Trigonometric Systems

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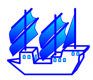
The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]



[1] Paine (2000) page 63 ⟨Golden Hind⟩

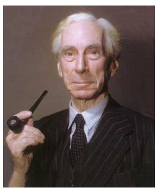
*❝Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night? ❞*

*❝Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine. ❞*

Alfred Edward Housman, English poet (1859–1936) [2]

❝The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning. ❞

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



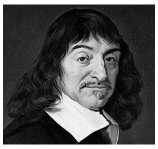
❝As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.❞

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

[2] quote: Housman (1936) page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg
[3] quote: Ewen (1961) page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg
[4] quote: Heijenoort (1967) page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.❞



❝Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.❞

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \dots$

…continued on next page…

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$
$\mathbb{Z}^{-}$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_{\mathsf{o}}$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_{\mathsf{e}}$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^{+}$	positive real numbers	$(0, \infty)$
$\mathbb{R}^{-}$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^{*}$	extended real numbers	$\mathbb{R}^{*} \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
Ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \triangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\nsubseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \setminus B$	set difference	
A^{c}	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_{A}(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	

...continued on next page...

symbol	description	
0	"false" condition	
$\neg$	logical NOT operation	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\implies$	"implies";	"only if"
$\impliedby$	"implied by";	"if"
$\iff$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\leq$	reflexive ordering relation	"less than or equal to"
$\geq$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\emptyset$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_\mathrm{j}(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_\mathrm{j}(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_\mathrm{j}(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		

...continued on next page...

symbol	description
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition 3.2 page 42)
$\hat{\mathbf{F}}$	*Fourier Series* operator (Definition 5.1 page 71)
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition 6.1 page 75)
$\mathbf{Z}$	*Z-Transform* operator (Definition D.4 page 114)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $\left(x_n \in \mathbb{C}\right)_{n\in\mathbb{Z}}$
$\breve{\mathsf{x}}(z)$	*Z-Transform* of a sequence $\left(x_n \in \mathbb{C}\right)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

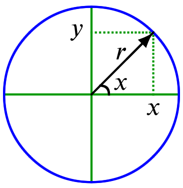
CHAPTER 1

TRIGONOMETRIC FUNCTIONS

1.1 Definition Candidates

There are several ways of defining the sine and cosine functions, including the following:[1]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[2]



$$\begin{aligned}\cos x &\triangleq \frac{x}{r}\\ \sin x &\triangleq \frac{y}{r}\end{aligned}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[3]

$$\cos x \triangleq \mathbf{R}_{\mathrm{e}} e^{ix} \qquad\qquad \sin x \triangleq \mathbf{I}_{\mathrm{m}}\left(e^{ix}\right)$$

3. **Polynomial:** Let $\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of *Taylor expansions* such that[4]

$$\begin{aligned}\cos(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\ \sin(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

[1] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means "bay" or "inlet"—thus resulting in the Latin translation *sinus*, which also means "bay" or "inlet". Reference: Boyer and Merzbach (1991) page 252

[2] Abramowitz and Stegun (1972) page 78

[3] Euler (1748)

[4] Rosenlicht (1968) page 157, Abramowitz and Stegun (1972) page 74

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \prod_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of a product of factors such that[5]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1-\left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[6]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-(n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-(n\pi)^2}\right)}_{\cot(x)}\sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{d}{dx}$ such that

$$\begin{array}{lclcccc}
\cos(x) & \triangleq & f(x) & \text{where} & \underbrace{\frac{d^2}{dx^2}f+f=0}_{\text{differential equation}} & \underbrace{f(0)=1}_{\text{1st initial condition}} & \underbrace{\left[\frac{d}{dx}f\right](0)=0}_{\text{2nd initial condition}}\\
\sin(x) & \triangleq & g(x) & \text{where} & \underbrace{\frac{d^2}{dx^2}g+g=0}_{\text{differential equation}} & \underbrace{g(0)=0}_{\text{1st initial condition}} & \underbrace{\left[\frac{d}{dx}g\right](0)=1}_{\text{2nd initial condition}}
\end{array}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[7]

$$\begin{array}{lclcll}
\cos(x) & \triangleq & f^{-1}(x) & \text{where} & f(x) \triangleq & \underbrace{\int_x^1 \sqrt{\frac{1}{1-y^2}}\,dy}_{\arccos(x)}\\
\sin(x) & \triangleq & g^{-1}(x) & \text{where} & g(x) \triangleq & \underbrace{\int_0^x \sqrt{\frac{1}{1-y^2}}\,dy}_{\arcsin(x)}
\end{array}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{d}{dx}$ (Definition 1.1 page 3). Support for such an approach includes the following:

- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{d}{dx}$ (Theorem 1.1 page 4).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem 1.3 page 6).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem 1.4 page 7).

[5] Abramowitz and Stegun (1972) page 75
[6] Abramowitz and Stegun (1972) page 75
[7] Abramowitz and Stegun (1972) page 79

- The complex exponential function is a solution of a second order homogeneous differential equation (Definition 1.4 page 8).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section 1.6 page 16).

1.2 Definitions

Definition 1.1. [8] *Let $\mathcal{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{\mathrm{d}}{\mathrm{d}x} \in \mathcal{C}^{\mathcal{C}}$ the differentiation operator.*

DEF *The function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ is the **cosine** function $\cos(x) \triangleq \mathsf{f}(x)$ if*

$$\begin{array}{llcll}
1. & \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} &=& 0 & \text{(second order homogeneous differential equation)} \quad \text{and} \\
2. & \mathsf{f}(0) &=& 1 & \text{(first initial condition)} \quad \text{and} \\
3. & \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) &=& 0 & \text{(second initial condition).}
\end{array}$$

Definition 1.2. [9] *Let $\mathcal{C}$ and $\frac{\mathrm{d}}{\mathrm{d}x} \in \mathcal{C}^{\mathcal{C}}$ be defined as in definition of $\cos(x)$ (Definition 1.1 page 3).*

DEF *The function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ is the **sine** function $\sin(x) \triangleq \mathsf{f}(x)$ if*

$$\begin{array}{llcll}
1. & \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} &=& 0 & \text{(second order homogeneous differential equation)} \quad \text{and} \\
2. & \mathsf{f}(0) &=& 0 & \text{(first initial condition)} \quad \text{and} \\
3. & \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) &=& 1 & \text{(second initial condition).}
\end{array}$$

Definition 1.3. [10]

DEF *Let π ("pi") be defined as the element in $\mathbb{R}$ such that*

$$\begin{array}{llcll}
(1). & \cos\left(\frac{\pi}{2}\right) &=& 0 & \text{and} \\
(2). & \pi &>& 0 & \text{and}
\end{array}$$

(3). *π is the **smallest** of all elements in $\mathbb{R}$ that satisfies (1) and (2).*

1.3 Basic properties

Lemma 1.1. [11] *Let $\mathcal{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{\mathrm{d}}{\mathrm{d}x} \in \mathcal{C}^{\mathcal{C}}$ the differentiation operator.*

LEM

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff$$

$$\left\{\begin{aligned}
\mathsf{f}(x) &= \underbrace{[\mathsf{f}](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \\
&= \left(\mathsf{f}(0) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)x\right) - \left(\frac{\mathsf{f}(0)}{2!}x^2 + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{3!}x^3\right) + \left(\frac{\mathsf{f}(0)}{4!}x^4 + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{5!}x^5\right)\cdots
\end{aligned}\right\}$$

[8] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[10] Rosenlicht (1968) page 158
[11] Rosenlicht (1968) page 156, Liouville (1839)

✎PROOF: Let $f'(x) \triangleq \frac{d}{dx}f(x)$.

$$\begin{aligned}
f'''(x) &= -\left[\frac{d}{dx}f\right](x) \\
f^{(4)}(x) &= -\left[\frac{d}{dx}f\right](x) && = -\left[\frac{d^2}{dx^2}f\right](x) = f(x)
\end{aligned}$$

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \implies f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n && \text{by Taylor expansion (Theorem B.13 page 105)} \\
&= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{\left[\frac{d^2}{dx^2}f\right](0)}{2!}x^2 - \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 - \cdots \\
&= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{f(0)}{2!}x^2 - \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3 + \frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5 - \cdots \\
&= f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]
\end{aligned}$$

2. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \impliedby f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\left[\frac{d^2}{dx^2}f\right](x) &= \frac{d}{dx}\frac{d}{dx}[f(x)] \\
&= \frac{d}{dx}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] && \text{by right hypothesis} \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)f(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n-1}\right] \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{f(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n-1)!}x^{2n-1}\right] \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \\
&= -f(x) && \text{by right hypothesis}
\end{aligned}$$

⇰

Theorem 1.1 (Taylor series for cosine/sine). [12]

THM

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && \forall x\in\mathbb{R} \\
\sin(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \forall x\in\mathbb{R}
\end{aligned}$$

[12] Rosenlicht (1968) page 157

✎PROOF:

$$\begin{aligned}
\cos(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma 1.1 page 3}\\
&= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by cos initial conditions (Definition 1.1 page 3)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
\sin(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma 1.1 page 3}\\
&= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by sin initial conditions (Definition 1.2 page 3)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

✏

Theorem 1.2. [13]

THM		
	$\cos(0) = 1$	$\cos(-x) = \cos(x) \quad \forall x\in\mathbb{R}$
	$\sin(0) = 0$	$\sin(-x) = -\sin(x) \quad \forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\cos(0) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\bigg|_{x=0} && \text{by } \textit{Taylor series for cosine} \text{ (Theorem 1.1 page 4)}\\
&= 1\\
\sin(0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\bigg|_{x=0} && \text{by } \textit{Taylor series for sine} \text{ (Theorem 1.1 page 4)}\\
&= 0\\
\cos(-x) &= 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \cdots && \text{by } \textit{Taylor series for cosine} \text{ (Theorem 1.1 page 4)}\\
&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\
&= \cos(x) && \text{by } \textit{Taylor series for cosine} \text{ (Theorem 1.1 page 4)}\\
\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots && \text{by } \textit{Taylor series for sine} \text{ (Theorem 1.1 page 4)}\\
&= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]\\
&= \sin(x) && \text{by } \textit{Taylor series for sine} \text{ (Theorem 1.1 page 4)}
\end{aligned}$$

✏

Lemma 1.2. [14]

LEM		
	$\cos(1) > 0$	$x \in (0:2) \implies \sin(x) > 0$
	$\cos(2) < 0$	

[13] Rosenlicht (1968) page 157
[14] Rosenlicht (1968) page 158

✎PROOF:

$$\begin{aligned}
\cos(1) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=1} && \text{by \textit{Taylor series for cosine}} && \text{(Theorem 1.1 page 4)}\\
&= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots \\
&> 0 \\
\cos(2) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=2} && \text{by \textit{Taylor series for cosine}} && \text{(Theorem 1.1 page 4)}\\
&= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots \\
&< 0
\end{aligned}$$

$$\begin{aligned}
x \in (0:2) &\implies \text{each term in the sequence } \left(\!\!\left(\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots \right)\!\!\right) \text{ is } > 0 \\
&\implies \sin(x) > 0
\end{aligned}$$

✏

Proposition 1.1. *Let π be defined as in Definition 1.3 (page 3).*

PRP
(A). *The value π **exists** in $\mathbb{R}$.*
(B). $2 < \pi < 4$.

✎PROOF:

$$\begin{aligned}
&\cos(1) > 0 && \text{by Lemma 1.2 page 5}\\
&\cos(2) < 0 && \text{by Lemma 1.2 page 5}\\
&\implies 1 < \frac{\pi}{2} < 2 \\
&\implies 2 < \pi < 4
\end{aligned}$$

✏

Theorem 1.3. [15] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{d}{dx} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $f'(0) \triangleq \left[\frac{d}{dx} f\right](0)$.*

THM
$$\left\{ \frac{d^2}{dx^2} f + f = 0 \right\} \iff \left\{ f(x) = f(0)\cos(x) + f'(0)\sin(x) \right\} \qquad \forall f \in \boldsymbol{C}, \forall x \in \mathbb{R}$$

✎PROOF:

1. Proof that $\left[\frac{d^2}{dx^2} f\right](x) = -f(x) \implies f(x) = f(0)\cos(x) + \left[\frac{d}{dx} f\right](0)\sin(x)$:

$$\begin{aligned}
f(x) &= f(0) \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx} f\right](0) \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by left hypothesis and Lemma 1.1 page 3}\\
&= f(0)\cos x + \left[\frac{d}{dx} f\right](0)\sin x && \text{by definitions of cos and sin (Definition 1.1 page 3, Definition 1.2 page 3)}
\end{aligned}$$

[15] Rosenlicht (1968) page 157. The general solution for the *non-homogeneous* equation $\frac{d^2}{dx^2} f(x) + f(x) = g(x)$ with initial conditions $f(a) = 1$ and $f'(a) = \rho$ is $f(x) = \cos(x) + \rho\sin(x) + \int_a^x g(y)\sin(x-y)\,dy$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992) page 371, Liouville (1839). Volterra equation references: Pedersen (2000) page 99, Lalescu (1908), Lalescu (1911)

2. Proof that $\frac{d^2}{dx^2}f = -f \impliedby f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$
\begin{aligned}
f(x) &= f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x && \text{by right hypothesis}\\
&= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)}\\
\implies \quad & \frac{d^2}{dx^2}f + f = 0 && \text{by Lemma 1.1 page 3}
\end{aligned}
$$

✎

Theorem 1.4. [16] *Let $\frac{d}{dx} \in C^{C}$ be the differentiation operator.*

THM: $\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x\in\mathbb{R}$ | $\frac{d}{dx}\sin(x) = \cos(x) \quad \forall x\in\mathbb{R}$ | $\cos^2(x) + \sin^2(x) = 1 \quad \forall x\in\mathbb{R}$

✎PROOF:

$$
\begin{aligned}
\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} && \text{by \textit{Taylor series} (Theorem 1.1 page 4)}\\
&= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}\\
&= -\sin(x) && \text{by \textit{Taylor series} (Theorem 1.1 page 4)}\\
\frac{d}{dx}\sin(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by \textit{Taylor series} (Theorem 1.1 page 4)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= \cos(x) && \text{by \textit{Taylor series} (Theorem 1.1 page 4)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\left[\cos^2(x) + \sin^2(x)\right] &= -2\cos(x)\sin(x) + 2\sin(x)\cos(x)\\
&= 0\\
&\implies \cos^2(x) + \sin^2(x) \text{ is \textit{constant}}\\
&\implies \cos^2(x) + \sin^2(x)\\
&= \cos^2(0) + \sin^2(0)\\
&= 1 + 0 = 1 && \text{by Theorem 1.2 page 5}
\end{aligned}
$$

✎

Proposition 1.2.

PRP: $\sin\left(\frac{\pi}{2}\right) = 1$

[16] Rosenlicht (1968) page 157

✎PROOF:

$$
\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2) + 0} \\
&= \pm\sqrt{\sin^2(\pi/2) + \cos^2(\pi/2)} && \text{by definition of } \pi && \text{(Definition 1.3 page 3)} \\
&= \pm\sqrt{1} && \text{by Theorem 1.4 page 7} \\
&= \pm 1 \\
&= 1 && \text{by Lemma 1.2 page 5}
\end{aligned}
$$

⇨

1.4 The complex exponential

Definition 1.4.

DEF *The function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq \mathsf{f}(x)$ *if*

1. $\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f} + \mathsf{f} = 0$ *(second order homogeneous differential equation)* *and*
2. $\mathsf{f}(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0) = i$ *(second initial condition).*

Theorem 1.5 (Euler's identity). [17]

THM
$$e^{ix} = \cos(x) + i\sin(x) \qquad \forall x \in \mathbb{R}$$

✎PROOF:

$$
\begin{aligned}
\exp(ix) &= \mathsf{f}(0)\cos(x) + \left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sin(x) && \text{by Theorem 1.3 page 6} \\
&= \cos(x) + i\sin(x) && \text{by Definition 1.4 page 8}
\end{aligned}
$$

⇨

Proposition 1.3.

PRP
$$e^{-i\pi/2} = -i \quad \Big| \quad e^{i\pi/2} = i$$

✎PROOF:

$$
\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem 1.5 page 8)} \\
&= 0 + i && \text{by Theorem 1.2 (page 5) and Proposition 1.2 (page 7)} \\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem 1.5 page 8)} \\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem 1.2 page 5} \\
&= 0 - i && \text{by Theorem 1.2 (page 5) and Proposition 1.2 (page 7)}
\end{aligned}
$$

⇨

Corollary 1.1.

COR
$$e^{ix} = \sum_{n \in \mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x \in \mathbb{R}$$

[17] Euler (1748), Bottazzini (1986) page 12

✎PROOF:

$$\begin{aligned}
\boxed{e^{ix}} &= \cos(x) + i\sin(x) && \text{by Euler's identity} && \text{(Theorem 1.5 page 8)}\\
&= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by Taylor series} && \text{(Theorem 1.1 page 4)}\\
&= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} && = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!}\\
&= \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} && = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}
\end{aligned}$$

✏

Corollary 1.2 (Euler formulas). [18]

COR

$$\cos(x) = \mathbf{R_e}\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \sin(x) = \mathbf{I_m}\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{R_e}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}+\left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} && \text{by definition of } \Re && \text{(Definition A.5 page 89)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} && \text{by Euler's identity} && \text{(Theorem 1.5 page 8)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} && = \frac{\cos(x)}{2} + \frac{\cos(x)}{2} && = \boxed{\cos(x)}\\
\boxed{\mathbf{I_m}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}-\left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} && \text{by definition of } \Im && \text{(Definition A.5 page 89)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} && \text{by Euler's identity} && \text{(Theorem 1.5 page 8)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} && = \frac{i\sin(x)}{2i} + \frac{i\sin(x)}{2i} && = \boxed{\sin(x)}
\end{aligned}$$

✏

Theorem 1.6. [19]

THM

$$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

✎PROOF:

$$\begin{aligned}
e^{\alpha}\, e^{\beta} &= \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) && \text{by Corollary 1.1 page 8}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{n!}{n!} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}
\end{aligned}$$

[18] Euler (1748), Bottazzini (1986) page 12

[19] Rudin (1987) page 1

$$
\begin{aligned}
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, \alpha^k\, \beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \alpha^k\, \beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the \textit{Binomial Theorem}} && \text{(Theorem B.14 page 105)} \\
&= e^{\alpha+\beta} && \text{by Corollary 1.1 page 8}
\end{aligned}
$$

✏

1.5 Trigonometric Identities

Theorem 1.7 (shift identities).

THM							
$\cos\left(x+\frac{\pi}{2}\right)$	$=$	$-\sin x$	$\forall x\in\mathbb{R}$	$\sin\left(x+\frac{\pi}{2}\right)$	$=$	$\cos x$	$\forall x\in\mathbb{R}$
$\cos\left(x-\frac{\pi}{2}\right)$	$=$	$\sin x$	$\forall x\in\mathbb{R}$	$\sin\left(x-\frac{\pi}{2}\right)$	$=$	$-\cos x$	$\forall x\in\mathbb{R}$

✎PROOF:

$$
\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)} + e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= \frac{e^{ix}e^{i\frac{\pi}{2}} + e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \frac{e^{ix}(i) + e^{-ix}(-i)}{2} && \text{by Proposition 1.3 page 8} \\
&= \frac{e^{ix} - e^{-ix}}{-2i} \\
&= -\sin x && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)} + e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}} + e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \frac{e^{ix}(-i) + e^{-ix}(i)}{2} && \text{by Proposition 1.3 page 8} \\
&= \frac{e^{ix} - e^{-ix}}{2i} \\
&= \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
\sin\left(x+\frac{\pi}{2}\right) &= \cos\left(\left[x+\frac{\pi}{2}\right] - \frac{\pi}{2}\right) && \text{by previous result} \\
&= \cos(x) \\
\sin\left(x-\frac{\pi}{2}\right) &= -\cos\left(\left[x-\frac{\pi}{2}\right] + \frac{\pi}{2}\right) && \text{by previous result} \\
&= -\cos(x)
\end{aligned}
$$

✏

Theorem 1.8 (product identities).

THM

$$\begin{array}{lllcllcll}
(A). & \cos x \cos y & = & & {}^1\!/\!_2\cos(x-y) & + & {}^1\!/\!_2\cos(x+y) & \forall x,y\in\mathbb{R}\\
(B). & \cos x \sin y & = & - & {}^1\!/\!_2\sin(x-y) & + & {}^1\!/\!_2\sin(x+y) & \forall x,y\in\mathbb{R}\\
(C). & \sin x \cos y & = & & {}^1\!/\!_2\sin(x-y) & + & {}^1\!/\!_2\sin(x+y) & \forall x,y\in\mathbb{R}\\
(D). & \sin x \sin y & = & & {}^1\!/\!_2\cos(x-y) & - & {}^1\!/\!_2\cos(x+y) & \forall x,y\in\mathbb{R}
\end{array}$$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary 1.2 page 9)
(algebraic method requiring *complex number system* $\mathbb{C}$):

$$\begin{aligned}
\cos x\cos y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right) && \text{by \textit{Euler formulas}} \quad \text{(Corollary 1.2 page 9)}\\
&= \frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(-x+y)}+e^{i(-x-y)}}{4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{4}+\frac{e^{i(x-y)}+e^{-i(x-y)}}{4}\\
&= \frac{2\cos(x+y)}{4}+\frac{2\cos(x-y)}{4} && \text{by \textit{Euler formulas}} \quad \text{(Corollary 1.2 page 9)}\\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem 1.3 page 6)
(differential equation method requiring only *real number system* $\mathbb{R}$):

$$\begin{aligned}
&\mathsf{f}(x) \triangleq {}^1\!/\!_2\cos(x-y)+{}^1\!/\!_2\cos(x+y)\\
&\implies \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) = -{}^1\!/\!_2\sin(x-y)-{}^1\!/\!_2\sin(x+y) && \text{by Theorem 1.4 page 7}\\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x) = -{}^1\!/\!_2\cos(x-y)-{}^1\!/\!_2\cos(x+y) && \text{by Theorem 1.4 page 7}\\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x)+\mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property}\\
&\implies \underbrace{{}^1\!/\!_2\cos(x-y)+{}^1\!/\!_2\cos(x+y)}_{\mathsf{f}(x)} = \underbrace{[{}^1\!/\!_2\cos(0-y)+{}^1\!/\!_2\cos(0+y)]}_{\mathsf{f}''(0)}\cos(x)+\underbrace{[-{}^1\!/\!_2\sin(0-y)-{}^1\!/\!_2\sin(0+y)]}_{\mathsf{f}'(0)}\sin(x)\\
&\implies {}^1\!/\!_2\cos(x-y)+{}^1\!/\!_2\cos(x+y) = \cos y\cos x+0\sin(x)\\
&\implies \cos x\cos y = {}^1\!/\!_2\cos(x-y)+{}^1\!/\!_2\cos(x+y)
\end{aligned}$$

3. Proof for (B) using *Euler formulas* (Corollary 1.2 page 9):

$$\begin{aligned}
\sin x\sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary 1.2 page 9}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary 1.2 page 9}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x \sin y &= \cos(x)\cos\left(y-\frac{\pi}{2}\right) && \text{by } \textit{shift identities} && \text{(Theorem 1.7 page 10)}\\
&= \frac{1}{2}\cos\left(x+y-\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(x-y+\frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y)-\frac{1}{2}\sin(x-y) && \text{by } \textit{shift identities} && \text{(Theorem 1.7 page 10)}\\
\sin x \cos y &= \cos y \sin x\\
&= \frac{1}{2}\sin(y+x)-\frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y)+\frac{1}{2}\sin(x-y) && \text{by Theorem 1.2 page 5}
\end{aligned}$$

✏

Proposition 1.4.

PRP												
	(A).	$\cos(\pi)$	$=$	-1	(C).	$\cos(2\pi)$	$=$	1	(E).	$e^{i\pi}$	$=$	-1
	(B).	$\sin(\pi)$	$=$	0	(D).	$\sin(2\pi)$	$=$	0	(F).	$e^{i2\pi}$	$=$	0

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1+1+\cos(\pi)\\
&= -1+2[{}^1\!/\!_2\cos({}^\pi\!/\!_2-{}^\pi\!/\!_2)+{}^1\!/\!_2\cos({}^\pi\!/\!_2+{}^\pi\!/\!_2)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= -1+2\cos({}^\pi\!/\!_2)\cos({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= -1+2(0)(0) && \text{by definition of } \pi && \text{(Definition 1.3 page 3)}\\
&= -1\\
\sin(\pi) &= 0+\sin(\pi)\\
&= 2[-{}^1\!/\!_2\sin({}^\pi\!/\!_2-{}^\pi\!/\!_2)+{}^1\!/\!_2\sin({}^\pi\!/\!_2+{}^\pi\!/\!_2)] && \text{by } \sin(0)=0 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= 2\cos({}^\pi\!/\!_2)\sin({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= 2(0)\sin({}^\pi\!/\!_2) && \text{by definition of } \pi && \text{(Definition 1.3 page 3)}\\
&= 0\\
\cos(2\pi) &= 1+\cos(2\pi)-1\\
&= 2[{}^1\!/\!_2\cos(\pi-\pi)+{}^1\!/\!_2\cos(\pi+\pi)]-1 && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= 2\cos(\pi)\cos(\pi)-1 && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= 2(-1)(-1)-1 && \text{by (A)}\\
&= 1\\
\sin(2\pi) &= 0+\sin(2\pi)\\
&= 2[{}^1\!/\!_2\sin(\pi-\pi)+{}^1\!/\!_2\sin(\pi+\pi)] && \text{by } \sin(0)=0 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= 2\sin(\pi)\cos(\pi) && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= 2(0)(-1) && \text{by (A) and (B)}\\
&= 0\\
e^{i\pi} &= \cos(\pi)+i\sin(\pi) && \text{by } \textit{Euler's identity} && \text{(Theorem 1.5 page 8)}\\
&= -1+0 && \text{by (A) and (B)}\\
&= -1\\
e^{i2\pi} &= \cos(2\pi)+i\sin(2\pi) && \text{by } \textit{Euler's identity} && \text{(Theorem 1.5 page 8)}\\
&= 1+0 && \text{by (C) and (D)}\\
&= 1
\end{aligned}$$

✏

Theorem 1.9 (double angle formulas). [20]

THM

(A).	$\cos(x+y)$	$=$	$\cos x\cos y - \sin x\sin y$	$\forall x,y\in\mathbb{R}$
(B).	$\sin(x+y)$	$=$	$\sin x\cos y + \cos x\sin y$	$\forall x,y\in\mathbb{R}$
(C).	$\tan(x+y)$	$=$	$\dfrac{\tan x+\tan y}{1-\tan x\tan y}$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *product identities* (Theorem 1.8 page 10).

$$\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x+y)}_{\cos(x+y)}+\underbrace{\frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x-y)}_{0}\\
&= \left[\frac{1}{2}\cos(x-y)+\frac{1}{2}\cos(x+y)\right]-\left[\frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)\right]\\
&= \cos x\cos y-\sin x\sin y && \text{by Theorem 1.8 page 10}
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem 1.3 page 6):

$$\begin{aligned}
\mathsf{f}(x)\triangleq\cos(x+y) &\implies \frac{d}{dx}\mathsf{f}(x)=-\sin(x+y) && \text{by Theorem 1.4 page 7}\\
&\implies \frac{d^2}{dx^2}\mathsf{f}(x)=-\cos(x+y) && \text{by Theorem 1.4 page 7}\\
&\implies \frac{d^2}{dx^2}\mathsf{f}(x)+\mathsf{f}(x)=0 && \text{by \textit{additive inverse} property}\\
&\implies \cos(x+y)=\cos y\cos x-\sin y\sin x && \text{by Theorem 1.3 page 6}\\
&\implies \cos(x+y)=\cos x\cos y-\sin x\sin y && \text{by \textit{commutative} property}
\end{aligned}$$

3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x-\frac{\pi}{2}+y\right) && \text{by \textit{shift identities} (Theorem 1.7 page 10)}\\
&= \cos\left(x-\frac{\pi}{2}\right)\cos(y)-\sin\left(x-\frac{\pi}{2}\right)\sin(y) && \text{by (A)}\\
&= \sin(x)\cos(y)+\cos(x)\sin(y) && \text{by \textit{shift identities} (Theorem 1.7 page 10)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)}\\
&= \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y} && \text{by (A)}\\
&= \left(\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\right)\left(\frac{\cos x\cos y}{\cos x\cos y}\right)\\
&= \frac{\frac{\sin x\cos y}{\cos x\cos y}+\frac{\cos x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y}-\frac{\sin x\sin y}{\cos x\cos y}} = \frac{\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}}{1-\frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x+\tan y}{1-\tan x\tan y}
\end{aligned}$$

⇰

Theorem 1.10 (trigonometric periodicity).

THM

(A).	$\cos(x+M\pi)$	$=$	$(-1)^M\cos(x)$	$\forall x\in\mathbb{R}$,	$M\in\mathbb{Z}$	(D).	$\cos(x+2M\pi)$	$=$	$\cos(x)$	$\forall x\in\mathbb{R}$,	$M\in\mathbb{Z}$
(B).	$\sin(x+M\pi)$	$=$	$(-1)^M\sin(x)$	$\forall x\in\mathbb{R}$,	$M\in\mathbb{Z}$	(E).	$\sin(x+2M\pi)$	$=$	$\sin(x)$	$\forall x\in\mathbb{R}$,	$M\in\mathbb{Z}$
(C).	$e^{i(x+M\pi)}$	$=$	$(-1)^M e^{ix}$	$\forall x\in\mathbb{R}$,	$M\in\mathbb{Z}$	(F).	$e^{i(x+2M\pi)}$	$=$	e^{ix}	$\forall x\in\mathbb{R}$,	$M\in\mathbb{Z}$

[20] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

✎PROOF:

1. Proof for (A):
   (a) $M = 0$ case: $\cos(x + 0\pi) = \cos(x) = (-1)^0\cos(x)$
   (b) Proof for $M > 0$ cases (by induction):
      i. Base case $M = 1$:

$$\begin{aligned}\cos(x+\pi) &= \cos x\cos\pi - \sin x\sin\pi && \text{by \textit{double angle formulas}} && \text{(Theorem 1.9 page 13)}\\ &= \cos x(-1) - \sin x(0) && \text{by } \cos\pi = -1 \text{ result} && \text{(Proposition 1.4 page 12)}\\ &= (-1)^1\cos x\end{aligned}$$

      ii. Inductive step...Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}\cos(x+[M+1]\pi) &= \cos([x+\pi]+M\pi)\\ &= (-1)^M\cos(x+\pi) && \text{by induction hypothesis (M case)}\\ &= (-1)^M(-1)\cos(x) && \text{by base case (item (1(b)i) page 14)}\\ &= (-1)^{M+1}\cos(x)\\ &\implies M+1 \text{ case}\end{aligned}$$

   (c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}\cos(x+M\pi) &\triangleq \cos(x-N\pi) && \text{by definition of } N\\ &= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem 1.9 page 13)}\\ &= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem 1.2 page 5}\\ &= \cos(x)\cos(0+N\pi) + \sin(x)\sin(0+N\pi)\\ &= \cos(x)(-1)^N\cos(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (1b) page 14)}\\ &= (-1)^N\cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem 1.2 page 5)}\\ &\triangleq (-1)^{-M}\cos(x) && \text{by definition of } N\\ &= (-1)^M\cos(x)\end{aligned}$$

   (d) Proof using complex exponential:

$$\begin{aligned}\cos(x+M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)}\\ &= e^{iM\pi}\left[\frac{e^{ix}+e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem 1.6 page 9)}\\ &= \left(e^{i\pi}\right)^M\cos x && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)}\\ &= (-1)^M\cos x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition 1.4 page 12)}\end{aligned}$$

2. Proof for (B):
   (a) $M = 0$ case: $\sin(x + 0\pi) = \sin(x) = (-1)^0\sin(x)$
   (b) Proof for $M > 0$ cases (by induction):
      i. Base case $M = 1$:

$$\begin{aligned}\sin(x+\pi) &= \sin x\cos\pi + \cos x\sin\pi && \text{by \textit{double angle formulas}} && \text{(Theorem 1.9 page 13)}\\ &= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} && \text{(Proposition 1.4 page 12)}\\ &= (-1)^1\sin x\end{aligned}$$

ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\sin(x+[M+1]\pi) &= \sin([x+\pi]+M\pi) \\
&= (-1)^M \sin(x+\pi) && \text{by induction hypothesis (M case)} \\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 14)} \\
&= (-1)^{M+1}\sin(x) \\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M<0$ cases: Let $N \triangleq -M \ldots \implies N>0$.

$$\begin{aligned}
\sin(x+M\pi) &\triangleq \sin(x-N\pi) && \text{by definition of } N \\
&= \sin(x)\sin(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem 1.9 page 13)} \\
&= \sin(x)\sin(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem 1.2 page 5} \\
&= \sin(x)\sin(0+N\pi) + \sin(x)\sin(0+N\pi) \\
&= \sin(x)(-1)^N\sin(0) + \sin(x)(-1)^N\sin(0) && \text{by } M\geq 0 \text{ results} && \text{(item (2b) page 14)} \\
&= (-1)^N\sin(x) && \text{by } \sin(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem 1.2 page 5)} \\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N \\
&= (-1)^M\sin(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)} - e^{-i(x+M\pi)}}{2i} && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= e^{iM\pi}\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \left(e^{i\pi}\right)^M \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= (-1)^M \sin x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition 1.4 page 12)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \left(e^{i\pi}\right)^M\left(e^{ix}\right) \\
&= (-1)^M e^{ix} && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition 1.4 page 12)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):

$$\begin{aligned}
\cos(i(x+2M\pi)) &= (-1)^{2M}\cos(ix) &&= \cos(ix) && \text{by (A)} \\
\sin(i(x+2M\pi)) &= (-1)^{2M}\sin(ix) &&= \sin(ix) && \text{by (B)} \\
e^{i(x+2M\pi)} &= (-1)^{2M}e^{ix} &&= e^{ix} && \text{by (C)}
\end{aligned}$$

✏

Theorem 1.11 (half-angle formulas/squared identities).

THM		
	(A). $\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$	(C). $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$
	(B). $\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$	

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x) + \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem 1.8 page 10)} \\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem 1.2 page 5)} \\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x) - \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem 1.8 page 10)} \\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem 1.2 page 5)} \\
\cos^2 x + \sin^2 x &= \tfrac{1}{2}[1+\cos(2x)] + \tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)} \\
& && \text{note: see also} && \text{Theorem 1.4 page 7}
\end{aligned}$$

✏

1.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

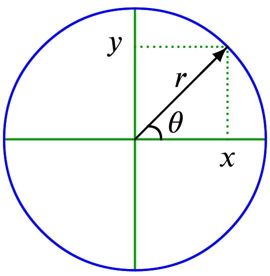
$$
\begin{aligned}
\langle \cos(x) \mid \sin(x) \rangle &= \int_{-\pi}^{+\pi} \cos(x)\sin(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x-x)\,\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x+x)\,\mathrm{d}x && \text{by Theorem 1.8 page 10} \\
&= \cancelto{0}{\frac{1}{2}\int_{-\pi}^{+\pi} \sin(0)\,\mathrm{d}x} + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(2x)\,\mathrm{d}x \\
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi} \cos(2x) \\
&= -\frac{1}{4}[\cos(2\pi) - \cos(-2\pi)] \\
&= 0
\end{aligned}
$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

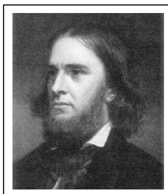
We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$



$$
\begin{array}{rcl@{\qquad}rcl}
\cos\theta & \triangleq & \frac{x}{r} & \sec\theta & \triangleq & \frac{r}{x} \\
\sin\theta & \triangleq & \frac{y}{r} & \csc\theta & \triangleq & \frac{r}{y} \\
\tan\theta & \triangleq & \frac{y}{x} & \cot\theta & \triangleq & \frac{x}{y}
\end{array}
$$

1.7 The power of the exponential



❝*Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.*❞
Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [21]

[21] quote: Kasner and Newman (1940) page 104
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

❝*Young man, in mathematics you don't understand things. You just get used to them.*❞
John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [22]

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary 1.3. [23]

COR
$$e^{i\pi} + 1 = 0$$

✎PROOF:

$$\begin{aligned}
e^{ix}\big|_{x=\pi} &= [\cos x + i\sin x]_{x=\pi} && \text{by Euler's identity (Theorem 1.5 page 8)}\\
&= -1 + i\cdot 0 && \text{by Proposition 1.4 page 12}\\
&= -1
\end{aligned}$$

✏

There are many transforms available, several of them integral transforms $[\mathbf{A}\mathsf{f}](s) \triangleq \int_t \mathsf{f}(s)\kappa(t,s)\,\mathrm{d}s$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:
① The *Laplace Transform* with kernel $\kappa(t,\ s) \triangleq e^{st}$
② The *Fourier Transform* with kernel $\kappa(t,\ \omega) \triangleq e^{i\omega t}$.
Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem 1.12 page 17).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem 1.12. [24] *Let* $\mathbf{L}$ *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*
$\check{\mathsf{h}}(s) \triangleq \left\langle \mathsf{h}(t,\omega) \,|\, e^{st} \right\rangle$ (LAPLACE TRANSFORM).

---

[22] quote: Zukav (1980) page 208
image: http://en.wikipedia.org/wiki/John_von_Neumann
The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."
[23] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html
[24] Mallat (1999) page 2, ...page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

THM
$$\left\{\begin{array}{ll}1. & \mathbf{L} \text{ is LINEAR and} \\ 2. & \mathbf{L} \text{ is TIME-INVARIANT}\end{array}\right\} \implies \left\{\mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{\text{eigenvalue}} \; \underbrace{e^{st}}_{\text{eigenvector}}\right\}$$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \,|\, \mathsf{h}((t;u),s)\rangle && \text{by linear hypothesis} \\
&= \langle e^{su} \,|\, \mathsf{h}((t-u),s)\rangle && \text{by time-invariance hypothesis} \\
&= \left\langle e^{s(t-v)} \,|\, \mathsf{h}(v,s)\right\rangle && \text{let } v = t-u \implies u = t-v \\
&= e^{st} \;\langle e^{-sv} \,|\, \mathsf{h}(v,s)\rangle && \text{by additivity of } \langle \triangle \,|\, \nabla\rangle \\
&= \langle \mathsf{h}(v,s) \,|\, e^{-sv}\rangle^* \; e^{st} && \text{by conjugate symmetry of } \langle \triangle \,|\, \nabla\rangle \\
&= \left\langle \mathsf{h}(v,s) \,|\, e^{(-s)v}\right\rangle^* \; e^{st} \\
&= \check{\mathsf{h}}^*(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}$$

✏

CHAPTER 2

TRIGONOMETRIC POLYNOMIALS

❝*I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives.*❞
Charles Hermite (1822 – 1901), French mathematician, in an 1893 letter to Stieltjes, in response to the “pathological” everywhere continuous but nowhere differentiable *Weierstrass functions* $\mathsf{f}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$.[1]

2.1 Trigonometric expansion

Theorem 2.1 (DeMoivre's Theorem)**.**

THM $$\left(re^{ix}\right)^n = r^n(\cos nx + i\sin nx) \qquad \forall r,x \in \mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\left(re^{ix}\right)^n &= r^n e^{inx} \\
&= r^n\,(\cos nx + i\sin nx) && \text{by Euler's identity (Theorem 1.5 page 8)}
\end{aligned}$$

✏

The cosine with argument nx can be expanded as a polynomial in $\cos(x)$ (next).

Theorem 2.2 (trigonometric expansion)**.** [2]

---

[1] quote: Hermite (1893)
translation: Lakatos (1976) page 19
image: http://www-groups.dcs.sx-and.ac.uk/~history/PictDisplay/Hermite.html

[2] Rivlin (1974) page 3 ⟨(1.8)⟩

THM

$$\cos(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

$$\sin(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\sin x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(nx) &= \Re\left(\cos nx + i\sin nx\right) \\
&= \Re\left(e^{inx}\right) \\
&= \Re\left[\left(e^{ix}\right)^n\right] \\
&= \Re\left[(\cos x + i\sin x)^n\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n}\binom{n}{k}(\cos x)^{n-k}(i\sin x)^k\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n} i^k\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \Re\left[\sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + i\sum_{k\in\{1,5,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right. \\
&\quad \left. -\sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + -i\sum_{k\in\{3,7,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x - \sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x \\
&= \sum_{k\in\{0,2,\ldots,n\}}\binom{n}{k}(-1)^{\frac{k}{2}}\cos^{n-k}x\sin^k x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\sin^{2k}x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x(1-\cos^2 x)^k \\
&= \left[\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\right]\left[\sum_{m=0}^{k}\binom{k}{m}(-1)^m\cos^{2m}x\right] \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}x
\end{aligned}$$

$$\begin{aligned}
\sin(nx) &= \cos\left(nx - \frac{\pi}{2}\right) \\
&= \cos\left(n\left[x - \frac{\pi}{2n}\right]\right) \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}\left(n\left[x - \frac{\pi}{2n}\right]\right)
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(nx - \frac{\pi}{2} \right)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \sin^{n-2(k-m)} (nx)$$

✏

Example 2.1.

