbol{x}_n\right) &= \mathrm{f}\left(\sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n + \lambda_{N+1}\boldsymbol{x}_{N+1}\right) \\
&= \mathrm{f}\left([1-\lambda_{N+1}]\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \boldsymbol{x}_n + \lambda_{N+1}\boldsymbol{x}_{N+1}\right) \\
&\le [1-\lambda_{N+1}]\mathrm{f}\left(\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \boldsymbol{x}_n\right) + \lambda_{N+1}\mathrm{f}(\boldsymbol{x}_{N+1}) && \text{by convexity hypothesis} \\
&\le [1-\lambda_{N+1}]\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \mathrm{f}(\boldsymbol{x}_n) + \lambda_{N+1}\mathrm{f}(\boldsymbol{x}_{N+1}) && \text{by “true for } N\text{” hypothesis} \\
&= \sum_{n=1}^{N} \lambda_n \mathrm{f}(\boldsymbol{x}_n) + \lambda_{N+1}\mathrm{f}(\boldsymbol{x}_{N+1}) \\
&= \sum_{n=1}^{N+1} \lambda_n \mathrm{f}(\boldsymbol{x}_n)
\end{aligned}$$

4. Since the statement is true for $N = 1$, $N = 2$, and true for $N \implies$ true for $N+1$, then it is true for $N = 1, 2, 3, 4, \ldots$

✏

The next theorem gives another form of convex functions that is a little less intuitive but provides powerful analytic results.

Theorem 10.2. *Let* $\mathrm{f} : \mathbb{R} \to \mathbb{R}$. *For every* $x_1, x_2 \in (a, b)$ *and* $\lambda \in [0, 1]$

THM f *is convex in* (a, b) $\iff$ $\mathrm{f}(\lambda x_1 + (1-\lambda)x_2) \le \lambda \mathrm{f}(x_1) + (1-\lambda)\mathrm{f}(x_2)$

✎PROOF:

1. prove f is convex $\implies \mathrm{f}(\lambda x_1 + (1-\lambda)x_2) \le \lambda \mathrm{f}(x_1) + (1-\lambda)\mathrm{f}(x_2)$:

$$\begin{aligned}
\mathrm{f}(\lambda x_1 + (1-\lambda)x_2) &\le \frac{\mathrm{f}(b)-\mathrm{f}(a)}{b-a}[\lambda x_1 + (1-\lambda)x_2 - a] + \mathrm{f}(a) \\
&= \frac{\mathrm{f}(x_2)-\mathrm{f}(x_1)}{x_2-x_1}[\lambda x_1 + (1-\lambda)x_2 - x_1] + \mathrm{f}(x_1) \\
&= \frac{\mathrm{f}(x_2)-\mathrm{f}(x_1)}{x_2-x_1}[(x_2-x_1)(1-\lambda)] + \mathrm{f}(x_1) \\
&= (1-\lambda)\mathrm{f}(x_2) - (1-\lambda)\mathrm{f}(x_1) + \mathrm{f}(x_1) \\
&= \lambda \mathrm{f}(x_1) + (1-\lambda)\mathrm{f}(x_2)
\end{aligned}$$

2. prove f is convex $\Longleftarrow$ $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda \mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)$:

   Let $x = \lambda(b-a) + a$ Notice that as λ varies from 0 to 1, x varies from b to a. So free variable λ works as a change of variable for free variable x.

$$\lambda = \frac{x-a}{b-a}$$

$$\begin{aligned}
\mathsf{f}(x) &= \mathsf{f}(\lambda(b-a)+a) \\
&\le \lambda\mathsf{f}(b) + (1-\lambda)\mathsf{f}(a) \\
&= \lambda[\mathsf{f}(b)-\mathsf{f}(a)] + \mathsf{f}(a) \\
&= \frac{\mathsf{f}(b)-\mathsf{f}(a)}{b-a}(x-a) + \mathsf{f}(a)
\end{aligned}$$

✎

Taking the second derivative of a function provides a convenient test for whether that function is convex.

Theorem 10.3. [11]

THM: $\mathsf{f}''(x) > 0 \implies \mathsf{f}$ *is convex*

✎PROOF:

$$\begin{aligned}
\mathsf{f}(x) &= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x-x_0) + \mathsf{f}''(c)(x-x_0)^2 \\
&\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x-x_0) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x - \lambda x_1 - (1-\lambda)x_2)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}(x_1) &\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x_1 - \lambda x_1 - (1-\lambda)x_2) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(1-\lambda)(x_1 - x_2) \\
&= \mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2 - x_1)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}(x_2) &\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x_2 - \lambda x_1 - (1-\lambda)x_2) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2 - x_1)
\end{aligned}$$

$$\begin{aligned}
\lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2) &\ge \lambda\left[\mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2-x_1)\right] + (1-\lambda)\left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] \\
&= \lambda\left[\mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2-x_1)\right] + \left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] - \lambda\left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] \\
&= \mathsf{f}(x_0) \\
&= \mathsf{f}(\lambda x_1 + (1-\lambda)x_2)
\end{aligned}$$

By Theorem 10.2 (page 155), $\mathsf{f}(x)$ is convex. ✎

10.4 Literature

LITERATURE SURVEY:

[11] Cover and Thomas (1991) pages 24–25

1. Abstract convexity:
   Edelman and Jamison (1985)
   van de Vel (1993)
   Hörmander (1994)

2. Order convexity (lattice theory):
   Edelman (1986)

3. Metric convexity:
   Menger (1928)
   Blumenthal (1970) page 41 ⟨?⟩
   Khamsi and Kirk (2001) pages 35–38

Part III

Structures on Spaces

CHAPTER 11

FINITE SUMS

❝ *I think that it was Harald Bohr who remarked to me that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."* ❞

G.H. Hardy (1877–1947) in his "Presidential Address" to the London Mathematical Society on November 8, 1928, about a remark that he suggested was from Harald Bohr (1887–1951), Danish mathematician pictured to the left. [1]

11.1 Summation

Definition 11.1. [2] *Let $+$ be an addition operator on a tuple $(\!(x_n)\!)_m^N$.*

DEF *The **summation** of $(\!(x_n)\!)$ from index m to index N with respect to $+$ is*

$$\sum_{n=m}^{N} x_n \triangleq \begin{cases} 0 & \text{for } N < m \\ \left(\displaystyle\sum_{n=m}^{N-1} x_n\right) + x_N & \text{for } N \geq m \end{cases}$$

Theorem 11.1 (Generalized associative property). [3] *Let $+$ be an addition operator on a tuple $(\!(x_n)\!)_m^N$.*

THM

$$+ \text{ is ASSOCIATIVE} \implies \underbrace{\sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right) = \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n \qquad \text{for } m < L < M \leq N}_{\sum_{n=m}^{N} \text{ is ASSOCIATIVE}}$$

---

[1] quote: Hardy (1929) page 64
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Bohr_Harald.html

[2] reference: Berberian (1961) page 8 ⟨Definition I.3.1⟩
"$\sum$" notation: Fourier (1820) page 280

[3] Berberian (1961) pages 9–10 ⟨Theorem I.3.1⟩

✎PROOF:

1. Proof for $N < m$ case: $\sum_{n=m}^{N} x_n = 0$.

2. Proof for $N = m$ case: $\sum_{n=m}^{m} x_n = \left(\sum_{n=m}^{m-1} x_n\right) + x_m = 0 + x_m = x_m$.

3. Proof for $N = m+1$ case: $\sum_{n=m}^{m+1} x_n = \left(\sum_{n=m}^{m} x_n\right) + x_{m+1} = x_m + x_{m+1}$

4. Proof for $N = m+2$ case:

$$\begin{aligned}
\sum_{n=m}^{m+2} x_n &= \left(\sum_{n=m}^{m+1} x_n\right) + x_{m+2} && \text{by Definition 11.1 page 161}\\
&= (x_m + x_{m+1}) + x_{m+2} && \text{by item (3)}\\
&= x_m + (x_{m+1} + x_{m+2}) && \text{by left hypothesis}
\end{aligned}$$

5. Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\sum_{n=m}^{N+1} x_n &= \underbrace{\left(\sum_{n=m}^{N} x_n\right)}_{\text{associative}} + x_{N+1} && \text{by Definition 11.1 page 161}\\
&= \left(\sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right)\right) + x_{N+1} &&= \left(\left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n\right) + x_{N+1}\\
&= \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N} x_n + x_{N+1}\right) &&= \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N+1} x_n\right)
\end{aligned}$$

✏

11.2 Means

11.2.1 Weighted ϕ-means

Definition 11.2. [4]

DEF

*The $(\!(\lambda_n)\!)_1^N$ weighted ϕ-**mean** of a tuple $(\!(x_n)\!)_1^N$ is defined as*

$$\mathsf{M}_\phi((\!(x_n)\!)) \triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right)$$

where ϕ is a CONTINUOUS *and* STRICTLY MONOTONIC *function in $\mathbb{R}^{\mathbb{R}}$ and $(\!(\lambda_n)\!)_{n=1}^N$ is a sequence of weights for which $\sum_{n=1}^{N} \lambda_n = 1$.*

Lemma 11.1. [5] *Let $\mathsf{M}_\phi((\!(x_n)\!))$ be the $(\!(\lambda_n)\!)_1^N$ weighted ϕ-mean of a tuple $(\!(x_n)\!)_1^N$. Let the property* CON-

[4] Bollobás (1999) page 5

[5] Pečarić et al. (1992) page 107, Bollobás (1999) page 5, Hardy et al. (1952) page 75

VEX *be defined as in Definition 10.7 (page 153).*

LEM

$$\begin{array}{llllll}
\phi\psi^{-1} \text{ is CONVEX} & \text{and} & \phi \text{ is INCREASING} & \implies & \mathsf{M}_\phi\left(⦇x_n⦈\right) \geq \mathsf{M}_\psi\left(⦇x_n⦈\right) \\
\phi\psi^{-1} \text{ is CONVEX} & \text{and} & \phi \text{ is DECREASING} & \implies & \mathsf{M}_\phi\left(⦇x_n⦈\right) \leq \mathsf{M}_\psi\left(⦇x_n⦈\right) \\
\phi\psi^{-1} \text{ is CONCAVE} & \text{and} & \phi \text{ is INCREASING} & \implies & \mathsf{M}_\phi\left(⦇x_n⦈\right) \leq \mathsf{M}_\psi\left(⦇x_n⦈\right) \\
\phi\psi^{-1} \text{ is CONCAVE} & \text{and} & \phi \text{ is DECREASING} & \implies & \mathsf{M}_\phi\left(⦇x_n⦈\right) \geq \mathsf{M}_\psi\left(⦇x_n⦈\right)
\end{array}$$

✎PROOF:

1. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *increasing*:

$$\begin{aligned}
\mathsf{M}_\phi(⦇x_n⦈) &\triangleq \phi^{-1}\left(\sum_{n=1}^N \lambda_n\phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi && \text{(Definition 11.2 page 162)}\\
&= \phi^{-1}\left(\sum_{n=1}^N \lambda_n\phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\geq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^N \lambda_n\psi(x_n)\right) && \text{by \textit{Jensen's Inequality}} && \text{(Theorem 10.1 page 154)}\\
&= \left(\psi^{-1}\sum_{n=1}^N \lambda_n\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\triangleq \mathsf{M}_\psi(⦇x_n⦈) && \text{by definition of } \mathsf{M}_\psi && \text{(Definition 11.2 page 162)}
\end{aligned}$$

2. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *decreasing*:

$$\begin{aligned}
\mathsf{M}_\phi(⦇x_n⦈) &\triangleq \phi^{-1}\left(\sum_{n=1}^N \lambda_n\phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi && \text{(Definition 11.2 page 162)}\\
&= \phi^{-1}\left(\sum_{n=1}^N \lambda_n\phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\leq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^N \lambda_n\psi(x_n)\right) && \text{by \textit{Jensen's Inequality}} && \text{(Theorem 10.1 page 154)}\\
& && \text{and because } \phi^{-1} \text{ is \textit{decreasing}} && \text{(by hypothesis)}\\
&= \left(\psi^{-1}\sum_{n=1}^N \lambda_n\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\triangleq \mathsf{M}_\psi(⦇x_n⦈) && \text{by definition of } \mathsf{M}_\psi && \text{(Definition 11.2 page 162)}
\end{aligned}$$

✎

One of the most well known inequalities in mathematics is *Minkowski's Inequality* (Theorem 11.5 page 169). In 1946, H.P. Mulholland submitted a result[6] that generalizes Minkowski's Inequality to an equal weighted ϕ-mean. And Milovanović and Milovanovć (1979) generalized this even further to a *weighted* ϕ-mean (Theorem 11.2, next).

Theorem 11.2. [7]

THM

$$\left\{\begin{array}{lll|lll}
(1). & \phi \text{ is CONVEX} & \text{and} & (3). & \phi(0) = 0 & \text{and}\\
(2). & \phi \text{ is STRICTLY MONOTONIC} & \text{and} & (4). & \log\circ\phi\circ\exp \text{ is CONVEX} &
\end{array}\right\} \implies$$

$$\left\{\phi^{-1}\left(\sum_{n=1}^N \lambda_n\phi(x_n+y_n)\right) \leq \phi^{-1}\left(\sum_{n=1}^N \lambda_n\phi(x_n)\right) + \phi^{-1}\left(\sum_{n=1}^N \lambda_n\phi(y_n)\right)\right\}$$

---

[6] Mulholland (1950)

[7] Milovanović and Milovanović (1979), Bullen (2003) page 306 ⟨Theorem 9⟩

11.2.2 Power means

Definition 11.3. [8] *Let* $\mathsf{M}_{\phi(x;r)}(\lparen x_n \rparen)$ *be the* $\lparen \lambda_n \rparen_1^N$ *weighted* ϕ*-mean of a* NON-NEGATIVE *tuple* $\lparen x_n \rparen_1^N$ *(Definition 11.2 page 162).*

DEF

A mean $\mathsf{M}_{\phi(x;r)}(\lparen x_n \rparen)$ *is a* ***power mean*** *with parameter* r *if* $\phi(x) \triangleq x^r$. *That is,*

$$\mathsf{M}_{\phi(x;r)}(\lparen x_n \rparen) = \left(\sum_{n=1}^{N} \lambda_n (x_n)^r\right)^{\frac{1}{r}}$$

Theorem 11.3. [9] *Let* $\mathsf{M}_{\phi(x;r)}(\lparen x_n \rparen)$ *be* POWER MEAN *with parameter* r *of an* N*-tuple* $\lparen x_n \rparen_1^N$. *Let* $\mathbb{R}^*$ *be the set of extended real numbers* $(\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\})$.[10]

THM

$$\mathsf{M}_{\phi(x;r)}(\lparen x_n \rparen) \triangleq \left(\sum_{n=1}^{N} \lambda_n (x_n)^r\right)^{\frac{1}{r}} \quad \textit{is } \text{CONTINUOUS } \textit{and } \text{STRICTLY INCREASING } \textit{in } \mathbb{R}^*.$$

$$\mathsf{M}_{\phi(x;r)}(\lparen x_n \rparen) = \begin{cases} \min\limits_{n=1,2,\ldots,N} \lparen x_n \rparen & \text{for } r = -\infty \\ \prod\limits_{n=1}^{N} x_n^{\lambda_n} & \text{for } r = 0 \\ \max\limits_{n=1,2,\ldots,N} \lparen x_n \rparen & \text{for } r = +\infty \end{cases}$$

✎PROOF:

1. Proof that $M_{\phi(x;r)}$ is *strictly increasing* in r:
   (a) Let r and s be such that $-\infty < r < s < \infty$.
   (b) Let $\phi_r \triangleq x^r$ and $\phi_s \triangleq x^s$. Then $\phi_r \phi_s^{-1} = x^{\frac{r}{s}}$.
   (c) The composite function $\phi_r \phi_s^{-1}$ is *convex* or *concave* depending on the values of r and s:

	$r < 0$ (ϕ_r decreasing)	$r > 0$ (ϕ_r increasing)
$s < 0$	*convex*	(not possible)
$s > 0$	*convex*	*concave*

   (d) Therefore by Lemma 11.1 (page 162),
$$-\infty < r < s < \infty \quad \Longrightarrow \quad \mathsf{M}_{\phi(x;r)}(\lparen x_n \rparen) < \mathsf{M}_{\phi(x;s)}(\lparen x_n \rparen).$$

2. Proof that $\mathsf{M}_{\phi(x;r)}$ is continuous in r for $r \in \mathbb{R}\backslash 0$: The sum of continuous functions is continuous. For the cases of $r \in \{-\infty, 0, \infty\}$, see the items that follow.

3. Lemma: $\mathsf{M}_{\phi(x;-r)}(\lparen x_n \rparen) = \left\{\mathsf{M}_{\phi(x;r)}(\lparen x_n^{-1} \rparen)\right\}^{-1}$. Proof:

$$\begin{aligned} \left\{\mathsf{M}_{\phi(x;r)}(\lparen x_n^{-1} \rparen)\right\}^{-1} &= \left\{\left(\sum_{n=1}^{N} \lambda_n (x_n^{-1})^r\right)^{\frac{1}{r}}\right\}^{-1} && \text{by definition of } \mathsf{M}_\phi \\ &= \left(\sum_{n=1}^{N} \lambda_n (x_n)^{-r}\right)^{\frac{1}{-r}} \\ &= \mathsf{M}_{\phi(x;-r)}(\lparen x_n \rparen) && \text{by definition of } \mathsf{M}_\phi \end{aligned}$$

[8] Bullen (2003) page 175, Bollobás (1999) page 6
[9] Bullen (2003) pages 175–177 ⟨see also page 203⟩, Bollobás (1999) pages 6–8, Besso (1879), Bienaymé (1840) page 68
[10] Rana (2002) pages 385–388 ⟨Appendix A⟩

4. Proof that $\lim_{r\to\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) = \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$:

   (a) Let $x_m \triangleq \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$

   (b) Note that $\lim_{r\to\infty} \mathsf{M}_\phi \le \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$ because

$$
\begin{aligned}
\lim_{r\to\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty}\left(\sum_{n=1}^{N} \lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi\\
&\le \lim_{r\to\infty}\left(\sum_{n=1}^{N} \lambda_n x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x) \triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing}\\
&= \lim_{r\to\infty}\left(x_m^r \underbrace{\sum_{n=1}^{N} \lambda_n}_{1}\right)^{\frac{1}{r}} && \text{because } x_m \text{ is a constant}\\
&= \lim_{r\to\infty}\left(x_m^r \cdot 1\right)^{\frac{1}{r}}\\
&= x_m\\
&= \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}
$$

   (c) But also note that $\lim_{r\to\infty} \mathsf{M}_\phi \ge \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$ because

$$
\begin{aligned}
\lim_{r\to\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty}\left(\sum_{n=1}^{N} \lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi\\
&\ge \lim_{r\to\infty}\left(w_m x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x) \triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing}\\
&= \lim_{r\to\infty} w_m^{\frac{1}{r}} x_m^{\frac{r}{r}}\\
&= x_m\\
&= \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}
$$

   (d) Combining items (b) and (c) we have $\lim_{r\to\infty} \mathsf{M}_\phi = \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$.

5. Proof that $\lim_{r\to-\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) = \min_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$:

$$
\begin{aligned}
\lim_{r\to-\infty} \mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty} \mathsf{M}_{\phi(x;-r)}\left(\llparenthesis x_n \rrparenthesis\right) && \text{by change of variable } r\\
&= \lim_{r\to\infty} \left\{\mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)\right\}^{-1} && \text{by Lemma in item (3) page 164}\\
&= \lim_{r\to\infty} \frac{1}{\mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)}\\
&= \frac{\lim_{r\to\infty} 1}{\lim_{r\to\infty} \mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)} && \text{by property of lim [11]}\\
&= \frac{1}{\max_{n\in\mathbb{Z}} \llparenthesis x_n^{-1} \rrparenthesis} && \text{by item (4)}
\end{aligned}
$$

$$= \frac{1}{\left(\min_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis\right)^{-1}}$$
$$= \min_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$$

6. Proof that $\lim_{r\to 0} \mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) = \prod_{n=1}^{N} x_n^{\lambda_n}$:

$$\begin{aligned}
\lim_{r\to 0} \mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) &= \lim_{r\to 0} \exp\left\{\ln\left\{\mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis)\right\}\right\} \\
&= \lim_{r\to 0} \exp\left\{\ln\left\{\left(\sum_{n=1}^{N} \lambda_n (x_n^r)\right)^{\frac{1}{r}}\right\}\right\} && \text{by definition of } \mathsf{M}_\phi \\
&= \exp\left\{\frac{\frac{\partial}{\partial r} \ln\left(\sum_{n=1}^{N} \lambda_n (x_n^r)\right)}{\frac{\partial}{\partial r} r}\right\}_{r=0} && \text{by l'Hôpital's rule[12]} \\
&= \exp\left\{\frac{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r} (x_n^r)}{\sum_{n=1}^{N} \lambda_n (x_n^r)}\right\}_{r=0} && = \exp\left\{\frac{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r} \exp\left(\ln\left(x_n^r\right)\right)}{\sum_{n=1}^{N} \lambda_n}\right\}_{r=0} \\
&= \exp\left\{\frac{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r} \exp\left(r \ln\left(x_n\right)\right)}{1}\right\}_{r=0} && = \exp\left\{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r} \exp\left(r \ln\left(x_n\right)\right)\right\}_{r=0} \\
&= \exp\left\{\sum_{n=1}^{N} \lambda_n \exp\left\{r \ln x_n\right\} \ln\left(x_n\right)\right\}_{r=0} && = \exp\left\{\sum_{n=1}^{N} \lambda_n \ln\left(x_n\right)\right\} \\
&= \exp\left\{\sum_{n=1}^{N} \ln\left(x_n^{\lambda_n}\right)\right\} && = \exp\left\{\ln \prod_{n=1}^{N} x_n^{\lambda_n}\right\} = \prod_{n=1}^{N} x_n^{\lambda_n}
\end{aligned}$$

✏

Definition 11.4. *Let $\llparenthesis x_n \rrparenthesis_1^N$ be a tuple. Let $\llparenthesis \lambda_n \rrparenthesis_1^N$ be a tuple of weighting values.*

---

[11] Rudin (1976) page 85 ⟨4.4 Theorem⟩
[12] Rudin (1976) page 109 ⟨5.13 Theorem⟩

DEF

*The **harmonic mean** of* $(\!|x_n|\!)$ *is defined as* $\mu_h \triangleq \left(\sum_{n=1}^{N}\lambda_n\frac{1}{x_n}\right)^{-1}$ *where* $\sum_{n=1}^{N}\lambda_n = 1$

*The **geometric mean** of* $(\!|x_n|\!)$ *is defined as* $\mu_g \triangleq \sum_{n=1}^{N} x_n^{\lambda_n}$ *where* $\sum_{n=1}^{N}\lambda_n = 1$

*The **arithmetic mean** of* $(\!|x_n|\!)$ *is defined as* $\mu_a \triangleq \sum_{n=1}^{N}\lambda_n x_n$ *where* $\sum_{n=1}^{N}\lambda_n = 1$

*The **average** of* $(\!|x_n|\!)$ *is defined as* $\mu_a \triangleq \frac{1}{N}\sum_{n=1}^{N} x_n$

11.3 Inequalities on power means

Corollary 11.1. [13] *Let* $(\!|x_n|\!)_1^N$ *be a tuple. Let* $(\!|\lambda_n|\!)_1^N$ *be a tuple of weighting values.*

COR

$$\min(\!|x_n|\!) \le \underbrace{\left(\sum_{n=1}^{N}\lambda_n\frac{1}{x_n}\right)^{-1}}_{\text{harmonic mean}} \le \underbrace{\prod_{n=1}^{N} x_n^{\lambda_n}}_{\text{geometric mean}} \le \underbrace{\sum_{n=1}^{N}\lambda_n x_n}_{\text{arithmetic mean}} \le \max(\!|x_n|\!) \qquad \text{where } \sum_{n=1}^{N}\lambda_n = 1$$

✎PROOF:

1. These five means are all special cases of the *power mean* $\mathsf{M}_{\phi(x:r)}$ (Definition 11.3 page 164):
   - $r=\infty$: $\max(\!|x_n|\!)$
   - $r=1$: arithmetic mean
   - $r=0$: geometric mean
   - $r=-1$: harmonic mean
   - $r=-\infty$: $\min(\!|x_n|\!)$
2. The inequalities follow directly from Theorem 11.3 (page 164).
3. Generalized AM-GM inequality: If one is only concerned with the arithmetic mean and geometric mean, their relationship can be established directly using *Jensen's Inequality*:

$$\begin{aligned}\sum_{n=1}^{N}\lambda_n x_n &= b^{\log_b\left(\sum_{n=1}^{N}\lambda_n x_n\right)} \ge b^{\left(\sum_{n=1}^{N}\lambda_n \log_b x_n\right)} && \text{by } \textit{Jensen's Inequality} \text{ (Theorem 10.1 page 154)}\\ &= \prod_{n=1}^{N} b^{(\lambda_n \log_b x_n)} = \prod_{n=1}^{N} b^{(\log_b x_n)\lambda_n} = \prod_{n=1}^{N} x_n^{\lambda_n}\end{aligned}$$

✏

Lemma 11.2 (Young's Inequality). [14]

[13] Bullen (2003) page 71, Bollobás (1999) page 5, Cauchy (1821) pages 457–459 ⟨Note II, theorem 17⟩, Jensen (1906) page 183

[14] Carothers (2000) page 43, Tolsted (1964) page 5, Maligranda (1995) page 257, Hardy et al. (1952) ⟨Theorem 24⟩, Young (1912) page 226

LEM

$$xy < \frac{x^p}{p} + \frac{y^q}{q} \quad \textit{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \textit{but } y \neq x^{p-1}$$
$$xy = \frac{x^p}{p} + \frac{y^q}{q} \quad \textit{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \textit{and } y = x^{p-1}$$

✎PROOF:

1. Proof that $\frac{1}{p-1} = q - 1$:

$$\begin{aligned}
\frac{1}{p} + \frac{1}{q} = 1 &\iff \frac{q}{q} + \frac{q}{p} = q \\
&\iff q\left(1 - \frac{1}{p}\right) = 1 \\
&\iff q = \frac{1}{1 - \frac{1}{p}} \\
&\iff q = \frac{p}{p-1} \\
&\iff q - 1 = \frac{p}{p-1} - \frac{p-1}{p-1} \\
&\iff q - 1 = \frac{p - (p-1)}{p-1} \\
&\iff q - 1 = \frac{1}{p-1}
\end{aligned}$$

2. Proof that $v = u^{p-1} \iff u = v^{q-1}$:

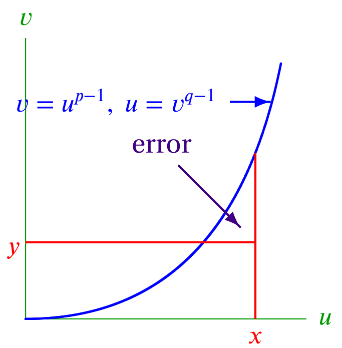
$$\begin{aligned}
u &= v^{\frac{1}{p-1}} && \text{by left hypothesis} \\
&= v^{q-1} && \text{by item (1)}
\end{aligned}$$

3. Proof that $v = u^{p-1}$ is propemonotonically increasing in u and $u = v^{q-1}$ is propemonotonically increasing in v:

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u}u^{p-1} && = (p-1)u^{p-2} && > 0 \\
\frac{\mathrm{d}u}{\mathrm{d}v} &= \frac{\mathrm{d}}{\mathrm{d}v}v^{q-1} && = (q-1)v^{q-2} && > 0
\end{aligned}$$

4. Proof that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$:

$$\begin{aligned}
xy &\leq \int_0^x u^{p-1}\,\mathrm{d}u + \int_0^y v^{q-1}\,\mathrm{d}v \\
&= \left.\frac{u^p}{p}\right|_0^x + \left.\frac{v^q}{q}\right|_0^y \\
&= \frac{x^p}{p} + \frac{y^q}{q}
\end{aligned}$$



✏

Theorem 11.4 (Hölder's Inequality). [15] *Let* $(\!(x_n \in \mathbb{C})\!)_1^N$ *and* $(\!(y_n \in \mathbb{C})\!)_1^N$ *be complex* N*-tuples.*

THM
$$\underbrace{\sum_{n=1}^{N} |x_n y_n|}_{\|\boldsymbol{x}\cdot\boldsymbol{y}\|_1} \le \underbrace{\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}}_{\|\boldsymbol{x}\|_p} \underbrace{\left(\sum_{n=1}^{N} |y_n|^q\right)^{\frac{1}{q}}}_{\|\boldsymbol{y}\|_q} \quad \text{with} \quad \frac{1}{p}+\frac{1}{q}=1 \quad \forall\, 1<p<\infty$$

✎PROOF: Let $\|x_n\|_p \triangleq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$.

$$\begin{aligned}
\sum_{n=1}^{N} |x_n y_n| &= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \sum_{n=1}^{N} \frac{|x_n y_n|}{\|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q} \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \sum_{n=1}^{N} \frac{|x_n|}{\|(\!(x_n)\!)\|_p} \frac{|y_n|}{\|(\!(y_n)\!)\|_q} \\
&\le \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \sum_{n=1}^{N} \left(\frac{1}{p} \frac{|x_n|^p}{\|(\!(x_n)\!)\|_p^p} + \frac{1}{q} \frac{|y_n|^q}{\|(\!(y_n)\!)\|_q^q} \right) && \text{by } \textit{Young's Inequality} \quad \text{(Lemma 11.2 page 167)} \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \left(\frac{1}{p} \cdot \frac{\sum |x_n|^p}{\|(\!(x_n)\!)\|_p^p} + \frac{1}{q} \cdot \frac{\sum |y_n|^q}{\|(\!(y_n)\!)\|_q^q} \right) \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \left(\frac{1}{p} \frac{\|(\!(x_n)\!)\|_p^p}{\|(\!(x_n)\!)\|_p^p} + \frac{1}{q} \frac{\|(\!(y_n)\!)\|_q^q}{\|(\!(y_n)\!)\|_q^q} \right) && \text{by definition of } \|\cdot\| \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \underbrace{\left(\frac{1}{p} + \frac{1}{q} \right)}_{1} \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q && \text{by } \frac{1}{p}+\frac{1}{q}=1 \text{ constraint}
\end{aligned}$$

✏

Theorem 11.5 (Minkowski's Inequality for sequences). [16] *Let* $(\!(x_n \in \mathbb{C})\!)_1^N$ *and* $(\!(y_n \in \mathbb{C})\!)_1^N$ *be complex* N*-tuples.*

THM
$$\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{\frac{1}{p}} \quad \forall\, 1<p<\infty$$

✎PROOF:

1. Define $q \triangleq \dfrac{p}{p-1}$

2. Define $\|\boldsymbol{x}\|_p \triangleq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$.

[15] Bullen (2003) page 178 ⟨2.1⟩, Carothers (2000) page 44, Tolsted (1964) page 6, Maligranda (1995) page 257, Hardy et al. (1952) ⟨Theorem 11⟩, Hölder (1889)

[16] Bullen (2003) page 179, Carothers (2000) page 44, Tolsted (1964) page 7, Maligranda (1995) page 258, Hardy et al. (1952) ⟨Theorem 24⟩, Minkowski (1910) page 115

3. Proof that $\|x_n + y_n\|_p \le \|x_n\|_p + \|y_n\|_p$:

$$
\begin{aligned}
&\boxed{\|x_n + y_n\|_p^p} \\
&= \sum_{n=1}^{N} |x_n + y_n|^p && \text{by definition of } \|\cdot\|_p && \text{(definition 2 page 169)} \\
&= \sum_{n=1}^{N} |x_n + y_n||x_n + y_n|^{p-1} && \text{by } \textit{homogeneous} \text{ property of } |\cdot| \\
&\boxed{\le} \sum_{n=1}^{N} |x_n||x_n + y_n|^{p-1} + \sum_{n=1}^{N} |y_n||x_n + y_n|^{p-1} && \text{by } \textit{subadditive} \text{ property of } |\cdot| \\
&= \sum_{n=1}^{N} \left|x_n(x_n + y_n)^{p-1}\right| + \sum_{n=1}^{N} \left|y_n(x_n + y_n)^{p-1}\right| && \text{by } \textit{homogeneous} \text{ property of } |\cdot| \\
&\boxed{\le} \|x_n\|_p \left\|(x_n + y_n)^{p-1}\right\|_q + \|y_n\|_p \left\|(x_n + y_n)^{p-1}\right\|_q && \text{by } \textit{Hölder's Inequality} && \text{(Theorem 11.4 page 169)} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left\|(x_n + y_n)^{p-1}\right\|_q \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left(\sum_{n=1}^{N} \left|(x_n + y_n)^{p-1}\right|^q\right)^{\frac{1}{q}} && \text{by definition of } \|\cdot\|_p && \text{(definition 2 page 169)} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left(\sum_{n=1}^{N} \left|(x_n + y_n)^{p-1}\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} && \text{by definition 1} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left(\sum_{n=1}^{N} \left|(x_n + y_n)\right|^{p}\right)^{\frac{p-1}{p}} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \|x_n + y_n\|^{p-1} && \text{by definition of } \|\cdot\|_p && \text{(definition 2 page 169)} \\
&\implies \boxed{\|x_n + y_n\|_p \le \|x_n\|_p + \|y_n\|_p}
\end{aligned}
$$

❝*Cauchy is the only one occupied with pure mathematics: Poisson, Fourier, Ampere, etc., busy themselves exclusively with magnetism and other physical subjects. Mr. Laplace writes nothing now, I believe.*❞
Niels Henrik Abel in an 1826 letter [17]

Theorem 11.6 (Cauchy-Schwarz Inequality for sequences). [18] *Let* $(\!(x_n \in \mathbb{C})\!)_1^N$ *and* $(\!(y_n \in \mathbb{C})\!)_1^N$ *be complex* N*-tuples.*

[17] quote: Bell (1986) page 318 ⟨Chapter 17. "Genius and Poverty" "Abel (1802–1829)"⟩, Boyer and Merzbach (2011) page 462 ⟨without "Mr. Laplace..." portion⟩. image: http://en.wikipedia.org/wiki/File:Augustin-Louis_Cauchy_1901.jpg, public domain

[18] Aliprantis and Burkinshaw (1998) page 278, Scharz (1885), Bouniakowsky (1859), Hardy et al. (1952) page 25 ⟨Theorem 11⟩, Cauchy (1821) page 455 ⟨???⟩

THM

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 \le \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \qquad \forall \boldsymbol{x},\boldsymbol{y}\in \boldsymbol{X}$$

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 = \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \iff \exists a \in \mathbb{C} \text{ such that } \boldsymbol{y} = a\boldsymbol{x} \qquad \forall \boldsymbol{x},\boldsymbol{y}\in \boldsymbol{X}$$

✎PROOF:

1. The *Cauchy-Schwarz Inequality for sequences* is a special case of the *Hölder inequality* (Theorem 11.4 page 169) for $p = q = 2$.

2. Alternatively, the *Cauchy-Schwarz inequality for sequences* is a special case of the *Cauchy-Schwarz inequality for inner-product spaces*:

   (a) $\langle x_n \mid y_n \rangle \triangleq \sum_{n=1}^{N} x_n y_n$ is an inner-product and $\left(\llparenthesis x_n \rrparenthesis, \langle \triangle \mid \triangledown \rangle\right)$ is an inner-product space.

   (b) By the more general *Cauchy-Schwarz Inequality for inner-product spaces*,

$$\begin{aligned}
\left(\sum_{n=1}^{N} a_n \lambda_n\right)^2 &\triangleq \langle a_n \mid \lambda_n \rangle^2 && \text{by definition of } \langle x_n \mid y_n \rangle\\
&\boxed{\le} \|x_n\|^2 \, \|y_n\|^2 && \text{by \textit{Cauchy-Schwarz Inequality for inner-product spaces} (Theorem 7.2 page 100)}\\
&\triangleq \left(\sum_{n=1}^{N} x_n^2\right)\left(\sum_{n=1}^{N} y_n^2\right) && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Not only does the *Hölder inequality* imply the *Cauchy-Schwarz inequality*, but somewhat surprisingly, the converse is also true: The Cauchy-Schwarz inequality implies the Hölder inequality.[19]

✏

Proposition 11.1. [20]

PRP $(x+y)^p \le 2^p(x^p + y^p) \qquad \forall x, y \ge 0,\ 1 < p < \infty$

✎PROOF:

$$\begin{aligned}
(x+y)^p &\le (2\max\{x,y\})^p\\
&= 2^p(\max\{x,y\})^p\\
&= 2^p(\max\{x^p,y^p\})\\
&\le 2^p(x^p+y^p)
\end{aligned}$$

✏

[19] Bullen (2003) pages 183–185 ⟨Theorem 5⟩
[20] Carothers (2000) page 43

11.4 Power Sums

Theorem 11.7 (Geometric Series). [21]

THM
$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \qquad \forall r \in \mathbb{C}\backslash 0$$

✎PROOF:

$$\boxed{\sum_{k=0}^{n-1} r^k} = \left(\frac{1}{1-r}\right)\left[(1-r)\sum_{k=0}^{n-1} r^k\right] = \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - r\sum_{k=0}^{n-1} r^k\right] = \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - \left(\sum_{k=0}^{n-1} r^k - 1 + r^n\right)\right]$$
$$= \left(\frac{1}{1-r}\right)[1-r^n] = \boxed{\frac{1-r^n}{1-r}}$$

✏

Lemma 11.3. *Let* $\mathsf{f}(x)$ *be a function.*

LEM
$$\mathsf{S}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) = \mathsf{S}(x+\tau) \qquad (\mathsf{S}(x) \text{ is PERIODIC with period } \tau)$$

✎PROOF:

$$\mathsf{S}(x+\tau) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+\tau+n\tau) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+(n+1)\tau) = \sum_{m\in\mathbb{Z}} \mathsf{f}(x+m\tau) \quad (\text{where } m \triangleq n+1)$$
$$\triangleq \mathsf{S}(x)$$

✏

Proposition 11.2 (Power Sums). [22]

PRP

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \forall n\in\mathbb{N}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \forall n\in\mathbb{N}$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \qquad \forall n\in\mathbb{N}$$
$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \qquad \forall n\in\mathbb{N}$$

✎PROOF:

1. Proof that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$: (proof by induction)

$$\sum_{k=1}^{n=1} k = 1 = \frac{1(1+1)}{2} = \left.\frac{n(n+1)}{2}\right|_{n=1}$$
$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + \left(n+1\right) = \underbrace{\left(\frac{n(n+1)}{2}\right) + \left(n+1\right)}_{\text{by left hypothesis}} = \left(n+1\right)\left(\frac{n}{2}+1\right)$$
$$= \left(n+1\right)\left(\frac{n+2}{2}\right) = \frac{(n+1)(n+2)}{2}$$

[21] Hall and Knight (1894) page 39 ⟨article 55⟩
[22] Amann and Escher (2008) pages 51–57, Menini and Oystaeyen (2004) page 91 ⟨Exercises 5.36–5.39⟩

2. Proof that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$: (proof by induction)

$$\sum_{k=1}^{n=1} k^2 = 1 = \frac{1(1+1)(2+1)}{6} = \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1}$$

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + \left(n+1\right)^2 = \underbrace{\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(n+1\right)^2}_{\text{by left hypothesis}} = \left(n+1\right)\left(\frac{n(2n+1)+6(n+1)}{6}\right)$$

$$= \left(n+1\right)\left(\frac{2n^2+7n+6}{6}\right) = \left(n+1\right)\left(\frac{(n+2)(2n+3)}{6}\right) = \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}$$

✏

CHAPTER 12

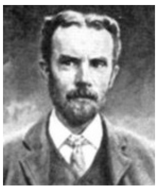
INFINITE SUMS

❝*If, for always increasing values of n, the sum s_n approaches a certain limit s, the series will be called convergent and the limit in question will be called the sum of the series.*❞

Cauchy [1]

❝*The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes...*❞

Niels Henrik Abel, in a January 16, 1826 letter to Holmboé [2]



❝*The series is divergent; therefore we may be able to do something with it.*❞
Oliver Heaviside (1850–1925) [3]

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[1] quote: Cauchy (1821) page 114, González-Velasco (1992) pages 429–430
image: http://en.wikipedia.org, public domain

[2] quote: Kline (1972) page 973 ⟨Chapter 47⟩
image: http://en.wikipedia.org/wiki/File:Niels_Henrik_Abel.jpg, public domain

[3] quote: Kline (1972) page 1096 ⟨Chapter 47⟩
image: http://en.wikipedia.org/wiki/File:Oliver_Heaviside2.jpg, public domain

> *"Some modern appraisals of the cavalier style of 18th-century mathematicians in handling infinite series conveys the impression that these poor men set their brains aside when confronted by them."*
>
> Ivor Grattan-Guinness (1990)[4]

12.1 Convergence

An infinite summation $\sum_{n=1}^{\infty} x_n$ is meaningless outside some topological space (e.g. metric space, normed space, etc.). The sum $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ is an abbreviation for $\lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{x}_n$ (next definition); and the concept of *limit* (Definition 9.3 page 132) is also itself meaningless outside of a *topological space* (Definition 1.1 page 3).

Definition 12.1. [5] *Let* $(X, \boldsymbol{T})$ *be a topological space and* lim *be the limit generated by the topology* $\boldsymbol{T}$.

DEF

$$\sum_{n=1}^{\infty} \boldsymbol{x}_n \triangleq \sum_{n\in\mathbb{N}} \boldsymbol{x}_n \triangleq \lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{x}_n$$

$$\sum_{n=-\infty}^{\infty} \boldsymbol{x}_n \triangleq \sum_{n\in\mathbb{Z}} \boldsymbol{x}_n \triangleq \lim_{N\to\infty} \left(\sum_{n=0}^{N} \boldsymbol{x}_n\right) + \left(\lim_{N\to-\infty} \sum_{n=-1}^{N} \boldsymbol{x}_n\right)$$

In general, the order of summation of an infinite series *does* matter.

Definition 12.2. [6] *Let P be the set of all* PERMUTATIONS *in* $\mathbb{N}^{\mathbb{N}}$.

DEF

A series $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ *is* ***absolutely convergent*** *if* $\quad \sum_{n=1}^{\infty} \boldsymbol{x}_n = \sum_{n=1}^{\infty} \boldsymbol{x}_{\mathsf{p}(n)} \quad \forall \mathsf{p} \in P$

A series is ***conditionally convergent*** *if it is* CONVERGENT *but not* ABSOLUTELY CONVERGENT.

Theorem 12.1 (Riemann Series Theorem). [7] *Let* $\mathsf{p}(n)$ *be a permutation on* $\mathbb{N}$. *Let* $(a_n)_{n\in\mathbb{Z}}$ *be a sequence of real numbers.*

THM

$$\left\{ \sum_{n=1}^{\infty} a_n \text{ is CONDITIONALLY CONVERGENT} \right\} \implies \left\{ \begin{array}{l} \text{For every } x \in \mathbb{R} \text{ there exists } \mathsf{p} \\ \text{such that } \sum_{n=1}^{\infty} a_{\mathsf{p}(n)} = x \\ \text{or such that } \sum_{n=1}^{\infty} a_{\mathsf{p}(n)} \text{ is DIVERGENT} \end{array} \right\}$$

Theorem 12.2. [8]

THM

$$\sum_{n=1}^{\infty} |\boldsymbol{x}_n| < \infty \implies \left\{ \sum_{n=1}^{\infty} \boldsymbol{x}_n \text{ is ABSOLUTELY CONVERGENT.} \right\}$$

---

[4] Grattan-Guinness (1990) page 163

[5] Klauder (2010) page 4, Kubrusly (2001) page 43, Bachman and Narici (1966) pages 3–4

[6] Kadets and Kadets (1997) page 5 ⟨THEOREM 1.1.1 (RIEMANN'S THEOREM)⟩, BROMWICH (1908) PAGE 64 ⟨IV. ABSOLUTE CONVERGENCE.⟩, SZÁSZ AND BARLAZ (1952) PAGE 2

[7] Kadets and Kadets (1997) page 5 ⟨THEOREM 1.1.1 (RIEMANN'S THEOREM)⟩, BROMWICH (1908) PAGE 68 ⟨ARTICLE 28. RIEMANN'S THEOREM⟩

[8] Kadets and Kadets (1997) page 5 ⟨THEOREM 1.1.1 (RIEMANN'S THEOREM)⟩

Example 12.1 (Logarithmic Series). [9] Consider the sum $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n}$. To which value this sum converges, or whether it even converges at all, depends on the order in which the terms are summed. This is demonstrated by the following series:

EX

If the series is added in the given order, the result is ln 2:

$$\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} \triangleq \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{(n-1)}\frac{1}{n} = \ln 2$$

But if the order is changed, the sum can be any real value:

Let x be any real value (even an irrational one such as π or $\sqrt{2}$).

$$\text{Let}\qquad \mathsf{p}(N) \triangleq \begin{cases} \text{next unused odd value} & \text{if } \sum_{n=1}^{N-1}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} \le x \\ \text{next unused even value} & \text{if } \sum_{n=1}^{N-1}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} > x \end{cases}$$

$$\sum_{n=1}^{\infty}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} \triangleq \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} = x$$

The series can even be summed in such a way that it does not converge at all:

Let $\mathsf{q}(n)$ be a permutation that partitions the natural numbers into odd and even values such that $\left(\!\left(x_{\mathsf{q}(n)}\right)\!\right) = \left(\!\left(1,3,5,\ldots,2,4,6,\ldots\right)\!\right)$.

$$\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} = \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n-1}}_{\infty} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n}}_{\infty} \implies \text{diverges}$$

✎PROOF:

1. Proof that $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} = \ln 2$ using polynomial expansion:[10]

   (a) Lemma: Proof that $\frac{1}{1+x} = \sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}$:

$$\begin{aligned}(1+x)\left(\sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}\right) &= \sum_{k=0}^{2n-1}(-1)^k x^k + \sum_{k=0}^{2n-1}(-1)^k x^{k+1} + x^{2n}\\ &= 1 + \sum_{k=1}^{2n-1}(-1)^k x^k - \sum_{k=1}^{2n-1}(-1)^k x^k + x^{2n}\\ &= 1 + x^{2n}\end{aligned}$$

   (b) Lemma: Proof that $\lim_{n\to\infty}\int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x = 0$:

$$\begin{aligned}0 &< \lim_{n\to\infty}\int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x\\ &< \lim_{n\to\infty}\int_0^1 x^{2n}\,\mathrm{d}x\end{aligned}$$

[9] Bromwich (1908) pages 51–52 ⟨Article 21 Example 1⟩, Hall and Knight (1894) page 191 ⟨article 223⟩, Jolley (1961) pages 14–15 ⟨item (71)⟩, Sloane (2014) ⟨http://oeis.org/A002939⟩⟨$2n(2n-1)$⟩, Graham et al. (1994) page 99 ⟨n.w. diagonal of *spiral function*⟩, Many many thanks to Po-Ning Chen (Chinese: ???, pinyin: Chén Bó Níng) for his consultation regarding this series.

[10] Bromwich (1908) pages 51–52 ⟨Article 21 Example 1⟩

$$= \lim_{n\to\infty} \left.\frac{x^{2n+1}}{2n+1}\right|_0^1$$
$$= \lim_{n\to\infty} \frac{1}{2n+1}$$
$$= 0$$

(c) Proof that sum = ln 2:

$$\begin{aligned}
\ln 2 &= \ln 2 - \ln 1 \\
&= \int_1^2 \frac{1}{x}\,\mathrm{d}x \\
&= \int_0^1 \frac{1}{x+1}\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_0^1 \left\{ \sum_{k=0}^{2n-1} (-1)^k x^k + \frac{x^{2n}}{1+x} \right\} \mathrm{d}x && \text{by item (1a)} \\
&= \sum_{k=0}^{2n-1} (-1)^k \int_0^1 x^k\,\mathrm{d}x + \lim_{n\to\infty} \int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{k=0}^{2n-1} (-1)^k \frac{1}{k} + 0 && \text{by item (1b)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^{k-1} \frac{1}{k} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}
\end{aligned}$$

2. Proof that $\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{1}{n} = \ln 2$ using Taylor expansion:[11]

(a) Lemma: Proof that $\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n} x^n$:

$$\begin{aligned}
\ln(x+1) &= \sum_{n=0}^{\infty} \frac{\left[\mathbf{D}^n \ln(x+1)\right](0)}{n!} x^n && \text{by } \textit{Taylor series expansion} \\
&= \frac{\ln(1+0)}{0!} x^0 + \frac{\frac{1}{1+0}}{1!} x^1 - \frac{\frac{1}{(1+0)^2}}{2!} x^2 + \frac{\frac{2}{(1+0)^3}}{3!} x^3 - \frac{\frac{6}{(1+0)^4}}{4!} x^4 + \frac{\frac{24}{(1+0)^5}}{5!} x^5 - \frac{\frac{120}{(1+0)^6}}{6!} x^6 + \cdots \\
&= 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n} x^n
\end{aligned}$$

11 Hall and Knight (1894) page 191 ⟨article 223⟩

(b) Proof that $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} = \ln 2$:

$$
\begin{aligned}
&= \underbrace{\left(1-\frac{1}{2}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{\frac{1}{12}} + \underbrace{\left(\frac{1}{5}-\frac{1}{6}\right)}_{\frac{1}{30}} + \underbrace{\left(\frac{1}{7}-\frac{1}{8}\right)}_{\frac{1}{56}} + \dots \\
&= \sum_{n=1}^{\infty}(-1)^{(n+1)}\frac{1}{n}x^n\bigg|_{x=1} \\
&= \ln 2 && \text{by Lemma in item 1} \\
&\approx 0.693147
\end{aligned}
$$

3. Proof that $\sum_{n=1}^{\infty}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} = x$: If the partial sum is less than x, positive values are added. If the partial sum is greater than x, negative values are added. The limit is x.

4. Proof that $\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} = \infty - \infty$:

$$
\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} &= \underbrace{\sum_{n=1}^{\infty}(-1)^{(2n-1-1)}\frac{1}{2n-1}}_{\text{odd indices}} + \underbrace{\sum_{n=1}^{\infty}(-1)^{(2n-1)}\frac{1}{2n}}_{\text{even indices}} \\
&= \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n-1}}_{\infty} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n}}_{\infty} \\
&\to \text{(diverges)}
\end{aligned}
$$

Divergent series could even result in decisions that may be considered extremely irrational, as demonstrated by *St. Petersberg Paradox* (next).

Example 12.2 (St. Petersburg Paradox). [12] There is a lottery with a prize pot of \$1. A coin is tossed. If the coin is a tail, the money in the lottery is doubled (\$2, \$4, \$8, \$16, …). If the coin is a head, you win the money and the game is finished.

How much money would you be willing to play this game? The answer to this question for some people may depend on the expected value of how much money would be won. But the expected value of the amount of money you would win is

$$\tfrac{1}{2}\times \$1 + \tfrac{1}{4}\times \$2 + \tfrac{1}{8}\times \$4 + \tfrac{1}{16}\times \$8 + \dots = \tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\dots = \infty$$

Since the expected value of the win is infinity, you should be willing to pay any finite amount of money to play this game (even trillions of dollars). But yet common sense would tell most people that this would be an unwise investment.

[12] Székely (1986) pages 27–28, Bernoulli (1783) pages 31–32 ⟨§17⟩, de Montmort (1713) page 402 ⟨1713 letter from Nicolas Bernoulli⟩

12.2 Multiplication

Theorem 12.3. [13] *Let* $(\!|x_n|\!)_1^N$ *and* $(\!|y_n|\!)_1^N$ *be sequences over a ring* $(\mathbb{X}, +, \times)$.

THM
$$\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) = \underbrace{\sum_{n=0}^{p+q}\left(\sum_{k=\max(0,n-q)}^{\min(n,p)} x_k y_{n-k}\right)}_{\text{Cauchy product}}$$

✎PROOF:

1.

$$\begin{aligned}
\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) &= \sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m} \\
&= \sum_{n=0}^{p}\sum_{k=n}^{q+n} x_n y_{k-n} && k = n+m \qquad m = k-n \\
&\;\vdots \\
&= \sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)
\end{aligned}$$

2. Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	y_0	y_1	y_2	y_3	$\cdots$	y_q
x_0	x_0y_0	x_0y_1	x_0y_2	x_0y_3	$\cdots$	x_0y_q
x_1	x_1y_0	x_1y_1	x_1y_2	x_1y_3	$\cdots$	x_1y_q
x_2	x_2y_0	x_2y_1	x_2y_2	x_2y_3	$\cdots$	x_2y_q
x_3	x_3y_0	x_3y_1	x_3y_2	x_3y_3	$\cdots$	x_3y_q
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_p	x_py_0	x_py_1	x_py_2	x_py_3	$\cdots$	x_py_q

(a) The expression $\sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

(b) If we switched the order of summation to $\sum_{m=0}^{q}\sum_{n=0}^{p} x_n y_m z^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

(c) However the final result expression $\sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)$ is equivalent to adding *diagonally* starting from the upper left corner and proceding to the lower right.

(d) Upper limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:
$$\left.\begin{array}{lll} k & \le & n \\ k & \le & p \end{array}\right\} \implies k \le \min(n,p)$$

(e) Lower limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:
$$\left.\begin{array}{lll} k & \ge & 0 \\ k & \ge & n-q \end{array}\right\} \implies k \ge \max(0, n-q)$$

✏

[13] Apostol (1975) page 204 ⟨Definition 8.45⟩

Corollary 12.1. *Let* $(\!(x_n \in \mathbb{C})\!)$ *and* $(\!(y_n \in \mathbb{C})\!)$.

COR

$$\left(\sum_{n=0}^{\infty} x_n\right)\left(\sum_{m=0}^{\infty} y_m\right) = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} x_k y_{n-k}\right)}_{\text{Cauchy product}}$$

✎PROOF:

$$\begin{aligned}
\left(\sum_{n=0}^{p=\infty} x_n\right)\left(\sum_{m=0}^{q=\infty} y_m\right) &= \sum_{n=0}^{\infty}\left[\sum_{k=\max(0,n-\infty)}^{\min(n,\infty)} x_k y_{n-k}\right] && \text{by Theorem 12.3 page 180}\\
&= \sum_{n=0}^{\infty}\left((\sum_{k=0}^{n} x_k y_{n-k}\right)
\end{aligned}$$

✏

Theorem 12.4. [14] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM

$$\left\{\begin{array}{ll} X \text{ is ABSOLUTELY CONVERGENT} & \text{and} \\ Y \text{ is ABSOLUTELY CONVERGENT} & \end{array}\right\} \implies \left\{\begin{array}{ll} Z \text{ is ABSOLUTELY CONVERGENT} & \text{and} \\ Z = XY. & \end{array}\right\}$$

Theorem 12.5. [15] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM

$$\left\{\begin{array}{lll} 1. & X \text{ is ABSOLUTELY CONVERGENT} & \text{and} \\ 2. & Y \text{ is CONVERGENT} & \end{array}\right\} \implies \left\{\begin{array}{lll} 1. & Z \text{ is CONVERGENT} & \text{and} \\ 2. & Z = XY & \end{array}\right\}$$

Theorem 12.6. [16] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM

$$\left\{\begin{array}{lll} 1. & X \text{ is CONVERGENT} & \text{and} \\ 2. & Y \text{ is CONVERGENT} & \text{and} \\ 3. & Z \text{ is CONVERGENT} & \end{array}\right\} \implies \{Z = XY\}$$

12.3 Summability

Cauchy and Abel, the 19th century champions of rigour in analysis, firmly rejected any and all divergent sums. However in more recent times, certain classes of divergent sums have been found to be extremely useful. Often such sums are ones that are said to be *summable*.

Definition 12.3. [17]

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[14] Hardy (1949) pages 227–228 ⟨THEOREM 160⟩, BROMWICH (1908) PAGE 66 ⟨ARTICLE 27.⟩, CAUCHY (1821) PAGES 147–148 ⟨6.^{e} THÉORÈME⟩

[15] Hardy (1949) page 228 ⟨THEOREM 161⟩, BROMWICH (1908) PAGES 85–86 ⟨ARTICLE 35.⟩, MERTENS (1875)

[16] Hardy (1949) page 228 ⟨THEOREM 162⟩, ABEL (1826)

[17] Zygmund (2002) pages 75–76, Hardy (1949) page 96 ⟨5.4 Cesàro means⟩, Whittaker and Watson (1920) pages 155–156 ⟨8.43, 8.431⟩, Cesàro (1890)

DEF

The series $\sum_{n=0}^{\infty} x_n$ *is* ***summable by the*** k***-th arithmetic mean of Cesàro to limit*** x, *or* ***summable*** (C, k) ***to the limit*** x, *if*

$$\lim_{n\to\infty} \frac{S_n^k}{A_n^k} = x \quad \text{for } n \in \mathbb{W} \text{ and where}$$

$$S_n^k \triangleq \begin{cases} \sum_{m=0}^{n} x_m & \text{for } k = 0 \\ \sum_{m=0}^{n} S_m^{k-1} & \text{for } k = 1, 2, 3, \ldots \end{cases} \quad \text{and} \quad A_n^k \triangleq \begin{cases} 1 & \text{for } k = 0 \\ \sum_{m=0}^{n} A_m^{k-1} & \text{for } k = 1, 2, 3, \ldots \end{cases}$$

Proposition 12.1. [18]

PRP

$$\sum_{n=0}^{\infty} x_n \text{ is summable } (C, 0) \text{ to the limit } x \text{ if } \lim_{N\to\infty} \sum_{n=0}^{N} x_n = x \qquad \text{(normal convergence)}$$

$$\sum_{n=0}^{\infty} x_n \text{ is summable } (C, 1) \text{ to the limit } x \text{ if } \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} s_n = x \qquad \text{(arthimetic mean)}$$

$$\text{where } s_n \triangleq \sum_{m=0}^{n} x_m$$

Definition 12.4. [19]

DEF

The series $\sum_{n=0}^{\infty} a_n$ *is* ***summable by Euler's method to limit*** a *if*

$$\lim_{x\to 1-0} \sum_{n=0}^{\infty} a_n x^n = a$$

Example 12.3. [20]

EX

The series $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ is *divergent*
(it is *not* summable $(C, 0)$),
but yet it *is* summable $(C, 1)$ to the limit $\frac{1}{2}$.
It is also summable by Euler's method to the limit $\frac{1}{2}$.

✎PROOF:

1. Proof for Cesàro summability:

   (a) Note that the sequence of partial sums s_n is $s_0 = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0$, $s_4 = 1$, ... That is

   $$s_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

[18] Zygmund (2002) pages 75–76, Thomson et al. (2008) page 129 ⟨Definition 3.54⟩, Szász and Barlaz (1952) page 13
[19] Whittaker and Watson (1920) page 155 ⟨8.42⟩
[20] Thomson et al. (2008) page 130 ⟨Example 3.56⟩, Whittaker and Watson (1920) page 155 ⟨8.42⟩

(b) Then

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} s_k &= \lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=0}^{2n} s_k \\
&= \lim_{n\to\infty}\frac{1}{2n+1}\left(\underbrace{\sum_{k=0}^{n} s_{2k}}_{\text{even terms}} + \underbrace{\sum_{k=0}^{n-1} s_{2k+1}}_{\text{odd terms}}\right) \\
&= \lim_{n\to\infty}\frac{1}{2n+1}\left(\sum_{k=0}^{n} 1 + \sum_{k=0}^{n-1} 0\right) \\
&= \lim_{n\to\infty}\frac{n+1}{2n+1} \\
&= \frac{1}{2}
\end{aligned}$$

2. Proof for Euler summability:

$$\begin{aligned}
\lim_{x\to 1-0}\sum_{n=0}^{\infty}(-1)^n &= \lim_{x\to 1-0}\lim_{n\to\infty}\left(\frac{1}{1+x} = \sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}\right) \\
&= \lim_{x\to 1-0}\frac{1}{1+x} && \text{by item (1a) (page 177) of Example 12.1} \\
&= \frac{1}{2}
\end{aligned}$$

✏

12.4 Convergence in Banach spaces

The properties of *strong convergence* and *weak convergence* are defined on *sequences* (Definition 9.9 page 145). An infinite sum $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ in a Banach space is the limit of a sequence of partial sums $\left(\!\left(\sum_{n=1}^{N} \boldsymbol{x}_n\right)\!\right)$, so the properties of strong and weak convergence apply to infinite sums as well. Definition 12.5 (next) assigns special equality symbols for these sums.

Definition 12.5. *Let* $\boldsymbol{B} \triangleq \left(X,\ +,\ \cdot,\ (\mathbb{F},\ \dot{+},\ \dot{\times}),\ \|\cdot\|\right)$ *be a Banach space.*

DEF

The expression $\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \boldsymbol{x}_n$ *denotes that the sum* ***converges strongly*** *to* $\boldsymbol{x}$.

The expression $\boldsymbol{x} \doteq \sum_{n=1}^{\infty} \boldsymbol{x}_n$ *denotes that the sum* ***converges weakly*** *to* $\boldsymbol{x}$.

12.5 Convergence tests for real sequences

Theorem 12.7 (*comparison test*).[21]

THM

$$\left\{\begin{array}{lll} 1. & \sum_{n=1}^{\infty} \left(\!\left(y_n \right)\!\right) \text{ CONVERGES} & \textit{and} \\ 2. & x_n \leq y_n & \forall n \in \mathbb{N} \end{array}\right\} \implies \sum_{n=1}^{\infty} \left(\!\left(x_n \right)\!\right) \text{ CONVERGES}$$

[21] Bonar et al. (2006) page 26 ⟨Theorem 1.53 (Limit Comparison Test Strengthened)⟩, Heinbockel (2010) page 152 ⟨Comparison Tests⟩

CHAPTER 13

DISTANCE SPACES WITH POWER TRIANGLE INEQUALITIES

13.1 Definitions

This chapter introduces a relation called the *power triangle inequality* (Definition 13.2 page 185). It is a generalization of other common relations, including the *triangle inequality* (Definition 13.3 page 186). The *power triangle inequality* is defined in terms of a function herein called the *power triangle function* (next definition). This function is a special case of the *power mean* with $N = 2$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$ (Definition 11.3 page 164). *Power means* have the attractive properties of being *continuous* and *strictly monontone* with respect to a free parameter $p \in \mathbb{R}^*$ (Theorem 11.3 page 164). This fact is inherited and exploited by the *power triangle inequality* (Corollary 13.1 page 186).

Definition 13.1. [1] *Let (X, d) be a* DISTANCE SPACE *(Definition 2.1 page 27). Let $\mathbb{R}^+$ be the set of all* POSITIVE REAL NUMBERS *and $\mathbb{R}^*$ be the set of* EXTENDED REAL NUMBERS $\left(\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}\right)$.

DEF *The **power triangle function** τ on (X, d) is defined as*

$$\tau(p, \sigma; x, y, z; \mathsf{d}) \triangleq 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x, z) + \frac{1}{2}\mathsf{d}^p(z, y)\right]^{\frac{1}{p}} \quad \forall (p,\sigma)\in\mathbb{R}^*\times\mathbb{R}^{\vdash}, \quad x,y,z\in X$$

Remark 13.1. [2] In the field of *probabilistic metric spaces*, a function called he *triangle function* was introduced by Sherstnev in 1962. However, the *power triangle function* as defined in this present paper is *not* a special case of (is not compatible with) the *triangle function* of Sherstnev. Another definition of *triangle function* has been offered by Bessenyei in 2014 with special cases of $\Phi(u, v) \triangleq c(u+v)$ and $\Phi(u, v) \triangleq (u^p + v^p)^{\frac{1}{p}}$, which *are* similar to the definition of *power triangle function* offered in this present paper.

Definition 13.2. *Let (X, d) be a* DISTANCE SPACE. *Let 2^{XXX} be the set of all trinomial* RELATIONS *on X.*

DEF *A relation $\triangle(p, \sigma; \mathsf{d})$ in 2^{XXX} is a **power triangle inequality** on (X, d) if*

$$\triangle(p, \sigma; \mathsf{d}) \triangleq \left\{(x, y, z) \in X^3 | \mathsf{d}(x, y) \leq \tau(p, \sigma; x, y, z; \mathsf{d})\right\} \qquad \textit{for some } (p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+.$$

*The tupple $(X, \mathsf{d}, p, \sigma)$ is a **power distance space** and d a **power distance** or **power distance function** if (X, d) is a* DISTANCE SPACE *in which the* TRIANGLE RELATION $\triangle(p, \sigma; \mathsf{d})$ *holds.*

[1] Greenhoe (2016)

[2] Sherstnev (1962) page 4, Schweizer and Sklar (1983) page 9 ⟨(1.6.1)–(1.6.4)⟩, Bessenyei and Pales (2014) page 2

The *power triangle function* can be used to define some standard inequalities (next definition). See Corollary 13.2 (page 187) for some justification of the definitions.

Definition 13.3. [3] *Let* $\triangle(p,\sigma;\mathsf{d})$ *be a* POWER TRIANGLE INEQUALITY *on a* DISTANCE SPACE (X,d).

DEF

1. $\triangle(\infty,\ \sigma/2;\mathsf{d})$ *is the* ***σ-inframetric inequality***
2. $\triangle(\infty,\ \frac{1}{2};\mathsf{d})$ *is the* ***inframetric inequality***
3. $\triangle(2,\ \sqrt{2}/2;\mathsf{d})$ *is the* ***quadratic inequality***
4. $\triangle(1,\ \sigma;\mathsf{d})$ *is the* ***relaxed triangle inequality***
5. $\triangle(1/2,\ 2;\mathsf{d})$ *is the* ***square mean root inequality***
6. $\triangle(1,\ 1;\mathsf{d})$ *is the* ***triangle inequality***
7. $\triangle(0,\ \frac{1}{2};\mathsf{d})$ *is the* ***geometric inequality***
8. $\triangle(-1,\ \frac{1}{4};\mathsf{d})$ *is the* ***harmonic inequality***
9. $\triangle(-\infty,\ \frac{1}{2};\mathsf{d})$ *is the* ***minimal inequality***

Definition 13.4. [4] *Let* (X,d) *be a* DISTANCE SPACE *(Definition 2.1 page 27).*

DEF

1. (X,d) *is a* ***metric space*** *if the* TRIANGLE INEQUALITY *holds in* X.
2. (X,d) *is a* ***near metric space*** *if the* RELAXED TRIANGLE INEQUALITY *holds in* X.
3. (X,d) *is an* ***inframetric space*** *if the* INFRAMETRIC INEQUALITY *holds in* X.
4. (X,d) *is a* ***σ-inframetric space*** *if the* σ-INFRAMETRIC INEQUALITY *holds in* X.

13.2 Properties

13.2.1 Relationships of the power triangle function

Corollary 13.1. *Let* $\tau(p,\sigma;x,y,z;\mathsf{d})$ *be the* POWER TRIANGLE FUNCTION *(Definition 13.1 page 185) in the* DISTANCE SPACE *(Definition 2.1 page 27)* (X,d). *Let* $(\mathbb{R},|\cdot|,\le)$ *be the* ORDERED METRIC SPACE *with the usual ordering relation* $\le$ *and usual metric* $|\cdot|$ *on* $\mathbb{R}$.

COR

The function $\tau(p,\sigma;x,y,z;\mathsf{d})$ *is* ***continuous*** *and* ***strictly monotone*** *in* $(\mathbb{R},|\cdot|,\le)$ *with respect to both the variables* p *and* σ.

✎PROOF:

1. Proof that $\tau(p,\sigma;x,y,z;\mathsf{d})$ is *continuous* and *strictly monotone* with respect to p: This follows directly from Theorem 11.3 (page 164).

[3] Bessenyei and Pales (2014) page 2, Czerwik (1993) page 5 ⟨*b-metric*; (1),(2),(5)⟩, Fagin et al. (2003b), Fagin et al. (2003a) ⟨Definition 4.2 (Relaxed metrics)⟩, Xia (2009) page 453 ⟨Definition 2.1⟩, Heinonen (2001) page 109 ⟨14.1 Quasimetric spaces.⟩, Kirk and Shahzad (2014) page 113 ⟨Definition 12.1⟩, Deza and Deza (2014) page 7, Hoehn and Niven (1985) page 151, Gibbons et al. (1977) page 51 ⟨*square-mean-root* (*SMR*) (2.4.1)⟩, Euclid (circa 300BC) ⟨triangle inequality—Book I Proposition 20⟩

[4] **metric space**: Dieudonné (1969) page 28, Copson (1968) page 21, Hausdorff (1937) page 109, Fréchet (1928), Fréchet (1906) page 30 **near metric space**: Czerwik (1993) page 5 ⟨*b-metric*; (1),(2),(5)⟩, Fagin et al. (2003b), Fagin et al. (2003a) ⟨Definition 4.2 (Relaxed metrics)⟩, Xia (2009) page 453 ⟨Definition 2.1⟩, Heinonen (2001) page 109 ⟨14.1 Quasimetric spaces.⟩, Kirk and Shahzad (2014) page 113 ⟨Definition 12.1⟩, Deza and Deza (2014) page 7

2. Proof that $\tau(p,\sigma;x,y,z;\mathsf{d})$ is *continuous* and *strictly monotone* with respect to σ:

$$\begin{aligned}
\tau(p,\sigma;x,y,z;\mathsf{d}) &\triangleq 2\sigma\underbrace{\left[\frac{1}{2}\mathsf{d}^p(x,z)+\frac{1}{2}\mathsf{d}^p(z,y)\right]^{\frac{1}{p}}}_{\mathsf{f}(p,x,y,z)} && \text{by definition of } \tau \text{ (Definition 13.1 page 185)}\\
&= 2\sigma\mathsf{f}(p,x,y,z) && \text{where f is defined as above}\\
&\implies \tau \text{ is } \textit{affine} \text{ with respect to } \sigma\\
&\implies \tau \text{ is } \textit{continuous} \text{ and } \textit{strictly monotone} \text{ with respect to } \sigma:
\end{aligned}$$

⇨

Corollary 13.2. *Let* $\tau(p,\sigma;x,y,z;\mathsf{d})$ *be the* POWER TRIANGLE FUNCTION *in the* DISTANCE SPACE (X,d).

