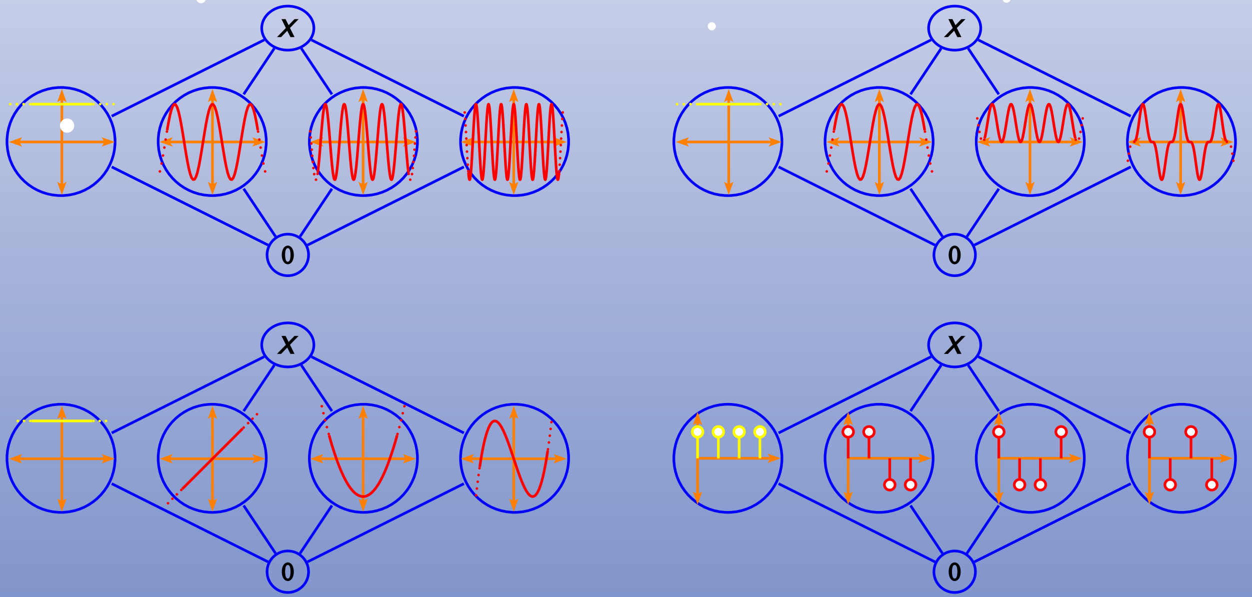
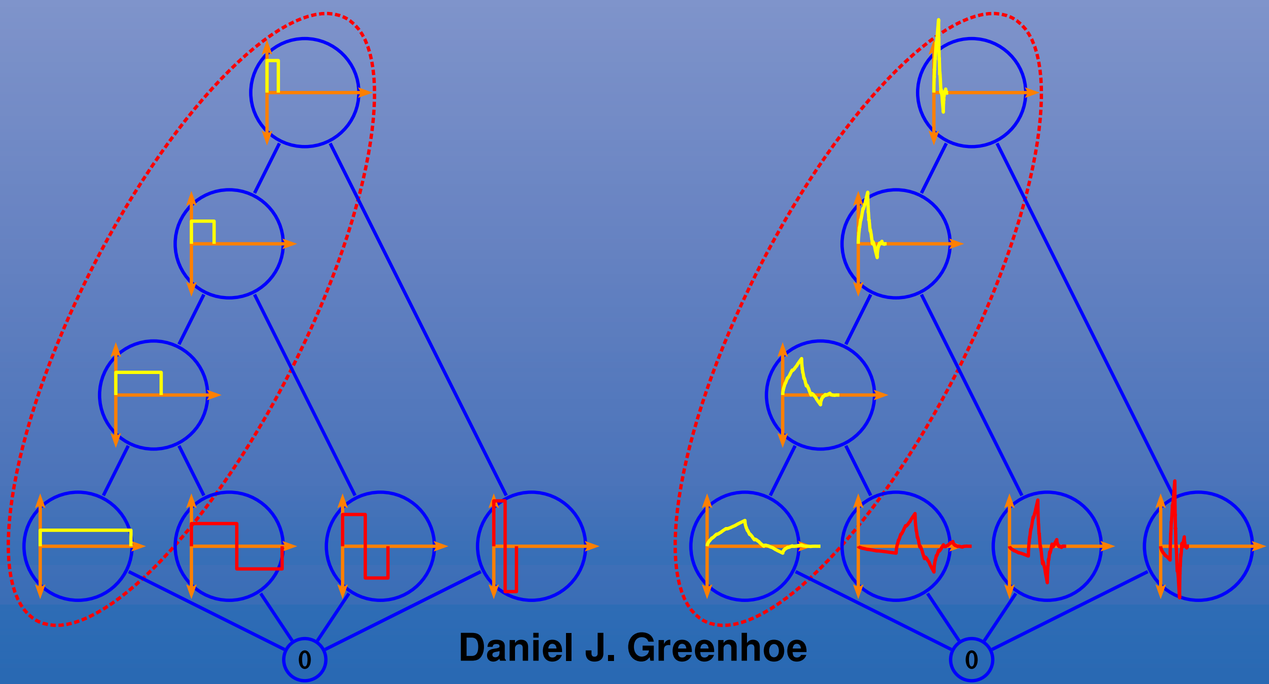
Frames and Bases
Structure and Design

VERSION 0.20





Daniel J. Greenhoe

***Mathematical Structure and Design* series**
volume $\boxed{4}$

title:	*Frames and Bases Structure and Design*
document type:	`book`
series:	*Mathematical Structure and Design*
volume:	$\boxed{4}$
author:	Daniel J. Greenhoe
version:	VERSION 0.20
time stamp:	2019 December 10 (Tuesday) 11:31am UTC
copyright:	Copyright © 2019 Daniel J. Greenhoe
license:	Creative Commons license CC BY-NC-ND 4.0
typesetting engine:	XƎLaTeX
document url:	`https://github.com/dgreenhoe/pdfs/blob/master/frames.pdf` `https://www.researchgate.net/project/Mathematical-Structure-and-Design`

This text was typeset using XeLaTeX, which is part of the TeXfamily of typesetting engines, which is arguably the greatest development since the Gutenberg Press. Graphics were rendered using the *pstricks* and related packages, and LaTeX graphics support.

The main roman, *italic*, and **bold** font typefaces used are all from the *Heuristica* family of typefaces (based on the *Utopia* typeface, released by *Adobe Systems Incorporated*). The math font is XITS from the XITS font project. The font used in quotation boxes is adapted from *Zapf Chancery Medium Italic*, originally from URW++ Design and Development Incorporated. The font used for the text in the title is Adventor (similar to *Avant-Garde*) from the *TeX-Gyre Project*. The font used for the ISBN in the footer of individual pages is LIQUID CRYSTAL (*Liquid Crystal*) from *FontLab Studio*. The Latin handwriting font is Lavi from the *Free Software Foundation*.

The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]

[1] Paine (2000) page 63 ⟨Golden Hind⟩

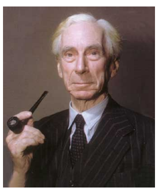
“Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?”

“Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.”

Alfred Edward Housman, English poet (1859–1936) [2]

“The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.”

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



“As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.”

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

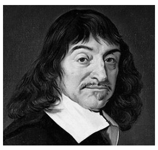
[2] quote: Housman (1936) page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961) page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967) page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.❞



❝Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.❞

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \ldots$

…continued on next page…

[5]quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6]quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$
$\mathbb{Z}^{-}$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_{\mathsf{o}}$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_{\mathsf{e}}$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^{+}$	positive real numbers	$(0, \infty)$
$\mathbb{R}^{-}$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^{*}$	extended real numbers	$\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \nabla)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}(®)$	domain of a relation ®	
$\mathcal{I}(®)$	image of a relation ®	
$\mathcal{R}(®)$	range of a relation ®	
$\mathcal{N}(®)$	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\not\subseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^{c}	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	

...continued on next page...

symbol	description	
0	"false" condition	
$\neg$	logical NOT operation	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\leq$	reflexive ordering relation	"less than or equal to"
$\geq$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle \triangle \mid \triangledown \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_{\mathsf{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_{\mathsf{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_{\mathsf{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		

...continued on next page...

symbol	description
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition H.2 page 192)
$\hat{\mathbf{F}}$	*Fourier Series* operator (Definition M.1 page 233)
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition L.1 page 223)
$\mathbf{Z}$	*Z-Transform* operator (Definition I.4 page 204)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\breve{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

CHAPTER 1

ANALYSES AND TRANSFORMS

❝*The analytical equations, unknown to the ancient geometers, which Descartes was the first to introduce into the study of curves and surfaces, ...they extend to all general phenomena. There cannot be a language more universal and more simple, more free from errors and from obscurities, ...mathematical analysis is as extensive as nature itself; it defines all perceptible relations, measures times, spaces, forces, temperatures ; this difficult science is formed slowly, but it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them.* ❞

Joseph Fourier (1768–1830) [1]

1.1 Abstract spaces

The **abstract space** was introduced by Maurice Fréchet in his 1906 Ph.D. thesis.[2] An *abstract space* in mathematics does not really have a rigorous definition; but in general it is a set together with some other unifying structure. Examples of spaces include *topological space*s, *metric space*s, and *linear space*s (*vector space*s).

[1] quote: Fourier (1878) pages 7–8 ⟨Preliminary Discourse⟩
image: `http://en.wikipedia.org/wiki/File:Fourier2.jpg`, public domain

[2] Fréchet (1906), Fréchet (1928). *"A collection of these abstract elements will be called an abstract set. If to this set there is added some rule of association of these elements, or some relation between them, the set will be called an abstract space."*—Maurice Fréchet

1.2 Lattice of subspaces

An abstract space can be decomposed into one or more *subspace*s. Roughly speaking, a subspace of an abstract space is simply a subset the abstract space that has the same properties of that abstract space. The subspaces can be ordered under the ordering relation $\subseteq$ (subset or equal to relation) to form a *lattice*.

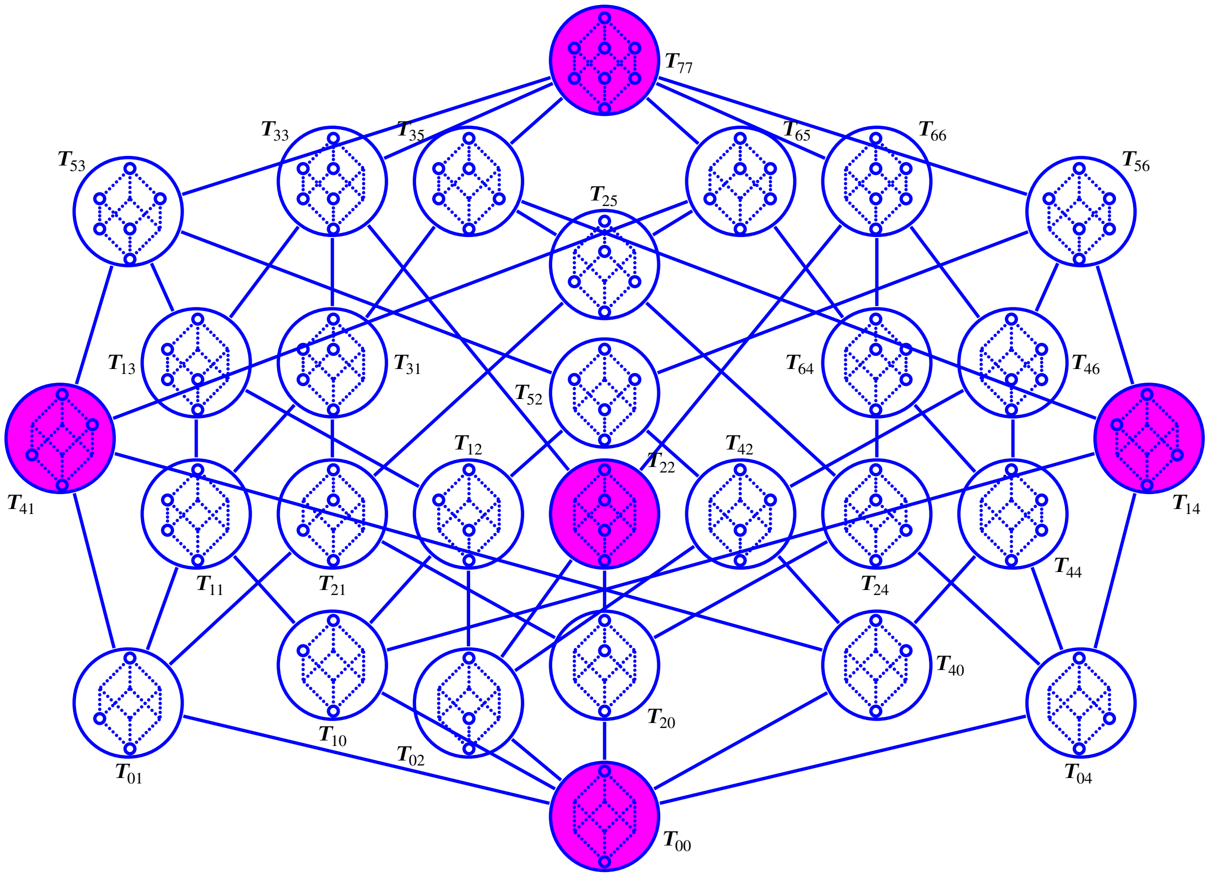
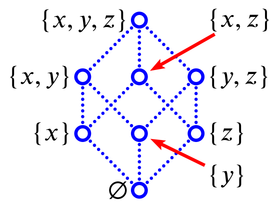


Figure 1.1: lattice of topologies on $X \triangleq \{x, y, z\}$ (Example 1.1 page 2)

Example 1.1. [3]The power set 2^X is a *topology* on the set X. But there are also 28 other topologies on $\{x, y, z\}$, and these are all *subspaces* of $2^{\{x,y,z\}}$. Let a given topology in $\mathcal{T}(\{x, y, z\})$ be represented by a Hasse diagram as illustrated to the right, where a circle present means the indicated set is in the topology, and a circle absent means the indicated set is not in the[4] topology. The lattice of the 29 topologies $(\mathcal{T}(\{x, y, z\}), \cup, \cap; \subseteq)$ is illustrated in Figure 1.1 (page 2). The lattice of these 29 topologies is *non-distributive* (it contains the *N5 lattice*). The five topologies illustrated by red shaded nodes are also *algebras of sets*.



Example 1.2. The power set 2^X is an *algebra of sets* on the set X. But there are also 14 other algebras of sets on $\{w, x, y, z\}$, and these are all *subspaces* of $2^{\{w,x,y,z\}}$. The *lattice of algebras of sets* on $\{w, x, y, z\}$ is illustrated in Figure 1.2 (page 3).

A *linear subspace* is a subspace of a *linear space* (*vector space*). Linear subspaces have some special properties: Every linear subspace contains the additive identity zero vector, and every linear subspace is *convex*.

[4] Isham (1999) page 44, Isham (1989) page 1516, Steiner (1966) page 386

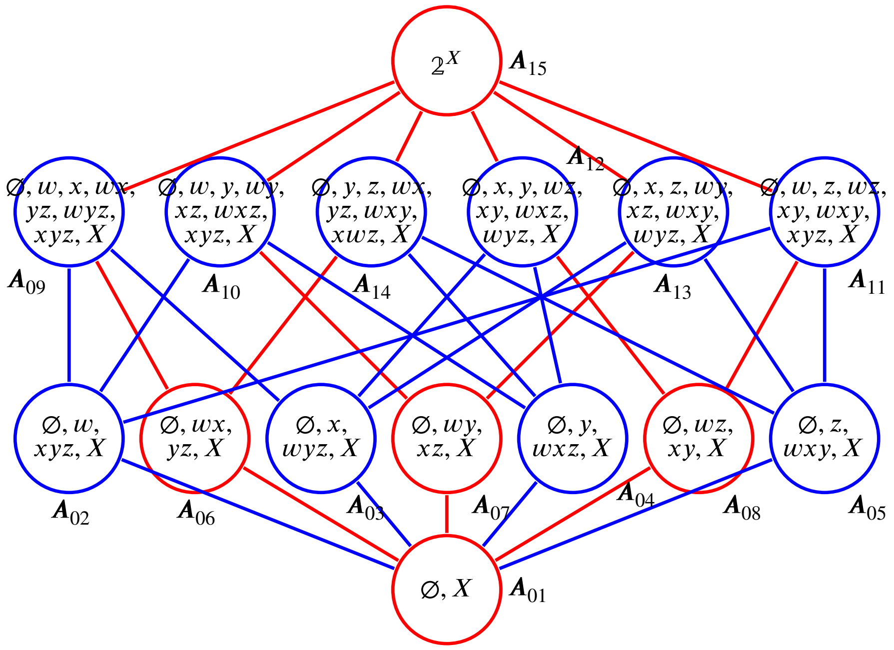
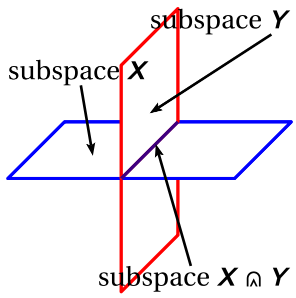


Figure 1.2: lattice of *algebras of sets* on $\{w, x, y, z\}$ (Example 1.2 page 2)

Example 1.3. The 3-dimensional Euclidean space $\mathbb{R}^3$ contains the 2-dimensional xy-plane and xz-plane subspaces, which in turn both contain the 1-dimensional x-axis subspace. These subspaces are illustrated in the figure to the right and in Figure B.1 (page 97).



