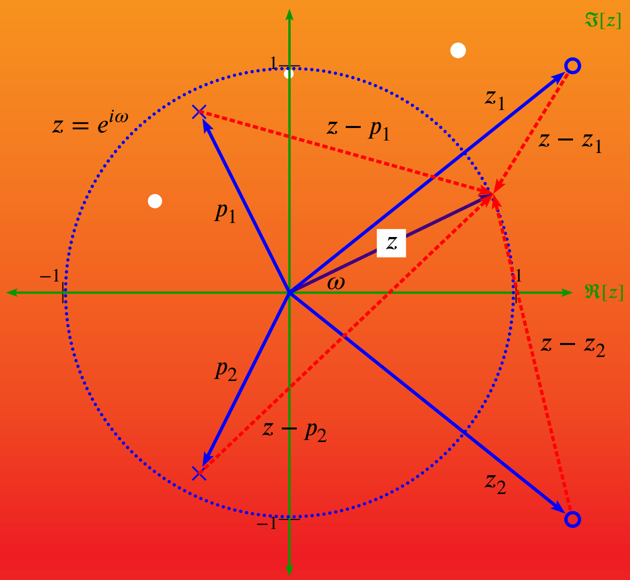
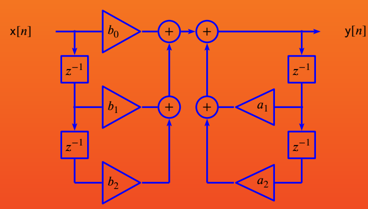
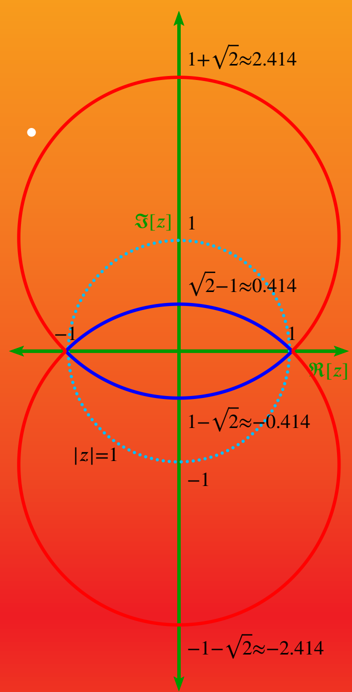
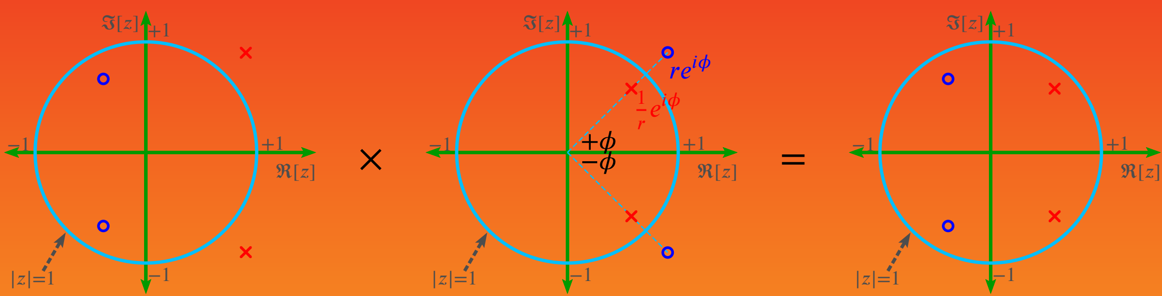
A Book Concerning Digital Signal Processing

VERSION 0.02X

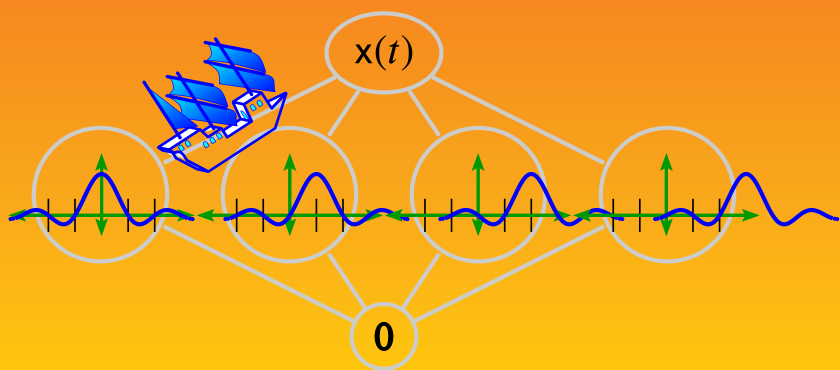








Daniel J. Greenhoe



Signal Processing ABCs series

volume $\boxed{1}$

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The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]

[1] Paine (2000) page 63 ⟨Golden Hind⟩

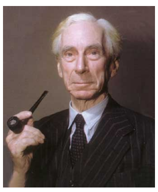
“*Here, on the level sand,*
Between the sea and land,
What shall I build or write
Against the fall of night?”

“*Tell me of runes to grave*
That hold the bursting wave,
Or bastions to design
For longer date than mine.”

Alfred Edward Housman, English poet (1859–1936) [2]

“*The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.*”

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



“*As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.*”

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

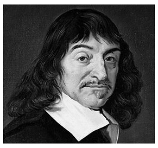
[2] quote: Housman (1936), page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961), page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967), page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝*rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.*❞



❝*Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.*❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝*In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.*❞
Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \ldots$
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$

…continued on next page…

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{Z}^-$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_o$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_e$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^+$	positive real numbers	$(0, \infty)$
$\mathbb{R}^-$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^*$	extended real numbers	$\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
Ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \bigtriangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\nsubseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^c	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	
0	"false" condition	
$\neg$	logical NOT operation	

...continued on next page...

symbol	description	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\leq$	reflexive ordering relation	"less than or equal to"
$\geq$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\mathfrak{R}$	real part of an element in a $*$-algebra	
$\mathfrak{I}$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_{\mathrm{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_{\mathrm{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_{\mathrm{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition J.2 page 149)	
$\hat{\mathbf{F}}$	*Fourier Series* operator (Definition I.1 page 145)	

...continued on next page...

symbol	description
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition 3.1 page 21)
$\mathbf{Z}$	*Z-Transform* operator (Definition 2.4 page 8)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\check{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

CHAPTER 1

SAMPLING

1.1 A basis for sampling

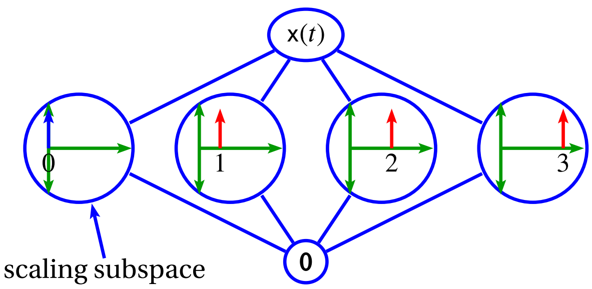
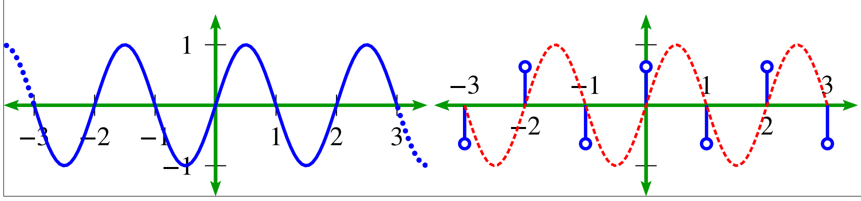
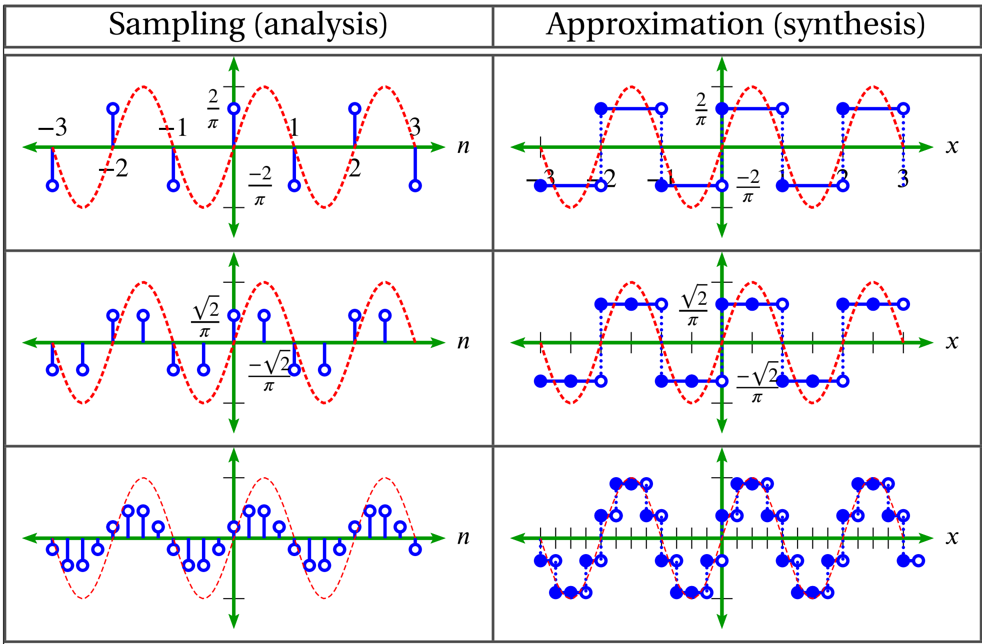


Figure 1.1: A basis for sampling

To perform **sampling**, we *project* continuous functions onto a very special basis to get a **sequence**, as illustrated in Figure 1.1 (page 1).

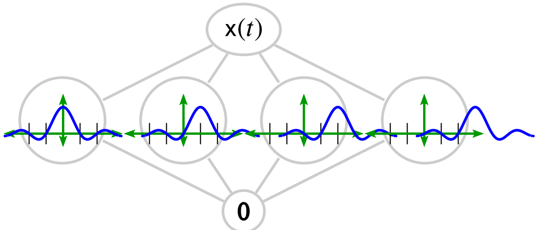
$$
\begin{aligned}
\dot{\mathsf{x}}(n) &\triangleq \langle \mathsf{x}(t) \mid \delta(t-n) \rangle \\
&\triangleq \int_{t=-\infty}^{t=\infty} \mathsf{x}(t)\delta(t-n)\, \mathrm{d}t \\
&= \mathsf{x}(n)
\end{aligned}
$$





Approximation getting closer with higher sample rate! But can we ever get back the original? If so, how fast do we need to sample?

The **Sample Theorem** (Theorem 1.3 page 4) answers this question:



- *If* your signal is **band-limited**, and
  if you sample at a rate of *at least* 2× the highest frequency (the *Nyquist frequency*), and
  if you happen to have an *ideal* low-pass filter,
  then you can get the original signal back (perfect synthesis!).
- *But* if you don't sample fast enough, you get **aliasing**.

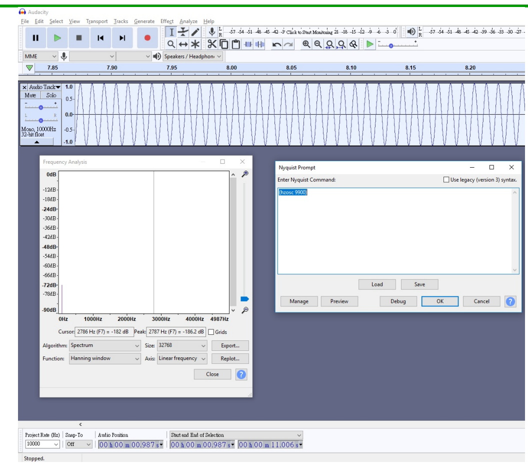
When aliasing occurs, a high frequency component can "masquerade" as ("pretend" to be, "impersonate", "assume the identity" of, or "take on the alias" of) an entirely different low frequency component. That is, is forces a high frequency component to take up residence as an *alien* (*alias* and *alien* have the same Latin root *al* meaning "beyond"[1]) in a low frequency location.

Example 1.1 (Aliasing using *Audacity*). Here is an experiment with aliasing you can try using the free program *Audacity* and the *Nyquist programming language plugin*.[2]

[1] https://www.etymonline.com/word/alias, https://www.etymonline.com/word/alien

[2] *Audacity*®: "Free, open source, cross-platform audio software". https://www.audacityteam.org/; *Nyquist plugin*: https://www.audacityteam.org/about/nyquist/

EX



1. Set Project Rate to 10000 (Hz)
2. Tracks → New Track → Mono Track
3. Select 1 second to 11 seconds
4. Effect → Nyquist prompt → (hzosc 9900)

In this case, the 9900 Hz sinusoid will be aliased to show up as a 100 Hz sinusoid (more impressive if you happen to have a good subwoofer handy).

1.2 Cardinal Series and Sampling

1.2.1 Cardinal series basis

The *Paley-Wiener* class of functions (next definition) are those with a bandlimited Fourier transform. The cardinal series forms an orthogonal basis for such a space (Theorem 1.2 page 4). In a *frame* $(\boldsymbol{x}_n)_{n\in\mathbb{Z}}$ with *frame operator* $\mathbf{S}$ on a *Hilbert Space* $\boldsymbol{H}$ with *inner product* $\langle \triangle \mid \nabla \rangle$, a function $\mathsf{f}(x)$ in the space spanned by the frame can be represented by

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \underbrace{\left\langle \mathsf{f} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle}_{\text{"Fourier coefficient"}} \boldsymbol{x}_n.$$

If the frame is *orthonormal* (giving an *orthonormal basis*), then $\mathbf{S} = \mathbf{S}^{-1} = \mathbf{I}$ and

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n.$$

In the case of the cardinal series, the *Fourier coefficients* are particularly simple—these coefficients are samples of f taken at regular intervals (Theorem 1.3 page 4). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \mid \delta(x - n\tau) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x - n\tau)\, \mathrm{d}t \triangleq \mathsf{f}(n\tau)$$

Definition 1.1. [3]

DEF

A function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is in the* ***Paley-Wiener*** *class of functions* $\boldsymbol{PW}_\sigma^p$ *if there exists* $\mathsf{F} \in \boldsymbol{L}^p(-\sigma : \sigma)$ *such that*

$$\mathsf{f}(x) = \int_{-\sigma}^{\sigma} \mathsf{F}(\omega)e^{ix\omega}\, \mathrm{d}\omega \qquad (\mathsf{f} \text{ has a BANDLIMITED Fourier transform } \mathsf{F} \text{ with bandwidth } \sigma)$$

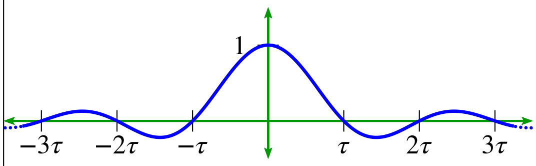
for $p \in [1 : \infty)$ *and* $\sigma \in (0 : \infty)$.

Theorem 1.1 (Paley-Wiener Theorem for Functions). [4] *Let* f *be an* ENTIRE FUNCTION *(the domain of* f *is the entire complex plane* $\mathbb{C}$*). Let* $\sigma \in \mathbb{R}^+$.

[3] Higgins (1996) page 52 ⟨Definition 6.15⟩

[4] Boas (1954) page 103 ⟨6.8.1 Theorem of Paley and Wiener⟩, Katznelson (2004) page 212 ⟨7.4 Theorem⟩, Zygmund (2002) pages 272–273 ⟨(7·2) Theorem of Paley-Wiener⟩, Yosida (1980) page 161, Rudin (1987) page 375 ⟨19.3 Theorem⟩, Young (2001) page 85 ⟨Theorem 18⟩

THM $\left\{\mathsf{f} \in \boldsymbol{PW}^2_\sigma\right\} \iff \left\{\begin{array}{ll} \text{1. } \exists C \in \mathbb{R}^+ \text{ such that } |\mathsf{f}(z)| \le Ce^{\sigma|z|} & \text{(EXPONENTIAL TYPE) and} \\ \text{2. } \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \end{array}\right\}$



Theorem 1.2 (Cardinal sequence). [5]

THM $\left\{\frac{1}{\tau} \ge 2\sigma\right\} \implies$ *The sequence* $\left(\!\!\left(\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)}\right)\!\!\right)_{n\in\mathbb{Z}}$ *is an* ORTHONORMAL BASIS *for* $\boldsymbol{PW}^2_\sigma$.

Theorem 1.3 (Sampling Theorem). [6]

THM $\left\{\begin{array}{ll} \text{1. } \mathsf{f} \in \boldsymbol{PW}^2_\sigma & \text{and} \\ \text{2. } \frac{1}{\tau} \ge 2\sigma \end{array}\right\} \implies \mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)}}_{\text{CARDINAL SERIES}}.$

✎PROOF:

$$\text{Let} \quad \mathsf{s}(x) \triangleq \frac{\sin\left[\frac{\pi}{\tau}x\right]}{\frac{\pi}{\tau}x} \iff \tilde{\mathsf{s}}(\omega) = \begin{cases} \tau & : |f| \le \frac{1}{2\tau} \\ 0 & : \text{otherwise} \end{cases}$$

1. Proof that the set is *orthonormal*: see Hardy (1941)
2. Proof that the set is a *basis*:

$$\begin{aligned}
\mathsf{f}(x) &= \int_\omega \tilde{\mathsf{f}}(\omega)e^{i\omega t}\,\mathsf{d}\omega && \text{by \textit{inverse Fourier transform}} && \text{(Theorem J.1 page 150)} \\
&= \int_\omega \mathbf{T}\tilde{\mathsf{f}}_{\mathsf{d}}(\omega)\tilde{\mathsf{s}}(\omega)e^{i\omega t}\,\mathsf{d}\omega && \text{if } W \le \frac{1}{2T} \\
&= \mathbf{T}\mathsf{f}_{\mathsf{d}}(x) \star \mathsf{s}(x) && \text{by \textit{Convolution theorem}} && \text{(Theorem J.6 page 152)} \\
&= \mathbf{T}\int_u [\mathsf{f}_{\mathsf{d}}(u)]\mathsf{s}(x-u)\,\mathsf{d}u && \text{by \textit{convolution} definition} && \text{(Definition J.3 page 152)} \\
&= \mathbf{T}\int_u \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(u)\delta(u-n\tau)\right]\mathsf{s}(x-u)\,\mathsf{d}u && \text{by \textit{sampling} definition} && \text{(Theorem 1.4 page 5)} \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}}\int_u \mathsf{f}(u)\mathsf{s}(x-u)\delta(u-n\tau)\,\mathsf{d}u \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\mathsf{s}(x-n\tau) && \text{by prop. of \textit{Dirac delta}}
\end{aligned}$$

[5] Higgins (1996) page 52 ⟨Definition 6.15⟩, Hardy (1941) ⟨*orthonormality*⟩, Higgins (1985), page 56 ⟨H1.; historical notes⟩

[6] Whittaker (1915), Kotelnikov (1933), Whittaker (1935), Shannon (1948) ⟨Theorem 13⟩, Shannon (1949) page 11 II (1991) page 1, Nashed and Walter (1991), Higgins (1996) page 5, Young (2001) pages 90–91 ⟨THE PALEY-WIENER SPACE⟩, Papoulis (1980) pages 418–419 ⟨The Sampling Theorem⟩. The *sampling theorem* was "discovered" and published by multiple people: Nyquist in 1928 (DSP?), Whittaker in 1935 (interpolation theory), and Shannon in 1949 (communication theory). references: Mallat (1999), page 43, Oppenheim and Schafer (1999), page 143.

$$= \mathbf{T} \sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau) \frac{\sin\left[\frac{\pi}{\tau}(x - n\tau)\right]}{\frac{\pi}{\tau}(x - n\tau)} \qquad \text{by definition of } \mathsf{s}(x)$$

1.2.2 Sampling

Definition 1.2. [7] *Let* $\delta(x)$ *be the* DIRAC DELTA *distribution.*

DEF: *The* ***Shah Function*** $\text{Ш}(x)$ *is defined as* $\text{Ш}(x) \triangleq \sum_{n\in\mathbb{Z}} \delta(x - n)$

If $\mathsf{f}_\mathsf{d}(x)$ is the function $\mathsf{f}(x)$ sampled at rate $1/T$, then $\tilde{\mathsf{f}}_\mathsf{d}(\omega)$ is simply $\tilde{\mathsf{f}}(\omega)$ *replicated* every $1/T$ Hertz and *scaled* by $1/T$. This is proven in Theorem 1.4 (next) and illustrated in Figure 1.2 (page 5).

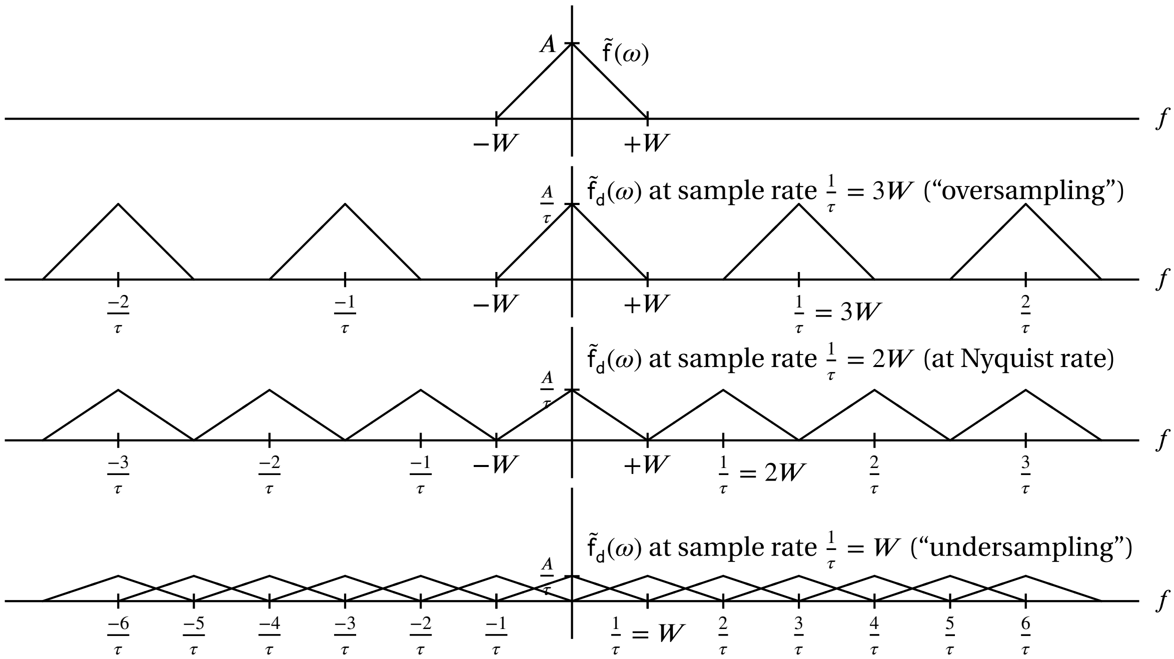


Figure 1.2: Sampling in frequency domain

Theorem 1.4. *Let* $\mathsf{f}, \mathsf{f}_\mathsf{d} \in \boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}, \tilde{\mathsf{f}}_\mathsf{d} \in \boldsymbol{L}^2_{\mathbb{R}}$ *be their respective fourier transforms. Let* $\mathsf{f}_\mathsf{d}(x)$ *be the* ***sampled*** $\mathsf{f}(x)$ *such that*

$$\mathsf{f}_\mathsf{d}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x - n\tau).$$

THM:
$$\left\{ \mathsf{f}_\mathsf{d}(x) \triangleq \mathsf{f}(x)\text{Ш}(x) \triangleq \mathsf{f}(x) \sum_{n\in\mathbb{Z}} \delta(x - n\tau) \right\} \implies \left\{ \tilde{\mathsf{f}}_\mathsf{d}(\omega) = \frac{2\pi}{\tau} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau} n\right) \right\}$$

[7] Bracewell (1978) page 77 ⟨The sampling or replicating symbol Ш(x)⟩, Córdoba (1989)191. Note: The symbol Ш is the Cyrillic upper case "sha" character, which has been assigned Unicode location U+0428. Reference: http://unicode.org/cldr/utility/character.jsp?a=0428

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{f}}_{\mathsf{d}}(\omega) &\triangleq \int_t \mathsf{f}_{\mathsf{d}}(x)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_t \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x-n\tau)\right] e^{-i\omega t}\,\mathrm{d}t \\
&= \sum_{n\in\mathbb{Z}} \int_t \mathsf{f}(x)\delta(x-n\tau)e^{-i\omega t}\,\mathrm{d}t \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau} && \text{by definition of } \delta \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right) && \text{by } \textit{IPSF} \quad \text{(Theorem A.3 page 68)} \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega-\frac{2\pi}{\tau}n\right)
\end{aligned}$$

✏

Suppose a waveform $\mathsf{f}(x)$ is sampled at every time T generating a sequence of sampled values $\mathsf{f}(n\tau)$. Then in general, we can *approximate* $\mathsf{f}(x)$ by using interpolation between the points $\mathsf{f}(n\tau)$. Interpolation can be performed using several interpolation techniques.

In general all techniques lead only to an approximation of $\mathsf{f}(x)$. However, if $\mathsf{f}(x)$ is *bandlimited* with bandwidth $W \le \frac{1}{2T}$, then $\mathsf{f}(x)$ is *perfectly reconstructed* (not just approximated) from the sampled values $\mathsf{f}(n\tau)$ (Theorem 1.3 page 4).

CHAPTER 2

OPERATIONS ON SEQUENCES

2.1 Convolution operator

Definition 2.1. [1] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF *A function f in X^Y is a **sequence** over X if $Y = \mathbb{Z}$.*
A sequence may be denoted in the form $(\!(x_n)\!)_{n\in\mathbb{Z}}$ or simply as $(\!(x_n)\!)$.

Definition 2.2. [2] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a* FIELD *(Definition B.5 page 74).*

DEF *The **space of all absolutely square summable sequences** $\boldsymbol{\ell}^2_{\mathbb{F}}$ over $\mathbb{F}$ is defined as*

$$\boldsymbol{\ell}^2_{\mathbb{F}} \triangleq \left\{ (\!(x_n)\!)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}$$

The space $\boldsymbol{\ell}^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is isomorphic to $\boldsymbol{L}^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition 2.3.

DEF *The **convolution** operation $\star$ is defined as*

$$(\!(x_n)\!) \star (\!(y_n)\!) \triangleq \left(\!\!\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m} \right)\!\!\right)_{n\in\mathbb{Z}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$$

Proposition 2.1. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition 2.3 page 7).*

PRP $$(\!(x_n)\!) \star (\!(y_n)\!) = (\!(y_n)\!) \star (\!(x_n)\!) \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

[1] Bromwich (1908), page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[2] Kubrusly (2011) page 347 ⟨Example 5.K⟩

✎PROOF:

$$\begin{aligned}
[x \star y](n) &\triangleq \sum_{m\in\mathbb{Z}} x_m y_{n-m} && \text{by Definition 2.3 page 7}\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{where } k = n-m \iff m = n-k\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{by change commutivity of addition}\\
&= \sum_{m\in\mathbb{Z}} x_{n-m} y_m && \text{by change of variables}\\
&= \sum_{m\in\mathbb{Z}} y_m x_{n-m} && \text{by commutative property of the field over } \mathbb{C}\\
&\triangleq (y \star x)_n && \text{by Definition 2.3 page 7}
\end{aligned}$$

✏

Proposition 2.2. *Let $\star$ be the* CONVOLUTION OPERATOR (Definition 2.3 page 7). *Let $\ell^2_{\mathbb{R}}$ be the set of* ABSOLUTELY SUMMABLE *sequences* (Definition 2.2 page 7).

PRP
$$\left\{\begin{array}{lrcl} \text{(A).} & \mathsf{x}(n) & \in & \ell^2_{\mathbb{R}} \quad \text{and} \\ \text{(B).} & \mathsf{y}(n) & \in & \ell^2_{\mathbb{R}} \end{array}\right\} \implies \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] \;=\; \mathsf{x}[-n] \star \mathsf{y}(n) \right\}$$

✎PROOF:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \implies k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by \textit{absolutely summable} hypothesis} && \text{(Definition 2.2 page 7)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \implies \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of \textit{convolution} } \star && \text{(Definition 2.3 page 7)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n]
\end{aligned}$$

✏

2.2 Z-transform

Definition 2.4. [3]

DEF
The ***z-transform*** $\mathbf{Z}$ *of* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *is defined as*

$$[\mathbf{Z}\,(\!(x_n)\!)](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\text{Laurent series}} \qquad \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}}$$

Theorem 2.1. *Let $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ be the* Z-TRANSFORM *of* $\mathsf{x}[n]$.

THM
$$\left\{ \check{\mathsf{x}}(z) \;\triangleq\; \mathbf{Z}\,(\!(\mathsf{x}[n])\!) \right\} \implies \left\{\begin{array}{lrcll}
\text{(1).} & \mathbf{Z}\,(\!(\alpha\mathsf{x}[n])\!) & = & \alpha\check{\mathsf{x}}(z) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(2).} & \mathbf{Z}\,(\!(\mathsf{x}[n-k])\!) & = & z^{-k}\check{\mathsf{x}}(z) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(3).} & \mathbf{Z}\,(\!(\mathsf{x}[-n])\!) & = & \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(4).} & \mathbf{Z}\,(\!(\mathsf{x}^*[n])\!) & = & \check{\mathsf{x}}^*\left(z^*\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(5).} & \mathbf{Z}\,(\!(\mathsf{x}^*[-n])\!) & = & \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}}
\end{array}\right\}$$

[3] *Laurent series*: Abramovich and Aliprantis (2002) page 49

✎PROOF:

$$
\begin{aligned}
\alpha\mathbb{Z}\check{x}(z) &\triangleq \alpha\mathbf{Z}\left(\!\left(x[n]\right)\!\right) && \text{by definition of } \check{x}(z) \\
&\triangleq \alpha\sum_{n\in\mathbb{Z}} x[n]z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator} \\
&\triangleq \sum_{n\in\mathbb{Z}} (\alpha x[n])z^{-n} && \text{by \textit{distributive} property} \\
&\triangleq \mathbf{Z}\left(\!\left(\alpha x[n]\right)\!\right) && \text{by definition of } \mathbf{Z} \text{ operator} \\
z^{-k}\check{x}(z) &= z^{-k}\mathbf{Z}\left(\!\left(x[n]\right)\!\right) && \text{by definition of } \check{x}(z) && \text{(left hypothesis)} \\
&\triangleq z^{-k}\sum_{n=-\infty}^{n=+\infty} x[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
&= \sum_{n=-\infty}^{n=+\infty} x[n]z^{-n-k} \\
&= \sum_{m-k=-\infty}^{m-k=+\infty} x[m-k]z^{-m} && \text{where } m \triangleq n+k && \implies n = m-k \\
&= \sum_{m=-\infty}^{m=+\infty} x[m-k]z^{-m} \\
&= \sum_{n=-\infty}^{n=+\infty} x[n-k]z^{-n} && \text{where } n \triangleq m \\
&\triangleq \mathbf{Z}\left(\!\left(x[n-k]\right)\!\right) && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
\mathbf{Z}\left(\!\left(x^*[n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
&\triangleq \left(\sum_{n\in\mathbb{Z}} x[n](z^*)^{-n}\right)^* && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
&\triangleq \check{x}^*(z^*) && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
\mathbf{Z}\left(\!\left(x[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
&= \sum_{-m\in\mathbb{Z}} x[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition 2.2 page 7)} \\
&= \sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition 2.2 page 7)} \\
&\triangleq \check{x}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
\mathbf{Z}\left(\!\left(x^*[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 2.4 page 8)} \\
&= \sum_{-m\in\mathbb{Z}} x^*[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x^*[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition 2.2 page 7)} \\
&= \sum_{m\in\mathbb{Z}} x^*[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition 2.2 page 7)} \\
&= \left(\sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z^*}\right)^{-m}\right)^* && \text{by \textit{absolutely summable} property} && \text{(Definition 2.2 page 7)}
\end{aligned}
$$

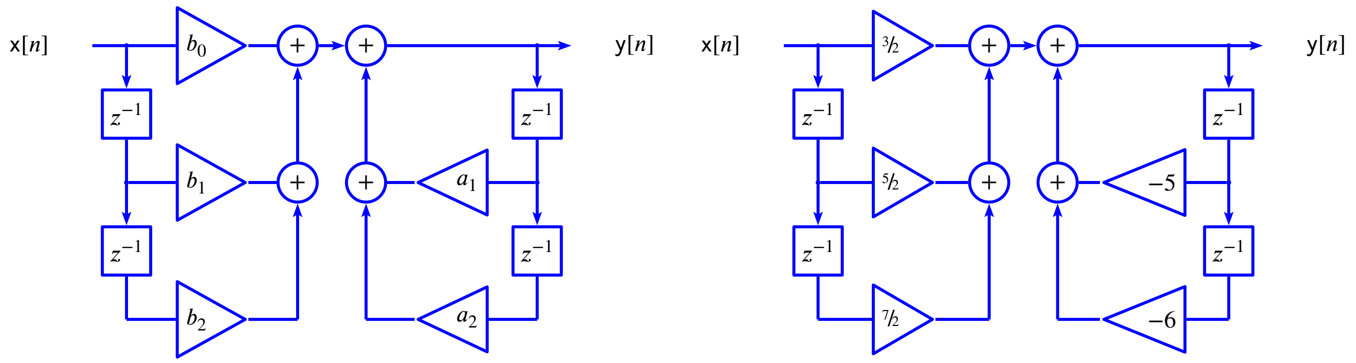


Figure 2.1: Direct form 1 order 2 IIR filters

$$\triangleq \check{x}^*\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition 2.4 page 8)}$$

Theorem 2.2 (convolution theorem). *Let $\star$ be the convolution operator* (Definition 2.3 page 7).

T H M

$$\mathbf{Z}\,\underbrace{\left(\langle\!\langle x_n\rangle\!\rangle \star \langle\!\langle y_n\rangle\!\rangle\right)}_{\text{sequence convolution}} = \underbrace{\left(\mathbf{Z}\,\langle\!\langle x_n\rangle\!\rangle\right)\ \left(\mathbf{Z}\,\langle\!\langle y_n\rangle\!\rangle\right)}_{\text{series multiplication}} \qquad \forall \langle\!\langle x_n\rangle\!\rangle_{n\in\mathbb{Z}}, \langle\!\langle y_n\rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x\star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by Definition 2.3 page 7}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by Definition 2.4 page 8}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k = n-m \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right]\\
&\triangleq \left(\mathbf{Z}\,\langle\!\langle x_n\rangle\!\rangle\right)\ \left(\mathbf{Z}\,\langle\!\langle y_n\rangle\!\rangle\right) && \text{by Definition 2.4 page 8}
\end{aligned}$$

2.3 From z-domain back to time-domain

$$\check{y}(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) - a_1z^{-1}Y(z) + a_2z^{-2}Y(z)$$

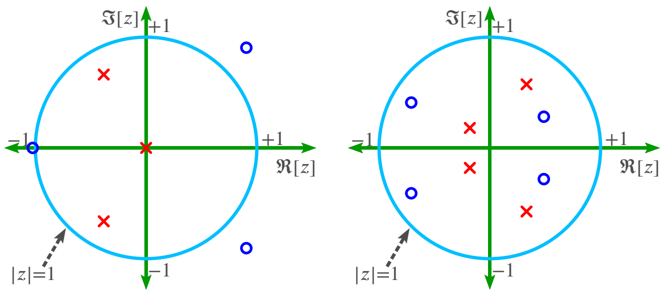
$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] - a_1y[n-1] - a_2y[n-2]$$

Example 2.1. See Figure 2.1 (page 10)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2\left(z^2+5z+6\right)} = \frac{\left({}^3\!/\!_2 z^2 + {}^5\!/\!_2 z + {}^7\!/\!_2\right)}{z^2+5z+6} = \frac{\left({}^3\!/\!_2 + {}^5\!/\!_2 z^{-1} + {}^7\!/\!_2 z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

2.4 Zero locations

The system property of *minimum phase* is defined in Definition 2.5 (next) and illustrated in Figure 2.2 (page 11).



not minimum phase *minimum phase*

Figure 2.2: Minimum Phase filter

Definition 2.5. [4] *Let* $\check{x}(z) \triangleq \mathbf{Z}\left(\!\left(x_n\right)\!\right)$ *be the* Z TRANSFORM (Definition 2.4 page 8) *of a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\left(\!\left(z_n\right)\!\right)_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\check{x}(z)$.

DEF *The sequence* $\left(\!\left(x_n\right)\!\right)$ *is* ***minimum phase*** *if*

$$\underbrace{\left|z_n\right| < 1 \qquad \forall n\in\mathbb{Z}}_{\check{x}(z) \text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem 2.3 (Robinson's Energy Delay Theorem). [5] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM

$$\left\{\begin{array}{lll} \mathsf{p} & \text{is MINIMUM PHASE} & \text{and} \\ \mathsf{q} & \text{is NOT } \textit{minimum phase} & \end{array}\right\} \implies \underbrace{\sum_{n=0}^{m-1}\left|a_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{p}(z)} \geq \underbrace{\sum_{n=0}^{m-1}\left|b_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{q}(z)} \qquad \forall 0 \leq m \leq N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle.

Example 2.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a

[4] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36
[5] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976), pages 52–53

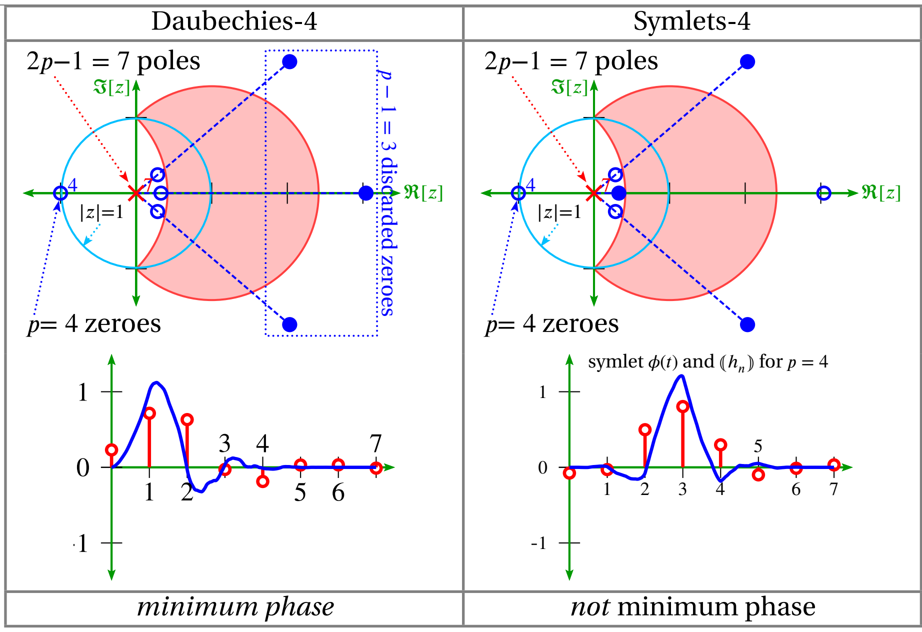


Figure 2.3: Daubechies-4 and Symlet-4 scaling functions pole-zero plots

scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure 2.3 (page 12).

2.5 Pole locations

Definition 2.6.

DEF *A filter (or system or operator)* **H** *is* ***causal*** *if its current output does not depend on future inputs.*

Definition 2.7.

DEF *A filter (or system or operator)* **H** *is* ***time-invariant*** *if the mapping it performs does not change with time.*

Definition 2.8.

DEF *An operation* **H** *is* ***linear*** *if any output y_n can be described as a linear combination of inputs x_n as in*

$$y_n = \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\mathsf{x}(n-m).$$

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem 2.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Example 2.3. Stable/unstable filters are illustrated in Figure 2.4 (page 13).

True or False? This filter has no poles:

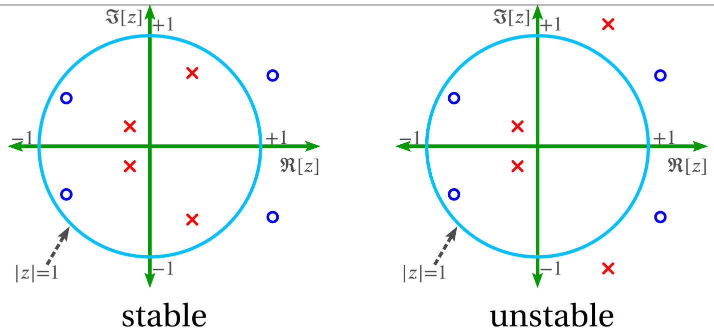
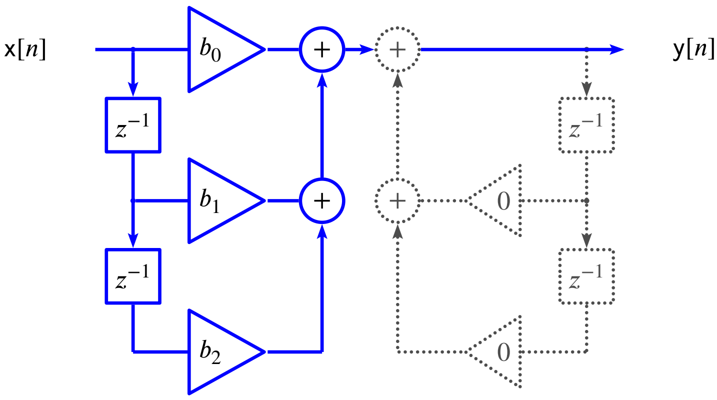
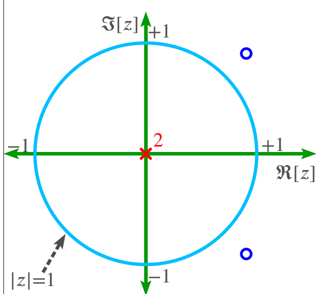


Figure 2.4: Pole-zero plot stable/unstable causal LTI filters (Example 2.3 page 12)

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$



$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$



2.6 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

Proposition 2.3.

PRP $\left\{\begin{array}{l}\text{ZEROS } \textit{and} \text{ POLES}\\ \textit{occur in} \text{ CONJUGATE PAIRS}\end{array}\right\} \implies \left\{\begin{array}{l}\text{COEFFICIENTS}\\ \textit{are} \text{ REAL.}\end{array}\right\}$

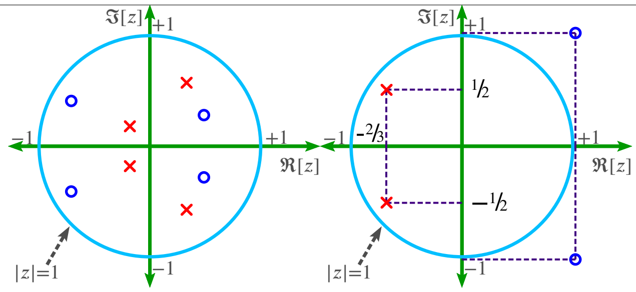


Figure 2.5: Conjugate pair structure yielding real coefficients

✎PROOF:

$$
\begin{aligned}
(z-p_1)(z-p_1^*) &= [z-(a+ib)][z-(a-ib)] \\
&= z^2 + [-a+ib-ib-a]z - [ib]^2 \\
&= z^2 - 2az + b^2
\end{aligned}
$$

⇨

Example 2.4. See Figure 2.5 (page 14).

$$
\begin{aligned}
H(z) &= G\frac{[z-z_1][z-z_2]}{[z-p_1][z-p_2]} = G\frac{[z-(1+i)][z-(1-i)]}{[z-(-\frac{2}{3}+i\frac{1}{2})][z-(-\frac{2}{3}-i\frac{1}{2})]} \\
&= G\frac{z^2 - z[(1-i)+(1+i)] + (1-i)(1+i)}{z^2 - z[(-\frac{2}{3}+i\frac{1}{2})+(-\frac{2}{3}+i\frac{1}{2})] + (-\frac{2}{3}+i\frac{1}{2})(-\frac{2}{3}+i\frac{1}{2})} \\
&= G\frac{z^2-2z+2}{z^2-\frac{4}{3}z+(\frac{4}{3}+\frac{1}{4})} = G\frac{z^2-2z+2}{z^2-\frac{4}{3}z+\frac{19}{12}}
\end{aligned}
$$

2.7 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma 2.1. *A causal LTI operator* **H** *can be expressed as a rational expression* $\check{h}(z)$.

$$
\begin{aligned}
\check{h}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \frac{\sum_{n=0}^{N} b_n z^{-n}}{1 + \sum_{n=1}^{N} a_n z^{-n}}
\end{aligned}
$$

A filter operation $\check{h}(z)$ can be expressed as a product of its roots (poles and zeros).

$$
\begin{aligned}
\check{h}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \alpha\frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}
\end{aligned}
$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with × and zeros with ○. An example is shown in Figure 2.6 page 15. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

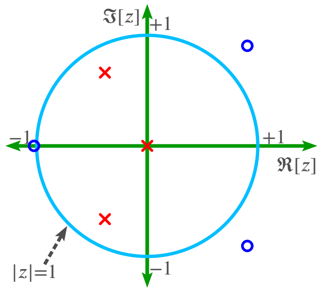


Figure 2.6: Pole-zero plot for rational expression with real coefficients

2.8 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{x}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{y}(z)$ is a *high-pass filter*.

Definition 2.9. [6] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES (Definition 2.1 page 7) *in* $\ell^2_{\mathbb{R}}$ (Definition 2.2 page 7).

DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter***.
Any triple $\left((\!(x_n)\!), (\!(y_n)\!), N\right)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition***.

Theorem 2.5 (CQF theorem). [7] *Let* $\check{y}(\omega)$ *and* $\check{x}(\omega)$ *be the* DTFT*s* (Definition 3.1 page 21) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$, *respectively, in* $\ell^2_{\mathbb{R}}$ (Definition 2.2 page 7).

THM

$$\begin{array}{rrcll}
\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{\text{(1) CQF in “time”}} & \iff \check{y}(z) & = & \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) & \text{(2) CQF in “z-domain”} \\
& \iff \check{y}(\omega) & = & \pm(-1)^N e^{-i\omega N} \check{x}^*(\omega+\pi) & \text{(3) CQF in “frequency”} \\
& \iff x_n & = & \pm(-1)^N (-1)^n y^*_{N-n} & \text{(4) “reversed” CQF in “time”} \\
& \iff \check{x}(z) & = & \pm z^{-N} \check{y}^*\left(\frac{-1}{z^*}\right) & \text{(5) “reversed” CQF in “z-domain”} \\
& \iff \check{x}(\omega) & = & \pm e^{-i\omega N} \check{y}^*(\omega+\pi) & \text{(6) “reversed” CQF in “frequency”}
\end{array}$$

$\forall N\in\mathbb{Z}$

✎PROOF:

[6] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

[7] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{y}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-}transform && \text{(Definition 2.4 page 8)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm \sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies && n = N-m\\
&= \pm(-1)^N z^{-N} \sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N} \sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*\\
&= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by definition of } z\text{-}transform && \text{(Definition 2.4 page 8)}
\end{aligned}$$

2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\check{y}(z) &= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by (2)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-}transform && \text{(Definition 2.4 page 8)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x^*_m \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-}transform && \text{(Definition 2.4 page 8)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x^*_m z^{-(N-m)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^n x^*_{N-n} z^{-n} && \text{where } n = N-m \implies && m \triangleq N-n\\
&\implies x_n = \pm(-1)^n x^*_{N-n}
\end{aligned}$$

3. Proof that (1) $\implies$ (3):

$$\begin{aligned}
\check{y}(\omega) &\triangleq \check{x}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition 3.1 page 21)}\\
&= \left[\pm(-1)^N z^{-N} \check{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)}\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i\pi} e^{i\omega}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i(\omega+\pi)}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition 3.1 page 21)}
\end{aligned}$$

4. Proof that (1) $\implies$ (6):

$$\begin{aligned}
\check{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 3.1 page 21)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x^*_{N-n}}_{CQF} e^{-i\omega n} && \text{by (1)}\\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x^*_m e^{-i\omega(N-m)} && \text{where } m \triangleq N-n \implies && n = N-m\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} (-1)^m x^*_m e^{i\omega m}
\end{aligned}$$

$$
\begin{aligned}
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} e^{i\pi m} x_m^* e^{i\omega m} \\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} x_m^* e^{i(\omega+\pi)m} \\
&= \pm(-1)^N e^{-i\omega N} \left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^* \\
&= \pm(-1)^N e^{-i\omega N} \breve{x}^*(\omega+\pi) && \text{by definition of } \textit{DTFT} && \text{(Definition 3.1 page 21)}
\end{aligned}
$$

5. Proof that (1) $\Longleftarrow$ (3):

$$
\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{y}(\omega) e^{i\omega n}\, d\omega && \text{by } \textit{inverse DTFT} && \text{(Theorem 3.3 page 27)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\breve{x}^*(\omega+\pi)}_{\text{right hypothesis}} e^{i\omega n}\, d\omega && \text{by right hypothesis} \\
&= \pm(-1)^N \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{x}^*(\omega+\pi) e^{i\omega(n-N)}\, d\omega && \text{by right hypothesis} \\
&= \pm(-1)^N \frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v) e^{i(v-\pi)(n-N)}\, dv && \text{where } v \triangleq \omega+\pi \implies && \omega = v-\pi \\
&= \pm(-1)^N e^{-i\pi(n-N)} \frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v) e^{iv(n-N)}\, dv \\
&= \pm(-1)^N \underbrace{(-1)^N}_{e^{i\pi N}} \underbrace{(-1)^n}_{e^{-i\pi n}} \left[\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}(v) e^{iv(N-n)}\, dv\right]^* \\
&= \pm(-1)^n x_{N-n}^* && \text{by } \textit{inverse DTFT} && \text{(Theorem 3.3 page 27)}
\end{aligned}
$$

6. Proof that (1) $\iff$ (4):

$$
\begin{aligned}
y_n = \pm(-1)^n x_{N-n}^* &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x_{N-n}^* \\
&\iff \pm(-1)^n y_n = x_{N-n}^* \\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x_{N-n}^*\right)^* \\
&\iff \pm(-1)^n y_n^* = x_{N-n} \\
&\iff x_{N-n} = \pm(-1)^n y_n^* \\
&\iff x_m = \pm(-1)^{N-m} y_{N-m}^* && \text{where } m \triangleq N-n \implies && n = N-m \\
&\iff x_m = \pm(-1)^{N-m} y_{N-m}^* \\
&\iff x_m = \pm(-1)^N(-1)^m y_{N-m}^* \\
&\iff x_n = \pm(-1)^N(-1)^n y_{N-n}^* && \text{by change of free variables}
\end{aligned}
$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

✏

Theorem 2.6. [8] *Let $\breve{y}(\omega)$ and $\breve{x}(\omega)$ be the* DTFT*s (Definition 3.1 page 21) of the sequences $(y_n)_{n\in\mathbb{Z}}$ and $(x_n)_{n\in\mathbb{Z}}$, respectively, in $\ell^2_{\mathbb{R}}$ (Definition 2.2 page 7).*

THM

Let $y_n = \pm(-1)^n x_{N-n}^*$ (CQF CONDITION, Definition 2.9 page 15). Then

$$
\left\{
\begin{array}{llcll}
\text{(A)} \quad \left[\frac{d}{d\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} = 0 & \iff & \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} & = 0 & \text{(B)} \\
& \iff & \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k & = 0 & \text{(C)} \\
& \iff & \sum_{k\in\mathbb{Z}} k^n y_k & = 0 & \text{(D)}
\end{array}
\right\} \quad \forall n\in\mathbb{W}
$$

[8] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

✎PROOF:

1. Proof that (A) $\implies$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} && \text{by (A)} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi)\Big|_{\omega=0} && \text{by \textit{CQF theorem}} \quad \text{(Theorem 2.5 page 15)} \\
&= (\pm)(-1)^N \sum_{\ell=0}^{n}\binom{n}{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{\ell}\left[e^{-i\omega N}\right]\cdot\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma F.2 page 103)} \\
&= (\pm)(-1)^N \sum_{\ell=0}^{n}\binom{n}{\ell}-iN^{\ell}e^{-i\omega N}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0} \\
&= (\pm)(-1)^N \cancel{e^{-i0N}} \sum_{\ell=0}^{n}\binom{n}{\ell}-iN^{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{x}^{(0)}(\pi) &= 0 \\
&\implies & \breve{x}^{(1)}(\pi) &= 0 \\
&\implies & \breve{x}^{(2)}(\pi) &= 0 \\
&\implies & \breve{x}^{(3)}(\pi) &= 0 \\
&\implies & \breve{x}^{(4)}(\pi) &= 0 \\
&\vdots & &\vdots \\
&\implies & \breve{x}^{(n)}(\pi) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

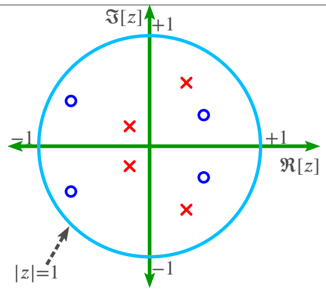
2. Proof that (A) $\impliedby$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} && \text{by (B)} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)e^{-i\omega N}\breve{y}^*(\omega+\pi)\Big|_{\omega=\pi} && \text{by \textit{CQF theorem}} \quad \text{(Theorem 2.5 page 15)} \\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{\ell}\left[e^{-i\omega N}\right]\cdot\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma F.2 page 103)} \\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}(-iN)^{\ell}e^{-i\omega N}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} \\
&= (\pm)\cancel{e^{-i\pi N}}\sum_{\ell=0}^{n}\binom{n}{\ell}-iN^{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} \\
&= (\pm)(-1)^N\sum_{\ell=0}^{n}\binom{n}{\ell}-iN^{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{y}^{(0)}(0) &= 0 \\
&\implies & \breve{y}^{(1)}(0) &= 0 \\
&\implies & \breve{y}^{(2)}(0) &= 0 \\
&\implies & \breve{y}^{(3)}(0) &= 0 \\
&\implies & \breve{y}^{(4)}(0) &= 0 \\
&\vdots & &\vdots \\
&\implies & \breve{y}^{(n)}(0) &= 0 \\
&\implies & \breve{y}^{(n)}(0) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

3. Proof that (B) $\iff$ (C): by Theorem 3.5 page 29

4. Proof that (A) $\iff$ (D): by Theorem 3.5 page 29



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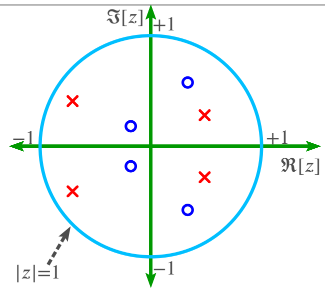


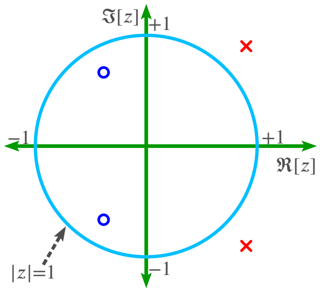
= 1

$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

5. Proof that (CQF) $\not\Leftarrow$ (A): Here is a counterexample: $\check{y}(\omega) = 0$.

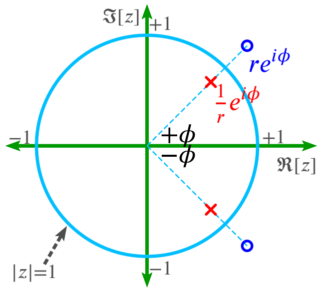
2.9 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.