EX

$$\begin{aligned} \cos 5x &= 16\cos^5 x - 20\cos^3 x + 5\cos x \\ \sin 5x &= 16\sin^5 x - 20\sin^3 x + 5\sin x. \end{aligned}$$

✎PROOF:

1. Proof using *DeMoivre's Theorem* (Theorem 2.1 page 19):

$$\begin{aligned}
&\cos 5x + i\sin 5x \\
&= e^{i5x} \\
&= \left(e^{ix}\right)^5 \\
&= (\cos x + i\sin x)^5 \\
&= \sum_{k=0}^{5} \binom{5}{k} [\cos x]^{5-k} [i\sin x]^k \\
&= \binom{5}{0}[\cos x]^{5-0}[i\sin x]^0 + \binom{5}{1}[\cos x]^{5-1}[i\sin x]^1 + \binom{5}{2}[\cos x]^{5-2}[i\sin x]^2 + \\
&\binom{5}{3}[\cos x]^{5-3}[i\sin x]^3 + \binom{5}{4}[\cos x]^{5-4}[i\sin x]^4 + \binom{5}{5}[\cos x]^{5-5}[i\sin x]^5 \\
&= 1\cos^5 x + i5\cos^4 x\sin x - 10\cos^3 x\sin^2 x - i10\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i1\sin^5 x \\
&= \left[\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\right] + i\left[5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)(1-\cos^2 x)\right] + \\
&i\left[5(1-\sin^2 x)(1-\sin^2 x)\sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)\right] + \\
&i\left[5(1 - 2\sin^2 x + \sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5(\cos x - 2\cos^3 x + \cos^5 x)\right] + \\
&i\left[5(\sin x - 2\sin^3 x + \sin^5 x) - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \underbrace{\left[16\cos^5 x - 20\cos^3 x + 5\cos x\right]}_{\cos 5x} + i\underbrace{\left[16\sin^5 x - 20\sin^3 x + 5\sin x\right]}_{\sin 5x}
\end{aligned}$$

2. Proof using trigonometric expansion (Theorem 2.2 page 19):

$$\begin{aligned}
\cos 5x &= \sum_{k=0}^{\lfloor \frac{5}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{2} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{5-2(k-m)} \\
&= (-1)^0 \binom{5}{0}\binom{0}{0}\cos^5 x + (-1)^1 \binom{5}{2}\binom{1}{0}\cos^3 x + (-1)^2 \binom{5}{2}\binom{1}{1}\cos^5 x + \\
&(-1)^2 \binom{5}{4}\binom{2}{0}\cos^1 x + (-1)^3 \binom{5}{4}\binom{2}{1}\cos^3 x + (-1)^4 \binom{5}{4}\binom{2}{2}\cos^5 x
\end{aligned}$$

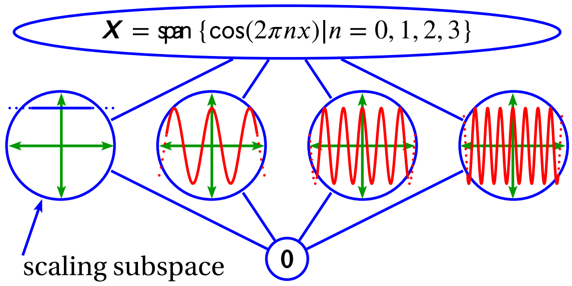


Figure 2.1: Lattice of harmonic cosines $\{\cos(nx) | n = 0, 1, 2, \ldots\}$

$$\begin{aligned}
&= +(1)(1)\cos^5 x - (10)(1)\cos^3 x + (10)(1)\cos^5 x + (5)(1)\cos x - (5)(2)\cos^3 x + (5)(1)\cos^5 x\\
&= +(1 + 10 + 5)\cos^5 x + (-10 - 10)\cos^3 x + 5\cos x\\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}$$

✏

Example 2.2. [3]

EX	n	$\cos nx$		polynomial in $\cos x$	n	$\cos nx$		polynomial in $\cos x$
	0	$\cos 0x$	$=$	1	4	$\cos 4x$	$=$	$8\cos^4 x - 8\cos^2 x + 1$
	1	$\cos 1x$	$=$	$\cos^1 x$	5	$\cos 5x$	$=$	$16\cos^5 x - 20\cos^3 x + 5\cos x$
	2	$\cos 2x$	$=$	$2\cos^2 x - 1$	6	$\cos 6x$	$=$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
	3	$\cos 3x$	$=$	$4\cos^3 x - 3\cos x$	7	$\cos 7x$	$=$	$64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

✎PROOF:

$$\begin{aligned}
\cos 2x &= \sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{2-2(k-m)}\\
&= (-1)^0 \binom{3}{0}\binom{0}{0}\cos^2 x + (-1)^1 \binom{3}{2}\binom{1}{0}\cos^0 x + (-1)^2 \binom{3}{2}\binom{1}{1}\cos^2 x\\
&= +(1)(1)\cos^2 x - (1)(1) + (1)(1)\cos^2 x\\
&= 2\cos^2 x - 1
\end{aligned}$$

$$\begin{aligned}
\cos 3x &= \sum_{k=0}^{\lfloor \frac{3}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{3-2(k-m)}\\
&= (-1)^0 \binom{3}{0}\binom{0}{0}\cos^3 x + (-1)^1 \binom{3}{2}\binom{1}{0}\cos^1 x + (-1)^2 \binom{3}{2}\binom{1}{1}\cos^3 x\\
&= +\binom{3}{0}\binom{0}{0}\cos^3 x - \binom{3}{2}\binom{1}{0}\cos^1 x + \binom{3}{2}\binom{1}{1}\cos^3 x\\
&= +(1)(1)\cos^3 x - (3)(1)\cos^1 x + (3)(1)\cos^3 x\\
&= 4\cos^3 x - 3\cos x
\end{aligned}$$

$$\cos 4x = \sum_{k=0}^{\lfloor \frac{4}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{4}{2k} \binom{k}{m} (\cos x)^{4-2(k-m)}$$

[3] Abramowitz and Stegun (1972) page 795, Guillemin (1957) page 593 ⟨(21)⟩, Sloane (2014) ⟨http://oeis.org/A039991⟩, Sloane (2014) ⟨http://oeis.org/A028297⟩

$$
\begin{aligned}
&= \sum_{k=0}^{2}\sum_{m=0}^{k}(-1)^{k+m}\binom{4}{2k}\binom{k}{m}(\cos x)^{4-2(k-m)}\\
&= (-1)^{0+0}\binom{4}{2\cdot 0}\binom{0}{0}(\cos x)^{4-2(0-0)} + (-1)^{1+0}\binom{4}{2\cdot 1}\binom{1}{0}(\cos x)^{4-2(1-0)}\\
&\quad + (-1)^{1+1}\binom{4}{2\cdot 1}\binom{1}{1}(\cos x)^{4-2(1-1)} + (-1)^{2+0}\binom{4}{2\cdot 2}\binom{2}{0}(\cos x)^{4-2(2-0)}\\
&\quad + (-1)^{2+1}\binom{4}{2\cdot 2}\binom{2}{1}(\cos x)^{4-2(2-1)} + (-1)^{2+2}\binom{4}{2\cdot 2}\binom{2}{2}(\cos x)^{4-2(2-2)}\\
&= (1)(1)\cos^4 x - (6)(1)\cos^2 x + (6)(1)\cos^4 x + (1)(1)\cos^0 x - (1)(2)\cos^2 x + (1)(1)\cos^4 x\\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}
$$

$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ see Example 2.1 page 21

$$
\begin{aligned}
\cos 6x &= \sum_{k=0}^{\lfloor\frac{6}{2}\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{6}{2k}\binom{k}{m}(\cos x)^{6-2(k-m)}\\
&= (-1)^0\binom{6}{0}\binom{0}{0}\cos^6 x + (-1)^1\binom{6}{2}\binom{1}{0}\cos^4 x + (-1)^2\binom{6}{2}\binom{1}{1}\cos^6 x + (-1)^2\binom{6}{4}\binom{2}{0}\cos^2 x +\\
&(-1)^3\binom{6}{4}\binom{2}{1}\cos^4 x + (-1)^4\binom{6}{4}\binom{2}{2}\cos^6 x + (-1)^3\binom{6}{6}\binom{3}{0}\cos^0 x + (-1)^4\binom{6}{6}\binom{3}{1}\cos^2 x +\\
&(-1)^5\binom{6}{6}\binom{3}{2}\cos^4 x + (-1)^6\binom{6}{6}\binom{3}{3}\cos^6 x\\
&= +(1)(1)\cos^6 x - (15)(1)\cos^4 x + (15)(1)\cos^6 x + (15)(1)\cos^2 x - (15)(2)\cos^4 x + (15)(1)\cos^6 x\\
&\quad - (1)(1)\cos^0 x + (1)(3)\cos^2 x - (1)(3)\cos^4 x + (1)(1)\cos^6 x\\
&= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 7x &= \sum_{k=0}^{\lfloor\frac{7}{2}\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)}\\
&= \sum_{k=0}^{3}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{7-2(k-m)}\\
&= (-1)^0\binom{7}{0}\binom{0}{0}\cos^7 x + (-1)^1\binom{7}{2}\binom{1}{0}\cos^5 x + (-1)^2\binom{7}{2}\binom{1}{1}\cos^7 x + (-1)^2\binom{7}{4}\binom{2}{0}\cos^3 x\\
&\quad + (-1)^3\binom{7}{4}\binom{2}{1}\cos^5 x + (-1)^4\binom{7}{4}\binom{2}{2}\cos^7 x + (-1)^3\binom{7}{6}\binom{3}{0}\cos^1 x + (-1)^4\binom{7}{6}\binom{3}{1}\cos^3 x\\
&\quad + (-1)^5\binom{7}{6}\binom{3}{2}\cos^5 x + (-1)^6\binom{7}{6}\binom{3}{3}\cos^7 x\\
&= (1)(1)\cos^7 x - (21)(1)\cos^5 x + (21)(1)\cos^7 x + (35)(1)\cos^3 x\\
&\quad - (35)(2)\cos^5 x + (35)(1)\cos^7 x - (7)(1)\cos^1 x + (7)(3)\cos^3 x\\
&\quad - (7)(3)\cos^5 x + (7)(1)\cos^7 x\\
&= (1 + 21 + 35 + 7)\cos^7 x - (21 + 70 + 21)\cos^5 x + (35 + 21)\cos^3 x - (7)\cos^1 x\\
&= 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x
\end{aligned}
$$

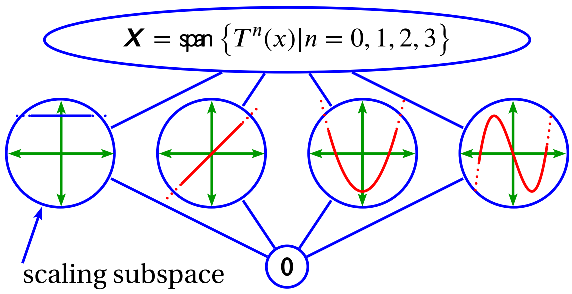


Figure 2.2: Lattice of Chebyshev polynomials $\{T_n(x)|n = 0, 1, 2, 3\}$

Note: Trigonometric expansion of $\cos(nx)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[4]

```
trigexpand(cos(2*x));
trigexpand(cos(3*x));
trigexpand(cos(4*x));
trigexpand(cos(5*x));
trigexpand(cos(6*x));
trigexpand(cos(7*x));
```

✏

Definition 2.1.

DEF *The nth **Chebyshev polynomial of the first kind** is defined as*

$$T_n(x) \triangleq \cos nx \qquad \textit{where} \qquad \cos x \triangleq x$$

Theorem 2.3. [5] *Let $T_n(x)$ be a* Chebyshev polynomial *with $n \in \mathbb{W}$.*

THM

$$\begin{array}{lcl} n \text{ is EVEN} & \implies & T_n(x) \text{ is EVEN.} \\ n \text{ is ODD} & \implies & T_n(x) \text{ is ODD.} \end{array}$$

Example 2.3. Let $T_n(x)$ be a *Chebyshev polynomial* with $n \in \mathbb{W}$.

EX

$T_0(x) = 1$	$T_4(x) = 8x^4 - 8x^2 + 1$
$T_1(x) = x$	$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_2(x) = 2x^2 - 1$	$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
$T_3(x) = 4x^3 - 3x$	

✎Proof: Proof of these equations follows directly from Example 2.2 (page 22). ✏

2.2 Trigonometric reduction

Theorem 2.2 (page 19) showed that $\cos nx$ can be expressed as a polynomial in $\cos x$. Conversely, Theorem 2.4 (next) shows that a polynomial in $\cos x$ can be expressed as a linear combination of $(\!(\cos nx)\!)_{n\in\mathbb{Z}}$.

Theorem 2.4 (trigonometric reduction).

[4] maxima pages 157–158 ⟨10.5 Trigonometric Functions⟩

[5] Rivlin (1974) page 5 ⟨(1.13)⟩, Süli and Mayers (2003) page 242 ⟨Lemma 8.2⟩, Davidson and Donsig (2010) page 222 ⟨exercise 10.7.A(a)⟩

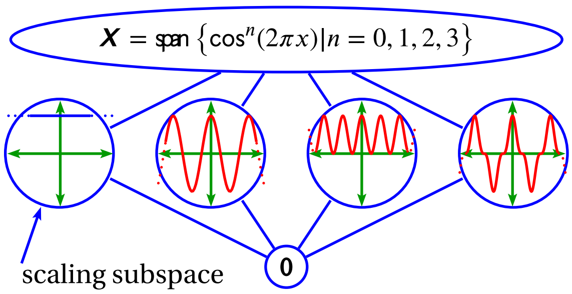


Figure 2.3: Lattice of exponential cosines $\{\cos^n x|n = 0,1,2,3\}$

THM

$$\begin{aligned}
\cos^n x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases} \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ even} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ odd} \end{cases}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\cos^n x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n \\
&= \mathbf{R_e}\left[\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)x}e^{-ikx}\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\cos[(n-2k)x]+i\sin[(n-2k)x]\right)\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] + i\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sin[(n-2k)x]\right] \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases} \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ even} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ odd} \end{cases}
\end{aligned}$$

✏

Example 2.4. [6]

[6] Abramowitz and Stegun (1972) page 795, Sloane (2014) ⟨http://oeis.org/A100257⟩, Sloane (2014) ⟨http://oeis.org/A008314⟩

EX

n	$\cos^n x$		trigonometric reduction	n	$\cos^n x$		trigonometric reduction
0	$\cos^0 x$	$=$	1	4	$\cos^4 x$	$=$	$\dfrac{\cos 4x + 4\cos 2x + 3}{2^3}$
1	$\cos^1 x$	$=$	$\cos x$	5	$\cos^5 x$	$=$	$\dfrac{\cos 5x + 5\cos 3x + 10\cos x}{2^4}$
2	$\cos^2 x$	$=$	$\dfrac{\cos 2x + 1}{2}$	6	$\cos^6 x$	$=$	$\dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{2^5}$
3	$\cos^3 x$	$=$	$\dfrac{\cos 3x + 3\cos x}{2^2}$	7	$\cos^7 x$	$=$	$\dfrac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{2^6}$

✎PROOF:

$$\begin{aligned}
\cos^0 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=0}\\
&= \frac{1}{2^0}\sum_{k=0}^{0}\binom{0}{k}\cos[(0-2k)x]\\
&= \binom{0}{0}\cos[(0-2\cdot 0)x]\\
&= 1\\
\cos^1 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=1}\\
&= \frac{1}{2^1}\sum_{k=0}^{1}\binom{1}{k}\cos[(1-2k)x]\\
&= \frac{1}{2}\left[\binom{1}{0}\cos[(1-2\cdot 0)x] + \binom{1}{1}\cos[(1-2\cdot 1)x]\right]\\
&= \frac{1}{2}\left[1\cos x + 1\cos(-x)\right]\\
&= \frac{1}{2}\left(\cos x + \cos x\right)\\
&= \cos x\\
\cos^2 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=2}\\
&= \frac{1}{2^2}\sum_{k=0}^{2}\binom{2}{k}\cos\left([2-2k]x\right)\\
&= \frac{1}{2^2}\left[\binom{2}{0}\cos\left([2-2\cdot 0]x\right) + \binom{2}{1}\cos\left([2-2\cdot 1]x\right) + \binom{2}{2}\cos\left([2-2\cdot 2]x\right) + \right]\\
&= \frac{1}{2^2}\left[1\cos(2x) + 2\cos(0x) + 1\cos(-2x)\right]\\
&= \frac{1}{2^2}\left[\cos(2x) + 2 + \cos(2x)\right]\\
&= \frac{1}{2}\left[\cos(2x) + 1\right]\\
\cos^3 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=3}\\
&= \frac{1}{2^3}\sum_{k=0}^{3}\binom{3}{k}\cos\left([3-2k]x\right)
\end{aligned}$$

$$= \frac{1}{2^3}\left[1\cos(3x) + 3\cos(1x) + 3\cos(-1x) + 1\cos(-3x)\right]$$
$$= \frac{1}{2^3}\left[\cos(3x) + 3\cos(x) + 3\cos(x) + \cos(3x)\right]$$
$$= \frac{1}{2^2}\left[\cos(3x) + 3\cos(x)\right]$$
$$\cos^4 x = \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=4}$$
$$= \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}\cos([4-2k]x)$$
$$= \frac{1}{2^4}\left[1\cos(4x) + 4\cos(2x) + 6\cos(0x) + 4\cos(-2x) + 1\cos(-4x)\right]$$
$$= \frac{1}{2^3}\left[\cos(4x) + 4\cos(2x) + 3\right]$$
$$\cos^5 x = \frac{1}{2^{5-1}}\sum_{k=0}^{\left\lfloor\frac{5}{2}\right\rfloor}\binom{5}{k}\cos[(5-2k)x]$$
$$= \frac{1}{16}\sum_{k=0}^{2}\binom{5}{k}\cos[(5-2k)x]$$
$$= \frac{1}{16}\left[\binom{5}{0}\cos 5x + \binom{5}{1}\cos 3x + \binom{5}{2}\cos x\right]$$
$$= \frac{1}{16}\left[\cos 5x + 5\cos 3x + 10\cos x\right]$$
$$\cos^6 x = \frac{1}{2^6}\binom{6}{\frac{6}{2}} + \frac{1}{2^{6-1}}\sum_{k=0}^{\frac{6}{2}-1}\binom{6}{k}\cos[(6-2k)x]$$
$$= \frac{1}{2^6}\binom{6}{3} + \frac{1}{2^5}\sum_{k=0}^{2}\binom{6}{k}\cos[(6-2k)x]$$
$$= \frac{1}{64}20 + \frac{1}{32}\left[\binom{6}{0}\cos 6x + \binom{6}{1}\cos 4x\binom{6}{2}\cos 2x\right]$$
$$= \frac{1}{32}\left[\cos 6x + 6\cos 4x + 15\cos 2x + 10\right]$$
$$\cos^7 x = \frac{1}{2^{7-1}}\sum_{k=0}^{\left\lfloor\frac{7}{2}\right\rfloor}\binom{7}{k}\cos[(7-2k)x]$$
$$= \frac{1}{64}\sum_{k=0}^{2}\binom{7}{k}\cos[(7-2k)x]$$
$$= \frac{1}{64}\left[\binom{7}{0}\cos 7x + \binom{7}{1}\cos 5x + \binom{7}{2}\cos 3x + \binom{7}{3}\cos x\right]$$
$$= \frac{1}{64}\left[\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right]$$

Note: Trigonometric reduction of $\cos^n(x)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[7]

```
trigreduce((cos(x))^2);
trigreduce((cos(x))^3);
trigreduce((cos(x))^4);
trigreduce((cos(x))^5);
trigreduce((cos(x))^6);
trigreduce((cos(x))^7);
```

[7] http://maxima.sourceforge.net/docs/manual/en/maxima_15.html
maxima page 158 ⟨10.5 Trigonometric Functions⟩

✏

2.3 Spectral Factorization

Theorem 2.5 (Fejér-Riesz spectral factorization). [8] *Let* $[0,\infty) \subsetneq \mathbb{R}$ *and*

$$\mathsf{p}\left(e^{ix}\right) \triangleq \sum_{n=-N}^{N} a_n e^{inx} \qquad \text{(Laurent trigonometric polynomial order } 2N\text{)}$$

$$\mathsf{q}\left(e^{ix}\right) \triangleq \sum_{n=1}^{N} b_n e^{inx} \qquad \text{(standard trigonometric polynomial order } N\text{)}$$

THM

$$\mathsf{p}\left(e^{ix}\right) \in [0,\infty) \quad \forall x \in [0,2\pi] \implies \begin{cases} \exists \left(\!\left(b_n\right)\!\right)_{n\in\mathbb{Z}} & \text{such that} \\ \mathsf{p}\left(e^{ix}\right) = \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right) & \forall x \in \mathbb{R} \end{cases}$$

✎PROOF:

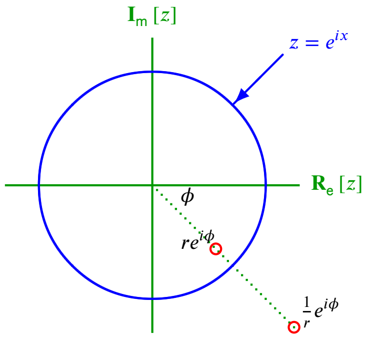
1. Proof that $a_n = a^*_{-n}$ ($\left(\!\left(a_n\right)\!\right)_{n\in\mathbb{Z}}$ is *Hermitian symmetric*):
   Let $a_n \triangleq r_n e^{i\phi_n}$, $r_n, \phi_n \in \mathbb{R}$. Then

$$\begin{aligned}
\mathsf{p}\left(e^{inx}\right) &\triangleq \sum_{n=-N}^{N} a_n e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{i\phi_n} e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{inx+\phi_n} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \sum_{n=-N}^{N} r_n \sin(nx+\phi_n) \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(0x+\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) + \sum_{n=1}^{N} r_{-n} \sin(-nx+\phi_{-n}) \right]}_{\text{imaginary part must equal 0 because } p(x)\in\mathbb{R}} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) - \sum_{n=1}^{N} r_{-n} \sin(nx-\phi_{-n}) \right]}_{\implies r_n = r_{-n},\ \phi_n = -\phi_{-n} \implies a_n = a^*_{-n},\ a_0\in\mathbb{R}}
\end{aligned}$$

2. Because the coefficients $\left(\!\left(c_n\right)\!\right)_{n\in\mathbb{Z}}$ are *Hermitian symmetric* and by Theorem B.7 (page 102), the zeros of $P(z)$ occur in *conjugate recipricol pairs*. This means that if $\sigma \in \mathbb{C}$ is a zero of $P(z)$ ($P(\sigma)=0$), then $\frac{1}{\sigma^*}$ is also a zero of $P(z)$ ($P\left(\frac{1}{\sigma^*}\right)=0$). In the complex z plane, this relationship means zeros are reflected across the unit circle such that

$$\frac{1}{\sigma^*} = \frac{1}{(re^{i\phi})^*} = \frac{1}{r}\frac{1}{e^{-i\phi}} = \frac{1}{r}e^{i\phi}$$

[8] Pinsky (2002) pages 330–331



3. Because the zeros of $\mathsf{p}(z)$ occur in conjugate recipricol pairs, $\mathsf{p}\left(e^{ix}\right)$ can be factored:

$$
\begin{aligned}
\mathsf{p}\left(e^{ix}\right) &= \mathsf{p}(z)\big|_{z=e^{ix}} \\
&= z^{-N} C \prod_{n=1}^{N} (z-\sigma_n) \prod_{n=1}^{N} \left(z - \frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N} (z-\sigma_n) \prod_{n=1}^{N} z^{-1}\left(z - \frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N} (z-\sigma_n) \prod_{n=1}^{N} \left(1 - \frac{1}{\sigma_n^*} z^{-1}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N} (z-\sigma_n) \prod_{n=1}^{N} \left(z^{-1} - \sigma_n^*\right)\left(-\frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= \left[C \prod_{n=1}^{N} \left(-\frac{1}{\sigma_n^*}\right)\right]\left[\prod_{n=1}^{N} (z-\sigma_n)\right]\left[\prod_{n=1}^{N} \left(\frac{1}{z^*} - \sigma_n\right)\right]^*\bigg|_{z=e^{ix}} \\
&= \left[C_2 \prod_{n=1}^{N} (z-\sigma_n)\right]\left[C_2 \prod_{n=1}^{N} \left(\frac{1}{z^*} - \sigma_n\right)\right]^*\bigg|_{z=e^{ix}} \\
&= \mathsf{q}(z)\mathsf{q}^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{ix}} \\
&= \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right)
\end{aligned}
$$

✏

2.4 Dirichlet Kernel

❝Dirichlet alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous proof is. Rather we learn it first from him. When Gauss says he has proved something it is clear; when Cauchy says it, one can wager as much pro as con; when Dirichlet says it, it is certain.❞

Carl Gustav Jacob Jacobi (1804–1851), Jewish-German mathematician [9]

[9] quote: Schubring (2005) page 558
image: http://en.wikipedia.org/wiki/File:Carl_Jacobi.jpg, public domain

The *Dirichlet Kernel* is critical in proving what is not immediately obvious in examining the Fourier Series—that for a broad class of periodic functions, a function can be recovered from (with uniform convergence) its Fourier Series analysis.

Definition 2.2. [10]

DEF: *The **Dirichlet Kernel** $\mathsf{D}_n \in \mathbb{R}^{\mathbb{W}}$ with period τ is defined as*

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}kx}$$

Proposition 2.1. [11] *Let D_n be the* Dirichlet Kernel *with period τ* (Definition 2.2 page 30).

PRP:
$$\mathsf{D}_n(x) = \frac{1}{\tau}\frac{\sin\left(\frac{\pi}{\tau}[2n+1]x\right)}{\sin\left(\frac{\pi}{\tau}x\right)}$$

✎Proof:

$$\begin{aligned}
\mathsf{D}_n(x) &\triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \mathsf{D}_n \quad \text{(Definition 2.2 page 30)}\\
&= \frac{1}{\tau}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}(k-n)x} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}kx} = \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\sum_{k=0}^{2n}\left(e^{i\frac{2\pi}{\tau}x}\right)^k\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\frac{1-\left(e^{i\frac{2\pi}{\tau}x}\right)^{2n+1}}{1-e^{i\frac{2\pi}{\tau}x}} && \text{by \textit{geometric series}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\frac{1-e^{i\frac{2\pi}{\tau}(2n+1)x}}{1-e^{i\frac{2\pi}{\tau}x}} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(\frac{e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{i\frac{\pi}{\tau}x}}\right)\frac{e^{-i\frac{\pi}{\tau}(2n+1)x}-e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{-i\frac{\pi}{\tau}x}-e^{i\frac{\pi}{\tau}x}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(e^{i\frac{2\pi n}{\tau}x}\right)\frac{-2i\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{-2i\sin\left[\frac{\pi}{\tau}x\right]} = \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}x\right]}
\end{aligned}$$

✎

Proposition 2.2. [12] *Let D_n be the* Dirichlet Kernel *with period τ* (Definition 2.2 page 30).

PRP:
$$\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x = 1$$

✎Proof:

$$\begin{aligned}
\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x &\triangleq \int_0^\tau \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \mathsf{D}_n \quad \text{(Definition 2.2 page 30)}\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right) + i\sin\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x
\end{aligned}$$

[10] Katznelson (2004) page 14, Heil (2011) pages 443–444, Folland (1992) pages 33–34
[11] Katznelson (2004) page 14, Heil (2011) page 444, Folland (1992) page 34
[12] Bruckner et al. (1997) pages 620–621

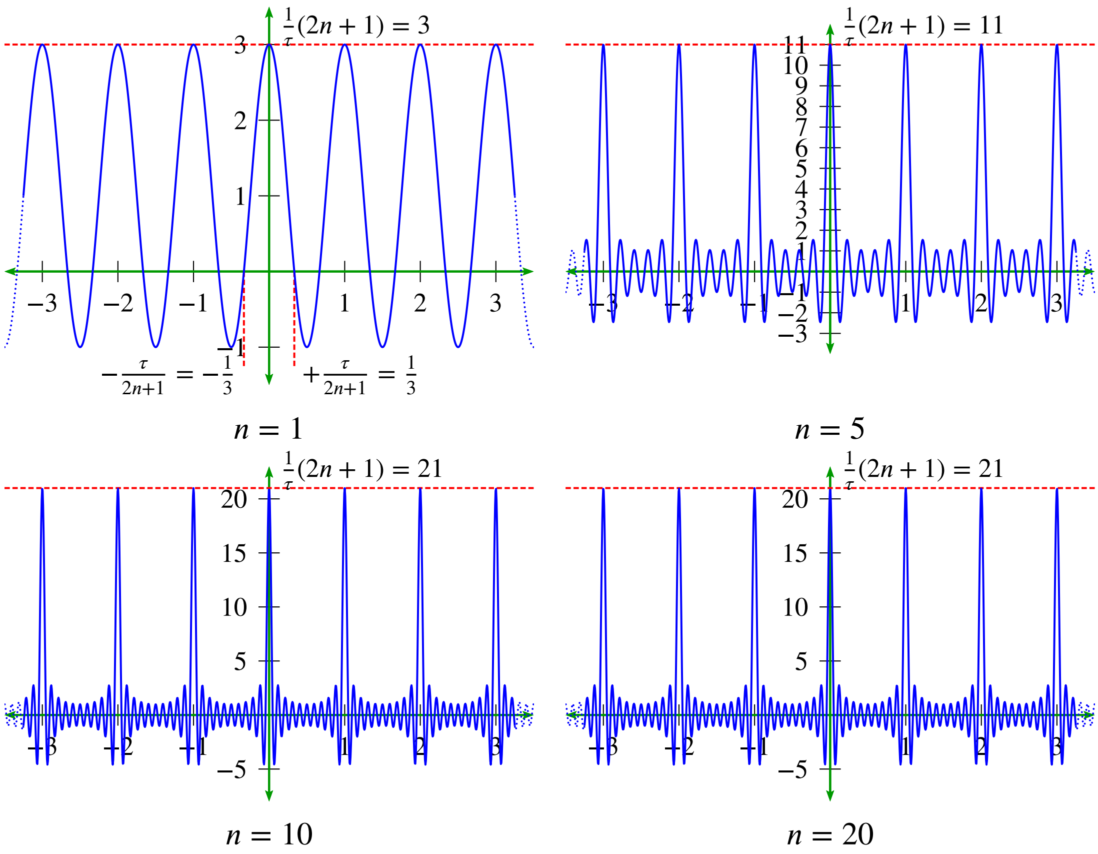


Figure 2.4: D_n function for $N = 1, 5, 10, 20$. $\mathsf{D}_n \to$ comb. (See Proposition 2.1 page 30).

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x \\
&= \frac{1}{\tau}\sum_{k=-n}^{n} \left.\frac{\sin\left(\frac{2\pi}{\tau}nx\right)}{\frac{2\pi}{\tau}n}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left[\frac{\sin\left(\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n} - \frac{\sin\left(-\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n}\right] \\
&= \frac{1}{\tau}\frac{\tau}{2}\sum_{k=-n}^{n}\left[\frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi n)}{\pi n}\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n)}{\pi n}\right]_{k=0} \\
&= 1
\end{aligned}
$$

✏

Proposition 2.3. *Let D_n be the* DIRICHLET KERNEL *with period τ* (Definition 2.2 page 30). *Let w_N (the "WIDTH" of $\mathsf{D}_n(x)$) be the distance between the two points where the center pulse of $\mathsf{D}_n(x)$ intersects the x axis.*

PRP

$$
\begin{aligned}
\mathsf{D}_n(0) &= \frac{1}{\tau}(2n+1) \\
w_n &= \frac{2\tau}{2n+1}
\end{aligned}
$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(0) &= \mathsf{D}_n(x)\big|_{t=0} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} && \text{by Proposition 2.1 page 30} \\
&= \frac{1}{\tau}\frac{\frac{\mathsf{d}}{\mathsf{dx}}\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\frac{\mathsf{d}}{\mathsf{dx}}\sin\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{\cos\left[\frac{\pi}{\tau}(2n+1)x\right]}{\cos\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{1}{1} \\
&= \frac{1}{\tau}(2n+1)
\end{aligned}$$

The center pulse of kernel $\mathsf{D}_n(x)$ intersects the x axis at

$$t = \pm\frac{\tau}{(2n+1)}$$

which implies

$$w_n = \frac{\tau}{2n+1} + \frac{\tau}{2n+1} = \frac{2\tau}{(2n+1)}.$$

✏

Proposition 2.4. [13] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition 2.2 page 30).

PRP: $\mathsf{D}_n(x) = \mathsf{D}_n(-x)$ (D_n *is an* EVEN *function*)

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]} && \text{by Proposition 2.1 page 30} \\
&= \frac{1}{\tau}\frac{-\sin\left[-\frac{\pi}{\tau}(2n+1)x\right]}{-\sin\left[-\frac{\pi}{\tau}t\right]} && \text{because } \sin x \text{ is an } \textit{odd} \text{ function} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)(-x)\right]}{\sin\left[\frac{\pi}{\tau}(-x)\right]} \\
&= \mathsf{D}_n(-x) && \text{by Proposition 2.1 page 30}
\end{aligned}$$

✏

[13] Bruckner et al. (1997) pages 620–621

2.5 Trigonometric summations

Theorem 2.6 (Lagrange trigonometric identities). [14]

THM

$$\sum_{n=0}^{N-1}\cos(nx) = \frac{1}{2} + \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)+\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)+\cos\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=0}^{N-1}\Re e^{inx} = \Re\sum_{n=0}^{N-1}e^{inx} = \Re\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Re\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \Re\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-i\frac{1}{2}x}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Re\left[\left(e^{i\frac{1}{2}(N-1)x}\right)\left(\frac{-i\frac{1}{2}\sin\left(\frac{1}{2}Nx\right)}{-i\frac{1}{2}\sin\left(\frac{1}{2}x\right)}\right)\right] \\
&= \cos\left(\frac{1}{2}(N-1)x\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{-\frac{1}{2}\sin\left(-\frac{1}{2}x\right)+\frac{1}{2}\sin\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by } \textit{product identities} \quad \text{(Theorem 1.8 page 10)} \\
&= \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=0}^{N-1}\Im e^{inx} = \Im\sum_{n=0}^{N-1}e^{inx} = \Im\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Im\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \Im\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-ix/2}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Im\left[\left(e^{i(N-1)x/2}\right)\left(\frac{-\frac{1}{2}i\sin\left(\frac{1}{2}Nx\right)}{-\frac{1}{2}i\sin\left(\frac{1}{2}x\right)}\right)\right]
\end{aligned}$$

[14] Muniz (1953) page 140 ⟨"Lagrange's Trigonometric Identities"⟩, Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (14), (13)⟩

$$
\begin{aligned}
&= \sin\left(\frac{(N-1)x}{2}\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{\frac{1}{2}\cos\left(-\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by \textit{product identities} (Theorem 1.8 page 10)} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}
$$

Note that these results (summed with indices from $n = 0$ to $n = N - 1$) are compatible with Muniz (1953) page 140 (summed with indices from $n = 1$ to $n = N$) as demonstrated next:

$$
\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=1}^{N}\cos(nx) + [\cos(0x) - \cos(Nx)] \\
&= \left[-\frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + [\cos(0x) - \cos(Nx)] && \text{by Muniz (1953) page 140} \\
&= \left(1 - \frac{1}{2}\right) + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\cos(Nx)}{2\sin\left(\frac{1}{2}x\right)} \\
&= \frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\left[\sin\left(\left[\frac{1}{2}-N\right]x\right) + \sin\left[\left(\frac{1}{2}+N\right)x\right]\right]}{2\sin\left(\frac{1}{2}x\right)} && \text{by Theorem 1.8 page 10} \\
&= \frac{1}{2} + \frac{\sin\left(\frac{1}{2}[2N-1]x\right)}{2\sin\left(\frac{1}{2}x\right)} && \implies \text{above result} \\
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=1}^{N}\sin(nx) + [\sin(0x) - \sin(Nx)] \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} + [0 - \sin(Nx)] && \text{by Muniz (1953) page 140} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\sin(Nx)}{2\sin\left(\frac{1}{2}x\right)} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - \left[\cos\left(\left[\frac{1}{2}-N\right]x\right) - \cos\left(\left[\frac{1}{2}+N\right]x\right)\right]}{2\sin\left(\frac{1}{2}x\right)} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} && \implies \text{above result}
\end{aligned}
$$

✐

Theorem 2.7. [15]

[15] Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (16) and (17)⟩

THM

$$\sum_{n=0}^{N-1}\cos(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]-\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]+\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)-\sin(nx)\sin(y)] && \text{by } \textit{double angle formulas} \text{ (Theorem 1.9 page 13)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx)-\sin(y)\sum_{n=0}^{N-1}\sin(nx)\\
\sum_{n=0}^{N-1}\sin(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)+\sin(nx)\sin(y)] && \text{by } \textit{double angle formulas} \text{ (Theorem 1.9 page 13)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx)+\sin(y)\sum_{n=0}^{N-1}\sin(nx)
\end{aligned}$$

✏

Corollary 2.1 (Summation around unit circle).

THM

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)=0 \qquad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

$$\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right)=\frac{N}{2} \qquad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

✎PROOF:

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\\
&= \cos(\theta)\sum_{n=0}^{N-1}\cos\left(\frac{2nM\pi}{N}\right)-\sin(\theta)\sum_{n=0}^{N-1}\sin\left(\frac{2nM\pi}{N}\right) && \text{by Theorem 1.9 page 13}\\
&= \cos(\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{2M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] && \text{by Theorem 2.6 page 33}\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{\cos\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{M\pi}{N}\right)}{\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)\right] && \text{by } \textit{trigonometric periodicity} \text{ (Theorem 1.10 page 13)}\\
&= \cos(\theta)[0]-\sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=0}^{N-1} \sin\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by \textit{shift identities}} \quad \text{(Theorem 1.7 page 10)} \\
&= \sum_{n=0}^{N-1} \cos\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2} \\
&= 0 && \text{by previous result}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \cos\left(\theta + \frac{2nM\pi}{N}\right)\sin\left(\theta + \frac{2nM\pi}{N}\right) \\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] - \left[\theta + \frac{2nM\pi}{N}\right]\right) + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] + \left[\theta + \frac{2nM\pi}{N}\right]\right) && \text{by Theorem 1.8 page 10} \\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \cancel{\sin(0)}\, 0 + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(2\theta + \frac{4nM\pi}{N}\right) \\
&= \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) && \text{by Theorem 1.9 page 13} \\
&= \cos(2\theta)\left[\frac{1}{2} + \frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{4M\pi}{N}\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{4M\pi}{N}\right)}\right] && \text{by Theorem 2.6 page 33} \\
&= \cos(2\theta)\left[\frac{1}{2} - \frac{\sin\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{\cos\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right] \\
&= \cos(\theta)\left[\frac{1}{2} - \frac{1}{2}\frac{\sin\left(\frac{2M\pi}{N}\right)}{\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity}} \quad \text{(Theorem 1.10 page 13)} \\
&= \cos(\theta)[0] - \sin(\theta)[0] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2nM\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos\left(2\theta + \frac{4nM\pi}{N}\right)\right] && \text{by Theorem 1.11 page 15} \\
&= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos(2\theta)\cos\left(\frac{4nM\pi}{N}\right) - \sin(2\theta)\sin\left(\frac{4nM\pi}{N}\right)\right] && \text{by Theorem 1.9 page 13} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} 1 + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\cos\left(\frac{4nM\pi}{N}\right) - \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right) \\
&= \left[\frac{1}{2}\sum_{n=0}^{N-1} 1\right] + \frac{1}{2}\cos(2\theta)0 - \frac{1}{2}\sin(2\theta)0 && \text{by previous results} \\
&= \frac{N}{2}
\end{aligned}
$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos^2\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by \textit{shift identities} (Theorem 1.7 page 10)} \\
&= \sum_{n=0}^{N-1} \cos^2\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2} \\
&= \frac{N}{2} && \text{by previous result}
\end{aligned}$$

✏

2.6 Summability Kernels

Definition 2.3. [16] *Let $(\kappa_n)_{n\in\mathbb{Z}}$ be a sequence of* CONTINUOUS 2π PERIODIC *functions.*

DEF

*The sequence $(\kappa_n)_{n\in\mathbb{Z}}$ is a **summability kernel** if*

1. $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(x)\, dx = 1 \quad \forall n\in\mathbb{Z}$ *and*
2. $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} |\kappa_n(x)|\, dx \in \mathbb{R} \quad \forall n\in\mathbb{Z}$ *and*
3. $\displaystyle \lim_{n\to\infty}\int_\delta^{2\pi-\delta} |\kappa_n(x)|\, dx = 0 \quad \forall n\in\mathbb{Z}, 0<\delta<\pi$

Theorem 2.8. [17] *Let $(\kappa_n)_{n\in\mathbb{Z}}$ be a sequence. Let $\mathbb{T}$ be the quotient $\mathbb{R}/2\pi\mathbb{Z}$.*

THM

$$\left.\begin{array}{ll} 1. & \mathsf{f} \in L^1(\mathbb{T}) \quad \textit{and} \\ 2. & (\kappa_n) \textit{ is a summability kernel} \end{array}\right\} \implies \mathsf{f}(x) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbb{T}} \kappa_n(x)\mathsf{f}(x-x)\, dx$$

The *Dirichlet kernel* (Definition 2.2 page 30) is *not* a *summability kernel*. Examples of kernels that *are* summability kernels include

1. *Fejér's kernel* (Definition 2.4 page 37)
2. *de la Vallée Poussin kernel* (Definition 2.5 page 39)
3. *Jackson kernel* (Definition 2.6 page 39)
4. *Poisson kernel* (Definition 2.7 page 39 .)