COR

$$\tau(p,\sigma;x,y,z;\mathsf{d}) = \begin{cases}
2\sigma\max\{\mathsf{d}(x,z),\ \mathsf{d}(z,y)\} & \text{for } p=\infty, & \text{(MAXIMUM, } \textit{corresponds to} \text{ INFRAMETRIC SPACE)}\\
2\sigma\left[\frac{1}{2}\mathsf{d}^2(x,z)+\frac{1}{2}\mathsf{d}^2(z,y)\right]^{\frac{1}{2}} & \text{for } p=2, & \text{(QUADRATIC MEAN)}\\
\sigma[\mathsf{d}(x,z)+\mathsf{d}(z,y)] & \text{for } p=1, & \text{(ARITHMETIC MEAN, } \textit{corresponds to} \text{ NEAR METRIC SPACE)}\\
2\sigma\sqrt{\mathsf{d}(x,z)}\sqrt{\mathsf{d}(z,y)} & \text{for } p=0 & \text{(GEOMETRIC MEAN)}\\
4\sigma\left[\frac{1}{\mathsf{d}(x,z)}+\frac{1}{\mathsf{d}(z,y)}\right]^{-1} & \text{for } p=-1 & \text{(HARMONIC MEAN)}\\
2\sigma\min\{\mathsf{d}(x,z),\ \mathsf{d}(z,y)\} & \text{for } p=-\infty, & \text{(MINIMUM)}
\end{cases}$$

✎PROOF: These follow directly from Theorem 11.3 (page 164). ⇨

Corollary 13.3. *Let* (X,d) *be a* DISTANCE SPACE.

COR

$$\begin{aligned}
2\sigma\min\{\mathsf{d}(x,z),\ \mathsf{d}(z,y)\} \;\leq\; 4\sigma\left[\frac{1}{\mathsf{d}(x,z)}+\frac{1}{\mathsf{d}(z,y)}\right]^{-1} &\;\leq\; 2\sigma\sqrt{\mathsf{d}(x,z)}\sqrt{\mathsf{d}(z,y)}\\
\leq\; \sigma[\mathsf{d}(x,z)+\mathsf{d}(z,y)] &\;\leq\; 2\sigma\max\{\mathsf{d}(x,z),\ \mathsf{d}(z,y)\}
\end{aligned}$$

✎PROOF: These follow directly from Corollary 11.1 (page 167). ⇨

13.2.2 Properties of power distance spaces

The *power triangle inequality* property of a *power distance space* axiomatically endows a metric with an upper bound. Lemma 13.1 (next) demonstrates that there is a complementary lower bound somewhat similar in form to the *power triangle inequality* upper bound. In the special case where $2\sigma = 2^{\frac{1}{p}}$, the lower bound helps provide a simple proof of the *continuity* of a large class of *power distance functions* (Theorem 13.5 page 192). The inequality $2\sigma \leq 2^{\frac{1}{p}}$ is a special relation in this paper and appears repeatedly in this paper; it appears as an inequality in Lemma 13.2 (page 190), Corollary 13.5 (page 190) and Corollary 13.6 (page 190), and as an equality in Lemma 13.1 (next) and Theorem 13.5 (page 192). It is plotted in Figure 13.1 (page 188).

Lemma 13.1. [5] *Let* (X,d,p,σ) *be a* POWER TRIANGLE TRIANGLE SPACE *(Definition 13.2 page 185). Let* $|\cdot|$ *be the* ABSOLUTE VALUE *function (Definition F.4 page 342). Let* $\max\{x,y\}$ *be the maximum and* $\min\{x,y\}$ *the*

[5] in *metric space* ($(p,\sigma)=(1,1)$): Dieudonné (1969) page 28, Michel and Herget (1993) page 266, Berberian (1961) page 37 ⟨Theorem II.4.1⟩

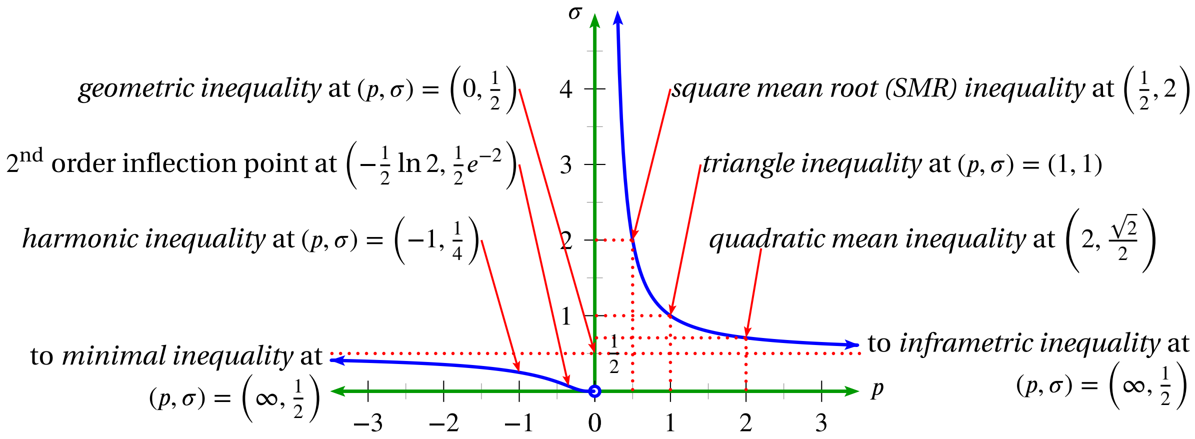

Figure 13.1: $\sigma = \frac{1}{2}(2^{\frac{1}{p}}) = 2^{\frac{1}{p}-1}$ or $p = \frac{\ln 2}{\ln(2\sigma)}$ (see Lemma 13.1 page 187, Lemma 13.2 page 190, Corollary 13.6 page 190, Corollary 13.5 page 190, and Theorem 13.5 page 192).

minimum of any $x, y \in \mathbb{R}^*$. Then, for all $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$,

LEM

1. $\mathsf{d}^p(x,y) \geq \max\left\{0, \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y), \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\right\}$ $\forall x,y,z \in X$ *and*
2. $\mathsf{d}(x,y) \geq |\mathsf{d}(x,z) - \mathsf{d}(z,y)|$ *if* $p \neq 0$ *and* $2\sigma = 2^{\frac{1}{p}}$ $\forall x,y,z \in X$.

✎PROOF:

1. lemma: $\frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y) \leq \mathsf{d}^p(x,y) \quad \forall (p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$: Proof:

$$\begin{aligned}
\frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y) &\leq \frac{2}{(2\sigma)^p}\left[2\sigma\left[\tfrac{1}{2}\mathsf{d}^p(x,y) + \tfrac{1}{2}\mathsf{d}^p(y,z)\right]^{\frac{1}{p}}\right]^p - \mathsf{d}^p(z,y) && \text{by \textit{power triangle inequality}}\\
&= \frac{2(2\sigma)^p}{(2\sigma)^p}\left[\tfrac{1}{2}\mathsf{d}^p(x,y) + \tfrac{1}{2}\mathsf{d}^p(y,z)\right] - \mathsf{d}^p(z,y)\\
&= \left[\mathsf{d}^p(x,y) + \mathsf{d}^p(y,z)\right] - \mathsf{d}^p(y,z) && \text{by \textit{symmetric} property of } \mathsf{d}\\
&= \mathsf{d}^p(x,y)
\end{aligned}$$

2. Proof for $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ case:

$$\begin{array}{lllll}
\mathsf{d}^p(x,y) & & \geq & \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y) & \text{by (1) lemma}\\
\mathsf{d}^p(x,y) & = \mathsf{d}^p(y,x) & \geq & \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x) & \text{by \textit{commutative} property of } \mathsf{d} \text{ and (1) lemma}\\
\mathsf{d}^p(x,y) & & \geq & 0 & \text{by \textit{non-negative} property of } \mathsf{d} \text{ (Definition 2.1 page 27)}
\end{array}$$

The rest follows because $\mathsf{g}(x) \triangleq x^{\frac{1}{p}}$ is *strictly monotone* in $\mathbb{R}^{\mathbb{R}}$.

3. Proof for $2\sigma = 2^{\frac{1}{p}}$ case:

$$\begin{aligned}
\mathsf{d}(x,y) &\geq \max\left\{0, \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y), \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\right\}^{\frac{1}{p}} && \text{by item (2) (page 188)}\\
&= \max\{0, \mathsf{d}(x,z) - \mathsf{d}(z,y), \mathsf{d}(y,z) - \mathsf{d}(z,x)\} && \text{by } 2\sigma = 2^{\frac{1}{p}} \text{ hyp.} \iff \frac{2}{(2\sigma)^p} = 1\\
&= \max\{0, (\mathsf{d}(x,z) - \mathsf{d}(z,y)), -(\mathsf{d}(x,z) - \mathsf{d}(z,y))\} && \text{by \textit{symmetric} property of } \mathsf{d}\\
&= |(\mathsf{d}(x,z) - \mathsf{d}(z,y))|
\end{aligned}$$

⇨

Theorem 13.1. *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE (Definition 13.2 page 185). *Let* B *be an* OPEN BALL (Definition 2.4 page 28) *on* (X, d). *Then for all* $(p, \sigma) \in (\mathbb{R}^* \setminus \{0\}) \times \mathbb{R}^+$,

$$\text{THM}\quad \left\{\begin{array}{lll} A. & 2\sigma \le 2^{\frac{1}{p}} & \text{and} \\ B. & q \in \mathsf{B}(\theta, r) & \end{array}\right\} \implies \left\{\begin{array}{ll} 1. & \exists r_q \in \mathbb{R}^+ \text{ such that} \\ & \mathsf{B}(q, r_q) \subseteq \mathsf{B}(\theta, r) \end{array}\right\} \implies \left\{\begin{array}{ll} B. & q \in \mathsf{B}(\theta, r) \end{array}\right\}$$

✎PROOF:

1. lemma:

$$\begin{aligned} q \in \mathsf{B}(\theta, r) &\iff \mathsf{d}(\theta, q) < r && \text{by definition of } \textit{open ball} \text{ (Definition 2.4 page 28)} \\ &\iff 0 < r - \mathsf{d}(\theta, q) && \text{by field property of real numbers} \\ &\iff \exists r_q \in \mathbb{R}^+ \text{ such that } 0 < r_q < r - \mathsf{d}(\theta, q) && \text{by } \textit{The Archimedean Property}^{6} \end{aligned}$$

2. Proof that $(A), (B) \implies (1)$:

$$\begin{aligned} \mathsf{B}(q, r_q) &\triangleq \left\{x \in X \mid \mathsf{d}(q, x) < r_q\right\} && \text{by definition of } \textit{open ball} \text{ (Definition 2.4 page 28)} \\ &= \left\{x \in X \mid \mathsf{d}^p(q, x) < r_q^p \in \mathbb{R}^+\right\} && \text{because } \mathsf{f}(x) \triangleq x^p \text{ is } \textit{monotone} \\ &\subseteq \{x \in X \mid \mathsf{d}^p(q, x) < r^p - \mathsf{d}^p(\theta, q)\} && \text{by hypothesis B and (1) lemma page 189} \\ &= \{x \in X \mid \mathsf{d}^p(\theta, q) + \mathsf{d}^p(q, x) < r^p\} && \text{by field property of real numbers} \\ &= \left\{x \in X \mid \left[\mathsf{d}^p(\theta, q) + \mathsf{d}^p(q, x)\right]^{\frac{1}{p}} < r\right\} && \text{because } \mathsf{f}(x) \triangleq x^{\frac{1}{p}} \text{ is } \textit{monotone} \\ &\subseteq \left\{x \in X \mid 2^{1-1/p}\sigma\left[\mathsf{d}^p(\theta, q) + \mathsf{d}^p(q, x)\right]^{\frac{1}{p}} < r\right\} && \text{by hypothesis A which implies } 2^{1-1/p}\sigma \le 1 \\ &= \left\{x \in X \mid 2\sigma\left[\tfrac{1}{2}\mathsf{d}(\theta, q)\,x + \tfrac{1}{2}\mathsf{d}^p(q, x)\right]^{\frac{1}{p}} < r\right\} && \text{because } 2^{1-1/p}\sigma = 2\sigma(\tfrac{1}{2})^{\frac{1}{p}} \\ &\triangleq \{x \in X \mid \tau(p, \sigma, \theta, x, q) < r\} && \text{by definition of } \tau \text{ (Definition 13.1 page 185)} \\ &\subseteq \{x \in X \mid \mathsf{d}(\theta, x) < r\} && \text{by definition of } (X, \mathsf{d}, p, \sigma) \text{ (Definition 13.2 page 185)} \\ &\triangleq \mathsf{B}(\theta, r) && \text{by definition of } \textit{open ball} \text{ (Definition 2.4 page 28)} \end{aligned}$$

3. Proof that $(B) \impliedby (1)$:

$$\begin{aligned} q &\in \{x \in X \mid \mathsf{d}(q, x) = 0\} && \text{by } \textit{nondegenerate} \text{ property (Definition 2.1 page 27)} \\ &\subseteq \left\{x \in X \mid \mathsf{d}(q, x) < r_q\right\} && \text{because } r_q > 0 \\ &\triangleq \mathsf{B}(q, r_q) && \text{by definition of } \textit{open ball} \text{ (Definition 2.4 page 28)} \\ &\subseteq \mathsf{B}(\theta, r) && \text{by hypothesis 2} \end{aligned}$$

✏

Corollary 13.4. *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE. *Then for all* $(p, \sigma) \in (\mathbb{R}^* \setminus \{0\}) \times \mathbb{R}^+$,

$$\text{THM}\quad \left\{2\sigma \le 2^{\frac{1}{p}}\right\} \implies \{\textit{every } \text{OPEN BALL } \textit{in } (X, \mathsf{d}) \textit{ is } \text{OPEN}\}$$

✎PROOF: This follows from Theorem 13.1 (page 189) and Theorem 2.2 (page 30). ✏

[6] Aliprantis and Burkinshaw (1998) page 17 ⟨Theorem 3.3 ("*The Archimedean Property*") and Theorem 3.4⟩, Zorich (2004) page 53 ⟨6° ("*The principle of Archimedes*") and 7°⟩

Corollary 13.5. *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE. *Let* $\boldsymbol{B}$ *be the set of all* OPEN BALLS *in* (X, d). *Then for all* $(p, \sigma) \in (\mathbb{R}^* \setminus \{0\}) \times \mathbb{R}^+$,

COR
$$\left\{ 2\sigma \le 2^{\frac{1}{p}} \right\} \implies \left\{ \boldsymbol{B} \text{ is a BASE for } (X, \boldsymbol{T}) \right\}$$

✎PROOF:

1. The set of all *open balls* in (X, d) is a *base* for $(X, \boldsymbol{T})$ by Corollary 13.4 (page 189) and Theorem 1.4 (page 8).

2. $\boldsymbol{T}$ is a topology on X by Definition 1.2 (page 8).

✏

Lemma 13.2 (next) demonstrates that every point in an open set is contained in an open ball that is contained in the original open set (Figure 3.3 page 38).

Lemma 13.2. *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE. *Let* B *be an* OPEN BALL *on* (X, d). *Then for all* $(p, \sigma) \in (\mathbb{R}^* \setminus \{0\}) \times \mathbb{R}^+$,

LEM
$$\left\{ \begin{array}{ll} \text{A.} & 2\sigma \le 2^{\frac{1}{p}} \quad \text{and} \\ \text{B.} & U \text{ is OPEN in } (X, \mathsf{d}) \end{array} \right\} \implies \left\{ \begin{array}{ll} \text{1.} & \forall x \in U,\ \exists r \in \mathbb{R}^+ \text{ such that} \\ & \mathsf{B}(x, r) \subseteq U \end{array} \right\} \implies \left\{ \begin{array}{ll} \text{B.} & U \text{ is} \\ & \text{OPEN in } (X, \mathsf{d}) \end{array} \right\}$$

✎PROOF:

1. Proof that for $((A), (B) \implies (1))$:

$$\begin{aligned} U &= \bigcup \left\{ \mathsf{B}(x_\gamma, r_\gamma) \,|\, \mathsf{B}(x_\gamma, r_\gamma) \subseteq U \right\} && \text{by left hypothesis and Corollary 13.5 page 190} \\ &\supseteq \mathsf{B}(x, r) && \text{because } x \text{ must be in one of those balls in } U \end{aligned}$$

2. Proof that $((B) \impliedby (1))$ case:

$$\begin{aligned} U &= \bigcup \{ x \in X | x \in U \} && \text{by definition of union operation } \bigcup \\ &= \bigcup \{ \mathsf{B}(x, r) \,|\, x \in U \text{ and } \mathsf{B}(x, r) \subseteq U \} && \text{by hypothesis (1)} \\ &\implies U \text{ is open} && \text{by Corollary 13.5 page 190 and Corollary 2.1 page 29} \end{aligned}$$

✏

Corollary 13.6. [7] *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE. *Let* B *be an* OPEN BALL *on* (X, d). *Then for all* $(p, \sigma) \in (\mathbb{R}^* \setminus \{0\}) \times \mathbb{R}^+$,

COR
$$\left\{ 2\sigma \le 2^{\frac{1}{p}} \right\} \implies \left\{ \text{every OPEN BALL } \mathsf{B}(x, r) \text{ in } (X, \mathsf{d}) \text{ is OPEN} \right\}$$

[7] in *metric space* $((p, \sigma) = (1, 1))$: Rosenlicht (1968) pages 40–41, Aliprantis and Burkinshaw (1998) page 35

✎PROOF:

The union of any set of open balls is open — by Corollary 13.5 page 190

$\implies$ the union of a set of just one open ball is open

$\implies$ every open ball is open.

✏

Theorem 13.2. [8] *Let $(X, \mathsf{d}, p, \sigma)$ be a* POWER DISTANCE SPACE. *Let (X, T) be a* TOPOLOGICAL SPACE INDUCED BY (X, d). *Let $(\!(x_n \in X)\!)_{n\in\mathbb{Z}}$ be a sequence in (X, d).*

THM

$$\underbrace{(\!(x_n)\!) \text{ converges to a limit } x}_{\text{(Definition 9.3 page 132)}} \iff \left\{\begin{array}{l}\text{for any } \varepsilon \in \mathbb{R}^+, \text{ there exists } N \in \mathbb{Z} \\ \text{such that for all } n > N, \quad \mathsf{d}(x_n, x) < \varepsilon\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
(\!(x_n)\!) \to x &\iff x_n \in U \quad \forall U \in N_x,\ n > N && \text{by Definition 9.3 page 132}\\
&\iff \exists \mathsf{B}(x, \varepsilon) \quad \text{such that} \quad x_n \in \mathsf{B}(x, \varepsilon)\, \forall n > N && \text{by Lemma 13.2 page 190}\\
&\iff \mathsf{d}(x_n, x) < \varepsilon && \text{by Definition 2.4 page 28}
\end{aligned}$$

✏

In *distance spaces* (Definition 2.1 page 27), not all *convergent* sequences are *Cauchy* (Example 9.4 page 138). However in a distance space with any *power triangle inequality* (Definition 13.2 page 185), all *convergent* sequences are *Cauchy* (next theorem).

Theorem 13.3. [9] *Let $(X, \mathsf{d}, p, \sigma)$ be a* POWER DISTANCE SPACE. *Let B be an* OPEN BALL *on (X, d).*

THM

For any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$,

$$\left\{\begin{array}{l}(\!(x_n)\!) \text{ is CONVERGENT} \\ \text{in } (X, \mathsf{d})\end{array}\right\} \implies \left\{\begin{array}{l}(\!(x_n)\!) \text{ is CAUCHY} \\ \text{in } (X, \mathsf{d})\end{array}\right\} \implies \left\{\begin{array}{l}(\!(x_n)\!) \text{ is BOUNDED} \\ \text{in } (X, \mathsf{d})\end{array}\right\}$$

✎PROOF:

1. Proof that *convergent* $\implies$ *Cauchy*:

$$\begin{aligned}
\mathsf{d}(x_n, x_m) &\le \tau(p, \sigma; x_n, x_m, x) && \text{by definition of } \textit{power triangle inequality} \text{ (Definition 13.2 page 185)}\\
&\triangleq 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x_n, x) + \frac{1}{2}\mathsf{d}^p(x_m, x)\right]^{\frac{1}{p}} && \text{by definition of } \textit{power triangle function} \text{ (Definition 13.1 page 185)}\\
&< 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} && \text{by } \textit{convergence} \text{ hypothesis (Definition 9.3 page 132)}\\
&= 2\sigma\varepsilon && \text{by definition of } \textit{convergence} \text{ (Definition 9.3 page 132)}\\
&\implies \textit{Cauchy} && \text{by definition of } \textit{Cauchy} \text{ (Definition 9.5 page 134)}\\
\mathsf{d}(x_n, x_m) &\le \tau(\infty, \sigma; x_n, x_m, x) && \text{by definition of } \textit{power triangle inequality} \text{ at } p = \infty\\
&= 2\sigma \max\left\{\mathsf{d}(x_n, x), \mathsf{d}(x_m, x)\right\} && \text{by Corollary 13.2 (page 187)}\\
&= 2\sigma \max\{\varepsilon, \varepsilon\} && \text{by } \textit{convergent} \text{ hypothesis (Definition 9.3 page 132)}\\
&= 2\sigma\varepsilon && \text{by definition of max}
\end{aligned}$$

[8] in *metric space*: Rosenlicht (1968) page 45, Giles (1987) page 37 ⟨3.2 Definition⟩

[9] in *metric space*: Giles (1987) page 49 ⟨Theorem 3.30⟩, Rosenlicht (1968) page 51, Apostol (1975) pages 72–73 ⟨Theorem 4.6⟩

$$\begin{aligned}
\mathsf{d}(x_n, x_m) &\le \tau(-\infty, \sigma; x_n, x_m, x) && \text{by definition of \textit{power triangle inequality} at } p = -\infty \\
&= 2\sigma \min\{\mathsf{d}(x_n, x),\ \mathsf{d}(x_m, x)\} && \text{by Corollary 13.2 (page 187)} \\
&= 2\sigma \min\{\varepsilon, \varepsilon\} && \text{by \textit{convergent} hypothesis (Definition 9.3 page 132)} \\
&= 2\sigma\varepsilon && \text{by definition of min} \\
\mathsf{d}(x_n, x_m) &\le \tau(0, \sigma; x_n, x_m, x) && \text{by definition of \textit{power triangle inequality} at } p = 0 \\
&= 2\sigma \sqrt{\mathsf{d}(x_n, x)}\sqrt{\mathsf{d}(x_m, x)} && \text{by Corollary 13.2 (page 187)} \\
&= 2\sigma\sqrt{\varepsilon}\sqrt{\varepsilon} && \text{by \textit{convergent} hypothesis (Definition 9.3 page 132)} \\
&= 2\sigma\varepsilon && \text{by property of } \mathbb{R}
\end{aligned}$$

2. Proof that *Cauchy* $\implies$ *bounded*: by Proposition 9.1 (page 135).

✏

Theorem 13.4. [10] *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE. *Let* $\mathsf{f} \in \mathbb{Z}^{\mathbb{Z}}$ *be a* STRICTLY MONOTONE *function such that* $\mathsf{f}(n) < \mathsf{f}(n+1)$.

THM *For any* $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$

$$\left\{\begin{array}{lll} 1. & (x_n)_{n\in\mathbb{Z}} \text{ is CAUCHY} & \text{and} \\ 2. & \left(x_{\mathsf{f}(n)}\right)_{n\in\mathbb{Z}} \text{ is CONVERGENT} & \end{array}\right\} \implies \left\{(x_n)_{n\in\mathbb{Z}} \text{ is CONVERGENT.}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{d}(x_n, x) &= \mathsf{d}(x, x_n) && \text{by \textit{symmetric} property of d} \\
&\le \tau(p, \sigma; x, x_n, x_{\mathsf{f}(n)}) && \text{by definition of \textit{power triangle inequality} (Definition 13.2 page 185)} \\
&\triangleq 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x, x_{\mathsf{f}(n)}) + \frac{1}{2}\mathsf{d}^p(x_{\mathsf{f}(n)}, x_n)\right]^{\frac{1}{p}} && \text{by definition of \textit{power triangle function} (Definition 13.1 page 185)} \\
&= 2\sigma\left[\frac{1}{2}\varepsilon + \frac{1}{2}\mathsf{d}^p(x_{\mathsf{f}(n)}, x_n)\right]^{\frac{1}{p}} && \text{by left hypothesis 2} \\
&= 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} && \text{by left hypothesis 1} \\
&= 2\sigma\varepsilon \\
&\implies \textit{convergent} && \text{by definition of \textit{convergent} (Definition 9.3 page 132)}
\end{aligned}$$

✏

Theorem 13.5. [11] *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE. *Let* $(\mathbb{R}, \mathsf{q})$ *be a metric space of real numbers with the usual metric* $\mathsf{q}(x, y) \triangleq |x - y|$. *Then*

THM $$\left\{2\sigma = 2^{\frac{1}{p}}\right\} \implies \{\ \mathsf{d} \text{ is CONTINUOUS in } (\mathbb{R}, \mathsf{q})\ \}$$

✎PROOF:

$$\begin{aligned}
|\mathsf{d}(x, y) - \mathsf{d}(x_n, y_n)| &\le |\mathsf{d}(x, y) - \mathsf{d}(x_n, y)| + |\mathsf{d}(x_n, y) - \mathsf{d}(x_n, y_n)| && \text{by \textit{triangle inequality} of } (\mathbb{R}, |x - y|) \\
&= |\mathsf{d}(x, y) - \mathsf{d}(y, x_n)| + |\mathsf{d}(y, x_n) - \mathsf{d}(x_n, y_n)| && \text{by \textit{commutative} property of d (Definition 2.1 page 27)} \\
&\le \mathsf{d}(x, x_n) + \mathsf{d}(y, y_n) && \text{by } 2\sigma = 2^{\frac{1}{p}} \text{ and Lemma 13.1 (page 187)} \\
&= 0 \qquad \text{as } n \to \infty
\end{aligned}$$

[10] in *metric space*: Rosenlicht (1968) page 52

[11] in *metric space* ($(p, \sigma) = (1, 1)$ case): Berberian (1961) page 37 ⟨Theorem II.4.1⟩

In *distance spaces* and *topological spaces*, limits of convergent sequences are in general *not unique* (Example 9.3 page 137, Example 9.1 page 132). However Theorem 13.6 (next) demonstrates that, in a *power distance space*, limits *are* unique.

Theorem 13.6 (Uniqueness of limit). [12] *Let* $(X, \mathsf{d}, p, \sigma)$ *be a* POWER DISTANCE SPACE. *Let* $x, y, \in X$ *and let* $(\!(x_n \in X)\!)$ *be an* X*-valued sequence.*

THM
$$\left\{\begin{array}{ll} 1. & \left\{\left((\!(x_n)\!), (\!(y_n)\!)\right) \to (x, y)\right\} \quad \text{and} \\ 2. & (p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+ \end{array}\right\} \implies \{x = y\}$$

✎PROOF:

1. lemma: Proof that for all $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ and for any $\varepsilon \in \mathbb{R}^+$, there exists N such that $\mathsf{d}(x, y) < 2\sigma\varepsilon$:

$$\begin{aligned}
\mathsf{d}(x, y) &\le \tau(p, \sigma; x, y, x_n) && \text{by definition of } \textit{power triangle inequality} \text{ (Definition 13.2 page 185)}\\
&\triangleq 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x, x_n) + \frac{1}{2}\mathsf{d}^p(x_n, y)\right]^{\frac{1}{p}} && \text{by definition of } \textit{power triangle function} \text{ (Definition 13.1 page 185)}\\
&< 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} && \text{by left hypothesis and for } p \in \mathbb{R}^* \setminus \{-\infty, 0, \infty\}\\
&= 2\sigma\varepsilon\\
\mathsf{d}(x, y) &\le \tau(\infty, \sigma; x, y, x_n) && \text{by definition of } \textit{power triangle inequality} \text{ at } p = \infty\\
&= 2\sigma \max\left\{\mathsf{d}(x, x_n),\ \mathsf{d}(x_n, y)\right\} && \text{by Corollary 13.2 (page 187)}\\
&< 2\sigma\varepsilon && \text{by left hypothesis}\\
\mathsf{d}(x, y) &\le \tau(-\infty, \sigma; x, y, x_n) && \text{by definition of } \textit{power triangle inequality} \text{ at } p = -\infty\\
&= 2\sigma \min\left\{\mathsf{d}(x, x_n),\ \mathsf{d}(x_n, y)\right\} && \text{by Corollary 13.2 (page 187)}\\
&< 2\sigma\varepsilon && \text{by left hypothesis}\\
\mathsf{d}(x, y) &\le \tau(0, \sigma; x, y, x_n) && \text{by definition of } \textit{power triangle inequality} \text{ at } p = 0\\
&= 2\sigma\sqrt{\mathsf{d}(x, x_n)}\,\sqrt{\mathsf{d}(x_n, y)} && \text{by Corollary 13.2 (page 187)}\\
&= 2\sigma\sqrt{\varepsilon}\,\sqrt{\varepsilon} && \text{by left hypothesis}\\
&< 2\sigma\varepsilon && \text{by property of real numbers}
\end{aligned}$$

2. Proof that $x = y$ (proof by contradiction):

$$\begin{aligned}
x \ne y &\implies \mathsf{d}(x, y) \ne 0 && \text{by the } \textit{nondegenerate} \text{ property of } \mathsf{d} \text{ (Definition 2.1 page 27)}\\
&\implies \mathsf{d}(x, y) > 0 && \text{by } \textit{non-negative} \text{ property of } \mathsf{d} \text{ (Definition 2.1 page 27)}\\
&\implies \exists\varepsilon \quad \text{such that} \quad \mathsf{d}(x, y) > 2\sigma\varepsilon\\
&\implies \textit{contradiction} \text{ to (1) lemma page 193}\\
&\implies \mathsf{d}(x, y) = 0\\
&\implies x = y
\end{aligned}$$

[12] in *metric space*: Rosenlicht (1968) page 46, Thomson et al. (2008) page 32 ⟨Theorem 2.8⟩

13.3 Examples

It is not always possible to find a *triangle relation* (Definition 13.2 page 185) $\circledtriangle(p, \sigma; \mathsf{d})$ that holds in every *distance space* (Definition 2.1 page 27), as demonstrated by Example 13.1 and Example 13.2 (next two examples).

Example 13.1. Let $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ be defined such that

$$\mathsf{d}(x, y) \triangleq \left\{\begin{array}{lll} y & \forall (x, y) \in \{4\} \times (0 : 2] & \text{(vertical half-open interval)} \\ x & \forall (x, y) \in (0 : 2] \times \{4\} & \text{(horizontal half-open interval)} \\ |x - y| & \text{otherwise} & \text{(Euclidean)} \end{array}\right\}.$$

Note the following about the pair $(\mathbb{R}, \mathsf{d})$:

1. By Example 9.3 (page 137), $(\mathbb{R}, \mathsf{d})$ is a *distance space*, but not a *metric space*—that is, the *triangle relation* $\circledtriangle(1, 1; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$.
2. Observe further that $(\mathbb{R}, \mathsf{d})$ is *not* a *power distance space*. In particular, the *triangle relation* $\circledtriangle(p, \sigma; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$ for any finite value of σ (does not hold for any $\sigma \in \mathbb{R}^+$):

$$\begin{aligned} \mathsf{d}(0, 4) = 4 \not\leq 0 = \lim_{\varepsilon\to 0} 2\sigma\varepsilon &= \lim_{\varepsilon\to 0} 2\sigma\left[\tfrac{1}{2}|0 - \varepsilon|^p + \tfrac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} \\ &\triangleq \lim_{\varepsilon\to 0} 2\sigma\left[\tfrac{1}{2}\mathsf{d}^p(0, \varepsilon) + \tfrac{1}{2}\mathsf{d}^p(\varepsilon, 4)\right]^{\frac{1}{p}} \triangleq \lim_{\varepsilon\to 0} \circledtriangle(p, \sigma; 0, 4, \varepsilon; \mathsf{d}) \end{aligned}$$

Example 13.2. Let $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ be defined such that

$$\mathsf{d}(x, y) \triangleq \left\{\begin{array}{lll} |x - y| & \text{for } x = 0 \text{ or } y = 0 \text{ or } x = y & \text{(Euclidean)} \\ 1 & \text{otherwise} & \text{(discrete)} \end{array}\right\}.$$

Note the following about the pair $(\mathbb{R}, \mathsf{d})$:

1. By Example 9.4 (page 138), $(\mathbb{R}, \mathsf{d})$ is a *distance space*, but not a *metric space*—that is, the *triangle relation* $\circledtriangle(1, 1; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$.
2. Observe further that $(\mathbb{R}, \mathsf{d})$ is *not* a *power distance space*—that is, the *triangle relation* $\circledtriangle(p, \sigma; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$ for any value of $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$.
   (a) Proof that $\circledtriangle(p, \sigma; \mathsf{d})$ does not hold for any $(p, \sigma) \in \{\infty\} \times \mathbb{R}^+$:

$$\begin{aligned} \lim_{n,m\to\infty} \mathsf{d}(1/n, 1/m) &\triangleq 1 \not\leq 0 = 2\sigma \max\{0, 0\} && \text{by definition of } \mathsf{d} \\ &= 2\sigma \lim_{n,m\to\infty} \max\{\mathsf{d}(1/n, 0), \mathsf{d}(0, 1/m)\} && \text{by Corollary 13.2 (page 187)} \\ &\geq \lim_{n,m\to\infty} 2\sigma\left[\tfrac{1}{2}\mathsf{d}^p(1/n, 0) + \tfrac{1}{2}\mathsf{d}^p(0, 1/m)\right]^{\frac{1}{p}} && \text{by Corollary 13.1 (page 186)} \\ &\triangleq \lim_{n,m\to\infty} \tau(p, \sigma, 1/n, 1/m, 0) && \text{by definition of } \tau \text{ (Definition 13.1 page 185)} \end{aligned}$$

   (b) Proof that $\circledtriangle(p, \sigma; \mathsf{d})$ does not hold for any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$: By Corollary 13.1 (page 186), the *triangle function* (Definition 13.1 page 185) $\tau(p, \sigma; x, y, z; \mathsf{d})$ is *continuous* and *strictly monotone* in $(\mathbb{R}, |\cdot|, \leq)$ with respect to the variable p. Item 2a demonstrates that $\circledtriangle(p, \sigma; \mathsf{d})$ fails to hold at the best case of $p = \infty$, and so by Corollary 13.1, it doesn't hold for any other value of $p \in \mathbb{R}^*$ either.

Example 13.3. Let d be a function in $\mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that

$$\mathsf{d}(x, y) \triangleq \left\{\begin{array}{lll} 2|x - y| & \forall (x, y) \in \{(0, 1), (1, 0)\} & \text{(dilated Euclidean)} \\ |x - y| & \text{otherwise} & \text{(Euclidean)} \end{array}\right\}.$$

Note the following about the pair $(\mathbb{R}, \mathsf{d})$:

1. By Example 9.5 (page 139), $(\mathbb{R}, \mathsf{d})$ is a *distance space*, but not a *metric space*—that is, the *triangle relation* $\triangle(1, 1; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$.
2. But observe further that $(\mathbb{R}, \mathsf{d}, 1, 2)$ *is* a *power distance space*:
   (a) Proof that $\triangle(1, 2; \mathsf{d})$ (Definition 13.2 page 185) holds for all $(x, y) \in \{(0, 1), (1, 0)\}$:

$$\begin{aligned}
\mathsf{d}(1,0) = \mathsf{d}(0,1) &\triangleq 2|0-1| = 2 && \text{by definition of } \mathsf{d}\\
&\le 2 \le 2(|0-z| + |z-1|) \quad \forall z \in \mathbb{R} && \text{by definition of } |\cdot| \text{ (Definition F.4 page 342)}\\
&= 2\sigma\left(\tfrac{1}{2}|0-z|^p + \tfrac{1}{2}|z-1|^p\right)^{\frac{1}{p}} \quad \forall z \in \mathbb{R} && \text{for } (p,\sigma) = (1,2)\\
&\triangleq 2\sigma\left(\tfrac{1}{2}\mathsf{d}^p(0,z) + \mathsf{d}^p(z,1)\right)^{\frac{1}{p}} \quad \forall z \in \mathbb{R} && \text{for } (p,\sigma) = (1,2) \text{ and by definition of } \mathsf{d}\\
&\triangleq \tau(1,2;0,1,z) && \text{by definition of } \tau \text{ (Definition 13.1 page 185)}
\end{aligned}$$

   (b) Proof that $\triangle(1, 2; \mathsf{d})$ holds for all other $(x, y) \in \mathbb{R}^* \times \mathbb{R}^+$:

$$\begin{aligned}
\mathsf{d}(x,y) &\triangleq 2|x-y| && \text{by definition of } \mathsf{d}\\
&\le (|x-z| + |z-y|) && \text{by property of \textit{Euclidean metric spaces}}\\
&= 2\sigma\left(\tfrac{1}{2}|0-z|^p + \tfrac{1}{2}|z-1|^p\right)^{\frac{1}{p}} && \text{for } (p,\sigma) = (1,1)\\
&\triangleq \tau(1,1;x,y,z) && \text{by definition of } \tau \text{ (Definition 13.1 page 185)}\\
&\le \tau(1,2;x,y,z) && \text{by Corollary 13.1 (page 186)}
\end{aligned}$$

3. In (X, d), the limits of *convergent* sequences are *unique*. This follows directly from the fact that $(\mathbb{R}, \mathsf{d}, 1, 2)$ is a *power distance space* (item (2) page 195) and by Theorem 13.6 page 193.
4. In (X, d), *convergent* sequences are *Cauchy*. This follows directly from the fact that $(\mathbb{R}, \mathsf{d}, 1, 2)$ is a *power distance space* (item (2) page 195) and by Theorem 13.3 page 191.

Example 13.4. Let d be a function in $\mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that $\mathsf{d}(x, y) \triangleq (x - y)^2$. Note the following about the pair $(\mathbb{R}, \mathsf{d})$:

1. It was demonstrated in Example 9.6 (page 140) that $(\mathbb{R}, \mathsf{d})$ is a *distance space*, but that it is *not* a *metric space* because the *triangle inequality* does not hold.
2. However, the tuple $(\mathbb{R}, \mathsf{d}, p, \sigma)$ *is* a *power distance space* (Definition 13.2 page 185) for any $(p, \sigma) \in \mathbb{R}^* \times [2 : \infty)$: In particular, for all $x, y, z \in \mathbb{R}$, the *power triangle inequality* (Definition 13.2 page 185) must hold. The "worst case" for this is when a third point z is exactly "halfway between" x and y in $\mathsf{d}(x, y)$; that is, when $z = \frac{x+y}{2}$:

$$\begin{aligned}
(x-y)^2 &\triangleq \mathsf{d}(x,y) && \text{by definition of } \mathsf{d}\\
&\le \tau(p,\sigma;x,y,z;\mathsf{d}) && \text{by definition \textit{power triangle inequality}}\\
&\triangleq 2\sigma\left[\tfrac{1}{2}\mathsf{d}^p(x,z) + \tfrac{1}{2}\mathsf{d}^p(z,y)\right]^{\frac{1}{p}} && \text{by definition } \tau \text{ (Definition 13.1 page 185)}\\
&\triangleq 2\sigma\left[\tfrac{1}{2}(x-z)^{2p} + \tfrac{1}{2}(z-y)^{2p}\right]^{\frac{1}{p}} && \text{by definition of } \mathsf{d}\\
&= 2\sigma\left[\tfrac{1}{2}|x-z|^{2p} + \tfrac{1}{2}|z-y|^{2p}\right]^{\frac{1}{p}} && \text{because } (x)^2 = |x|^2 \text{ for all } x \in \mathbb{R}\\
&= 2\sigma\left[\tfrac{1}{2}\left|x - \frac{x+y}{2}\right|^{2p} + \tfrac{1}{2}\left|\frac{x+y}{2} - y\right|^{2p}\right]^{\frac{1}{p}} && \text{because } z = \frac{x+y}{2} \text{ is the ``worst case'' scenario}\\
&= 2\sigma\left[\tfrac{1}{2}\left|\frac{y-x}{2}\right|^{2p} + \tfrac{1}{2}\left|\frac{x-y}{2}\right|^{2p}\right]^{\frac{1}{p}}
\end{aligned}$$

$$= 2\sigma\left[\left|\frac{x-y}{2}\right|^{2p}\right]^{\frac{1}{p}} = \frac{2\sigma}{4}|x-y|^2$$
$$\Longrightarrow\ (p,\sigma) \in \mathbb{R}^* \times [2 : \infty)$$

3. The *power distance function* d is *continuous* in $(\mathbb{R}, \mathsf{d}, p, \sigma)$ for any (p, σ) such that $\sigma \geq 2$ and $2\sigma = p^{\frac{1}{p}}$. This follows directly from Theorem 13.5 (page 192).

Part IV

Structures between Spaces

CHAPTER 14

LINEAR FUNCTIONALS

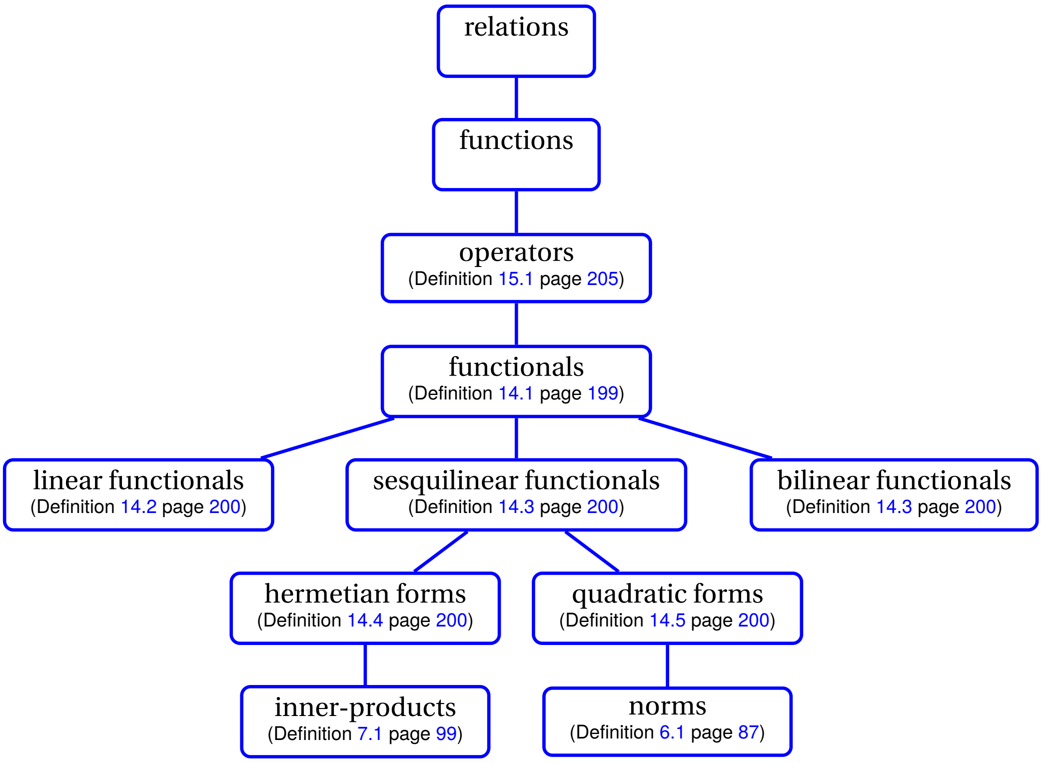

Figure 14.1: Relations

14.1 Definitions

Definition 14.1. [1] *Let* $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\big)$ *be a* LINEAR SPACE *(Definition 4.1 page 71). Let* $\mathbb{F}^{\boldsymbol{X}}$ *be the set of all functions from the set* X *into the* FIELD *(Definition F.5 page 342)* $\mathbb{F}$.

DEF	*A function* f *is a* ***functional*** *on* $\boldsymbol{\Omega}$ *if* f *is in* $\mathbb{F}^{\boldsymbol{X}}$.

[1] Bachman and Narici (1966) page 5, Michel and Herget (1993) pages 109–114 ⟨Definitions 3.5.1, 3.6.1⟩

Definition 14.2. [2] *Let* $\mathsf{f} \in \mathbb{F}^{\boldsymbol{X}}$ *be a* FUNCTIONAL *on a* LINEAR SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF

$$\begin{array}{llcll}
\mathsf{f} \text{ is } \textbf{linear} \text{ if} & \mathsf{f}(\alpha\boldsymbol{x}+\beta\boldsymbol{y}) & = & \alpha\mathsf{f}(\boldsymbol{x})+\beta\mathsf{f}(\boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y}\in X, \forall\alpha,\beta\in\mathbb{F}. \\
\mathsf{f} \text{ is a } \textbf{conjugate linear} \text{ if} & \mathsf{f}(\alpha\boldsymbol{x}+\beta\boldsymbol{y}) & = & \overline{\alpha}\mathsf{f}(\boldsymbol{x})+\overline{\beta}\mathsf{f}(\boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y}\in X, \forall\alpha,\beta\in\mathbb{F}. \\
\mathsf{f} \text{ is } \textbf{subaddtive} \text{ if} & \mathsf{f}(\boldsymbol{x}+\boldsymbol{y}) & \le & \mathsf{f}(\boldsymbol{x})+\mathsf{f}(\boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y}\in X.
\end{array}$$

Definition 14.3. [3] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition 4.1 page 71).*

DEF

A function $\mathsf{b} \in \mathbb{F}^{X\times X}$ *is a* ***bilinear functional*** *or* ***bilinear form*** *on* $\mathbb{F}^{X\times X}$ *if*

1. $\mathsf{b}(\alpha\boldsymbol{x}+\beta\boldsymbol{y},\boldsymbol{u}) = \alpha\mathsf{b}(\boldsymbol{x},\boldsymbol{u})+\beta\mathsf{b}(\boldsymbol{y},\boldsymbol{u}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X, \alpha,\beta\in\mathbb{F}$ (LINEAR *in first variable) and*
2. $\mathsf{b}(\boldsymbol{u},\alpha\boldsymbol{x}+\beta\boldsymbol{y}) = \alpha\mathsf{b}(\boldsymbol{u},\boldsymbol{x})+\beta\mathsf{b}(\boldsymbol{u},\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X, \alpha,\beta\in\mathbb{F}$ (LINEAR *in second variable).*

A function $\mathsf{s} \in \mathbb{F}^{X\times X}$ *is a* ***sesquilinear functional*** *or* ***sesquilinear form*** *on* $\mathbb{F}^{X\times X}$ *if*[4]

1. $\mathsf{b}(\alpha\boldsymbol{x}+\beta\boldsymbol{y},\boldsymbol{u}) = \alpha\mathsf{b}(\boldsymbol{u},\boldsymbol{x})+\beta\mathsf{b}(\boldsymbol{u},\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X, \alpha,\beta\in\mathbb{F}$ (LINEAR *in first variable) and*
2. $\mathsf{b}(\boldsymbol{u},\alpha\boldsymbol{x}+\beta\boldsymbol{y}) = \overline{\alpha}\mathsf{b}(\boldsymbol{u},\boldsymbol{x})+\overline{\beta}\mathsf{b}(\boldsymbol{u},\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X, \alpha,\beta\in\mathbb{F}$ (CONJUGATE LINEAR *in second variable).*

Definition 14.4. [5] *Let* $\phi \in \mathbb{F}^{X\times X}$ *be a* BILINEAR FUNCTIONAL *or a* SESQUILINEAR FUNCTIONAL *(Definition 14.3 page 200) on a* LINEAR SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF

1. ϕ *is* ***symmetric*** *if* $\phi(\boldsymbol{x},\boldsymbol{y}) = \phi(\boldsymbol{y},\boldsymbol{x}) \quad \forall \boldsymbol{x},\boldsymbol{y}\in X.$
2. ϕ *is* ***Hermitian symmetric*** *if* $\phi(\boldsymbol{x},\boldsymbol{y}) = \overline{\phi(\boldsymbol{y},\boldsymbol{x})} \quad \forall \boldsymbol{x},\boldsymbol{y}\in X.$
3. ϕ *is* ***nonnegative*** *if* $\phi(\boldsymbol{x},\boldsymbol{y}) \ge 0 \quad \forall \boldsymbol{x},\boldsymbol{y}\in X.$
4. ϕ *is* ***positive*** *if* $\phi(\boldsymbol{x},\boldsymbol{x}) > 0 \quad \forall \boldsymbol{x}\in X\backslash 0.$
5. *If* ϕ *is* HERMITIAN SYMMETRIC, *then* ϕ *is a* ***hermitian form***.

Definition 14.5. [6] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition 4.1 page 71).*

DEF

A function $\mathsf{q} \in \mathbb{F}^{\boldsymbol{X}}$ *is the* ***quadratic form*** *induced (or generated) by* $\mathsf{s} \in \mathbb{F}^{X\times X}$ *if*

1. s *is a* SESQUILINEAR FUNCTIONAL *and*
2. $\mathsf{q}(\boldsymbol{x}) \triangleq \mathsf{s}(\boldsymbol{x},\boldsymbol{x}) \quad \forall \boldsymbol{x}\in X$

14.2 Basic results

Lemma 14.1. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition 4.1 page 71).*

LEM

$$\mathsf{q} \in \mathbb{F}^{\boldsymbol{X}} \text{ is a QUADRATIC FORM} \implies \left\{\begin{array}{llcll} 1. & \mathsf{q}(-\boldsymbol{x}) & = & \mathsf{q}(\boldsymbol{x}) & \forall \boldsymbol{x}\in X \quad \text{and} \\ 2. & \mathsf{q}(i\boldsymbol{x}) & = & \mathsf{q}(\boldsymbol{x}) & \forall \boldsymbol{x}\in X. \end{array}\right\}$$

[2] Michel and Herget (1993) pages 109–114 ⟨Definitions 3.5.1, 3.6.1⟩, Nikol'skiĭ (1992) pages 109–110 ⟨subadditive property⟩

[3] Kubrusly (2001) page 312, Brown and Pearcy (1995) page 58

[4] The prefix *sesqui–* is Latin for "one and a half". Reference: merriamwebsterdictionary ⟨http://www.merriam-webster.com/dictionary/sesqui-⟩

[5] Kubrusly (2001) page 312, Brown and Pearcy (1995) page 58, Michel and Herget (1993) page 115 ⟨Definition 3.6.10⟩, Debnath and Mikusiński (2005) page 152 ⟨Theorem 4.3.5⟩, Grove (2002) page 85 ⟨item 4⟩

[6] Kubrusly (2001) page 312, Michel and Herget (1993) page 115 ⟨Definition 3.6.11⟩, Debnath and Mikusiński (2005) page 152 ⟨Theorem 4.3.6⟩

✎PROOF: Let $\mathsf{s} \in \mathbb{F}^{X \times X}$ be the *sesquilinear functional* in $\mathbb{F}^{X \times X}$ that induces q.

$$\begin{aligned}
\mathsf{q}(\boldsymbol{x}) &= \mathsf{s}(\boldsymbol{x}, \boldsymbol{x}) \\
&= (-1)(-1)\mathsf{s}(\boldsymbol{x}, \boldsymbol{x}) \\
&= \mathsf{s}((-1)\boldsymbol{x}, \overline{(-1)}\boldsymbol{x}) \\
&= \mathsf{s}(-\boldsymbol{x}, -\boldsymbol{x}) \\
&= \mathsf{q}(-\boldsymbol{x}) \\
\mathsf{q}(\boldsymbol{x}) &= \mathsf{s}(\boldsymbol{x}, \boldsymbol{x}) \\
&= (i)(-i)\mathsf{s}(\boldsymbol{x}, \boldsymbol{x}) \\
&= \mathsf{s}(i\boldsymbol{x}, \overline{(-i)}\boldsymbol{x}) \\
&= \mathsf{s}(i\boldsymbol{x}, i\boldsymbol{x}) \\
&= \mathsf{q}(i\boldsymbol{x})
\end{aligned}$$

✏

Theorem 14.1. [7] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition 4.1 page 71). *Let* $\mathsf{q} \in \mathbb{F}^{\boldsymbol{X}}$ *be the* QUADRATIC FORM (Definition 14.5 page 200) *induced by a function* $\mathsf{s} \in \mathbb{F}^{X \times X}$.

THM

s *is a* SESQUILINEAR FUNCTIONAL $\implies$

$$2\mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) + 2\mathsf{s}(\boldsymbol{y}, \boldsymbol{x}) = \mathsf{q}(\boldsymbol{x} + \boldsymbol{y}) - \mathsf{q}(\boldsymbol{x} - \boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X$$

✎PROOF:

$$\begin{aligned}
\mathsf{q}(\boldsymbol{x}+\boldsymbol{y}) - \mathsf{q}(\boldsymbol{x}-\boldsymbol{y}) &= \mathsf{s}(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}) - \mathsf{s}(\boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y}) \\
&= \underbrace{\{\mathsf{s}(\boldsymbol{x}, \boldsymbol{x}) + \mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) + \mathsf{s}(\boldsymbol{y}, \boldsymbol{x}) + \mathsf{s}(\boldsymbol{y}, \boldsymbol{y})\}}_{\mathsf{s}(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y})} - \underbrace{\{\mathsf{s}(\boldsymbol{x}, \boldsymbol{x}) - \mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) - \mathsf{s}(\boldsymbol{y}, \boldsymbol{x}) + \mathsf{s}(\boldsymbol{y}, \boldsymbol{y})\}}_{\mathsf{s}(\boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y})} \\
&= \{\mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) + \mathsf{s}(\boldsymbol{y}, \boldsymbol{x})\} - \{-\mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) - \mathsf{s}(\boldsymbol{y}, \boldsymbol{x})\} \\
&= 2\mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) + 2\mathsf{s}(\boldsymbol{y}, \boldsymbol{x})
\end{aligned}$$

✏

Theorem 14.2 (polarization identities). [8] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition 4.1 page 71). *Let* $\mathsf{q} \in \mathbb{F}^{\boldsymbol{X}}$ *be the* QUADRATIC FORM (Definition 14.5 page 200) *induced by a function* $\mathsf{s} \in \mathbb{F}^{X \times X}$.

THM

1. s *is a* SESQUILINEAR FUNCTIONAL $\implies$
$$4\mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) = \mathsf{q}(\boldsymbol{x}+\boldsymbol{y}) - \mathsf{q}(\boldsymbol{x}-\boldsymbol{y}) + i\mathsf{q}(\boldsymbol{x}+i\boldsymbol{y}) - i\mathsf{q}(\boldsymbol{x}-i\boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X$$
2. s *is a* HERMITIAN SYMMETRIC *sesquilinear functional and* $\mathbb{F} = \mathbb{R}$ $\implies$
$$4\mathsf{s}(\boldsymbol{x}, \boldsymbol{y}) = \mathsf{q}(\boldsymbol{x}+\boldsymbol{y}) - \mathsf{q}(\boldsymbol{x}-\boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X$$

✎PROOF:

[7] Michel and Herget (1993) pages 115–116 ⟨Theorem 3.6.12⟩

[8] Brown and Pearcy (1995) pages 62–63 ⟨Problem P(ii)⟩, Halmos (1998b) pages 13–14 ⟨Theorem 1⟩, Michel and Herget (1993) page 116 ⟨Theorem 3.6.13⟩, Debnath and Mikusiński (2005) pages 152–153 ⟨Theorem 4.3.7⟩

1. Proof that $4\mathsf{s}(x,y) = \mathsf{q}(x+y) - \mathsf{q}(x-y) + i\mathsf{q}(x+iy) - i\mathsf{q}(x-iy)$:

$$\begin{aligned}
&\mathsf{q}(x+y) - \mathsf{q}(x-y) + i\mathsf{q}(x+iy) - i\mathsf{q}(x-iy)\\
&= \mathsf{s}(x+y,x+y) - \mathsf{s}(x-y,x-y) + i\mathsf{s}(x+iy,x+iy) - i\mathsf{s}(x-iy,x-iy)\\
&= \underbrace{\{\mathsf{s}(x,x)+\mathsf{s}(x,y)+\mathsf{s}(y,x)+\mathsf{s}(y,y)\}}_{\mathsf{s}(x+y,x+y)} - \underbrace{\{\mathsf{s}(x,x)-\mathsf{s}(x,y)-\mathsf{s}(y,x)+\mathsf{s}(y,y)\}}_{\mathsf{s}(x-y,x-y)}\\
&\quad + i\underbrace{\left\{\mathsf{s}(x,x)-i\mathsf{s}(x,y)+i\mathsf{s}(y,x)-i^2\mathsf{s}(y,y)\right\}}_{\mathsf{s}(x+iy,x+iy)} - i\underbrace{\left\{\mathsf{s}(x,x)+i\mathsf{s}(x,y)-i\mathsf{s}(y,x)-i^2\mathsf{s}(y,y)\right\}}_{\mathsf{s}(x-iy,x-iy)}\\
&= \{\mathsf{s}(x,y)+\mathsf{s}(y,x)\} - \{-\mathsf{s}(x,y)-\mathsf{s}(y,x)\} + i\{-i\mathsf{s}(x,y)+i\mathsf{s}(y,x)\} - i\{+i\mathsf{s}(x,y)-i\mathsf{s}(y,x)\}\\
&= \{\mathsf{s}(x,y)+\mathsf{s}(y,x)\} + \{\mathsf{s}(x,y)+\mathsf{s}(y,x)\} + \{\mathsf{s}(x,y)-\mathsf{s}(y,x)\} + \{\mathsf{s}(x,y)-\mathsf{s}(y,x)\}\\
&= 4\mathsf{s}(x,y)
\end{aligned}$$

2. Proof that s is a *Hermitian symmetric* sesquilinear functional and $\mathbb{F}=\mathbb{R} \implies 4\mathsf{s}(x,y) = \mathsf{q}(x+y)-\mathsf{q}(x-y)$:

$$\begin{aligned}
\mathsf{q}(x+y) - \mathsf{q}(x-y) &= 2\mathsf{s}(x,y) + 2\mathsf{s}(y,x) && \text{by Theorem 14.1 page 201}\\
&= 2\mathsf{s}(x,y) + 2\overline{\mathsf{s}(x,y)} && \text{by } \textit{symmetric} \text{ hypothesis}\\
&= 2\mathsf{s}(x,y) + 2\mathsf{s}(x,y) && \text{by } \textit{real} \text{ hypothesis}\\
&= 4\mathsf{s}(x,y)
\end{aligned}$$

Theorem 14.3. [9] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* $\mathsf{q}_1 \in \mathbb{F}^{\boldsymbol{X}}$ *be the* QUADRATIC FORM (Definition 14.5 page 200) *induced by a sesquilinear functional* (Definition 14.3 page 200) $\mathsf{s}_1 \in \mathbb{F}^{X\times X}$ *and* $\mathsf{q}_2 \in \mathbb{F}^{\boldsymbol{X}}$ *be the* QUADRATIC FORM *induced by a sesquilinear functional* $\mathsf{s}_2 \in \mathbb{F}^{X\times X}$.

THM
$$\left\{\mathsf{q}_1(x) = \mathsf{q}_2(x) \quad \forall x\in X\right\} \iff \left\{\mathsf{s}_1(x,y) = \mathsf{s}_2(x,y) \quad \forall x,y\in X\right\}$$

✎PROOF:

1. Proof that $\mathsf{q}_1(x) = \mathsf{q}_2(x) \implies \mathsf{s}_1(x,y) = \mathsf{s}_2(x,y)$:

$$\begin{aligned}
\mathsf{s}_1(x,y) &= \mathsf{q}_1(x+y) - \mathsf{q}_1(x-y) + i\mathsf{q}_1(x+iy) - i\mathsf{q}_1(x-iy) && \text{by } \textit{polarization identity} \text{ (Theorem 14.2 page 201)}\\
&= \mathsf{q}_2(x+y) - \mathsf{q}_2(x-y) + i\mathsf{q}_2(x+iy) - i\mathsf{q}_2(x-iy) && \text{by left hypothesis}\\
&= \mathsf{s}_2(x,y)
\end{aligned}$$

2. Proof that $\mathsf{q}_1(x) = \mathsf{q}_2(x) \implies \mathsf{s}_1(x,y) = \mathsf{s}_2(x,y)$:

$$\begin{aligned}
\mathsf{q}_1(x) &= \mathsf{s}_1(x,x) && \text{by Definition 14.5 page 200}\\
&= \mathsf{s}_2(x,x) && \text{by right hypothesis}\\
&= \mathsf{q}_2(x) && \text{by Definition 14.5 page 200}
\end{aligned}$$

Theorem 14.4. [10] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* $\mathsf{q} \in \mathbb{F}^{\boldsymbol{X}}$ *be the* QUADRATIC FORM (Definition 14.5 page 200) *induced by a* SESQUILINEAR FUNCTIONAL (Definition 14.3 page 200) $\mathsf{s} \in \mathbb{F}^{X\times X}$.

THM
$$\underbrace{\mathsf{q} \in \mathbb{R}}_{\mathsf{q} \textit{ is } \text{REAL}} \iff \underbrace{\mathsf{s}(x,y) = \overline{\mathsf{s}(y,x)}}_{\mathsf{s} \textit{ is } \text{HERMITIAN SYMMETRIC}}$$

[9] Michel and Herget (1993) page 117 ⟨Theorem 3.6.15⟩, Debnath and Mikusiński (2005) page 153 ⟨Corollary 4.3.8⟩

[10] Kubrusly (2001) page 312, Michel and Herget (1993) pages 115–116 ⟨Theorem 3.6.12⟩, Debnath and Mikusiński (2005) page 153 ⟨Theorem 4.3.9⟩

✎PROOF:

1. Proof that q is *real* $\implies$ s is *hermitian*:

$$\begin{aligned}
\overline{4\mathsf{s}(\boldsymbol{y},\boldsymbol{x})} &= \overline{\mathsf{q}(\boldsymbol{y}+\boldsymbol{x})} - \overline{\mathsf{q}(\boldsymbol{y}-\boldsymbol{x})} + \overline{i\mathsf{q}(\boldsymbol{y}+i\boldsymbol{x})} - \overline{i\mathsf{q}(\boldsymbol{y}-i\boldsymbol{x})} && \text{by \textit{polarization id.} (Theorem 14.2 page 201)}\\
&= \mathsf{q}(\boldsymbol{y}+\boldsymbol{x}) - \mathsf{q}(\boldsymbol{y}-\boldsymbol{x}) - i\mathsf{q}(\boldsymbol{y}+i\boldsymbol{x}) + i\mathsf{q}(\boldsymbol{y}-i\boldsymbol{x}) && \text{by left hypothesis}\\
&= \mathsf{q}(\boldsymbol{y}+\boldsymbol{x}) - \mathsf{q}(-\boldsymbol{y}+\boldsymbol{x}) - i\mathsf{q}(-i\boldsymbol{y}-i^2\boldsymbol{x}) + i\mathsf{q}(i\boldsymbol{y}-i^2\boldsymbol{x}) && \text{by Lemma 14.1 page 200}\\
&= \mathsf{q}(\boldsymbol{x}+\boldsymbol{y}) - \mathsf{q}(\boldsymbol{x}-\boldsymbol{y}) - i\mathsf{q}(\boldsymbol{x}-i\boldsymbol{y}) + i\mathsf{q}(\boldsymbol{x}+i\boldsymbol{y})\\
&= 4\mathsf{s}(\boldsymbol{x},\boldsymbol{y}) && \text{by \textit{polarization id.} (Theorem 14.2 page 201)}
\end{aligned}$$

2. Proof that q is *real* $\impliedby$ s is *Hermitian symmetric*:

$$\begin{aligned}
\overline{\mathsf{q}(\boldsymbol{x})} &= \overline{\mathsf{s}(\boldsymbol{x},\boldsymbol{x})} && \text{by definition of q (Definition 14.5 page 200)}\\
&= \overline{\overline{\mathsf{s}(\boldsymbol{x},\boldsymbol{x})}} && \text{by right hypothesis}\\
&= \mathsf{s}(\boldsymbol{x},\boldsymbol{x})\\
&= \mathsf{q}(\boldsymbol{x}) && \text{by definition of q (Definition 14.5 page 200)}\\
\implies & \quad \mathsf{q} \in \mathbb{R}
\end{aligned}$$

✏

Theorem 14.5 (Riesz Representation Theorem).[11] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be a* HILBERT SPACE *(Definition 9.14 page 149) and* f *a* BOUNDED LINEAR FUNCTIONAL *in* $\mathscr{B}(\boldsymbol{H}, \mathbb{F})$.

THM *For every* $\mathsf{f} \in \mathscr{B}(\boldsymbol{X}, \mathbb{F})$, *there exists a* UNIQUE $\boldsymbol{y}$ *such that*

$$\mathsf{f}(\boldsymbol{x}) = \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle \quad \forall x \in X;$$

Moreover $$|||\mathsf{f}||| = \|\boldsymbol{y}\|$$

✎PROOF:

1. Proof that $\boldsymbol{y}$ is unique:

$$\begin{aligned}
& & \mathsf{f}(\boldsymbol{x}) &= \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \langle \boldsymbol{x} \mid \boldsymbol{z} \rangle\\
&\implies & 0 &= \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle - \langle \boldsymbol{x} \mid \boldsymbol{z} \rangle\\
& & &= \langle \boldsymbol{x} \mid \boldsymbol{y}-\boldsymbol{z} \rangle && \text{by Definition 7.1 page 99}\\
&\implies & \boldsymbol{y} &= \boldsymbol{z} && \text{by Theorem 7.1}
\end{aligned}$$

2. Proof that $\exists \boldsymbol{y}$ such that $\mathsf{f}(\boldsymbol{x}) = \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$:

[11] Yosida (1980) page 90, Schechter (2002) pages 29–30 ⟨Theorem 2.1⟩, Kubrusly (2001) page 374 ⟨Proposition 5.62⟩

3. Proof that $\|\|\mathsf{f}\|\| = \|\boldsymbol{y}\|$:

$$\begin{aligned}
\|\|\mathsf{f}\|\| &\triangleq \sup_{\|\boldsymbol{x}\|\le 1} |\mathsf{f}(\boldsymbol{x})| \\
&\ge \left|\mathsf{f}\left(\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right)\right| \\
&= \left\langle \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \mid \boldsymbol{y} \right\rangle \\
&= \frac{1}{\|\boldsymbol{y}\|} \langle \boldsymbol{y} \mid \boldsymbol{y} \rangle \\
&= \|\boldsymbol{y}\| \\
\|\|\mathsf{f}\|\| &\triangleq \sup_{\|\boldsymbol{x}\|\le 1} |\mathsf{f}(\boldsymbol{x})| \\
&= \sup_{\|\boldsymbol{x}\|\le 1} |\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle| \\
&\le \sup_{\|\boldsymbol{x}\|\le 1} \|\boldsymbol{x}\| \, \|\boldsymbol{y}\| \\
&= \|\boldsymbol{y}\|
\end{aligned}$$

✏

CHAPTER 15

OPERATORS ON LINEAR SPACES

> *"And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients. . . . we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly."*
>
> Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

15.1 Operators on linear spaces

15.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition 15.1. [2]

DEF *A function* $\mathbf{A}$ *in* Y^X *is an* ***operator*** *in* Y^X *if* X *and* Y *are both* LINEAR SPACES *(Definition 4.1 page 71).*

Two operators $\mathbf{A}$ and $\mathbf{B}$ in Y^X are **equal** if $\mathbf{A}x = \mathbf{B}x$ for all $x \in X$. The inverse relation of an operator $\mathbf{A}$ in Y^X always exists as a *relation* in 2^{XY}, but may not always be a *function* (may not always be an operator) in Y^X.

The operator $\mathbf{I} \in X^X$ is the *identity* operator if $\mathbf{I}x = \mathbf{I}$ for all $x \in X$.