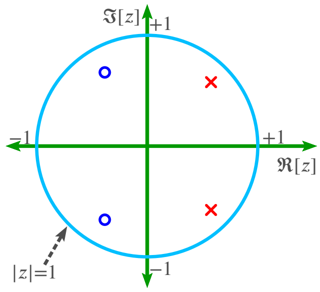
unstable

×



all-pass

=



stable!

$$\begin{aligned}
|A(z)|_{z=e^{i\omega}} &= \frac{1}{r}\left|\frac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|e^{i\phi}\left(\frac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|-z\left(\frac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\frac{1}{e^{-iv}}\left(\frac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right| \\
&= 1
\end{aligned}$$

$$\begin{aligned}
&= \left|\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|z\left(\frac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\frac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
&= \left|\frac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right|
\end{aligned}$$

CHAPTER 3

DISCRETE TIME FOURIER TRANSFORM

3.1 Definition

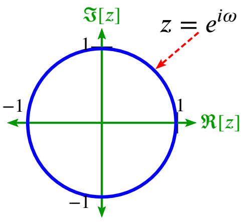
Definition 3.1.

DEF

*The **discrete-time Fourier transform** $\breve{\mathbf{F}}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$\left[\breve{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition 3.1 page 21) to the definition of the Z-transform (Definition 2.4 page 8), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.



3.2 Properties

Proposition 3.1 (DTFT periodicity). *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 3.1 page 21) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP

$$\underbrace{\breve{x}(\omega) = \breve{x}(\omega + 2\pi n)}_{\text{PERIODIC with period } 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned} \breve{x}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} && = \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi nm}} \\ &= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} && = \breve{x}(\omega) \end{aligned}$$

✑

Theorem 3.1. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 3.1 page 21) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM
$$\left\{ \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\,(\!(x[n])\!) \right\} \implies \left\{ \begin{array}{llcll} (1). & \breve{\mathbf{F}}\,(\!(x[-n])\!) & = & \tilde{x}(-\omega) & \text{and} \\ (2). & \breve{\mathbf{F}}\,(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}\,(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 3.1 page 21)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n=-m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}\\
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x^*[n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 3.1 page 21)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{i\omega n}\right)^* && \text{by \textit{distributive} property of } *\text{-\textbf{algebras}} && \text{(Definition C.3 page 76)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{-i(-\omega)n}\right)^*\\
&\triangleq \tilde{x}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x^*[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 3.1 page 21)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[-n]e^{i\omega n}\right)^* && \text{by \textit{distributive} property of } *\text{-\textbf{algebras}} && \text{(Definition C.3 page 76)}\\
&= \left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n=-m\\
&\triangleq \tilde{x}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

✎

Theorem 3.2. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 3.1 page 21) of a sequence* $(\!(x[n])\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM
$$\left\{ \begin{array}{lll} (1). & \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\,(\!(x[n])\!) & \text{and} \\ (2). & (\!(x[n])\!) \text{ is REAL-VALUED} & \end{array} \right\} \implies \left\{ \begin{array}{llcllll} (1). & \breve{\mathbf{F}}\,(\!(x[-n])\!) & = & \tilde{x}(-\omega) & & & \text{and} \\ (2). & \breve{\mathbf{F}}\,(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & = & \tilde{x}(\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}\,(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & = & \tilde{x}(-\omega) & \end{array} \right\}$$

✎PROOF:

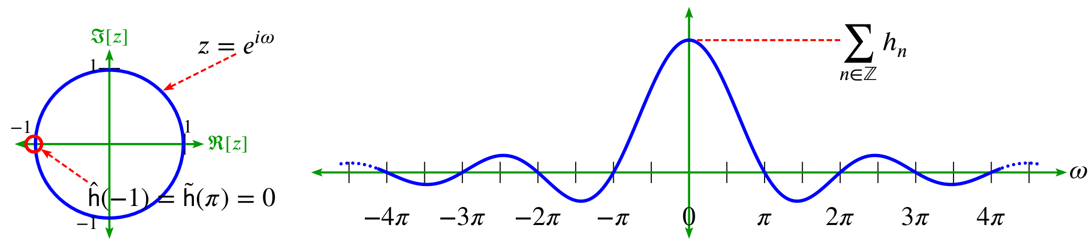
$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 3.1 page 21)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n=-m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}
\end{aligned}$$

$$\triangleq \tilde{x}(-\omega) \qquad \text{by left hypothesis}$$

$$\begin{aligned}
\boxed{\tilde{x}^*(-\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(x^*[n])\!)} && \text{by Theorem 3.1 page 22}\\
&= \breve{\mathbf{F}}\,(\!(x[n])\!) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{x}(\omega)} && \text{by definition of } \tilde{x}(\omega) && \text{(Definition 3.1 page 21)}
\end{aligned}$$

$$\begin{aligned}
\boxed{\tilde{x}^*(\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(x^*[-n])\!)} && \text{by Theorem 3.1 page 22}\\
&= \breve{\mathbf{F}}\,(\!(x[-n])\!) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{x}(-\omega)} && \text{by result (1)}
\end{aligned}$$

⇨



Proposition 3.2. *Let* $\check{x}(z)$ *be the* Z-TRANSFORM (*Definition 2.4 page 8*) *and* $\tilde{x}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM (*Definition 3.1 page 21*) *of* $(\!(x_n)\!)$.

PRP

$$\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\check{x}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\check{x}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}},\, c\in\mathbb{R}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{x}(z)\Big|_{z=1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \text{by definition of } \check{x}(z) \text{ (Definition 2.4 page 8)}\\
&= \sum_{n\in\mathbb{Z}} x_n && \text{because } z^n = 1 \text{ for all } n\in\mathbb{Z}\\
&= c && \text{by hypothesis (1)}
\end{aligned}$$

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\check{x}(\omega)\Big|_{\omega=0} &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} && \text{by definition of } \check{x}(\omega) && \text{(Definition 3.1 page 21)}\\
&= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1}\\
&= \check{x}(z)\Big|_{z=1} && \text{by definition of } \check{x}(z) && \text{(Definition 2.4 page 8)}\\
&= c && \text{by hypothesis (2)}
\end{aligned}$$

3. Proof that (3) $\Longrightarrow$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=0} \\
&= \breve{\mathsf{x}}(\omega) && \text{by definition of } \breve{\mathsf{x}}(\omega) && \text{(Definition 3.1 page 21)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

⇨

Proposition 3.3. *If the coefficients are **real**, then the magnitude response (MR) is **symmetric**.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathsf{h}}(-\omega)\right| &\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} &&\triangleq \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m] z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m] e^{i\omega m}\right| &&= \left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}^*[m] e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m] e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } real} &&= \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m] e^{-i\omega m}\right| \\
&\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} &&\triangleq \left|\tilde{\mathsf{h}}(\omega)\right|
\end{aligned}$$

⇨

Proposition 3.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}} (-1)^n x_n = c}_{\text{(1) in "time"}} \iff \underbrace{\breve{\mathsf{x}}(z)|_{z=-1} = c}_{\text{(2) in "z domain"}} \iff \underbrace{\breve{\mathsf{x}}(\omega)|_{\omega=\pi} = c}_{\text{(3) in "frequency"}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}
\breve{\mathsf{x}}(z)|_{z=-1} &= \left.\sum_{n\in\mathbb{Z}} x_n z^{-n}\right|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\bigg|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \sum_{n\in\mathbb{Z}} z^{-n} x_n \bigg|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n \bigg|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\implies$ (4):

(a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

(b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

(e) Solutions to these equations give

$$\sum_{n\in\mathbb{Z}} x_{2n} \triangleq A = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)$$
$$\sum_{n\in\mathbb{Z}} x_{2n+1} \triangleq B = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1} \\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1} \\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)} \\
&= c
\end{aligned}$$

✏

Lemma 3.1. *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* DTFT (Definition 3.1 page 21) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

LEM

$$\underbrace{(x_n \in \mathbb{R})_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{\mathsf{x}}(\omega)|^2 = |\breve{\mathsf{x}}(-\omega)|^2}_{\text{EVEN}} \qquad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
|\breve{\mathsf{x}}(\omega)|^2 &= |\breve{\mathsf{x}}(z)|^2\Big|_{z=e^{i\omega}} \\
&= \breve{\mathsf{x}}(z)\breve{\mathsf{x}}^*(z)\Big|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^*(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right] \\
&= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2\sum_{n\in\mathbb{Z}}\sum_{m>n} x_n x_m \cos[\omega(m-n)]
\end{aligned}
$$

Since cos is real and even, then $|\breve{x}(\omega)|^2$ must also be real and even. ✏

Theorem 3.3 (inverse DTFT). [2] *Let* $\breve{x}(\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM (Definition 3.1 page 21) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. *Let* $\breve{x}^{-1}$ *be the inverse of* $\breve{x}$.

THM

$$
\underbrace{\left\{ \breve{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\breve{x}(\omega)\triangleq\breve{\mathbf{F}}(\!(x_n)\!)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega)e^{i\omega n}\,d\omega \quad \forall\alpha\in\mathbb{R} \right\}}_{(\!(x_n)\!)=\breve{\mathbf{F}}^{-1}\breve{\mathbf{F}}(\!(x_n)\!)} \quad \forall(\!(x_n)\!)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}
$$

✎PROOF:

$$
\begin{aligned}
\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega)e^{i\omega n}\,d\omega &= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m}\right]}_{\breve{x}(\omega)} e^{i\omega n}\,d\omega && \text{by definition of } \breve{x}(\omega) \\
&= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\,d\omega \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\,d\omega \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \left[2\pi\bar{\delta}_{m-n}\right] \\
&= x_n
\end{aligned}
$$

✏

Theorem 3.4 (orthonormal quadrature conditions). [3] *Let* $\breve{x}(\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM (Definition 3.1 page 21) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION *at* n.

THM

$$
\begin{array}{rclcrcll}
\displaystyle\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} &=& 0 & \iff & \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) &=& 0 & \forall n\in\mathbb{Z}, \forall(\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}} \\
\displaystyle\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} &=& \bar{\delta}_n & \iff & |\breve{x}(\omega)|^2 + |\breve{x}(\omega+\pi)|^2 &=& 2 & \forall n\in\mathbb{Z}, \forall(\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}}
\end{array}
$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

1. Proof that $2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)$:

$$
\begin{aligned}
&2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} \\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n}
\end{aligned}
$$

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩

[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

$$
\begin{aligned}
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} \left(1+e^{i\pi n}\right) \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_{m} z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_{m} e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n \\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_{m} z^{m} + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_{m} e^{+i(\omega+\pi)m} \\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k} \left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^* \\
&\triangleq \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)
\end{aligned}
$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$
\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}
$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$
\begin{aligned}
2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 0 && \text{by right hypothesis}
\end{aligned}
$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$
\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n} \\
&= 2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}
$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$
\begin{aligned}
2\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 2 && \text{by right hypothesis}
\end{aligned}
$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

3.3 Derivatives

Theorem 3.5. [4] *Let* $\breve{x}(\omega)$ *be the* DTFT (Definition 3.1 page 21) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

THM

(A)	$\left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=0} = 0$	$\iff$	$\sum_{k\in\mathbb{Z}} k^n x_k = 0$	(B)	$\forall n\in\mathbb{W}$
(C)	$\left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} = 0$	$\iff$	$\sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0$	(D)	$\forall n\in\mathbb{W}$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=0} && \text{by hypothesis (A)}\\
&= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=0} && \text{by definition of } \breve{x}(\omega) \text{ (Definition 3.1 page 21)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\Big|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=0} &= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=0} && \text{by definition of } \tilde{g}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\right]\Big|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k\\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

3. Proof that $(C) \implies (D)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} && \text{by hypothesis (C)}\\
&= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=\pi} && \text{by definition of } \breve{x} \text{ (Definition 3.1 page 21)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=\pi}
\end{aligned}$$

[4] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right] \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k
\end{aligned}
$$

4. Proof that $(C) \impliedby (D)$:

$$
\begin{aligned}
\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=\pi} &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=\pi} && \text{by definition of } \breve{\mathsf{x}} \text{ (Definition 3.1 page 21)} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=\pi} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=\pi} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right] \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k \\
&= 0 && \text{by hypothesis (D)}
\end{aligned}
$$

⇰

CHAPTER 4

SAMPLE RATE CONVERSION

Theorem 4.1 (upsampling). *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES *(Definition 2.1 page 7) in* $\ell^2_{\mathbb{F}}$ *(Definition 2.2 page 7) over a* FIELD $\mathbb{F}$.

$$\text{THM}\quad y_n = \left\{\begin{array}{ll} x_{(n/N)} & \text{for } n \mod N = 0 \\ 0 & \text{otherwise} \end{array}\right\} \implies \check{y}(z) = \check{x}\left(z^N\right)$$

✎PROOF:

$$\begin{aligned}
\check{y}(z) &\triangleq \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-}transform && \text{(Definition 2.4 page 8)}\\
&= \sum_{n \mod N=0} y_n z^{-n} + \sum_{n \mod N\neq 0} y_n z^{-n} \\
&= \sum_{n \mod N=0} x_{n/N} z^{-n} + \cancel{\sum_{n \mod N\neq 0} 0 z^{-n}} 0 && \text{by definition of } (\!(y_n)\!) \\
&= \sum_{m\in\mathbb{Z}} x_m z^{-mN} && \text{where } m \triangleq n/N \implies n = mN \\
&= \sum_{m\in\mathbb{Z}} x_m \left(z^N\right)^{-m} \\
&\triangleq \check{x}\left(z^N\right) && \text{by definition of } z\text{-}transform && \text{(Definition 2.4 page 8)}
\end{aligned}$$

✏

Theorem 4.2 (downsampling). *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES *(Definition 2.1 page 7) in* $\ell^2_{\mathbb{F}}$ *(Definition 2.2 page 7) over a* FIELD $\mathbb{F}$.

$$\text{THM}\quad \left\{y_n = x_{(Nn)}\right\} \implies \left\{\check{y}(z) = \frac{1}{N}\sum_{m=0}^{N-1} \check{x}\left(e^{i\frac{2\pi m}{N}} z^{\frac{1}{N}}\right)\right\}$$

✎PROOF:

$$\begin{aligned}
\check{y}(z) &\triangleq \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-}transform && \text{(Definition 2.4 page 8)}\\
&= \sum_{n\in\mathbb{Z}} x_{(nN)} z^{-n} && \text{by definition of } (\!(y_n)\!)
\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} x_n \left[\bar{\delta}_{(n \mod N)}\right] z^{-\frac{n}{N}}$$

$$= \sum_{n\in\mathbb{Z}} x_n \left[\frac{1}{N}\sum_{m=0}^{N-1} e^{-i\frac{2\pi nm}{N}}\right] z^{-\frac{n}{N}}$$ by *Summation around unit circle* (Corollary H.1 page 139)

$$= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n\in\mathbb{Z}} x_n \left(e^{i\frac{2\pi m}{N}}\right)^{-n}\left(z^{\frac{1}{N}}\right)^{-n}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n\in\mathbb{Z}} x_n \left(e^{i\frac{2\pi m}{N}} z^{\frac{1}{N}}\right)^{-n}$$

$$\triangleq \frac{1}{N}\sum_{m=0}^{N-1} \check{x}\left(e^{i\frac{2\pi m}{N}} z^{\frac{1}{N}}\right)$$ by definition of *z-transform* (Definition 2.4 page 8)

CHAPTER 5

MAGNITUDE CHARACTERISTICS OF Z-FILTERS

5.1 The 0Hz and $F_s/2$ Gain

Proposition 5.1.

PRP

$$\left(\check{\mathsf{h}}(z) = \frac{\sum_{n=0}^{N} b_n z^{-n}}{\sum_{n=0}^{N} a_n z^{-n}}\right) \implies \left(\tilde{\mathsf{h}}(0) = \frac{\sum_{n=0}^{N} b_n}{\sum_{n=0}^{N} a_n}\right)$$

$$\check{\mathsf{x}}(z) \longrightarrow \boxed{\frac{b_0 z^2 + b_1 z + b_2}{a_0 z^2 + a_1 z + a_2}} \longrightarrow \check{\mathsf{y}}(z)$$

✎PROOF:

$$\tilde{\mathsf{h}}(0) = \tilde{\mathsf{h}}(\omega)\big|_{\omega=0} = \check{\mathsf{H}}\left(e^{i\omega}\right)\big|_{\omega=0} = \check{\mathsf{h}}(z)\big|_{z=1} = \left.\frac{\sum_{n=0}^{N} b_n z^{-n}}{\sum_{n=0}^{N} a_n z^{-n}}\right|_{z=1} = \frac{\sum_{n=0}^{N} b_n}{\sum_{n=0}^{N} a_n}$$

✏

Proposition 5.2.

PRP

$$\left(\check{\mathsf{h}}(z) = \frac{\sum_{n=0}^{N} b_n z^{-n}}{\sum_{n=0}^{N} a_n z^{-n}}\right) \implies \left(\tilde{\mathsf{h}}(\omega)\big|_{\omega=\frac{F_s}{2}} = \frac{\sum_{n=0}^{N} (-1)^n b_n}{\sum_{n=0}^{N} (-1)^n a_n}\right)$$

✎PROOF:

$$\tilde{\mathsf{h}}(\omega)\Big|_{\omega=\frac{F_s}{2}} = \check{\mathsf{h}}(z)\Big|_{z=e^{i\pi}} = \check{\mathsf{h}}(z)\Big|_{z=-1} = \left.\frac{\sum\limits_{n=0}^{N}(-1)^n b_n}{\sum\limits_{n=0}^{N}(-1)^n a_n}\right|_{z=-1} = \frac{\sum\limits_{n=0}^{N}(-1)^n b_n}{\sum\limits_{n=0}^{N}(-1)^n a_n}$$

⇨

5.2 Pole and zero location analysis

Note the following:

- The frequency response of $\check{\mathsf{h}}(z)$ **repeats** every 2π. Proposition 3.1 page 21
- If the coefficients are **real**, then the magnitude response is **symmetric** Proposition 3.3 page 24
- Moments and derivatives are related: Theorem 3.5 page 29

The pole zero locations of a digital filter determine the magnitude and phase frequency response of the digital filter.[1] This can be seen by representing the pole and zero vectors in the complex z-plane. Each of these vectors has a magnitude M and a direction θ. Also, each factor $(z - z_i)$ and $(z - p_i)$ can be represented as vectors as well (the difference of two vectors). Each of these factors can be represented by a magnitude/phase factor $M_i e^{i\theta_i}$. The overall magnitude and phase of $H(z)$ can then be analyzed.

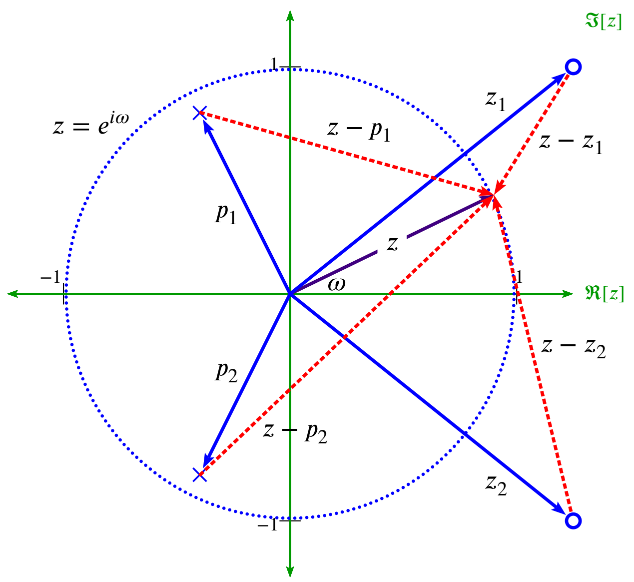


Figure 5.1: Vector response of digital filter (Example 5.1 page 35)

[1] Cadzow (1987), pages 90–91, Ifeachor and Jervis (1993) pages 134–136 ⟨§“1.5.3 Geometric evaluation of frequency response”⟩, Ifeachor and Jervis (2002) pages 201–203 ⟨§“4.5.3 Geometric evaluation of frequency response”⟩

Example 5.1. Take the following filter for example.

$$
\begin{aligned}
H(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \\
&= \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} \\
&= \frac{M_1 e^{i\theta_1} \; M_2 e^{i\theta_2}}{M_3 e^{i\theta_3} \; M_4 e^{i\theta_4}} \\
&= \left(\frac{M_1 M_2}{M_3 M_4}\right)\left(\frac{e^{i\theta_1} e^{i\theta_2}}{e^{i\theta_3} e^{i\theta_4}}\right)
\end{aligned}
$$

This is illustrated in Figure 5.1 (page 34). The unit circle represents frequency in the Fourier domain. The frequency response of a filter is just a rotating vector on this circle. The magnitude response of the filter is just then a *vector sum*. For example, the magnitude of any $H(z)$ is as follows:

$$
|H(z)| = \frac{|(z - z_1)| \; |(z - z_2)|}{|(z - p_1)| \; |(z - p_2)|}
$$

5.3 Coefficient analysis

Lemma 5.1.

$$
\boxed{\sum_{n=0}^{N}\sum_{m=0}^{N} a_n a_m e^{-i\omega(n-m)} = \sum_{n=0}^{N} \left|a_n\right|^2 + 2\sum_{n=0}^{N}\sum_{m=n+1}^{N} \Re\left[a_n a_m^*\right]\cos[\omega(n-m)]}
$$

Example 5.2. This example graphically illustrates Lemma 5.1 (page 35) for the case $N = 4$.

$$
\begin{aligned}
\sum_{n=0}^{4}\sum_{m=0}^{4} a_n a_m e^{-i\omega(n-m)} &= \left(\begin{array}{c|ccccc}
 & m=0 & m=1 & m=2 & m=3 & m=4 \\
\hline
n=0 & a_0 a_0^* & a_0 a_1^* e^{+i\omega} & a_0 a_2^* e^{+i2\omega} & a_0 a_3^* e^{+i3\omega} & a_0 a_4^* e^{+i4\omega} \\
n=1 & a_1 a_0^* e^{-i\omega} & a_1 a_1^* & a_1 a_2^* e^{+i\omega} & a_1 a_3^* e^{+i2\omega} & a_1 a_4^* e^{+i3\omega} \\
n=2 & a_2 a_0^* e^{-i2\omega} & a_2 a_1^* e^{-i\omega} & a_2 a_2^* & a_2 a_3^* e^{+i\omega} & a_2 a_4^* e^{+i2\omega} \\
n=3 & a_3 a_0^* e^{-i3\omega} & a_3 a_1^* e^{-i2\omega} & a_3 a_2^* e^{-i\omega} & a_3 a_3^* & a_3 a_4^* e^{+i\omega} \\
n=4 & a_4 a_0^* e^{-i4\omega} & a_4 a_1^* e^{-i3\omega} & a_4 a_2^* e^{-i2\omega} & a_4 a_3^* e^{-i\omega} & a_4 a_4^*
\end{array}\right) \\
&= \sum_{n=0}^{4} a_n a_n^* + 2\sum_{n=0}^{4}\sum_{m=n+1}^{4} \left[\left(a_n a_m^* e^{i\omega}\right) + \left(a_n^* a_m e^{-i\omega}\right)\right] \\
&= \sum_{n=0}^{4} a_n a_n^* + 2\sum_{n=0}^{4}\sum_{m=n+1}^{4} \left[\left(a_n a_m^* e^{i\omega}\right) + \left(a_n a_m^* e^{i\omega}\right)^*\right] \\
&= \sum_{n=0}^{4} a_n a_n^* + \sum_{n=0}^{4}\sum_{m=n+1}^{4} 2\Re\left[\left(a_n a_m^* e^{i\omega}\right)\right] \\
&= \sum_{n=0}^{4} \left|a_n\right|^2 + 2\sum_{n=0}^{4}\sum_{m=n+1}^{4} \Re\left[a_n a_m^*\right]\Re\left[e^{i\omega(n-m)}\right]
\end{aligned}
$$

$$= \sum_{n=0}^{4} |a_n|^2 + 2\sum_{n=0}^{4}\sum_{m=n+1}^{4} \Re\left[a_n a_m^*\right]\cos[\omega(n-m)]$$

Lemma 5.2.

LEM

$$\left\{ \breve{q}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n} \right\} \implies \left\{ |\tilde{q}(\omega)|^2 = \sum_{n=0}^{N} |a_n|^2 + 2\sum_{n=0}^{N}\sum_{m=n+1}^{N} a_n a_m^* \cos[\omega(n-m)] \right\}$$

✎PROOF:

$$\begin{aligned}
\boxed{|\tilde{q}(\omega)|^2} &= \left[|\breve{q}(z)|^2\right]_{z=e^{i\omega}} \\
&= \left[\breve{q}(z)\breve{q}^*(z)\right]_{z=e^{i\omega}} \\
&= \left[\left(\sum_{n=0}^{N} a_n z^{-n}\right)\left(\sum_{n=0}^{N} a_n^* z^{n}\right)\right]_{z=e^{i\omega}} \\
&= \left[\sum_{m=0}^{N}\sum_{n=0}^{N} a_m a_n^* z^{n-m}\right]_{z=e^{i\omega}} \\
&= \sum_{m=0}^{N}\sum_{n=0}^{N} a_m a_n^* e^{i\omega(n-m)} \\
&= \sum_{n=0}^{N} a_n^2 + 2\sum_{n=0}^{N}\sum_{m=n+1}^{N} a_n a_m \cos[\omega(n-m)] && \text{by Lemma 5.1 page 35}
\end{aligned}$$

✏

Theorem 5.1.

THM

$$\left|\tilde{h}(\omega)\right|^2 = \frac{\sum_{n=0}^{N} b_n^2 + 2\sum_{n=0}^{N}\sum_{m=n+1}^{N} b_n b_m \cos[\omega(n-m)]}{\sum_{n=0}^{N} a_n^2 + 2\sum_{n=0}^{N}\sum_{m=n+1}^{N} a_n a_m \cos[\omega(n-m)]}$$

✎PROOF:

$$\begin{aligned}
\boxed{\left|\tilde{h}(\omega)\right|^2} &= \left|\breve{h}(z)\right|^2_{z=e^{i\omega}} \\
&= \left[\breve{h}(z)\breve{h}^*(z)\right]_{z=e^{i\omega}} \\
&= \left|\frac{\sum_{n=0}^{N} b_n z^{-n}}{\sum_{n=0}^{N} a_n z^{-n}}\right|^2_{z=e^{i\omega}} \\
&= \boxed{\frac{\sum_{n=0}^{N} b_n^2 + 2\sum_{n=0}^{N}\sum_{m=n+1}^{N} b_n b_m \cos[\omega(n-m)]}{\sum_{n=0}^{N} a_n^2 + 2\sum_{n=0}^{N}\sum_{m=n+1}^{N} a_n a_m \cos[\omega(n-m)]}} && \text{by Lemma 5.2 page 36}
\end{aligned}$$

✏

Theorem 5.2.

THM $$\frac{\mathrm{d}}{\mathrm{d}\omega}\left|\tilde{\mathsf{h}}(\omega)\right|^2_{\omega=0} = 0$$

✎PROOF:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\omega}\left|\check{\mathsf{h}}(z)\right|^2_{z=e^{i\omega},\omega=0} &= \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\check{\mathsf{h}}(z)\check{\mathsf{h}}^*(z)\right]_{z=e^{i\omega},\omega=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{\sum_{n=0}^{N} b_n^2 + 2\sum_{n=0}^{N}\sum_{m=n}^{N} b_n b_m \cos[\omega(n-m)]}{\sum_{n=0}^{N} a_n^2 + 2\sum_{n=0}^{N}\sum_{m=n}^{N} a_n a_m \cos[\omega(n-m)]}\right]_{\omega=0} \\
&\triangleq \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{\mathsf{f}(\omega)}{\mathsf{g}(\omega)}\right]_{\omega=0} \\
&= \left[\frac{\mathsf{f}'(\omega)\mathsf{g}(\omega)-\mathsf{f}(\omega)\mathsf{g}'(\omega)}{\mathsf{g}^2(\omega)}\right]_{\omega=0} && \text{by the \textbf{Quotient Rule}} \\
&= 0 \quad \left(\text{because} \quad \begin{array}{ll} \frac{\mathrm{d}}{\mathrm{d}\omega}\text{constant}=0 & \text{and} \\ \frac{\mathrm{d}}{\mathrm{d}\omega}\cos(k\omega) = -\sin(k\omega) = 0 & \text{at } \omega = 0,\ \pi \end{array}\right)
\end{aligned}$$

⇰

5.4 Conversion from low-pass to high-pass

Theorem 5.3.

THM $\left\{ \check{\mathsf{h}}(z) \text{ is } \textbf{\textit{low-pass}} \right\} \implies \left\{ \check{\mathsf{h}}(-z) \text{ is } \textbf{\textit{high-pass}} \right\}$

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathsf{g}}(\omega)\right|^2 &\triangleq \left|\check{\mathsf{h}}(-z)\right|_{z=e^{i\omega}} && \text{by definition of } \tilde{\mathsf{g}}(\omega) \\
&= \left|\check{\mathsf{h}}(e^{-i\pi}z)\right|_{z=e^{i\omega}} \\
&= \left|\check{\mathsf{h}}(z)\right|_{z=e^{i\omega}e^{-i\pi}} \\
&= \left|\check{\mathsf{h}}(z)\right|_{z=e^{i(\omega-\pi)}} \\
&\triangleq \left|\tilde{\mathsf{h}}(\omega-\pi)\right|^2 && \text{by definition of } \tilde{\mathsf{h}}(\omega)
\end{aligned}$$

⇰

CHAPTER 6

COEFFICIENT CALCULATION

6.1 IIR order 1 filter

Lemma 6.1 (order 1 filter). *Let* $\check{\mathsf{h}}(z) \triangleq \frac{a+bz^{-1}}{1+cz^{-1}}$ *be the* Z-TRANSFORM $[\mathbf{Z}\mathsf{h}(n)](z)$ (Definition 2.4 page 8) *and* $\tilde{\mathsf{h}}(\omega)$ *be the* DTFT $\left[\check{\mathbf{F}}\mathsf{h}(n)\right](\omega)$ (Definition 3.1 page 21) *of a sequence* $\mathsf{h}(n)$.

LEM

$$\left\{\begin{array}{llcll} (1). & \{a,b,c\} & \in & \mathbb{R} & \text{and} \\ (2). & c & \notin & \{-1,+1\} & \text{and} \\ (3). & b & \neq & ac & \end{array}\right\} \implies \left\{\begin{array}{lllll} (A). & \left|\tilde{\mathsf{h}}(\omega)\right|^2 & = & \dfrac{a^2+2ab\cos(\omega)+b^2}{1+2c\cos(\omega)+c^2} & \text{and} \\ (B). & \check{\mathsf{h}}(z) \text{ has a ZERO at } z = -\frac{b}{a} & & & \text{and} \\ (C). & \check{\mathsf{h}}(z) \text{ has a POLE at } z = -c & & & \end{array}\right\}$$

✎PROOF:

1. Proof for (A):

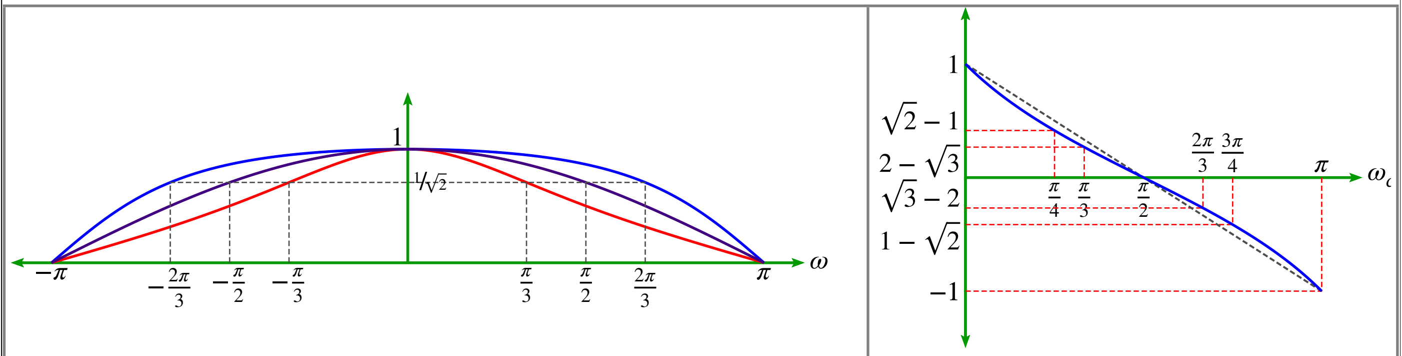
$$\begin{aligned} \boxed{\left|\check{\mathsf{h}}(z)\right|^2_{z=e^{i\omega}}} &= \check{\mathsf{h}}(z)\check{\mathsf{h}}^*(z)\Big|_{z=e^{i\omega}} \\ &= \check{\mathsf{h}}(z)\check{\mathsf{h}}(z^{-1})\Big|_{z=e^{i\omega}} && \text{by left hypothesis (1)} \\ &= \left(\frac{a+be^{-i\omega}}{1+ce^{-i\omega}}\right)\left(\frac{a+be^{i\omega}}{1+ce^{i\omega}}\right) && = \frac{a^2+abe^{-i\omega}+abe^{i\omega}+b^2}{1+ce^{i\omega}+ce^{-i\omega}+c^2} \\ &= \frac{a^2+2ab\cos(\omega)+b^2}{1+2c\cos(\omega)+c^2} && \text{by } \textit{Euler formulas} \text{ (Corollary G.2 page 113)} \end{aligned}$$

2. Proof for (B):

$$\begin{aligned} \check{\mathsf{h}}(z)\Big|_{z=-b/a} &\triangleq \left.\frac{a+bz^{-1}}{1+cz^{-1}}\right|_{z=-b/a} = 0 \\ &= \frac{a+b\left(-\frac{b}{a}\right)^{-1}}{1+c\left(-\frac{b}{a}\right)^{-1}} \\ &= \frac{a-a}{1-\frac{ac}{b}} \end{aligned}$$

$$= \frac{0}{b - ac}$$

6.2 1st Order Low-Pass calculation



magnitude $\left|\tilde{\mathsf{h}}(\omega)\right|$ for $\omega_c \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}\right\}$ | *pole* locations of $\check{\mathsf{h}}(z)$ for $\omega_c \in [0 : \pi]$

Figure 6.1: order 1 low pass filter of Theorem 6.1 (page 40) characteristics

Theorem 6.1 (order 1 low-pass filter). *Let* $\check{\mathsf{h}}(z) \triangleq \frac{a+bz^{-1}}{1+cz^{-1}}$ *be the* Z-TRANSFORM $[\mathbf{Z}\mathsf{h}(n)](z)$ (Definition 2.4 page 8) *and* $\tilde{\mathsf{h}}(\omega)$ *be the* DTFT $\left[\check{\mathbf{F}}\mathsf{h}(n)\right](\omega)$ (Definition 3.1 page 21) *of a sequence* $\mathsf{h}(n)$.

THM

$$\left\{\begin{array}{llcll}
(1). & \tilde{\mathsf{h}}(0) & = & 1 & \text{and}\\
(2). & \tilde{\mathsf{h}}(\pi) & = & 0 & \text{and}\\
(3). & \left|\tilde{\mathsf{h}}(\omega_c)\right|^2 & = & \frac{1}{2} & \text{and}\\
(4). & \{a,b,c\} & \in & \mathbb{R} & \text{and}\\
(5). & c & \notin & \{-1,+1\} &
\end{array}\right\}
\implies
\left\{\begin{array}{llcll}
(A). & c & = & \dfrac{-1+\sin(\omega_c)}{\cos(\omega_c)} & \text{and}\\
(B). & b & = & (c+1)/2 & \text{and}\\
(C). & a & = & b & \text{and}\\
(D). & \multicolumn{3}{l}{\check{\mathsf{h}}(z) \text{ has a ZERO at } z=-1} & \text{and}\\
(E). & \multicolumn{3}{l}{\check{\mathsf{h}}(z) \text{ has a POLE at } z=-c} & \text{and}\\
(F). & \left|\tilde{\mathsf{h}}(\omega)\right|^2 & = & \dfrac{1}{2}\left(\dfrac{(c+1)^2[1+\cos(\omega)]}{c^2+2c\cos(\omega)+1}\right) &
\end{array}\right\}$$

✎PROOF:

1. Proof that $a = b$:

$$\begin{aligned}
0 &= \tilde{\mathsf{h}}(\pi) && \text{by left hypothesis (2)}\\
&= \left\{\check{\mathsf{h}}(z)\right\}_{z=e^{i\pi}} = \left\{\check{\mathsf{h}}(z)\right\}_{z=-1} = \left\{\frac{a+bz^{-1}}{1+cz^{-1}}\right\}_{z=-1} = \frac{a-b}{1-c}\\
\implies & \boxed{a = b} && \text{by left hypothesis (5)}
\end{aligned}$$

2. Proof that $a = (c+1)/2$:

$$\begin{aligned}
1 &= \tilde{\mathsf{h}}(0) && \text{by left hypothesis (1)} = \left\{\check{\mathsf{h}}(z)\right\}_{z=e^{i0}} = \left\{\check{\mathsf{h}}(z)\right\}_{z=1} = \left\{\frac{a+bz^{-1}}{1+cz^{-1}}\right\}_{z=1} = \frac{a+b}{1+c}\\
&= \frac{2a}{1+c} && \text{by item (1) page 40}\\
\implies & \boxed{a = \frac{c+1}{2}} && \text{by left hypothesis (5)}
\end{aligned}$$

3. Proof for (F):

$$\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \frac{a^2 + 2ab\cos(\omega) + b^2}{1 + 2c\cos(\omega) + c^2} && \text{by Lemma 6.1 page 39}\\
&= \frac{2a^2[1+\cos(\omega)]}{c^2 + 2c\cos(\omega) + 1} && \text{by item (1) page 40}\\
&= \frac{2\left(\frac{c+1}{2}\right)^2[1+\cos(\omega)]}{c^2 + 2c\cos(\omega) + 1} && \text{by item (2) page 40}\\
&= \left(\frac{1}{2}\right)\frac{(c+1)^2[1+\cos(\omega)]}{c^2 + 2c\cos(\omega) + 1}
\end{aligned}$$

4. lemma. $c^2\cos(\omega_c) + 2c + \cos(\omega_c) = 0$. Proof:

$$\begin{aligned}
\frac{1}{2} &= \left|\tilde{\mathsf{h}}(\omega_c)\right|^2 && \text{by left hypothesis (3)}\\
&= \frac{1}{2}\left(\frac{(c+1)^2\left[1+\cos(\omega_c)\right]}{c^2 + 2c\cos(\omega_c) + 1}\right) && \text{by item (3) page 41}
\end{aligned}$$

$$\implies c^2 + 2c\cos(\omega_c) + 1 = (c+1)^2\left[1+\cos(\omega_c)\right]$$

$$\implies c^2\left[1 - 1 - \cos(\omega_c)\right] + c\left[2\cos(\omega_c) - 2 - 2\cos(\omega_c)\right] + \left[1 - 1 - \cos(\omega_c)\right] = 0$$

$$\implies c^2\cos(\omega_c) + 2c + \cos(\omega_c) = 0$$

5. Proof for (A):

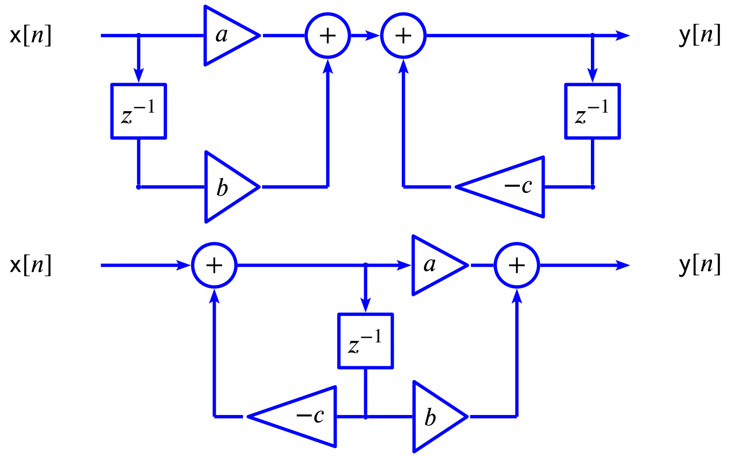
$$\begin{aligned}
c &= \frac{-2 \pm \sqrt{(2)^2 - 4\cos^2(\omega_c)}}{2\cos(\omega_c)} && \text{by (4) lemma page 41 and \textit{Quadratic Equation}}\\
&= \frac{-1 \pm \sqrt{1 - \cos^2(\omega_c)}}{\cos(\omega_c)}\\
&= \frac{-1 \pm \sin(\omega_c)}{\cos(\omega_c)} && \text{by Theorem G.4 page 111}
\end{aligned}$$

$$\implies \boxed{c = \frac{-1 + \sin(\omega_c)}{\cos(\omega_c)}}$$

6. Proof that the zero is at $z = -1$:

$$\begin{aligned}
z &= -\frac{b}{a} && \text{by Lemma 6.1 page 39}\\
&= -\frac{a}{a} && \text{by item (1) page 40}\\
&= -1
\end{aligned}$$

7. Proof that the pole is at $-c$: by Lemma 6.1 page 39



Example 6.1 (order 1 low-pass filter with corner frequency $\omega_c = \frac{2}{3}\pi$).

$$
\begin{aligned}
c &= \frac{-1+\sin(\omega_c)}{\cos(\omega_c)} = \frac{-1+\sin(^1\!/\!_3\pi)}{\cos(^1\!/\!_3\pi)} = \frac{-1+\frac{\sqrt{3}}{2}}{\frac{1}{2}} && = \sqrt{3}-2 \\
a &= \frac{c+1}{2} = \frac{\left(\sqrt{3}-2\right)+1}{2} && = \frac{\sqrt{3}+1}{2} \\
b &= a && = \frac{\sqrt{3}+1}{2} \\
|H(\omega)|^2 &= \left|\check{\mathsf{h}}(z)\right|^2_{z=e^{i\omega}} = \check{\mathsf{h}}(z)\check{\mathsf{h}}^*(z)\Big|_{z=e^{i\omega}} && = \left(\frac{1}{2}\right)\frac{(c+1)^2[1+\cos(\omega)]}{c^2+2c\cos(\omega)+1}
\end{aligned}
$$

$|H(\omega)|^2$

For a C++ implementation, see Section L.1 (page 167).

6.3 1st Order High-Pass calculation

$$\boxed{\check{\mathsf{h}}(z) = \frac{a+bz^{-1}}{1+cz^{-1}}}$$

$$
\begin{aligned}
0 &= \left|\check{\mathsf{h}}(z)\right|_{z=e^{i0}=1} &&= \left.\frac{a+bz^{-1}}{1+cz^{-1}}\right|_{z=1} &&= \frac{a+b}{1+c} &&\implies \boxed{a=-b} \\
1 &= \left|\check{\mathsf{h}}(z)\right|_{z=e^{i\pi}=-1} &&= \left.\frac{a+bz^{-1}}{1+cz^{-1}}\right|_{z=-1} &&= \frac{a-b}{1-c} = \frac{2a}{1-c} &&\implies \boxed{a=\frac{1-c}{2}}
\end{aligned}
$$

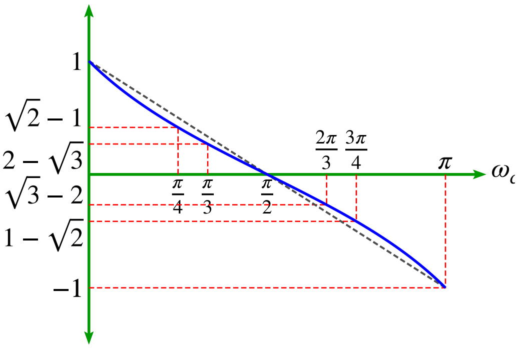
$$
\begin{aligned}
\boxed{\left|\check{\mathsf{h}}(z)\right|^2_{z=e^{i\omega}}} &= \left[\check{\mathsf{h}}(z)\check{\mathsf{h}}^*(z)\right]_{z=e^{i\omega}} \\
&= \left(\frac{a+be^{-i\omega}}{1+ce^{-i\omega}}\right)\left(\frac{a+be^{i\omega}}{1+ce^{i\omega}}\right) && = \frac{a^2+abe^{-i\omega}+abe^{i\omega}+b^2}{1+ce^{i\omega}+ce^{-i\omega}+c^2} \\
&= \frac{a^2+2ab\cos(\omega)+b^2}{1+2c\cos(\omega)+c^2} \\
&= \frac{2a^2[1-\cos(\omega)]}{c^2+2c\cos(\omega)+1} && \text{because } a=-b \\
&= \frac{2\left(\frac{1-c}{2}\right)^2[1-\cos(\omega)]}{c^2+2c\cos(\omega)+1} && \text{because } a=\frac{1-c}{2} \\
&= \boxed{\left(\frac{1}{2}\right)\frac{(1-c)^2[1-\cos(\omega)]}{c^2+2c\cos(\omega)+1}}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2} = \left|\check{\mathsf{h}}(z)\right|^2_{z=e^{i\omega_c}} = \left(\frac{1}{2}\right)\frac{(1-c)^2[1-\cos(\omega)]}{c^2+2c\cos(\omega)+1} \\
&\implies c^2+2c\cos(\omega_c)+1 = (1-c)^2\left[1-\cos(\omega_c)\right] \\
&\implies c^2\left[1-1+\cos(\omega_c)\right]+ \\
&\qquad c\left[2\cos(\omega_c)+2-2\cos(\omega_c)\right]+ \\
&\qquad \left[1-1+\cos(\omega_c)\right] \\
&\qquad =0 \\
&\implies c^2\cos(\omega_c)+2c+\cos(\omega_c)=0
\end{aligned}
$$

$$
\begin{aligned}
\implies c &= \frac{-2\pm\sqrt{(2)^2-4\cos^2(\omega_c)}}{2\cos(\omega_c)} && \text{by \textit{Quadratic Equation}} \\
&= \frac{-1\pm\sqrt{1-\cos^2(\omega_c)}}{\cos(\omega_c)} \\
&= \frac{-1\pm\sin(\omega_c)}{\cos(\omega_c)} && \text{because } \sin^2 x+\cos^2 x = 1 \text{ for all } x\in\mathbb{R} \\
\implies c &= \boxed{\frac{-1+\sin(\omega_c)}{\cos(\omega_c)}} && \text{because want pole inside unit circle}
\end{aligned}
$$

Where is the zero? Where is the pole?