1.3 Analyses

An **analysis** of a space $\boldsymbol{X}$ is any lattice of subspaces of $\boldsymbol{X}$. The partial or complete reconstruction of $\boldsymbol{X}$ from this set is a **synthesis**.[5]

Example 1.4. The lattices of subspaces illustrated in Figure 1.4 (page 4) are all *analyses* of $\mathbb{R}^3$.

1.4 Transform

Definition 1.1. *A* ***transform*** *on a space* $\boldsymbol{X}$ *is a sequence of projection operators that induces an* ANALYSIS *on* $\boldsymbol{X}$.

Section 1.3 defined an **analysis** of a space $\boldsymbol{X}$ as is any lattice of subspaces of $\boldsymbol{X}$. In like manner, an **analysis** of a function $\mathsf{f}(x)$ with respect to a transform $\mathbf{T}$ is simply the transform $\mathbf{T}$ of f ($\mathbf{T}\mathsf{f}$). Such

[5]The word *analysis* comes from the Greek word ἀνάλυσις, meaning "dissolution" (Perschbacher (1990) page 23 ⟨entry 359⟩), which in turn means "the resolution or separation into component parts" (Black et al. (2009), `http://dictionary.reference.com/browse/dissolution`)

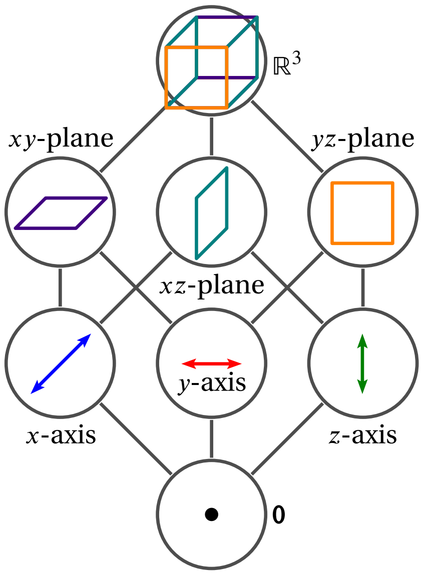


Figure 1.3: lattice of subspaces of $\mathbb{R}^3$ (Example B.1 page 97)

linearly ordered analysis of $\mathbb{R}^3$ M-3 analysis of $\mathbb{R}^3$ wavelet-like analysis of $\mathbb{R}^3$

Figure 1.4: some anayses of $\mathbb{R}^3$ (Example 1.4 page 3)

an analysis or transform is often represented as the sequence of coefficients $(\!(\lambda_n)\!)$ multiplying the basis vectors $(\!(\psi_n(x))\!)$ such that

$$\mathsf{f}(x) = \mathbf{T}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \lambda_n \psi_n(x)$$

Example 1.5. A *Fourier analysis* is a sequence of subspaces with sinusoidal bases. Examples of subspaces in a Fourier analysis include $\boldsymbol{V}_1 = \operatorname{span}\{e^{ix}\}$, $\boldsymbol{V}_{2.3} = \operatorname{span}\{e^{i2.3x}\}$, $\boldsymbol{V}_{\sqrt{2}} = \operatorname{span}\left\{e^{i\sqrt{2}x}\right\}$, etc. A **transform** is a set of *projection operators* that maps a family of functions (e.g. $\boldsymbol{L}^2_{\mathbb{R}}$) into an analysis. The *Fourier transform*" for Fourier Analysis is (Definition H.2 page 192)

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x$$

1.5 Properties of subspace order structures

The ordered set of all linear subspaces of a *Hilbert space* is an *orthomodular lattice*. Orthomodular lattices (and hence Hilbert subspaces) have some special properties (next theorem). One is that they satisfy *de Morgan's law*.

Theorem 1.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an algebraic structure.*

THM

$$\left.\begin{array}{l}\boldsymbol{L} \textit{ is an } \text{ORTHOMODULAR}\\ \text{LATTICE}\end{array}\right\} \implies \left\{\begin{array}{lllll}
1.\ (x \vee y)^\perp &=& x^\perp \wedge y^\perp & \forall x,y\in X & (\text{DE MORGAN}) \textit{ and}\\
2.\ (x \wedge y)^\perp &=& x^\perp \vee y^\perp & \forall x,y\in X & (\text{DE MORGAN}) \textit{ and}\\
3.\ (z^\perp \wedge y^\perp)^\perp \vee x &=& (x \vee y) \vee z & \forall x,y,z\in X & \textit{and}\\
4.\ x \wedge (x \vee y) &=& x & \forall x,y\in X & \textit{and}\\
5.\ x \vee (y \wedge y^\perp) &=& x & \forall x,y\in X. &
\end{array}\right.$$

[6] Beran (1985) pages 30–33, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Beran (1976) pages 251–252

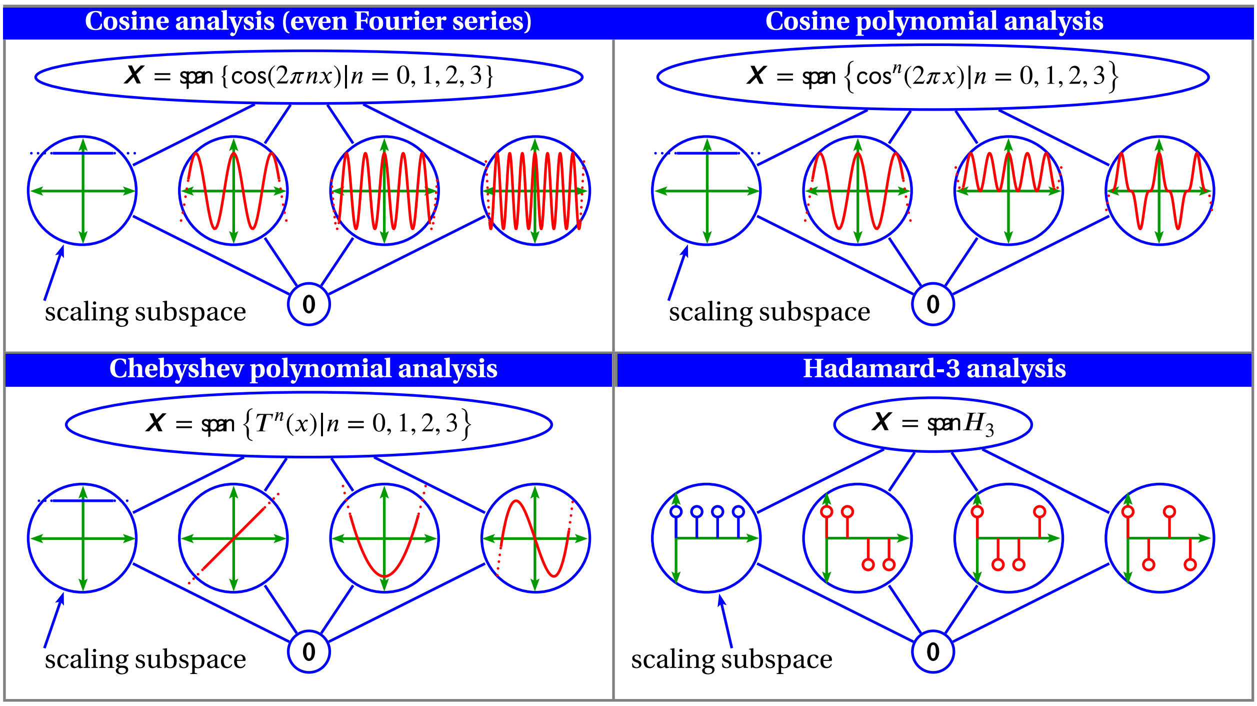
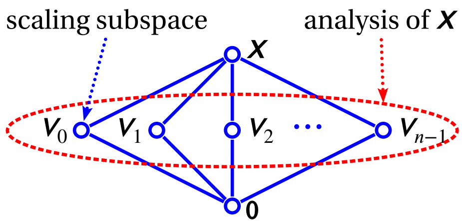
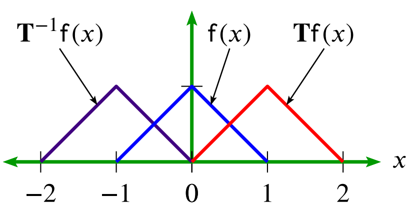


Figure 1.5: some common transforms

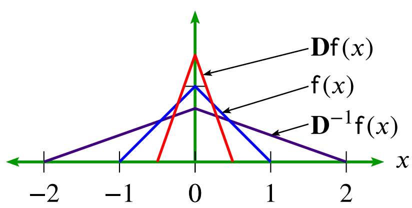
Most transforms have a very simple M-n order structure, as illustrated to the right and in Figure 1.5 page 5. The M-n lattices for $n \geq 3$ are *modular* but not *distributive*. Analyses typically have one subspace that is a *scaling* subspace; and this subspace is often simply a family of constants (as is the case with *Fourier Analysis*). There is one noteable exception to this—MRA induced *wavelet analysis* (Definition 5.1 page 81).



1.6 Operator inducing analyses



An *analysis* is often defined in terms of a small number (e.g. 2) operators. Two such operators are the *translation operator* and the *dilation operator* (Definition 3.3 page 40).



Example 1.6. In *Fourier analysis*, continuous dilations (Definition 3.3 page 40) of the *complex exponential* form a *basis* (Definition 2.7 page 14) for the *space of square integrable functions* $L^2_{\mathbb{R}}$ (Definition D.1 page 141) such that $L^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}_\omega e^{ix} | \omega\in\mathbb{R}\right\}$.

Example 1.7. In *Fourier series analysis* (Theorem M.1 page 234), discrete dilations of the complex exponential form a basis for $L^2_{\mathbb{R}}(0:2\pi)$ such that $L^2_{\mathbb{R}}(0:2\pi) = \operatorname{span}\left\{\mathbf{D}_j e^{ix} \middle| j\in\mathbb{Z}\right\}$.

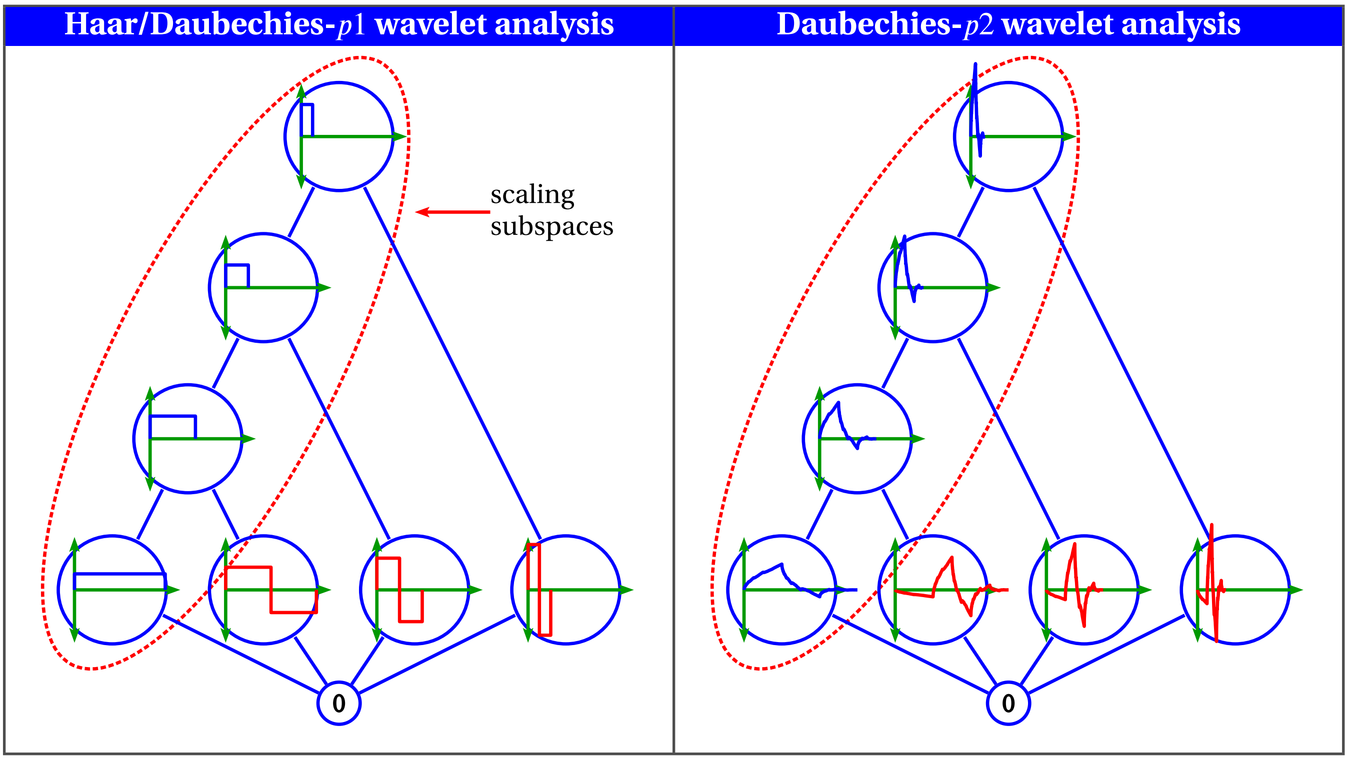
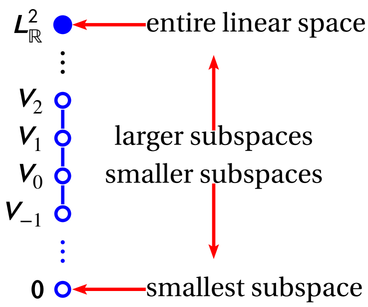


Figure 1.6: some wavelet transforms

1.7 Wavelet analyses

The term “wavelet” comes from the French word “*ondelette*”, meaning “small wave”. And in essence, wavelets are “small waves” (as opposed to the “long waves” of Fourier analysis) that form a basis for the Hilbert space $L^2_{\mathbb{R}}$.[8]

A special characteristic of wavelet analysis is that there is not just one scaling subspace, (as is with the case of Fourier and several other analyses), but an entire sequence of scaling subspaces (Figure 1.6 page 6). These scaling subspaces are *linearly ordered* with respect to the ordering relation $\subseteq$. In wavelet theory, this structure is called a *multiresolution analysis*, or *MRA* (Definition 4.1 page 54). The MRA was introduced by Stéphane G. Mallat in 1989. The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan, and later as the *Gaussian Pyramid* by Burt and Adelson in the 1980s in the West.[9]



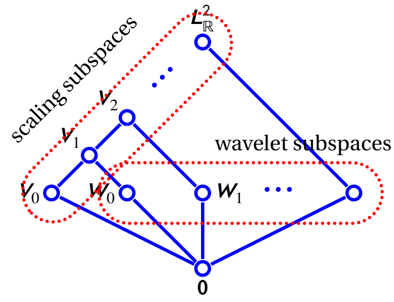
The MRA has become the dominate wavelet construction method. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA.[11]

[8] Strang and Nguyen (1996) page ix; Atkinson and Han (2009) page 191

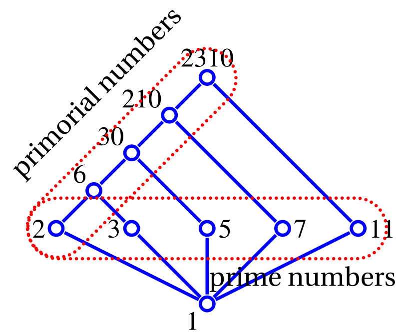
[10] Mallat (1989) page 70; Iijima (1959); Burt and Adelson (1983); Adelson and Burt (1981); Lindeberg (1993); Alvarez et al. (1993); Guichard et al. (2012); Weickert (1999) ⟨historical survey⟩

[11] Lemarié (1990), Mallat (1999) page 240

A second special characteristic of wavelet analysis is that it's order structure with respect to the $\subseteq$ relation is not a simple M-n lattice (as is with the case of Fourier and several other analyses). Rather, it is a lattice of the form illustrated to the right and in Figure 1.6 (page 6). This lattice is *non-complemented*, *non-distributive*, *non-modular*, and *non-Boolean* (Proposition 5.1 page 83).[12]



In the world of mathematical structures, the order structure of wavelet analyses is quite rare, but not completely unique. One example of a system with similar structure is the set of *Primorial*[14] *numbers* together with the | ("divides") ordering relation[13] as illustrated to the right.



The basis sequence of most transform are fixed with no design freedom For example, the Fourier Transform uses the complex exponential, Taylor Expansion uses monomials of the form $(x-a)^n$. However, there are an infinite number of wavelet basis sequences—lots and lots of design freedom. For information regarding designing wavelet basis sequences, see Greenhoe (2013).

However, one arguable disadvantage is that wavelets do not support a **convolution theorem**—a theorem enjoyed by the Fourier transforms, Laplace Transform, and Z Transform. These other transforms induce a convolution theorem because they are defined in terms of an exponential (e.g. $e^{-i\omega t}$, $e^{-i\omega n}$, e^{-st}, z^{-n}), and exponentials sport the property $a^{x+y} = a^x a^y$.

[12] Greenhoe (2013) page 72 ⟨Section 2.4.3 Order structure⟩
[14] Sloane (2014) ⟨http://oeis.org/A002110⟩, Greenhoe (2013) page 30

CHAPTER 2

LINEAR COMBINATIONS

2.1 Linear combinations in linear spaces

A *linear space* (Definition C.1 page 111) in general is not equipped with a *topology*. Without a topology, it is not possible to determine whether an *infinite sum* of vectors converges. Therefore in this section (dealing with linear spaces), all definitions related to sums of vectors will be valid for *finite* sums only (finite "N").