Definition 15.2. [3] *Let* X^X *be the set of all operators with from a* LINEAR SPACE X *to* X*. Let* $\mathbf{I}$ *be an*

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[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Heil (2011) page 42

[3] Michel and Herget (1993) page 411

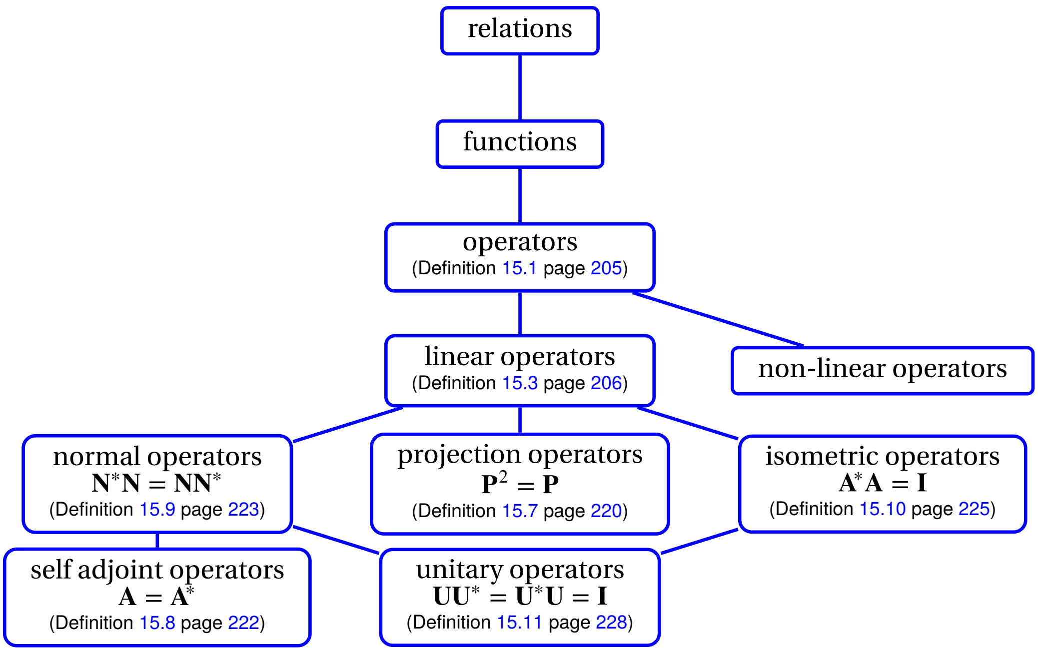

Figure 15.1: Some operator types

operator in $\boldsymbol{X}^{\boldsymbol{X}}$. Let $①(\boldsymbol{X})$ be the IDENTITY ELEMENT *in $\boldsymbol{X}^{\boldsymbol{X}}$.*

DEF: $\mathbf{I}$ *is the **identity operator** in $\boldsymbol{X}^{\boldsymbol{X}}$ if* $\quad \mathbf{I} = ①(\boldsymbol{X})$.

15.1.2 Linear operators

Definition 15.3. [4] *Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be linear spaces.*

DEF: *An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ is **linear** if*

1. $\mathbf{L}(\boldsymbol{x} + \boldsymbol{y}) = \mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$ (ADDITIVE) *and*
2. $\mathbf{L}(\alpha\boldsymbol{x}) = \alpha\mathbf{L}\boldsymbol{x} \quad \forall \boldsymbol{x} \in \boldsymbol{X}, \ \forall \alpha \in \mathbb{F}$ (HOMOGENEOUS).

The set of all linear operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is denoted $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ such that
$\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \{\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}} | \mathbf{L} \text{ is linear}\}$.

Theorem 15.1. [5] *Let $\mathbf{L}$ be an operator from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$, both over a field $\mathbb{F}$.*

THM:
$$\{\mathbf{L} \text{ is LINEAR}\} \implies \left\{\begin{array}{llcll}
1. & \mathbf{L}\mathbb{0} & = & \mathbb{0} & & \text{and}\\
2. & \mathbf{L}(-\boldsymbol{x}) & = & -(\mathbf{L}\boldsymbol{x}) & \forall \boldsymbol{x} \in \boldsymbol{X} & \text{and}\\
3. & \mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X} & \text{and}\\
4. & \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) & = & \sum_{n=1}^{N} \alpha_n (\mathbf{L}\boldsymbol{x}_n) & \boldsymbol{x}_n \in \boldsymbol{X}, \alpha_n \in \mathbb{F} &
\end{array}\right\}$$

✎PROOF:

[4] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[5] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned}
\mathbf{L}\mathbb{0} &= \mathbf{L}(0 \cdot \mathbb{0}) && \text{by additive identity property} && \text{(Theorem 4.1 page 73)}\\
&= 0 \cdot (\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition 15.3 page 206)}\\
&= \mathbb{0} && \text{by } \textit{additive identity} \text{ property} && \text{(Theorem 4.1 page 73)}
\end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned}
\mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1 \cdot \boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} && \text{(Theorem 4.2 page 74)}\\
&= -1 \cdot (\mathbf{L}\boldsymbol{x}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition 15.3 page 206)}\\
&= -(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} && \text{(Theorem 4.2 page 74)}
\end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) = \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y}$:

$$\begin{aligned}
\mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x} + (-\boldsymbol{y})) && \text{by } \textit{additive inverse} \text{ property} && \text{(Theorem 4.2 page 74)}\\
&= \mathbf{L}(\boldsymbol{x}) + \mathbf{L}(-\boldsymbol{y}) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} && \text{(Definition 15.3 page 206)}\\
&= \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} && \text{by item (2)}
\end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n \left(\mathbf{L}\boldsymbol{x}_n\right)$:

(a) Proof for $N = 1$:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_1 \boldsymbol{x}_1\right) && \text{by } N = 1 \text{ hypothesis}\\
&= \alpha_1 \left(\mathbf{L}\boldsymbol{x}_1\right) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition 15.3 page 206)}
\end{aligned}$$

(b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1} \alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1} + \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right)\\
&= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1}\right) + \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} && \text{(Definition 15.3 page 206)}\\
&= \alpha_{N+1}\mathbf{L}\left(\boldsymbol{x}_{N+1}\right) + \sum_{n=1}^{N} \mathbf{L}\left(\alpha_n \boldsymbol{x}_n\right) && \text{by left } N+1 \text{ hypothesis}\\
&= \sum_{n=1}^{N+1} \mathbf{L}\left(\alpha_n \boldsymbol{x}_n\right)
\end{aligned}$$

⇨

Theorem 15.2. [6] *Let $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* NULL SPACE *of an operator* $\mathbf{L}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *and* $\mathscr{I}(\mathbf{L})$ *the* IMAGE SET *of* $\mathbf{L}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$.

THM			
$\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathscr{N}(\mathbf{L})$	*is a linear subspace of* $\boldsymbol{X}$	$\forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathscr{I}(\mathbf{L})$	*is a linear subspace of* $\boldsymbol{Y}$	$\forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$	

✎PROOF:

[6] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩

1. Proof that $\mathscr{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0} \in \mathscr{N}(\mathbf{L}) \implies \mathscr{N}(\mathbf{L}) \neq \varnothing$
   (b) $\mathscr{N}(\mathbf{L}) \triangleq \{\boldsymbol{x} \in \boldsymbol{X} | \mathbf{L}\boldsymbol{x} = \mathbb{0}\} \subseteq \boldsymbol{X}$
   (c) $\boldsymbol{x} + \boldsymbol{y} \in \mathscr{N}(\mathbf{L}) \implies \mathbb{0} = \mathbf{L}(\boldsymbol{x} + \boldsymbol{y}) = \mathbf{L}(\boldsymbol{y} + \boldsymbol{x}) \implies \boldsymbol{y} + \boldsymbol{x} \in \mathscr{N}(\mathbf{L})$
   (d) $\alpha \in \mathbb{F},\ \boldsymbol{x} \in \boldsymbol{X} \implies \mathbb{0} = \mathbf{L}\boldsymbol{x} \implies \mathbb{0} = \alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0} = \mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x} \in \mathscr{N}(\mathbf{L})$
2. Proof that $\mathscr{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0} \in \mathscr{I}(\mathbf{L}) \implies \mathscr{I}(\mathbf{L}) \neq \varnothing$
   (b) $\mathscr{I}(\mathbf{L}) \triangleq \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \text{ such that } \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\} \subseteq \boldsymbol{Y}$
   (c) $\boldsymbol{x} + \boldsymbol{y} \in \mathscr{I}(\mathbf{L}) \implies \exists \boldsymbol{v} \in \boldsymbol{X} \text{ such that } \mathbf{L}\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x} \implies \boldsymbol{y} + \boldsymbol{x} \in \mathscr{I}(\mathbf{L})$
   (d) $\alpha \in \mathbb{F},\ \boldsymbol{x} \in \mathscr{I}(\mathbf{L}) \implies \exists \boldsymbol{x} \in \boldsymbol{X} \text{ such that } \boldsymbol{y} = \mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y} = \alpha\mathbf{L}\boldsymbol{x} = \mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x} \in \mathscr{I}(\mathbf{L})$

✏

Example 15.1. [7] Let $\mathscr{C}([a:b], \mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX	$\mathscr{C}([a:b], \mathbb{R})$ is a linear space.

Theorem 15.3. [8] *Let $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* Null Space *of a linear operator $\mathbf{L} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$.*

THM			
	$\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x} - \boldsymbol{y} \in \mathscr{N}(\mathbf{L})$
	$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathscr{N}(\mathbf{L}) = \{\mathbb{0}\}$

✎PROOF:

1. Proof that $\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \implies \boldsymbol{x} - \boldsymbol{y} \in \mathscr{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) &= \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} && \text{by Theorem 15.1 page 206} \\
&= \mathbb{0} && \text{by left hypothesis} \\
&\implies \quad \boldsymbol{x} - \boldsymbol{y} \in \mathscr{N}(\mathbf{L}) && \text{by definition of } \textit{Null Space}
\end{aligned}$$

2. Proof that $\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \impliedby \boldsymbol{x} - \boldsymbol{y} \in \mathscr{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}\boldsymbol{y} &= \mathbf{L}\boldsymbol{y} + \mathbb{0} && \text{by definition of linear space (Definition 4.1 page 71)} \\
&= \mathbf{L}\boldsymbol{y} + \mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) && \text{by right hypothesis} \\
&= \mathbf{L}\boldsymbol{y} + (\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y}) && \text{by Theorem 15.1 page 206} \\
&= (\mathbf{L}\boldsymbol{y} - \mathbf{L}\boldsymbol{y}) + \mathbf{L}\boldsymbol{x} && \text{by } \textit{associative} \text{ and } \textit{commutative} \text{ properties (Definition 4.1 page 71)} \\
&= \mathbf{L}\boldsymbol{x}
\end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathscr{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned}
\mathbf{L} \text{ is } \textit{injective} &\iff \{(\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \iff \boldsymbol{x} = \boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X\} \\
&\iff \left\{\left[\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} = \mathbb{0} \iff (\boldsymbol{x} - \boldsymbol{y}) = \mathbb{0}\right] \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X\right\} \\
&\iff \left\{\left[\mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) = \mathbb{0} \iff (\boldsymbol{x} - \boldsymbol{y}) = \mathbb{0}\right] \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X\right\} \\
&\iff \mathscr{N}(\mathbf{L}) = \{\mathbb{0}\}
\end{aligned}$$

[7] Eidelman et al. (2004) page 3
[8] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

Theorem 15.4. [9] *Let $\boldsymbol{W}$, $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ be linear spaces over a field $\mathbb{F}$.*

THM

1.	$\mathbf{L}(\mathbf{MN})$	$=$	$(\mathbf{LM})\mathbf{N}$	$\forall\mathbf{L}\in\mathscr{L}(\boldsymbol{Z},\boldsymbol{W}), \mathbf{M}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(ASSOCIATIVE)
2.	$\mathbf{L}(\mathbf{M}\dotplus\mathbf{N})$	$=$	$(\mathbf{LM})\dotplus(\mathbf{LN})$	$\forall\mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(LEFT DISTRIBUTIVE)
3.	$(\mathbf{L}\dotplus\mathbf{M})\mathbf{N}$	$=$	$(\mathbf{LN})\dotplus(\mathbf{MN})$	$\forall\mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{LM})$	$=$	$(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall\mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{MN}) = (\mathbf{LM})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}(\mathbf{M}\dotplus\mathbf{N}) = (\mathbf{LM})\dotplus(\mathbf{LN})$:

$$\begin{aligned}
[\mathbf{L}(\mathbf{M}\dotplus\mathbf{N})]\boldsymbol{x} &= \mathbf{L}[(\mathbf{M}\dotplus\mathbf{N})\boldsymbol{x}] \\
&= \mathbf{L}[(\mathbf{M}\boldsymbol{x})\dotplus(\mathbf{N}\boldsymbol{x})] \\
&= [\mathbf{L}(\mathbf{M}\boldsymbol{x})]\dotplus[\mathbf{L}(\mathbf{N}\boldsymbol{x})] && \text{by } \textit{additive} \text{ property Definition 15.3 page 206} \\
&= [(\mathbf{LM})\boldsymbol{x}]\dotplus[(\mathbf{LN})\boldsymbol{x}]
\end{aligned}$$

3. Proof that $(\mathbf{L}\dotplus\mathbf{M})\mathbf{N} = (\mathbf{LN})\dotplus(\mathbf{MN})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{LM}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{LM}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned}
[\alpha(\mathbf{LM})]\boldsymbol{x} &= \alpha[(\mathbf{LM})\boldsymbol{x}] \\
&= \mathbf{L}[\alpha(\mathbf{M}\boldsymbol{x})] && \text{by } \textit{homogeneous} \text{ property Definition 15.3 page 206} \\
&= \mathbf{L}[(\alpha\mathbf{M})\boldsymbol{x}] \\
&= [\mathbf{L}(\alpha\mathbf{M})]\boldsymbol{x}
\end{aligned}$$

✏

Theorem 15.5 (Fundamental theorem of linear equations). Michel and Herget (1993) page 99 *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathscr{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\dim\mathscr{I}(\mathbf{L}) + \dim\mathscr{N}(\mathbf{L}) = \dim\boldsymbol{X} \qquad \forall\mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1, 2, \ldots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for

[9] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

$\mathcal{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\} \\
&= \dim \left\{\boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\sum_{k=1}^{p} \alpha_k \psi_k\right\} \\
&= \dim \left\{\boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L}\psi_k\right\} \\
&= \dim \left\{\boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L}\psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L}\psi_k\right\} \\
&= \dim \left\{\boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L}\psi_k + \mathbb{0}\right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ✏

15.2 Operators on Normed linear spaces

15.2.1 Operator norm

Definition 15.4. [10] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [11]

DEF *The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{\boldsymbol{x} \in \boldsymbol{X}} \{\|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \qquad \forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition 15.1 (next) shows that the functional defined in Definition 15.4 (previous) is a *norm* (Definition 6.1 page 87).

Proposition 15.1. [12] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

PRP *The functional $|||\cdot|||$ is a **norm** on $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1.	$|||\mathbf{A}||| \ge 0$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NON-NEGATIVE)	*and*
2.	$|||\mathbf{A}||| = 0 \iff \mathbf{A} \doteq \mathbb{0}$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NONDEGENERATE)	*and*
3.	$|||\alpha\mathbf{A}||| = |\alpha|\, |||\mathbf{A}|||$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), \alpha \in \mathbb{F}$	(HOMOGENEOUS)	*and*
4.	$|||\mathbf{A} \mathring{+} \mathbf{B}||| \le |||\mathbf{A}||| + |||\mathbf{B}|||$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

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[10] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225

[11] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990) page 290

[12] Rudin (1991) page 93

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \neq \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x\in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \le 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \\ &> 0 \end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \doteq \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x\in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \le 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \\ &= \sup_{x\in \boldsymbol{X}} \{ \|\mathbb{0}x\| \mid \|x\| \le 1 \} \\ &= 0 \end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha|\ |||\mathbf{A}|||$:

$$\begin{aligned} |||\alpha\mathbf{A}||| &\triangleq \sup_{x\in \boldsymbol{X}} \{ \|\alpha\mathbf{A}x\| \mid \|x\| \le 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \\ &= \sup_{x\in \boldsymbol{X}} \{ |\alpha|\ \|\mathbf{A}x\| \mid \|x\| \le 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \\ &= |\alpha| \sup_{x\in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \le 1 \} && \text{by definition of sup} \\ &= |\alpha|\ \ |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \end{aligned}$$

4. Proof that $|||\mathbf{A} \dotplus \mathbf{B}||| \le |||\mathbf{A}||| + |||\mathbf{B}|||$:

$$\begin{aligned} |||\mathbf{A} \dotplus \mathbf{B}||| &\triangleq \sup_{x\in \boldsymbol{X}} \left\{ \|(\mathbf{A} \dotplus \mathbf{B})x\| \mid \|x\| \le 1 \right\} && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \\ &= \sup_{x\in \boldsymbol{X}} \{ \|\mathbf{A}x + \mathbf{B}x\| \mid \|x\| \le 1 \} \\ &\le \sup_{x\in \boldsymbol{X}} \{ \|\mathbf{A}x\| + \|\mathbf{B}x\| \mid \|x\| \le 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \\ &\le \sup_{x\in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \le 1 \} + \sup_{x\in \boldsymbol{X}} \{ \|\mathbf{B}x\| \mid \|x\| \le 1 \} \\ &\triangleq |||\mathbf{A}||| + |||\mathbf{B}||| && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \end{aligned}$$

✏

Lemma 15.1. *Let* $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dotplus, \dot\times), \|\cdot\|)$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dotplus, \dot\times), \|\cdot\|)$.

LEM

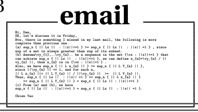
$$|||\mathbf{L}||| = \sup_{x} \{ \|\mathbf{L}x\| \mid \|x\| = 1 \} \qquad \forall x \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF: [13]

1. Proof that $\sup_x \{ \|\mathbf{L}x\| \mid \|x\| \le 1 \} \ge \sup_x \{ \|\mathbf{L}x\| \mid \|x\| = 1 \}$:

$$\sup_{x} \{ \|\mathbf{L}x\| \mid \|x\| \le 1 \} \ge \sup_{x} \{ \|\mathbf{L}x\| \mid \|x\| = 1 \} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

[13] **email**



Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ y \in X \,\middle|\, \begin{array}{ll} 1. & \|\mathbf{L}y\| = \sup_{x \in X} \{\|\mathbf{L}x\| \mid \|x\| \le 1\} \text{ and} \\ 2. & 0 < \|y\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_x \{\|\mathbf{L}x\| \mid \|x\| \le 1\} \le \sup_x \{\|\mathbf{L}x\| \mid \|x\| = 1\}$:

$$\begin{aligned}
\sup_{x \in X} \{\|\mathbf{L}x\| \mid \|x\| \le 1\} &= \|\mathbf{L}y\| && \text{by definition of set } Y \\
&= \frac{\|y\|}{\|y\|} \|\mathbf{L}y\| \\
&= \|y\| \left\| \frac{1}{\|y\|} \mathbf{L}y \right\| && \text{by homogeneous property (page 87)} \\
&= \|y\| \left\| \mathbf{L} \frac{y}{\|y\|} \right\| && \text{by homogeneous property (page 206)} \\
&\le \|y\| \sup_{y \in Y} \left\{ \left\| \mathbf{L} \frac{y}{\|y\|} \right\| \right\} && \text{by definition of supremum} \\
&= \|y\| \sup_{y \in Y} \left\{ \left\| \mathbf{L} \frac{y}{\|y\|} \right\| \,\middle|\, \left\| \frac{y}{\|y\|} \right\| = 1 \right\} && \text{because } \left\| \frac{y}{\|y\|} \right\| = 1 \text{ for all } y \in Y \\
&\le \sup_{y \in Y} \left\{ \left\| \mathbf{L} \frac{y}{\|y\|} \right\| \,\middle|\, \left\| \frac{y}{\|y\|} \right\| = 1 \right\} && \text{because } 0 < \|y\| \le 1 \\
&\le \sup_{x \in X} \{\|\mathbf{L}x\| \mid \|x\| = 1\} && \text{because } \frac{y}{\|y\|} \in X \quad \forall y \in Y
\end{aligned}$$

4. By (1) and (3),

$$\sup_{x \in X} \{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \sup_{x \in X} \{\|\mathbf{L}x\| \mid \|x\| = 1\}$$

⇨

Proposition 15.2. [14] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{X}), |||\cdot|||)$.

PRP	$\|\|\|\mathbf{I}\|\|\| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup \{\|\mathbf{I}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition 15.4 page 210)} \\
&= \sup \{\|x\| \mid \|x\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition 15.2 page 205)} \\
&= 1
\end{aligned}$$

⇨

Theorem 15.6. [15] *Let* $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM			
$\|\mathbf{L}x\|$	$\le$	$\|\|\|\mathbf{L}\|\|\| \; \|x\|$	$\forall \mathbf{L} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}),\ x \in \boldsymbol{X}$
$\|\|\|\mathbf{KL}\|\|\|$	$\le$	$\|\|\|\mathbf{K}\|\|\| \; \|\|\|\mathbf{L}\|\|\|$	$\forall \mathbf{K}, \mathbf{L} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$

[14] Michel and Herget (1993) page 410

[15] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

✎PROOF:

1. Proof that $\|\mathbf{L}\boldsymbol{x}\| \le |||\mathbf{L}|||\ \|\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{L}\boldsymbol{x}\| &= \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\|\mathbf{L}\boldsymbol{x}\| \\
&= \|\boldsymbol{x}\|\left\|\frac{1}{\|\boldsymbol{x}\|}\mathbf{L}\boldsymbol{x}\right\| && \text{by property of norms} \\
&= \|\boldsymbol{x}\|\left\|\mathbf{L}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\| && \text{by property of linear operators} \\
&\triangleq \|\boldsymbol{x}\|\ \|\mathbf{L}\boldsymbol{y}\| && \text{where } \boldsymbol{y} \triangleq \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \\
&\le \|\boldsymbol{x}\| \sup_{\boldsymbol{y}} \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of supremum} \\
&= \|\boldsymbol{x}\| \sup_{\boldsymbol{y}} \{\|\mathbf{L}\boldsymbol{y}\| \mid \|\boldsymbol{y}\| = 1\} && \text{because } \|\boldsymbol{y}\| = \left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\| = \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|} = 1 \\
&\triangleq \|\boldsymbol{x}\|\ \|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

2. Proof that $|||\mathbf{KL}||| \le |||\mathbf{K}|||\ |||\mathbf{L}|||$:

$$\begin{aligned}
|||\mathbf{KL}||| &\triangleq \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{\|(\mathbf{KL})\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by Definition 15.4 page 210 } (|||\cdot|||) \\
&= \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{\|\mathbf{K}(\mathbf{L}\boldsymbol{x})\| \mid \|\boldsymbol{x}\| \le 1\} \\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{|||\mathbf{K}|||\ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by 1.} \\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{|||\mathbf{K}|||\ |||\mathbf{L}|||\ \|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by 1.} \\
&= \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{|||\mathbf{K}|||\ |||\mathbf{L}|||\ 1 \mid \|\boldsymbol{x}\| \le 1\} && \text{by definition of sup} \\
&= |||\mathbf{K}|||\ |||\mathbf{L}||| && \text{by definition of sup}
\end{aligned}$$

✏

15.2.2 Bounded linear operators

Definition 15.5. [16] *Let $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ be a normed space of linear operators.*

DEF

*An operator $\mathbf{B}$ is **bounded** if $|||\mathbf{B}||| < \infty$.*
*The quantity $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ is the set of all **bounded linear operators** on $(\boldsymbol{X},\boldsymbol{Y})$ such that*

$$\mathscr{B}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \{\mathbf{L} \in \mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \mid |||\mathbf{L}||| < \infty\}.$$

Theorem 15.7. [17] *Let $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ be the set of linear operators over normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

THM

The following conditions are all EQUIVALENT:

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $\boldsymbol{x}_0 \in X$ $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point $\boldsymbol{x} \in X$)* $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
3. $|||\mathbf{L}||| < \infty$ *($\mathbf{L}$ is* BOUNDED*)* $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}\boldsymbol{x}\| \le M\|\boldsymbol{x}\|$ $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \boldsymbol{x}\in X$

[16] Rudin (1991) pages 92–93
[17] Aliprantis and Burkinshaw (1998) page 227

✎PROOF:

1. Proof that 1 $\implies$ 2:

$$\begin{aligned}
\epsilon &> \left\|\mathbf{L}x - \mathbf{L}x_0\right\| && \text{by hypothesis 1}\\
&= \left\|\mathbf{L}(x - x_0)\right\| && \text{by linearity (Definition 15.3 page 206)}\\
&= \left\|\mathbf{L}(x + y - x_0 - y)\right\| \\
&= \left\|\mathbf{L}(x + y) - \mathbf{L}(x_0 + y)\right\| && \text{by linearity (Definition 15.3 page 206)}\\
&\implies \mathbf{L} \text{ is continuous at point } x + y \\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}$$

2. Proof that 2 $\implies$ 1: obvious:

3. Proof that 4 $\implies$ 2:[18]

$$\begin{aligned}
|||\mathbf{L}x||| \le M\,\|x\| &\implies |||\mathbf{L}(x-y)||| \le M\,\|x-y\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}x - \mathbf{L}y||| \le M\,\|x-y\| && \text{by linearity of } \mathbf{L} \text{ (Definition 15.3 page 206)}\\
&\implies |||\mathbf{L}x - \mathbf{L}y||| \le \epsilon \text{ whenever } M\,\|x-y\| < \epsilon \\
&\implies |||\mathbf{L}x - \mathbf{L}y||| \le \epsilon \text{ whenever } \|x-y\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that 3 $\implies$ 4:

$$\begin{aligned}
\|\mathbf{L}x\| &\le \underbrace{|||\mathbf{L}|||}_{M}\,\|x\| && \text{by Theorem 15.6 page 212}\\
&= M\,\|x\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that 1 $\implies$ 3:[19]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \infty \\
&\implies \exists\,(x_n) \text{ such that } \left\|x_n\right\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\left\|\mathbf{L}x_n\right\| \mid \left\|x_n\right\| \le 1\right\} = \infty \\
&\implies \left\|x_n\right\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \left\|\mathbf{L}x_n\right\| \\
&\implies \left\|x_n\right\| = 1 \text{ and } \left\|\mathbf{L}x_n\right\| \ge n \\
&\implies \frac{1}{n}\left\|x_n\right\| = \frac{1}{n} \text{ and } \frac{1}{n}\left\|\mathbf{L}x_n\right\| \ge 1 \\
&\implies \left\|\frac{x_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1 \\
&\implies \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1 \\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

⇰

[18] Bollobás (1999) page 29

[19] Aliprantis and Burkinshaw (1998) page 227

15.2.3 Adjoints on normed linear spaces

Definition 15.6. *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *(Definition 5.4 page 84) of $\boldsymbol{X}$.*

DEF

$\mathbf{B}^*$ *is the **adjoint** of an operator* $\mathbf{B} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *if*

$$\mathsf{f}(\mathbf{B}\boldsymbol{x}) = \left[\mathbf{B}^*\mathsf{f}\right](\boldsymbol{x}) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ \boldsymbol{x} \in \boldsymbol{X}$$

Theorem 15.8. [20] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *$\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM

$$\begin{array}{rcll}
(\mathbf{A} \dotplus \mathbf{B})^* &=& \mathbf{A}^* \dotplus \mathbf{B}^* & \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{Y}) \\
(\lambda\mathbf{A})^* &=& \lambda\mathbf{A}^* & \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{Y}) \\
(\mathbf{A}\mathbf{B})^* &=& \mathbf{B}^*\mathbf{A}^* & \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{Y})
\end{array}$$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{A} \dotplus \mathbf{B}\right]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \dotplus \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition 15.6 page 215)} \\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition 15.3 page 206)} \\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} && \text{(Definition 14.1 page 199)} \\
&= \mathbf{A}^*\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition 15.6 page 215)} \\
&= \left[\mathbf{A}^* + \mathbf{B}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{linear functional} && \text{(Definition 14.1 page 199)}
\end{aligned}$$

$$\begin{aligned}
[\lambda\mathbf{A}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition 15.6 page 215)} \\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} && \text{(Definition 14.1 page 199)} \\
&= \left[\lambda\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition 15.6 page 215)}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{A}\mathbf{B}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition 15.6 page 215)} \\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of } \textit{linear operators} && \text{(Definition 15.3 page 206)} \\
&= \left[\mathbf{A}^*\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition 15.6 page 215)} \\
&= \mathbf{B}^*\left[\mathbf{A}^*\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition 15.6 page 215)} \\
&= \left[\mathbf{B}^*\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition 15.6 page 215)}
\end{aligned}$$

✏

Theorem 15.9. [21] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^*$ be the adjoint of an operator $\mathbf{B}$.*

THM

$$|||\mathbf{B}||| = |||\mathbf{B}^*||| \qquad \forall \mathbf{B} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$$

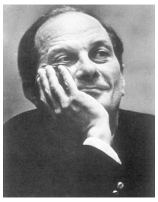
✎PROOF:

$$\begin{aligned}
|||\mathbf{B}||| &\triangleq \sup\left\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\right\} && \text{by Definition 15.4 page 210} \\
&\overset{?}{=} \sup\left\{|\mathsf{g}(\mathbf{B}\boldsymbol{x};\boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x};\mathbf{B}^*\boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&\triangleq \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&= \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&\triangleq |||\mathbf{B}^*||| && \text{by Definition 15.4 page 210}
\end{aligned}$$

✏

[20] Bollobás (1999) page 156
[21] Rudin (1991) page 98

15.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞
Stanislaus M. Ulam (1909–1984), Polish mathematician [22]

Theorem 15.10 (Mazur-Ulam theorem). [23] *Let $\phi \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ be a function on normed linear spaces $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. Let $\mathbf{I} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$ be the identity operator on $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.*

THM

$$\left.\begin{array}{lll}1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} & \text{and}\\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}}_{\text{isometric}} & \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$

   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$

2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$

3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned}
\left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}}\\
&= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis}\\
&= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}}\\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1}
\end{aligned}$$

[22] quote: Ulam (1991) page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[23] Oikhberg and Rosenthal (2007) page 598, Väisälä (2003) page 634, Giles (2000) page 11, Dunford and Schwartz (1957) page 91, Mazur and Ulam (1932)

4. Proof that $gz = z$:

$$\begin{aligned}
2\lambda &= 2\sup\{\|gz - z\|\} && \text{by definition of } \lambda \text{ item (2)}\\
&\le 2\|gz - z\| && \text{by definition of sup}\\
&= \|2z - 2gz\|\\
&= \|\psi gz - gz\| && \text{by definition of } \psi \text{ item (1)}\\
&= \left\|g^{-1}\psi gz - g^{-1}gz\right\| && \text{by item (3)}\\
&= \left\|g^{-1}\psi gz - z\right\| && \text{by definition of } g^{-1}\\
&= \left\|\psi g^{-1}\psi gz - z\right\|\\
&= \left\|g^{*}z - z\right\|\\
&\le \lambda && \text{by definition of } \lambda \text{ item (2)}\\
&\implies 2\lambda \le \lambda\\
&\implies \lambda = 0\\
&\implies gz = z
\end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}\phi x + \frac{1}{2}\phi y$:

$$\begin{aligned}
\phi\left(\frac{1}{2}x + \frac{1}{2}y\right) &=\\
&= \frac{1}{2}\phi x + \frac{1}{2}\phi y
\end{aligned}$$

6. Proof that $\phi([1-\lambda]x + \lambda y) = [1-\lambda]\phi x + \lambda\phi y$:

$$\begin{aligned}
\phi([1-\lambda]x + \lambda y) &=\\
&= [1-\lambda]\phi x + \lambda\phi y
\end{aligned}$$

⇨

Theorem 15.11 (Neumann Expansion Theorem). [24] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{ll}1. & \mathbf{A} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X}) \quad (\mathbf{A}\text{ is bounded})\\ 2. & |||\mathbf{A}||| < 1\end{array}\right\} \implies \left\{\begin{array}{lccl}1. & (\mathbf{I}-\mathbf{A})^{-1} & & \textit{exists}\\ 2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \le & \dfrac{1}{1-|||\mathbf{A}|||}\\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \displaystyle\sum_{n=0}^{\infty}\mathbf{A}^n\\ & \multicolumn{3}{l}{\textit{with uniform convergence}}\end{array}\right.$$

15.3 Operators on Inner product spaces

15.3.1 General Results

Theorem 15.12. [25] *Let* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM

$$\begin{array}{lclcl}
\langle \mathbf{B}x \mid x\rangle = 0 \quad \forall x \in X & \iff & \mathbf{B}x = \mathbb{0} \quad \forall x \in X\\
\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \quad \forall x \in X & \iff & \mathbf{A} = \mathbf{B}
\end{array}$$

[24] Michel and Herget (1993) page 415

[25] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

✎PROOF:

1. Proof that $\langle \mathbf{B}x \mid x \rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x + \mathbf{B}x) \mid (x + \mathbf{B}x) \rangle + i \langle \mathbf{B}(x + i\mathbf{B}x) \mid (x + i\mathbf{B}x) \rangle && \text{by left hypothesis} \\
&= \left\{ \left\langle \mathbf{B}x + \mathbf{B}^2 x) \mid x + \mathbf{B}x \right\rangle \right\} + i \left\{ \left\langle \mathbf{B}x + i\mathbf{B}^2 x) \mid x + i\mathbf{B}x \right\rangle \right\} && \text{by Definition 15.3 page 206} \\
&= \left\{ \langle \mathbf{B}x \mid x \rangle + \langle \mathbf{B}x \mid \mathbf{B}x \rangle + \left\langle \mathbf{B}^2 x \mid x \right\rangle + \left\langle \mathbf{B}^2 x \mid \mathbf{B}x \right\rangle \right\} && \text{by Definition 7.1 page 99} \\
&\quad + i \left\{ \langle \mathbf{B}x \mid x \rangle - i \langle \mathbf{B}x \mid \mathbf{B}x \rangle + i \left\langle \mathbf{B}^2 x \mid x \right\rangle - i^2 \left\langle \mathbf{B}^2 x \mid \mathbf{B}x \right\rangle \right\} \\
&= \left\{ 0 + \langle \mathbf{B}x \mid \mathbf{B}x \rangle + \left\langle \mathbf{B}^2 x \mid x \right\rangle + 0 \right\} + i \left\{ 0 - i \langle \mathbf{B}x \mid \mathbf{B}x \rangle + i \left\langle \mathbf{B}^2 x \mid x \right\rangle - i^2 0 \right\} && \text{by left hypothesis} \\
&= \left\{ \langle \mathbf{B}x \mid \mathbf{B}x \rangle + \left\langle \mathbf{B}^2 x \mid x \right\rangle \right\} + \left\{ \langle \mathbf{B}x \mid \mathbf{B}x \rangle - \left\langle \mathbf{B}^2 x \mid x \right\rangle \right\} \\
&= 2 \langle \mathbf{B}x \mid \mathbf{B}x \rangle \\
&= 2 \|\mathbf{B}x\|^2 \\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition 6.1 page 87}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \mid x \rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products (Theorem 7.1 page 99).

3. Proof that $\langle \mathbf{A}x \mid x \rangle = \langle \mathbf{B}x \mid x \rangle \implies \mathbf{A} \doteq \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \mid x \rangle - \langle \mathbf{B}x \mid x \rangle && \text{by left hypothesis} \\
&= \langle \mathbf{A}x - \mathbf{B}x \mid x \rangle && \text{by } \textit{additivity} \text{ property of } \langle \triangle \mid \nabla \rangle \text{ (Definition 7.1 page 99)} \\
&= \langle (\mathbf{A} - \mathbf{B})x \mid x \rangle && \text{by definition of operator addition} \\
\implies (\mathbf{A} - \mathbf{B})x &= \mathbb{0} && \text{by item 1} \\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \mid x \rangle = \langle \mathbf{B}x \mid x \rangle \impliedby \mathbf{A} \doteq \mathbf{B}$:

$$\langle \mathbf{A}x \mid x \rangle = \langle \mathbf{B}x \mid x \rangle \qquad \text{by } \mathbf{A} \doteq \mathbf{B} \text{ hypothesis}$$

✏

15.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition 15.3 page 218). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem 15.13 page 219).
- Both support decomposition into "real" and "imaginary" parts (Theorem 17.3 page 254).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *Null Space* of an operator (Theorem 15.14 page 220).

Proposition 15.3. [26] *Let $\mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *(Definition 15.5 page 213) on a* HILBERT SPACE $\boldsymbol{H}$.

PRP *An operator $\mathbf{B}^*$ is the **adjoint** of $\mathbf{B} \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ if*

$$\langle \mathbf{B}x \mid y \rangle = \langle x \mid \mathbf{B}^* y \rangle \qquad \forall x, y \in \boldsymbol{H}.$$

[26] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000) page 182, von Neumann (1929) page 49, Stone (1932) page 41

✎PROOF:

1. For fixed y, $\mathsf{f}(\boldsymbol{x}) \triangleq \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is a *functional* in $\mathbb{F}^{\boldsymbol{X}}$ (Definition 14.1 page 199).
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}
\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle &\triangleq \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by Definition 14.1 (page 199)} \\
&\triangleq \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{operator adjoint} && \text{(Definition 15.6 page 215)} \\
&= \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle && \text{by Definition 14.1 (page 199)}
\end{aligned}$$

✏

Example 15.2.

EX	
	In matrix algebra ("linear algebra")
	The inner product operation $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is represented by $\boldsymbol{y}^H\boldsymbol{x}$.
	The linear operator is represented as a matrix $\boldsymbol{A}$.
	The operation of $\boldsymbol{A}$ on a vector $\boldsymbol{x}$ is represented as $\boldsymbol{A}\boldsymbol{x}$.
	The adjoint of matrix $\boldsymbol{A}$ is the Hermitian matrix $\boldsymbol{A}^H$.

✎PROOF:

$$\langle \boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H\boldsymbol{A}\boldsymbol{x} = [(\boldsymbol{A}\boldsymbol{x})^H\boldsymbol{y}]^H = [\boldsymbol{x}^H\boldsymbol{A}^H\boldsymbol{y}]^H = (\boldsymbol{A}^H\boldsymbol{y})^H\boldsymbol{x} \triangleq \left\langle \boldsymbol{x} \mid \boldsymbol{A}^H\boldsymbol{y} \right\rangle$$

✏

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition 17.3 page 252). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[27]

Theorem 15.13 (operator star-algebra). [28] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM							
	The pair $(\boldsymbol{H}, *)$ *is a* $*$-ALGEBRA (STAR-ALGEBRA). *In particular,*						
	1.	$(\mathbf{A} \dotplus \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
	2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
	3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
	4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(INVOLUTARY)	

✎PROOF:

$$\begin{aligned}
\left\langle \boldsymbol{x} \mid (\mathbf{A} \dotplus \mathbf{B})^*\boldsymbol{y} \right\rangle &= \left\langle (\mathbf{A} \dotplus \mathbf{B})\boldsymbol{x} \mid \boldsymbol{y} \right\rangle && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)} \\
&= \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition 7.1 page 99)} \\
&= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of operator addition} \\
&= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} + \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition 7.1 page 99)} \\
&= \langle \boldsymbol{x} \mid (\mathbf{A}^* + \mathbf{B}^*)\boldsymbol{y} \rangle && \text{by definition of operator addition}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \mid (\alpha\mathbf{A})^*\boldsymbol{y} \rangle &= \langle (\alpha\mathbf{A})\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)} \\
&= \langle \alpha(\mathbf{A}\boldsymbol{x}) \mid \boldsymbol{y} \rangle && \text{by definition of scalar multiplication} \\
&= \alpha \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition 7.1 page 99)}
\end{aligned}$$

[27] Sakai (1998) page 1
[28] Halmos (1998a) pages 39–40, Rudin (1991) page 311

$$
\begin{aligned}
&= \alpha \langle \boldsymbol{x} \,|\, \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)} \\
&= \langle \boldsymbol{x} \,|\, \alpha^* \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition 7.1 page 99)}
\end{aligned}
$$

$$
\begin{aligned}
\langle \boldsymbol{x} \,|\, (\mathbf{AB})^* \boldsymbol{y} \rangle &= \langle (\mathbf{AB}) \boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)} \\
&= \langle \mathbf{A}(\mathbf{B}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle && \text{by definition of operator multiplication} \\
&= \langle (\mathbf{B}\boldsymbol{x}) \,|\, \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{B}^* \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)}
\end{aligned}
$$

$$
\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{A}^{**} \boldsymbol{y} \rangle &= \langle \mathbf{A}^* \boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)} \\
&= \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle^* && \text{by definition of inner product} && \text{(Definition 7.1 page 99)} \\
&= \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* && \text{by definition of adjoint} && \text{(Proposition 15.3 page 218)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A} \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition 7.1 page 99)}
\end{aligned}
$$

⇰

Theorem 15.14. [29] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathscr{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$
\mathscr{N}(\mathbf{A}) = \mathscr{I}(\mathbf{A}^*)^\perp
$$

$$
\mathscr{N}(\mathbf{A}^*) = \mathscr{I}(\mathbf{A})^\perp
$$

✎PROOF:

$$
\begin{aligned}
\mathscr{I}(\mathbf{A}^*)^\perp &= \left\{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathscr{I}(\mathbf{A}^*) \right\} \\
&= \left\{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} \\
&= \{ y \in \boldsymbol{H} | \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathbf{A}^* && \text{(Proposition 15.3 page 218)} \\
&= \{ y \in \boldsymbol{H} | \mathbf{A} \boldsymbol{y} = 0 \} \\
&= \mathscr{N}(\mathbf{A}) && \text{by definition of } \mathscr{N}(\mathbf{A})
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{I}(\mathbf{A})^\perp &= \{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathscr{I}(\mathbf{A}) \} \\
&= \{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \mathbf{A} \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathscr{I} \\
&= \left\{ y \in \boldsymbol{H} | \langle \mathbf{A}^* \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition 15.3 page 218)} \\
&= \left\{ y \in \boldsymbol{H} | \mathbf{A}^* \boldsymbol{y} = 0 \right\} \\
&= \mathscr{N}(\mathbf{A}^*) && \text{by definition of } \mathscr{N}(\mathbf{A})
\end{aligned}
$$

⇰

15.4 Special Classes of Operators

15.4.1 Projection operators

Definition 15.7. [30] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

[29] Rudin (1991) page 312
[30] Rudin (1991) page 133 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 6, Halmos (1958) page 73 ⟨§41. Projections⟩

DEF $\mathbf{P}$ *is a* ***projection*** *operator if* $\mathbf{P}^2 = \mathbf{P}$.

Theorem 15.15. [31] *Let* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{P}$ *be a bounded linear operator in* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *with* NULL SPACE $\mathscr{N}(\mathbf{P})$ *and* IMAGE SET $\mathscr{I}(\mathbf{P})$.

THM
$$\left.\begin{array}{llcll} 1. & \mathbf{P}^2 & = & \mathbf{P} & (\mathbf{P}\text{ is a projection operator}) \quad \text{and} \\ 2. & \boldsymbol{\Omega} & = & \boldsymbol{X} \dotplus \boldsymbol{Y} & (\boldsymbol{Y}\text{ compliments } \boldsymbol{X} \text{ in } \boldsymbol{\Omega}) \quad \text{and} \\ 3. & \mathbf{P}\boldsymbol{\Omega} & = & \boldsymbol{X} & (\mathbf{P}\text{ projects onto } \boldsymbol{X}) \end{array}\right\} \implies \left\{\begin{array}{llcll} 1. & \mathscr{I}(\mathbf{P}) & = & \boldsymbol{X} & \text{and} \\ 2. & \mathscr{N}(\mathbf{P}) & = & \boldsymbol{Y} & \text{and} \\ 3. & \boldsymbol{\Omega} & = & \mathscr{I}(\mathbf{P}) \dotplus \mathscr{N}(\mathbf{P}) & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} \mathscr{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\ &= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\ &= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\ &= \boldsymbol{\Omega}_1 \end{aligned}$$

$$\begin{aligned} \mathscr{N}(\mathbf{P}) &= \{x \in \boldsymbol{\Omega} | \mathbf{P}x = \mathbb{0}\} \\ &= \left\{x \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}x = \mathbb{0}\right\} \\ &= \left\{x \in \boldsymbol{\Omega}_1 | \mathbf{P}x = \mathbb{0}\right\} + \left\{x \in \boldsymbol{\Omega}_2 | \mathbf{P}x = \mathbb{0}\right\} \\ &= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_2 \end{aligned}$$

✏

Theorem 15.16. [32] *Let* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{P}$ *be a bounded linear operator in* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.

THM
$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P}\text{ is a projection operator}} \iff \underbrace{(\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})}_{(\mathbf{I}-\mathbf{P})\text{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})$:

$$\begin{aligned} (\mathbf{I}-\mathbf{P})^2 &= (\mathbf{I}-\mathbf{P})(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{I}(\mathbf{I}-\mathbf{P}) + (-\mathbf{P})(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{I}-\mathbf{P}-\mathbf{P}\mathbf{I}+\mathbf{P}^2 \\ &= \mathbf{I}-\mathbf{P}-\mathbf{P}+\mathbf{P} && \text{by left hypothesis} \\ &= \mathbf{I}-\mathbf{P} \end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})$:

$$\begin{aligned} \mathbf{P}^2 &= \underbrace{\mathbf{I}-\mathbf{P}-\mathbf{P}+\mathbf{P}^2}_{(\mathbf{I}-\mathbf{P})^2} - (\mathbf{I}-\mathbf{P}-\mathbf{P}) \\ &= (\mathbf{I}-\mathbf{P})^2 - (\mathbf{I}-\mathbf{P}-\mathbf{P}) \\ &= (\mathbf{I}-\mathbf{P}) - (\mathbf{I}-\mathbf{P}-\mathbf{P}) && \text{by right hypothesis} \\ &= \mathbf{P} \end{aligned}$$

[31] Michel and Herget (1993) pages 120–121
[32] Michel and Herget (1993) page 121

Theorem 15.17. [33] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathscr{N}(\mathbf{P})$, *and* IMAGE SET $\mathscr{I}(\mathbf{P})$.

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

1. $\mathbf{P}^* = \mathbf{P}$ ($\mathbf{P}$ *is* SELF-ADJOINT) $\iff$
2. $\mathbf{P}^*\mathbf{P} = \mathbf{P}\mathbf{P}^*$ ($\mathbf{P}$ *is* NORMAL) $\iff$
3. $\mathscr{I}(\mathbf{P}) = \mathscr{N}(\mathbf{P})^\perp$ $\iff$
4. $\langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle = \|\mathbf{P}\boldsymbol{x}\|^2 \quad \forall x \in X$

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned} \mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)} \\ &= \mathbf{P}\mathbf{P}^* && \text{by Theorem 15.13 page 219} \end{aligned}$$

Proof that (1) $\Longrightarrow$ (3):

$$\begin{aligned} \mathscr{I}(\mathbf{P}) &= \mathscr{N}(\mathbf{P}^*)^\perp && \text{by Theorem 15.14 page 220} \\ &= \mathscr{N}(\mathbf{P})^\perp && \text{by (1)} \end{aligned}$$

Proof that (3) $\Longrightarrow$ (4):

Proof that (4) $\Longrightarrow$ (1):

15.4.2 Self Adjoint Operators

Definition 15.8. [34] *Let* $\mathbf{B} \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be* ***self-adjoint*** *or* ***hermitian*** *if* $\mathbf{B} \triangleq \mathbf{B}^*$.

Example 15.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}}_{\text{expectation}}[\mathsf{x}(t)\mathsf{x}^*(u)].$$

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}} \mathsf{f}(u)\, \mathrm{d}u$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

[33] Rudin (1991) page 314

[34] Historical works regarding self-adjoint operators: von Neumann (1929) page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932) page 50 ⟨"self-adjoint transformations"⟩

Theorem 15.18. [35] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}$.

THM

$$\left\{ \underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S} \textit{ is self adjoint}} \right\} \implies \left\{ \begin{array}{lll} 1. & \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \in \mathbb{R} & \textit{(the hermitian quadratic form of } \mathbf{S} \textit{ is REAL-VALUED)} \\ 2. & \lambda_n \in \mathbb{R} & \textit{(eigenvalues of } \mathbf{S} \textit{ are REAL-VALUED)} \\ 3. & \lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0 & \textit{(eigenvectors are ORTHOGONAL)} \end{array} \right\}$$

✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \in \mathbb{R}$:

$$\begin{aligned} \langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\ &= \langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition 7.1 page 99} \end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned} \lambda_n \|\psi_n\|^2 &= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition} \\ &= \langle \lambda_n\psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition 7.1 page 99} \\ &= \langle \mathbf{S}\psi_n \,|\, \psi_n \rangle && \text{by definition of eigenpairs} \\ &= \langle \psi_n \,|\, \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\ &= \langle \psi_n \,|\, \lambda_n\psi_n \rangle && \text{by definition of eigenpairs} \\ &= \lambda_n^* \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition 7.1 page 99} \\ &= \lambda_n^* \|\psi_n\|^2 && \text{by definition} \end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned} \lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n\psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition 7.1 page 99} \\ &= \langle \mathbf{S}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\ &= \langle \psi_n \,|\, \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\ &= \langle \psi_n \,|\, \lambda_m\psi_m \rangle && \text{by definition of eigenpairs} \\ &= \lambda_m^* \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition 7.1 page 99} \\ &= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{because } \lambda_m \text{ is real} \end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

15.4.3 Normal Operators

Definition 15.9. [36] *Let* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{N}^*$ *be the adjoint of an operator* $\mathbf{N} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.

DEF

$\mathbf{N}$ *is* ***normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

[35] Lax (2002) pages 315–316, Keener (1988) pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩, Bertero and Boccacci (1998) page 225 ⟨§"9.2 SVD of a matrix …If all eigenvectors are normalized…"⟩
[36] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969) page 167, Frobenius (1878), Frobenius (1968) page 391

Theorem 15.19. [37] *Let* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ *be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathcal{N}(\mathbf{N})$ *be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ *and* $\mathcal{I}(\mathbf{N})$ *the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\underbrace{\mathbf{N}^*\mathbf{N}=\mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \iff \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\| \qquad \forall \boldsymbol{x}\in\boldsymbol{H}$$

✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N}=\mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition 15.3 page 218 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (}\mathbf{N}\text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition 15.3 page 218 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N}=\mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition 15.3 page 218 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem 15.13 page 219 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis (}\|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by Proposition 15.3 page 218 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem 15.20. [38] *Let* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ *be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathcal{N}(\mathbf{N})$ *be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ *and* $\mathcal{I}(\mathbf{N})$ *the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\underbrace{\mathbf{N}^*\mathbf{N}=\mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same Null Space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \left\{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{by definition of } \textit{Null Space}\\
&= \left\{\boldsymbol{x} | \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition 6.1 page 87)}\\
&= \left\{\boldsymbol{x} | \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\}\\
&= \left\{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition 6.1 page 87)}\\
&= \mathcal{N}(\mathbf{N}) && \text{by definition of } \textit{Null Space } \mathcal{N}
\end{aligned}$$

✏

Theorem 15.21. [39] *Let* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ *be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathcal{N}(\mathbf{N})$ *be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ *and* $\mathcal{I}(\mathbf{N})$ *the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N}=\mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m\rangle = 0}_{\text{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

[37] Rudin (1991) pages 312–313
[38] Rudin (1991) pages 312–313
[39] Rudin (1991) pages 312–313

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. [Rudin] claims both to be true.(Rudin, 1991)313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\quad\; = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem 15.13 page 219} \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem 15.13 page 219} \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\Longrightarrow \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\iff \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \mid \psi_m \rangle &= \langle \lambda_n\psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \triangledown \rangle \text{ Definition 7.1 page 99} \\
&= \langle \mathbf{N}\psi_n \mid \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \mid \mathbf{N}^*\psi_m \rangle && \text{by Proposition 15.3 page 218 (definition of adjoint)} \\
&= \langle \psi_n \mid \lambda_m^*\psi_m \rangle && \text{by (4.)} \\
&= \lambda_m \langle \psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \triangledown \rangle \text{ Definition 7.1 page 99}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \mid \psi_m \rangle = 0$.

✏

15.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition 15.10. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES *(Definition 6.1 page 87).*

DEF: *An operator* $\mathbf{M} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is* ***isometric*** *if*
$$\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \qquad \forall x \in X.$$

Theorem 15.22. [40] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM:
$$\underbrace{\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall x \in X}_{\text{isometric in length}} \iff \underbrace{\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall x,y \in X}_{\text{isometric in distance}}$$

✎PROOF:

[40] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

1. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \implies \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| &= \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\| && \text{by definition of linear operators (Definition 15.3 page 206)} \\
&= \|\mathbf{M}\boldsymbol{u}\| && \text{let } \boldsymbol{u} \triangleq \boldsymbol{x} - \boldsymbol{y} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by left hypothesis}
\end{aligned}$$

2. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \impliedby \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \|\mathbf{M}(\boldsymbol{x} - \mathbb{0})\| \\
&= \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\mathbb{0}\| && \text{by definition of linear operators (Definition 15.3 page 206)} \\
&= \|\boldsymbol{x} - \mathbb{0}\| && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\|
\end{aligned}$$

⇨

Isometric operators have already been defined (Definition 15.10 page 225) in the more general normed linear spaces, while Theorem 15.22 (page 225) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \iff \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem 15.23 (next) demonstrates two additional equivalent properties.

Theorem 15.23. [41] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle}$.*

THM *The following conditions are all* ***equivalent****:*

1.	$\mathbf{M}^*\mathbf{M}$	$=$	$\mathbf{I}$			$\iff$
2.	$\langle \mathbf{M}\boldsymbol{x} \,\vert\, \mathbf{M}\boldsymbol{y}\rangle$	$=$	$\langle \boldsymbol{x} \,\vert\, \boldsymbol{y}\rangle$	$\forall x,y \in X$	($\mathbf{M}$ *is surjective*)	$\iff$
3.	$\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|$	$=$	$\|x - y\|$	$\forall x,y \in X$	(*isometric in distance*)	$\iff$
4.	$\|\mathbf{M}\boldsymbol{x}\|$	$=$	$\|\boldsymbol{x}\|$	$\forall x \in X$	(*isometric in length*)	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y}\rangle &= \langle \boldsymbol{x} \,|\, \mathbf{M}^*\mathbf{M}\boldsymbol{y}\rangle && \text{by Proposition 15.3 page 218 (definition of adjoint)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{I}\boldsymbol{y}\rangle && \text{by (1)} \\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by Definition 15.2 page 205 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \sqrt{\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{x}\rangle} && \text{by definition of } \|\cdot\| \\
&= \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle} && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y}\rangle &= \|\mathbf{M}\boldsymbol{x} + \mathbf{M}\boldsymbol{y}\|^2 - \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|^2 + i\,\|\mathbf{M}\boldsymbol{x} + i\mathbf{M}\boldsymbol{y}\|^2 - i\,\|\mathbf{M}\boldsymbol{x} - i\mathbf{M}\boldsymbol{y}\|^2 && \text{by polarization id.} \\
&= \|\mathbf{M}(\boldsymbol{x} + \boldsymbol{y})\|^2 - \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\|^2 + i\,\|\mathbf{M}(\boldsymbol{x} + i\boldsymbol{y})\|^2 - i\,\|\mathbf{M}(\boldsymbol{x} - i\boldsymbol{y})\|^2 && \text{by Definition 15.3} \\
&= \|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 + i\,\|\boldsymbol{x} + i\boldsymbol{y}\|^2 - i\,\|\boldsymbol{x} - i\boldsymbol{y}\|^2 && \text{by left hypothesis}
\end{aligned}$$

[41] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

4. Proof that (3) $\iff$ (4): by Theorem 15.22 page 225

5. Proof that (4) $\implies$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition 15.3 page 218 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem 15.13 page 219 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y = 0\\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Definition 15.2 page 205 (definition of } \mathbf{I})\\
\implies \qquad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}$$

✏

Theorem 15.24. [42] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M} \text{ is isometric}} \implies \left\{\begin{array}{lcll} |||\mathbf{M}||| &=& 1 & \text{(UNIT LENGTH) and}\\ |\lambda| &=& 1 & \forall \lambda \in \Lambda \end{array}\right.$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \implies |||\mathbf{M}||| = 1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{\boldsymbol{x}\in X}\{\|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Definition 15.4 page 210}\\
&= \sup_{\boldsymbol{x}\in X}\{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Theorem 15.23 page 226}\\
&= \sup_{\boldsymbol{x}\in X}\{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x}, \lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|}\ \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem 15.23 page 226}\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|}\ |\lambda|\ \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

✏

Example 15.4 (One sided shift operator). [43] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ (0, 1, 2, ...) and shift operators defined as

$$\begin{array}{llcll}
1. & \mathbf{S}_r\,(x_0, x_1, x_2, \ldots) & \triangleq & (0, x_0, x_1, x_2, \ldots) & \text{(right shift operator)}\\
2. & \mathbf{S}_l\,(x_0, x_1, x_2, \ldots) & \triangleq & (x_1, x_2, x_3, \ldots) & \text{(left shift operator)}
\end{array}$$

[42] Michel and Herget (1993) page 432
[43] Michel and Herget (1993) page 441

EX
1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r (\!(x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_0, y_1, y_2, \ldots)\!) \rangle &= \langle (\!(0, x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_0, y_1, y_2, \ldots)\!) \rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\, y_n^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \langle (\!(x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_1, y_2, y_3, \ldots)\!) \rangle \\
&= \left\langle (\!(x_0, x_1, x_2, \ldots)\!) \,\middle|\, \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} (\!(y_0, y_1, y_2, \ldots)\!) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✏

15.4.5 Unitary operators

Definition 15.11. [44] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF *The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition 15.4. *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP $\left.\begin{array}{l}\mathbf{U} \textit{ is } \text{UNITARY} \quad \textit{and} \\ \mathbf{V} \textit{ is } \text{UNITARY}\end{array}\right\} \implies (\mathbf{U}\mathbf{V}) \textit{ is } \text{UNITARY}.$

[44] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\left(\mathbf{V}^*\mathbf{U}^*\right) && \text{by Theorem 15.8 page 215}\\
&= \mathbf{U}\left(\mathbf{VV}^*\right)\mathbf{U}^* && \text{by associative property}\\
&= \mathbf{UIU}^* && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition 15.11 page 228)}\\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition 15.11 page 228)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \left(\mathbf{V}^*\mathbf{U}^*\right)(\mathbf{UV}) && \text{by Theorem 15.8 page 215}\\
&= \mathbf{V}^*\left(\mathbf{U}^*\mathbf{U}\right)\mathbf{V} && \text{by associative property}\\
&= \mathbf{V}^*\mathbf{IV} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition 15.11 page 228)}\\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition 15.11 page 228)}
\end{aligned}$$

✏

Theorem 15.25. [45] *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathscr{I}(\mathbf{U})$ be the* IMAGE SET *of* $\mathbf{U}$.

THM

*If $\mathbf{U}$ is a **bounded linear operator** ($\mathbf{U} \in \mathscr{B}(\boldsymbol{H},\boldsymbol{H})$), then the following conditions are **equivalent**:*

1. $\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ (UNITARY) $\iff$
2. $\langle \mathbf{U}x \,|\, \mathbf{U}y\rangle = \langle \mathbf{U}^*x \,|\, \mathbf{U}^*y\rangle = \langle x \,|\, y\rangle$ *and* $\mathscr{I}(\mathbf{U}) = X$ (SURJECTIVE) $\iff$
3. $\|\mathbf{U}x - \mathbf{U}y\| = \left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x - y\|$ *and* $\mathscr{I}(\mathbf{U}) = X$ (ISOMETRIC IN DISTANCE) $\iff$
4. $\|\mathbf{U}x\| = \|x\|$ *and* $\mathscr{I}(\mathbf{U}) = X$ (ISOMETRIC IN LENGTH)

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) $\langle \mathbf{U}x \,|\, \mathbf{U}y\rangle = \langle \mathbf{U}^*x \,|\, \mathbf{U}^*y\rangle = \langle x \,|\, y\rangle$ by Theorem 15.23 (page 226).
   (b) Proof that $\mathscr{I}(\mathbf{U}) = X$:

$$\begin{aligned}
X &\supseteq \mathscr{I}(\mathbf{U}) && \text{because } \mathbf{U} \in X^X\\
&\supseteq \mathscr{I}(\mathbf{UU}^*)\\
&= \mathscr{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{UU}^* = \mathbf{I})\\
&= X && \text{by Definition 15.2 page 205 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\iff$ (3) $\iff$ (4): by Theorem 15.23 page 226.
3. Proof that (3) $\implies$ (1):
   (a) Proof that $\|\mathbf{U}x - \mathbf{U}y\| = \|x - y\| \implies \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem 15.23 page 226
   (b) Proof that $\left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x - y\| \implies \mathbf{UU}^* = \mathbf{I}$:

$$\begin{aligned}
\left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x - y\| \implies \quad \mathbf{U}^{**}\mathbf{U}^* &= \mathbf{I} && \text{by Theorem 15.23 page 226}\\
\mathbf{UU}^* &= \mathbf{I} && \text{by Theorem 15.13 page 219}
\end{aligned}$$

✏

Theorem 15.26. *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathbf{U}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathscr{N}(\mathbf{U})$ the* NULL SPACE *of $\mathbf{U}$, and $\mathscr{I}(\mathbf{U})$ the* IMAGE SET

[45] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

of **U**.

THM

$$\underbrace{\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U} \text{ is unitary}} \implies \left\{\begin{array}{lclclr} \mathbf{U}^{-1} & = & \mathbf{U}^* & & & \text{and} \\ \mathcal{I}(\mathbf{U}) & = & \mathcal{I}(\mathbf{U}^*) & = & X & \text{and} \\ \mathcal{N}(\mathbf{U}) & = & \mathcal{N}(\mathbf{U}^*) & = & \{\mathbb{0}\} & \text{and} \\ |||\mathbf{U}||| & = & |||\mathbf{U}^*||| & = & 1 & \text{(unit length)} \end{array}\right\}$$

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both *isometric* and *normal*:

$$\begin{array}{lclclcl} \mathbf{U}^*\mathbf{U} & & & = & \mathbf{I} & \implies & \mathbf{U} \text{ is } \textit{isometric} \\ \mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & \mathbf{I} & \implies & \mathbf{U}^* \text{ is } \textit{isometric} \\ \mathbf{U}^{-1} & = & \mathbf{U}^* & & & \implies & \mathbf{U}^{-1} \text{ is } \textit{isometric} \\ \\ \mathbf{U}^*\mathbf{U} & = & \mathbf{U}\mathbf{U}^* & = & & \implies & \mathbf{U} \text{ is } \textit{normal} \\ \mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & & \implies & \mathbf{U}^* \text{ is } \textit{normal} \\ \mathbf{U}^{-1} & = & \mathbf{U}^* & & & \implies & \mathbf{U}^{-1} \text{ is } \textit{normal} \end{array}$$

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem 15.25 page 229.

3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}^{-1})$:

$$\begin{aligned} \mathcal{N}(\mathbf{U}^*) &= \mathcal{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both } \textit{normal} \text{ and by Theorem 15.20 page 224} \\ &= \mathcal{I}(\mathbf{U})^\perp && \text{by Theorem 15.14 page 220} \\ &= X^\perp && \text{by above result} \\ &= \{\mathbb{0}\} && \text{by Proposition 8.6 page 121} \end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
   Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem 15.24 page 227.

✏

Example 15.5 (Rotation matrix). [46]

EX

$$\underbrace{\left\{\mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right\}}_{\text{rotation matrix } \mathbf{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2} \implies \left\{\begin{array}{llcll} (1). & \mathbf{R}^{-1}{}_\theta & = & \mathbf{R}_{-\theta} & \text{and} \\ (2). & \mathbf{R}^*{}_\theta & = & \mathbf{R}^{-1}{}_\theta & (\mathbf{R} \text{ is unitary}) \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} \mathbf{R}^* &= \mathbf{R}^H \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of } \textit{Hermetian transpose} \text{ operator } H \\ &= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R} \\ &= \mathbf{R}^{-1} && \text{by 1.} \end{aligned}$$

✏

[46] Noble and Daniel (1988) page 311

Example 15.6. [47] Let $\mathbf{A}$ and $\mathbf{B}$ be matrix operators.

EX

$$\underbrace{\begin{array}{c|c} \mathbf{A} \triangleq \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} & \mathbf{B} \triangleq \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \mathbf{A} \text{ is a } \textit{rotation operator}. & \mathbf{B} \text{ is a } \textit{reflection operator}. \end{array}}_{\text{Both } \mathbf{A} \text{ and } \mathbf{B} \text{ are } \textbf{unitary}.}$$

Example 15.7. Examples of *Fredholm integral operators* include

EX

1.	**Fourier Transform**	$[\tilde{\mathbf{F}}\mathsf{x}](f) = \int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-i2\pi f t}\,\mathrm{d}t$	$\kappa(t, f) = e^{-i2\pi f t}$
2.	**Inverse Fourier Transform**	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t) = \int_{f\in\mathbb{R}} \tilde{\mathsf{x}}(f)e^{i2\pi f t}\,\mathrm{d}f$	$\kappa(f, t) = e^{i2\pi f t}$
3.	**Laplace operator**	$[\mathbf{L}\mathsf{x}](s) = \int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t, s) = e^{-st}$

Example 15.8 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathsf{f}(x) \triangleq \mathsf{f}(x-1) \qquad \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

EX

$$\begin{array}{rrcll} 1. & \mathbf{T}^{-1}\mathsf{f}(x) &=& \mathsf{f}(x+1) \quad \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \text{(inverse translation operator)} \\ 2. & \mathbf{T}^* &=& \mathbf{T}^{-1} & (\mathbf{T} \text{ is invertible}) \\ 3. & \mathbf{T}^*\mathbf{T} &=& \mathbf{T}\mathbf{T}^* = \mathbf{I} & (\mathbf{T} \text{ is unitary}) \end{array}$$

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$
$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned} \langle \mathbf{T}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \langle \mathsf{f}(x-1) \,|\, \mathsf{g}(x)\rangle && \text{by definition of } \mathbf{T} \\ &= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathrm{d}x \\ &= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathrm{d}x \\ &= \langle \mathsf{f}(x) \,|\, \mathsf{g}(x+1)\rangle \\ &= \left\langle \mathsf{f}(x) \,\middle|\, \underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x)\right\rangle && \text{by 1.} \end{aligned}$$

✏

Example 15.9 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX

$$\begin{array}{rrcll} 1. & \mathbf{D}^{-1}\mathsf{f}(x) &=& \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right) \quad \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \text{(inverse dilation operator)} \\ 2. & \mathbf{D}^* &=& \mathbf{D}^{-1} & (\mathbf{D} \text{ is invertible}) \\ 3. & \mathbf{D}^*\mathbf{D} &=& \mathbf{D}\mathbf{D}^* = \mathbf{I} & (\mathbf{D} \text{ is unitary}) \end{array}$$

[47] Gel'fand (1963) page 4, Gelfand et al. (2018) page 4

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x) \,|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\,\mathrm{d}u\\
&= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example 15.10 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n-1}\right)\!\right)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n+1}\right)\!\right)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\left(\!\left(x_n\right)\!\right) \,|\, \left(\!\left(y_n\right)\!\right)\rangle &= \langle \left(\!\left(x_{n-1}\right)\!\right) \,|\, \left(\!\left(y_n\right)\!\right)\rangle && \text{by definition of } \mathbf{D}\\
&= \sum_n \boldsymbol{x}_{n-1}\,\boldsymbol{y}_n^*\\
&= \sum_n \boldsymbol{x}_n\,\boldsymbol{y}_{n+1}^*\\
&= \left\langle \left(\!\left(x_n\right)\!\right) \,|\, \left(\!\left(y_{n+1}\right)\!\right)\right\rangle\\
&= \left\langle \left(\!\left(x_n\right)\!\right) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\left(\!\left(y_n\right)\!\right)\right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary. ✏

Example 15.11 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator

$$[\tilde{\mathbf{F}}\mathsf{x}](f) \triangleq \int_t \mathsf{x}(t)\underbrace{e^{-i2\pi ft}}_{\kappa(t,f)}\,\mathrm{d}t \qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](t) \triangleq \int_f \tilde{\mathsf{x}}(f)\underbrace{e^{i2\pi ft}}_{\kappa^*(t,f)}\,\mathrm{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$
\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathrm{x} \,|\, \tilde{\mathrm{y}} \right\rangle &= \left\langle \int_t \mathrm{x}(t)e^{-i2\pi ft}\,\mathrm{d}t \,|\, \tilde{\mathrm{y}}(f) \right\rangle \\
&= \int_t \mathrm{x}(t) \left\langle e^{-i2\pi ft} \,|\, \tilde{\mathrm{y}}(f) \right\rangle \,\mathrm{d}t \\
&= \int_t \mathrm{x}(t) \int_f e^{-i2\pi ft}\tilde{\mathrm{y}}^*(f)\,\mathrm{d}f\,\mathrm{d}t \\
&= \int_t \mathrm{x}(t) \left[\int_f e^{i2\pi ft}\tilde{\mathrm{y}}(f)\,\mathrm{d}f\right]^* \,\mathrm{d}t \\
&= \left\langle \mathrm{x}(t) \,|\, \int_f \tilde{\mathrm{y}}(f)e^{i2\pi ft}\,\mathrm{d}f \right\rangle \\
&= \left\langle \mathrm{x} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathrm{y}} \right\rangle
\end{aligned}
$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✏

15.5 Operator order

Definition 15.12. [48] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF: $\mathbf{P}$ ***is positive*** *if* $\langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle \geq 0 \;\forall \boldsymbol{x} \in \boldsymbol{X}$.
This condition is denoted $\mathbf{P} \geq 0$.