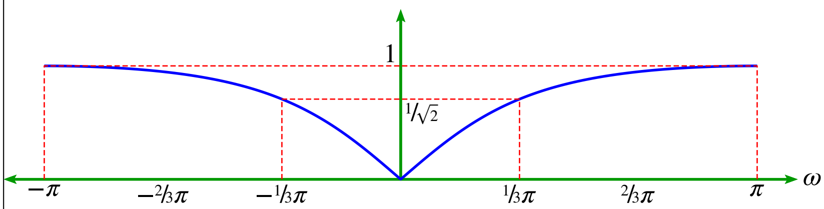
The zero is at z=+1. The pole is at $\boxed{z = -c = \dfrac{1-\sin(\omega_c)}{\cos(\omega_c)}}$

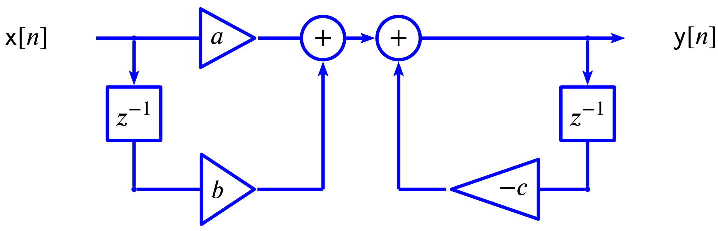


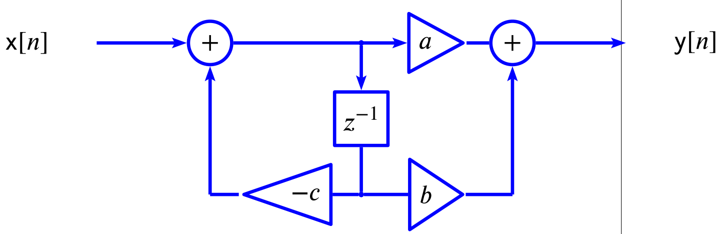
Example 6.2. order 1 **high-pass** with corner frequency $\omega_c = \frac{1}{3}\pi$

$$
\begin{aligned}
c &= \frac{-1+\sin(\omega_c)}{\cos(\omega_c)} = \frac{-1+\sin(1/3\pi)}{\cos(1/3\pi)} = \frac{-1+\frac{\sqrt{3}}{2}}{\frac{1}{2}} && = \sqrt{3}-2 \\
a &= \frac{1-c}{2} = \frac{1-\left(\sqrt{3}-2\right)}{2} && = \frac{1-\sqrt{3}}{2} \\
b &= -a && = \frac{\sqrt{3}-1}{2}
\end{aligned}
$$

$$
|H(\omega)|^2 = \left|\breve{\mathsf{h}}(z)\right|^2_{z=e^{i\omega}} = \breve{\mathsf{h}}(z)\breve{\mathsf{h}}^*(z)\Big|_{z=e^{i\omega}} = \left(\frac{1}{2}\right)\frac{2\left(\frac{1-c}{2}\right)^2[1-\cos(\omega)]}{c^2+2c\cos(\omega)+1}
$$







So $c = \sqrt{3}-2$, $\quad a = \frac{1-c}{2} = \frac{3-\sqrt{3}}{2}$, $\quad b = -a = \frac{\sqrt{3}-3}{2}$

$$
\begin{aligned}
H_{hp}(z) &= \frac{a+bz^{-1}}{1+cz^{-1}} \\
&= \frac{\left(\frac{3-\sqrt{3}}{2}\right)+\left(\frac{\sqrt{3}-3}{2}\right)z^{-1}}{1+\left(\sqrt{3}-2\right)z^{-1}}
\end{aligned}
$$

$$= \frac{\left(\frac{[2-\sqrt{3}]+1}{2}\right) + \left(\frac{[2-\sqrt{3}]+1}{2}\right)(-z)^{-1}}{1+\left(2-\sqrt{3}\right)(-z)^{-1}}$$
$$= H_{lp}(-z)$$

6.4 2nd Order low-pass calculation—polynomial form

$$\begin{aligned}
\boxed{\tilde{\mathsf{h}}(\omega)} &= \check{\mathsf{h}}(z)\big|_{z=e^{i\omega}} \\
&= G\left[\frac{\left(1+z^{-1}\right)^2}{1+az^{-1}+bz^{-2}}\right]\Bigg|_{z=e^{i\omega}} \\
&= \boxed{G\left[\frac{1+2z^{-1}+z^{-2}}{1+az^{-1}+bz^{-2}}\right]\Bigg|_{z=e^{i\omega}}}
\end{aligned}$$

We need 3 equations to solve for the 3 unkowns G, a, and b

Equation 1: Gain=1 at DC

$$\begin{aligned}
1 &= \left|\tilde{\mathsf{h}}(0)\right| \\
&= \check{\mathsf{h}}(z)\big|_{z=e^{i\omega},\,\omega=0} \\
&= G\left[\frac{\left(1+z^{-1}\right)^2}{1+az^{-1}+bz^{-2}}\right]\Bigg|_{z=1} \\
&= \frac{4G}{1+a+b} \\
&\implies \boxed{G = \frac{a+b+1}{4}}
\end{aligned}$$

Equation 2: Gain = ½ at corner frequency

$$\boxed{\left|\tilde{\mathsf{h}}(\omega)\right|^2} = \left|\check{\mathsf{h}}(z)\right|^2_{z=e^{i\omega}} = \left|G\left[\frac{\left(1+z^{-1}\right)^2}{1+az^{-1}+bz^{-2}}\right]\right|^2_{z=e^{i\omega}}$$

$$= \frac{\sum_{n=0}^{2} b_n^2 + 2\sum_{n=0}^{2}\sum_{m=n+1}^{2} b_n b_m \cos[\omega(n-m)]}{\sum_{n=0}^{2} a_n^2 + 2\sum_{n=0}^{2}\sum_{m=n+1}^{2} a_n a_m \cos[\omega(n-m)]}$$

$$= G^2\left[\frac{\left(1^2+2^2+1^2\right)+2[2\cos(\omega)+\cos(2\omega)+2\cos(\omega)]}{\left(1^2+a^2+b^2\right)+2b\cos(2\omega)+2a(b+1)\cos(\omega)+(1+a^2+b^2)}\right]$$

$$= G^2\left[\frac{2\cos(2\omega)+8\cos(\omega)+6}{2b\cos(2\omega)+2a(b+1)\cos(\omega)+(a^2+b^2+1)}\right]$$

$$\boxed{G^2\left[\frac{2\cos(2\omega_c)+8\cos(\omega_c)+6}{2b\cos(2\omega_c)+2a(b+1)\cos(\omega_c)+(a^2+b^2+1)}\right] = \frac{1}{2}}$$

Equation 3: For more smoothness in passband, set 2nd derivative to 0:

$$\begin{aligned}
0 &= \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left|\tilde{\mathsf{h}}(\omega)\right|^2_{\omega=0} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}\omega^2} G^2\left[\frac{\mathsf{f}(\omega)}{\mathsf{g}(\omega)}\right]_{\omega=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\omega} G^2\left[\frac{\mathsf{f}'\mathsf{g}+\mathsf{f}\mathsf{g}'}{\mathsf{g}^2}\right]_{\omega=0} && \text{by \textit{product rule}} \\
&= G^2\left[\frac{(\mathsf{f}''\mathsf{g}+\mathsf{f}'\mathsf{g}'-\mathsf{f}'\mathsf{g}'-\mathsf{f}\mathsf{g}'')\mathsf{g}^2-(\mathsf{f}'\mathsf{g}-\mathsf{f}\mathsf{g}')(2\mathsf{g}\mathsf{g}')}{\mathsf{g}^4}\right]_{\omega=0} \\
&= G^2\left[\frac{\mathsf{f}''\mathsf{g}-\mathsf{f}\mathsf{g}''}{\mathsf{g}^2}\right]_{\omega=0} \qquad \Longrightarrow \quad \boxed{\mathsf{f}''\mathsf{g} = \mathsf{f}\mathsf{g}''\big|_{\omega=0}}
\end{aligned}$$

...because $\mathsf{f}'(0) = \mathsf{g}'(0) = 0$

$$\begin{aligned}
&\Longrightarrow \underbrace{[-8\cos(2\omega)-8\cos(\omega)]}_{\mathsf{f}''}\underbrace{\left[2b\cos(2\omega)+2a(b+1)\cos(\omega)+(a^2+b^2+1)\right]}_{\mathsf{g}} \\
&\qquad = \left.\underbrace{[2\cos(2\omega)+8\cos(\omega)+6]}_{\mathsf{f}}+\underbrace{[-8b\cos(2\omega)-2a(b+1)\cos(\omega)]}_{\mathsf{g}''}\right|_{\omega=0} \\
&\Longrightarrow [-8-8]\left[2b+2a(b+1)+(a^2+b^2+1)\right] = [2+8+6]+[-8b-2a(b+1)] \\
&\Longrightarrow 16\left(a^2+b^2+2ab+2a+2b+1\right) = 2ab+2a+8b+16 \\
&\Longrightarrow 16a^2+16b^2+30ab+30a+24b = 0 \\
&\Longrightarrow \boxed{8a^2+8b^2+15ab+15a+12b = 0}
\end{aligned}$$

Example 6.3. 2nd order **low-pass** with corner frequency $\omega_c = \frac{2}{3}\pi$

$$1 = \tilde{\mathsf{h}}(0) = \check{\mathsf{h}}(z)\big|_{z=e^{i0}} = \frac{G(1+1)^2}{1+a+b} \implies \boxed{4G = a+b+1}$$

$$\begin{aligned}\frac{1}{2} &= \left|G\left[\frac{(1+z^{-1})^2}{1+az^{-1}+bz^{-2}}\right]\right|^2_{z=e^{i2\pi/3}} = G^2\left[\frac{2\cos(4\pi/3)+8\cos(2\pi/3)+6}{2b\cos(4\pi/3)+2a(b+1)\cos(2\pi/3)+(a^2+b^2+1)}\right]\\
&= G^2\left[\frac{-\sqrt{3}-4+6}{-b\sqrt{3}-a(b+1)+(a^2+b^2+1)}\right]\\
&= \frac{G^2\left(2-\sqrt{3}\right)}{a^2+b^2-ab-a-\sqrt{3}b} \implies \boxed{2\left(2-\sqrt{3}\right)G^2 = a^2+b^2-ab-a-\sqrt{3}b}\end{aligned}$$

We can combine the previous two $\boxed{\text{boxed}}$ equations to eliminate G

$$\begin{aligned}0 &= 8\times 0\\
&= 8\left[(1-c)a^2+(1-c)b^2-(2c+1)ab-(2c+1)a-(2c-\sqrt{3})b-3c\right] \quad \text{where } c = \frac{2-\sqrt{3}}{8}\\
&= 8\left[\left(\frac{6+\sqrt{3}}{8}\right)a^2+\left(\frac{6+\sqrt{3}}{8}\right)b^2-\left(\frac{6-\sqrt{3}}{4}\right)ab-\left(\frac{6-\sqrt{3}}{4}\right)a-\left(\frac{2-3\sqrt{3}}{4}\right)b-\left(\frac{6-3\sqrt{3}}{8}\right)c\right]\\
&= \left(6+\sqrt{3}\right)a^2+\left(6+\sqrt{3}\right)b^2-\left(12-2\sqrt{3}\right)ab-\left(12-2\sqrt{3}\right)a-\left(4-6\sqrt{3}\right)b-\left(6-3\sqrt{3}\right)c\end{aligned}$$

Combined equations:

$$\boxed{\left(6+\sqrt{3}\right)a^2+\left(6+\sqrt{3}\right)b^2-\left(12-2\sqrt{3}\right)ab-\left(12-2\sqrt{3}\right)a-\left(4-6\sqrt{3}\right)b-\left(6-3\sqrt{3}\right)c = 0}$$

2nd derivative equation:

$$\boxed{8a^2+8b^2+15ab+15a+12b = 0}$$

6.5 2nd Order low-pass calculation—polar form

$$\begin{aligned}\boxed{\tilde{\mathsf{h}}(\omega)} &= \check{\mathsf{h}}(z)\big|_{z=e^{i\omega}}\\
&= G\left[\frac{(z+1)^2}{(z-p)(z-p^*)}\right]_{z=e^{i\omega}}\\
&= G\left[\frac{(z+1)^2}{\left(z-re^{i\phi}\right)\left(z-(re^{i\phi})^*\right)}\right]_{z=e^{i\omega}}\\
&= G\left[\frac{z^2+2z+1}{z^2-2r\cos(\phi)z+r^2}\right]_{z=e^{i\omega}}\end{aligned}$$

We need 3 equations to solve for the 3 unkowns G, r, and ϕ

Equation 1: Gain=1 at DC

$$
\begin{aligned}
1 &= \left|\tilde{\mathsf{h}}(0)\right| \\
&= G\left[\frac{z^2+2z+1}{z^2-2r\cos(\phi)z+r^2}\right]_{z=e^{i\omega},\,\omega=0} \\
&= G\left[\frac{1+2+1}{1-2r\cos(\phi)+r^2}\right]
\end{aligned}
$$

$$\implies \boxed{G = \frac{r^2-2r\cos(\phi)+1}{4}}$$

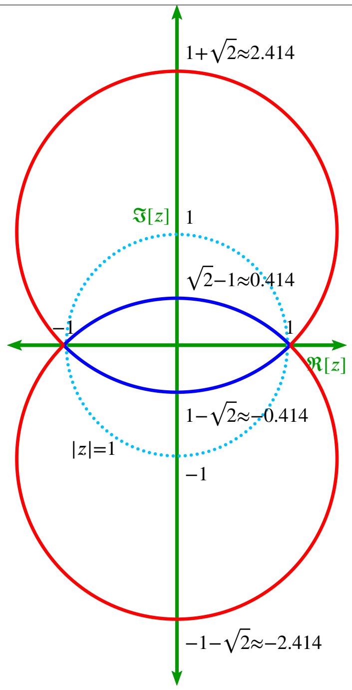
Equation 2: Gain = ½ at corner frequency

$$
\begin{aligned}
\boxed{\left|\tilde{\mathsf{h}}(\omega)\right|^2} &= \left|\check{\mathsf{h}}(z)\right|^2_{z=e^{i\omega}} = G^2\left[\frac{z^2+2z+1}{z^2-2r\cos(\phi)z+r^2}\right]^2_{z=e^{i\omega}} \\
&= G^2\left.\left[\frac{z^2+2z+1}{z^2-2r\cos(\phi)z+r^2}\right]\left[\frac{z^2+2z+1}{z^2-2r\cos(\phi)z+r^2}\right]^*\right|_{z=e^{i\omega}} \\
&= G^2\left.\left[\frac{z^2+2z+1}{z^2-2r\cos(\phi)z+r^2}\right]\left[\frac{z^{2*}+2z^*+1}{z^{2*}-2r\cos(\phi)z^*+r^2}\right]\right|_{z=e^{i\omega}} \\
&= G^2\left.\frac{\left[|z|^4+2|z|^2z+z^2\right]+\left[2|z|^2z^*+4|z|^2+2z\right]+\left[z^{2*}+2z^*+1\right]}{\begin{array}{rl} & \left[|z|^4-2r\cos(\phi)z|z|^2+r^2z^2\right] \\ + & \left[-2r\cos(\phi)|z|^2z^*+4r^2\cos^2(\phi)|z|^2-2r^3\cos(\phi)z\right] \\ + & \left[r^2z^{2*}-2r^3\cos(\phi)z^*+r^4\right]\end{array}}\right|_{z=e^{i\omega}} \\
&= G^2\left.\frac{|z|^4+2|z|^2(z+z^*)+(z^2+z^{2*})+4|z|^2+2(z+z^*)+1}{|z|^4-2r\cos(\phi)|z|^2(z+z^*)+r^2(z^2+z^{2*})+4r^2\cos^2(\phi)|z|^2-2r^3\cos(\phi)(z+z^*)+r^4}\right|_{z=e^{i\omega}} \\
&= G^2\left[\frac{1+4\cos(\omega)+2\cos(2\omega)+4+4\cos(\omega)+1}{1-4r\cos(\phi)\cos(\omega)+2r^2\cos(2\omega)+4r^2\cos^2(\phi)-4r^3\cos(\phi)\cos(\omega)+r^4}\right] \\
&= G^2\left[\frac{2\boxed{\cos(2\omega)}+8\boxed{\cos(\omega)}+\boxed{6}}{2r^2\boxed{\cos(2\omega)}-4r\cos(\phi)[1+r^2]\boxed{\cos(\omega)}+\boxed{r^4+4r^2\cos^2(\phi)+1}}\right]
\end{aligned}
$$

$$\boxed{G^2\left[\frac{2\boxed{\cos(2\omega_c)}+8\boxed{\cos(\omega_c)}+\boxed{6}}{2r^2\boxed{\cos(2\omega_c)}-4r\cos(\phi)[1+r^2]\boxed{\cos(\omega_c)}+\boxed{r^4+4r^2\cos^2(\phi)+1}}\right]=\frac{1}{2}}$$

Equation 3: For more smoothness in passband, set 2nd derivative to 0.

Remark 6.1. Who cares about the second derivative? In mathematics, **smoothness** of a function $\mathsf{f}(x)$ is the number of derivatives $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ that are continuous. For an example, consider *Hermite interpolation* (Section K.2 page 163, Theorem K.1 page 164).



$$\begin{aligned}
0 &= \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left|\tilde{\mathsf{h}}(\omega)\right|^2_{\omega=0} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}G^2\left[\frac{\mathsf{f}(\omega)}{\mathsf{g}(\omega)}\right]_{\omega=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\omega}G^2\left[\frac{\mathsf{f}'\mathsf{g}+\mathsf{f}\mathsf{g}'}{\mathsf{g}^2}\right]_{\omega=0} && \text{by } \textit{product rule} \\
&= G^2\left[\frac{(\mathsf{f}''\mathsf{g}+\mathsf{f}'\mathsf{g}'-\mathsf{f}'\mathsf{g}'-\mathsf{f}\mathsf{g}'')\mathsf{g}^2-(\mathsf{f}'\mathsf{g}-\mathsf{f}\mathsf{g}')(2\mathsf{g}\mathsf{g}')}{\mathsf{g}^4}\right]_{\omega=0} \\
&= G^2\left[\frac{\mathsf{f}''\mathsf{g}-\mathsf{f}\mathsf{g}''}{\mathsf{g}^2}\right]_{\omega=0} \qquad \Longrightarrow \boxed{\mathsf{f}''\mathsf{g}=\mathsf{f}\mathsf{g}''\big|_{\omega=0}}
\end{aligned}$$

...because $\mathsf{f}'(0)=\mathsf{g}'(0)=0$

$$\begin{aligned}
&\Longrightarrow \underbrace{[-8\cos(2\omega)-8\cos(\omega)]}_{\mathsf{f}''}\underbrace{\left[2r^2\cos(2\omega)-4r\cos(\phi)[1+r^2]\cos(\omega)+r^4+4r^2\cos^2(\phi)+1\right]}_{\mathsf{g}} \\
&\qquad = \left[\underbrace{[2\cos(2\omega)+8\cos(\omega)+6]}_{\mathsf{f}}\underbrace{\left[-8r^2\cos(2\omega)+4r\cos(\phi)[1+r^2]\cos(\omega)\right]}_{\mathsf{g}''}\right]_{\omega=0} \\
&\Longrightarrow [-8-8]\left[2r^2-4r\cos(\phi)[1+r^2]+r^4+4r^2\cos^2(\phi)+1\right] \\
&\qquad = [2+8+6]\left[-8r^2+4r\cos(\phi)[1+r^2]\right] \\
&\Longrightarrow \boxed{r^4+\left[4\cos^2(\phi)-6\right]r^2+1=0}
\end{aligned}$$

$$
\begin{aligned}
r^2 &= \frac{-\left[4\cos^2\phi - 6\right] \pm \sqrt{\left[4\cos^2\phi - 6\right]^2 - 4}}{2} \\
&= \frac{-\left[4\cos^2\phi - 6\right] \pm \sqrt{\left[16\cos^4\phi - 48\cos^2\phi + 36\right] - 4}}{2} \\
&= \frac{-2\left[2\cos^2\phi - 3\right] \pm 2\sqrt{4\cos^4\phi - 12\cos^2\phi + 8}}{2} \\
&= \left[3 - 2\cos^2\phi\right] \pm \sqrt{4\cos^4\phi - 12\cos^2\phi + 8} \\
&= \left[3 - 2\cos^2\phi\right] \pm 2\sqrt{\cos^4\phi - 3\cos^2\phi + 2} \\
&= \left[3 - 2\cos^2\phi\right] \pm 2\sqrt{(2-\cos^2\phi)(1-\cos^2)} \\
&= \left[\sqrt{1-\cos^2\phi} \pm \sqrt{2-\cos^2\phi}\right]^2 \\
&= \left[\sin\phi \pm \sqrt{2-\cos^2\phi}\right]^2
\end{aligned}
$$

Example 6.4. 2nd order **low-pass** with corner frequency $\omega_c = \frac{2}{3}\pi$

$$
\begin{aligned}
1 &= \left|\tilde{\mathsf{h}}(0)\right| \\
&= G\left[\frac{z^2+2z+1}{z^2 - 2r\cos(\phi)z + r^2}\right]_{z=e^{i\omega},\,\omega=0} \\
&= G\left[\frac{1+2+1}{1-2r\cos(\phi)+r^2}\right]
\end{aligned}
$$

$$
\implies \boxed{G = \frac{r^2 - 2r\cos(\phi) + 1}{4}}
$$

$$
\begin{aligned}
\frac{1}{2} &= G^2\left[\frac{2\boxed{\cos(2\omega_c)} + 8\boxed{\cos(\omega_c)} + \boxed{6}}{2r^2\boxed{\cos(2\omega_c)} - 4r\cos(\phi)[1+r^2]\boxed{\cos(\omega_c)} + \boxed{r^4 + 4r^2\cos^2(\phi+1}}\right] \\
&= \boxed{G^2\left[\frac{2-\sqrt{3}}{r^4 + 2\cos(\phi)r^3 + [1-\sqrt{3}]r^2 + 2r\cos(\phi) + 1}\right]}
\end{aligned}
$$

We can combine these equations to eliminate G

$$
\begin{aligned}
1/2 &= \left[\frac{r^2 - 2r\cos(\phi) + 1}{4}\right]^2\left[\frac{2-\sqrt{3}}{r^4 + 2\cos(\phi)r^3 + [1-\sqrt{3}]r^2 + 2r\cos(\phi) + 1}\right] \\
&= \left[\frac{2-\sqrt{3}}{16}\right]\frac{\left[r^2 - 2r\cos(\phi) + 1\right]^2}{r^4 + 2\cos(\phi)r^3 + [1-\sqrt{3}]r^2 + 2\cos(\phi)r + 1}
\end{aligned}
$$

$$
8\left[r^4 + 2\cos(\phi)r^3 + [1-\sqrt{3}]r^2 + 2\cos(\phi)r + 1\right] = \left[2-\sqrt{3}\right]\left[r^2 - 2r\cos(\phi) + 1\right]^2
$$

$$
8\left[r^4 + 2\cos(\phi)r^3 + [1-\sqrt{3}]r^2 + 2\cos(\phi)r + 1\right] = \left[2-\sqrt{3}\right]\left[r^4 - 4\cos(\phi)r^3 + 6\cos^2(\phi)r^2 - 4\cos(\phi)r + 1\right]
$$

$$
\boxed{\left[6+\sqrt{3}\right]r^4 + 4\left[6-\sqrt{3}\right]\cos(\phi)r^3 + \left[8 - 8\sqrt{3} - 6\left(2-\sqrt{3}\right)\cos^2(\phi)\right]r^2 + 4\left[18-\sqrt{3}\right]\cos(\phi)r + \left[6+\sqrt{3}\right] = 0}
$$

CHAPTER 7

DSP CALCULUS

7.1 Fourier Transform calculus

THM

$$\begin{aligned}
\mathsf{x}(t) &= \int_{u=0}^{t} \mathsf{v}(u)\,\mathsf{d}u + \underbrace{\mathsf{x}(0)}_{\text{initial condition}} \\
\mathsf{v}(t) &= \int_{u=0}^{t} \mathsf{a}(u)\,\mathsf{d}u + \underbrace{\mathsf{v}(0)}_{\text{initial condition}}
\end{aligned}$$

✎PROOF: *Fundamental Theorem of Calculus* ✏

Proposition 7.1.

PRP *The* FOURIER TRANSFORM *of the* DIFFERENTIAL OPERATOR *is*

$$\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right] = i\omega\tilde{\mathsf{X}}(\omega)$$

✎PROOF:

$$\begin{aligned}
\boxed{\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right]} &\triangleq \int_{t=-\infty}^{t=+\infty} \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right]}_{\mathsf{d}v} \underbrace{e^{-i\omega t}}_{u}\,\mathsf{d}t \\
&= \underbrace{e^{-i\omega t}}_{u}\underbrace{\mathsf{x}(t)}_{v}\Big|_{t=-\infty}^{t=+\infty} - \int_{t=-\infty}^{t=+\infty} \underbrace{\mathsf{x}(t)}_{v}\underbrace{(-i\omega)e^{-i\omega t}}_{\mathsf{d}u}\,\mathsf{d}t && \text{by } \textit{Integration by Parts} \\
&= \cancelto{0}{e^{-i\omega\infty}}\mathsf{x}(\infty) - e^{-i\omega\infty}\cancelto{0}{\mathsf{x}(-\infty)} - (-i\omega)\underbrace{\int_{t=-\infty}^{t=+\infty} \mathsf{x}(t)e^{-i\omega t}\,\mathsf{d}t}_{\textit{Fourier Transform} \text{ of } \mathsf{x}(t)} && \text{assuming } \mathsf{x}(t) \text{ started at } 0 \\
&= \boxed{i\omega X(\omega)}
\end{aligned}$$

✏

Proposition 7.2.

PRP *The* FOURIER TRANSFORM *of the* INTEGRATION OPERATOR *is*

$$\tilde{\mathbf{F}} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\, \mathrm{d}u = \frac{1}{i\omega} \tilde{\mathsf{X}}(\omega)$$

✎PROOF:

$$\begin{aligned}
\boxed{\tilde{\mathbf{F}} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\, \mathrm{d}u} &\triangleq \int_{t=-\infty}^{t=+\infty} \left[\int_{u=-\infty}^{u=t} \mathsf{x}(u)\, \mathrm{d}u \right] e^{-i\omega t}\, \mathrm{d}t \\
&= \int_{t=-\infty}^{t=+\infty} \left[\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)\mathsf{h}(t-u)\, \mathrm{d}u \right] e^{-i\omega t}\, \mathrm{d}t && \left(\begin{array}{l}\text{where } \mathsf{h}(t) \text{ is the} \\ \textit{Heaviside function}\end{array} \right) \\
&= \int_{v=-\infty}^{v=+\infty} \int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)\mathsf{h}(v) e^{-i\omega(u+v)}\, \mathrm{d}u\, \mathrm{d}v && \left(\begin{array}{lrcl}\text{where} & v &=& t-u \\ \Longrightarrow & t &=& u+v\end{array} \right) \\
&= \left[\int_{v=-\infty}^{v=+\infty} \mathsf{h}(v) e^{-i\omega v}\, \mathrm{d}v \right] \underbrace{\left[\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u) e^{-i\omega u}\, \mathrm{d}u \right]}_{\textit{Fourier Transform } X(\omega) \text{ of } \mathsf{x}(t)} \\
&= \left[\int_{v=0}^{v=+\infty} e^{-i\omega v}\, \mathrm{d}v \right] X(\omega) \\
&= \frac{1}{-i\omega} e^{-i\omega v} \Big|_{v=0}^{v=+\infty} X(\omega) \quad = \quad \boxed{\frac{1}{i\omega} X(\omega)}
\end{aligned}$$

⇨

7.2 Digital differentiation methods

Digital Differentiation Method #1: *Difference*[1]

$$\mathsf{y}[n] \triangleq \mathsf{x}[n] - \mathsf{x}[n-1]$$

$$\begin{aligned}
\mathbf{Z}\{\mathsf{y}[n]\} &= \mathbf{Z}\{\mathsf{x}[n] - \mathsf{x}[n-1]\} \\
\check{\mathsf{Y}}(z) &= \check{\mathsf{X}}(z) + z^{-1}\check{\mathsf{X}}(z)
\end{aligned}$$

$$\frac{\check{\mathsf{Y}}(z)}{\check{\mathsf{X}}(z)} = 1 - z^{-1} \quad = \quad \boxed{\frac{z-1}{z}} \qquad \left\{ \begin{array}{ll}\text{How many zeros?} & \text{Where?} \\ \text{How many poles?} & \text{Where?}\end{array} \right\}$$

Is digital differentiation equivalent to continuous differentiation?[2]

[1] Williams (1986) page 69 ⟨Difference⟩
[2] Williams (1986) page 70 ⟨Figure 2.14(a)⟩

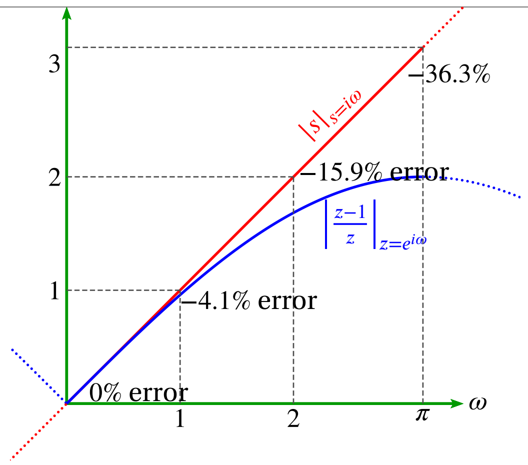


Figure 7.1: Digital differentiation methods

$$
\begin{aligned}
\left|\frac{z-1}{z}\right|_{z=e^{i\omega}} &= \left|\frac{e^{i\omega}-1}{e^{i\omega}}\right| \\
&= \left|\frac{e^{i\omega/2}\left(e^{i\omega/2}-e^{-i\omega/2}\right)}{e^{i\omega}}\right| &&= \left|\underbrace{e^{-i\omega/2}}_{\text{phase}}\underbrace{2\sin\left(\frac{\omega}{2}\right)}_{\text{magnitude}}\right| \\
&= \boxed{2\sin\left(\frac{\omega}{2}\right)} &&\text{for } 0 \le \omega \le \pi
\end{aligned}
$$

7.2.1 Digital Differentiation Method #2: *Central Difference*

[3]

$$
\begin{aligned}
\mathsf{y}[n] &\triangleq \frac{\mathsf{x}[n]-\mathsf{x}[n-2]}{2} \\
Y(z) &= \frac{X(z)+z^{-1}X(z)}{2} \\
\frac{Y(z)}{X(z)} &= \frac{1-z^{-1}}{2} \quad = \quad \frac{z^2-1}{2z^2} \\
&= \boxed{\frac{(z+1)(z-1)}{2z^2}} \qquad \left\{\begin{array}{ll}\text{How many zeros?} & \text{Where?}\\ \text{How many poles?} & \text{Where?}\end{array}\right\}
\end{aligned}
$$

Central Difference = Continuous Differentiation?[4]

[3] Williams (1986) page 69 ⟨Difference⟩
[4] Williams (1986) page 70 ⟨Figure 2.14(b)⟩

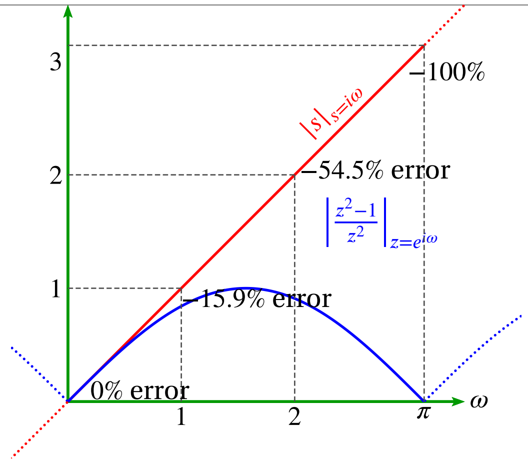


Figure 7.2: Central difference

$$\begin{aligned}
\left|\frac{z^2-1}{2z^2}\right|_{z=e^{i\omega}} &= \left|\frac{e^{2i\omega}-1}{2e^{2i\omega}}\right| = \left|\left(\frac{e^{i\omega}}{e^{2i\omega}}\right)\frac{\left(e^{i\omega}-e^{-i\omega}\right)}{2}\right| \\
&= \left|\left(e^{-i\omega}\right)\frac{[\cos(\omega)+i\sin(\omega)]-[\cos(\omega)+i\sin(-\omega)]}{2}\right| \\
&= \left|\left(e^{-i\omega}\right)\frac{[\cos(\omega)+i\sin(\omega)]-[\cos(\omega)-i\sin(\omega)]}{2}\right| \\
&= \left|\left(e^{-i\omega+\pi/2}\right)\frac{2\sin(\omega)}{2}\right| = \boxed{|\sin(\omega)|}
\end{aligned}$$

7.3 Digital integration

7.3.1 Digital Integration Method #1: Summation

$$\begin{aligned}
\mathsf{y}[n] &\triangleq \mathsf{x}[n] + \underbrace{\mathsf{x}[n-1]+\mathsf{x}[n-2]+\mathsf{x}[n-3]+\mathsf{x}[n-4]+\mathsf{x}[n-5]+\cdots}_{\mathsf{y}[n-1]} \\
\mathsf{y}[n] &= \mathsf{x}[n] + \mathsf{y}[n-1]
\end{aligned}$$

$$\begin{aligned}
\mathbf{Z}\{\mathsf{y}[n]\} &= \mathbf{Z}\{\mathsf{x}[n]+\mathsf{y}[n-1]\} \\
Y(z) &= X(z) + z^{-1}Y(z) \\
Y(z)\left[1-z^{-1}\right] &= X(z)
\end{aligned}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}} = \boxed{\frac{z}{z-1}} \qquad \left\{\begin{array}{ll}\text{How many zeros?} & \text{Where?}\\ \text{How many poles?} & \text{Where?}\end{array}\right\}$$

7.3.2 Digital Integration Method #2: Trapezoid

$$\begin{aligned}
\mathsf{y}[n] &\triangleq \frac{\mathsf{x}[n]+\mathsf{x}[n-1]}{2} + \frac{\mathsf{x}[n-1]+\mathsf{x}[n-2]}{2} + \frac{\mathsf{x}[n-2]+\mathsf{x}[n-3]}{2} + \cdots \\
&= \tfrac{1}{2}\mathsf{x}[n] + \underbrace{\mathsf{x}[n-1]+\mathsf{x}[n-2]+\mathsf{x}[n-3]+\mathsf{x}[n-4]+\mathsf{x}[n-5]+\cdots}_{\mathsf{y}[n-1]+\frac{1}{2}\mathsf{x}[n-1]} \\
&= \tfrac{1}{2}\mathsf{x}[n] + \mathsf{y}[n-1] + \tfrac{1}{2}\mathsf{x}[n-1]
\end{aligned}$$

$$\mathsf{y}[n] - \mathsf{y}[n-1] = \tfrac{1}{2}[\mathsf{x}[n] + +\mathsf{x}[n-1]]$$

$$Y(z)\left[1-z^{-1}\right] = \tfrac{1}{2}X(z)\left[1+z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \left(\frac{1}{2}\right)\frac{1+z^{-1}}{1-z^{-1}} = \boxed{\left(\frac{1}{2}\right)\frac{z+1}{z-1}} \qquad \left\{\begin{array}{ll}\text{How many zeros?} & \text{Where?}\\ \text{How many poles?} & \text{Where?}\end{array}\right\}$$

7.3.3 Digital Integration Method #3: Simpson's Rule

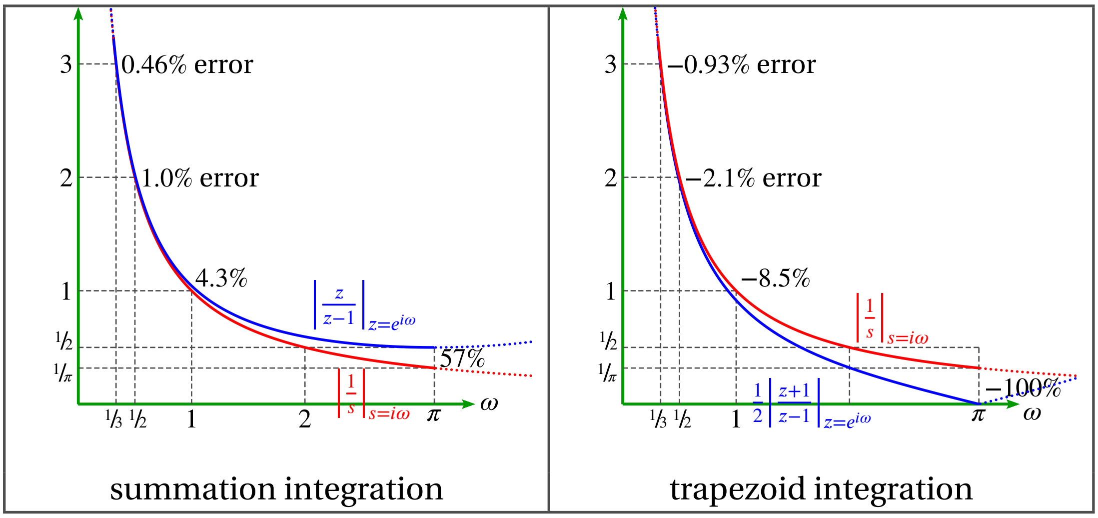


Figure 7.3: Comparison of digital integration methods to analytic integration

Is digital summation integration equivalent to continuous integration? Not really (Figure 7.3 page 55).

$$\left|\frac{z}{z-1}\right|_{z=e^{i\omega}} = \left|\frac{e^{i\omega}}{e^{i\omega}-1}\right|$$

$$= \left|\frac{e^{i\omega}}{e^{i\omega/2}\left(e^{i\omega/2}-e^{-i\omega/2}\right)}\right| \qquad = \left|\underbrace{e^{i\omega/2}}_{\text{phase}}\underbrace{\frac{1}{2\sin\left(\frac{\omega}{2}\right)}}_{\text{magnitude}}\right|$$

$$= \boxed{\frac{1}{2\sin\left(\frac{\omega}{2}\right)}} \qquad \text{for } 0 \le \omega \le \pi$$

Is digital trapezoid integration equivalent to continuous integration? Not really (Figure 7.3 page 55).

$$\left|\frac{1}{2}\left(\frac{z+1}{z-1}\right)\right|_{z=e^{i\omega}} = \frac{1}{2}\left|\frac{e^{i\omega}+1}{e^{i\omega}-1}\right|$$

$$= \frac{1}{2}\left|\frac{e^{i\omega/2}\left(e^{i\omega/2}+e^{-i\omega/2}\right)}{e^{i\omega/2}\left(e^{i\omega/2}-e^{-i\omega/2}\right)}\right| \qquad = \frac{1}{2}\left|\frac{2\cos\left(\frac{\omega}{2}\right)}{2\sin\left(\frac{\omega}{2}\right)}\right|$$

$$= \boxed{\frac{1}{2}\left|\cot\left(\frac{\omega}{2}\right)\right|} \qquad \text{for } 0 \le \omega \le \pi$$

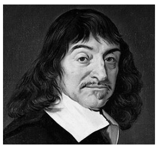
Part I

Appendices

APPENDIX A

TRANSVERSAL OPERATORS

❝*Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:*❞



❝*I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them.*❞

René Descartes, philosopher and mathematician (1596–1650) [1]

A.1 Families of Functions

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition F.1 page 101). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition A.1 page 59). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition A.1. *Let X and Y be sets.*

DEF *The space Y^X represents the set of all functions with* DOMAIN *X and* RANGE *Y such that*

$$Y^X \triangleq \{\mathsf{f}(x) | \mathsf{f}(x) : X \to Y\}$$

Definition A.2. [2] *Let X be a set.*

[1] quote: Descartes (1637b)
translation: Descartes (1637c) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

[2] Aliprantis and Burkinshaw (1998), page 126, Hausdorff (1937), page 22, de la Vallée-Poussin (1915) page 440

DEF: *The* ***indicator function*** $\mathbb{1} \in \{0,1\}^{2^X}$ *is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \qquad \forall x \in X,\ A \in 2^X \\ 0 & \text{for } x \notin A \qquad \forall x \in X,\ A \in 2^X \end{cases}$$

The indicator function $\mathbb{1}$ *is also called the* ***characteristic function***.

A.2 Definitions and algebraic properties

Much of the wavelet theory developed in this text is constructed using the **translation operator T** and the **dilation operator D** (next).

Definition A.3. [3]

DEF:

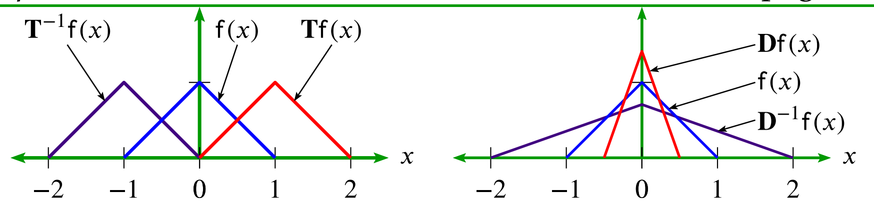
$\mathbf{T}_\tau$ *is a* ***translation operator*** *on* $\mathbb{C}^{\mathbb{C}}$ *if* $\quad \mathbf{T}_\tau \mathsf{f}(x) \triangleq \mathsf{f}(x-\tau) \quad \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}}$.

$\mathbf{D}_\alpha$ *is a* ***dilation operator*** *on* $\mathbb{C}^{\mathbb{C}}$ *if* $\quad \mathbf{D}_\alpha \mathsf{f}(x) \triangleq \mathsf{f}(\alpha x) \quad \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}}$.

Moreover, $\mathbf{T} \triangleq \mathbf{T}_1$ *and* $\mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2$.

Example A.1. Let **T** and **D** be defined as in Definition A.3 (page 60).

EX



Proposition A.1. *Let* $\mathbf{T}_\tau$ *be a* TRANSLATION OPERATOR *(Definition A.3 page 60).*

PRP

$$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) \qquad \forall \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) \text{ is PERIODIC with period } \tau\right)$$

✎PROOF:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n\tau+\tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition A.3 page 60)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m\tau) && \text{where } m \triangleq n-1 && \implies n = m+1 \\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m \mathsf{f}(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition A.3 page 60)}
\end{aligned}$$

⇰

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exist as operators, as demonstrated next.

Proposition A.2 (transversal operator inverses). *Let* **T** *and* **D** *be as defined in Definition A.3 page 60.*

PRP

T *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}} \qquad \textit{(translation operator inverse)}.$$

D *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}} \qquad \textit{(dilation operator inverse)}.$$

---

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2000) page 473, Packer (2004) page 260, Benedetto and Zayed (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a), page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996), page 81, Dai and Larson (1998) page 2

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} && \text{(Definition A.3 page 60)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} && \text{(Definition A.3 page 60)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\implies \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} && \text{(Definition A.3 page 60)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} && \text{(Definition A.3 page 60)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\implies \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

⇨

Proposition A.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition A.3 page 60.*
Let $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP
$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^jx-n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

A.3 Linear space properties

Proposition A.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 page 60.*

PRP
$$\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}\mathsf{g}\right] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

✎PROOF:

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}(x)\mathsf{g}(x)\right] &= 2^{j/2}\mathsf{f}\left(2^jx-n\right)\mathsf{g}\left(2^jx-n\right) && \text{by Proposition A.3 page 61}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^jx-n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^jx-n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition A.3 page 61}
\end{aligned}$$

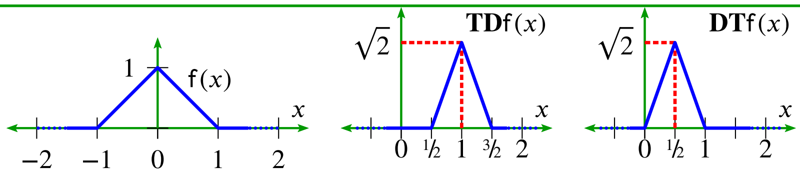
⇨

In general the operators $\mathbf{T}$ and $\mathbf{D}$ are *noncommutative* ($\mathbf{TD} \neq \mathbf{DT}$), as demonstrated by Counterexample A.1 (next) and Proposition A.5 (page 62).

Counterexample A.1.

CNT

As illustrated to the right, it is **not** always true that $\mathbf{TD} = \mathbf{DT}$:



Proposition A.5 (commutator relation). [4] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 page 60.*

PRP

$$\begin{array}{rcll} \mathbf{D}^j\mathbf{T}^n &=& \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j & \forall j,n\in\mathbb{Z} \\ \mathbf{T}^n\mathbf{D}^j &=& \mathbf{D}^j\mathbf{T}^{2^j n} & \forall n,j\in\mathbb{Z} \end{array}$$

✎PROOF:

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^{2^j n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - 2^j n) && \text{by Proposition A.4 page 61} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) \quad \text{(Definition B.6 page 74)} \\ &= \mathbf{T}^n 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} \quad \text{(Definition A.3 page 60)} \\ &= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} \quad \text{(Definition A.3 page 60)} \end{aligned}$$

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - n) && \text{by Proposition A.4 page 61} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j\left[x-2^{-j/2}n\right]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) \quad \text{(Definition B.6 page 74)} \\ &= \mathbf{T}^{2^{-j/2}n} 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} \quad \text{(Definition A.3 page 60)} \\ &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} \quad \text{(Definition A.3 page 60)} \end{aligned}$$

✏

A.4 Inner product space properties

In an inner product space, every operator has an *adjoint* and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator $\mathbf{U}$ coincide, then $\mathbf{U}$ is said to be *unitary*. And in this case, $\mathbf{U}$ has several nice properties (see Proposition A.9 and Theorem A.1 page 65). Proposition A.6 (next) gives the adjoints of $\mathbf{D}$ and $\mathbf{T}$, and Proposition A.7 (page 63) demonstrates that both $\mathbf{D}$ and $\mathbf{T}$ are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ (Corollary J.1 page 151) and the *rotation matrix operator*.

Proposition A.6. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition A.3 page 60) with* ADJOINT $\mathbf{T}^*$ *and* $\mathbf{D}$ *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$.

PRP

$$\begin{array}{rcll} \mathbf{T}^*\mathsf{f}(x) &=& \mathsf{f}(x+1) & \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \quad \text{(TRANSLATION OPERATOR ADJOINT)} \\ \mathbf{D}^*\mathsf{f}(x) &=& \dfrac{\sqrt{2}}{2}\,\mathsf{f}\left(\dfrac{1}{2}x\right) & \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \quad \text{(DILATION OPERATOR ADJOINT)} \end{array}$$

✎PROOF:

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a), page 641, Goodman et al. (1993b), page 110

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u\\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^*\\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} && \text{(Definition A.3 page 60)}\\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1\\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u\\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^*\\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition A.3 page 60)}\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\,\mathrm{d}u && \text{by definition of } \langle \triangle \,|\, \nabla\rangle\\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^*\mathrm{d}x && \text{where } x = 2u\\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle\\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}$$

⇨

Proposition A.7. [5] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 (page 60). Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition A.2 (page 60).*

PRP
$\mathbf{T}$ *is* UNITARY *in* $\boldsymbol{L}^2_{\mathbb{R}}$ $\quad(\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $\boldsymbol{L}^2_{\mathbb{R}})$.
$\mathbf{D}$ *is* UNITARY *in* $\boldsymbol{L}^2_{\mathbb{R}}$ $\quad(\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $\boldsymbol{L}^2_{\mathbb{R}})$.

✎PROOF:

$$\begin{aligned}
\mathbf{T}^{-1} &= \mathbf{T}^* && \text{by Proposition A.2 page 60 and Proposition A.6 page 62}\\
&\implies \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}^{-1} &= \mathbf{D}^* && \text{by Proposition A.2 page 60 and Proposition A.6 page 62}\\
&\implies \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators}
\end{aligned}$$

⇨

A.5 Normed linear space properties

Proposition A.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR (*Definition A.3 page 60*).

PRP
$$\left\{\begin{array}{lll} (1). & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) & \textit{and}\\ (2). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} & \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$\begin{aligned}\mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition A.3 page 60)}\\ &= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis}\end{aligned}$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$\begin{aligned}\|\mathsf{f}(x)-\mathsf{f}(x+h)\| &= \|\mathsf{f}(x)-\mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis}\\ &= \|0\|\\ &= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|\\ &\le \varepsilon\\ &\implies \forall h>0,\ \exists\varepsilon \quad \text{such that} \quad \|\mathsf{f}(x)-\mathsf{f}(x+h)\|<\varepsilon\\ &\overset{\text{def}}{\iff} \mathsf{f}(x) \text{ is } \textit{continuous}\end{aligned}$$

3. Proof that (1,2) $\implies$ *constant* property:
   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \neq \mathsf{f}(y)$.
   (b) Let $(\!(x_n)\!)_{n\in\mathbb{N}}$ be a sequence with limit x and $(\!(y_n)\!)_{n\in\mathbb{N}}$ a sequence with limit y
   (c) Then

$$\begin{aligned}0 &< \|\mathsf{f}(x)-\mathsf{f}(y)\| && \text{by assumption in item (3a) page 64}\\ &= \lim_{n\to\infty}\left\|\mathsf{f}(x_n)-\mathsf{f}(y_n)\right\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 64}\\ &= \lim_{n\to\infty}\left\|\mathsf{f}(2^m x_n)-\mathsf{f}(2^\ell y_n)\right\| \quad \forall m,\ell\in\mathbb{Z} && \text{by (1)}\\ &= 0\end{aligned}$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

✏

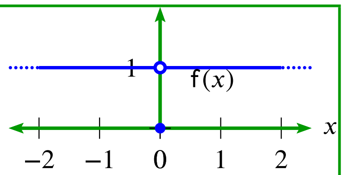
Remark A.1.

REM In Proposition A.8 page 63, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample A.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x=0\\ 1 & \text{otherwise.}\end{cases}$

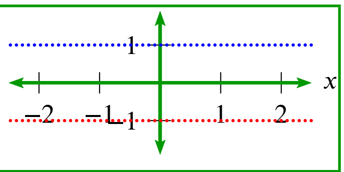
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Counterexample A.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\backslash\mathbb{Q}$ the set of *irrational numbers*.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x\in\mathbb{Q}\\ -1 & \text{for } x\in\mathbb{R}\backslash\mathbb{Q}.\end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Proposition A.9 (Operator norm). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 page 60. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition A.2 page 60. Let* $\mathbf{T}^*$ *and* $\mathbf{D}^*$ *be as in Proposition A.6 page 62. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition F.1 page 101. Let* $|\!|\!|\cdot|\!|\!|$ *be the operator norm induced by* $\|\cdot\|$.

PRP $$|\!|\!|\mathbf{T}|\!|\!| = |\!|\!|\mathbf{D}|\!|\!| = |\!|\!|\mathbf{T}^*|\!|\!| = |\!|\!|\mathbf{D}^*|\!|\!| = |\!|\!|\mathbf{T}^{-1}|\!|\!| = |\!|\!|\mathbf{D}^{-1}|\!|\!| = 1$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* and from properties of unitary operators. ⇨

Theorem A.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 page 60.*
Let $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition A.2 page 60. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition F.1 page 101.*

1.	$\|\mathbf{T}\mathsf{f}\| = \|\mathbf{D}\mathsf{f}\| = \|\mathsf{f}\|$	$\forall \mathsf{f}\in\mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}\mathsf{f}-\mathbf{T}\mathsf{g}\| = \|\mathbf{D}\mathsf{f}-\mathbf{D}\mathsf{g}\| = \|\mathsf{f}-\mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
3.	$\left\|\mathbf{T}^{-1}\mathsf{f}-\mathbf{T}^{-1}\mathsf{g}\right\| = \left\|\mathbf{D}^{-1}\mathsf{f}-\mathbf{D}^{-1}\mathsf{g}\right\| = \|\mathsf{f}-\mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
4.	$\langle \mathbf{T}\mathsf{f} \mid \mathbf{T}\mathsf{g}\rangle = \langle \mathbf{D}\mathsf{f} \mid \mathbf{D}\mathsf{g}\rangle = \langle \mathsf{f} \mid \mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(SURJECTIVE)
5.	$\left\langle \mathbf{T}^{-1}\mathsf{f} \mid \mathbf{T}^{-1}\mathsf{g}\right\rangle = \left\langle \mathbf{D}^{-1}\mathsf{f} \mid \mathbf{D}^{-1}\mathsf{g}\right\rangle = \langle \mathsf{f} \mid \mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(SURJECTIVE)

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition A.7 page 63) and from properties of unitary operators. ⇨

Proposition A.10. *Let* $\mathbf{T}$ *be as in Definition A.3 page 60. Let* $\mathbf{A}^*$ *be the* ADJOINT *of an operator* $\mathbf{A}$.

$$\left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* \iff \left(\text{The operator } \left[\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right] \text{ is SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)\mathsf{f}(x) \,\middle|\, \mathsf{g}(x)\right\rangle &= \left\langle \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) \,\middle|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition A.3 page 60)}\\
&= \left\langle \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n) \,\middle|\, \mathsf{g}(x)\right\rangle && \text{by } \textit{commutative} \text{ property} && \text{(Definition B.5 page 74)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x+n) \mid \mathsf{g}(x)\rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(u) \mid \mathsf{g}(u-n)\rangle && \text{where } u \triangleq x+n\\
&= \left\langle \mathsf{f}(u) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathsf{g}(u-n)\right\rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathsf{g}(x-n)\right\rangle && \text{by change of variable: } u \to x\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition A.3 page 60)}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) &= \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* && \text{by definition of } \textit{adjoint}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) &\text{ is } \textit{self-adjoint} && \text{by definition of } \textit{self-adjoint}
\end{aligned}$$

⇨

A.6 Fourier transform properties

Proposition A.11. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 page 60.*
Let $\mathbf{B}$ *be the* TWO-SIDED LAPLACE TRANSFORM *defined as* $[\mathbf{B}\mathsf{f}](s) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-sx}\, dx$.