Definition 2.1. [1] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in a* LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

DEF: *A vector* $\boldsymbol{x} \in X$ *is a* ***linear combination*** *of the vectors in* $\{\boldsymbol{x}_n\}$ *if there exists* $\left\{ \alpha_n \in \mathbb{F} \,\middle|\, n=1,2,\ldots,N \right\}$ *such that*

$$\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n .$$

Definition 2.2. [2] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space and* Y *be a subset of* X.

DEF: *The* ***linear span*** *of* Y *is defined as*

$$\operatorname{span} Y \triangleq \left\{ \sum_{\gamma \in \Gamma} \alpha_\gamma \boldsymbol{y}_\gamma \,\middle|\, \alpha_\gamma \in \mathbb{F},\ \boldsymbol{y}_\gamma \in Y \right\}.$$

The set Y ***spans*** *a set* A *if* $A \subseteq \operatorname{span} Y$.

Proposition 2.1. [3] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

PRP:
1. $\operatorname{span}\{\boldsymbol{x}_n\}$ *is a* LINEAR SPACE (Definition C.1 page 111) *and*
2. $\operatorname{span}\{\boldsymbol{x}_n\}$ *is a* LINEAR SUBSPACE *of* $\boldsymbol{L}$.

Definition 2.3. [4] *Let* $\boldsymbol{L} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a* LINEAR SPACE.

DEF: *The set* $Y \triangleq \left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *is* ***linearly independent*** *in* $\boldsymbol{L}$ *if*

$$\left\{ \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = 0 \right\} \implies \left\{ \alpha_1 = \alpha_2 = \cdots = \alpha_N = 0 \right\}.$$

The set Y *is* ***linearly dependent*** *in* $\boldsymbol{L}$ *if* Y *is not linearly independent in* $\boldsymbol{L}$.

[1] Berberian (1961) page 11 ⟨Definition I.4.1⟩, Kubrusly (2001) page 46

[2] Michel and Herget (1993) page 86 ⟨3.3.7 Definition⟩, Kurdila and Zabarankin (2005) page 44, Searcóid (2002) page 71 ⟨Definition 3.2.5—more general definition⟩

[3] Kubrusly (2001) page 46

[4] Bachman and Narici (1966) pages 3–4, Christensen (2003) page 2, Heil (2011) page 156 ⟨Definition 5.7⟩

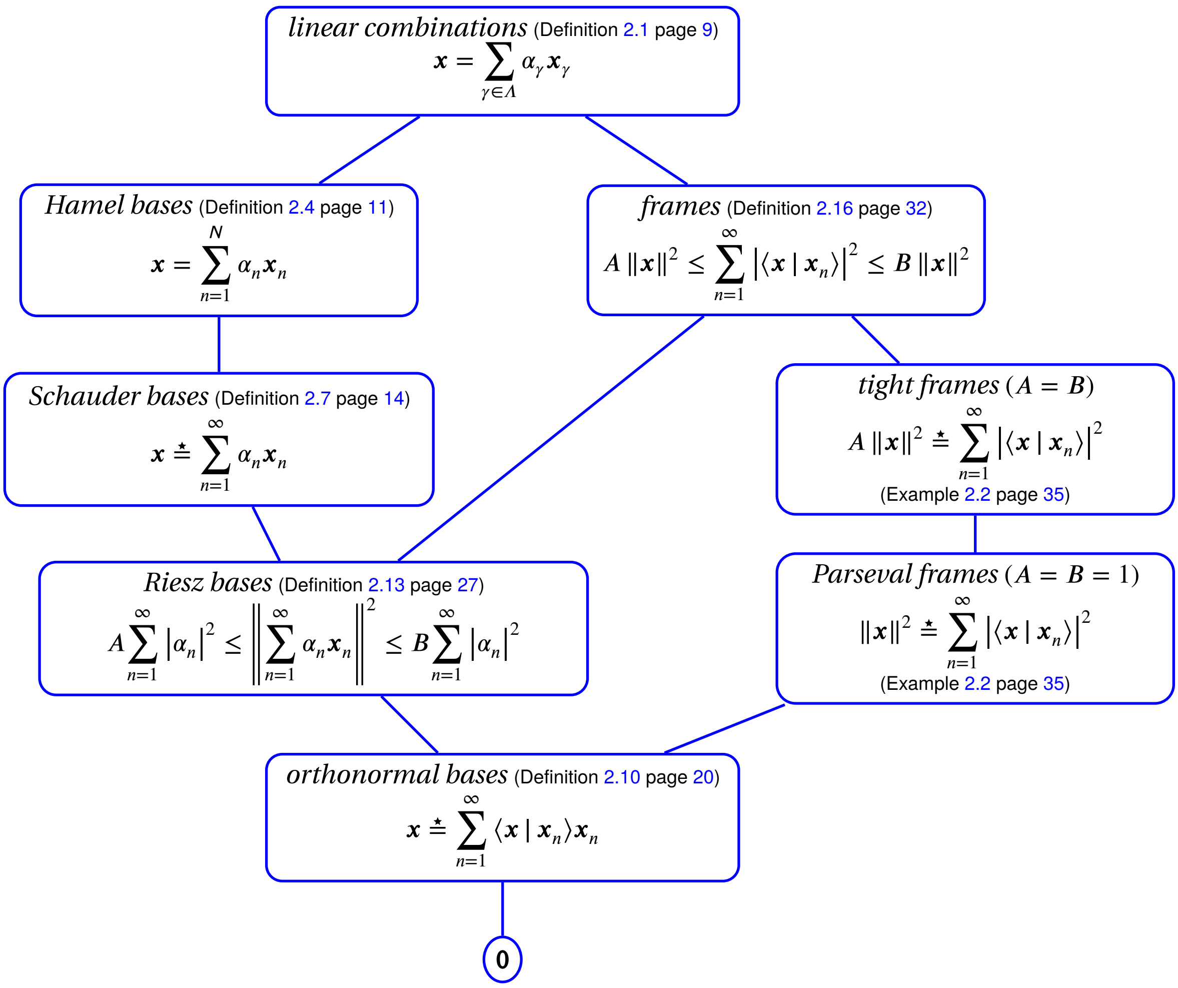


Figure 2.1: Lattice of *linear combinations*

Definition 2.4. [5] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF

The set $\{\boldsymbol{x}_n\}$ *is a* ***Hamel basis*** *for* $\boldsymbol{L}$ *if*

1. $\{\boldsymbol{x}_n\}$ SPANS $\boldsymbol{L}$ (Definition 2.2 page 9) *and*
2. $\{\boldsymbol{x}_n\}$ *is* LINEARLY INDEPENDENT *in* $\boldsymbol{L}$ (Definition 2.1 page 9) .

A HAMEL BASIS *is also called a* ***linear basis***.

Definition 2.5. [6] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* $\boldsymbol{x}$ *be a* VECTOR *in* $\boldsymbol{L}$ *and* $Y \triangleq \{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in* $\boldsymbol{L}$.

DEF

The expression $\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$ *is the* ***expansion*** *of* $\boldsymbol{x}$ *on* Y *in* $\boldsymbol{L}$ *if* $\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$.

In this case, the sequence $(\!(\alpha_n)\!)_{n=1}^{N}$ *is the* ***coordinates*** *of* $\boldsymbol{x}$ *with respect to* Y *in* $\boldsymbol{L}$.

If $\alpha_N \neq 0$, *then* N *is the* ***dimension*** $\dim \boldsymbol{L}$ *of* $\boldsymbol{L}$.

Theorem 2.1. [7] *Let* $\{\boldsymbol{x}_n \mid n=1,2,\ldots,N\}$ *be a* HAMEL BASIS (Definition 2.4 page 11) *for a* LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

THM

$$\left\{\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = \sum_{n=1}^{N} \beta_n \boldsymbol{x}_n\right\} \implies \underbrace{\alpha_n = \beta_n \quad \forall n=1,2,\ldots,N}_{\text{coordinates of } \boldsymbol{x} \text{ are UNIQUE}} \qquad \forall \boldsymbol{x} \in X$$

✎PROOF:

$$\begin{aligned}
\mathbb{0} &= \boldsymbol{x} - \boldsymbol{x} \\
&= \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n - \sum_{n=1}^{N} \beta_n \boldsymbol{x}_n \\
&= \sum_{n=1}^{N} (\alpha_n - \beta_n) \boldsymbol{x}_n \\
&\implies \{\boldsymbol{x}_n\} \text{ is } \textit{linearly dependent} \text{ if } (\alpha_n - \beta_n) \neq 0 \qquad \forall n = 1, 2, \ldots, N \\
&\implies (\alpha_n - \beta_n) = 0 \qquad \forall n = 1, 2, \ldots, N \qquad \text{(because } \{\boldsymbol{x}_n\} \text{ is a } \textit{basis} \text{ and therefore must be } \textit{linearly independent}) \\
&\implies \alpha_n = \beta_n \text{ for } n = 1, 2, \ldots, N
\end{aligned}$$

✏

Theorem 2.2. [8] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

THM

$$\left\{\begin{array}{ll} 1. & \{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \quad \text{and} \\ 2. & \{\boldsymbol{y}_n \in X \mid n=1,2,\ldots,M\} \text{ is a set of LINEARLY INDEPENDENT vectors in } \boldsymbol{L} \end{array}\right\}$$
$$\implies \left\{\begin{array}{ll} 1. & M \leq N \quad \text{and} \\ 2. & M = N \implies \{\boldsymbol{y}_n \mid n=1,2,\ldots,M\} \text{ is a BASIS for } \boldsymbol{L} \quad \text{and} \\ 3. & M \neq N \implies \{\boldsymbol{y}_n \mid n=1,2,\ldots,M\} \text{ is NOT a basis for } \boldsymbol{L} \end{array}\right\}$$

✎PROOF:

1. Proof that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is a *basis* for $\boldsymbol{L}$:

---

[5] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[6] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[7] Michel and Herget (1993) pages 89–90 ⟨Theorem 3.3.25⟩

[8] Michel and Herget (1993) pages 90–91 ⟨Theorem 3.3.26⟩

(a) Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ *spans* $\mathbf{L}$:

i. Because $\{x_n | n=1,2,\ldots,N\}$ is a *basis* for $\mathbf{L}$, there exists $\beta \in \mathbb{F}$ and $\{\alpha_n \in \mathbb{F} | n=1,2,\ldots,N\}$ such that $\beta y_1 + \sum_{n=1}^{N} \alpha_n x_n = 0$.

ii. Select an n such that $\alpha_n \neq 0$ and renumber (if necessary) the above indices such that $x_n = -\frac{\beta}{\alpha_n} y_1 + \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n} x_n$.

iii. Then, for any $y \in X$, we can write

$$
\begin{aligned}
y &= \sum_{n=1}^{N} \gamma_{n\in\mathbb{Z}} x_n \\
&= \left(\sum_{n=1}^{N-1} \gamma_{n\in\mathbb{Z}} x_n \right) + \gamma_{n\in\mathbb{Z}} \left(-\frac{\beta}{\alpha_n} y_1 - \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n} x_n \right) \\
&= -\frac{\beta\gamma_n}{\alpha_n} y_1 + \sum_{n=1}^{N-1} \left(\gamma_n - \frac{\alpha_n \gamma_n}{\alpha_n} \right) x_n \\
&= \delta y_1 + \sum_{n=1}^{N-1} \delta_{n\in\mathbb{Z}} x_n
\end{aligned}
$$

iv. This implies that $\{y_1, x_1, \ldots, x_{N-1}\}$ *spans* $\mathbf{L}$:

(b) Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly independent*:

i. If $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly dependent*, then there exists $\{\epsilon, \epsilon_1, \ldots, \epsilon_{N-1}\}$ such that $\epsilon y_1 + \left(\sum_{n=1}^{N-1} \epsilon_{n\in\mathbb{Z}} x_n \right) + 0 x_n = 0$.

ii. item (1(b)i) implies that the coordinate of y_1 associated with x_n *is* 0. $y_1 = -\left(\sum_{n=1}^{N-1} \frac{\epsilon_n}{\epsilon} x_n \right) + 0 x_n = 0$.

iii. item (1(a)i) implies that the coordinate of y_1 associated with x_n is *not* 0. $y_1 = -\sum_{n=1}^{N} \frac{\alpha_n}{\beta} x_n$.

iv. This implies that item (1(b)i) (that the set is linearly dependent) is *false* because item (1(b)ii) and item (1(b)iii) *contradict* each other.

v. This implies $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly independent*.

2. Proof that $\{y_1, y_2, x_1, \ldots, x_{N-2}\}$ is a *basis*: Repeat item (1).

3. Suppose $m = n$. Proof that $\{y_1, y_2, \ldots, y_M\}$ is a *basis*: Repeat item (1) $M - 1$ times.

4. Proof that $M \not> N$:

(a) Suppose that $M = N + 1$.

(b) Then because $\{y_n | n=1,2,\ldots,N\}$ is a *basis*, there exists $\{\zeta_n | n=1,2,\ldots,N+1\}$ such that $\sum_{n=1}^{N+1} \zeta_{n\in\mathbb{Z}} y_{n\in\mathbb{Z}} = 0$.

(c) This implies that $\{y_n | n=1,2,\ldots,N+1\}$ is *linearly dependent*.

(d) This implies that $\{y_n | n=1,2,\ldots,N+1\}$ is *not* a basis.

(e) This implies that $M \not> N$.

5. Proof that $M \neq N \implies \{y_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$:

(a) Proof that $M > N \implies \{y_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$: same as in item (4).

(b) Proof that $M < N \implies \{y_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$:

i. Suppose $m = N - 1$.

ii. Then $\{y_n|_{n=1,2,\ldots,N-1}\}$ is a *basis* and there exists λ such that

$$\sum_{n=1}^{N} \lambda_{n\in\mathbb{Z}} y_{n\in\mathbb{Z}} = 0.$$

iii. This implies that $\{y_n|_{n=1,2,\ldots,N}\}$ is *linearly dependent* and is *not* a basis.

iv. But this contradicts item (3), therefore $M \neq N - 1$.

v. Because $M = N$ yields a basis but $M = N - 1$ does not, $M < N - 1$ also does not yield a basis.

✏

Corollary 2.1. [9] *Let $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a linear space.*

COR

$$\underbrace{\left\{\begin{array}{ll} 1. & \{x_n \in X|_{n=1,2,\ldots,N}\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \quad \text{and} \\ 2. & \{y_n \in X|_{n=1,2,\ldots,M}\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \end{array}\right\} \implies \{N = M\}}_{\text{(all Hamel bases for } \boldsymbol{L} \text{ have the same number of vectors)}}$$

✎PROOF: This follows from Theorem 2.2 (page 11). ✏

2.2 Bases in topological linear spaces

A linear space supports the concept of the *span* of a set of vectors (Definition 2.2 page 9). In a topological linear space $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), T)$, a set A is said to be *total* in $\boldsymbol{\Omega}$ if the span of A is *dense* in $\boldsymbol{\Omega}$. In this case, A is said to be a *total set* or a *complete set*. However, this use of "complete" in a "*complete set*" is not equivalent to the use of "complete" in a "*complete metric space*" .[10] In this text, except for these comments and Definition 2.6, "complete" refers to the metric space definition only.

If a set is both *total* and *linearly independent* (Definition 2.3 page 9) in $\boldsymbol{\Omega}$, then that set is a *Hamel basis* (Definition 2.4 page 11) for $\boldsymbol{\Omega}$.

Definition 2.6. [11] *Let A^- be the* CLOSURE *of a A in a* TOPOLOGICAL LINEAR SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), T)$*. Let $\operatorname{span} A$ be the* SPAN *(Definition 2.2 page 9) of a set A.*

DEF

*A set of vectors A is **total** (or **complete** or **fundamental**) in $\boldsymbol{\Omega}$ if*
$(\operatorname{span} A)^- = \boldsymbol{\Omega}$ (SPAN *of A is* DENSE *in $\boldsymbol{\Omega}$*).

[9] Kubrusly (2001) page 52 ⟨Theorem 2.7⟩, Michel and Herget (1993) page 91 ⟨Theorem 3.3.31⟩
[10] Haaser and Sullivan (1991) pages 296–297 ⟨6·Orthogonal Bases⟩, Rynne and Youngson (2008) page 78 ⟨Remark 3.50⟩, Heil (2011) page 21 ⟨Remark 1.26⟩
[11] Young (2001) page 19 ⟨Definition 1.5.1⟩, Sohrab (2003) page 362 ⟨Definition 9.2.3⟩, Gupta (1998) page 134 ⟨Definition 2.4⟩, Bachman and Narici (1966) pages 149–153 ⟨Definition 9.3, Theorems 9.9 and 9.10⟩

2.3 Schauder bases in Banach spaces

Definition 2.7. [12] *Let* $\mathbf{B} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* BANACH SPACE. *Let* $\stackrel{\star}{=}$ *represent* STRONG CONVERGENCE *in* $\mathbf{B}$.