Theorem 15.27. [49]

THM:
$$
\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \implies \left\{\begin{array}{lcll}
(\mathbf{P}+\mathbf{Q}) & \geq & 0 & & ((\mathbf{P}+\mathbf{Q}) \text{ is positive}) \\
\mathbf{A}^*\mathbf{P}\mathbf{A} & \geq & 0 & \forall \mathbf{A}\in\mathscr{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive}) \\
\mathbf{A}^*\mathbf{A} & \geq & 0 & \forall \mathbf{A}\in\mathscr{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{A} \text{ is positive})
\end{array}\right.
$$

✎PROOF:

$$
\begin{aligned}
\langle(\mathbf{P}+\mathbf{Q})\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle\mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle + \langle\mathbf{Q}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by additive property of } \langle\triangle \,|\, \nabla\rangle \text{ (Definition 7.1 page 99)} \\
&\geq \langle\mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by left hypothesis} \\
&\geq 0 && \text{by left hypothesis} \\
\langle\mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle\mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \mathbf{A}\boldsymbol{x}\rangle && \text{by definition of adjoint (Proposition 15.3 page 218)} \\
&= \langle\mathbf{P}\boldsymbol{y} \,|\, \boldsymbol{y}\rangle && \text{where } \boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x} \\
&\geq 0 && \text{by left hypothesis} \\
\langle\mathbf{I}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by definition of } \mathbf{I} \text{ (Definition 15.2 page 205)} \\
&\geq 0 && \text{by non-negative property of } \langle\triangle \,|\, \nabla\rangle \text{ (Definition 7.1 page 99)} \\
&\implies \mathbf{I} \text{ is positive} \\
\langle\mathbf{A}^*\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle\mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by definition of } \mathbf{I} \text{ (Definition 15.2 page 205)} \\
&\geq 0 && \text{by two previous results}
\end{aligned}
$$

✏

[48] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩
[49] Michel and Herget (1993) page 429

Definition 15.13. [50] *Let* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be* BOUNDED *operators.*

DEF

$\mathbf{A} \geq \mathbf{B}$ *("$\mathbf{A}$ is greater than or equal to $\mathbf{B}$") if*
$\mathbf{A} - \mathbf{B} \geq 0$ *("$(\mathbf{A} - \mathbf{B})$ is positive")*

[50] Michel and Herget (1993) page 429

Part V

Structure of Spaces

CHAPTER 16

ORTHOCOMPLEMENTED LATTICES

*Orthocomplemented lattice*s (Definition 16.1 page 238) are a kind of generalization of *Boolean algebra*s. The relationship between lattices of several types, including orthocomplemented and Boolean lattices, is stated in Theorem 16.7 (page 249) and illustrated in Figure 16.1 (page 237).

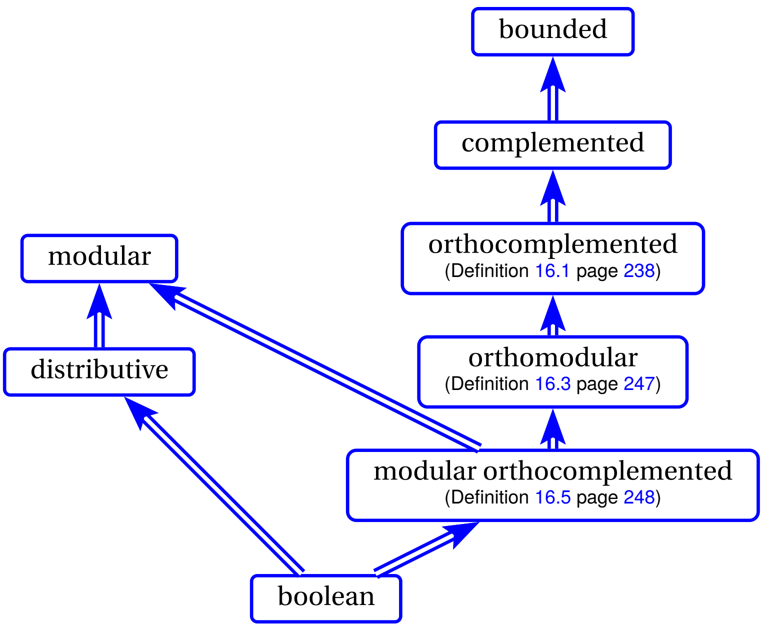


Figure 16.1: lattice of orthocomplemented lattices

16.1 Orthocomplemented Lattices

16.1.1 Definition

Definition 16.1. [1] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE.

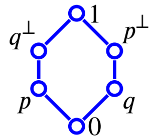
DEF

An element $x^{\perp} \in X$ *is an* ***orthocomplement*** *of an element* $x \in X$ *if*

1. $x^{\perp\perp} = x$ (INVOLUTORY) *and*
2. $x \wedge x^{\perp} = 0$ (NON-CONTRADICTION) *and*
3. $x \leq y \implies y^{\perp} \leq x^{\perp} \quad \forall y \in X$ (ANTITONE).

The LATTICE $\mathbf{L}$ *is* ***orthocomplemented*** *(*$\mathbf{L}$ *is an* ***orthocomplemented lattice****)* *if every element* x *in* X *has an* ORTHOCOMPLEMENT $x^{\perp}$ *in* X.

Definition 16.2. [2]



DEF

The $\mathbf{O}_6$ ***lattice*** *is the ordered set* $(\{0, p, q, p^{\perp}, q^{\perp}, 1\}, \leq)$ *with cover relation* $\lessdot = \{(0,p), (0,q), (p, q^{\perp}), (q, p^{\perp}), (p^{\perp}, 1), (q^{\perp}, 1)\}$.

The O_6 *lattice is illustrated by the Hasse diagram to the right.*

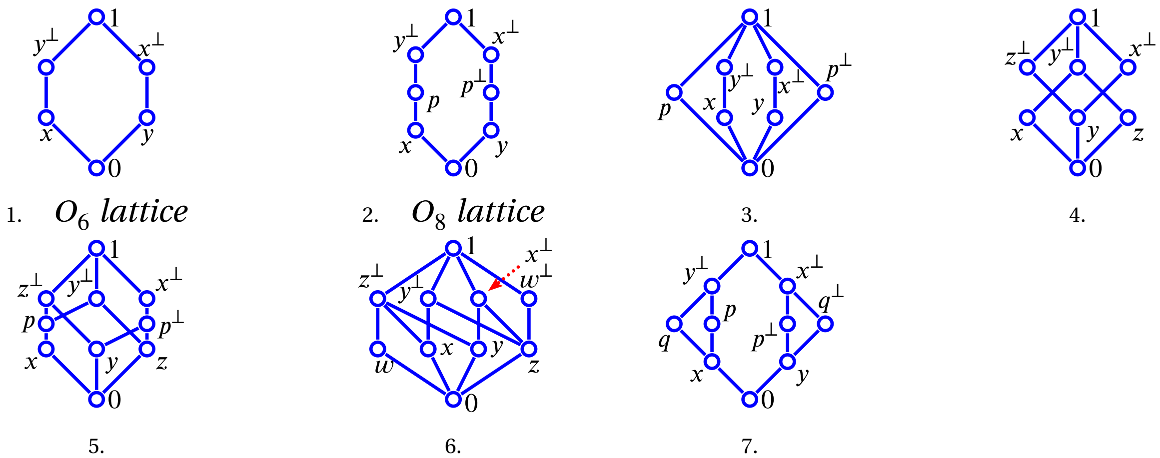
Example 16.1. [3]

EX

The O_6 **lattice** (Definition 16.2 page 238) is an **orthocomplemented lattice** (Definition 16.1 page 238).

Example 16.2. [4]There are a total of 10 **orthocomplemented lattices** with 8 elements or less. These 10, along with 3 other orthocomplemented lattices with 10 elements, are illustrated next:

Lattices that are **orthocomplemented** but *non-orthomodular* and hence also *not modular orthocomplemented* and *non-Boolean*:



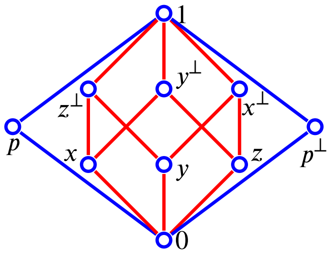
Lattices that are **orthocomplemented** and **orthomodular** but *not modular orthocomplemented* and hence also *non-Boolean*:

[1] Stern (1999) page 11, Beran (1985) page 28, Kalmbach (1983) page 16, Gudder (1988) page 76, Loomis (1955) page 3, Birkhoff and Neumann (1936) page 830 ⟨L71–L73⟩

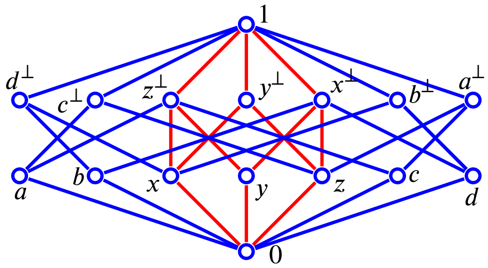
[2] Kalmbach (1983) page 22, Holland (1970) page 50, Beran (1985) page 33, Stern (1999) page 12, The O_6 *lattice* is also called the **Benzene ring** or the **hexagon**.

[3] Holland (1963) page 50

[4] Beran (1985) pages 33–42, Maeda (1966) page 250, Kalmbach (1983) page 24 ⟨Figure 3.2⟩, Stern (1999) page 12, Holland (1970) page 50

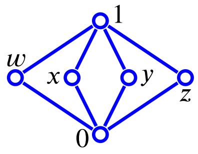


8.

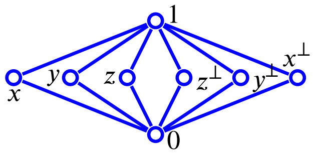


9.

Lattices that are **orthocomplemented**, **orthomodular**, and **modular orthocomplemented** but *non-Boolean*:



10. M_4 *lattice*

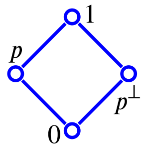


11. M_6 *lattice*

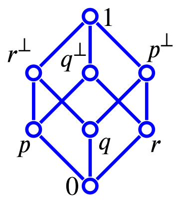
Lattices that are **orthocomplemented**, **orthomodular**, **modular orthocomplemented** and **Boolean**:

12. L_1 *lattice*

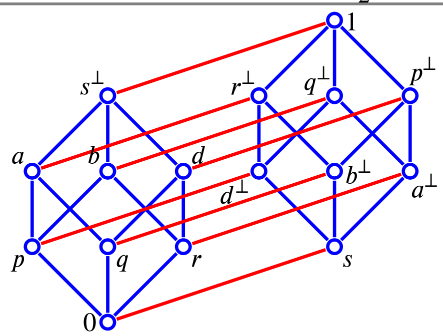
13. L_2 *lattice*



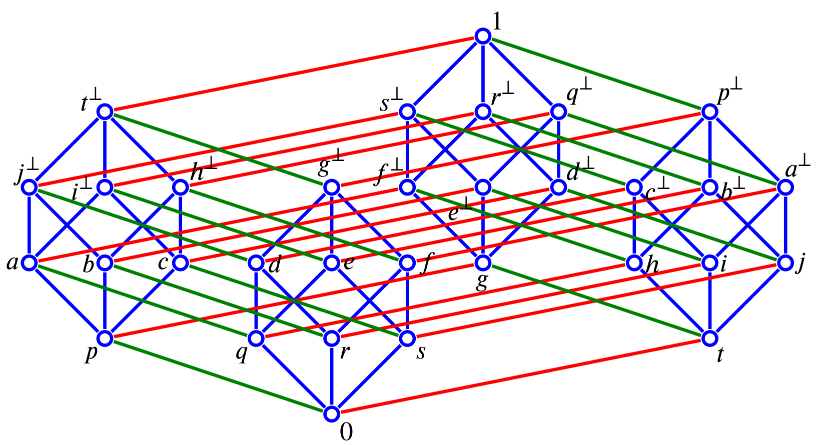
14. L_2^2 *lattice*



15. L_2^3 *lattice*



16. L_2^4 *lattice*

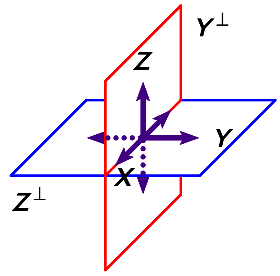


17. L_2^5 *lattice*

Example 16.3.

EX

The structure $\left(2^{\mathbb{R}^N},\, +,\, \cap,\, \varnothing,\, \boldsymbol{H}\,;\, \subseteq\right)$
is an **orthocomplemented lattice** where
- $\mathbb{R}^N$ is an **Euclidean space** with dimension N
- $2^{\mathbb{R}^N}$ is the set of all subspaces of $\mathbb{R}^N$
- $\boldsymbol{V} + \boldsymbol{W}$ is the *Minkowski sum* of subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$
- $\boldsymbol{V} \cap \boldsymbol{W}$ is the *intersection* of subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$



Example 16.4.

EX

The structure $\left(2^{\boldsymbol{H}},\, \oplus,\, \cap,\, \varnothing,\, \boldsymbol{H}\,;\, \subseteq\right)$ is an **orthocomplemented lattice** where
- $\boldsymbol{H}$ is a **Hilbert space**
- $2^{\boldsymbol{H}}$ is the set of all closed subspaces of $\boldsymbol{H}$
- $\boldsymbol{X} + \boldsymbol{Y}$ is the *Minkowski sum* of subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$
- $\boldsymbol{X} \oplus \boldsymbol{Y} \triangleq (\boldsymbol{X} + \boldsymbol{Y})^-$ is the *closure* of $\boldsymbol{X} + \boldsymbol{Y}$
- $\boldsymbol{X} \cap \boldsymbol{Y}$ is the *intersection* of subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$

16.1.2 Properties

Theorem 16.1. [5] *Let* $\boldsymbol{L} \triangleq (X,\, \vee,\, \wedge,\, 0,\, 1\,;\, \leq)$ *be a* BOUNDED LATTICE.

THM

$$\left\{\begin{array}{l}\boldsymbol{L}\text{ is}\\ \text{ORTHOCOMPLEMENTED}\\ \text{(Definition 16.1 page 238)}\end{array}\right\} \implies \left\{\begin{array}{lrcllll}
(1). & 0^\perp & = & 1 & & \text{(BOUNDARY CONDITION)} & \text{and}\\
(2). & 1^\perp & = & 0 & & \text{(BOUNDARY CONDITION)} & \text{and}\\
(3). & (x \vee y)^\perp & = & x^\perp \wedge y^\perp & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN)} & \text{and}\\
(4). & (x \wedge y)^\perp & = & x^\perp \vee y^\perp & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)} & \text{and}\\
(5). & x \vee x^\perp & = & 1 & \forall x \in X & \text{(EXCLUDED MIDDLE).} &
\end{array}\right.$$

✎PROOF: Let $x^\perp \triangleq \neg x$, where $\neg$ is an *ortho negation* function (Definition D.3 page 318). Then, this theorem follows directly from Theorem D.5 (page 322). ⇨

Corollary 16.1. *Let* $\boldsymbol{L} \triangleq (X,\, \vee,\, \wedge,\, 0,\, 1\,;\, \leq)$ *be a* BOUNDED LATTICE.

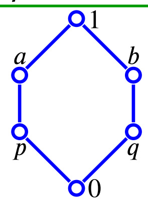
COR

$$\left\{\begin{array}{l}\boldsymbol{L}\textit{ is orthocomplemented}\\ \text{(Definition 16.1 page 238)}\end{array}\right\} \implies \left\{\boldsymbol{L}\textit{ is complemented}\right\}$$

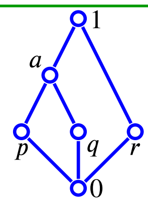
✎PROOF: This follows directly from the definition of *orthocomplemented lattices* (Definition 16.1 page 238) and *complemented lattices*. ⇨

Example 16.5.

EX



The O_6 *lattice* (Definition 16.2 page 238) illustrated to the left is both **orthocomplemented** (Definition 16.1 page 238) and **multiply complemented**. The lattice illustrated to the right is **multiply complemented**, but is **non-orthocomplemented**.



✎PROOF:

1. Proof that O_6 *lattice* is multiply complemented: b and q are both *complements* of p.

[5] Beran (1985) pages 30–31, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Cohen (1989) page 37 ⟨3B.13. Theorem⟩

2. Proof that the right side lattice is multiply complemented: a, p, and q are all *complements* of r.

✏

Lemma 16.1 (next) is useful in proving that *de Morgan*'s laws (Theorem A.7 page 274) hold in orthocomplemented lattices (Theorem 16.1 page 240) and in proving the characterization of Theorem 16.2 (page 242).

Lemma 16.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238).*

LEM

$$\underbrace{x \leq y \implies y^\perp \leq x^\perp}_{\text{ANTITONE}} \iff \underbrace{\left\{\begin{array}{rcll} (x \vee y)^\perp &=& x^\perp \wedge y^\perp & \forall x,y\in X \quad \text{and} \\ (x \wedge y)^\perp &=& x^\perp \vee y^\perp & \forall x,y\in X \end{array}\right.}_{\text{DE MORGAN}}$$

✎PROOF: This follows directly from Lemma D.2 (page 320). ✏

Lemma 16.2. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238).*

LEM

The set $\{0, x, x^\perp\}$ *is* DISTRIBUTIVE *for all* $x \in X$.

✎PROOF:

$$\begin{array}{rcll}
0 \wedge (x \vee x^\perp) &=& 0 & \text{by } \textit{lower bounded} \text{ property} \\
&=& 0 \vee 0 & \text{by } \textit{join identity} \\
&=& (0 \wedge x) \vee (0 \wedge x^\perp) & \text{by } \textit{lower bounded} \text{ property} \\
0 \wedge (x^\perp \vee x) &=& 0 & \text{by } \textit{lower bounded} \text{ property} \\
&=& 0 \vee 0 & \text{by } \textit{join identity} \\
&=& (0 \wedge x^\perp) \vee (0 \wedge x) & \text{by } \textit{lower bounded} \text{ property} \\
x \wedge (x^\perp \vee 0) &=& x \wedge x^\perp & \text{by } \textit{join identity} \\
&=& 0 & \text{by } \textit{non-contradiction} \text{ property} \quad \text{(Definition 16.1 page 238)} \\
&=& 0 \vee 0 & \text{by } \textit{join identity} \\
&=& (x \wedge x^\perp) \vee 0 & \text{by } \textit{non-contradiction} \text{ property} \quad \text{(Definition 16.1 page 238)} \\
&=& (x \wedge x^\perp) \vee (x \wedge 0) & \text{by } \textit{lower bounded} \text{ property} \\
x \wedge (0 \vee x^\perp) &=& x \wedge (x^\perp \vee 0) & \text{by } \textit{commutative} \text{ property of } \textit{lattices} \quad \text{(Theorem C.3 page 302)} \\
&=& (x \wedge x^\perp) \vee (x \wedge 0) & \text{by previous result} \\
&=& (x \wedge 0) \vee (x \wedge x^\perp) & \text{by } \textit{commutative} \text{ property of } \textit{lattices} \quad \text{(Theorem C.3 page 302)} \\
x^\perp \wedge (x \vee 0) &=& (x^\perp \wedge x) \vee (x^\perp \wedge 0) & \text{by } x \wedge (x^\perp \vee 0) \text{ result} \\
x^\perp \wedge (0 \vee x) &=& (x^\perp \wedge 0) \vee (x^\perp \wedge x) & \text{by } x \wedge (0 \vee x^\perp) \text{ result}
\end{array}$$

✏

[6] Beran (1985) pages 30–31, Fáy (1967) ⟨cf Beran 1985 page 30⟩, Nakano and Romberger (1971) ⟨cf Beran 1985⟩

16.1.3 Characterization

Theorem 16.2. [7] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be an algebraic structure.*

THM

$$\left.\begin{array}{l}\boldsymbol{L} \text{ is an} \\ \textbf{orthocomplemented lattice}\end{array}\right\} \iff \left\{\begin{array}{llcllll} 1. & (z^\perp \wedge y^\perp)^\perp \vee x & = & (x \vee y) \vee z & \forall x,y,z \in X & \text{and} \\ 2. & x \wedge (x \vee y) & = & x & \forall x,y \in X & \text{and} \\ 3. & x \vee (y \wedge y^\perp) & = & x & \forall x,y \in X. & \end{array}\right.$$

✎PROOF:

1. Proof that orthocomplemented lattice $\implies$ 3 properties:

$$\begin{aligned}
(z^\perp \wedge y^\perp)^\perp \vee x &= \left[(z^\perp)^\perp \vee (y^\perp)^\perp\right] \vee x && \text{by \textit{de Morgan} property (Theorem 16.1 page 240)} \\
&= (z \vee y) \vee x && \text{by \textit{involutory} property (Definition 16.1 page 238)} \\
&= x \vee (z \vee y) && \text{by \textit{commutative} property (Theorem C.3 page 302)} \\
&= x \vee (y \vee z) && \text{by \textit{commutative} property (Theorem C.3 page 302)} \\
&= (x \vee y) \vee z && \text{by \textit{associative} property (Theorem C.3 page 302)}
\end{aligned}$$

$$x \wedge (x \vee y) = x \qquad \text{by \textit{absorptive} property (Theorem C.3 page 302)}$$

$$\begin{aligned}
x \vee (y \wedge y^\perp) &= x \vee 0 && \text{by \textit{complemented} property (Definition 16.1 page 238)} \\
&= x
\end{aligned}$$

2. Proof that orthocomplemented lattice $\impliedby$ 3 properties:

   (a) Proof that $\boldsymbol{L}$ is *meet-idempotent*:

$$\begin{aligned}
x \wedge x &= x \wedge \left[x \vee (y \wedge y^\perp)\right] && \text{by (3)} \\
&= x \wedge \left[x \vee (y \wedge y^\perp)\right] && \text{by (3)} \\
&= x && \text{by (2)}
\end{aligned}$$

   (b) Define $0 \triangleq xx^\perp$ for some $x \in X$. Proof that 0 is the *greatest lower bound* of $\boldsymbol{L}$: The element 0 is the greatest lower bound if and only if $xx^\perp = yy^\perp \quad \forall x, y \in X \ldots$

      i. Proof that $(xx^\perp)^{\perp\perp} = (xx^\perp) \quad \forall x \in X$:

$$\begin{aligned}
(xx^\perp)^{\perp\perp} &= (xx^\perp)^{\perp\perp} + (xx^\perp) && \text{by (3)} \\
&= \left[(xx^\perp)^\perp (xx^\perp)^\perp\right]^\perp + (xx^\perp) && \text{by item (2a)} \\
&= \left[(xx^\perp) + (xx^\perp)\right] + (xx^\perp) && \text{by (1)} \\
&= \left[(xx^\perp)\right] + (xx^\perp) && \text{by (3)} \\
&= (xx^\perp) && \text{by (3)}
\end{aligned}$$

      ii. Proof that $a = (xx^\perp) + a \quad \forall a, x \in X$:

$$\begin{aligned}
a &= a + (xx^\perp) && \text{by (3)} \\
&= \left[a + (xx^\perp)\right] + (xx^\perp) && \text{by (3)} \\
&= \left[(xx^\perp)^\perp (xx^\perp)^\perp\right]^\perp + a && \text{by (1)} \\
&= \left[(xx^\perp)^\perp\right]^\perp + a && \text{by item (2a)} \\
&= (xx^\perp) + a && \text{by item (2(b)i)}
\end{aligned}$$

[7] Beran (1985) pages 31–33, Beran (1976) pages 251–252

iii. Proof that $(xx^\perp) = (yy^\perp) \quad \forall x, y \in X$:

$$\begin{aligned}
(xx^\perp) &= (xx^\perp) + (yy^\perp) && \text{by (3)}\\
&= (yy^\perp) && \text{by item (2(b)ii)}
\end{aligned}$$

(c) Proof that $x + 0 = 0 + x = x \quad \forall x \in X$ (*join identity*):

$$\begin{aligned}
x + 0 &= x + (yy^\perp) && \text{by item (2(b)iii)}\\
&= x && \text{by (3)}\\
0 + x &= (uu^\perp) + x && \text{by item (2(b)iii)}\\
&= x && \text{by item (2(b)ii)}
\end{aligned}$$

(d) Proof that $x + y = (y^\perp x^\perp)^\perp \quad \forall x, y \in X$:

$$\begin{aligned}
(y^\perp x^\perp)^\perp &= (y^\perp x^\perp)^\perp + 0 && \text{by item (2c)}\\
&= (0 + x) + y && \text{by (1)}\\
&= x + y && \text{by item (2c)}
\end{aligned}$$

(e) Proof that $x + x = x^{\perp\perp} \quad \forall x \in X$:

$$\begin{aligned}
x + x &= (x^\perp x^\perp)^\perp && \text{by item (2d)}\\
&= (x^\perp)^\perp && \text{by item (2a)}
\end{aligned}$$

(f) Proof that $x + y = y + x \quad \forall x, y \in X \quad$ (*join-commutative*):

$$\begin{aligned}
x + y &= (x + 0) + y && \text{by item (2c)}\\
&= (0^\perp x^\perp)^\perp + y && \text{by item (2d)}\\
&= (y + x) + 0 && \text{by (1)}\\
&= y + x && \text{by item (2c)}
\end{aligned}$$

(g) Proof that $(x + y) + z = x + (y + z) \quad \forall x, y, z \in X \quad$ (*join-associative*):

$$\begin{aligned}
(x + y) + z &= (z^\perp y^\perp)^\perp + x && \text{by (1)}\\
&= (y + z) + x && \text{by item (2d)}\\
&= x + (y + z) && \text{by item (2f)}
\end{aligned}$$

(h) Proof that $x^{\perp\perp} = x \quad \forall x \in X \quad$ (*involutory*):

$$\begin{aligned}
x^{\perp\perp} &= \left(x^\perp\right) \perp && \text{by definition of } x^{\perp\perp}\\
&= \left[x^\perp\left(x^\perp + x\right)\right] \perp && \text{by (2)}\\
&= \left[x^\perp\left(x^\perp x^{\perp\perp}\right)^\perp\right] \perp && \text{by item (2d)}\\
&= \left(x^\perp x^{\perp\perp}\right) + x && \text{by item (2d)}\\
&= (0) + x && \text{by item (2b)}\\
&= x && \text{by item (2c)}
\end{aligned}$$

(i) Proof of *de Morgan*'s laws:

$$\begin{aligned}
(x + y)^\perp &= (y + x)^\perp && \text{by item (2g)}\\
&= \left[\left(x^\perp y^\perp\right)^\perp\right]^\perp && \text{by item (2d)}\\
&= x^\perp y^\perp && \text{by item (2h)}
\end{aligned}$$

$$\begin{aligned}
(xy)^\perp &= \left(x^{\perp\perp} y^{\perp\perp}\right)^\perp && \text{by item (2h)}\\
&= y^\perp + x^\perp && \text{by item (2d)}\\
&= x^\perp + y^\perp && \text{by item (2g)}
\end{aligned}$$

(j) Proof that $(xy)z = x(yz) \quad \forall x, y, z \in X \quad$ (*meet-commutative*):

$$\begin{aligned}
xy &= (xy)^{\perp\perp} && \text{by item (2h)}\\
&= \left(x^{\perp} + y^{\perp}\right)^{\perp} && \text{by item (2i)}\\
&= \left(y^{\perp} + x^{\perp}\right)^{\perp} && \text{by item (2g)}\\
&= y^{\perp\perp}x^{\perp\perp} && \text{by item (2i)}\\
&= yx && \text{by item (2i)}
\end{aligned}$$

(k) Proof that $(xy)z = x(yz) \quad \forall x, y, z \in X \quad$ (*meet-associative*):

$$\begin{aligned}
(xy)z &= [(xy)z]^{\perp}\ \perp && \text{by item (2h)}\\
&= \left[(xy)^{\perp} + z^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[\left(x^{\perp} + y^{\perp}\right) + z^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp} + \left(y^{\perp} + z^{\perp}\right)\right]^{\perp} && \text{by item (2g)}\\
&= x^{\perp\perp}\left(y^{\perp} + z^{\perp}\right)^{\perp} && \text{by item (2i)}\\
&= x^{\perp\perp}\left(y^{\perp\perp}z^{\perp\perp}\right) && \text{by item (2i)}\\
&= x(yz) && \text{by item (2h)}
\end{aligned}$$

(l) Proof that $x + (xz) = x$ (*join-meet-absorptive*):

$$\begin{aligned}
x \vee (xz) &= [x + (xz)]^{\perp\perp} && \text{by item (2h)}\\
&= \left[x^{\perp}(xz)^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp}\left(x^{\perp} + z^{\perp}\right)\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp}\right]^{\perp} && \text{by (2)}\\
&= x && \text{by item (2h)}
\end{aligned}$$

(m) Because $\boldsymbol{L}$ is *commutative* (item (2f) and item (2j)), *associative* (item (2g) and item (2k)), and *absorptive* ((2) and item (2l)), and by Theorem C.8 (page 310), $\boldsymbol{L}$ is a *lattice*.

(n) Define $1 \triangleq x + x^{\perp}$ for some $x \in X$. Proof that 1 is the *least upper bound* of $\boldsymbol{L}$: The element 1 is the least upper bound if and only if $x + x^{\perp} = y + y^{\perp} \quad \forall x, y \in X\ldots$

$$\begin{aligned}
1 &= \left(x + x^{\perp}\right) && \text{by definition of 1}\\
&= \left(x + x^{\perp}\right)^{\perp\perp} && \text{by item (2h)}\\
&= \left(x^{\perp}x\right)^{\perp} && \text{by item (2h)}\\
&= \left(xx^{\perp}\right)^{\perp} && \text{by item (2j)}\\
&= \left(yy^{\perp}\right)^{\perp} && \text{by item (2(b)iii)}\\
&= y^{\perp} + y^{\perp\perp} && \text{by item (2i)}\\
&= y^{\perp} + y && \text{by item (2h)}\\
&= y + y^{\perp} && \text{by item (2f)}
\end{aligned}$$

(o) Proof that $\boldsymbol{L}$ is *antitone*: by Theorem D.4 (page 322).

(p) Proof that $\boldsymbol{L}$ is *complemented*: by item (2(b)iii) and item (2n).

(q) Because $\boldsymbol{L}$ is a *bounded* (item (2b) and item (2n)) lattice (item (2m)), and because $\boldsymbol{L}$ is *complemented* (item (2p)), is *involutory* (item (2h)), and is *antitone* (item (2o)), and by Definition 16.1 (page 238), $\boldsymbol{L}$ is an *orthocomplemented lattice*.

✏

16.1.4 Restrictions resulting in Boolean algebras

Proposition 16.1. [8] *Let* $\boldsymbol{L} = (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* LATTICE *(Definition C.3 page 301).*

PRP: $\left\{\begin{array}{ll} 1. & \boldsymbol{L} \text{ is } \textbf{orthocomplemented} \text{ (Definition 16.1 page 238) and} \\ 2. & \boldsymbol{L} \text{ is } \textbf{distributive} \end{array}\right\} \implies \left\{ \boldsymbol{L} \text{ is } \textbf{Boolean} \right\}$

✎PROOF: To be a *Boolean algebra*, $\boldsymbol{L}$ must satisfy the 8 requirements of *boolean algebras*:

1. Proof for *commutative* properties: These are true for *all* lattices (Definition C.3 page 301).
2. Proof for *join-distributive* property: by hypothesis (2).
3. Proof for *meet-distributive* property: by *join-distributive* property and the *Principle of duality* (Theorem C.4 page 303) for lattices.
4. Proof for *identity* properties: because $\boldsymbol{L}$ is a *bounded lattice* and by definitions of 1 (*least upper bound*), 0 (*greatest lower bound*), $\vee$, and $\wedge$.
5. Proof for *complemented* properties: by hypothesis (1) and definition of *orthocomplemented lattices* (Definition 16.1 page 238).

✏

Proposition 16.2. *Let* $\boldsymbol{L} = (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* LATTICE *(Definition C.3 page 301).*

PRP: $\left\{\begin{array}{ll} 1. & \boldsymbol{L} \text{ is } \textbf{orthocomplemented} \text{ (Definition 16.1 page 238) and} \\ 2. & \text{Every } x \in \boldsymbol{L} \text{ is in the } \textbf{center} \text{ of } \boldsymbol{L} \text{ (Definition E.4 page 336)} \end{array}\right\} \iff \left\{ \boldsymbol{L} \text{ is } \textbf{Boolean} \right\}$

✎PROOF:

1. Proof that (1,2) $\implies$ *Boolean*: $\boldsymbol{L}$ is *Boolean* because it satisfies *Huntington's Fourth Set*, as demonstrated by the following ...
   (a) Proof that $x \vee x = x$ (*idempotent*): $\boldsymbol{L}$ is a *lattice* (by definition of $\boldsymbol{L}$), and all lattices are *idempotent* (Definition C.3 page 301).
   (b) Proof that $x \vee y = y \vee x$ (*commutative*): $\boldsymbol{L}$ is a *lattice* (by definition of $\boldsymbol{L}$), and all lattices are *commutative* (Definition C.3 page 301).
   (c) Proof that $(x \vee y) \vee z = x \vee (y \vee z)$ (*associative*): $\boldsymbol{L}$ is a *lattice* (by definition of $\boldsymbol{L}$), and all lattices are *associative* (Definition C.3 page 301).
   (d) Proof that $(x^\perp \vee y^\perp)^\perp \vee (x^\perp \vee y)^\perp = x$ (*Huntington's axiom*):

$$\begin{aligned}
(x^\perp \vee y^\perp)^\perp \vee (x^\perp \vee y)^\perp &= (x^\perp \perp \wedge y^\perp \perp) \vee (x^\perp \perp \wedge y^\perp) && \text{by \textit{de Morgan} property (Theorem 16.1 page 240)}\\
&= (x \wedge y) \vee (x \wedge y^\perp) && \text{by \textit{involution} property (Definition 16.1 page 238)}\\
&= x && \text{by definition of \textit{center} (Definition E.4 page 336)}
\end{aligned}$$

2. Proof that (1) $\impliedby$ *Boolean*:
   (a) Proof that $x \vee x^\perp = 1$: by definition of *Boolean algebras*.
   (b) Proof that $x \wedge x^\perp = 0$: by definition of *Boolean algebras*.

[8] Kalmbach (1983) page 22

(c) Proof that $x^{\perp\perp} = x$: by *involutory* property of *Boolean algebra*.

(d) Proof that $x \le y \implies y^{\perp} \le x^{\perp}$:

$$\begin{aligned}
y^{\perp} \le x^{\perp} &\iff y^{\perp} &&= y^{\perp} \wedge x^{\perp} && \text{by Lemma C.1 page 303}\\
&\iff y^{\perp\perp} &&= \left(y^{\perp} \wedge x^{\perp}\right)^{\perp} &&\\
&\iff y^{\perp\perp} &&= y^{\perp\perp} \vee x^{\perp\perp} && \text{by \textit{de Morgan} property}\\
&\iff y &&= y \vee x && \text{by \textit{involutory} property}\\
&\iff y &&= y && \text{by } x \le y \text{ hypothesis}
\end{aligned}$$

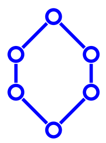
3. Proof that (2) $\impliedby$ *Boolean*: for all $x, y \in \boldsymbol{L}$

$$\begin{aligned}
(x \wedge y) \vee \left(x \wedge y^{\perp}\right) &= [(x \wedge y) \vee x] \wedge \left[(x \wedge y) \vee y^{\perp}\right] && \text{by \textit{distributive} property}\\
&= x \wedge \left[(x \wedge y) \vee y^{\perp}\right] && \text{by \textit{absorptive} property}\\
&= x \wedge \left[\left(x \vee y^{\perp}\right) \wedge \left(y \vee y^{\perp}\right)\right] && \text{by \textit{distributive} property}\\
&= x \wedge \left(x \vee y^{\perp}\right) \wedge 1 && \text{by \textit{complement} property}\\
&= x && \text{by \textit{absorptive} property}\\
&\implies x \copyright y \quad \forall x, y \in \boldsymbol{L} && \text{by Definition E.2 page 333}\\
&\implies x \text{ is in the \textit{center} of } \boldsymbol{L} \text{ for all } x \in \boldsymbol{L} && \text{by Definition E.4 page 336}
\end{aligned}$$

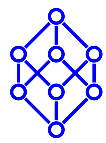
✏

Example 16.6.

EX



The O_6 *lattice* (Definition 16.2 page 238) illustrated to the left is **orthocomplemented** (Definition 16.1 page 238) but **non-join-distributive**,and hence *non-Boolean*. The lattice illustrated to the right is **orthocomplemented** *and* **distributive** and hence also **Boolean** (Proposition 16.1 page 245). Alternatively, the right side lattice is **orthocomplemented** *and* every element is in the *center*, and hence also **Boolean** (Proposition 16.2 page 245).



✎PROOF:

1. Proof that the O_6 *lattice* is *non-join-distributive*:

$$\begin{aligned}
x \vee (x^{\perp} \wedge z^{\perp}) &= x \vee 0\\
&= x\\
&\neq z^{\perp}\\
&= 1 \wedge z^{\perp}\\
&= (x \vee x^{\perp}) \wedge (x \vee z^{\perp})
\end{aligned}$$

2. Proof that the O_6 *lattice* is also *non-meet-distributive*:

$$\begin{aligned}
z^{\perp} \wedge (x \vee z) &= z^{\perp} \wedge 1\\
&= z^{\perp}\\
&\neq x\\
&= x \vee 1\\
&= (z^{\perp} \wedge x) \vee (z^{\perp} \wedge z)
\end{aligned}$$

✏

16.2 Orthomodular lattices

16.2.1 Properties

Definition 16.3. [9] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an algebraic structure.*

DEF

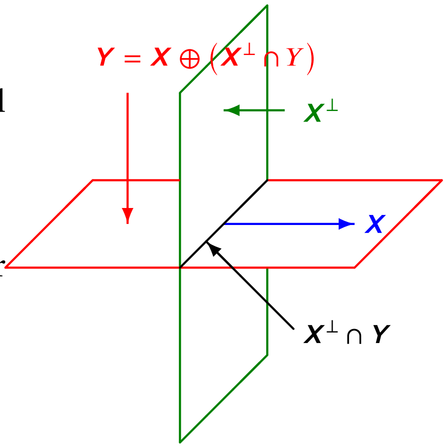
*$\mathbf{L}$ is an **orthomodular lattice** if*

1. *$\mathbf{L}$ is an* ORTHOCOMPLEMENTED LATTICE *and*
2. $x \leq y \implies x \vee (x^{\perp} \wedge y) = y \quad \forall x,y \in X$ *(ORTHOMODULAR IDENTITY)*

Example 16.7.

EX

The O_6 *lattice* (Definition 16.2 page 238) is *orthocomplemented*, but *non-orthomodular* (and hence, *non-modular* and *non-Boolean*).



Example 16.8. [10]Let $\boldsymbol{H}$ be a Hilbert space and $2^{\boldsymbol{H}}$ the set of closed linear subspaces of $\boldsymbol{H}$.

EX

$(2^{\boldsymbol{H}}, \oplus, \cap, \emptyset, \boldsymbol{H}; \subseteq)$ is an orthomodular lattice.

This concept is illustrated to the right where $\boldsymbol{X}, \boldsymbol{Y} \in 2^{\boldsymbol{H}}$ are linear subspaces of the linear space $\boldsymbol{H}$ and

$$\boldsymbol{X} \subseteq \boldsymbol{Y} \implies \boldsymbol{Y} = \boldsymbol{X} \oplus (\boldsymbol{X}^{\perp} \cap Y).$$

Theorem 16.3. [11] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a lattice.*

THM

$$\left.\begin{array}{ll} 1. & \mathbf{L} \text{ is ORTHOMODULAR and} \\ 2. & y \copyright x \text{ and } z \copyright x \end{array}\right\} \implies (x, y, z) \in Ⓓ$$

16.2.2 Characterizations

Theorem 16.4. [12] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238). Let Ⓜ and Ⓜ* be the modularity relation and dual modularity relation, respectively, $\perp$ the orthogonality relation (Definition E.1 page 331), and © the commutes relation (Definition E.2 page 333).*

THM

The following statements are EQUIVALENT*:*

	1.	$\mathbf{L}$ *is* ORTHOMODULAR
$\iff$	2.	$x \leq y$ *and* $y \wedge x^{\perp} = 0 \implies x = y$
$\iff$	3.	$\mathbf{L}$ *does* NOT *contain the* O_6 *lattice*
$\iff$	4.	$x \copyright y \iff y \copyright x$ (© *is* SYMMETRIC)
$\iff$	5.	$x Ⓜ x^{\perp} \quad \forall x \in X$
$\iff$	6.	$x Ⓜ^* x^{\perp} \quad \forall x \in X$
$\iff$	7.	$x \vee [x^{\perp} \wedge (x \vee y)] = x \vee y \quad \forall x, y \in X$
$\iff$	8.	$x \leq y \implies \exists p \in X$ such that $x \perp p$ *and* $x \vee p = y$

[9] Kalmbach (1983) page 22, Lidl and Pilz (1998) page 90, Husimi (1937)
[10] Iturrioz (1985) pages 56–57
[11] Kalmbach (1983) page 25, Holland (1963) pages 69–70 ⟨THEOREM 3⟩, Foulis (1962) page 68 ⟨THEOREM 5⟩
[12] Kalmbach (1983) page 22, Stern (1999) page 12, Nakamura (1957), Holland (1963), Foulis (1962), Maeda and Maeda (1970) page 132 ⟨Theorem 29.13⟩

✎PROOF:

1. Proof that *orthomodular* $\iff$ *symmetric*: by Proposition E.3 (page 334).

✏

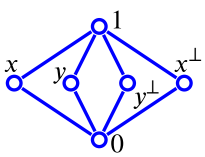
16.2.3 Restrictions resulting in Boolean algebras

Theorem 16.5. [13] *Let* $\boldsymbol{L} = (X, \vee, \wedge, 0, 1; \leq)$ *be an algebraic structure.*

THM
$$\left\{\begin{array}{ll} \boldsymbol{L} \text{ is an } \textbf{orthomodular lattice} & \text{and} \\ \underbrace{\left(x \wedge y^\perp\right)^\perp = y \vee \left(x^\perp \wedge y^\perp\right)}_{\text{Elkan's law}} & \forall x, y \in X \end{array}\right\} \implies \left\{\begin{array}{l} \boldsymbol{L} \text{ is a} \\ \textbf{Boolean algebra} \end{array}\right\}$$

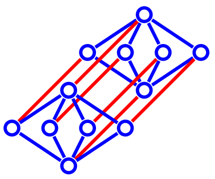
Definition 16.4. [14]

DEF
The $\textbf{MO}_2$ ***lattice*** *is the ordered set* $\left(\{0, x, y, x^\perp, y^\perp, 1\}, \leq\right)$ *with cover relation*
$$\lessdot = \left\{(0,x), (0,y), \left(0,x^\perp\right), \left(0,y^\perp\right), (x,1), (y,1), \left(x^\perp,1\right), \left(y^\perp,1\right)\right\}$$
This lattice is also called the ***Chinese lantern***.



MO_2

L_2



$MO_2 \times L_2$

Theorem 16.6. [15] *Let* $\boldsymbol{M} = (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOMODULAR *lattice.*

THM
$$\left\{\begin{array}{l} \boldsymbol{M} \text{ is} \\ \text{Boolean} \end{array}\right\} \iff \left\{\begin{array}{lll} 1. & \boldsymbol{M} \text{ does not contain the } MO_2 \text{ lattice (Definition 16.4 page 248)} & \text{and} \\ 2. & \boldsymbol{M} \text{ does not contain the } MO_2 \times L_2 \text{ lattice.} & \end{array}\right\}$$

16.3 Modular orthocomplemented lattices

Definition 16.5. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE.

DEF
$\boldsymbol{L}$ *is a* ***modular orthocomplemeted lattice*** *if*

1. $\boldsymbol{L}$ *is* ***orthocomplemented*** (Definition 16.1 page 238) *and*
2. $\boldsymbol{L}$ *is* ***modular***

[13] Renedo et al. (2003) page 72
[14] Iturrioz (1985) page 57, Davey and Priestley (2002) pages 18–19 ⟨1.25 Products⟩
[15] Iturrioz (1985) page 57, Carrega (1982) ⟨cf Iturrioz 1985 page 57⟩

16.4 Relationships between orthocomplemented lattices

Theorem 16.7. [16] *Let $\mathbf{L}$ be a lattice.*

THM

$$\left\{\begin{array}{l}\textit{L is}\\ \text{BOOLEAN}\end{array}\right\} \implies \left\{\begin{array}{l}\textit{L is}\\ \text{MODULAR}\\ \text{ORTHOCOM-}\\ \text{PLEMENTED}\end{array}\right\} \implies \left\{\begin{array}{l}\textit{L is}\text{ ORTHO-}\\ \text{MODULAR}\end{array}\right\} \implies \left\{\begin{array}{l}\textit{L is}\\ \text{ORTHOCOM-}\\ \text{PLEMENTED}\end{array}\right\}$$

Remark 16.1. [17]Lattice number 8 in Example 16.2 (page 238) was originally introduced by Dilworth as a counterexample to *Husimi's conjecture* (1937). Kalmbach(1983) points out that this lattice was the first example of a *finite orthomodular* lattice.

[16] Kalmbach (1983) page 32 ⟨20.⟩, Iturrioz (1985) page 57
[17] Dilworth (1940), Dilworth (1990), Kalmbach (1983) page 9

CHAPTER 17

NORMED ALGEBRAS

17.1 Algebras

All *linear spaces* (Definition 4.1 page 71) are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition 17.1. [3] *Let* $\boldsymbol{A}$ *be an* ALGEBRA.

DEF

An algebra $\boldsymbol{A}$ *is* ***unital*** *if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition 17.2. [4] *Let* $\boldsymbol{A}$ *be an* UNITAL ALGEBRA *(Definition 17.1 page 251) with unit* e.

DEF

$$\begin{array}{llll}
\text{The \textbf{spectrum} of } x \in \boldsymbol{A} \text{ is} & \sigma(x) & \triangleq & \{\lambda \in \mathbb{C} | \lambda e - x \text{ is not invertible}\}. \\
\text{The \textbf{resolvent} of } x \in \boldsymbol{A} \text{ is} & \rho_x(\lambda) & \triangleq & (\lambda e - x)^{-1} \qquad \forall \lambda \notin \sigma(x). \\
\text{The \textbf{spectral radius} of } x \in \boldsymbol{A} \text{ is} & r(x) & \triangleq & \sup\{|\lambda| | \lambda \in \sigma(x)\}.
\end{array}$$

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[1] Fuchs (1995) page 2
[2] Hazewinkel (2000) page v
[3] Folland (1995) page 1
[4] Folland (1995) pages 3–4

17.2 Star-Algebras

Definition 17.3. [5] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF

The pair $(\boldsymbol{A}, *)$ is a ***-algebra**, *or* **star-algebra**, *if*

1.	$(x+y)^* = x^* + y^*$	$\forall x,y \in \boldsymbol{A}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha x)^* = \bar{\alpha} x^*$	$\forall x \in \boldsymbol{A}, \alpha \in \mathbb{C}$	(CONJUGATE LINEAR)	*and*
3.	$(xy)^* = y^* x^*$	$\forall x,y \in \boldsymbol{A}$	(ANTIAUTOMORPHIC)	*and*
4.	$x^{**} = x$	$\forall x \in \boldsymbol{A}$	(INVOLUTORY)	

The operator $*$ is called an **involution** *on the algebra $\boldsymbol{A}$.*

Proposition 17.1. [6] *Let $(\boldsymbol{A}, *)$ be an* UNITAL $*$-ALGEBRA.

PRP

$$x \text{ is invertible} \implies \begin{cases} 1. & x^* \text{ is INVERTIBLE} \quad \forall x \in \boldsymbol{A} \quad \text{and} \\ 2. & (x^*)^{-1} = (x^{-1})^* \quad \forall x \in \boldsymbol{A} \end{cases}$$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= (x\,e^*)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition 17.3 page 252)}\\
&= (x^*\,e^{**})^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition 17.3 page 252)}\\
&= (x^*\,e)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition 17.3 page 252)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition 17.3 page 252)}\\
e^*\,x &= (e^*\,x)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition 17.3 page 252)}\\
&= (e^{**}\,x^*)^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition 17.3 page 252)}\\
&= (e\,x^*)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition 17.3 page 252)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition 17.3 page 252)}
\end{aligned}$$

2. Proof that $(x^*)^{-1} = (x^{-1})^*$:

$$\begin{aligned}
(x^{-1})^*\,(x^*) &= [x\,(x^{-1})]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition 17.3 page 252)}\\
&= e^*\\
&= e && \text{by item (1) page 252}\\
(x^*)\,(x^{-1})^* &= [x^{-1}\,x]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition 17.3 page 252)}\\
&= e^*\\
&= e && \text{by item (1) page 252}
\end{aligned}$$

⇨

Definition 17.4. [7] *Let $(\boldsymbol{A}, \|\cdot\|)$ be a* $*$-ALGEBRA *(Definition 17.3 page 252).*

DEF

- *An element $x \in \boldsymbol{A}$ is* **hermitian** *or* **self-adjoint** *if $x^* = x$.*
- *An element $x \in \boldsymbol{A}$ is* **normal** *if $xx^* = x^*x$.*
- *An element $x \in \boldsymbol{A}$ is a* **projection** *if $xx = x$ (INVOLUTORY) and $x^* = x$ (HERMITIAN).*

[5] Rickart (1960) page 178, Gelfand and Naimark (1964), page 241
[6] Folland (1995) page 5
[7] Rickart (1960) page 178, Gelfand and Naimark (1964), page 242

Theorem 17.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (Definition 17.3 page 252).

THM

$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are HERMITIAN}} \implies \left\{\begin{array}{lcll} x + y & = & (x+y)^* & (x+y \text{ is self adjoint}) \\ x^* & = & (x^*)^* & (x^* \text{ is self adjoint}) \\ \multicolumn{3}{l}{\underbrace{xy = (xy)^*}_{(xy) \text{ is HERMITIAN}} \iff \underbrace{xy = yx}_{\text{commutative}}} \end{array}\right.$$

✎PROOF:

$$\begin{aligned} (x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition 17.3 page 252)} \\ &= x + y && \text{by left hypothesis} \end{aligned}$$

$$\left(x^*\right)^* = x \quad \text{by } \textit{involutory} \text{ property of } * \quad \text{(Definition 17.3 page 252)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned} xy &= (xy)^* && \text{by left hypothesis} \\ &= y^* x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition 17.3 page 252)} \\ &= yx && \text{by left hypothesis} \end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned} (xy)^* &= (yx)^* && \text{by left hypothesis} \\ &= x^* y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition 17.3 page 252)} \\ &= xy && \text{by left hypothesis} \end{aligned}$$

✏

Definition 17.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (Definition 17.3 page 252).

DEF

The ***real part*** *of* x *is defined as* $\mathbf{R}_{\mathrm{e}} x \triangleq \frac{1}{2}\left(x + x^*\right)$

The ***imaginary part*** *of* x *is defined as* $\mathbf{I}_{\mathrm{m}} x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem 17.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA (Definition 17.3 page 252).

THM

$$\begin{array}{lcll} \mathbf{R}_{\mathrm{e}} x & = & \left(\mathbf{R}_{\mathrm{e}} x\right)^* \quad \forall x \in \boldsymbol{A} & (\mathbf{R}_{\mathrm{e}} x \text{ is HERMITIAN}) \\ \mathbf{I}_{\mathrm{m}} x & = & \left(\mathbf{I}_{\mathrm{m}} x\right)^* \quad \forall x \in \boldsymbol{A} & (\mathbf{I}_{\mathrm{m}} x \text{ is HERMITIAN}) \end{array}$$

✎PROOF:

$$\begin{aligned} \left(\mathbf{R}_{\mathrm{e}} x\right)^* &= \left(\frac{1}{2}\left(x + x^*\right)\right)^* && \text{by definition of } \Re && \text{(Definition 17.5 page 253)} \\ &= \frac{1}{2}\left(x^* + x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition 17.3 page 252)} \\ &= \frac{1}{2}\left(x^* + x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition 17.3 page 252)} \\ &= \mathbf{R}_{\mathrm{e}} x && \text{by definition of } \Re && \text{(Definition 17.5 page 253)} \end{aligned}$$

$$\left(\mathbf{I}_{\mathrm{m}} x\right)^* = \left(\frac{1}{2i}\left(x - x^*\right)\right)^* \quad \text{by definition of } \Im \quad \text{(Definition 17.5 page 253)}$$

[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998a) page 42

$$= \frac{1}{2i}\left(x^* - x^{**}\right) \quad \text{by } \textit{distributive} \text{ property of } * \quad \text{(Definition 17.3 page 252)}$$
$$= \frac{1}{2i}\left(x^* - x\right) \quad \text{by } \textit{involutory} \text{ property of } * \quad \text{(Definition 17.3 page 252)}$$
$$= \mathbf{I}_{\mathrm{m}}x \quad \text{by definition of } \mathfrak{I} \quad \text{(Definition 17.5 page 253)}$$

✏

Theorem 17.3 (Hermitian representation). [11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition 17.3 page 252).*

THM
$$a = x + iy \iff x = \mathbf{R}_{\mathrm{e}}a \quad \textit{and} \quad y = \mathbf{I}_{\mathrm{m}}a$$

✎PROOF:

Proof that $a = x + iy \implies x = \mathbf{R}_{\mathrm{e}}a$ and $y = \mathbf{I}_{\mathrm{m}}a$:

$$\begin{aligned}
& a = x + iy && \text{by left hypothesis} \\
\implies & a^* = (x + iy)^* && \text{by definition of } \textit{adjoint} && \text{(Definition 17.4 page 252)} \\
& = x^* - iy^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition 17.3 page 252)} \\
& = x - iy && \text{by Theorem 17.2 page 253} \\
\implies & x = a - iy && \text{by solving for } x \text{ in } a = x + iy \text{ equation} \\
& x = a^* + iy && \text{by solving for } x \text{ in } a^* = x - iy \text{ equation} \\
\implies & x + x = a + a^* && \text{by adding previous 2 equations} \\
\implies & 2x = a + a^* && \text{by solving for } x \text{ in previous equation} \\
\implies & x = \frac{1}{2}\left(a + a^*\right) \\
& = \mathbf{R}_{\mathrm{e}}a && \text{by definition of } \mathfrak{R} && \text{(Definition 17.5 page 253)}
\end{aligned}$$

$$\begin{aligned}
& iy = a - x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation} \\
& iy = -a^* + x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation} \\
\implies & iy + iy = a - a^* && \text{by adding previous 2 equations} \\
\implies & y = \frac{1}{2i}\left(a - a^*\right) && \text{by solving for } iy \text{ in previous equations} \\
& = \mathbf{I}_{\mathrm{m}}a && \text{by definition of } \mathfrak{I} && \text{(Definition 17.5 page 253)}
\end{aligned}$$

Proof that $a = x + iy \impliedby x = \mathbf{R}_{\mathrm{e}}a$ and $y = \mathbf{I}_{\mathrm{m}}a$:

$$\begin{aligned}
x + iy &= \mathbf{R}_{\mathrm{e}}a + i\,\mathbf{I}_{\mathrm{m}}a && \text{by right hypothesis} \\
&= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\mathbf{R}_{\mathrm{e}}a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\mathbf{I}_{\mathrm{m}}a} && \text{by definition of } \mathfrak{R} \text{ and } \mathfrak{I} && \text{(Definition 17.5 page 253)} \\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \cancelto{0}{\left(\frac{1}{2}a^* - \frac{1}{2}a^*\right)} \\
&= a
\end{aligned}$$

✏

[11] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Neumark (1943b) page 7

17.3 Normed Algebras

Definition 17.6. [12] *Let $\boldsymbol{A}$ be an algebra.*

DEF

*The pair $(\boldsymbol{A}, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in \boldsymbol{A} \qquad \text{(multiplicative condition)}$$

*A normed algebra $(\boldsymbol{A}, \|\cdot\|)$ is a **Banach algebra** if $(\boldsymbol{A}, \|\cdot\|)$ is also a Banach space.*

Proposition 17.2.

PRP

$(\boldsymbol{A}, \|\cdot\|)$ *is a normed algebra* $\implies$ *multiplication is **continuous** in* $(\boldsymbol{A}, \|\cdot\|)$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x - y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(\boldsymbol{A}, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 255)}\\
&= \|z(x-y)\| \\
&\le \|z\|\ \|x-y\| && \text{by definition of \textit{normed algebra}} && \text{(Definition 17.6 page 255)}\\
&\triangleq \|z\|\ \delta && \text{by definition of } \delta && \text{(item (2) page 255)}\\
&\le \epsilon && \text{for some value of } \delta > 0
\end{aligned}$$

✏

Theorem 17.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l} (\boldsymbol{A}, \|\cdot\|) \text{ is a Banach algebra} \\ \text{every nonzero } x \in \boldsymbol{A} \text{ is invertible} \end{array}\right\} \implies \boldsymbol{A} \equiv \mathbb{C} \quad (\boldsymbol{A} \text{ is isomorphic to } \mathbb{C})$$

17.4 C* Algebras

Definition 17.7. [14]

DEF

*The triple $(\boldsymbol{A}, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(\boldsymbol{A}, \|\cdot\|)$ *is a Banach algebra* and
2. $(\boldsymbol{A}, *)$ *is a $*$-algebra* and
3. $\left\|x^* x\right\| = \|x\|^2 \quad \forall x \in \boldsymbol{A}$.

*A C^* **algebra** $(\boldsymbol{A}, \|\cdot\|, *)$ is also called a **C star algebra**.*

[12] Rickart (1960) page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩
[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩
[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem 17.5. [15] *Let $\boldsymbol{A}$ be an algebra.*

THM	$(\boldsymbol{A}, \|\cdot\|, *)$ *is a* C^* ***algebra*** $\implies$ $\|x^*\| = \|x\|$

✎PROOF:

$$
\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\|x\|^2 \\
&= \frac{1}{\|x\|}\|x^* x\| && \text{by definition of } C^*\text{-}\textit{algebra} && \text{(Definition 17.7 page 255)} \\
&\le \frac{1}{\|x\|}\|x^*\|\,\|x\| && \text{by definition of } \textit{normed algebra} && \text{(Definition 17.6 page 255)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by } \textit{involution} \text{ property of } * && \text{(Definition 17.3 page 252)}
\end{aligned}
$$

✏

[15] Folland (1995) page 1, Gelfand and Neumark (1943b) page 4, Gelfand and Neumark (1943a)

Part VI

Appendices

APPENDIX A

SET STRUCTURES

A.1 General set structures

Similar to the definition of a *relation* on a set X as being any subset of the *Cartesian product* $X \times X$, a *set structure* on a set X is simply any subset of the *power set* 2^X (next) of the set X.

Definition A.1.

DEF *The **power set** 2^X on a set X is defined as*

$$2^X \triangleq \{A | A \subseteq X\} \qquad \text{(the set of all subsets of } X\text{)}$$

Definition A.2. [1] *Let 2^X be the* POWER SET *(Definition A.1 page 259) of a set X.*

DEF *A set $\mathcal{S}(X)$ is a **set structure** on X if* $\quad \mathcal{S}(X) \subseteq 2^X$.
A SET STRUCTURE $\mathcal{Q}(X)$ *is a **paving** on X if* $\quad \varnothing \in \mathcal{Q}(X)$.

Definition A.3. [2] *Let $\mathcal{Q}(X)$ be a* PAVING *(Definition A.2 page 259) on a set X. Let Y be a set containing the element* 0.

DEF *A function $\mathsf{m} \in Y^{\mathcal{Q}(X)}$ is a **set function** if*

$$\mathsf{m}(\varnothing) = 0.$$

A.2 Operations on the power set

A.2.1 Standard operations

Definition A.4. [3] *Let 2^X be a set. Let $|X|$ be a function in the function space* $[0 : +\infty]^X$.

DEF $|X|$ *is the **cardinality** or **order** of X if*

$$|X| \triangleq \begin{cases} \text{number of elements in } X & \text{if } X \text{ is FINITE} \\ +\infty & \text{otherwise} \end{cases}$$

---

[1] Molchanov (2005) page 389, Pap (1995) page 7, Hahn and Rosenthal (1948) page 254

[2] Pap (1995) page 8 ⟨Definition 2.3: extended real-valued set function⟩, Halmos (1950) page 30 ⟨§7. MEASURE ON RINGS⟩, Hahn and Rosenthal (1948), Choquet (1954)

[3] Tao (2011) page 12 ⟨Example 3.6⟩, Tao (2010) page 7 ⟨Example 1.1.14⟩

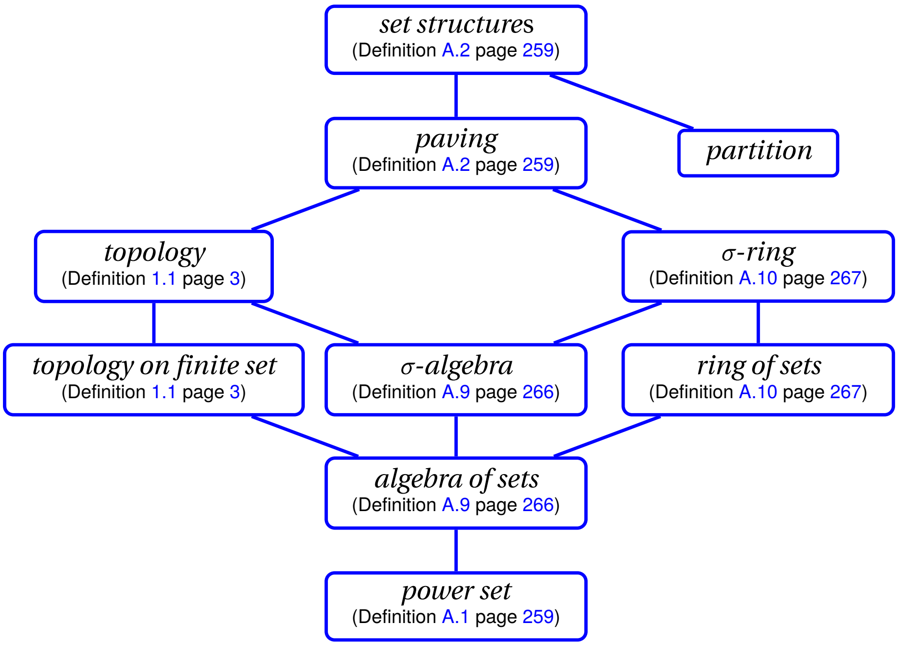


Figure A.1: some standard set structures

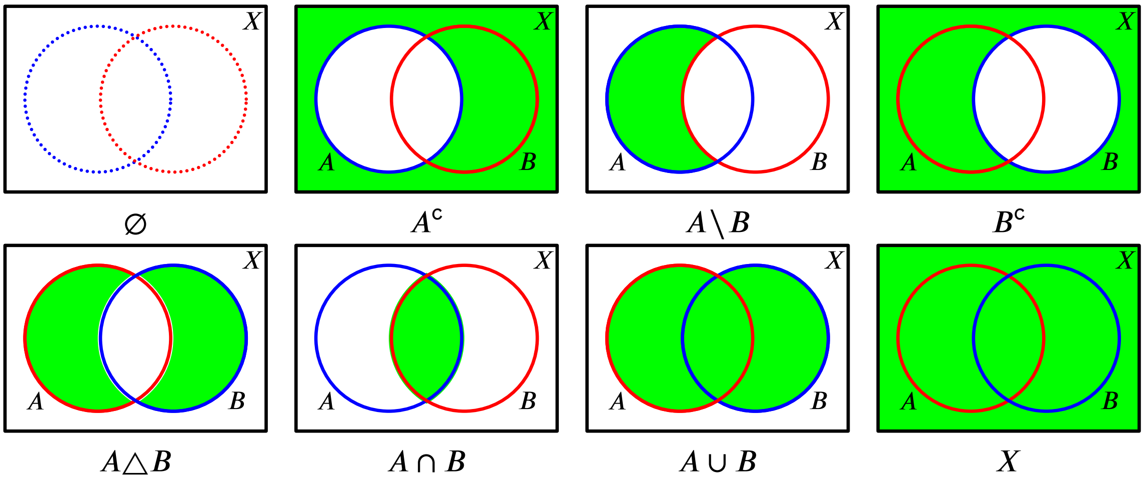


Figure A.2: Venn diagrams for standard set operations (Definition A.5 page 260)

Definition A.5 (next) introduces seven standard set operations: two *nullary* operations, one *unary* operation, and four *binary operations*.

Definition A.5. [4] *Let 2^X be the* POWER SET *(Definition A.1 page 259) on a set X. Let $\neg$ represent the* LOGICAL NOT *operation, $\vee$ represent the* LOGICAL OR *operation, $\wedge$ represent the* LOGICAL AND *operation, and $\oplus$ represent the* LOGICAL EXCLUSIVE-OR *operation.*

DEF

name/symbol		arity	definition	domain
emptyset	$\varnothing$	0	$\varnothing \triangleq \{x \in X \mid x \neq x\}$	
universal set	X	0	$X \triangleq \{x \in X \mid x = x\}$	
complement	c	1	$A^\mathsf{c} \triangleq \{x \in X \mid \neg(x \in A)\}$	$\forall A \in 2^X$
union	$\cup$	2	$A \cup B \triangleq \{x \in X \mid (x \in A) \vee (x \in B)\}$	$\forall A,B \in 2^X$
intersection	$\cap$	2	$A \cap B \triangleq \{x \in X \mid (x \in A) \wedge (x \in B)\}$	$\forall A,B \in 2^X$
difference	$\setminus$	2	$A \setminus B \triangleq \{x \in X \mid (x \in A) \wedge \neg(x \in B)\}$	$\forall A,B \in 2^X$
symmetric difference	$\triangle$	2	$A \triangle B \triangleq \{x \in X \mid (x \in A) \oplus (x \in B)\}$	$\forall A,B \in 2^X$

[4] Aliprantis and Burkinshaw (1998) pages 2–4

With regards to the standard seven set operations only, Theorem A.1 (next) expresses each of the set operations in terms of pairs of other operations.

Theorem A.1.

THM

$$\begin{array}{rclclclcl}
X &=& \varnothing^c \\
\varnothing &=& X^c = \left(A \cup A^c\right)^c &=& A \cap A^c &=& A \setminus A &=& A \triangle A \\
X &=& A \cup A^c &=& \left(A \cap A^c\right)^c \\
A^c &=& X \setminus A &=& X \triangle A \\
A \cup B &=& \left(A^c \cap B^c\right)^c &=& (A \triangle B) \triangle (A \cap B) &=& (A \setminus B) \triangle B \\
A \cap B &=& \left(A^c \cup B^c\right)^c &=& (A \cup B) \triangle A \triangle B &=& A \setminus (A \setminus B) \\
A \setminus B &=& \left(A^c \cup B\right)^c &=& A \cap B^c &=& (A \cup B) \triangle B &=& (A \triangle B) \cap A \\
A \triangle B &=& \left[\left(A^c \cup B\right)^c\right] \cup \left[\left(A \cup B^c\right)^c\right] &&&=& \left[\left(A^c \cap B^c\right)^c\right] \cap (A \cap B)^c \\
&=& (A \setminus B) \cup (B \setminus A)
\end{array}$$

Proposition A.1. *Let X be a set and 2^X the power set of X. Let $\boldsymbol{R} \subseteq X$ such that $\boldsymbol{R}$ is closed with respect to the set symmetric difference operator $\triangle$.*

PRP

$(\boldsymbol{R}, \triangle)$ *is a* GROUP. *In particular,*

1. $\varnothing \triangle A = A \triangle \varnothing = A$ $\forall A \in \boldsymbol{R}$ *($\varnothing$ is the* IDENTITY *element)*
2. $A \triangle A = \varnothing$ $\forall A \in \boldsymbol{R}$ *(A is the* INVERSE *of A)*
3. $A \triangle (B \triangle C) = (A \triangle B) \triangle C$ $\forall A, B, C \in \boldsymbol{R}$ (ASSOCIATIVE)

✎PROOF: The definition of a group is given by Definition F.1 (page 341).

Proof that $\varnothing$ is the *identity* element:

1a. Proof that $\varnothing \in \boldsymbol{R}$:

$$\begin{aligned}
\varnothing &= A \triangle A && \triangle \text{ closed with respect to } \boldsymbol{R} \\
&\in \boldsymbol{R}
\end{aligned}$$

1b. Proof that $\varnothing \triangle A = A$:

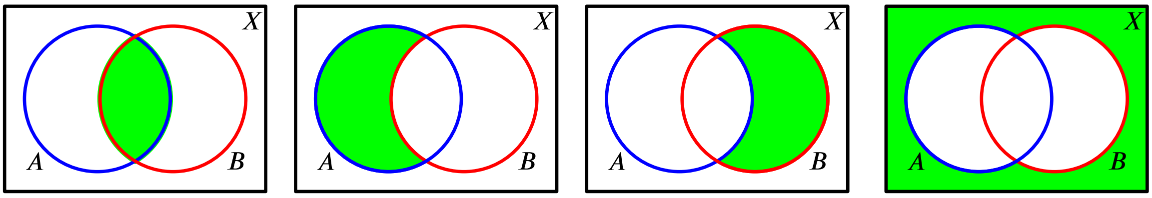
$$\begin{aligned}
\varnothing \triangle A &= \{x \in X \mid (x \in \varnothing) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 260} \\
&= \{x \in X \mid (x \in \{x \in X \mid x \neq x\}) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 260} \\
&= \{x \in X \mid 0 \oplus (x \in A)\} \\
&= \{x \in X \mid (x \in A)\} && \text{by definition of } \oplus \\
&= A
\end{aligned}$$

1c. Proof that $A \triangle \varnothing = A$:

$$\begin{aligned}
A \triangle \varnothing &= \{x \in X \mid (x \in A) \oplus (x \in \varnothing)\} && \text{by definition of } \triangle \text{ page 260} \\
&= \{x \in X \mid (x \in A) \oplus (x \in \{x \in X \mid x \neq x\})\} && \text{by definition of } \triangle \text{ page 260} \\
&= \{x \in X \mid (x \in A) \oplus 0\} \\
&= \{x \in X \mid (x \in A)\} && \text{by definition of } \oplus \\
&= A
\end{aligned}$$

2. Proof that $A \triangle A$:

$$\begin{aligned}
A \triangle A &= \{x \in X \mid (x \in A) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 260} \\
&= \{x \in X \mid 0\} && \text{by definition of } \triangle \text{ page 260} \\
&= \varnothing && \text{by definition of } \triangle \text{ page 260}
\end{aligned}$$



$A \cap B$ (in both A and B) — $A \cap B^c$ (in A but not in B) — $A^c \cap B$ (not in A but in B) — $A^c \cap B^c$ (neither in A nor B)

Figure A.3: The partition of a set X into 4 regions by subsets A and B

3. Proof that $A\triangle(B\triangle C) = (A\triangle B)\triangle C$:

$$\begin{aligned}
A\triangle(B\triangle C) &= \{x \in X | (x \in A) \oplus [x \in (B\triangle C)]\} && \text{by definition of } \triangle \text{ page 260}\\
&= \{x \in X | (x \in A) \oplus [(x \in B) \oplus (x \in C)]\} && \text{by definition of } \triangle \text{ page 260}\\
&= \{x \in X | [(x \in A) \oplus (x \in B)] \oplus (x \in C)\}\\
&= (A\triangle B)\triangle C
\end{aligned}$$

⇰

A.2.2 Non-standard operations

Two subsets A and B of a set X that are intersecting but yet one is not contained in the other, partition the set X into four regions, as illustrated in Figure A.3 (page 262). Because there are four regions, the number of ways we can select one or more of them is $2^4 = 16$. Therefore, a binary operator on sets A and B can likewise result in one of $2^4 = 16$ possibilities. Definition A.6 (page 262) presents 7 set operations. Therefore, there should be an additional $16-7 = 9$ operations. Definition A.6 (next definition) attempts to define these additional operations. Some definitions are adapted from logic. But in general these definitions are non-standard definitions with respect to set theory. The 16 set operations under the inclusion relation $\subseteq$ form a lattice; this lattice is illustrated by a *Hasse diagram* in Figure A.4 (page 263).