Definition 2.4. [18]

DEF

***Fejér's kernel** K_n is defined as*

$$\mathsf{K}_n(x) \triangleq \sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx}$$

Proposition 2.5. [19] *Let K_n be Fejér's kernel* (Definition 2.4 page 37).

PRP

$$\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$$

[16] Cerdà (2010) page 56, Katznelson (2004) page 10, de Reyna (2002) page 21, Walnut (2002) pages 40–41, Heil (2011) page 440, Istrăţescu (1987) page 309

[17] Katznelson (2004) page 11

[18] Katznelson (2004) page 12

[19] Katznelson (2004) page 12, Heil (2011) page 448

✎PROOF:

1. Lemma: Proof that $\sin^2\frac{x}{2} \equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right)$:

$$\begin{aligned}
\sin^2\frac{x}{2} &\equiv \left(\frac{e^{-i\frac{x}{2}} - e^{+i\frac{x}{2}}}{2i}\right)^2 && \text{by Euler Formulas (Corollary 1.2 page 9)}\\
&\equiv \frac{-1}{4}\left(e^{-2i\frac{x}{2}} - 2e^{-i\frac{x}{2}}e^{i\frac{x}{2}} + e^{2i\frac{x}{2}}\right)\\
&\equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right) \;:
\end{aligned}$$

2. Lemma:

$$2|k| - |k+1| - |k-1| = \begin{cases} -2 & \text{for } k = 0\\ 0 & \text{for } k \in \mathbb{Z}\backslash 0 \end{cases}$$

3. Proof that $\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$:

$$\begin{aligned}
&-4(n+1)\left(\sin\frac{1}{2}x\right)^2 \mathsf{K}_n(x)\\
&= -4(n+1)\left(\frac{-1}{4}\right)\left(e^{-ix} - 2 + e^{ix}\right)\mathsf{K}_n(x) && \text{by item (1)}\\
&= (n+1)\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx} && \text{by Definition 2.4}\\
&= (n+1)\frac{1}{n+1}\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= e^{-ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}e^{ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= \sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k-1)x} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k+1)x}\\
&= \sum_{k=-n-1}^{k=n-1}(n+1-|k+1|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n+1}^{k=n+1}(n+1-|k-1|)e^{ikx}\\
&= \underbrace{e^{-i(n+1)x}}_{k=-n-1} + \underbrace{2e^{-inx}}_{k=-n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad \underbrace{-2e^{-inx}}_{k=-n} + \underbrace{-2e^{inx}}_{k=n} - 2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad \underbrace{e^{i(n+1)x}}_{k=n+1} + \underbrace{2e^{inx}}_{k=n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}\\
&= e^{-i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad -2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}
\end{aligned}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} [(n+1-|k+1|) - 2(n+1-|k|) + (n+1-|k-1|)]\, e^{ikx}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} (2|k| - |k+1| - |k-1|)\, e^{ikx}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} - 2 \qquad \text{by item (2)}$$

$$= -4\left(\sin\frac{n+1}{2}x\right)^2 \qquad \text{by item (1)}$$

✏

Definition 2.5. [20] *Let* K_n *be* FEJÉR'S KERNEL *(Definition 2.4 page 37).*

DEF: *The **de la Vallée Poussin kernel** V_n is defined as*

$$\mathsf{V}_n(x) \triangleq 2K_{2n+1}(x) - K_n(x)$$

Definition 2.6. [21] *Let* K_n *be* FEJÉR'S KERNEL *(Definition 2.4 page 37).*

DEF: *The **Jackson kernel** J_n is defined as*

$$\mathsf{J}_n(x) \triangleq \|\mathsf{K}_n\|^{-2}\, \mathsf{K}_n^2(x)$$

Definition 2.7. [22]

DEF: *The **Poisson kernel** P is defined as*

$$P(r,x) \triangleq \sum_{k\in\mathbb{Z}} r^{|k|} e^{ikx}$$

[20] Katznelson (2004) page 16
[21] Katznelson (2004) page 17
[22] Katznelson (2004) page 16

CHAPTER 3

FOURIER TRANSFORM

❝ *Up to this point we have supposed that the function whose development is required in a series of sines of multiple arcs can be developed in a series arranged according to powers of the variable x, ... We can extend the same results to any functions, even to those which are discontinuous and entirely arbitrary. ... even entirely arbitrary functions may be developed in series of sines of multiple arcs.* ❞

Joseph Fourier (1768–1830) [1]

3.1 Introduction

Historically, before the Fourier Transform was the Taylor Expansion (transform). The Taylor Expansion demonstrates that for **analytic** functions knowledge of the derivatives of a function at a location $x = a$ allows you to determine (predict) arbitrarily closely all the points $\mathsf{f}(x)$ in the vicinity of $x = a$ (Chapter **??** page **??**). But analytic functions are by definition functions for which all their derivatives exist. Thus, if a function is *discontinuous*, it is simply not a candidate for a Taylor Expansion. And some 300 years ago, mathematician giants of the day were fairly content with this.

But then in came an engineer named Joseph Fourier whose day job was working as a governor of lower Egypt under Napolean. He claimed that, rather than expansion based on derivatives, one could expand based on integrals over sinusoids, and that this would work not just for analytic functions, but for **discontinuous** ones as well![2]

Needless to say, this did not go over too well initially in the mathematical community. But over time (on the order of 200 or so years), the Fourier Transform has in many ways won the day.


[3]

[1] quote: Fourier (1878) page 184,186 ⟨§219,220⟩
image: `http://en.wikipedia.org/wiki/File:Fourier2.jpg`, public domain

[2] **?** page 886

[3] Caricature of Legendre (left) and Fourier (right), 1820, by Julien-Léopold Boilly (1796–1874). "*Album de 73*

3.2 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on ℝ, and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \middle| \int_{\mathbb{R}} |\mathsf{f}|^2 \, \mathrm{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \mid \triangledown \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \mathrm{d}\mu$ such that

$$\langle \mathsf{f} \mid \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\, \mathrm{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \mid \triangledown \rangle \right)$ is a *Hilbert space*.

Definition 3.1. *Let κ be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF: *The function κ is the **Fourier kernel** if* $\quad \kappa(x,\omega) \triangleq e^{i\omega x} \quad \forall x,\omega\in\mathbb{R}$

Definition 3.2. [4] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF: *The **Fourier Transform** operator $\tilde{\mathbf{F}}$ is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

*This definition of the Fourier Transform is also called the **unitary Fourier Transform**.*

Remark 3.1 (**Fourier transform scaling factor**). [5] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B \int_{\mathbb{R}} \mathsf{F}(\omega) e^{i\omega x}\, \mathrm{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x}\, \mathrm{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x}\, \mathrm{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int\cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem 3.2 page 43)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

Portraits-Charge Aquarelle's des Membres de I'Institute (watercolor portrait #29). Biliotheque de l'Institut de France." Public domain. https://en.wikipedia.org/wiki/File:Legendre_and_Fourier_(1820).jpg

[4] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965) page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

[5] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005b) pages 374–375

3.3 Operator properties

Theorem 3.1 (Inverse Fourier transform). [6] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition 3.2 page 42). *The inverse* $\tilde{\mathbf{F}}^{-1}$ *of* $\tilde{\mathbf{F}}$ *is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\,\mathrm{d}\omega \qquad \forall \tilde{\mathsf{f}} \in \mathcal{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem 3.2. *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with inverse* $\tilde{\mathbf{F}}^{-1}$ *and adjoint* $\tilde{\mathbf{F}}^{*}$.

THM
$$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \mathsf{g}\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathrm{d}x \,\middle|\, \mathsf{g}(\omega)\right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 42}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,\left\langle e^{-i\omega x} \,|\, \mathsf{g}(\omega)\right\rangle\,\mathrm{d}x && \text{by \textit{additive property} of } \langle \triangle \,|\, \nabla\rangle\\
&= \int_{\mathbb{R}} \mathsf{f}(x)\,\frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x}\right\rangle^*\,\mathrm{d}x && \text{by \textit{conjugate symmetric property} of } \langle \triangle \,|\, \nabla\rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x}\right\rangle\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle\\
&= \left\langle \mathsf{f} \,\middle|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\mathsf{g}\right\rangle && \text{by Theorem 3.1 page 43}
\end{aligned}$$

✏

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* [7] (Corollary 3.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem 3.3 page 43).

Corollary 3.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

COR
$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem 3.2 page 43). ✏

Theorem 3.3. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $\left(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \nabla\rangle\right)$. *Let* $\mathcal{R}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

THM
$$\begin{array}{rclcll}
\mathcal{R}(\mathbf{F}\tau) & = & \mathcal{R}(\tilde{\mathbf{F}}^{-1}) & = & \mathcal{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| & = & |||\tilde{\mathbf{F}}^{-1}||| & = & 1 & \text{(UNITARY)}\\
\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g}\rangle & = & \langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g}\rangle & = & \langle \mathsf{f} \,|\, \mathsf{g}\rangle & \text{(PARSEVAL'S EQUATION)}\\
\|\tilde{\mathbf{F}}\mathsf{f}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f}\| & = & \|\mathsf{f}\| & \text{(PLANCHEREL'S FORMULA)}\\
\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\| & = & \|\mathsf{f} - \mathsf{g}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary 3.1 page 43) and from the properties of unitary operators (Theorem E.26 page 152). ✏

[6] Chorin and Hald (2009) page 13
[7] *unitary operators*: Definition E.14 page 151

3.4 Shift relations

Theorem 3.4 (Shift relations)**.** *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition 3.2 page 42).

THM
$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega)\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r)
\end{aligned}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator.

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= \mathbf{L}[\mathsf{f}(x-y)](s)|_{s=i\omega} && \text{by definition of } \mathbf{L}\\
&= e^{-sy}\,[\mathbf{L}\mathsf{f}(x)](s)|_{s=i\omega} && \text{by } \textit{Laplace shift relation}\\
&= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 42)}\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\mathbf{L}\left(e^{irx}\mathsf{g}(x)\right)\right](s)\Big|_{s=i\omega} && \text{by definition of } \mathbf{L}\\
&= \left[[\mathbf{L}\mathsf{g}(x)](s-r)\right]\Big|_{s=i\omega} && \text{by } \textit{Laplace shift relation}\\
&= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 42)}
\end{aligned}$$

✏

Theorem 3.5 (Complex conjugate)**.** *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

THM
$$\begin{aligned}
& & \tilde{\mathbf{F}}\mathsf{f}^*(-x) &= -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \forall \mathsf{f}\in\mathcal{L}^2_{(\mathbb{R},\mathscr{B},\mu)}\\
\mathsf{f} \textit{ is real} &\implies & \tilde{\mathsf{f}}(-\omega) &= \left[\tilde{\mathsf{f}}(\omega)\right]^* && \forall\omega\in\mathbb{R} && \text{REALITY CONDITION}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(-x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 42)}\\
&= \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(u)e^{i\omega u}(-1)\,\mathrm{d}u && \text{where } u\triangleq -x \implies \mathrm{d}x = -\,\mathrm{d}u\\
&= -\left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right]^*\\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 42)}\\
\tilde{\mathsf{f}}(-\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i(-\omega)x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 42)}\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(x)e^{-i\omega x}\,\mathrm{d}x\right]^*\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]^* && \text{by } \mathsf{f} \text{ is real hypothesis}\\
&\triangleq \tilde{\mathsf{f}}^*(\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 42)}
\end{aligned}$$

✏

3.5 Convolution relations

Definition 3.3. [8]

DEF *The* ***convolution operation*** *is defined as*

$$\left[\mathsf{f} \star \mathsf{g}\right](x) \triangleq \mathsf{f}(x) \star \mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem D.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem 3.6 (convolution theorem). [9] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition 3.2 page 42) *and* $\star$ *the convolution operator* (Definition 3.3 page 45).

THM

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator.

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega) &= \mathbf{L}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](s)\Big|_{s=i\omega} && \text{by definition of } \mathbf{L}\\
&= \sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s)\Big|_{s=i\omega} && \text{by } \textit{Laplace convolution} \text{ result} \quad \text{(Theorem ?? page ??)}\\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)\\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s)|_{s=i\omega}\\
&= \frac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s) \star \left[\mathbf{L}\mathsf{g}\right](s)\Big|_{s=i\omega}\\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)
\end{aligned}$$

✏

3.6 Calculus relations

Theorem 3.7. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* (Definition 3.2 page 42).

THM

$$\left\{\lim_{t\to-\infty} \mathsf{x}(t) = 0\right\} \implies \left\{\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right] = i\omega\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)\right\}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator.

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right] &\triangleq \mathbf{L}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right](s)\Big|_{s=i\omega} && \text{by definitions of } \mathbf{L} \text{ and } \tilde{\mathbf{F}}\\
&= s[\mathbf{L}\mathsf{x}(t)](s)|_{s=i\omega} && \text{by Theorem ?? page ??}\\
&= i\omega\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)
\end{aligned}$$

✏

[8] Bachman (1964) page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩
[9] Bracewell (1978) page 110

Theorem 3.8. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* (Definition 3.2 page 42).

THM
$$\tilde{\mathbf{F}}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u = \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)$$

Let $\mathbf{L}$ be the *Laplace Transform* operator. ✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u &\triangleq \mathbf{L}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u\bigg|_{s=i\omega} \\
&= \frac{1}{s}[\mathbf{L}\mathsf{x}(t)](s)\bigg|_{s=i\omega} && \text{by Theorem ?? page ??} \\
&= \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}(t)\right](\omega)
\end{aligned}$$

✏

3.7 Real valued functions

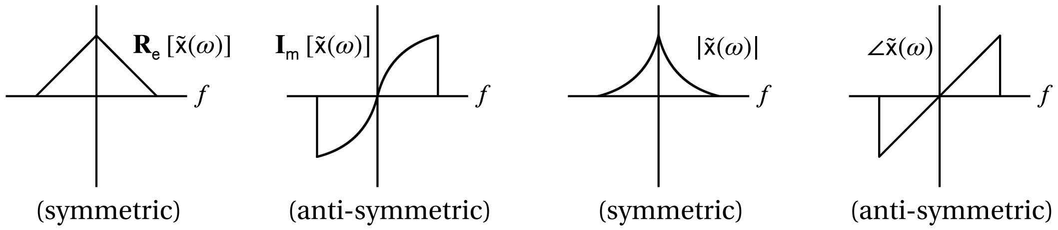


Figure 3.1: Fourier transform components of real-valued signal

Theorem 3.9. *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM
$$\left\{\begin{array}{l}\mathsf{f}(x) \text{ is REAL-VALUED} \\ (\mathsf{f}\in\mathbb{R}^{\mathbb{R}})\end{array}\right\} \implies \left\{\begin{array}{rcll}
\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & \text{(HERMITIAN SYMMETRIC)} \\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(SYMMETRIC)} \\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(ANTI-SYMMETRIC)} \\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| & \text{(SYMMETRIC)} \\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & \text{(ANTI-SYMMETRIC).}
\end{array}\right\}$$

✎PROOF:

$$\begin{array}{lclclclcl}
\tilde{\mathsf{f}}(\omega) & \triangleq & [\tilde{\mathbf{F}}\mathsf{f}(x)](\omega) & \triangleq & \left\langle \mathsf{f}(x)\,|\,e^{i\omega x}\right\rangle & = & \left\langle \mathsf{f}(x)\,|\,e^{i(-\omega)x}\right\rangle^* & \triangleq & \tilde{\mathsf{f}}^*(-\omega) \\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] & = & \mathbf{R_e}\left[\tilde{\mathsf{f}}^*(-\omega)\right] & = & \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] & = & \mathbf{I_m}\left[\tilde{\mathsf{f}}^*(-\omega)\right] & = & -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
|\tilde{\mathsf{f}}(\omega)| & = & |\tilde{\mathsf{f}}^*(-\omega)| & = & |\tilde{\mathsf{f}}(-\omega)| \\
\angle\tilde{\mathsf{f}}(\omega) & = & \angle\tilde{\mathsf{f}}^*(-\omega) & = & -\angle\tilde{\mathsf{f}}(-\omega)
\end{array}$$

✏

3.8 Moment properties

Definition 3.4. [10]

DEF
The quantity M_n *is the* n**th moment** *of a function* $\mathsf{f}(x)\in\boldsymbol{L}^2_{\mathbb{R}}$ *if*
$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n\mathsf{f}(x)\,\mathsf{d}x \qquad \text{for } n\in\mathbb{W}.$$

[10] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83

Lemma 3.1. [11] *Let* M_n *be the* nTH MOMENT (Definition 3.4 page 46) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition 3.2 page 42) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition C.1 page 109).

LEM

$$\begin{aligned} \mathsf{M}_n &= \sqrt{2\pi}(i)^n \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \\ \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} &= \frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \end{aligned}$$

✎PROOF:

$$\begin{aligned} \sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 3.2 page 42)} \\ &= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\right] \mathrm{d}x\bigg|_{\omega=0} \\ &= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathrm{d}x\bigg|_{\omega=0} \\ &= (-i^2)^n \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\ &\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \quad \text{(Definition 3.4 page 46)} \end{aligned}$$

✑

Lemma 3.2. [12] *Let* M_n *be the* nTH MOMENT (Definition 3.4 page 46) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition 3.2 page 42) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition C.1 page 109).

LEM

$$\mathsf{M}_n = 0 \quad\iff\quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof for ($\implies$) case:

$$\begin{aligned} 0 &= \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by left hypothesis} \\ &= \sqrt{2\pi}(-i)^{-n} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by Lemma 3.1 page 47} \\ \implies & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} = 0 \end{aligned}$$

2. Proof for ($\impliedby$) case:

$$\begin{aligned} 0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by right hypothesis} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\bigg|_{\omega=0} && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\,\mathrm{d}x\bigg|_{\omega=0} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathrm{d}x\bigg|_{\omega=0} \\ &= (-i)^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\ &= (-i)^n \frac{1}{\sqrt{2\pi}} \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by definition of } \langle \cdot \,|\, \cdot \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{(Definition C.1 page 109)} \end{aligned}$$

[11] Goswami and Chan (1999) pages 38–39
[12] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

Lemma 3.3 (Strang-Fix condition). [13] *Let* $\mathsf{f}(x)$ *be a function in* $\mathbf{L}^2_{\mathbb{R}}$ *and* M_n *the* nTH MOMENT *(Definition 3.4 page 46) of* $\mathsf{f}(x)$. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 4.3 page 54).*

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in “time”}} \iff \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in “frequency”}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by definition of } \tilde{\mathsf{f}}(\omega) \quad \text{(Definition 3.2 page 42)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k)\bar{\delta}_k && \text{by } \textit{PSF} \quad \text{(Theorem 4.2 page 62)} \\
&= \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[(-i)^n \bar{\delta}_k \mathsf{M}_n\right] e^{-i2\pi kx} && \text{by definition of } \bar{\delta} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } \textit{PSF} \quad \text{(Theorem 4.2 page 62)}
\end{aligned}$$

⇨

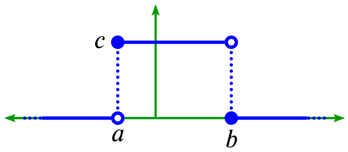
[13] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83, Mallat (1999) pages 241–243, Fix and Strang (1969)

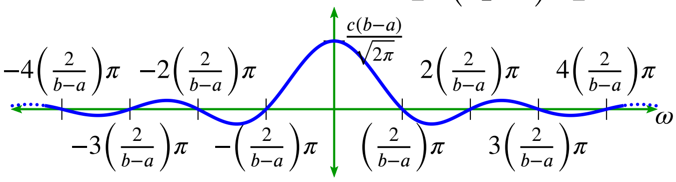
3.9 Examples

Example 3.1 (rectangular pulse). Let $\tilde{f}(\omega)$ be the *Fourier transform* of a function $f(x) \in L^2_{\mathbb{R}}$.

EX

$$f(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right]$$





✎PROOF:

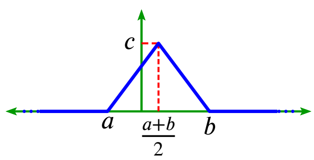
$$\begin{aligned}
\tilde{f}(\omega) &= \tilde{\mathbf{F}}[f(x)](\omega) && \text{by definition of } \tilde{f}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[f\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by \textit{shift relation}} \quad \text{(Theorem 3.4 page 44)} \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by definition of } f(x) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} \quad \text{(Definition 4.2 page 54)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)e^{-i\omega x}\,dx && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 3.2 page 42)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} ce^{-i\omega x}\,dx && \text{by definition of } \mathbb{1} \quad \text{(Definition 4.2 page 54)} \\
&= \frac{c}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\left.\frac{1}{-i\omega}e^{-i\omega x}\right|_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \\
&= \frac{2c}{\sqrt{2\pi}\omega}e^{-i\left(\frac{a+b}{2}\right)\omega}\left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i}\right] \\
&= \frac{c(b-a)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right] && \text{by \textit{Euler formulas}} \quad \text{(Corollary 1.2 page 9)}
\end{aligned}$$

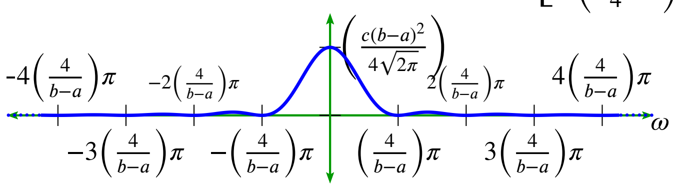
✏

Example 3.2 (triangle). Let $\tilde{f}(\omega)$ be the *Fourier transform* of a function $f(x) \in L^2_{\mathbb{R}}$.

EX

$$f(x) = \begin{Bmatrix} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{Bmatrix} \qquad \tilde{f}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$





✎PROOF:

$$\tilde{f}(\omega) = \tilde{\mathbf{F}}[f(x)](\omega) \qquad \text{by definition of } \tilde{f}(\omega)$$

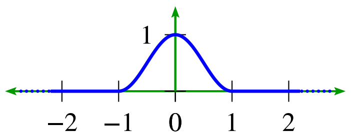
$$\begin{aligned}
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} && \text{(Theorem 3.4 page 44)}\\
&= \tilde{\mathbf{F}}\left[c\left(1 - \frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) && \text{by definition of } \mathsf{f}(x)\\
&= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x) \star \mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega)\\
&= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] && \text{by } \textit{convolution theorem} && \text{(Theorem D.2 page 116)}\\
&= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2\\
&= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 && \text{by } \textit{Rectangular pulse} \text{ ex.} && \text{Example 3.1 page 49}\\
&= \frac{c(b-a)^2}{4\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2
\end{aligned}$$

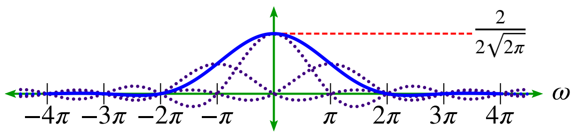
✏

Example 3.3. Let a function f be defined in terms of the cosine function (Definition 1.1 page 3) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1\\ 0 & \text{otherwise}\end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$





✎**Proof:** Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 4.2 page 54) on a set A.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition 3.2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathsf{f}(x)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \text{ (Definition 4.2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left[\frac{e^{i\frac{\pi}{2}x}+e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\,\mathrm{d}x && \text{by Corollary 1.2 page 9}\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1}\left[2+e^{i\pi x}+e^{-i\pi x}\right]e^{-i\omega x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1} 2e^{-i\omega x}+e^{-i(\omega+\pi)x}+e^{-i(\omega-\pi)x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\left[2\frac{e^{-i\omega x}}{-i\omega}+\frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}\\
&= \frac{1}{2\sqrt{2\pi}}\left[2\frac{e^{-i\omega}-e^{+i\omega}}{-2i\omega}+\frac{e^{-i(\omega+\pi)}-e^{+i(\omega+\pi)}}{-2i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)}-e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}
\end{aligned}$$

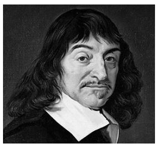
$$= \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$

✏

CHAPTER 4

TRANSVERSAL OPERATORS

❝Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:❞



❝I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them.❞

René Descartes, philosopher and mathematician (1596–1650) [1]

4.1 Families of Functions

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition C.1 page 109). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition 4.1 page 53). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition 4.1. *Let X and Y be sets.*

DEF *The space Y^X represents the set of all functions with* DOMAIN X *and* RANGE Y *such that*
$$Y^X \triangleq \{\mathsf{f}(x) | \mathsf{f}(x) : X \to Y\}$$

[1] quote: Descartes (1637b)
translation: Descartes (1637c) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

Definition 4.2. [2] *Let X be a set.*

DEF

The ***indicator function*** $\mathbb{1} \in \{0,1\}^{2^X}$ *is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases} \qquad \begin{matrix} \forall x{\in}X,\ A{\in}2^X \\ \forall x{\in}X,\ A{\in}2^X \end{matrix}$$

The indicator function $\mathbb{1}$ *is also called the* ***characteristic function***.

4.2 Definitions and algebraic properties

Much of the wavelet theory developed in this text is constructed using the **translation operator T** and the **dilation operator D** (next).

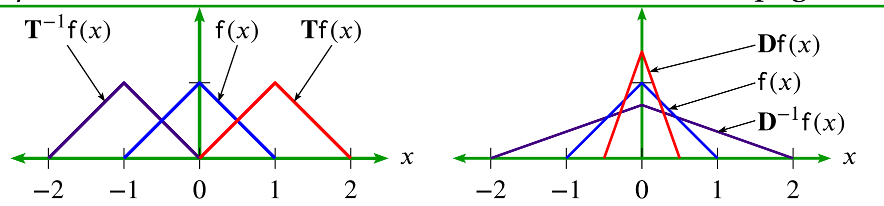
Definition 4.3. [3]

DEF

$$\begin{array}{llll} \mathbf{T}_\tau \text{ is a } \textbf{translation operator} \text{ on } \mathbb{C}^\mathbb{C} \text{ if} & \mathbf{T}_\tau \mathsf{f}(x) & \triangleq \mathsf{f}(x-\tau) & \forall \mathsf{f}{\in}\mathbb{C}^\mathbb{C}. \\ \mathbf{D}_\alpha \text{ is a } \textbf{dilation operator} \text{ on } \mathbb{C}^\mathbb{C} \text{ if} & \mathbf{D}_\alpha \mathsf{f}(x) & \triangleq \mathsf{f}(\alpha x) & \forall \mathsf{f}{\in}\mathbb{C}^\mathbb{C}. \\ \text{Moreover, } \mathbf{T} \triangleq \mathbf{T}_1 \quad \text{and} \quad \mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2 & & & . \end{array}$$

Example 4.1. Let **T** and **D** be defined as in Definition 4.3 (page 54).

EX



Proposition 4.1. *Let* $\mathbf{T}_\tau$ *be a* TRANSLATION OPERATOR *(Definition 4.3 page 54).*

PRP

$$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) \qquad \forall \mathsf{f}{\in}\mathbb{R}^\mathbb{R} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) \text{ is PERIODIC with period } \tau\right)$$

✎PROOF:

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n\tau+\tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 4.3 page 54)} \\ &= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m\tau) && \text{where } m \triangleq n-1 && \implies n = m+1 \\ &= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m \mathsf{f}(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 4.3 page 54)} \end{aligned}$$

⇰

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exist as operators, as demonstrated next.

[2] Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) page 22, de la Vallée-Poussin (1915) page 440

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2000) page 473, Packer (2004) page 260, zay (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a) page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996) page 81, Dai and Larson (1998) page 2

Proposition 4.2 (transversal operator inverses). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 4.3 page 54.*

PRP

$\mathbf{T}$ *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(translation operator inverse).}$$

$\mathbf{D}$ *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(dilation operator inverse).}$$

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} && \text{(Definition 4.3 page 54)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} && \text{(Definition 4.3 page 54)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\implies \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} && \text{(Definition 4.3 page 54)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} && \text{(Definition 4.3 page 54)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\implies \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

✏

Proposition 4.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 4.3 page 54. Let* $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP

$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^jx-n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

4.3 Linear space properties

Proposition 4.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 54.*

PRP

$$\mathbf{D}^j\mathbf{T}^n[\mathsf{f}\mathsf{g}] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

✎PROOF:

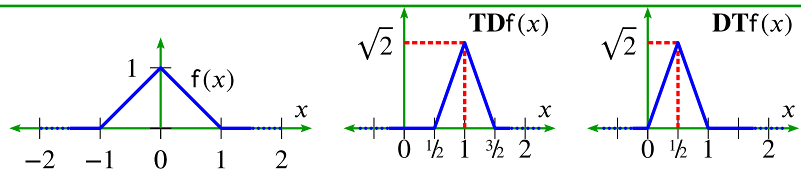
$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n[\mathsf{f}(x)\mathsf{g}(x)] &= 2^{j/2}\mathsf{f}\left(2^jx-n\right)\mathsf{g}\left(2^jx-n\right) && \text{by Proposition 4.3 page 55}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^jx-n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^jx-n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition 4.3 page 55}
\end{aligned}$$

⇨

In general the operators $\mathbf{T}$ and $\mathbf{D}$ are *noncommutative* ($\mathbf{TD} \neq \mathbf{DT}$), as demonstrated by Counterexample 4.1 (next) and Proposition 4.5 (page 56).

Counterexample 4.1.

CNT As illustrated to the right, it is **not** always true that $\mathbf{TD} = \mathbf{DT}$:



Proposition 4.5 (commutator relation). [4] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 54.*

PRP
$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j && \forall j,n\in\mathbb{Z} \\ \mathbf{T}^n\mathbf{D}^j &= \mathbf{D}^j\mathbf{T}^{2^j n} && \forall n,j\in\mathbb{Z} \end{aligned}$$

✎PROOF:

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^{2^j n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - 2^j n) && \text{by Proposition 4.4 page 55} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition ?? page ??)} \\ &= \mathbf{T}^n 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 54)} \\ &= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 54)} \end{aligned}$$

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - n) && \text{by Proposition 4.4 page 55} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j\left[x - 2^{-j/2}n\right]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition ?? page ??)} \\ &= \mathbf{T}^{2^{-j/2}n}2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 54)} \\ &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 54)} \end{aligned}$$

⇨

4.4 Inner product space properties

In an inner product space, every operator has an *adjoint* (Proposition E.3 page 141) and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator $\mathbf{U}$ coincide, then $\mathbf{U}$ is said to be *unitary* (Definition E.14 page 151). And in this case, $\mathbf{U}$ has several nice properties (see Proposition 4.9 and Theorem 4.1 page 59). Proposition 4.6 (next) gives the adjoints of $\mathbf{D}$ and $\mathbf{T}$, and Proposition 4.7 (page 57) demonstrates that both $\mathbf{D}$ and $\mathbf{T}$ are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ (Corollary 3.1 page 43) and the *rotation matrix operator* (Example E.5 page 153).

Proposition 4.6. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 4.3 page 54) with* ADJOINT $\mathbf{T}^*$ *and* $\mathbf{D}$ *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$ *(Definition E.8 page 137).*

PRP
$$\begin{aligned} \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1) && \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(TRANSLATION OPERATOR ADJOINT)} \\ \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) && \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(DILATION OPERATOR ADJOINT)} \end{aligned}$$

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a) page 641, Goodman et al. (1993b) page 110

✎PROOF:

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^* && \text{(Definition E.8 page 137)} \\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 54)} \\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1 \\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^* && \text{(Definition E.8 page 137)} \\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 54)} \\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\,\mathrm{d}u && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^*\,\mathrm{d}x && \text{where } x = 2u \\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}$$

✏

Proposition 4.7. [5] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 (page 54). Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 4.2 (page 55).*

PRP
$\mathbf{T}$ *is* UNITARY *in* $\boldsymbol{L}^2_{\mathbb{R}}$ ($\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$).
$\mathbf{D}$ *is* UNITARY *in* $\boldsymbol{L}^2_{\mathbb{R}}$ ($\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$).

✎PROOF:

$$\begin{aligned}
\mathbf{T}^{-1} &= \mathbf{T}^* && \text{by Proposition 4.2 page 55 and Proposition 4.6 page 56} \\
&\implies \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition E.14 page 151)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}^{-1} &= \mathbf{D}^* && \text{by Proposition 4.2 page 55 and Proposition 4.6 page 56} \\
&\implies \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition E.14 page 151)}
\end{aligned}$$

✏

4.5 Normed linear space properties

Proposition 4.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR *(Definition 4.3 page 54).*

PRP
$$\left\{\begin{array}{lll} (1). & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) & \textit{and} \\ (2). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} & \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$$

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

✎PROOF:

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 54)}\\
&= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis}
\end{aligned}$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(x+h)\| &= \|\mathsf{f}(x) - \mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis}\\
&= \|0\|\\
&= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|\\
&\le \varepsilon\\
&\implies \forall h > 0, \exists \varepsilon \quad \text{such that} \quad \|\mathsf{f}(x) - \mathsf{f}(x+h)\| < \varepsilon\\
&\overset{\text{def}}{\iff} \mathsf{f}(x) \text{ is } \textit{continuous}
\end{aligned}$$

3. Proof that (1,2) $\implies$ *constant* property:

   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \neq \mathsf{f}(y)$.

   (b) Let $(\!(x_n)\!)_{n\in\mathbb{N}}$ be a sequence with limit x and $(\!(y_n)\!)_{n\in\mathbb{N}}$ a sequence with limit y

   (c) Then

$$\begin{aligned}
0 &< \|\mathsf{f}(x) - \mathsf{f}(y)\| && \text{by assumption in item (3a) page 58}\\
&= \lim_{n\to\infty} \|\mathsf{f}(x_n) - \mathsf{f}(y_n)\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 58}\\
&= \lim_{n\to\infty} \left\|\mathsf{f}(2^m x_n) - \mathsf{f}(2^\ell y_n)\right\| \quad \forall m, \ell \in \mathbb{Z} && \text{by (1)}\\
&= 0
\end{aligned}$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

✏

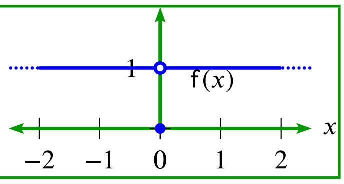
Remark 4.1.

REM: In Proposition 4.8 page 57, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample 4.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT: Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x = 0\\ 1 & \text{otherwise.}\end{cases}$

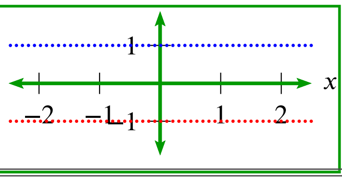
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Counterexample 4.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\backslash\mathbb{Q}$ the set of *irrational numbers*.

CNT: Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x \in \mathbb{Q}\\ -1 & \text{for } x \in \mathbb{R}\backslash\mathbb{Q}.\end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Proposition 4.9 (Operator norm). *Let $\mathbf{T}$ and $\mathbf{D}$ be as in Definition 4.3 page 54. Let $\mathbf{T}^{-1}$ and $\mathbf{D}^{-1}$ be as in Proposition 4.2 page 55. Let $\mathbf{T}^*$ and $\mathbf{D}^*$ be as in Proposition 4.6 page 56. Let $\|\cdot\|$ and $\langle \triangle \mid \nabla \rangle$ be as in Definition C.1 page 109. Let $|||\cdot|||$ be the operator norm* (Definition E.6 page 133) *induced by $\|\cdot\|$.*

PRP
$$|||\mathbf{T}||| = |||\mathbf{D}||| = |||\mathbf{T}^*||| = |||\mathbf{D}^*||| = |||\mathbf{T}^{-1}||| = |||\mathbf{D}^{-1}||| = 1$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 4.7 page 57) and from Theorem E.25 page 152 and Theorem E.26 page 152. ✐

Theorem 4.1. *Let $\mathbf{T}$ and $\mathbf{D}$ be as in Definition 4.3 page 54.*
Let $\mathbf{T}^{-1}$ and $\mathbf{D}^{-1}$ be as in Proposition 4.2 page 55. Let $\|\cdot\|$ and $\langle \triangle \mid \nabla \rangle$ be as in Definition C.1 page 109.

THM

1.	$\|\mathbf{T}\mathsf{f}\|$	$=$	$\|\mathbf{D}\mathsf{f}\|$	$=$	$\|\mathsf{f}\|$	$\forall \mathsf{f}\in\mathcal{L}^2_{\mathbb{R}}$		(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}\mathsf{f}-\mathbf{T}\mathsf{g}\|$	$=$	$\|\mathbf{D}\mathsf{f}-\mathbf{D}\mathsf{g}\|$	$=$	$\|\mathsf{f}-\mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$		(ISOMETRIC IN DISTANCE)
3.	$\|\mathbf{T}^{-1}\mathsf{f}-\mathbf{T}^{-1}\mathsf{g}\|$	$=$	$\|\mathbf{D}^{-1}\mathsf{f}-\mathbf{D}^{-1}\mathsf{g}\|$	$=$	$\|\mathsf{f}-\mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$		(ISOMETRIC IN DISTANCE)
4.	$\langle\mathbf{T}\mathsf{f}\mid\mathbf{T}\mathsf{g}\rangle$	$=$	$\langle\mathbf{D}\mathsf{f}\mid\mathbf{D}\mathsf{g}\rangle$	$=$	$\langle\mathsf{f}\mid\mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$		(SURJECTIVE)
5.	$\langle\mathbf{T}^{-1}\mathsf{f}\mid\mathbf{T}^{-1}\mathsf{g}\rangle$	$=$	$\langle\mathbf{D}^{-1}\mathsf{f}\mid\mathbf{D}^{-1}\mathsf{g}\rangle$	$=$	$\langle\mathsf{f}\mid\mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$		(SURJECTIVE)

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 4.7 page 57) and from Theorem E.25 page 152 and Theorem E.26 page 152. ✐

Proposition 4.10. *Let $\mathbf{T}$ be as in Definition 4.3 page 54. Let $\mathbf{A}^*$ be the* ADJOINT (Definition E.8 page 137) *of an operator $\mathbf{A}$. Let the property "*SELF ADJOINT*" be defined as in Definition E.11 (page 145).*

PRP
$$\left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)^* \qquad \left(\text{The operator } \left[\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right] \text{ is SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle\left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)\mathsf{f}(x)\,\middle|\,\mathsf{g}(x)\right\rangle &= \left\langle\sum_{n\in\mathbb{Z}}\mathsf{f}(x-n)\,\middle|\,\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 54)}\\
&= \left\langle\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n)\,\middle|\,\mathsf{g}(x)\right\rangle && \text{by } \textit{commutative} \text{ property} && \text{(Definition ?? page ??)}\\
&= \sum_{n\in\mathbb{Z}}\langle\mathsf{f}(x+n)\mid\mathsf{g}(x)\rangle && \text{by } \textit{additive} \text{ property of } \langle\triangle\mid\nabla\rangle\\
&= \sum_{n\in\mathbb{Z}}\langle\mathsf{f}(u)\mid\mathsf{g}(u-n)\rangle && \text{where } u\triangleq x+n\\
&= \left\langle\mathsf{f}(u)\,\middle|\,\sum_{n\in\mathbb{Z}}\mathsf{g}(u-n)\right\rangle && \text{by } \textit{additive} \text{ property of } \langle\triangle\mid\nabla\rangle\\
&= \left\langle\mathsf{f}(x)\,\middle|\,\sum_{n\in\mathbb{Z}}\mathsf{g}(x-n)\right\rangle && \text{by change of variable: } u\to x\\
&= \left\langle\mathsf{f}(x)\,\middle|\,\sum_{n\in\mathbb{Z}}\mathbf{T}^n\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 54)}\\
\iff \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) &= \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)^* && \text{by definition of } \textit{adjoint} && \text{(Proposition E.3 page 141)}\\
\iff \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) &\text{ is } \textit{self-adjoint} && \text{by definition of } \textit{self-adjoint} && \text{(Definition E.11 page 145)}
\end{aligned}$$

✐

4.6 Fourier transform properties

Proposition 4.11. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 54.*
Let $\mathbf{B}$ *be the* TWO-SIDED LAPLACE TRANSFORM *defined as* $[\mathbf{B}\mathsf{f}](s) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-sx}\,\mathrm{d}x$.