DEF

The countable set $\{\mathbf{x}_n \in X | n \in \mathbb{N}\}$ *is a **Schauder basis** for* $\mathbf{B}$ *if for each* $\mathbf{x} \in X$

1. $\exists (\alpha_n \in \mathbb{F})_{n\in\mathbb{N}}$ such that $\mathbf{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \alpha_n \mathbf{x}_n$ (STRONG CONVERGENCE *in* $\mathbf{B}$) *and*
2. $\left\{ \sum_{n=1}^{\infty} \alpha_n \mathbf{x}_n \stackrel{\star}{=} \sum_{n=1}^{\infty} \beta_n \mathbf{x}_n \right\} \implies \{(\alpha_n) = (\beta_n)\}$ (COEFFICIENT FUNCTIONALS *are* UNIQUE)

In this case, $\sum_{n=1}^{\infty} \alpha_n \mathbf{x}_n$ *is the **expansion** of* $\mathbf{x}$ *on* $\{\mathbf{x}_n | n\in\mathbb{N}\}$ *and the elements of* (α_n) *are the **coefficient functionals** associated with the basis* $\{\mathbf{x}_n\}$. *Coefficient functionals are also called **coordinate functionals**.*

In a Banach space, the existence of a Schauder basis implies that the space is *separable* (Theorem 2.3 page 14). The question of whether the converse is also true was posed by Banach himself in 1932,[13] and became know as "*The basis problem*". This remained an open question for many years. The question was finally answered some 41 years later in 1973 by Per Enflo (University of California at Berkley), with the answer being "no". Enflo constructed a counterexample in which a separable Banach space does *not* have a Schauder basis.[14] Life is simpler in Hilbert spaces where the converse *is* true: a Hilbert space has a Schauder basis *if and only if* it is separable (Theorem 2.11 page 27).

Theorem 2.3. [15] *Let* $\mathbf{B} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* BANACH SPACE. *Let* $\mathbb{Q}$ *be the field of rational numbers.*

THM

$$\left\{ \begin{array}{ll} 1. & \mathbf{B} \text{ has a SCHAUDER BASIS and} \\ 2. & \mathbb{Q} \text{ is DENSE in } \mathbb{F}. \end{array} \right\} \implies \{ \mathbf{B} \text{ is SEPARABLE} \}$$

✎PROOF:

1. lemma:

$$\begin{aligned} \left| \left\{ \mathbf{x} | \exists (\alpha_n \mathbb{Q})_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty} \left\| \mathbf{x} - \sum_{n=1}^{N} \alpha_n \mathbf{x}_n \right\| = 0 \right\} \right| &= |\mathbb{Q} \times \mathbb{N}| \\ &= |\mathbb{Z} \times \mathbb{Z}| \\ &= |\mathbb{Z}| \\ &= \textit{countably infinite} \end{aligned}$$

[12] Carothers (2005) pages 24–25, Christensen (2003) pages 46–49 ⟨Definition 3.1.1 and page 49⟩, Young (2001) page 19 ⟨Section 6⟩, Singer (1970) page 17, Schauder (1927), Schauder (1928)

[13] Banach (1932a) page 111

[14] Enflo (1973), Lindenstrauss and Tzafriri (1977) pages 84–95 ⟨Section 2.d⟩

[15] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Giles (2000) page 17, Heil (2011) page 21 ⟨Theorem 1.27⟩

2. remainder of proof:

$\boldsymbol{B}$ has a *Schauder basis* $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$

$\implies$ for every $\boldsymbol{x}\in\boldsymbol{B}$, there exists $(\alpha_n\in\mathbb{F})_{n\in\mathbb{N}}$ such that $\boldsymbol{x}\doteq\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n$ by Definition 2.7 page 14

$\implies$ for every $\boldsymbol{x}\in\boldsymbol{B}$, there exists $(\alpha_n\in\mathbb{F})_{n\in\mathbb{N}}$ such that $\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\|=0$

$\implies$ for every $\boldsymbol{x}\in\boldsymbol{B}$, there exists $(\alpha_n\in\mathbb{Q})_{n\in\mathbb{N}}$ such that $\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\|=0$ because $\mathbb{Q}^-=\mathbb{F}$

$\implies \boldsymbol{B}=\left\{\boldsymbol{x}\,|\,\exists\,(\alpha_n\mathbb{Q})_{n\in\mathbb{N}}\text{ such that }\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\|=0\right\}$

$\implies \boldsymbol{B}=\left\{\boldsymbol{x}\,|\,\exists\,(\alpha_n\mathbb{Q})_{n\in\mathbb{N}}\text{ such that }\boldsymbol{x}=\lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\}^-$

$\implies \boldsymbol{B}$ is *separable* by (1) lemma page 14

✏

Definition 2.8. [16] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\|\cdot\|)$.

DEF
$\{\boldsymbol{x}_n\}$ *is **equivalent** to* $\{\boldsymbol{y}_n\}$
if there exists a BOUNDED INVERTIBLE *operator* $\mathbf{R}$ *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *such that* $\mathbf{R}\boldsymbol{x}_n=\boldsymbol{y}_n \quad \forall n\in\mathbb{Z}$

Theorem 2.4. [17] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\|\cdot\|)$.

THM
$$\left\{\{\boldsymbol{x}_n\}\text{ is EQUIVALENT to }\{\boldsymbol{y}_n\}\right\}\iff\left\{\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\text{ is CONVERGENT}\iff\sum_{n=1}^{\infty}\alpha_n\boldsymbol{y}_n\text{ is CONVERGENT}\right\}$$

Lemma 2.1. [18] *Let* $\boldsymbol{\Omega}\triangleq(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\boldsymbol{T})$ *be a topological linear space. Let* $\operatorname{span}A$ *be the* SPAN *of a set* A (Definition 2.2 page 9). *Let* $\tilde{\mathsf{f}}(\omega)$ *and* $\tilde{\mathsf{g}}(\omega)$ *be the* FOURIER TRANSFORMS (Definition H.2 page 192) *of the functions* $\mathsf{f}(x)$ *and* $\mathsf{g}(x)$*, respectively, in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141). *Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT (Definition L.1 page 223) *of a sequence* $(a_n)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ (Definition I.2 page 203).

LEM
$$\left\{\begin{array}{ll}(1). & \{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}\text{ is a SCHAUDER BASIS for }\boldsymbol{\Omega}\quad\text{and}\\ (2). & \{\mathbf{T}^n\mathsf{g}\,|\,n\in\mathbb{Z}\}\text{ is a SCHAUDER BASIS for }\boldsymbol{\Omega}\end{array}\right\}\implies\left\{\begin{array}{l}\exists\,(a_n)_{n\in\mathbb{Z}}\quad\text{such that}\\ \tilde{\mathsf{f}}(\omega)=\breve{\mathsf{a}}(\omega)\tilde{\mathsf{g}}(\omega)\end{array}\right\}$$

✎PROOF: Let $\boldsymbol{V}_0'$ be the space spanned by $\{\mathbf{T}^n\phi\,|\,n\in\mathbb{Z}\}$.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \tilde{\mathbf{F}}\mathsf{f} && \text{by definition of }\tilde{\mathbf{F}} \quad \text{(Definition H.2 page 192)}\\
&=\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}}a_n\mathbf{T}\mathsf{g} && \text{by (2)}\\
&=\sum_{n\in\mathbb{Z}}a_n\tilde{\mathbf{F}}\mathbf{T}\mathsf{g}
\end{aligned}$$

[16] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[17] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[18] Daubechies (1992) page 140

$$= \underbrace{\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}}_{\breve{a}(\omega)} \underbrace{\tilde{\mathbf{F}}\mathsf{g}}_{\tilde{\mathsf{g}}(\omega)} \qquad \text{by Corollary 3.1 page 47}$$

$$= \breve{a}(\omega)\tilde{\mathsf{g}}(\omega) \qquad \text{by definition of } \breve{\mathbf{F}} \text{ and } \tilde{\mathbf{F}} \quad \text{by (Definition L.1 page 223, Definition H.2 page 192)}$$

$$\begin{aligned}
V_0 &\triangleq \left\{ \mathsf{f}(x) | \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathbf{F}}\mathsf{f}(x) = \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{b}}(\omega)\tilde{\mathsf{g}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{b}}(\omega)\breve{a}(\omega)\tilde{\mathsf{f}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{c}}(\omega)\tilde{\mathsf{f}}(\omega) \right\} && \text{where } \tilde{\mathsf{c}}(\omega) \triangleq \tilde{\mathsf{b}}(\omega)\breve{a}(\omega) \\
&= \left\{ \mathsf{f}(x) | \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{c}_n \mathsf{f}(x-n) \right\} \\
&\triangleq V_0'
\end{aligned}$$

⇰

2.4 Linear combinations in inner product spaces

Definition 2.9. *Let* $(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be an* INNER PRODUCT SPACE *(Definition C.9 page 124).*

DEF: *Two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* X *are* ***orthogonal*** *if*

$$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \begin{cases} 0 & \text{for } \boldsymbol{x} \neq \boldsymbol{y} \\ c \in \mathbb{F}\backslash 0 & \text{for } \boldsymbol{x} = \boldsymbol{y} \end{cases}$$

In an *inner product space, orthogonality* is a special case of *linear independence*; or alternatively, linear independence is a generalization of orthogonality (next theorem).

Theorem 2.5. [19] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition C.9 page 124)* $(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM: $\left\{ \{\boldsymbol{x}_n\} \text{ is ORTHOGONAL (Definition 2.9 page 16)} \right\} \implies \left\{ \{\boldsymbol{x}_n\} \text{ is LINEARLY INDEPENDENT (Definition 2.1 page 9)} \right\}$

✎PROOF:

1. Proof using *Pythagorean theorem* :
   Let $(\!(\alpha_n)\!)_{n\in\mathbb{N}}$ be a sequence with at least one nonzero element.

[19] Aliprantis and Burkinshaw (1998) page 283 ⟨Corollary 32.8⟩, Kubrusly (2001) page 352 ⟨Proposition 5.34⟩

$$\begin{aligned}
\left\|\sum_{n=1}^{N} \alpha_n x_n\right\|^2 &= \sum_{n=1}^{N} \|\alpha_n x_n\|^2 && \text{by left hypoth. and } \textit{Pythagorean Theorem} \\
&= \sum_{n=1}^{N} |\alpha_n|^2 \|x_n\|^2 && \text{by definition of } \|\cdot\| \quad \text{(Definition C.5 page 116)} \\
&> 0 \\
\implies \sum_{n=1}^{N} \alpha_n x_n &\neq 0 \\
\implies (x_n)_{n\in\mathbb{N}} &\text{ is } \textit{linearly independent} && \text{by definition of } \textit{linear independence} \quad \text{(Definition 2.3 page 9)}
\end{aligned}$$

2. Alternative proof:

$$\begin{aligned}
\sum_{n=1}^{N} \alpha_n x_n = \mathbb{0} &\implies \left\langle \sum_{n=1}^{N} \alpha_n x_n \,\middle|\, x_m \right\rangle = \langle \mathbb{0} \,|\, x_m \rangle \\
&\implies \sum_{n=1}^{N} \alpha_n \langle x_n \,|\, x_m \rangle = 0 \\
&\implies \sum_{n=1}^{N} \alpha_n \bar{\delta}(k-m) = 0 \\
&\implies \alpha_m = 0 \qquad \text{for } m = 1, 2, \ldots, N
\end{aligned}$$

✏

Theorem 2.6 (Bessel's Equality). [20] *Let* $\{x_n \in X \,|\, n=1,2,\ldots,N\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition C.9 page 124)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ *and with* $\|x\| \triangleq \sqrt{\langle x \,|\, x \rangle}$.

THM

$$\left\{ \{x_n\} \text{ is ORTHONORMAL (Definition 2.9 page 16)} \right\} \implies \left\{ \underbrace{\left\| x - \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2}_{\text{approximation error}} = \|x\|^2 - \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 \quad \forall x \in X \right\}$$

✎PROOF:

$$\begin{aligned}
&\left\| x - \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2 \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2 - 2\Re \left\langle x \,\middle|\, \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\rangle && \text{by } \textit{polar identity} \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle x \,|\, x_n \rangle \right)^* \langle x \,|\, x_n \rangle \right] && \text{by property of } \langle \triangle \,|\, \nabla \rangle \quad \text{(Definition C.9 page 124)} \\
&= \|x\|^2 + \sum_{n=1}^{N} \|\langle x \,|\, x_n \rangle x_n\|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle x \,|\, x_n \rangle \right)^* \langle x \,|\, x_n \rangle \right] && \text{by } \textit{Pythagorean Theorem}
\end{aligned}$$

[20] Bachman et al. (2000) page 103, Pedersen (2000) pages 38–39

$$
\begin{aligned}
&= \|x\|^2 + \sum_{n=1}^{N} \left\|\langle x \mid x_n\rangle x_n\right\|^2 - 2\Re\left(\sum_{n=1}^{N} \langle x \mid x_n\rangle^* \ \langle x \mid x_n\rangle\right) \\
&= \|x\|^2 + \sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 \ \underbrace{\|x_n\|^2}_{1} - 2\Re\left(\sum_{n=1}^{N} \langle x \mid x_n\rangle^* \ \langle x \mid x_n\rangle\right) && \text{by property of } \|\cdot\| && \text{(Definition C.5 page 116)} \\
&= \|x\|^2 + \sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 \ \cdot 1 - 2\Re\left(\sum_{n=1}^{N} \langle x \mid x_n\rangle^* \ \langle x \mid x_n\rangle\right) && \text{by def. of } \textit{orthonormality} && \text{(Definition 2.9 page 16)} \\
&= \|x\|^2 + \sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 - 2\Re\sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 - 2\sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 && \text{because } |\cdot| \text{ is real} \\
&= \|x\|^2 - \sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2
\end{aligned}
$$

✎

Theorem 2.7 (Bessel's inequality). [21] *Let* $\left\{x_n \in X \,\middle|\, n=1,2,\ldots,N\right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition C.9 page 124)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown\rangle\right)$ *and with* $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

THM

$$\left\{ \begin{array}{c}\{x_n\} \text{ is ORTHONORMAL} \\ \text{(Definition 2.9 page 16)}\end{array} \right\} \implies \left\{ \sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 \leq \|x\|^2 \quad \forall x \in X \right\}$$

✎PROOF:

$$
\begin{aligned}
0 &\leq \left\| x - \sum_{n=1}^{N} \langle x \mid x_n\rangle x_n \right\|^2 && \text{by definition of } \|\cdot\| && \text{(Definition C.5 page 116)} \\
&= \|x\|^2 - \sum_{n=1}^{N} \left|\langle x \mid x_n\rangle\right|^2 && \text{by } \textit{Bessel's Equality} && \text{(Theorem 2.6 page 17)}
\end{aligned}
$$

✎

The *Best Approximation Theorem* (next) shows that

- the best sequence for representing a vector is the sequence of projections of the vector onto the sequence of basis functions
- the error of the projection is orthogonal to the projection.

Theorem 2.8 (Best Approximation Theorem). [22] *Let* $\left\{x_n \in X \,\middle|\, n=1,2,\ldots,N\right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition C.9 page 124)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown\rangle\right)$ *and with* $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

[21] Giles (2000) pages 54–55 ⟨3.13 Bessel's inequality⟩, Bollobás (1999) page 147, Aliprantis and Burkinshaw (1998) page 284

[22] Walter and Shen (2001) pages 3–4, Pedersen (2000) page 39, Edwards (1995) pages 94–100, Weyl (1940)

THM $\left\{\begin{array}{c}\{x_n\} \text{ is} \\ \text{ORTHONORMAL} \\ \text{(Definition 2.9 page 16)}\end{array}\right\} \implies \left\{\begin{array}{lll}1. & \arg\min\limits_{(\!(\alpha_n)\!)_{n=1}^N} \left\|x - \sum_{n=1}^N \alpha_n x_n\right\| = \underbrace{(\!(\langle x \,|\, x_n\rangle)\!)_{n=1}^N}_{\text{best } \alpha_n = \langle x | x_n\rangle} & \forall x \in X \quad \text{and} \\ 2. & \underbrace{\left(\sum_{n=1}^N \langle x \,|\, x_n\rangle x_n\right)}_{\text{approximation}} \perp \underbrace{\left(x - \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n\right)}_{\text{approximation error}} & \forall x \in X\end{array}\right\}$

✎PROOF:

1. Proof that $(\!(\langle x \,|\, x_n\rangle)\!)$ is the best sequence:

$$\begin{aligned}
&\left\|x - \sum_{n=1}^N \alpha_n x_n\right\|^2 \\
&= \|x\|^2 - 2\Re\left\langle x \,\middle|\, \sum_{n=1}^N \alpha_n x_n\right\rangle + \left\|\sum_{n=1}^N \alpha_n x_n\right\|^2 \\
&= \|x\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^* \langle x \,|\, x_n\rangle\right) + \sum_{n=1}^N \|\alpha_n x_n\|^2 && \text{by \textit{Pythagorean Theorem}} \\
&= \|x\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^* \langle x \,|\, x_n\rangle\right) + \sum_{n=1}^N |\alpha_n|^2 + \underbrace{\left[\sum_{n=1}^N |\langle x \,|\, x_n\rangle|^2 - \sum_{n=1}^N |\langle x \,|\, x_n\rangle|^2\right]}_{0} \\
&= \left[\|x\|^2 - \sum_{n=1}^N |\langle x \,|\, x_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle x \,|\, x_n\rangle|^2 - 2\mathbf{R}_e\left[\alpha_n^* \langle x \,|\, x_n\rangle\right] + |\alpha_n|^2\right] \\
&= \left[\|x\|^2 - \sum_{n=1}^N |\langle x \,|\, x_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle x \,|\, x_n\rangle|^2 - \alpha_n^* \langle x \,|\, x_n\rangle - \alpha_n \langle x \,|\, x_n\rangle^* + |\alpha_n|^2\right] \\
&= \left\|x - \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n\right\|^2 + \sum_{n=1}^N |\langle x \,|\, x_n\rangle - \alpha_n|^2 && \text{by \textit{Bessel's Equality}} \quad \text{(Theorem 2.6 page 17)} \\
&\geq \left\|x - \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n\right\|^2
\end{aligned}$$

2. Proof that the approximation and approximation error are orthogonal:

$$\begin{aligned}
\left\langle \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n \,\middle|\, x - \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n\right\rangle &= \left\langle \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n \,\middle|\, x\right\rangle - \left\langle \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n \,\middle|\, \sum_{n=1}^N \langle x \,|\, x_n\rangle x_n\right\rangle \\
&= \sum_{n=1}^N \langle x \,|\, x_n\rangle^* \langle x \,|\, x_n\rangle - \sum_{n=1}^N \sum_{m=1}^N \langle x \,|\, x_n\rangle \langle x \,|\, x_m\rangle^* \langle x_n \,|\, x_m\rangle \\
&= \sum_{n=1}^N |\langle x \,|\, x_n\rangle|^2 - \sum_{n=1}^N \sum_{m=1}^N \langle x \,|\, x_n\rangle \langle x \,|\, x_m\rangle^* \bar{\delta}_{nm} \\
&= \sum_{n=1}^N |\langle x \,|\, x_n\rangle|^2 - \sum_{n=1}^N |\langle x \,|\, x_n\rangle|^2 \\
&= 0
\end{aligned}$$

2.5 Orthonormal bases in Hilbert spaces

Definition 2.10. *Let* $\{ \boldsymbol{x}_n \in X \,|\, n=1,2,\ldots,N \}$ *be a set of vectors in an* INNER PRODUCT SPACE (Definition C.9 page 124) $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$.