PRP

1. $\mathbf{B}\mathbf{T}^n = e^{-sn}\mathbf{B} \quad \forall n\in\mathbb{Z}$
2. $\mathbf{B}\mathbf{D}^j = \mathbf{D}^{-j}\mathbf{B} \quad \forall j\in\mathbb{Z}$
3. $\mathbf{D}\mathbf{B} = \mathbf{B}\mathbf{D}^{-1} \quad \forall n\in\mathbb{Z}$
4. $\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} = \mathbf{D} \quad \forall n\in\mathbb{Z}$ ($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5. $\mathbf{D}\mathbf{B}\mathbf{D} = \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1} = \mathbf{B} \quad \forall n\in\mathbb{Z}$

✎PROOF:

$$
\begin{aligned}
\mathbf{B}\mathbf{T}^n\mathsf{f}(x) &= \mathbf{B}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition A.3 page 60)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathrm{d}u && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right]\\
&= e^{-sn}\,\mathbf{B}\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}\mathbf{D}^j\mathsf{f}(x) &= \mathbf{B}\left[2^{j/2}\,\mathsf{f}\left(2^jx\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition A.3 page 60)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}\left(2^jx\right)\right]e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathrm{d}u && \text{let } u \triangleq 2^jx \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathrm{d}u\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right] && \text{by Proposition A.6 page 62 and} && \text{Proposition A.7 page 63}\\
&= \mathbf{D}^{-j}\,\mathbf{B}\,\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{D}\mathbf{B}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x\right] && \text{by definition of } \mathbf{B}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathrm{d}x && \text{by definition of } \mathbf{D} && \text{(Definition A.3 page 60)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{u}{2}\right)\right]e^{-su}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\,e^{-su}\,\mathrm{d}u && \text{by Proposition A.6 page 62 and} && \text{Proposition A.7 page 63}\\
&= \mathbf{B}\mathbf{D}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{B}\mathbf{D} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse}\\
\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} &= \mathbf{D}\mathbf{B}\mathbf{B}^{-1} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse}\\
\mathbf{D}\mathbf{B}\mathbf{D} &= \mathbf{D}\mathbf{D}^{-1}\mathbf{B} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse}\\
\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1} &= \mathbf{D}^{-1}\mathbf{D}\mathbf{B} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse}
\end{aligned}
$$

Corollary A.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 page 60. Let* $\tilde{f}(\omega) \triangleq \tilde{\mathbf{F}}f(x)$ *be the* FOURIER TRANSFORM *(Definition J.2 page 149) of some function* $f \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition F.1 page 101).*

COR

$$\begin{array}{lrcll}
1. & \tilde{\mathbf{F}}\mathbf{T}^n & = & e^{-i\omega n}\tilde{\mathbf{F}} & \\
2. & \tilde{\mathbf{F}}\mathbf{D}^j & = & \mathbf{D}^{-j}\tilde{\mathbf{F}} & \\
3. & \mathbf{D}\tilde{\mathbf{F}} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1} & \\
4. & \mathbf{D} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} & = \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}} \\
5. & \tilde{\mathbf{F}} & = & \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} & = \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}
\end{array}$$

✎PROOF: These results follow directly from Proposition A.11 page 65 with $\tilde{\mathbf{F}} = \mathbf{B}|_{s=i\omega}$. ⇨

Proposition A.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition A.3 page 60. Let* $\tilde{f}(\omega) \triangleq \tilde{\mathbf{F}}f(x)$ *be the* FOURIER TRANSFORM *(Definition J.2 page 149) of some function* $f \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition F.1 page 101).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n f(x) = \frac{1}{2^{j/2}} e^{-i\frac{\omega}{2^j}n}\tilde{f}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n f(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n f(x) && \text{by Corollary A.1 page 67 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}f(x) && \text{by Corollary A.1 page 67 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{f}(\omega) \\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{f}\left(2^{-j}\omega\right) && \text{by Proposition A.2 page 60}
\end{aligned}$$

⇨

Proposition A.13. *Let* $\mathbf{T}$ *be the translation operator (Definition A.3 page 60). Let* $\tilde{f}(\omega) \triangleq \tilde{\mathbf{F}}f(x)$ *be the* FOURIER TRANSFORM *(Definition J.2 page 149) of a function* $f \in \boldsymbol{L}^2_{\mathbb{R}}$*. Let* $\breve{a}(\omega)$ *be the* DTFT *(Definition 3.1 page 21) of a sequence* $(a_n)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition 2.2 page 7).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{a}(\omega)\tilde{\phi}(\omega) \qquad \forall (a_n)\in\boldsymbol{\ell}^2_{\mathbb{R}}, \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\tilde{\mathbf{F}}\phi(x) && \text{by Corollary A.1 page 67}\\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega)\\
&= \breve{a}(\omega)\tilde{\phi}(\omega) && \text{by definition of } DTFT \text{ (Definition 3.1 page 21)}
\end{aligned}$$

⇨

Definition A.4. *Let* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition F.1 page 101). Let* $\boldsymbol{\ell}^2_{\mathbb{R}}$ *be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition F.1 page 101).*

DEF

$\mathbf{S}$ *is the **sampling operator** in* $\boldsymbol{\ell}^{2\,\boldsymbol{L}^2_{\mathbb{R}}}_{\mathbb{R}}$ *if* $\quad [\mathbf{S}f(x)](n) \triangleq f\left(\frac{2\pi}{\tau}n\right) \qquad \forall f\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \tau\in\mathbb{R}^+$

Theorem A.2 (Poisson Summation Formula—PSF). [6] *Let $\tilde{\mathsf{f}}(\omega)$ be the* FOURIER TRANSFORM *(Definition J.2 page 149) of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$. Let $\mathbf{S}$ be the* SAMPLING OPERATOR *(Definition A.4 page 67).*

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_\tau \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)$ then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:

   Note that $\mathsf{h}(x)$ is *periodic* with period τ. Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of PSF (this theorem—Theorem A.2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) && \text{by (1) lemma page 68} \\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau\left(\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}[\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition I.1 page 145)} \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau \mathsf{f}(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,\mathrm{d}u\right] && \text{where } u \triangleq x+n\tau \implies x = u-n\tau \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi kn}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}ku}\,\mathrm{d}u\right] \\
&= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,\mathrm{d}u\right]}_{[\tilde{\mathbf{F}}\mathsf{f}]\left(\frac{2\pi}{\tau}k\right)} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem I.1 page 146)} \\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}\mathsf{f}(x)]\left(\frac{2\pi}{\tau}k\right)\right] && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition J.2 page 149)} \\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \mathbf{S} \quad \text{(Definition A.4 page 67)} \\
&= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem I.1 page 146)}
\end{aligned}$$

✎

Theorem A.3 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{\mathsf{f}}(\omega)$ be the FOURIER TRANSFORM *(Definition J.2 page 149) of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.*

[6] Andrews et al. (2001), page 624, Knapp (2005) page 389, Lasser (1996), page 254, Rudin (1987), pages 194–195, Folland (1992), page 337

[7] Gauss (1900), page 88

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)}_{\text{summation in "frequency"}} = \underbrace{\frac{\tau}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\text{summation in "time"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)$, then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:
   Note that $\mathsf{h}(\omega)$ is periodic with period $2\pi/T$:
   $$\mathsf{h}\left(\omega + \frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau} + \frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + (n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \triangleq \mathsf{h}(\omega)$$
   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of IPSF (this theorem—Theorem A.3):

$$\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)$$

$$= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \qquad \text{by (1) lemma page 69}$$

$$= \hat{\mathbf{F}}^{-1} \underbrace{\left[\sqrt{\frac{\tau}{2\pi}} \int_0^{\frac{2\pi}{\tau}} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\,\mathsf{d}\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)\right]} \qquad \text{by definition of } \hat{\mathbf{F}} \qquad \text{(Definition I.1 page 145)}$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \sum_{n\in\mathbb{Z}} \int_0^{\frac{2\pi}{\tau}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) e^{-i\omega Tk}\,\mathsf{d}\omega\right]$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \sum_{n\in\mathbb{Z}} \int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u) e^{-i\left(u-\frac{2\pi}{\tau}n\right)Tk}\,\mathsf{d}u\right] \qquad \text{where } u \triangleq \omega + \frac{2\pi}{\tau}n \implies \omega = u - \frac{2\pi}{\tau}n$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \sum_{n\in\mathbb{Z}} \cancelto{1}{e^{i2\pi nk}} \int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u) e^{-iu\tau k}\,\mathsf{d}u\right]$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(u) e^{-iu\tau k}\,\mathsf{d}u\right]$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(u) e^{iu(-\tau k)}\,\mathsf{d}u\right]}_{[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)\right] \qquad \text{by value of } \tilde{\mathbf{F}}^{-1} \qquad \text{(Theorem J.1 page 150)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\,\tilde{\mathsf{f}} \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition A.4 page 67)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\mathsf{f}(x) \qquad \text{by definition of } \tilde{\mathbf{F}} \qquad \text{(Definition J.2 page 149)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathsf{f}(-k\tau) \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition A.4 page 67)}$$

$$= \sqrt{\tau}\,\frac{1}{\sqrt{\frac{2\pi}{\tau}}} \sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau) e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem I.1 page 146)}$$

$$= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}} \sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau) e^{ik\tau\omega} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem I.1 page 146)}$$

$$= \frac{\tau}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \mathsf{f}(m\tau) e^{-i\omega m\tau} \qquad \text{let } m \triangleq -k$$

Remark A.2. The left hand side of the *Poisson Summation Formula* (Theorem A.2 page 68) is very similar to the *Zak Transform* $\mathbf{Z}$: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark A.3. A generalization of the *Poisson Summation Formula* (Theorem A.2 page 68) is the **Selberg Trace Formula**.[9]

A.7 Examples

Example A.2 (linear functions). [10] Let $\mathbf{T}$ be the *translation operator* (Definition A.3 page 60). Let $\mathcal{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX
1. $\{x,\ \mathbf{T}x\}$ is a *basis* for $\mathcal{L}(\mathbb{C},\mathbb{C})$ and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x$ $\quad\forall \mathsf{f}\in\mathcal{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition A.3 page 60}\\
&= (ax+b)|_{x=1}\,x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of } \mathsf{f}\\
&= (a+b)x - b(x-1)\\
&= ax+bx-bx+b\\
&= ax+b\\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of } \mathsf{f}
\end{aligned}$$

⇨

Example A.3 (Cardinal Series). Let $\mathbf{T}$ be the *translation operator* (Definition A.3 page 60). The *Paley-Wiener* class of functions $\boldsymbol{PW}^2_\sigma$ (Definition 1.1 page 3) are those functions which are "*bandlimited*" with respect to their Fourier transform (Definition J.2 page 149). The cardinal series forms an orthogonal basis for such a space (Theorem 1.2 page 4). The *Fourier coefficients* for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of $\mathsf{f}(x)$ taken at regular intervals (Theorem 1.3 page 4). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x)\,|\,\mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\,\mathrm{d}x \triangleq \mathsf{f}(n)$$

EX
1. $\left\{\mathbf{T}^n\frac{\sin(\pi x)}{\pi x}\,\middle|\,n\in\mathbb{N}\right\}$ is a *basis* for $\boldsymbol{PW}^2_\sigma$ and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty}\mathsf{f}(n)\mathbf{T}^n\frac{\sin(\pi x)}{\pi x}}_{\textit{Cardinal series}}$ $\quad\forall \mathsf{f}\in\boldsymbol{PW}^2_\sigma,\ \sigma\le\frac{1}{2}$

✎PROOF: See Theorem 1.2 page 4. ⇨

[8] Janssen (1988), page 24, Zayed (1996), page 482
[9] Lax (2002), page 349, Selberg (1956), Terras (1999)
[10] Higgins (1996) page 2

Example A.4 (Fourier Series).

EX

1. $\left\{ \mathbf{D}_n e^{ix} \middle| n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix} \qquad \forall x\in(0:2\pi), \mathsf{f}\in\boldsymbol{L}(0:2\pi)$ where
3. $\alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x)\mathbf{D}_n e^{-ix}\, \mathrm{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}(0:2\pi)$

✎PROOF: See Theorem I.1 page 146. ✏

Example A.5 (Fourier Transform). [11]

EX

1. $\left\{ \mathbf{D}_\omega e^{ix} | \omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)\mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}_\omega e^{-ix}\, \mathrm{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example A.6 (Gabor Transform). [12]

EX

1. $\left\{ \left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{ix}\right) \middle| \tau,\omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \int_{\mathbb{R}} \mathsf{G}(\tau,\omega)\, \mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \qquad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\mathsf{G}(\tau,\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{-ix}\right)\, \mathrm{d}x \qquad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example A.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX

1. $\left\{ \mathbf{D}^k\mathbf{T}^n\psi(x) \middle| k,n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha_{k,n} \mathbf{D}^k\mathbf{T}^n\psi(x) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\alpha_n \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}^k\mathbf{T}^n\psi^*(x)\, \mathrm{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

[11] cross reference: Definition J.2 page 149

[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

APPENDIX B

ALGEBRAIC STRUCTURES

"In this book, learned reader, you have the rules of algebra... It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one.... this most abstruse and unsurpassed treasury of the entire arithmetic being brought to light and, as in a theater, exposed to the sight of all..."

Gerolamo Cardano (1501–1576), Italian mathematician, physician, and astrologer [1]

A set together with one or more operations forms several standard mathematical structures:

group $\supseteq$ *ring* $\supseteq$ *commutative ring* $\supseteq$ *integral domain* $\supseteq$ *field*

Definition B.1. [2] *Let X be a set and $\diamond : X \times X \to X$ be an operation on X.*

DEF *The pair $(X, \diamond)$ is a **group** if*

1.	$\exists e \in X$	such that	$e \diamond x = x \diamond e = x$	$\forall x \in X$	(IDENTITY *element*)	*and*
2.	$\exists(-x) \in X$	such that	$(-x) \diamond x = x \diamond (-x) = e$	$\forall x \in X$	(INVERSE *element*)	*and*
3.			$x \diamond (y \diamond z) = (x \diamond y) \diamond z$	$\forall x,y,z \in X$	(ASSOCIATIVE)	

Definition B.2. [3] *Let $+ : X \times X \to X$ and $* : X \times X \to X$ be operations on a set X. Furthermore, let the operation $*$ also be represented by juxtapostion as in $a * b \equiv ab$.*

DEF *The triple $(X, +, *)$ is a **ring** if*

1.	$(X, +)$ is a group.				*(additive group)*	*and*
2.	$x(yz)$	$=$	$(xy)z$	$\forall x,y,z \in X$	*(associative with respect to $*$)*	*and*
3.	$x(y+z)$	$=$	$(xy)+(xz)$	$\forall x,y,z \in X$	*($*$ is left distributive over $+$)*	*and*
4.	$(x+y)z$	$=$	$(xz)+(yz)$	$\forall x,y,z \in X$	*($*$ is right distributive over $+$).*	

Definition B.3. [4]

[1] quote: Cardano (1545), page 1
image: http://en.wikipedia.org/wiki/Image:Cardano.jpg
[2] Durbin (2000), page 29
[3] Durbin (2000), pages 114–115
[4] Durbin (2000), page 118

DEF *A triple $(X, +, *)$ is a* ***commutative ring*** *if*

1. $(X, +, *)$ *is a ring* (ring) and
2. $xy = yx \quad \forall x, y \in X$ (commutative).

Definition B.4. [5] *Let R be a* COMMUTATIVE RING *(Definition B.3 page 73).*

DEF *A function $|\cdot|$ in $\mathbb{R}^{\mathbb{R}}$ is an* ***absolute value*** *(or* ***modulus****) if*

1. $|x| \geq 0 \quad x \in \mathbb{R}$ (NON-NEGATIVE) and
2. $|x| = 0 \iff x = 0 \quad x \in \mathbb{R}$ (NONDEGENERATE) and
3. $|xy| = |x| \cdot |y| \quad x, y \in \mathbb{R}$ (HOMOGENEOUS / SUBMULTIPLICATIVE) and
4. $|x + y| \leq |x| + |y| \quad x, y \in \mathbb{R}$ (SUBADDITIVE / TRIANGLE INEQUALITY)

Definition B.5. [6]

DEF *The structure $F \triangleq (X, +, \cdot, 0, 1)$ is a* ***field*** *if*

1. $(X, +, *)$ *is a ring* (ring) and
2. $xy = yx \quad \forall x, y \in X$ (commutative with respect to $*$) and
3. $(X \setminus \{0\}, *)$ *is a group* (group with respect to $*$).

Definition B.6. [7] *Let $\boldsymbol{V} = (F, +, \cdot)$ be a vector space and $\otimes : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ be a vector-vector multiplication operator.*
An ***algebra*** *is any pair $(\boldsymbol{V}, \otimes)$ that satisfies ($\otimes$ is represented by juxtaposition)*

DEF

1. $(\boldsymbol{ux})\boldsymbol{y} = \boldsymbol{u}(\boldsymbol{xy}) \quad \forall \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ (ASSOCIATIVE) and
2. $\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{y}) = (\boldsymbol{ux}) + (\boldsymbol{uy}) \quad \forall \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ (LEFT DISTRIBUTIVE) and
3. $(\boldsymbol{u} + \boldsymbol{x})\boldsymbol{y} = (\boldsymbol{uy}) + (\boldsymbol{xy}) \quad \forall \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ (RIGHT DISTRIBUTIVE) and
4. $\alpha(\boldsymbol{xy}) = (\alpha\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x}(\alpha\boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ and $\alpha \in F$ (SCALAR COMMUTATIVE) .

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[5] Cohn (2002) page 312

[6] Durbin (2000), page 123, Weber (1893)

[7] Abramovich and Aliprantis (2002), page 3, Michel and Herget (1993), page 56

APPENDIX C

NORMED ALGEBRAS

C.1 Algebras

All *linear spaces* are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition C.1. [3] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF *An algebra $\boldsymbol{A}$ is **unital** if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition C.2. [4] *Let $\boldsymbol{A}$ be an* UNITAL ALGEBRA *(Definition C.1 page 75) with unit e.*

DEF

$$\begin{array}{llcll}
\textit{The } \textbf{\textit{spectrum}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \sigma(x) & \triangleq & \left\{\lambda \in \mathbb{C} | \lambda e - x \textit{ is not invertible}\right\}. & \\
\textit{The } \textbf{\textit{resolvent}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \rho_x(\lambda) & \triangleq & (\lambda e - x)^{-1} & \forall \lambda \notin \sigma(x). \\
\textit{The } \textbf{\textit{spectral radius}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & r(x) & \triangleq & \sup\{|\lambda| | \lambda \in \sigma(x)\}. &
\end{array}$$

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[1] Fuchs (1995) page 2
[2] Hazewinkel (2000) page v
[3] Folland (1995) page 1
[4] Folland (1995) pages 3–4

C.2 Star-Algebras

Definition C.3. [5] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF

*The pair $(\boldsymbol{A}, *)$ is a **$*$-algebra**, or **star-algebra**, if*

1.	$(x+y)^* = x^* + y^*$	$\forall x,y \in \boldsymbol{A}$	(DISTRIBUTIVE)	and
2.	$(\alpha x)^* = \bar{\alpha} x^*$	$\forall x \in \boldsymbol{A}, \alpha \in \mathbb{C}$	(CONJUGATE LINEAR)	and
3.	$(xy)^* = y^* x^*$	$\forall x,y \in \boldsymbol{A}$	(ANTIAUTOMORPHIC)	and
4.	$x^{**} = x$	$\forall x \in \boldsymbol{A}$	(INVOLUTORY)	

*The operator $*$ is called an **involution** on the algebra $\boldsymbol{A}$.*

Proposition C.1. [6] *Let $(\boldsymbol{A}, *)$ be an* UNITAL $*$-ALGEBRA.

PRP

x is invertible $\implies$ $\left\{\begin{array}{lll} 1. & x^* \text{ is INVERTIBLE} & \forall x \in \boldsymbol{A} \quad \text{and} \\ 2. & (x^*)^{-1} = (x^{-1})^* & \forall x \in \boldsymbol{A} \end{array}\right.$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= (x\,e^*)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition C.3 page 76)}\\
&= (x^*\,e^{**})^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition C.3 page 76)}\\
&= (x^*\,e)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition C.3 page 76)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition C.3 page 76)}\\
e^*\,x &= (e^*\,x)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition C.3 page 76)}\\
&= (e^{**}\,x^*)^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition C.3 page 76)}\\
&= (e\,x^*)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition C.3 page 76)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition C.3 page 76)}
\end{aligned}$$

2. Proof that $(x^*)^{-1} = (x^{-1})^*$:

$$\begin{aligned}
(x^{-1})^*\,(x^*) &= \left[x\,(x^{-1})\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition C.3 page 76)}\\
&= e^*\\
&= e && \text{by item (1) page 76}\\
(x^*)\,(x^{-1})^* &= \left[x^{-1}\,x\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition C.3 page 76)}\\
&= e^*\\
&= e && \text{by item (1) page 76}
\end{aligned}$$

✏

Definition C.4. [7] *Let $(\boldsymbol{A}, \|\cdot\|)$ be a* $*$-ALGEBRA *(Definition C.3 page 76).*

DEF

- *An element $x \in \boldsymbol{A}$ is **hermitian** or **self-adjoint** if $x^* = x$.*
- *An element $x \in \boldsymbol{A}$ is **normal** if $xx^* = x^*x$.*
- *An element $x \in \boldsymbol{A}$ is a **projection** if $xx = x$ (INVOLUTORY) and $x^* = x$ (HERMITIAN).*

[5] Rickart (1960), page 178, Gelfand and Naimark (1964), page 241
[6] Folland (1995) page 5
[7] Rickart (1960), page 178, Gelfand and Naimark (1964), page 242

Theorem C.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (Definition C.3 page 76).

THM
$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are hermitian}} \implies \left\{\begin{array}{lcll} x+y &=& (x+y)^* & (x+y \text{ is self adjoint}) \\ x^* &=& (x^*)^* & (x^* \text{ is self adjoint}) \\ \multicolumn{4}{l}{\underbrace{xy = (xy)^*}_{(xy) \text{ is hermitian}} \iff \underbrace{xy = yx}_{\text{commutative}}} \end{array}\right.$$

✎PROOF:

$$\begin{aligned} (x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition C.3 page 76)} \\ &= x + y && \text{by left hypothesis} \end{aligned}$$

$$\left(x^*\right)^* = x \qquad \text{by } \textit{involutory} \text{ property of } * \qquad \text{(Definition C.3 page 76)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned} xy &= (xy)^* && \text{by left hypothesis} \\ &= y^*x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition C.3 page 76)} \\ &= yx && \text{by left hypothesis} \end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned} (xy)^* &= (yx)^* && \text{by left hypothesis} \\ &= x^*y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition C.3 page 76)} \\ &= xy && \text{by left hypothesis} \end{aligned}$$

⇨

Definition C.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (Definition C.3 page 76).

DEF
The ***real part*** *of* x *is defined as* $\mathbf{R}_{\mathrm{e}}x \triangleq \frac{1}{2}\left(x + x^*\right)$

The ***imaginary part*** *of* x *is defined as* $\mathbf{I}_{\mathrm{m}}x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem C.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA (Definition C.3 page 76).

THM
$$\begin{array}{lcllr} \Re x &=& (\Re x)^* & \forall x \in \boldsymbol{A} & (\Re x \text{ is hermitian}) \\ \Im x &=& (\Im x)^* & \forall x \in \boldsymbol{A} & (\Im x \text{ is hermitian}) \end{array}$$

✎PROOF:

$$\begin{aligned} (\Re x)^* &= \left(\frac{1}{2}(x + x^*)\right)^* && \text{by definition of } \Re && \text{(Definition C.5 page 77)} \\ &= \frac{1}{2}(x^* + x^{**}) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition C.3 page 76)} \\ &= \frac{1}{2}(x^* + x) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition C.3 page 76)} \\ &= \Re x && \text{by definition of } \Re && \text{(Definition C.5 page 77)} \\ (\Im x)^* &= \left(\frac{1}{2i}(x - x^*)\right)^* && \text{by definition of } \Im && \text{(Definition C.5 page 77)} \end{aligned}$$

[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960), page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998) page 42

$= \frac{1}{2i}\left(x^* - x^{**}\right)$	by *distributive* property of $*$	(Definition C.3 page 76)
$= \frac{1}{2i}\left(x^* - x\right)$	by *involutory* property of $*$	(Definition C.3 page 76)
$= \Im x$	by definition of $\Im$	(Definition C.5 page 77)

✎

Theorem C.3 (Hermitian representation). [11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition C.3 page 76).*

THM

$$a = x + iy \quad\iff\quad x = \Re a \quad \textit{and} \quad y = \Im a$$

✎PROOF:

Proof that $a = x + iy \implies x = \Re a$ and $y = \Im a$:

	$a = x + iy$	by left hypothesis	
$\implies$	$a^* = (x+iy)^*$	by definition of *adjoint*	(Definition C.4 page 76)
	$= x^* - iy^*$	by *distributive* property of $*$	(Definition C.3 page 76)
	$= x - iy$	by Theorem C.2 page 77	
$\implies$	$x = a - iy$	by solving for x in $a = x + iy$ equation	
	$x = a^* + iy$	by solving for x in $a^* = x - iy$ equation	
$\implies$	$x + x = a + a^*$	by adding previous 2 equations	
$\implies$	$2x = a + a^*$	by solving for x in previous equation	
$\implies$	$x = \frac{1}{2}\left(a + a^*\right)$		
	$= \Re a$	by definition of $\Re$	(Definition C.5 page 77)
	$iy = a - x$	by solving for iy in $a = x + iy$ equation	
	$iy = -a^* + x$	by solving for iy in $a = x + iy$ equation	
$\implies$	$iy + iy = a - a^*$	by adding previous 2 equations	
$\implies$	$y = \frac{1}{2i}\left(a - a^*\right)$	by solving for iy in previous equations	
	$= \Im a$	by definition of $\Im$	(Definition C.5 page 77)

Proof that $a = x + iy \impliedby x = \Re a$ and $y = \Im a$:

$x + iy = \Re a + i\,\Im a$	by right hypothesis	
$= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\Re a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\Im a}$	by definition of $\Re$ and $\Im$	(Definition C.5 page 77)
$= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \cancelto{0}{\left(\frac{1}{2}a^* - \frac{1}{2}a^*\right)}$		
$= a$		

✎

[11] Michel and Herget (1993) page 430, Rickart (1960), page 179, Gelfand and Neumark (1943b), page 7

C.3 Normed Algebras

Definition C.6. [12] *Let $\boldsymbol{A}$ be an algebra.*

DEF

*The pair $(\boldsymbol{A}, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in \boldsymbol{A} \qquad \text{(multiplicative condition)}$$

*A normed algebra $(\boldsymbol{A}, \|\cdot\|)$ is a **Banach algebra** if $(\boldsymbol{A}, \|\cdot\|)$ is also a Banach space.*

Proposition C.2.

PRP

$(\boldsymbol{A}, \|\cdot\|)$ *is a normed algebra* $\implies$ *multiplication is **continuous** in* $(\boldsymbol{A}, \|\cdot\|)$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x - y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(\boldsymbol{A}, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 79)}\\
&= \|z(x-y)\| \\
&\le \|z\|\,\|x-y\| && \text{by definition of } \textit{normed algebra} && \text{(Definition C.6 page 79)}\\
&\triangleq \|z\|\,\delta && \text{by definition of } \delta && \text{(item (2) page 79)}\\
&\le \epsilon && \text{for some value of } \delta > 0
\end{aligned}$$

✏

Theorem C.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l} (\boldsymbol{A}, \|\cdot\|) \text{ is a Banach algebra} \\ \text{every nonzero } x \in \boldsymbol{A} \text{ is invertible} \end{array}\right\} \implies \boldsymbol{A} \equiv \mathbb{C} \quad (\boldsymbol{A} \text{ is isomorphic to } \mathbb{C})$$

C.4 C* Algebras

Definition C.7. [14]

DEF

*The triple $(\boldsymbol{A}, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(\boldsymbol{A}, \|\cdot\|)$ *is a Banach algebra* and
2. $(\boldsymbol{A}, *)$ *is a $*$-algebra* and
3. $\left\|x^* x\right\| = \|x\|^2 \quad \forall x \in \boldsymbol{A}$.

*A C^* **algebra** $(\boldsymbol{A}, \|\cdot\|, *)$ is also called a **C star algebra**.*

[12] Rickart (1960), page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩
[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩
[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem C.5. [15] *Let $\boldsymbol{A}$ be an algebra.*

THM: $(\boldsymbol{A}, \|\cdot\|, *)$ *is a* C^* ***algebra*** $\implies$ $\|x^*\| = \|x\|$

✎PROOF:

$$
\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\ \|x\|^2 \\
&= \frac{1}{\|x\|}\ \|x^* x\| && \text{by definition of } C^*\text{-}\textit{algebra} && \text{(Definition C.7 page 79)} \\
&\le \frac{1}{\|x\|}\ \|x^*\|\ \|x\| && \text{by definition of } \textit{normed algebra} && \text{(Definition C.6 page 79)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by } \textit{involution} \text{ property of } * && \text{(Definition C.3 page 76)}
\end{aligned}
$$

✏

[15] Folland (1995) page 1, Gelfand and Neumark (1943b), page 4, Gelfand and Neumark (1943a)

APPENDIX D

POLYNOMIALS

D.1 Definitions

Definition D.1. [1] *Let* $(\mathbb{F},+,\cdot,0,1)$ *be a* FIELD.

DEF

A function p *in* $\mathbb{F}^{\mathbb{F}}$ *is a **polynomial** over* $(\mathbb{F},+,\cdot,0,1)$ *if it is of the form*

$$\mathsf{p}(x) \triangleq \sum_{n=0}^{N} \alpha_n x^n \qquad \alpha_n \in \mathbb{F},\ \alpha_N \neq 0.$$

*The **degree** of* p *is* N. *A **coefficient** of* p *is any element of* $\left(\!\left(\alpha_n\right)\!\right)_1^N$.
*The **leading coefficient** of* p *is* α_N.

Definition D.2. [2] *Let* $(\mathbb{F},+,\cdot,0,1)$ *be a* FIELD.

DEF

A polynomial p *of degree* N *over the field* $\mathbb{F}$ *and*
a polynomial q *of degree* M *over the field* $\mathbb{F}$ *are **equal** if*

1. $N = M$ *and*
2. $\alpha_n = \beta_n$ *for* $n = 0, 1, \ldots, N$.

The expression $\mathsf{p}(x) = \mathsf{q}(x)$ *(or* $\mathsf{p} = \mathsf{q}$*) denotes that* p *and* q *are* EQUAL.

[1] Barbeau (1989) page 1, Fuhrmann (2012) page 11, Borwein and Erdélyi (1995) page 2
[2] Fuhrmann (2012) page 11

D.2 Ring properties

D.2.1 Polynomial Arithmetic

Theorem D.1 (polynomial addition). [3] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

THM

$$\underbrace{\left(\sum_{n=0}^{N} \alpha_n x^n\right)}_{\mathsf{p}(x)} + \underbrace{\left(\sum_{n=0}^{M} \beta_n x^n\right)}_{\mathsf{q}(x)} = \underbrace{\sum_{n=0}^{\max(N,M)} \gamma_n x^n}_{\mathsf{p}(x)+\mathsf{q}(x)} \quad \textit{where} \quad \gamma_n \triangleq \begin{cases} \alpha_n + \beta_n & \text{for } n \le \min(N,M) \\ \alpha_n & \text{for } n > M \\ \beta_n & \text{for } n > N \end{cases}$$

$$\textit{for all} \quad x, \alpha_n, \beta_n \in \mathbb{F}$$

Polynomial multiplication is equivalent to convolution (Definition 2.3 page 7) of the coefficients (Definition D.1 page 81). [4]

Theorem D.2 (polynomial multiplication). [5] *Let* $(\alpha_n \in \mathbb{C})$, $(b_n \in \mathbb{C})$, *and* $x \in \mathbb{C}$.

THM

$$\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{n=0}^{M} \beta_n x^n\right) = \sum_{n=0}^{N+M} \underbrace{\left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right)}_{\textit{Cauchy product}} x^n$$

✎PROOF:

$$\begin{aligned}
\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{m=0}^{M} \beta_m x^m\right) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n \beta_m x^{n+m} \\
&= \sum_{n=0}^{N}\sum_{k=n}^{M+n} \alpha_n \beta_{k-n} x^k && k \triangleq n+m \iff m = k-n \\
&= \sum_{n=0}^{N+M}\left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right) x^n
\end{aligned}$$

Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	β_0	β_1	β_2	β_3	$\cdots$	β_M
α_0	$\alpha_0\beta_0$	$\alpha_0\beta_1 x$	$\alpha_0\beta_2 x^2$	$\alpha_0\beta_3 x^3$	$\cdots$	$\alpha_0\beta_M x^M$
α_1	$\alpha_1\beta_0 x$	$\alpha_1\beta_1 x^2$	$\alpha_1\beta_2 x^3$	$\alpha_1\beta_3 x^4$	$\cdots$	$\alpha_1\beta_M x^{1+M}$
α_2	$\alpha_2\beta_0 x^2$	$\alpha_2\beta_1 x^3$	$\alpha_2\beta_2 x^4$	$\alpha_2\beta_3 x^5$	$\cdots$	$\alpha_2\beta_M x^{2+M}$
α_3	$\alpha_3\beta_0 x^3$	$\alpha_3\beta_1 x^4$	$\alpha_3\beta_2 x^5$	$\alpha_3\beta_3 x^6$	$\cdots$	$\alpha_3\beta_M x^{3+M}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_N	$\alpha_N\beta_0 x^N$	$\alpha_N\beta_1 x^{N+1}$	$\alpha_N\beta_2 x^{N+2}$	$\alpha_N\beta_3 x^{N+3}$	$\cdots$	$\alpha_N\beta_M x^{N+M}$

1. The expression $\sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n\beta_m x^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

[3] Fuhrmann (2012) page 11

[4] *Convolution*: In fact, using *GNU Octave*™ or *MatLab*™, polynomial multiplication can be performed using convolution. For example, the operation $(x^3 + 5x^2 + 7x + 9)(4x^2 + 11)$ can be calculated in *GNU Octave*™ or *MatLab*™ with conv([1 5 7 9],[4 0 11])

[5] Apostol (1975), page 237

2. If we switched the order of summation to $\sum_{m=0}^{M}\sum_{n=0}^{N}\alpha_n\beta_m x^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

3. For $N = M = \infty$, the expression $\sum_{n=0}^{N+M}\left(\sum_{k=0}^{n}\alpha_k\beta_{n-k}\right)x^n$ is equivalent to adding *diagonally* starting from the upper left corner and proceeding towards the lower right.

4. For finite N and M...

   (a) The upper limit on the inner summation puts two constraints on k:
   $$\left\{\begin{array}{lll} k & \le & n \quad \text{and} \\ k & \le & N \end{array}\right\} \implies k \le \min(n, N)$$

   (b) The lower limit on the inner summation also puts two constraints on k:
   $$\left\{\begin{array}{lll} k & \ge & 0 \quad \text{and} \\ k & \ge & n - M \end{array}\right\} \implies k \ge \max(0, n - M)$$

✏

Polynomial division can be performed in a manner very similar to integer division (both integers and polynomials are *rings*).

Definition D.3 (Polynomial division). *The quantities of polynomial division are defined as follows:*

DEF
$$\frac{\mathsf{d}(x)}{\mathsf{p}(x)} = \mathsf{q}(x) + \frac{\mathsf{r}(x)}{\mathsf{p}(x)} \qquad \textit{where} \qquad \left\{\begin{array}{lll} \mathsf{d}(x) & \textit{is the } \textbf{\textit{dividend}} & \textit{and} \\ \mathsf{p}(x) & \textit{is the } \textbf{\textit{divisor}} & \textit{and} \\ \mathsf{q}(x) & \textit{is the } \textbf{\textit{quotient}} & \textit{and} \\ \mathsf{r}(x) & \textit{is the } \textbf{\textit{remainder}}. & \end{array}\right\}$$

The ring of integers $\mathbb{Z}$ contains some special elements called *primes* which can only be divided[6] by themselves or 1.

Rings of polynomials have a similar elements called *primitive polynomials.*

Definition D.4.

DEF
A ***primitive polynomial*** *is any polynomial* $\mathsf{p}(x)$ *that satisfies*
1. $\mathsf{p}(x)$ *cannot be factored*
2. *the smallest order polynomial that* $\mathsf{p}(x)$ *can divide is* $x^{2^n-1} + 1 = 0$.

Example D.1. [7] Some examples of primitive polynomials over $GF(2)$ are

EX

order	primitive polynomial
2	$\mathsf{p}(x) = x^2 + x + 1$
3	$\mathsf{p}(x) = x^3 + x + 1$
4	$\mathsf{p}(x) = x^4 + x + 1$
5	$\mathsf{p}(x) = x^5 + x^2 + 1$
5	$\mathsf{p}(x) = x^5 + x^4 + x^2 + x + 1$
16	$\mathsf{p}(x) = x^{16} + x^{15} + x^{13} + x^4 + 1$
31	$\mathsf{p}(x) = x^{31} + x^{28} + 1$

An m-sequence is the remainder when dividing any non-zero polynomial by a primitive polynomial. We can define an *equivalence relation* on polynomials which defines two polynomials as *equivalent with respect to* $p(x)$ when their remainders are equal.

[6] The expression "a divides b" means that b/a has remainder 0.

[7] Wicker (1995), pages 465–475

Definition D.5 (Equivalence relation). *Let* $\dfrac{\alpha_1(x)}{p(x)} = \mathsf{q}_1(x) + \dfrac{r_1(x)}{p(x)}$ *and* $\dfrac{\alpha_2(x)}{p(x)} = \mathsf{q}_2(x) + \dfrac{r_2(x)}{p(x)}$.

Then $\alpha_1(x) \equiv \alpha_2(x)$ *with respect to* $p(x)$ *if* $r_1(x) = r_2(x)$.

Using the equivalence relation of Definition D.5, we can develop two very useful equivalent representations of polynomials over GF(2). We will call these two representations the *exponential* representation and the *polynomial* representation.

Example D.2. By Definition D.5 and under $p(x) = x^3 + x + 1$, we have the following equivalent representations:

EX

$$\begin{array}{rcrcrcrcl}
\frac{x^0}{x^3+x+1} &=& 0 &+& \frac{1}{x^3+x+1} &\Longrightarrow& x^0 &\equiv& 1\\
\frac{x^1}{x^3+x+1} &=& 0 &+& \frac{x}{x^3+x+1} &\Longrightarrow& x^1 &\equiv& x\\
\frac{x^2}{x^3+x+1} &=& 0 &+& \frac{x^2}{x^3+x+1} &\Longrightarrow& x^2 &\equiv& x^2\\
\frac{x^3}{x^3+x+1} &=& 1 &+& \frac{x+1}{x^3+x+1} &\Longrightarrow& x^3 &\equiv& x+1\\
\frac{x^4}{x^3+x+1} &=& x &+& \frac{x^2+x}{x^3+x+1} &\Longrightarrow& x^4 &\equiv& x^2+x\\
\frac{x^5}{x^3+x+1} &=& x^2+1 &+& \frac{x^2+x+1}{x^3+x+1} &\Longrightarrow& x^5 &\equiv& x^2+x+1\\
\frac{x^6}{x^3+x+1} &=& x^3+x+1 &+& \frac{x^2+1}{x^3+x+1} &\Longrightarrow& x^6 &\equiv& x^2+1\\
\frac{x^7}{x^3+x+1} &=& x^4+x^2+x+1 &+& \frac{1}{x^3+x+1} &\Longrightarrow& x^7 &\equiv& 1
\end{array}$$

Notice that $x^7 \equiv x^0$, and so a cycle is formed with $2^3 - 1 = 7$ elements in the cycle. The monomials to the left of the $\equiv$ are the *exponential* representation and the polynomials to the right are the *polynomial* representation. Additionally, the polynomial representation may be put in a vector form giving a *vector* representation. The vectors may be interpreted as a binary number and represented as a decimal numeral.

EX

exponential	polynomial	vector	decimal
x^0	1	[001]	1
x^1	x	[010]	2
x^2	x^2	[100]	4
x^3	$x+1$	[011]	3
x^4	x^2+x	[110]	6
x^5	x^2+x+1	[111]	7
x^6	x^2+1	[101]	5

Example D.3. We can generate an m-sequence of length $2^3 - 1 = 7$ by dividing 1 by the primitive polynomial $x^3 + x + 1$.

$$
\begin{array}{r|l}
 & x^{-3}+x^{-5}+x^{-6}+\quad x^{-7}+x^{-10}+x^{-12}+x^{-13}+x^{-14}+x^{-17}+\cdots \\ \cline{2-2}
x^3+x+1 & 1 \\
 & 1+x^{-2}+x^{-3} \\ \cline{2-2}
 & x^{-2}+x^{-3} \\
 & x^{-2}+x^{-4}+x^{-5} \\ \cline{2-2}
 & x^{-3}+x^{-4}+x^{-5} \\
 & x^{-3}+x^{-5}+x^{-6} \\ \cline{2-2}
 & x^{-4}+x^{-6} \\
 & x^{-4}+x^{-6}+x^{-7} \\ \cline{2-2}
 & x^{-7} \\
 & x^{-7}+x^{-9}+x^{-10} \\ \cline{2-2}
 & x^{-9}+x^{-10} \\
 & x^{-9}+x^{-11}+x^{-12} \\ \cline{2-2}
 & x^{-10}+x^{-11}+x^{-12} \\
 & x^{-10}+x^{-12}+x^{-13} \\ \cline{2-2}
 & x^{-11}+x^{-13} \\
 & x^{-11}+x^{-13}+x^{-14} \\ \cline{2-2}
 & x^{-14} \\
 & \vdots
\end{array}
$$

The coefficients, starting with the x^{-1} term, of the resulting polynomial form the m-sequence

$$0010111\ 0010111\ \cdots$$

which repeats every $2^3-1=7$ elements.

Note that the division operation in Example D.3 can be performed using vector notation rather than polynomial notation.

Example D.4. Generate an m-sequence of length $2^3-1=7$ by dividing 1 by the primitive polynomial x^3+x+1 using vector notation.

$$
\begin{array}{r|ccccccccccccccccccc}
 & & . & 0&0&1&0&1&1&1&0&0&1&0&1&1&1&0&\cdots \\ \hline
1011 & 1 & . & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0& \\
 & 1 & & 0&1&1 \\
 & 0 & & 0&1&1&0 \\
 & & & 0&0&0&0 \\
 & & & 0&1&1&0&0 \\
 & & & & 1&0&1&1 \\
 & & & & 0&1&1&1&0 \\
 & & & & & 1&0&1&1 \\
 & & & & & 0&1&0&1&0 \\
 & & & & & & 1&0&1&1 \\
 & & & & & & 0&0&0&1&0 \\
 & & & & & & & 0&0&0&0 \\
 & & & & & & & 0&0&1&0&0 \\
 & & & & & & & & 0&0&0&0 \\
 & & & & & & & & 0&1&0&0&0 \\
 & & & & & & & & & 1&0&1&1 \\
 & & & & & & & & & 0&0&1&1&0 \\
 & & & & & & & & & & 0&0&0&0 \\
 & & & & & & & & & & 0&1&1&0&0 \\
 & & & & & & & & & & & 1&0&1&1 \\
 & & & & & & & & & & & 0&1&1&1&0 \\
 & & & & & & & & & & & & 0&0&0&0 \\
 & & & & & & & & & & & & & \vdots
\end{array}
$$

The coefficients, starting to the right of the binary point, is again the sequence

$$0010111\ 0010111\ \cdots.$$

D.2.2 Greatest common divisor

Theorem D.3 (Extended Euclidean Algorithm). [8]

Let $r_1(x)$ *and* $r_2(x)$ *be polynomials. The following algorithm computes their greatest common divisor* $\gcd(r_1(x), r_2(x))$, *and factors* $a(x)$ *and* $b(x)$ *such that*

$$r_1(x)a(x) + r_2(x)b(x) = \gcd(r_1, r_2)$$

THM

n	*remainder* $r_n = r_{n-2} - q_n r_{n-1}$	*quotient* q_n	*factor* $\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	*factor* $\beta_n = b_{n-2} - q_n\beta_{n-1}$
1	$r_1(x)$	–	1	0
2	$r_2(x)$	–	0	1
3	$r_1 - q_3 r_2$	q_3	1	$-q_3$
4	$r_2 - q_4 r_3$	q_4	$-q_4$	$1 + q_4 q_1$
5	$r_1 - q_5 r_2$	q_5	$1 + q_5 q_4$	$-q_3 - q_5(1 + q_4 q_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\gcd(r_1(x), r_2(x))$	q_n	$a(x) = a_{n-2} - q_n\alpha_{n-1}$	$b(x) = b_{n-2} - q_n\beta_{n-1}$
$n+1$	0	q_{n+1}		

✎PROOF:

$$\begin{aligned} r_1 &= q_3 r_2 + r_3 \\ &= q_3 r_2 + r_3 \end{aligned}$$

✏

Example D.5. Let

$$u(x) \triangleq (1-x)^2 \qquad v(x) \triangleq x^2.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^2}_{u(x)}\underbrace{(1+2x)}_{a(x)} + \underbrace{(x^2)}_{v(x)}\underbrace{(3-2x)}_{b(x)} = \underbrace{1}_{\gcd}$$

Because $\gcd(u, v) = 1$, $u(x)$ and $v(x)$ are said to be *relatively prime.*

✎PROOF:

n	$r_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^2 = 1 - 2x + x^2 = u(x)$	–	1	0
0	$x^2 = v(x)$	–	0	1
1	$1 - 2x$	1	1	-1
2	$\frac{1}{2}x$	$-\frac{1}{2}x$	$\frac{1}{2}x$	$1 - \frac{1}{2}x$
3	$1 = \gcd(\ (1-x)^2,\ x^2\ )$	-4	$1 + 2x = a(x)$	$3 - 2x = b(x)$
4	0	$\frac{1}{2}x$	–	–

✏

[8] Wicker (1995), page 53, Fuhrmann (2012) page 11

Example D.6. Let

$$\mathsf{u}(x) \triangleq (1-x)^3 \qquad \mathsf{v}(x) \triangleq x^3.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^3}_{\mathsf{u}(x)}\underbrace{(1+3x+6x^2)}_{\mathsf{a}(x)} + \underbrace{(x^3)}_{\mathsf{v}(x)}\underbrace{(10-15x+6x^2)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^3 = 1-3x+3x^2-x^3$	–	1	0
0	x^3	–	0	1
1	$1-3x+3x^2$	-1	1	1
2	$-\frac{1}{3}x+x^2$	$\frac{1}{3}x$	$-\frac{1}{3}x$	$1-\frac{1}{3}x$
3	$1-2x$	3	$1+x$	$-2+x$
4	$\frac{1}{6}x$	$-\frac{1}{2}x$	$\frac{1}{6}x+\frac{1}{2}x^2$	$1-\frac{4}{3}y+\frac{1}{2}x^2$
5	$1 = \gcd((1-x)^3, x^3)$	-12	$1+3x+6x^2 = \mathsf{a}(x)$	$10-15x+6x^2 = \mathsf{b}(x)$
6	0	$\frac{1}{6}x$		

✏

Example D.7. Let

$$\mathsf{u}(x) \triangleq (1-x)^4 \qquad \mathsf{v}(x) \triangleq x^4.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^4}_{\mathsf{u}(x)}\underbrace{(1+4x+10x^2+20x^3)}_{\mathsf{a}(x)} + \underbrace{(x^4)}_{\mathsf{v}(x)}\underbrace{(35-84x+70x^2-20x^3)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^4 = 1-4x+6^2-4x^3+x^4$	–	1	0
0	x^4	–	0	1
1	$1-4x+6x^2-4x^3$	1	1	-1
2	$\frac{1}{4}x-x^2+\frac{3}{2}x^3$	$-\frac{1}{4}x$	$\frac{1}{4}x$	$1-\frac{1}{4}x$
3	$1-\frac{10}{3}x+\frac{10}{3}x^2$	$-\frac{8}{3}$	$1+\frac{2}{3}x$	$\frac{5}{3}-\frac{2}{3}x$
4	$-\frac{1}{5}x+\frac{1}{2}x^2$	$\frac{3}{2}\cdot\frac{3}{10}x$	$-\frac{1}{5}x-\frac{3}{10}x^2$	$1-x+\frac{3}{10}x^2$
5	$1-2x$	$\frac{20}{3}$	$1+2x+2x^2$	$-5+6x-2x^2$
6	$\frac{1}{20}x$	$-\frac{1}{4}x$	$\frac{1}{20}x+\frac{1}{5}x^2+\frac{1}{2}x^3$	$1-\frac{9}{4}x+\frac{18}{10}x^2-\frac{1}{2}x^3$
7	$1 = \gcd((1-x)^4, x^4)$	-40	$1+4x+10x^2+20x^3$	$35-84x+70x^2-20x^3$
8	0	$\frac{1}{20}x$	–	–

✏

❝*Infinitesimal analysis was considered so attractive and important because of its numerous and useful applications; as such, it attracted upon itself all research attention and efforts. Concurrently, algebraic analysis appeared to be a field where nothing remained to be done, or where whatever remained to be done would have only been worthless speculation. …Nevertheless, the major contributors to infinitesimal analysis are well aware of the need to improve algebraic analysis: Their own progress depends upon it.*❞

Étienne Bézout, 1779 [9]

Theorem D.4 (Bézout's Identity). [10] [11] *Let* $\mathrm{p}_1(x)$ *be a polynomial of degree* n_1 *and* $\mathrm{p}_2(x)$ *be a polynomial of degree* n_2.

THM

$$\underbrace{\gcd(\mathrm{p}_1(x),\ \mathrm{p}_2(x)) = 1}_{\mathrm{p}_1(x)\text{ and }\mathrm{p}_2(x)\text{ are relatively prime}} \implies \left\{\begin{array}{ll} 1. & \exists\mathrm{q}_1(x),\mathrm{q}_2(x)\quad\text{such that} \\ & \begin{array}{ccccc} \text{degree } n_2-1 & & \text{degree } n_1-1 & & \\ \downarrow & & \downarrow & & \\ \mathrm{p}_1(x)\mathrm{q}_1(x) & + & \mathrm{p}_2(x)\mathrm{q}_2(x) & = & 1 \\ \uparrow & & \uparrow & & \\ \text{degree } n_1 & & \text{degree } n_2 & & \end{array} \\ 2. & \text{order of } \mathrm{q}_1(x) = n_2-1 \\ 3. & \text{order of } \mathrm{q}_2(x) = n_1-1 \end{array}\right.$$

✎PROOF: No proof at this time. ⇨

D.3 Roots

❝*Neither the true nor the false roots are always real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation* $x^3 - 6x^2 + 13x - 10 = 0$ *as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remain always imaginary.*❞

René Descartes (1596–1650), French philosopher and mathematician[12]

Theorem D.5 (Fundamental Theorem of Algebra). [13] *Let* $\mathrm{p}(x)$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

THM

$$\{\text{degree of } \mathrm{p}(x) \text{ is } N\} \implies \left\{\begin{array}{l} \underbrace{\exists\,(\!|x_n|\!)_1^N \text{ such that } \mathrm{p}(x_n) = 0 \text{ for } n = 1,2,\ldots,N}_{\mathrm{p}(x)\text{ has } N \text{ zeros}} \\ \text{where } x_n \text{ and } x_m \text{ are not necessarily distinct for } n \neq m. \end{array}\right\}$$

[9] quote: Bézout (1779a)
translation: Bézout (1779b), page xv
image: http://en.wikipedia.org/wiki/File:Etienne_Bezout2.jpg, public domain

[10] Bourbaki (2003b) page 2 ⟨Theorem 1 Chapter VII⟩, Fuhrmann (2012) pages 15–17 ⟨Corollary 1.31, Corollary 1.38⟩, Adhikari and Adhikari (2003) page 182, Warner (1990) page 381, Daubechies (1992), page 169, Mallat (1999), page 250

[11] Historical information: Bézout (1779a) ⟨???⟩, Bézout (1779b) ⟨???⟩, Bachet (1621) ⟨???⟩, Childs (2009) pages 37–46 ⟨some history on page 46⟩, http://serge.mehl.free.fr/chrono/Bachet.html, http://serge.mehl.free.fr/chrono/Bezout.html

[12] quote: Descartes (1637a)
English: Descartes (1954), page 175
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

[13] Prasolov (2004) pages 1–2 ⟨Section 1.1.1⟩, Borwein and Erdélyi (1995) page 11 ⟨Theorem 1.2.1⟩

Corollary D.1. *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

COR

$$\left\{ \underbrace{\begin{array}{l}\text{There exists } \langle x_n \rangle_1^N \\ \text{such that } \mathsf{p}(x_n) = 0 \text{ for } n = 0, 1, \ldots, N \\ \text{and where } x_n \text{ and } x_m \text{ are} \\ \text{not necessarily distinct for } n \neq m.\end{array}}_{N \text{ zeros of } \mathsf{p}(x)} \right\} \implies \left\{ \mathsf{p}(x) = \left(\frac{\alpha_0}{\prod_{n=1}^{N} (-x_n)} \right) \underbrace{\prod_{n=1}^{N} (x - x_n)}_{N \text{ factors}} \right\}$$

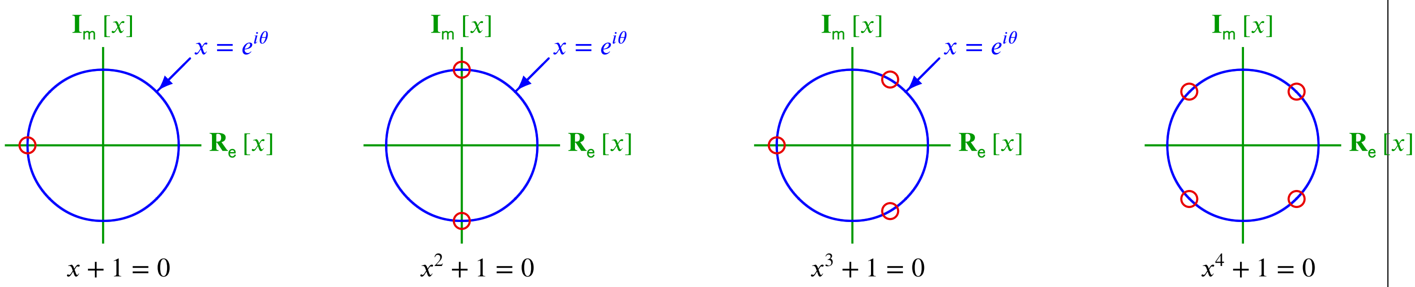


Figure D.1: Roots of $x^n + 1 = 0$

Lemma D.1.