PRP

1.	$\mathbf{B}\mathbf{T}^n$	$=$	$e^{-sn}\mathbf{B}$			$\forall n\in\mathbb{Z}$	
2.	$\mathbf{B}\mathbf{D}^j$	$=$	$\mathbf{D}^{-j}\mathbf{B}$			$\forall j\in\mathbb{Z}$	
3.	$\mathbf{D}\mathbf{B}$	$=$	$\mathbf{B}\mathbf{D}^{-1}$			$\forall n\in\mathbb{Z}$	
4.	$\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1}$	$=$	$\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B}$	$=$	$\mathbf{D}$	$\forall n\in\mathbb{Z}$	($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5.	$\mathbf{D}\mathbf{B}\mathbf{D}$	$=$	$\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1}$	$=$	$\mathbf{B}$	$\forall n\in\mathbb{Z}$	

✎PROOF:

$$\begin{aligned}
\mathbf{B}\mathbf{T}^n\mathsf{f}(x) &= \mathbf{B}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 54)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathrm{d}u && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right]\\
&= e^{-sn}\,\mathbf{B}\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}\mathbf{D}^j\mathsf{f}(x) &= \mathbf{B}\left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 54)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right]e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathrm{d}u && \text{let } u \triangleq 2^j x \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathrm{d}u\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right] && \text{by Proposition 4.6 page 56 and} && \text{Proposition 4.7 page 57}\\
&= \mathbf{D}^{-j}\,\mathbf{B}\,\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{D}\mathbf{B}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x\right] && \text{by definition of } \mathbf{B}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathrm{d}x && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 54)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\tfrac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{u}{2}\right)\right]e^{-su}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\,e^{-su}\,\mathrm{d}u && \text{by Proposition 4.6 page 56 and} && \text{Proposition 4.7 page 57}\\
&= \mathbf{B}\mathbf{D}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{B}\mathbf{D} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition E.3 page 128)}\\
\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} &= \mathbf{D}\mathbf{B}\mathbf{B}^{-1} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition E.3 page 128)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{DBD} &= \mathbf{DD}^{-1}\mathbf{B} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse} && \text{(Definition E.3 page 128)}\\
\mathbf{D}^{-1}\mathbf{BD}^{-1} &= \mathbf{D}^{-1}\mathbf{DB} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse} && \text{(Definition E.3 page 128)}
\end{aligned}$$

⇨

Corollary 4.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 54. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 42) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition C.1 page 109).*

COR

$$\begin{array}{llcl cl}
1. & \tilde{\mathbf{F}}\mathbf{T}^n & = & e^{-i\omega n}\tilde{\mathbf{F}} & & \\
2. & \tilde{\mathbf{F}}\mathbf{D}^j & = & \mathbf{D}^{-j}\tilde{\mathbf{F}} & & \\
3. & \mathbf{D}\tilde{\mathbf{F}} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1} & & \\
4. & \mathbf{D} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} & = & \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}} \\
5. & \tilde{\mathbf{F}} & = & \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} & = & \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}
\end{array}$$

✎PROOF: These results follow directly from Proposition 4.11 page 60 with $\tilde{\mathbf{F}} = \mathbf{B}|_{s=i\omega}$. ⇨

Proposition 4.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 54. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 42) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition C.1 page 109).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = \frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{\mathsf{f}}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n\mathsf{f}(x) && \text{by Corollary 4.1 page 61 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}\mathsf{f}(x) && \text{by Corollary 4.1 page 61 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathsf{f}}(\omega)\\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{\mathsf{f}}\left(2^{-j}\omega\right) && \text{by Proposition 4.2 page 55}
\end{aligned}$$

⇨

Proposition 4.13. *Let* $\mathbf{T}$ *be the translation operator (Definition 4.3 page 54). Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 42) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\breve{a}(\omega)$ *be the* DTFT *(Definition 6.1 page 75) of a sequence* $(a_n)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition D.2 page 113).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{a}(\omega)\tilde{\phi}(\omega) \qquad \forall (a_n)\in\boldsymbol{\ell}^2_{\mathbb{R}},\ \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}^n\phi(x)\\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\tilde{\mathbf{F}}\phi(x) && \text{by Corollary 4.1 page 61}\\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega)\\
&= \breve{a}(\omega)\tilde{\phi}(\omega) && \text{by definition of } \textit{DTFT} \text{ (Definition 6.1 page 75)}
\end{aligned}$$

⇨

Definition 4.4. *Let* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition C.1 page 109). Let* $\boldsymbol{\ell}^2_{\mathbb{R}}$ *be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition C.1 page 109).*

DEF

$\mathbf{S}$ *is the* ***sampling operator*** *in* $\boldsymbol{\ell}^{2\,\boldsymbol{L}^2_{\mathbb{R}}}_{\mathbb{R}}$ *if* $\quad [\mathbf{S}\mathsf{f}(x)](n) \triangleq \mathsf{f}\left(\frac{2\pi}{\tau}n\right) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \tau\in\mathbb{R}^+$

Theorem 4.2 (Poisson Summation Formula—PSF). [6] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 42) of a function* $\mathsf{f}(x)\in\boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\mathbf{S}$ *be the* SAMPLING OPERATOR *(Definition 4.4 page 62).*

THM

$$\underbrace{\sum_{n\in\mathbb{Z}}\mathbf{T}^n_\tau\mathsf{f}(x) = \sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)}_{\text{summation in “time”}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in “frequency”}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(x)\triangleq\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)$ then $\mathsf{h}\equiv\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:

   Note that $\mathsf{h}(x)$ is *periodic* with period τ. Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $\mathsf{h}\equiv\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of PSF (this theorem—Theorem 4.2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau) && \text{by (1) lemma page 62}\\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau\left(\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}[\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition 5.1 page 71)}\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau\mathsf{f}(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right]\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau}\mathsf{f}(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,\mathrm{d}u\right] && \text{where } u\triangleq x+n\tau \implies x=u-n\tau\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi kn}}\int_{u=n\tau}^{u=(n+1)\tau}\mathsf{f}(u)e^{-i\frac{2\pi}{\tau}ku}\,\mathrm{d}u\right]\\
&= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,\mathrm{d}u\right]}_{[\tilde{\mathbf{F}}\mathsf{f}]\left(\frac{2\pi}{\tau}k\right)} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 5.1 page 72)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}\mathsf{f}(x)]\left(\frac{2\pi}{\tau}k\right)\right] && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 3.2 page 42)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \mathbf{S} \quad \text{(Definition 4.4 page 62)}\\
&= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 5.1 page 72)}
\end{aligned}$$

✏

[6] Andrews et al. (2001) page 624, Knapp (2005b) page 389, Lasser (1996) page 254, Rudin (1987) pages 194–195, Folland (1992) page 337

Theorem 4.3 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition 3.2 page 42) *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM
$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)}_{\text{summation in “frequency”}} \quad = \quad \underbrace{\frac{\tau}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\text{summation in “time”}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)$, then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:
   Note that $\mathsf{h}(\omega)$ is periodic with period $2\pi/T$:
   $$\mathsf{h}\left(\omega + \frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau} + \frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + (n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \triangleq \mathsf{h}(\omega)$$
   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of IPSF (this theorem—Theorem 4.3):
$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \\
&= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) && \text{by (1) lemma page 63} \\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\sqrt{\frac{\tau}{2\pi}}\int_0^{\frac{2\pi}{\tau}} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\,\mathrm{d}\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)\right]} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 5.1 page 71)} \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{\tau}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega Tk}\,\mathrm{d}\omega\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-i\left(u-\frac{2\pi}{\tau}n\right)Tk}\,\mathrm{d}u\right] && \text{where } u \triangleq \omega + \frac{2\pi}{\tau}n \implies && \omega = u - \frac{2\pi}{\tau}n \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi nk}}\int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{d}u\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{d}u\right] \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{iu(-\tau k)}\,\mathrm{d}u\right]}_{[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)\right] && \text{by value of } \tilde{\mathbf{F}}^{-1} && \text{(Theorem 3.1 page 43)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\,\tilde{\mathsf{f}} && \text{by definition of } \mathbf{S} && \text{(Definition 4.4 page 62)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\mathsf{f}(x) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 42)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathsf{f}(-k\tau) && \text{by definition of } \mathbf{S} && \text{(Definition 4.4 page 62)} \\
&= \sqrt{\tau}\,\frac{1}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}}\mathsf{f}(-k\tau)e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem 5.1 page 72)}
\end{aligned}$$

[7] Gauss (1900) page 88

$$= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}} \sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau)e^{ik\tau\omega} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem 5.1 page 72)}$$

$$= \frac{\tau}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \mathsf{f}(m\tau)e^{-i\omega m\tau} \qquad \text{let } m \triangleq -k$$

Remark 4.2. The left hand side of the *Poisson Summation Formula* (Theorem 4.2 page 62) is very similar to the *Zak Transform* **Z**: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark 4.3. A generalization of the *Poisson Summation Formula* (Theorem 4.2 page 62) is the **Selberg Trace Formula**.[9]

4.7 Examples

Example 4.2 (linear functions). [10] Let **T** be the *translation operator* (Definition 4.3 page 54). Let $\mathcal{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX

1. $\{x, \mathbf{T}x\}$ is a *basis* for $\mathcal{L}(\mathbb{C},\mathbb{C})$ and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x \qquad \forall \mathsf{f} \in \mathcal{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition 4.3 page 54}\\
&= (ax+b)|_{x=1}\, x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of f}\\
&= (a+b)x - b(x-1)\\
&= ax + bx - bx + b\\
&= ax + b\\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of f}
\end{aligned}$$

⇨

Example 4.3 (Cardinal Series). Let **T** be the *translation operator* (Definition 4.3 page 54). The *Paley-Wiener* class of functions $\boldsymbol{PW}^2_\sigma$ are those functions which are "*bandlimited*" with respect to their Fourier transform (Definition 3.2 page 42). The cardinal series forms an orthogonal basis for such a space. The *Fourier coefficients* for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of f(x) taken at regular intervals. In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \,|\, \mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\, \mathrm{d}x \triangleq \mathsf{f}(n)$$

EX

1. $\left\{ \mathbf{T}^n \frac{\sin(\pi x)}{\pi x} \middle| n\in\mathbb{N} \right\}$ is a *basis* for $\boldsymbol{PW}^2_\sigma$ and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin(\pi x)}{\pi x}}_{\text{Cardinal series}} \qquad \forall \mathsf{f} \in \boldsymbol{PW}^2_\sigma,\ \sigma \le \frac{1}{2}$

[8] Janssen (1988) page 24, Zayed (1996) page 482
[9] Lax (2002) page 349, Selberg (1956), Terras (1999)
[10] Higgins (1996) page 2

Example 4.4 (Fourier Series).

EX
1. $\left\{ \mathbf{D}_n e^{ix} \,\middle|\, n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix} \quad \forall x\in(0:2\pi), \mathsf{f}\in\boldsymbol{L}(0:2\pi)$ where
3. $\alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x) \mathbf{D}_n e^{-ix} \, dx \quad \forall \mathsf{f}\in\boldsymbol{L}(0:2\pi)$

✎PROOF: See Theorem 5.1 page 72. ✏

Example 4.5 (Fourier Transform). [11]

EX
1. $\left\{ \mathbf{D}_\omega e^{ix} \,|\, \omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) \mathbf{D}_x e^{i\omega} \, d\omega \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \mathbf{D}_\omega e^{-ix} \, dx \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example 4.6 (Gabor Transform). [12]

EX
1. $\left\{ \left(\mathbf{T}_\tau e^{-\pi x^2} \right) \left(\mathbf{D}_\omega e^{ix} \right) \,\middle|\, \tau,\omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \int_{\mathbb{R}} \mathsf{G}(\tau,\omega) \mathbf{D}_x e^{i\omega} \, d\omega \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\mathsf{G}(\tau,\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x) \left(\mathbf{T}_\tau e^{-\pi x^2} \right) \left(\mathbf{D}_\omega e^{-ix} \right) dx \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example 4.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX
1. $\left\{ \mathbf{D}^k \mathbf{T}^n \psi(x) \,\middle|\, k,n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha_{k,n} \mathbf{D}^k \mathbf{T}^n \psi(x) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\alpha_n \triangleq \int_{\mathbb{R}} \mathsf{f}(x) \mathbf{D}^k \mathbf{T}^n \psi^*(x) \, dx \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

[11] cross reference: Definition 3.2 page 42

[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

4.8 Cardinal Series and Sampling

4.8.1 Cardinal series basis

The *Paley-Wiener* class of functions (next definition) are those with a bandlimited Fourier transform. The cardinal series forms an orthogonal basis for such a space (Theorem 4.5 page 66). In a *frame* $(\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{Z}}$ with *frame operator* $\mathbf{S}$ on a *Hilbert Space* $\boldsymbol{H}$ with *inner product* $\langle \triangle \mid \nabla \rangle$, a function $\mathsf{f}(x)$ in the space spanned by the frame can be represented by

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \underbrace{\left\langle \mathsf{f} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle}_{\text{“Fourier coefficient”}} \boldsymbol{x}_n.$$

If the frame is *orthonormal* (giving an *orthonormal basis*), then $\mathbf{S} = \mathbf{S}^{-1} = \mathbf{I}$ and

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n.$$

In the case of the cardinal series, the *Fourier coefficients* are particularly simple—these coefficients are samples of f taken at regular intervals (Theorem 4.6 page 67). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \mid \delta(x - n\tau) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x - n\tau)\, \mathrm{d}t \triangleq \mathsf{f}(n\tau)$$

Definition 4.5. [13]

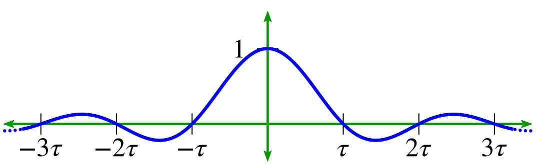
DEF: *A function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is in the* ***Paley-Wiener*** *class of functions* $\boldsymbol{PW}^p_\sigma$ *if there exists* $\mathsf{F} \in \boldsymbol{L}^p(-\sigma : \sigma)$ *such that*

$$\mathsf{f}(x) = \int_{-\sigma}^{\sigma} \mathsf{F}(\omega) e^{ix\omega}\, \mathrm{d}\omega \qquad (\mathsf{f} \text{ has a BANDLIMITED Fourier transform } \mathsf{F} \text{ with bandwidth } \sigma)$$

for $p \in [1 : \infty)$ *and* $\sigma \in (0 : \infty)$.

Theorem 4.4 (Paley-Wiener Theorem for Functions). [14] *Let* f *be an* ENTIRE FUNCTION *(the domain of* f *is the entire complex plane* $\mathbb{C}$*). Let* $\sigma \in \mathbb{R}^+$.

THM:
$$\left\{\mathsf{f} \in \boldsymbol{PW}^2_\sigma\right\} \iff \left\{\begin{array}{lll} 1. & \exists C \in \mathbb{R}^+ \text{ such that } |\mathsf{f}(z)| \le Ce^{\sigma|z|} & (\text{EXPONENTIAL TYPE}) \text{ and} \\ 2. & \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} & \end{array}\right\}$$



Theorem 4.5 (Cardinal sequence). [15]

THM:
$$\left\{\frac{1}{\tau} \ge 2\sigma\right\} \implies \text{The sequence } \left(\!\!\left(\frac{\sin\left[\frac{\pi}{\tau}(x - n\tau)\right]}{\frac{\pi}{\tau}(x - n\tau)}\right)\!\!\right)_{n\in\mathbb{Z}} \text{ is an ORTHONORMAL BASIS for } \boldsymbol{PW}^2_\sigma.$$

[13] Higgins (1996) page 52 ⟨Definition 6.15⟩

[14] Boas (1954) page 103 ⟨6.8.1 Theorem of Paley and Wiener⟩, Katznelson (2004) page 212 ⟨7.4 Theorem⟩, Zygmund (2002) pages 272–273 ⟨(7·2) Theorem of Paley-Wiener⟩, Yosida (1980) page 161, Rudin (1987) page 375 ⟨19.3 Theorem⟩, Young (2001) page 85 ⟨Theorem 18⟩

[15] Higgins (1996) page 52 ⟨Definition 6.15⟩, Hardy (1941) ⟨*orthonormality*⟩, Higgins (1985) page 56 ⟨H1.; historical notes⟩

Theorem 4.6 (Sampling Theorem). [16]

THM
$$\left\{\begin{array}{ll} 1. & \mathsf{f} \in \boldsymbol{PW}_\sigma^2 \quad \text{and} \\ 2. & \dfrac{1}{\tau} \geq 2\sigma \end{array}\right\} \implies \mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)}}_{\text{CARDINAL SERIES}}.$$

✎PROOF:

$$\text{Let} \quad \mathsf{s}(x) \triangleq \frac{\sin\left[\frac{\pi}{\tau}x\right]}{\frac{\pi}{\tau}x} \iff \tilde{\mathsf{s}}(\omega) = \begin{cases} \tau & : |f| \leq \frac{1}{2\tau} \\ 0 & : \text{otherwise} \end{cases}$$

1. Proof that the set is *orthonormal*: see Hardy (1941)
2. Proof that the set is a *basis*:

$$\begin{aligned}
\mathsf{f}(x) &= \int_\omega \tilde{\mathsf{f}}(\omega)e^{i\omega t}\,\mathrm{d}\omega && \text{by } \textit{inverse Fourier transform} && \text{(Theorem 3.1 page 43)} \\
&= \int_\omega \mathbf{T}\tilde{\mathsf{f}}_\mathsf{d}(\omega)\tilde{\mathsf{s}}(\omega)e^{i\omega t}\,\mathrm{d}\omega && \text{if } W \leq \frac{1}{2T} \\
&= \mathbf{T}\mathsf{f}_\mathsf{d}(x) \star \mathsf{s}(x) && \text{by } \textit{Convolution theorem} && \text{(Theorem D.2 page 116)} \\
&= \mathbf{T}\int_u [\mathsf{f}_\mathsf{d}(u)]\mathsf{s}(x-u)\,\mathrm{d}u && \text{by } \textit{convolution} \text{ definition} && \text{(Definition 3.3 page 45)} \\
&= \mathbf{T}\int_u \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(u)\delta(u-n\tau)\right]\mathsf{s}(x-u)\,\mathrm{d}u && \text{by } \textit{sampling} \text{ definition} && \text{(Theorem 4.7 page 68)} \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}}\int_u \mathsf{f}(u)\mathsf{s}(x-u)\delta(u-n\tau)\,\mathrm{d}u \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\mathsf{s}(x-n\tau) && \text{by prop. of } \textit{Dirac delta} \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)} && \text{by definition of } \mathsf{s}(x)
\end{aligned}$$

✏

4.8.2 Sampling

Definition 4.6. [17] *Let $\delta(x)$ be the* DIRAC DELTA *distribution.*

DEF
*The **Shah Function** $\text{Ш}(x)$ is defined as* $\text{Ш}(x) \triangleq \sum_{n\in\mathbb{Z}} \delta(x-n)$

[16] Whittaker (1915), Kotelnikov (1933), Whittaker (1935), Shannon (1948) ⟨Theorem 13⟩, Shannon (1949) page 11 Il (1991) page 1, Nashed and Walter (1991), Higgins (1996) page 5, Young (2001) pages 90–91 ⟨THE PALEY-WIENER SPACE⟩, Papoulis (1980) pages 418–419 ⟨The Sampling Theorem⟩. The *sampling theorem* was "discovered" and published by multiple people: Nyquist in 1928 (DSP?), Whittaker in 1935 (interpolation theory), and Shannon in 1949 (communication theory). references: Mallat (1999) page 43, Oppenheim and Schafer (1999) page 143.

[17] Bracewell (1978) page 77 ⟨The sampling or replicating symbol Ш(x)⟩, Córdoba (1989)191. Note: The symbol Ш is the Cyrillic upper case "sha" character, which has been assigned Unicode location U+0428. Reference: http://unicode.org/cldr/utility/character.jsp?a=0428

If $\mathsf{f}_d(x)$ is the function $\mathsf{f}(x)$ sampled at rate $1/T$, then $\tilde{\mathsf{f}}_d(\omega)$ is simply $\tilde{\mathsf{f}}(\omega)$ *replicated* every $1/T$ Hertz and *scaled* by $1/T$. This is proven in Theorem 4.7 (next) and illustrated in Figure 4.1 (page 68).

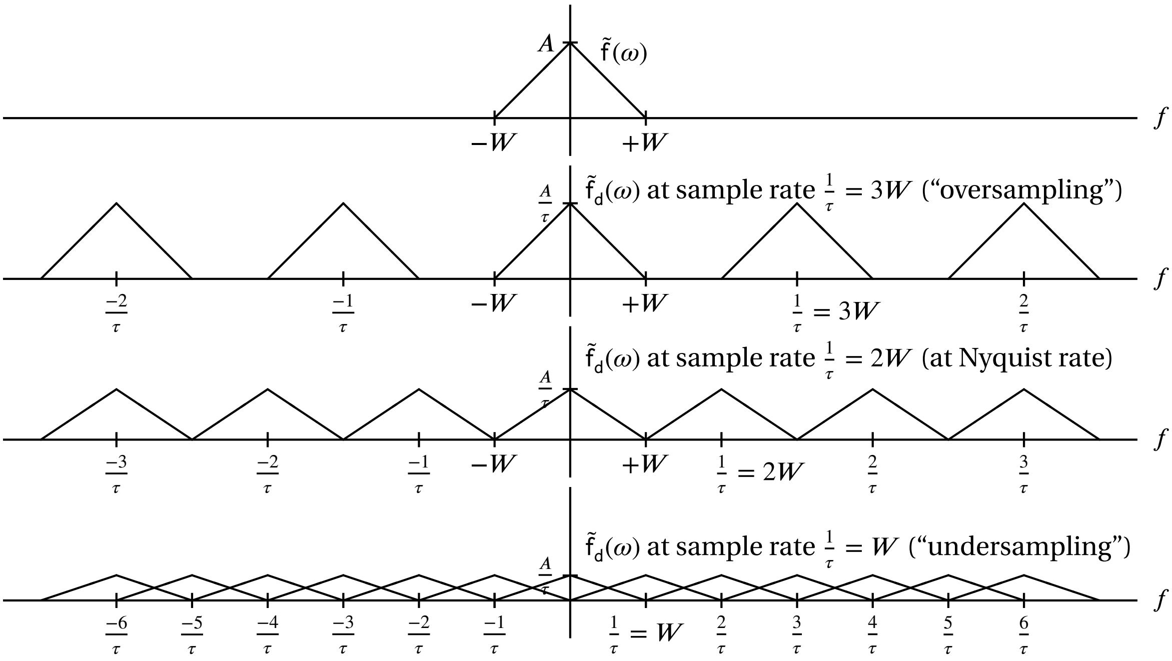


Figure 4.1: Sampling in frequency domain

Theorem 4.7. *Let* $\mathsf{f}, \mathsf{f}_d \in \boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}, \tilde{\mathsf{f}}_d \in \boldsymbol{L}^2_{\mathbb{R}}$ *be their respective fourier transforms. Let* $\mathsf{f}_d(x)$ *be the* ***sampled*** $\mathsf{f}(x)$ *such that*

$$\mathsf{f}_d(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x - n\tau).$$

THM

$$\left\{ \mathsf{f}_d(x) \triangleq \mathsf{f}(x)\text{Ш}(x) \triangleq \mathsf{f}(x) \sum_{n\in\mathbb{Z}} \delta(x - n\tau) \right\} \implies \left\{ \tilde{\mathsf{f}}_d(\omega) = \frac{2\pi}{\tau} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau} n\right) \right\}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{f}}_d(\omega) &\triangleq \int_t \mathsf{f}_d(x) e^{-i\omega t}\, dt \\
&= \int_t \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x - n\tau) \right] e^{-i\omega t}\, dt \\
&= \sum_{n\in\mathbb{Z}} \int_t \mathsf{f}(x)\delta(x - n\tau) e^{-i\omega t}\, dt \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau) e^{-i\omega n\tau} && \text{by definition of } \delta \\
&= \frac{2\pi}{\tau} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau} n\right) && \text{by } \textit{IPSF} \quad \text{(Theorem 4.3 page 63)} \\
&= \frac{2\pi}{\tau} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau} n\right)
\end{aligned}$$

✏

Suppose a waveform $\mathsf{f}(x)$ is sampled at every time T generating a sequence of sampled values $\mathsf{f}(n\tau)$. Then in general, we can *approximate* $\mathsf{f}(x)$ by using interpolation between the points $\mathsf{f}(n\tau)$. Interpolation can be performed using several interpolation techniques.

In general all techniques lead only to an approximation of $\mathsf{f}(x)$. However, if $\mathsf{f}(x)$ is *bandlimited* with bandwidth $W \leq \frac{1}{2T}$, then $\mathsf{f}(x)$ is *perfectly reconstructed* (not just approximated) from the sampled values $\mathsf{f}(n\tau)$ (Theorem 4.6 page 67).

CHAPTER 5

FOURIER SERIES

❝ ...et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur.❞


MÉMOIRE
SUR LA
PROPAGATION DE LA CHALEUR
DANS LES CORPS SOLIDES.

❝ ...and the innovation of the subject, together with its importance, convinced the committee to crown this work. By observing however that the way in which the author arrives at his equations is not free from difficulties, and the analysis of which, to integrate them, still leaves something to be desired, either relative to generality, or even on the side of rigour.❞

A competition awards committee consisting of the mathematical giants Lagrange, Laplace, Legendre, and others, commenting on Fourier's 1807 landmark paper *Dissertation on the propagation of heat in solid bodies* that introduced the *Fourier Series*. [1]

5.1 Definition

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition 5.1. [2]

DEF *The **Fourier Series operator** $\hat{\mathbf{F}} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{R}}$ is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{f}(x) e^{-i\frac{2\pi}{\tau}nx}\, \mathsf{d}x \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{f} \text{ is periodic with period } \tau\right\}$$

[1] quote: Lagrange et al. (1812b) page 374, Lagrange et al. (1812a) page 112, Kahane (2008) page 199
translation: assisted by Google Translate, Castanedo (2005) ⟨chapter 2 footnote 5⟩
paper: Fourier (1807)

[2] Katznelson (2004) page 3

5.2 Inverse Fourier Series operator

Theorem 5.1. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

THM

*The **inverse Fourier Series** operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{x}}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall \left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)\in\ell^2_{\mathbb{R}}$$

✎PROOF: The proof of the pointwise convergence of the Fourier Series is notoriously difficult. It was conjectured in 1913 by Nokolai Luzin that the Fourier Series for all square summable periodic functions are pointwise convergent: Luzin (1913)

Fifty-three years later (1966) at a conference in Moscow, Lennart Axel Edvard Carleson presented one of the most spectacular results ever in mathematics; he demonstrated that the Luzin conjecture is indeed correct. Carleson formally published his result that same year: Carleson (1966)

Carleson's proof is expounded upon in Reyna's (2002) 175 page book: de Reyna (2002)

Interestingly enough, Carleson started out trying to disprove Luzin's conjecture. Carleson said this in an interview published in 2001:[3] ❝*Well, the problem of course presents itself already when you are a student and I was thinking of the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I suddenly realized why there should be a counter-example and how you should produce it. I thought I really understood what was the background and then to my amazement I could prove that this "correct" counter-example couldn't exist and therefore I suddenly realized that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. The most important aspect in solving a mathematical problem is the conviction of what is the true result! Then it took like 2 or 3 years using the technique that had been developed during the past 20 years or so. It is actually a problem related to analytic functions basically even though it doesn't look that way.*❞

For now, if you just want some intuitive justification for the Fourier Series, and you can somehow imagine that the Dirichlet kernel generates a *comb function* of *Dirac delta* functions, then perhaps what follows may help (or not). It is certainly not mathematically rigorous and is by no means a real proof (but at least it is less than 175 pages).

$$\begin{aligned}
\left[\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{x}\right](x) &= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}\mathsf{x}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 5.1 page 71)}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}\,\mathrm{d}u\right]e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \hat{\mathbf{F}}^{-1}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{i\frac{2\pi}{\tau}n(x-u)}\,\mathrm{d}u
\end{aligned}$$

[3] Carleson and Engquist (2001)

$$
\begin{aligned}
&= \int_0^\tau \mathsf{x}(u) \underbrace{\frac{1}{\tau}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi}{\tau}n(x-u)}}_{\lim_{N\to\infty}\mathsf{D}_n(x)}\,\mathsf{d}u \\
&= \int_0^\tau \mathsf{x}(u)\left[\sum_{n\in\mathbb{Z}}\delta(x-u-n\tau)\right]\mathsf{d}u \\
&= \sum_{n\in\mathbb{Z}}\int_{u=0}^{u=\tau} \mathsf{x}(u)\delta(x-u-n\tau)\,\mathsf{d}u \\
&= \sum_{n\in\mathbb{Z}}\int_{v-n\tau=0}^{v-n\tau=\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathsf{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}}\int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathsf{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}}\int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v)\delta(x-v)\,\mathsf{d}v && \text{because } \mathsf{x} \text{ is periodic with period } \tau \\
&= \int_{\mathbb{R}} \mathsf{x}(v)\delta(x-v)\,\mathsf{d}v \\
&= \mathsf{x}(x) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} && \text{(Definition E.3 page 128)}
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](n) &= \hat{\mathbf{F}}\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right]e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{d}x && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 5.1 page 71)} \\
&= \frac{1}{\tau}\int_0^\tau\left[\sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}(k-n)x}\right]\mathsf{d}x \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\left[\frac{1}{\tau}\int_0^\tau e^{i\frac{2\pi}{\tau}(k-n)x}\,\mathsf{d}x\right] \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\frac{1}{\tau}\left[\frac{1}{i\frac{2\pi}{\tau}(k-n)}e^{i\frac{2\pi}{\tau}(k-n)x}\right]_0^\tau \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\frac{1}{i2\pi(k-n)}\left[e^{i2\pi(k-n)}-1\right] \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\,\bar{\delta}(k-n)\lim_{x\to 0}\left[\frac{e^{i2\pi x}-1}{i2\pi x}\right] \\
&= \tilde{\mathsf{x}}(n)\left.\frac{\frac{\mathsf{d}}{\mathsf{d}x}\left(e^{i2\pi x}-1\right)}{\frac{\mathsf{d}}{\mathsf{d}x}(i2\pi x)}\right|_{x=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \tilde{\mathsf{x}}(n)\left.\frac{i2\pi e^{i2\pi x}}{i2\pi}\right|_{x=0} \\
&= \tilde{\mathsf{x}}(n) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} && \text{(Definition E.3 page 128)}
\end{aligned}
$$

✏

Theorem 5.2.

THM *The **Fourier Series adjoint** operator $\hat{\mathbf{F}}^*$ is given by*

$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathsf{x}(x) \mid \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{x}(x) e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x \mid \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 5.1 page 71)} \\
&= \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{x}(x) \left\langle e^{-i\frac{2\pi}{\tau}nx} \mid \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} \mathrm{d}x && \text{by additivity property of } \langle \triangle \mid \nabla \rangle \\
&= \int_0^\tau \mathsf{x}(x) \frac{1}{\sqrt{\tau}} \left\langle \tilde{\mathsf{y}}(n) \mid e^{-i\frac{2\pi}{\tau}nx} \right\rangle_{\mathbb{Z}}^* \mathrm{d}x && \text{by property of } \langle \triangle \mid \nabla \rangle \\
&= \int_0^\tau \mathsf{x}(x) \left[\hat{\mathbf{F}}^{-1}\tilde{\mathsf{y}}(n)\right]^* \mathrm{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem 5.1 page 72)} \\
&= \left\langle \mathsf{x}(x) \mid \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^*} \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{R}}
\end{aligned}$$

⇰

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary*[4] (Corollary 5.1 page 74).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary 5.2 page 74).

Corollary 5.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\hat{\mathbf{F}}$ *be the Fourier Series operator with adjoint* $\hat{\mathbf{F}}^*$.

COR: $\left\{ \hat{\mathbf{F}}\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^*\hat{\mathbf{F}} = \mathbf{I} \right\}$ ($\hat{\mathbf{F}}$ *is* ***unitary*** *…and thus also* NORMAL *and* ISOMETRIC)

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$ (Theorem 5.2 page 73). ⇰

Corollary 5.2. *Let* $\hat{\mathbf{F}}$ *be the Fourier series operator with adjoint* $\hat{\mathbf{F}}^*$ *and inverse* $\hat{\mathbf{F}}^{-1}$.

COR:

$$\begin{array}{rclcll}
\mathcal{R}(\hat{\mathbf{F}}) &=& \mathcal{R}(\hat{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
\left|\!\left|\!\left|\hat{\mathbf{F}}\right|\!\right|\!\right| &=& \left|\!\left|\!\left|\hat{\mathbf{F}}^{-1}\right|\!\right|\!\right| &=& 1 & \text{(UNITARY)} \\
\left\langle \hat{\mathbf{F}}\boldsymbol{x} \mid \hat{\mathbf{F}}\boldsymbol{y} \right\rangle &=& \left\langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \mid \hat{\mathbf{F}}^{-1}\boldsymbol{y} \right\rangle &=& \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle & \text{(PARSEVAL'S EQUATION)} \\
\left\|\hat{\mathbf{F}}\boldsymbol{x}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\right\| &=& \|\boldsymbol{x}\| & \text{(PLANCHEREL'S FORMULA)} \\
\left\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\right\| &=& \|\boldsymbol{x} - \boldsymbol{y}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary 5.1 page 74) and from the properties of unitary operators (Theorem E.26 page 152). ⇰

5.3 Fourier series for compactly supported functions

Theorem 5.3.

THM: *The set*

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i\frac{2\pi}{\tau}nx} \middle| n \in \mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathsf{f}(x)$ *with support in* $[0 : \tau]$.

[4] *unitary operators*: Definition E.14 page 151

CHAPTER 6

DISCRETE TIME FOURIER TRANSFORM

6.1 Definition

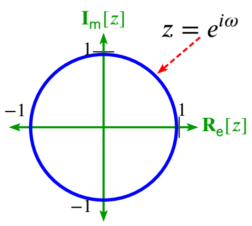
Definition 6.1.

DEF

*The **discrete-time Fourier transform** $\breve{\mathbf{F}}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$\left[\breve{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition 6.1 page 75) to the definition of the Z-transform (Definition D.4 page 114), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.



6.2 Properties

Proposition 6.1 (DTFT periodicity). *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 75) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP

$$\underbrace{\breve{x}(\omega) = \breve{x}(\omega + 2\pi n)}_{\text{PERIODIC with period } 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\breve{x}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} &&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi nm}} \\
&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} &&= \breve{x}(\omega)
\end{aligned}$$

✏

Theorem 6.1. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[\left(\!\left(x[n]\right)\!\right)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 75) of a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$.

THM
$$\left\{ \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\left(\!\left(x[n]\right)\!\right) \right\} \implies \left\{ \begin{array}{llll} (1). & \breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &= \tilde{x}(-\omega) & \text{and} \\ (2). & \breve{\mathbf{F}}\left(\!\left(x^*[n]\right)\!\right) &= \tilde{x}^*(-\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}\left(\!\left(x^*[-n]\right)\!\right) &= \tilde{x}^*(\omega) & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 75)} \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m} \\
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x^*[n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 75)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\mathbf{algebras} && \text{(Definition A.3 page 88)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{-i(-\omega)n}\right)^* \\
&\triangleq \tilde{x}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x^*[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 75)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[-n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\mathbf{algebras} && \text{(Definition A.3 page 88)} \\
&= \left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n = -m \\
&\triangleq \tilde{x}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

✏

Theorem 6.2. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[\left(\!\left(x[n]\right)\!\right)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 75) of a sequence* $\left(\!\left(x[n]\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$.

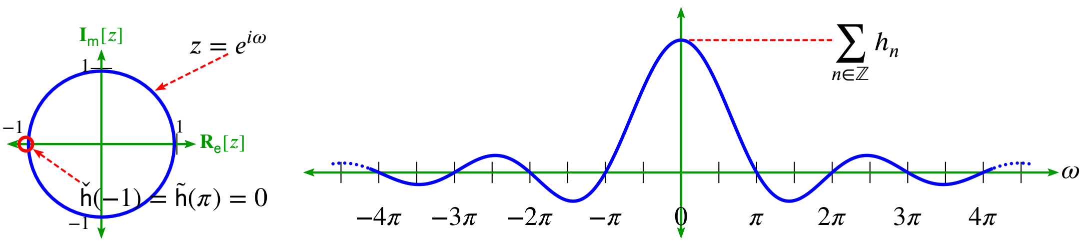
THM
$$\left\{ \begin{array}{lll} (1). & \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\left(\!\left(x[n]\right)\!\right) & \text{and} \\ (2). & \left(\!\left(x[n]\right)\!\right) \text{ is REAL-VALUED} & \end{array} \right\} \implies \left\{ \begin{array}{llllll} (1). & \breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &= \tilde{x}(-\omega) & & & \text{and} \\ (2). & \breve{\mathbf{F}}\left(\!\left(x^*[n]\right)\!\right) &= \tilde{x}^*(-\omega) &= \tilde{x}(\omega) & & \text{and} \\ (3). & \breve{\mathbf{F}}\left(\!\left(x^*[-n]\right)\!\right) &= \tilde{x}^*(\omega) &= \tilde{x}(-\omega) & & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 75)} \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}
\end{aligned}$$

$$
\begin{aligned}
&\triangleq \tilde{\mathsf{x}}(-\omega) && \text{by left hypothesis}\\[1em]
\boxed{\tilde{\mathsf{x}}^*(-\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(\mathsf{x}^*[n])\!)} && \text{by Theorem 6.1 page 76}\\
&= \breve{\mathbf{F}}\,(\!(\mathsf{x}[n])\!) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(\omega)} && \text{by definition of } \tilde{\mathsf{x}}(\omega) \quad \text{(Definition 6.1 page 75)}\\[1em]
\boxed{\tilde{\mathsf{x}}^*(\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(\mathsf{x}^*[-n])\!)} && \text{by Theorem 6.1 page 76}\\
&= \breve{\mathbf{F}}\,(\!(\mathsf{x}[-n])\!) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(-\omega)} && \text{by result (1)}
\end{aligned}
$$

✏



Proposition 6.2. *Let* $\breve{\mathsf{x}}(z)$ *be the* Z-TRANSFORM *(Definition D.4 page 114) and* $\breve{\mathsf{x}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 75) of* $(\!(x_n)\!)$.

PRP

$$
\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}},\ c\in\mathbb{R}
$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$
\begin{aligned}
\breve{\mathsf{x}}(z)\Big|_{z=1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \text{by definition of } \breve{\mathsf{x}}(z) \text{ (Definition D.4 page 114)}\\
&= \sum_{n\in\mathbb{Z}} x_n && \text{because } z^n = 1 \text{ for all } n\in\mathbb{Z}\\
&= c && \text{by hypothesis (1)}
\end{aligned}
$$

2. Proof that (2) $\implies$ (3):

$$
\begin{aligned}
\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} && \text{by definition of } \breve{\mathsf{x}}(\omega) \quad \text{(Definition 6.1 page 75)}\\
&= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \\
&= \breve{\mathsf{x}}(z)\Big|_{z=1} && \text{by definition of } \breve{\mathsf{x}}(z) \quad \text{(Definition D.4 page 114)}\\
&= c && \text{by hypothesis (2)}
\end{aligned}
$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=0} \\
&= \breve{\mathsf{x}}(\omega) && \text{by definition of } \breve{\mathsf{x}}(\omega) && \text{(Definition 6.1 page 75)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

✏

Proposition 6.3. *If the coefficients are* ***real****, then the magnitude response (MR) is* ***symmetric****.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathsf{h}}(-\omega)\right| &\triangleq \left|\breve{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{i\omega m}\right| && = \left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}^*[m]e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } real} && = \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right| \\
&\triangleq \left|\breve{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\tilde{\mathsf{h}}(\omega)\right|
\end{aligned}$$

✏

Proposition 6.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}}(-1)^n x_n = c}_{\text{(1) in "time"}} \iff \underbrace{\breve{\mathsf{x}}(z)|_{z=-1} = c}_{\text{(2) in "z domain"}} \iff \underbrace{\breve{\mathsf{x}}(\omega)|_{\omega=\pi} = c}_{\text{(3) in "frequency"}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{\mathsf{x}}(z)|_{z=-1} &= \left.\sum_{n\in\mathbb{Z}} x_n z^{-n}\right|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \left.\sum_{n\in\mathbb{Z}} z^{-n} x_n\right|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \left.\sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n\right|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\implies$ (4):

   (a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

   (b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

(e) Solutions to these equations give

$$\sum_{n\in\mathbb{Z}} x_{2n} \triangleq A = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)$$
$$\sum_{n\in\mathbb{Z}} x_{2n+1} \triangleq B = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1} \\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1} \\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)} \\
&= c
\end{aligned}$$

✏

Lemma 6.1. *Let* $\tilde{\mathrm{f}}(\omega)$ *be the* DTFT (Definition 6.1 page 75) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

LEM

$$\underbrace{(x_n \in \mathbb{R})_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{\mathrm{x}}(\omega)|^2 = |\breve{\mathrm{x}}(-\omega)|^2}_{\text{EVEN}} \qquad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
|\breve{\mathrm{x}}(\omega)|^2 &= |\breve{\mathrm{x}}(z)|^2\Big|_{z=e^{i\omega}} \\
&= \breve{\mathrm{x}}(z)\breve{\mathrm{x}}^*(z)\Big|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^*(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right]$$
$$= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2 \sum_{n\in\mathbb{Z}} \sum_{m>n} x_n x_m \cos[\omega(m-n)]$$

Since cos is real and even, then $|\breve{x}(\omega)|^2$ must also be real and even. ✏

Theorem 6.3 (inverse DTFT). [2] *Let* $\breve{x}(\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 75) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. *Let* $\breve{x}^{-1}$ *be the inverse of* $\breve{x}$.