DEF

The set $\{\boldsymbol{x}_n\}$ *is an* ***orthogonal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{\boldsymbol{x}_n\}$ *is* ORTHOGONAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.
The set $\{\boldsymbol{x}_n\}$ *is an* ***orthonormal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{\boldsymbol{x}_n\}$ *is* ORTHONORMAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.

Definition 2.11. [23] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ *be a Hilbert space.*

DEF

Suppose there exists a set $\{ \boldsymbol{x}_n \in X \,|\, n\in\mathbb{N} \}$ *such that* $\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n$.

Then the quantities $\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle$ *are called the* ***Fourier coefficients*** *of* $\boldsymbol{x}$ *and the sum* $\sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n$ *is called the* ***Fourier expansion*** *of* $\boldsymbol{x}$ *or the* ***Fourier series*** *for* $\boldsymbol{x}$.

Definition 2.12.

DEF

The ***Kronecker delta function*** $\bar{\delta}_n$ *is defined as* $\bar{\delta}_n \triangleq \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$ *and* $\forall n\in\mathbb{Z}$

Lemma 2.2 (Perfect reconstruction). *Let* $\{ \boldsymbol{x}_n \in X \,|\, n\in\mathbb{N} \}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$.

LEM

$$\left\{ \begin{array}{lll} (1). & (\boldsymbol{x}_n) \text{ is a BASIS for } \boldsymbol{H} & \text{and} \\ (2). & (\boldsymbol{x}_n) \text{ is ORTHONORMAL} & \end{array} \right\} \implies \overbrace{\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \underbrace{\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle}_{\text{Fourier coefficient}} \boldsymbol{x}_n}^{\text{Fourier expansion}} \quad \forall \boldsymbol{x}\in X$$

✎PROOF:

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle &= \left\langle \sum_{m\in\mathbb{Z}} \alpha_m \boldsymbol{x}_m \,\middle|\, \boldsymbol{x}_n \right\rangle && \text{by left hypothesis (1)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \langle \boldsymbol{x}_m \,|\, \boldsymbol{x}_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \,|\, \nabla \rangle \quad \text{(Definition C.9 page 124)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \bar{\delta}_{n-m} && \text{by left hypothesis (2)} \quad \text{(Definition 2.9 page 16)} \\
&= \alpha_n
\end{aligned}$$

Proposition 2.2. [24] *Let* $\{ \boldsymbol{x}_n \in X \,|\, n\in\mathbb{N} \}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$.

[23] Fabian et al. (2010) page 27 ⟨Theorem 1.55⟩, Young (2001) page 6, Young (1980) page 6
[24] Han et al. (2007) pages 93–94 ⟨Proposition 3.11⟩

PRP

$$\underbrace{\|\boldsymbol{x}\|^2 \stackrel{\star}{=} \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2}_{\text{Parseval frame}} \iff \underbrace{\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n}_{\text{Fourier expansion (Definition 2.11 page 20)}} \qquad \forall x \in X$$

✎PROOF:

1. Proof that *Parseval frame* $\Longleftarrow$ *Fourier expansion*

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &\triangleq \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \|\cdot\| \\
&= \left\langle \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x} \,\middle|\, \boldsymbol{x}_n \right\rangle && \text{by right hypothesis} \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle^* && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by property of } \mathbb{C} \quad \text{(Definition E.7 page 149)}
\end{aligned}$$

2. Proof that *Parseval frame* $\Longrightarrow$ *Fourier expansion*

   (a) Let $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ be the *standard othornormal basis* such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.

   (b) Let $\mathbf{M}$ be an operator in $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{x} \triangleq \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{e}_n$.

   (c) lemma: $\mathbf{M}$ is *isometric*. Proof:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{e}_n \right\|^2 && \text{by definition of } \mathbf{M} \quad \text{(item (2b) page 21)} \\
&= \sum_{n=1}^{\infty} \|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{e}_n\|^2 && \text{by \textit{Pythagorean Theorem}} \\
&= \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 \|\boldsymbol{e}_n\|^2 && \text{by \textit{homogeneous} property of } \|\cdot\| \quad \text{(Definition C.5 page 116)} \\
&= \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by definition of \textit{orthonormal}} \quad \text{(Definition 2.9 page 16)} \\
&= \|\boldsymbol{x}\|^2 && \text{by Parseval frame hypothesis} \\
\Longrightarrow &\quad \mathbf{M} \text{ is \textit{isometric}} && \text{by definition of \textit{isometric}} \quad \text{(Definition C.13 page 132)}
\end{aligned}$$

   (d) Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthornormal basis* for $\boldsymbol{H}$.

(e) Proof for *Fourier expansion*:

$$\begin{aligned}
x &= \sum_{n=1}^{\infty} \langle x \mid u_n \rangle u_n && \text{by \textit{Fourier expansion} (Proposition 2.3 page 24)}\\
&= \sum_{n=1}^{\infty} \langle \mathbf{M}x \mid \mathbf{M}u_n \rangle u_n && \text{by (2c) lemma page 21 and Theorem C.23 page 133}\\
&= \sum_{n=1}^{\infty} \left\langle \sum_{m=1}^{\infty} \langle x \mid x_m \rangle e_m \,\middle|\, \sum_{k=1}^{\infty} \langle u_n \mid x_k \rangle e_k \right\rangle u_n && \text{by item (2b) page 21}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle x \mid x_m \rangle \sum_{k=1}^{\infty} \langle u_n \mid x_k \rangle^* \langle e_m \mid e_k \rangle u_n && \text{by Definition C.9 page 124}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle x \mid x_m \rangle \langle u_n \mid x_m \rangle^* u_n && \text{by item (2a) page 21 and Definition 2.9 page 16}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle x \mid x_m \rangle \langle x_m \mid u_n \rangle u_n && \text{by Definition C.9 page 124}\\
&= \sum_{m=1}^{\infty} \langle x \mid x_m \rangle \sum_{n=1}^{\infty} \langle x_m \mid u_n \rangle u_n\\
&= \sum_{m=1}^{\infty} \langle x \mid x_m \rangle x_m && \text{by item (2d) page 21}
\end{aligned}$$

⇒

When is a set of orthonormal vectors in a Hilbert space H *total*? Theorem 2.9 (next) offers some help.

Theorem 2.9 (The Fourier Series Theorem). [25] *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $H \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and let* $\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$.

THM

(A) $\{x_n\}$ *is* ORTHONORMAL *in* H $\implies$

$$\left\{\begin{array}{llrcll}
 & (1). & (\operatorname{span}\{x_n\})^- & = & H & (\{x_n\} \text{ is TOTAL in } H)\\
\iff & (2). & \langle x \mid y \rangle & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle x \mid x_n \rangle \langle y \mid x_n \rangle^* \quad \forall x,y\in X & (\text{GENERALIZED PARSEVAL'S IDENTITY})\\
\iff & (3). & \|x\|^2 & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} |\langle x \mid x_n \rangle|^2 \quad \forall x\in X & (\text{PARSEVAL'S IDENTITY})\\
\iff & (4). & x & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n \quad \forall x\in X & (\text{FOURIER SERIES EXPANSION})
\end{array}\right\}$$

✎PROOF:

[25] Bachman and Narici (1966) pages 149–155 ⟨Theorem 9.12⟩, Kubrusly (2001) pages 360–363 ⟨Theorem 5.48⟩, Aliprantis and Burkinshaw (1998) pages 298–299 ⟨Theorem 34.2⟩, Christensen (2003) page 57 ⟨Theorem 3.4.2⟩, Berberian (1961) pages 52–53 ⟨Theorem II§8.3⟩, Heil (2011) pages 34–35 ⟨Theorem 1.50⟩, Bracewell (1978) page 112 ⟨Rayleigh's theorem⟩

1. Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle &\stackrel{\star}{=} \left\langle \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \,\middle|\, \sum_{m=1}^{\infty} \langle \boldsymbol{y} \,|\, \boldsymbol{x}_m \rangle \boldsymbol{x}_m \right\rangle && \text{by (A) and (1)} \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \left\langle \boldsymbol{x}_n \,\middle|\, \sum_{m=1}^{\infty} \langle \boldsymbol{y} \,|\, \boldsymbol{x}_m \rangle \boldsymbol{x}_m \right\rangle && \text{by property of } \langle \cdot \,|\, \cdot \rangle \quad \text{(Definition C.9 page 124)} \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \sum_{m=1}^{\infty} \langle \boldsymbol{y} \,|\, \boldsymbol{x}_m \rangle^* \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_m \rangle && \text{by property of } \langle \cdot \,|\, \cdot \rangle \quad \text{(Definition C.9 page 124)} \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \sum_{m=1}^{\infty} \langle \boldsymbol{y} \,|\, \boldsymbol{x}_m \rangle^* \bar{\delta}_{mn} && \text{by (A)} \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{y} \,|\, \boldsymbol{x}_n \rangle^* && \text{by definition of } \bar{\delta}_n \quad \text{(Definition 2.12 page 20)}
\end{aligned}$$

2. Proof that (2) $\Longrightarrow$ (3):

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &\triangleq \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \textit{induced norm} \\
&= \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle^* && \text{by (2)} \\
&= \sum_{n=1}^{\infty} \left| \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right|^2
\end{aligned}$$

3. Proof that (3) $\Longleftrightarrow$ (4) *not* using (A): by Proposition 2.2 page 20

4. Proof that (3) $\Longrightarrow$ (1) (proof by contradiction):
   (a) Suppose $\{\boldsymbol{x}_n\}$ is *not total*.
   (b) Then there must exist a vector $\boldsymbol{y}$ in $\boldsymbol{H}$ such that the set $B \triangleq \{\boldsymbol{x}_n\} \cup \boldsymbol{y}$ is *orthonormal*.
   (c) Then $1 = \|y\|^2 \neq \sum_{n=1}^{\infty} \left| \langle \boldsymbol{y} \,|\, \boldsymbol{x}_n \rangle \right|^2 = 0$.
   (d) But this contradicts (3), and so $\{\boldsymbol{x}_n\}$ must be *total* and (3) $\Longrightarrow$ (1).

5. Extraneous proof that (3) $\Longrightarrow$ (4) (this proof is not really necessary here):

$$\begin{aligned}
\left\| \boldsymbol{x} - \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 &= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{\infty} | \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle |^2 && \text{by } \textit{Bessel's Equality} \quad \text{(Theorem 2.6 page 17)} \\
&= 0 && \text{by (3)} \\
\Longrightarrow \quad \boldsymbol{x} &\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by definition of } \stackrel{\star}{=}
\end{aligned}$$

6. Extraneous proof that (A) $\Longrightarrow$ (4) (this proof is not really necessary here)
   (a) The sequence $\sum_{n=1}^{N} \left| \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right|^2$ is *monotonically increasing* in n.
   (b) By Bessel's inequality (page 18), the sequence is upper bounded by $\|\boldsymbol{x}\|^2$:

$$\sum_{n=1}^{N} \left| \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right|^2 \leq \|\boldsymbol{x}\|^2$$

(c) Because this sequence is both monotonically increasing and bounded in n, it must equal its bound in the limit as n approaches infinity:

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle\right|^2 = \|\boldsymbol{x}\|^2 \tag{2.1}$$

(d) If we combine this result with *Bessel's Equality* (Theorem 2.6 page 17) we have

$$\begin{aligned}
\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2 &= \|\boldsymbol{x}\|^2 - \lim_{N\to\infty}\sum_{n=1}^{N}|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle|^2 && \text{by \textit{Bessel's Equality} (Theorem 2.6 page 17)}\\
&= \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 && \text{by equation (2.1) page 24}\\
&= 0
\end{aligned}$$

✏

Proposition 2.3 (Fourier expansion). *Let* $\left\{\boldsymbol{x}_n \in X \,\middle|\, n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla\rangle\right)$.

$$\text{PRP}\quad \underbrace{\left\{\boldsymbol{x}_n\right\} \textit{ is an } \text{ORTHONORMAL BASIS} \textit{ for } \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{\boldsymbol{x} \doteq \sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n}_{(1)} \iff \underbrace{\alpha_n = \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle}_{(2)}\right\}$$

✎PROOF:

1. Proof that (1) $\implies$ (2): by Lemma 2.2 page 20
2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\left\|\boldsymbol{x}-\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_{n\in\mathbb{Z}}\right\|^2 &= \left\|\boldsymbol{x}-\sum_{n=1}^{\infty}\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle\,\boldsymbol{x}_{n\in\mathbb{Z}}\right\|^2 && \text{by right hypothesis}\\
&= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{\infty}|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle|^2 && \text{by Bessel's equality (Theorem 2.6 page 17)}\\
&= 0 && \text{by \textit{Parseval's Identity} (Theorem 2.9 page 22)}\\
&\overset{\text{def}}{\iff} \boldsymbol{x} \doteq \sum_{n=1}^{\infty}\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n && \text{by definition of \textit{strong convergence}}
\end{aligned}$$

✏

Proposition 2.4 (Riesz-Fischer Theorem). [26] *Let* $\left\{\boldsymbol{x}_n \in X \,\middle|\, n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla\rangle\right)$.

$$\text{PRP}\quad \underbrace{\left\{\boldsymbol{x}_n\right\} \textit{ is an } \text{ORTHONORMAL BASIS} \textit{ for } \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{\sum_{n=1}^{\infty}|\alpha_n|^2 < \infty}_{(1)} \iff \underbrace{\exists \boldsymbol{x}\in\boldsymbol{H} \text{ such that } \alpha_n = \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle}_{(2)}\right\}$$

✎PROOF:

[26] Young (2001) page 6

1. Proof that (1) $\implies$ (2):

   (a) If (1) is true, then let $\boldsymbol{x} \triangleq \sum_{n\in\mathbb{N}} \alpha_n \boldsymbol{x}_n$.

   (b) Then

$$
\begin{aligned}
\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle &= \left\langle \sum_{m\in\mathbb{N}} \alpha_m \boldsymbol{x}_m \mid \boldsymbol{x}_n \right\rangle && \text{by definition of } \boldsymbol{x} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \langle \boldsymbol{x}_m \mid \boldsymbol{x}_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \nabla \rangle && \text{(Definition C.9 page 124)} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \bar{\delta}_{mn} && \text{by (A)} \\
&= \sum_{m\in\mathbb{N}} \alpha_n && \text{by definition of } \bar{\delta} && \text{(Definition 2.12 page 20)}
\end{aligned}
$$

2. Proof that (1) $\impliedby$ (2):

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} |\alpha_n|^2 &= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 && \text{by (2)} \\
&\le \|\boldsymbol{x}\|^2 && \text{by } \textit{Bessel's Inequality} && \text{(Theorem 2.7 page 18)} \\
&\le \infty
\end{aligned}
$$

✏

Theorem 2.10. [27]

THM *All* SEPARABLE HILBERT SPACES *are* ISOMORPHIC. *That is,*

$$
\left\{ \begin{array}{l} \boldsymbol{X} \textit{ is a separable} \\ \textit{Hilbert space} \quad \textit{and} \\ \boldsymbol{Y} \textit{ is a separable} \\ \textit{Hilbert space} \end{array} \right\} \implies \left\{ \begin{array}{lllll} \multicolumn{5}{l}{\textit{there is a } \text{BIJECTIVE} \textit{ operator } \mathbf{M} \in \boldsymbol{Y}^{\boldsymbol{X}} \textit{ such that}} \\ \text{(1).} & \boldsymbol{y} = & \mathbf{M}\boldsymbol{x} & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} & \textit{and} \\ \text{(2).} & \|\mathbf{M}\boldsymbol{x}\| = & \|\boldsymbol{x}\| & \forall x\in\boldsymbol{X} & \textit{and} \\ \text{(3).} & \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle = & \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} & \end{array} \right\}
$$

✎PROOF:

1. Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{ \boldsymbol{x}_n | n\in\mathbb{N} \}$.
   Let $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{ \boldsymbol{y}_n | n\in\mathbb{N} \}$.