LEM

$$x^N + 1 = 0 \quad \implies \quad x \in \left\{ e^{i\theta_n} | \theta_n = \frac{\pi}{N}(2n+1), n = 0, 1, \ldots, N-1 \right\}$$

✎PROOF:

$$\begin{aligned} e^{iN\theta_n - i2\pi n} &= -1 & n &\in \mathbb{Z} \\ N\theta_n - 2\pi n &= \pi & n &= 0, 1, \ldots, N-1 \\ N\theta_n &= 2\pi n + \pi \\ \theta_n &= \frac{\pi}{N}(2n+1) \end{aligned}$$

✏

Theorem D.6. *Let* $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \leq n \leq N\}$ *and* $\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$.

THM

$$\underbrace{\alpha_n = \alpha_{-n}^* \quad \forall n \in I}_{\langle \alpha_n \rangle \text{ is Hermitian symmetric}} \iff \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C}$$

✎PROOF:

1. Proof that $\alpha_n = \alpha^*_{-n} \implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\mathsf{p}(x) &\triangleq \sum_{n=-N}^{N} \alpha_n x^n && \text{by definition of } \mathsf{p}(x)\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha_{-n} x^{-n}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha^*_n x^{-n} && \text{by left hypothesis}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n x^{-n} + \sum_{n=1}^{N} \alpha_n x^n\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x}\right)^{-n}\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x^*}\right)^{-n}\right]^*\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha_{-n} \left(\frac{1}{x^*}\right)^{-n}\right]^* && \text{by left hypothesis}\\
&= \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^*\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by definition of } \mathsf{p}(x)
\end{aligned}$$

2. Proof that $\alpha_n = \alpha^*_{-n} \impliedby \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\sum_{n=-N}^{N} \alpha_n x^n &\triangleq \mathsf{p}(x) && \text{by definition of } \mathsf{p}(x)\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by right hypothesis}\\
&\triangleq \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^* && \text{by definition of } \mathsf{p}(x)\\
&= \sum_{n=-N}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n\\
&= \sum_{n=-N}^{N} \alpha^*_{-n} x^n && \text{by symmetry of summation indices}\\
\implies \quad \alpha_n &= \alpha^*_{-n} && \text{by matching of polynomial coefficients}
\end{aligned}$$

⇨

Theorem D.7. *Let $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \le n \le N\}$ and*

$$\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$$

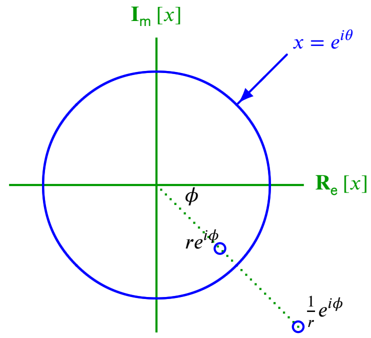


Figure D.2: Recipricol conjugate zero pairs

THM

$$\underbrace{\alpha_n = \alpha^*_{-n} \quad \forall n \in I}_{(\alpha_n) \text{ is Hermitian symmetric}} \quad\Longrightarrow\quad \underbrace{\left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]}_{\text{roots occur in conjugate recipricol pairs}}$$

✎PROOF:

$$\begin{aligned}
&\alpha_n = \alpha^*_{-n} \quad \forall n \in I && \text{by left hypothesis}\\
&\Longrightarrow \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C} && \text{by Theorem D.6 page 89}\\
&\Longrightarrow \left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]
\end{aligned}$$

If σ is a zero of $\mathsf{p}(x)$, then so is $\frac{1}{\sigma^*}$ because

$$\mathsf{p}\left(\frac{1}{\sigma^*}\right) = \mathsf{p}^*(\sigma) = 0^* = 0.$$

⇨

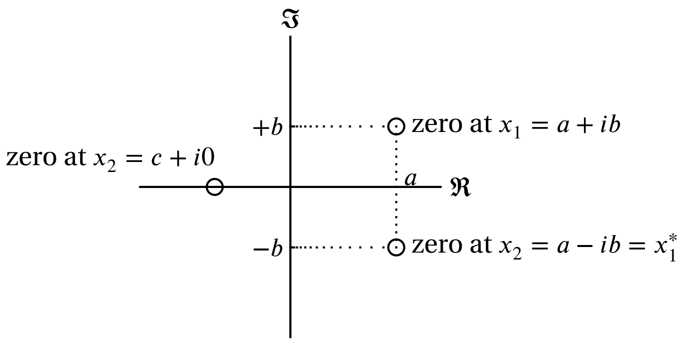


Figure D.3: Conjugate pairs of roots

Theorem D.8 page 91 (next) states that the roots of real polynomials occur in complex conjugate pairs. This is illustrated in Figure D.3.

Theorem D.8. [14] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial.*

[14] Korn and Korn (1968), page 17

THM
$$\left[\underbrace{(\alpha_n \in \mathbb{R})_{n=0,1,\ldots,N}}_{\text{coefficients are real}}\right] \implies \left[\underbrace{\mathsf{p}(x_0)=0 \iff \mathsf{p}(x_0^*)=0}_{\text{zeros occur in conjugate pairs}}\right]$$

Theorem D.9 (Routh-Hurwitz Criterion). [15] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$ *and*

$$d_0 \triangleq \alpha_0 \qquad d_1 \triangleq \alpha_1 \qquad d_2 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} \qquad d_3 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} \qquad d_4 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 \end{vmatrix}$$

$$d_n \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ \alpha_{2n-3} & \alpha_{2n-4} & \cdots & \alpha_{n-2} \\ \alpha_{2n-1} & \alpha_{2n-2} & \cdots & \alpha_n \end{vmatrix}$$

Let $\mathbf{S}(\!(x_n)\!)$ *be the number of sign changes of some sequence* $(\!(x_n)\!)$ *after eliminating all zero elements* $(x_n = 0)$.

THM
$$\begin{aligned}\underbrace{\left|\left\{x_n | \mathsf{p}(x_n)=0,\ \Re[x_n]>0\right\}\right|}_{\text{number of roots in right half plane}} &= \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ d_1d_2,\ d_2d_3,\ \ldots,\ d_{p-2}d_{p-1},\ \alpha_p\right)\!\right)}_{\text{number of sign changes}} \\ &= \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ \frac{d_2}{d_1},\ \frac{d_3}{d_2},\ \ldots,\ \frac{d_p}{d_{p-1}}\right)\!\right)}_{\text{number of sign changes}}\end{aligned}$$

Theorem D.10 (Descartes rule of signs). [16] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$.

THM
$$\underbrace{\left|\left\{x_n | \mathsf{p}(x_n)=0,\ \Re[x_n]>0\right\},\ \Im[x_n]=0\right|}_{\text{number of roots on right real axis}} = \underbrace{\mathbf{S}(\!(\alpha_n)\!) - 2m}_{\text{number of sign changes}-\text{even integer}} \qquad \text{where } m \in \mathbb{W}$$

Theorem D.11. [17] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$.

THM
$$\underbrace{\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1} \geq 0}_{\text{first } k \text{ coefficients are nonnegative}} \implies \left\{\begin{array}{l} \underbrace{\left|\left\{x_n | \mathsf{p}(x_n)=0,\ \Im[x_n]=0\right\}\right|}_{\text{number of real roots}} < \underbrace{1+\left(\frac{q}{\alpha_0}\right)^{\frac{1}{k}}}_{\text{upper bound}} \\ \text{where } q \triangleq \underbrace{\max\left\{\ |\alpha_n|\ | \alpha_n < 0\right\}}_{\text{largest negative coefficient}} \end{array}\right.$$

Theorem D.12 (Rolle's Theorem). [18] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$. *The number of real zeros of* $\mathsf{p}'(x)$ *between any two real consecutive real zeros of* $\mathsf{p}(x)$ *is* ***odd***.

Definition D.6. [19] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

DEF
$\dfrac{\mathsf{p}(x)}{\mathsf{q}(x)}$ *is a* ***rational function***
if $\mathsf{p}(x)$ *and* $\mathsf{q}(x)$ *are* POLYNOMIALS *over* $(\mathbb{F}, +, \cdot, 0, 1)$.

[15] Korn and Korn (1968), page 17
[16] Korn and Korn (1968), page 17
[17] Korn and Korn (1968), page 18
[18] Korn and Korn (1968), page 18
[19] Fuhrmann (2012) page 22

Example D.8.

An example of a rational function using polynomials in x^{-1} is

$$A(x) = \frac{b_0 + \beta_1 x^{-1} + \beta_2 x^{-2} + \beta_3 x^{-3}}{1 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \alpha_3 x^{-3}}$$

This can be expressed as a rational function using polynomials in x by multiplying numerator and denominator by x^3:

$$A(x) = \frac{x^3}{x^3} A(x) = \frac{b_0 x^3 + \beta_1 x^2 + \beta_2 x + \beta_3}{x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3}$$

Definition D.7.

*The **zeros** of a rational function $H(x) = \frac{B(x)}{A(x)}$ are the roots of $B(x)$.*

*The **poles** of a rational function $H(x) = \frac{B(x)}{A(x)}$ are the roots of $A(x)$.*

D.4 Polynomial expansions

❝*Thus, if a straight-line is cut at random, then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces.*❞
Euclid (~300BC), Greek mathematician, demonstrating the *Binomial theorem* for exponent $n = 2$ as in $(x+y)^2 = x^2 + 2xy + y^2$. [20]

Theorem D.13 (Taylor Series). [21] *Let* $\boldsymbol{C}$ *be the space of all continuously differentiable real functions and* $\frac{\mathsf{d}}{\mathsf{dx}}$ *in* $\boldsymbol{C}^{\boldsymbol{C}}$ *the differentiation operator.*

$$\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\left[\frac{\mathsf{d}}{\mathsf{dx}}^n \mathsf{f}\right](a)}{n!}(x-a)^n \qquad \forall a \in \mathbb{R}, \mathsf{f} \in \boldsymbol{C} \qquad \text{(Taylor series \textit{about the point} } a\text{)}$$

*A **Maclaurin series** is a* Taylor series *about the point* $a = 0$.

Theorem D.14 (Binomial Theorem). [22]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad \textit{where} \qquad \binom{n}{k} \triangleq \frac{n!}{(n-k)!\,k!}$$

✎Proof: This theorem is proven using two different techniques. Either is sufficient. The first requires the Maclaurin series resulting in a more compact proof, but requires the additional (here unproven) Maclaurin series. The second proof uses induction resulting in a longer proof, but does not require any external theorem.

[20] quote: Euclid (circa 300BC) ⟨Book II, Proposition 4⟩, Coolidge (1949), page 147
image: http://commons.wikimedia.org/wiki/File:Euklid-von-Alexandria_1.jpg, public domain

[21] Flanigan (1983) page 221 ⟨Theorem 15⟩, Strichartz (1995) page 281, Sohrab (2003) page 317 ⟨Theorem 8.4.9⟩, Taylor (1715), Maclaurin (1742)

[22] Graham et al. (1994) page 162 ⟨(5.12)⟩, Rotman (2010) page 84 ⟨Proposition 2.5⟩, Bourbaki (2003a) page 99 ⟨Corollary 1⟩, Warner (1990) pages 189–190 ⟨Theorem 21.1⟩, Metzler et al. (1908), page 169 ⟨any real exponent⟩, Coolidge (1949)

1. Proof using Maclaurin series:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}y^k}\left[(x+y)^n\right]_{y=0} y^k && \text{by Maclaurin series (Theorem D.13 page 93)}\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left[n(n-1)(n-2)\cdots(n-k+1)(x+y)^{n-k}\right]_{y=0} y^k\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{n!}{(n-k)!} x^{n-k}y^k\\
&= \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k}y^k && \text{by definition of } \binom{n}{k}\\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k + \underbrace{\cancel{\sum_{k=n+1}^{\infty} \binom{n}{k} x^{n-k}y^k}}_{0}\\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k && \text{because } (x+y)^n \text{ has order } n
\end{aligned}$$

2. Proof using induction:

(a) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ is true for $n=0$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right|_{n=0} &= \binom{0}{0}x^0y^{0-0}\\
&= 1\\
&= (x+y)^n|_{n=0}
\end{aligned}$$

(b) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ is true for $n=1$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right|_{n=1} &= \binom{1}{0}x^0y^{1-0} + \binom{1}{1}x^1y^{1-1}\\
&= y+x\\
&= (x+y)^n|_{n=1}
\end{aligned}$$

(c) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \implies (x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n}{k}x^k y^{n+1-k}$:

$$\begin{aligned}
&\sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n+1-k}\\
&= x^{n+1}+y^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n+1-k}\\
&= x^{n+1}+y^{n+1}+\sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]x^k y^{n+1-k} && \text{by \textit{Pascal's Rule}}\\
&= x^{n+1}+y^{n+1}+\sum_{k=1}^{n}\binom{n}{k-1}x^k y^{n+1-k}+\sum_{k=1}^{n}\binom{n}{k}x^k y^{n+1-k}\\
&= x^{n+1}+y^{n+1}+\left[\sum_{k=0}^{n}\binom{n}{k}x^{k+1} y^{n+1-(k+1)} - x^{n+1}\right]+\left[\sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k} - y^{n+1}\right]
\end{aligned}$$

$$= x\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} + y\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

$$= x(x+y)^n + y(x+y)^n \qquad \text{by left hypothesis}$$

$$= (x+y)(x+y)^n$$

$$= (x+y)^{n+1}$$

APPENDIX E

INTEGRATION

[1]

Definition E.1. [2] *Let $(X, \boldsymbol{S}, \mu)$ be a measure space. Let $(\!(\boldsymbol{P}_n)\!)_{n\in\mathbb{Z}}$ be sequence of increasingly fine partitions of a set $E \subseteq X$.*

DEF

*The **Cauchy integral operator** $\int$ of a function f over a set $E \subseteq X$ on the measure space $(X, \boldsymbol{S}, \mu)$ is*

$$\int_E \mathsf{f}(x)\,\mathsf{d}\mu \triangleq \sum_n \mathsf{f}(x_n)\mu(E_n)$$

where $x_n \in E_n$ and $E_n \in \lim_{m\to\infty} \boldsymbol{P}_m \triangleq \{E_n | n \in \mathbb{Z}\}$

Definition E.2. [3] *Let $(X, \boldsymbol{S}, \mu)$ be a measure space. Let $(\!(\boldsymbol{P}_n)\!)_{n\in\mathbb{Z}}$ be sequence of increasingly fine partitions of a set $E \subseteq X$.*

DEF

$$\begin{aligned}
\int_E \mathsf{f}\,\mathsf{d}\mu &= \inf_{x_i\in P_i}\left\{\sum_n \mathsf{f}(x_n)\mu(E_n)\right\} && \textit{(lower integral)}\\
\int_E^* \mathsf{f}\,\mathsf{d}\mu &= \sup_{x_i\in P_i}\left\{\sum_n \mathsf{f}(x_n)\mu(E_n)\right\} && \textit{(upper integral)}
\end{aligned}$$

*where $x_n \in E_n$ and $E_n \in \lim_{m\to\infty} \boldsymbol{P}_m \triangleq \{E_n | n \in \mathbb{Z}\}$. The sum $\int_E \mathsf{f}\,\mathsf{d}\mu$ is **Riemann integrable** if $\int_E \mathsf{f}\,\mathsf{d}\mu = \int_E^* \mathsf{f}\,\mathsf{d}\mu$ and in this case the **Riemann integral operator** $\int$ of f over $E \subseteq X$ on $(X, \boldsymbol{S}, \mu)$ is $\int_E \mathsf{f}\,\mathsf{d}\mu$.*

Definition E.3. [4] *Let $(X, \boldsymbol{S}, \mu)$ be a measure space. Let $(\!(\boldsymbol{P}_n)\!)_{n\in\mathbb{Z}}$ be sequence of increasingly fine partitions of a set $E \subseteq X$.*

---

[1] The name *integral calculus* and its operational symbol $\int$ were the product of a collaboration between Gottfried Leibnitz (1646–1716) and Johann Bernoulli (1667–1748). Leibnitz preferred the terminology *calculus summatorius* (summation calculus) and the operational symbol $\int$ (an enlongated "S"). Bernoulli preferred the terminology *calculus integralis* (integral calculs) and the operational symbol I. In the end, a compromise was reached which is the currently used terminology "integral calculs" with symbol $\int$. Reference: Cajori (1993), pages 181–182

[2] Jahnke (2003), page 262, Cauchy (1823)

[3] Jahnke (2003), page 264, Riemann (1854)

[4] Lebesgue (1902), Lebesgue (1972)

DEF

*The **Lebesgue integral operator** $\int$ of* f *over* $E \subseteq X$ *on* $(X, \boldsymbol{S}, \mu)$ *is*

$$\int_E \mathsf{f}\, \mathrm{d}\mu \triangleq \sum_{y \in Y} y\mu\left(\mathsf{f}^{-1}(y)\right)$$

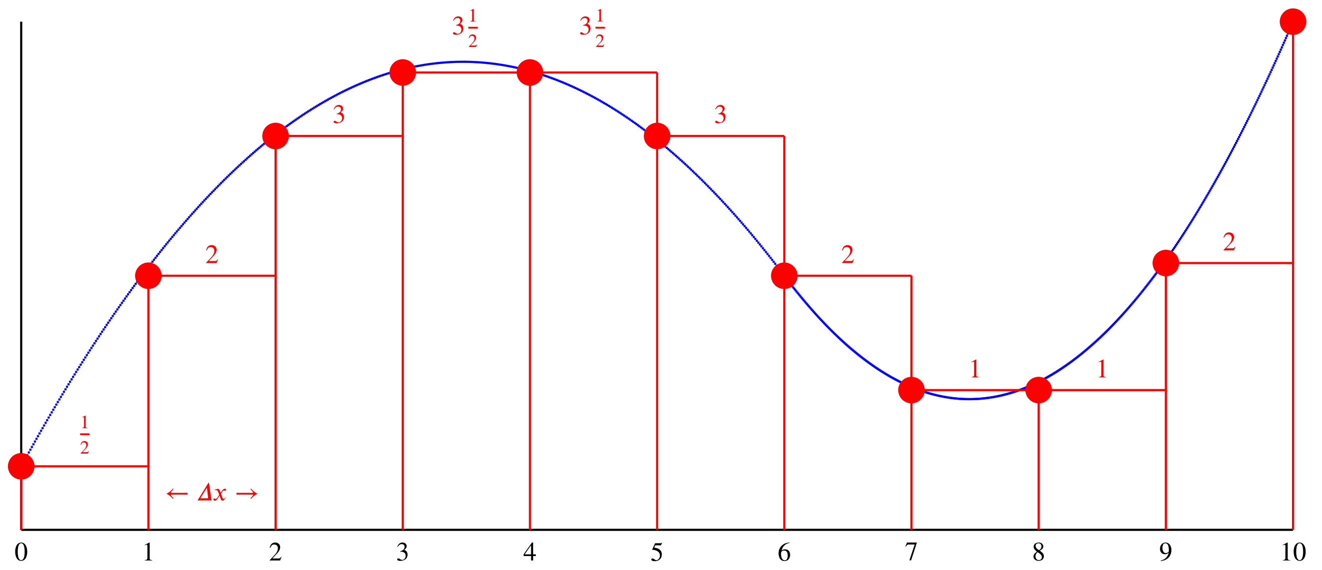


Figure E.1: Curve for example Example E.1 (page 98)

Example E.1. Suppose we want to compute the area under the curve in Figure E.1 (page 98) from $x = 0$ to $x = 10$. This can be accomplished using either Riemann or Lebesgue integration. Riemann integration adds up the areas one by one, starting from $x = 0$ and ending with $x = 10$ such that

$$\begin{aligned}
\oint_0^{10} &= \sum_{n=0}^{n=9} x_n \underbrace{\mu\left\{x \in E | x_n \le x < x_{n+1}\right\}}_{\Delta x = 1} \\
&= \sum_{n=0}^{n=9} x_n \cdot 1 \\
&= \frac{1}{2} + 2 + 3 + 3\frac{1}{2} + 3\frac{1}{2} + 3 + 2 + 1 + 1 + 2 \\
&= 21\frac{1}{2}
\end{aligned}$$

On the other hand, Lebesgue integration first groups together all equal values into their own set and then sums the value of each set times the size of the set such that

$$\begin{aligned}
\int_0^{10} &= \sum_{k=1}^{n=5} y_k \mu\left\{x \in E | \mathsf{f}(x) = y_k\right\} \\
&= \underbrace{\frac{1}{2}}_{y_1} \times 1 + \underbrace{1}_{y_2} \times 2 + \underbrace{2}_{y_3} \times 3 + \underbrace{3}_{y_4} \times 2 + \underbrace{3\frac{1}{2}}_{y_5} \times 2 \\
&= 21\frac{1}{2}
\end{aligned}$$

Of course in this case and in the case of all other "well behaved" functions, the two approaches yield the same result.

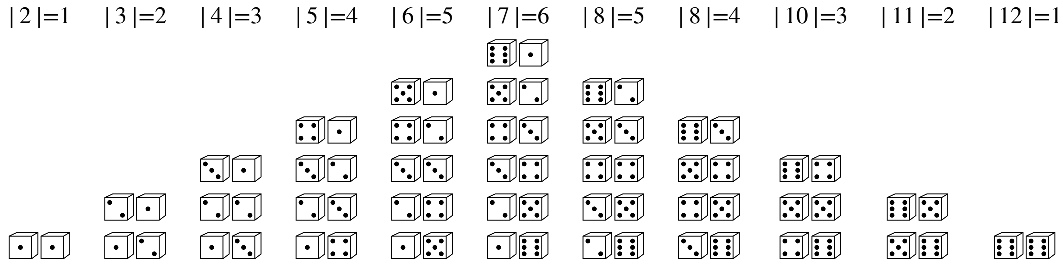


Figure E.2: Pair of dice distribution for Example E.2 (page 99)

Example E.2. Suppose we want to find the sum of all possible outcomes of the sum of a pair of dice. All the possible outcomes are summarized in the table at the left. **Riemann integration** would start in the upper left hand corner (⚀⚀) and sum across each row such that:

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

$$\begin{aligned}\oint_E \mathsf{f}(x)\,\mathsf{d}x &= \sum_{n=1}^{36} \mathsf{f}(x_n)\cdot 1\\ &= \text{⚀⚀} + \text{⚀⚁} + \text{⚀⚂} + \cdots + \text{⚅⚄} + \text{⚅⚅}\\ &= \underbrace{2+3+4+\cdots+11+12}_{36 \text{ terms}}\\ &= 250\end{aligned}$$

Lebesgue integration, on the other hand, groups like values into sets and thus actually adds diagonally—because like values occur along diagonal lines. This organization of like values is illustrated in Figure E.2 (page 99) and calculated below:

$$\begin{aligned}\int_E \mathsf{f}\,\mathsf{d}\mu &= \sum_{k=2}^{k=12} k\mu\left\{\text{sum of dice pair}|\text{sum } = k\right\}\\ &= \underbrace{2\times1+3\times2+4\times3+5\times4+6\times5+7\times6+8\times5+9\times4+10\times3+11\times2+12\times1}_{11 \text{ terms}}\\ &= 250\end{aligned}$$

Example E.3 (Salt and pepper function/Dirichlet monster). [5]

EX $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x \text{ rational}\\ 1 & \text{for } x \text{ irrational}\end{cases} \implies \{\mathsf{f} \text{ is } \textit{not} \text{ Riemann integrable}\}$

[5] Jahnke (2003), page 263, Dirichlet (1829a), Dirichlet (1829b)

APPENDIX F

CALCULUS

Definition F.1. *Let $\mathbb{R}$ be the set of real numbers, $\mathscr{B}$ the set of* Borel sets *on $\mathbb{R}$, and μ the standard* Borel measure *on $\mathscr{B}$. Let $\mathbb{R}^{\mathbb{R}}$ be as in Definition A.1 page 59.*

DEF

*The **space of Lebesgue square-integrable functions** $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ (or $\boldsymbol{L}^2_{\mathbb{R}}$) is defined as*

$$\boldsymbol{L}^2_{\mathbb{R}} \triangleq \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \mathsf{d}\mu < \infty \right\}.$$

*The **standard inner product** $\langle \triangle \mid \nabla \rangle$ on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\,\mathsf{d}x.$$

*The **standard norm** $\|\cdot\|$ on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$*

Definition F.2. *Let $\mathsf{f}(x)$ be a* function *in $\mathbb{R}^{\mathbb{R}}$.*

DEF

$$\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon \to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition F.1.

PRP

$$\left\{ \begin{array}{lll} (1). & \mathsf{f}(x) \text{ is CONTINUOUS} & \text{and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC about a point } a} & \end{array} \right\} \implies \left\{ \begin{array}{llll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) \quad \text{(ANTI-SYMMETRIC about } a\text{)} \\ (2). & \mathsf{f}'(a) & = & 0 \end{array} \right\}$$

✎Proof:

$$\begin{aligned}
\mathsf{f}'(a+x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\
&= -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\
&= -\mathsf{f}(a-x)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\
&= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result}
\end{aligned}$$

$$= 0$$

✎

Lemma F.1.

LEM $\mathsf{f}(x)$ *is* INVERTIBLE $\implies \left\{ \frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) = \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to 0} \frac{\mathsf{f}^{-1}(y+\varepsilon) - \mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}y} && \text{(Definition F.2 page 101)}\\
&= \lim_{\delta\to 0} \left.\frac{1}{\left[\frac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]}\right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1}\\
&\triangleq \left.\frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x)}\right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition F.2 page 101)}\\
&= \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✎

Theorem F.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

THM $$\int_{[0:1)^n} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n} x_k\right) \mathsf{d}x_1\, \mathsf{d}x_2 \cdots \mathsf{d}x_n = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n \in \mathbb{N}$$

✎PROOF: Proof by induction:

1. Base case …proof for $n = 1$ case:

$$\begin{aligned}
\int_{[0:1)} \mathsf{f}^{(1)}(x)\, \mathsf{d}x &= \mathsf{f}(1) - \mathsf{f}(0) && \text{by \textit{Fundamental theorem of calculus}}\\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1) + (-1)^{1+0}\binom{1}{0}\mathsf{f}(0)\\
&= \sum_{k=0}^{1} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

2. Induction step ...proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_{n+1} \\
&= \int_{[0:1)^n} \left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right) \mathrm{d}x_{n+1}\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n && \text{by \textit{Fundamental theorem of calculus}} \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(1 + \sum_{k=1}^{n} x_k\right) - \mathsf{f}^{(n)}\left(0 + \sum_{k=1}^{n} x_k\right)\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1) - \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by induction hypothesis} \\
&= \sum_{m=1}^{m=n+1}(-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m) + \sum_{k=0}^{n}(-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{where } m \triangleq k+1 \implies k = m-1 \\
&= \left[\mathsf{f}(n+1) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right] + \left[(-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right] \\
&= \mathsf{f}(n+1) + (-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\text{use \textit{Stifel formula}}}\mathsf{f}(k) \\
&= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1) + (-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by \textit{Stifel formula}} \\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)
\end{aligned}$$

✎

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma F.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots,$ *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example F.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\mathsf{f}(x)\mathsf{g}(x)\right] = \mathsf{f}'''(x)\mathsf{g}(x) + 3\mathsf{f}''(x)\mathsf{g}'(x) + 3\mathsf{f}'(x)\mathsf{g}''(x) + \mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem F.2 (Leibniz integration rule). [3]

THM
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathsf{a}(x)}^{\mathsf{b}(x)} \mathsf{g}(t)\, \mathrm{d}t = \mathsf{g}[\mathsf{b}(x)]\mathsf{b}'(x) - \mathsf{g}[\mathsf{a}(x)]\mathsf{a}'(x)$$

[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)

[3] Flanders (1973) page 615 ⟨(1.1)⟩ Talvila (2001), Knapp (2005) page 389 ⟨Chapter VII⟩, ? page 422 ⟨Leibniz Rule. Theorem 1.⟩, http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

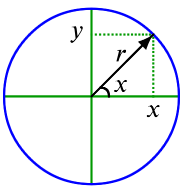
APPENDIX G

TRIGONOMETRIC FUNCTIONS

G.1 Definition Candidates

There are several ways of defining the sine and cosine functions, including the following:[1]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[2]



$$\begin{aligned} \cos x &\triangleq \frac{x}{r} \\ \sin x &\triangleq \frac{y}{r} \end{aligned}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[3]

$$\cos x \triangleq \mathbf{R}_{\mathrm{e}} e^{ix} \qquad\qquad \sin x \triangleq \mathbf{I}_{\mathrm{m}}\left(e^{ix}\right)$$

3. **Polynomial:** Let $\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of *Taylor expansions* such that[4]

$$\begin{aligned} \cos(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \sin(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

[1] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means "bay" or "inlet"—thus resulting in the Latin translation *sinus*, which also means "bay" or "inlet". Reference: Boyer and Merzbach (1991) page 252

[2] Abramowitz and Stegun (1972), page 78

[3] Euler (1748)

[4] Rosenlicht (1968), page 157, Abramowitz and Stegun (1972), page 74

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \prod_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of a product of factors such that[5]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1-\left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[6]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-(n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-(n\pi)^2}\right)}_{\cot(x)}\sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ such that

$$\begin{array}{llllll}
\cos(x) & \triangleq & \mathrm{f}(x) & \text{where} & \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}+\mathrm{f}=0}_{\text{differential equation}} & \underbrace{\mathrm{f}(0)=1}_{\text{1st initial condition}} & \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)=0}_{\text{2nd initial condition}}\\
\sin(x) & \triangleq & \mathrm{g}(x) & \text{where} & \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{g}+\mathrm{g}=0}_{\text{differential equation}} & \underbrace{\mathrm{g}(0)=0}_{\text{1st initial condition}} & \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{g}\right](0)=1}_{\text{2nd initial condition}}
\end{array}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[7]

$$\begin{array}{llllll}
\cos(x) & \triangleq & \mathrm{f}^{-1}(x) & \text{where} \quad \mathrm{f}(x) & \triangleq & \underbrace{\int_x^1 \sqrt{\frac{1}{1-y^2}}\,\mathrm{d}y}_{\arccos(x)}\\
\sin(x) & \triangleq & \mathrm{g}^{-1}(x) & \text{where} \quad \mathrm{g}(x) & \triangleq & \underbrace{\int_0^x \sqrt{\frac{1}{1-y^2}}\,\mathrm{d}y}_{\arcsin(x)}
\end{array}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ (Definition G.1 page 107). Support for such an approach includes the following:

- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ (Theorem G.1 page 108).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem G.3 page 110).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem G.4 page 111).

[5] Abramowitz and Stegun (1972), page 75
[6] Abramowitz and Stegun (1972), page 75
[7] Abramowitz and Stegun (1972), page 79

- The complex exponential function is a solution of a second order homogeneous differential equation (Definition G.4 page 112).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section G.6 page 120).

G.2 Definitions

Definition G.1. [8] *Let C be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{\mathsf{d}}{\mathsf{dx}} \in C^C$ the differentiation operator.*

DEF

*The function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ is the **cosine** function $\cos(x) \triangleq \mathsf{f}(x)$ if*

$$\begin{array}{llcll}
1. & \frac{\mathsf{d}^2}{\mathsf{dx}^2}\mathsf{f} + \mathsf{f} & = & 0 & \text{(second order homogeneous differential equation) and}\\
2. & \mathsf{f}(0) & = & 1 & \text{(first initial condition) and}\\
3. & \left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0) & = & 0 & \text{(second initial condition).}
\end{array}$$

Definition G.2. [9] *Let C and $\frac{\mathsf{d}}{\mathsf{dx}} \in C^C$ be defined as in definition of $\cos(x)$* (Definition G.1 page 107).

DEF

*The function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ is the **sine** function $\sin(x) \triangleq \mathsf{f}(x)$ if*

$$\begin{array}{llcll}
1. & \frac{\mathsf{d}^2}{\mathsf{dx}^2}\mathsf{f} + \mathsf{f} & = & 0 & \text{(second order homogeneous differential equation) and}\\
2. & \mathsf{f}(0) & = & 0 & \text{(first initial condition) and}\\
3. & \left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0) & = & 1 & \text{(second initial condition).}
\end{array}$$

Definition G.3. [10]

DEF

Let π ("pi") be defined as the element in $\mathbb{R}$ such that

$$\begin{array}{llcll}
(1). & \cos\left(\frac{\pi}{2}\right) & = & 0 & \text{and}\\
(2). & \pi & > & 0 & \text{and}
\end{array}$$

(3). *π is the **smallest** of all elements in $\mathbb{R}$ that satisfies (1) and (2).*

G.3 Basic properties

Lemma G.1. [11] *Let C be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{\mathsf{d}}{\mathsf{dx}} \in C^C$ the differentiation operator.*

LEM

$$\left\{\frac{\mathsf{d}^2}{\mathsf{dx}^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\begin{array}{rcl}
\mathsf{f}(x) & = & \underbrace{[\mathsf{f}](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}}\\
 & = & \left(\mathsf{f}(0) + \left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)x\right) - \left(\frac{\mathsf{f}(0)}{2!}x^2 + \frac{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)}{3!}x^3\right) + \left(\frac{\mathsf{f}(0)}{4!}x^4 + \frac{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)}{5!}x^5\right)\cdots
\end{array}\right\}$$

[8] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229

[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229

[10] Rosenlicht (1968) page 158

[11] Rosenlicht (1968), page 156, Liouville (1839)

✎PROOF: Let $f'(x) \triangleq \frac{d}{dx}f(x)$.

$$\begin{aligned}
f'''(x) &= -\left[\frac{d}{dx}f\right](x) \\
f^{(4)}(x) &= -\left[\frac{d}{dx}f\right](x) && = -\left[\frac{d^2}{dx^2}f\right](x) = f(x)
\end{aligned}$$

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \implies f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n && \text{by Taylor expansion (Theorem D.13 page 93)} \\
&= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{\left[\frac{d^2}{dx^2}f\right](0)}{2!}x^2 - \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 - \cdots \\
&= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{f(0)}{2!}x^2 - \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3 + \frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5 - \cdots \\
&= f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]
\end{aligned}$$

2. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \impliedby f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\left[\frac{d^2}{dx^2}f\right](x) &= \frac{d}{dx}\frac{d}{dx}[f(x)] \\
&= \frac{d}{dx}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] && \text{by right hypothesis} \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)f(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n-1}\right] \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{f(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n-1)!}x^{2n-1}\right] \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \\
&= -f(x) && \text{by right hypothesis}
\end{aligned}$$

✏

Theorem G.1 (Taylor series for cosine/sine). [12]

THM

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && \forall x\in\mathbb{R} \\
\sin(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \forall x\in\mathbb{R}
\end{aligned}$$

[12] Rosenlicht (1968), page 157

PROOF:

$$\begin{aligned}
\cos(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma G.1 page 107}\\
&= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by cos initial conditions (Definition G.1 page 107)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
\sin(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma G.1 page 107}\\
&= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by sin initial conditions (Definition G.2 page 107)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Theorem G.2. [13]

THM		
	$\cos(0) = 1$	$\cos(-x) = \cos(x) \quad \forall x\in\mathbb{R}$
	$\sin(0) = 0$	$\sin(-x) = -\sin(x) \quad \forall x\in\mathbb{R}$

PROOF:

$$\begin{aligned}
\cos(0) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\bigg|_{x=0} && \text{by } \textit{Taylor series for cosine} \quad \text{(Theorem G.1 page 108)}\\
&= 1\\
\sin(0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\bigg|_{x=0} && \text{by } \textit{Taylor series for sine} \quad \text{(Theorem G.1 page 108)}\\
&= 0\\
\cos(-x) &= 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \cdots && \text{by } \textit{Taylor series for cosine} \quad \text{(Theorem G.1 page 108)}\\
&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\
&= \cos(x) && \text{by } \textit{Taylor series for cosine} \quad \text{(Theorem G.1 page 108)}\\
\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots && \text{by } \textit{Taylor series for sine} \quad \text{(Theorem G.1 page 108)}\\
&= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]\\
&= \sin(x) && \text{by } \textit{Taylor series for sine} \quad \text{(Theorem G.1 page 108)}
\end{aligned}$$

Lemma G.2. [14]

LEM		
	$\cos(1) > 0$	$x \in (0:2) \implies \sin(x) > 0$
	$\cos(2) < 0$	

[13] Rosenlicht (1968), page 157
[14] Rosenlicht (1968), page 158

✎PROOF:

$$\begin{aligned}
\cos(1) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=1} && \text{by } \textit{Taylor series for cosine} && \text{(Theorem G.1 page 108)}\\
&= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots\\
&> 0\\
\cos(2) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=2} && \text{by } \textit{Taylor series for cosine} && \text{(Theorem G.1 page 108)}\\
&= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots\\
&< 0
\end{aligned}$$

$$\begin{aligned}
x \in (0:2) &\implies \text{each term in the sequence } \left(\!\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots\right) \text{ is} > 0\\
&\implies \sin(x) > 0
\end{aligned}$$

✏

Proposition G.1. *Let π be defined as in Definition G.3 (page 107).*

PRP	
(A).	*The value π* ***exists*** *in $\mathbb{R}$.*
(B).	$2 < \pi < 4$.

✎PROOF:

$$\begin{aligned}
&\cos(1) > 0 && \text{by Lemma G.2 page 109}\\
&\cos(2) < 0 && \text{by Lemma G.2 page 109}\\
&\implies 1 < \frac{\pi}{2} < 2\\
&\implies 2 < \pi < 4
\end{aligned}$$

✏

Theorem G.3. [15] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $\mathrm{f}'(0) \triangleq \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)$.*

THM
$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f} + \mathrm{f} = 0\right\} \iff \left\{\mathrm{f}(x) = \mathrm{f}(0)\cos(x) + \mathrm{f}'(0)\sin(x)\right\} \qquad \forall \mathrm{f}\in\boldsymbol{C}, \forall x\in\mathbb{R}$$

✎PROOF:

1. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}\right](x) = -\mathrm{f}(x) \implies \mathrm{f}(x) = \mathrm{f}(0)\cos(x) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)\sin(x)$:

$$\begin{aligned}
\mathrm{f}(x) &= \mathrm{f}(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by left hypothesis and Lemma G.1 page 107}\\
&= \mathrm{f}(0)\cos x + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)\sin x && \text{by definitions of cos and sin (Definition G.1 page 107, Definition G.2 page 107)}
\end{aligned}$$

[15] Rosenlicht (1968), page 157. The general solution for the *non-homogeneous* equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = \mathrm{g}(x)$ with initial conditions $\mathrm{f}(a) = 1$ and $\mathrm{f}'(a) = \rho$ is $\mathrm{f}(x) = \cos(x) + \rho\sin(x) + \int_a^x \mathrm{g}(y)\sin(x-y)\,\mathrm{d}y$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992), page 371, Liouville (1839). Volterra equation references: Pedersen (2000), page 99, Lalescu (1908), Lalescu (1911)

2. Proof that $\frac{d^2}{dx^2}f = -f \iff f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$\begin{aligned}
f(x) &= f(0)\cos x + \left[\tfrac{d}{dx}f\right](0)\sin x && \text{by right hypothesis}\\
&= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\tfrac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)}\\
\implies \quad & \frac{d^2}{dx^2}f + f = 0 && \text{by Lemma G.1 page 107}
\end{aligned}$$

Theorem G.4. [16] *Let $\frac{d}{dx} \in C^C$ be the differentiation operator.*

THM		
$\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x\in\mathbb{R}$	$\frac{d}{dx}\sin(x) = \cos(x) \quad \forall x\in\mathbb{R}$	$\cos^2(x) + \sin^2(x) = 1 \quad \forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} && \text{by Taylor series (Theorem G.1 page 108)}\\
&= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}\\
&= -\sin(x) && \text{by Taylor series (Theorem G.1 page 108)}\\
\frac{d}{dx}\sin(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by Taylor series (Theorem G.1 page 108)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= \cos(x) && \text{by Taylor series (Theorem G.1 page 108)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left[\cos^2(x) + \sin^2(x)\right] &= -2\cos(x)\sin(x) + 2\sin(x)\cos(x)\\
&= 0\\
&\implies \cos^2(x) + \sin^2(x) \text{ is } \textit{constant}\\
&\implies \cos^2(x) + \sin^2(x)\\
&= \cos^2(0) + \sin^2(0)\\
&= 1 + 0 = 1 && \text{by Theorem G.2 page 109}
\end{aligned}$$

Proposition G.2.

PRP
$\sin\left(\frac{\pi}{2}\right) = 1$

[16] Rosenlicht (1968), page 157

✎PROOF:

$$\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2) + 0} \\
&= \pm\sqrt{\sin^2(\pi/2) + \cos^2(\pi/2)} && \text{by definition of } \pi && \text{(Definition G.3 page 107)} \\
&= \pm\sqrt{1} && \text{by Theorem G.4 page 111} \\
&= \pm 1 \\
&= 1 && \text{by Lemma G.2 page 109}
\end{aligned}$$

⇨

G.4 The complex exponential

Definition G.4.

DEF *The function* $f \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq f(x)$ *if*

1. $\frac{d^2}{dx^2} f + f = 0$ *(second order homogeneous differential equation)* *and*
2. $f(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{d}{dx} f\right](0) = i$ *(second initial condition).*

Theorem G.5 (Euler's identity). [17]

THM $$e^{ix} = \cos(x) + i\sin(x) \qquad \forall x \in \mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\exp(ix) &= f(0)\cos(x) + \left[\frac{d}{dx} f\right](0)\sin(x) && \text{by Theorem G.3 page 110} \\
&= \cos(x) + i\sin(x) && \text{by Definition G.4 page 112}
\end{aligned}$$

⇨

Proposition G.3.

PRP $$e^{-i\pi/2} = -i \quad \Big| \quad e^{i\pi/2} = i$$

✎PROOF:

$$\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem G.5 page 112)} \\
&= 0 + i && \text{by Theorem G.2 (page 109) and Proposition G.2 (page 111)} \\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem G.5 page 112)} \\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem G.2 page 109} \\
&= 0 - i && \text{by Theorem G.2 (page 109) and Proposition G.2 (page 111)}
\end{aligned}$$

⇨

Corollary G.1.

COR $$e^{ix} = \sum_{n \in \mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x \in \mathbb{R}$$

[17] Euler (1748), Bottazzini (1986), page 12

✎PROOF:

$$\boxed{e^{ix}} = \cos(x) + i\sin(x) \qquad \text{by } \textit{Euler's identity} \qquad \text{(The}$$

$$= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} \qquad \text{by } \textit{Taylor series} \qquad \text{(The}$$

$$= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!} = \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}$$

✏

Corollary G.2 (Euler formulas). [18]

COR

$$\cos(x) = \mathbf{R_e}\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \sin(x) = \mathbf{I_m}\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\boxed{\mathbf{R_e}\left(e^{ix}\right)} \triangleq \frac{e^{ix}+\left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} \qquad \text{by definition of } \Re \qquad \text{(Definition C.5 page 77)}$$

$$= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} \qquad \text{by } \textit{Euler's identity} \qquad \text{(Theorem G.5 page 112)}$$

$$= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} = \frac{\cos(x)}{2} + \frac{\cos(x)}{2} = \boxed{\cos(x)}$$

$$\boxed{\mathbf{I_m}\left(e^{ix}\right)} \triangleq \frac{e^{ix}-\left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} \qquad \text{by definition of } \Im \qquad \text{(Definition C.5 page 77)}$$

$$= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} \qquad \text{by } \textit{Euler's identity} \qquad \text{(Theorem G.5 page 112)}$$

$$= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} = \frac{i\sin(x)}{2i} + \frac{i\sin(x)}{2i} = \boxed{\sin(x)}$$

✏

Theorem G.6. [19]

THM

$$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

✎PROOF:

$$e^{\alpha}\, e^{\beta} = \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) \qquad \text{by Corollary G.1 page 112}$$

$$= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}$$

$$= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{n!}{n!} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}$$

$$= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, \alpha^k\, \beta^{n-k}$$

[18] Euler (1748), Bottazzini (1986), page 12
[19] Rudin (1987) page 1

$$
\begin{aligned}
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the } \textit{Binomial Theorem} && \text{(Theorem D.14 page 93)} \\
&= e^{\alpha+\beta} && \text{by Corollary G.1 page 112}
\end{aligned}
$$

⇨

G.5 Trigonometric Identities

Theorem G.7 (shift identities).

THM	$\cos\left(x+\frac{\pi}{2}\right) = -\sin x \quad \forall x\in\mathbb{R}$	$\sin\left(x+\frac{\pi}{2}\right) = \cos x \quad \forall x\in\mathbb{R}$
	$\cos\left(x-\frac{\pi}{2}\right) = \sin x \quad \forall x\in\mathbb{R}$	$\sin\left(x-\frac{\pi}{2}\right) = -\cos x \quad \forall x\in\mathbb{R}$

✎PROOF:

$$
\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)} + e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by } \textit{Euler formulas} && \text{(Corollary G.2 page 113)} \\
&= \frac{e^{ix}e^{i\frac{\pi}{2}} + e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem G.6 page 113)} \\
&= \frac{e^{ix}(i) + e^{-ix}(-i)}{2} && \text{by Proposition G.3 page 112} \\
&= \frac{e^{ix} - e^{-ix}}{-2i} \\
&= -\sin x && \text{by } \textit{Euler formulas} && \text{(Corollary G.2 page 113)} \\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)} + e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by } \textit{Euler formulas} && \text{(Corollary G.2 page 113)} \\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}} + e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem G.6 page 113)} \\
&= \frac{e^{ix}(-i) + e^{-ix}(i)}{2} && \text{by Proposition G.3 page 112} \\
&= \frac{e^{ix} - e^{-ix}}{2i} \\
&= \sin x && \text{by } \textit{Euler formulas} && \text{(Corollary G.2 page 113)} \\
\sin\left(x+\frac{\pi}{2}\right) &= \cos\left(\left[x+\frac{\pi}{2}\right] - \frac{\pi}{2}\right) && \text{by previous result} \\
&= \cos(x) \\
\sin\left(x-\frac{\pi}{2}\right) &= -\cos\left(\left[x-\frac{\pi}{2}\right] + \frac{\pi}{2}\right) && \text{by previous result} \\
&= -\cos(x)
\end{aligned}
$$

⇨

Theorem G.8 (product identities).

THM	(A).	$\cos x \cos y$	$=$	$\tfrac{1}{2}\cos(x-y)$	$+$	$\tfrac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$
	(B).	$\cos x \sin y$	$=$	$-\tfrac{1}{2}\sin(x-y)$	$+$	$\tfrac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
	(C).	$\sin x \cos y$	$=$	$\tfrac{1}{2}\sin(x-y)$	$+$	$\tfrac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
	(D).	$\sin x \sin y$	$=$	$\tfrac{1}{2}\cos(x-y)$	$-$	$\tfrac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary G.2 page 113)
(algebraic method requiring *complex number system* $\mathbb{C}$):

$$\begin{aligned}
\cos x \cos y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right) && \text{by \textit{Euler formulas} (Corollary G.2 page 113)}\\
&= \frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(-x+y)}+e^{i(-x-y)}}{4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{4}+\frac{e^{i(x-y)}+e^{-i(x-y)}}{4}\\
&= \frac{2\cos(x+y)}{4}+\frac{2\cos(x-y)}{4} && \text{by \textit{Euler formulas} (Corollary G.2 page 113)}\\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem G.3 page 110)
(differential equation method requiring only *real number system* $\mathbb{R}$):

$$\begin{aligned}
&\mathsf{f}(x) \triangleq \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)\\
\implies &\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) = -\tfrac{1}{2}\sin(x-y)-\tfrac{1}{2}\sin(x+y) && \text{by Theorem G.4 page 111}\\
\implies &\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x) = -\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y) && \text{by Theorem G.4 page 111}\\
\implies &\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x)+\mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property}\\
\implies &\underbrace{\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)}_{\mathsf{f}(x)} = \underbrace{[\tfrac{1}{2}\cos(0-y)+\tfrac{1}{2}\cos(0+y)]}_{\mathsf{f}''(0)}\cos(x)+\underbrace{[-\tfrac{1}{2}\sin(0-y)-\tfrac{1}{2}\sin(0+y)]}_{\mathsf{f}'(0)}\sin(x)\\
\implies &\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y) = \cos y\cos x + 0\sin(x)\\
\implies &\cos x\cos y = \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)
\end{aligned}$$

3. Proof for (B) using *Euler formulas* (Corollary G.2 page 113):

$$\begin{aligned}
\sin x\sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary G.2 page 113}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary G.2 page 113}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x\sin y &= \cos(x)\cos\left(y-\frac{\pi}{2}\right) && \text{by \textit{shift identities} (Theorem G.7 page 114)}\\
&= \frac{1}{2}\cos\left(x+y-\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(x-y+\frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y)-\frac{1}{2}\sin(x-y) && \text{by \textit{shift identities} (Theorem G.7 page 114)}
\end{aligned}$$

$\sin x\cos y = \cos y\sin x$

$$\begin{aligned}
&= \frac{1}{2}\sin(y+x) - \frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y) && \text{by Theorem G.2 page 109}
\end{aligned}$$

Proposition G.4.