Definition A.6. [5] *Let 2^X be the power set on a set X. For any sets $A, B \in 2^X$, let $AB \triangleq (A \cap B)$.*

DEF

name/symbol		*arity*	*definition*	*domain*
empty set	$\oslash$	2	$A \oslash B \triangleq \varnothing$	$\forall A,B \in 2^X$
rejection	$\downarrow$	2	$A \downarrow B \triangleq A^c B^c$	$\forall A,B \in 2^X$
inhibit x	$\ominus$	2	$A \ominus B \triangleq A^c B$	$\forall A,B \in 2^X$
complement x	c_x	2	$A\, c_x\, B \triangleq A^c B \cup A^c B^c$	$\forall A,B \in 2^X$
difference	$\setminus$	2	$A \setminus B \triangleq AB^c$	$\forall A,B \in 2^X$
complement y	c_y	2	$A\, c_y\, B \triangleq AB^c \cup A^c B^c$	$\forall A,B \in 2^X$
symmetric difference	$\triangle$	2	$A \triangle B \triangleq AB^c \cup A^c B$	$\forall A,B \in 2^X$
Sheffer stroke	$\mid$	2	$A \mid B \triangleq AB^c \cup A^c B \cup A^c B^c$	$\forall A,B \in 2^X$
intersection	$\cap$	2	$A \cap B \triangleq AB \cup$	$\forall A,B \in 2^X$
equivalence	$\Leftrightarrow$	2	$A \Leftrightarrow B \triangleq AB \cup A^c B^c$	$\forall A,B \in 2^X$
projection y	$\Vdash$	2	$A \Vdash B \triangleq AB \cup A^c B$	$\forall A,B \in 2^X$
implication	$\Rightarrow$	2	$A \Rightarrow B \triangleq AB \cup A^c B \cup A^c B^c$	$\forall A,B \in 2^X$
projection x	$\dashv\mid$	2	$A \dashv\mid B \triangleq AB \cup AB^c$	$\forall A,B \in 2^X$
adjunction	$\div$	2	$A \div B \triangleq AB \cup AB^c \cup A^c B^c$	$\forall A,B \in 2^X$
union	$\cup$	2	$A \cup B \triangleq AB \cup AB^c \cup A^c B$	$\forall A,B \in 2^X$
universal set	$\otimes$	2	$A \otimes B \triangleq AB \cup AB^c \cup A^c B \cup A^c B^c$	$\forall A,B \in 2^X$

[5] standard ops: Aliprantis and Burkinshaw (1998) pages 2–4

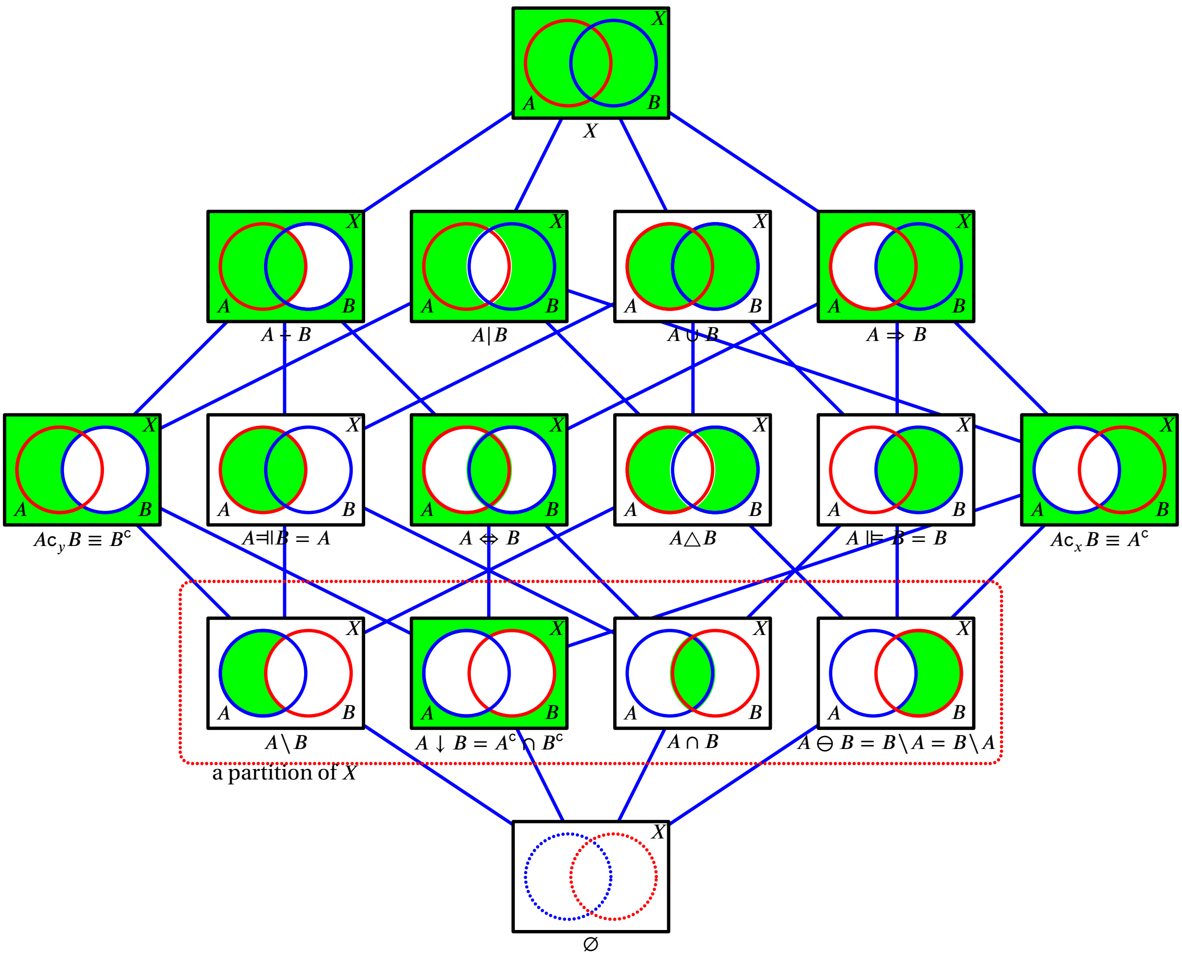


Figure A.4: lattice of set operations

A.2.3 Generated operations

Definition A.5 (page 260) defines set operations in terms of logical operations. However, it is also possible to express set operations in terms of two or more other set operations. When all the set operations can be expressed in terms of a set of operations, then that set of operations is *functionally complete* (next definition).

Definition A.7. [6] *Let S be a set structure.*

DEF

*A set of operations Φ is **functionally complete** in S if*

$\cup, \cap, \mathsf{c}, \varnothing$, *and* X

can all be expressed in terms of elements of Φ.

Example A.1. Here are some examples of *functionally complete* sets:

EX

	$\{\downarrow\}$	(*rejection*)
	$\{\mid\}$	(*Sheffer stroke*)
	$\{\dot{+}, \varnothing\}$	(*adjunction* and $\varnothing$)
	$\{\setminus, X\}$	(*set difference* and X)
	$\{\cup, \mathsf{c}\}$	(*union* and *complement*)
	$\{\cap, \mathsf{c}\}$	(*intersection* and *complement*)
	$\{\triangle, \cap, X\}$	(*symmetric difference*, *intersection*, and X)
	$\{\triangle, \cup, X\}$	(*symmetric difference*, *union*, and X)
	$\{\triangle, \setminus, \mathsf{c}\}$	(*symmetric difference*, *set difference*, and *complement*)

A.2.4 Set multiplication

The *Cartesian product* operation $\times$ (next definition) is a kind of *set multiplication* operation.

Definition A.8. [7] *Let X and Y be sets, and let (x, y) be an* ORDERED PAIR.

DEF

*The **Cartesian product** $X \times Y$ of X and Y is*

$$X \times Y \triangleq \big\{(x, y) \,|\, (x \in X) \text{ and } (y \in Y)\big\}$$

Theorem A.2 (next theorem) demonstrates how this set operation interacts with certain other set operations. The Cartesian product is of critical importance in general because, for example, relations and functions are subsets of Cartesian products.

Theorem A.2. [8] *Let X, Y, Z be sets.*

THM

$$\begin{array}{rcll}
X \times (Y \cup Z) &=& (X \times Y) \cup (X \times Z) & (\times \text{ distributes over } \cup)\\
X \times (Y \cap Z) &=& (X \times Y) \cap (X \times Z) & (\times \text{ distributes over } \cap)\\
X \times (Y \setminus Z) &=& (X \times Y) \setminus (X \times Z) & (\times \text{ distributes over } \setminus)\\
(X \times Y) \cap (Y \times X) &=& (X \cap Y) \times (Y \cap X) & \\
(X \times X) \cap (Y \times Y) &=& (X \cap Y) \times (X \cap Y) &
\end{array}$$

[6] Whitesitt (1995) page 69

[7] Halmos (1960) page 24
G. Frege, 2007 August 25, http://groups.google.com/group/sci.logic/msg/3b3294f5ac3a76f0

[8] Menini and Oystaeyen (2004) page 50, Halmos (1960) page 25

✎PROOF:

$$
\begin{aligned}
X\times(Y\cup Z) &= \{(a,b)\,|(a\in X)\wedge(b\in Y\cup Z)\} \\
&= \{(a,b)\,|(a\in X)\wedge[(b\in Y)\vee(b\cup Z)]\} && \text{by Definition A.5} \\
&= \{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\vee[(a\in X)\wedge(b\in Z)]\} \\
&= \underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\}}_{X\times Y}\cup\underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Z)]\}}_{X\times Z} && \text{by Definition A.5} \\
&= (X\times Y)\cup(X\times Z)
\end{aligned}
$$

$$
\begin{aligned}
X\times(Y\cap Z) &= \{(a,b)\,|(a\in X)\wedge(b\in Y\cap Z)\} \\
&= \{(a,b)\,|(a\in X)\wedge[(b\in Y)\wedge(b\cup Z)]\} && \text{by Definition A.5} \\
&= \{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\wedge[(a\in X)\wedge(b\in Z)]\} \\
&= \underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\}}_{X\times Y}\cap\underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Z)]\}}_{X\times Z} && \text{by Definition A.5} \\
&= (X\times Y)\cap(X\times Z)
\end{aligned}
$$

$$
\begin{aligned}
X\times(Y\setminus Z) &= \{(a,b)\,|(a\in X)\wedge(b\in Y\setminus Z)\} \\
&= \left\{(a,b)\,|(a\in X)\wedge(b\in Y\cap Z^c)\right\} \\
&= \left\{(a,b)\,|(a\in X)\wedge\left[(b\in Y)\wedge(b\in Z^c)\right]\right\} && \text{by Definition A.5} \\
&= \left\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\wedge\left[(a\in X)\wedge(b\in Z^c)\right]\right\} \\
&= \underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\}}_{X\times Y}\cap\underbrace{\left\{(a,b)\,|\,\left[(a\in X)\wedge(b\in Z^c)\right]\right\}}_{X\times Z^c} && \text{by Definition A.5} \\
&= (X\times Y)\cap(X\times Z^c) \\
&\neq (X\times Y)\setminus(X\times Z)
\end{aligned}
$$

$$
\begin{aligned}
(X\times Y)\cap(Y\times X) &= \{(a,b)\,|(a\in X)\wedge(b\in Y)\}\cap\{(a,b)\,|(a\in Y)\wedge(b\in X)\} \\
&= \{(a,b)\,|[(a\in X)\wedge(b\in Y)]\wedge[(a\in Y)\wedge(b\in X)]\} && \text{by Definition A.5} \\
&= \{(a,b)\,|[(a\in X)\wedge(a\in Y)]\wedge[(b\in Y)\wedge(b\in X)]\} \\
&= \{(a,b)\,|(a\in X\cap Y)\wedge(b\in Y\cap X)\} \\
&= (X\cap Y)\times(Y\cap X)
\end{aligned}
$$

$$
\begin{aligned}
(X\times X)\cap(Y\times Y) &= \{(a,b)\,|(a\in X)\wedge(b\in X)\}\cap\{(a,b)\,|(a\in Y)\wedge(b\in Y)\} \\
&= \{(a,b)\,|[(a\in X)\wedge(b\in X)]\ \wedge\ [(a\in Y)\wedge(b\in Y)]\} && \text{by Definition A.5} \\
&= \{(a,b)\,|[(a\in X)\wedge(a\in Y)]\ \wedge\ [(b\in X)\wedge(b\in Y)]\} \\
&= \{(a,b)\,|(a\in X\cap Y)\ \wedge\ (b\in X\cap Y)\} \\
&= (X\cap Y)\times(X\cap Y)
\end{aligned}
$$

✏

A.3 Standard set structures

Set structures are typically designed to satisfy some special properties— such as being closed with respect to certain set operations. Examples of commonly occurring set structures include

A.3.1 Topologies

See CHAPTER 1 (page 3)

A.3.2 Algebras of sets

Definition A.9. [9] *Let X be a set with* POWER SET 2^X (Definition A.1 page 259).

DEF

*$\boldsymbol{A} \subseteq 2^X$ is an **algebra of sets** on X if*

1. $A \in \boldsymbol{A} \implies A^c \in \boldsymbol{A}$ (closed under complement operation) and
2. $A, B \in \boldsymbol{A} \implies A \cap B \in \boldsymbol{A}$ (closed under $\cap$)

The set of all algebra of sets on a set X is denoted $\mathscr{A}(X)$ such that

$$\mathscr{A}(X) \triangleq \left\{\boldsymbol{A} \subseteq 2^X \,|\, \boldsymbol{A} \text{ is an algebra of sets}\right\}.$$

An ALGEBRA OF SETS $\boldsymbol{A}$ *on X is a **σ-algebra** on X if*

3. $\left\{A_n \,\middle|\, n\in\mathbb{Z}\right\} \subseteq \boldsymbol{A} \implies \bigcup_{n\in\mathbb{Z}} A_n \in \boldsymbol{A}$ (closed under countable union operations).

On every set X with at least 2 elements, there are always two particular algebras of sets: the *smallest algebra* and the *largest algebra*, as demonstrated by Example A.2 (next).

Example A.2. [10] Let $\mathscr{A}(X)$ be the set of *algebras of sets* (Definition A.9 page 266) on a set X and 2^X the *power set* (Definition A.1 page 259) on X.

EX

$$\begin{array}{lll} \{\varnothing, X\} & \in & \mathscr{A}(X) \quad \text{(smallest algebra)} \\ 2^X & \in & \mathscr{A}(X) \quad \text{(largest algebra)} \end{array}$$

Isomorphically, all *algebras of sets* are *boolean algebras* and all boolean algebras are algebras of sets (next theorem).

Theorem A.3 (Stone Representation Theorem). [11] *Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ be a* LATTICE.

THM

$$\boldsymbol{L} \text{ is BOOLEAN} \iff \left\{ \begin{array}{l} \boldsymbol{L} \text{ is isomorphic to } (\boldsymbol{A}, \cup, \cap, \varnothing, X; \subseteq) \\ \text{for some ALGEBRA OF SETS (Definition A.9 page 266) } \boldsymbol{A} \end{array} \right\}$$

✎PROOF:

1. Proof that *algebra of sets* $\implies$ *Boolean algebra*:
   (a) Proof that $\boldsymbol{S}$ is closed under $\cup$ and $\cap$: by hypothesis.

[9] Aliprantis and Burkinshaw (1998) page 95, Aliprantis and Burkinshaw (1998) page 151, Halmos (1950) page 21, Hausdorff (1937) page 91

[10] Stroock (1999) page 33, Aliprantis and Burkinshaw (1998) pages 95–96

[11] Levy (2002) page 257, Grätzer (2003) page 85, Joshi (1989) page 224, Saliĭ (1988) page 32 ⟨"Stone's Theorem"⟩, Stone (1936a)

(b) By item (1b) and by Theorem A.5 (page 273), $\boldsymbol{L}$ is a *distributive* lattice.

(c) By item (1b) and properties of *lattices* (Theorem C.3 page 302), $\boldsymbol{L}$ is *idempotent*, *commutative*, *associative*, and *absorptive*.

(d) Proof that $\boldsymbol{L}$ has *identity*:

$$\begin{aligned}
A \cup \varnothing &= \{x \in X | (x \in A) \vee (x \in \varnothing)\} && \text{by definition of } \cup \text{ Definition A.5 page 260}\\
&= \{x \in X | x \in A\} && \text{by definition of } \varnothing \text{ Definition A.5 page 260}\\
&= A\\
A \cap X &= \{x \in X | (x \in A) \wedge (x \in X)\} && \text{by definition of } \cap \text{ Definition A.5 page 260}\\
&= \{x \in X | x \in A\} && \text{by definition of } \varnothing \text{ Definition A.5 page 260}\\
&= A
\end{aligned}$$

(e) Proof that $\boldsymbol{L}$ is *complemented*: by hypothesis.

(f) Because $\boldsymbol{L}$ is *commutative* (item (1c) page 267), *distributive* (item (1b) page 267), has *identity* (item (1d) page 267), and is *complemented* (item (1e) page 267), and by the definition of *Boolean algebras*, $\boldsymbol{L}$ is a *Boolean algebra*.

2. Proof that *Boolean algebra* $\implies$ *algebra of sets*: not included at this time.

⇨

A.3.3 Rings of sets

A *ring of sets* (next definition) is a family of subsets that is closed under an "addition-like" set union operator $\cup$ and "subtraction-like" set difference operator $\setminus$. Using these two operations, it is not difficult to show that a ring of sets is also closed under a "multiplication-like" set intersection operator $\cap$. Because of this, a ring of sets behaves like an *algebraic ring*. Note however that a ring of sets is not necessarily a *topology* (Definition 1.1 page 3) because it does not necessarily include X itself.

Definition A.10. [12] *Let X be a set with* POWER SET 2^X *(Definition A.1 page 259).*

DEF

$\boldsymbol{R} \subseteq 2^X$ is a ***ring of sets*** *on X if*

1. $A, B \in \boldsymbol{R} \implies A \cup B$ *(closed under $\cup$)* *and*
2. $A, B \in \boldsymbol{R} \implies A \setminus B \in \boldsymbol{R}$ *(closed under $\setminus$)*

The set of all rings of sets on a set X is denoted $\mathscr{R}(X)$ such that

$$\mathscr{R}(X) \triangleq \left\{\boldsymbol{R} \subseteq 2^X | \boldsymbol{R} \text{ is a ring of sets}\right\}.$$

A RING OF SETS $\boldsymbol{R}$ *on X is a* ***σ-ring*** *on X if*

3. $\left\{A_n \middle| n \in \mathbb{Z}\right\} \subseteq \boldsymbol{R} \implies \bigcup_{n\in\mathbb{Z}} A_n \in \boldsymbol{R}$ *(closed under countable union operations).*

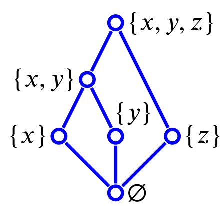
Example A.3. Table A.1 (page 268) lists some *rings of sets* on a finite set X.

Example A.4. Let $X \triangleq \{x, y, z\}$ be a set and $\boldsymbol{R}$ be the family of sets

$\boldsymbol{R} \triangleq \{\varnothing, X, \{x\}, \{y\}, \{z\}, \{x, y\}\}$.

Note that $(\boldsymbol{R}, \subseteq, \cup, \cap)$ is a lattice as illustrated in the figure to the right. However, $\boldsymbol{R}$ is *not* a ring of sets on X because, for example,

$\{x, y, z\} \setminus \{x\} = \{y, z\} \notin \boldsymbol{R}$.



[12] Berezansky et al. (1996) page 4, Halmos (1950) page 19, Hausdorff (1937) page 90

rings $\mathscr{R}(X)$ on a set X

$$\mathscr{R}(\varnothing) = \left\{ \boldsymbol{R}_1 = \{\varnothing\} \right\}$$

$$\mathscr{R}(\{x\}) = \left\{ \begin{array}{lclllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\} & \} \end{array} \right\}$$

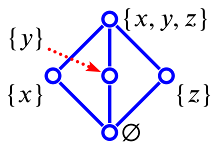
$$\mathscr{R}(\{x,y\}) = \left\{ \begin{array}{lcllllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\}, & & & \} \\ \boldsymbol{R}_3 & = & \{ & \varnothing, & & \{y\}, & & \} \\ \boldsymbol{R}_4 & = & \{ & \varnothing, & & & \{x,y\} & \} \\ \boldsymbol{R}_5 & = & \{ & \varnothing, & \{x\}, & \{y\}, & \{x,y\} & \} \end{array} \right\}$$

$$\mathscr{R}(\{x,y,z\}) = \left\{ \begin{array}{lcllllllllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & & & & & & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\}, & & & & & & & \} \\ \boldsymbol{R}_3 & = & \{ & \varnothing, & & \{y\}, & & & & & & \} \\ \boldsymbol{R}_4 & = & \{ & \varnothing, & & & \{z\}, & & & & & \} \\ \boldsymbol{R}_5 & = & \{ & \varnothing, & & & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_6 & = & \{ & \varnothing, & & & & & \{x,z\}, & & & \} \\ \boldsymbol{R}_7 & = & \{ & \varnothing, & & & & & & \{y,z\}, & & \} \\ \boldsymbol{R}_8 & = & \{ & \varnothing, & \{x\}, & \{y\}, & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_9 & = & \{ & \varnothing, & \{x\}, & & \{z\}, & & \{x,z\}, & & & \} \\ \boldsymbol{R}_{10} & = & \{ & \varnothing, & & \{y\}, & \{z\}, & & & \{y,z\}, & & \} \\ \boldsymbol{R}_{11} & = & \{ & \varnothing, & & & & & & & X & \} \\ \boldsymbol{R}_{12} & = & \{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X & \} \\ \boldsymbol{R}_{13} & = & \{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X & \} \\ \boldsymbol{R}_{14} & = & \{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X & \} \\ \boldsymbol{R}_{15} & = & \{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X & \} \end{array} \right\}$$

Table A.1: some *rings of sets* on a finite set X (Example A.3 page 267)

Example A.5. Let $X \triangleq \{x, y, z\}$ be a set and $\boldsymbol{R}$ be the family of sets $\boldsymbol{R} \triangleq \{\emptyset, X, \{x\}, \{y\}, \{z\}\}$. Note that $(\boldsymbol{T}, \subseteq)\cup\cap$ is a lattice as illustrated in the figure to the right. However, $\boldsymbol{R}$ is *not* a ring of sets on X because, for example,

$$\{x, y, z\}\backslash\{x\} = \{y, z\} \notin \boldsymbol{R}.$$



Proposition A.2. [13] *Let $\mathscr{R}(X)$ be the set of* RINGS OF SETS (*Definition A.10 page 267*) *on a set X.*

PRP

$$\left\{\begin{array}{l}\boldsymbol{R}_1 \text{ and } \boldsymbol{R}_2\\ \text{are \textbf{rings of sets}}\end{array}\right\} \implies \left\{\begin{array}{l}(\boldsymbol{R}_1 \cap \boldsymbol{R}_2)\\ \text{is a \textbf{ring of sets}}\end{array}\right\}$$

A.3.4 Partitions

The following definition is a special case of *partition* defined on lattices.

Definition A.11. [14]

DEF

A SET STRUCTURE $\left\{P_n \in 2^X \,\middle|\, n=1,2,\ldots,N\right\}$ *is a* ***partition*** *of the set X if*

1. $P_n \neq \emptyset \quad \forall n\in\{1,2,\ldots,N\}$ NON-EMPTY *and*
2. $P_n \cap P_m = \emptyset \quad \forall n\neq m$ MUTUALLY EXCLUSIVE *and*
3. $\bigcup_{n\in\mathbb{Z}} P_n = X$

Example A.6. Let $A, B \subseteq X$, as illustrated in Figure A.3 (page 262). There are a total of 15 partitions of X induced by A and B (Proposition A.6 page 271). Here are 5 of these partitions:

EX

1. $\{X\}$ (1 region)
2. $\{A, A^c\}$ (2 regions)
3. $\{A\cup B, A^c\cap B^c\}$ (2 regions)
4. $\{A\cap B, A\triangle B, A^c\cap B^c\}$ (3 regions)
5. $\{A\cap B, A\cap B^c, A^c\cap B, A^c\cap B^c\}$ (4 regions) [see also Figure A.3 page 262 and Figure A.4 page 263]

Proposition A.3. [15] *Let $\mathscr{P}(X)$ be the set of partitions on a set X.*

PRP

The relation $\trianglelefteq \in 2^{\mathbb{PP}}$ defined as

$$\boldsymbol{P} \trianglelefteq \boldsymbol{Q} \quad \overset{\text{def}}{\iff} \quad \forall B\in\boldsymbol{Q}, \exists A\in\boldsymbol{P} \quad \text{such that} \quad B\subseteq A$$

is an ordering relation on $\mathscr{P}(X)$.

Example A.7. Table A.2 (page 270) lists some partitions $\boldsymbol{P}(X)$ on a finite set X.

A.4 Numbers of set structures

Proposition A.4. [16]

PRP

The ***number of topologies*** *t_n on a finite set X_n with n elements is*

n	0	1	2	3	4	5	6	7	8
t_n	1	1	4	29	355	6942	209,527	9,535,241	642,779,354

n	9	10
t_n	63,260,289,423	8,977,053,873,043

[13] Kolmogorov and Fomin (1975) page 32, Bartle (2001) page 318

[14] Munkres (2000) page 23, Rota (1964) page 498, Halmos (1950) page 31

[15] Roman (2008) page 111, Comtet (1974) page 220, Grätzer (2007) page 697

[16] Sloane (2014) ⟨http://oeis.org/A000798⟩, Brown and Watson (1996) page 31, Comtet (1974) page 229, Comtet (1966), Chatterji (1967) page 7, Evans et al. (1967), Krishnamurthy (1966) page 157

partitions $\mathscr{P}(X)$ on a set X

$$\mathscr{P}(\varnothing) = \left\{ \boldsymbol{P}_1 = \varnothing \right\}$$

$$\mathscr{P}(\{x\}) = \left\{ \boldsymbol{P}_1 = \{ \{x\} \} \right\}$$

$$\mathscr{P}(\{x,y\}) = \left\{ \begin{array}{lcl} \boldsymbol{P}_1 &=& \{ \{x\}, \{y\}, \} \\ \boldsymbol{P}_2 &=& \{ \{x,y\} \} \end{array} \right\}$$

$$\mathscr{P}(\{x,y,z\}) = \left\{ \begin{array}{lcl} \boldsymbol{P}_1 &=& \{ \{x,y,z\} \} \\ \boldsymbol{P}_2 &=& \{ \{x\}, \{y,z\}, \} \\ \boldsymbol{P}_3 &=& \{ \{y\}, \{x,z\}, \} \\ \boldsymbol{P}_4 &=& \{ \{z\}, \{x,y\} \} \\ \boldsymbol{P}_5 &=& \{ \{x\}, \{y\}, \{z\} \} \end{array} \right\}$$

$$\mathscr{P}(\{w,x,y,z\}) = \left\{ \begin{array}{lcllll} \boldsymbol{P}_1 &=& \{ & & & X \; \} \\ \boldsymbol{P}_2 &=& \{ & \{w\}, & & \{x,y,z\} \; \} \\ \boldsymbol{P}_3 &=& \{ & \{x\}, & & \{w,y,z\} \; \} \\ \boldsymbol{P}_4 &=& \{ & \{y\}, & & \{w,x,z\} \; \} \\ \boldsymbol{P}_5 &=& \{ & \{z\}, & & \{w,x,y\} \; \} \\ \boldsymbol{P}_6 &=& \{ & & \{w,x\},\{y,z\} & \; \} \\ \boldsymbol{P}_7 &=& \{ & & \{w,y\},\{x,z\} & \; \} \\ \boldsymbol{P}_8 &=& \{ & & \{w,z\},\{x,y\}, & \; \} \\ \boldsymbol{P}_9 &=& \{ & \{w\},\{x\} & \{y,z\} & \; \} \\ \boldsymbol{P}_{10} &=& \{ & \{w\},\{y\}, & \{x,z\} & \; \} \\ \boldsymbol{P}_{11} &=& \{ & \{w\},\{z\}, & \{x,y\} & \; \} \\ \boldsymbol{P}_{12} &=& \{ & \{x\},\{y\}, & \{w,z\} & \; \} \\ \boldsymbol{P}_{13} &=& \{ & \{x\},\{z\}, & \{w,y\} & \; \} \\ \boldsymbol{P}_{14} &=& \{ & \{y\},\{z\}, & \{w,x\} & \; \} \\ \boldsymbol{P}_{15} &=& \{ & \{w\},\{x\},\{y\},\{z\}, & & \; \} \end{array} \right\}$$

Table A.2: some partitions $\boldsymbol{P}(X)$ on a finite set X (Example A.7 page 269)

Proposition A.5. [17] *Let t_n be the number of topologies on a finite set with n elements.*

PRP

$$\begin{aligned}
\lim_{n\to\infty} \frac{t_n}{2^{\frac{n^2}{4}}} &= \infty && && \text{(lower bound)}\\
\lim_{n\to\infty} \frac{t_n}{2^{\left(\frac{1}{2}+\epsilon\right)n^2}} &= 0 && \forall\epsilon>0 && \text{(upper bound)}\\
t_n &> nt_{n-1} && && \text{(rate of growth)}
\end{aligned}$$

Similar to the amazing relationship between e, π, i, 1, and 0 given by $e^{i\pi} + 1 = 0$, we find another relationship between e and the number of partitions, rings of sets, and algebras of sets (Theorem A.4 page 272).

Definition A.12. [18]

DEF

*The **Bell numbers** are the elements of the sequence $\left(B_n\right)_{n\in\mathbb{W}}$ defined as the solution to the following equation:*

$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

*The Bell numbers are also called the **exponential numbers**.*

Proposition A.6. [19] *Let $\left(B_n\right)_{n\in\mathbb{W}}$ be the sequence of Bell numbers. Then $\left(B_n\right)$ has the following values:*

PRP

n	0	1	2	3	4	5	6	7	8	9	10	11
B_n	1	1	2	5	15	52	203	877	4140	21,147	115,975	678,570

✎PROOF: By Definition A.12 (page 271), the sequence $\left(B_n\right)$ is the solution to

$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Let $\mathsf{f}^{(n)}(x)$ be the nth derivative of a function $\mathsf{f} : \mathbb{R} \to \mathbb{R}$. The Maclaurin expansion of $\mathsf{f}(x)$ is

$$\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\mathsf{f}^{n}(0)}{n!} x^n$$

Let $\mathsf{f}(x) \triangleq e^{e^x}$. Then

$$\begin{aligned}
\mathsf{f}^{(0)}(0) &= \mathsf{f}^{(0)}(x)\big|_{x=0}\\
&= e^{e^0}\\
&= e\\
\mathsf{f}^{(1)}(0) &= \mathsf{f}^{(1)}(x)\big|_{x=0}\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\right|_{x=0}\\
&= \left. e^{e^x} e^x \right|_{x=0}\\
&= e\\
\mathsf{f}^{(2)}(0) &= \left.\frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(1)}(x)\right|_{x=0}
\end{aligned}$$

[17] Chatterji (1967) pages 6–7, Kleitman and Rothschild (1970)
[18] Comtet (1974) pages 210–211, Rota (1964) page 499, Bell (1934) page 417, d'Ocagne (1887) page 371
[19] Sloane (2014) ⟨http://oeis.org/A000110⟩

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x} e^x \Big|_{x=0} \\
&= \left(e^{e^x} e^x\right) e^x + e^{e^x} e^x \Big|_{x=0} \\
&= e^{e^x}\left(e^{2x} + e^x\right)\Big|_{x=0} \\
&= 2e \\
\mathsf{f}^{(3)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(2)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x} e^x\left(e^{2x} + e^x\right) + e^{e^x}\left(2e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x}\left(e^{3x} + 3e^{2x} + e^x\right)\Big|_{x=0} \\
&= 5e \\
\mathsf{f}^{(4)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(3)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{3x} + 3e^{2x} + e^x\right)\Big|_{x=0} \\
&= \left(e^{e^x} e^x\right)\left(e^{3x} + 3e^{2x} + e^x\right) + e^{e^x}\left(3e^{3x} + 6e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x}\left(e^{4x} + 6e^{3x} + 7e^{2x} + e^x\right)\Big|_{x=0} \\
&= 15e \\
\mathsf{f}^{(5)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(4)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{4x} + 6e^{3x} + 7e^{2x} + e^x\right)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{e^x} e^x\right)\left(e^{4x} + 6e^{3x} + 7e^{2x} + e^x\right) + e^{e^x}\left(4e^{4x} + 18e^{3x} + 14e^{2x} + e^x\right)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{5x} + 10e^{4x} + 25e^{3x} + 15e^{2x} + e^x\right)\Big|_{x=0} \\
&= 52e \\
\mathsf{f}^{(6)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(5)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{5x} + 10e^{4x} + 25e^{3x} + 15e^{2x} + e^x\right)\Big|_{x=0} \\
&= \left(e^{e^x} e^x\right)\left(e^{5x} + 10e^{4x} + 25e^{3x} + 15e^{2x} + e^x\right) + e^{e^x}\left(5e^{5x} + 40e^{4x} + 75e^{3x} + 30e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x}\left(e^{6x} + 15e^{5x} + 65e^{4x} + 90e^{3x} + 31e^{2x} + e^x\right)\Big|_{x=0} \\
&= 203e
\end{aligned}
$$

Thus, e^{e^x} has Maclaurin expansion

$$
e^{e^x} = e\left(1 + x + \frac{2}{2}x^2 + \frac{5}{3!}x^3 + \frac{15}{4!}x^4 + \frac{52}{5!}x^5 + \frac{203}{6!}x^6 + \cdots\right) = e\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n
$$

✎

Theorem A.4. [20] *Let X_n be a finite set with n elements. Let $(\!(B_n)\!)_{n\in\mathbb{W}}$ be the sequence of Bell numbers.*

[20] http://groups.google.com/group/sci.math/browse_thread/thread/70a73e734b69a6ec/

THM

The number of PARTITIONS *on* X_n *is* B_n.
The number of RINGS OF SETS *on* X_n *is* B_{n+1}.
The number of ALGEBRAS OF SETS *on* X_n *is* B_n.

A.5 Operations on set structures

Proposition A.7.

PRP

closed under	*partition*	*ring of sets*	*algebra of sets*	*topology*
$\varnothing$		✓	✓	✓
X	✓		✓	✓
c			✓	
$\cup$		✓	✓	✓
$\cap$		✓	✓	✓
$\triangle$		✓	✓	
$\setminus$		✓	✓	

✎PROOF:

1. Proof for closure in a *topology*: Definition 1.1 (page 3)
2. Proof for closure in a *ring of sets*: Definition A.10 (page 267) and Theorem A.9 (page 275)
3. Proof for closure in an *algebra of sets*: Definition A.9 (page 266) and Theorem A.8 (page 273)

⇒

Theorem A.5. *Let* $\boldsymbol{T}$ *be a* SET STRUCTURE *(Definition A.2 page 259) on a set* X.

THM

$\boldsymbol{T}$ *is a* ***topology*** $\implies$ $\forall A, B, C \in \boldsymbol{T}$

property with emphasis on $\cup$	dual property with emphasis on $\cap$	property name
$A \cup A = A$	$A \cap A = A$	(IDEMPOTENT)
$A \cup B = B \cup A$	$A \cap B = B \cap A$	(COMMUTATIVE)
$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	(ASSOCIATIVE)
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	(ABSORPTIVE)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(DISTRIBUTIVE)

✎PROOF:

1. By Definition 1.1 (page 3), $\boldsymbol{T}$ is a *topology*.
2. By Theorem A.3 (page 266), $(\boldsymbol{T}, \cup, \cap; \subseteq)$ is a *distributive lattice*.
3. The properties listed are all properties of *distributive lattices*.

⇒

Proposition A.8. *Let* $\boldsymbol{A}$ *be a* SET STRUCTURE *(Definition A.2 page 259) on a set* X.

PRP

$$\left\{ \begin{array}{l} \boldsymbol{A} \text{ is an} \\ \textbf{\textit{algebra of sets}} \end{array} \right\} \implies \left\{ \begin{array}{llllll} 1. & \varnothing & \in \boldsymbol{A} & & (\boldsymbol{A} \text{ includes the } \cup \text{ identity element}) \\ 2. & X & \in \boldsymbol{A} & & (\boldsymbol{A} \text{ includes the } \cap \text{ identity element}) \\ 3. & A^{\mathsf{c}} & \in \boldsymbol{A} & \forall A \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \mathsf{c}) \\ 4. & A \cup B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \cup) \\ 5. & A \cap B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \cap) \\ 6. & A \setminus B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \setminus) \\ 7. & A \triangle B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \triangle) \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\varnothing &= A \cap A^c \\
X &= \mathsf{c}\varnothing \\
A \cup B &= \mathsf{c}(A^c \cap B^c) && \text{by } \textit{de Morgan's Law} \text{ (Theorem A.7 page 274)} \\
A \setminus B &= A \cap B^c \\
A \triangle B &= (A \setminus B^c) \cup (B \setminus A)
\end{aligned}$$

$(\boldsymbol{A}, \cup, \setminus)$ is a ring of sets because $\cup$ and $\setminus$ are closed in $\boldsymbol{A}$ (as shown above). ✏

Theorem A.6. [21] *Let $\boldsymbol{A}$ be a* SET STRUCTURE *(Definition A.2 page 259) on a set X.*

THM

$\boldsymbol{A}$ *is an* ***algebra of sets*** $\implies$ $\forall A, B, C \in \boldsymbol{A}$

property emphasizing $\cup$			dual property emphasizing $\cap$			property name
$A \cup A$	$=$	A	$A \cap A$	$=$	A	(IDEMPOTENT)
$A \cup B$	$=$	$B \cup A$	$A \cap B$	$=$	$B \cap A$	(COMMUTATIVE)
$A \cup (B \cup C)$	$=$	$(A \cup B) \cup C$	$A \cap (B \cap C)$	$=$	$(A \cap B) \cap C$	(ASSOCIATIVE)
$A \cup (A \cap B)$	$=$	A	$A \cap (A \cup B)$	$=$	A	(ABSORPTIVE)
$A \cup (B \cap C)$	$=$	$(A \cup B) \cap (A \cup C)$	$A \cap (B \cup C)$	$=$	$(A \cap B) \cup (A \cap C)$	(DISTRIBUTIVE)
$A \cup \varnothing$	$=$	A	$A \cap X$	$=$	A	(IDENTITY)
$A \cup X$	$=$	X	$A \cap \varnothing$	$=$	$\varnothing$	(BOUNDED)
$A \cup A^c$	$=$	X	$A \cap A^c$	$=$	$\varnothing$	(COMPLEMENTED)
$(A^c)^c$	$=$	A				(UNIQUELY COMPLEMENTED)
$(A \cup B)^c$	$=$	$A^c \cap B^c$	$(A \cap B)^c$	$=$	$A^c \cup B^c$	(DE MORGAN)

✎PROOF:

1. By Definition A.9 (page 266), $\boldsymbol{S}$ is an *algebra of sets*.
2. By the *Stone Representation Theorem* (Theorem A.3 page 266), $(\boldsymbol{S}, \cup, \cap, \varnothing, X\, ; \subseteq)$ is a *Boolean algebra*.
3. The properties listed are all properties of *Boolean algebras*.

✏

Theorem A.7. [22] *Let $\boldsymbol{A}$ be an* ALGEBRA OF SETS *(Definition A.9 page 266) on a set X.*

THM

$\boldsymbol{A}$ *is an* ***algebra of sets*** $\implies$ $\forall A_1, A_2, \ldots, A_N, B \in \boldsymbol{A}$ *and* $\forall N \in \mathbb{N}$

property emphasizing $\cup$	dual property emphasizing $\cap$	property name
$\left(\bigcup_{n=1}^{N} A_n\right)^c = \bigcap_{n=1}^{N} A^c{}_n$	$\left(\bigcap_{n=1}^{N} A_n\right)^c = \bigcup_{n=1}^{N} A^c{}_n$	(DE MORGAN)
$\left(\bigcup_{n=1}^{N} A_n\right) \cap B = \bigcup_{n=1}^{N} (A_n \cap B)$	$\left(\bigcap_{n=1}^{N} A_n\right) \cup B = \bigcap_{n=1}^{N} (A_n \cup B)$	(DISTRIBUTIVE *with respect to* $\cup$ *and* $\cap$)
$\left(\bigcup_{n=1}^{N} A_n\right) \setminus B = \bigcup_{n=1}^{N} (A_n \setminus B)$	$\left(\bigcap_{n=1}^{N} A_n\right) \setminus B = \bigcap_{n=1}^{N} (A_n \setminus B)$	(DISTRIBUTIVE *with respect to* $\setminus$ *and* $\cap$)

✎PROOF:

1. By Theorem A.3 (page 266), the lattice $(X, \cup, \cap\, ; \subseteq)$ is *Boolean*.

2. The first four properties are true any Boolean system.

[21] Dieudonné (1969) pages 3–4, Copson (1968) page 9

[22] Michel and Herget (1993) page 12, Aliprantis and Burkinshaw (1998) page 4, Vaidyanathaswamy (1960) pages 3–4

3. Proof for the remaining two:

$$\begin{aligned}
\left(\bigcap_{n=1}^{N} A_n\right)\setminus B &= \left(\bigcap_{n=1}^{N} A_n\right)\cap B^c && \text{by Theorem A.1 page 261}\\
&= \bigcap_{n=1}^{N}(A_n \cap B^c) && \text{by previous result}\\
&= \bigcap_{n=1}^{N}(A_n \setminus B) && \text{by Theorem A.1 page 261}
\end{aligned}$$

$$\begin{aligned}
\left(\bigcup_{n=1}^{N} A_n\right)\setminus B &= \left(\bigcup_{n=1}^{N} A_n\right)\cap B^c && \text{by Theorem A.1 page 261}\\
&= \bigcup_{n=1}^{N}(A_n \cap B^c) && \text{by previous result}\\
&= \bigcup_{n=1}^{N}(A_n \setminus B) && \text{by Theorem A.1 page 261}
\end{aligned}$$

✏

Proposition A.9. [23] *Let $\boldsymbol{R}$ be a* SET STRUCTURE (Definition A.2 page 259) *on a set X.*

PRP

$$\left\{\begin{array}{l}\boldsymbol{R} \text{ is a}\\ \textbf{ring of sets}\\ \text{on } X\end{array}\right\} \implies \left\{\begin{array}{lllll}
1. & \varnothing & \in \boldsymbol{R} & & (\boldsymbol{R} \text{ includes the } \cup \text{ identity element}) \quad \text{and}\\
2. & A \cup B & \in \boldsymbol{R} & \forall A,B \in \boldsymbol{R} & (\boldsymbol{R} \text{ is closed under } \cup) \quad \text{and}\\
3. & A \cap B & \in \boldsymbol{R} & \forall A,B \in \boldsymbol{R} & (\boldsymbol{R} \text{ is closed under } \cap) \quad \text{and}\\
4. & A \setminus B & \in \boldsymbol{R} & \forall A,B \in \boldsymbol{R} & (\boldsymbol{R} \text{ is closed under } \setminus) \quad \text{and}\\
5. & A \triangle B & \in \boldsymbol{R} & \forall A,B \in \boldsymbol{R} & (\boldsymbol{R} \text{ is closed under } \triangle)
\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
A \triangle B &= (A \setminus B) \cup (B \setminus A)\\
A \cap B &= (A \cup B) \setminus (A \triangle B)\\
A \setminus A &= \varnothing
\end{aligned}$$

✏

Theorem A.8. [24] *Let $\boldsymbol{R}$ be a* SET STRUCTURE (Definition A.2 page 259) *on a set X.*

THM

*If $\boldsymbol{R}$ is an **ring of sets** on X, then $(\boldsymbol{R}, \triangle, \cap)$ is an* ALGEBRAIC RING; *in particular,*

$$\begin{array}{lll|lll}
A \triangle \varnothing = A & \forall A \in \boldsymbol{R} & & A \cap \varnothing = \varnothing & \forall A \in \boldsymbol{R} &\\
A \triangle X = A^c & \forall A \in \boldsymbol{R} & & A \cap X = A & \forall A \in \boldsymbol{R} &\\
A \triangle \varnothing = A & \forall A \in \boldsymbol{R} & & A \cap A = A & \forall A \in \boldsymbol{R} &
\end{array}$$

$$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C) \qquad \forall A,B,C \in \boldsymbol{R}$$

properties emphasizing $\triangle$ | *properties emphasizing $\cap$*

✎PROOF:

[23] Berezansky et al. (1996) page 4, Halmos (1950) pages 19–20

[24] Vaidyanathaswamy (1960) pages 17–18, Kelley and Srinivasan (1988) page 22, Wilker (1982) page 211, Vaidyanathaswamy (1960) page 19

1. Proof that $(\boldsymbol{R}, \cup, \setminus)$ is an *algebraic ring*: by Theorem A.8 (page 275)
2. Proof that a ring of sets is equivalent to $(\boldsymbol{R}, \cup, \setminus)$: This is proven simply by noting that $\cup$ and $\setminus$ (the two operations in a ring of sets $(\boldsymbol{R}, \cup, \setminus)$) can be expressed in terms of $\triangle$ and $\cap$ (the two operations in the algebraic ring $(\boldsymbol{R}, \triangle, \cap)$) and vice-versa. And this is demonstrated by Theorem A.1 (page 261).

The definition of an algebraic ring is given in Definition F.2 (page 341).

1. Proof that $(\boldsymbol{S}, \triangle)$ is a group: see Proposition A.1 (page 261).

2. Proof that $A \cap (B \cap C) = (A \cap B) \cap C$:

$$\begin{aligned}
A \cap (B \cap C) &= \{x \in X | (x \in A) \wedge [(x \in B) \wedge (x \in C)]\} && \text{by definition of } \cap \text{ page 260} \\
&= \{x \in X | [(x \in A) \wedge (x \in B)] \wedge (x \in C)\} \\
&= (A \cap B) \cap C && \text{by definition of } \cap \text{ page 260}
\end{aligned}$$

3. Proof that $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$:

$$\begin{aligned}
A \cap (B \triangle C) &= \{x \in X | (x \in A) \wedge [(x \in B) \oplus (x \in C)]\} && \text{by definition of } \cap, \triangle \text{ page 260} \\
&= \{x \in X | [(x \in A) \wedge (x \in B)] \oplus [(x \in A) \wedge (x \in C)]\} \\
&= (A \cap B) \triangle (A \cap C) && \text{by definition of } \cap, \triangle \text{ page 260}
\end{aligned}$$

4. Proof that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$:

$$\begin{aligned}
(A \triangle B) \cap C &= \{x \in X | [(x \in A) \oplus (x \in B)] \wedge (x \in C)\} && \text{by definition of } \cap, \triangle \text{ page 260} \\
&= \{x \in X | [(x \in A) \wedge (x \in C)] \oplus [(x \in B) \wedge (x \in C)]\} \\
&= (A \cap C) \triangle (B \cap C) && \text{by definition of } \cap, \triangle \text{ page 260}
\end{aligned}$$

⇰

A.6 Lattices of set structures

The *set inclusion* relation $\subseteq$ (Definition A.13 page 276) is an *order relation* (Definition B.2 page 286) on set structures, as demonstrated by Proposition A.10 (next proposition).

Definition A.13. *Let $\boldsymbol{S}$ be a* SET STRUCTURE *(Definition A.2 page 259) on a set X.*

DEF

The relation $\subseteq \in 2^{\boldsymbol{SS}}$ is defined as

$$A \subseteq B \quad \text{if} \quad x \in A \implies x \in B \qquad \forall x \in X$$

Proposition A.10 (order properties)**.** *Let $\boldsymbol{S}$ be a* SET STRUCTURE *(Definition A.2 page 259) on a set X.*

PRP

The pair $(\boldsymbol{S}, \subseteq)$ is an ORDERED SET. *In particular,*

$$\begin{array}{lllllllll}
A \subseteq A & & & & \forall A \in \boldsymbol{S} & \text{(REFLEXIVE)} & \text{and} \\
A \subseteq B & \text{and} & B \subseteq C & \implies & A \subseteq C & \forall A,B,C \in \boldsymbol{S} & \text{(TRANSITIVE)} & \text{and} \\
A \subseteq B & \text{and} & B \subseteq A & \implies & A = B & \forall A,B \in \boldsymbol{S} & \text{(ANTI-SYMMETRIC).}
\end{array}$$

✎PROOF: By Definition B.2 (page 286), a relation is an *order relation* if it is *reflexive*, *transitive*, and *anti-symmetric*.

1. Proof that $\subseteq$ is *reflexive* on 2^X:

$$\begin{aligned}
x \in A &\implies x \in A \\
&\implies A \subseteq A
\end{aligned}$$

2. Proof that $\subseteq$ is *transitive* on 2^X:

$$\begin{aligned}
x \in A &\implies x \in B && \text{by first left hypothesis}\\
&\implies x \in C && \text{by second left hypothesis}\\
&\implies A \subseteq C
\end{aligned}$$

3. Proof that $\subseteq$ is *anti-symmetric* on 2^X:

$$\begin{aligned}
A \subseteq B &\implies (x \in A \implies x \in B)\\
B \subseteq A &\implies (x \in B \implies x \in A)\\
A \subseteq B \text{ and } B \subseteq A &\implies (x \in A \iff x \in B)\\
&\implies A = B
\end{aligned}$$

✏

In a set structure that is *closed* under the *union* operation $\cup$ and *intersection* operation $\cap$, the *greatest lower bound* of any two elements A and B is simply $A \cap B$ and *least upper bound* is simply $A \cup B$ (Proposition A.11 page 277). However, this may not be true for a set structure that is *not* closed under these operations (Example A.8 page 278).

Proposition A.11. *Let S be a* SET STRUCTURE *(Definition A.2 page 259) on a set X.*

PRP

If** S is closed under $\cup$ and $\cap$ **then

$A \cup B$	*is the* LEAST UPPER BOUND	*of A and B in $(S, \subseteq)$*	$(\cup = \vee)$ *and*
$A \cap B$	*is the* GREATEST LOWER BOUND	*of A and B in $(S, \subseteq)$*	$(\cap = \wedge)$.

✎PROOF:

1. Proof that $A \cup B$ is the least upper bound:

$$\begin{aligned}
A &= \{x \in X | x \in A\}\\
&\subseteq \{x \in X | x \in A \text{ or } x \in B\}\\
&= A \cup B && \text{by Definition A.5 page 260}\\
B &= \{x \in X | x \in B\}\\
&\subseteq \{x \in X | x \in A \text{ or } x \in B\}\\
&= A \cup B && \text{by Definition A.5 page 260}
\end{aligned}$$

$$\begin{aligned}
A \subseteq C \text{ and } B \subseteq C &\implies \left\{x \in A \text{ and } y \in B \implies x, y \in C\right\}\\
&\implies \{x \in A \text{ or } x \in B \implies x \in C\}\\
&\implies \{x \in A \cup B \implies x \in C\}\\
&\implies A \cup B \subseteq C
\end{aligned}$$

2. Proof that $A \cap B$ is the greatest lower bound:

$$\begin{aligned}
A \cap B &= \left\{x \in X | x \in A \text{ and } x \in B\right\} && \text{by Definition A.5 page 260}\\
&\subseteq \{x \in X | x \in A\}\\
&= A\\
A \cap B &= \left\{x \in X | x \in A \text{ and } x \in B\right\} && \text{by Definition A.5 page 260}\\
&\subseteq \{x \in X | x \in B\}\\
&= B
\end{aligned}$$

$$
\begin{aligned}
C \subseteq A \text{ and } C \subseteq B &\implies \left\{x \in C \implies x \in A \quad \text{and} \quad x \in C \implies x \in B\right\} \\
&\implies \{x \in C \implies x \in A \text{ or } x \in B\} \\
&\implies \{x \in C \implies x \in A \cap B\} \\
&\implies C \subseteq A \cap B
\end{aligned}
$$

✎

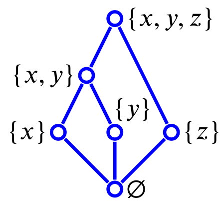
Example A.8. The set structure

$\boldsymbol{S} \triangleq \{\varnothing,\ \{x\},\ \{y\},\ \{z\},\ \{x,y\},\ \{x,y,z\}\}$

ordered by the set inclusion relation $\subseteq$ is illustrated by the Hasse diagram to the right. Note that

$\{x\} \vee \{z\} = \{x,y,z\} \neq \{x,z\} = \{x\} \cup \{z\}.$

That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.



A.6.1 Lattices of topologies

See Section 1.1.6 (page 10)

A.6.2 Lattices of algebra of sets

Example A.9. The following table lists some algebras of sets on a finite set X. Lattices of algebras of sets are illustrated in Figure A.7 (page 280) and Figure A.5 (page 279).

algebra of sets $\mathscr{A}(X)$ on a set X		
$\mathscr{A}(\varnothing)$	$=$	$\left\{\ \boldsymbol{A}_1 = \{\varnothing\}\ \right\}$
$\mathscr{A}(\{x\})$	$=$	$\left\{\ \boldsymbol{A}_1 = \{\varnothing,\ \{x\}\}\ \right\}$
$\mathscr{A}(\{x,y\})$	$=$	$\left\{\begin{array}{lcl} \boldsymbol{A}_1 &=& \{\varnothing,\ X\} \\ \boldsymbol{A}_2 &=& \{\varnothing,\ \{x\},\ \{y\},\ X\} \end{array}\right\}$
$\mathscr{A}(\{x,y,z\})$	$=$	$\left\{\begin{array}{lclllllllll} \boldsymbol{A}_1 &=& \{ & \varnothing, & & & & & & & X\ \} \\ \boldsymbol{A}_2 &=& \{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X\ \} \\ \boldsymbol{A}_3 &=& \{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X\ \} \\ \boldsymbol{A}_4 &=& \{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X\ \} \\ \boldsymbol{A}_5 &=& \{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X\ \} \end{array}\right\}$
$\mathscr{A}(\{w,x,y,z\})$	$=$	

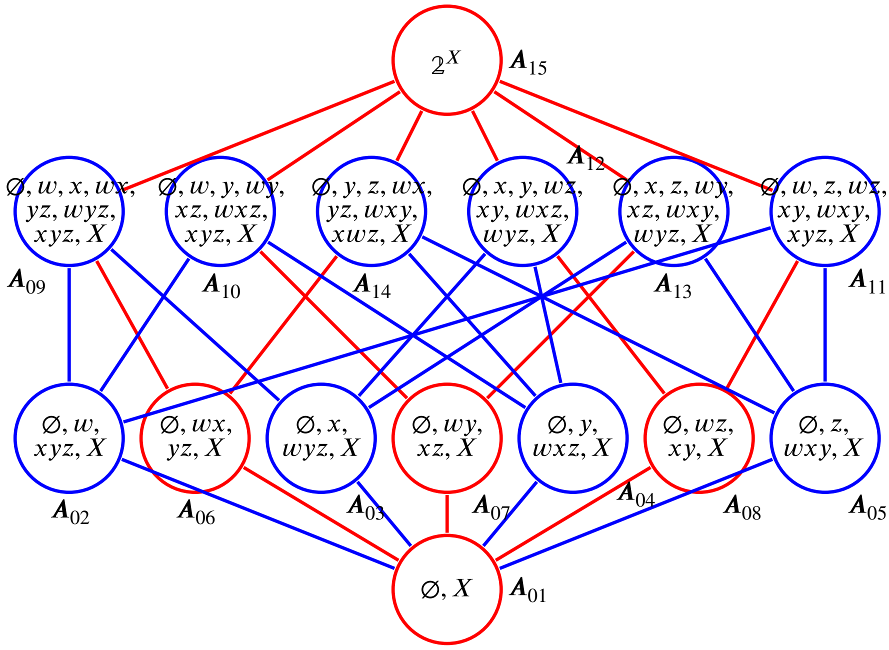


Figure A.5: lattice of *algebras of sets* on $\{w, x, y, z\}$ (Example A.9 page 278)

$$
\left\{
\begin{array}{lclllllll}
\boldsymbol{A}_1 & = & \{ & \varnothing, & & & & X & \} \\
\boldsymbol{A}_2 & = & \{ & \varnothing, & \{w\}, & & \{x, y, z\}, & X & \} \\
\boldsymbol{A}_3 & = & \{ & \varnothing, & \{x\}, & & \{w, y, z\}, & X & \} \\
\boldsymbol{A}_4 & = & \{ & \varnothing, & \{y\}, & & \{w, x, z\}, & X & \} \\
\boldsymbol{A}_5 & = & \{ & \varnothing, & \{z\}, & & \{w, x, y\}, & X & \} \\
\boldsymbol{A}_6 & = & \{ & \varnothing, & & \{w, x\}, \{y, z\}, & & X & \} \\
\boldsymbol{A}_7 & = & \{ & \varnothing, & & \{w, y\}, \{x, z\}, & & X & \} \\
\boldsymbol{A}_8 & = & \{ & \varnothing, & & \{w, z\}, \{x, y\}, & & X & \} \\
\boldsymbol{A}_9 & = & \{ & \varnothing, & \{w\}, \{x\}, & \{w, x\}, \{y, z\}, & \{w, y, z\}, \{x, y, z\}, & X & \} \\
\boldsymbol{A}_{10} & = & \{ & \varnothing, & \{w\}, \{y\}, & \{w, y\}, \{x, z\}, & \{w, x, z\}, \{x, y, z\}, & X & \} \\
\boldsymbol{A}_{11} & = & \{ & \varnothing, & \{w\}, \{z\}, & \{w, z\}, \{x, y\}, & \{w, x, y\}, \{x, y, z\}, & X & \} \\
\boldsymbol{A}_{12} & = & \{ & \varnothing, & \{x\}, \{y\}, & \{w, z\}, \{x, y\}, & \{w, x, z\}, \{w, y, z\}, & X & \} \\
\boldsymbol{A}_{13} & = & \{ & \varnothing, & \{x\}, \{z\}, & \{w, y\}, \{x, z\}, & \{w, x, y\}, \{w, y, z\}, & X & \} \\
\boldsymbol{A}_{14} & = & \{ & \varnothing, & \{y\}, \{z\}, & \{w, x\}, \{y, z\}, & \{w, x, y\}, \{w, x, z\}, & X & \} \\
\boldsymbol{A}_{15} & = & 2^X & & & & & &
\end{array}
\right\}
$$

A.6.3 Lattices of rings of sets

Example A.10. There are a total of **15** rings of sets on the set $X \triangleq \{x, y, z\}$. These rings of sets are listed in Example A.3 (page 267) and illustrated in Figure A.6 (page 280). The five rings containing X ($\boldsymbol{R}_{11}$–$\boldsymbol{R}_{15}$) are also *algebras of sets* (Proposition A.13 page 283), and thus also *Boolean algebras* (Theorem A.3 page 266). The five algebras of sets are shaded Figure A.6.

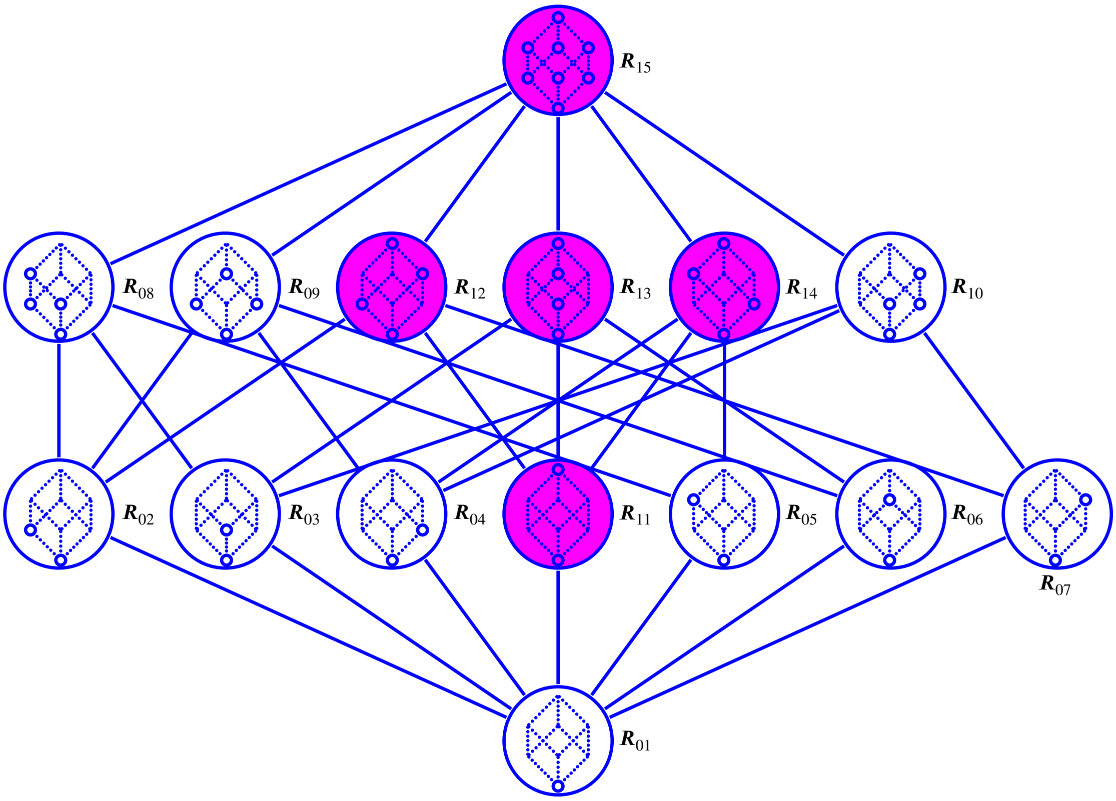


Figure A.6: Lattice of rings of sets on $X \triangleq \{x, y, z\}$ (Example A.10 page 279)

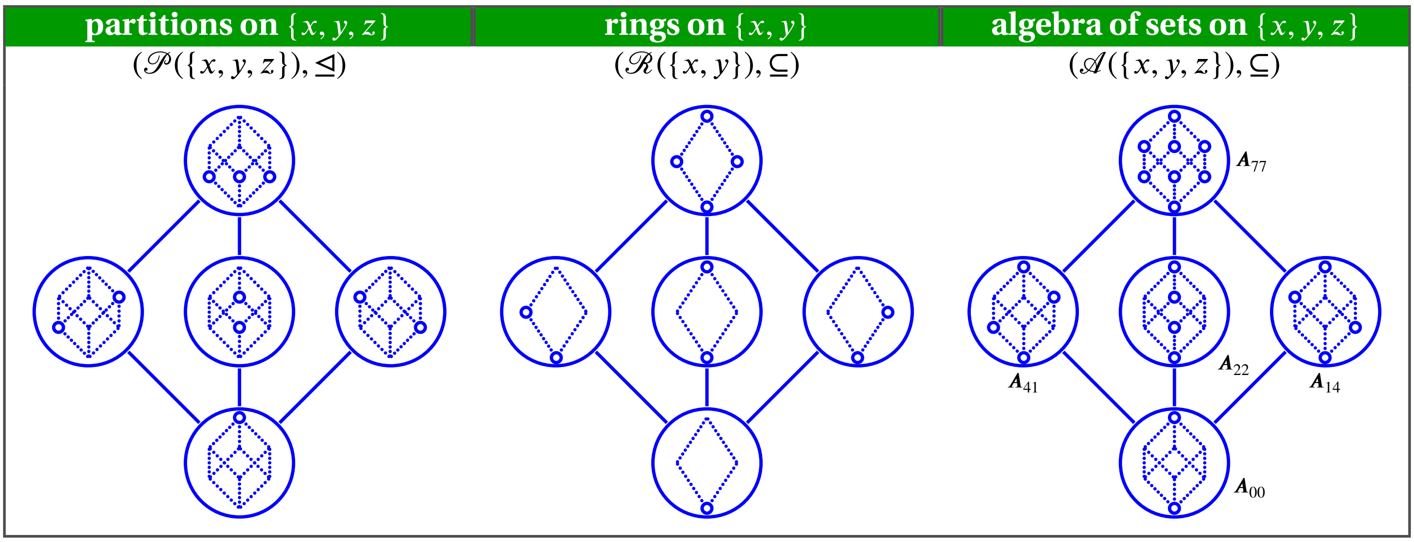


Figure A.7: Lattices of set structures (see Example A.11 (page 281), Example A.3 (page 267), and Example A.9 (page 278))

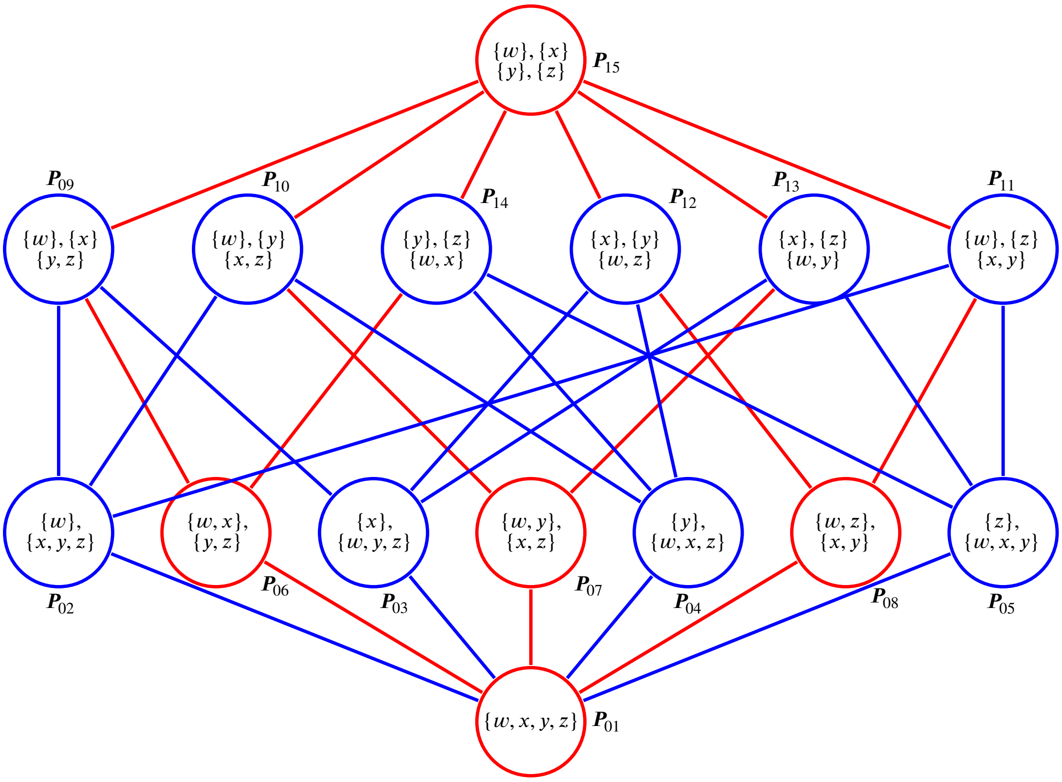


Figure A.8: Lattice of partitions of sets on $X \triangleq \{w, x, y, z\}$ (Example A.12 page 281)

A.6.4 Lattices of partitions of sets

Example A.11. There are a total of **5** partitions of sets on the set $X \triangleq \{x, y, z\}$. These sets are listed in Example A.7 (page 269) and illustrated in Figure A.7 (page 280).

Example A.12. There are a total of **15** partitions of sets on the set $X \triangleq \{w, x, y, z\}$. These sets are listed in Example A.7 (page 269) and illustrated in Figure A.8 (page 281).

In 1946, Philip Whitman proposed an amazing conjecture—that all finite lattices are isomorphic to a lattice of partitions. A proof for this was published some 30 years later by Pavel Pudlák and Jiří Tůma (next theorem).