2. Proof that there exists *bijective* operator $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$:

   (a) Let $\mathbf{M}$ be defined such that $\boldsymbol{y}_n \triangleq \mathbf{M}\boldsymbol{x}_n$.

   (b) Thus $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$.

   (c) Because $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$, $\mathbf{M}$ has an inverse operator $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$ such that $\boldsymbol{x}_n = \mathbf{M}^{-1}\boldsymbol{y}_n$.

3. Proof that $\mathbf{M}$ and $\mathbf{M}^{-1}$ are *bijective* operators between $\boldsymbol{X}$ and $\boldsymbol{Y}$:

[27] Young (2001) page 6

(a) Proof that $\mathbf{M}$ maps $\boldsymbol{X}$ *into* $\boldsymbol{Y}$:

$$
\begin{aligned}
\boldsymbol{x} \in \boldsymbol{X} \iff \boldsymbol{x} &\triangleq \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem 2.9 page 22)}\\
&\implies \exists \boldsymbol{y} \in \boldsymbol{Y} \quad \text{such that} \quad \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle = \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition 2.4 page 24)}\\
&\implies\\
\boldsymbol{y} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem 2.9 page 22)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{y}_n && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition 2.4 page 24)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n && \text{by definition of } \mathbf{M} && \text{(item (2a) page 25)}\\
&= \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by prop. of linear ops.} && \text{(Theorem C.1 page 113)}\\
&= \mathbf{M}\boldsymbol{x} && \text{by definition of } \boldsymbol{x}
\end{aligned}
$$

(b) Proof that $\mathbf{M}^{-1}$ maps $\boldsymbol{Y}$ *into* $\boldsymbol{X}$:

$$
\begin{aligned}
\boldsymbol{y} \in \boldsymbol{Y} \iff \boldsymbol{y} &\triangleq \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem 2.9 page 22)}\\
&\implies \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle = \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition 2.4 page 24)}\\
&\implies\\
\boldsymbol{x} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem 2.9 page 22)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{x}_n && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition 2.4 page 24)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \mathbf{M}^{-1}\boldsymbol{y}_n && \text{by definition of } \mathbf{M}^{-1} && \text{(item (2c) page 25)}\\
&= \mathbf{M}^{-1} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by prop. of \textit{linear ops.}} && \text{(Theorem C.1 page 113)}\\
&= \mathbf{M}^{-1}\boldsymbol{y} && \text{by definition of } \boldsymbol{y}
\end{aligned}
$$

4. Proof for (2):

$$
\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem 2.9 page 22)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n \right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem C.1 page 113)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{y}_n \right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2a) page 25)}\\
&= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition 2.4 page 24)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition 2.4 page 24)}\\
&= \|\boldsymbol{x}\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem 2.9 page 22)}
\end{aligned}
$$

5. Proof for (3): by (2) and Theorem C.23 page 133

Theorem 2.11. [28] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has a* SCHAUDER BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

Theorem 2.12. [29] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has an* ORTHONORMAL BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

2.6 Riesz bases in Hilbert spaces

Definition 2.13. [30] *Let* $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+},$

DEF: $\{\boldsymbol{x}_n\}$ *is a* ***Riesz basis*** *for* $\boldsymbol{H}$ *if* $\{\boldsymbol{x}_n\}$ *is* EQUIVALENT (Definition 2.8 page 15) *to some* ORTHONORMAL BASIS (Definition 2.10 page 20) *in* $\boldsymbol{H}$.

Definition 2.14. [31] *Let* $(\boldsymbol{x}_n \in X)_{n\in\mathbb{N}}$ *be a sequence of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

DEF: *The sequence* $(\boldsymbol{x}_n)$ *is a* ***Riesz sequence*** *for* $\boldsymbol{H}$ *if*

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 \qquad \forall(\alpha_n)\in\ell^2_{\mathbb{F}}.$$

Definition 2.15. *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be an* INNER PRODUCT SPACE (Definition C.9 page 124).

DEF: *The sequences* $(\boldsymbol{x}_n \in X)_{n\in\mathbb{Z}}$ *and* $(\boldsymbol{y}_n \in X)_{n\in\mathbb{Z}}$ *are* ***biorthogonal*** *with respect to each other in* $\boldsymbol{X}$ *if* $\langle \boldsymbol{x}_n \mid \boldsymbol{y}_m \rangle = \bar{\delta}_{nm}$

Lemma 2.3. [32] *Let* $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ *be a sequence in a* HILBERT SPACE $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$. *Let* $\{\boldsymbol{y}_n \mid n\in\mathbb{N}\}$ *be a sequence in a* HILBERT SPACE $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$. *Let*

LEM:

$$\left\{\begin{array}{lll}
\text{(i).} & \{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{X} & \text{and}\\
\text{(ii).} & \text{There exists } A>0 \text{ such that } A\sum_{n\in C}|a_n|^2 \le \left\|\sum_{n\in C}a_n\boldsymbol{x}_n\right\|^2 \quad \text{for finite } C & \text{and}\\
\text{(iii).} & \text{There exists } B>0 \text{ such that } \left\|\sum_{n=1}^{\infty}b_n\boldsymbol{y}_n\right\|^2 \le B\sum_{n=1}^{\infty}|b_n|^2 \quad \forall(b_n)_{n\in\mathbb{N}}\in\ell^2_{\mathbb{F}} &
\end{array}\right\} \implies$$

$$\left\{\begin{array}{lll}
\text{(1).} & \mathbf{R}^\circ \text{ is a linear bounded operator that maps from } \operatorname{span}\{\boldsymbol{x}_n\} \text{ to } \operatorname{span}\{\boldsymbol{y}_n\} & \\
 & \quad \text{where } \mathbf{R}^\circ\sum_{n\in C}c_n\boldsymbol{x}_n \triangleq \sum_{n\in C}c_n\boldsymbol{y}_n, \text{ for some sequence } (c_n) \text{ and finite set } C & \text{and}\\
\text{(2).} & \mathbf{R} \text{ has a unique extension to a bounded operator } \mathbf{R} \text{ that maps from } \boldsymbol{X} \text{ to } \boldsymbol{Y} & \text{and}\\
\text{(3).} & |||\mathbf{R}^\circ||| \le \frac{B}{A} & \text{and}\\
\text{(4).} & |||\mathbf{R}||| \le \frac{B}{A} &
\end{array}\right\}$$

[28] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Berberian (1961) page 53 ⟨Theorem II§8.3⟩

[29] Kubrusly (2001) page 357 ⟨Proposition 5.43⟩

[30] Young (2001) page 27 ⟨Definition 1.8.2⟩, Christensen (2003) page 63 ⟨Definition 3.6.1⟩, Heil (2011) page 196 ⟨Definition 7.9⟩

[31] Christensen (2003) pages 66–68 ⟨page 68 and (3.24) on page 66⟩, Wojtaszczyk (1997) page 20 ⟨Definition 2.6⟩

[32] Christensen (2003) pages 65–66 ⟨Lemma 3.6.5⟩

Theorem 2.13. [33] *Let* $\left\{ \boldsymbol{x}_n \in X \middle| n\in\mathbb{N} \right\}$ *be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM

$$\left\{ \begin{array}{l} \{\boldsymbol{x}_n\} \text{ is a RIESZ BASIS} \\ \text{for } \boldsymbol{H} \end{array} \right\} \iff \left\{ \begin{array}{lll} (1). & \{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{H} & \text{and} \\ (2). & \exists A,B\in\mathbb{R}^+ \text{ such that } & \forall (\alpha_n)\in\ell^2_{\mathbb{F}}, \\ & A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 & \end{array} \right\}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:
   (a) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (1): all bases for $\boldsymbol{H}$ are *total* in $\boldsymbol{H}$.
   (b) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (2):
      i. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthonormal basis* for $\boldsymbol{H}$.
      ii. Let $\mathbf{R}$ be a *bounded bijective* operator such that $\boldsymbol{x}_n = \mathbf{R}\boldsymbol{u}_n$.
      iii. Proof for upper bound B:

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \left\|\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii))}\\
&= \left\|\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem C.1 page 113}\\
&\le |||\mathbf{R}|||^2\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem C.6 page 119}\\
&= |||\mathbf{R}|||^2\sum_{n=1}^{\infty}\left\|\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by \textit{Pythagorean Theorem}}\\
&= |||\mathbf{R}|||^2\sum_{n=1}^{\infty}|\alpha|^2\left\|\boldsymbol{u}_n\right\|^2 && \text{by \textit{homogeneous} property of norms} && \text{(Definition C.5 page 116)}\\
&= \underbrace{|||\mathbf{R}|||^2}_{B}\sum_{n=1}^{\infty}|\alpha|^2 && \text{by definition of \textit{orthonormality}} && \text{(Definition 2.9 page 16)}
\end{aligned}$$

      iv. Proof for lower bound A:

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \frac{|||\mathbf{R}^{-1}|||^2}{|||\mathbf{R}^{-1}|||^2}\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{because } |||\mathbf{R}^{-1}||| > 0 && \text{(Proposition C.1 page 117)}\\
&\ge \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{by Theorem C.6 page 119}\\
&= \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii) page 28)}\\
&= \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem C.1 page 113)}
\end{aligned}$$

[33] Young (2001) page 27 ⟨Theorem 1.8.9⟩, Christensen (2003) page 66 ⟨Theorem 3.6.6⟩, Heil (2011) pages 197–198 ⟨Theorem 7.13⟩, Christensen (2008) pages 61–62 ⟨Theorem 3.3.7⟩

$$
\begin{aligned}
&= \frac{1}{\|\mathbf{R}^{-1}\|^2}\left\|\sum_{n=1}^{\infty}\alpha_n \boldsymbol{u}_n\right\|^2 && \text{by definition of inverse op.} && \text{(Definition C.3 page 112)}\\
&= \frac{1}{\|\mathbf{R}^{-1}\|^2}\sum_{n=1}^{\infty}\|\alpha_n \boldsymbol{u}_n\|^2 && \text{by \textit{Pythagorean Theorem}}\\
&= \frac{1}{\|\mathbf{R}^{-1}\|^2}\sum_{n=1}^{\infty}|\alpha_n|^2\,\|\boldsymbol{u}_n\|^2 && \text{by } \|\cdot\| \text{ \textit{homogeneous} prop.} && \text{(Definition C.5 page 116)}\\
&= \underbrace{\frac{1}{\|\mathbf{R}^{-1}\|^2}}_{A}\sum_{n=1}^{\infty}|\alpha_n|^2 && \text{by def. of \textit{orthonormality}} && \text{(Definition 2.9 page 16)}
\end{aligned}
$$

2. Proof for ($\Longrightarrow$) case:
   (a) Let $\{\boldsymbol{u}_n | n\in\mathbb{N}\}$ be an *orthonormal basis* for $\boldsymbol{H}$.
   (b) Using (2) and Lemma 2.3 (page 27), construct an bounded extension operator $\mathbf{R}$ such that $\mathbf{R}\boldsymbol{u}_n = \boldsymbol{x}_n$ for all $n\in\mathbb{N}$.
   (c) Using (2) and Lemma 2.3 (page 27), construct an bounded extension operator $\mathbf{S}$ such that $\mathbf{S}\boldsymbol{x}_n = \boldsymbol{u}_n$ for all $n\in\mathbb{N}$.
   (d) Then, $\mathbf{RV}\boldsymbol{x} = \mathbf{VR}\boldsymbol{x} \implies \mathbf{V} = \mathbf{R}^{-1}$, and so $\mathbf{R}$ is a bounded invertible operator
   (e) and $\{\boldsymbol{x}_n\}$ is a *Riesz sequence*.

✏

Theorem 2.14. [34] *Let* $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\triangledown\rangle\right)$ *be a* SEPARABLE HILBERT SPACE.

THM

$$
\left\{\begin{array}{l}(\boldsymbol{x}_n \in \boldsymbol{H})_{n\in\mathbb{Z}} \textit{ is a}\\ \text{RIESZ BASIS } \textit{for } \boldsymbol{H}\end{array}\right\} \implies \left\{\begin{array}{l}\textit{There exists } (\boldsymbol{y}_n\in\boldsymbol{H})_{n\in\mathbb{Z}} \textit{ such that}\\
\quad (1).\ (\boldsymbol{x}_n) \textit{ and } (\boldsymbol{y}_n) \textit{ are } \text{BIORTHOGONAL} \quad \textit{and}\\
\quad (2).\ (\boldsymbol{y}_n) \textit{ is also a } \text{RIESZ BASIS } \textit{for } \boldsymbol{H} \quad \textit{and}\\
\quad (3).\ \exists B>A>0 \text{ such that}\\
\qquad A\sum_{n=1}^{\infty}|a_n|^2 \le \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{x}_n\right\|^2 = \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{y}_n\right\|^2 \le B\sum_{n=1}^{\infty}|a_n|^2\\
\qquad \forall (a_n)_{n\in\mathbb{N}}\in\ell^2_{\mathbb{F}}
\end{array}\right\}
$$

✎PROOF:

1. Proof for (1):
   (a) Let $\boldsymbol{e}_n$ be the *unit vector* in $\boldsymbol{H}$ such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.
   (b) Let $\mathbf{M}$ be an operator on $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{e}_n = \boldsymbol{x}_n$.
   (c) Note that $\mathbf{M}$ is *isometric*, and as such $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall \boldsymbol{x}\in\boldsymbol{H}$.
   (d) Let $\boldsymbol{y}_n \triangleq (\mathbf{M}^{-1})^*$.
   (e) Then,

$$
\begin{aligned}
\langle \boldsymbol{y}_n \,|\, \boldsymbol{x}_m\rangle &= \left\langle (\mathbf{M}^{-1})^*\boldsymbol{e}_n \,|\, \mathbf{M}\boldsymbol{e}_m\right\rangle\\
&= \left\langle \boldsymbol{e}_n \,|\, \mathbf{M}^{-1}\mathbf{M}\boldsymbol{e}_m\right\rangle\\
&= \langle \boldsymbol{e}_n \,|\, \boldsymbol{e}_m\rangle\\
&= \bar{\delta}_{nm}\\
&\implies \{\boldsymbol{x}_n\} \text{ and } \{\boldsymbol{y}_n\} \text{ are \textit{biorthogonal}} && \text{by Definition 2.9 page 16}
\end{aligned}
$$

[34] Wojtaszczyk (1997) page 20 ⟨Lemma 2.7(a)⟩

2. Proof for (3):

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{y}_n\right\| &= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \left(\mathbf{M}^{-1}\right)^* \boldsymbol{e}_n\right\| && \text{by definition of } \boldsymbol{y}_n && \text{(Proposition 1d page 29)}\\
&= \left\|\left(\mathbf{M}^{-1}\right)^* \sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{by property of } \textit{linear ops.}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \left(\mathbf{M}^{-1}\right)^* \text{ is } \textit{isometric} && \text{(Definition C.13 page 132)}\\
&= \left\|\mathbf{M}\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \mathbf{M} \text{ is } \textit{isometric} && \text{(Definition C.13 page 132)}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \mathbf{M}\boldsymbol{e}_n\right\| && \text{by property of } \textit{linear operators}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{x}_n\right\| && \text{by definition of } \mathbf{M}\\
&\implies \{\boldsymbol{y}_n\} \text{ is a } \textit{Riesz basis} && \text{by left hypothesis}
\end{aligned}$$

3. Proof for (2): by (3) and definition of *Riesz basis* (Definition 2.13 page 27)

Proposition 2.5. [35] *Let* $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$.

PRP

$$\left\{\begin{array}{l}\{\boldsymbol{x}_n\} \text{ is a } \textsc{Riesz basis} \text{ for } \boldsymbol{H} \text{ with}\\ A\sum_{n=1}^{\infty}|a_n|^2 \le \left\|\sum_{n=1}^{\infty} a_n \boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|a_n|^2\\ \forall \{a_n\}\in\ell^2_{\mathbb{F}}\end{array}\right\} \implies \left\{\begin{array}{l}\{\boldsymbol{x}_n\} \text{ is a } \textsc{frame} \text{ for } \boldsymbol{H} \text{ with}\\ \underbrace{\frac{1}{B}\|\boldsymbol{x}\|^2 \le \sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \le \frac{1}{A}\|\boldsymbol{x}\|^2}_{\text{STABILITY CONDITION}}\\ \forall \boldsymbol{x}\in\boldsymbol{H}\end{array}\right\}$$

✎PROOF:

1. Let $\left\{\boldsymbol{y}_n \middle| n\in\mathbb{N}\right\}$ be a *Riesz basis* that is *biorthogonal* to $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ (Theorem 2.14 page 29).