PRP

$$\begin{array}{llcl|llcl|llcl}
(A). & \cos(\pi) & = & -1 & (C). & \cos(2\pi) & = & 1 & (E). & e^{i\pi} & = & -1\\
(B). & \sin(\pi) & = & 0 & (D). & \sin(2\pi) & = & 0 & (F). & e^{i2\pi} & = & 0
\end{array}$$

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1 + 1 + \cos(\pi)\\
&= -1 + 2[\tfrac{1}{2}\cos(\tfrac{\pi}{2} - \tfrac{\pi}{2}) + \tfrac{1}{2}\cos(\tfrac{\pi}{2} + \tfrac{\pi}{2})] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem G.2 page 109)}\\
&= -1 + 2\cos(\tfrac{\pi}{2})\cos(\tfrac{\pi}{2}) && \text{by \textit{product identities}} && \text{(Theorem G.8 page 114)}\\
&= -1 + 2(0)(0) && \text{by definition of } \pi && \text{(Definition G.3 page 107)}\\
&= -1\\
\sin(\pi) &= 0 + \sin(\pi)\\
&= 2[-\tfrac{1}{2}\sin(\tfrac{\pi}{2} - \tfrac{\pi}{2}) + \tfrac{1}{2}\sin(\tfrac{\pi}{2} + \tfrac{\pi}{2})] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem G.2 page 109)}\\
&= 2\cos(\tfrac{\pi}{2})\sin(\tfrac{\pi}{2}) && \text{by \textit{product identities}} && \text{(Theorem G.8 page 114)}\\
&= 2(0)\sin(\tfrac{\pi}{2}) && \text{by definition of } \pi && \text{(Definition G.3 page 107)}\\
&= 0\\
\cos(2\pi) &= 1 + \cos(2\pi) - 1\\
&= 2[\tfrac{1}{2}\cos(\pi - \pi) + \tfrac{1}{2}\cos(\pi + \pi)] - 1 && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem G.2 page 109)}\\
&= 2\cos(\pi)\cos(\pi) - 1 && \text{by \textit{product identities}} && \text{(Theorem G.8 page 114)}\\
&= 2(-1)(-1) - 1 && \text{by (A)}\\
&= 1\\
\sin(2\pi) &= 0 + \sin(2\pi)\\
&= 2[\tfrac{1}{2}\sin(\pi - \pi) + \tfrac{1}{2}\sin(\pi + \pi)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem G.2 page 109)}\\
&= 2\sin(\pi)\cos(\pi) && \text{by \textit{product identities}} && \text{(Theorem G.8 page 114)}\\
&= 2(0)(-1) && \text{by (A) and (B)}\\
&= 0\\
e^{i\pi} &= \cos(\pi) + i\sin(\pi) && \text{by \textit{Euler's identity}} && \text{(Theorem G.5 page 112)}\\
&= -1 + 0 && \text{by (A) and (B)}\\
&= -1\\
e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) && \text{by \textit{Euler's identity}} && \text{(Theorem G.5 page 112)}\\
&= 1 + 0 && \text{by (C) and (D)}\\
&= 1
\end{aligned}$$

Theorem G.9 (double angle formulas). [20]

THM

$$\begin{array}{llcll}
(A). & \cos(x+y) & = & \cos x \cos y - \sin x \sin y & \forall x,y \in \mathbb{R}\\
(B). & \sin(x+y) & = & \sin x \cos y + \cos x \sin y & \forall x,y \in \mathbb{R}\\
(C). & \tan(x+y) & = & \dfrac{\tan x + \tan y}{1 - \tan x \tan y} & \forall x,y \in \mathbb{R}
\end{array}$$

[20] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

✎PROOF:

1. Proof for (A) using *product identities* (Theorem G.8 page 114).

$$\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x+y)}_{\cos(x+y)} + \underbrace{\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x-y)}_{0} \\
&= \left[\frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\right] - \left[\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\right] \\
&= \cos x\cos y - \sin x\sin y && \text{by Theorem G.8 page 114}
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem G.3 page 110):

$$\begin{aligned}
\mathrm{f}(x) \triangleq \cos(x+y) &\implies \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x) = -\sin(x+y) && \text{by Theorem G.4 page 111} \\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) = -\cos(x+y) && \text{by Theorem G.4 page 111} \\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = 0 && \text{by } \textit{additive inverse} \text{ property} \\
&\implies \cos(x+y) = \cos y\cos x - \sin y\sin x && \text{by Theorem G.3 page 110} \\
&\implies \cos(x+y) = \cos x\cos y - \sin x\sin y && \text{by } \textit{commutative} \text{ property}
\end{aligned}$$

3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x - \frac{\pi}{2} + y\right) && \text{by } \textit{shift identities} \text{ (Theorem G.7 page 114)} \\
&= \cos\left(x - \frac{\pi}{2}\right)\cos(y) - \sin\left(x - \frac{\pi}{2}\right)\sin(y) && \text{by (A)} \\
&= \sin(x)\cos(y) + \cos(x)\sin(y) && \text{by } \textit{shift identities} \text{ (Theorem G.7 page 114)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\
&= \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y} && \text{by (A)} \\
&= \left(\frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y}\right)\left(\frac{\cos x\cos y}{\cos x\cos y}\right) \\
&= \frac{\frac{\sin x\cos y}{\cos x\cos y} + \frac{\cos x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y} - \frac{\sin x\sin y}{\cos x\cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x\tan y}
\end{aligned}$$

✏

Theorem G.10 (trigonometric periodicity).

THM

(A).	$\cos(x+M\pi)$	$=$	$(-1)^M\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		
(B).	$\sin(x+M\pi)$	$=$	$(-1)^M\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		
(C).	$e^{i(x+M\pi)}$	$=$	$(-1)^M e^{ix}$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		
(D).	$\cos(x+2M\pi)$	$=$	$\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		
(E).	$\sin(x+2M\pi)$	$=$	$\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		
(F).	$e^{i(x+2M\pi)}$	$=$	e^{ix}	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		

✎PROOF:

1. Proof for (A):

   (a) $M=0$ case: $\cos(x+0\pi) \;=\; \cos(x) \;=\; (-1)^0\cos(x)$

   (b) Proof for $M>0$ cases (by induction):

i. Base case $M = 1$:

$$\begin{aligned}
\cos(x+\pi) &= \cos x \cos\pi - \sin x \sin\pi && \text{by *double angle formulas*} && \text{(Theorem G.9 page 116)}\\
&= \cos x(-1) - \sin x(0) && \text{by } \cos\pi = -1 \text{ result} && \text{(Proposition G.4 page 116)}\\
&= (-1)^1 \cos x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\cos(x+[M+1]\pi) &= \cos([x+\pi]+M\pi)\\
&= (-1)^M \cos(x+\pi) && \text{by induction hypothesis (M case)}\\
&= (-1)^M(-1)\cos(x) && \text{by base case (item 1(b)i) page 118)}\\
&= (-1)^{M+1}\cos(x)\\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M$ … $\implies$ $N > 0$.

$$\begin{aligned}
\cos(x+M\pi) &\triangleq \cos(x-N\pi) && \text{by definition of } N\\
&= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by *double angle formulas*} && \text{(Theorem G.9 page 116)}\\
&= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem G.2 page 109}\\
&= \cos(x)\cos(0+N\pi) + \sin(x)\sin(0+N\pi)\\
&= \cos(x)(-1)^N\cos(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (1b) page 117)}\\
&= (-1)^N\cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem G.2 page 109)}\\
&\triangleq (-1)^{-M}\cos(x) && \text{by definition of } N\\
&= (-1)^M\cos(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\cos(x+M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by *Euler formulas*} && \text{(Corollary G.2 page 113)}\\
&= e^{iM\pi}\left[\frac{e^{ix}+e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem G.6 page 113)}\\
&= \left(e^{i\pi}\right)^M \cos x && \text{by *Euler formulas*} && \text{(Corollary G.2 page 113)}\\
&= (-1)^M \cos x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition G.4 page 116)}
\end{aligned}$$

2. Proof for (B):

(a) $M = 0$ case: $\sin(x+0\pi) = \sin(x) = (-1)^0\sin(x)$

(b) Proof for $M > 0$ cases (by induction):

i. Base case $M = 1$:

$$\begin{aligned}
\sin(x+\pi) &= \sin x \cos\pi + \cos x \sin\pi && \text{by *double angle formulas*} && \text{(Theorem G.9 page 116)}\\
&= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} && \text{(Proposition G.4 page 116)}\\
&= (-1)^1 \sin x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\sin(x+[M+1]\pi) &= \sin([x+\pi]+M\pi)\\
&= (-1)^M \sin(x+\pi) && \text{by induction hypothesis (M case)}\\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 118)}\\
&= (-1)^{M+1}\sin(x)\\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\sin(x+M\pi) &\triangleq \sin(x-N\pi) && \text{by definition of } N\\
&= \sin(x)\sin(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem G.9 page 116)}\\
&= \sin(x)\sin(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem G.2 page 109}\\
&= \sin(x)\sin(0+N\pi) + \sin(x)\sin(0+N\pi)\\
&= \sin(x)(-1)^N\sin(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (2b) page 118)}\\
&= (-1)^N\sin(x) && \text{by } \sin(0){=}1,\ \sin(0){=}0 \text{ results} && \text{(Theorem G.2 page 109)}\\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N\\
&= (-1)^M\sin(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)} - e^{-i(x+M\pi)}}{2i} && \text{by \textit{Euler formulas}} && \text{(Corollary G.2 page 113)}\\
&= e^{iM\pi}\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem G.6 page 113)}\\
&= \left(e^{i\pi}\right)^M \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary G.2 page 113)}\\
&= (-1)^M \sin x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition G.4 page 116)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem G.6 page 113)}\\
&= \left(e^{i\pi}\right)^M\left(e^{ix}\right)\\
&= (-1)^M e^{ix} && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition G.4 page 116)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):
$$\begin{array}{rclcll}
\cos(i(x+2M\pi)) &=& (-1)^{2M}\cos(ix) &=& \cos(ix) & \text{by (A)}\\
\sin(i(x+2M\pi)) &=& (-1)^{2M}\sin(ix) &=& \sin(ix) & \text{by (B)}\\
e^{i(x+2M\pi)} &=& (-1)^{2M}e^{ix} &=& e^{ix} & \text{by (C)}
\end{array}$$

✎

Theorem G.11 (half-angle formulas/squared identities).

THM

(A). $\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$

(B). $\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$

(C). $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x) + \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem G.8 page 114)}\\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem G.2 page 109)}\\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x) - \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem G.8 page 114)}\\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem G.2 page 109)}\\
\cos^2 x + \sin^2 x &= \tfrac{1}{2}[1+\cos(2x)] + \tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)}\\
& && \text{note: see also} && \text{Theorem G.4 page 111}
\end{aligned}$$

✎

G.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

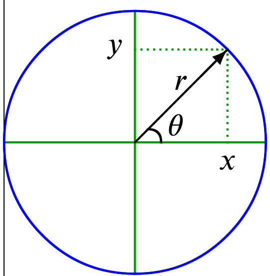
$$\begin{aligned}
\langle \cos(x) \mid \sin(x) \rangle &= \int_{-\pi}^{+\pi} \cos(x)\sin(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x-x)\,\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x+x)\,\mathrm{d}x && \text{by Theorem G.8 page 114} \\
&= \cancelto{0}{\frac{1}{2}\int_{-\pi}^{+\pi} \sin(0)\,\mathrm{d}x} + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(2x)\,\mathrm{d}x \\
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi} \cos(2x) \\
&= -\frac{1}{4}[\cos(2\pi) - \cos(-2\pi)] \\
&= 0
\end{aligned}$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

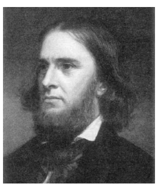
We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$



$$\begin{aligned}
\cos\theta &\triangleq \frac{x}{r} & \sec\theta &\triangleq \frac{r}{x} \\
\sin\theta &\triangleq \frac{y}{r} & \csc\theta &\triangleq \frac{r}{y} \\
\tan\theta &\triangleq \frac{y}{x} & \cot\theta &\triangleq \frac{x}{y}
\end{aligned}$$

G.7 The power of the exponential



❝*Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.*❞
Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [21]

[21] quote: Kasner and Newman (1940), page 104
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

❝*Young man, in mathematics you don't understand things. You just get used to them.*❞
John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [22]

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary G.3. [23]

COR $\boxed{e^{i\pi} + 1 = 0}$

✎PROOF:

$$\begin{aligned} e^{ix}\big|_{x=\pi} &= [\cos x + i\sin x]_{x=\pi} && \text{by Euler's identity (Theorem G.5 page 112)} \\ &= -1 + i \cdot 0 && \text{by Proposition G.4 page 116} \\ &= -1 \end{aligned}$$

⇒

There are many transforms available, several of them integral transforms $[\mathbf{A}\mathsf{f}](s) \triangleq \int_t \mathsf{f}(s)\kappa(t,s)\,\mathrm{d}s$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:

① The *Laplace Transform* with kernel $\kappa(t,\ s) \triangleq e^{st}$
② The *Fourier Transform* with kernel $\kappa(t,\ \omega) \triangleq e^{i\omega t}$.

Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem G.12 page 121).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem G.12. [24] *Let* $\mathbf{L}$ *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*
$\check{\mathsf{h}}(s) \triangleq \left\langle \mathsf{h}(t,\omega) \,|\, e^{st} \right\rangle$ (LAPLACE TRANSFORM).

[22] quote: Zukav (1980), page 208
image: http://en.wikipedia.org/wiki/John_von_Neumann
The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."

[23] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

[24] Mallat (1999), page 2, …page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

THM $\left\{\begin{array}{ll} 1. & \mathbf{L} \textit{ is } \text{LINEAR } \textit{and} \\ 2. & \mathbf{L} \textit{ is } \text{TIME-INVARIANT} \end{array}\right\} \implies \left\{ \mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{\textit{eigenvalue}} \underbrace{e^{st}}_{\textit{eigenvector}} \right\}$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \,|\, \mathsf{h}((t;u),s)\rangle && \text{by linear hypothesis} \\
&= \langle e^{su} \,|\, \mathsf{h}((t-u),s)\rangle && \text{by time-invariance hypothesis} \\
&= \left\langle e^{s(t-v)} \,|\, \mathsf{h}(v,s)\right\rangle && \text{let } v = t-u \implies u = t-v \\
&= e^{st}\ \langle e^{-sv} \,|\, \mathsf{h}(v,s)\rangle && \text{by additivity of } \langle \triangle \,|\, \triangledown\rangle \\
&= \langle \mathsf{h}(v,s) \,|\, e^{-sv}\rangle^*\ e^{st} && \text{by conjugate symmetry of } \langle \triangle \,|\, \triangledown\rangle \\
&= \left\langle \mathsf{h}(v,s) \,|\, e^{(-s)v}\right\rangle^*\ e^{st} \\
&= \check{\mathsf{h}}^*(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}$$

✏

APPENDIX H

TRIGONOMETRIC POLYNOMIALS

❝*I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives.*❞
Charles Hermite (1822 – 1901), French mathematician, in an 1893 letter to Stieltjes, in response to the "pathological" everywhere continuous but nowhere differentiable *Weierstrass functions* $\mathsf{f}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$.[1]

H.1 Trigonometric expansion

Theorem H.1 (DeMoivre's Theorem)**.**

THM
$$\left(re^{ix}\right)^n = r^n(\cos nx + i\sin nx) \qquad \forall r,x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\left(re^{ix}\right)^n &= r^n e^{inx} \\
&= r^n\left(\cos nx + i\sin nx\right) && \text{by Euler's identity (Theorem G.5 page 112)}
\end{aligned}$$

⇒

The cosine with argument nx can be expanded as a polynomial in $\cos(x)$ (next).

Theorem H.2 (trigonometric expansion)**.** [2]

[1] quote: Hermite (1893)
translation: Lakatos (1976), page 19
image: http://www-groups.dcs.sx-and.ac.uk/~history/PictDisplay/Hermite.html

[2] Rivlin (1974) page 3 ⟨(1.8)⟩

THM

$$\cos(nx) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \qquad \forall n \in \mathbb{W} \text{ and } x \in \mathbb{R}$$

$$\sin(nx) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\sin x)^{n-2(k-m)} \qquad \forall n \in \mathbb{W} \text{ and } x \in \mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(nx) &= \Re\left(\cos nx + i \sin nx\right) \\
&= \Re\left(e^{inx}\right) \\
&= \Re\left[\left(e^{ix}\right)^n\right] \\
&= \Re\left[(\cos x + i \sin x)^n\right] \\
&= \Re\left[\sum_{k \in \mathbb{Z}}^{n} \binom{n}{k} (\cos x)^{n-k} (i \sin x)^k\right] \\
&= \Re\left[\sum_{k \in \mathbb{Z}}^{n} i^k \binom{n}{k} \cos^{n-k} x \sin^k x\right] \\
&= \Re\left[\sum_{k \in \{0,4,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x + i \sum_{k \in \{1,5,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x\right. \\
&\quad \left. - \sum_{k \in \{2,6,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x + -i \sum_{k \in \{3,7,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x\right] \\
&= \sum_{k \in \{0,4,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x - \sum_{k \in \{2,6,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x \\
&= \sum_{k \in \{0,2,\ldots,n\}} \binom{n}{k} (-1)^{\frac{k}{2}} \cos^{n-k} x \sin^k x \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \cos^{n-2k} x \sin^{2k} x \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \cos^{n-2k} x (1 - \cos^2 x)^k \\
&= \left[\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \cos^{n-2k} x\right] \left[\sum_{m=0}^{k} \binom{k}{m} (-1)^m \cos^{2m} x\right] \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} x
\end{aligned}$$

$$\begin{aligned}
\sin(nx) &= \cos\left(nx - \frac{\pi}{2}\right) \\
&= \cos\left(n\left[x - \frac{\pi}{2n}\right]\right) \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(n\left[x - \frac{\pi}{2n}\right]\right)
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(nx - \frac{\pi}{2} \right)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \sin^{n-2(k-m)} (nx)$$

Example H.1.

EX	$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ $\sin 5x = 16\sin^5 x - 20\sin^3 x + 5\sin x.$

✎PROOF:

1. Proof using *DeMoivre's Theorem* (Theorem H.1 page 123):

$$\begin{aligned}
&\cos 5x + i\sin 5x \\
&= e^{i5x} \\
&= \left(e^{ix}\right)^5 \\
&= (\cos x + i\sin x)^5 \\
&= \sum_{k=0}^{5} \binom{5}{k} [\cos x]^{5-k} [i\sin x]^k \\
&= \binom{5}{0}[\cos x]^{5-0}[i\sin x]^0 + \binom{5}{1}[\cos x]^{5-1}[i\sin x]^1 + \binom{5}{2}[\cos x]^{5-2}[i\sin x]^2 + \\
&\binom{5}{3}[\cos x]^{5-3}[i\sin x]^3 + \binom{5}{4}[\cos x]^{5-4}[i\sin x]^4 + \binom{5}{5}[\cos x]^{5-5}[i\sin x]^5 \\
&= 1\cos^5 x + i5\cos^4 x\sin x - 10\cos^3 x\sin^2 x - i10\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i1\sin^5 x \\
&= \left[\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\right] + i\left[5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)(1-\cos^2 x)\right] + \\
&i\left[5(1-\sin^2 x)(1-\sin^2 x)\sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)\right] + \\
&i\left[5(1-2\sin^2 x + \sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5(\cos x - 2\cos^3 x + \cos^5 x)\right] + \\
&i\left[5(\sin x - 2\sin^3 x + \sin^5 x) - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \underbrace{\left[16\cos^5 x - 20\cos^3 x + 5\cos x\right]}_{\cos 5x} + i\underbrace{\left[16\sin^5 x - 20\sin^3 x + 5\sin x\right]}_{\sin 5x}
\end{aligned}$$

2. Proof using trigonometric expansion (Theorem H.2 page 123):

$$\begin{aligned}
\cos 5x &= \sum_{k=0}^{\lfloor \frac{5}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k}\binom{k}{m} (\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{2} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k}\binom{k}{m} (\cos x)^{5-2(k-m)} \\
&= (-1)^0\binom{5}{0}\binom{0}{0}\cos^5 x + (-1)^1\binom{5}{2}\binom{1}{0}\cos^3 x + (-1)^2\binom{5}{2}\binom{1}{1}\cos^5 x + \\
&(-1)^2\binom{5}{4}\binom{2}{0}\cos^1 x + (-1)^3\binom{5}{4}\binom{2}{1}\cos^3 x + (-1)^4\binom{5}{4}\binom{2}{2}\cos^5 x
\end{aligned}$$

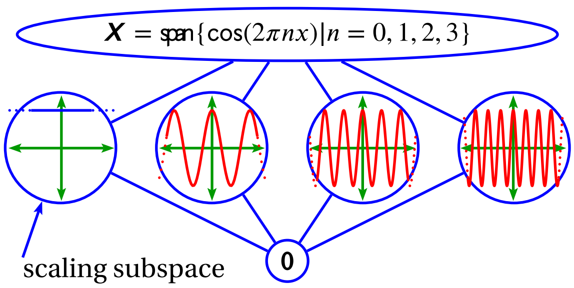


Figure H.1: Lattice of harmonic cosines $\{\cos(nx)|n = 0, 1, 2, \ldots\}$

$$
\begin{aligned}
&= +(1)(1)\cos^5 x - (10)(1)\cos^3 x + (10)(1)\cos^5 x + (5)(1)\cos x - (5)(2)\cos^3 x + (5)(1)\cos^5 x \\
&= +(1 + 10 + 5)\cos^5 x + (-10 - 10)\cos^3 x + 5\cos x \\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}
$$

Example H.2. [3]

	n	$\cos nx$		polynomial in $\cos x$	n	$\cos nx$		polynomial in $\cos x$
EX	0	$\cos 0x$	$=$	1	4	$\cos 4x$	$=$	$8\cos^4 x - 8\cos^2 x + 1$
	1	$\cos 1x$	$=$	$\cos^1 x$	5	$\cos 5x$	$=$	$16\cos^5 x - 20\cos^3 x + 5\cos x$
	2	$\cos 2x$	$=$	$2\cos^2 x - 1$	6	$\cos 6x$	$=$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
	3	$\cos 3x$	$=$	$4\cos^3 x - 3\cos x$	7	$\cos 7x$	$=$	$64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

✎PROOF:

$$
\begin{aligned}
\cos 2x &= \sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{2-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^2 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^0 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^2 x \\
&= +(1)(1)\cos^2 x - (1)(1) + (1)(1)\cos^2 x \\
&= 2\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 3x &= \sum_{k=0}^{\lfloor \frac{3}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{3-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^3 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^1 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^3 x \\
&= +\binom{3}{0}\binom{0}{0} \cos^3 x - \binom{3}{2}\binom{1}{0} \cos^1 x + \binom{3}{2}\binom{1}{1} \cos^3 x \\
&= +(1)(1)\cos^3 x - (3)(1)\cos^1 x + (3)(1)\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}
$$

$$
\cos 4x = \sum_{k=0}^{\lfloor \frac{4}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{4}{2k} \binom{k}{m} (\cos x)^{4-2(k-m)}
$$

[3] Abramowitz and Stegun (1972), page 795, Guillemin (1957), page 593 ⟨(21)⟩, Sloane (2014) ⟨http://oeis.org/A039991⟩, Sloane (2014) ⟨http://oeis.org/A028297⟩

$$
\begin{aligned}
&= \sum_{k=0}^{2}\sum_{m=0}^{k}(-1)^{k+m}\binom{4}{2k}\binom{k}{m}(\cos x)^{4-2(k-m)} \\
&= (-1)^{0+0}\binom{4}{2\cdot 0}\binom{0}{0}(\cos x)^{4-2(0-0)} + (-1)^{1+0}\binom{4}{2\cdot 1}\binom{1}{0}(\cos x)^{4-2(1-0)} \\
&\quad + (-1)^{1+1}\binom{4}{2\cdot 1}\binom{1}{1}(\cos x)^{4-2(1-1)} + (-1)^{2+0}\binom{4}{2\cdot 2}\binom{2}{0}(\cos x)^{4-2(2-0)} \\
&\quad + (-1)^{2+1}\binom{4}{2\cdot 2}\binom{2}{1}(\cos x)^{4-2(2-1)} + (-1)^{2+2}\binom{4}{2\cdot 2}\binom{2}{2}(\cos x)^{4-2(2-2)} \\
&= (1)(1)\cos^4 x - (6)(1)\cos^2 x + (6)(1)\cos^4 x + (1)(1)\cos^0 x - (1)(2)\cos^2 x + (1)(1)\cos^4 x \\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}
$$

$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ see Example H.1 page 125

$$
\begin{aligned}
\cos 6x &= \sum_{k=0}^{\left\lfloor\frac{6}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{6}{2k}\binom{k}{m}(\cos x)^{6-2(k-m)} \\
&= (-1)^0\binom{6}{0}\binom{0}{0}\cos^6 x + (-1)^1\binom{6}{2}\binom{1}{0}\cos^4 x + (-1)^2\binom{6}{2}\binom{1}{1}\cos^6 x + (-1)^2\binom{6}{4}\binom{2}{0}\cos^2 x + \\
&(-1)^3\binom{6}{4}\binom{2}{1}\cos^4 x + (-1)^4\binom{6}{4}\binom{2}{2}\cos^6 x + (-1)^3\binom{6}{6}\binom{3}{0}\cos^0 x + (-1)^4\binom{6}{6}\binom{3}{1}\cos^2 x + \\
&(-1)^5\binom{6}{6}\binom{3}{2}\cos^4 x + (-1)^6\binom{6}{6}\binom{3}{3}\cos^6 x \\
&= +(1)(1)\cos^6 x - (15)(1)\cos^4 x + (15)(1)\cos^6 x + (15)(1)\cos^2 x - (15)(2)\cos^4 x + (15)(1)\cos^6 x \\
&\quad - (1)(1)\cos^0 x + (1)(3)\cos^2 x - (1)(3)\cos^4 x + (1)(1)\cos^6 x \\
&= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 7x &= \sum_{k=0}^{\left\lfloor\frac{7}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{3}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{7-2(k-m)} \\
&= (-1)^0\binom{7}{0}\binom{0}{0}\cos^7 x + (-1)^1\binom{7}{2}\binom{1}{0}\cos^5 x + (-1)^2\binom{7}{2}\binom{1}{1}\cos^7 x + (-1)^2\binom{7}{4}\binom{2}{0}\cos^3 x \\
&\quad + (-1)^3\binom{7}{4}\binom{2}{1}\cos^5 x + (-1)^4\binom{7}{4}\binom{2}{2}\cos^7 x + (-1)^3\binom{7}{6}\binom{3}{0}\cos^1 x + (-1)^4\binom{7}{6}\binom{3}{1}\cos^3 x \\
&\quad + (-1)^5\binom{7}{6}\binom{3}{2}\cos^5 x + (-1)^6\binom{7}{6}\binom{3}{3}\cos^7 x \\
&= (1)(1)\cos^7 x - (21)(1)\cos^5 x + (21)(1)\cos^7 x + (35)(1)\cos^3 x \\
&\quad - (35)(2)\cos^5 x + (35)(1)\cos^7 x - (7)(1)\cos^1 x + (7)(3)\cos^3 x \\
&\quad - (7)(3)\cos^5 x + (7)(1)\cos^7 x \\
&= (1 + 21 + 35 + 7)\cos^7 x - (21 + 70 + 21)\cos^5 x + (35 + 21)\cos^3 x - (7)\cos^1 x \\
&= 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x
\end{aligned}
$$

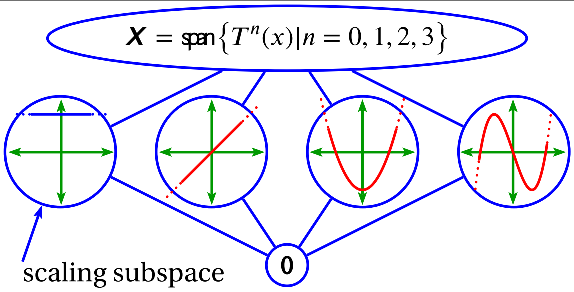


Figure H.2: Lattice of Chebyshev polynomials $\{T_n(x)|n=0,1,2,3\}$

Note: Trigonometric expansion of $\cos(nx)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[4]

```
trigexpand(cos(2*x));
trigexpand(cos(3*x));
trigexpand(cos(4*x));
trigexpand(cos(5*x));
trigexpand(cos(6*x));
trigexpand(cos(7*x));
```

Definition H.1.

DEF *The nth **Chebyshev polynomial of the first kind** is defined as*

$$T_n(x) \triangleq \cos nx \qquad \textit{where} \qquad \cos x \triangleq x$$

Theorem H.3. [5] *Let $T_n(x)$ be a* Chebyshev polynomial *with $n \in \mathbb{W}$.*

THM

$$\begin{array}{lcl} n \text{ is EVEN} & \Longrightarrow & T_n(x) \text{ is EVEN.} \\ n \text{ is ODD} & \Longrightarrow & T_n(x) \text{ is ODD.} \end{array}$$

Example H.3. Let $T_n(x)$ be a *Chebyshev polynomial* with $n \in \mathbb{W}$.

EX

$$\begin{array}{rcl|rcl} T_0(x) &=& 1 & T_4(x) &=& 8x^4 - 8x^2 + 1 \\ T_1(x) &=& x & T_5(x) &=& 16x^5 - 20x^3 + 5x \\ T_2(x) &=& 2x^2 - 1 & T_6(x) &=& 32x^6 - 48x^4 + 18x^2 - 1 \\ T_3(x) &=& 4x^3 - 3x & && \end{array}$$

✎Proof: Proof of these equations follows directly from Example H.2 (page 126). ⇒

H.2 Trigonometric reduction

Theorem H.2 (page 123) showed that $\cos nx$ can be expressed as a polynomial in $\cos x$. Conversely, Theorem H.4 (next) shows that a polynomial in $\cos x$ can be expressed as a linear combination of $(\!(\cos nx)\!)_{n\in\mathbb{Z}}$.

Theorem H.4 (trigonometric reduction)**.**

[4] maxima, pages 157–158 ⟨10.5 Trigonometric Functions⟩

[5] Rivlin (1974) page 5 ⟨(1.13)⟩, Süli and Mayers (2003) page 242 ⟨Lemma 8.2⟩, Davidson and Donsig (2010) page 222 ⟨exercise 10.7.A(a)⟩

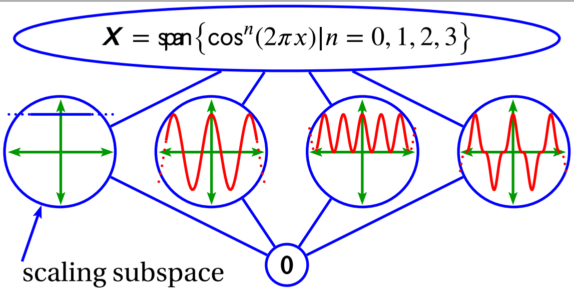


Figure H.3: Lattice of exponential cosines $\left\{\cos^n x | n = 0, 1, 2, 3\right\}$

THM

$$\begin{aligned}
\cos^n x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases}
\frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}}\sum\limits_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ even} \\
\frac{1}{2^{n-1}}\sum\limits_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ odd}
\end{cases}
\end{aligned}$$

✎ PROOF:

$$\begin{aligned}
\cos^n x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n \\
&= \mathbf{R}_e\left[\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)x}e^{-ikx}\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\cos[(n-2k)x] + i\sin[(n-2k)x]\right)\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] + i\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sin[(n-2k)x]\right] \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases}
\frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}}\sum\limits_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ even} \\
\frac{1}{2^{n-1}}\sum\limits_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ odd}
\end{cases}
\end{aligned}$$

⇒

Example H.4. [6]

[6] Abramowitz and Stegun (1972), page 795, Sloane (2014) ⟨http://oeis.org/A100257⟩, Sloane (2014) ⟨http://oeis.org/A008314⟩

EX

n	$\cos^n x$	trigonometric reduction	n	$\cos^n x$	trigonometric reduction
0	$\cos^0 x$	$= 1$	4	$\cos^4 x$	$= \dfrac{\cos 4x + 4\cos 2x + 3}{2^3}$
1	$\cos^1 x$	$= \cos x$	5	$\cos^5 x$	$= \dfrac{\cos 5x + 5\cos 3x + 10\cos x}{2^4}$
2	$\cos^2 x$	$= \dfrac{\cos 2x + 1}{2}$	6	$\cos^6 x$	$= \dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{2^5}$
3	$\cos^3 x$	$= \dfrac{\cos 3x + 3\cos x}{2^2}$	7	$\cos^7 x$	$= \dfrac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{2^6}$

✎PROOF:

$$
\begin{aligned}
\cos^0 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=0}\\
&= \frac{1}{2^0}\sum_{k=0}^{0}\binom{0}{k}\cos[(0-2k)x]\\
&= \binom{0}{0}\cos[(0-2\cdot 0)x]\\
&= 1\\
\cos^1 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=1}\\
&= \frac{1}{2^1}\sum_{k=0}^{1}\binom{1}{k}\cos[(1-2k)x]\\
&= \frac{1}{2}\left[\binom{1}{0}\cos[(1-2\cdot 0)x] + \binom{1}{1}\cos[(1-2\cdot 1)x]\right]\\
&= \frac{1}{2}\left[1\cos x + 1\cos(-x)\right]\\
&= \frac{1}{2}\left(\cos x + \cos x\right)\\
&= \cos x\\
\cos^2 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=2}\\
&= \frac{1}{2^2}\sum_{k=0}^{2}\binom{2}{k}\cos\left([2-2k]x\right)\\
&= \frac{1}{2^2}\left[\binom{2}{0}\cos\left([2-2\cdot 0]x\right) + \binom{2}{1}\cos\left([2-2\cdot 1]x\right) + \binom{2}{2}\cos\left([2-2\cdot 2]x\right) + \right]\\
&= \frac{1}{2^2}\left[1\cos(2x) + 2\cos(0x) + 1\cos(-2x)\right]\\
&= \frac{1}{2^2}\left[\cos(2x) + 2 + \cos(2x)\right]\\
&= \frac{1}{2}\left[\cos(2x) + 1\right]\\
\cos^3 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\right|_{n=3}\\
&= \frac{1}{2^3}\sum_{k=0}^{3}\binom{3}{k}\cos\left([3-2k]x\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2^3}\left[1\cos(3x) + 3\cos(1x) + 3\cos(-1x) + 1\cos(-3x)\right] \\
&= \frac{1}{2^3}\left[\cos(3x) + 3\cos(x) + 3\cos(x) + \cos(3x)\right] \\
&= \frac{1}{2^2}\left[\cos(3x) + 3\cos(x)\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^4 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=4} \\
&= \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}\cos([4-2k]x) \\
&= \frac{1}{2^4}\left[1\cos(4x) + 4\cos(2x) + 6\cos(0x) + 4\cos(-2x) + 1\cos(-4x)\right] \\
&= \frac{1}{2^3}\left[\cos(4x) + 4\cos(2x) + 3\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^5 x &= \frac{1}{2^{5-1}}\sum_{k=0}^{\lfloor\frac{5}{2}\rfloor}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\sum_{k=0}^{2}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\left[\binom{5}{0}\cos 5x + \binom{5}{1}\cos 3x + \binom{5}{2}\cos x\right] \\
&= \frac{1}{16}\left[\cos 5x + 5\cos 3x + 10\cos x\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^6 x &= \frac{1}{2^6}\binom{6}{\frac{6}{2}} + \frac{1}{2^{6-1}}\sum_{k=0}^{\frac{6}{2}-1}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{2^6}\binom{6}{3} + \frac{1}{2^5}\sum_{k=0}^{2}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{64}20 + \frac{1}{32}\left[\binom{6}{0}\cos 6x + \binom{6}{1}\cos 4x\binom{6}{2}\cos 2x\right] \\
&= \frac{1}{32}\left[\cos 6x + 6\cos 4x + 15\cos 2x + 10\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^7 x &= \frac{1}{2^{7-1}}\sum_{k=0}^{\lfloor\frac{7}{2}\rfloor}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\sum_{k=0}^{2}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\left[\binom{7}{0}\cos 7x + \binom{7}{1}\cos 5x + \binom{7}{2}\cos 3x + \binom{7}{3}\cos x\right] \\
&= \frac{1}{64}\left[\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right]
\end{aligned}
$$

Note: Trigonometric reduction of $\cos^n(x)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[7]

```
trigreduce((cos(x))^2);
trigreduce((cos(x))^3);
trigreduce((cos(x))^4);
trigreduce((cos(x))^5);
trigreduce((cos(x))^6);
trigreduce((cos(x))^7);
```

[7] http://maxima.sourceforge.net/docs/manual/en/maxima_15.html
maxima, page 158 ⟨10.5 Trigonometric Functions⟩

H.3 Spectral Factorization

Theorem H.5 (Fejér-Riesz spectral factorization). [8] *Let* $[0,\infty) \subsetneq \mathbb{R}$ *and*

$$\mathsf{p}\left(e^{ix}\right) \triangleq \sum_{n=-N}^{N} a_n e^{inx} \qquad \text{(Laurent trigonometric polynomial order } 2N\text{)}$$

$$\mathsf{q}\left(e^{ix}\right) \triangleq \sum_{n=1}^{N} b_n e^{inx} \qquad \text{(standard trigonometric polynomial order } N\text{)}$$

THM

$$\mathsf{p}\left(e^{ix}\right) \in [0,\infty) \quad \forall x \in [0,2\pi] \quad \Longrightarrow \quad \left\{\begin{array}{ll} \exists \left(\!\left(b_n\right)\!\right)_{n\in\mathbb{Z}} & \text{such that} \\ \mathsf{p}\left(e^{ix}\right) = \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right) & \forall x \in \mathbb{R} \end{array}\right.$$

✎PROOF:

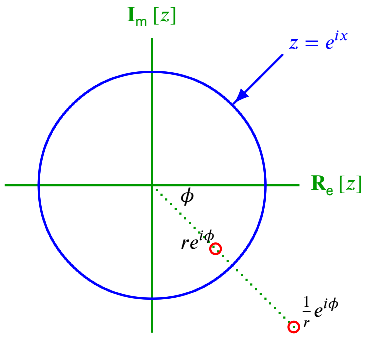
1. Proof that $a_n = a^*_{-n}$ ($\left(\!\left(a_n\right)\!\right)_{n\in\mathbb{Z}}$ is *Hermitian symmetric*):
   Let $a_n \triangleq r_n e^{i\phi_n}$, $r_n, \phi_n \in \mathbb{R}$. Then

$$\begin{aligned}
\mathsf{p}\left(e^{inx}\right) &\triangleq \sum_{n=-N}^{N} a_n e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{i\phi_n} e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{inx+\phi_n} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\sum_{n=-N}^{N} r_n \sin(nx+\phi_n) \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\underbrace{\left[r_0\sin(0x+\phi_0) + \sum_{n=1}^{N} r_n\sin(nx+\phi_n) + \sum_{n=1}^{N} r_{-n}\sin(-nx+\phi_{-n})\right]}_{\text{imaginary part must equal 0 because } p(x)\in\mathbb{R}} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\underbrace{\left[r_0\sin(\phi_0) + \sum_{n=1}^{N} r_n\sin(nx+\phi_n) - \sum_{n=1}^{N} r_{-n}\sin(nx-\phi_{-n})\right]}_{\Longrightarrow\ r_n = r_{-n},\ \ \phi_n = -\phi_{-n}\ \ \Longrightarrow\ a_n = a^*_{-n},\ a_0\in\mathbb{R}}
\end{aligned}$$

2. Because the coefficients $\left(\!\left(c_n\right)\!\right)_{n\in\mathbb{Z}}$ are *Hermitian symmetric* and by Theorem D.7 (page 90), the zeros of $P(z)$ occur in *conjugate recipricol pairs*. This means that if $\sigma \in \mathbb{C}$ is a zero of $P(z)$ ($P(\sigma) = 0$), then $\frac{1}{\sigma^*}$ is also a zero of $P(z)$ ($P\left(\frac{1}{\sigma^*}\right) = 0$). In the complex z plane, this relationship means zeros are reflected across the unit circle such that

$$\frac{1}{\sigma^*} = \frac{1}{(re^{i\phi})^*} = \frac{1}{r}\frac{1}{e^{-i\phi}} = \frac{1}{r}e^{i\phi}$$

[8] Pinsky (2002), pages 330–331



3. Because the zeros of $\mathsf{p}(z)$ occur in conjugate recipricol pairs, $\mathsf{p}\left(e^{ix}\right)$ can be factored:

$$\begin{aligned}
\mathsf{p}\left(e^{ix}\right) &= \mathsf{p}(z)\big|_{z=e^{ix}} \\
&= z^{-N} C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N}\left(z-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N} z^{-1}\left(z-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N}\left(1-\frac{1}{\sigma_n^*}z^{-1}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N}\left(z^{-1}-\sigma_n^*\right)\left(-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= \left[C\prod_{n=1}^{N}\left(-\frac{1}{\sigma_n^*}\right)\right]\left[\prod_{n=1}^{N}(z-\sigma_n)\right]\left[\prod_{n=1}^{N}\left(\frac{1}{z^*}-\sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \left[C_2\prod_{n=1}^{N}(z-\sigma_n)\right]\left[C_2\prod_{n=1}^{N}\left(\frac{1}{z^*}-\sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}(z)\mathsf{q}^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right)
\end{aligned}$$

H.4 Dirichlet Kernel

❝Dirichlet alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous proof is. Rather we learn it first from him. When Gauss says he has proved something it is clear; when Cauchy says it, one can wager as much pro as con; when Dirichlet says it, it is certain.❞

Carl Gustav Jacob Jacobi (1804–1851), Jewish-German mathematician [9]

[9] quote: Schubring (2005), page 558
image: http://en.wikipedia.org/wiki/File:Carl_Jacobi.jpg, public domain

The *Dirichlet Kernel* is critical in proving what is not immediately obvious in examining the Fourier Series—that for a broad class of periodic functions, a function can be recovered from (with uniform convergence) its Fourier Series analysis.

Definition H.2. [10]

DEF: *The **Dirichlet Kernel** $\mathsf{D}_n \in \mathbb{R}^{\mathbb{W}}$ with period τ is defined as*

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}kx}$$

Proposition H.1. [11] *Let D_n be the* DIRICHLET KERNEL *with period τ* (Definition H.2 page 134).

PRP:
$$\mathsf{D}_n(x) = \frac{1}{\tau}\frac{\sin\left(\frac{\pi}{\tau}[2n+1]x\right)}{\sin\left(\frac{\pi}{\tau}x\right)}$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &\triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \mathsf{D}_n \quad \text{(Definition H.2 page 134)}\\
&= \frac{1}{\tau}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}(k-n)x} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}kx} = \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\sum_{k=0}^{2n}\left(e^{i\frac{2\pi}{\tau}x}\right)^k\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\frac{1-\left(e^{i\frac{2\pi}{\tau}x}\right)^{2n+1}}{1-e^{i\frac{2\pi}{\tau}x}} && \text{by \textit{geometric series}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\frac{1-e^{i\frac{2\pi}{\tau}(2n+1)x}}{1-e^{i\frac{2\pi}{\tau}x}} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(\frac{e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{i\frac{\pi}{\tau}x}}\right)\frac{e^{-i\frac{\pi}{\tau}(2n+1)x}-e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{-i\frac{\pi}{\tau}x}-e^{i\frac{\pi}{\tau}x}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(e^{i\frac{2\pi n}{\tau}x}\right)\frac{-2i\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{-2i\sin\left[\frac{\pi}{\tau}x\right]} = \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}x\right]}
\end{aligned}$$

✏

Proposition H.2. [12] *Let D_n be the* DIRICHLET KERNEL *with period τ* (Definition H.2 page 134).

PRP:
$$\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x = 1$$

✎PROOF:

$$\begin{aligned}
\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x &\triangleq \int_0^\tau \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \mathsf{D}_n \text{ (Definition H.2 page 134)}\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right) + i\sin\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x
\end{aligned}$$

[10] Katznelson (2004) page 14, Heil (2011) pages 443–444, Folland (1992), pages 33–34
[11] Katznelson (2004) page 14, Heil (2011) page 444, Folland (1992), page 34
[12] Bruckner et al. (1997) pages 620–621

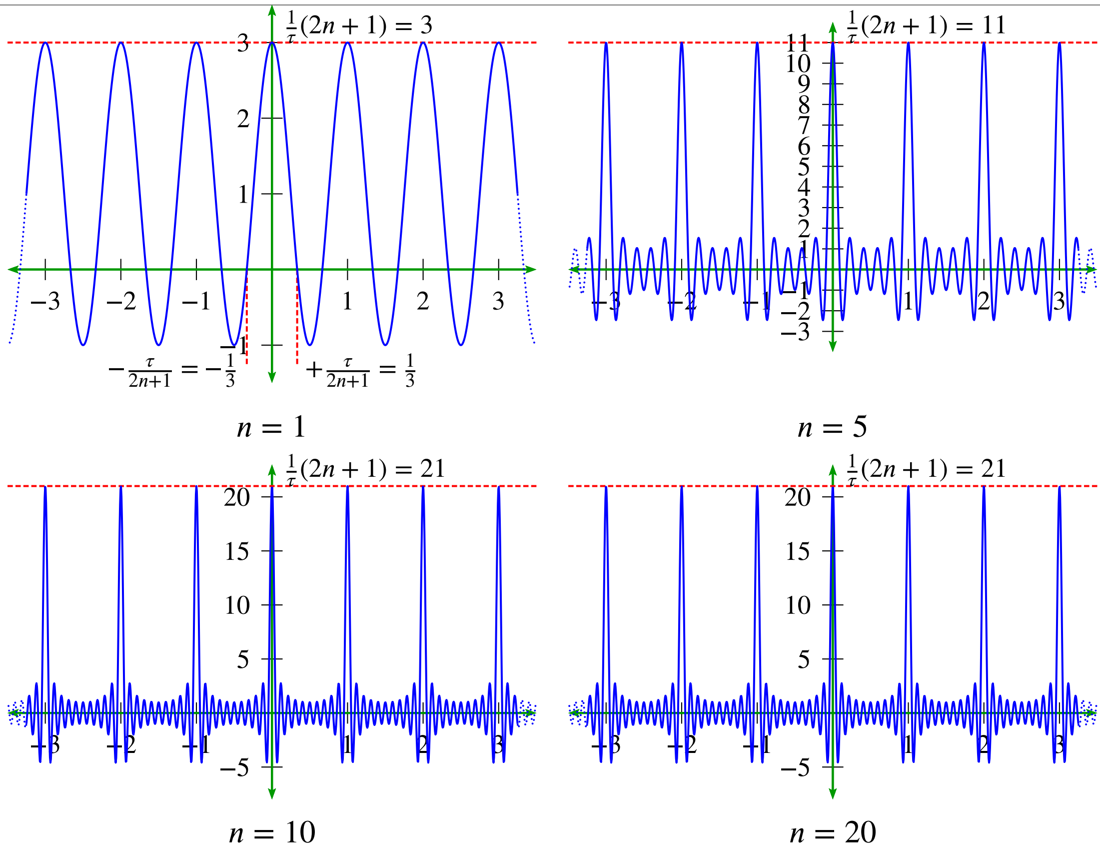


Figure H.4: D_n function for $N = 1, 5, 10, 20$. $\mathsf{D}_n \to$ comb. (See Proposition H.1 page 134).

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x \\
&= \frac{1}{\tau}\sum_{k=-n}^{n} \left.\frac{\sin\left(\frac{2\pi}{\tau}nx\right)}{\frac{2\pi}{\tau}n}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left[\frac{\sin\left(\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n} - \frac{\sin\left(-\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n}\right] \\
&= \frac{1}{\tau}\frac{\tau}{2}\sum_{k=-n}^{n}\left[\frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi n)}{\pi n}\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n)}{\pi n}\right]_{k=0} \\
&= 1
\end{aligned}
$$

Proposition H.3. *Let D_n be the* DIRICHLET KERNEL *with period τ* (Definition H.2 page 134). *Let w_N (the "*WIDTH*" of $\mathsf{D}_n(x)$) be the distance between the two points where the center pulse of $\mathsf{D}_n(x)$ intersects the x axis.*

PRP

$$
\begin{aligned}
\mathsf{D}_n(0) &= \frac{1}{\tau}(2n+1) \\
w_n &= \frac{2\tau}{2n+1}
\end{aligned}
$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(0) &= \mathsf{D}_n(x)\big|_{t=0} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} && \text{by Proposition H.1 page 134} \\
&= \frac{1}{\tau}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\frac{\mathrm{d}}{\mathrm{d}x}\sin\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{\cos\left[\frac{\pi}{\tau}(2n+1)x\right]}{\cos\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{1}{1} \\
&= \frac{1}{\tau}(2n+1)
\end{aligned}$$

The center pulse of kernel $\mathsf{D}_n(x)$ intersects the x axis at

$$t = \pm\frac{\tau}{(2n+1)}$$

which implies

$$w_n = \frac{\tau}{2n+1} + \frac{\tau}{2n+1} = \frac{2\tau}{(2n+1)}.$$

✏

Proposition H.4. [13] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition H.2 page 134).

PRP $\mathsf{D}_n(x) = \mathsf{D}_n(-x)$ (D_n *is an* EVEN *function*)

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]} && \text{by Proposition H.1 page 134} \\
&= \frac{1}{\tau}\frac{-\sin\left[-\frac{\pi}{\tau}(2n+1)x\right]}{-\sin\left[-\frac{\pi}{\tau}t\right]} && \text{because } \sin x \text{ is an } \textit{odd} \text{ function} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)(-x)\right]}{\sin\left[\frac{\pi}{\tau}(-x)\right]} \\
&= \mathsf{D}_n(-x) && \text{by Proposition H.1 page 134}
\end{aligned}$$

✏

[13] Bruckner et al. (1997) pages 620–621

H.5 Trigonometric summations

Theorem H.6 (Lagrange trigonometric identities). [14]

THM

$$\sum_{n=0}^{N-1}\cos(nx) = \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)+\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \quad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)+\cos\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=0}^{N-1}\Re e^{inx} = \Re\sum_{n=0}^{N-1}e^{inx} = \Re\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Re\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by \textit{geometric series}} \\
&= \Re\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-i\frac{1}{2}x}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Re\left[\left(e^{i\frac{1}{2}(N-1)x}\right)\left(\frac{-i\frac{1}{2}\sin\left(\frac{1}{2}Nx\right)}{-i\frac{1}{2}\sin\left(\frac{1}{2}x\right)}\right)\right] \\
&= \cos\left(\frac{1}{2}(N-1)x\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{-\frac{1}{2}\sin\left(-\frac{1}{2}x\right)+\frac{1}{2}\sin\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by \textit{product identities}} \quad \text{(Theorem G.8 page 114)} \\
&= \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=0}^{N-1}\Im e^{inx} = \Im\sum_{n=0}^{N-1}e^{inx} = \Im\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Im\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by \textit{geometric series}} \\
&= \Im\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-ix/2}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Im\left[\left(e^{i(N-1)x/2}\right)\left(\frac{-\frac{1}{2}i\sin\left(\frac{1}{2}Nx\right)}{-\frac{1}{2}i\sin\left(\frac{1}{2}x\right)}\right)\right]
\end{aligned}$$

[14] Muniz (1953) page 140 ⟨"Lagrange's Trigonometric Identities"⟩, Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (14), (13)⟩

$$= \sin\left(\frac{(N-1)x}{2}\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right)$$

$$= \frac{\frac{1}{2}\cos\left(-\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} \qquad \text{by } \textit{product identities} \qquad \text{(Theorem G.8 page 114)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}$$

Note that these results (summed with indices from $n = 0$ to $n = N-1$) are compatible with Muniz (1953) page 140 (summed with indices from $n = 1$ to $n = N$) as demonstrated next:

$$\sum_{n=0}^{N-1}\cos(nx) = \sum_{n=1}^{N}\cos(nx) + [\cos(0x) - \cos(Nx)]$$

$$= \left[-\frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + [\cos(0x) - \cos(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \left(1-\frac{1}{2}\right) + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\cos(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\left[\sin\left(\left[\frac{1}{2}-N\right]x\right) + \sin\left[\left(\frac{1}{2}+N\right)x\right]\right]}{2\sin\left(\frac{1}{2}x\right)} \qquad \text{by Theorem G.8 page 114}$$

$$= \frac{1}{2} + \frac{\sin\left(\frac{1}{2}[2N-1]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \sum_{n=1}^{N}\sin(nx) + [\sin(0x) - \sin(Nx)]$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} + [0 - \sin(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\sin(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - \left[\cos\left(\left[\frac{1}{2}-N\right]x\right) - \cos\left(\left[\frac{1}{2}+N\right]x\right)\right]}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

Theorem H.7. [15]

[15] Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (16) and (17)⟩

THM

$$\sum_{n=0}^{N-1}\cos(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]-\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]+\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)-\sin(nx)\sin(y)] && \text{by \textit{double angle formulas} (Theorem G.9 page 116)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx)-\sin(y)\sum_{n=0}^{N-1}\sin(nx)\\
\sum_{n=0}^{N-1}\sin(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)+\sin(nx)\sin(y)] && \text{by \textit{double angle formulas} (Theorem G.9 page 116)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx)+\sin(y)\sum_{n=0}^{N-1}\sin(nx)
\end{aligned}$$

✏

Corollary H.1 (Summation around unit circle).