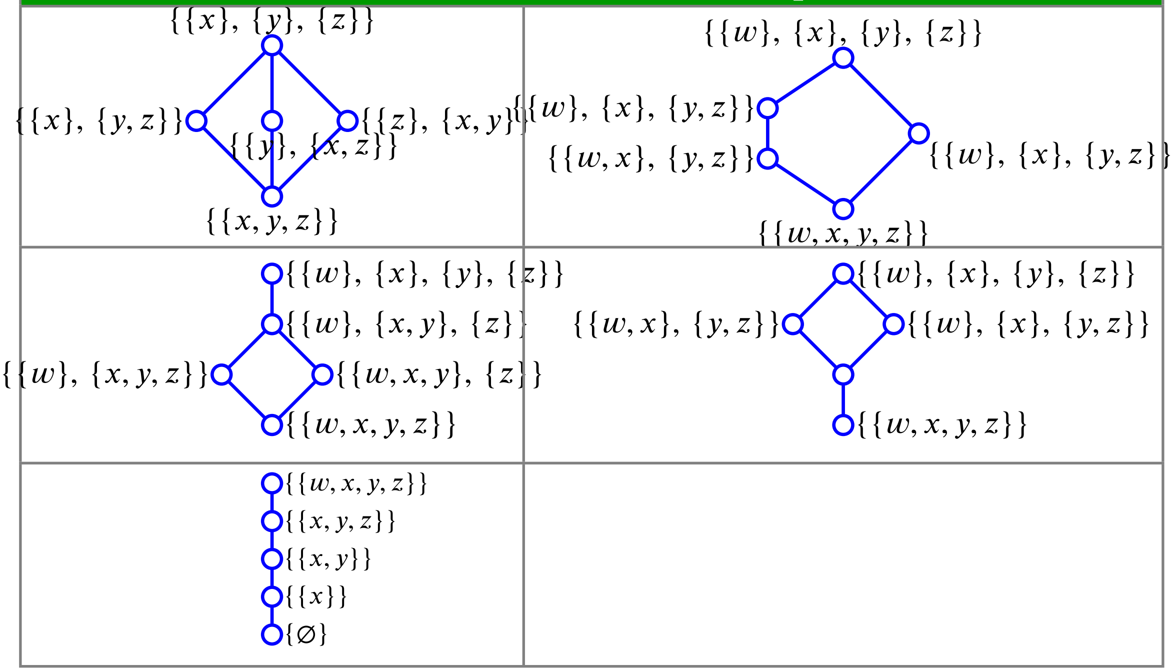
Theorem A.9. [25] *Let* **L** *be a lattice.*

THM			
	L *is* FINITE	$\Longrightarrow$	**L** *is isomorphic to a* LATTICE OF PARTITIONS

Example A.13. There are five unlabeled lattices on a five element set as stated in Proposition C.2 (page 307) and illustrated in Example C.11 (page 308). All of these lattices are isomorphic to a lattice of partitions (Theorem A.9 page 281), as illustrated next.

[25] Pudlák and Tůma (1980) ⟨improved proof⟩, Pudlák and Tůma (1977) ⟨proof⟩, Whitman (1946) ⟨conjecture⟩, Saliĭ (1988) page vii ⟨list of lattice theory breakthroughs⟩

lattices on 5 element set as lattices of partitions



A.7 Relationships between set structures

Proposition A.12. [26]

PRP: $\left\{\begin{array}{l} \boldsymbol{R} \textit{ is a } \textbf{\textit{ring of sets}} \\ \textit{on a set } X \end{array}\right\} \implies \left\{\begin{array}{l} \boldsymbol{R} \cup X \textit{ is an } \textbf{\textit{algebra of sets}} \\ \textit{on } X \end{array}\right\}$

Theorem A.10. *Let X be a set.*

THM: $\left\{\begin{array}{l} \boldsymbol{A} \textit{ is an } \textbf{\textit{algebra of sets}} \\ \textit{on } X \end{array}\right\} \iff \left\{\begin{array}{lll} 1. & \boldsymbol{A} \textit{ is a } \textbf{\textit{topology}} \textit{ on } X & \textit{and} \\ 2. & \boldsymbol{A} \textit{ is a } \textbf{\textit{ring of sets}} \textit{ on } X & \end{array}\right\}$

✎PROOF:

$$\begin{aligned} \boldsymbol{A} \text{ is an algebra of sets on } X &\implies \boldsymbol{A} \text{ is closed under } \cup, \cap, \mathsf{c}, \setminus, \varnothing, X && \text{by Theorem A.7 page 273} \\ &\implies \left\{\begin{array}{ll} 1. & \boldsymbol{A} \text{ is a topology on } X \\ & \text{AND} \\ 2. & \boldsymbol{A} \text{ is a ring of sets on } X \end{array}\right\} \end{aligned}$$

$$\begin{aligned} \left\{\begin{array}{ll} 1. & \boldsymbol{A} \text{ is a topology on } X \\ & \text{AND} \\ 2. & \boldsymbol{A} \text{ is a ring of sets on } X \end{array}\right\} &\implies \boldsymbol{A} \text{ is closed under } \mathsf{c} \text{ and } \cap && \text{by Theorem A.7 page 273} \\ &\implies \boldsymbol{A} \text{ is a ring of sets} \end{aligned}$$

✏

Corollary A.1. *Let X be a set and 2^X the power set of X.*

COR: $$\left\{\boldsymbol{A} \subseteq 2^X \,|\, \boldsymbol{A} \textit{ is an algebra of sets on } X\right\} = \left\{\boldsymbol{T} \subseteq 2^X \,|\, \boldsymbol{T} \textit{ is a topology on } X\right\} \cap \left\{\boldsymbol{R} \subseteq 2^X \,|\, \boldsymbol{R} \textit{ is a ring of sets on } X\right\}$$

[26] Berezansky et al. (1996) page 4, Halmos (1950) page 21

✎PROOF:

$$
\begin{aligned}
&\{\boldsymbol{T} | \boldsymbol{T} \text{ is a topology}\} \cap \{\boldsymbol{R} | \boldsymbol{R} \text{ is a ring of sets}\} && \\
&= \{\boldsymbol{Y} | \boldsymbol{Y} \text{ is a topology AND a ring of sets}\} && \text{by Definition A.5 page 260} \\
&= \{\boldsymbol{Y} | \boldsymbol{Y} \text{ is an algebra of sets}\} && \text{by Theorem A.10 page 282} \\
&= \{\boldsymbol{A} | \boldsymbol{A} \text{ is an algebra of sets}\} && \text{by change of variable}
\end{aligned}
$$

✏

Example A.14. Note that the *intersection* of the lattice of topologies on $\{x, y, z\}$ (Figure 1.1 page 11) *and* the lattice of rings of sets on $\{x, y, z\}$ (Figure A.6 page 280) is *equal to* the lattice of algebras of sets on $\{x, y, z\}$ (Figure A.7 page 280).

Proposition A.13. *Let $\mathscr{R}(X)$ be the set of* RINGS OF SETS *(Definition A.10 page 267) and $\mathscr{A}(X)$ the set of* ALGEBRAS OF SETS *(Definition A.9 page 266) on a set X.*

PRP

$$
\left\{\begin{array}{ll} 1. & \boldsymbol{R} \text{ is a } \textbf{\textit{ring of sets}} \quad \text{and} \\ 2. & X \in \boldsymbol{R} \end{array}\right\} \iff \left\{ \boldsymbol{R} \text{ is an } \textbf{\textit{algebra of sets}} \right\}
$$

✎PROOF:

$$
\begin{aligned}
A^{c} &= X \backslash A && \text{by Theorem A.1 page 261} \\
A \cap B &= A \backslash (A \backslash B) && \text{by Theorem A.1 page 261}
\end{aligned}
$$

Therefore, $(\boldsymbol{R} \cup X)$ is closed under c and $\cap$, and thus by the definition of algebras of sets (Definition A.9 page 266), $(\boldsymbol{R} \cup X)$ is an algebra of sets. ✏

Definition A.14. [27]

DEF *The **Borel set** $\mathrm{B}(X, \boldsymbol{T})$ generated by the topological space $(X, \boldsymbol{T})$ is the σ-algebra generated by the topology $\boldsymbol{T}$.*

Example A.15. Suppose we have a dice with the standard six possible outcomes X. Suppose also we construct the following topology $\boldsymbol{T}$ on X, and this in turn generates the following Borel set (σ-algebra) B on X:

EX

$$
\begin{aligned}
X &= \{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\} \\
\boldsymbol{T} &= \left\{ \underbrace{\{\ \}}_{\varnothing}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}}_{\Omega}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\}}_{\text{first four}}, \underbrace{\{\text{⚃},\text{⚄},\text{⚅}\}}_{\text{last three}}, \underbrace{\{\text{⚃}\}}_{\{1234\}\cap\{456\}}, \right\} \\
B &= \left\{ \underbrace{\{\ \}}_{\varnothing}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}}_{\Omega}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\}}_{\text{first four}}, \underbrace{\{\text{⚃},\text{⚄},\text{⚅}\}}_{\text{last three}}, \underbrace{\{\text{⚃}\}}_{\{1234\}\cap\{456\}}, \right. \\
&\qquad \left. \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚄},\text{⚅}\}}_{\{4\}}, \underbrace{\{\text{⚄},\text{⚅}\}}_{(\{4\})\cap\{456\}}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂}\}}_{\{1234\}\cap\{4\}}, \right\}
\end{aligned}
$$

Example A.16. There are a total of 29 *topologies* on the set $X \triangleq \{x, y, z\}$ (Theorem 1.2 page 6); and of these, 5 are also *algebras of sets*, 24 are not. Figure A.9 (page 284) illustrates the 24 topologies on the set $\{x, y, z\}$ that are *not* algebras of sets and the 5 algebras of sets that they generate.

[27] Aliprantis and Burkinshaw (1998) page 97

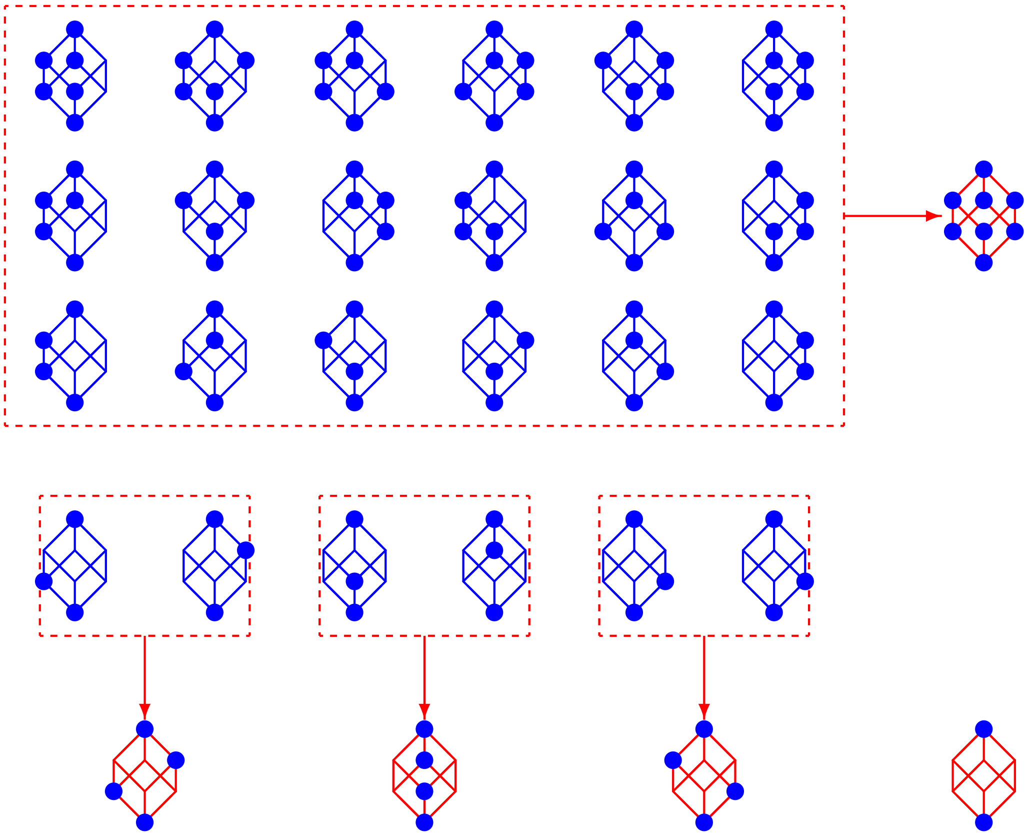


Figure A.9: Algebras of sets generated by topologies on the set $X \triangleq \{x, y, z\}$ (see Example A.16 page 283)

APPENDIX B

ORDER

Equivalence relations require *symmetry* ($x \asymp y \iff y \asymp x$). However another very important type of relation, the *order relation*, actually requires *anti-symmetry*. This chapter presents some useful structures regarding order relations. Ordering relations on a set allow us to *compare* some pairs of elements in a set and determine whether or not one element is *less than* another. In this case, we say that those two elements are *comparable*; otherwise, they are *incomparable*. A set together with an order relation is called an *ordered set*, a *partially ordered set*, or a *poset* (Definition B.2 page 286).

B.1 Preordered sets

Definition B.1. [1] *Let X be a set.*

DEF

*A relation $\sqsubseteq$ is a **preorder relation** on X if*

1. $x \sqsubseteq x$ $\quad \forall x \in X \quad$ (REFLEXIVE) *and*
2. $x \sqsubseteq y$ *and* $y \sqsubseteq z \implies x \sqsubseteq z$ $\quad \forall x,y,z \in X \quad$ (TRANSITIVE)

*A **preordered set** is the pair* $(X, \sqsubseteq)$.

Example B.1. [2]

EX

$\sqsubseteq$ is a *preorder relation* on the set of *positive integers* $\mathbb{N}$ if

$$n \sqsubseteq m \iff (p \text{ is a prime factor of } n \implies p \text{ is a prime factor of } m)$$

[1] Schröder (2003) page 115, Brown and Watson (1991) page 317
[2] Shen and Vereshchagin (2002) page 43

B.2 Order relations

Definition B.2. [3] *Let X be a set. Let 2^{XX} be the set of all relations on X.*

DEF

*A relation $\le$ is an **order relation** in 2^{XX} if*

1. $x \le x \quad \forall x \in X$ (REFLEXIVE) *and* — preorder
2. $x \le y$ *and* $y \le z \implies x \le z \quad \forall x,y,z \in X$ (TRANSITIVE) *and* — preorder
3. $x \le y$ *and* $y \le x \implies x = y \quad \forall x,y \in X$ (ANTI-SYMMETRIC)

*An **ordered set** is the pair $(X, \le)$. The set X is called the **base set** of $(X, \le)$. If $x \le y$ or $y \le x$, then elements x and y are said to be **comparable**, denoted $x \sim y$. Otherwise they are **incomparable**, denoted $x \| y$. The relation $\lneq$ is the relation $\le \setminus =$ ("less than but not equal to"), where $\setminus$ is the* SET DIFFERENCE *operator, and $=$ is the equality relation. An order relation is also called a **partial order relation**. An ordered set is also called a **partially ordered set** or **poset**.*

The familiar relations $\ge$, $<$, and $>$ (next) can be defined in terms of the order relation $\le$ (Definition B.2—previous).

Definition B.3. [4] *Let $(X, \le)$ be an ordered set.*

DEF

The relations $\ge$, $<$, $> \in 2^{XX}$ are defined as follows:

$$x \ge y \overset{\text{def}}{\iff} y \le x \qquad \forall x,y \in X$$

$$x \lneq y \overset{\text{def}}{\iff} x \le y \text{ and } x \ne y \qquad \forall x,y \in X$$

$$x \gneq y \overset{\text{def}}{\iff} x \ge y \text{ and } x \ne y \qquad \forall x,y \in X$$

*The relation $\ge$ is called the **dual** of $\le$.*

Theorem B.1. [5] *Let X be a set.*

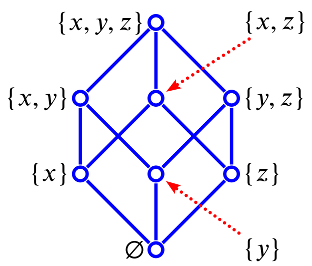
THM

$(X, \le)$ *is an ordered set* $\iff$ $(X, \ge)$ *is an ordered set*

Example B.2.

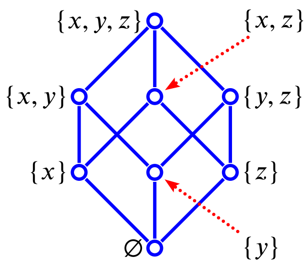
EX

order relation		dual order relation	
$\le$	(integer less than or equal to)	$\ge$	(integer greater than or equal to)
$\subseteq$	(subset)	$\supseteq$	(super set)
$\vert$	(divides)		(divided by)
$\implies$	(implies)	$\impliedby$	(implied by)



Example B.3. The Hasse diagram to the left illustrates the ordered set $\left(2^{\{x,y,z\}}, \subseteq\right)$ and the Hasse diagram to the right illustrates its dual $\left(2^{\{x,y,z\}}, \supseteq\right)$.

[3] MacLane and Birkhoff (1999) page 470, Beran (1985) page 1, Korselt (1894) page 156 ⟨I, II, (1)⟩, Dedekind (1900) page 373 ⟨I–III⟩
[4] Peirce (1880) page 2
[5] Grätzer (1998) page 3



B.3 Linearly ordered sets

In an ordered set we can say that some element is less than or equal to some other element. That is, we can say that these two elements are *comparable*—we can *compare* them to see which one is lesser or equal to the other. But it is very possible that there are two elements that are not comparable, or *incomparable*. That is, we cannot say that one element is less than the other—it is simply not possible to compare them because their ordered pair is not an element of the order relation.

For example, in the ordered set $(2^{\{x,y,z\}}, \subseteq)$ of Example B.9, we can say that $\{x\} \subseteq \{x, z\}$ (we can compare these two sets with respect to the order relation $\subseteq$), but we cannot say $\{y\} \subseteq \{x, z\}$, nor can we say $\{x, z\} \subseteq \{y\}$. Rather, these two elements $\{y\}$ and $\{x, z\}$ are simply *incomparable*.

However, there are some ordered sets in which every element is comparable with every other element; and in this special case we say that this ordered set is a *totally ordered* set or is *linearly ordered* (next definition).

Definition B.4. [6]

DEF

*A relation $\le$ is a **linear order relation** on X if*

1. $\le$ *is an* ORDER RELATION (*Definition B.2 page 286*) *and*
2. $x \le y$ *or* $y \le x \quad \forall x,y \in X$ (COMPARABLE).

*A **linearly ordered set** is the pair $(X, \le)$.*
*A linearly ordered set is also called a **totally ordered set**, a **fully ordered set**, and a **chain**.*

Definition B.5 (poset product). [7]

DEF

*The **product** $P \times Q$ of ordered pairs $P \triangleq (X, \precsim)$ and $Q \triangleq (Y, \trianglelefteq)$ is the ordered pair $(X \times Y, \le)$ where*

$$(x_1, y_1) \le (x_2, y_2) \quad \overset{\text{def}}{\Longleftrightarrow} \quad x_1 \precsim x_2 \text{ and } y_1 \trianglelefteq y_2 \qquad \forall x_1, x_2 \in X;\ y_1, y_2 \in Y$$

B.4 Representation

Definition B.6. [8]

DEF

*y **covers** x in the ordered set $(X, \le)$ if*

1. $x \le y$ (*y is greater than x*) *and*
2. $(x \le z \le y) \implies (z = x \text{ or } z = y)$ (*there is no element between x and y*).

The case in which y covers x is denoted

$x \prec y$.

[6] MacLane and Birkhoff (1999) page 470, Ore (1935) page 410
[7] Birkhoff (1948) page 7, MacLane and Birkhoff (1967) page 489
[8] Birkhoff (1933a) page 445

Example B.4. Let $(\{x, y, z\}, \leq)$ be an ordered set with cover relation $\lessdot$.

EX
$$\{x < y < z\} \implies \left\{\begin{array}{lll} y & \text{covers} & x \\ z & \text{covers} & y \\ z & \text{does **not** cover} & x \end{array}\right\}$$

An ordered set can be represented in four ways:

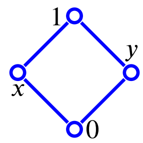
1. Hasse diagram
2. tables
3. set of ordered pairs of order relations
4. set of ordered pairs of cover relations

Definition B.7. *Let $(X, \leq)$ be an ordered pair.*

DEF
A diagram is a ***Hasse diagram*** *of $(X, \leq)$ if it satisfies the following criteria:*
- *Each element in X is represented by a dot or small circle.*
- *For each $x, y \in X$, if $x \lessdot y$, then y appears at a higher position than x and a line connects x and y.*

Example B.5. Here are three ways of representing the ordered set $\left(2^{\{x,y\}}, \subseteq\right)$;

1. **Hasse diagrams**: If two elements are comparable, then the lesser of the two is drawn lower on the page than the other with a line connecting them.

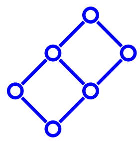


2. Sets of ordered pairs specifying *order relations* (Definition B.2 page 286):

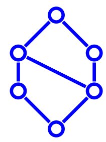
$$\subseteq = \left\{\begin{array}{ccccc} (\emptyset, \emptyset), & (\{x\}, \{x\}), & (\{y\}, \{y\}), & & (\{x, y\}, \{x, y\}), \\ (\emptyset, \{x\}), & (\emptyset, \{y\}), & (\emptyset, \{x, y\}), & (\{x\}, \{x, y\}), & (\{y\}, \{x, y\}) \end{array}\right\}$$

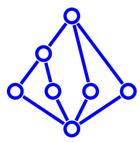
3. Sets of ordered pairs specifying *covering relations*:

$$\lessdot = \left\{ (\emptyset, \{x\}), \quad (\emptyset, \{y\}), \quad (\{x\}, \{x, y\}), (\{y\}, \{x, y\}) \right\}$$

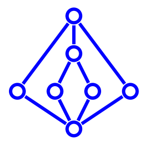


Example B.6. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.

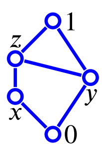




Example B.7. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.

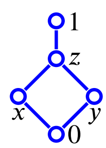


Example B.8. The Hasse diagrams to the left and right represent *equivalent* ordered sets.



In particular, the line extending from 1 to y in the diagram to the left is redundant because other lines already indicate that $z \leq 1$ and $y \leq z$; and thus by the *transitive* property (Definition B.2 page 286), these two relations imply $1 \leq y$. A more concise explanation is that both have the same convering relation:

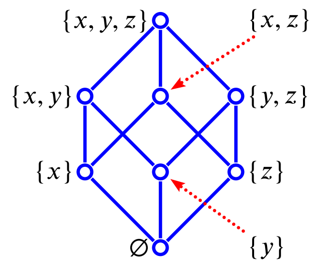
$$\lessdot = \{(z, 1), (x, z), (0, x), (y, z), (0, y)\}$$



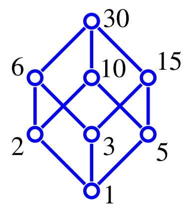
B.5 Examples

Examples of order relations include the following:

Example B.9 (Set inclusion order relation). [9]Let X be a set, 2^X the power set of X, and $\subseteq$ the set inclusion relation. Then, $\subseteq$ is an *order relation* on the set 2^X and the pair $(2^X, \subseteq)$ is an ordered set. The ordered set $(2^{\{x,y,z\}}, \subseteq)$ is illustrated to the right by its *Hasse diagram*.



Example B.10 (Integer divides order relation). [10]Let $|$ be the "divides" relation on the set $\mathbb{N}$ of positive integers such that $n|m$ represents m divides n. Then $|$ is an *order relation* on $\mathbb{N}$ and the pair $(\mathbb{N}, |)$ is an *ordered set*. The ordered set $(\{n \in \mathbb{N} | n|2 \text{ or } n|3 \text{ or } n|5\}, |)$ is illustrated by a *Hasse diagram* to the right.



Example B.11 (Operator order relation). [11] Let $\boldsymbol{X}$ be an inner-product space. We can define the order relation $\lessapprox$ on the linear operators $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3 \ldots \in X^X$ as follows:

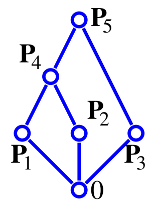
$$\mathbf{L}_1 \lessapprox \mathbf{L}_2 \quad \overset{\text{def}}{\iff} \quad \langle \mathbf{L}_2 \boldsymbol{x} - \mathbf{L}_1 \boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X}$$

Example B.12 (Projection operator order relation). [12]Let $(\!(\boldsymbol{V}_n)\!)$ be a sequence of subspaces in a Hilbert space $\boldsymbol{X}$. We can define a projection operator $\mathbf{P}_n$ for every subspace $\boldsymbol{V}_n \subseteq \boldsymbol{X}$ in a subspace lattice such that

$$\boldsymbol{V}_n = \mathbf{P}_n \boldsymbol{X} \qquad \forall n \in \mathbb{Z}.$$

Each projection operator $\mathbf{P}_n$ in the lattice "projects" the range space $\boldsymbol{X}$ onto a subspace $\boldsymbol{V}_n$. We can define an order relation on the projection operators as follows:

$$\mathbf{P}_1 \leq \mathbf{P}_2 \quad \overset{\text{def}}{\iff} \quad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1$$



[9] Menini and Oystaeyen (2004) pages 56–57
[10] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩
[11] Michel and Herget (1993) page 429, Pedersen (2000) page 87
[12] Isham (1999) pages 21–22, Dunford and Schwartz (1957) page 481, Svozil (1994) page 72

Example B.13 (Integer order relation). Let $\le$ be the standard order relation on the set of integers $\mathbb{Z}$. Then the ordered pair $(\mathbb{Z}, \le)$ is a totally ordered set. The totally ordered set $(\{1,2,3,4\}, \le)$ is illustrated to the right. Other familiar examples of totally ordered sets include the pair $(\mathbb{Q}, \le)$ (where $\mathbb{Q}$ is the set of rational numbers) and $(\mathbb{R}, \le)$ (where $\mathbb{R}$ is the set of real numbers).

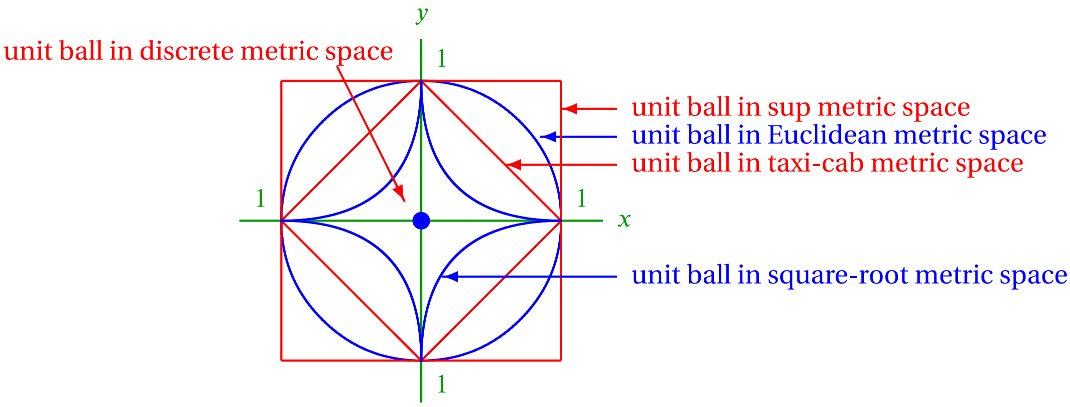
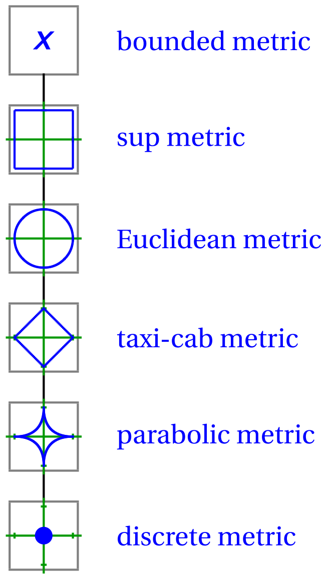


Figure B.1: Balls on the set $\mathbb{R}^2$ using different metrics

Example B.14 (Metric order relation). [13] Let d_n be a metric on the set X and B_n be the unit ball centered at "0" in the metric space (X, d_n). Define an order relation $\le$ on the set of metric spaces $\{(X, \mathsf{d}_n) \,|\, n = 1, 2, \ldots\}$ such that

$$(X, \mathsf{d}_n) \le (X, \mathsf{d}_m) \qquad \Longleftrightarrow \qquad \mathsf{B}_n \subseteq \mathsf{B}_m.$$

The the tuple $(\{(X, \mathsf{d}_n) \,|\, n = 1, 2, \ldots\}, \le)$ is an ordered set. The ordered set of several common metric spaces is a *totally ordered* set, as illustrated to the right and with associated unit balls illustrated in Figure B.1 (page 290).



Example B.15 (Coordinatewise order relation). [14] Let $(X, \le)$ be an ordered set. Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

EX The **coordinatewise order relation** $\lesssim$ on the Cartesian product X^n is defined for all $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \lesssim \boldsymbol{y} \qquad \stackrel{\text{def}}{\Longleftrightarrow} \qquad \{x_1 \le y_1 \text{ and } x_2 \le y_2 \text{ and } \ldots \text{ and } x_n \le y_n\}$$

Example B.16 (Lexicographical order relation). [15] Let $(X, \le)$ be an ordered set. Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

[13] Michel and Herget (1993) page 354, Giles (1987) page 29
[14] Shen and Vereshchagin (2002) page 43
[15] Shen and Vereshchagin (2002) page 44, Halmos (1960) page 58, Hausdorff (1937) page 54

EX

The **lexicographical order relation** $\precsim$ on the Cartesian product X^n is defined forall $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \precsim \boldsymbol{y} \quad \overset{\text{def}}{\iff} \quad \left\{ \begin{array}{llll} (& x_1 < y_1 & &) \text{ or} \\ (& x_2 < y_2 & \text{and} \quad x_1 = y_1 &) \text{ or} \\ (& x_3 < y_3 & \text{and} \quad (x_1, x_2) = (y_1, y_2) &) \text{ or} \\ (& \dots & \dots \quad \dots &) \text{ or} \\ (& x_{n-1} < y_{n-1} & \text{and} \quad (x_1, x_2, \dots, x_{n-2}) = (y_1, y_2, \dots, y_{n-2}) &) \text{ or} \\ (& x_n \le y_n & \text{and} \quad (x_1, x_2, \dots, x_{n-1}) = (y_1, y_2, \dots, y_{n-1}) &) \end{array} \right\}$$

The lexicographical order relation is also called the **dictionary order relation** or **alphabetic order relation**.

Definition B.8.

DEF

*An ordered set is **labeled** if the labels on the elements are significant.*
*An ordered set is **unlabeled** if the labels on the elements are not significant.*

Proposition B.1. [16] *Let X_n be a finite set with order $n = |X_n|$. Let P_n be the number of labeled ordered sets on X_n and p_n the number of unlabeled ordered sets.*

PRP

n	0	1	2	3	4	5	6	7	8	9
P_n	1	1	3	19	219	4231	130,023	6,129,859	431,723,379	44,511,042,511
p_n	1	1	2	5	16	63	318	2045	16,999	183,231

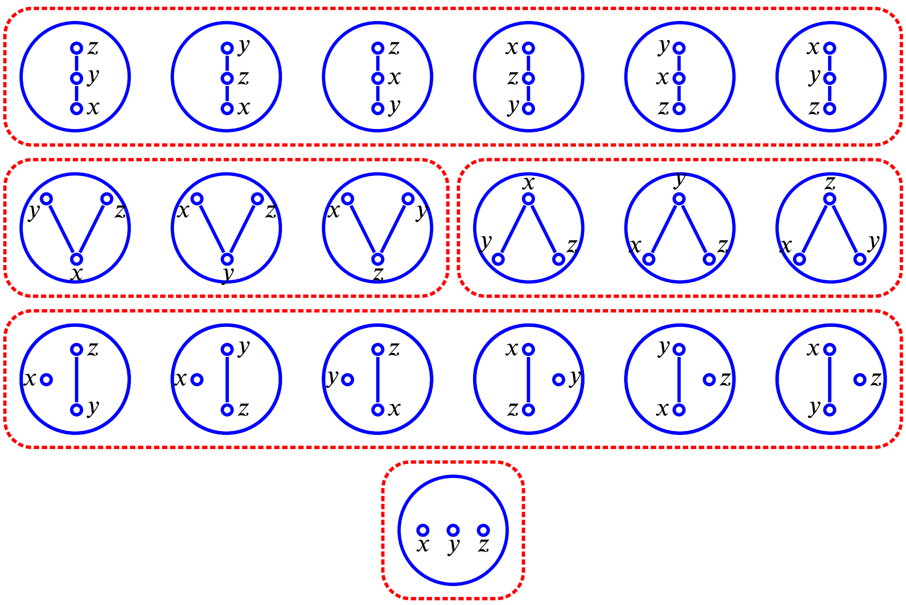


Figure B.2: All possible orderings of the set $\{x, y, z\}$ (Example B.17 page 291).

Example B.17. Proposition B.1 (page 291) indicates that there are exactly 19 labeled order relations on the set $\{x, y, z\}$ and 5 unlabeled order relations.

EX

The 19 labeled order relations on $\{x, y, z,\}$ are represented here using three methods:

1. Hasse diagrams: Figure B.2 page 291
2. order relations: Table B.2 page 292
3. covering relations: Table B.3 page 292

In each of these three methods, the 19 *labeled* order relations are arranged into 5 groups, each group representing one of the 5 *unlabeled* order relations.

[16] Sloane (2014) ⟨http://oeis.org/A001035⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Comtet (1974) page 60, Brinkmann and McKay (2002)

labeled order relations on $\{x, y, z\}$		
$\leq_1$	$=$	$\{ (x, x), (y, y), (z, z) \}$
$\leq_2$	$=$	$\{ (x, x), (y, y), (z, z), (y, z) \}$
$\leq_3$	$=$	$\{ (x, x), (y, y), (z, z), (z, y) \}$
$\leq_4$	$=$	$\{ (x, x), (y, y), (z, z), (x, z) \}$
$\leq_5$	$=$	$\{ (x, x), (y, y), (z, z), (z, x) \}$
$\leq_6$	$=$	$\{ (x, x), (y, y), (z, z), (x, y) \}$
$\leq_7$	$=$	$\{ (x, x), (y, y), (z, z), (y, x) \}$
$\leq_8$	$=$	$\{ (x, x), (y, y), (z, z), (x, y), (x, z) \}$
$\leq_9$	$=$	$\{ (x, x), (y, y), (z, z), (x, y), (y, z) \}$
$\leq_{10}$	$=$	$\{ (x, x), (y, y), (z, z), (z, x), (z, y) \}$
$\leq_{11}$	$=$	$\{ (x, x), (y, y), (z, z), (y, x), (z, x) \}$
$\leq_{12}$	$=$	$\{ (x, x), (y, y), (z, z), (x, y), (z, y) \}$
$\leq_{13}$	$=$	$\{ (x, x), (y, y), (z, z), (x, z), (y, z) \}$
$\leq_{14}$	$=$	$\{ (x, x), (y, y), (z, z), (x, y), (y, z), (x, z) \}$
$\leq_{15}$	$=$	$\{ (x, x), (y, y), (z, z), (x, z), (x, y), (z, y) \}$
$\leq_{16}$	$=$	$\{ (x, x), (y, y), (z, z), (y, x), (y, z), (x, z) \}$
$\leq_{17}$	$=$	$\{ (x, x), (y, y), (z, z), (y, z), (y, x), (z, x) \}$
$\leq_{18}$	$=$	$\{ (x, x), (y, y), (z, z), (z, x), (z, y), (x, y) \}$
$\leq_{19}$	$=$	$\{ (x, x), (y, y), (z, z), (z, y), (z, x), (y, x) \}$

Table B.2: labeled order relations on $\{x, y, z\}$

labeled cover relations on $\{x, y, z\}$					
$\prec_1$	$=$	$\varnothing$	$\prec_{11}$	$=$	$\{ (y, x), (z, x) \}$
$\prec_2$	$=$	$\{ (y, z) \}$	$\prec_{12}$	$=$	$\{ (x, y), (z, y) \}$
$\prec_3$	$=$	$\{ (z, y) \}$	$\prec_{13}$	$=$	$\{ (x, z), (y, z) \}$
$\prec_4$	$=$	$\{ (x, z) \}$	$\prec_{14}$	$=$	$\{ (x, y), (y, z) \}$
$\prec_5$	$=$	$\{ (z, x) \}$	$\prec_{15}$	$=$	$\{ (x, z), (x, y) \}$
$\prec_6$	$=$	$\{ (x, y) \}$	$\prec_{16}$	$=$	$\{ (y, x), (y, z) \}$
$\prec_7$	$=$	$\{ (y, x) \}$	$\prec_{17}$	$=$	$\{ (y, z), (y, x) \}$
$\prec_8$	$=$	$\{ (x, y), (x, z) \}$	$\prec_{18}$	$=$	$\{ (z, x), (z, y) \}$
$\prec_9$	$=$	$\{ (x, y), (y, z) \}$	$\prec_{19}$	$=$	$\{ (z, y), (z, x) \}$
$\prec_{10}$	$=$	$\{ (z, x), (z, y) \}$			

Table B.3: labeled cover relations on $\{x, y, z\}$

B.6 Functions on ordered sets

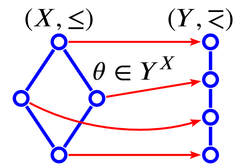
Definition B.9. [17] *Let $(X, \leq)$ and $(Y, \precsim)$ be ordered sets.*

DEF

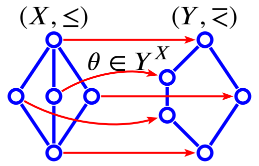
*A function $\theta \in Y^X$ is **order preserving** with respect to $\leq$ and $\precsim$ if*

$$x \leq y \implies \theta(x) \precsim \theta(y) \qquad \forall x, y \in X.$$

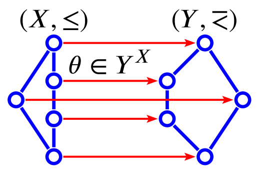
Example B.18. [18] In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} is *not* order preserving. This example also illustrates the fact that that order preserving does not imply *isomorphic*.



Example B.19. In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} is *not* order preserving. Like Example B.18 (page 293), this example also illustrates the fact that that order preserving does not imply *isomorphic*.



Example B.20. In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} *is also* order preserving. In this case, θ is an *isomorphism* and the ordered sets $(X, \leq)$ and $(Y, \precsim)$ are *isomorphic*.



Example B.21. [19]

EX

The function $\mathsf{f}(x) \triangleq \dfrac{x}{1 - x^2}$ in $\mathbb{R}^{(-1:1)}$ is *bijective* and *order preserving*.

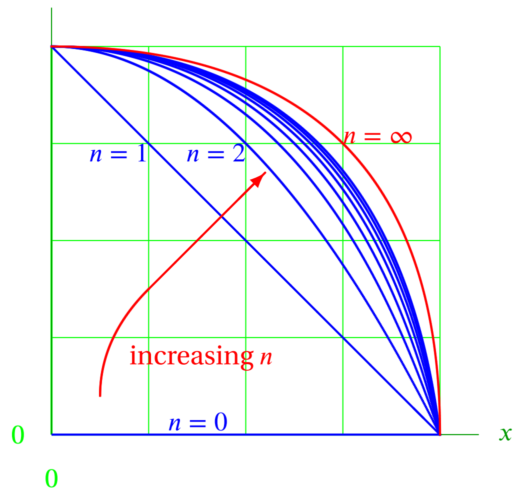
Theorem B.2 (Pointwise ordering relation). [20] *Let X be a set, $(Y, \leq)$ an ordered set, and $\mathsf{f}, \mathsf{g} \in Y^X$.*

THM

$$\mathsf{f}(x) \leq \mathsf{g}(x) \forall x \in X \implies \left(Y^X, \precsim\right) \text{ is an ordered set.}$$

In this case we say f is "dominated by" g in X, or we say g "dominates" f in X.

Example B.22 (Pointwise ordering relation). [21] Let $\mathsf{f} \precsim \mathsf{g}$ represent that $\mathsf{f}(x) \leq \mathsf{g}(x)$ for all $0 \leq x \leq 1$ (we say f is "dominated by" g in the region $[0, 1]$, or we say g "dominates" f in the region $[0, 1]$). The pair $\left(\left\{\mathsf{f}_n(x) = 1 - x^n | n \in \mathbb{N}\right\}, \precsim\right)$ is a totally ordered set.



[17] Burris and Sankappanavar (2000) page 10
[18] Burris and Sankappanavar (2000) page 10
[19] Munkres (2000) page 25 ⟨Example 1§3.9⟩
[20] Shen and Vereshchagin (2002) page 43, Giles (2000) page 252
[21] Shen and Vereshchagin (2002) page 43, Giles (2000) page 252, Aliprantis and Burkinshaw (2006) page 2

B.7 Decomposition

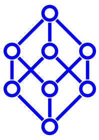
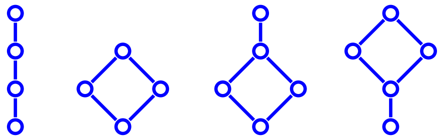
B.7.1 Subposets

Definition B.10. [22]

DEF *The tupple $(Y, \eqslantless)$ is a **subposet** of the ordered set $(X, \leq)$ if*

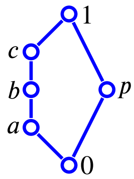
1. $Y \subseteq X$ *(Y is a subset of X)* *and*
2. $\eqslantless = \leq \cap Y^2$ *($\eqslantless$ is the relation $\leq$ restricted to $Y \times Y$)*

Example B.23.

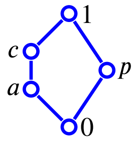
Subposets of  include 

Example B.24. Let

$$(X, \leq) \triangleq \Big(\{0, a, b, c, p, 1\}, \quad \Big\{ (0,0), (a,a), (b,b), (c,c), (p,p), (1,1), (0,a), (0,b), (0,c), (0,p), (0,1), (a,b), (a,c), (a,1), (p,1), (b,c), (b,1), (c,1), (p,1) \Big\}\Big)$$



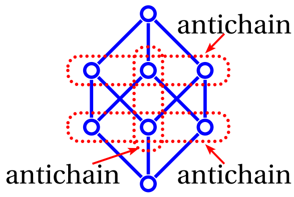
$$(Y, \eqslantless) \triangleq \Big(\{0, a, c, p, 1\}, \quad \Big\{ (0,0), (a,a), (c,c), (p,p), (1,1), (0,a), (0,c), (0,p), (0,1), (a,c), (a,1), (p,1), (c,1), (p,1) \Big\}\Big).$$



Then $(Y, \eqslantless)$ is a subposet of $(X, \leq)$ because $Y \subseteq X$ and $\eqslantless = (\leq \cap Y^2)$.

A *chain* is an ordered set in which every pair of elements is *comparable* (Definition B.4 page 287). An *antichain* is just the opposite—it is an ordered set in which *no* pair of elements is comparable (next definition).

Definition B.11. [23]



DEF *The subposet $(A, \leq)$ in the ordered set $(X, \leq)$ is an **antichain** if*

$a \| b \qquad \forall a, b \in A$

(all elements in A are INCOMPARABLE*).*

Definition B.12. [24]

DEF
- *The **length** of a chain $(C, \leq)$ equals $|C| - 1$.*
- *The **length** of a poset $(X, \leq)$ is the length of the longest chain in the ordered set.*
- *The **width** of a poset $(X, \leq)$ is number of elements in the largest antichain in the ordered set.*

Theorem B.3 (Dilworth's theorem). [25] *Let $(X, \leq)$ be an ordered set with width n.*

[22] Grätzer (2003) page 2
[23] Grätzer (2003) page 2
[24] Grätzer (2003) page 2, Birkhoff (1967) page 5
[25] Dilworth (1950a) page 161, Dilworth (1950b), Farley (1997) page 4

THM $\left\{\begin{array}{l}\text{WIDTH } n \textit{ of } (X, \le) \\ \textit{is } \text{FINITE}\end{array}\right\} \implies \left\{\begin{array}{ll}1. & \textit{there exists a } \text{PARTITION} \textit{ of } (X,\le) \textit{ into } n \textit{ chains} \quad \textit{and} \\ 2. & \textit{there does not exist any } \text{PARTITION} \\ & \textit{of } (X,\le) \textit{ into less than } n \textit{ chains}\end{array}\right\}$

B.7.2 Operations on posets

Definition B.13. [26] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **direct sum** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} + \boldsymbol{Q} \triangleq (X \cup Y, \le)$$

where $x \le y$ if

1. $x, y \in X$ and $x \precsim y$ or
2. $x, y \in Y$ and $x \trianglelefteq y$

*The direct sum operation is also called the **disjoint union**. The notation $n\boldsymbol{P}$ is defined as*

$$n\boldsymbol{P} \triangleq \underbrace{\boldsymbol{P} + \boldsymbol{P} + \cdots + \boldsymbol{P}}_{n-1 \text{ “}+\text{” operations}}.$$

Definition B.14. [27] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **direct product** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} \times \boldsymbol{Q} \triangleq (X \times Y, \le)$$

where $(x_1, y_1) \le (x_2, y_2)$ if $x_1 \precsim x_2$ and $y_1 \precsim y_2$.

*The direct product operation is also called the **cartesian product**. The order relation $\le$ is called a **coordinate wise order relation**. The notation $\boldsymbol{P}^n$ is defined as*

$$\boldsymbol{P}^n \triangleq \underbrace{\boldsymbol{P} \times \boldsymbol{P} \times \cdots \times \boldsymbol{P}}_{n-1 \text{ “}\times\text{” operations}}.$$

Definition B.15. [28] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **ordinal sum** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} \oplus \boldsymbol{Q} \triangleq (X \cup Y, \le)$$

where $x \le y$ if

1. $x, y \in X$ and $x \precsim y$ or
2. $x, y \in Y$ and $x \trianglelefteq y$ or
3. $x \in X$ and $y \in Y$.

Definition B.16. [29] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **ordinal product** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} \otimes \boldsymbol{Q} \triangleq (X \times Y, \le)$$

where $(x_1, y_1) \le (x_2, y_2)$ if

1. $x_1 \ne x_2$ and $x_1 \precsim x_2$ or
2. $x_1 = x_2$ and $y_1 \trianglelefteq y_2$

*The order relation $\le$ is called a **lexicographical** order relation, **dictionary order relation**, or **alphabetic order relation**.*

[26] Stanley (1997) page 100
[27] Stanley (1997) pages 100–101, Shen and Vereshchagin (2002) page 43
[28] Stanley (1997) page 100
[29] Stanley (1997) page 101, Shen and Vereshchagin (2002) page 44, Halmos (1960) page 58, Hausdorff (1937) page 54

Definition B.17. [30] *Let* $\mathbf{P} \triangleq (X, \leq)$ *be an ordered set. Let* $\geq$ *be the dual order relation of* $\leq$.

*The **dual** of* $\mathbf{P}$ *is defined as*

$$\mathbf{P}^* \triangleq (X, \geq)$$

Definition B.18. [31] *Let* X *and* Y *be disjoint sets. Let* $\mathbf{P} \triangleq (X, \precsim)$ *and* $\mathbf{Q} \triangleq (Y, \trianglelefteq)$ *be ordered sets on* X *and* Y.

*The **ordinal product** of* $\mathbf{P}$ *and* $\mathbf{Q}$ *is defined as*

$$\mathbf{Q}^{\mathbf{P}} \triangleq \left(\left\{\mathsf{f} \in Y^X | \mathsf{f} \text{ is ORDER PRESERVING}\right\}, \leq\right)$$

where $\mathsf{f} \leq \mathsf{g}$ *iff* $\mathsf{f}(x) \leq \mathsf{g}(x) \quad \forall x \in X$.

The order relation $\leq$ *is called a **pointwise order relation*** (Example B.22 page 293).

Theorem B.4 (cardinal arithmetic). [32] *Let* $\mathbf{P} \triangleq (X, \leq)$ *be an ordered set.*

$$\begin{array}{llcll}
1. & \mathbf{P} + \mathbf{Q} & = & \mathbf{Q} + \mathbf{P} & \textit{commutative} \\
2. & \mathbf{P} \times \mathbf{Q} & = & \mathbf{Q} \times \mathbf{P} & \textit{commutative} \\
3. & (\mathbf{P} + \mathbf{Q}) + (\mathbb{R}, \leq) & = & \mathbf{P} + (\mathbf{Q} + (\mathbb{R}, \leq)) & \textit{associative} \\
4. & (\mathbf{P} \times \mathbf{Q}) \times (\mathbb{R}, \leq) & = & \mathbf{P} \times (\mathbf{Q} \times (\mathbb{R}, \leq)) & \textit{associative} \\
5. & \mathbf{P} \times (\mathbf{Q} + (\mathbb{R}, \leq)) & = & (\mathbf{P} \times \mathbf{Q}) + (\mathbf{P} \times (\mathbb{R}, \leq)) & \textit{distributive} \\
6. & (\mathbb{R}, \leq)^{\mathbf{P}+\mathbf{Q}} & = & (\mathbb{R}, \leq)^{\mathbf{P}} \times (\mathbb{R}, \leq)^{\mathbf{Q}} & \\
7. & \left(\mathbf{P}^{\mathbf{Q}}\right)^{(\mathbb{R}, \leq)} & = & \mathbf{P}^{\mathbf{Q} \times (\mathbb{R}, \leq)} &
\end{array}$$

B.7.3 Primitive subposets

Definition B.19.

The ordered set $\mathbf{L}_1$ *is defined as* $(\{x\}, \leq)$, *for some value* x.

The $\mathbf{L}_1$ *ordered set is illustrated by the Hasse diagram to the right.*

Definition B.20.

The ordered set $\mathbb{2}$ *is defined as* $\mathbb{2} \triangleq \mathbb{1}^2$.

The $\mathbb{2}$ *ordered set is illustrated by the Hasse diagram to the right.*

B.7.4 Decomposition examples

Example B.25. Figure B.3 (page 297) illustrates the four ordered set operations $+$, $\times$, $\oplus$, and $\otimes$.

Example B.26. [33]The ordered set $n\mathbb{1}$ is the *anti-chain* with n elements. The ordered set $4\mathbb{1}$ is illustrated to the right.

[30] Stanley (1997) page 101
[31] Stanley (1997) page 101
[32] Stanley (1997) page 102
[33] Stanley (1997) page 100

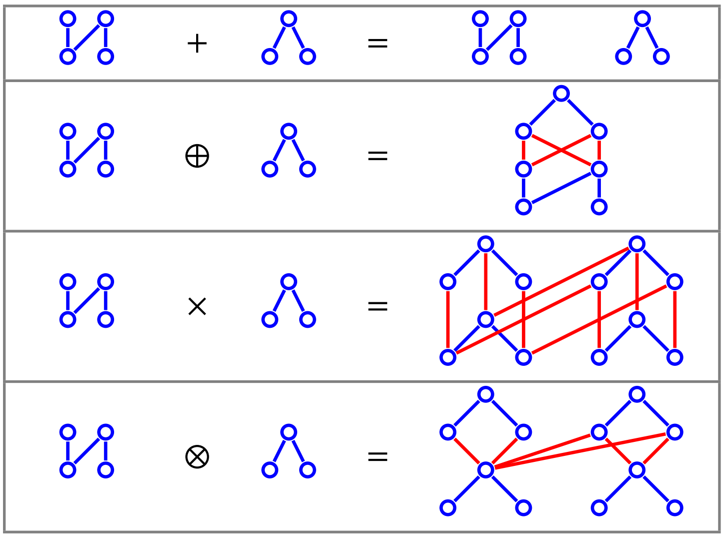
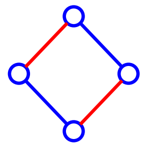
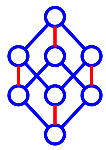


Figure B.3: Operations on ordered sets (Example B.25 page 296)

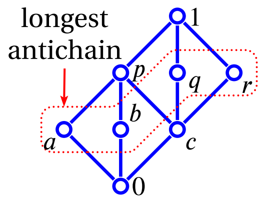
Example B.27. The ordered set $\mathbb{1}^n$ is the *chain* with n elements. The ordered set $\mathbb{1}^4$ is illustrated to the right.



Example B.28. The ordered set $\mathbb{2}^2$ is the 4 element *Boolean algebra* illustrated to the right.



Example B.29. The ordered set $\mathbb{2}^3$ is the 8 element *Boolean algebra* illustrated to the right.



Example B.30. [34]The longest *antichain* (Definition B.11 page 294) in the figure to the right has 4 elements giving this ordered set a *width* (Definition B.12 page 294) of 4. The longest chain also has 4 elements, giving the ordered set a *length* (Definition B.12 page 294) of 3. By *Dilworth's theorem* (Theorem B.3 page 294), the smallest *partition* consists of four *chains* (Definition B.4 page 287). One such *partition* is

$$\{\{0, a, p, 1\}, \{b\}, \{c, q\}, \{r\}\}.$$

B.8 Bounds on ordered sets

In an *ordered set* (Definition B.2 page 286), a pair of elements $\{x, y\}$ may not be *comparable*. Despite this, we may still be able to find elements that *are* comparable to both x and y and are "*greater*" than both of them. Such a greater element is called an *upper bound* of x and y. There may be many elements that are upper bounds of x and y. But if one of these upper bounds is comparable with and is smaller than all the other upper bounds, than this "smallest" of the "greater" elements is called the *least upper bound* (*lub*) of x and y, and is denoted $x \vee y$ (Definition B.21 page 298). Likewise,

[34] Farley (1997) page 4

we may also be able to find elements that are comparable to $\{x, y\}$ and are "*lesser*" than both of them. Such a lesser element is called a *lower bound* of x and y. If one of these lower bounds is comparable with and is larger than all the other lower bounds, than this "largest" of the "lesser" elements is called the *greatest lower bound* (*glb*) of $\{x, y\}$ and is denoted $x \wedge y$ (Definition B.22 page 298). If every pair of elements in an ordered set has both a least upper bound and a greatest lower bound in the ordered set, then that ordered set is a *lattice* (Definition C.3 page 301).

Definition B.21. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

*For any set $A \in 2^X$, c is an **upper bound** of A in $(X, \leq)$ if*
1. $x \leq c \quad \forall x \in A.$

*An element b is the **least upper bound**, or LUB, of A in $(X, \leq)$ if*
2. b *and* c *are* UPPER BOUNDS *of* $A \implies b \leq c.$

*The least upper bound of the set A is denoted $\bigvee A$. It is also called the **supremum** of A, which is denoted $\sup A$. The **join** $x \vee y$ of x and y is defined as $x \vee y \triangleq \bigvee \{x, y\}$.*

Definition B.22. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

*For any set $A \in 2^X$, p is a **lower bound** of A in $(X, \leq)$ if*
1. $p \leq x \quad \forall x \in A.$

*An element a is the **greatest lower bound**, or GLB, of A in $(X, \leq)$ if*
2. a *and* p *are* LOWER BOUNDS *of* $A \implies p \leq a.$

*The greatest lower bound of the set A is denoted $\bigwedge A$. It is also called the **infimum** of A, which is denoted $\inf A$. The **meet** $x \wedge y$ of x and y is defined as $x \wedge y \triangleq \bigwedge \{x, y\}$.*

Definition B.23 (least upper bound property). [35] *Let X be a set. Let $\sup A$ be the supremum (least upper bound) of a set A.*

DEF

*A set X satisfies the **least upper bound property** if*

$$\left.\begin{array}{lll} 1. & A \subseteq X & \text{and} \\ 2. & A \neq \varnothing & \text{and} \\ 3. & \exists b \in X \text{ such that } \forall a \in A,\ a \leq b & (A \text{ is bounded above in } X) \end{array}\right\} \implies \exists \sup A \in X$$

*A set X that satisfies the least upper bound property is also said to be **complete**.*

Proposition B.2. *Let $(X, \vee, \wedge; \leq)$ be an* ORDERED SET *(Definition B.2 page 286).*

PRP

$$x \leq y \iff \left\{\begin{array}{llll} 1. & x \wedge y & = & x \quad \text{and} \\ 2. & x \vee y & = & y \end{array}\right\} \quad \forall x,y \in X$$

Proposition B.3. *Let 2^X be the* POWER SET *of a set X.*

PRP

$$A \subseteq B \implies \left\{\begin{array}{llll} 1. & \bigvee A & \leq & \bigvee B \quad \text{and} \\ 2. & \bigwedge A & \leq & \bigwedge B \end{array}\right\} \quad \forall A,B \in 2^X$$

[35] Pugh (2002) page 13, Rudin (1976) page 4

APPENDIX C

LATTICES

C.1 Semi-lattices

Definition B.21 (page 298) defined the least upper bound $\vee$ of pairs of elements in terms of an ordering relation $\le$. However, the converse development is also possible— we can first define a binary operation $\otimes$ with a handful of "least upper bound like properties", and then define an ordering relation $\precsim$ in terms of $\otimes$ (Definition C.1 page 299). In fact, Theorem C.1 (page 299) shows that under Definition C.1, $(X, \precsim)$ *is* a partially ordered set and $\otimes$ is a least upper bound on that ordered set.

The same development is performed with regards to a greatest lower bound $\oslash$ with the result that $(X, \precsim)$ *is* a partially ordered set and $\oslash$ is a greatest lower bound on that ordered set (Theorem C.2 page 300).

Definition C.1. [1] *Let* $\otimes, \precsim \colon X^2 \to X$ *be binary operators on a set* X.

DEF: *The algebraic structure* $(X, \precsim, \otimes)$ *is a* ***join semilattice*** *if*

1.	$x \otimes x = x$	$\forall x \in X$	(IDEMPOTENT)	*and*
2.	$x \otimes y = y \otimes x$	$\forall x, y \in X$	(COMMUTATIVE)	*and*
3.	$(x \otimes y) \otimes z = x \otimes (y \otimes z)$	$\forall x, y, z \in X$	(ASSOCIATIVE).	

Definition C.2. [2] *Let* $\oslash, \precsim \colon X^2 \to X$ *be binary operators on a set* X.

DEF: *The algebraic structure* $(X, \precsim, \oslash)$ *is a* ***meet semilattice*** *if*

1.	$x \oslash x = x$	$\forall x \in X$	(IDEMPOTENT)	*and*
2.	$x \oslash y = y \oslash x$	$\forall x, y \in X$	(COMMUTATIVE)	*and*
3.	$(x \oslash y) \oslash z = x \oslash (y \oslash z)$	$\forall x, y, z \in X$	(ASSOCIATIVE).	

Theorem C.1. [3] *Let* $\otimes, \precsim \colon X^2 \to X$ *be binary operators over a set* X.

THM: $\left\{ (X, \precsim, \otimes) \text{ is a JOIN SEMILATTICE} \right\} \implies \left\{ \begin{array}{lll} 1. & (X, \precsim) \text{ is a PARTIALLY ORDERED SET} & \text{and} \\ 2. & x \otimes y \text{ is a LEAST UPPER BOUND of } x \text{ and } y & \forall x, y \in X. \end{array} \right\}$

✎PROOF: In order for $(X, \le)$ to be an ordered set, $\le$ must be, according to Definition B.2 (page 286), *reflexive*, *antisymmetric*, and *transitive*;

[1] MacLane and Birkhoff (1999) page 475, Birkhoff (1967) page 22
[2] MacLane and Birkhoff (1999) page 475
[3] MacLane and Birkhoff (1999) page 475

Proof that $\le$ is reflexive:

$$\begin{aligned}
& x = x \ovee x && \text{by idempotent hypothesis}\\
\iff & x \le x && \text{by definition of } \le\\
\implies & \le \text{ is reflexive}
\end{aligned}$$

Proof that $\le$ is antisymmetric:

$$\begin{aligned}
x \le y \text{ and } y \le x \iff & x \ovee y = y \text{ and } y \ovee x = x && \text{by definition of } \le\\
\implies & x \ovee y = y \text{ and } x \ovee y = x && \text{by commutative hypothesis}\\
\implies & x = y\\
\implies & \le \text{ is antisymmetric}
\end{aligned}$$

Proof that $\le$ is transitive:

$$\begin{aligned}
x \le y \text{ and } y \le z \iff & x \ovee y = y \text{ and } y \ovee z = z && \text{by definition of } \le\\
\implies & (x \ovee y) \ovee z = z\\
\iff & x \ovee (y \ovee z) = z && \text{by associative hypothesis}\\
\implies & x \ovee z = z\\
\iff & x \le z\\
\iff & \le \text{ is transitive}
\end{aligned}$$

Proof that $x \ovee y$ is a lub of x and y:

$$\begin{aligned}
x \ovee y = y \iff & x \le y && \text{by definition of } \le\\
\iff & x \vee y = y && \text{by definition of } \vee\\
\implies & x \ovee y = x \vee y\\
\implies & x \ovee y \text{ is the lub of } x \text{ and } y
\end{aligned}$$

✑

Theorem C.2. [4] *Let $\owedge, \lesssim\colon X^2 \to X$ be binary operators over a set X.*

THM
$$\left\{\begin{array}{l}(X, \lesssim, \owedge) \text{ is a}\\ \text{MEET SEMILATTICE}\end{array}\right\} \implies \left\{\begin{array}{lll}1. & (X, \lesssim) \text{ is a PARTIALLY ORDERED SET} & \text{and}\\ 2. & x \owedge y \text{ is a GREATEST LOWER BOUND of } x \text{ and } y & \forall x, y \in X.\end{array}\right\}$$

✎PROOF: In order for $(X, \le)$ to be an ordered set, $\le$ must be, according to Definition B.2 (page 286), *reflexive*, *antisymmetric*, and *transitive*;

Proof that $\le$ is reflexive:

$$\begin{aligned}
& x = x \owedge x && \text{by idempotent hypothesis}\\
\iff & x \le x && \text{by definition of } \le\\
\implies & \le \text{ is reflexive}
\end{aligned}$$

Proof that $\le$ is antisymmetric:

$$\begin{aligned}
x \le y \text{ and } y \le x \iff & x \owedge y = x \text{ and } y \owedge x = y && \text{by definition of } \le\\
\implies & x \owedge y = x \text{ and } x \owedge y = y && \text{by commutative hypothesis}\\
\implies & x = y\\
\implies & \le \text{ is antisymmetric}
\end{aligned}$$

[4] MacLane and Birkhoff (1999) page 475

Proof that $\le$ is transitive:

$$\begin{aligned}
x \le y \text{ and } y \le z &\iff x \oslash y = x \text{ and } y \oslash z = y && \text{by definition of } \le\\
&\implies x \oslash (y \oslash z) = x\\
&\iff (x \oslash y) \oslash z = x && \text{by associative hypothesis}\\
&\implies x \oslash z = x\\
&\iff x \le z\\
&\iff \le \text{ is transitive}
\end{aligned}$$

Proof that $x \oslash y$ is a glb of x and y:

$$\begin{aligned}
x \oslash y = x &\iff x \le y && \text{by definition of } \le\\
&\iff x \wedge y = x && \text{by definition of } \wedge\\
&\implies x \oslash y = x \wedge y\\
&\implies x \oslash y \text{ is the glb of } x \text{ and } y
\end{aligned}$$

C.2 Lattices

An *ordered set* is a set together with the additional structure of an ordering relation (Definition B.2 page 286). However, this amount of structure tends to be insufficient to ensure "well-behaved" mathematical systems. This situation is greatly remedied if every pair of elements in an ordered set (partially or linearly ordered) has both a *least upper bound* and a *greatest lower bound* (Definition B.22 page 298) in the ordered set; in this case, that ordered set is a *lattice* (next definition). Gian-Carlo Rota (1932–1999) illustrates the advantage of lattices over simple ordered sets by pointing out that the *ordered set* of partitions of an integer "is fraught with pathological properties", while the *lattice* of partitions of a set "remains to this day rich in pleasant surprises".[5] Further examples of lattices follow in Section C.3 (page 306).

Definition C.3. [6]

DEF

An algebraic structure $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ *is a* ***lattice*** *if*

1. $(X, \le)$ *is an ordered set* *and*
2. $x, y \in X \implies x \vee y \in X$ *and*
3. $x, y \in X \implies x \wedge y \in X$

The algebraic structure $\mathbf{L}^* \triangleq (X, \obslash, \oslash; \ge)$ *is the* ***dual*** *lattice of* $\mathbf{L}$, *where* $\obslash$ *and* $\oslash$ *are determined by* $\ge$. *The* LATTICE $\mathbf{L}$ *is* ***linear*** *if* $(X, \le)$ *is a* CHAIN *(Definition B.4 page 287)*.

Definition C.3 (previous) characterizes lattices in terms of *order properties*. Under this definition, lattices have an equivalent characterization in terms of *algebraic properties*. In particular, all lattices have four basic algebraic properties: all lattices are *idempotent*, *commutative*, *associative*, and *absorptive*. Conversely, any structure that possesses these four properties *is* a lattice. These results are demonstrated by Theorem C.3 (next). However, note that the four properties are not *independent*, as it is possible to prove that any structure $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ that is *commutative*, *associative*, and *absorptive*, is also *idempotent* (Theorem C.8 page 310). Thus, when proving that $\mathbf{L}$ is a lattice, it is only necessary to prove that it is *commutative*, *associative*, and *absorptive*.

[5] Rota (1997) page 1440 ⟨Introduction⟩, Rota (1964) page 498 ⟨partitions of a set⟩

[6] MacLane and Birkhoff (1999) page 473, Birkhoff (1948) page 16, Ore (1935), Birkhoff (1933a) page 442, Maeda and Maeda (1970) page 1

Theorem C.3. [7]

THM

$(X, \vee, \wedge; \leq)$ *is a* LATTICE $\iff$

$$\left\{\begin{array}{rcl|rcl|lll} x \vee x &=& x & x \wedge x &=& x & \forall x \in X & \text{(IDEMPOTENT)} & \text{and} \\ x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \forall x,y \in X & \text{(COMMUTATIVE)} & \text{and} \\ (x \vee y) \vee z &=& x \vee (y \vee z) & (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & \forall x,y,z \in X & \text{(ASSOCIATIVE)} & \text{and} \\ x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \forall x,y \in X & \text{(ABSORPTIVE).} & \end{array}\right\}$$

✎PROOF:

1. Proof that $(X, \vee, \wedge; \leq)$ is a lattice $\implies$ 4 properties:
   These follow directly from the definitions of least upper bound $\vee$ and greatest lower bound $\wedge$. For the absorptive property,

$$\begin{aligned} x \leq y &\implies x \vee (x \wedge y) = x \vee x = x \\ y \leq x &\implies x \vee (x \wedge y) = x \vee y = x \\ x \leq y &\implies x \wedge (x \vee y) = x \wedge y = x \\ y \leq x &\implies x \wedge (x \vee y) = x \wedge x = x \end{aligned}$$

2. Proof that $(X, \vee, \wedge; \leq)$ is a lattice $\impliedby$ 4 properties:

   According to Definition C.3 (page 301), in order for $(X, \vee, \wedge; \leq)$ to be a lattice, $(X, \vee, \wedge; \leq)$ must be an ordered set, $x \vee y$ must be the least upper bound for any $x, y \in X$ and $x \wedge y$ must be the greatest lower bound for any $x, y \in X$.

   (a) By Theorem C.1 (page 299), $(X, \vee, \wedge; \leq)$ is an ordered set.
   (b) By Theorem C.1 (page 299), $x \vee y$ is the least upper bound for any $x, y \in X$.
   (c) Proof that $x \wedge y$ is the greatest lower bound for any $x, y \in X$: To prove this, we must show that $x \leq y \iff x \wedge y = x$.

   Proof that $x \leq y \implies x \wedge y = x$:

$$\begin{aligned} x &= x \wedge (x \vee y) && \text{by absorptive hypothesis} \\ &= x \wedge y && \text{by } x \leq y \text{ hypothesis and definition of } \leq \end{aligned}$$

   Proof that $x \leq y \impliedby x \wedge y = x$:

$$\begin{aligned} y &= y \vee (y \wedge x) && \text{by absorptive hypothesis} \\ &= y \vee (x \wedge y) && \text{by commutative hypothesis} \\ &= y \vee x && \text{by } x \wedge y = x \text{ hypothesis} \\ &= x \vee y && \text{by commutative hypothesis} \\ \implies \quad & x \leq y && \text{by definition of } \leq \end{aligned}$$

⇨

[7] MacLane and Birkhoff (1999) pages 473–475 ⟨LEMMA 1, THEOREM 4⟩, Burris and Sankappanavar (1981) pages 4–7, Birkhoff (1938) pages 795–796, Ore (1935) page 409 ⟨(α)⟩, Birkhoff (1933a) page 442, Dedekind (1900) pages 371–372 ⟨(1)–(4)⟩. Peirce (1880) credits Boole and Jevons with the *commutative* property: Peirce (1880) page 33 ⟨"(5)"⟩. Peirce (1880) credits Boole and Jevons with the *associative* property. Peirce (1880) credits Jevons (1864) with the *idempotent* property: Jevons (1864) page 41
$A + A = A$ "Law of Unity"
$AA = A$ "Law of Simplicity"

Lemma C.1. [8] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition C.3 page 301)*.

LEM
$$x \leq y \iff x = x \wedge y \qquad \forall x,y \in \boldsymbol{L}$$

✎PROOF:

1. Proof for $\implies$ case: by left hypothesis and definition of $\wedge$ (Definition B.22 page 298).
2. Proof for $\impliedby$ case: by right hypothesis and definition of $\wedge$ (Definition B.22 page 298).

✏

The identities of Theorem C.3 (page 302) occur in pairs that are *duals* of each other. That is, for each identity, if you swap the join and meet operations, you will have the other identity in the pair. Thus, the characterization of lattices provided by Theorem C.3 (page 302) is called *self-dual*. And because of this, lattices support the *principle of duality* (next theorem).