THM
$$\underbrace{\left\{ \breve{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\breve{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x_n)\!)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi} \int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega) e^{i\omega n}\, d\omega \quad \forall \alpha\in\mathbb{R} \right\}}_{(\!(x_n)\!) = \breve{\mathbf{F}}^{-1}\breve{\mathbf{F}}(\!(x_n)\!)} \quad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega) e^{i\omega n}\, d\omega = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \right]}_{\breve{x}(\omega)} e^{i\omega n}\, d\omega \qquad \text{by definition of } \breve{x}(\omega)$$
$$= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\, d\omega$$
$$= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\, d\omega$$
$$= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} x_m \left[2\pi \bar{\delta}_{m-n} \right]$$
$$= x_n$$

✏

Theorem 6.4 (orthonormal quadrature conditions). [3] *Let* $\breve{x}(\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 75) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION *at* n.

THM
$$\begin{array}{lllll} \sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 & \iff & \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0 & \forall n\in\mathbb{Z}, \forall (\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}} \\ \sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n & \iff & |\breve{x}(\omega)|^2 + |\breve{x}(\omega+\pi)|^2 = 2 & \forall n\in\mathbb{Z}, \forall (\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}} \end{array}$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

1. Proof that $2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} \right] e^{-i2\omega n} = \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)$:

$$2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} \right] e^{-i2\omega n}$$
$$= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n}$$

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩
[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

$$
\begin{aligned}
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} \left(1 + e^{i\pi n}\right) \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n \\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_m z^{m} + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_m e^{+i(\omega+\pi)m} \\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k} \left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^* \\
&\triangleq \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)
\end{aligned}
$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$
\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}
$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$
\begin{aligned}
2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 0 && \text{by right hypothesis}
\end{aligned}
$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$
\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n} \\
&= 2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}
$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$
\begin{aligned}
2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 2 && \text{by right hypothesis}
\end{aligned}
$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

6.3 Derivatives

Theorem 6.5. [4] *Let* $\breve{x}(\omega)$ *be the* DTFT (Definition 6.1 page 75) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

THM

$$\text{(A)}\quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} = 0 \quad\iff\quad \sum_{k\in\mathbb{Z}} k^n x_k = 0 \quad \text{(B)}\quad \forall n\in\mathbb{W}$$

$$\text{(C)}\quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=\pi} = 0 \quad\iff\quad \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 \quad \text{(D)}\quad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} && \text{by hypothesis (A)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \breve{x}(\omega) \text{ (Definition 6.1 page 75)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \tilde{g}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k\\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

3. Proof that $(C) \implies (D)$:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=\pi} && \text{by hypothesis (C)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=\pi} && \text{by definition of } \breve{x} \text{ (Definition 6.1 page 75)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=\pi}
\end{aligned}$$

[4] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

$$= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]$$
$$= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k$$

4. Proof that $(C) \Longleftarrow (D)$:

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=\pi} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=\pi} && \text{by definition of } \breve{x} \text{ (Definition 6.1 page 75)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k\\
&= 0 && \text{by hypothesis (D)}
\end{aligned}$$

✏

APPENDIX A

NORMED ALGEBRAS

A.1 Algebras

All *linear spaces* are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition A.1. [3] *Let* $\boldsymbol{A}$ *be an* ALGEBRA.

DEF — *An algebra* $\boldsymbol{A}$ *is* ***unital*** *if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition A.2. [4] *Let* $\boldsymbol{A}$ *be an* UNITAL ALGEBRA *(Definition A.1 page 87) with unit* e.

DEF —

The ***spectrum*** *of* $x \in \boldsymbol{A}$ *is* $\sigma(x) \triangleq \left\{\lambda \in \mathbb{C} | \lambda e - x \text{ is not invertible}\right\}.$

The ***resolvent*** *of* $x \in \boldsymbol{A}$ *is* $\rho_x(\lambda) \triangleq (\lambda e - x)^{-1} \qquad \forall \lambda \notin \sigma(x).$

The ***spectral radius*** *of* $x \in \boldsymbol{A}$ *is* $r(x) \triangleq \sup\{|\lambda| | \lambda \in \sigma(x)\}.$

---

[1] Fuchs (1995) page 2
[2] Hazewinkel (2000) page v
[3] Folland (1995) page 1
[4] Folland (1995) pages 3–4

A.2 Star-Algebras

Definition A.3. [5] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF

*The pair $(\boldsymbol{A}, *)$ is a **$*$-algebra**, or **star-algebra**, if*

1.	$(x+y)^* = x^* + y^*$	$\forall x,y \in \boldsymbol{A}$	(DISTRIBUTIVE)	and
2.	$(\alpha x)^* = \bar{\alpha} x^*$	$\forall x \in \boldsymbol{A}, \alpha \in \mathbb{C}$	(CONJUGATE LINEAR)	and
3.	$(xy)^* = y^* x^*$	$\forall x,y \in \boldsymbol{A}$	(ANTIAUTOMORPHIC)	and
4.	$x^{**} = x$	$\forall x \in \boldsymbol{A}$	(INVOLUTORY)	

*The operator $*$ is called an **involution** on the algebra $\boldsymbol{A}$.*

Proposition A.1. [6] *Let $(\boldsymbol{A}, *)$ be an* UNITAL $*$-ALGEBRA.

PRP

x is invertible $\implies$ $\begin{cases} 1. & x^* \text{ is INVERTIBLE} \quad \forall x \in \boldsymbol{A} \quad \text{and} \\ 2. & (x^*)^{-1} = (x^{-1})^* \quad \forall x \in \boldsymbol{A} \end{cases}$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= (x\,e^*)^{**} && \text{by \textit{involutory} property of } * && \text{(Definition A.3 page 88)}\\
&= (x^*\,e^{**})^* && \text{by \textit{antiautomorphic} property of } * && \text{(Definition A.3 page 88)}\\
&= (x^*\,e)^* && \text{by \textit{involutory} property of } * && \text{(Definition A.3 page 88)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by \textit{involutory} property of } * && \text{(Definition A.3 page 88)}\\
e^*\,x &= (e^*\,x)^{**} && \text{by \textit{involutory} property of } * && \text{(Definition A.3 page 88)}\\
&= (e^{**}\,x^*)^* && \text{by \textit{antiautomorphic} property of } * && \text{(Definition A.3 page 88)}\\
&= (e\,x^*)^* && \text{by \textit{involutory} property of } * && \text{(Definition A.3 page 88)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by \textit{involutory} property of } * && \text{(Definition A.3 page 88)}
\end{aligned}$$

2. Proof that $(x^*)^{-1} = (x^{-1})^*$:

$$\begin{aligned}
(x^{-1})^*\,(x^*) &= \left[x\,(x^{-1})\right]^* && \text{by \textit{antiautomorphic} and \textit{involution} properties of } * && \text{(Definition A.3 page 88)}\\
&= e^*\\
&= e && \text{by item (1) page 88}\\
(x^*)\,(x^{-1})^* &= \left[x^{-1}\,x\right]^* && \text{by \textit{antiautomorphic} and \textit{involution} properties of } * && \text{(Definition A.3 page 88)}\\
&= e^*\\
&= e && \text{by item (1) page 88}
\end{aligned}$$

⇨

Definition A.4. [7] *Let $(\boldsymbol{A}, \|\cdot\|)$ be a* $*$-ALGEBRA *(Definition A.3 page 88).*

DEF

- *An element $x \in \boldsymbol{A}$ is **hermitian** or **self-adjoint** if $x^* = x$.*
- *An element $x \in \boldsymbol{A}$ is **normal** if $xx^* = x^*x$.*
- *An element $x \in \boldsymbol{A}$ is a **projection** if $xx = x$* (INVOLUTORY) *and $x^* = x$* (HERMITIAN).

[5] Rickart (1960) page 178, Gelfand and Naimark (1964), page 241
[6] Folland (1995) page 5
[7] Rickart (1960) page 178, Gelfand and Naimark (1964), page 242

Theorem A.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition A.3 page 88).*

THM

$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are HERMITIAN}} \implies \left\{\begin{array}{lll} x+y &= (x+y)^* & (x+y \text{ is self adjoint}) \\ x^* &= (x^*)^* & (x^* \text{ is self adjoint}) \\ \underbrace{xy = (xy)^*}_{(xy) \text{ is HERMITIAN}} & \iff \underbrace{xy = yx}_{\text{commutative}} & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} (x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 88)} \\ &= x + y && \text{by left hypothesis} \end{aligned}$$

$$\left(x^*\right)^* = x \quad \text{by } \textit{involutory} \text{ property of } * \quad \text{(Definition A.3 page 88)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned} xy &= (xy)^* && \text{by left hypothesis} \\ &= y^*x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition A.3 page 88)} \\ &= yx && \text{by left hypothesis} \end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned} (xy)^* &= (yx)^* && \text{by left hypothesis} \\ &= x^*y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition A.3 page 88)} \\ &= xy && \text{by left hypothesis} \end{aligned}$$

✏

Definition A.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition A.3 page 88).*

DEF

*The **real part** of* x *is defined as* $\mathbf{R}_e x \triangleq \frac{1}{2}\left(x + x^*\right)$

*The **imaginary part** of* x *is defined as* $\mathbf{I}_m x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem A.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition A.3 page 88).*

THM

$$\begin{array}{llll} \mathbf{R}_e x &= \left(\mathbf{R}_e x\right)^* & \forall x \in \boldsymbol{A} & (\mathbf{R}_e x \text{ is HERMITIAN}) \\ \mathbf{I}_m x &= \left(\mathbf{I}_m x\right)^* & \forall x \in \boldsymbol{A} & (\mathbf{I}_m x \text{ is HERMITIAN}) \end{array}$$

✎PROOF:

$$\begin{aligned} \left(\mathbf{R}_e x\right)^* &= \left(\frac{1}{2}\left(x + x^*\right)\right)^* && \text{by definition of } \Re && \text{(Definition A.5 page 89)} \\ &= \frac{1}{2}\left(x^* + x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 88)} \\ &= \frac{1}{2}\left(x^* + x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 88)} \\ &= \mathbf{R}_e x && \text{by definition of } \Re && \text{(Definition A.5 page 89)} \end{aligned}$$

$$\left(\mathbf{I}_m x\right)^* = \left(\frac{1}{2i}\left(x - x^*\right)\right)^* \quad \text{by definition of } \Im \quad \text{(Definition A.5 page 89)}$$

[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998) page 42

$$
\begin{aligned}
&= \frac{1}{2i}\left(x^* - x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 88)}\\
&= \frac{1}{2i}\left(x^* - x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 88)}\\
&= \mathbf{I}_{\text{m}} x && \text{by definition of } \mathfrak{I} && \text{(Definition A.5 page 89)}
\end{aligned}
$$

✏

Theorem A.3 (Hermitian representation). [11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition A.3 page 88).*

THM

$$a = x + iy \iff x = \mathbf{R}_{\text{e}} a \quad \textit{and} \quad y = \mathbf{I}_{\text{m}} a$$

✎PROOF:

Proof that $a = x + iy \implies x = \mathbf{R}_{\text{e}} a$ and $y = \mathbf{I}_{\text{m}} a$:

$$
\begin{aligned}
& & a &= x + iy && \text{by left hypothesis}\\
&\implies & a^* &= (x + iy)^* && \text{by definition of } \textit{adjoint} && \text{(Definition A.4 page 88)}\\
& & &= x^* - iy^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 88)}\\
& & &= x - iy && \text{by Theorem A.2 page 89}\\
&\implies & x &= a - iy && \text{by solving for } x \text{ in } a = x + iy \text{ equation}\\
& & x &= a^* + iy && \text{by solving for } x \text{ in } a^* = x - iy \text{ equation}\\
&\implies & x + x &= a + a^* && \text{by adding previous 2 equations}\\
&\implies & 2x &= a + a^* && \text{by solving for } x \text{ in previous equation}\\
&\implies & x &= \frac{1}{2}\left(a + a^*\right)\\
& & &= \mathbf{R}_{\text{e}} a && \text{by definition of } \mathfrak{R} && \text{(Definition A.5 page 89)}
\end{aligned}
$$

$$
\begin{aligned}
& & iy &= a - x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation}\\
& & iy &= -a^* + x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation}\\
&\implies & iy + iy &= a - a^* && \text{by adding previous 2 equations}\\
&\implies & y &= \frac{1}{2i}\left(a - a^*\right) && \text{by solving for } iy \text{ in previous equations}\\
& & &= \mathbf{I}_{\text{m}} a && \text{by definition of } \mathfrak{I} && \text{(Definition A.5 page 89)}
\end{aligned}
$$

Proof that $a = x + iy \impliedby x = \mathbf{R}_{\text{e}} a$ and $y = \mathbf{I}_{\text{m}} a$:

$$
\begin{aligned}
x + iy &= \mathbf{R}_{\text{e}} a + i\,\mathbf{I}_{\text{m}} a && \text{by right hypothesis}\\
&= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\mathbf{R}_{\text{e}} a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\mathbf{I}_{\text{m}} a} && \text{by definition of } \mathfrak{R} \text{ and } \mathfrak{I} && \text{(Definition A.5 page 89)}\\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \left(\cancelto{0}{\frac{1}{2}a^* - \frac{1}{2}a^*}\right)\\
&= a
\end{aligned}
$$

✏

[11] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Neumark (1943b) page 7

A.3 Normed Algebras

Definition A.6. [12] *Let $\boldsymbol{A}$ be an algebra.*

DEF

*The pair $(\boldsymbol{A}, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in \boldsymbol{A} \qquad \text{(multiplicative condition)}$$

*A normed algebra $(\boldsymbol{A}, \|\cdot\|)$ is a **Banach algebra** if $(\boldsymbol{A}, \|\cdot\|)$ is also a Banach space.*

Proposition A.2.

PRP

$(\boldsymbol{A}, \|\cdot\|)$ *is a normed algebra* $\implies$ *multiplication is **continuous** in* $(\boldsymbol{A}, \|\cdot\|)$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x-y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(\boldsymbol{A}, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 91)}\\
&= \|z(x-y)\| \\
&\le \|z\|\,\|x-y\| && \text{by definition of } \textit{normed algebra} && \text{(Definition A.6 page 91)}\\
&\triangleq \|z\|\,\delta && \text{by definition of } \delta && \text{(item (2) page 91)}\\
&\le \epsilon && \text{for some value of } \delta > 0
\end{aligned}$$

✏

Theorem A.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l}(\boldsymbol{A}, \|\cdot\|) \text{ is a Banach algebra}\\ \text{every nonzero } x \in \boldsymbol{A} \text{ is invertible}\end{array}\right\} \implies \boldsymbol{A} \equiv \mathbb{C} \quad (\boldsymbol{A} \text{ is isomorphic to } \mathbb{C})$$

A.4 C* Algebras

Definition A.7. [14]

DEF

*The triple $(\boldsymbol{A}, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(\boldsymbol{A}, \|\cdot\|)$ *is a Banach algebra* and
2. $(\boldsymbol{A}, *)$ *is a $*$-algebra* and
3. $\|x^*x\| = \|x\|^2 \quad \forall x \in \boldsymbol{A}$.

*A C^* **algebra** $(\boldsymbol{A}, \|\cdot\|, *)$ is also called a **C star algebra**.*

---

[12] Rickart (1960) page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩

[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩

[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem A.5. [15] *Let $\boldsymbol{A}$ be an algebra.*

THM: $(\boldsymbol{A}, \|\cdot\|, *)$ *is a* C^* ***algebra*** $\implies$ $\|x^*\| = \|x\|$

✎PROOF:

$$
\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\ \|x\|^2 \\
&= \frac{1}{\|x\|}\ \|x^* x\| && \text{by definition of } C^*\text{-}\textit{algebra} && \text{(Definition A.7 page 91)} \\
&\le \frac{1}{\|x\|}\ \|x^*\|\ \|x\| && \text{by definition of } \textit{normed algebra} && \text{(Definition A.6 page 91)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by } \textit{involution} \text{ property of } * && \text{(Definition A.3 page 88)}
\end{aligned}
$$

⇒

[15] Folland (1995) page 1, Gelfand and Neumark (1943b) page 4, Gelfand and Neumark (1943a)

APPENDIX B

POLYNOMIALS

B.1 Definitions

Definition B.1. [1] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

D E F

A function p *in* $\mathbb{F}^{\mathbb{F}}$ *is a **polynomial** over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if it is of the form*

$$\mathsf{p}(x) \triangleq \sum_{n=0}^{N} \alpha_n x^n \qquad \alpha_n \in \mathbb{F},\ \alpha_N \neq 0.$$

*The **degree** of* p *is* N. *A **coefficient** of* p *is any element of* $\left(\!\!\left(\alpha_n\right)\!\!\right)_1^N$.
*The **leading coefficient** of* p *is* α_N.

Definition B.2. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

D E F

A polynomial p *of degree* N *over the field* $\mathbb{F}$ *and*
a polynomial q *of degree* M *over the field* $\mathbb{F}$ *are **equal** if*

1. $N = M$ *and*
2. $\alpha_n = \beta_n$ *for* $n = 0, 1, \ldots, N$.

The expression $\mathsf{p}(x) = \mathsf{q}(x)$ *(or* $\mathsf{p} = \mathsf{q}$*) denotes that* p *and* q *are* EQUAL.

[1] Barbeau (1989) page 1, Fuhrmann (2012) page 11, Borwein and Erdélyi (1995) page 2
[2] Fuhrmann (2012) page 11

B.2 Ring properties

B.2.1 Polynomial Arithmetic

Theorem B.1 (polynomial addition). [3] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

$$\underbrace{\left(\sum_{n=0}^{N} \alpha_n x^n\right)}_{\mathsf{p}(x)} + \underbrace{\left(\sum_{n=0}^{M} \beta_n x^n\right)}_{\mathsf{q}(x)} = \underbrace{\sum_{n=0}^{\max(N,M)} \gamma_n x^n}_{\mathsf{p}(x)+\mathsf{q}(x)} \quad \textit{where} \quad \gamma_n \triangleq \begin{cases} \alpha_n + \beta_n & \textit{for } n \le \min(N,M) \\ \alpha_n & \textit{for } n > M \\ \beta_n & \textit{for } n > N \end{cases}$$

$$\textit{for all} \quad x,\ \alpha_n,\ \beta_n \in \mathbb{F}$$

Polynomial multiplication is equivalent to convolution (Definition D.3 page 113) of the coefficients (Definition B.1 page 93). [4]

Theorem B.2 (polynomial multiplication). [5] *Let* $(\alpha_n \in \mathbb{C})$, $(b_n \in \mathbb{C})$, *and* $x \in \mathbb{C}$.

$$\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{n=0}^{M} \beta_n x^n\right) = \sum_{n=0}^{N+M} \underbrace{\left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right)}_{\textit{Cauchy product}} x^n$$

✎PROOF:

$$\begin{aligned}
\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{m=0}^{M} \beta_m x^m\right) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n \beta_m x^{n+m} \\
&= \sum_{n=0}^{N}\sum_{k=n}^{M+n} \alpha_n \beta_{k-n} x^k && k \triangleq n+m \iff m = k-n \\
&= \sum_{n=0}^{N+M} \left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right) x^n
\end{aligned}$$

Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	β_0	β_1	β_2	β_3	$\cdots$	β_M
α_0	$\alpha_0\beta_0$	$\alpha_0\beta_1 x$	$\alpha_0\beta_2 x^2$	$\alpha_0\beta_3 x^3$	$\cdots$	$\alpha_0\beta_M x^M$
α_1	$\alpha_1\beta_0 x$	$\alpha_1\beta_1 x^2$	$\alpha_1\beta_2 x^3$	$\alpha_1\beta_3 x^4$	$\cdots$	$\alpha_1\beta_M x^{1+M}$
α_2	$\alpha_2\beta_0 x^2$	$\alpha_2\beta_1 x^3$	$\alpha_2\beta_2 x^4$	$\alpha_2\beta_3 x^5$	$\cdots$	$\alpha_2\beta_M x^{2+M}$
α_3	$\alpha_3\beta_0 x^3$	$\alpha_3\beta_1 x^4$	$\alpha_3\beta_2 x^5$	$\alpha_3\beta_3 x^6$	$\cdots$	$\alpha_3\beta_M x^{3+M}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_N	$\alpha_N\beta_0 x^N$	$\alpha_N\beta_1 x^{N+1}$	$\alpha_N\beta_2 x^{N+2}$	$\alpha_N\beta_3 x^{N+3}$	$\cdots$	$\alpha_N\beta_M x^{N+M}$

1. The expression $\sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n\beta_m x^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

[3] Fuhrmann (2012) page 11

[4] *Convolution*: In fact, using *GNU Octave*™ or *MatLab*™, polynomial multiplication can be performed using convolution. For example, the operation $(x^3 + 5x^2 + 7x + 9)(4x^2 + 11)$ can be calculated in *GNU Octave*™ or *MatLab*™ with conv([1 5 7 9],[4 0 11])

[5] Apostol (1975) page 237

2. If we switched the order of summation to $\sum_{m=0}^{M}\sum_{n=0}^{N}\alpha_n\beta_m x^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

3. For $N = M = \infty$, the expression $\sum_{n=0}^{N+M}\left(\sum_{k=0}^{n}\alpha_k\beta_{n-k}\right)x^n$ is equivalent to adding *diagonally* starting from the upper left corner and proceeding towards the lower right.

4. For finite N and M…

   (a) The upper limit on the inner summation puts two constraints on k:
   $$\left\{\begin{array}{lllc} k & \le & n & \text{and} \\ k & \le & N & \end{array}\right\} \implies k \le \min(n, N)$$

   (b) The lower limit on the inner summation also puts two constraints on k:
   $$\left\{\begin{array}{lllc} k & \ge & 0 & \text{and} \\ k & \ge & n-M & \end{array}\right\} \implies k \ge \max(0, n-M)$$

✎

Polynomial division can be performed in a manner very similar to integer division (both integers and polynomials are *rings*).

Definition B.3 (Polynomial division). *The quantities of polynomial division are defined as follows:*

DEF
$$\frac{\mathsf{d}(x)}{\mathsf{p}(x)} = \mathsf{q}(x) + \frac{\mathsf{r}(x)}{\mathsf{p}(x)} \qquad \textit{where} \qquad \left\{\begin{array}{lll} \mathsf{d}(x) & \textit{is the } \textbf{\textit{dividend}} & \textit{and} \\ \mathsf{p}(x) & \textit{is the } \textbf{\textit{divisor}} & \textit{and} \\ \mathsf{q}(x) & \textit{is the } \textbf{\textit{quotient}} & \textit{and} \\ \mathsf{r}(x) & \textit{is the } \textbf{\textit{remainder}}. & \end{array}\right\}$$

The ring of integers $\mathbb{Z}$ contains some special elements called *primes* which can only be divided[6] by themselves or 1.

Rings of polynomials have a similar elements called *primitive polynomials.*

Definition B.4.

DEF
*A **primitive polynomial** is any polynomial $p(x)$ that satisfies*
1. *$p(x)$ cannot be factored*
2. *the smallest order polynomial that $p(x)$ can divide is $x^{2^n-1} + 1 = 0$.*

Example B.1. [7] Some examples of primitive polynomials over $GF(2)$ are

EX

order	primitive polynomial
2	$\mathsf{p}(x) = x^2 + x + 1$
3	$\mathsf{p}(x) = x^3 + x + 1$
4	$\mathsf{p}(x) = x^4 + x + 1$
5	$\mathsf{p}(x) = x^5 + x^2 + 1$
5	$\mathsf{p}(x) = x^5 + x^4 + x^2 + x + 1$
16	$\mathsf{p}(x) = x^{16} + x^{15} + x^{13} + x^4 + 1$
31	$\mathsf{p}(x) = x^{31} + x^{28} + 1$

An m-sequence is the remainder when dividing any non-zero polynomial by a primitive polynomial. We can define an *equivalence relation* on polynomials which defines two polynomials as *equivalent with respect to* $p(x)$ when their remainders are equal.

[6] The expression "a divides b" means that b/a has remainder 0.
[7] Wicker (1995) pages 465–475

Definition B.5 (Equivalence relation). *Let* $\dfrac{\alpha_1(x)}{p(x)} = \mathsf{q}_1(x) + \dfrac{r_1(x)}{p(x)}$ *and* $\dfrac{\alpha_2(x)}{p(x)} = \mathsf{q}_2(x) + \dfrac{r_2(x)}{p(x)}$.

Then $\alpha_1(x) \equiv \alpha_2(x)$ *with respect to* $p(x)$ *if* $r_1(x) = r_2(x)$.

Using the equivalence relation of Definition B.5, we can develop two very useful equivalent representations of polynomials over GF(2). We will call these two representations the *exponential* representation and the *polynomial* representation.

Example B.2. By Definition B.5 and under $p(x) = x^3 + x + 1$, we have the following equivalent representations:

$$\begin{array}{rcrcccccl}
\frac{x^0}{x^3+x+1} & = & 0 & + & \frac{1}{x^3+x+1} & \Longrightarrow & x^0 & \equiv & 1 \\
\frac{x^1}{x^3+x+1} & = & 0 & + & \frac{x}{x^3+x+1} & \Longrightarrow & x^1 & \equiv & x \\
\frac{x^2}{x^3+x+1} & = & 0 & + & \frac{x^2}{x^3+x+1} & \Longrightarrow & x^2 & \equiv & x^2 \\
\frac{x^3}{x^3+x+1} & = & 1 & + & \frac{x+1}{x^3+x+1} & \Longrightarrow & x^3 & \equiv & x+1 \\
\frac{x^4}{x^3+x+1} & = & x & + & \frac{x^2+x}{x^3+x+1} & \Longrightarrow & x^4 & \equiv & x^2+x \\
\frac{x^5}{x^3+x+1} & = & x^2+1 & + & \frac{x^2+x+1}{x^3+x+1} & \Longrightarrow & x^5 & \equiv & x^2+x+1 \\
\frac{x^6}{x^3+x+1} & = & x^3+x+1 & + & \frac{x^2+1}{x^3+x+1} & \Longrightarrow & x^6 & \equiv & x^2+1 \\
\frac{x^7}{x^3+x+1} & = & x^4+x^2+x+1 & + & \frac{1}{x^3+x+1} & \Longrightarrow & x^7 & \equiv & 1
\end{array}$$

Notice that $x^7 \equiv x^0$, and so a cycle is formed with $2^3 - 1 = 7$ elements in the cycle. The monomials to the left of the $\equiv$ are the *exponential* representation and the polynomials to the right are the *polynomial* representation. Additionally, the polynomial representation may be put in a vector form giving a *vector* representation. The vectors may be interpreted as a binary number and represented as a decimal numeral.

exponential	polynomial	vector	decimal
x^0	1	[001]	1
x^1	x	[010]	2
x^2	x^2	[100]	4
x^3	$x+1$	[011]	3
x^4	x^2+x	[110]	6
x^5	x^2+x+1	[111]	7
x^6	x^2+1	[101]	5

Example B.3. We can generate an m-sequence of length $2^3 - 1 = 7$ by dividing 1 by the primitive polynomial $x^3 + x + 1$.

$$
\begin{array}{r|l}
 & x^{-3}+x^{-5}+x^{-6}+\quad x^{-7}+x^{-10}+x^{-12}+x^{-13}+x^{-14}+x^{-17}+\cdots \\ \hline
x^3+x+1 & 1 \\
 & 1+x^{-2}+x^{-3} \\ \cline{2-2}
 & x^{-2}+x^{-3} \\
 & x^{-2}+x^{-4}+x^{-5} \\ \cline{2-2}
 & x^{-3}+x^{-4}+x^{-5} \\
 & x^{-3}+x^{-5}+x^{-6} \\ \cline{2-2}
 & x^{-4}+x^{-6} \\
 & x^{-4}+x^{-6}+x^{-7} \\ \cline{2-2}
 & x^{-7} \\
 & x^{-7}+x^{-9}+x^{-10} \\ \cline{2-2}
 & x^{-9}+x^{-10} \\
 & x^{-9}+x^{-11}+x^{-12} \\ \cline{2-2}
 & x^{-10}+x^{-11}+x^{-12} \\
 & x^{-10}+x^{-12}+x^{-13} \\ \cline{2-2}
 & x^{-11}+x^{-13} \\
 & x^{-11}+x^{-13}+x^{-14} \\ \cline{2-2}
 & x^{-14} \\
 & \vdots
\end{array}
$$

The coefficients, starting with the x^{-1} term, of the resulting polynomial form the m-sequence

$$0010111\ 0010111\ \cdots$$

which repeats every $2^3-1=7$ elements.

Note that the division operation in Example B.3 can be performed using vector notation rather than polynomial notation.

Example B.4. Generate an m-sequence of length $2^3-1=7$ by dividing 1 by the primitive polynomial x^3+x+1 using vector notation.

$$
\begin{array}{r|cccccccccccccccccc}
 & & . & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots \\ \hline
1011 & 1 & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & 1 & & 0 & 1 & 1 & & & & & & & & & & & & & \\
 & 0 & & 0 & 1 & 1 & 0 & & & & & & & & & & & & \\
 & & & 0 & 0 & 0 & 0 & & & & & & & & & & & & \\
 & & & 0 & 1 & 1 & 0 & 0 & & & & & & & & & & & \\
 & & & & 1 & 0 & 1 & 1 & & & & & & & & & & & \\
 & & & & 0 & 1 & 1 & 1 & 0 & & & & & & & & & & \\
 & & & & & 1 & 0 & 1 & 1 & & & & & & & & & & \\
 & & & & & 0 & 1 & 0 & 1 & 0 & & & & & & & & & \\
 & & & & & & 1 & 0 & 1 & 1 & & & & & & & & & \\
 & & & & & & 0 & 0 & 0 & 1 & 0 & & & & & & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & & & & & & & & \\
 & & & & & & & 0 & 0 & 1 & 0 & 0 & & & & & & & \\
 & & & & & & & & 0 & 0 & 0 & 0 & & & & & & & \\
 & & & & & & & & 0 & 1 & 0 & 0 & 0 & & & & & & \\
 & & & & & & & & & 1 & 0 & 1 & 1 & & & & & & \\
 & & & & & & & & & 0 & 0 & 1 & 1 & 0 & & & & & \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & & & & & \\
 & & & & & & & & & & 0 & 1 & 1 & 0 & 0 & & & & \\
 & & & & & & & & & & & 1 & 0 & 1 & 1 & & & & \\
 & & & & & & & & & & & 0 & 1 & 1 & 1 & 0 & & & \\
 & & & & & & & & & & & & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & & & & & & & \vdots & & & & &
\end{array}
$$

The coefficients, starting to the right of the binary point, is again the sequence

$$0010111\ 0010111\ \cdots.$$

B.2.2 Greatest common divisor

Theorem B.3 (Extended Euclidean Algorithm). [8]

Let $\mathsf{r}_1(x)$ *and* $\mathsf{r}_2(x)$ *be polynomials. The following algorithm computes their greatest common divisor* $\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$*, and factors* $\mathsf{a}(x)$ *and* $\mathsf{b}(x)$ *such that*

$$\mathsf{r}_1(x)\mathsf{a}(x) + \mathsf{r}_2(x)\mathsf{b}(x) = \gcd(\mathsf{r}_1, \mathsf{r}_2)$$

THM

n	*remainder* $\mathsf{r}_n = \mathsf{r}_{n-2} - \mathsf{q}_n\mathsf{r}_{n-1}$	*quotient* q_n	*factor* $\alpha_n = a_{n-2} - \mathsf{q}_n\alpha_{n-1}$	*factor* $\beta_n = b_{n-2} - \mathsf{q}_n\beta_{n-1}$
1	$\mathsf{r}_1(x)$	–	1	0
2	$\mathsf{r}_2(x)$	–	0	1
3	$\mathsf{r}_1 - q_3\mathsf{r}_2$	q_3	1	$-q_3$
4	$\mathsf{r}_2 - q_4 r_3$	q_4	$-\mathsf{q}_4$	$1 + \mathsf{q}_4\mathsf{q}_1$
5	$\mathsf{r}_1 - \mathsf{q}_5\mathsf{r}_2$	q_5	$1 + \mathsf{q}_5\mathsf{q}_4$	$-q_3 - \mathsf{q}_5(1 + \mathsf{q}_4\mathsf{q}_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$	q_n	$\mathsf{a}(x) = a_{n-2} - q_n\alpha_{n-1}$	$\mathsf{b}(x) = b_{n-2} - q_n\beta_{n-1}$
$n+1$	0	q_{n+1}		

✎PROOF:

$$\begin{aligned}\mathsf{r}_1 &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3\\ &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3\end{aligned}$$

✏

Example B.5. Let

$$\mathsf{u}(x) \triangleq (1-x)^2 \qquad \mathsf{v}(x) \triangleq x^2.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^2}_{\mathsf{u}(x)}\underbrace{(1+2x)}_{\mathsf{a}(x)} + \underbrace{(x^2)}_{\mathsf{v}(x)}\underbrace{(3-2x)}_{\mathsf{b}(x)} = \underbrace{1}_{\gcd}$$

Because $\gcd(\mathsf{u}, \mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$r_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^2 = 1 - 2x + x^2 = \mathsf{u}(x)$	–	1	0
0	$x^2 = \mathsf{v}(x)$	–	0	1
1	$1 - 2x$	1	1	-1
2	$\frac{1}{2}x$	$-\frac{1}{2}x$	$\frac{1}{2}x$	$1 - \frac{1}{2}x$
3	$1 = \gcd(\,(1-x)^2,\ x^2\,)$	-4	$1 + 2x = \mathsf{a}(x)$	$3 - 2x = \mathsf{b}(x)$
4	0	$\frac{1}{2}x$	–	–

✏

8 Wicker (1995) page 53, Fuhrmann (2012) page 11

Example B.6. Let

$$\mathsf{u}(x) \triangleq (1-x)^3 \qquad \mathsf{v}(x) \triangleq x^3.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^3}_{\mathsf{u}(x)}\underbrace{(1+3x+6x^2)}_{\mathsf{a}(x)} + \underbrace{(x^3)}_{\mathsf{v}(x)}\underbrace{(10-15x+6x^2)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^3 = 1-3x+3x^2-x^3$	–	1	0
0	x^3	–	0	1
1	$1-3x+3x^2$	-1	1	1
2	$-\frac{1}{3}x+x^2$	$\frac{1}{3}x$	$-\frac{1}{3}x$	$1-\frac{1}{3}x$
3	$1-2x$	3	$1+x$	$-2+x$
4	$\frac{1}{6}x$	$-\frac{1}{2}x$	$\frac{1}{6}x+\frac{1}{2}x^2$	$1-\frac{4}{3}y+\frac{1}{2}x^2$
5	$1 = \gcd((1-x)^3, x^3)$	-12	$1+3x+6x^2 = \mathsf{a}(x)$	$10-15x+6x^2 = \mathsf{b}(x)$
6	0	$\frac{1}{6}x$		

✏

Example B.7. Let

$$\mathsf{u}(x) \triangleq (1-x)^4 \qquad \mathsf{v}(x) \triangleq x^4.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^4}_{\mathsf{u}(x)}\underbrace{(1+4x+10x^2+20x^3)}_{\mathsf{a}(x)} + \underbrace{(x^4)}_{\mathsf{v}(x)}\underbrace{(35-84x+70x^2-20x^3)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^4 = 1-4x+6^2-4x^3+x^4$	–	1	0
0	x^4	–	0	1
1	$1-4x+6x^2-4x^3$	1	1	-1
2	$\frac{1}{4}x-x^2+\frac{3}{2}x^3$	$-\frac{1}{4}x$	$\frac{1}{4}x$	$1-\frac{1}{4}x$
3	$1-\frac{10}{3}x+\frac{10}{3}x^2$	$-\frac{8}{3}$	$1+\frac{2}{3}x$	$\frac{5}{3}-\frac{2}{3}x$
4	$-\frac{1}{5}x+\frac{1}{2}x^2$	$\frac{3}{2}\cdot\frac{3}{10}x$	$-\frac{1}{5}x-\frac{3}{10}x^2$	$1-x+\frac{3}{10}x^2$
5	$1-2x$	$\frac{20}{3}$	$1+2x+2x^2$	$-5+6x-2x^2$
6	$\frac{1}{20}x$	$-\frac{1}{4}x$	$\frac{1}{20}x+\frac{1}{5}x^2+\frac{1}{2}x^3$	$1-\frac{9}{4}x+\frac{18}{10}x^2-\frac{1}{2}x^3$
7	$1 = \gcd((1-x)^4, x^4)$	-40	$1+4x+10x^2+20x^3$	$35-84x+70x^2-20x^3$
8	0	$\frac{1}{20}x$	–	–

✏

❝*Infinitesimal analysis was considered so attractive and important because of its numerous and useful applications; as such, it attracted upon itself all research attention and efforts. Concurrently, algebraic analysis appeared to be a field where nothing remained to be done, or where whatever remained to be done would have only been worthless speculation. …Nevertheless, the major contributors to infinitesimal analysis are well aware of the need to improve algebraic analysis: Their own progress depends upon it.*❞

Étienne Bézout, 1779 [9]

Theorem B.4 (Bézout's Identity). [10] [11] *Let* $\mathsf{p}_1(x)$ *be a polynomial of degree* n_1 *and* $\mathsf{p}_2(x)$ *be a polynomial of degree* n_2.

THM

$$\underbrace{\gcd(\mathsf{p}_1(x),\ \mathsf{p}_2(x)) = 1}_{\mathsf{p}_1(x) \text{ and } \mathsf{p}_2(x) \text{ are relatively prime}} \implies \left\{\begin{array}{ll} 1. & \exists \mathsf{q}_1(x), \mathsf{q}_2(x) \quad \text{such that} \\ & \begin{array}{ccccc} \text{degree } n_2 - 1 & & \text{degree } n_1 - 1 & & \\ \downarrow & & \downarrow & & \\ \mathsf{p}_1(x)\mathsf{q}_1(x) & + & \mathsf{p}_2(x)\mathsf{q}_2(x) & = & 1 \\ \uparrow & & \uparrow & & \\ \text{degree } n_1 & & \text{degree } n_2 & & \end{array} \\ 2. & \text{order of } \mathsf{q}_1(x) = n_2 - 1 \\ 3. & \text{order of } \mathsf{q}_2(x) = n_1 - 1 \end{array}\right.$$

✎PROOF: No proof at this time. ✏

B.3 Roots

❝*Neither the true nor the false roots are always real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation* $x^3 - 6x^2 + 13x - 10 = 0$ *as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remain always imaginary.*❞

René Descartes (1596–1650), French philosopher and mathematician[12]

---

[9] quote: Bézout (1779a)
translation: Bézout (1779b) page xv
image: http://en.wikipedia.org/wiki/File:Etienne_Bezout2.jpg, public domain

[10] Bourbaki (2003b) page 2 ⟨Theorem 1 Chapter VII⟩, Fuhrmann (2012) pages 15–17 ⟨Corollary 1.31, Corollary 1.38⟩, Adhikari and Adhikari (2003) page 182, Warner (1990) page 381, Daubechies (1992) page 169, Mallat (1999) page 250

[11] Historical information: Bézout (1779a) ⟨???⟩, Bézout (1779b) ⟨???⟩, Bachet (1621) ⟨???⟩, Childs (2009) pages 37–46 ⟨some history on page 46⟩, http://serge.mehl.free.fr/chrono/Bachet.html, http://serge.mehl.free.fr/chrono/Bezout.html

[12] quote: Descartes (1637a)
English: Descartes (1954) page 175
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

Theorem B.5 (Fundamental Theorem of Algebra).[13] *Let* $\mathsf{p}(x)$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

THM
$$\{\text{degree of } \mathsf{p}(x) \text{ is } N\} \implies \left\{ \begin{array}{l} \underbrace{\exists \, (\!(x_n)\!)_1^N \text{ such that } \mathsf{p}(x_n) = 0 \text{ for } n = 1, 2, \ldots, N}_{\mathsf{p}(x) \text{ has } N \text{ zeros}} \\ \text{where } x_n \text{ and } x_m \text{ are not necessarily distinct for } n \neq m. \end{array} \right\}$$

Corollary B.1. *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

COR
$$\left\{ \underbrace{\begin{array}{l} \text{There exists } (\!(x_n)\!)_1^N \\ \text{such that } \mathsf{p}(x_n) = 0 \text{ for } n = 0, 1, \ldots, N \\ \text{and where } x_n \text{ and } x_m \text{ are} \\ \text{not necessarily distinct for } n \neq m. \end{array}}_{N \text{ zeros of } \mathsf{p}(x)} \right\} \implies \left\{ \mathsf{p}(x) = \left(\frac{\alpha_0}{\prod_{n=1}^{N}(-x_n)} \right) \underbrace{\prod_{n=1}^{N}(x - x_n)}_{N \text{ factors}} \right\}$$

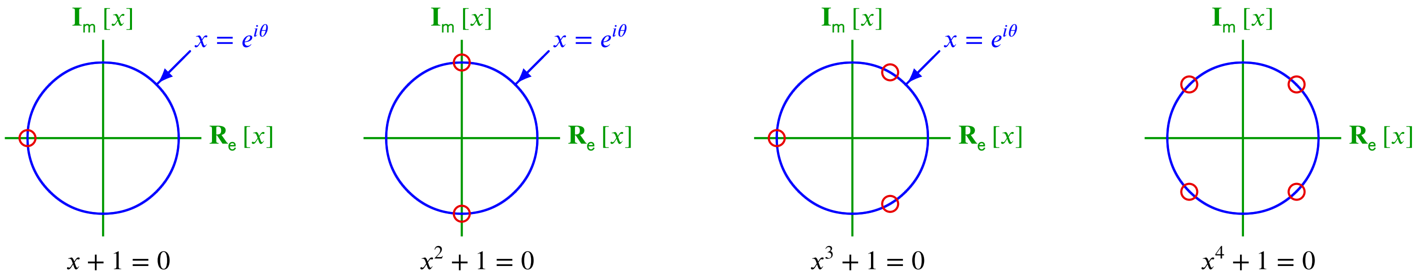


Figure B.1: Roots of $x^n + 1 = 0$

Lemma B.1.