THM

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)=0 \qquad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

$$\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right)=\frac{N}{2} \qquad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

✎PROOF:

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\\
&= \cos(\theta)\sum_{n=0}^{N-1}\cos\left(\frac{2nM\pi}{N}\right)-\sin(\theta)\sum_{n=0}^{N-1}\sin\left(\frac{2nM\pi}{N}\right) && \text{by Theorem G.9 page 116}\\
&= \cos(\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{2M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] && \text{by Theorem H.6 page 137}\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{\cos\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{M\pi}{N}\right)}{\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity} (Theorem G.10 page 117)}\\
&= \cos(\theta)[0]-\sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \sin\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by \textit{shift identities}} \quad \text{(Theorem G.7 page 114)}\\
&= \sum_{n=0}^{N-1} \cos\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2}\\
&= 0 && \text{by previous result}
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{N-1} \cos\left(\theta + \frac{2nM\pi}{N}\right)\sin\left(\theta + \frac{2nM\pi}{N}\right)\\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] - \left[\theta + \frac{2nM\pi}{N}\right]\right) + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] + \left[\theta + \frac{2nM\pi}{N}\right]\right) && \text{by Theorem G.8 page 114}\\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \cancelto{0}{\sin(0)} + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(2\theta + \frac{4nM\pi}{N}\right)\\
&= \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) && \text{by Theorem G.9 page 116}\\
&= \cos(2\theta)\left[\frac{1}{2} + \frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{4M\pi}{N}\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{4M\pi}{N}\right)}\right] && \text{by Theorem H.6 page 137}\\
&= \cos(2\theta)\left[\frac{1}{2} - \frac{\sin\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{\cos\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2} - \frac{1}{2}\frac{\sin\left(\frac{2M\pi}{N}\right)}{\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity}} \quad \text{(Theorem G.10 page 117)}\\
&= \cos(\theta)[0] - \sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2nM\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos\left(2\theta + \frac{4nM\pi}{N}\right)\right] && \text{by Theorem G.11 page 119}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos(2\theta)\cos\left(\frac{4nM\pi}{N}\right) - \sin(2\theta)\sin\left(\frac{4nM\pi}{N}\right)\right] && \text{by Theorem G.9 page 116}\\
&= \frac{1}{2}\sum_{n=0}^{N-1} 1 + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\cos\left(\frac{4nM\pi}{N}\right) - \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)\\
&= \left[\frac{1}{2}\sum_{n=0}^{N-1} 1\right] + \frac{1}{2}\cos(2\theta)0 - \frac{1}{2}\sin(2\theta)0 && \text{by previous results}\\
&= \frac{N}{2}
\end{aligned}$$

$$\sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\cos^2\left(\theta-\frac{\pi}{2}+\frac{2nM\pi}{N}\right)$$ by *shift identities* (Theorem G.7 page 114)

$$=\sum_{n=0}^{N-1}\cos^2\left(\phi+\frac{2nM\pi}{N}\right)$$ where $\phi \triangleq \theta-\frac{\pi}{2}$

$$=\frac{N}{2}$$ by previous result

H.6 Summability Kernels

Definition H.3. [16] *Let $(\kappa_n)_{n\in\mathbb{Z}}$ be a sequence of* CONTINUOUS 2π PERIODIC *functions. The sequence $(\kappa_n)_{n\in\mathbb{Z}}$ is a* ***summability kernel*** *if*

DEF

1. $\frac{1}{2\pi}\int_0^{2\pi}\kappa_n(x)\,dx = 1 \quad \forall n\in\mathbb{Z}$ *and*
2. $\frac{1}{2\pi}\int_0^{2\pi}|\kappa_n(x)|\,dx \in \mathbb{R} \quad \forall n\in\mathbb{Z}$ *and*
3. $\lim_{n\to\infty}\int_\delta^{2\pi-\delta}|\kappa_n(x)|\,dx = 0 \quad \forall n\in\mathbb{Z}, 0<\delta<\pi$

Theorem H.8. [17] *Let $(\kappa_n)_{n\in\mathbb{Z}}$ be a sequence. Let $\mathbb{T}$ be the quotient $\mathbb{R}/2\pi\mathbb{Z}$.*

THM

$$\left.\begin{array}{ll}1.\ \mathsf{f}\in L^1(\mathbb{T}) & \textit{and}\\ 2.\ (\kappa_n) \textit{ is a summability kernel} & \end{array}\right\} \implies \mathsf{f}(x)=\lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbb{T}}\kappa_n(x)\mathsf{f}(x-x)\,dx$$

The *Dirichlet kernel* (Definition H.2 page 134) is *not* a *summability kernel*. Examples of kernels that *are* summability kernels include

1. *Fejér's kernel* (Definition H.4 page 141)
2. *de la Vallée Poussin kernel* (Definition H.5 page 143)
3. *Jackson kernel* (Definition H.6 page 143)
4. *Poisson kernel* (Definition H.7 page 143 .)

Definition H.4. [18]

DEF

Fejér's kernel K_n *is defined as*

$$\mathsf{K}_n(x) \triangleq \sum_{k=-n}^{k=n}\left(1-\frac{|k|}{n+1}\right)e^{ikx}$$

Proposition H.5. [19] *Let K_n be Fejér's kernel (Definition H.4 page 141).*

PRP

$$\mathsf{K}_n(x)=\frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$$

[16] Cerdà (2010) page 56, Katznelson (2004) page 10, de Reyna (2002) page 21, Walnut (2002) pages 40–41, Heil (2011) page 440, Istrăţescu (1987) page 309
[17] Katznelson (2004) page 11
[18] Katznelson (2004) page 12
[19] Katznelson (2004) page 12, Heil (2011) page 448

✎PROOF:

1. Lemma: Proof that $\sin^2\frac{x}{2} \equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right)$:

$$\begin{aligned}
\sin^2\frac{x}{2} &\equiv \left(\frac{e^{-i\frac{x}{2}} - e^{+i\frac{x}{2}}}{2i}\right)^2 && \text{by Euler Formulas (Corollary G.2 page 113)}\\
&\equiv \frac{-1}{4}\left(e^{-2i\frac{x}{2}} - 2e^{-i\frac{x}{2}}e^{i\frac{x}{2}} + e^{2i\frac{x}{2}}\right)\\
&\equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right) \;:
\end{aligned}$$

2. Lemma:

$$2|k| - |k+1| - |k-1| = \begin{cases} -2 & \text{for } k=0\\ 0 & \text{for } k \in \mathbb{Z}\backslash 0\end{cases}$$

3. Proof that $\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$:

$$\begin{aligned}
&-4(n+1)\left(\sin\frac{1}{2}x\right)^2 \mathsf{K}_n(x)\\
&= -4(n+1)\left(\frac{-1}{4}\right)\left(e^{-ix} - 2 + e^{ix}\right)\mathsf{K}_n(x) && \text{by item (1)}\\
&= (n+1)\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx} && \text{by Definition H.4}\\
&= (n+1)\frac{1}{n+1}\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= e^{-ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}e^{ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= \sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k-1)x} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k+1)x}\\
&= \sum_{k=-n-1}^{k=n-1}(n+1-|k+1|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n+1}^{k=n+1}(n+1-|k-1|)e^{ikx}\\
&= \underbrace{e^{-i(n+1)x}}_{k\,=\,-n-1} + \underbrace{2e^{-inx}}_{k\,=\,-n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad \underbrace{-2e^{-inx}}_{k\,=\,-n} + \underbrace{-2e^{inx}}_{k\,=\,n} - 2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad \underbrace{e^{i(n+1)x}}_{k\,=\,n+1} + \underbrace{2e^{inx}}_{k\,=\,n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}\\
&= e^{-i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad -2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}
\end{aligned}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} [(n+1-|k+1|) - 2(n+1-|k|) + (n+1-|k-1|)]\, e^{ikx}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} (2|k| - |k+1| - |k-1|)\, e^{ikx}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} - 2 \qquad \text{by item (2)}$$

$$= -4\left(\sin\frac{n+1}{2}x\right)^2 \qquad \text{by item (1)}$$

✏

Definition H.5. [20] *Let* K_n *be* FEJÉR'S KERNEL *(Definition H.4 page 141).*

DEF: *The* ***de la Vallée Poussin kernel*** V_n *is defined as*

$$\mathsf{V}_n(x) \triangleq 2K_{2n+1}(x) - K_n(x)$$

Definition H.6. [21] *Let* K_n *be* FEJÉR'S KERNEL *(Definition H.4 page 141).*

DEF: *The* ***Jackson kernel*** J_n *is defined as*

$$\mathsf{J}_n(x) \triangleq \|\mathsf{K}_n\|^{-2}\, \mathsf{K}_n^2(x)$$

Definition H.7. [22]

DEF: *The* ***Poisson kernel*** P *is defined as*

$$P(r,x) \triangleq \sum_{k\in\mathbb{Z}} r^{|k|} e^{ikx}$$

[20] Katznelson (2004) page 16
[21] Katznelson (2004) page 17
[22] Katznelson (2004) page 16

APPENDIX I

FOURIER SERIES

> ❝ …et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur.❞

MÉMOIRE

SUR LA

PROPAGATION DE LA CHALEUR

DANS LES CORPS SOLIDES,

PAR M. FOURIER (*).

L'auteur de ce Mémoire s'est proposé de soumettre la Théorie de la chaleur à l'Analyse mathématique et de vérifier, par l'expérience, les résultats du calcul. Pour exposer l'état de la question, supposons une barre de métal, cylindrique et d'une longueur indéterminée, plongée par une de ses extrémités dans un fluide entretenu à une température constante : la chaleur se répandra successivement dans la barre; et, sans la perte qui a lieu à sa surface et à son autre extrémité, elle prendroit dans toute son étendue la température constante du foyer; mais, à cause de cette perte, la chaleur ne s'étendra d'une manière sensible que jusqu'à une distance du foyer dépendante de la grosseur de la barre, de la conductibilité du métal et de son degré de poli, qui influe sur le rayonnement; de sorte que des thermomètres placés dans l'éten-

> ❝ …and the innovation of the subject, together with its importance, convinced the committee to crown this work. By observing however that the way in which the author arrives at his equations is not free from difficulties, and the analysis of which, to integrate them, still leaves something to be desired, either relative to generality, or even on the side of rigour.❞

A competition awards committee consisting of the mathematical giants Lagrange, Laplace, Legendre, and others, commenting on Fourier's 1807 landmark paper *Dissertation on the propagation of heat in solid bodies* that introduced the *Fourier Series*. [1]

I.1 Definition

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition I.1. [2]

DEF

*The **Fourier Series operator** $\hat{\mathbf{F}} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{R}}$ is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{f}(x) e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{f} \text{ is periodic with period } \tau\right\}$$

---

[1] quote: Lagrange et al. (1812b), page 374, Lagrange et al. (1812a), page 112, Kahane (2008) page 199
translation: assisted by Google Translate, Castanedo (2005) ⟨chapter 2 footnote 5⟩
paper: Fourier (1807)

[2] Katznelson (2004) page 3

I.2 Inverse Fourier Series operator

Theorem I.1. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

THM *The **inverse Fourier Series** operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{x}}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)\in\ell^2_{\mathbb{R}}$$

✎PROOF: The proof of the pointwise convergence of the Fourier Series is notoriously difficult. It was conjectured in 1913 by Nokolai Luzin that the Fourier Series for all square summable periodic functions are pointwise convergent: Luzin (1913)

Fifty-three years later (1966) at a conference in Moscow, Lennart Axel Edvard Carleson presented one of the most spectacular results ever in mathematics; he demonstrated that the Luzin conjecture is indeed correct. Carleson formally published his result that same year: Carleson (1966)

Carleson's proof is expounded upon in Reyna's (2002) 175 page book: de Reyna (2002)

Interestingly enough, Carleson started out trying to disprove Luzin's conjecture. Carleson said this in an interview published in 2001:[3]
"...the problem of course presents itself already when you are a student and I was thinking about the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I realised why there should be a counter-example and how you should produce it. I thought I really understood what was the background and then to my amazement I could prove that this "correct" counter-example couldn't exist and I suddenly realised that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so."

For now, if you just want some intuitive justification for the Fourier Series, and you can somehow imagine that the Dirichlet kernel generates a *comb function* of *Dirac delta* functions, then perhaps what follows may help (or not). It is certainly not mathematically rigorous and is by no means a real proof (but at least it is less than 175 pages).

$$\begin{aligned}
\left[\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{x}\right](x) &= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{d}x\right]}_{\hat{\mathbf{F}}\mathsf{x}} && \text{by definition of } \hat{\mathbf{F}} \text{ Definition I.1 page 145}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}\,\mathsf{d}u\right]e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \hat{\mathbf{F}}^{-1}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}e^{i\frac{2\pi}{\tau}nx}\,\mathsf{d}u\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{i\frac{2\pi}{\tau}n(x-u)}\,\mathsf{d}u
\end{aligned}$$

[3] Carleson and Engquist (2001), http://www.gap-system.org/~history/Biographies/Carleson.html

$$
\begin{aligned}
&= \int_0^\tau \mathrm{x}(u) \underbrace{\frac{1}{\tau}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi}{\tau}n(x-u)}}_{\lim_{N\to\infty}\mathrm{D}_n(x)} \,\mathrm{d}u \\
&= \int_0^\tau \mathrm{x}(u) \left[\sum_{n\in\mathbb{Z}} \delta(x-u-n\tau)\right] \mathrm{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{u=0}^{u=\tau} \mathrm{x}(u)\delta(x-u-n\tau)\,\mathrm{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{v-n\tau=0}^{v-n\tau=\tau} \mathrm{x}(v-n\tau)\delta(x-v)\,\mathrm{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathrm{x}(v-n\tau)\delta(x-v)\,\mathrm{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathrm{x}(v)\delta(x-v)\,\mathrm{d}v && \text{because x is periodic with period } \tau \\
&= \int_{\mathbb{R}} \mathrm{x}(v)\delta(x-v)\,\mathrm{d}v \\
&= \mathrm{x}(x) \\
&= \mathbf{I}\tilde{\mathrm{x}}(n) && \text{by definition of } \mathbf{I}
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\tilde{\mathrm{x}}\right](n) &= \hat{\mathbf{F}}\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathrm{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau \left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathrm{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \hat{\mathbf{F}} \text{ (Definition I.1 page 145)} \\
&= \frac{1}{\tau}\int_0^\tau \left[\sum_{k\in\mathbb{Z}} \tilde{\mathrm{x}}(k)e^{i\frac{2\pi}{\tau}(k-n)x}\right] \mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathrm{x}}(k)\left[\frac{1}{\tau}\int_0^\tau e^{i\frac{2\pi}{\tau}(k-n)x}\,\mathrm{d}x\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathrm{x}}(k)\frac{1}{\tau}\left[\frac{1}{i\frac{2\pi}{\tau}(k-n)}e^{i\frac{2\pi}{\tau}(k-n)x}\right]_0^\tau \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathrm{x}}(k)\frac{1}{i2\pi(k-n)}\left[e^{i2\pi(k-n)}-1\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathrm{x}}(k)\,\bar{\delta}(k-n)\lim_{x\to 0}\left[\frac{e^{i2\pi x}-1}{i2\pi x}\right] \\
&= \tilde{\mathrm{x}}(n)\left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{i2\pi x}-1\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(i2\pi x)}\right|_{x=0} && \text{by \textit{l'Hôpital's rule}} \\
&= \tilde{\mathrm{x}}(n)\left.\frac{i2\pi e^{i2\pi x}}{i2\pi}\right|_{x=0} \\
&= \tilde{\mathrm{x}}(n) \\
&= \mathbf{I}\tilde{\mathrm{x}}(n) && \text{by definition of } \mathbf{I}
\end{aligned}
$$

⇨

Theorem I.2.

THM *The **Fourier Series adjoint operator** $\hat{\mathbf{F}}^*$ is given by*
$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathsf{x}(x) \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}} \int_0^{\tau} \mathsf{x}(x) e^{-i\frac{2\pi}{\tau}nx}\, \mathrm{d}x \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} \text{ Definition I.1 page 145} \\
&= \frac{1}{\sqrt{\tau}} \int_0^{\tau} \mathsf{x}(x) \left\langle e^{-i\frac{2\pi}{\tau}nx} \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} \mathrm{d}x && \text{by additivity property of } \langle \triangle \,|\, \nabla \rangle \\
&= \int_0^{\tau} \mathsf{x}(x) \frac{1}{\sqrt{\tau}} \left\langle \tilde{\mathsf{y}}(n) \,|\, e^{-i\frac{2\pi}{\tau}nx} \right\rangle_{\mathbb{Z}}^{*} \mathrm{d}x && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&= \int_0^{\tau} \mathsf{x}(x) \left[\hat{\mathbf{F}}^{-1} \tilde{\mathsf{y}}(n) \right]^{*} \mathrm{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} \text{ page 146} \\
&= \left\langle \mathsf{x}(x) \,|\, \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^{*}} \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{R}}
\end{aligned}$$

✏

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary* (Corollary I.1 page 148).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary I.2 page 148).

Corollary I.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\hat{\mathbf{F}}$ *be the Fourier Series operator with adjoint* $\hat{\mathbf{F}}^*$.

COR

$$\hat{\mathbf{F}}\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^*\hat{\mathbf{F}} = \mathbf{I} \qquad (\hat{\mathbf{F}} \text{ is unitary...and thus also normal and isometric})$$

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$ (Theorem I.2 (page 147)). ✏

Corollary I.2. *Let* $\hat{\mathbf{F}}$ *be the Fourier series operator,* $\hat{\mathbf{F}}^*$ *be its adjoint, and* $\hat{\mathbf{F}}^{-1}$ *be its inverse.*

COR

$$\begin{array}{rclcll}
\mathcal{R}(\hat{\mathbf{F}}) &=& \mathcal{R}(\hat{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
\left\|\left|\hat{\mathbf{F}}\right|\right\| &=& \left\|\left|\hat{\mathbf{F}}^{-1}\right|\right\| &=& 1 & (\text{UNITARY}) \\
\left\langle \hat{\mathbf{F}}\boldsymbol{x} \,|\, \hat{\mathbf{F}}\boldsymbol{y} \right\rangle &=& \left\langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \,|\, \hat{\mathbf{F}}^{-1}\boldsymbol{y} \right\rangle &=& \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle & (\text{PARSEVAL'S EQUATION}) \\
\left\|\hat{\mathbf{F}}\boldsymbol{x}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\right\| &=& \|\boldsymbol{x}\| & (\text{PLANCHEREL'S FORMULA}) \\
\left\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\right\| &=& \|\boldsymbol{x} - \boldsymbol{y}\| & (\text{ISOMETRIC})
\end{array}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary I.1 page 148) and from the properties of unitary operators. ✏

I.3 Fourier series for compactly supported functions

Theorem I.3.

THM

The set

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i\frac{2\pi}{\tau}nx} \middle| n \in \mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathsf{f}(x)$ *with support in* $[0 : \tau]$.

APPENDIX J

FOURIER TRANSFORM

❝ *The analytical equations . . . extend to all general phenomena. There cannot be a language more universal and more simple, more free from errors and from obscurities, . . . mathematical analysis is as extensive as nature itself; . . . it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them.* ❞

Joseph Fourier (1768–1830) [1]

J.1 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on ⊠, and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \middle| \int_{\mathbb{R}} |\mathsf{f}|^2 \, \mathrm{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \mid \nabla \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \, \mathrm{d}\mu$ such that

$$\langle \mathsf{f} \mid \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, \mathrm{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \mid \nabla \rangle\right)$ is a *Hilbert space*.

Definition J.1. *Let κ be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF	*The function κ is the **Fourier kernel** if*	$\kappa(x,\omega) \triangleq e^{i\omega x}$	$\forall x,\omega \in \mathbb{R}$

Definition J.2. [2] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

[1] quote: Fourier (1878), pages 7–8 ⟨Preliminary Discourse⟩
image: http://en.wikipedia.org/wiki/File:Fourier2.jpg, public domain

[2] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965), page 144, Knapp (2005) pages 374–375, Fourier (1822), Fourier (1878) page 336?

DEF *The* ***Fourier Transform*** *operator* $\tilde{\mathbf{F}}$ *is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \qquad \forall \mathsf{f}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

This definition of the Fourier Transform is also called the ***unitary Fourier Transform****.*

Remark J.1 (**Fourier transform scaling factor**). [3] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B\int_{\mathbb{R}} \mathsf{F}(\omega)e^{i\omega x}\,\mathrm{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\,\mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x}\,\mathrm{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x}\,\mathrm{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int\cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\,\mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem J.2 page 150)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

J.2 Operator properties

Theorem J.1 (Inverse Fourier transform). [4] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition J.2 page 149). *The inverse* $\tilde{\mathbf{F}}^{-1}$ *of* $\tilde{\mathbf{F}}$ *is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\,\mathrm{d}\omega \qquad \forall \tilde{\mathsf{f}}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem J.2. *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with inverse* $\tilde{\mathbf{F}}^{-1}$ *and adjoint* $\tilde{\mathbf{F}}^*$*.*

THM
$$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \mathsf{g}\right\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\,\mathrm{d}x \,|\, \mathsf{g}(\omega)\right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 149}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,\left\langle e^{-i\omega x} \,|\, \mathsf{g}(\omega)\right\rangle\,\mathrm{d}x && \text{by } \textit{additive property} \text{ of } \langle\triangle\,|\,\nabla\rangle\\
&= \int_{\mathbb{R}} \mathsf{f}(x)\,\frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x}\right\rangle^*\,\mathrm{d}x && \text{by } \textit{conjugate symmetric property} \text{ of } \langle\triangle\,|\,\nabla\rangle
\end{aligned}$$

[3] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005) pages 374–375
[4] Chorin and Hald (2009) page 13

$$= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2\pi}} \left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle \right\rangle \qquad \text{by definition of } \langle \triangle \,|\, \nabla \rangle$$

$$= \left\langle \mathsf{f} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*} \mathsf{g} \right\rangle \qquad \text{by Theorem J.1 page 150}$$

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* (Corollary J.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem J.3 page 151).

Corollary J.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

COR

$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem J.2 page 150).

Theorem J.3. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \nabla \rangle)$. *Let* $\mathcal{R}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

THM

$$\begin{array}{rclcll}
\mathcal{R}(\mathbf{F}\tau) & = & \mathcal{R}(\tilde{\mathbf{F}}^{-1}) & = & \mathcal{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| & = & |||\tilde{\mathbf{F}}^{-1}||| & = & 1 & \textit{(unitary)} \\
\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g} \rangle & = & \langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g} \rangle & = & \langle \mathsf{f} \,|\, \mathsf{g} \rangle & \textit{(Parseval's equation)} \\
\|\tilde{\mathbf{F}}\mathsf{f}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f}\| & = & \|\mathsf{f}\| & \textit{(Plancherel's formula)} \\
\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\| & = & \|\mathsf{f} - \mathsf{g}\| & \textit{(isometric)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary J.1 page 151) and from the properties of unitary operators.

Theorem J.4 (Shift relations)**.** *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator.*

THM

$$\begin{array}{rcl}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) & = & e^{-i\omega u} \left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \\
\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) & = & \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v)
\end{array}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x-u) e^{-i\omega x} \, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149))} \\
&= \frac{1}{\sqrt{2\pi}} \int_{v \in \mathbb{R}} \mathsf{f}(v) e^{-i\omega(u+v)} \, \mathrm{d}v && \text{where } v \triangleq x - u \implies t = u + v && \\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{v \in \mathbb{R}} \mathsf{f}(v) e^{-i\omega v} \, \mathrm{d}v && && \\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x} \, \mathrm{d}x && \text{by change of variable } t = v && \\
&= e^{-i\omega u} \left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149))} \\
\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ivx} \mathsf{g}(x) e^{-i\omega x} \, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149))} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{g}(x) e^{-i(\omega - v)x} \, \mathrm{d}x && && \\
&= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149))}
\end{aligned}$$

Theorem J.5 (Complex conjugate). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

THM
$$\begin{array}{rrcll}
 & \tilde{\mathbf{F}}\mathsf{f}^*(-x) &=& -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* & \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \\
\mathsf{f} \text{ is real} \implies & \tilde{\mathsf{f}}(-\omega) &=& \left[\tilde{\mathsf{f}}(\omega)\right]^* & \forall \omega\in\mathbb{R} \qquad \text{REALITY CONDITION}
\end{array}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(-x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149)}\\
&= \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(u)e^{i\omega u}(-1)\,\mathrm{d}u && \text{where } u \triangleq -x \implies \mathrm{d}x = -\,\mathrm{d}u\\
&= -\left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right]^*\\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149)}\\
\tilde{\mathsf{f}}(-\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i(-\omega)x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149)}\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(x)e^{-i\omega x}\,\mathrm{d}x\right]^*\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]^* && \text{by } \mathsf{f} \text{ is real hypothesis}\\
&\triangleq \tilde{\mathsf{f}}^*(\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149)}
\end{aligned}$$

J.3 Convolution

Definition J.3. [5]

DEF
*The **convolution operation** is defined as*
$$\left[\mathsf{f}\star\mathsf{g}\right](x) \triangleq \mathsf{f}(x)\star\mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem J.6 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem J.6 (convolution theorem). [6] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $\star$ the convolution operator.*

THM
$$\begin{array}{rcll}
\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} &=& \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} & \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}\\
\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} &=& \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} & \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.
\end{array}$$

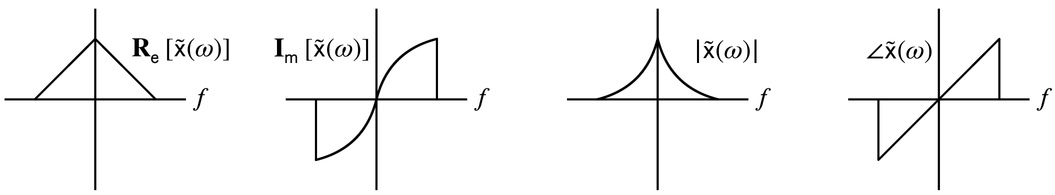
[5] Bachman (1964), page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩
[6] Bracewell (1978) page 110

✎PROOF:

$$
\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega) &= \tilde{\mathbf{F}}\left[\int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u\right](\omega) && \text{by definition of } \star \text{ (Definition J.3 page 152)}\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\tilde{\mathbf{F}}\mathsf{g}(x-u)\right](\omega)\,\mathrm{d}u \\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega)\,\mathrm{d}u && \text{by Theorem J.4 page 151}\\
&= \sqrt{2\pi}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right)}_{[\tilde{\mathbf{F}}\mathsf{f}](\omega)}\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)\\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition J.2 page 149)}\\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \tilde{\mathbf{F}}\left[\left(\tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathsf{f}(x)\right)\mathsf{g}(x)\right](\omega) && \text{by definition of operator inverse}\\
&= \tilde{\mathbf{F}}\left[\left(\frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)e^{ivx}\,\mathrm{d}v\right)\mathsf{g}(x)\right](\omega) && \text{by Theorem J.1 page 150}\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega,v)\,\mathrm{d}v\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-v)\,\mathrm{d}v && \text{by Theorem J.4 page 151}\\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \star \text{ (Definition J.3 page 152)}
\end{aligned}
$$

✏

J.4 Real valued functions



(symmetric) (anti-symmetric) (symmetric) (anti-symmetric)

Figure J.1: Fourier transform components of real-valued signal

Theorem J.7. *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM

$$
\left\{\begin{array}{l}\mathsf{f}(x) \textit{ is } \text{REAL-VALUED}\\ (\mathsf{f}\in\mathbb{R}^{\mathbb{R}})\end{array}\right\}
\implies
\left\{\begin{array}{rcll}
\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & \text{(HERMITIAN SYMMETRIC)}\\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(SYMMETRIC)}\\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(ANTI-SYMMETRIC)}\\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| & \text{(SYMMETRIC)}\\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & \text{(ANTI-SYMMETRIC).}
\end{array}\right\}
$$

✎PROOF:

$$
\begin{array}{lclclcl}
\tilde{\mathsf{f}}(\omega) &\triangleq& \left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) &\triangleq& \left\langle \mathsf{f}(x)\,|\,e^{i\omega x}\right\rangle = \left\langle \mathsf{f}(x)\,|\,e^{i(-\omega)x}\right\rangle^* &\triangleq& \tilde{\mathsf{f}}^*(-\omega)\\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right]\\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{I_m}\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right]\\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}^*(-\omega)| &=& |\tilde{\mathsf{f}}(-\omega)|\\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}^*(-\omega) &=& -\angle\tilde{\mathsf{f}}(-\omega)
\end{array}
$$

✏

J.5 Moment properties

Definition J.4. [7]

DEF *The quantity* M_n *is the n***th moment** *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\,\mathrm{d}x \qquad \text{for } n \in \mathbb{W}.$$

Lemma J.1. [8] *Let* M_n *be the n*TH MOMENT *(Definition J.4 page 154) and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM *(Definition J.2 page 149) of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ *(Definition F.1 page 101).*

LEM

$$\begin{aligned}
\mathsf{M}_n &= \sqrt{2\pi}(i)^n \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \\
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} &= \frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition J.2 page 149)} \\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\right]\mathrm{d}x\bigg|_{\omega=0} \\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right]\mathrm{d}x\bigg|_{\omega=0} \\
&= (-i^2)^n \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\
&\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \text{ (Definition J.4 page 154)}
\end{aligned}$$

✏

Lemma J.2. [9] *Let* M_n *be the n*TH MOMENT *(Definition J.4 page 154) and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM *(Definition J.2 page 149) of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ *(Definition F.1 page 101).*

LEM

$$\mathsf{M}_n = 0 \quad\iff\quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
0 &= \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by left hypothesis} \\
&= \sqrt{2\pi}(-i)^{-n} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by Lemma J.1 page 154} \\
&\Longrightarrow \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} = 0
\end{aligned}$$

[7] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83

[8] Goswami and Chan (1999), pages 38–39

[9] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathrm{f}}(\omega)\Big|_{\omega=0} && \text{by right hypothesis}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{f}(x)e^{-i\omega x}\,\mathrm{d}x\Big|_{\omega=0} && \text{by definition of } \tilde{\mathrm{f}}(\omega)\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{f}(x)\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\,\mathrm{d}x\Big|_{\omega=0}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{f}(x)\left[(-i)^n x^n e^{-i\omega x}\right]\,\mathrm{d}x\Big|_{\omega=0}\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{f}(x)x^n\,\mathrm{d}x\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\langle \mathrm{f}(x)\,|\,x^n\rangle && \text{by definition of } \langle\cdot\,|\,\cdot\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \text{ (Definition F.1 page 101)}
\end{aligned}$$

✏

Lemma J.3 (Strang-Fix condition). [10] *Let* $\mathrm{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* M_n *the* nTH MOMENT *(Definition J.4 page 154) of* $\mathrm{f}(x)$*. Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition A.3 page 60).*

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathrm{f}(x) = \mathsf{M}_n}_{\text{Strang-Fix condition in "time"}} \quad\Longleftrightarrow\quad \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathrm{f}}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k \mathsf{M}_n}_{\text{Strang-Fix condition in "frequency"}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathrm{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathrm{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}} \mathrm{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by Definition J.2 page 149}\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\int_{\mathbb{R}} \mathrm{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\int_{\mathbb{R}} x^n\mathrm{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}(x-k)^n\mathrm{f}(x-k)\bar{\delta}_k && \text{by } \textit{PSF} \text{ (Theorem A.2 page 68)}\\
&= \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k\mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n\mathsf{M}_n &= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[(-i)^n\bar{\delta}_k\mathsf{M}_n\right]e^{-i2\pi kx} && \text{by definition of } \bar{\delta}\\
&= \sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathrm{f}}(\omega)\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis}\\
&= \sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}} \mathrm{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx}
\end{aligned}$$

[10] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83, Mallat (1999), pages 241–243, Fix and Strang (1969)

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \left[\left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x\right]\right]_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]\right]_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } \textit{PSF} && \text{(Theorem A.2 page 68)}
\end{aligned}
$$

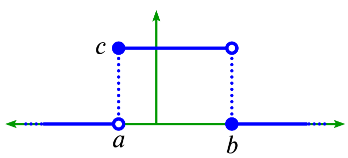
⇒

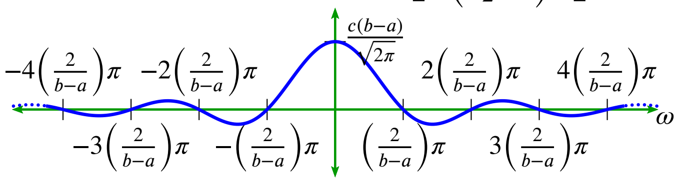
J.6 Examples

Example J.1 (rectangular pulse). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$
\mathsf{f}(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases}
\qquad
\tilde{\mathsf{f}}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right]
$$





✎PROOF:

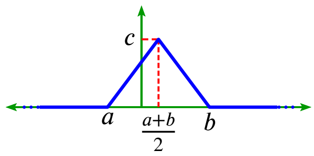
$$
\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[\mathsf{f}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by \textit{shift relation}} && \text{(Theorem J.4 page 151)} \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by definition of } \mathsf{f}(x) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} && \text{(Definition A.2 page 59)} \\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x) e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition J.2 page 149)} \\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} c e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} && \text{(Definition A.2 page 59)} \\
&= \frac{c}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \frac{1}{-i\omega} e^{-i\omega x}\Big|_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \\
&= \frac{2c}{\sqrt{2\pi}\omega} e^{-i\left(\frac{a+b}{2}\right)\omega} \left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i}\right] \\
&= \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right] && \text{by \textit{Euler formulas}} && \text{(Corollary G.2 page 113)}
\end{aligned}
$$

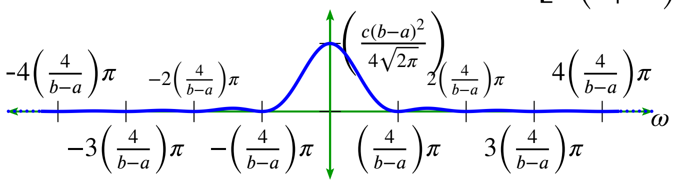
⇒

Example J.2 (triangle). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \begin{cases} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$





✎PROOF:

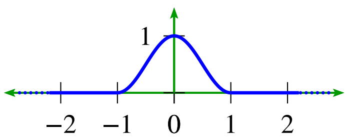
$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} \quad \text{(Theorem J.4 page 151)} \\
&= \tilde{\mathbf{F}}\left[c\left(1 - \frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) && \text{by definition of } \mathsf{f}(x) \\
&= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x) \star \mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega) \\
&= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] && \text{by } \textit{convolution theorem} \quad \text{(Theorem J.6 page 152)} \\
&= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2 \\
&= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2} - \frac{a}{2}\right)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 && \text{by } \textit{Rectangular pulse} \text{ ex.} \quad \text{Example J.1 page 156} \\
&= \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2
\end{aligned}$$

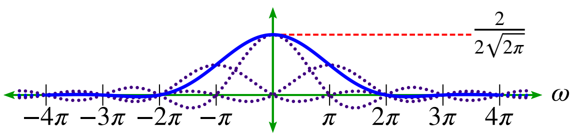
✎

Example J.3. Let a function f be defined in terms of the cosine function (Definition G.1 page 107) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$





✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition A.2 page 59) on a set A.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition J.2)} \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathsf{f}(x) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \text{ (Definition A.2)}
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left[\frac{e^{i\frac{\pi}{2}x} + e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\, \mathrm{d}x$$

by Corollary G.2 page 113

$$= \frac{1}{4\sqrt{2\pi}} \int_{-1}^{1} \left[2 + e^{i\pi x} + e^{-i\pi x}\right] e^{-i\omega x}\, \mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}} \int_{-1}^{1} 2e^{-i\omega x} + e^{-i(\omega+\pi)x} + e^{-i(\omega-\pi)x}\, \mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}} \left[2\frac{e^{-i\omega x}}{-i\omega} + \frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)} + \frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2\sqrt{2\pi}} \left[2\frac{e^{-i\omega} - e^{+i\omega}}{-2i\omega} + \frac{e^{-i(\omega+\pi)} - e^{+i(\omega+\pi)}}{-2i(\omega+\pi)} + \frac{e^{-i(\omega-\pi)} - e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$

✏

APPENDIX K

INTERPOLATION

K.1 Polynomial interpolation

Definition K.1. [1] *The **Lagrange polynomial** $L_{P,n}(x)$ with respect to the $n+1$ points $P = \{(x_k, y_k) | k = 0, 1, 2, \ldots, n\}$ is defined as*

DEF

$$L_{P,n}(x) \triangleq \sum_{k=0}^{n} y_k \prod_{m \neq n} \frac{x - x_m}{x_k - x_m}$$

Proposition K.1. *Let $L_{P,n}(x)$ be the Lagrange polynomial with respect to the points $P = \{(x_k, y_k) | k = 0, 1, 2, \ldots, n\}$.*

PRP

1. *$L_{P,n}(x)$ is an nth order polynomial.*
2. *$L_{P,n}(x)$ intersects all $n+1$ points in P.*

Example K.1 (Lagrange interpolation). The Lagrange polynomial $L_{P,3}(x)$ with respect to the 4 points $P = \{(-2, 1),\ (-1, 3),\ (3, 2),\ (5, 4)\}$ is

EX

$$L_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$\begin{aligned}
L_{P,3}(x) &= \sum_{k=0}^{n} y_k \prod_{m \neq n} \frac{x - x_m}{x_k - x_m} && \text{by Definition K.1} \\
&= y_0 \frac{(x+1)(x-3)(x-5)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{(x+2)(x-3)(x-5)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
&\quad + y_2 \frac{(x+2)(x+1)(x-5)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x+2)(x+1)(x-3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \\
&= 1 \frac{(x+1)(x-3)(x-5)}{(-2+1)(-2-3)(-2-5)} + 3 \frac{(x+2)(x-3)(x-5)}{(-1+2)(-1-3)(-1-5)} \\
&\quad + 2 \frac{(x+2)(x+1)(x-5)}{(3+2)(3+1)(3-5)} + 4 \frac{(x+2)(x+1)(x-3)}{(5+2)(5+1)(5-3)}
\end{aligned}$$

[1] Matthews and Fink (1992), page 206

$$
\begin{aligned}
&= 1\underbrace{\frac{x^3-7x^2+7x+15}{-35}}_{\text{roots}=-1,3,5} + 3\underbrace{\frac{x^3-6x^2-x+30}{24}}_{\text{roots}=-2,3,5} + 2\underbrace{\frac{x^3-2x^2-13x-10}{-40}}_{\text{roots}=-2,-1,5} + 4\underbrace{\frac{x^3-7x-6}{84}}_{\text{roots}=-2,-1,3}\\
&= -\frac{x^3-7x^2+7x+15}{35} + \frac{x^3-6x^2-x+30}{8} - \frac{x^3-2x^2-13x-10}{20} + \frac{x^3-7x-6}{21}\\
&= x^3\left(\frac{-8\cdot20\cdot21+35\cdot20\cdot21-35\cdot8\cdot21+35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right)\\
&\quad + x^2\left(\frac{7\cdot8\cdot20\cdot21-6\cdot35\cdot20\cdot21+2\cdot35\cdot8\cdot21+0\cdot35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right)\\
&\quad + x\left(\frac{-7\cdot8\cdot20\cdot21-35\cdot20\cdot21+13\cdot35\cdot8\cdot21-7\cdot35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right)\\
&\quad + \left(\frac{-15\cdot8\cdot20\cdot21+30\cdot35\cdot20\cdot21+10\cdot35\cdot8\cdot21-6\cdot35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right)\\
&= \frac{11060}{117600}x^3 + \frac{-52920}{117600}x^2 + \frac{-980}{117600}x + \frac{415800}{117600}\\
&= \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}
\end{aligned}
$$

✏

Definition K.2. [2] *The **Newton polynomial** $N_{P,n}(x)$ with respect to the $n+1$ points $P = \{(x_k, y_k)|k = 0, 1, 2, \ldots, n\}$ is defined as*

DEF
$$N_{P,n}(x) \triangleq \sum_{k=0}^{n} \alpha_k \prod_{m=0}^{k}(x - x_m)$$

Proposition K.2. *Let $N_{P,n}(x)$ be the Newton polynomial with respect to the points $P = \{(x_k, y_k)|k = 0, 1, 2, \ldots, n\}$.*

PRP
1. *$N_{P,n}(x)$ is an nth order polynomial.*
2. *$N_{P,n}(x)$ intersects all $n+1$ points in P.*

Example K.2 (Newton polynomial interpolation). The Newton polynomial $N_{P,3}(x)$ with respect to the 4 points
$P = \{(-2, 1),\ (-1, 3),\ (3, 2),\ (5, 4)\}$ is

EX
$$N_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$
\begin{aligned}
N_{P,3}(x) &= \sum_{k=0}^{n} \alpha_k \prod_{m=1}^{k}(x - x_m)\\
&= \alpha_0 + \alpha_1(x-x_0) + \alpha_2(x-x_0)(x-x_1) + \alpha_3(x-x_0)(x-x_1)(x-x_2)\\
&= \alpha_0 + \alpha_1(x+2) + \alpha_2(x+2)(x+1) + \alpha_3(x+2)(x+1)(x-3)\\
&= \alpha_0 + \alpha_1(x+2) + \alpha_2(x^2+3x+2) + \alpha_3(x^3-7x-6)\\
&= x^3(\alpha_3) + x^2(\alpha_2) + x(-7\alpha_3+3\alpha_2+\alpha_1) + (-6\alpha_3+2\alpha_2+2\alpha_1+\alpha_0)\\
&= \begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}\begin{bmatrix}1&0&0&0\\2&1&0&0\\2&3&1&0\\-6&-7&0&1\end{bmatrix}\begin{bmatrix}1\\x\\x^2\\x^3\end{bmatrix}
\end{aligned}
$$

[2] Matthews and Fink (1992), page 220

$$
\begin{aligned}
\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} &= \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & 0 \\ 1 & (x_3 - x_0) & (x_3 - x_0)(x_3 - x_1) & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (-1+2) & 0 & 0 \\ 1 & (3+2) & (3+2)(3+1) & 0 \\ 1 & (5+2) & (5+2)(5+1) & (5+2)(5+1)(5-3) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 \\ 1 & 7 & 42 & 84 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 & 0 & 0 & 1 & 0 \\ 1 & 7 & 42 & 84 & 0 & 0 & 0 & 1 \end{bmatrix}
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 5 & 20 & 0 & -1 & 0 & 1 & 0 \\ 0 & 7 & 42 & 84 & -1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 20 & 0 & 4 & -5 & 1 & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & 6 - \frac{42}{5} & -7 + \frac{42}{4} & -\frac{42}{20} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{12}{5} & \frac{14}{4} & -\frac{42}{20} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{24}{10} & \frac{35}{10} & -\frac{21}{10} & \frac{10}{10} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 1 & -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840} \end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0\\ -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840}\end{bmatrix}\begin{bmatrix}1\\ 3\\ 2\\ 4\end{bmatrix}$$

$$= \begin{bmatrix}1\\ 2\\ -\frac{9}{20}\\ \frac{79}{840}\end{bmatrix}$$

$$N_{P,3}(x) = \begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 2 & 3 & 1 & 0\\ -6 & -7 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ x\\ x^2\\ x^3\end{bmatrix}$$

$$= \begin{bmatrix}1 \mid 2 \mid -\frac{9}{20} \mid \frac{79}{840}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 2 & 3 & 1 & 0\\ -6 & -7 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ x\\ x^2\\ x^3\end{bmatrix}$$

$$= \begin{bmatrix}1+4-\frac{9}{10}-\frac{79}{140} \mid 2-\frac{27}{20}-\frac{79}{120} \mid -\frac{9}{20} \mid \frac{79}{840}\end{bmatrix}\begin{bmatrix}1\\ x\\ x^2\\ x^3\end{bmatrix}$$

$$= \frac{79}{840}x^3 - \frac{378}{840}x^2 - \frac{7}{840}x + \frac{2970}{840}$$

Example K.3 (Least squares polynomial interpolation). [3] The best 3rd order polynomial in the **least squares** $S_{P,3}(x)$ sense with respect to the 4 points
$P = \{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is

EX $\boxed{S_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}}$

✎PROOF:

$$\boldsymbol{x} = \begin{bmatrix}x_0\\ x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-2\\ -1\\ 3\\ 5\end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix}1\\ 3\\ 2\\ 4\end{bmatrix} = \begin{bmatrix}1\\ 3\\ 2\\ 4\end{bmatrix}$$

We want to find a third order polynomial

$$dx^3 + cx^2 + bx + a$$

that best approximates the 4 points in the least squares sense. We define the matrix U (known) and vector $\hat{\theta}$ (to be computed) as follows:

$$U^H \triangleq \begin{bmatrix}1 & x_0 & x_0^2 & x_0^3\\ 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 1 & x_3 & x_3^2 & x_3^3\end{bmatrix} \qquad \hat{\theta} \triangleq \begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}$$

[3] van Overschee and de Moor (2012) page 14 ⟨Table 1.1; historical perspective as relates to "subspace identification"⟩

EX

p	$(1-y)^p \mathsf{P}_m(y) = (1-y)^p \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k$
1	$1 - y$
2	$1 - 3y^2 + 2y^3$
3	$1 - 10y^3 + 15y^4 - 6y^5$
4	$1 - 35y^4 + 84y^5 - 70y^6 + 20y^7$
5	$1 - 126y^5 + 420y^6 - 540y^7 + 315y^8 - 70y^9$
6	$1 - 462y^6 + 1980y^7 - 3465y^8 + 3080y^9 - 1386y^{10} + 252y^{11}$

Table K.1: Low-pass term $(1-y)^p \mathsf{P}_m(y)$

Then, using *Least squares*, the best coefficients for the polynomial are

$$
\begin{aligned}
\hat{\theta} &= \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
&= R^{-1} W \\
&= (U U^H)^{-1} (U \boldsymbol{y}) \\
&= \left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}^H \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}^H \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix}^H \begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix}^H \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix}^H \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix}^H \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \right) \\
&= \begin{bmatrix} 2970 \\ -7 \\ -378 \\ 79 \end{bmatrix}
\end{aligned}
$$

K.2 Hermite interpolation

The quadrature condition can be expressed as a polynomial in $y = \sin^2\frac{\omega}{2}$. The first term in this polynomial quadrature condition is a low-pass response and the second term is a high pass; and they meet in the middle at $\omega = \frac{\pi}{2}$.

$$\underbrace{(1-y)^p \mathsf{P}(y)}_{\text{low-pass}} + \underbrace{y^p \mathsf{P}(1-y)}_{\text{high-pass}} = 1$$

The low-pass and high-pass terms are especially smooth at $\omega = 0$ ($y = 0$) and $\omega = \pi$ ($y = 1$) in that the first $p-1$ derivatives at both points are zero for both terms. This is illustrated in Figure K.1 (page

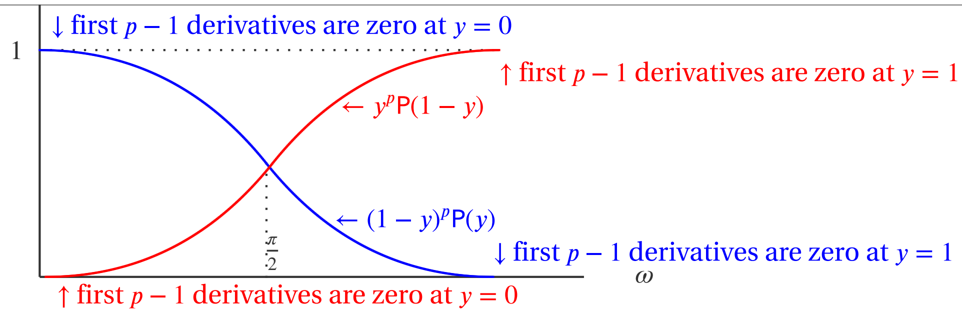


Figure K.1: Polynomial quadrature condition low-pass and high-pass terms

164).

Theorem K.1 (*Hermite Interpolation*).

THM

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}y^k\right]\Bigg|_{y=0} &= \bar{\delta}_n \quad \text{for } n=0,1,2,\ldots,p-1\\
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}y^k\right]\Bigg|_{y=1} &= 0 \quad \text{for } n=0,1,2,\ldots,p-1\\
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[y^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}(1-y)^k\right]\Bigg|_{y=0} &= 0 \quad \text{for } n=0,1,2,\ldots,p-1\\
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[y^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}(1-y)^k\right]\Bigg|_{y=1} &= \bar{\delta}_n \quad \text{for } n=0,1,2,\ldots,p-1
\end{aligned}$$

✎PROOF: Let

$$\begin{aligned}
\mathsf{f}(y) &\triangleq (1-y)^p\sum_{n=0}^{p-1}\binom{p-1+n}{n}y^n\\
\mathsf{g}(y) &\triangleq y^p\sum_{n=0}^{p-1}\binom{p-1+n}{n}(1-y)^n\\
q &\triangleq p-1
\end{aligned}$$

1. Proof that $\mathsf{f}(0)=1$:

$$\begin{aligned}
\mathsf{f}(0) &= (1-y)^p\sum_{m=0}^{p-1}\binom{p-1+m}{m}y^m\Bigg|_{y=0}\\
&= (1-y)^p\left[\binom{p-1}{0}+\sum_{m=1}^{p-1}\binom{p-1+m}{m}y^m\right]\Bigg|_{y=0}\\
&= 1
\end{aligned}$$

2. Proof that $\mathsf{f}(y) = p\sum_{n=0}^{2p-1}\left[\sum_{k=\max(0,n-q)}^{\min(n,p)}(-1)^k\frac{(p+n-k-1)!}{(p-k)!(n-k)!\,k!}\right]y^n$:

$$\begin{aligned}
(1-y)^p\mathsf{P}_m(y) &= \sum_{n=0}^{p}\binom{p}{n}(-1)^n y^n\sum_{m=0}^{p-1}\binom{p-1+m}{m}y^m \\
&= \sum_{n=0}^{2p-1}\sum_{k=\max(0,n-q)}^{\min(n,p)}\binom{p}{k}(-1)^k\binom{p-1+n-k}{n-k}y^n && \text{by Theorem D.2 page 82} \\
&= \sum_{n=0}^{2p-1}\sum_{k=\max(0,n-q)}^{\min(n,p)}(-1)^k\frac{p!}{(p-k)!k!}\frac{(p-1+n-k)!}{(p-1)!(n-k)!}y^n \\
&= p\sum_{n=0}^{2p-1}\left[\sum_{k=\max(0,n-q)}^{\min(n,p)}(-1)^k\frac{(p+n-k-1)!}{(p-k)!(n-k)!\,k!}\right]y^n
\end{aligned}$$

3. Proof that $\mathsf{f}^{(n)}(0) = \bar{\delta}_n$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\mathsf{P}_m(y)\right]\bigg|_{y=0} &= \frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[p\sum_{m=0}^{2p-1}\left[\sum_{k=\max(0,m-q)}^{\min(m,p)}(-1)^k\frac{(p+m-k-1)!}{(p-k)!(m-k)!\,k!}\right]y^m\right]\Bigg|_{y=0} && \text{by 1.} \\
&= p\sum_{m=n}^{2p-1}\sum_{k=\max(0,m-q)}^{\min(m,p)}(-1)^k\frac{(p-1+m-k)!}{(p-k)!(m-k)!k!}\frac{m!}{(m-n)!}y^{m-n}\bigg|_{y=0} \\
&= p\sum_{k=\max(0,n-q)}^{\min(n,p)}(-1)^k\frac{(p-1+n-k)!}{(p-k)!}\frac{n!}{(n-k)!k!} \\
&= p\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(p+n-k-1)!}{(p-k)!} \\
&\overset{?}{=} \bar{\delta}_n \qquad \text{for } n = 0, 1, 2, \ldots, p-1
\end{aligned}$$

4. Proof that $\mathsf{f}^{(n)}(0) = \bar{\delta}_n$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
&\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\mathsf{P}_m(y)\right]\bigg|_{y=0} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\mathrm{d}^{n-k}}{\mathrm{d}y^{n-k}}(1-y)^p\right]\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}\mathsf{P}_m(y)\right]\bigg|_{y=0} && \text{by Lemma F.2 (Leibnitz rule)} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\mathrm{d}^{n-k}}{\mathrm{d}y^{n-k}}(1-y)^p\right]\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}\sum_{m=0}^{p-1}\binom{p-1+m}{m}y^m\right]\Bigg|_{y=0} && \text{by definition of } \mathsf{P}_m(y) \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}(1-y)^{(p-n+k)}\right]\left[\sum_{m=k}^{p-1}\binom{p-1+m}{m}\frac{m!}{(m-k)!}y^{m-k}\right]\Bigg|_{y=0} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}\right]\left[\binom{p-1+k}{k}k!\right] \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}\right]\left[\frac{(p-1+k)!}{(p-1)!k!}\,k!\right] \\
&= (-1)^n p\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(p+k-1)!}{(p+k-n)!} \\
&\overset{?}{=} \bar{\delta}_n \qquad \text{for } k = 0, 1, 2, \ldots, p-1
\end{aligned}$$

5. Proof that $\mathsf{f}^{(n)}(1) = 0$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p \mathsf{P}_m(y)\right]\Big|_{y=1} &= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}(1-y)^p\right] \mathsf{P}_m^{(n-k)}(y)\Big|_{y=1} && \text{by Lemma F.2 (Leibnitz rule)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left[(-1)^k \frac{p!}{(p-k)!}(1-y)^{p-k}\right] \mathsf{P}_m^{(n-k)}(y)\Big|_{y=1} \\
&= \sum_{k=0}^{n} \binom{n}{k}\, 0 \cdot \mathsf{P}_m^{(n-k)}(y)\Big|_{y=1} && \text{by Lemma F.2} \\
&= 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1
\end{aligned}$$

APPENDIX L

SOURCE CODE

The source code in this appendix for *GNU Octave*. [1] Octave is similar to *MatLab* with some differences:

1. GNU Octave is free.
2. GNU Octave is open-source.
3. GNU Octave uses a separate graphics engine called *GNU-Plot* for all graphing.