Theorem C.4 (Principle of duality). [9] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

THM
$$\left\{\begin{array}{l}\phi \text{ is an identity on } \boldsymbol{L} \text{ in terms}\\ \text{of the operations } \vee \text{ and } \wedge\end{array}\right\} \implies \mathbf{T}\phi \text{ is also an identity on } \boldsymbol{L}$$
where the operator $\mathbf{T}$ *performs the following mapping on the operations of* ϕ:
$$\vee \to \wedge, \qquad \wedge \to \vee$$

✎PROOF: For each of the identities in Theorem C.3 (page 302), the operator $\mathbf{T}$ produces another identity that is also in the set of identities:

$$\begin{array}{lclclclclcl}
\mathbf{T}(1a) &=& \mathbf{T}[x \vee y &=& y \vee x] &=& [x \wedge y &=& y \wedge x] &=& (1b)\\
\mathbf{T}(1b) &=& \mathbf{T}[x \wedge y &=& y \wedge x] &=& [x \vee y &=& y \vee x] &=& (1a)\\
\mathbf{T}(2a) &=& \mathbf{T}[x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& [x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& (2b)\\
\mathbf{T}(2b) &=& \mathbf{T}[x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& [x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& (2a)
\end{array}$$

Therefore, if the statement ϕ is consistent with regards to the lattice $\boldsymbol{L}$, then $\mathbf{T}\phi$ is also consistent with regards to the lattice $\boldsymbol{L}$. ✏

Proposition C.1 (Monotony laws). [10] *Let* $(X, \vee, \wedge; \leq)$ *be a lattice.*

PRP
$$\left.\begin{array}{rcl} a &\leq& b \quad \text{and}\\ x &\leq& y. \end{array}\right\} \implies \left\{\begin{array}{rcl} a \wedge x &\leq& b \wedge y \quad \text{and}\\ a \vee x &\leq& b \vee y. \end{array}\right.$$

8 Holland (1970) page ???
9 Padmanabhan and Rudeanu (2008) pages 7–8, Beran (1985) pages 29–30
10 Givant and Halmos (2009) page 39, Doner and Tarski (1969) pages 97–99

✎PROOF:

$$\begin{aligned}
1.(a \wedge x) &\le a && \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition B.22 page 298}\\
&\le b && \text{by left hypothesis}\\
2.(a \wedge x) &\le x && \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition B.22 page 298}\\
&\le y && \text{by left hypothesis}\\
3.(a \wedge x) &= \underbrace{(a \wedge x)}_{\le b} \wedge \underbrace{(a \wedge x)}_{\le y} && \text{by } \textit{idempotent} \text{ property Theorem C.3 page 302}\\
&\le b \wedge y && \text{by 1 and 2}\\
4.(a \vee x) &= \underbrace{(a \vee x)}_{\le b} \vee \underbrace{(a \vee x)}_{\le y} && \text{by } \textit{idempotent} \text{ property Theorem C.3 page 302}\\
&\le b \vee y && \text{by 1 and 2}
\end{aligned}$$

✏

Minimax inequality. Suppose we arrange a finite sequence of values into m groups of n elements per group. This could be represented as an $m \times n$ matrix. Suppose now we find the minimum value in each row, and the maximum value in each column. We can call the maximum of all the minimum row values the *maximin*, and the minimum of all the maximum column values the *minimax*. Now, which is greater, the maximin or the minimax? The *minimax inequality* demonstrates that the maximin is always less than or equal to the minimax. The minimax inequality is illustrated below and stated formerly in Theorem C.5 (page 304).

$$\underbrace{\bigvee_1^m \left\{ \begin{array}{c}
\bigwedge_1^n \left\{ x_{11} \quad x_{12} \quad \cdots \quad x_{1n} \right\} \\ \hline
\bigwedge_1^n \left\{ x_{21} \quad x_{22} \quad \cdots \quad x_{2n} \right\} \\ \hline
\bigwedge_1^n \left\{ \vdots \quad \ddots \quad \ddots \quad \vdots \right\} \\ \hline
\bigwedge_1^n \left\{ x_{m1} \quad x_{m2} \quad \cdots \quad x_{mn} \right\}
\end{array} \right\}}_{\text{maximin}}
\le
\underbrace{\bigwedge_1^n \left\{ \begin{array}{c|c|c|c}
\bigvee_1^m & \bigvee_1^m & \bigvee_1^m & \bigvee_1^m \\
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{array} \right\}}_{\text{minimax}}$$

Theorem C.5 (Minimax inequality). [11] *Let* $(X, \vee, \wedge; \le)$ *be a lattice.*

THM

$$\underbrace{\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} x_{ij}}_{\text{maxmini: largest of the smallest}} \le \underbrace{\bigwedge_{j=1}^{n} \bigvee_{i=1}^{m} x_{ij}}_{\text{minimax: smallest of the largest}} \qquad \forall x_{ij} \in X$$

[11] Birkhoff (1948) pages 19–20

✎PROOF:

$$\begin{aligned}
&\underbrace{\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{smallest for any given } i} \le x_{ij} \le \underbrace{\left(\bigvee_{k=1}^{n} x_{kj}\right)}_{\text{largest for any given } j} && \forall i,j \\
\implies\quad & \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{largest amoung all } i\text{s of the smallest values}} \le \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{k=1}^{m} x_{kj}\right)}_{\text{smallest amoung all } j\text{s of the largest values}} \\
\implies\quad & \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{j=1}^{n} x_{ij}\right)}_{\text{maxmini}} \le \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{i=1}^{m} x_{ij}\right)}_{\text{minimax}} && \text{(change of variables)}
\end{aligned}$$

✑

Distributive inequalities. Special cases of the minimax inequality include three distributive *inequalities* (next theorem). If for some lattice any *one* of these inequalities is an *equality*, then *all three* are *equalities*; and in this case, the lattice is a called a *distributive* lattice.

Theorem C.6 (distributive inequalities). [12]

THM

$(X, \vee, \wedge; \le)$ *is a lattice* $\implies$ *for all x,y,z∈X*

$$\begin{array}{rcll}
x \wedge (y \vee z) & \ge & (x \wedge y) \vee (x \wedge z) & \text{(JOIN SUPER-DISTRIBUTIVE)} \quad \textit{and} \\
x \vee (y \wedge z) & \le & (x \vee y) \wedge (x \vee z) & \text{(MEET SUB-DISTRIBUTIVE)} \quad \textit{and} \\
(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \le & (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \text{(MEDIAN INEQUALITY).}
\end{array}$$

✎PROOF:

1. Proof that $\wedge$ sub-distributes over $\vee$:

$$\begin{aligned}
(x \wedge y) \vee (x \wedge z) &\le (x \vee x) \wedge (y \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem C.5 page 304)} \\
&= x \wedge (y \vee z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem C.3 page 302)}
\end{aligned}$$

$$\bigvee\left\{\begin{array}{c}\bigwedge\left\{\begin{array}{cc} x & y \end{array}\right\} \\ \hline \bigwedge\left\{\begin{array}{cc} x & z \end{array}\right\}\end{array}\right\} \le \bigwedge\left\{\begin{array}{c|c} \bigvee & \bigvee \\ x & y \\ x & z \end{array}\right\}$$

2. Proof that $\vee$ super-distributes over $\wedge$:

$$\begin{aligned}
x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem C.3 page 302)} \\
&\le (x \vee y) \wedge (x \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem C.5 page 304)}
\end{aligned}$$

$$\bigvee\left\{\begin{array}{c}\bigwedge\left\{\begin{array}{cc} x & x \end{array}\right\} \\ \hline \bigwedge\left\{\begin{array}{cc} y & z \end{array}\right\}\end{array}\right\} \le \bigwedge\left\{\begin{array}{c|c} \bigvee & \bigvee \\ x & x \\ y & z \end{array}\right\}$$

3. Proof that of median inequality: by *minimax inequality* (Theorem C.5 page 304)

✑

[12] Davey and Priestley (2002) page 85, Grätzer (2003) page 38, Birkhoff (1933a) page 444, Korselt (1894) page 157, Müller-Olm (1997) page 13 ⟨terminology⟩

Modular inequalities. Besides the distributive property, another consequence of the minimax inequality is the *modularity inequality* (next theorem). A lattice in which this inequality becomes equality is said to be *modular*.

Theorem C.7 (Modular inequality). [13] *Let* $(X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition C.3 page 301).*

THM

$$x \leq y \qquad \Longrightarrow \qquad x \vee (y \wedge z) \leq y \wedge (x \vee z)$$

✎PROOF:

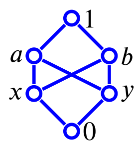
$$\begin{aligned}
x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by *absorptive* property (Theorem C.3 page 302)}\\
&\leq (x \vee y) \wedge (x \vee z) && \text{by the *minimax inequality* (Theorem C.5 page 304)}\\
&= y \wedge (x \vee z) && \text{by left hypothesis}
\end{aligned}$$

$$\bigvee\left\{\begin{array}{c}\bigwedge\left\{\begin{array}{cc} x & x \end{array}\right\}\\ \hline \bigwedge\left\{\begin{array}{cc} y & z \end{array}\right\}\end{array}\right\} \quad\leq\quad \bigwedge\left\{\begin{array}{c|c} \bigvee & \bigvee \\ x & x \\ y & z \end{array}\right\}$$

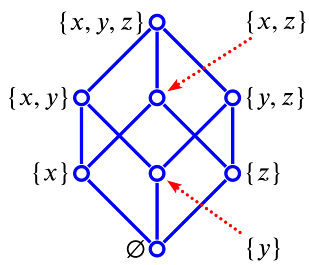
✏

C.3 Examples

Example C.1. [14] the ordered set illustrated to the right is **not** a lattice because, for example, while x and y have *upper bounds* a, b, and 1, x and y have no *least upper bound*. Obviously 1 is not the least upper bound because $a \leq 1$ and $b \leq 1$. And neither a nor b is a least upper bound because $a \not\leq b$ and $b \not\leq a$; rather, a and b are incomparable $(a \| b)$. Note that if we remove either or both of the two lines crossing the center, the ordered set becomes a lattice.



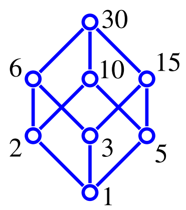
Example C.2 (Discrete lattice). Let 2^A be the power set of a set A, $\subseteq$ the set inclusion relation, $\cup$ the set union operation, and $\cap$ the set intersection operation. Then the tupple $\left(2^{\{x,y,z\}}, \cup, \cap; \subseteq\right)$ is a lattice.



Examples of least upper bounds	Examples of greatest lower bounds
$\{x\} \cup \{z\} = \{x, z\}$	$\{x\} \cap \{z\} = \varnothing$
$\{x, y\} \cup \{y\} = \{x, y\}$	$\{x, y\} \cap \{y\} = \{y\}$
$\{x, z\} \cup \{y, z\} = \{x, y, z\}$	$\{x, z\} \cap \{y, z\} = \{z\}$

Example C.3 (Integer factor lattice). [15] For any pair of natural numbers $n, m \in \mathbb{N}$, let $n|m$ represent the relation "m divides n", $\text{lcm}(n, m)$ the *least common multiple* of n and m, and $\text{gcd}(n, m)$ the *greatest common divisor* of n and m.

EX $(\{1, 2, 3, 5, 6, 10, 15, 30\}, \text{gcd}, \text{lcm}; |)$ is a lattice.



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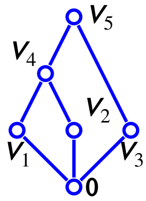
[13] Birkhoff (1948) page 19, Burris and Sankappanavar (1981) page 11, Dedekind (1900) page 374

[14] Oxley (2006) page 54, Farley (1997) page 3, Farley (1996) page 5, Birkhoff (1967) pages 15–16

[15] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩

Example C.4 (Linear lattice). Let $\le$ be the standard counting ordering relation on the set of integers; and for any pair of integers $n, m \in \mathbb{N}$, let $\max(n, m)$ be the maximum of n and m, and $\min(n, m)$ be the minimum of n and m. Then the tupple $(\{1, 2, 3, 4\}, \max, \min\,;\, \le)$ is a lattice.

Example C.5 (Subspace lattices). [16]Let $(\!(\boldsymbol{V}_n)\!)$ be a sequence of subspaces, $\subseteq$ be the set inclusion relation, $+$ the subspace addition operator, and $\cap$ the set intersection operator. Then the tuple $(\{\boldsymbol{V}_n\}, +, \cap\,;\, \subseteq)$ is a lattice.

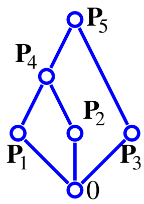


Example C.6 (Projection operator lattices). [17]Let $(\!(\mathbf{P}_n)\!)$ be a sequence of projection operators in a Hilbert space X.

EX

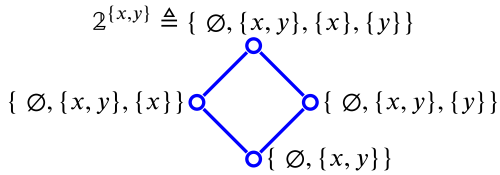
$(\{\mathbf{P}_n\}, \vee, \wedge\,;\, \le)$ is a lattice

$$\begin{array}{rrcl} \text{where} & \mathbf{P}_1 \le \mathbf{P}_2 & \stackrel{\text{def}}{\iff} & \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1 \\ & \mathbf{P}_1 \vee \mathbf{P}_2 & = & \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 \\ & \mathbf{P}_1 \wedge \mathbf{P}_2 & = & \mathbf{P}_1\mathbf{P}_2 \end{array}$$



Example C.7 (Lattice of a single topology). [18] Let X be a set, τ a topology on X, $\subseteq$ the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $(\tau, \cup, \cap\,;\, \subseteq)$ is a lattice.

Example C.8 (Lattice of topologies). [19]Let X be a set and $\{\tau_1, \tau_2, \tau_3, \ldots\}$ all the possible topologies on X. Let $\subseteq$ be the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $(\{(X, \tau_n)\}, \cup, \cap\,;\, \subseteq)$ is a lattice.

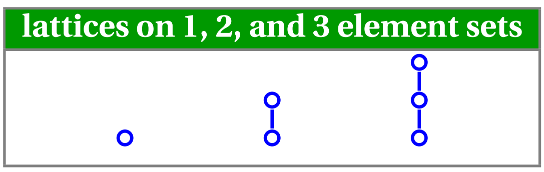


Proposition C.2. [20] *Let X_n be a finite set with order $n = |X_n|$. Let L_n be the number of labeled lattices on X_n, l_n the number of unlabeled lattices, and p_n the number of unlabeled posets.*

PRP

n	0	1	2	3	4	5	6	7	8	9	10
L_n	1	1	2	6	36	380	6390	157962	5396888	243, 179, 064	13, 938, 711, 210
l_n	1	1	1	1	2	5	15	53	222	1078	5994
p_n	1	1	2	5	16	63	318	2045	16, 999	183, 231	2, 567, 284

Example C.9 (lattices on 1–3 element sets). [21]There is only one unlabeled lattice for finite sets with 3 or fewer elements (Proposition C.2 page 307). Thus, these lattices are all linearly ordered. These 3 lattices are illustrated to the right.



[16] Isham (1999) pages 21–22

[17] Isham (1999) pages 21–22, Dunford and Schwartz (1957) pages 481–482

[18] Burris and Sankappanavar (1981) page 9, Birkhoff (1936a) page 161

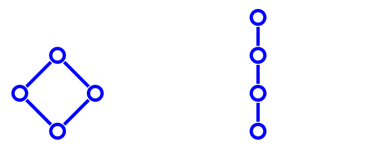
[19] Isham (1999) page 44, Isham (1989) page 1515

[20] Sloane (2014) ⟨http://oeis.org/A055512⟩, Sloane (2014) ⟨http://oeis.org/A006966⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Heitzig and Reinhold (2002)

[21] Kyuno (1979) page 412, Stanley (1997) page 102

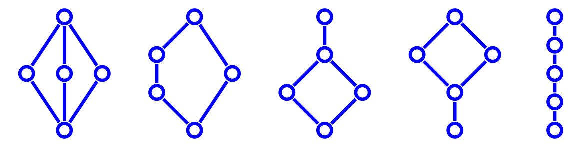
Example C.10 (lattices on a 4 element set). [22]There are 2 unlabeled lattices on a 4 element set (Proposition C.2 page 307). These are illustrated to the right.

lattices on 4 element sets



Example C.11 (lattices on a 5 element set). [23]There are 5 unlabeled lattices on a 5 element set (Proposition C.2 page 307). These are illustrated to the right.

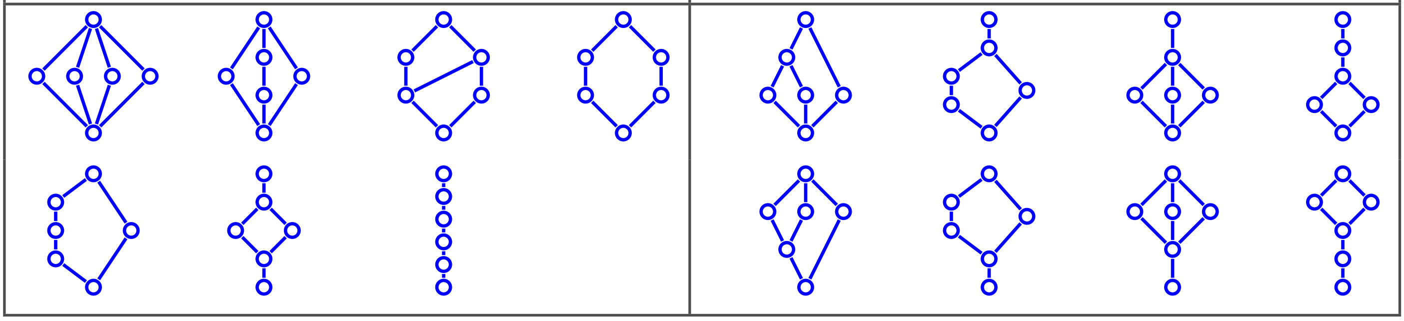
lattices on 5 element sets



Example C.12 (lattices on a 6 element set). [24] There are 15 *unlabeled lattices* on a 6 element set (Proposition C.2 page 307). These are illustrated in the following table. Notice that the lattices in the second row are simply generated from the 5 element lattices (Example C.11 page 308) with a "head" or "tail" added to each one.

lattices on 6 element sets

self-dual | *non-self dual*



Example C.13 (lattices on a 7 element set). [25] There are 53 unlabeled lattices on a 7 element set (Proposition C.2 page 307). These are illustrated in Figure C.1 (page 309).

Example C.14 (lattices on 8 element sets). There are 222 unlabeled lattices on a 8 element set (Proposition C.2 page 307). See Kyuno's paper[26] for Hasse diagrams of all 222 lattices.

C.4 Characterizations

Theorem C.3 (page 302)gave eight equations in three variables and two operators that are true of all lattices. But the converse is also true: that is, if the eight equations of Theorem C.3 are true for all values of the underlying set, then that set together with the two operators are a lattice.

That is, the eight equations in three variables of Theorem C.3 *characterize* lattices, or serve as an *equational basis* for lattices.[27] And this is not the only system of equations that characterize a lattice. There are other systems that use fewer equations in more variables. Here are some examples of lattice characterizations:

[22] Kyuno (1979) page 412, Stanley (1997) page 102
[23] Kyuno (1979) page 413, Stanley (1997) page 102
[24] Kyuno (1979) page 413, Stanley (1997) page 102
[25] Kyuno (1979) pages 413–414
[26] Kyuno (1979) pages 415–421
[27] McKenzie (1970) page 24, Tarski (1966)

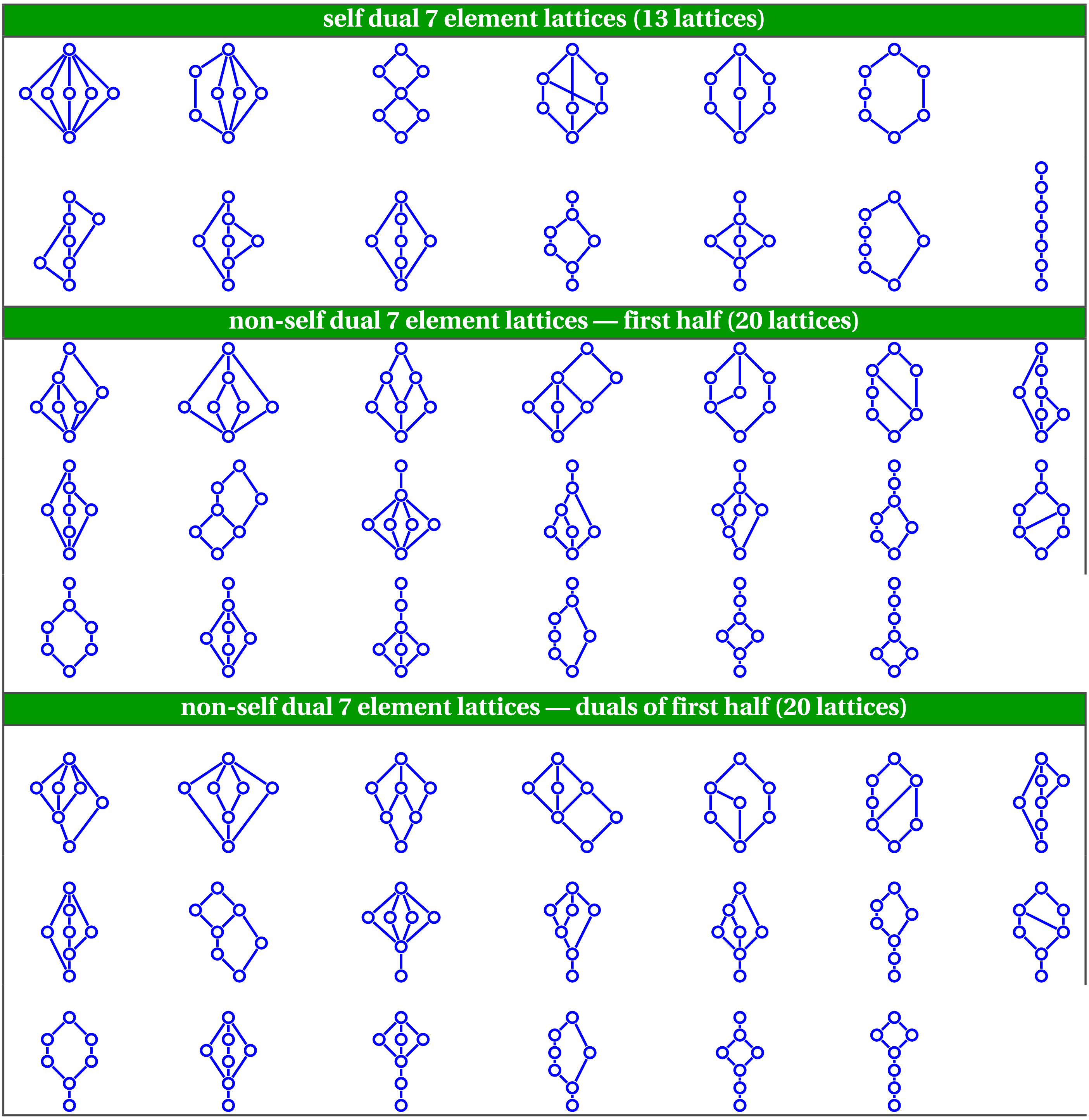


Figure C.1: The 53 unlabeled lattices on a 7 element set (Example C.13 page 308)

- 8 equations in 3 variables — Theorem C.3 — page 302
- 6 equations in 3 variables — Theorem C.8 — page 310
- 2 equations in 5 variables — Theorem C.9 — page 310
- 1 equation in 8 variables with length 29 — Theorem C.10 — page 310
- 1 equation in 7 variables with length 79 — Theorem C.10 — page 310

Since these characterizations are equivalent to the definition of the lattice, we could in fact change things around and essentially make any of these characterizations into the definition, and make the definition into a theorem.[28]

Theorem C.3 (page 302) gave 4 necessary and sufficient pairs of properties for a structure $(X, \vee, \wedge; \leq)$ to be a *lattice*. However, these 4 pairs are actually *overly* sufficient (they are not *independent*), as demonstrated next.

Theorem C.8. [29]

THM $(X, \vee, \wedge; \leq)$ *is a lattice* $\iff$

$$\left\{\begin{array}{rcl|rcl|lll} x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \forall x,y \in X & (\text{COMMUTATIVE}) & \textit{and} \\ (x \vee y) \vee z &=& x \vee (y \vee z) & (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & \forall x,y,z \in X & (\text{ASSOCIATIVE}) & \textit{and} \\ x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \forall x,y \in X & (\text{ABSORPTIVE}) & \end{array}\right\}$$

✎PROOF: Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$.

1. Proof that $\boldsymbol{L}$ is a *lattice* $\implies$ 3 properties: by Theorem C.3 page 302
2. Proof that $\boldsymbol{L}$ is a *lattice* $\impliedby$ 3 properties:
   (a) Proof that 3 properties $\implies$ $\boldsymbol{L}$ is *idempotent*:

$$\begin{aligned} x \vee x &= x \vee [x \wedge (x \vee y)] && \text{by } \textit{absorptive} \text{ property} \\ &= x \vee [x \wedge z] && \text{where } z \triangleq x \vee y \\ &= x && \text{by } \textit{absorptive} \text{ property} \\ x \wedge x &= x \wedge [x \vee (x \wedge y)] && \text{by } \textit{absorptive} \text{ property} \\ &= x \wedge [x \vee z] && \text{where } z \triangleq x \wedge y \\ &= x && \text{by } \textit{absorptive} \text{ property} \end{aligned}$$

   (b) By Theorem C.3 page 302 and because $\boldsymbol{L}$ is *commutative*, *associative*, *absorptive*, and *idempotent* with respect to $\vee$ and $\wedge$, $\boldsymbol{L}$ is a *lattice*.

✏

Theorem C.9 (Lattice characterization in 2 equations and 5 variables). [30] *Let X be a set and $\vee$ and $\wedge$ be two binary operators on X.*

THM $(X, \leq, \vee, \wedge)$ *is a lattice* ***if and only if***

$$\begin{array}{rcll} x &=& (x \wedge y) \vee x & \forall x,y \in X \quad \textit{and} \\ [(x \wedge y) \wedge z \vee u] \vee w &=& [(y \wedge z) \wedge x \vee w] \vee (y \vee u) \wedge u & \forall x,y,z,u,w \in X \end{array}$$

Theorem C.10 (Lattice characterizations in 1 equation). [31] *Let X be a set and $\vee$ and $\wedge$ be two binary*

[28] Burris and Sankappanavar (1981) pages 6–7,
[29] Padmanabhan and Rudeanu (2008) page 8, Beran (1985) page 5, McKenzie (1970) page 24
[30] Tamura (1975) page 137
[31] McCune et al. (2003b) page 2, McCune et al. (2003a), McCune and Padmanabhan (1996) page 144, http://www.cs.unm.edu/%7Everoff/LT/

operators on X.

THM *The following four statements are all equivalent:*

1. $(X, \vee, \wedge ; \leq)$ *is a* ***lattice***
2. $(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x \quad \forall x,y,z,u,v,w,s,t \in X$ *(1 equation, 8 variables, length 29)*
3. $(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (((w \vee x) \wedge (s \vee x)) \vee t) = x \quad \forall x,y,z,u,v,w,s,t \in X$ *(1 equation, 8 variables, length 29)*
4. $(((x \wedge y) \vee (y \wedge (x \vee y))) \wedge z) \vee (((x \wedge (((x_1 \wedge y) \vee (y \wedge x_2))\vee y)) \vee (((y \wedge (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)) \vee (u \wedge (y\vee (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)))) \wedge (x \vee (((x_1 \wedge y) \vee (y\wedge x_2)) \vee y)))) \wedge (((x \wedge y) \vee (y \wedge (x \vee y))) \vee z)) = y \quad \forall x,y,z,x_1,x_2,x_3,u \in X$ *(1 equation, 7 variables, length 79)*

C.5 Functions on lattices

C.5.1 Isomorphisms

Lattices and *ordered set* (Definition B.2 page 286) are examples of mathematical *order structures*. Often we are interested in similarities between two lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ with respect to order. Similarities between lattices can be described by defining a function θ that maps from the first lattice to the second. The degree of similarity can be roughly described in terms of the mapping θ as follows:

1. If there exists a mapping that is *bijective* then the number of elements in $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ is the same. However, their order structure may still be very different.

2. Lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are more similar if there exists a mapping that is *bijective* and *order preserving* (Definition B.9 page 293). Despite having this property however, their order structure may still be remarkably different, as illustrated by Example B.18 (page 293) and Example B.19 (page 293).

3. Lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are essentially identical (except possibly for their labeling) if there exists a mapping θ that is not only *bijective* and *order preserving*, but whose *inverse* is *also bijective* (Theorem C.11 page 311). In this case, the lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are *isomorphic* and the mapping θ is an *isomorphism*. An isomorphism between $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ implies that the two lattices have an identical order structure. In particular, the isomorphism θ preserves joins and meets (next definition).

Definition C.4. *Let* $\boldsymbol{L}_1 \triangleq (X, \vee, \wedge ; \leq)$ *and* $\boldsymbol{L}_2 \triangleq (Y, \varovee, \varowedge ; \precsim)$ *be lattices.*

DEF $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_2$ *are* ***algebraically isomorphic****, or simply* ***isomorphic****, if there exists a function* $\theta \in Y^X$ *such that*

1. $\theta(x \vee y) = \theta(x) \varovee \theta(y) \quad \forall x, y \in X$ (PRESERVES JOINS) *and*
2. $\theta(x \wedge y) = \theta(x) \varowedge \theta(y) \quad \forall x, y \in X$ (PRESERVES MEETS).

In this case, the function θ *is said to be an* ***isomorphism*** *from* $\boldsymbol{L}_1$ *to* $\boldsymbol{L}_2$*, and the isomorphic relationship between* $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_2$ *is denoted as*

$$\boldsymbol{L}_1 \equiv \boldsymbol{L}_2.$$

Theorem C.11. [32] *Let* $(X, \vee, \wedge ; \leq)$ *and* $(Y, \varovee, \varowedge ; \precsim)$ *be lattices and* $\theta \in Y^X$ *be a* BIJECTIVE *function with inverse* $\theta^{-1} \in X^Y$. *Let* $(X, \vee, \wedge ; \leq) \equiv (Y, \varovee, \varowedge ; \precsim)$ *represent the condition that the two lattices*

[32] Burris and Sankappanavar (2000) page 10

are ISOMORPHIC.

THM

$$\underbrace{\left.\begin{array}{lll} x_1 \le x_2 & \Longrightarrow \quad \theta(x_1) \precsim \theta(x_2) & \forall x_1,x_2 \in X \\ y_1 \precsim y_2 & \Longrightarrow \quad \theta^{-1}(y_1) \precsim \theta^{-1}(y_2) & \forall y_1,y_2 \in Y \end{array}\right\}}_{\theta \text{ and } \theta^{-1} \text{ are ORDER PRESERVING with respect to } \le \text{ and } \precsim[33]} \iff \underbrace{(X, \vee, \wedge; \le) \equiv (Y, \ovee, \owedge; \precsim)}_{\text{isomorphic}}$$

✎PROOF: Let $\theta \in Y^X$ be the isomorphism between lattices $(X, \vee, \wedge; \le)$ and $(Y, \ovee, \owedge; \precsim)$.

1. Proof that *order preserving* $\Longrightarrow$ *preserves joins*:

   (a) Proof that $\theta(x_1 \vee x_2) \succsim \theta(x_1) \ovee \theta(x_2)$:

   i. Note that

   $$\begin{aligned} x_1 &\le x_1 \vee x_2 \\ x_2 &\le x_1 \vee x_2. \end{aligned}$$

   ii. Because θ is *order preserving*

   $$\begin{aligned} \theta(x_1) &\precsim \theta(x_1 \vee x_2) \\ \theta(x_2) &\precsim \theta(x_1 \vee x_2). \end{aligned}$$

   iii. We can then finish the proof of item (1a):

   $$\begin{aligned} \theta(x_1) \ovee \theta(x_2) &\precsim \underbrace{\theta(x_1 \vee x_2)}_{x_1 \le x_1 \vee x_2} \ovee \underbrace{\theta(x_1 \vee x_2)}_{x_2 \le x_1 \vee x_2} && \text{by *order preserving* hypothesis} \\ &= \theta(x_1 \vee x_2) && \text{by *idempotent* property page 302} \end{aligned}$$

   (b) Proof that $\theta(x_1 \vee x_2) \precsim \theta(x_1) \ovee \theta(x_2)$:

   i. Just as in item (1a), note that $\theta^{-1}(y_1) \vee \theta^{-1}(y_2) \le \theta^{-1}(y_1 \ovee y_2)$:

   $$\begin{aligned} \theta^{-1}(y_1) \vee \theta^{-1}(y_2) &\le \underbrace{\theta^{-1}(y_1 \ovee y_2)}_{y_1 \precsim y_1 \ovee y_2} \vee \underbrace{\theta^{-1}(y_1 \ovee y_2)}_{y_2 \precsim y_1 \ovee y_2} && \text{by *order preserving* hypothesis} \\ &= \theta^{-1}(y_1 \ovee y_2) && \text{by *idempotent* property page 302} \end{aligned}$$

   ii. Because θ is *order preserving*

   $$\begin{aligned} \theta\left[\theta^{-1}(y_1) \vee \theta^{-1}(y_2)\right] &\precsim \theta\theta^{-1}(y_1 \ovee y_2) && \text{by item (1(b)i) page 312} \\ &= y_1 \ovee y_2 && \text{by definition of inverse function } \theta^{-1} \end{aligned}$$

   iii. Let $u_1 \triangleq \theta(x_1)$ and $u_2 \triangleq \theta(x_2)$.

   iv. We can then finish the proof of item (1b):

   $$\begin{aligned} \theta(x_1 \vee x_2) &= \theta\left[\theta^{-1}\theta(x_1) \vee \theta^{-1}\theta(x_2)\right] && \text{by definition of inverse function } \theta^{-1} \\ &= \theta\left[\theta^{-1}(u_1) \vee \theta^{-1}(u_2)\right] && \text{by definition of } u_1, u_2 \text{, item (1(b)iii)} \\ &\precsim u_1 \ovee u_2 && \text{by item (1(b)ii)} \\ &= \theta(x_1) \ovee \theta(x_2) && \text{by definition of } u_1, u_2 \text{, item (1(b)iii)} \end{aligned}$$

   (c) And so, combining item (1a) and item (1b), we have

   $$\left.\begin{array}{llll} \theta(x_1 \vee x_2) & \succsim & \theta(x_1) \ovee \theta(x_2) & \text{(item (1a) page 312)} \quad \text{and} \\ \theta(x_1 \vee x_2) & \precsim & \theta(x_1) \ovee \theta(x_2) & \text{(item (1b) page 312)} \end{array}\right\} \Longrightarrow \quad \theta(x_1 \vee x_2) = \theta(x_1) \ovee \theta(x_2)$$

[33] *order preserving*: Definition B.9 page 293

2. Proof that *order preserving* $\Longrightarrow$ *preserves meets*:

   (a) Proof that $\theta(x_1 \wedge x_2) \precsim \theta(x_1) \owedge \theta(x_2)$:

$$\begin{aligned} \theta(x_1) \owedge \theta(x_2) &\oslash \underbrace{\theta(x_1 \wedge x_2)}_{x_1 \geq x_1 \wedge x_2} \owedge \underbrace{\theta(x_1 \wedge x_2)}_{x_2 \geq x_1 \wedge x_2} && \text{by *order preserving* hypothesis} \\ &= \theta(x_1 \wedge x_2) && \text{by *idempotent* property page 302} \end{aligned}$$

   (b) Proof that $\theta(x_1 \wedge x_2) \oslash \theta(x_1) \owedge \theta(x_2)$:

      i. Just as in item (2a), note that $\theta^{-1}(y_1) \wedge \theta^{-1}(y_2) \geq \theta^{-1}(y_1 \owedge y_2)$:

$$\begin{aligned} \theta^{-1}(y_1) \wedge \theta^{-1}(y_2) &\geq \underbrace{\theta^{-1}(y_1 \owedge y_2)}_{y_1 \oslash y_1 \owedge y_2} \owedge \underbrace{\theta^{-1}(y_1 \owedge y_2)}_{y_2 \oslash y_1 \owedge y_2} && \text{by *order preserving* hypothesis} \\ &= \theta^{-1}(y_1 \owedge y_2) && \text{by *idempotent* property page 302} \end{aligned}$$

      ii. Because θ is *order preserving*

$$\begin{aligned} \theta\left[\theta^{-1}(y_1) \wedge \theta^{-1}(y_2)\right] &\oslash \theta\theta^{-1}(y_1 \owedge y_2) && \text{by item (2(b)i)} \\ &= y_1 \owedge y_2 \end{aligned}$$

      iii. Let $v_1 \triangleq \theta(x_1)$ and $v_2 \triangleq \theta(x_2)$.

      iv. We can then finish the proof of item (2a):

$$\begin{aligned} \theta(x_1 \wedge x_2) &= \theta\left[\theta^{-1}\theta(x_1) \wedge \theta^{-1}\theta(x_2)\right] \\ &= \theta\left[\theta^{-1}(v_1) \wedge \theta^{-1}(v_2)\right] && \text{by item (2(b)iii)} \\ &\oslash v_1 \owedge v_2 && \text{by item (2(b)ii)} \\ &= \theta(x_1) \owedge \theta(x_2) && \text{by item (2(b)iii)} \end{aligned}$$

   (c) And so, combining item (2a) and item (2b), we have

$$\left.\begin{array}{lcll} \theta(x_1 \wedge x_2) & \precsim & \theta(x_1) \owedge \theta(x_2) & \text{(item (2a) page 313)} \quad \text{and} \\ \theta(x_1 \wedge x_2) & \oslash & \theta(x_1) \owedge \theta(x_2) & \text{(item (2b) page 313)} \end{array}\right\} \Longrightarrow \theta(x_1 \wedge x_2) = \theta(x_1) \owedge \theta(x_2)$$

3. Proof that *order preserving* $\Longleftarrow$ *isomorphic*:

$$\begin{aligned} x \leq y &\Longrightarrow \theta(y) = \theta(x \vee y) = \theta(x) \ovee \theta(y) && \text{by right hypothesis} \\ &\Longrightarrow \theta(x) \precsim \theta(y) \\ x \leq y &\Longrightarrow \theta(x) = \theta(x \wedge y) = \theta(x) \owedge \theta(y) && \text{by right hypothesis} \\ &\Longrightarrow \theta(x) \precsim \theta(y) \end{aligned}$$

✏

Example C.15. Let $\boldsymbol{L} \equiv \boldsymbol{M}$ represent the condition that a lattice $\boldsymbol{L}$ and a lattice $\boldsymbol{M}$ are *isomorphic*.

EX

$$\left(2^{\{x,y,z\}}, \cup, \cap; \subseteq\right) \equiv \left(\{1,2,3,5,6,10,15,30\}, \text{lcm}, \text{gcd}; |\right)$$

with isomorphism

$$\theta(A) = 5^{\mathbb{1}_A(z)} \cdot 3^{\mathbb{1}_A(y)} \cdot 2^{\mathbb{1}_A(x)} \qquad \forall A \in 2^{\{a,b,c\}}$$

Explicit cases are listed below and illustrated in Example B.9 (page 289) and Example B.10 (page 289).

$$\begin{aligned} \theta(\emptyset) &= 5^0 \cdot 3^0 \cdot 2^0 &&= 1 & \theta(\{z\}) &= 5^1 \cdot 3^0 \cdot 2^0 &&= 5 \\ \theta(\{x\}) &= 5^0 \cdot 3^0 \cdot 2^1 &&= 2 & \theta(\{x,z\}) &= 5^1 \cdot 3^0 \cdot 2^1 &&= 10 \\ \theta(\{y\}) &= 5^0 \cdot 3^1 \cdot 2^0 &&= 3 & \theta(\{y,z\}) &= 5^1 \cdot 3^1 \cdot 2^0 &&= 15 \\ \theta(\{x,y\}) &= 5^0 \cdot 3^1 \cdot 2^1 &&= 6 & \theta(\{x,y,z\}) &= 5^1 \cdot 3^1 \cdot 2^1 &&= 30 \end{aligned}$$

✎PROOF:

$$\begin{aligned}
\theta(A \cup B) &= 5^{\mathbb{1}_{A\cup B}(a)} \cdot 3^{\mathbb{1}_{A\cup B}(b)} \cdot 2^{\mathbb{1}_{A\cup B}(c)} \\
&= 5^{\mathbb{1}_A(a)\vee\mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_A(b)\vee\mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_A(c)\vee\mathbb{1}_B(c)} \\
&= \operatorname{lcm}\left(5^{\mathbb{1}_A(a)},\ 5^{\mathbb{1}_B(a)}\right) \cdot \operatorname{lcm}\left(3^{\mathbb{1}_A(b)},\ 3^{\mathbb{1}_B(b)}\right) \cdot \operatorname{lcm}\left(2^{\mathbb{1}_A(c)},\ 2^{\mathbb{1}_B(c)}\right) \\
&= \operatorname{lcm}\left(5^{\mathbb{1}_A(a)} \cdot 3^{\mathbb{1}_A(b)} \cdot 2^{\mathbb{1}_A(c)},\ 5^{\mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_B(c)}\right) \\
&= \operatorname{lcm}(\theta(A),\ \theta(B))
\end{aligned}$$

$$\begin{aligned}
\theta(A \cap B) &= 5^{\mathbb{1}_{A\cap B}(a)} \cdot 3^{\mathbb{1}_{A\cap B}(b)} \cdot 2^{\mathbb{1}_{A\cap B}(c)} \\
&= 5^{\mathbb{1}_A(a)\wedge\mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_A(b)\wedge\mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_A(c)\wedge\mathbb{1}_B(c)} \\
&= \gcd\left(5^{\mathbb{1}_A(a)},\ 5^{\mathbb{1}_B(a)}\right) \cdot \gcd\left(3^{\mathbb{1}_A(b)},\ 3^{\mathbb{1}_B(b)}\right) \cdot \gcd\left(2^{\mathbb{1}_A(c)},\ 2^{\mathbb{1}_B(c)}\right) \\
&= \gcd\left(5^{\mathbb{1}_A(a)} \cdot 3^{\mathbb{1}_A(b)} \cdot 2^{\mathbb{1}_A(c)},\ 5^{\mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_B(c)}\right) \\
&= \gcd(\theta(A),\ \theta(B))
\end{aligned}$$

✏

C.5.2 Metrics

Definition C.5. [34] *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ be a lattice.*

DEF

A function $\mathsf{v} \in \mathbb{R}^X$ is a ***subvaluation*** *if*

1. $\mathsf{v}(x) \geq 0 \quad \forall x \in X$ *and*
2. $\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) \leq \mathsf{v}(x) + \mathsf{v}(y) \quad \forall x, y \in X$

A subvaluation v *is* ***isotone*** *if* $x \leq y \implies \mathsf{v}(x) \leq \mathsf{v}(y)$.

A subvaluation v *is* ***positive*** *if* $x < y \implies \mathsf{v}(x) < \mathsf{v}(y)$.

Definition C.6. [35] *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ be a lattice.*

DEF

A function $\mathsf{v} \in \mathbb{R}^X$ is a ***valuation*** *if*

1. $\mathsf{v}(x) \geq 0 \quad \forall x \in X$ *and*
2. $\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) = \mathsf{v}(x) + \mathsf{v}(y) \quad \forall x, y \in X$ *and*
3. $x \leq y \implies \mathsf{v}(x) \leq \mathsf{v}(y) \quad \forall x, y \in X$ (ISOTONE).

Proposition C.3 (lattice subvaluation metric). [36] *Let $\mathbf{L}$ be a lattice.*

PRP

$$\left\{ \mathsf{v} \text{ is a positive SUBVALUATION on } \mathbf{L} \right\} \implies \left\{ \mathsf{d}(x, y) = 2\mathsf{v}(x \vee y) - \mathsf{v}(x) - \mathsf{v}(y) \text{ is a metric.} \right\}$$

Proposition C.4 (lattice valuation metric). [37] *Let $\mathbf{L}$ be a lattice.*

PRP

$$\left\{ \mathsf{v} \text{ is a positive VALUATION on } \mathbf{L} \right\} \implies \left\{ \mathsf{d}(x, y) = \mathsf{v}(x) + \mathsf{v}(y) - 2\mathsf{v}(x \wedge y) \text{ is a metric.} \right\}$$

---

[34] Deza and Deza (2006) page 143

[35] Deza and Deza (2006) page 143, Istrăţescu (1987) page 127 ⟨differs from Deza⟩, Birkhoff (1948) page 74 ⟨not compatible with Deza⟩

[36] Deza and Deza (2006) page 143

[37] Deza and Deza (2006) page 143

C.5.3 Lattice products

Theorem C.12 (lattice product). [38] *Let* $(X \times Y, \leq)$ *be the* POSET PRODUCT[39] *of* $(X, \lessdot)$ *and* $(Y, \trianglelefteq)$.

THM
$$\left.\begin{array}{lll} (X, \varovee, \varowedge; \lessdot) & \text{is a lattice} & \text{and} \\ (Y, \veebar, \barwedge; \trianglelefteq) & \text{is a lattice} & \end{array}\right\} \implies (X \times Y, \vee, \wedge; \leq) \text{ is also a lattice}$$

C.6 Literature

LITERATURE SURVEY:

1. Early lattice theory concepts:
   Dedekind (1900)
   Ore (1935)
2. Garrett Birkhoff's contribution:
   (a) The modern concept of the lattice was introduced by Garrett Birkhoff in 1933:
   Birkhoff (1933a)
   Birkhoff (1933b)
   (b) However, Birkhoff came to realize that the concept of the lattice had actually already been published in 1900 by Richard Dedekind. Birkhoff later remarked in an interview "My ideas about lattice theory developed gradually ...It was my father who, when he told Ore at Yale about what I was doing some time in 1933, found out from Ore that my lattices coincided with Dedekind's Dualgruppen ...I was lucky to have gone beyond Dedekind before I discovered his work. It would have been quite discouraging if I had discovered all my results anticipated by Dedekind."[40]
   (c) Birkhoff wrote a book in 1940 called *Lattice Theory*. There are basically three editions:
   Birkhoff (1940)
   Birkhoff (1948)
   Birkhoff (1967) With regards to his *Lattice Theory* book and another book entitled *A Survey of Modern Algebra* written with Saunders MacLane, Birkhoff remarked, "Morse had told me that no one under 30 should write a book. So I thought it over and wrote two!"[41]
3. Standard text books of lattice theory:
   Birkhoff (1967)
   Grätzer (1998)
   Crawley and Dilworth (1973)
4. Characterizations / equational bases:
   (a) General discussion:
   Tarski (1966)
   Baker (1969)
   McKenzie (1970)
   McKenzie (1972)
   Pigozzi (1975)
   Taylor (1979)
   Taylor (2008)
   Jipsen and Rose (1992) pages 115–127 ⟨Chapter 5⟩
   Padmanabhan and Rudeanu (2008)
   (b) Characterizations for lattices:
   Kalman (1968)
   Tamura (1975)
   Sobociński (1979)

[38] MacLane and Birkhoff (1967) page 489
[39] *poset product*: Definition B.5 page 287
[40] Albers and Alexanderson (1985) page 4
[41] Albers and Alexanderson (1985) page 4

(c) Specific characterizations:
- Padmanabhan (1969) ⟨2 equations in 7 variables⟩
- McCune and Padmanabhan (1996) page 144 ⟨1 equation, 7 variables, length 79⟩
- McCune et al. (2003a) ⟨1 equation, 8 variables, length 29⟩
- McCune et al. (2003b) ⟨1 equation, 8 variables, length 29⟩

5. Lattice drawing program:
Ralph Freese, http://www.math.hawaii.edu/~ralph/LatDraw/

APPENDIX D NEGATION

> ❝ *When we say not-being, we speak, I think, not of something that is the opposite of being, but only of something different. …Then when we are told that the negative signifies the opposite, we shall not admit it; we shall admit only that the particle "not" indicates something different from the words to which it is prefixed, or rather from the things denoted by the words that follow the negative.* ❞
>
> Plato's the *Sophist* (circa 360 B.C.) [1]

> ❝ *Clearly, then, it is a principle of this kind that is the most certain of all principles.…Let us next state what this principle is. "It is impossible for the same attribute at once to belong and not to belong to the same thing and in the same relation"; …This is the most certain of all principles,…for it is impossible for anyone to suppose that the same thing is and and is not,…it is by nature the starting-point of all the other axioms as well.* ❞
>
> Aristotle (384BC–322BC), Greek philosopher [2]

D.1 Definitions

Definition D.1. [3] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \le)$ *be a* BOUNDED LATTICE .

DEF *A* FUNCTION $\neg \in X^X$ *is a* ***subminimal negation*** *on* $\mathbf{L}$ *if*

$$x \le y \implies \neg y \le \neg x \quad \forall x,y \in X \qquad \text{(ANTITONE)}$$ [4]

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[1] Plato (circa 360 B.C.) ⟨257b–257c⟩, Horn (2001) page 5

[2] Aristotle page 4.1005b

[3] Dunn (1996) pages 4–6, Dunn (1999) pages 24–26 ⟨2 THE KITE OF NEGATIONS⟩

[4] The *antitone* property may also be referred to as *antitonic*, *order-reversing*, or *contrapositive*.

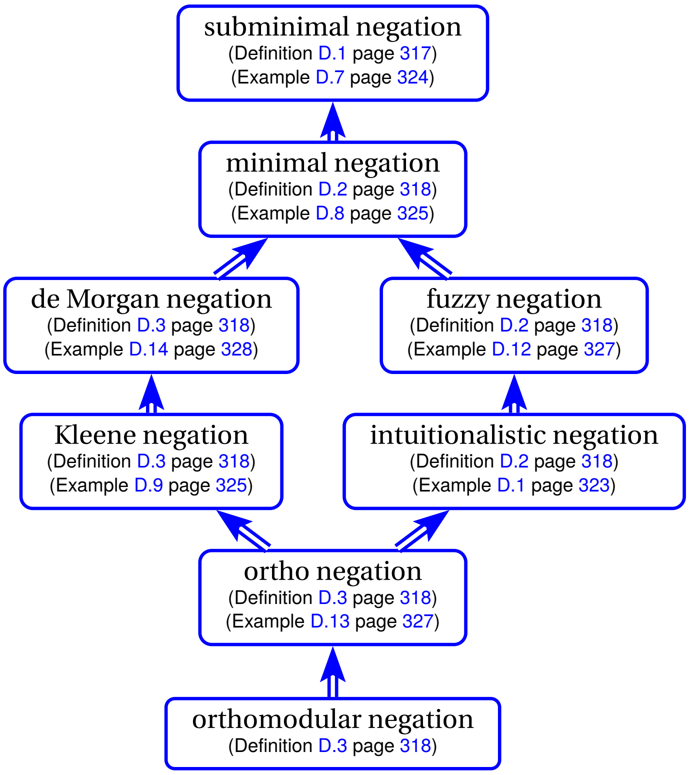

Figure D.1: lattice of negations

Remark D.1. [5] In the context of natural language, D. Devidi argues that, *subminimal negation* (Definition D.1 page 317) is "difficult to take seriously as" a negation. He essentially gives this example: Let $x \triangleq$"p is a fish" and $y \triangleq$"p has gills". Suppose "p is a fish" implies "p has gills" ($x \le y$). Now let $p \triangleq$ "many dogs". Then the *antitone* property and $x \le y$ tells us ($\implies$) that "Not many dogs have gills" implies that "Not many dogs are fish".

Definition D.2. [6] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE.

DEF

A FUNCTION $\neg \in X^X$ *is a* ***negation****, or* ***minimal negation****, on* $\mathbf{L}$ *if*

1. $x \le y \implies \neg y \le \neg x \quad \forall x,y \in X$ (ANTITONE) *and*
2. $x \le \neg\neg x \quad \forall x \in X$ (WEAK DOUBLE NEGATION).

A MINIMAL NEGATION $\neg$ *is an* ***intuitionistic negation*** *if*

3. $x \wedge \neg x = 0 \quad \forall x,y \in X$ (NON-CONTRADICTION).

A MINIMAL NEGATION $\neg$ *is a* ***fuzzy negation*** *if*

4. $\neg 1 = 0$ (BOUNDARY CONDITION).

Definition D.3. [7] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE.

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[5] Devidi (2010) page 511, Devidi (2006) page 568

[6] Dunn (1996) pages 4–6, Dunn (1999) pages 24–26 ⟨2 The Kite of Negations⟩, Troelstra and van Dalen (1988) page 4 ⟨1.6 Intuitionism. (b)⟩, de Vries (2007) page 11 ⟨Definition 16⟩, Gottwald (1999) page 21 ⟨Definition 3.3⟩, Novák et al. (1999) page 50 ⟨Definition 2.26⟩, Nguyen and Walker (2006) pages 98–99 ⟨5.4 Negations⟩, Höhle (1978) ⟨???⟩, Bellman and Giertz (1973) pages 155–156 ⟨(N1) $\neg 0 = 1$ and $\neg 1 = 0$, (N3) $\neg\neg x = x$⟩

[7] Dunn (1999) pages 24–26 ⟨2 The Kite of Negations⟩, Jenei (2003) page 283, Kalmbach (1983) page 22, Lidl and Pilz (1998) page 90, Husimi (1937)

DEF

A MINIMAL NEGATION $\neg$ *is a* ***de Morgan negation*** *if*

5. $x = \neg\neg x \quad \forall x \in X$ (INVOLUTORY).

A DE MORGAN NEGATION $\neg$ *is a* ***Kleene negation*** *if*

6. $x \wedge \neg x \leq y \vee \neg y \quad \forall x,y \in X$ (KLEENE CONDITION).

A DE MORGAN NEGATION $\neg$ *is an* ***ortho negation*** *if*

7. $x \wedge \neg x = 0 \quad \forall x,y \in X$ (NON-CONTRADICTION).

A DE MORGAN NEGATION $\neg$ *is an* ***orthomodular negation*** *if*

8. $x \wedge \neg x = 0 \quad \forall x,y \in X$ (NON-CONTRADICTION) *and*
9. $x \leq y \implies x \vee (y \wedge \neg x) = y \quad \forall x,y \in X$ (ORTHOMODULAR).

Remark D.2. [8] The *Kleene condition* is basically a weakened form of the *non-contradiction* and *excluded middle* properties because $\underbrace{x \wedge \neg x = 0}_{\textit{non-contradiction}} \leq \underbrace{1 = y \vee \neg y}_{\textit{excluded middle}}$.

Definition D.4. [9]

DEF

A MINIMAL NEGATION $\neg \in X^X$ *is* ***strict*** *($\neg$ is a* ***strict negation****) if*

1. $x \lneq y \implies \neg y \lneq \neg x \quad \forall x,y \in X$ (STRICTLY ANTITONE) *and*
2. $\neg$ *is* CONTINUOUS

A STRICT NEGATION $\neg$ *is* ***strong*** *($\neg$ is a* ***strong negation****) if*

3. $\neg\neg x = x \quad \forall x \in X$ (INVOLUTORY).

Definition D.5. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \leq)$ *be a* BOUNDED LATTICE *with a function* $\neg$ *in* X^X.

DEF

If $\neg$ *is a* MINIMAL NEGATION, *then* $\mathbf{L}$ *is a* ***lattice with negation***.

D.2 Properties of negations

Lemma D.1. [10] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$.

LEM

$$\underbrace{x \leq y \implies \neg y \leq \neg x\}}_{\text{ANTITONE}} \implies \begin{cases} \neg x \vee \neg y \leq \neg(x \wedge y) & \forall x,y \in X \quad \text{(CONJUNCTIVE DE MORGAN INEQ.) and} \\ \neg(x \vee y) \leq \neg x \wedge \neg y & \forall x,y \in X \quad \text{(DISJUNCTIVE DE MORGAN INEQ.) and} \end{cases}$$

✎PROOF:

1. Proof that *antitone* $\implies$ *conjunctive de Morgan*:

$x \wedge y \leq x$ and $x \wedge y \leq y$	by definition of $\wedge$
$\implies \neg(x \wedge y) \geq \neg x$ and $\neg(x \wedge y) \geq \neg y$	by *antitone*
$\implies \neg(x \wedge y) \geq \neg x \vee \neg y$	by definition of $\vee$

2. Proof that *antitone* $\implies$ *disjunctive de Morgan*:

$x \leq x \vee y$ and $y \leq x \vee y$	by definition of $\vee$
$\implies \neg x \geq \neg(x \vee y)$ and $\neg y \geq \neg(x \vee y)$	by *antitone*
$\implies \neg x \wedge \neg y \geq \neg(x \vee y)$	by definition of $\wedge$
$\implies \neg(x \vee y) \leq \neg x \wedge \neg y$	

[8] Cattaneo and Ciucci (2009) page 78
[9] Fodor and Yager (2000), pages 127–128, Bellman and Giertz (1973)
[10] Beran (1985) page 31 ⟨Theorem 1.2 Proof⟩, Fáy (1967) page 268 ⟨Lemma 1 Proof⟩, de Vries (2007) page 12 ⟨Theorem 18⟩

Lemma D.2. [11] *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ (*Definition C.3 page 301*).

LEM

If $x = (\neg\neg x)$ *for all* $x \in X$ (INVOLUTORY), *then*

$$\underbrace{x \leq y \implies \neg y \leq \neg x\}}_{\text{ANTITONE}} \iff \underbrace{\left\{\begin{array}{lclll} \neg(x \vee y) & = & \neg x \wedge \neg y & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN)} \quad \textit{and} \\ \neg(x \wedge y) & = & \neg x \vee \neg y & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)} \end{array}\right.}_{\text{DE MORGAN}}$$

✎PROOF:

1. Proof that *antitone* $\implies$ *de Morgan* equalities:

   (a) Proof that $\neg(\neg x \wedge \neg y) \geq x \vee y$:

   $$\begin{aligned} \neg(\neg x \wedge \neg y) &\geq \neg\neg x \vee \neg\neg y && \text{by Lemma D.1} \\ &= x \vee y && \text{by } \textit{involutory} \text{ property (Definition D.5 page 319)} \end{aligned}$$

   (b) Proof that $\neg(\neg x \vee \neg y) \leq x \wedge y$:

   $$\begin{aligned} \neg(\neg x \vee \neg y) &\leq \neg\neg x \wedge \neg\neg y && \text{by Lemma D.1} \\ &= x \wedge y && \text{by } \textit{involutory} \text{ property (Definition D.5 page 319)} \end{aligned}$$

   (c) Proof that $\neg(x \wedge y) = \neg x \vee \neg y$:

   $$\begin{aligned} \neg(x \wedge y) &\geq \neg x \vee \neg y && \text{by Lemma D.1} \\ \neg(x \wedge y) &= \neg[\neg\neg x \wedge \neg\neg y] && \text{by } \textit{involutory} \text{ property (Definition D.5 page 319)} \\ &\leq \neg x \vee \neg y && \text{by item (1b)} \end{aligned}$$

   (d) Proof that $\neg(x \vee y) = \neg x \wedge \neg y$:

   $$\begin{aligned} \neg(x \vee y) &\geq \neg x \wedge \neg y && \text{by Lemma D.1} \\ \neg(x \vee y) &= \neg[\neg\neg x \vee \neg\neg y] && \text{by } \textit{involutory} \text{ property (Definition D.5 page 319)} \\ &\leq \neg x \wedge \neg y && \text{by item (1a)} \end{aligned}$$

2. Proof that *antitone* $\impliedby$ *de Morgan*:

   $$\begin{aligned} x \leq y \implies \neg y &= \neg(x \vee y) && \text{because } x \leq y \\ &= \neg x \wedge \neg y && \text{by } \textit{de Morgan} \\ &\leq \neg x && \text{by definition of } \wedge \end{aligned}$$

Lemma D.3. *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ (*Definition C.3 page 301*).

LEM

$$\left\{\begin{array}{llll} 1. & x \leq \neg\neg x & \forall x \in X & \text{(WEAK DOUBLE NEGATION)} \quad \textit{and} \\ 2. & \neg 1 = 0 & & \text{(BOUNDARY CONDITION)} \end{array}\right\} \implies \left\{ \neg 0 = 1 \quad \text{(BOUNDARY CONDITION)} \right\}$$

✎PROOF:

$$\begin{aligned} \neg 0 &= \neg\neg 1 && \text{by } \textit{boundary condition} \text{ hypothesis (2)} \\ &\geq 1 && \text{by } \textit{weak double negation} \text{ hypothesis (1)} \\ &\implies \neg 0 = 1 && \text{by } \textit{upper bound} \text{ property} \end{aligned}$$

[11] Beran (1985) pages 30–31 ⟨Theorem 1.2⟩, Fáy (1967) page 268 ⟨Lemma 1⟩, Nakano and Romberger (1971) ⟨cf Beran 1985⟩

Lemma D.4. *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ (Definition C.3 page 301).

LEM

$$\left\{ (x \wedge \neg x = 0 \quad \forall x \in X \quad \text{(NON-CONTRADICTION)} \right\} \implies \left\{ \neg 1 = 0 \quad \text{(BOUNDARY CONDITION)} \right\}$$

✎PROOF:

$$\begin{aligned} 0 &= 1 \wedge \neg 1 && \text{by } \textit{non-contradiction} \text{ hypothesis} \\ &= \neg 1 && \text{by definition of g.u.b. } 1 \text{ and } \wedge \end{aligned}$$

✏

Lemma D.5. [12] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$.

LEM

$$\left\{ \begin{array}{lll} \text{(A).} & \neg \text{ is BIJECTIVE} & \text{and} \\ \text{(B).} & x \leq y \implies \neg y \leq \neg x \quad \forall x,y \in X & \text{(ANTITONE)} \end{array} \right\} \implies \underbrace{\left\{ \begin{array}{llll} \text{(1).} & \neg 0 = 1 & \text{and} \\ \text{(2).} & \neg 1 = 0 & \end{array} \right\}}_{\text{BOUNDARY CONDITIONS}}$$

✎PROOF:

1. Proof that $\neg 0 = 1$:

$$\begin{aligned} & x \leq 1 && \forall x \in X && \text{by definition of l.u.b. } 1 \\ \implies & \neg 1 \leq \neg x && \forall x \in X && \text{by } \textit{antitone} \text{ hypothesis} \\ \implies & \neg 1 \leq y && \forall y \in X && \text{by } \textit{bijective} \text{ hypothesis} \\ \implies & \neg 1 = 0 && && \text{by definition of g.l.b. } 0 \end{aligned}$$

2. Proof that $\neg 0 = 1$:

$$\begin{aligned} & 0 \leq x && \forall x \in X && \text{by definition of g.l.b. } 0 \\ \implies & \neg x \leq \neg 0 && \forall x \in X && \text{by } \textit{antitone} \text{ hypothesis} \\ \implies & \neg x \leq y && \forall y \in X && \text{by } \textit{bijective} \text{ hypothesis} \\ \implies & \neg 0 = 1 && && \text{by definition of l.u.b. } 1 \end{aligned}$$

✏

Theorem D.1. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$.

THM

$$\left\{ \begin{array}{l} \neg \text{ is an} \\ \text{INTUITIONISTIC NEGATION} \end{array} \right\} \implies \left\{ \neg 1 = 0 \quad \text{(BOUNDARY CONDITION)} \right\}$$

✎PROOF: This follows directly from Definition D.5 (page 319) and Lemma D.4 (page 321). ✏

Theorem D.2. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$.

THM

$$\left\{ \begin{array}{l} \neg \text{ is a} \\ \text{FUZZY NEGATION} \end{array} \right\} \implies \left\{ \neg 0 = 1 \quad \text{(BOUNDARY CONDITION)} \right\}$$

✎PROOF: This follows directly from Definition D.2 (page 318) and Lemma D.3 (page 320). ✏

[12] Varadarajan (1985) page 42

Theorem D.3. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$.

$$\text{THM}\quad \left\{\begin{array}{l}\neg \text{ is a}\\ \text{minimal}\\ \text{negation}\end{array}\right\} \implies \left\{\begin{array}{llll}\neg x \vee \neg y & \leq & \neg(x \wedge y) \quad \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN INEQUALITY) and}\\ \neg(x \vee y) & \leq & \neg x \wedge \neg y \quad \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN INEQUALITY)}\end{array}\right\}$$

✎PROOF: This follows directly from Definition D.5 (page 319) and Lemma D.1 (page 319). ⇨

Theorem D.4. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$.

$$\text{THM}\quad \left\{\begin{array}{l}\neg \text{ is a}\\ \text{de Morgan negation}\end{array}\right\} \implies \left\{\begin{array}{llll}\neg(x \vee y) & = & \neg x \wedge \neg y \quad \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN) and}\\ \neg(x \wedge y) & = & \neg x \vee \neg y \quad \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)}\end{array}\right.$$

✎PROOF: This follows directly from Definition D.5 (page 319) and Lemma D.2 (page 320). ⇨

Theorem D.5. [13] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$.

$$\text{THM}\quad \left\{\begin{array}{l}\neg \text{ is an}\\ \text{ortho negation}\end{array}\right\} \implies \left\{\begin{array}{llllll}1. & \neg 0 & = & 1 & & \text{(BOUNDARY CONDITION) and}\\ 2. & \neg 1 & = & 0 & & \text{(BOUNDARY CONDITION) and}\\ 3. & \neg(x \vee y) & = & \neg x \wedge \neg y & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN) and}\\ 4. & \neg(x \wedge y) & = & \neg x \vee \neg y & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN) and}\\ 5. & x \vee \neg x & = & 1 & \forall x \in X & \text{(EXCLUDED MIDDLE) and}\\ 6. & x \wedge \neg x & \leq & y \vee \neg y & \forall x,y \in X & \text{(KLEENE CONDITION).}\end{array}\right\}$$

✎PROOF:

1. Proof for $0 = \neg 1$ boundary condition: by Lemma D.4 (page 321)
2. Proof for boundary conditions:

$$\begin{aligned} 1 &= \neg\neg 1 && \text{by } \textit{involutory} \text{ property}\\ &= \neg 0 && \text{by previous result}\end{aligned}$$

3. Proof for *de Morgan* properties:
   (a) By Definition D.5 (page 319), *ortho negation* is *involutory* and *antitone*.
   (b) Therefore by Lemma D.2 (page 320), *de Morgan* properties hold.
4. Proof for *excluded middle* property:

$$\begin{aligned} x \vee \neg x &= (x \vee \neg x)^{\neg\neg} && \text{by } \textit{involutory} \text{ property of } \textit{ortho negation} \text{ (Definition D.5 page 319)}\\ &= \neg(x\neg \wedge x^{\neg\neg}) && \text{by } \textit{disjunctive de Morgan} \text{ property}\\ &= \neg(\neg x \wedge x) && \text{by } \textit{involutory} \text{ property of } \textit{ortho negation} \text{ (Definition D.5 page 319)}\\ &= \neg(x \wedge \neg x) && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Definition C.3 page 301)}\\ &= \neg 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{ortho negation} \text{ (Definition D.5 page 319)}\\ &= 1 && \text{by } \textit{boundary condition} \text{ (item (2) page 322) of } \textit{minimal negation}\end{aligned}$$

5. Proof for *Kleene condition*:

$$\begin{aligned} x \wedge \neg x &= 0 && \text{by } \textit{non-contradiction} \text{ property (Definition D.5 page 319)}\\ &\leq 1 && \text{by definition of 0 and 1}\\ &= y \vee \neg y && \text{by } \textit{excluded middle} \text{ property (item (4) page 322)}\end{aligned}$$

⇨

[13] Beran (1985) pages 30–31, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Cohen (1989) page 37 ⟨3B.13. Theorem⟩

D.3 Examples

Example D.1 (discrete negation). [14] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \leq)$ be a *bounded lattice* with a function $\neg \in X^X$.

EX The function $\neg x$ defined as

$$\neg x \triangleq \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

is an *intuitionistic negation* (Definition D.2 page 318) and a *fuzzy negation* (Definition D.2 page 318).

✎PROOF: To be an *intuitionistic negation*, $\neg x$ must be *antitone*, have *weak double negation*, and have the *non-contradiction* property (Definition D.2 page 318). To be a *fuzzy negation*, $\neg x$ must be *antitone*, have *weak double negation*, and have the *boundary condition* $\neg 1 = 0$.

$$\left\{\begin{array}{lclll} \neg y \leq \neg x & \iff & 1 \leq 1 & \text{for } 0 = x = y \\ \neg y \leq \neg x & \iff & 0 \leq 1 & \text{for } 0 = x \lneq y \\ \neg y \leq \neg x & \iff & 0 \leq 0 & \text{for } 0 \neq x \leq y \end{array}\right\} \implies \neg x \text{ is } \textit{antitone}$$

$$\left\{\begin{array}{lllllllll} \neg\neg x & = & \neg 1 & = & 0 & \geq & 0 & = & x & \text{for } x = 0 \\ \neg\neg x & = & \neg 0 & = & 1 & \geq & x & = & x & \text{for } x \neq 0 \end{array}\right\} \implies \neg x \text{ has } \textit{weak double negation}$$

$$\left\{\begin{array}{lllllll} x \wedge \neg x & = & x \wedge 1 & = & 0 \wedge 0 & = & 0 & \text{for } x = 0 \\ x \wedge \neg x & = & x \wedge 0 & = & x \wedge 0 & = & 0 & \text{for } x \neq 0 \end{array}\right\} \implies \neg x \text{ has } \textit{non-contradiction} \text{ property}$$

$$\neg 1 = 0 \implies \neg x \text{ has the } \textit{boundary condition} \text{ property}$$

✏

Example D.2 (dual discrete negation). [15] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \leq)$ be a *bounded lattice* with a function $\neg \in X^X$.