LEM
$$x^N + 1 = 0 \implies x \in \left\{ e^{i\theta_n} | \theta_n = \frac{\pi}{N}(2n+1), n = 0, 1, \ldots, N-1 \right\}$$

✎PROOF:

$$\begin{aligned} e^{iN\theta_n - i2\pi n} &= -1 && n \in \mathbb{Z} \\ N\theta_n - 2\pi n &= \pi && n = 0, 1, \ldots, N-1 \\ N\theta_n &= 2\pi n + \pi \\ \theta_n &= \frac{\pi}{N}(2n+1) \end{aligned}$$

✏

Theorem B.6. *Let* $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \leq n \leq N\}$ *and* $\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$.

THM
$$\underbrace{\alpha_n = \alpha^*_{-n} \quad \forall n \in I}_{(\!(\alpha_n)\!) \text{ is Hermitian symmetric}} \iff \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C}$$

✎PROOF:

[13] Prasolov (2004) pages 1–2 ⟨Section 1.1.1⟩, Borwein and Erdélyi (1995) page 11 ⟨Theorem 1.2.1⟩

1. Proof that $\alpha_n = \alpha^*_{-n} \implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\mathsf{p}(x) &\triangleq \sum_{n=-N}^{N} \alpha_n x^n && \text{by definition of } \mathsf{p}(x)\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha_{-n} x^{-n}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha^*_n x^{-n} && \text{by left hypothesis}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n x^{-n} + \sum_{n=1}^{N} \alpha_n x^n\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x}\right)^{-n}\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x^*}\right)^{-n}\right]^*\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha_{-n} \left(\frac{1}{x^*}\right)^{-n}\right]^* && \text{by left hypothesis}\\
&= \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^*\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by definition of } \mathsf{p}(x)
\end{aligned}$$

2. Proof that $\alpha_n = \alpha^*_{-n} \impliedby \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\sum_{n=-N}^{N} \alpha_n x^n &\triangleq \mathsf{p}(x) && \text{by definition of } \mathsf{p}(x)\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by right hypothesis}\\
&\triangleq \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^* && \text{by definition of } \mathsf{p}(x)\\
&= \sum_{n=-N}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n\\
&= \sum_{n=-N}^{N} \alpha^*_{-n} x^n && \text{by symmetry of summation indices}\\
\implies \quad \alpha_n &= \alpha^*_{-n} && \text{by matching of polynomial coefficients}
\end{aligned}$$

⊫

Theorem B.7. *Let $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \le n \le N\}$ and*

$$\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$$

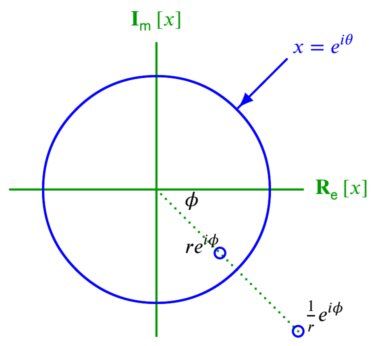


Figure B.2: Recipricol conjugate zero pairs

THM

$$\underbrace{\alpha_n = \alpha^*_{-n} \quad \forall n \in I}_{(\!(\alpha_n)\!) \text{ is Hermitian symmetric}} \implies \underbrace{\left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]}_{\text{roots occur in conjugate reciprical pairs}}$$

✎PROOF:

$$\begin{aligned}
&\alpha_n = \alpha^*_{-n} \quad \forall n \in I && \text{by left hypothesis}\\
&\implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C} && \text{by Theorem B.6 page 101}\\
&\implies \left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]
\end{aligned}$$

If σ is a zero of $\mathsf{p}(x)$, then so is $\frac{1}{\sigma^*}$ because

$$\mathsf{p}\left(\frac{1}{\sigma^*}\right) = \mathsf{p}^*(\sigma) = 0^* = 0.$$

✏

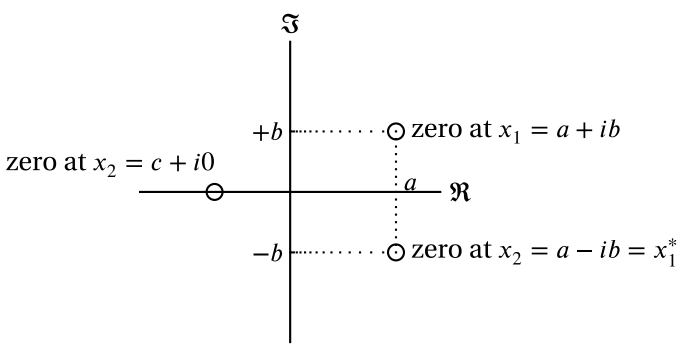


Figure B.3: Conjugate pairs of roots

Theorem B.8 page 103 (next) states that the roots of real polynomials occur in complex conjugate pairs. This is illustrated in Figure B.3.

Theorem B.8. [14] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a Nth order polynomial.*

[14] Korn and Korn (1968) page 17

T H M $$\left[\underbrace{(\alpha_n \in \mathbb{R})_{n=0,1,\ldots,N}}_{\text{coefficients are real}}\right] \implies \left[\underbrace{\mathsf{p}(x_0) = 0 \iff \mathsf{p}(x_0^*) = 0}_{\text{zeros occur in conjugate pairs}}\right]$$

Theorem B.9 (Routh-Hurwitz Criterion). [15] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$ *and*

$$d_0 \triangleq \alpha_0 \qquad d_1 \triangleq \alpha_1 \qquad d_2 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} \qquad d_3 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} \qquad d_4 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 \end{vmatrix}$$

$$d_n \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ \alpha_{2n-3} & \alpha_{2n-4} & \cdots & \alpha_{n-2} \\ \alpha_{2n-1} & \alpha_{2n-2} & \cdots & \alpha_n \end{vmatrix}$$

Let $\mathbf{S}(\!(x_n)\!)$ *be the number of sign changes of some sequence* $(\!(x_n)\!)$ *after eliminating all zero elements* ($x_n = 0$).

T H M
$$\begin{aligned} \underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Re[x_n] > 0\right\}\right|}_{\text{number of roots in right half plane}} &= \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ d_1 d_2,\ d_2 d_3,\ \ldots,\ d_{p-2} d_{p-1},\ \alpha_p\right)\!\right)}_{\text{number of sign changes}} \\ &= \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ \frac{d_2}{d_1},\ \frac{d_3}{d_2},\ \ldots,\ \frac{d_p}{d_{p-1}}\right)\!\right)}_{\text{number of sign changes}} \end{aligned}$$

Theorem B.10 (Descartes rule of signs). [16] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$.

T H M $$\underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Re[x_n] > 0\right\},\ \Im[x_n] = 0\right|}_{\text{number of roots on right real axis}} = \underbrace{\mathbf{S}(\!(\alpha_n)\!) - 2m}_{\text{number of sign changes – even integer}} \qquad \text{where } m \in \mathbb{W}$$

Theorem B.11. [17] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$.

T H M $$\underbrace{\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1}\ \geq 0}_{\text{first } k \text{ coefficients are nonnegative}} \implies \left\{ \begin{array}{l} \underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Im[x_n] = 0\right\}\right|}_{\text{number of real roots}} < \underbrace{1 + \left(\frac{q}{\alpha_0}\right)^{\frac{1}{k}}}_{\text{upper bound}} \\ \text{where } q \triangleq \underbrace{\max\left\{\ |\alpha_n|\ |\alpha_n < 0\right\}}_{\text{largest negative coefficient}} \end{array} \right.$$

Theorem B.12 (Rolle's Theorem). [18] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$. *The number of real zeros of* $\mathsf{p}'(x)$ *between any two real consecutive real zeros of* $\mathsf{p}(x)$ *is* ***odd***.

Definition B.6. [19] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

D E F $\dfrac{\mathsf{p}(x)}{\mathsf{q}(x)}$ *is a* ***rational function***
if $\mathsf{p}(x)$ *and* $\mathsf{q}(x)$ *are* POLYNOMIALS *over* $(\mathbb{F}, +, \cdot, 0, 1)$.

[15] Korn and Korn (1968) page 17
[16] Korn and Korn (1968) page 17
[17] Korn and Korn (1968) page 18
[18] Korn and Korn (1968) page 18
[19] Fuhrmann (2012) page 22

Example B.8.

EX

An example of a rational function using polynomials in x^{-1} is

$$A(x) = \frac{b_0 + \beta_1 x^{-1} + \beta_2 x^{-2} + \beta_3 x^{-3}}{1 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \alpha_3 x^{-3}}$$

This can be expressed as a rational function using polynomials in x by multiplying numerator and denominator by x^3:

$$A(x) = \frac{x^3}{x^3} A(x) = \frac{b_0 x^3 + \beta_1 x^2 + \beta_2 x + \beta_3}{x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3}$$

Definition B.7.

DEF

*The **zeros** of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $B(x)$.

*The **poles** of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $A(x)$.

B.4 Polynomial expansions

❝*Thus, if a straight-line is cut at random, then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces.*❞

Euclid (~300BC), Greek mathematician, demonstrating the *Binomial theorem* for exponent $n = 2$ as in $(x+y)^2 = x^2 + 2xy + y^2$. [20]

Theorem B.13 (Taylor Series). [21] *Let* $\mathbf{C}$ *be the space of all* ANALYTIC *functions and* $\frac{\mathrm{d}}{\mathrm{dx}}$ *in* $\mathbf{C}^{\mathbf{C}}$ *the* DIFFERENTIATION OPERATOR.

THM

*A **Taylor series** about the point* a *of a function* $\mathsf{f} \in \mathbf{C}^{\mathbf{C}}$ *is* $\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\left[\frac{\mathrm{d}}{\mathrm{dx}}^n \mathsf{f}\right](a)}{n!}(x-a)^n \qquad \forall a \in \mathbb{R}, \mathsf{f} \in \mathbf{C}$

*A **Maclaurin series** is a* TAYLOR SERIES *about the point* $a = 0$.

Theorem B.14 (Binomial Theorem). [22]

THM

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad \textit{where} \qquad \binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$$

✎PROOF: This theorem is proven using two different techniques. Either is sufficient. The first requires the Maclaurin series resulting in a more compact proof, but requires the additional (here unproven) Maclaurin series. The second proof uses induction resulting in a longer proof, but does not require any external theorem.

[20] quote: Euclid (circa 300BC) ⟨Book II, Proposition 4⟩, Coolidge (1949) page 147
image: http://commons.wikimedia.org/wiki/File:Euklid-von-Alexandria_1.jpg, public domain

[21] Flanigan (1983) page 221 ⟨Theorem 15⟩, Strichartz (1995) page 281, Sohrab (2003) page 317 ⟨Theorem 8.4.9⟩, Taylor (1715), Maclaurin (1742)

[22] Graham et al. (1994) page 162 ⟨(5.12)⟩, Rotman (2010) page 84 ⟨Proposition 2.5⟩, Bourbaki (2003a) page 99 ⟨Corollary 1⟩, Warner (1990) pages 189–190 ⟨Theorem 21.1⟩, Metzler et al. (1908) page 169 ⟨any real exponent⟩, Coolidge (1949)

1. Proof using Maclaurin series:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}y^k}\Big[(x+y)^n\Big]_{y=0} y^k && \text{by Maclaurin series (Theorem B.13 page 105)}\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\Big[n(n-1)(n-2)\cdots(n-k+1)(x+y)^{n-k}\Big]_{y=0} y^k\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{n!}{(n-k)!} x^{n-k}y^k\\
&= \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k}y^k && \text{by definition of } \binom{n}{k}\\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k + \cancelto{0}{\sum_{k=n+1}^{\infty} \binom{n}{k} x^{n-k}y^k}\\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k && \text{because } (x+y)^n \text{ has order } n
\end{aligned}$$

2. Proof using induction:

(a) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ is true for $n=0$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right|_{n=0} &= \binom{0}{0}x^0y^{0-0}\\
&= 1\\
&= (x+y)^n|_{n=0}
\end{aligned}$$

(b) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ is true for $n=1$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right|_{n=1} &= \binom{1}{0}x^0y^{1-0} + \binom{1}{1}x^1y^{1-1}\\
&= y+x\\
&= (x+y)^n|_{n=1}
\end{aligned}$$

(c) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \implies (x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n}{k}x^k y^{n+1-k}$:

$$\begin{aligned}
&\sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n+1-k}\\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n+1-k}\\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]x^k y^{n+1-k} && \text{by \textit{Pascal's Rule}}\\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}x^k y^{n+1-k} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n+1-k}\\
&= x^{n+1} + y^{n+1} + \left[\sum_{k=0}^{n}\binom{n}{k}x^{k+1} y^{n+1-(k+1)} - x^{n+1}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k} - y^{n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= x \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= x(x+y)^n + y(x+y)^n && \text{by left hypothesis} \\
&= (x+y)(x+y)^n \\
&= (x+y)^{n+1}
\end{aligned}
$$

✏

APPENDIX C

CALCULUS

Definition C.1. *Let $\mathbb{R}$ be the set of real numbers, $\mathscr{B}$ the set of* BOREL SETS *on $\mathbb{R}$, and μ the standard* BOREL MEASURE *on $\mathscr{B}$. Let $\mathbb{R}^{\mathbb{R}}$ be as in Definition 4.1 page 53.*

DEF

*The **space of Lebesgue square-integrable functions** $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ (or $\boldsymbol{L}^2_{\mathbb{R}}$) is defined as*

$$\boldsymbol{L}^2_{\mathbb{R}} \triangleq \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \mathrm{d}\mu < \infty \right\}.$$

*The **standard inner product** $\langle \triangle \mid \nabla \rangle$ on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\,\mathrm{d}x.$$

*The **standard norm** $\|\cdot\|$ on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$*

Definition C.2. *Let $\mathsf{f}(x)$ be a* FUNCTION *in $\mathbb{R}^{\mathbb{R}}$.*

DEF

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon\to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition C.1.

PRP

$$\left\{ \begin{array}{lll} (1). & \mathsf{f}(x) \text{ is CONTINUOUS} & \text{and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC about a point } a} & \end{array} \right\} \implies \left\{ \begin{array}{lrcll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) & \text{(ANTI-SYMMETRIC about } a) \\ (2). & \mathsf{f}'(a) & = & 0 & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{f}'(a+x) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\
&= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\
&= -\mathsf{f}(a-x)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\
&= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result}
\end{aligned}$$

$$= 0$$

✏

Lemma C.1.

LEM $\mathsf{f}(x)$ *is* INVERTIBLE $\implies \left\{ \frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) = \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to 0} \frac{\mathsf{f}^{-1}(y+\varepsilon) - \mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}y} && \text{(Definition C.2 page 109)} \\
&= \lim_{\delta\to 0} \left.\frac{1}{\left[\frac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1} \\
&\triangleq \left.\frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x)}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition C.2 page 109)} \\
&= \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✏

Theorem C.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

THM $$\int_{[0:1)^n} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n} x_k\right) \mathsf{d}x_1\,\mathsf{d}x_2 \cdots \mathsf{d}x_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n \in \mathbb{N}$$

✎PROOF: Proof by induction:

1. Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\int_{[0:1)} \mathsf{f}^{(1)}(x)\,\mathsf{d}x &= \mathsf{f}(1) - \mathsf{f}(0) && \text{by \textit{Fundamental theorem of calculus}} \\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1) + (-1)^{1+0}\binom{1}{0}\mathsf{f}(0) \\
&= \sum_{k=0}^{1}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

2. Induction step …proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \,\mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_{n+1} \\
&= \int_{[0:1)^n} \left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right) \mathrm{d}x_{n+1}\right] \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right] \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n && \text{by \textit{Fundamental theorem of calculus}} \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(1 + \sum_{k=1}^{n} x_k\right) - \mathsf{f}^{(n)}\left(0 + \sum_{k=1}^{n} x_k\right)\right] \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1) - \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by induction hypothesis} \\
&= \sum_{m=1}^{m=n+1}(-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m) + \sum_{k=0}^{n}(-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{where } m \triangleq k+1 \implies k = m-1 \\
&= \left[\mathsf{f}(n+1) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right] + \left[(-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right] \\
&= \mathsf{f}(n+1) + (-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\text{use \textit{Stifel formula}}}\mathsf{f}(k) \\
&= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1) + (-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by \textit{Stifel formula}} \\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)
\end{aligned}$$

✎

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma C.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots,$ *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example C.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\big[\mathsf{f}(x)\mathsf{g}(x)\big] = \mathsf{f}'''(x)\mathsf{g}(x) + 3\mathsf{f}''(x)\mathsf{g}'(x) + 3\mathsf{f}'(x)\mathsf{g}''(x) + \mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem C.2 (Leibniz integration rule). [3]

THM
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathsf{a}(x)}^{\mathsf{b}(x)} \mathsf{g}(t)\,\mathrm{d}t = \mathsf{g}[\mathsf{b}(x)]\mathsf{b}'(x) - \mathsf{g}[\mathsf{a}(x)]\mathsf{a}'(x)$$

[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)

[3] Flanders (1973) page 615 ⟨(1.1)⟩ Talvila (2001), Knapp (2005b) page 389 ⟨Chapter VII⟩, ? page 422 ⟨Leibniz Rule. Theorem 1.⟩, http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

APPENDIX D

Z TRANSFORM

D.1 Convolution operator

Definition D.1. [1] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF: *A function f in X^Y is a **sequence** over X if $Y = \mathbb{Z}$.*
A sequence may be denoted in the form $(\!(x_n)\!)_{n\in\mathbb{Z}}$ or simply as $(\!(x_n)\!)$.

Definition D.2. [2] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a* FIELD *(Definition ?? page ??).*

DEF: *The **space of all absolutely square summable sequences** $\boldsymbol{\ell}^2_{\mathbb{F}}$ over $\mathbb{F}$ is defined as*

$$\boldsymbol{\ell}^2_{\mathbb{F}} \triangleq \left\{ (\!(x_n)\!)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}$$

The space $\boldsymbol{\ell}^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is isomorphic to $\boldsymbol{L}^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition D.3.

DEF: *The **convolution** operation $\star$ is defined as*

$$(\!(x_n)\!) \star (\!(y_n)\!) \triangleq \left(\!\!\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m} \right)\!\!\right)_{n\in\mathbb{Z}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$$

Proposition D.1. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition D.3 page 113).*

PRP: $$(\!(x_n)\!) \star (\!(y_n)\!) = (\!(y_n)\!) \star (\!(x_n)\!) \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

[1] Bromwich (1908) page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[2] Kubrusly (2011) page 347 ⟨Example 5.K⟩

✎PROOF:

$$
\begin{aligned}
[x \star y](n) &\triangleq \sum_{m\in\mathbb{Z}} x_m y_{n-m} && \text{by Definition D.3 page 113}\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{where } k \triangleq n-m \implies m = n-k\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{by \textit{commutativity} of addition}\\
&= \sum_{m\in\mathbb{Z}} x_{n-m} y_m && \text{by change of variables}\\
&= \sum_{m\in\mathbb{Z}} y_m x_{n-m} && \text{by commutative property of the field over } \mathbb{C}\\
&\triangleq (y \star x)_n && \text{by Definition D.3 page 113}
\end{aligned}
$$

⇰

Proposition D.2. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition D.3 page 113). Let $\ell^2_{\mathbb{R}}$ be the set of* ABSOLUTELY SUMMABLE *sequences (Definition D.2 page 113).*

PRP
$$
\left\{\begin{array}{llll} \text{(A).} & \mathsf{x}(n) & \in & \ell^2_{\mathbb{R}} \quad \text{and} \\ \text{(B).} & \mathsf{y}(n) & \in & \ell^2_{\mathbb{R}} \end{array}\right\} \implies \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] = \mathsf{x}[-n] \star \mathsf{y}(n) \right\}
$$

✎PROOF:

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \implies k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by \textit{absolutely summable} hypothesis} && \text{(Definition D.2 page 113)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \implies \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of \textit{convolution} } \star && \text{(Definition D.3 page 113)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n]
\end{aligned}
$$

⇰

D.2 Z-transform

Definition D.4. [3]

DEF *The **z-transform** $\mathbf{Z}$ of $(x_n)_{n\in\mathbb{Z}}$ is defined as*

$$
\left[\mathbf{Z}\,(x_n)\right](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\text{Laurent series}} \qquad \forall (x_n)\in\ell^2_{\mathbb{R}}
$$

Theorem D.1. *Let $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ be the* Z-TRANSFORM *of $\mathsf{x}[n]$.*

THM
$$
\left\{ \check{\mathsf{x}}(z) \triangleq \mathbf{Z}\,(\mathsf{x}[n]) \right\} \implies \left\{\begin{array}{lrcll}
\text{(1).} & \mathbf{Z}\,(\alpha\mathsf{x}[n]) & = & \alpha\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(2).} & \mathbf{Z}\,(\mathsf{x}[n-k]) & = & z^{-k}\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(3).} & \mathbf{Z}\,(\mathsf{x}[-n]) & = & \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(4).} & \mathbf{Z}\,(\mathsf{x}^*[n]) & = & \check{\mathsf{x}}^*\left(z^*\right) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(5).} & \mathbf{Z}\,(\mathsf{x}^*[-n]) & = & \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}}
\end{array}\right\}
$$

[3] *Laurent series*: Abramovich and Aliprantis (2002) page 49

✎PROOF:

$$\begin{aligned}
\alpha\mathbb{Z}\check{\mathsf{x}}(z) &\triangleq \alpha\mathbf{Z}\left(\!\left(\mathsf{x}[n]\right)\!\right) && \text{by definition of } \check{\mathsf{x}}(z)\\
&\triangleq \alpha\sum_{n\in\mathbb{Z}}\mathsf{x}[n]z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator}\\
&\triangleq \sum_{n\in\mathbb{Z}}(\alpha\mathsf{x}[n])z^{-n} && \text{by \textit{distributive} property}\\
&\triangleq \mathbf{Z}\left(\!\left(\alpha\mathsf{x}[n]\right)\!\right) && \text{by definition of } \mathbf{Z} \text{ operator}\\
z^{-k}\check{\mathsf{x}}(z) &= z^{-k}\mathbf{Z}\left(\!\left(\mathsf{x}[n]\right)\!\right) && \text{by definition of } \check{\mathsf{x}}(z) && \text{(left hypothesis)}\\
&\triangleq z^{-k}\sum_{n=-\infty}^{n=+\infty}\mathsf{x}[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
&= \sum_{n=-\infty}^{n=+\infty}\mathsf{x}[n]z^{-n-k}\\
&= \sum_{m-k=-\infty}^{m-k=+\infty}\mathsf{x}[m-k]z^{-m} && \text{where } m\triangleq n+k && \implies n=m-k\\
&= \sum_{m=-\infty}^{m=+\infty}\mathsf{x}[m-k]z^{-m}\\
&= \sum_{n=-\infty}^{n=+\infty}\mathsf{x}[n-k]z^{-n} && \text{where } n\triangleq m\\
&\triangleq \mathbf{Z}\left(\!\left(\mathsf{x}[n-k]\right)\!\right) && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
\mathbf{Z}\left(\!\left(\mathsf{x}^*[n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}}\mathsf{x}^*[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
&\triangleq \left(\sum_{n\in\mathbb{Z}}\mathsf{x}[n](z^*)^{-n}\right)^* && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
&\triangleq \check{\mathsf{x}}^*(z^*) && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
\mathbf{Z}\left(\!\left(\mathsf{x}[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}}\mathsf{x}[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
&= \sum_{-m\in\mathbb{Z}}\mathsf{x}[m]z^{m} && \text{where } m\triangleq -n && \implies n=-m\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition D.2 page 113)}\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition D.2 page 113)}\\
&\triangleq \check{\mathsf{x}}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
\mathbf{Z}\left(\!\left(\mathsf{x}^*[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}}\mathsf{x}^*[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
&= \sum_{-m\in\mathbb{Z}}\mathsf{x}^*[m]z^{m} && \text{where } m\triangleq -n && \implies n=-m\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}^*[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition D.2 page 113)}\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}^*[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition D.2 page 113)}\\
&= \left(\sum_{m\in\mathbb{Z}}\mathsf{x}[m]\left(\frac{1}{z^*}\right)^{-m}\right)^* && \text{by \textit{absolutely summable} property} && \text{(Definition D.2 page 113)}
\end{aligned}$$

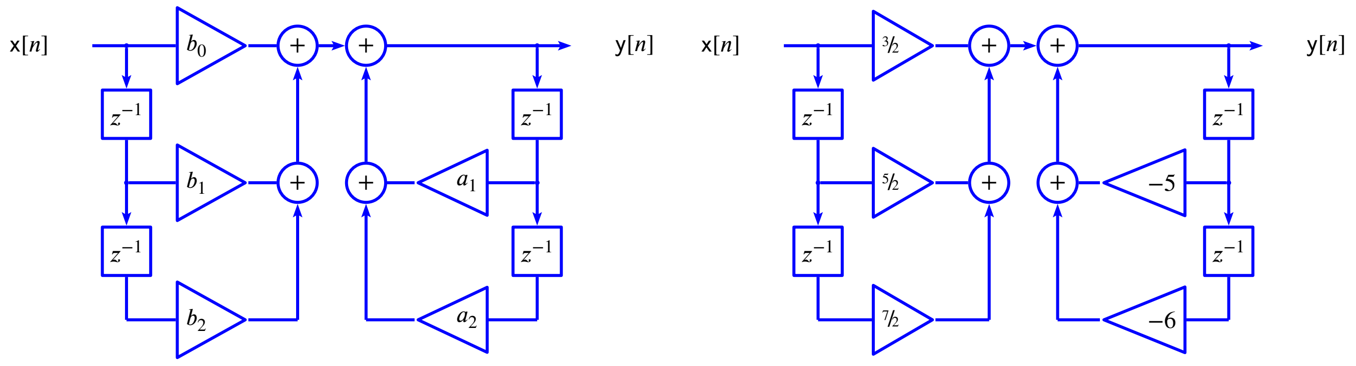


Figure D.1: Direct form 1 order 2 IIR filters

$$\triangleq \check{x}^*\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition D.4 page 114)}$$

Theorem D.2 (convolution theorem)**.** *Let $\star$ be the convolution operator* (Definition D.3 page 113)*.*

THM

$$\underbrace{\mathbf{Z}\left(\langle\!\langle x_n\rangle\!\rangle \star \langle\!\langle y_n\rangle\!\rangle\right)}_{\text{sequence convolution}} = \underbrace{\left(\mathbf{Z}\,\langle\!\langle x_n\rangle\!\rangle\right)\left(\mathbf{Z}\,\langle\!\langle y_n\rangle\!\rangle\right)}_{\text{series multiplication}} \qquad \forall \langle\!\langle x_n\rangle\!\rangle_{n\in\mathbb{Z}}, \langle\!\langle y_n\rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x\star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by definition of } \star && \text{(Definition D.3 page 113)}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k \triangleq n-m && \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right]\\
&\triangleq \left[\mathbf{Z}\,\langle\!\langle x_n\rangle\!\rangle\right]\left[\mathbf{Z}\,\langle\!\langle y_n\rangle\!\rangle\right] && \text{by definition of } \mathbf{Z} && \text{(Definition D.4 page 114)}
\end{aligned}$$

D.3 From z-domain back to time-domain

$$\check{y}(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) - a_1z^{-1}Y(z) + a_2z^{-2}Y(z)$$

$$\mathsf{y}[n] = b_0\mathsf{x}[n] + b_1\mathsf{x}[n-1] + b_2\mathsf{x}[n-2] - a_1\mathsf{y}[n-1] - a_2\mathsf{y}[n-2]$$

Example D.1. See Figure D.1 (page 116)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2\left(z^2+5z+6\right)} = \frac{\left(\tfrac{3}{2}z^2+\tfrac{5}{2}z+\tfrac{7}{2}\right)}{z^2+5z+6} = \frac{\left(\tfrac{3}{2}+\tfrac{5}{2}z^{-1}+\tfrac{7}{2}z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

D.4 Zero locations

The system property of *minimum phase* is defined in Definition D.5 (next) and illustrated in Figure D.2 (page 117).

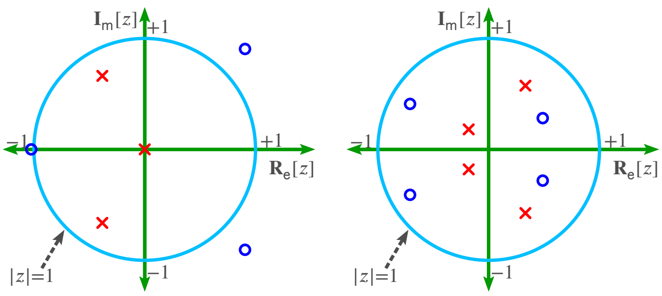


Figure D.2: Minimum Phase filter

Definition D.5. [4] *Let* $\check{x}(z) \triangleq \mathbf{Z}\left(x_n\right)$ *be the* Z TRANSFORM (Definition D.4 page 114) *of a sequence* $\left(x_n\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\left(z_n\right)_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\check{x}(z)$.

DEF: *The sequence* $\left(x_n\right)$ *is* ***minimum phase*** *if*

$$\underbrace{\left|z_n\right| < 1 \qquad \forall n \in \mathbb{Z}}_{\check{x}(z) \text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem D.3 (Robinson's Energy Delay Theorem). [5] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM:

$$\left\{\begin{array}{lll} \mathsf{p} & \text{is MINIMUM PHASE} & \text{and} \\ \mathsf{q} & \text{is NOT } \textit{minimum phase} & \end{array}\right\} \implies \underbrace{\sum_{n=0}^{m-1}\left|a_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{p}(z)} \geq \underbrace{\sum_{n=0}^{m-1}\left|b_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{q}(z)} \qquad \forall 0 \leq m \leq N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle (Figure D.3 page 118).

Example D.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure D.3 (page 118).

[4] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36
[5] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976) pages 52–53

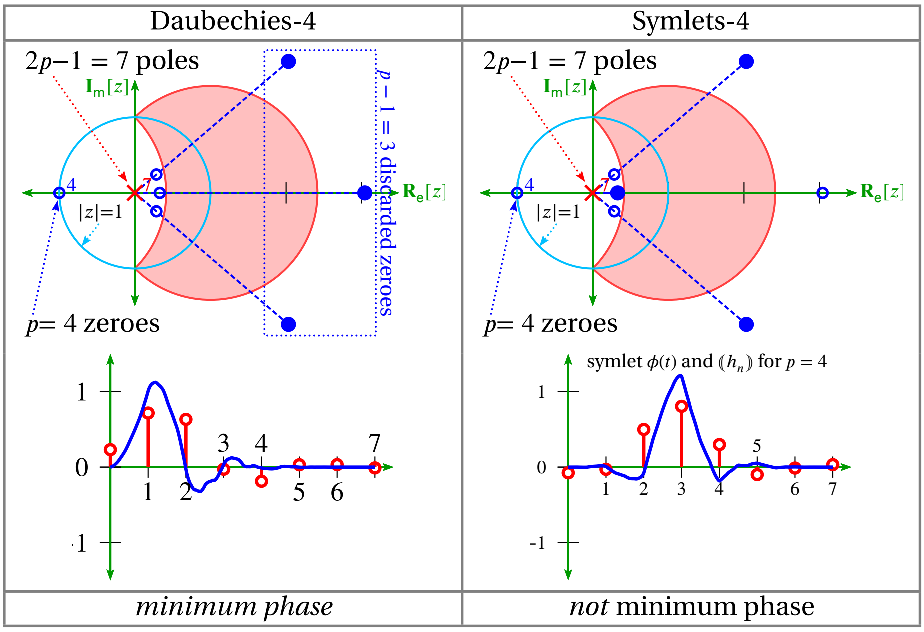


Figure D.3: Daubechies-4 and Symlet-4 scaling functions pole-zero plots

D.5 Pole locations

Definition D.6.

> DEF *A filter (or system or operator)* **H** *is* ***causal***
> *if its current output does not depend on future inputs.*

Definition D.7.

> DEF *A filter (or system or operator)* **H** *is* ***time-invariant***
> *if the mapping it performs does not change with time.*

Definition D.8.

> DEF *An operation* **H** *is* ***linear*** *if any output* y_n *can be described as a linear combination of inputs* x_n *as in*
> $$\mathsf{y}_n = \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\mathsf{x}(n-m)\,.$$

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem D.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Example D.3. Stable/unstable filters are illustrated in Figure D.4 (page 119).

True or False? This filter has no poles:

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

$$\mathsf{H}(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$

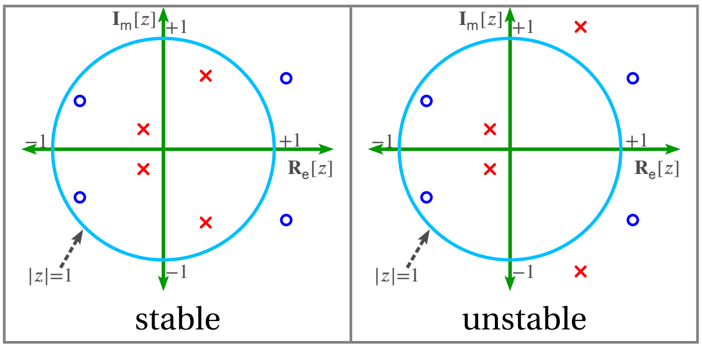


Figure D.4: Pole-zero plot stable/unstable causal LTI filters (Example D.3 page 118)

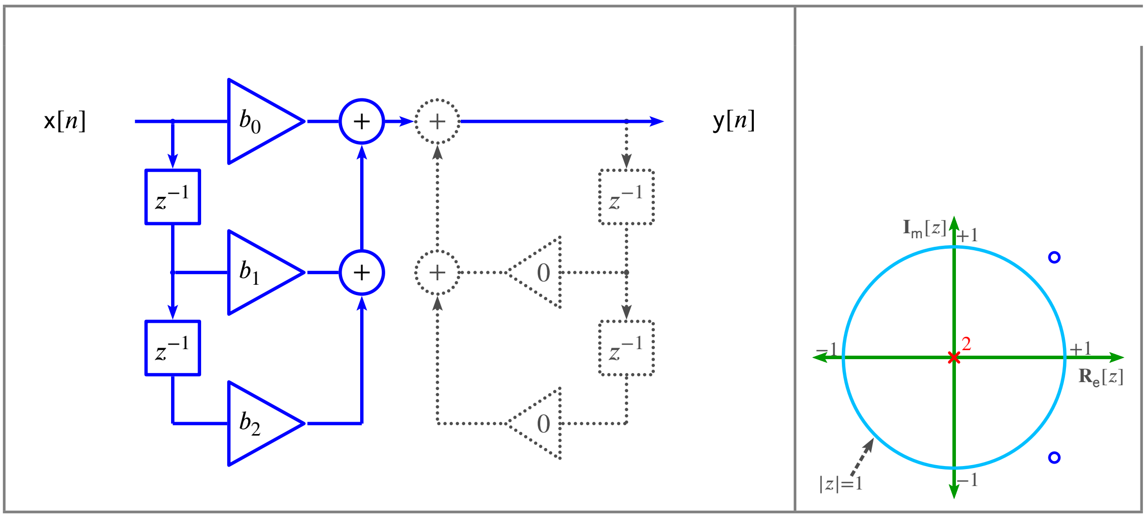


Figure D.5: FIR filters

D.6 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

Proposition D.3.

PRP	$\left\{\begin{array}{l}\text{ZEROS } \textit{and} \text{ POLES}\\ \textit{occur in} \text{ CONJUGATE PAIRS}\end{array}\right\} \implies \left\{\begin{array}{l}\text{COEFFICIENTS}\\ \textit{are} \text{ REAL.}\end{array}\right\}$

✎PROOF:

$$\begin{aligned}\left(z - p_1\right)\left(z - p_1^*\right) &= [z - (a + ib)][z - (a - ib)]\\ &= z^2 + [-a + ib - ib - a]z - [ib]^2\end{aligned}$$

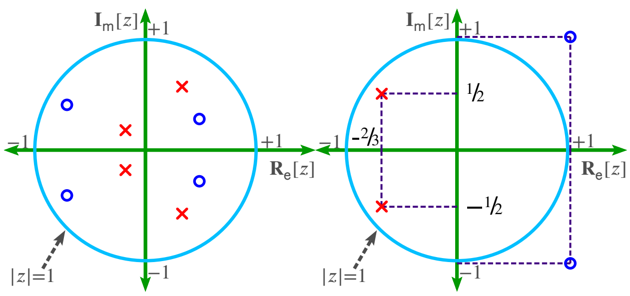


Figure D.6: Conjugate pair structure yielding real coefficients

$$= z^2 - 2az + b^2$$

✏

Example D.4. See Figure D.6 (page 119).

$$\begin{aligned}
H(z) &= G\frac{[z-z_1][z-z_2]}{[z-p_1][z-p_2]} = G\frac{[z-(1+i)][z-(1-i)]}{[z-(-2/3+i1/2)][z-(-2/3-i1/2)]} \\
&= G\frac{z^2 - z[(1-i)+(1+i)] + (1-i)(1+i)}{z^2 - z[(-2/3+i1/2)+(-2/3+i1/2)] + (-2/3+i1/2)(-2/3+i1/2)} \\
&= G\frac{z^2-2z+2}{z^2 - 4/3z + (4/3+1/4)} = G\frac{z^2-2z+2}{z^2-4/3z+19/12}
\end{aligned}$$

D.7 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma D.1. *A causal LTI operator* **H** *can be expressed as a rational expression* $\check{\mathsf{h}}(z)$.

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \frac{\sum\limits_{n=0}^{N} b_n z^{-n}}{1 + \sum\limits_{n=1}^{N} a_n z^{-n}}
\end{aligned}$$

A filter operation $\check{\mathsf{h}}(z)$ can be expressed as a product of its roots (poles and zeros).

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \alpha\frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}
\end{aligned}$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with $\times$ and zeros with $\bigcirc$. An example is shown in Figure D.7 page 121. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

D.8 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{\mathsf{x}}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{\mathsf{y}}(z)$ is a *high-pass filter*.

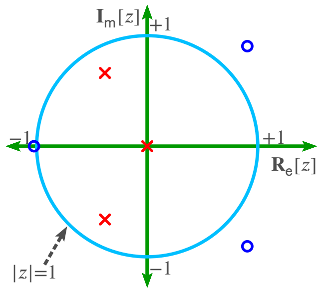


Figure D.7: Pole-zero plot for rational expression with real coefficients

Definition D.9. [6] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES (Definition D.1 page 113) *in* $\ell^2_{\mathbb{R}}$ (Definition D.2 page 113).

DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter***.
Any triple $\left((\!(x_n)\!), (\!(y_n)\!), N\right)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition***.

Theorem D.5 (CQF theorem). [7] *Let* $\check{\mathsf{y}}(\omega)$ *and* $\check{\mathsf{x}}(\omega)$ *be the* DTFT*s* (Definition 6.1 page 75) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$*, respectively, in* $\ell^2_{\mathbb{R}}$ (Definition D.2 page 113).