Octave code can easily be adapted to MatLab code and vice-versa.

L.1 IIR filter code

```
//=========================================================================
//! Daniel J. Greenhoe
//! \brief DFII order 1 filter
//! \code{.markdown}
//!                               p      |\ a
//!    x[n] -->--(+)---->-------o-----| |---(+)-->-- y[n]
//!              |              |      |/     |
//!              |             \|/            |
//!              |            __|__           |
//!              |           | -1  |          |
//!             /|\          |z    |         /|\
//!              |           |_____|          |
//!              |              |             |
//!              |    -c /|     |      |\ b   |
//!              o-----| |--<--o-->--| |----o
//!                     \|   state     |/
//! \endcode
//=========================================================================
double df2_order1_filter(//! \return      Return state value
  const double a,        //! \param[in]  a      filter coefficient a
  const double b,        //! \param[in]  b      filter coefficient b
  const double c,        //! \param[in]  c      filter coefficient c
  const double state,    //! \param[in]  state  state of state-machine filter
  const float *x,        //! \param[in]  x      pointer to input  data
        float *y,        //! \param[out] y      pointer to output data
  const long   N         //! \param[in]  N      length of x, y;
  )
{
  long n;
```

[1] *GNU Octave*: http://www.octave.org/

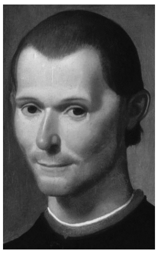
```
  double xn,yn;
  for(n=0; n<N; n++)
  {
    xn    = (double)x[n];     // convert float to double
    p     = xn  - c*state;
    yn    = a*p + b*state;
    y[n]  = (float)yn;        // convert double to float
    state = p;                // update state
  }
  return state;
}
```

L.2 IIR filter code

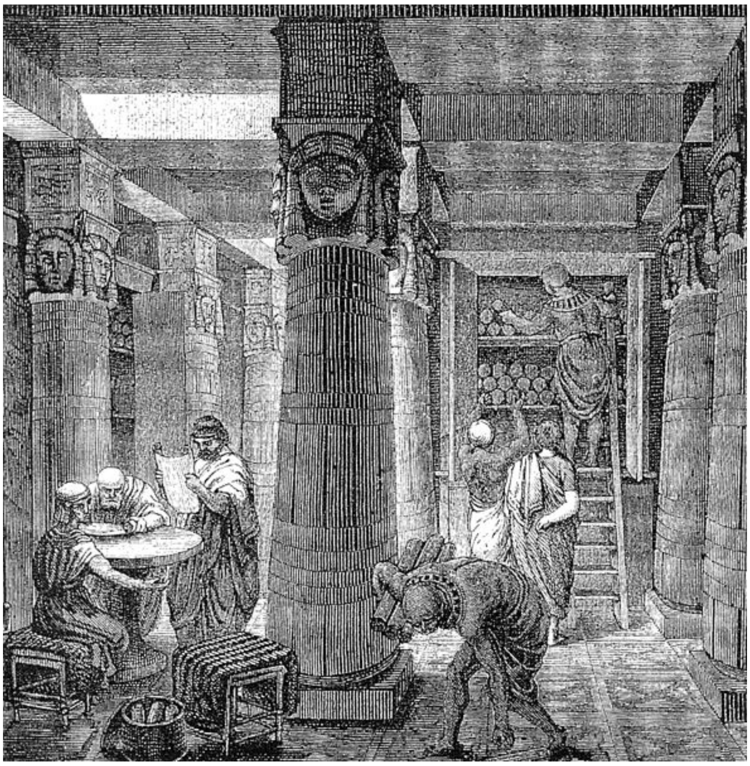
```
//=============================================================================
//! Daniel J. Greenhoe
//! \brief DFII order 1 filter
//! \code{.markdown}
//!                             p     |\b0
//!    x[n] -->--(+)---->-------o-----| |---(+)-->-- y[n]
//!               |             |     |/      |
//!               |            \|/            |
//!               |           __|__           |
//!               |          | -1  |          |
//!              /|\         |z    |         /|\
//!               |          |_____|          |
//!               |             |s1           |
//!               | -a1 /|      |     |\b1    |
//!              (+)---| |--<---o-->--| |---(+)
//!               |     \|      |     |/      |
//!               |            \|/            |
//!               |           __|__           |
//!               |          | -1  |          |
//!              /|\         |z    |         /|\
//!               |          |_____|          |
//!               |             |s2           |
//!               | -a2 /|      |     |\b2    |
//!               o----| |--<---o-->--| |----o
//!                     \|            |/
//! \endcode
//=============================================================================
void df2_order1_filter(//! \return     Return state value
  const double a1,       //! \param[in]  a1     filter coefficient a1
  const double a2,       //! \param[in]  a2     filter coefficient a2
  const double b0,       //! \param[in]  b0     filter coefficient b0
  const double b1,       //! \param[in]  b1     filter coefficient b1
  const double b2,       //! \param[in]  b2     filter coefficient b2
        double *s1,      //! \param[in]  state  state of state-machine filter
        double *s2,      //! \param[in]  state  state of state-machine filter
  const float  *x,       //! \param[in]  x      pointer to input  data
        float  *y,       //! \param[out] y      pointer to output data
  const long    N        //! \param[in]  N      length of x, y;
  )
{
  long n;
  double xn,yn;
  for(n=0; n<N; n++)
  {
    xn    = (double)x[n];       // convert float to double
    p     = xn  -a1*s1 -a2*s2;
    yn    = b0*p + b1*s1 + b2*s2;
    y[n]  = (float)y n;       // convert double to float
    *s2   = *s1;              // update state
    *s1   = p;                   // update state
  }
}
```

Back Matter

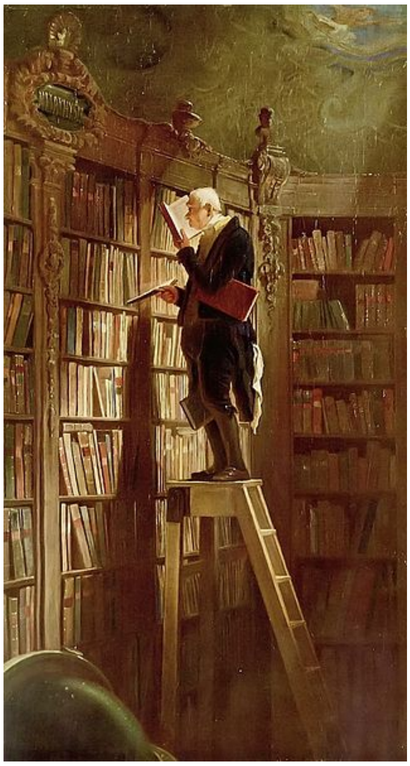
"It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils."
Niels Henrik Abel (1802–1829), Norwegian mathematician [2]



"When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world."
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [3]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850 [4]

"To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare."
Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [5]

2 quote: Simmons (2007), page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

3 quote: Machiavelli (1961), page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

4 http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

5 quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

BIBLIOGRAPHY

Yuri A. Abramovich and Charalambos D. Aliprantis. *An Invitation to Operator Theory*. American Mathematical Society, Providence, Rhode Island, 2002. ISBN 0-8218-2146-6. URL http://books.google.com/books?vid=ISBN0821821466.

Milton Abramowitz and Irene A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards, 1972. URL http://www.cs.bham.ac.uk/~aps/research/projects/as/book.php.

M.R. Adhikari and A. Adhikari. *Groups, Rings And Modules With Applications*. Universities Press, Hyderabad, 2 edition, 2003. ISBN 978-8173714290. URL http://books.google.com/books?vid=ISBN8173714290.

Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis*. Acedemic Press, London, 3 edition, 1998. ISBN 9780120502578. URL http://www.amazon.com/dp/0120502577.

George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*, volume 71 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, Cambridge, U.K., new edition, February 15 2001. ISBN 0521789885. URL http://books.google.com/books?vid=ISBN0521789885.

Tom M. Apostol. *Mathematical Analysis*. Addison-Wesley series in mathematics. Addison-Wesley, Reading, 2 edition, 1975. ISBN 986-154-103-9. URL http://books.google.com/books?vid=ISBN0201002884.

Claude Gaspard Bachet. *Arithmétiques de Diophante*. 1621. URL http://www.bsb-muenchen-digital.de/~web/web1008/bsb10081407/images/index.html?digID=bsb10081407.

George Bachman. *Elements of Abstract Harmonic Analysis*. Academic paperbacks. Academic Press, New York, 1964. URL http://books.google.com/books?id=ZP8-AAAAIAAJ.

George Bachman, Lawrence Narici, and Edward Beckenstein. *Fourier and Wavelet Analysis*. Universitext Series. Springer, 2000. ISBN 9780387988993. URL http://books.google.com/books?vid=ISBN0387988998.

Edward Barbeau. *Polynomials*. Problem Books in Mathematics. Springer, New York, 1989. ISBN 0-387-96919-5. URL http://books.google.com/books?vid=ISBN0387406271.

Adi Ben-Israel and Robert P. Gilbert. *Computer-supported calculus*. Texts and monographs in symbolic computation. Springer, 2002. ISBN 3-211-82924-5. URL http://books.google.com/books?vid=ISBN3211829245.

John Benedetto and Ahmed I. Zayed, editors. *A Prelude to Sampling, Wavelets, and Tomography*, pages 1–32. Applied and Numerical Harmonic Analysis. Springer, 2004. ISBN 9780817643041. URL http://books.google.com/books?vid=ISBN0817643044.

Sterling Khazag Berberian. *Introduction to Hilbert Space*. Oxford University Press, New York, 1961. URL http://books.google.com/books?vid=ISBN0821819127.

Etienne Bézout. *Théorie Générale des équations Algébriques*. 1779a. URL http://books.google.com.tw/books?id=RDEVAAAAQAAJ. (General Theory of Algebraic Equations).

Etienne Bézout. *General Theory of Algebraic Equations*. Princeton University Press, 1779b. ISBN 978-0691114323. URL http://books.google.com/books?vid=ISBN0691114323. 2006 translation of Théorie générale des équations algébriques (1779).

Ralph Philip Boas. *Entire Functions*, volume 5 of *Pure and Applied Mathematics. A Series of Monographs and Text Books*. Academic Press, 1954. ISBN 9780123745828. URL http://books.google.com/books?vid=ISBN0123745829.

Peter B. Borwein and Tamás Erdélyi. *Polynomials and Polynomial Inequalities*. Graduate Texts in Mathematics Series. Springer, 1995. ISBN 9780387945095. URL http://books.google.com/books?vid=ISBN0387945091.

Umberto Bottazzini. *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*. Springer-Verlag, New York, 1986. ISBN 0-387-96302-2. URL http://books.google.com/books?vid=ISBN0387963022.

Nicolas Bourbaki. *Algebra I*. Elements of Mathematics. Springer, 2003a. ISBN 3-540-64243-9. URL http://books.google.com/books?vid=ISBN3540642439.

Nicolas Bourbaki. *Algebra II*. Elements of Mathematics. Springer, 2003b. ISBN 978-3540007067. URL http://books.google.com/books?vid=ISBN3540007067.

Carl Benjamin Boyer and Uta C. Merzbach. *A History of Mathematics*. Wiley, New York, 2 edition, 1991. ISBN 0471543977. URL http://books.google.com/books?vid=ISBN0471543977.

Ronald Newbold Bracewell. *The Fourier transform and its applications*. McGraw-Hill electrical and electronic engineering series. McGraw-Hill, 2, illustrated, international student edition edition, 1978. ISBN 9780070070134. URL http://books.google.com/books?vid=ISBN007007013X.

Thomas John I'Anson Bromwich. *An Introduction to the Theory of Infinite Series*. Macmillan and Company, 1 edition, 1908. ISBN 9780821839768. URL http://www.archive.org/details/anintroductiont00bromgoog.

Andrew M. Bruckner, Judith B. Bruckner, and Brian S. Thomson. *Real Analysis*. Prentice-Hall, Upper Saddle River, N.J., 1997. ISBN 9780134588865. URL http://books.google.com/books?vid=ISBN013458886X.

James A. Cadzow. *Foundations of Digital Signal Processing and Data Analysis*. Macmillan Publishing Company, New York, 1987. ISBN 0023180102. URL http://frontweb.vuse.vanderbilt.edu/vuse_web/directory/facultybio.asp?FacultyID=9.

Florian Cajori. A history of mathematical notations; notations mainly in higher mathematics. In *A History of Mathematical Notations; Two Volumes Bound as One*, volume 2. Dover, Mineola, New York, USA, 1993. ISBN 0-486-67766-4. URL http://books.google.com/books?vid=ISBN0486677664. reprint of 1929 edition by *The Open Court Publishing Company*.

Gerolamo Cardano. *Ars Magna or the Rules of Algebra*. Dover Publications, Mineola, New York, 1545. ISBN 0486458733. URL http://www.amazon.com/dp/0486458733. English translation of the Latin *Ars Magna* edition, published in 2007.

Lennart Axel Edvard Carleson. Convergence and growth of partial sums of fourier series. *Acta Mathematica*, 116(1):135–157, 1966. doi: 10.1007/BF02392815. URL http://dx.doi.org/10.1007/BF02392815.

Lennart Axel Edvard Carleson and Björn Engquist. *After the 'golden age': what next? Lennart Carleson interviewed by Björn Engquist*, pages 455–461. Springer, Berlin, 2001. ISBN 978-3-540-66913-5. URL http://books.google.com/books?vid=ISBN3540669132.

Peter G. Casazza and Mark C. Lammers. *Bracket Products for Weyl-Heisenberg Frames*, pages 71–98. Applied and Numerical Harmonic Analysis. Birkhäuser, 1998. ISBN 9780817639594.

Clémente Ibarra Castanedo. *Quantitative subsurface defect evaluation by pulsed phase thermography: depth retrieval with the phase*. PhD thesis, Université Laval, October 2005. URL http://archimede.bibl.ulaval.ca/archimede/fichiers/23016/23016.html. Faculte' Des Sciences Et De Génie.

Augustin-Louis Cauchy. *Résumé des le/,cons données à l'Ecole Royale Polytechnique sur le calcul infinitésimal*, volume 1. Imprimerie royale, Paris, 1823. URL https://books.google.com/books?id=m8RRAAAAcAAJ. English translation of title (with assistance of Google Translate): “Summary of the lessons given to the polytechnic royal school on the infinitesimal calculus”; cross-reference https://www.encyclopediaofmath.org/index.php/Cauchy_integral.

Joan Cerdà. *Linear functional analysis*, volume 116 of *Graduate studies in mathematics*. American Mathematical Society, July 16 2010. ISBN 0821851152. URL http://books.google.com/books?vid=ISBN0821851152.

Lindsay Childs. *A Concrete Introduction To Higher Algebra*. Undergraduate texts in mathematics. Springer, New York, 3 edition, 2009. ISBN 978-0-387-74527-5. URL http://books.google.com/books?vid=ISBN0387745270.

Alexandre J. Chorin and Ole H. Hald. *Stochastic Tools in Mathematics and Science*, volume 1 of *Surveys and Tutorials in the Applied Mathematical Sciences*. Springer, New York, 2 edition, 2009. ISBN 978-1-4419-1001-1. URL http://books.google.com/books?vid=ISBN9781441910011.

Ole Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston/Basel/Berlin, 2003. ISBN 0-8176-4295-1. URL http://books.google.com/books?vid=ISBN0817642951.

Charles K. Chui. *An Introduction to Wavelets*. Academic Press, San Diego, California, USA, January 3 1992. ISBN 9780121745844. URL http://books.google.com/books?vid=ISBN0121745848.

Jon F. Claerbout. *Fundamentals of Geophysical Data Processing with Applications to Petroleum Prospecting*. International series in the earth and planetary sciences, Tab Mastering Electronics Series. McGraw-Hill, New York, 1976. ISBN 9780070111172. URL http://sep.stanford.edu/sep/prof/.

Paul M. Cohn. *Basic Algebra; Groups, Rings and Fields*. Springer, December 6 2002. ISBN 1852335874. URL http://books.google.com/books?vid=isbn1852335874.

J. L. Coolidge. The story of the binomial theorem. *The American Mathematical Monthly*, 56(3): 147–157, March 1949. URL http://www.jstor.org/stable/2305028.

A. Córdoba. Dirac combs. *Letters in Mathematical Physics*, 17(3):191–196, 1989. URL https://doi.org/10.1007/BF00401584. print ISSN 0377-9017 online ISSN 1573-0530.

Xingde Dai and David R. Larson. *Wandering vectors for unitary systems and orthogonal wavelets*. Number 640 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence R.I., July 1998. ISBN 0821808001. URL http://books.google.com/books?vid=ISBN0821808001.

Xingde Dai and Shijie Lu. Wavelets in subspaces. *Michigan Math. J.*, 43(1):81–98, 1996. doi: 10.1307/mmj/1029005391. URL http://projecteuclid.org/euclid.mmj/1029005391.

Ingrid Daubechies. *Ten Lectures on Wavelets*. Society for Industrial and Applied Mathematics, Philadelphia, 1992. ISBN 0-89871-274-2. URL http://www.amazon.com/dp/0898712742.

Kenneth R. Davidson and Allan P. Donsig. *Real Analysis and Applications*. Springer, 2010. ISBN 9781441900050. URL http://books.google.com/books?vid=ISBN1441900055.

Charles Jean de la Vallée-Poussin. Sur l'intégrale de lebesgue. *Transactions of the American Mathematical Society*, 16(4):435–501, October 1915. URL http://www.jstor.org/stable/1988879.

Juan-Arias de Reyna. *Pointwise Convergence of Fourier Series*, volume 1785 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin/Heidelberg/New York, 2002. ISBN 3540432701. URL http://books.google.com/books?vid=ISBN3540432701.

René Descartes. *La géométrie*. 1637a. URL http://historical.library.cornell.edu/math/math_D.html.

René Descartes. *Discours de la méthode pour bien conduire sa raison, et chercher la verite' dans les sciences*. Jan Maire, Leiden, 1637b. URL http://www.gutenberg.org/etext/13846.

René Descartes. *Discourse on the Method of Rightly Conducting the Reason in the Search for Truth in the Sciences*. 1637c. URL http://www.gutenberg.org/etext/59.

René Descartes. *Regulae ad directionem ingenii*. 1684a. URL http://www.fh-augsburg.de/~harsch/Chronologia/Lspost17/Descartes/des_re00.html.

René Descartes. *Rules for Direction of the Mind*. 1684b. URL http://en.wikisource.org/wiki/Rules_for_the_Direction_of_the_Mind.

René Descartes. *The Geometry of Rene Descartes*. Courier Dover Publications, June 1 1954. ISBN 0486600688. URL http://books.google.com/books?vid=isbn0486600688. orginally published by Open Court Publishing, Chicago, 1925; translation of La géométrie.

Johann Peter Gustav Lejeune Dirichlet. Sur la convergence des séries trigonométriques qui servent représenter une fonction arbitraire entre des limites données. *Journal für die reine und angewandte Mathematik*, 4(2):157–169, 1829a. URL http://arxiv.org/abs/0806.1294.

Johann Peter Gustav Lejeune Dirichlet. *Sur la convergence des séries trigonométriques qui servent représenter une fonction arbitraire entre des limites données.*, pages 117–132. 1829b.

Bogdan Dumitrescu. *Positive Trigonometric Polynomials and Signal Processing Applications*. Signals and Communication Technology. Springer, 2007. ISBN 978-1-4020-5124-1. URL gen.lib.rus.ec/get?md5=5346e169091b2d928d8333cd053300f9.

John R. Durbin. *Modern Algebra; An Introduction*. John Wiley & Sons, Inc., 4 edition, 2000. ISBN 0-471-32147-8. URL http://www.worldcat.org/isbn/0471321478.

Euclid. *Elements*. circa 300BC. URL http://farside.ph.utexas.edu/euclid.html.

Leonhard Euler. *Introductio in analysin infinitorum*, volume 1. Marcum-Michaelem Bousquet & Socios, Lausannæ, 1748. URL http://www.math.dartmouth.edu/~euler/pages/E101.html. Introduction to the Analysis of the Infinite.

Leonhard Euler. *Introduction to the Analysis of the Infinite*. Springer, 1988. ISBN 0387968245. URL http://books.google.com/books?vid=ISBN0387968245. translation of 1748 Introductio in analysin infinitorum.

David Ewen. *The Book of Modern Composers*. Alfred A. Knopf, New York, 1950. URL http://books.google.com/books?id=yHw4AAAAIAAJ.

David Ewen. *The New Book of Modern Composers*. Alfred A. Knopf, New York, 3 edition, 1961. URL http://books.google.com/books?id=bZIaAAAAMAAJ.

Lorenzo Farina and Sergio Rinaldi. *Positive Linear Systems: Theory and Applications*. Pure and applied mathematics. John Wiley & Sons, 1 edition, July 3 2000. ISBN 9780471384564. URL http://books.google.com/books?vid=ISBN0471384569.

G.L. Fix and G. Strang. Fourier analysis of the finite element method in ritz-galerkin theory. *Studies in Applied Mathematics*, 48:265–273, 1969.

Harley Flanders. Differentiation under the integral sign. *The American Mathematical Monthly*, 80(6):615–627, June–July 1973. URL http://sgpwe.izt.uam.mx/files/users/uami/jdf/proyectos/Derivar_inetegral.pdf. http://www.jstor.org/pss/2319163.

Francis J. Flanigan. *Complex Variables; Harmonic and Analytic Functions*. Dover, New York, 1983. ISBN 9780486613888. URL http://books.google.com/books?vid=ISBN0486613887.

Gerald B. Folland. *Fourier Analysis and its Applications*. Wadsworth & Brooks / Cole Advanced Books & Software, Pacific Grove, California, USA, 1992. ISBN 0-534-17094-3. URL http://www.worldcat.org/isbn/0534170943.

Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1995. ISBN 0-8493-8490-7. URL http://books.google.com/books?vid=ISBN0849384907.

Brigitte Forster and Peter Massopust, editors. *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*. Applied and Numerical Harmonic Analysis. Springer, November 19 2009. ISBN 9780817648909. URL http://books.google.com/books?vid=ISBN0817648909.

Jean-Baptiste-Joseph Fourier. Mémoire sur la propagation de la chaleur dans les corps solides (dissertation on the propagation of heat in solid bodies). In M. Gaston Darboux, editor, *Œuvres De Fourier*, pages 215–221. Ministère de L'instruction Publique, Paris, France, 2 edition, December 21 1807. URL http://gallica.bnf.fr/ark:/12148/bpt6k33707/f220n7.

Jean-Baptiste-Joseph Fourier. *Théorie Analytique de la Chaleur (The Analytical Theory of Heat)*. Chez Firmin Didot, pere et fils, Paris, 1822. URL http://books.google.com/books?vid=04X2vlqZx7hydlQUWEq&id=TDQJAAAAIAAJ.

Jean-Baptiste-Joseph Fourier. *The Analytical Theory of Heat (Théorie Analytique de la Chaleur)*. Cambridge University Press, Cambridge, February 20 1878. URL http://www.archive.org/details/analyticaltheory00fourrich. 1878 English translation of the original 1822 French edition. A 2003 Dover edition is also available: isbn 0486495310.

Jürgen Fuchs. *Affine Lie Algebras and Quantum Groups: An Introduction, With Applications in Conformal Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1995. ISBN 052148412X. URL http://books.google.com/books?vid=ISBN052148412X.

Paul Abraham Fuhrmann. *A Polynomial Approach to Linear Algebra*. Springer Science+Business Media, LLC, 2 edition, 2012. ISBN 978-1461403371. URL http://books.google.com/books?vid=ISBN1461403375.

Dennis Gabor. Theory of communication. *Journal of the Institution of Electrical Engineers*, 93(26): 429–457, November 1946. URL http://bigwww.epfl.ch/chaudhury/gabor.pdf.

Carl Friedrich Gauss. *Carl Friedrich Gauss Werke*, volume 8. Königlichen Gesellschaft der Wissenschaften, B.G. Teubneur In Leipzig, Göttingen, 1900. URL http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN236010751.

Israel M. Gelfand. Normierte ringe. *Mat. Sbornik*, 9(51):3–24, 1941.

Israel M. Gelfand and Mark A. Naimark. *Normed Rings with an Involution and their Representations*, pages 240–274. Chelsea Publishing Company, Bronx, 1964. ISBN 0821820222. URL http://books.google.com/books?vid=ISBN0821820222.

Israel M. Gelfand and Mark A. Neumark. On the imbedding of normed rings into the ring of operators in hilbert space. *Mat. Sbornik*, 12(54:2):197–217, 1943a.

Israel M. Gelfand and Mark A. Neumark. *On the imbedding of normed rings into the ring of operators in Hilbert Space*, pages 3–19. 1943b. ISBN 0821851756. URL http://books.google.com/books?vid=ISBN0821851756.

T. N. T. Goodman, S. L. Lee, and W. S. Tang. Wavelets in wandering subspaces. *Transactions of the A.M.S.*, 338(2):639–654, August 1993a. URL http://www.jstor.org/stable/2154421. Transactions of the American Mathematical Society.

T. N. T. Goodman, S. L. Lee, and W. S. Tang. Wavelets in wandering subspaces. *Advances in Computational Mathematics 1*, pages 109–126, February 1993b.

Jaideva C. Goswami and Andrew K. Chan. *Fundamentals of Wavelets; Theory, Algorithms, and Applications*. John Wiley & Sons, Inc., 1999. ISBN 0-471-19748-3. URL http://vadkudr.boom.ru/Collection/fundwave_contents.html.

Ronald L. Graham, Donald Ervin Knuth, and Oren Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, 2 edition, 1994. ISBN 0201558025. URL http://books.google.com/books?vid=ISBN0201558025.

Ernst Adolph Guillemin. *Synthesis of Passive Networks: Theory and Methods Appropriate to the Realization and Approximation Problems*. John Wiley & Sons, 1957. ISBN 9780882754819. URL http://books.google.com.tw/books?id=JQ4nAAAAMAAJ.

Paul R. Haddad and Ali N. Akansu. *Multiresolution Signal Decomposition: Transforms, Subbands, and Wavelets*. Acedemic Press, October 1 1992. ISBN 0323138365. URL http://books.google.com/books?vid=ISBN0323138365.

Paul R. Halmos. *Intoduction to Hilbert Space and the Theory of Spectral Multiplicity*. Chelsea Publishing Company, New York, 2 edition, 1998. ISBN 0821813781. URL http://books.google.com/books?vid=ISBN0821813781.

Godfrey H. Hardy. *A Mathematician's Apology*. Cambridge University Press, Cambridge, 1940. URL http://www.math.ualberta.ca/~mss/misc/A%20Mathematician's%20Apology.pdf.

Godfrey Harold Hardy. Notes on special systems of orthogonal functions (iv): the orthogonal functions of whittaker's cardinal series. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37(4):331–348, October 1941. URL http://dx.doi.org/10.1017/S0305004100017977.

Felix Hausdorff. *Set Theory*. Chelsea Publishing Company, New York, 3 edition, 1937. ISBN 0828401195. URL http://books.google.com/books?vid=ISBN0828401195. 1957 translation of the 1937 German *Grundzüge der Mengenlehre*.

Michiel Hazewinkel, editor. *Handbook of Algebras*, volume 2. North-Holland, Amsterdam, 1 edition, 2000. ISBN 044450396X. URL http://books.google.com/books?vid=ISBN044450396X.

Jean Van Heijenoort. *From Frege to Gödel : A Source Book in Mathematical Logic, 1879-1931*. Harvard University Press, Cambridge, Massachusetts, 1967. URL http://www.hup.harvard.edu/catalog/VANFGX.html.

Christopher Heil. *A Basis Theory Primer*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, expanded edition edition, 2011. ISBN 9780817646868. URL http://books.google.com/books?vid=ISBN9780817646868.

Christopher E. Heil and David F. Walnut. Continuous and discrete wavelet transforms. *Society for Industrial and Applied Mathematics*, 31(4), December 1989. URL http://citeseer.ist.psu.edu/viewdoc/download?doi=10.1.1.132.1241&rep=rep1&type=pdf.

Charles Hermite. Lettre à stieltjes. In B. Baillaud and H. Bourget, editors, *Correspondance d'Hermite et de Stieltjes*, volume 2, pages 317–319. Gauthier-Villars, Paris, May 20 1893. published in 1905.

J. R. Higgins. Five short stories about the cardinal series. *Bulletin of the American Mathematical Society*, 12(1):45–89, 1985. URL http://www.ams.org/journals/bull/1985-12-01/S0273-0979-1985-15293-0/.

John Rowland Higgins. *Sampling Theory in Fourier and Signal Analysis: Foundations*. Oxford Science Publications. Oxford University Press, August 1 1996. ISBN 9780198596998. URL http://books.google.com/books?vid=ISBN0198596995.

Alfred Edward Housman. *More Poems*. Alfred A. Knopf, 1936. URL http://books.google.com/books?id=rTMiAAAAMAAJ.

Emmanuel C. Ifeachor and Barrie W. Jervis. *Digital Signal Processing: A Practical Approach*. Electronic systems engineering series. Prentice Hall, 1993. ISBN 020154413X. URL http://www.amazon.com/dp/020154413X.

Emmanuel C. Ifeachor and Barrie W. Jervis. *Digital Signal Processing: A Practical Approach*. Electronic systems engineering series. Prentice Hall, 2002. ISBN 9780201596199. URL http://www.amazon.com/dp/0201596199.

Robert J. Marks II. *Introduction to Shannon Sampling and Interpolation Theory*. Springer-Verlag, New York, 1991. ISBN 0-387-97391-5,3-540-97391-5. URL http://marksmannet.com/RobertMarks/REPRINTS/1999_IntroductionToShannonSamplingAndInterpolationTheory.pdf.

Vasile I. Istrăţescu. *Inner Product Structures: Theory and Applications*. Mathematics and Its Applications. D. Reidel Publishing Company, 1987. ISBN 9789027721822. URL http://books.google.com/books?vid=ISBN9027721823.

Hans Niels Jahnke. *A History of Analysis*, volume 24 of *History of mathematics*. American Mathematical Society, Providence, RI, 2003. ISBN 0821826239. URL http://books.google.com/books?vid=ISBN0821826239.

A. J. E. M. Janssen. The zak transform: A signal transform for sampled time-continuous signals. *Philips Journal of Research*, 43(1):23–69, 1988.

Bjorn Jawerth and Wim Sweldens. An overview of wavelet based multiresolutional analysis. *SIAM Review*, 36:377–412, September 1994. URL http://cm.bell-labs.com/who/wim/papers/papers.html#overview.

Alan Jeffrey and Hui Hui Dai. *Handbook of Mathematical Formulas and Integrals*. Handbook of Mathematical Formulas and Integrals Series. Academic Press, 4 edition, January 18 2008. ISBN 9780080556840. URL http://books.google.com/books?vid=ISBN0080556841.

K. D. Joshi. *Applied Discrete Structures*. New Age International, New Delhi, 1997. ISBN 8122408265. URL http://books.google.com/books?vid=ISBN8122408265.

J.S.Chitode. *Signals And Systems*. Technical Publications, 2009. ISBN 9788184316780. URL http://books.google.com/books?vid=ISBN818431678X.

Jean-Pierre Kahane. Partial differential equations, trigonometric series, and the concept of function around 1800: a brief story about lagrange and fourier. In Dorina Mitrea and V. G. Mazía, editors, *Perspectives In Partial Differential Equations, Harmonic Analysis And Applications: a Volume in Honor of Vladimir G. Maz'ya's 70th birthday*, volume 79 of *Proceedings of Symposia in Pure Mathematics*, pages 187–206. American Mathematical Society, 2008. ISBN 0821844245. URL http://books.google.com/books?vid=ISBN0821844245.

David W. Kammler. *A First Course in Fourier Analysis*. Cambridge University Press, 2 edition, 2008. ISBN 9780521883405. URL http://books.google.com/books?vid=ISBN0521883407.

Edward Kasner and James Roy Newman. *Mathematics and the Imagination*. Simon and Schuster, 1940. ISBN 0486417034. URL http://books.google.com/books?vid=ISBN0486417034. “unabridged and unaltered republication” available from Dover.

Yitzhak Katznelson. *An Introduction to Harmonic Analysis*. Cambridge mathematical library. Cambridge University Press, 3 edition, 2004. ISBN 0521543592. URL http://books.google.com/books?vid=ISBN0521543592.

Yoshida Kenko. *The Tsuredzure Gusa of Yoshida No Kaneyoshi. Being the meditations of a recluse in the 14th Century (Essays in Idleness)*. circa 1330. URL http://www.humanistictexts.org/kenko.htm. 1911 translation of circa 1330 text.

Anthony W Knapp. *Basic Real Analysis*. Cornerstones. Birkhäuser, Boston, Massachusetts, USA, 1 edition, July 29 2005. ISBN 0817632506. URL http://books.google.com/books?vid=ISBN0817632506.

Granino A. Korn and Theresea M. Korn. *Mathematical Handbook for Scientists and Engineers; Definitions, Theorems, and Formulas for Reference and Review*. 2 edition, 1968. ISBN 0486411478. URL http://books.google.com/books?vid=ISBN0486411478.

Vladimir Aleksandrovich Kotelnikov. On the transmission capacity of the 'ether' and of cables in electrical communications. In *Proceedings of the first All-Union Conference on the technological reconstruction of the communications sector and the development of low-current engineering*, Moscow, 1933. URL http://ict.open.ac.uk/classics/1.pdf.

Carlos S. Kubrusly. *The Elements of Operator Theory*. Springer, 2 edition, 2011. ISBN 9780817649975. URL http://books.google.com/books?vid=ISBN0817649972.

Joseph-Louis Lagrange, Pierre-Simon Laplace, Étienne Louis Malus, René Just Haüy, and Adrien-Marie Legendre. Proclamation des prix décernés dans la séance publique de 6 janvier 1812. *Esprit des Journaux, Francais et étrangers par Une Societe de Gens Delettres*, 2:111–112, January 6 1812a. URL http://books.google.com/books?id=QpUUAAAAQAAJ.

Joseph-Louis Lagrange, Pierre-Simon Laplace, Étienne Louis Malus, René Just Haüy, and Adrien-Marie Legendre. Proclamation des prix décernés dans la séance publique de 6 janvier 1812. *Mercure De France, Journal Littéraire et Politique*, 50:374–375, January 6 1812b. URL http://books.google.com/books?id=8HxBAAAAcAAJ.

Imre Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge University Press, Cambridge, 1976. ISBN 0521290384. URL http://books.google.com/books?vid=ISBN0521290384.

Traian Lalescu. *Sur les équations de Volterra*. PhD thesis, University of Paris, 1908. advisor was Émile Picard.

Traian Lalescu. *Introduction à la théorie des équations intégrales (Introduction to the Theory of Integral Equations)*. Librairie Scientifique A. Hermann, Paris, 1911. URL http://www.worldcat.org/oclc/1278521. first book about integral equations ever published.

Rupert Lasser. *Introduction to Fourier Series*, volume 199 of *Monographs and textbooks in pure and applied mathematics*. Marcel Dekker, New York, New York, USA, February 8 1996. ISBN 978-0824796105. URL http://books.google.com/books?vid=ISBN0824796101. QA404.L33 1996.

Peter D. Lax. *Functional Analysis*. John Wiley & Sons Inc., USA, 2002. ISBN 0-471-55604-1. URL http://www.worldcat.org/isbn/0471556041. QA320.L345 2002.

Henri Lebesgue. Intégrale, longueur, aire (integral, length, area). *Annali Di Matematica Pura Ed Applicata*, 7:231–359, 1902. ISSN 0373-3114. URL http://www.emis.de/cgi-bin/JFM-item?33.0307.02. publication of Lebesgue's 1902 doctoral dissertation.

Henri Lebesgue. *Introduction, Intégration et dérivation*, volume 1 of *Œuvres scientifiques de Henri Lebesgue*, chapter Intégrale, Longueur, Aire, pages 201–331. L'Enseignement Mathématique, Institut de mathématiques, Université Genève, Genève, Switzerland, 1972. URL http://www.worldcatlibraries.org/oclc/789977. publication of Lebesgue's 1902 doctoral dissertation.

Gottfried W. Leibniz. Symbolismus memorabilis calculi algebraici et infinitesimalis, in comparatione potentiarum et differentiarum; et de lege homogeneorum transcendentali. *Miscellanea Berolinensia ad incrementum scientiarum, ex scriptis Societati Regiae scientarum*, pages 160–165, 1710. URL http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/anzeige/index_html?band=01-misc/1&seite:int=184.

J. Liouville. Sur l'integration d'une classe d'équations différentielles du second ordre en quantités finies explicites. *Journal De Mathematiques Pures Et Appliquees*, 4:423–456, 1839. URL http://gallica.bnf.fr/ark:/12148/bpt6k16383z.

Lynn H. Loomis and Ethan D. Bolker. *Harmonic analysis*. Mathematical Association of America, 1965. URL http://books.google.com/books?id=MEfvAAAAMAAJ.

Nikolai Luzin. Sur la convergence des séries trigonom etriers de fourier. *C. R. Acad. Sci.*, 156:1655–1658, 1913.

Niccolò Machiavelli. *The Literary Works of Machiavelli: Mandragola, Clizia, A Dialogue on Language, and Belfagor, with Selections from the Private Correspondence*. Oxford University Press, 1961. ISBN 0313212481. URL http://www.worldcat.org/isbn/0313212481.

Colin Maclaurin. *Treatise of Fluxions*. W. Baynes, 1742. URL http://www.amazon.com/dp/B000863E7M.

Stéphane G. Mallat. *A Wavelet Tour of Signal Processing*. Elsevier, 2 edition, September 15 1999. ISBN 0-12-466606-X. URL http://books.google.com/books?vid=ISBN012466606X.

John H. Matthews and Kurtis D. Fink. *Numerical Methods Using MatLab 3rd edition*. Prentice-Hall, Inc., Upper River, NJ, 1992. ISBN 0-13-013164-4.

maxima. *Maxima Manual version 5.28.0*. 5.28.0 edition. URL http://maxima.sourceforge.net/documentation.html.

Stefan Mazur. Sur les anneaux linéaires. *Comptes rendus de l'Académie des sciences*, 207:1025–1027, 1938.

William Henry Metzler, Edward Drake Roe, and Warren Gardner Bullard. *College Algebra*. Longsmans, Green, & Co., New York, 1908. URL http://archive.org/details/collegealgebra00metzrich.

Anthony N. Michel and Charles J. Herget. *Applied Algebra and Functional Analysis*. Dover Publications, Inc., 1993. ISBN 0-486-67598-X. URL http://books.google.com/books?vid=ISBN048667598X. original version published by Prentice-Hall in 1981.

Fred Mintzer. Filters for distortion-free two-band multi-rate filter banks. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 32, 1985.

Eddie Ortiz Muniz. A method for deriving various formulas in electrostatics and electromagnetism using lagrange's trigonometric identities. *American Journal of Physics*, 21(140), 1953. doi: 10.1119/1.1933371. URL http://dx.doi.org/10.1119/1.1933371.

M. Zuhair Nashed and Gilbert G. Walter. General sampling theorems for functions in reproducing kernel hilbert spaces. *Mathematics of Control, Signals and Systems*, 4(4):363–390, 1991. URL http://link.springer.com/article/10.1007/BF02570568.

Alan V. Oppenheim and Ronald W. Schafer. *Discrete-Time Signal Processing*. Prentice Hall, 2 edition, 1999. ISBN 9780137549207. URL http://www.amazon.com/dp/0137549202.

Judith Packer. Applications of the work of stone and von neumann to wavelets. In Robert S. Doran and Richard V. Kadison, editors, *Operator Algebras, Quantization, and Noncommutative Geometry: A Centennial Celebration Honoring John Von Neumann and Marshall H. Stone : AMS Special Session on Operator Algebras, Quantization, and Noncommutative Geometry, a Centennial Celebration Honoring John Von Neumann and Marshall H. Stone, January 15-16, 2003, Baltimore,* *Maryland*, volume 365 of *Contemporary mathematics—American Mathematical Society*, pages 253–280, Baltimore, Maryland, 2004. American Mathematical Society. ISBN 9780821834022. URL http://books.google.com/books?vid=isbn0821834029.

Lincoln P. Paine. *Warships of the World to 1900*. Ships of the World Series. Houghton Mifflin Harcourt, 2000. ISBN 9780395984147. URL http://books.google.com/books?vid=ISBN9780395984149.

Athanasios Papoulis. *Circuits and Systems: A Modern Approach*. HRW series in electrical and computer engineering. Holt, Rinehart, and Winston, 1980. ISBN 9780030560972. URL http://books.google.com/books?vid=ISBN0030560977.

Michael Pedersen. *Functional Analysis in Applied Mathematics and Engineering*. Chapman & Hall/CRC, New York, 2000. ISBN 9780849371691. URL http://books.google.com/books?vid=ISBN0849371694. Library QA320.P394 1999.

Mark A. Pinsky. *Introduction to Fourier Analysis and Wavelets*. Brooks/Cole, Pacific Grove, 2002. ISBN 0-534-37660-6. URL http://www.amazon.com/dp/0534376606.

Lakshman Prasad and Sundararaja S. Iyengar. *Wavelet Analysis with Applications to Image Processing*. CRC Press LLC, Boca Raton, 1997. ISBN 978-0849331695. URL http://books.google.com/books?vid=ISBN0849331692. Library TA1637.P7 1997.

Victor V. Prasolov. *Polynomials*, volume 11 of *Algorithms and Computation in Mathematics*. Springer, 2004. ISBN 978-3-540-40714-0. URL http://books.google.com/books?vid=ISBN3540407146. translated from Russian (2001, 2nd edition).

Ptolemy. *Ptolemy's Almagest*. Springer-Verlag (1984), New York, circa 100AD. ISBN 0387912207. URL http://gallica.bnf.fr/ark:/12148/bpt6k3974x.

Shie Qian and Dapang Chen. *Joint time-frequency analysis: methods and applications*. PTR Prentice Hall, 1996. ISBN 9780132543842. URL http://books.google.com/books?vid=ISBN0132543842.

Charles Earl Rickart. *General Theory of Banach Algebras*. University series in higher mathematics. D. Van Nostrand Company, Yale University, 1960. URL http://books.google.com/books?id=PVrvAAAAMAAJ.

Bernard Riemann. Ueber die darstellbarkeit einer function durch eine trigonometrische reihe. 13?, 1854. URL http://www.emis.de/classics/Riemann/.

Theodore J. Rivlin. *The Chebyshev Polynomials*. Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts. John Wiley & Sons, New York, 1974. ISBN 0-471-72470-X. URL http://books.google.com/books?vid=ISBN047172470X.

Enders A. Robinson. *Random Wavelets and Cybernetic Systems*, volume 9 of *Griffins Statistical Monographs & Courses*. Lubrecht & Cramer Limited, London, June 1962. ISBN 0852640757. URL http://books.google.com/books?vid=ISBN0852640757.

Enders A. Robinson. Multichannel z-transforms and minimum delay. *Geophysics*, 31(3):482–500, June 1966. doi: 10.1190/1.1439788. URL http://dx.doi.org/10.1190/1.1439788.

Maxwell Rosenlicht. *Introduction to Analysis*. Dover Publications, New York, 1968. ISBN 0-486-65038-3. URL http://books.google.com/books?vid=ISBN0486650383.

Joseph J. Rotman. *Advanced Modern Algebra*, volume 114 of *Graduate studies in mathematics*. American Mathematical Society, 2 edition, 2010. ISBN 978-0-8218-4741-1. URL http://books.google.com/books?vid=ISBN0821847414.

Walter Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, New York, New York, USA, 3 edition, 1987. ISBN 9780070542341. URL http://www.amazon.com/dp/0070542341. Library QA300.R8 1976.

Gert Schubring. *Conflicts Between Generalization, Rigor, and Intuition: Number Concepts Underlying the Development of Analysis in 17th–19th Century France and Germany*. Sources and studies in the history of mathematics and physical sciences. Springer, New York, 1 edition, June 2005. ISBN 0387228365. URL http://books.google.com/books?vid=ISBN0387228365.

Atle Selberg. Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to dirichlet series. *Journal of the Indian Mathematical Society*, 20:47–87, 1956.

Claude E. Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27:379–343,623–656, July,October 1948. URL http://cm.bell-labs.com/cm/ms/what/shannonday/shannon1948.pdf.

Claude E. Shannon. Communication in the presence of noise. *Proceedings of the IRE*, 37:10–21, January 1949. ISSN 0096-8390. doi: 10.1109/JRPROC.1949.232969. URL www.stanford.edu/class/ee104/shannonpaper.pdf.

George Finlay Simmons. *Calculus Gems: Brief Lives and Memorable Mathematicians*. Mathematical Association of America, Washington DC, 2007. ISBN 0883855615. URL http://books.google.com/books?vid=ISBN0883855615.

Neil J. A. Sloane. On-line encyclopedia of integer sequences. World Wide Web, 2014. URL http://oeis.org/.

M.J.T. Smith and T.P. Barnwell. A procedure for designing exact reconstruction filter banks for tree-structured subband coders. *IEEE International Conference on Acoustics, Speech and Signal Processing*, 9:421–424, 1984a. T.P. Barnwell is T.P. Barnwell III.

M.J.T. Smith and T.P. Barnwell. The design of digital filters for exact reconstruction in subband coding. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 34(3):434–441, June 1984b. ISSN 0096-3518. doi: 10.1109/TASSP.1986.1164832. T.P. Barnwell is T.P. Barnwell III.

Houshang H. Sohrab. *Basic Real Analysis*. Birkhäuser, Boston, 1 edition, 2003. ISBN 978-0817642112. URL http://books.google.com/books?vid=ISBN0817642110.

Gilbert Strang and Truong Nguyen. *Wavelets and Filter Banks*. Wellesley-Cambridge Press, Wellesley, MA, 1996. ISBN 9780961408879. URL http://books.google.com/books?vid=ISBN0961408871.

Robert S. Strichartz. *The Way of Analysis*. Jones and Bartlett Publishers, Boston-London, 1995. ISBN 978-0867204711. URL http://books.google.com/books?vid=ISBN0867204710.

Endre Süli and David F. Mayers. *An Introduction to Numerical Analysis*. Cambridge University Press, August 28 2003. ISBN 9780521007948. URL http://books.google.com/books?vid=ISBN0521007941.

Wim Sweldens and Robert Piessens. Wavelet sampling techniques. In *1993 Proceedings of the Statistical Computing Section*, pages 20–29. American Statistical Association, August 1993. URL http://citeseer.ist.psu.edu/18531.html.

Erik Talvila. Necessary and sufficient conditions for differentiating under the integral sign. *The American Mathematical Monthly*, 108(6):544–548, June–July 2001. URL http://arxiv.org/abs/math/0101012.

Brook Taylor. *Methodus Incrementorum Directa et Inversa*. London, 1715.

Audrey Terras. *Fourier Analysis on Finite Groups and Applications*. Number 43 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1999. ISBN 0-521-45718-1. URL http://books.google.com/books?vid=ISBN0521457181.

Brian S. Thomson, Andrew M. Bruckner, and Judith B. Bruckner. *Elementary Real Analysis*. www.classicalrealanalysis.com, 2 edition, 2008. ISBN 9781434843678. URL http://classicalrealanalysis.info/com/Elementary-Real-Analysis.php.

P.P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall Signal Processing Series. Prentice Hall, 1993. ISBN 0136057187. URL http://books.google.com/books?vid=ISBN0136057187.

Peter van Overschee and B. L. de Moor. *Subspace Identification for Linear Systems: Theory — Implementation — Applications*. Springer Science & Business Media, illustrated edition, 2012. ISBN 9781461304654. URL http://books.google.com/books?vid=isbn1461304652.

Brani Vidakovic. *Statistical Modeling by Wavelets*. John Wiley & Sons, Inc, New York, 1999. ISBN 0-471-29365-2. URL http://www.amazon.com/dp/0471293652.

David F. Walnut. *An Introduction to Wavelet Analysis*. Applied and numerical harmonic analysis. Springer, 2002. ISBN 0817639624. URL http://books.google.com/books?vid=ISBN0817639624.

Seth Warner. *Modern Algebra*. Dover, Mineola, 1990. ISBN 0-486-66341-8. URL http://books.google.com/books?vid=ISBN0486663418. “An unabridged, corrected republication of the work originally published in two volumes by Prentice Hall, Inc., Englewood Cliffs, New Jersey, in 1965”.

Heinrich Martin Weber. Die allgemeinen grundlagen der galois'schen gleichungstheorie. *Mathematische Annalen*, 43(4):521–549, December 1893. URL http://resolver.sub.uni-goettingen.de/purl?GDZPPN002254670. The general foundation of Galois' equation theory.

Edmund Taylor Whittaker. On the functions which are represented by the expansions of the interpolation theory. *Proceedings of the Royal Society Edinburgh*, 35:181–194, 1915.

John Macnaghten Whittaker. *Interpolatory Function Theory*, volume 33 of *Cambridge tracts in mathematics and mathematical physics*. Cambridge University Press, 1935. URL http://books.google.com.tw/books?id=yyPvAAAAMAAJ.

Stephen B. Wicker. *Error Control Systems for Digital Communication and Storage*. Prentice Hall, Upper Saddle River, 1995. ISBN 0-13-200809-2. URL http://www.worldcat.org/isbn/0132008092.

Charles Sumner Williams. *Designing digital filters*. Prentice-Hall information and system sciences series. Prentice-Hall, 1986. ISBN 9780132018562. URL http://books.google.com/books?vid=ISBN9780132018562.

P. Wojtaszczyk. *A Mathematical Introduction to Wavelets*, volume 37 of *London Mathematical Society student texts*. Cambridge University Press, February 13 1997. ISBN 9780521578943. URL http://books.google.com/books?vid=ISBN0521578949.

Kôsaku Yosida. *Functional Analysis*, volume 123 of *Classics in Mathematics*. Springer, 6 reprint revised edition, 1980. ISBN 9783540586548. URL http://books.google.com/books?vid=ISBN3540586547.

Robert M. Young. *An introduction to nonharmonic Fourier series*, volume 93 of *Pure and applied mathematics*. Academic Press, revised first edition, May 16 2001. ISBN 0127729550. URL http://books.google.com/books?vid=ISBN0127729550.

Ahmed I. Zayed. *Handbook of Function and Generalized Function Transformations*. Mathematical Sciences Reference Series. CRC Press, Boca Raton, 1996. ISBN 0849378516. URL http://books.google.com/books?vid=ISBN0849378516.

Gary Zukav. *The Dancing Wu Li Masters : An Overview of the New Physics*. Bantam Books, New York, 1980. ISBN 055326382X. URL http://books.google.com/books?vid=ISBN055326382X.

Antoni Zygmund. Trigonometric series volume ii. In *Trigonometric Series*, page 364. Cambridge University Press, London/New York/Melbourne, 3 edition, 2002. ISBN 0-521-89053-5. URL http://books.google.com/books?vid=ISBN0521890535.

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