EX The function $\neg x$ defined as

$$\neg x \triangleq \begin{cases} 0 & \text{for } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

is a *subminimal negation* (Definition D.1 page 317) but it is *not* a *minimal negation* (Definition D.2 page 318) (and not any other negation defined here).

✎PROOF: To be an *subminimal negation*, $\neg x$ must be *antitone* (Definition D.1 page 317). To be a *minimal negation*, $\neg x$ must be *antitone* and have *weak double negation* (Definition D.2 page 318).

$$\left\{\begin{array}{lclll} \neg y \leq \neg x & \iff & 0 \leq 0 & \text{for } x = y = 1 \\ \neg y \leq \neg x & \iff & 0 \leq 1 & \text{for } x \lneq y = 1 \\ \neg y \leq \neg x & \iff & 1 \leq 1 & \text{for } x \leq y \neq 1 \end{array}\right\} \implies \neg x \text{ is } \textit{antitone}$$

$$\left\{\begin{array}{lllllll} \neg\neg x & = & \neg 0 & = & 1 & \geq & x & \text{for } x = 1 \\ \neg\neg x & = & \neg 1 & = & 0 & \leq & x & \text{for } x \neq 1 \end{array}\right\} \implies \neg x \text{ does } \textit{not} \text{ have } \textit{weak double negation}$$

✏

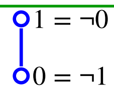
Example D.3. [16] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ be a *bounded lattice*

EX The function $\neg x$ is an *intuitionistic negation* (Definition D.2 page 318) if

$$\neg x \triangleq \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example D.4.

EX The function $\neg$ illustrated to the right is an *ortho negation* (Definition D.3 page 318).



[14] Fodor and Yager (2000) page 128, Yager (1980) pages 256–257, Yager (1979) ⟨cf Fodor⟩
[15] Fodor and Yager (2000) page 128, Ovchinnikov (1983) page 235 ⟨Example 4⟩
[16] Fodor and Yager (2000) page 128

✎PROOF:

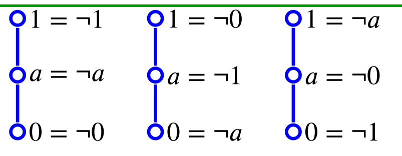
1. Proof that ¬ is *antitone*: $0 \le 1 \implies \neg 1 = 0 \le x = \neg 0 \implies \neg$ is *antitone* over $(0,1)$
2. Proof that ¬ is *involutory*: $1 = \neg 0 = \neg\neg 1$
3. Proof that ¬ has the *non-contradiction* property: $$\begin{array}{rcl} 1 \wedge \neg 1 &=& 1 \wedge 0 = 0 \\ 0 \wedge \neg 0 &=& 0 \wedge 1 = 0 \end{array}$$

✏

Example D.5.

EX

The functions ¬ illustrated to the right are *not* any negation defined here. In particular, they are *not antitone.*



(a) (b) (c)

✎PROOF:

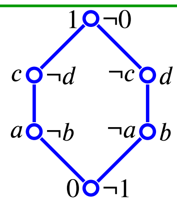
1. Proof that (a) is *not antitone*: $a \le 1 \implies \neg 1 = 1 \not\le a = \neg a$
2. Proof that (b) is *not antitone*: $a \le 1 \implies \neg 1 = a \not\le 0 = \neg a$
3. Proof that (c) is *not antitone*: $0 \le a \implies \neg a = 1 \not\le a = \neg 0$

✏

Example D.6.

EX

The function ¬ as illustrated to the right is *not* a *subminimal negation* (it is *not antitone*) and so is *not* any negation defined here. Note however that the problem is *not* the O_6 *lattice*—it is possible to define a negation on an O_6 *lattice* (Example D.16 page 329).



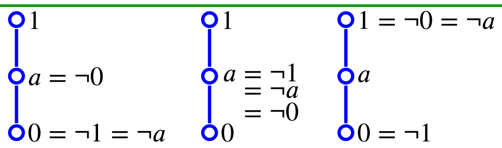
✎PROOF: Proof that ¬ is *not antitone*: $a \le c \implies \neg c = d \not\le b = \neg a$ ✏

Remark D.3. The concept of a *complement* and the concept of a *negation* are fundamentally different. A *complement* is a *relation* on a lattice $\boldsymbol{L}$ and a *negation* is a *function*. In Example D.6 (page 324), b and d are both complements of a, but yet ¬ is *not* a negation. In the right side lattice of Example D.16 (page 329), both b and d are complements of a (and so the lattice is *multipy complemented*), but yet only d is equal to the negation of a ($d = \neg a$). It can also be said that complementation is a property *of* a lattice, whereas negation is a function defined *on* a lattice.

Example D.7.

EX

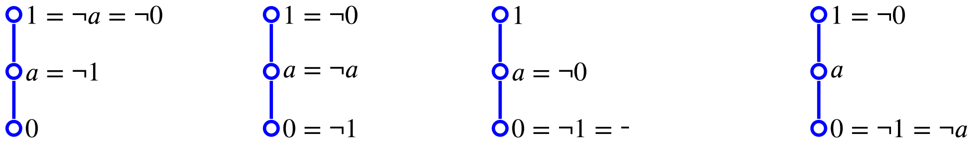
Each of the functions ¬ illustrated to the right is a *subminimal negation* (Definition D.1 page 317); *none* of them is a *minimal negation* (each fails to have *weak double negation*).



(a) (b) (c)

✎PROOF:

1. Proof that (a) ¬ is *antitone*: $$\begin{array}{rcl} a \le 1 &\implies& \neg 1 = 0 \le 0 = \neg a \implies \neg \text{ is antitone over } (a,1) \\ 0 \le 1 &\implies& \neg 1 = 0 \le a = \neg 0 \implies \neg \text{ is antitone over } (0,1) \\ 0 \le a &\implies& \neg a = 0 \le a = \neg 0 \implies \neg \text{ is antitone over } (0,a) \end{array}$$



(A) *minimal negation* (Example D.8 page 325) | (B) *Kleene negation* (Example D.9 page 325) | (C) *intuitionistic negation* (Example D.10 page 326) | (D) *fuzzy* and *intuitionistic* negation (Example D.11 page 326)

Figure D.2: negations on $\boldsymbol{L}_3$

2. Proof that (a) $\neg$ *fails* to have *weak double negation*:
   $1 \not\leq a = \neg 0 = \neg\neg 1$

3. Proof that (b) $\neg$ is *antitone*:
$$\begin{array}{lllllllllll}
a & \leq & 1 & \implies & \neg 1 & = & a & \leq & a & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ over } (a,1)\\
0 & \leq & 1 & \implies & \neg 1 & = & a & \leq & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,1)\\
0 & \leq & a & \implies & \neg a & = & a & \leq & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,a)
\end{array}$$

4. Proof that (b) $\neg$ *fails* to have *weak double negation*: $1 \not\leq a = \neg a = \neg\neg 1$

5. (c) is a special case of the *dual discrete negation* (Example D.2 page 323).

✎

Example D.8. The function $\neg$ illustrated in Figure D.2 page 325 (A) is a **minimal negation** (Definition D.2 page 318); it is *not* an *intuitionistic negation* (it does not have the *non-contradiction* property), it is *not* a *de Morgan negation* (it is not *involutory*), and it is *not* a *fuzzy negation* ($\neg 1 \neq 0$).

✎PROOF:

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{lllllllllll}
a & \leq & 1 & \implies & \neg 1 & = & a & \leq & 1 & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ over } (a,1)\\
0 & \leq & 1 & \implies & \neg 1 & = & a & \leq & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,1)\\
0 & \leq & a & \implies & \neg a & = & 1 & \leq & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,a)
\end{array}$$

2. Proof that $\neg$ is a *weak double negation* (and so is a *minimal negation*, but is *not* a *de Morgan negation*):
$$\begin{array}{lllllllll}
1 & = & 1 & = & \neg a & = & \neg\neg 1 & \implies & \neg \text{ is } \textit{involutory} \text{ at } 1\\
a & = & a & = & \neg 1 & = & \neg\neg a & \implies & \neg \text{ is } \textit{involutory} \text{ at } a\\
0 & \leq & a & = & \neg 1 & = & 0^{\neg\neg} & \implies & \neg \text{ is a } \textit{weak double negation} \text{ at } 0
\end{array}$$

3. Proof that $\neg$ does *not* have the *non-contradiction* property (and so is not an *intuitionistic negation*):
   $1 \wedge \neg 1 = 1 \wedge a = a \neq 0$

4. Proof that $\neg$ is not a *fuzzy negation*: $\neg 1 = a \neq 0$

✎

Example D.9 (Łukasiewicz 3-valued logic/Kleene 3-valued logic/RM_3 logic). [17] The function $\neg$ illustrated in Figure D.2 page 325 (B) is a **Kleene negation** (Definition D.3 page 318), and is also a *fuzzy negation* (Definition D.2 page 318); but it is *not* an *ortho negation* and is *not* an *intuitionistic negation* (it does not have the *non-contradiction* property).

✎PROOF:

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{lllllllllll}
a & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & a & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ over } (a,1)\\
0 & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,1)\\
0 & \leq & a & \implies & \neg a & = & a & \leq & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,a)
\end{array}$$

[17] Łukasiewicz (1920), Avron (1991) pages 277–278, Kleene (1938) page 153, Kleene (1952) pages 332–339 ⟨§64. The 3-valued logic⟩, Sobociński (1952)

2. Proof that ¬ is *involutory* (and so is a *de Morgan negation*):

$$\begin{array}{ccccccl} 1 &=& \neg 0 &=& \neg\neg 1 & \implies & \neg \text{ is } \textit{involutory} \text{ at } 1\\ a &=& \neg a &=& \neg\neg a & \implies & \neg \text{ is } \textit{involutory} \text{ at } a\\ 0 &=& \neg 0 &=& 0^{\neg\neg} & \implies & \neg \text{ is } \textit{involutory} \text{ at } 0 \end{array}$$

3. Proof that ¬ does *not* have the *non-contradiction* property (and so is not an *ortho negation*):
$x \wedge \neg x = x \wedge x = x \neq 0$

4. Proof that ¬ satisfies the *Kleene condition* (and so is a *Kleene negation*):

$$\begin{array}{ccccccccccccccccc} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 & \le & a & = & a & \vee & a & = & a & \vee & \neg a\\ 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 & \le & 1 & = & 0 & \vee & 1 & = & 0 & \vee & \neg 0\\ a & \wedge & \neg a & = & 1 & \wedge & a & = & a & \le & 1 & = & 1 & \vee & 0 & = & 1 & \vee & \neg 1\\ a & \wedge & \neg a & = & 1 & \wedge & a & = & a & \le & 1 & = & 0 & \vee & 1 & = & 0 & \vee & \neg 0\\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 & \le & 1 & = & 1 & \vee & 0 & = & 1 & \vee & \neg 1\\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 & \le & a & = & a & \vee & a & = & a & \vee & \neg a \end{array}$$

✏

Example D.10. The function ¬ illustrated in Figure D.2 page 325 (C) an **intuitionistic negation** (Definition D.2 page 318); but it is *not* a *fuzzy negation* ($1 \neq \neg 0$), and it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF:

1. Proof that ¬ is *antitone*:

$$\begin{array}{cccccccccccl} a & \le & 1 & \implies & \neg 1 & = & 0 & \le & 0 & = & \neg a & \implies \neg \text{ is } \textit{antitone} \text{ at } (a,1)\\ 0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & a & = & \neg 0 & \implies \neg \text{ is } \textit{antitone} \text{ at } (0,1)\\ 0 & \le & a & \implies & \neg a & = & 0 & \le & a & = & \neg 0 & \implies \neg \text{ is } \textit{antitone} \text{ at } (0,a) \end{array}$$

2. Proof that ¬ has *weak double negation* property (and so is a *minimal negation*, but *not* a *de Morgan negation*):

$$\begin{array}{cccccccl} 1 & \le & a & = & \neg 0 & = & \neg\neg 1 & \implies \neg \text{ has } \textit{weak double negation} \text{ at } 1\\ a & = & \neg 0 & = & \neg\neg a & & & \implies \neg \text{ has } \textit{weak double negation} \text{ at } a\\ 0 & = & \neg a & = & 0^{\neg\neg} & & & \implies \neg \text{ is } \textit{involutory} \text{ at } 0 \end{array}$$

3. Proof that ¬ has the *non-contradiction* property (and so is an *intuitionistic negation*):

$$\begin{array}{ccccccccc} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0\\ a & \wedge & \neg a & = & a & \wedge & 0 & = & 0\\ 0 & \wedge & \neg 0 & = & 0 & \wedge & a & = & 0 \end{array}$$

4. Proof that ¬ is *not* a *fuzzy negation*: $\neg 1 \neq 0$

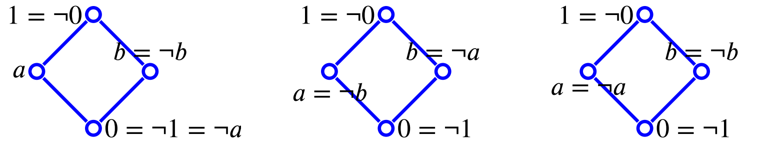
✏

Example D.11 (Heyting 3-valued logic/Jaśkowski's first matrix). [18] The function ¬ illustrated in Figure D.2 page 325 (D) is an **intuitionistic negation** (Definition D.2 page 318), and is also a **fuzzy negation** (Definition D.2 page 318), but it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF: This is simply a special case of the *discrete negation* (Example D.1 page 323). ✏

Remark D.4. There is only one linearly ordered (Definition B.4 page 287) 3-element lattice ($\boldsymbol{L}_3$) that is a *fuzzy negation* (Example D.11 page 326). However, this lattice is also an *intuitionistic negation*. There are no $\boldsymbol{L}_3$ lattices that are *fuzzy* but yet not *intuitionistic*. In fact, there are only three linearly ordered 3-element lattices with with $1 = \neg 0$ and $0 = \neg 1$. Of these three, only one is both *fuzzy* and *intuitionistic*

[18] Karpenko (2006) page 45, Johnstone (1982) page 9 ⟨§1.12⟩, Heyting (1930a), Heyting (1930b), Heyting (1930c), Heyting (1930d), Jaskowski (1936), Mancosu (1998)



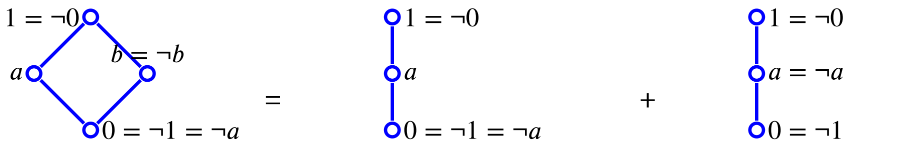
(A) *fuzzy negation* (Example D.12 page 327) (B) *ortho negation* (Example D.13 page 327) (C) *de Morgan negation* (Example D.14 page 328)

Figure D.3: negations on $\boldsymbol{M}_2$

(Example D.11 page 326), one is *Kleene* but not *fuzzy* (Example D.9 page 325), and one is *subminimal* but not *fuzzy* (Example D.7 page 324). It can be claimed that the "simplist" *fuzzy negation* that is not *de Morgan* and *not intuitionistic* is the $\boldsymbol{M}_2$ lattice of Example D.12 (next).

Example D.12. The function ¬ illustrated in Figure D.3 page 327 (A) is a **fuzzy negation** (Definition D.2 page 318). It is not an *intuitionistic negation* (it does not have the *non-contradiction* property) and it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF: Note that



(Example D.12 page 327) | *fuzzy* and *intuitionistic* (Example D.11 page 326) | *Kleene negation* (Example D.9 page 325)

1. Proof that ¬ is *antitone*:
$$\begin{array}{ccccccccccc}
a & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & 0 & = & \neg a & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (a,1)\\
0 & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & 1 & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,1)\\
0 & \le & a & \Longrightarrow & \neg a & = & 0 & \le & 1 & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,a)\\
b & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & b & = & \neg b & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (b,1)\\
0 & \le & b & \Longrightarrow & \neg b & = & b & \le & 1 & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,b)
\end{array}$$

2. Proof that ¬ has *weak double negation* property (and so is a *minimal negation*, but *not* a *de Morgan negation*):
$$\begin{array}{cccccccl}
1 & = & \neg 0 & = & \neg\neg 1 & & & \Longrightarrow \neg \text{ is } \textit{involutory} \text{ at } 1\\
a & \le & 1 & = & \neg 0 & = & \neg\neg a & \Longrightarrow \neg \text{ has } \textit{weak double negation} \text{ at } a\\
0 & = & \neg 1 & = & 0^{\neg\neg} & & & \Longrightarrow \neg \text{ is } \textit{involutory} \text{ at } 0\\
b & = & \neg b & = & \neg\neg b & = & & \Longrightarrow \neg \text{ is } \textit{involutory} \text{ at } b
\end{array}$$

3. Proof that ¬ does *not* have the *non-contradiction* property (and so is *not* an *intuitionistic negation*): $b \wedge \neg b = b \wedge b = b \neq 0$

4. Proof that ¬ is has *boundary conditions* (and so is a *fuzzy negation*): $\neg 1 = 0, \quad \neg 0 = 1$

✏

Example D.13. [19] The function ¬ illustrated in Figure D.3 page 327 (B) is an *ortho negation* (Definition D.3 page 318).

✎PROOF:

[19] Belnap (1977) page 13 Restall (2000) page 177 ⟨Example 8.44⟩, Pavičić and Megill (2008) page 28 ⟨Definition 2, *classical implication*⟩

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{ccccccccccc}
a & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & b & = & \neg a\\
0 & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & 1 & = & \neg 0\\
0 & \leq & a & \implies & \neg a & = & b & \leq & 1 & = & \neg 0\\
b & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & a & = & \neg b\\
0 & \leq & b & \implies & \neg b & = & a & \leq & 1 & = & \neg 0
\end{array}$$

2. Proof that $\neg$ is *involutory* (and so is a *de Morgan negation*):
$$\begin{array}{ccccc}
1 & = & \neg 0 & = & \neg\neg 1\\
a & = & \neg a & = & \neg\neg a\\
b & = & \neg b & = & \neg\neg b\\
0 & = & \neg 0 & = & 0^{\neg\neg}
\end{array}$$

3. Proof that $\neg$ is has the *non-contradiction* property (and so is an *ortho negation*):
$$\begin{array}{ccccccc}
1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0\\
a & \wedge & \neg a & = & a & \wedge & b & = & 0\\
b & \wedge & \neg b & = & b & \wedge & a & = & 0\\
0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0
\end{array}$$

⇨

Example D.14 (BN_4). [20] The function $\neg$ illustrated in Figure D.3 page 327 (C) is a **de Morgan negation** (Definition D.3 page 318), but it is *not* a *Kleene negation* and not an *ortho negation* (it does *not* satisfy the *Kleene condition*).

✎PROOF:

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{ccccccccccc}
a & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & b & = & \neg a\\
0 & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & 1 & = & \neg 0\\
0 & \leq & a & \implies & \neg a & = & a & \leq & 1 & = & \neg 0\\
b & \leq & 1 & \implies & \neg 1 & = & 0 & \leq & b & = & \neg b\\
0 & \leq & b & \implies & \neg b & = & b & \leq & 1 & = & \neg 0
\end{array}$$

2. Proof that $\neg$ is *involutory* (and so is a *de Morgan negation*):
$$\begin{array}{ccccc}
1 & = & \neg 0 & = & \neg\neg 1\\
a & = & \neg a & = & \neg\neg a\\
b & = & \neg b & = & \neg\neg b\\
0 & = & \neg 0 & = & 0^{\neg\neg}
\end{array}$$

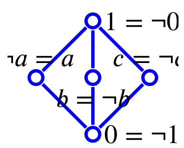
3. Proof that $\neg$ does *not* have the *non-contradiction* property (and so is *not* an *ortho negation*):
$$\begin{array}{ccccccccc}
a & \wedge & \neg a & = & a & \wedge & a & = & a & \neq & 0\\
b & \wedge & \neg b & = & b & \wedge & b & = & b & \neq & 0
\end{array}$$

4. Proof that $\neg$ does *not* satisfy the *Kleene condition* (and so is a *de Morgan negation*):
$$a \wedge \neg a = a \wedge a = a \not\leq b \wedge \neg b = b$$

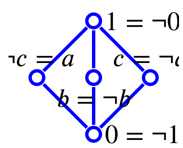
⇨

Example D.15.

EX

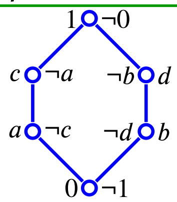


The function $\neg$ illustrated to the left is a *de Morgan negation* (Definition D.3 page 318), but it is *not* a *Kleene negation* and not an *ortho negation* (it does *not* satisfy the *Kleene condition*). The *negation* illustrated to the right is a *Kleene negation* (Definition D.3 page 318), but it is *not* an *ortho negation* (it does *not* have the *non-contradiction* property).

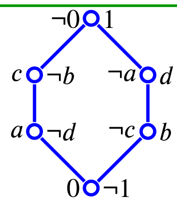


[20] Cignoli (1975) page 270, Restall (2000) page 171 ⟨Example 8.39⟩, de Vries (2007) pages 15–16 ⟨Example 26⟩, Dunn (1976), Belnap (1977)

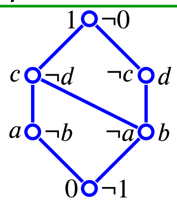
Example D.16.



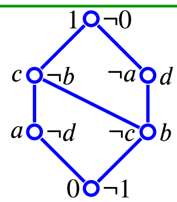
The function ¬ illustrated to the left is a *de Morgan negation* (Definition D.3 page 318); it is *not* a *Kleene negation* (it does not satisfy the Kleene condition). The *negation* illustrated to the right is an *ortho negation* (Definition D.3 page 318).



Example D.17.



The function ¬ illustrated to the left is *not antitone* and therefore is not a *negation* (Definition D.2 page 318). The function ¬ illustrated to the right is a *Kleene negation* (Definition D.3 page 318); it is *not* an *ortho negation* (it does not have the *non-contradiction* property).

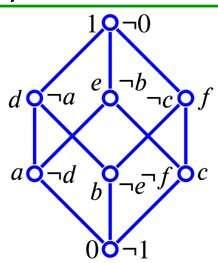


✎PROOF:

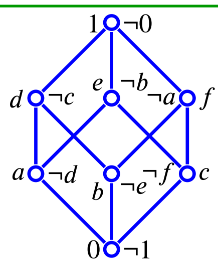
1. Proof that left ¬ is *not antitone*: $a \le c$ but $\neg c \not\le \neg a$.
2. Proof that right ¬ satisfies the *Kleene condition*:
$$x \wedge \neg x = \begin{cases} b & \text{for } x = b \\ 0 & \text{otherwise} \end{cases} \quad \forall x \in X \qquad \text{and} \qquad y \wedge \neg y = \begin{cases} c & \text{for } y = c \\ 0 & \text{otherwise} \end{cases} \quad \forall y \in X$$
$$\Longrightarrow \qquad x \wedge \neg x \le y \vee \neg y \qquad \forall x, y \in X$$
3. Proof that right ¬ does not have the *non-contradiction* property: $b \wedge \neg b = b \wedge c = b \neq 0$

✏

Example D.18.



The lattices illustrated to the left and right are *Boolean*. The function ¬ illustrated to the left is a *Kleene negation* (Definition D.3 page 318), but it is *not* an *ortho negation* (it does *not* have the *non-contradiction* property). The *negation* illustrated to the right is an *ortho negation* (Definition D.3 page 318).



✎PROOF:

1. Proof that left side negation does *not* have *non-contradiction* property (and so is *not* an *ortho negation*):
$a \wedge \neg a = a \wedge d = a \neq 0$
2. Proof that left side negation does *not* satisfy *Kleene condition* (and so is *not* a *Kleene negation*):
$a \wedge \neg a = a \wedge d = a \not\le f = c \vee f = c \vee \neg c$

✏

APPENDIX E

RELATIONS ON LATTICES WITH NEGATION

The relations in this chapter are typically defined on an *orthocomplemented lattice* (Definition 16.1 page 238). Here, some relations are generalized to a *lattice with negation* (Definition D.5 page 319). A *lattice* (Definition C.3 page 301) with an *ortho negation* negation successfully defined on it is an *orthocomplemented lattice* (Definition 16.1 page 238). In many cases, these relations only work well on an *orthocomplemented lattice*, and thus many results are restricted to orthocomplemented lattices.

E.1 Orthogonality

Proposition E.1. *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238).*

PRP
$$x \leq y \implies \left\{\begin{array}{ccccc} x^{\perp} & \vee & y & = & 1 \quad \text{and} \\ x & \wedge & y^{\perp} & = & 0 \end{array}\right\} \qquad \forall x,y \in X$$

✎PROOF:

$$\begin{aligned}
x \leq y &\implies x \vee x^{\perp} \leq y \vee x^{\perp} && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition C.1 page 303)}\\
&\implies 1 \leq y \vee x^{\perp} && \text{by } \textit{excluded middle} \text{ property of } \textit{ortho lattices} \text{ (Definition 16.1 page 238)}\\
&\implies x^{\perp} \vee y = 1 && \text{by } \textit{upper bounded} \text{ property of } \textit{bounded lattices}\\
x \leq y &\implies x \wedge y^{\perp} \leq y \wedge y^{\perp} && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition C.1 page 303)}\\
&\implies x \wedge y^{\perp} \leq 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{ortho lattices} \text{ (Definition 16.1 page 238)}\\
&\implies x \wedge y^{\perp} = 0 && \text{by } \textit{lower bounded} \text{ property of } \textit{bounded lattices}
\end{aligned}$$

✏

Definition E.1. [1] *Let* $(X, \vee, \wedge, \neg, 0, 1; \leq)$ *be a* LATTICE WITH NEGATION *(Definition D.5 page 319).*

DEF
*The **orthogonality** relation* $\perp \in 2^{XX}$ *is defined as*

$$x \perp y \quad \overset{\text{def}}{\iff} \quad x \leq \neg y$$

If $x \perp y$*, we say that* x *is **orthogonal** to* y*.*

[1] Stern (1999) page 12, Loomis (1955) page 3

Lemma E.1. *Let* $(X, \vee, \wedge, \neg, 0, 1\,;\, \leq)$ *be a* LATTICE WITH NEGATION *(Definition D.5 page 319)*.

LEM

$$\left\{\, x \perp y \quad (\text{ORTHOGONAL Definition E.1 page 331}) \,\right\} \implies \left\{\, y \perp x \quad (\text{SYMMETRIC}) \,\right\}$$

✎PROOF:

$$\begin{aligned}
x \perp y &\implies x \leq \neg y && \text{by definition of } \perp \text{ (Definition E.1 page 331)}\\
&\implies (\neg\neg y) \leq \neg x && \text{by } \textit{antitone} \text{ property (Definition 16.1 page 238)}\\
&\implies y \leq \neg x && \text{by } \textit{weak double negation} \text{ property of } \textit{negation} \text{ (Definition D.2 page 318)}\\
&\implies y \perp x && \text{by definition of } \perp \text{ (Definition E.1 page 331)}
\end{aligned}$$

✏

Lemma E.2. [2] *Let* $(X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238)*.

LEM

$$\underbrace{x \perp y}_{\text{ORTHOGONAL (Definition E.1 page 331)}} \implies \left\{\begin{array}{llll} 1. & x \wedge y & = & 0 \quad \textit{and} \\ 2. & x^{\perp} \vee y^{\perp} & = & 1 \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
x \perp y &\implies x \leq y^{\perp} && \text{by definition of } \perp \text{ (Definition E.1 page 331)}\\
&\implies x \wedge y \leq y^{\perp} \wedge y && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition C.1 page 303)}\\
&\implies x \wedge y \leq y \wedge y^{\perp} && \text{by } \textit{commutative} \text{ property of lattices (Theorem C.3 page 302)}\\
&\implies x \wedge y \leq 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{ortho negation} \text{ (Definition D.3 page 318)}\\
&\implies x \wedge y = 0 && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices}
\end{aligned}$$

$$\begin{aligned}
x \perp y &\implies x \leq y^{\perp} && \text{by definition of } \perp \text{ (Definition E.1 page 331)}\\
&\implies x^{\perp} \vee x \leq x^{\perp} \vee y^{\perp} && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition C.1 page 303)}\\
&\implies x \vee x^{\perp} \leq x^{\perp} \vee y^{\perp} && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Theorem C.3 page 302)}\\
&\implies 1 \leq x^{\perp} \vee y^{\perp} && \text{by } \textit{excluded middle} \text{ property of } \textit{ortho lattices} \text{ (Theorem D.5 page 322)}\\
&\implies x^{\perp} \vee y^{\perp} && \text{by } \textit{upper bound} \text{ property of } \textit{bounded lattices}
\end{aligned}$$

✏

Remark E.1. In an *orthocomplemented lattice* $\boldsymbol{L}$, the *orthogonality* relation $\perp$ is in general *non-associative*. That is

$$\left\{\begin{array}{llll} x & \perp & y & \text{and} \\ y & \perp & z & \end{array}\right\} \not\Longrightarrow x \perp z$$

✎PROOF: Consider the $\boldsymbol{L}_2^4$ *Boolean lattice* in Example 16.2 (page 238).

- $a^{\perp} \perp p$ because $a^{\perp} \leq p^{\perp}$.
- $p \perp r$ because $p \leq r^{\perp}$.
- But yet $a^{\perp}$ is *not* orthogonal to r because $a^{\perp} \not\leq r^{\perp}$.

✏

Example E.1.

EX

In the O_6 *lattice* (Definition 16.2 page 238), there are a total of $\binom{6}{2} = \frac{6!}{(6-2)!2!} = \frac{6\times 5}{2} = 15$ distinct unordered (the $\perp$ relation is *symmetric* by Lemma E.1 page 332 so the order doesn't matter) pairs of elements.

Of these 15 pairs, 8 are orthogonal to each other, and 0 is orthogonal to itself, making a total of 9 orthogonal pairs:

$x \perp y$	$x \perp 0$	$y^{\perp} \perp 0$
$x \perp x^{\perp}$	$y \perp 0$	$1 \perp 0$
$y \perp y^{\perp}$	$x^{\perp} \perp 0$	$0 \perp 0$

[2] Holland (1963) page 67

Example E.2.

EX

In lattice 5 of Example 16.2 (page 238), there are a total of $\binom{10}{2} = \frac{10!}{(10-2)!2!} = \frac{10\times 9}{2} = 45$ distinct unordered pairs of elements.

Of these 45 pairs, 18 are orthogonal to each other, and 0 is orthogonal to itself, making a total of 19 orthogonal pairs:

$p \perp p^\perp$	$x \perp x^\perp$	$y \perp z$	$x^\perp \perp 0$
$p \perp x^\perp$	$x \perp y$	$y \perp 0$	$y^\perp \perp 0$
$p \perp y$	$x \perp z$	$z \perp z^\perp$	$z^\perp \perp 0$
$p \perp z$	$x \perp 0$	$z \perp 0$	$0 \perp 0$
$p \perp 0$	$y \perp y^\perp$	$p^\perp \perp 0$	

Example E.3.

EX

In the $\mathbb{R}^3$ **Euclidean space** illustrated in Example 16.3 (page 239),

$$\begin{array}{llll} \mathbf{X} \subseteq \mathbf{Y}^\perp \implies \mathbf{X} \perp \mathbf{Y} & & \mathbf{Y} \subseteq \mathbf{X}^\perp \implies \mathbf{Y} \perp \mathbf{X} \\ \mathbf{X} \subseteq \mathbf{Z}^\perp \implies \mathbf{X} \perp \mathbf{Z} & & \mathbf{Y} \subseteq \mathbf{Z}^\perp \implies \mathbf{Y} \perp \mathbf{Z} \\ \mathbf{X} \wedge \mathbf{Y} = \mathbf{X} \wedge \mathbf{Z} = \mathbf{Y} \wedge \mathbf{Z} = \mathbf{0} \end{array}$$

E.2 Commutativity

The *commutes* relation is defined next. Motivation for the name "commutes" is provided by Proposition E.4 (page 336) which shows that if x commutes with y in a lattice $\mathbf{L}$, then x and y commute in the *Sasaki projection* $\phi_x(y)$ on $\mathbf{L}$.

Definition E.2. [3] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \leq)$ *be a* LATTICE WITH NEGATION (Definition D.5 page 319).

DEF

*The **commutes** relation* © *is defined as*

$$x © y \quad \overset{\text{def}}{\iff} \quad x = (x \wedge y) \vee (x \wedge \neg y) \qquad \forall x,y \in X,$$

*in which case we say, "x **commutes** with y in $\mathbf{L}$".*
That is, © *is a relation in 2^{XX} such that*

$$© \triangleq \{(x,y) \in X^2 | x = (x \wedge y) \vee (x \wedge \neg y)\}$$

Proposition E.2. [4] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE.

PRP

$$\begin{array}{llll|llll} x©0 & \text{and} & 0©x & \forall x \in X & x©y & \iff & x©y^\perp & \forall x,y \in X \\ x©1 & \text{and} & 1©x & \forall x \in X & x \leq y & \implies & x©y & \forall x,y \in X \\ x©x & & & \forall x \in X & x \perp y & \implies & x©y & \forall x,y \in X \end{array}$$

✎PROOF:

$$\begin{aligned} (x \wedge 0) \vee (x \wedge 0^\perp) &= 0 \vee (x \wedge 0^\perp) && \text{by *lower bound* property of *bounded lattices*} \\ &= 0 \vee (x \wedge 1) && \text{by *boundary condition* of *ortho negation* (Theorem D.5 page 322)} \\ &= 0 \vee (x) && \text{by *upper bound* property of *bounded lattices*} \\ &= x && \text{by *lower bound* property of *bounded lattices*} \\ &\implies x©0 && \text{by definition of © relation (Definition E.2 page 333)} \\ (0 \wedge x) \vee (0 \wedge x^\perp) &= 0 \vee (0) && \text{by *lower bound* property of *bounded lattices*} \\ &= 0 && \text{by *lower bound* property of *bounded lattices*} \\ &\implies 0©x && \text{by definition of © relation (Definition E.2 page 333)} \end{aligned}$$

[3] Kalmbach (1983) page 20, Holland (1970) page 79 ⟨A. Commutativity⟩, Maeda (1958) page 227 ⟨Hilfssatz (Lemma) XII.1.2⟩, Sasaki (1954) page 301 ⟨Def.5.2, cf Foulis 1962⟩, Birkhoff (1936b) page 833 ⟨"$a = (a \cap x) \cup (a \cap x')$"⟩

[4] Holland (1963) page 67

$(x \wedge 1) \vee (x \wedge 1^{\perp}) = x \vee (x \wedge 1^{\perp})$	by *lower bound* property of *bounded lattices*
$= x \vee (x \wedge 0)$	by *boundary condition* of *ortho negation* (Theorem D.5 page 322)
$= (x) \vee (0)$	by *lower bound* property of *bounded lattices*
$= x$	by *lower bound* property of *bounded lattices*
$\implies x \copyright 1$	by definition of © relation (Definition E.2 page 333)
$(1 \wedge x) \vee (1 \wedge x^{\perp}) = (x) \vee (x^{\perp})$	by *non-contradiction* prop. of *ortho negation* (Definition D.3 page 318)
$= 1$	by *excluded middle* property of *ortho negation* (Theorem D.5 page 322)
$\implies 1 \copyright x$	by definition of © relation (Definition E.2 page 333)
$(x \wedge x) \vee (x \wedge x^{\perp}) = x \vee (x \wedge x^{\perp})$	by *idempotent* property of *lattices* (Theorem C.3 page 302)
$= x \vee (0)$	by *non-contradiction* prop. of *ortho negation* (Definition D.3 page 318)
$= x$	by *lower bound* property of *bounded lattices*
$\implies x \copyright x$	by definition of © relation (Definition E.2 page 333)
$x \copyright y \implies (x \wedge y^{\perp}) \vee (x \wedge y^{\perp\perp})$	by definition of © (Definition E.2 page 333)
$= (x \wedge y^{\perp}) \vee (x \wedge y)$	by *involution* property of $\perp$ (Definition 16.1 page 238)
$= (x \wedge y) \vee (x \wedge y^{\perp})$	by *commutative* property of *lattices* (Definition C.3 page 301)
$= x$	by $x \copyright y$ hypothesis and Definition E.2 page 333
$\implies x \copyright y^{\perp}$	by definition of © relation (Definition E.2 page 333)
$x \copyright y^{\perp} \implies (x \wedge y) \vee (x \wedge y^{\perp})$	by definition of © (Definition E.2 page 333)
$= (x \wedge y^{\perp\perp}) \vee (x \wedge y^{\perp})$	by *involution* property of $\perp$ (Definition 16.1 page 238)
$= (x \wedge y^{\perp}) \vee (x \wedge y^{\perp\perp})$	by *commutative* property of *lattices* (Definition C.3 page 301)
$= x$	by $x \copyright y^{\perp}$ hypothesis and Definition E.2 page 333
$\implies x \copyright y$	by definition of © relation (Definition E.2 page 333)
$x \leq y \implies (x \wedge y) \vee (x \wedge y^{\perp})$	by definition of © (Definition E.2 page 333)
$= x \vee (x \wedge y^{\perp})$	by $x \leq y$ hypothesis
$= x$	by *absorptive* property (Theorem C.3 page 302)
$\implies x \copyright y$	by definition of © (Definition E.2 page 333)
$x \perp y \implies (x \wedge y) \vee (x \wedge y^{\perp})$	by definition of © (Definition E.2 page 333)
$= 0 \vee (x \wedge y^{\perp})$	by Lemma E.2 page 332
$= 0 \vee x$	by $x \perp y$ hypothesis ($x \perp y \implies x \leq y^{\perp}$)
$= x \vee 0$	by *commutative* property (Theorem C.3 page 302)
$= x$	by *identity* property of *bounded lattices*
$\implies x \copyright y$	by definition of © (Definition E.2 page 333)

⇨

Definition E.3. *Let © be the* COMMUTES *relation (Definition E.2 page 333) on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1\,;\, \leq)$ *(Definition D.5 page 319).*

DEF | $\boldsymbol{L}$ *is* ***symmetric*** *if*

$$x \copyright y \implies y \copyright x \qquad \forall x,y \in X$$

In general, the commutes relation is not *symmetric*. But Proposition E.3 (next) describes some conditions under which it *is* symmetric.

Proposition E.3. [5] *Let* $(X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238).*

[5] Holland (1963) page 68, Nakamura (1957) page 158

PRP

$$\underbrace{\{x \circledcirc y \implies y \circledcirc x\}}_{\circledcirc \text{ is SYMMETRIC at } (x,y)\ (1)} \iff \{x \le y \implies y = x \vee (x^\perp \wedge y)\} \quad \text{(ORTHOMODULAR IDENTITY)} \quad (2)$$

$$\iff \{x \le y \implies x = y \wedge (x \vee y^\perp)\} \quad (x = \phi_y(x) \text{ (SASAKI PROJECTION)}) \quad (3)$$

$$\iff \{y = (x \wedge y) \vee [y \wedge (x \wedge y)^\perp]\} \quad (4)$$

$$\iff \{x = (x \vee y) \wedge [x \vee (x \vee y)^\perp]\} \quad (5)$$

✎PROOF:

1. Proof that (2) $\iff$ (3):

$$\begin{aligned}
x \le y \implies y^\perp &\le x^\perp && \text{by \emph{antitone} property (Definition 16.1 page 238)}\\
\implies x^\perp &= y^\perp \vee (y^{\perp\perp} \wedge x^\perp) && \text{by left hypothesis}\\
\implies (x^\perp)^\perp &= [y^\perp \vee (y^{\perp\perp} \wedge x^\perp)]^\perp &&\\
\implies x &= [y^\perp \vee (y^{\perp\perp} \wedge x^\perp)]^\perp && \text{by \emph{involutory} property (Definition 16.1 page 238)}\\
&= y^{\perp\perp} \wedge (y^{\perp\perp} \wedge x^\perp)^\perp && \text{by \emph{de Morgan} property (Theorem 16.1 page 240)}\\
&= y \wedge (y \wedge x^\perp)^\perp && \text{by \emph{involutory} property (Definition 16.1 page 238)}\\
&= y \wedge (y^\perp \vee x^{\perp\perp}) && \text{by \emph{de Morgan} property (Theorem 16.1 page 240)}\\
&= y \wedge (y^\perp \vee x) && \text{by \emph{involutory} property (Definition 16.1 page 238)}\\
&= y \wedge (x \vee y^\perp) && \text{by \emph{commutative} property (Theorem C.3 page 302)}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies y^\perp &\le x^\perp && \text{by \emph{antitone} property (Definition 16.1 page 238)}\\
\implies y^\perp &= x^\perp \wedge (y^\perp \vee x^{\perp\perp}) && \text{by right hypothesis}\\
\implies (y^\perp)^\perp &= [x^\perp \wedge (y^\perp \vee x^{\perp\perp})]^\perp &&\\
\implies y &= [x^\perp \wedge (y^\perp \vee x^{\perp\perp})]^\perp && \text{by \emph{involutory} property (Definition 16.1 page 238)}\\
&= x^{\perp\perp} \vee (y^\perp \vee x^{\perp\perp})^\perp && \text{by \emph{de Morgan} property (Theorem 16.1 page 240)}\\
&= x \vee (y^\perp \vee x)^\perp && \text{by \emph{involutory} property (Definition 16.1 page 238)}\\
&= x \vee (y^{\perp\perp} \wedge x^\perp) && \text{by \emph{de Morgan} property (Theorem 16.1 page 240)}\\
&= x \vee (y \wedge x^\perp) && \text{by \emph{involutory} property (Definition 16.1 page 238)}\\
&= x \vee (x^\perp \wedge y) && \text{by \emph{commutative} property (Theorem C.3 page 302)}
\end{aligned}$$

2. Proof that (2) $\iff$ (4):

$$\begin{aligned}
(xy) \vee [y(xy)^\perp] &= u \vee [yu^\perp] && \text{where } u \triangleq xy \le y\\
&= u \vee [u^\perp y] && \text{by \emph{commutative} property of lattices (Theorem C.3 page 302)}\\
&= y && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies x \vee (x^\perp y) &= xy \vee [(xy)^\perp y] && \text{by } x \le y \text{ hypothesis}\\
&= xy \vee [y(xy)^\perp] && \text{by \emph{commutative} property of lattices (Theorem C.3 page 302)}\\
&= y && \text{by right hypothesis}
\end{aligned}$$

3. Proof that (3) $\iff$ (5):

$$\begin{aligned}
(x \vee y)[x \vee (x \vee y)^\perp] &= u[x \vee u^\perp] && \text{where } x \le u \triangleq x \vee y\\
&= x && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies y(x \vee y^\perp) &= (x \vee y)[x \vee (x \vee y)^\perp] && \text{by } x \le y \text{ hypothesis}\\
&= x && \text{by right hypothesis}
\end{aligned}$$

4. Proof that (1) $\implies$ (2):

$$\begin{aligned}
x \le y \implies\; & x \copyright y && \text{by Proposition E.2 page 333}\\
\implies\; & y \copyright x && \text{by } \textit{symmetry} \text{ hypothesis (left hypothesis)}\\
\implies\; & y = (y \wedge x) \vee \left(y \wedge x^{\perp}\right) && \text{by definition of } \copyright \text{ (Definition E.2 page 333)}\\
\implies\; & y = x \vee \left(y \wedge x^{\perp}\right) && \text{by } x \le y \text{ hypothesis}\\
\implies\; & y = x \vee \left(x^{\perp} \wedge y\right) && \text{by } \textit{commutative} \text{ property of lattices (Theorem C.3 page 302)}
\end{aligned}$$

5. Proof that (2) $\implies$ (4):

   (a) lemma: proof that $x\copyright y \implies x^{\perp}y = (xy)^{\perp}y$:

$$\begin{aligned}
x\copyright y \implies x^{\perp}y &= \left(xy \vee xy^{\perp}\right)^{\perp}y && \text{by definition of } \copyright \text{ (Definition E.2 page 333)}\\
&= (xy)^{\perp}\left(xy^{\perp}\right)^{\perp}y && \text{by } \textit{de Morgan}\text{'s law (Theorem D.4 page 322)}\\
&= (xy)^{\perp}\left[\left(x^{\perp} \vee y^{\perp\perp}\right)y\right] && \text{by } \textit{de Morgan}\text{'s law (Theorem D.4 page 322)}\\
&= (xy)^{\perp}\left[\left(x^{\perp} \vee y\right)y\right] && \text{by } \textit{involutory}\text{'s property (Definition 16.1 page 238)}\\
&= (xy)^{\perp}y && \text{by } \textit{absorptive} \text{ property of lattices (Theorem C.3 page 302)}
\end{aligned}$$

   (b) Completion of proof for (2) $\implies$ (4):

$$\begin{aligned}
x\copyright y \implies xy \vee y(xy)^{\perp} &= xy \vee (xy)^{\perp}y && \text{by } \textit{commutative} \text{ property (Theorem C.3 page 302)}\\
&= xy \vee x^{\perp}y && \text{by } x\copyright y \text{ hypothesis and item (5a)}\\
&= (yx) \vee \left[yx^{\perp}\right] && \text{by } \textit{commutative} \text{ property (Theorem C.3 page 302)}\\
&\implies y\copyright x && \text{by definition of } \copyright \text{ (Definition E.2 page 333)}
\end{aligned}$$

✏

Theorem E.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238).*

THM:
$$\{x\copyright c \quad \forall x \in X\} \iff \left\{\boldsymbol{L} \text{ is ISOMORPHIC to } [0:c] \times \left[0:c^{\perp}\right]\right\}$$
with isomorphism $\theta(x) \triangleq \left([0:c], \left[0:c^{\perp}\right]\right)$.

Proposition E.4. [7] *Let* $(X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOMODULAR *lattice.*

PRP:
$$x\copyright y \iff \phi_x(y) = \phi_y(x) = x \wedge y \qquad \forall x,y \in X$$

E.3 Center

An element in an *orthocomplemented lattice* (Definition 16.1 page 238) is in the *center* of the lattice if that element *commutes* (Definition E.2 page 333) with every other element in the lattice (next definition). *All* the elements of an *orthocomplemented lattice* are in the *center* if and only if that lattice is *Boolean* (Proposition 16.2 page 245).

Definition E.4. [8] *Let* $\copyright$ *be the* COMMUTES *relation (Definition E.2 page 333) on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ *(Definition D.5 page 319).*

DEF: *The* **center** *of* $\boldsymbol{L}$ *is defined as*
$$\{x \in X \mid x\copyright y \quad \forall y \in X\}$$

[6] Kalmbach (1983) page 20, MacLaren (1964)
[7] Foulis (1962) page 66, Sasaki (1954) ⟨cf Foulis 1962⟩
[8] Holland (1970) page 80

Proposition E.5. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238).*

PRP

0 *and* 1 *are in the* ***center*** *of* $\mathbf{L}$.

✎PROOF: This follows directly from Definition E.2 (page 333) and Proposition E.2 (page 333). ✏

Theorem E.2. [9] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 16.1 page 238).*

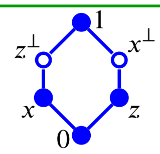
THM

The CENTER *of* $\mathbf{L}$ *is* BOOLEAN.

Example E.4.

EX

The **center** of the O_6 **lattice** (Definition 16.2 page 238) is the set $\{0, x, z, 1\}$. The elements $x^\perp$ and $z^\perp$ are **not** in the center of $\mathbf{L}$. The O_6 lattice is illustrated to the right, with the center elements as solid dots. Note that the center is the *Boolean* lattice $\mathbf{L}_2^2$ (Proposition 16.2 page 245).



✎PROOF:

1. Proof that 0 and 1 are in the *center* of $\mathbf{L}$: by Proposition E.5 (page 337).
2. Proof that x is in the *center* of $\mathbf{L}$:

$$\begin{aligned}
(x \wedge x) \vee (x \wedge x^\perp) &= x \vee 0 &&= x &&\implies x \copyright x\\
(x \wedge z) \vee (x \wedge z^\perp) &= 0 \vee x &&= x &&\implies x \copyright z
\end{aligned}$$

$x \copyright x$, $x \copyright x^\perp$, $x \copyright z^\perp$, $x \copyright 0$, and $x \copyright 1$ by Proposition E.2 (page 333).

3. Proof that z is in the *center* of $\mathbf{L}$:

$$\begin{aligned}
(z \wedge z) \vee (z \wedge z^\perp) &= z \vee 0 &&= z &&\implies z \copyright z\\
(z \wedge x) \vee (z \wedge x^\perp) &= 0 \vee z &&= z &&\implies z \copyright x
\end{aligned}$$

$z \copyright z$, $z \copyright x^\perp$, $z \copyright z^\perp$, $z \copyright 0$, and $z \copyright 1$ by Proposition E.2 (page 333).

4. Proof that $x^\perp$ and $z^\perp$ are *not* in the *center* of $\mathbf{L}$:

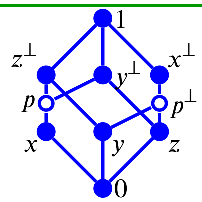
$$\begin{aligned}
(x^\perp \wedge y) \vee (x^\perp \wedge y^\perp) &= y \vee 0 &&= y &&\implies x^\perp \not\copyright y\\
(z^\perp \wedge x) \vee (z^\perp \wedge x^\perp) &= x \vee 0 &&= x &&\implies z^\perp \not\copyright x
\end{aligned}$$

✏

Example E.5.

EX

The **center** the lattice illustrated to the right (Example 16.2 page 238), with center elements as solid dots, is the set $\{0, 1, p, y, z, x^\perp, y^\perp, z^\perp,\}$. The elements x and $p^\perp$ are *not* in the *center* of $\mathbf{L}$. Note that the center is the *Boolean* lattice $\mathbf{L}_2^3$ (Proposition 16.2 page 245).



[9] Jeffcott (1972) page 645 ⟨§5. Main theorem⟩

✎PROOF:

1. Proof that 0 and 1 are in the *center* of $\boldsymbol{L}$: by Proposition E.5 (page 337).

2. Proof that x is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(x \wedge p) \vee (x \wedge p^{\perp}) &= x \vee 0 && = x && \Longrightarrow && x \copyright p \\
(x \wedge y) \vee (x \wedge y^{\perp}) &= 0 \vee x && = x && \Longrightarrow && x \copyright y \\
(x \wedge z) \vee (x \wedge z^{\perp}) &= 0 \vee x && = x && \Longrightarrow && x \copyright z
\end{aligned}$$

$x \copyright x$, $x \copyright x^{\perp}$, $x \copyright p^{\perp}$, $x \copyright y^{\perp}$, $x \copyright z^{\perp}$, $x \copyright 0$, and $x \copyright 1$ by Proposition E.2 (page 333).

3. Proof that y is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(y \wedge x) \vee (y \wedge x^{\perp}) &= 0 \vee y && = y && \Longrightarrow && y \copyright x \\
(y \wedge p) \vee (y \wedge p^{\perp}) &= 0 \vee y && = y && \Longrightarrow && y \copyright p \\
(y \wedge z) \vee (y \wedge z^{\perp}) &= 0 \vee y && = y && \Longrightarrow && y \copyright z
\end{aligned}$$

$y \copyright y$, $y \copyright x^{\perp}$, $y \copyright p^{\perp}$, $y \copyright y^{\perp}$, $y \copyright z^{\perp}$, $y \copyright 0$, and $y \copyright 1$ by Proposition E.2 (page 333).

4. Proof that z is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(z \wedge x) \vee (z \wedge x^{\perp}) &= 0 \vee z && = z && \Longrightarrow && z \copyright x \\
(z \wedge p) \vee (z \wedge p^{\perp}) &= 0 \vee z && = z && \Longrightarrow && z \copyright p \\
(z \wedge y) \vee (z \wedge y^{\perp}) &= 0 \vee z && = z && \Longrightarrow && z \copyright y
\end{aligned}$$

$z \copyright z$, $z \copyright x^{\perp}$, $z \copyright p^{\perp}$, $z \copyright y^{\perp}$, $z \copyright z^{\perp}$, $z \copyright 0$, and $z \copyright 1$ by Proposition E.2 (page 333).

5. Proof that $x^{\perp}$ is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(p^{\perp} \wedge x) \vee (p^{\perp} \wedge x^{\perp}) &= 0 \vee p^{\perp} && = p^{\perp} && \Longrightarrow && p^{\perp} \copyright x \\
(p^{\perp} \wedge y) \vee (p^{\perp} \wedge y^{\perp}) &= y \vee z && = p^{\perp} && \Longrightarrow && p^{\perp} \copyright y \\
(p^{\perp} \wedge z) \vee (p^{\perp} \wedge z^{\perp}) &= z \vee y && = p^{\perp} && \Longrightarrow && p^{\perp} \copyright z
\end{aligned}$$

$p^{\perp} \copyright x^{\perp}$, $p^{\perp} \copyright p^{\perp}$, $p^{\perp} \copyright y^{\perp}$, $p^{\perp} \copyright z^{\perp}$, $p^{\perp} \copyright 0$, and $p^{\perp} \copyright 1$ by Proposition E.2 (page 333).

6. Proof that $y^{\perp}$ is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(y^{\perp} \wedge x) \vee (y^{\perp} \wedge x^{\perp}) &= x \vee z && = y^{\perp} && \Longrightarrow && y^{\perp} \copyright x \\
(y^{\perp} \wedge p) \vee (y^{\perp} \wedge p^{\perp}) &= p \vee z && = y^{\perp} && \Longrightarrow && y^{\perp} \not\copyright p \\
(y^{\perp} \wedge z) \vee (y^{\perp} \wedge z^{\perp}) &= z \vee p && = y^{\perp} && \Longrightarrow && y^{\perp} \copyright z
\end{aligned}$$

$p^{\perp} \copyright x^{\perp}$, $p^{\perp} \copyright p^{\perp}$, $p^{\perp} \copyright y^{\perp}$, $p^{\perp} \copyright z^{\perp}$, $p^{\perp} \copyright 0$, and $p^{\perp} \copyright 1$ by Proposition E.2 (page 333).

7. Proof that $z^{\perp}$ is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(z^{\perp} \wedge x) \vee (z^{\perp} \wedge x^{\perp}) &= x \vee y && = z^{\perp} && \Longrightarrow && z^{\perp} \copyright x \\
(z^{\perp} \wedge p) \vee (z^{\perp} \wedge p^{\perp}) &= p \vee y && = z^{\perp} && \Longrightarrow && y^{\perp} \not\copyright p \\
(z^{\perp} \wedge y) \vee (z^{\perp} \wedge y^{\perp}) &= z \vee p && = z^{\perp} && \Longrightarrow && y^{\perp} \copyright z
\end{aligned}$$

$z^{\perp} \copyright x^{\perp}$, $z^{\perp} \copyright p^{\perp}$, $z^{\perp} \copyright y^{\perp}$, $z^{\perp} \copyright z^{\perp}$, $z^{\perp} \copyright 0$, and $z^{\perp} \copyright 1$ by Proposition E.2 (page 333).

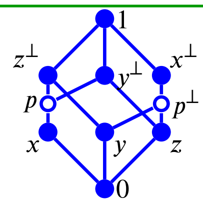
8. Proof that p and $x^\perp$ are *not* in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(p \wedge x) \vee (p \wedge x^\perp) = x \vee 0 &= x && \Longrightarrow && p \not\mathrm{C} x \\
(x^\perp \wedge p) \vee (x^\perp \wedge p^\perp) = 0 \vee p^\perp &= p^\perp && \Longrightarrow && x^\perp \not\mathrm{C} p
\end{aligned}$$

Example E.6.

EX

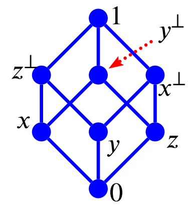
The **center** of the lattice illustrated to the right is illustrated with solid dots. Note that the center is the *Boolean* lattice $\boldsymbol{L}_2^2$ (Proposition 16.2 page 245).



Example E.7.

EX

In a *Boolean* lattice, such as the one illustrated to the right, every element is in the center (Proposition 16.2 page 245).



APPENDIX F

ALGEBRAIC STRUCTURES

❝*In this book, learned reader, you have the rules of algebra... It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one.... this most abstruse and unsurpassed treasury of the entire arithmetic being brought to light and, as in a theater, exposed to the sight of all...*❞

Gerolamo Cardano (1501–1576), Italian mathematician, physician, and astrologer [1]

A set together with one or more operations forms several standard mathematical structures:

group $\supseteq$ *ring* $\supseteq$ *commutative ring* $\supseteq$ *integral domain* $\supseteq$ *field*

Definition F.1. [2] *Let X be a set and $\diamond : X \times X \to X$ be an operation on X.*

DEF

*The pair $(X, \diamond)$ is a **group** if*

1.	$\exists e \in X$	such that	$e \diamond x = x \diamond e = x$	$\forall x \in X$	(IDENTITY *element*)	*and*
2.	$\exists (-x) \in X$	such that	$(-x) \diamond x = x \diamond (-x) = e$	$\forall x \in X$	(INVERSE *element*)	*and*
3.			$x \diamond (y \diamond z) = (x \diamond y) \diamond z$	$\forall x,y,z \in X$	(ASSOCIATIVE)	

Definition F.2. [3] *Let $+ : X \times X \to X$ and $* : X \times X \to X$ be operations on a set X. Furthermore, let the operation $*$ also be represented by juxtapostion as in $a * b \equiv ab$.*

DEF

*The triple $(X, +, *)$ is a **ring** if*

1.	$(X, +)$ *is a group.*				*(additive group)*	*and*
2.	$x(yz)$	$=$	$(xy)z$	$\forall x,y,z \in X$	(ASSOCIATIVE *with respect to* $*$)	*and*
3.	$x(y+z)$	$=$	$(xy) + (xz)$	$\forall x,y,z \in X$	($*$ *is* LEFT DISTRIBUTIVE *over* $+$)	*and*
4.	$(x+y)z$	$=$	$(xz) + (yz)$	$\forall x,y,z \in X$	($*$ *is* RIGHT DISTRIBUTIVE *over* $+$).	

Definition F.3. [4]

---

[1] quote: Cardano (1545) page 1
image: http://en.wikipedia.org/wiki/Image:Cardano.jpg

[2] Durbin (2000) page 29

[3] Durbin (2000) pages 114–115

[4] Durbin (2000) page 118

DEF

*A triple $(X, +, *)$ is a **commutative ring** if*

1. $(X, +, *)$ is a RING and
2. $xy = yx \quad \forall x,y \in X$ (COMMUTATIVE).

Definition F.4. [5] *Let R be a* COMMUTATIVE RING *(Definition F.3 page 341).*

DEF

*A function $|\cdot|$ in $\mathbb{R}^{\mathbb{R}}$ is an **absolute value** (or **modulus**) if*

1. $|x| \geq 0 \quad x \in \mathbb{R}$ (NON-NEGATIVE) and
2. $|x| = 0 \iff x = 0 \quad x \in \mathbb{R}$ (NONDEGENERATE) and
3. $|xy| = |x| \cdot |y| \quad x,y \in \mathbb{R}$ (HOMOGENEOUS / SUBMULTIPLICATIVE) and
4. $|x+y| \leq |x| + |y| \quad x,y \in \mathbb{R}$ (SUBADDITIVE / TRIANGLE INEQUALITY)

Definition F.5. [6]

DEF

*The structure $F \triangleq (X, +, \cdot, 0, 1)$ is a **field** if*

1. $(X, +, *)$ is a ring (ring) and
2. $xy = yx \quad \forall x,y \in X$ (commutative with respect to $*$) and
3. $(X \setminus \{0\}, *)$ is a group (group with respect to $*$).

Definition F.6. [7] *Let $\boldsymbol{V} = (F, +, \cdot)$ be a vector space and $\otimes : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ be a vector-vector multiplication operator.*
*An **algebra** is any pair $(\boldsymbol{V}, \otimes)$ that satisfies ($\otimes$ is represented by juxtaposition)*

DEF

1. $(\boldsymbol{ux})\boldsymbol{y} = \boldsymbol{u}(\boldsymbol{xy}) \quad \forall \boldsymbol{u,x,y} \in \boldsymbol{V}$ (ASSOCIATIVE) and
2. $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}) = (\boldsymbol{ux}) + (\boldsymbol{uy}) \quad \forall \boldsymbol{u,x,y} \in \boldsymbol{V}$ (LEFT DISTRIBUTIVE) and
3. $(\boldsymbol{u}+\boldsymbol{x})\boldsymbol{y} = (\boldsymbol{uy}) + (\boldsymbol{xy}) \quad \forall \boldsymbol{u,x,y} \in \boldsymbol{V}$ (RIGHT DISTRIBUTIVE) and
4. $\alpha(\boldsymbol{xy}) = (\alpha\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x}(\alpha\boldsymbol{y}) \quad \forall \boldsymbol{x,y} \in \boldsymbol{V}$ and $\alpha \in F$ (SCALAR COMMUTATIVE) .

[5] Cohn (2002) page 312
[6] Durbin (2000) page 123, Weber (1893)
[7] Abramovich and Aliprantis (2002) page 3, Michel and Herget (1993) page 56

APPENDIX G

CALCULUS

Definition G.1. *Let X and Y be sets.*

DEF

The space Y^X represents the set of all functions with DOMAIN *X and* RANGE *Y such that*

$$Y^X \triangleq \{\mathsf{f}(x) | \mathsf{f}(x) : X \to Y\}$$

Definition G.2. *Let $\mathbb{R}$ be the set of real numbers, $\mathscr{B}$ the set of* BOREL SETS *on $\mathbb{R}$, and μ the standard* BOREL MEASURE *on $\mathscr{B}$. Let $\mathbb{R}^{\mathbb{R}}$ be as in Definition G.1* *page 343*.

DEF

The ***space of Lebesgue square-integrable functions*** $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *(or $\boldsymbol{L}^2_{\mathbb{R}}$) is defined as*

$$\boldsymbol{L}^2_{\mathbb{R}} \triangleq \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \, \mathsf{d}\mu < \infty \right\}.$$

The ***standard inner product*** $\langle \triangle \mid \nabla \rangle$ *on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, \mathsf{d}x.$$

The ***standard norm*** $\|\cdot\|$ *on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as* $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$

Definition G.3. *Let $\mathsf{f}(x)$ be a* FUNCTION *in $\mathbb{R}^{\mathbb{R}}$.*

DEF

$$\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon \to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition G.1.

PRP

$$\left\{ \begin{array}{ll} (1). & \mathsf{f}(x) \text{ is CONTINUOUS} \quad \textit{and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC about a point } a} \end{array} \right\} \implies \left\{ \begin{array}{llll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) \quad (\text{ANTI-SYMMETRIC about } a) \\ (2). & \mathsf{f}'(a) & = & 0 \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{f}'(a+x) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\
&= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\
&= -\mathsf{f}(a-x)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\
&= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result} \\
&= 0
\end{aligned}$$

✏

Lemma G.1.

LEM: $\mathsf{f}(x)$ *is* INVERTIBLE $\implies \left\{ \frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) = \dfrac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to 0}\frac{\mathsf{f}^{-1}(y+\varepsilon) - \mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}y} && \text{(Definition G.3 page 343)} \\
&= \lim_{\delta\to 0}\left.\frac{1}{\left[\frac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]}\right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1} \\
&\triangleq \left.\frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x)}\right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition G.3 page 343)} \\
&= \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✏

Theorem G.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

THM: $$\int_{[0:1)^n} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n} x_k\right) \mathsf{d}x_1\, \mathsf{d}x_2 \cdots \mathsf{d}x_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n \in \mathbb{N}$$

✎PROOF: Proof by induction:

1. Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\int_{[0:1)} \mathsf{f}^{(1)}(x)\, \mathsf{d}x &= \mathsf{f}(1) - \mathsf{f}(0) && \text{by \textit{Fundamental theorem of calculus}} \\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1) + (-1)^{1+0}\binom{1}{0}\mathsf{f}(0) \\
&= \sum_{k=0}^{1}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

2. Induction step …proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_{n+1}\\
&= \int_{[0:1)^n}\left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1}+\sum_{k=1}^{n} x_k\right)\mathrm{d}x_{n+1}\right]\mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n\\
&= \int_{[0:1)^n}\left[\mathsf{f}^{(n)}\left(x_{n+1}+\sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right]\mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n && \text{by \textit{Fundamental theorem of calculus}}\\
&= \int_{[0:1)^n}\left[\mathsf{f}^{(n)}\left(1+\sum_{k=1}^{n} x_k\right)-\mathsf{f}^{(n)}\left(0+\sum_{k=1}^{n} x_k\right)\right]\mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n\\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1)-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by induction hypothesis}\\
&= \sum_{m=1}^{m=n+1}(-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m)+\sum_{k=0}^{n}(-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{where } m \triangleq k+1 \implies k=m-1\\
&= \left[\mathsf{f}(n+1)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right]+\left[(-1)^{n+1}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right]\\
&= \mathsf{f}(n+1)+(-1)^{n+1}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1}+\binom{n}{k}\right]}_{\text{use \textit{Stifel formula}}}\mathsf{f}(k)\\
&= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1)+(-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by \textit{Stifel formula}}\\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)
\end{aligned}$$

✎

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma G.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots,$ *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example G.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\mathsf{f}(x)\mathsf{g}(x)\right] = \mathsf{f}'''(x)\mathsf{g}(x)+3\mathsf{f}''(x)\mathsf{g}'(x)+3\mathsf{f}'(x)\mathsf{g}''(x)+\mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem G.2 (Leibniz integration rule). [3]

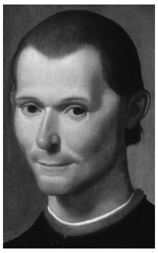
[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)
[3] Flanders (1973) page 615 ⟨(1.1)⟩ Talvila (2001), Knapp (2005b) page 389 ⟨Chapter VII⟩, Protter and Morrey (2012) page 422 ⟨Leibniz Rule. Theorem 1.⟩, http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

THM

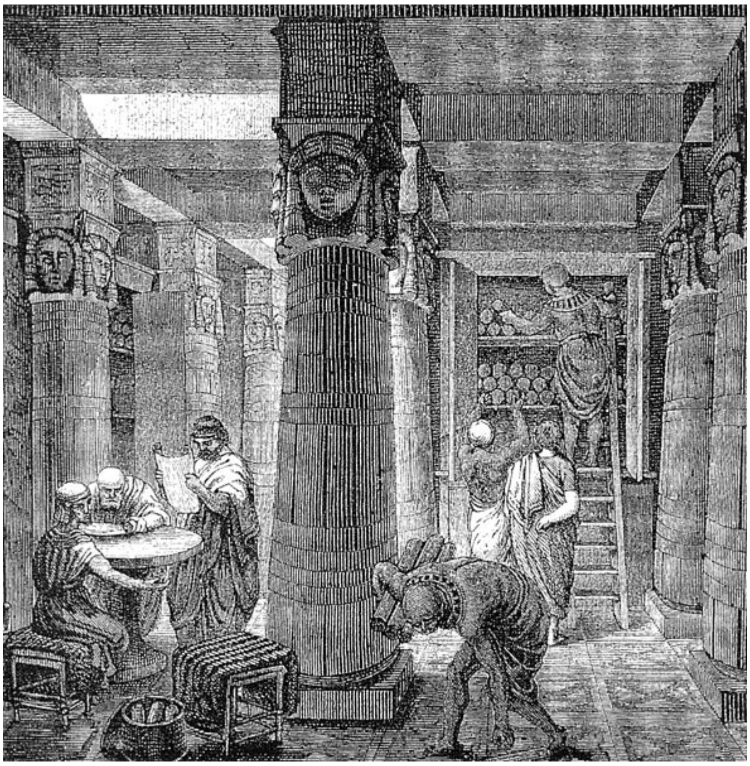
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathsf{a}(x)}^{\mathsf{b}(x)} \mathsf{g}(t)\,\mathrm{d}t = \mathsf{g}[\mathsf{b}(x)]\mathsf{b}'(x) - \mathsf{g}[\mathsf{a}(x)]\mathsf{a}'(x)$$

Back Matter

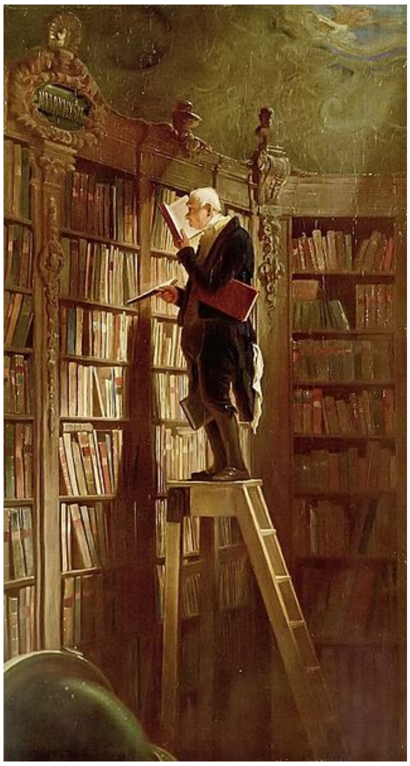
❝It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.❞
Niels Henrik Abel (1802–1829), Norwegian mathematician [4]



❝When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.❞
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [5]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850

[6]

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[4] quote: Simmons (2007) page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

[5] quote: Machiavelli (1961) page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

[6] http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

❝To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.❞

Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [7]

[7] quote: Kenko (circa 1330)
image: `http://en.wikipedia.org/wiki/Yoshida_Kenko`

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REFERENCE INDEX

SUBJECT INDEX

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