2. Let $\boldsymbol{x} \triangleq \sum_{n=1}^{\infty} a_n \boldsymbol{y}_n$.

3. lemma:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 &= \sum_{n=1}^{\infty}\left|\left\langle \sum_{m=1}^{\infty} a_n \boldsymbol{y}_m \,\middle|\, \boldsymbol{x}_n\right\rangle\right|^2 && \text{by definition of } \boldsymbol{x} && \text{(item (2) page 30)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \langle \boldsymbol{y}_m \mid \boldsymbol{x}_n\rangle\right|^2 && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \nabla \rangle && \text{(Definition C.9 page 124)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \bar{\delta}_{mn}\right|^2 && \text{by definition of } \textit{biorthogonal} && \text{(Definition 2.15 page 27)}\\
&= \sum_{n=1}^{\infty}|a_n|^2 && \text{by definition of } \bar{\delta} && \text{(Definition 2.12 page 20)}
\end{aligned}$$

[35] Igari (1996) page 220 ⟨Lemma 9.8⟩, Wojtaszczyk (1997) pages 20–21 ⟨Lemma 2.7(a)⟩

4. Then

$$\begin{array}{llllll}
 & A\sum_{n=1}^{\infty}|a_n|^2 & \le & \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{x}_n\right\|^2 & \le & B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 30)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le & \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{y}_n\right\|^2 & \le & B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 30)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le & \|\boldsymbol{x}\|^2 & \le & B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \boldsymbol{x} \text{ (item (2) page 30)}\\
\implies & A\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \le & \|\boldsymbol{x}\|^2 & \le & B\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \text{by (3) lemma}\\
\implies & \frac{1}{B}\|\boldsymbol{x}\|^2 & \le & \sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \le & \frac{1}{A}\|\boldsymbol{x}\|^2 &
\end{array}$$

✏

Theorem 2.15 (Battle-Lemarié orthogonalization). [36] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* Fourier Transform *(Definition H.2 page 192) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM

$$\left\{\begin{array}{ll}1. & \{\mathbf{T}^n\mathsf{g}\,|\,n\in\mathbb{Z}\} \text{ is a Riesz basis for } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{and}\\ 2. & \tilde{\mathsf{f}}(\omega) \triangleq \dfrac{\tilde{\mathsf{g}}(\omega)}{\sqrt{2\pi\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2}}\end{array}\right\} \implies \left\{\begin{array}{l}\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}\\ \text{is an orthonormal basis for } \boldsymbol{L}^2_{\mathbb{R}}\end{array}\right\}$$

✎Proof:

1. Proof that $\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}$ is orthonormal:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\phi\phi}(\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem O.1 page 241}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{2\pi\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi(n-m))|^2}}\right|^2 && \text{by left hypothesis}\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{1}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2|\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= \frac{1}{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= 1\\
\implies & \{\mathsf{f}_n\,|\,n\in\mathbb{Z}\} \text{ is orthonormal} && \text{by Theorem O.3 page 247}
\end{aligned}$$

[36] Wojtaszczyk (1997) page 25 ⟨Remark 2.4⟩, Vidakovic (1999) page 71, Mallat (1989) page 72, Mallat (1999) page 225, Daubechies (1992) page 140 ⟨(5.3.3)⟩

2. Proof that $\{\mathbf{T}^n \mathsf{f} \mid n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_0$: by Lemma 2.1 page 15.

⇨

2.7 Frames in Hilbert spaces

Definition 2.16. [37] *Let $\{x_n \in X \mid n\in\mathbb{N}\}$ be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

DEF

*The set $\{x_n\}$ is a **frame** for $\boldsymbol{H}$ if* (STABILITY CONDITION)

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\,\|x\|^2 \le \sum_{n=1}^{\infty} |\langle x \mid x_n\rangle|^2 \le B\,\|x\|^2 \qquad \forall x\in X.$$

*The quantities A and B are **frame bounds**.*
*The quantity A' is the **optimal lower frame bound** if*
$A' = \sup\{A \in \mathbb{R}^+ \mid A \text{ is a lower frame bound}\}$.
*The quantity B' is the **optimal upper frame bound** if*
$B' = \inf\{B \in \mathbb{R}^+ \mid B \text{ is an upper frame bound}\}$.
*A frame is a **tight frame** if $A = B$.*
*A frame is a **normalized tight frame** (or a **Parseval frame**) if $A = B = 1$.*
*A frame $\{x_n \mid n\in\mathbb{N}\}$ is an **exact frame** if for some $m \in \mathbb{Z}$, $\{x_n \mid n\in\mathbb{N}\}\setminus\{x_m\}$ is* NOT *a frame.*

A frame is a *Parseval frame* (Definition 2.16) if it satisfies *Parseval's Identity* (Theorem 2.9 page 22). All orthonormal bases are Parseval frames (Theorem 2.9 page 22); but not all Parseval frames are orthonormal bases.

Definition 2.17. *Let $\{x_n\}$ be a **frame** (Definition 2.16 page 32) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$. *Let* $\mathbf{S}$ *be an* OPERATOR *on* $\boldsymbol{H}$.

DEF

$\mathbf{S}$ *is a **frame operator** for $\{x_n\}$ if* $\quad \mathbf{S}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid x_n\rangle\, x_n \qquad \forall \mathsf{f}\in\boldsymbol{H}.$

Theorem 2.16. [38] *Let* $\mathbf{S}$ *be a* FRAME OPERATOR *(Definition 2.17 page 32) of a* FRAME $\{x_n\}$ *(Definition 2.16 page 32) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

THM

(1). $\mathbf{S}$ *is* INVERTIBLE. *and*
(2). $$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \mathbf{S}^{-1} x_n\rangle\, x_n = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid x_n\rangle\, \mathbf{S}^{-1} x_n \qquad \forall \mathsf{f} \in \boldsymbol{H}$$

Theorem 2.17. [39] *Let $\{x_n \in X \mid n=1,2,\ldots,N\}$ be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

THM

$\{x_n\}$ *is a* FRAME *for* $\operatorname{span}\{x_n\}$.

✎PROOF:

[37] Young (2001) pages 154–155, Christensen (2003) page 88 ⟨Definitions 5.1.1, 5.1.2⟩, Heil (2011) pages 204–205 ⟨Definition 8.2⟩, Jørgensen et al. (2008) page 267 ⟨Definition 12.22⟩, Duffin and Schaeffer (1952) page 343, Daubechies et al. (1986) page 1272

[38] Christensen (2008) pages 100–102 ⟨Theorem 5.1.7⟩

[39] Christensen (2003) page 3

1. Upper bound: Proof that there exists B such that $\sum_{n=1}^{N} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 \le B \|\boldsymbol{x}\|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

$$\begin{aligned}
\sum_{n=1}^{N} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 &\le \sum_{n=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by \textit{Cauchy-Schwarz inequality}}\\
&= \underbrace{\left\{ \sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2 \right\}}_{B} \|\boldsymbol{x}\|^2
\end{aligned}$$

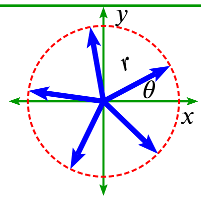
2. Lower bound: Proof that there exists A such that $A \|\boldsymbol{x}\|^2 \le \sum_{n=1}^{N} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

$$\begin{aligned}
\sum_{n=1}^{N} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 &= \sum_{n=1}^{N} \left| \left\langle \boldsymbol{x}_n \,\middle|\, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right\rangle \right|^2 \|\boldsymbol{x}\|^2\\
&\ge \underbrace{\left(\inf_{\boldsymbol{y}} \left\{ \sum_{n=1}^{N} |\langle \boldsymbol{x}_n \,|\, \boldsymbol{y} \rangle|^2 \,|\, \|\boldsymbol{y}\| = 1 \right\} \right)}_{A} \|\boldsymbol{x}\|^2
\end{aligned}$$

✏

Example 2.1. Let $(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \triangledown \rangle)$ be an inner product space with $\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \,\middle|\, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1 x_2 + y_1 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition 2.17 page 32) with *inverse* $\mathbf{S}^{-1}$.

EX



Let $N \in \{3, 4, 5, \ldots\}$, $\theta \in \mathbb{R}$, and $r \in \mathbb{R}^+$ $(r > 0)$.
Let $\boldsymbol{x}_n \triangleq r \begin{bmatrix} \cos(\theta + 2n\pi/N) \\ \sin(\theta + 2n\pi/N) \end{bmatrix} \quad \forall n \in \{0, 1, \ldots, N-1\}$.
Then, $(\!(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})\!)$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{Nr^2}{2}$.

Moreover, $\quad \mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{S}^{-1} = \frac{2}{Nr^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad$ and $\quad \boldsymbol{v} = \dfrac{2}{Nr^2} \sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \quad \forall \boldsymbol{v} \in \mathbb{R}^2$.

✎PROOF:

1. Proof that $(\!(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})\!)$ is a *tight frame* with *frame bound* $A = \frac{Nr^2}{2}$: Let $\boldsymbol{v} \triangleq (x, y) \in \mathbb{R}^2$.

$$\begin{aligned}
\sum_{n=0}^{N-1} |\langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle|^2 &\triangleq \sum_{n=0}^{N-1} \left| \boldsymbol{v}^{\mathbf{H}} r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \right|^2 && \text{by definitions of } \boldsymbol{v} \text{ of } \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle\\
&\triangleq \sum_{n=0}^{N-1} r^2 \left| x\cos\left(\theta + \frac{2n\pi}{N}\right) + y\sin\left(\theta + \frac{2n\pi}{N}\right) \right|^2 && \text{by definition of } \boldsymbol{y}^{\mathbf{H}}\boldsymbol{x} \text{ operation}\\
&= r^2 x^2 \sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 y^2 \sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 xy \sum_{n=0}^{N-1} \cos\left(\theta + \frac{2n\pi}{N}\right) \sin\left(\theta + \frac{2n\pi}{N}\right)\\
&= r^2 x^2 \frac{N}{2} + r^2 y^2 \frac{N}{2} + r^2 xy0 && \text{by Corollary G.1 page 185}\\
&= (x^2 + y^2) \frac{Nr^2}{2} = \left(\frac{Nr^2}{2} \right) \boldsymbol{v}^{\mathbf{H}} \boldsymbol{v} \triangleq \underbrace{\left(\frac{Nr^2}{2} \right)}_{A} \|\boldsymbol{v}\|^2 && \text{by definition of } \|\boldsymbol{v}\|
\end{aligned}$$

2. Proof that $\mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

(a) Let $\boldsymbol{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) lemma: $\mathbf{S}\boldsymbol{e}_1 = \frac{Nr^2}{2}\boldsymbol{e}_1$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_1 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\cos\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \cos^2\left(\theta + \frac{2n\pi}{N}\right) \\ \cos\left(\theta + \frac{2n\pi}{N}\right)\sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} N/2 \\ 0 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_1 \qquad \text{by \textit{Summation around unit circle} (Corollary G.1 page 185)}
\end{aligned}$$

(c) lemma: $\mathbf{S}\boldsymbol{e}_2 = \frac{Nr^2}{2}\boldsymbol{e}_2$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_2 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_2 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\sin\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \sin\left(\theta + \frac{2n\pi}{N}\right)\cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin^2\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} 0 \\ N/2 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_2 \qquad \text{by \textit{Summation around unit circle} (Corollary G.1 page 185)}
\end{aligned}$$

(d) Complete the proof of item (2) using *Eigendecomposition* $\mathbf{S} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \frac{Nr^2}{2}\boldsymbol{e}_1 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_1 \text{ is an \textit{eigenvector} of } \mathbf{S} \text{ with \textit{eigenvalue} } \frac{Nr^2}{2} \\
\mathbf{S}\boldsymbol{e}_2 &= \frac{Nr^2}{2}\boldsymbol{e}_2 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_2 \text{ is an \textit{eigenvector} of } \mathbf{S} \text{ with \textit{eigenvalue} } \frac{Nr^2}{2}
\end{aligned}$$

$$\mathbf{S} = \overbrace{\underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}^{-1}}_{\mathbf{Q}^{-1}}}^{\textit{Eigendecomposition} \text{ of } \mathbf{S}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Proof that $\mathbf{S}^{-1} = \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$\begin{aligned}
\mathbf{S}\mathbf{S}^{-1} &= \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)} \\
\mathbf{S}^{-1}\mathbf{S} &= \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)}
\end{aligned}$$

4. Proof that $\boldsymbol{v} = \frac{2}{Nr^2}\sum_{n=0}^{N-1} (\boldsymbol{v}^{\mathsf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n = \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,\middle|\, \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n && \text{by item (3)} \\
&= \frac{2}{Nr^2}\sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \frac{2}{Nr^2}\sum_{n=0}^{N-1} (\boldsymbol{v}^{\mathsf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n && \text{by definition of } \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle
\end{aligned}$$

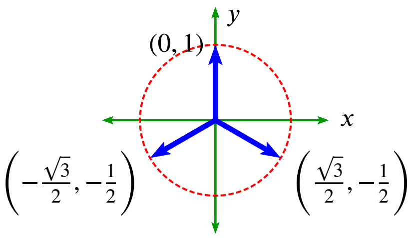
Example 2.2 (Peace Frame/Mercedes Frame). [40] Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \bigtriangledown \rangle\right)$ be an inner product space with $\left\langle (x_1, y_1) \mid (x_2, y_2) \right\rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition 2.17 page 32) with *inverse* $\mathbf{S}^{-1}$.

Let $\boldsymbol{x}_1 \triangleq \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\boldsymbol{x}_2 \triangleq \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}$, and $\boldsymbol{x}_3 \triangleq \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$.

Then, $\left(\!\left(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\right)\!\right)$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{3}{2}$.

Moreover, $\mathbf{S} = \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \mathbf{S}^{-1} = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

and $\quad \boldsymbol{v} = \frac{2}{3}\sum_{n=1}^{3} \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \triangleq \frac{2}{3}\sum_{n=1}^{3} \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_n\right) \boldsymbol{x}_n \qquad \forall \boldsymbol{v} \in \mathbb{R}^2.$



✎PROOF:

1. This frame is simply a special case of the frame presented in Example 2.1 (page 33) with $r = 1$, $N = 3$, and $\theta = \pi/2$.

2. Let's give it a try! Let $\boldsymbol{v} \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned}
\sum_{n=1}^{3} \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n &= \frac{2}{3}\sum_{n=1}^{3} \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_n\right) \boldsymbol{x}_n && \text{by Example 2.1 page 33} \\
&= \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_1\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_2\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_3\right)\boldsymbol{x}_3 \\
&= \frac{2}{3}\left(\left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{2}{3}\cdot\frac{1}{2}\left(\left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{1}{3}\left((2)\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \left(-\sqrt{3}-1\right)\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix} + \left(\sqrt{3}-1\right)\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right) \\
&= \frac{1}{6}\begin{bmatrix} 2(0) & + & (-\sqrt{3}-1)(-\sqrt{3}) & + & (\sqrt{3}-1)(\sqrt{3}) \\ 2(2) & + & (-\sqrt{3}-1)(-1) & + & (\sqrt{3}-1)(-1) \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 0 & + & (3+\sqrt{3}) & + & (3-\sqrt{3}) \\ 4 & + & (1+\sqrt{3}) & + & (1-\sqrt{3}) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \triangleq \boldsymbol{v}
\end{aligned}$$

In Example 2.1 (page 33) and Example 2.2 (page 35), the frame operator $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ were computed. In general however, it is not always necessary or even possible to compute these, as illustrated in Example 2.3 (next).

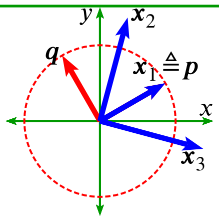
Example 2.3. [41] Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \bigtriangledown \rangle\right)$ be an inner product space with $\left\langle (x_1, y_1) \mid (x_2, y_2) \right\rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition 2.17 page 32) with *inverse* $\mathbf{S}^{-1}$.

[40] Heil (2011) pages 204–205 ⟨$r = 1$ case⟩, Byrne (2005) page 80 ⟨$r = 1$ case⟩, Han et al. (2007) page 91 ⟨Example 3.9, $r = \sqrt{2/3}$ case⟩

[41] Christensen (2003) pages 7–8 ⟨?⟩

EX

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be *orthonormal* vectors in $\boldsymbol{X} \triangleq \operatorname{span}\{\boldsymbol{p}, \boldsymbol{q}\}$.
Let $\boldsymbol{x}_1 \triangleq \boldsymbol{p}$, $\boldsymbol{x}_2 \triangleq \boldsymbol{p}+\boldsymbol{q}$, and $\boldsymbol{x}_3 \triangleq \boldsymbol{p}-\boldsymbol{q}$.
Then, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, , \boldsymbol{x}_3\}$ is a **frame** for $\boldsymbol{X}$ with *frame bounds* $A = 0$ and $B = 5$.



Moreover,
$\mathbf{S}^{-1}\boldsymbol{x}_1 = {}^1\!/\!_3\boldsymbol{p}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_2 = {}^1\!/\!_3\boldsymbol{p} + {}^1\!/\!_2\boldsymbol{q}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_3 = {}^1\!/\!_3\boldsymbol{p} - {}^1\!/\!_2\boldsymbol{q}$.