THM

$$\begin{array}{rcrcll}
\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{\text{(1) CQF in "time"}} & \iff & \check{\mathsf{y}}(z) & = & \pm(-1)^N z^{-N}\check{\mathsf{x}}^*\left(\frac{-1}{z^*}\right) & \text{(2) CQF in "z-domain"}\\
& \iff & \check{\mathsf{y}}(\omega) & = & \pm(-1)^N e^{-i\omega N}\check{\mathsf{x}}^*(\omega+\pi) & \text{(3) CQF in "frequency"}\\
& \iff & x_n & = & \pm(-1)^N(-1)^n y^*_{N-n} & \text{(4) "reversed" CQF in "time"}\\
& \iff & \check{\mathsf{x}}(z) & = & \pm z^{-N}\check{\mathsf{y}}^*\left(\frac{-1}{z^*}\right) & \text{(5) "reversed" CQF in "z-domain"}\\
& \iff & \check{\mathsf{x}}(\omega) & = & \pm e^{-i\omega N}\check{\mathsf{y}}^*(\omega+\pi) & \text{(6) "reversed" CQF in "frequency"}
\end{array}$$

$\forall N\in\mathbb{Z}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{\mathsf{y}}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-transform} && \text{(Definition D.4 page 114)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm\sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies && n = N-m\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*
\end{aligned}$$

[6] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

[7] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

$$= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) \qquad \text{by definition of } z\text{-transform} \quad \text{(Definition D.4 page 114)}$$

2. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\check{y}(z) &= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by (2)}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-transform} \quad \text{(Definition D.4 page 114)}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m^* \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-transform} \quad \text{(Definition D.4 page 114)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x_m^* z^{-(N-m)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{n} x_{N-n}^* z^{-n} && \text{where } n = N-m \implies \quad m \triangleq N-n\\
&\implies x_n = \pm(-1)^n x_{N-n}^*
\end{aligned}$$

3. Proof that (1) $\Longrightarrow$ (3):

$$\begin{aligned}
\check{y}(\omega) &\triangleq \check{x}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition 6.1 page 75)}\\
&= \left[\pm(-1)^N z^{-N}\check{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)}\\
&= \pm(-1)^N e^{-i\omega N}\check{x}\left(e^{i\pi}e^{i\omega}\right)\\
&= \pm(-1)^N e^{-i\omega N}\check{x}\left(e^{i(\omega+\pi)}\right)\\
&= \pm(-1)^N e^{-i\omega N}\check{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition 6.1 page 75)}
\end{aligned}$$

4. Proof that (1) $\Longrightarrow$ (6):

$$\begin{aligned}
\check{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT \quad \text{(Definition 6.1 page 75)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x_{N-n}^*}_{CQF} e^{-i\omega n} && \text{by (1)}\\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x_m^* e^{-i\omega(N-m)} && \text{where } m \triangleq N-n \implies \quad n = N-m\\
&= \pm(-1)^N e^{-i\omega N}\sum_{m\in\mathbb{Z}} (-1)^m x_m^* e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N}\sum_{m\in\mathbb{Z}} e^{i\pi m} x_m^* e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N}\sum_{m\in\mathbb{Z}} x_m^* e^{i(\omega+\pi)m}\\
&= \pm(-1)^N e^{-i\omega N}\left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^*\\
&= \pm(-1)^N e^{-i\omega N}\check{x}^*(\omega+\pi) && \text{by definition of } DTFT \quad \text{(Definition 6.1 page 75)}
\end{aligned}$$

5. Proof that (1) $\Longleftarrow$ (3):

$$\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{y}(\omega)e^{i\omega n}\,\mathrm{d}\omega && \text{by } \textit{inverse DTFT} \quad \text{(Theorem 6.3 page 81)}\\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\breve{x}^*(\omega+\pi)}_{\text{right hypothesis}} e^{i\omega n}\,\mathrm{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{x}^*(\omega+\pi)e^{i\omega(n-N)}\,\mathrm{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{i(v-\pi)(n-N)}\,\mathrm{d}v && \text{where } v \triangleq \omega+\pi \implies \omega = v-\pi\\
&= \pm(-1)^N e^{-i\pi(n-N)}\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{iv(n-N)}\,\mathrm{d}v\\
&= \pm(-1)^N\underbrace{(-1)^N}_{e^{i\pi N}}\underbrace{(-1)^n}_{e^{-i\pi n}}\left[\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}(v)e^{iv(N-n)}\,\mathrm{d}v\right]^*\\
&= \pm(-1)^n x^*_{N-n} && \text{by } \textit{inverse DTFT} \quad \text{(Theorem 6.3 page 81)}
\end{aligned}$$

6. Proof that (1) $\Longleftrightarrow$ (4):

$$\begin{aligned}
y_n = \pm(-1)^n x^*_{N-n} &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x^*_{N-n}\\
&\iff \pm(-1)^n y_n = x^*_{N-n}\\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x^*_{N-n}\right)^*\\
&\iff \pm(-1)^n y^*_n = x_{N-n}\\
&\iff x_{N-n} = \pm(-1)^n y^*_n\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m} && \text{where } m \triangleq N-n \implies n = N-m\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m}\\
&\iff x_m = \pm(-1)^{N}(-1)^m y^*_{N-m}\\
&\iff x_n = \pm(-1)^{N}(-1)^n y^*_{N-n} && \text{by change of free variables}
\end{aligned}$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

⇒

Theorem D.6. [8] *Let $\breve{y}(\omega)$ and $\breve{x}(\omega)$ be the* DTFT*s (Definition 6.1 page 75) of the sequences $(\!(y_n)\!)_{n\in\mathbb{Z}}$ and $(\!(x_n)\!)_{n\in\mathbb{Z}}$, respectively, in $\ell^2_{\mathbb{R}}$ (Definition D.2 page 113).*

THM *Let $y_n = \pm(-1)^n x^*_{N-n}$* (CQF CONDITION, *Definition D.9 page 121*). *Then*

$$\left\{\begin{array}{llcll}
\text{(A)} & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} = 0 & \iff & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} = 0 & \text{(B)}\\
& & \iff & \sum_{k\in\mathbb{Z}}(-1)^k k^n x_k = 0 & \text{(C)}\\
& & \iff & \sum_{k\in\mathbb{Z}} k^n y_k = 0 & \text{(D)}
\end{array}\right\} \quad \forall n\in\mathbb{W}$$

✎PROOF:

[8] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

1. Proof that (A) $\implies$ (B):

$$
\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} && \text{by (A)} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi)\Big|_{\omega=0} && \text{by } \textit{CQF theorem} \quad \text{(Theorem D.5 page 121)} \\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^\ell \left[e^{-i\omega N}\right] \cdot \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma C.2 page 111)} \\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell e^{-i\omega N} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0} \\
&= (\pm)(-1)^N \cancel{e^{-i0N}} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0}
\end{aligned}
$$

$$
\begin{aligned}
&\implies & \breve{x}^{(0)}(\pi) &= 0 \\
&\implies & \breve{x}^{(1)}(\pi) &= 0 \\
&\implies & \breve{x}^{(2)}(\pi) &= 0 \\
&\implies & \breve{x}^{(3)}(\pi) &= 0 \\
&\implies & \breve{x}^{(4)}(\pi) &= 0 \\
&\ \vdots & &\vdots \\
&\implies & \breve{x}^{(n)}(\pi) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}
$$

2. Proof that (A) $\impliedby$ (B):

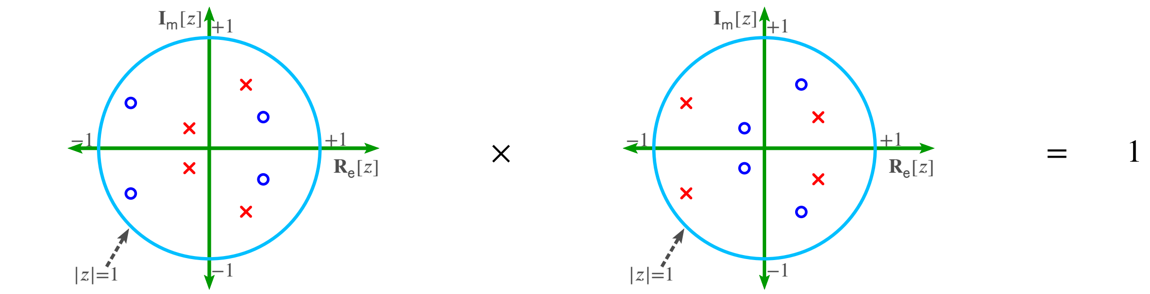
$$
\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} && \text{by (B)} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)e^{-i\omega N}\breve{y}^*(\omega+\pi)\Big|_{\omega=\pi} && \text{by } \textit{CQF theorem} \quad \text{(Theorem D.5 page 121)} \\
&= (\pm) \sum_{\ell=0}^{n} \binom{n}{\ell} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^\ell \left[e^{-i\omega N}\right] \cdot \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma C.2 page 111)} \\
&= (\pm) \sum_{\ell=0}^{n} \binom{n}{\ell} (-iN)^\ell e^{-i\omega N} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} \\
&= (\pm)\cancel{e^{-i\pi N}} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} \\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}
\end{aligned}
$$

$$
\begin{aligned}
&\implies & \breve{y}^{(0)}(0) &= 0 \\
&\implies & \breve{y}^{(1)}(0) &= 0 \\
&\implies & \breve{y}^{(2)}(0) &= 0 \\
&\implies & \breve{y}^{(3)}(0) &= 0 \\
&\implies & \breve{y}^{(4)}(0) &= 0 \\
&\ \vdots & &\vdots \\
&\implies & \breve{y}^{(n)}(0) &= 0 \\
&\implies & \breve{y}^{(n)}(0) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}
$$

3. Proof that (B) $\iff$ (C): by Theorem 6.5 page 83

4. Proof that (A) $\iff$ (D): by Theorem 6.5 page 83

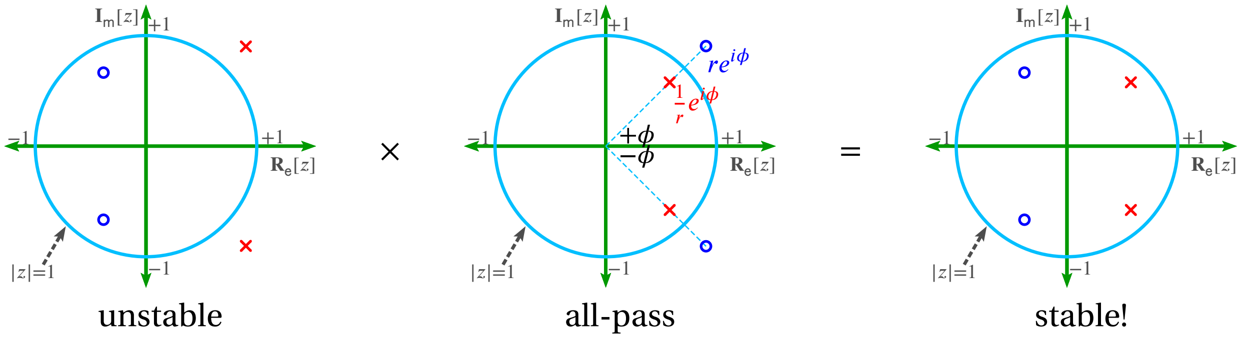
5. Proof that (CQF) $\not\Leftarrow$ (A): Here is a counterexample: $\breve{y}(\omega) = 0$.



$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

D.9 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.



$$\begin{aligned}
|A(z)|_{z=e^{i\omega}} &= \frac{1}{r}\left|\frac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|e^{i\phi}\left(\frac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|-z\left(\frac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\frac{1}{e^{-iv}}\left(\frac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right| \\
&= 1
\end{aligned}$$

$$\begin{aligned}
&= \left|\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|z\left(\frac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\frac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
&= \left|\frac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right|
\end{aligned}$$

APPENDIX E

OPERATORS ON LINEAR SPACES

❝*And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.*❞
Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

E.1 Operators on linear spaces

E.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition E.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD. *Let* X *be a set, let* $+$ *be an* OPERATOR (Definition E.2 page 128) *in* X^{X^2}*, and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991) page 41, Halmos (1948) pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

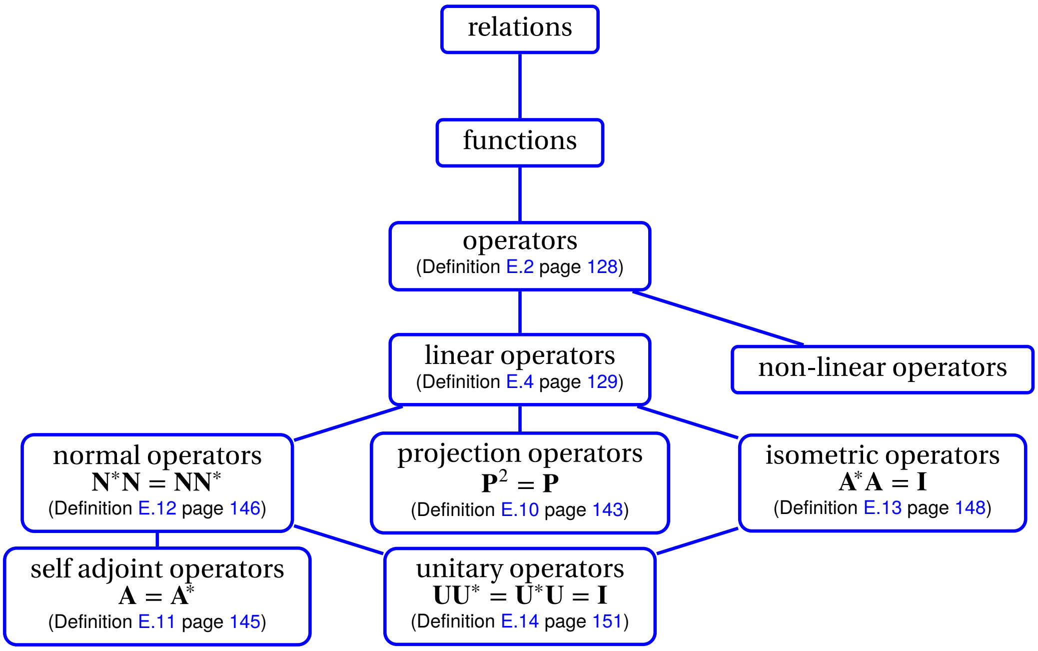


Figure E.1: Some operator types

DEF *The structure* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1.	$\exists \mathbb{0} \in X$ such that	$x + \mathbb{0}$	$=$	x	$\forall x \in X$	(+ IDENTITY)
2.	$\exists y \in X$ such that	$x + y$	$=$	$\mathbb{0}$	$\forall x \in X$	(+ INVERSE)
3.		$(x + y) + z$	$=$	$x + (y + z)$	$\forall x,y,z \in X$	(+ *is* ASSOCIATIVE)
4.		$x + y$	$=$	$y + x$	$\forall x,y \in X$	(+ *is* COMMUTATIVE)
5.		$1 \cdot x$	$=$	x	$\forall x \in X$	($\cdot$ IDENTITY)
6.		$\alpha \cdot (\beta \cdot x)$	$=$	$(\alpha \cdot \beta) \cdot x$	$\forall \alpha,\beta \in S$ *and* $x \in X$	($\cdot$ ASSOCIATES *with* $\cdot$)
7.		$\alpha \cdot (x + y)$	$=$	$(\alpha \cdot x) + (\alpha \cdot y)$	$\forall \alpha \in S$ *and* $x,y \in X$	($\cdot$ DISTRIBUTES *over* +)
8.		$(\alpha + \beta) \cdot x$	$=$	$(\alpha \cdot x) + (\beta \cdot x)$	$\forall \alpha,\beta \in S$ *and* $x \in X$	($\cdot$ PSEUDO-DISTRIBUTES *over* +)

*

The set X *is called the* ***underlying set****. The elements of* X *are called* ***vectors****. The elements of* $\mathbb{F}$ *are called* ***scalars****. A linear space is also called a* ***vector space****. If* $\mathbb{F} \triangleq \mathbb{R}$*, then* $\boldsymbol{\Omega}$ *is a* ***real linear space****. If* $\mathbb{F} \triangleq \mathbb{C}$*, then* $\boldsymbol{\Omega}$ *is a* ***complex linear space****.*

Definition E.2. [3]

DEF *A function* $\mathbf{A}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *is an* ***operator*** *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *if* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both* LINEAR SPACES *(Definition E.1 page 127).*

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}x = \mathbf{B}x$ for all $x \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{XY}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}x = \mathbf{I}$ for all $x \in \boldsymbol{X}$.

Definition E.3. [4] *Let* $\boldsymbol{X}^{\boldsymbol{X}}$ *be the set of all operators with from a* LINEAR SPACE $\boldsymbol{X}$ *to* $\boldsymbol{X}$*. Let* $\mathbf{I}$ *be an operator in* $\boldsymbol{X}^{\boldsymbol{X}}$*. Let* $①(\boldsymbol{X})$ *be the* IDENTITY ELEMENT *in* $\boldsymbol{X}^{\boldsymbol{X}}$*.*

DEF $\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\quad \mathbf{I} = ①(\boldsymbol{X})$.

[3] Heil (2011) page 42
[4] Michel and Herget (1993) page 411

E.1.2 Linear operators

Definition E.4. [5] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be linear spaces.*

DEF

An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ ***is linear*** *if*

$$\begin{array}{llcll}
1. & \mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x}+\mathbf{L}\boldsymbol{y} & \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X} \quad (\text{ADDITIVE}) \quad \text{and}\\
2. & \mathbf{L}(\alpha\boldsymbol{x}) & = & \alpha\mathbf{L}\boldsymbol{x} & \forall \boldsymbol{x}\in\boldsymbol{X},\ \forall\alpha\in\mathbb{F} \quad (\text{HOMOGENEOUS}).
\end{array}$$

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *such that*

$$\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \left\{\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}} | \mathbf{L} \text{ is linear}\right\}.$$

Theorem E.1. [6] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$.

THM

$$\{\mathbf{L} \text{ is LINEAR}\} \implies \left\{\begin{array}{llcll}
1. & \mathbf{L}\mathbb{0} & = & \mathbb{0} & \text{and}\\
2. & \mathbf{L}(-\boldsymbol{x}) & = & -(\mathbf{L}\boldsymbol{x}) \quad \forall \boldsymbol{x}\in\boldsymbol{X} & \text{and}\\
3. & \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X} & \text{and}\\
4. & \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) & = & \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}\boldsymbol{x}_n\right) \quad \boldsymbol{x}_n\in\boldsymbol{X}, \alpha_n\in\mathbb{F} &
\end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned}
\mathbf{L}\mathbb{0} &= \mathbf{L}(0\cdot\mathbb{0}) && \text{by additive identity property}\\
&= 0\cdot(\mathbf{L}\mathbb{0}) && \text{by \textit{homogeneous} property of } \mathbf{L} && \text{(Definition E.4 page 129)}\\
&= \mathbb{0} && \text{by \textit{additive identity} property}
\end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned}
\mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1\cdot\boldsymbol{x}) && \text{by \textit{additive inverse} property}\\
&= -1\cdot(\mathbf{L}\boldsymbol{x}) && \text{by \textit{homogeneous} property of } \mathbf{L} && \text{(Definition E.4 page 129)}\\
&= -(\mathbf{L}\boldsymbol{x}) && \text{by \textit{additive inverse} property}
\end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}$:

$$\begin{aligned}
\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x}+(-\boldsymbol{y})) && \text{by \textit{additive inverse} property}\\
&= \mathbf{L}(\boldsymbol{x})+\mathbf{L}(-\boldsymbol{y}) && \text{by \textit{linearity} property of } \mathbf{L} && \text{(Definition E.4 page 129)}\\
&= \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} && \text{by item (2)}
\end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) = \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}\boldsymbol{x}_n\right)$:

   (a) Proof for $N = 1$:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_1\boldsymbol{x}_1\right) && \text{by } N=1 \text{ hypothesis}\\
&= \alpha_1\left(\mathbf{L}\boldsymbol{x}_1\right) && \text{by \textit{homogeneous} property of } \mathbf{L} && \text{(Definition E.4 page 129)}
\end{aligned}$$

[5] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[6] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

(b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1}\alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1} + \sum_{n=1}^{N}\alpha_n \boldsymbol{x}_n\right) \\
&= \mathbf{L}(\alpha_{N+1}\boldsymbol{x}_{N+1}) + \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n \boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \quad \text{(Definition E.4 page 129)} \\
&= \alpha_{N+1}\mathbf{L}(\boldsymbol{x}_{N+1}) + \sum_{n=1}^{N}\mathbf{L}(\alpha_n \boldsymbol{x}_n) && \text{by left } N+1 \text{ hypothesis} \\
&= \sum_{n=1}^{N+1}\mathbf{L}(\alpha_n \boldsymbol{x}_n)
\end{aligned}$$

✏

Theorem E.2. [7] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathcal{N}(\mathbf{L})$	*is a linear subspace of* $\boldsymbol{X}$	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathcal{I}(\mathbf{L})$	*is a linear subspace of* $\boldsymbol{Y}$	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	

✎PROOF:

1. Proof that $\mathcal{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0}\in\mathcal{N}(\mathbf{L}) \implies \mathcal{N}(\mathbf{L})\neq\emptyset$
   (b) $\mathcal{N}(\mathbf{L}) \triangleq \{\boldsymbol{x}\in\boldsymbol{X} | \mathbf{L}\boldsymbol{x}=\mathbb{0}\} \subseteq \boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{N}(\mathbf{L}) \implies \mathbb{0}=\mathbf{L}(\boldsymbol{x}+\boldsymbol{y})=\mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\boldsymbol{X} \implies \mathbb{0}=\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
2. Proof that $\mathcal{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0}\in\mathcal{I}(\mathbf{L}) \implies \mathcal{I}(\mathbf{L})\neq\emptyset$
   (b) $\mathcal{I}(\mathbf{L}) \triangleq \{\boldsymbol{y}\in\boldsymbol{Y} | \exists\boldsymbol{x}\in\boldsymbol{X}$ such that $\boldsymbol{y}=\mathbf{L}\boldsymbol{x}\} \subseteq \boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{v}\in\boldsymbol{X}$ such that $\mathbf{L}\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{x}\in\boldsymbol{X}$ such that $\boldsymbol{y}=\mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y}=\alpha\mathbf{L}\boldsymbol{x}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$

✏

Example E.1. [8] Let $C([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX
$C([a:b],\mathbb{R})$ is a linear space.

Theorem E.3. [9] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of a linear operator $\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM		
$\mathbf{L}\boldsymbol{x}=\mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y}\in\mathcal{N}(\mathbf{L})$
$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathcal{N}(\mathbf{L})=\{\mathbb{0}\}$

[7] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩
[8] Eidelman et al. (2004) page 3
[9] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

✎PROOF:

1. Proof that $\mathbf{L}x = \mathbf{L}y \implies x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}(x-y) &= \mathbf{L}x - \mathbf{L}y && \text{by Theorem E.1 page 129}\\
&= \mathbb{0} && \text{by left hypothesis}\\
&\implies x - y \in \mathcal{N}(\mathbf{L}) && \text{by definition of } \textit{null space}
\end{aligned}$$

2. Proof that $\mathbf{L}x = \mathbf{L}y \impliedby x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}y &= \mathbf{L}y + \mathbb{0} && \text{by definition of linear space (Definition E.1 page 127)}\\
&= \mathbf{L}y + \mathbf{L}(x-y) && \text{by right hypothesis}\\
&= \mathbf{L}y + (\mathbf{L}x - \mathbf{L}y) && \text{by Theorem E.1 page 129}\\
&= (\mathbf{L}y - \mathbf{L}y) + \mathbf{L}x && \text{by } \textit{associative} \text{ and } \textit{commutative} \text{ properties (Definition E.1 page 127)}\\
&= \mathbf{L}x
\end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned}
\mathbf{L} \text{ is } \textit{injective} &\iff \{(\mathbf{L}x = \mathbf{L}y \iff x = y) \quad \forall x, y \in X\}\\
&\iff \left\{\left[\mathbf{L}x - \mathbf{L}y = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \left\{\left[\mathbf{L}(x-y) = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}
\end{aligned}$$

✏

Theorem E.4. [10] *Let W, X, Y, and Z be linear spaces over a field $\mathbb{F}$.*

THM 1.	$\mathbf{L}(\mathbf{MN})$	$=$	$(\mathbf{LM})\mathbf{N}$	$\forall \mathbf{L} \in \mathcal{L}(Z,W), \mathbf{M} \in \mathcal{L}(Y,Z), \mathbf{N} \in \mathcal{L}(X,Y)$	(ASSOCIATIVE)
2.	$\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})$	$=$	$(\mathbf{LM}) \dotplus (\mathbf{LN})$	$\forall \mathbf{L} \in \mathcal{L}(Y,Z), \mathbf{M} \in \mathcal{L}(X,Y), \mathbf{N} \in \mathcal{L}(X,Y)$	(LEFT DISTRIBUTIVE)
3.	$(\mathbf{L} \dotplus \mathbf{M})\mathbf{N}$	$=$	$(\mathbf{LN}) \dotplus (\mathbf{MN})$	$\forall \mathbf{L} \in \mathcal{L}(Y,Z), \mathbf{M} \in \mathcal{L}(Y,Z), \mathbf{N} \in \mathcal{L}(X,Y)$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{LM})$	$=$	$(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L} \in \mathcal{L}(Y,Z), \mathbf{M} \in \mathcal{L}(X,Y), \alpha \in \mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{MN}) = (\mathbf{LM})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}(\mathbf{M} \dotplus \mathbf{N}) = (\mathbf{LM}) \dotplus (\mathbf{LN})$:

$$\begin{aligned}
[\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})]x &= \mathbf{L}[(\mathbf{M} \dotplus \mathbf{N})x]\\
&= \mathbf{L}[(\mathbf{M}x) \dotplus (\mathbf{N}x)]\\
&= [\mathbf{L}(\mathbf{M}x)] \dotplus [\mathbf{L}(\mathbf{N}x)] && \text{by } \textit{additive} \text{ property Definition E.4 page 129}\\
&= [(\mathbf{LM})x] \dotplus [(\mathbf{LN})x]
\end{aligned}$$

3. Proof that $(\mathbf{L} \dotplus \mathbf{M})\mathbf{N} = (\mathbf{LN}) \dotplus (\mathbf{MN})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{LM}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{LM}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned}
[\alpha(\mathbf{LM})]x &= \alpha[(\mathbf{LM})x]\\
&= \mathbf{L}[\alpha(\mathbf{M}x)] && \text{by } \textit{homogeneous} \text{ property Definition E.4 page 129}\\
&= \mathbf{L}[(\alpha\mathbf{M})x]\\
&= [\mathbf{L}(\alpha\mathbf{M})]x
\end{aligned}$$

[10] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

Theorem E.5 (Fundamental theorem of linear equations). *Michel and Herget (1993) page 99 Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ (Definition ?? page ??).*

THM
$$\dim \mathcal{I}(\mathbf{L}) + \dim \mathcal{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$$

PROOF: Let $\{\psi_k | k = 1, 2, \ldots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for $\mathcal{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \mathbb{0} \right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail.

E.2 Operators on Normed linear spaces

E.2.1 Operator norm

Definition E.5. [11] *Let $\boldsymbol{V} = (X, \mathbb{F}, \dot{+}, \cdot)$ be a linear space and $\mathbb{F}$ be a field with absolute value function $|\cdot| \in \mathbb{R}^{\mathbb{F}}$.*

DEF

*A **norm** is any functional $\|\cdot\|$ in $\mathbb{R}^X$ that satisfies*

1.	$\|\boldsymbol{x}\|$	$\geq$	0	$\forall x \in X$	(STRICTLY POSITIVE)	*and*
2.	$\|\boldsymbol{x}\|$	$=$	$0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall x \in X$	(NONDEGENERATE)	*and*
3.	$\|a\boldsymbol{x}\|$	$=$	$\lvert a\rvert \, \|\boldsymbol{x}\|$	$\forall x \in X,\ a \in \mathbb{C}$	(HOMOGENEOUS)	*and*
4.	$\|\boldsymbol{x} + \boldsymbol{y}\|$	$\leq$	$\|\boldsymbol{x}\| + \|\boldsymbol{y}\|$	$\forall x, y \in X$	(SUBADDITIVE/*triangle inquality*).	

*A **normed linear space** is the pair $(\boldsymbol{V}, \|\cdot\|)$.*

[11] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

Definition E.6. [12] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [13]

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} \qquad \forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition E.1 (next) shows that the functional defined in Definition E.6 (previous) is a *norm* (Definition E.5 page 132).

Proposition E.1. [14] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ and $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1.	$\|\|\|\mathbf{A}\|\|\| \ge 0$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(NON-NEGATIVE)	and
2.	$\|\|\|\mathbf{A}\|\|\| = 0 \iff \mathbf{A} \doteq \mathbb{0}$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(NONDEGENERATE)	and
3.	$\|\|\|\alpha\mathbf{A}\|\|\| = \|\alpha\| \, \|\|\|\mathbf{A}\|\|\|$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)	and
4.	$\|\|\|\mathbf{A} \dot{+} \mathbf{B}\|\|\| \le \|\|\|\mathbf{A}\|\|\| + \|\|\|\mathbf{B}\|\|\|$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \ne \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)}\\
&> 0
\end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \doteq \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)}\\
&= \sup_{x\in\boldsymbol{X}} \{\|\mathbb{0}x\| \mid \|x\| \le 1\}\\
&= 0
\end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha| \, |||\mathbf{A}|||$:

$$\begin{aligned}
|||\alpha\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\alpha\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)}\\
&= \sup_{x\in\boldsymbol{X}} \{|\alpha| \, \|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)}\\
&= |\alpha| \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of sup}\\
&= |\alpha| \; |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)}
\end{aligned}$$

[12] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225
[13] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990) page 290
[14] Rudin (1991) page 93

4. Proof that $\|\mathbf{A} \mathring{+} \mathbf{B}\| \leq |||\mathbf{A}||| + |||\mathbf{B}|||$:

$$\begin{aligned}
|||\mathbf{A} \mathring{+} \mathbf{B}||| &\triangleq \sup_{x \in \boldsymbol{X}} \left\{ \left\|(\mathbf{A} \mathring{+} \mathbf{B})\boldsymbol{x}\right\| \mid \|\boldsymbol{x}\| \leq 1 \right\} && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)} \\
&= \sup_{x \in \boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1 \right\} \\
&\leq \sup_{x \in \boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x}\| + \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1 \right\} && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)} \\
&\leq \sup_{x \in \boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1 \right\} + \sup_{x \in \boldsymbol{X}} \left\{ \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1 \right\} \\
&\triangleq |||\mathbf{A}||| + |||\mathbf{B}||| && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)}
\end{aligned}$$

⇨

Lemma E.1. *Let* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *and* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.

LEM

$$|||\mathbf{L}||| = \sup_{\boldsymbol{x}} \left\{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \right\} \qquad \forall \boldsymbol{x} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF: [15]

1. Proof that $\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1\} \geq \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$:

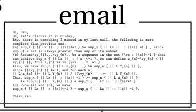
$$\sup_{\boldsymbol{x}} \left\{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1 \right\} \geq \sup_{\boldsymbol{x}} \left\{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \right\} \qquad \text{because } A \subseteq B \implies \sup A \leq \sup B$$

2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ \boldsymbol{y} \in X \,\middle|\, \begin{array}{ll} 1. & \|\mathbf{L}\boldsymbol{y}\| = \sup_{x \in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1\} \text{ and} \\ 2. & 0 < \|\boldsymbol{y}\| \leq 1 \end{array} \right\}$$

3. Proof that $\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1\} \leq \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$:

$$\begin{aligned}
\sup_{x \in X} \left\{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \leq 1 \right\} &= \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of set } Y \\
&= \frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \|\mathbf{L}\boldsymbol{y}\| \\
&= \|\boldsymbol{y}\| \left\| \frac{1}{\|\boldsymbol{y}\|} \mathbf{L}\boldsymbol{y} \right\| && \text{by homogeneous property (page 132)} \\
&= \|\boldsymbol{y}\| \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| && \text{by homogeneous property (page 129)} \\
&\leq \|\boldsymbol{y}\| \sup_{y \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \right\} && \text{by definition of supremum} \\
&= \|\boldsymbol{y}\| \sup_{y \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \text{ for all } \boldsymbol{y} \in Y \\
&\leq \sup_{y \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } 0 < \|\boldsymbol{y}\| \leq 1 \\
&\leq \sup_{x \in X} \left\{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \right\} && \text{because } \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in X \quad \forall \boldsymbol{y} \in Y
\end{aligned}$$

[15] email  Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

4. By (1) and (3),

$$\sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| = 1\}$$

✏

Proposition E.2. [16] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{X}), |||\cdot|||)$.

PRP: $|||\mathbf{I}||| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition E.6 page 133)}\\
&= \sup\{\|x\| \mid \|x\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition E.3 page 128)}\\
&= 1
\end{aligned}$$

✏

Theorem E.6. [17] *Let* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM:

$$\begin{array}{rcll}
\|\mathbf{L}x\| & \le & |||\mathbf{L}|||\ \|x\| & \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}),\ x\in\boldsymbol{X}\\
|||\mathbf{K}\mathbf{L}||| & \le & |||\mathbf{K}|||\ |||\mathbf{L}||| & \forall \mathbf{K},\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})
\end{array}$$

✎PROOF:

1. Proof that $\|\mathbf{L}x\| \le |||\mathbf{L}|||\ \|x\|$:

$$\begin{aligned}
\|\mathbf{L}x\| &= \frac{\|x\|}{\|x\|}\|\mathbf{L}x\|\\
&= \|x\|\left\|\frac{1}{\|x\|}\mathbf{L}x\right\| && \text{by property of norms}\\
&= \|x\|\left\|\mathbf{L}\frac{x}{\|x\|}\right\| && \text{by property of linear operators}\\
&\triangleq \|x\|\,\|\mathbf{L}y\| && \text{where } y \triangleq \frac{x}{\|x\|}\\
&\le \|x\|\sup_y \|\mathbf{L}y\| && \text{by definition of supremum}\\
&= \|x\|\sup_y\{\|\mathbf{L}y\| \mid \|y\| = 1\} && \text{because } \|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1\\
&\triangleq \|x\|\,\|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

[16] Michel and Herget (1993) page 410
[17] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

2. Proof that $|||\mathbf{KL}||| \le |||\mathbf{K}|||\ |||\mathbf{L}|||$:

$$\begin{aligned}
|||\mathbf{KL}||| &\triangleq \sup_{x\in X}\{\|(\mathbf{KL})x\| \mid \|x\|\le 1\} && \text{by Definition E.6 page 133 } (|||\cdot|||)\\
&= \sup_{x\in X}\{\|\mathbf{K}(\mathbf{L}x)\| \mid \|x\|\le 1\}\\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ \|\mathbf{L}x\| \mid \|x\|\le 1\} && \text{by 1.}\\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ \|x\| \mid \|x\|\le 1\} && \text{by 1.}\\
&= \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ 1 \mid \|x\|\le 1\} && \text{by definition of sup}\\
&= |||\mathbf{K}|||\ |||\mathbf{L}||| && \text{by definition of sup}
\end{aligned}$$

✑

E.2.2 Bounded linear operators

Definition E.7. [18] *Let $(\mathcal{L}(X,Y), |||\cdot|||)$ be a normed space of linear operators.*

DEF
*An operator $\mathbf{B}$ is **bounded** if $|||\mathbf{B}||| < \infty$.*
*The quantity $\mathcal{B}(X,Y)$ is the set of all **bounded linear operators** on (X,Y) such that*
$$\mathcal{B}(X,Y) \triangleq \{\mathbf{L}\in\mathcal{L}(X,Y) \mid |||\mathbf{L}||| < \infty\}.$$

Theorem E.7. [19] *Let $(\mathcal{L}(X,Y), |||\cdot|||)$ be the set of linear operators over normed linear spaces $X \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ and $Y \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.*

THM
The following conditions are all EQUIVALENT:

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $x_0 \in X$ $\quad \forall \mathbf{L}\in\mathcal{L}(X,Y)$ $\iff$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point $x \in X$)* $\quad \forall \mathbf{L}\in\mathcal{L}(X,Y)$ $\iff$
3. $|||\mathbf{L}||| < \infty$ *($\mathbf{L}$ is* BOUNDED*)* $\quad \forall \mathbf{L}\in\mathcal{L}(X,Y)$ $\iff$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}x\| \le M\|x\|$ $\quad \forall \mathbf{L}\in\mathcal{L}(X,Y), x\in X$

✎PROOF:

1. Proof that $1 \implies 2$:

$$\begin{aligned}
\epsilon &> \left\|\mathbf{L}x - \mathbf{L}x_0\right\| && \text{by hypothesis 1}\\
&= \left\|\mathbf{L}(x - x_0)\right\| && \text{by linearity (Definition E.4 page 129)}\\
&= \left\|\mathbf{L}(x + y - x_0 - y)\right\|\\
&= \left\|\mathbf{L}(x+y) - \mathbf{L}(x_0+y)\right\| && \text{by linearity (Definition E.4 page 129)}\\
&\implies \mathbf{L} \text{ is continuous at point } x+y\\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}$$

2. Proof that $2 \implies 1$: obvious:

[18] Rudin (1991) pages 92–93
[19] Aliprantis and Burkinshaw (1998) page 227

3. Proof that $4 \implies 2$:[20]

$$\begin{aligned}
|||\mathbf{L}x||| \le M\,\|x\| &\implies |||\mathbf{L}(x-y)||| \le M\,\|x-y\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le M\,\|x-y\| && \text{by linearity of } \mathbf{L} \text{ (Definition E.4 page 129)}\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le \epsilon \text{ whenever } M\,\|x-y\| < \epsilon\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le \epsilon \text{ whenever } \|x-y\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that $3 \implies 4$:

$$\begin{aligned}
\|\mathbf{L}x\| &\le \underbrace{|||\mathbf{L}|||}_{M}\ \|x\| && \text{by Theorem E.6 page 135}\\
&= M\ \|x\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that $1 \implies 3$:[21]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \infty\\
&\implies \exists\,(x_n) \quad \text{such that} \quad \|x_n\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\|\mathbf{L}x_n\| \mid \|x_n\| \le 1\right\} = \infty\\
&\implies \|x_n\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \|\mathbf{L}x_n\|\\
&\implies \|x_n\| = 1 \text{ and } \|\mathbf{L}x_n\| \ge n\\
&\implies \frac{1}{n}\|x_n\| = \frac{1}{n} \text{ and } \frac{1}{n}\|\mathbf{L}x_n\| \ge 1\\
&\implies \left\|\frac{x_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

✏

E.2.3 Adjoints on normed linear spaces

Definition E.8. *Let $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *of $\boldsymbol{X}$.*

DEF: $\mathbf{B}^*$ *is the* ***adjoint*** *of an operator $\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ if*
$$\mathsf{f}(\mathbf{B}x) = \left[\mathbf{B}^*\mathsf{f}\right](x) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ x \in \boldsymbol{X}$$

Theorem E.8. [22] *Let $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *$\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM:
$$\begin{array}{rcll}
(\mathbf{A} \dotplus \mathbf{B})^* &=& \mathbf{A}^* \dotplus \mathbf{B}^* & \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\lambda\mathbf{A})^* &=& \lambda\mathbf{A}^* & \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\mathbf{A}\mathbf{B})^* &=& \mathbf{B}^*\mathbf{A}^* & \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})
\end{array}$$

[20] Bollobás (1999) page 29
[21] Aliprantis and Burkinshaw (1998) page 227
[22] Bollobás (1999) page 156

✎PROOF:

$$
\begin{aligned}
\left[\mathbf{A} \dotplus \mathbf{B}\right]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \dotplus \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition E.8 page 137)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition E.4 page 129)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of \textit{linear functional}}\\
&= \mathbf{A}^{*}\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^{*}\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition E.8 page 137)}\\
&= \left[\mathbf{A}^{*} + \mathbf{B}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{linear functional}}
\end{aligned}
$$

$$
\begin{aligned}
[\lambda\mathbf{A}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition E.8 page 137)}\\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of \textit{linear functional}}\\
&= \left[\lambda\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition E.8 page 137)}
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{A}\mathbf{B}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition E.8 page 137)}\\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of \textit{linear operators}} && \text{(Definition E.4 page 129)}\\
&= \left[\mathbf{A}^{*}\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition E.8 page 137)}\\
&= \mathbf{B}^{*}\left[\mathbf{A}^{*}\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition E.8 page 137)}\\
&= \left[\mathbf{B}^{*}\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition E.8 page 137)}
\end{aligned}
$$

✏

Theorem E.9. [23] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^{*}$ be the adjoint of an operator $\mathbf{B}$.*

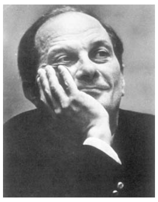
THM
$$|||\mathbf{B}||| = \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| \qquad \forall \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF:

$$
\begin{aligned}
|||\mathbf{B}||| &\triangleq \sup\left\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\right\} && \text{by Definition E.6 page 133}\\
&\overset{?}{=} \sup\left\{|\mathsf{g}(\mathbf{B}\boldsymbol{x}; \boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x}; \mathbf{B}^{*}\boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&\triangleq \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&= \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&\triangleq \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| && \text{by Definition E.6 page 133}
\end{aligned}
$$

✏

E.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞
Stanislaus M. Ulam (1909–1984), Polish mathematician [24]

[23] Rudin (1991) page 98

Theorem E.10 (Mazur-Ulam theorem). [25] *Let $\phi \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be a function on normed linear spaces $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. Let $\mathbf{I} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$ be the identity operator on $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.*

THM

$$\left.\begin{array}{lll}1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} & \text{and}\\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}}_{\text{isometric}} & \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$
   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$
2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$
3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned}
\left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}}\\
&= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis}\\
&= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}}\\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1}
\end{aligned}$$

4. Proof that $\mathrm{g}\boldsymbol{z} = \boldsymbol{z}$:

$$\begin{aligned}
2\lambda &= 2\sup\{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\} && \text{by definition of } \lambda \text{ item (2)}\\
&\le 2\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\| && \text{by definition of sup}\\
&= \|2\boldsymbol{z} - 2\mathrm{g}\boldsymbol{z}\| &&\\
&= \|\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}\boldsymbol{z}\| && \text{by definition of } \psi \text{ item (1)}\\
&= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}^{-1}\mathrm{g}\boldsymbol{z}\right\| && \text{by item (3)}\\
&= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| && \text{by definition of } \mathrm{g}^{-1}\\
&= \left\|\psi\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| &&\\
&= \left\|\mathrm{g}^{*}\boldsymbol{z} - \boldsymbol{z}\right\| &&\\
&\le \lambda && \text{by definition of } \lambda \text{ item (2)}\\
&\implies 2\lambda \le \lambda &&\\
&\implies \lambda = 0 &&\\
&\implies \mathrm{g}\boldsymbol{z} = \boldsymbol{z} &&
\end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) = \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}$:

$$\begin{aligned}
\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) &=\\
&= \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}
\end{aligned}$$

[24] quote: Ulam (1991) page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[25] Oikhberg and Rosenthal (2007) page 598, Väisälä (2003) page 634, Giles (2000) page 11, Dunford and Schwartz (1957) page 91, Mazur and Ulam (1932)

6. Proof that $\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}$:

$$\begin{aligned}\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) &= \\ &= [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}\end{aligned}$$

✏

Theorem E.11 (Neumann Expansion Theorem). [26] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{ll}1. & \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) \quad (\mathbf{A}\text{ is bounded}) \\ 2. & \|\|\mathbf{A}\|\| < 1\end{array}\right\} \implies \left\{\begin{array}{llll}1. & (\mathbf{I}-\mathbf{A})^{-1} & & \textit{exists} \\ 2. & \|\|(\mathbf{I}-\mathbf{A})^{-1}\|\| & \leq & \dfrac{1}{1-\|\|\mathbf{A}\|\|} \\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \displaystyle\sum_{n=0}^{\infty}\mathbf{A}^n \\ & \multicolumn{3}{l}{\textit{with uniform convergence}}\end{array}\right.$$

E.3 Operators on Inner product spaces

E.3.1 General Results

Definition E.9. [27] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space.*

DEF

A function $\langle \triangle \mid \triangledown \rangle \in \mathbb{F}^{X\times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

1.	$\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle \geq 0$	$\forall x \in X$	*(non-negative)*	*and*
2.	$\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall x \in X$	*(nondegenerate)*	*and*
3.	$\langle \alpha\boldsymbol{x} \mid \boldsymbol{y} \rangle = \alpha\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$\forall x,y \in X,\ \forall \alpha \in \mathbb{C}$	*(homogeneous)*	*and*
4.	$\langle \boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{u} \rangle = \langle \boldsymbol{x} \mid \boldsymbol{u} \rangle + \langle \boldsymbol{y} \mid \boldsymbol{u} \rangle$	$\forall x,y,u \in X$	*(additive)*	*and*
5.	$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \langle \boldsymbol{y} \mid \boldsymbol{x} \rangle^*$	$\forall x,y \in X$	*(conjugate symmetric).*	

An inner product is also called a ***scalar product****.*
An ***inner product space*** *is the pair* $(\boldsymbol{\Omega}, \langle \triangle \mid \triangledown \rangle)$.

Theorem E.12. [28] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

THM

$$\begin{array}{lllllllll}\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x} \rangle & = & 0 & \forall x \in X & \iff & \mathbf{B}\boldsymbol{x} & = & \mathbb{0} & \forall x \in X \\ \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x} \rangle & = & \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x} \rangle & \forall x \in X & \iff & \mathbf{A} & = & \mathbf{B} & \end{array}$$

✎PROOF:

[26] Michel and Herget (1993) page 415
[27] Haaser and Sullivan (1991) page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72
[28] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

1. Proof that $\langle \mathbf{B}x \,|\, x\rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x+\mathbf{B}x) \,|\, (x+\mathbf{B}x)\rangle + i\,\langle \mathbf{B}(x+i\mathbf{B}x) \,|\, (x+i\mathbf{B}x)\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}x+\mathbf{B}^2x) \,|\, x+\mathbf{B}x\right\rangle\right\} + i\left\{\left\langle \mathbf{B}x+i\mathbf{B}^2x) \,|\, x+i\mathbf{B}x\right\rangle\right\} && \text{by Definition E.4 page 129}\\
&= \left\{\langle \mathbf{B}x \,|\, x\rangle + \langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \,|\, x\right\rangle + \left\langle \mathbf{B}^2x \,|\, \mathbf{B}x\right\rangle\right\} && \text{by Definition E.9 page 140}\\
&\quad + i\left\{\langle \mathbf{B}x \,|\, x\rangle - i\,\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \,|\, x\right\rangle - i^2\left\langle \mathbf{B}^2x \,|\, \mathbf{B}x\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \,|\, x\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \,|\, x\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \,|\, x\right\rangle\right\} + \left\{\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle - \left\langle \mathbf{B}^2x \,|\, x\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle\\
&= 2\,\|\mathbf{B}x\|^2\\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition E.5 page 132}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \,|\, x\rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products.

3. Proof that $\langle \mathbf{A}x \,|\, x\rangle = \langle \mathbf{B}x \,|\, x\rangle \implies \mathbf{A} \stackrel{\circ}{=} \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \,|\, x\rangle - \langle \mathbf{B}x \,|\, x\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}x - \mathbf{B}x \,|\, x\rangle && \text{by } \textit{additivity} \text{ property of } \langle \triangle \,|\, \triangledown\rangle \text{ (Definition E.9 page 140)}\\
&= \langle (\mathbf{A}-\mathbf{B})x \,|\, x\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A}-\mathbf{B})x &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \,|\, x\rangle = \langle \mathbf{B}x \,|\, x\rangle \impliedby \mathbf{A} \stackrel{\circ}{=} \mathbf{B}$:

$$\langle \mathbf{A}x \,|\, x\rangle = \langle \mathbf{B}x \,|\, x\rangle \qquad \text{by } \mathbf{A} \stackrel{\circ}{=} \mathbf{B} \text{ hypothesis}$$

✏

E.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition E.3 page 141). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem E.13 page 142).
- Both support decomposition into "real" and "imaginary" parts (Theorem A.3 page 90).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *null space* of an operator (Theorem E.14 page 143).

Proposition E.3. [29] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *(Definition E.7 page 136) on a* HILBERT SPACE $\boldsymbol{H}$.

PRP *An operator $\mathbf{B}^*$ is the **adjoint** of $\mathbf{B} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ if*

$$\langle \mathbf{B}x \,|\, y\rangle = \langle x \,|\, \mathbf{B}^* y\rangle \qquad \forall x, y \in \boldsymbol{H}.$$

✎PROOF:

[29] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000) page 182, von Neumann (1929) page 49, Stone (1932) page 41

1. For fixed y, $f(x) \triangleq \langle x \mid y\rangle$ is a *functional* in $\mathbb{F}^X$.
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}
\langle \mathbf{B}x \mid y\rangle &\triangleq f(\mathbf{B}x) \\
&\triangleq \mathbf{B}^* f(x) && \text{by definition of } \textit{operator adjoint} && \text{(Definition E.8 page 137)}\\
&= \langle x \mid \mathbf{B}^* y\rangle
\end{aligned}$$

Example E.2.

EX

In matrix algebra ("linear algebra")

- The inner product operation $\langle x \mid y\rangle$ is represented by $y^H x$.
- The linear operator is represented as a matrix A.
- The operation of A on a vector x is represented as Ax.
- The adjoint of matrix A is the Hermitian matrix A^H.

✎PROOF:

$$\langle Ax \mid y\rangle \triangleq y^H Ax = [(Ax)^H y]^H = [x^H A^H y]^H = (A^H y)^H x \triangleq \left\langle x \mid A^H y\right\rangle$$

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition A.3 page 88). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[30]

Theorem E.13 (operator star-algebra). [31] *Let H be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(H, H)$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathcal{B}(H, H)$. *Let $\bar{\alpha}$ be the complex conjugate of some $\alpha \in \mathbb{C}$.*

THM

The pair $(H, *)$ is a *-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \dotplus \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in H$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in H$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in H$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in H$	(INVOLUTARY)	

✎PROOF:

$$\begin{aligned}
\left\langle x \mid (\mathbf{A} \dotplus \mathbf{B})^* y\right\rangle &= \left\langle (\mathbf{A} \dotplus \mathbf{B})x \mid y\right\rangle && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}\\
&= \langle \mathbf{A}x \mid y\rangle + \langle \mathbf{B}x \mid y\rangle && \text{by definition of inner product} && \text{(Definition E.9 page 140)}\\
&= \langle x \mid \mathbf{A}^* y\rangle + \langle x \mid \mathbf{B}^* y\rangle && \text{by definition of operator addition}\\
&= \langle x \mid \mathbf{A}^* y + \mathbf{B}^* y\rangle && \text{by definition of inner product} && \text{(Definition E.9 page 140)}\\
&= \langle x \mid (\mathbf{A}^* + \mathbf{B}^*) y\rangle && \text{by definition of operator addition}
\end{aligned}$$

$$\begin{aligned}
\langle x \mid (\alpha\mathbf{A})^* y\rangle &= \langle (\alpha\mathbf{A})x \mid y\rangle && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}\\
&= \langle \alpha(\mathbf{A}x) \mid y\rangle && \text{by definition of scalar multiplication}\\
&= \alpha \langle \mathbf{A}x \mid y\rangle && \text{by definition of inner product} && \text{(Definition E.9 page 140)}\\
&= \alpha \langle x \mid \mathbf{A}^* y\rangle && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}\\
&= \langle x \mid \alpha^* \mathbf{A}^* y\rangle && \text{by definition of inner product} && \text{(Definition E.9 page 140)}
\end{aligned}$$

[30] Sakai (1998) page 1
[31] Halmos (1998) pages 39–40, Rudin (1991) page 311

$$\begin{aligned}
\langle x \,|\, (\mathbf{AB})^* y\rangle &= \langle (\mathbf{AB})x \,|\, y\rangle && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}\\
&= \langle \mathbf{A}(\mathbf{B}x) \,|\, y\rangle && \text{by definition of operator multiplication}\\
&= \langle (\mathbf{B}x) \,|\, \mathbf{A}^* y\rangle && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}\\
&= \langle x \,|\, \mathbf{B}^*\mathbf{A}^* y\rangle && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}
\end{aligned}$$

$$\begin{aligned}
\langle x \,|\, \mathbf{A}^{**} y\rangle &= \langle \mathbf{A}^* x \,|\, y\rangle && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}\\
&= \langle y \,|\, \mathbf{A}^* x\rangle^* && \text{by definition of inner product} && \text{(Definition E.9 page 140)}\\
&= \langle \mathbf{A} y \,|\, x\rangle^* && \text{by definition of adjoint} && \text{(Proposition E.3 page 141)}\\
&= \langle x \,|\, \mathbf{A} y\rangle && \text{by definition of inner product} && \text{(Definition E.9 page 140)}
\end{aligned}$$

⇨

Theorem E.14. [32] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\mathcal{N}(\mathbf{A}) = \mathcal{I}(\mathbf{A}^*)^\perp$$
$$\mathcal{N}(\mathbf{A}^*) = \mathcal{I}(\mathbf{A})^\perp$$

✎PROOF:

$$\begin{aligned}
\mathcal{I}(\mathbf{A}^*)^\perp &= \left\{ y \in \boldsymbol{H} \,|\, \langle y \,|\, u\rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}^*)\right\}\\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle y \,|\, \mathbf{A}^* x\rangle = 0 \quad \forall x \in \boldsymbol{H}\right\}\\
&= \{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A} y \,|\, x\rangle = 0 \quad \forall x \in \boldsymbol{H}\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition E.3 page 141)}\\
&= \{ y \in \boldsymbol{H} \,|\, \mathbf{A} y = 0\}\\
&= \mathcal{N}(\mathbf{A}) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\mathbf{A})^\perp &= \{ y \in \boldsymbol{H} \,|\, \langle y \,|\, u\rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A})\}\\
&= \{ y \in \boldsymbol{H} \,|\, \langle y \,|\, \mathbf{A} x\rangle = 0 \quad \forall x \in \boldsymbol{H}\} && \text{by definition of } \mathcal{I}\\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A}^* y \,|\, x\rangle = 0 \quad \forall x \in \boldsymbol{H}\right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition E.3 page 141)}\\
&= \left\{ y \in \boldsymbol{H} \,|\, \mathbf{A}^* y = 0\right\}\\
&= \mathcal{N}(\mathbf{A}^*) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

⇨

E.4 Special Classes of Operators

E.4.1 Projection operators

Definition E.10. [33] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF

$\mathbf{P}$ *is a* ***projection*** *operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[32] Rudin (1991) page 312

[33] Rudin (1991) page 133 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 6, Halmos (1958) page 73 ⟨§41. Projections⟩

Theorem E.15. [34] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ with* NULL SPACE $\mathcal{N}(\mathbf{P})$ *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

$$\left.\begin{array}{lrcll} 1. & \mathbf{P}^2 &=& \mathbf{P} & (\mathbf{P}\text{ is a projection operator}) \text{ and} \\ 2. & \boldsymbol{\Omega} &=& \boldsymbol{X} \dotplus \boldsymbol{Y} & (\boldsymbol{Y}\text{ compliments }\boldsymbol{X}\text{ in }\boldsymbol{\Omega}) \text{ and} \\ 3. & \mathbf{P}\boldsymbol{\Omega} &=& \boldsymbol{X} & (\mathbf{P}\text{ projects onto }\boldsymbol{X}) \end{array}\right\} \implies \left\{\begin{array}{lrcll} 1. & \mathcal{I}(\mathbf{P}) &=& \boldsymbol{X} & \text{and} \\ 2. & \mathcal{N}(\mathbf{P}) &=& \boldsymbol{Y} & \text{and} \\ 3. & \boldsymbol{\Omega} &=& \mathcal{I}(\mathbf{P}) \dotplus \mathcal{N}(\mathbf{P}) & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\ &= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\ &= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\ &= \boldsymbol{\Omega}_1 \end{aligned}$$

$$\begin{aligned} \mathcal{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\} \\ &= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_2 \end{aligned}$$

✏

Theorem E.16. [35] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

THM

$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P}\text{ is a projection operator}} \iff \underbrace{(\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})}_{(\mathbf{I}-\mathbf{P})\text{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})$:

$$\begin{aligned} (\mathbf{I}-\mathbf{P})^2 &= (\mathbf{I}-\mathbf{P})(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{I}(\mathbf{I}-\mathbf{P}) + (-\mathbf{P})(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2 \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} && \text{by left hypothesis} \\ &= \mathbf{I} - \mathbf{P} \end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})$:

$$\begin{aligned} \mathbf{P}^2 &= \underbrace{\mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P}^2}_{(\mathbf{I}-\mathbf{P})^2} - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I}-\mathbf{P})^2 - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I}-\mathbf{P}) - (\mathbf{I} - \mathbf{P} - \mathbf{P}) && \text{by right hypothesis} \\ &= \mathbf{P} \end{aligned}$$

✏

[34] Michel and Herget (1993) pages 120–121
[35] Michel and Herget (1993) page 121

Theorem E.17. [36] *Let $\boldsymbol{H}$ be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathcal{N}(\mathbf{P})$, *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

$$\begin{array}{llcll} 1. & \mathbf{P}^* & = & \mathbf{P} & (\mathbf{P} \text{ is SELF-ADJOINT}) \iff \\ 2. & \mathbf{P}^*\mathbf{P} & = & \mathbf{P}\mathbf{P}^* & (\mathbf{P} \text{ is NORMAL}) \iff \\ 3. & \mathcal{I}(\mathbf{P}) & = & \mathcal{N}(\mathbf{P})^\perp & \iff \\ 4. & \langle \mathbf{P}x \,|\, x \rangle & = & \|\mathbf{P}x\|^2 \quad \forall x \in X & \end{array}$$

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\implies$ (2):

$$\begin{aligned} \mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)} \\ &= \mathbf{P}\mathbf{P}^* && \text{by Theorem E.13 page 142} \end{aligned}$$

Proof that (1) $\implies$ (3):

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathcal{N}(\mathbf{P}^*)^\perp && \text{by Theorem E.14 page 143} \\ &= \mathcal{N}(\mathbf{P})^\perp && \text{by (1)} \end{aligned}$$

Proof that (3) $\implies$ (4):

Proof that (4) $\implies$ (1):

✎

E.4.2 Self Adjoint Operators

Definition E.11. [37] *Let* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be **self-adjoint** or **hermitian** if* $\mathbf{B} \doteq \mathbf{B}^*$.

Example E.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}.$$

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}} \mathsf{f}(u)\, \mathsf{d}u$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

Theorem E.18. [38] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|x\| \triangleq \sqrt{\langle x \,|\, x\rangle}$.

THM

$$\left\{ \underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S} \text{ is self adjoint}} \right\} \implies \left\{ \begin{array}{lll} 1. & \langle \mathbf{S}x \,|\, x \rangle \in \mathbb{R} & (\text{the hermitian quadratic form of } \mathbf{S} \text{ is REAL-VALUED}) \\ 2. & \lambda_n \in \mathbb{R} & (\text{eigenvalues of } \mathbf{S} \text{ are REAL-VALUED}) \\ 3. & \lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0 & (\text{eigenvectors are ORTHOGONAL}) \end{array} \right\}$$

[36] Rudin (1991) page 314

[37] Historical works regarding self-adjoint operators: von Neumann (1929) page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932) page 50 ⟨"self-adjoint transformations"⟩

[38] Lax (2002) pages 315–316, Keener (1988) pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩, Bertero and Boccacci (1998) page 225 ⟨§"9.2 SVD of a matrix …If all eigenvectors are normalized…"⟩

✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle \in \mathbb{R}$:

$$\begin{aligned}
\langle \boldsymbol{x} \mid \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by left hypothesis}\\
&= \langle \boldsymbol{x} \mid \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition E.9 page 140}
\end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned}
\lambda_n \|\psi_n\|^2 &= \lambda_n \langle \psi_n \mid \psi_n \rangle && \text{by definition}\\
&= \langle \lambda_n \psi_n \mid \psi_n \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition E.9 page 140}\\
&= \langle \mathbf{S}\psi_n \mid \psi_n \rangle && \text{by definition of eigenpairs}\\
&= \langle \psi_n \mid \mathbf{S}\psi_n \rangle && \text{by left hypothesis}\\
&= \langle \psi_n \mid \lambda_n \psi_n \rangle && \text{by definition of eigenpairs}\\
&= \lambda_n^* \langle \psi_n \mid \psi_n \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition E.9 page 140}\\
&= \lambda_n^* \|\psi_n\|^2 && \text{by definition}
\end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \mid \psi_m \rangle &= \langle \lambda_n \psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition E.9 page 140}\\
&= \langle \mathbf{S}\psi_n \mid \psi_m \rangle && \text{by definition of eigenpairs}\\
&= \langle \psi_n \mid \mathbf{S}\psi_m \rangle && \text{by left hypothesis}\\
&= \langle \psi_n \mid \lambda_m \psi_m \rangle && \text{by definition of eigenpairs}\\
&= \lambda_m^* \langle \psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition E.9 page 140}\\
&= \lambda_m \langle \psi_n \mid \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \mid \psi_m \rangle = 0$.

✏

E.4.3 Normal Operators

Definition E.12. [39] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{N}^*$ be the adjoint of an operator $\mathbf{N} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF: $\mathbf{N}$ ***is normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

Theorem E.19. [40] *Let $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE *$\boldsymbol{H}$. Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$.*

THM: $$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \text{ is normal}} \iff \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\| \qquad \forall \boldsymbol{x} \in \boldsymbol{H}$$

[39] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969) page 167, Frobenius (1878), Frobenius (1968) page 391

[40] Rudin (1991) pages 312–313

✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \mid \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition E.3 page 141 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \mid \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (}\mathbf{N}\text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition E.3 page 141 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition E.3 page 141 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem E.13 page 142 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis } (\|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Proposition E.3 page 141 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem E.20. [41] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ *and* $\mathcal{I}(\mathbf{N})$ *the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same null space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \left\{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{(definition of } \mathcal{N})\\
&= \left\{\boldsymbol{x} |\ \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition E.5 page 132)}\\
&= \{\boldsymbol{x} |\ \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\}\\
&= \{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\} && \text{by definition of } \|\cdot\| \text{ (Definition E.5 page 132)}\\
&= \mathcal{N}(\mathbf{N}) && \text{(definition of } \mathcal{N})
\end{aligned}$$

✏

Theorem E.21. [42] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ *and* $\mathcal{I}(\mathbf{N})$ *the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m\rangle = 0}_{\text{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. [Rudin] claims both to be true.(Rudin, 1991)313

[41] Rudin (1991) pages 312–313
[42] Rudin (1991) pages 312–313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$
\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\quad\ = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\quad\ = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem E.13 page 142} \\
&\quad\ = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem E.13 page 142} \\
&\quad\ = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\implies \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\iff \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}
$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$
\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition E.9 page 140} \\
&= \langle \mathbf{N}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{N}^*\psi_m \rangle && \text{by Proposition E.3 page 141 (definition of adjoint)} \\
&= \langle \psi_n \,|\, \lambda_m^*\psi_m \rangle && \text{by (4.)} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition E.9 page 140}
\end{aligned}
$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

E.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition E.13. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES (Definition E.5 page 132).

DEF: *An operator* $\mathbf{M} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is* ***isometric*** *if*

$$\|\mathbf{M}x\| = \|x\| \qquad \forall x \in X.$$

Theorem E.22. [43] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM:

$$\underbrace{\|\mathbf{M}x\| = \|x\| \quad \forall x \in X}_{\textit{isometric in length}} \iff \underbrace{\|\mathbf{M}x - \mathbf{M}y\| = \|x - y\| \quad \forall x, y \in X}_{\textit{isometric in distance}}$$

✎PROOF:

1. Proof that $\|\mathbf{M}x\| = \|x\| \implies \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$:

$$
\begin{aligned}
\|\mathbf{M}x - \mathbf{M}y\| &= \|\mathbf{M}(x - y)\| && \text{by definition of linear operators (Definition E.4 page 129)} \\
&= \|\mathbf{M}u\| && \text{let } u \triangleq x - y \\
&= \|x - y\| && \text{by left hypothesis}
\end{aligned}
$$

[43] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

2. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \impliedby \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \|\mathbf{M}(\boldsymbol{x} - \mathbb{0})\| \\
&= \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition E.4 page 129)} \\
&= \|\boldsymbol{x} - \mathbb{0}\| && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition E.13 page 148) in the more general normed linear spaces, while Theorem E.22 (page 148) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \iff \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem E.23 (next) demonstrates two additional equivalent properties.

Theorem E.23. [44] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.*

THM *The following conditions are all* ***equivalent***:

1.	$\mathbf{M}^*\mathbf{M}$	$=$	$\mathbf{I}$			$\iff$
2.	$\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle$	$=$	$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	($\mathbf{M}$ *is surjective*)	$\iff$
3.	$\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|$	$=$	$\|x - y\|$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(*isometric in distance*)	$\iff$
4.	$\|\mathbf{M}\boldsymbol{x}\|$	$=$	$\|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X$	(*isometric in length*)	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle &= \langle \boldsymbol{x} \mid \mathbf{M}^*\mathbf{M}\boldsymbol{y} \rangle && \text{by Proposition E.3 page 141 (definition of adjoint)} \\
&= \langle \boldsymbol{x} \mid \mathbf{I}\boldsymbol{y} \rangle && \text{by (1)} \\
&= \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by Definition E.3 page 128 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \sqrt{\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x} \rangle} && \text{by definition of } \|\cdot\| \\
&= \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle} && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle &= \|\mathbf{M}\boldsymbol{x} + \mathbf{M}\boldsymbol{y}\|^2 - \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|^2 + i\|\mathbf{M}\boldsymbol{x} + i\mathbf{M}\boldsymbol{y}\|^2 - i\|\mathbf{M}\boldsymbol{x} - i\mathbf{M}\boldsymbol{y}\|^2 && \text{by polarization id.} \\
&= \|\mathbf{M}(\boldsymbol{x} + \boldsymbol{y})\|^2 - \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\|^2 + i\|\mathbf{M}(\boldsymbol{x} + i\boldsymbol{y})\|^2 - i\|\mathbf{M}(\boldsymbol{x} - i\boldsymbol{y})\|^2 && \text{by Definition E.4} \\
&= \|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 + i\|\boldsymbol{x} + i\boldsymbol{y}\|^2 - i\|\boldsymbol{x} - i\boldsymbol{y}\|^2 && \text{by left hypothesis}
\end{aligned}$$

4. Proof that (3) $\iff$ (4): by Theorem E.22 page 148

[44] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

5. Proof that (4) $\implies$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition E.3 page 141 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem E.13 page 142 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y=0\\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Definition E.3 page 128 (definition of } \mathbf{I})\\
\implies \quad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}$$

✏

Theorem E.24. [45] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M}\text{ is isometric}} \quad \implies \quad \begin{cases} |||\mathbf{M}||| = 1 & \text{(UNIT LENGTH)} \quad \text{and} \\ |\lambda| = 1 \quad \forall \lambda \in \Lambda \end{cases}$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \implies |||\mathbf{M}||| = 1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{x\in X} \{\|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Definition E.6 page 133}\\
&= \sup_{x\in X} \{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Theorem E.23 page 149}\\
&= \sup_{x\in X}\{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x}, \lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|} \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|} \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem E.23 page 149}\\
&= \frac{1}{\|\boldsymbol{x}\|} \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|} |\lambda| \; \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

✏

Example E.4 (One sided shift operator). [46] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ $(0, 1, 2, \ldots)$ and shift operators defined as

$$\begin{aligned}
&1. \quad \mathbf{S}_r \left(x_0, x_1, x_2, \ldots\right) \triangleq \left(0, x_0, x_1, x_2, \ldots\right) && \text{(right shift operator)}\\
&2. \quad \mathbf{S}_l \left(x_0, x_1, x_2, \ldots\right) \triangleq \left(x_1, x_2, x_3, \ldots\right) && \text{(left shift operator)}
\end{aligned}$$

EX

1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

[45] Michel and Herget (1993) page 432

[46] Michel and Herget (1993) page 441

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r \,(\!(x_0, x_1, x_2, \dots)\!) \,|\, (\!(y_0, y_1, y_2, \dots)\!) \rangle &= \langle (\!(0, x_0, x_1, x_2, \dots)\!) \,|\, (\!(y_0, y_1, y_2, \dots)\!) \rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\, y_n^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \langle (\!(x_0, x_1, x_2, \dots)\!) \,|\, (\!(y_1, y_2, y_3, \dots)\!) \rangle \\
&= \left\langle (\!(x_0, x_1, x_2, \dots)\!) \,\middle|\, \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} (\!(y_0, y_1, y_2, \dots)\!) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✏

E.4.5 Unitary operators

Definition E.14. [47] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF: *The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} \;=\; \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition E.4. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP: $\left.\begin{array}{l} \mathbf{U} \textit{ is } \text{UNITARY} \quad \textit{and} \\ \mathbf{V} \textit{ is } \text{UNITARY} \end{array}\right\} \implies (\mathbf{UV}) \textit{ is } \text{UNITARY}.$

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\big(\mathbf{V}^*\mathbf{U}^*\big) && \text{by Theorem E.8 page 137} \\
&= \mathbf{U}\big(\mathbf{VV}^*\big)\mathbf{U}^* && \text{by associative property} \\
&= \mathbf{UIU}^* && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition E.14 page 151)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition E.14 page 151)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \big(\mathbf{V}^*\mathbf{U}^*\big)(\mathbf{UV}) && \text{by Theorem E.8 page 137} \\
&= \mathbf{V}^*\big(\mathbf{U}^*\mathbf{U}\big)\mathbf{V} && \text{by associative property} \\
&= \mathbf{V}^*\mathbf{IV} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition E.14 page 151)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition E.14 page 151)}
\end{aligned}$$

[47] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

Theorem E.25. [48] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{I}(\mathbf{U})$ be the* IMAGE SET *of* $\mathbf{U}$.

THM *If* $\mathbf{U}$ *is a* ***bounded linear operator*** ($\mathbf{U} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H})$), *then the following conditions are* ***equivalent***:

1. $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ (UNITARY) $\iff$
2. $\langle \mathbf{U}x \mid \mathbf{U}y \rangle = \langle \mathbf{U}^*x \mid \mathbf{U}^*y \rangle = \langle x \mid y \rangle$ *and* $\mathcal{I}(\mathbf{U}) = X$ (SURJECTIVE) $\iff$
3. $\|\mathbf{U}x - \mathbf{U}y\| = \|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x - y\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN DISTANCE) $\iff$
4. $\|\mathbf{U}x\| = \|x\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN LENGTH)

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) $\langle \mathbf{U}x \mid \mathbf{U}y \rangle = \langle \mathbf{U}^*x \mid \mathbf{U}^*y \rangle = \langle x \mid y \rangle$ by Theorem E.23 (page 149).
   (b) Proof that $\mathcal{I}(\mathbf{U}) = X$:
$$\begin{aligned} X &\supseteq \mathcal{I}(\mathbf{U}) && \text{because } \mathbf{U} \in X^X \\ &\supseteq \mathcal{I}(\mathbf{U}\mathbf{U}^*) \\ &= \mathcal{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}) \\ &= X && \text{by Definition E.3 page 128 (definition of } \mathbf{I}) \end{aligned}$$
2. Proof that (2) $\iff$ (3) $\iff$ (4): by Theorem E.23 page 149.
3. Proof that (3) $\implies$ (1):
   (a) Proof that $\|\mathbf{U}x - \mathbf{U}y\| = \|x - y\| \implies \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem E.23 page 149
   (b) Proof that $\|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x - y\| \implies \mathbf{U}\mathbf{U}^* = \mathbf{I}$:
$$\begin{aligned} \|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x - y\| \implies && \mathbf{U}^{**}\mathbf{U}^* = \mathbf{I} && \text{by Theorem E.23 page 149} \\ && \mathbf{U}\mathbf{U}^* = \mathbf{I} && \text{by Theorem E.13 page 142} \end{aligned}$$

Theorem E.26. *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathbf{U}$ *be a bounded linear operator in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathcal{N}(\mathbf{U})$ the* NULL SPACE *of* $\mathbf{U}$, *and $\mathcal{I}(\mathbf{U})$ the* IMAGE SET *of* $\mathbf{U}$.

THM
$$\underbrace{\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U} \text{ is unitary}} \implies \left\{ \begin{array}{rclcll} \mathbf{U}^{-1} &=& \mathbf{U}^* & & & \text{and} \\ \mathcal{I}(\mathbf{U}) &=& \mathcal{I}(\mathbf{U}^*) &=& X & \text{and} \\ \mathcal{N}(\mathbf{U}) &=& \mathcal{N}(\mathbf{U}^*) &=& \{\mathbb{0}\} & \text{and} \\ \|\mathbf{U}\| &=& \|\mathbf{U}^*\| &=& 1 & (\text{UNIT LENGTH}) \end{array} \right\}$$

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both *isometric* and *normal*:
$$\begin{array}{lclclcl} \mathbf{U}^*\mathbf{U} & & &=& \mathbf{I} & \implies & \mathbf{U} \text{ is } \textit{isometric} \\ \mathbf{U}\mathbf{U}^* &=& \mathbf{U}^*\mathbf{U} &=& \mathbf{I} & \implies & \mathbf{U}^* \text{ is } \textit{isometric} \\ \mathbf{U}^{-1} &=& \mathbf{U}^* & & & \implies & \mathbf{U}^{-1} \text{ is } \textit{isometric} \\ \\ \mathbf{U}^*\mathbf{U} &=& \mathbf{U}\mathbf{U}^* &=& & \implies & \mathbf{U} \text{ is } \textit{normal} \\ \mathbf{U}\mathbf{U}^* &=& \mathbf{U}^*\mathbf{U} &=& & \implies & \mathbf{U}^* \text{ is } \textit{normal} \\ \mathbf{U}^{-1} &=& \mathbf{U}^* & & & \implies & \mathbf{U}^{-1} \text{ is } \textit{normal} \end{array}$$

[48] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem E.25 page 152.
3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}^{-1})$:

$$\begin{aligned}
\mathcal{N}(\mathbf{U}^*) &= \mathcal{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both } \textit{normal} \text{ and by Theorem E.21 page 147}\\
&= \mathcal{I}(\mathbf{U})^\perp && \text{by Theorem E.14 page 143}\\
&= X^\perp && \text{by above result}\\
&= \{\mathbb{0}\}
\end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem E.24 page 150.

✏

Example E.5 (Rotation matrix). [49]

EX
$$\underbrace{\left\{ \mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}}_{\text{rotation matrix } \mathbf{R}_\theta \,:\, \mathbb{R}^2 \to \mathbb{R}^2} \implies \left\{ \begin{array}{lrcll} (1). & \mathbf{R}^{-1}{}_\theta &=& \mathbf{R}_{-\theta} & \text{and} \\ (2). & \mathbf{R}^*{}_\theta &=& \mathbf{R}^{-1}{}_\theta & (\mathbf{R} \text{ is unitary}) \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R}\\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of } \textit{Hermetian transpose} \text{ operator } H\\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{by Theorem 1.2 page 5}\\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R}\\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✏

Example E.6. [50] Let $\mathbf{A}$ and $\mathbf{B}$ be matrix operators.

EX
$$\underbrace{\underset{\mathbf{A} \text{ is a } \textit{rotation operator}.}{\mathbf{A} \triangleq \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{\mathbf{B} \text{ is a } \textit{reflection operator}.}{\mathbf{B} \triangleq \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}}_{\text{Both } \mathbf{A} \text{ and } \mathbf{B} \text{ are } \textbf{unitary}.}$$

Example E.7. Examples of *Fredholm integral operators* include

EX

1.	**Fourier Transform**	$[\tilde{\mathbf{F}}\mathsf{x}](f) = \int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-i2\pi ft}\,\mathrm{d}t$	$\kappa(t,f) = e^{-i2\pi ft}$
2.	**Inverse Fourier Transform**	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t) = \int_{f\in\mathbb{R}} \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathrm{d}f$	$\kappa(f,t) = e^{i2\pi ft}$
3.	**Laplace operator**	$[\mathbf{L}\mathsf{x}](s) = \int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t,s) = e^{-st}$

Example E.8 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathsf{f}(x) \triangleq \mathsf{f}(x-1) \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

[49] Noble and Daniel (1988) page 311
[50] Gel'fand (1963) page 4, Gelfand et al. (2018) page 4

EX

1.	$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(inverse translation operator)
2.	$\mathbf{T}^* = \mathbf{T}^{-1}$		($\mathbf{T}$ is invertible)
3.	$\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^* = \mathbf{I}$		($\mathbf{T}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$
$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned}
\langle \mathbf{T}\mathsf{f}(x)\,|\,\mathsf{g}(x)\rangle &= \langle \mathsf{f}(x-1)\,|\,\mathsf{g}(x)\rangle && \text{by definition of } \mathbf{T}\\
&= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathrm{d}x\\
&= \langle \mathsf{f}(x)\,|\,\mathsf{g}(x+1)\rangle\\
&= \left\langle \mathsf{f}(x)\,|\,\underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example E.9 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX

1.	$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(inverse dilation operator)
2.	$\mathbf{D}^* = \mathbf{D}^{-1}$		($\mathbf{D}$ is invertible)
3.	$\mathbf{D}^*\mathbf{D} = \mathbf{D}\mathbf{D}^* = \mathbf{I}$		($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x)\,|\,\mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x)\,|\,\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\,\mathrm{d}u\\
&= \left\langle \mathsf{f}(x)\,|\,\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x)\,|\,\underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

Example E.10 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n-1}\right)\!\right)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n+1}\right)\!\right)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\left(\!\left(x_n\right)\!\right) \mid \left(\!\left(y_n\right)\!\right)\rangle &= \langle \left(\!\left(x_{n-1}\right)\!\right) \mid \left(\!\left(y_n\right)\!\right)\rangle && \text{by definition of } \mathbf{D}\\
&= \sum_n \boldsymbol{x}_{n-1}\, \boldsymbol{y}_n^*\\
&= \sum_n \boldsymbol{x}_n\, \boldsymbol{y}_{n+1}^*\\
&= \left\langle \left(\!\left(x_n\right)\!\right) \mid \left(\!\left(y_{n+1}\right)\!\right) \right\rangle\\
&= \left\langle \left(\!\left(x_n\right)\!\right) \mid \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\left(\!\left(y_n\right)\!\right) \right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary. ✏

Example E.11 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator (Theorem 5.1 page 72)

$$[\tilde{\mathbf{F}}\mathrm{x}](f) \triangleq \int_t \mathrm{x}(t)\underbrace{e^{-i2\pi f t}}_{\kappa(t,f)}\,\mathrm{d}t \qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathrm{x}}\right](t) \triangleq \int_f \tilde{\mathrm{x}}(f)\underbrace{e^{i2\pi f t}}_{\kappa^*(t,f)}\,\mathrm{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathrm{x} \mid \tilde{\mathrm{y}} \right\rangle &= \left\langle \int_t \mathrm{x}(t)e^{-i2\pi f t}\,\mathrm{d}t \mid \tilde{\mathrm{y}}(f) \right\rangle\\
&= \int_t \mathrm{x}(t)\left\langle e^{-i2\pi f t} \mid \tilde{\mathrm{y}}(f)\right\rangle \mathrm{d}t\\
&= \int_t \mathrm{x}(t)\int_f e^{-i2\pi f t}\tilde{\mathrm{y}}^*(f)\,\mathrm{d}f\,\mathrm{d}t\\
&= \int_t \mathrm{x}(t)\left[\int_f e^{i2\pi f t}\tilde{\mathrm{y}}(f)\,\mathrm{d}f\right]^*\mathrm{d}t\\
&= \left\langle \mathrm{x}(t) \mid \int_f \tilde{\mathrm{y}}(f)e^{i2\pi f t}\,\mathrm{d}f \right\rangle\\
&= \left\langle \mathrm{x} \mid \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathrm{y}} \right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✏

E.5 Operator order

Definition E.15. [51] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF: $\mathbf{P}$ *is* ***positive*** *if* $\langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \; \forall \boldsymbol{x} \in \boldsymbol{X}$.
This condition is denoted $\mathbf{P} \geq 0$.

Theorem E.27. [52]

THM:
$$\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \implies \left\{ \begin{array}{llll} (\mathbf{P}+\mathbf{Q}) \geq 0 & & & ((\mathbf{P}+\mathbf{Q}) \text{ is positive}) \\ \mathbf{A}^*\mathbf{P}\mathbf{A} \geq 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) & & (\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive}) \\ \mathbf{A}^*\mathbf{A} \geq 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) & & (\mathbf{A}^*\mathbf{A} \text{ is positive}) \end{array} \right.$$

✎PROOF:

$$\begin{aligned}
\langle (\mathbf{P}+\mathbf{Q})\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle + \langle \mathbf{Q}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by additive property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition E.9 page 140)} \\
&\geq \langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \mathbf{A}\boldsymbol{x} \rangle && \text{by definition of adjoint (Proposition E.3 page 141)} \\
&= \langle \mathbf{P}\boldsymbol{y} \,|\, \boldsymbol{y} \rangle && \text{where } \boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{I}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition E.3 page 128)} \\
&\geq 0 && \text{by non-negative property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition E.9 page 140)} \\
&\implies \mathbf{I} \text{ is positive} \\
\langle \mathbf{A}^*\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition E.3 page 128)} \\
&\geq 0 && \text{by two previous results}
\end{aligned}$$

✏

Definition E.16. [53] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be* BOUNDED *operators.*

DEF: $\mathbf{A} \geq \mathbf{B}$ *("$\mathbf{A}$ is greater than or equal to $\mathbf{B}$") if*
$\mathbf{A} - \mathbf{B} \geq 0$ *("$(\mathbf{A}-\mathbf{B})$ is positive")*

[51] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩
[52] Michel and Herget (1993) page 429
[53] Michel and Herget (1993) page 429

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