✎PROOF:

1. Proof that $(\!(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\!)$ is a *frame* with *frame bounds* $A = 0$ and $B = 5$:

$$\begin{aligned}
\sum_{n=1}^{3} \left|\langle \boldsymbol{v} \,|\, \boldsymbol{x}_n\rangle\right|^2 &\triangleq |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle|^2 && \text{by definitions of } \boldsymbol{x}_1, \boldsymbol{x}_2, \text{ and } \boldsymbol{x}_3 \\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 && \text{by } \textit{additivity} \text{ of } \langle \triangle \,|\, \triangledown\rangle \text{ (Definition C.9 page 124)} \\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right) \\
&\quad + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 - \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right) \\
&= 3|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + 2|\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 \\
&\boxed{\leq} 3\,\|\boldsymbol{v}\|\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{v}\|\,\|\boldsymbol{q}\| && \text{by } \textit{CS Inequality} \\
&= \|\boldsymbol{v}\|\,(3\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{q}\|) \\
&= \boxed{5}\,\|\boldsymbol{v}\| && \text{by } \textit{orthonormality} \text{ of } \boldsymbol{p} \text{ and } \boldsymbol{q}
\end{aligned}$$

2. lemma: $\mathbf{S}\boldsymbol{p} = 3\boldsymbol{p}$, $\mathbf{S}\boldsymbol{q} = 2\boldsymbol{q}$, $\mathbf{S}^{-1}\boldsymbol{p} = \frac{1}{3}\boldsymbol{p}$, and $\mathbf{S}^{-1}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{p} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{p} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n \\
&= \langle \boldsymbol{p} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{p} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\,(\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{p} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\,(\boldsymbol{p}-\boldsymbol{q}) \\
&= (1)\boldsymbol{p} + (1+0)(\boldsymbol{p}+\boldsymbol{q}) + (1-0)(\boldsymbol{p}-\boldsymbol{q}) \\
&= 3\boldsymbol{p} \\
\implies \mathbf{S}^{-1}\boldsymbol{p} &= \frac{1}{3}\boldsymbol{p} \\
\mathbf{S}\boldsymbol{q} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{q} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n \\
&= \langle \boldsymbol{q} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{q} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\,(\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{q} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\,(\boldsymbol{p}-\boldsymbol{q}) \\
&= (0)\boldsymbol{q} + (0+1)(\boldsymbol{p}+\boldsymbol{q}) + (0-1)(\boldsymbol{p}-\boldsymbol{q}) \\
&= 2\boldsymbol{q} \\
\implies \mathbf{S}^{-1}\boldsymbol{q} &= \frac{1}{2}\boldsymbol{q}
\end{aligned}$$

3. Remark: Without knowing $\boldsymbol{p}$ and $\boldsymbol{q}$, from (2) lemma it follows that it is not possible to compute $\mathbf{S}$ or $\mathbf{S}^{-1}$ explicitly.

4. Proof that $\mathbf{S}^{-1}\boldsymbol{x}_1 = {}^1\!/\!_3\boldsymbol{p}$, $\mathbf{S}^{-1}\boldsymbol{x}_2 = {}^1\!/\!_3\boldsymbol{p} + {}^1\!/\!_2\boldsymbol{q}$ and $\mathbf{S}^{-1}\boldsymbol{x}_3 = {}^1\!/\!_3\boldsymbol{p} - {}^1\!/\!_2\boldsymbol{q}$:

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_1 &\triangleq \mathbf{S}^{-1}\boldsymbol{p} && \text{by definition of } \boldsymbol{x}_1 \\
&= \frac{1}{3}\boldsymbol{p} && \text{by (2) lemma} \\
\mathbf{S}^{-1}\boldsymbol{x}_2 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2 \\
&= \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_3 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2 \\
&= \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

5. Check that $\boldsymbol{v} = \sum_n \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}$:

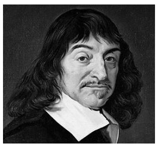
$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=1}^{3} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n \\
&= \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q}) \\
&= \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q}) \\
&= \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \left(\frac{1}{3}-\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{q} + \left(\frac{1}{2}-\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{p} + \left(\frac{1}{2}+\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q} \\
&= \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}
\end{aligned}$$

✏

CHAPTER 3

TRANSVERSAL OPERATORS

❝Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:❞



❝I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them.❞

René Descartes, philosopher and mathematician (1596–1650) [1]

3.1 Families of Functions

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition 3.1 page 39). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition 3.1. *Let X and Y be sets.*

D E F *The space Y^X represents the set of all functions with* DOMAIN X *and* RANGE Y *such that*

$$Y^X \triangleq \{\mathsf{f}(x) | \mathsf{f}(x) : X \to Y\}$$

---

[1] quote: Descartes (1637b)
translation: Descartes (1637c) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

Definition 3.2. [2] *Let X be a set.*

DEF

*The **indicator function** $\mathbb{1} \in \{0,1\}^{2^X}$ is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases} \qquad \begin{matrix} \forall x\in X,\ A\in 2^X \\ \forall x\in X,\ A\in 2^X \end{matrix}$$

*The indicator function $\mathbb{1}$ is also called the **characteristic function**.*

3.2 Definitions and algebraic properties

Much of the wavelet theory developed in this text is constructed using the **translation operator T** and the **dilation operator D** (next).

Definition 3.3. [3]

DEF

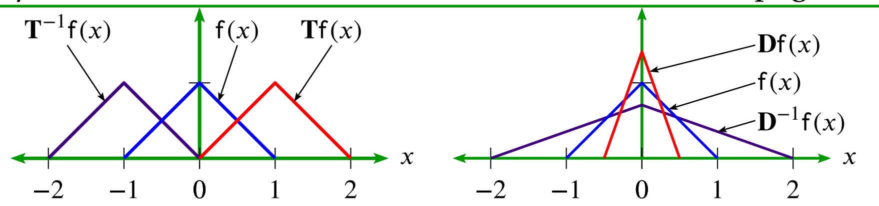
$\mathbf{T}_\tau$ *is a **translation operator** on $\mathbb{C}^\mathbb{C}$ if* $\quad \mathbf{T}_\tau \mathsf{f}(x) \triangleq \mathsf{f}(x-\tau) \quad \forall \mathsf{f}\in\mathbb{C}^\mathbb{C}$.

$\mathbf{D}_\alpha$ *is a **dilation operator** on $\mathbb{C}^\mathbb{C}$ if* $\quad \mathbf{D}_\alpha \mathsf{f}(x) \triangleq \mathsf{f}(\alpha x) \quad \forall \mathsf{f}\in\mathbb{C}^\mathbb{C}$.

Moreover, $\mathbf{T} \triangleq \mathbf{T}_1$ *and* $\mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2$.

Example 3.1. Let **T** and **D** be defined as in Definition 3.3 (page 40).

EX



Proposition 3.1. *Let $\mathbf{T}_\tau$ be a* TRANSLATION OPERATOR *(Definition 3.3 page 40).*

PRP

$$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) \qquad \forall \mathsf{f}\in\mathbb{R}^\mathbb{R} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) \text{ is PERIODIC with period } \tau\right)$$

✎PROOF:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n\tau+\tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 3.3 page 40)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m\tau) && \text{where } m \triangleq n-1 && \implies n = m+1\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m \mathsf{f}(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 3.3 page 40)}
\end{aligned}$$

⇒

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exist as operators, as demonstrated next.

[2] Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) page 22, de la Vallée-Poussin (1915) page 440

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2000) page 473, Packer (2004) page 260, zay (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a) page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996) page 81, Dai and Larson (1998) page 2

Proposition 3.2 (transversal operator inverses)**.** *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 3.3 page 40.*

PRP

$\mathbf{T}$ *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(translation operator inverse)}.$$

$\mathbf{D}$ *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(dilation operator inverse)}.$$

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} && \text{(Definition 3.3 page 40)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} && \text{(Definition 3.3 page 40)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\implies \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} && \text{(Definition 3.3 page 40)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} && \text{(Definition 3.3 page 40)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\implies \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

✏

Proposition 3.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 3.3 page 40.*
Let $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP

$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^j x - n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

3.3 Linear space properties

Proposition 3.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 3.3 page 40.*

PRP

$$\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}\mathsf{g}\right] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

✎PROOF:

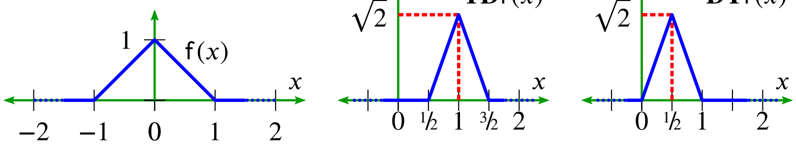
$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}(x)\mathsf{g}(x)\right] &= 2^{j/2}\mathsf{f}\left(2^j x - n\right)\mathsf{g}\left(2^j x - n\right) && \text{by Proposition 3.3 page 41}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^j x - n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^j x - n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition 3.3 page 41}
\end{aligned}$$

⇨

In general the operators **T** and **D** are *noncommutative* (**TD** ≠ **DT**), as demonstrated by Counterexample 3.1 (next) and Proposition 3.5 (page 42).

Counterexample 3.1.

CNT As illustrated to the right, it is **not** always true that **TD** = **DT**:



Proposition 3.5 (commutator relation). [4] *Let* **T** *and* **D** *be as in Definition 3.3 page 40.*

PRP
$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j && \forall j,n\in\mathbb{Z} \\ \mathbf{T}^n\mathbf{D}^j &= \mathbf{D}^j\mathbf{T}^{2^j n} && \forall n,j\in\mathbb{Z} \end{aligned}$$

✎PROOF:

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^{2^j n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - 2^j n) && \text{by Proposition 3.4 page 41} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 96)} \\ &= \mathbf{T}^n 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)} \\ &= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 3.3 page 40)} \end{aligned}$$

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^{n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - n) && \text{by Proposition 3.4 page 41} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j\left[x-2^{-j/2}n\right]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 96)} \\ &= \mathbf{T}^{2^{-j/2}n} 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)} \\ &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 3.3 page 40)} \end{aligned}$$

⇨

3.4 Inner product space properties

In an inner product space, every operator has an *adjoint* (Proposition C.3 page 125) and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator **U** coincide, then **U** is said to be *unitary* (Definition C.14 page 135). And in this case, **U** has several nice properties (see Proposition 3.9 and Theorem 3.1 page 45). Proposition 3.6 (next) gives the adjoints of **D** and **T**, and Proposition 3.7 (page 43) demonstrates that both **D** and **T** are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ (Corollary H.1 page 193) and the *rotation matrix operator* (Example C.5 page 137).

Proposition 3.6. *Let* **T** *be the* TRANSLATION OPERATOR *(Definition 3.3 page 40) with* ADJOINT $\mathbf{T}^*$ *and* **D** *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$ *(Definition C.8 page 121).*

PRP
$$\begin{aligned} \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(TRANSLATION OPERATOR ADJOINT)} \\ \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(DILATION OPERATOR ADJOINT)} \end{aligned}$$

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a) page 641, Goodman et al. (1993b) page 110

✎PROOF:

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^* && \text{(Definition C.8 page 121)} \\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)} \\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1 \\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^* && \text{(Definition C.8 page 121)} \\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition 3.3 page 40)} \\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\,\mathrm{d}u && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^*\mathrm{d}x && \text{where } x = 2u \\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}$$

✏

Proposition 3.7. [5] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 3.3 (page 40). Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 3.2 (page 41).*

PRP:
$\mathbf{T}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $(\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $L^2_{\mathbb{R}})$.
$\mathbf{D}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $(\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $L^2_{\mathbb{R}})$.

✎PROOF:

$$\begin{aligned}
\mathbf{T}^{-1} &= \mathbf{T}^* && \text{by Proposition 3.2 page 41 and Proposition 3.6 page 42} \\
&\implies \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition C.14 page 135)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}^{-1} &= \mathbf{D}^* && \text{by Proposition 3.2 page 41 and Proposition 3.6 page 42} \\
&\implies \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition C.14 page 135)}
\end{aligned}$$

✏

3.5 Normed linear space properties

Proposition 3.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR *(Definition 3.3 page 40).*

PRP:
$$\left\{\begin{array}{ll} \text{(1).} & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) \quad \textit{and} \\ \text{(2).} & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f}\in L^2_{\mathbb{R}}$$

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

✎PROOF:

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition 3.3 page 40)}\\
&= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis}
\end{aligned}$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\|\mathsf{f}(x)-\mathsf{f}(x+h)\| &= \|\mathsf{f}(x)-\mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis}\\
&= \|0\|\\
&= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|\\
&\le \varepsilon\\
&\implies \forall h>0,\ \exists\varepsilon \quad \text{such that} \quad \|\mathsf{f}(x)-\mathsf{f}(x+h)\|<\varepsilon\\
&\overset{\text{def}}{\iff} \mathsf{f}(x) \text{ is } \textit{continuous}
\end{aligned}$$

3. Proof that (1,2) $\implies$ *constant* property:
   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \ne \mathsf{f}(y)$.
   (b) Let $(\!(x_n)\!)_{n\in\mathbb{N}}$ be a sequence with limit x and $(\!(y_n)\!)_{n\in\mathbb{N}}$ a sequence with limit y
   (c) Then

$$\begin{aligned}
0 &< \|\mathsf{f}(x)-\mathsf{f}(y)\| && \text{by assumption in item (3a) page 44}\\
&= \lim_{n\to\infty}\left\|\mathsf{f}(x_n)-\mathsf{f}(y_n)\right\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 44}\\
&= \lim_{n\to\infty}\left\|\mathsf{f}(2^m x_n)-\mathsf{f}(2^\ell y_n)\right\| \quad \forall m,\ell\in\mathbb{Z} && \text{by (1)}\\
&= 0
\end{aligned}$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

✏

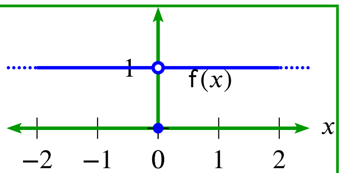
Remark 3.1.

REM In Proposition 3.8 page 43, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample 3.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x=0\\ 1 & \text{otherwise.}\end{cases}$

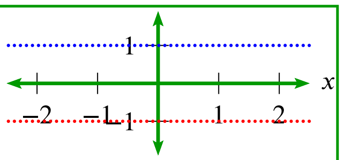
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Counterexample 3.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\backslash\mathbb{Q}$ the set of *irrational numbers*.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x\in\mathbb{Q}\\ -1 & \text{for } x\in\mathbb{R}\backslash\mathbb{Q}.\end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Proposition 3.9 (Operator norm). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 3.3 page 40. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 3.2 page 41. Let* $\mathbf{T}^*$ *and* $\mathbf{D}^*$ *be as in Proposition 3.6 page 42. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition D.1 page 141. Let* $|||\cdot|||$ *be the operator norm* (Definition C.6 page 117) *induced by* $\|\cdot\|$.

PRP
$$|||\mathbf{T}||| = |||\mathbf{D}||| = |||\mathbf{T}^*||| = |||\mathbf{D}^*||| = |||\mathbf{T}^{-1}||| = |||\mathbf{D}^{-1}||| = 1$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 3.7 page 43) and from Theorem C.25 page 136 and Theorem C.26 page 136. ✏

Theorem 3.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 3.3 page 40.*
Let $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 3.2 page 41. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition D.1 page 141.*

THM

1.	$\|\mathbf{T}\mathsf{f}\|$	=	$\|\mathbf{D}\mathsf{f}\|$	=	$\|\mathsf{f}\|$	$\forall \mathsf{f} \in \mathscr{L}^2_{\mathbb{R}}$	(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}\mathsf{f} - \mathbf{T}\mathsf{g}\|$	=	$\|\mathbf{D}\mathsf{f} - \mathbf{D}\mathsf{g}\|$	=	$\|\mathsf{f} - \mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g} \in \mathscr{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
3.	$\|\mathbf{T}^{-1}\mathsf{f} - \mathbf{T}^{-1}\mathsf{g}\|$	=	$\|\mathbf{D}^{-1}\mathsf{f} - \mathbf{D}^{-1}\mathsf{g}\|$	=	$\|\mathsf{f} - \mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g} \in \mathscr{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
4.	$\langle \mathbf{T}\mathsf{f} \mid \mathbf{T}\mathsf{g} \rangle$	=	$\langle \mathbf{D}\mathsf{f} \mid \mathbf{D}\mathsf{g} \rangle$	=	$\langle \mathsf{f} \mid \mathsf{g} \rangle$	$\forall \mathsf{f},\mathsf{g} \in \mathscr{L}^2_{\mathbb{R}}$	(SURJECTIVE)
5.	$\langle \mathbf{T}^{-1}\mathsf{f} \mid \mathbf{T}^{-1}\mathsf{g} \rangle$	=	$\langle \mathbf{D}^{-1}\mathsf{f} \mid \mathbf{D}^{-1}\mathsf{g} \rangle$	=	$\langle \mathsf{f} \mid \mathsf{g} \rangle$	$\forall \mathsf{f},\mathsf{g} \in \mathscr{L}^2_{\mathbb{R}}$	(SURJECTIVE)

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 3.7 page 43) and from Theorem C.25 page 136 and Theorem C.26 page 136. ✏

Proposition 3.10. *Let* $\mathbf{T}$ *be as in Definition 3.3 page 40. Let* $\mathbf{A}^*$ *be the* ADJOINT (Definition C.8 page 121) *of an operator* $\mathbf{A}$*. Let the property "*SELF ADJOINT*" be defined as in Definition C.11 (page 129).*

PRP
$$\left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* \iff \left(\textit{The operator } \left[\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right] \textit{ is } \text{SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)\mathsf{f}(x) \,\middle|\, \mathsf{g}(x) \right\rangle
&= \left\langle \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) \,\middle|\, \mathsf{g}(x) \right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)}\\
&= \left\langle \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n) \,\middle|\, \mathsf{g}(x) \right\rangle && \text{by \textit{commutative} property} && \text{(Definition A.5 page 96)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x+n) \mid \mathsf{g}(x) \rangle && \text{by \textit{additive} property of } \langle \triangle \mid \nabla \rangle &&\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(u) \mid \mathsf{g}(u-n) \rangle && \text{where } u \triangleq x+n &&\\
&= \left\langle \mathsf{f}(u) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathsf{g}(u-n) \right\rangle && \text{by \textit{additive} property of } \langle \triangle \mid \nabla \rangle &&\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathsf{g}(x-n) \right\rangle && \text{by change of variable: } u \to x &&\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{g}(x) \right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) &= \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* && \text{by definition of \textit{adjoint}} && \text{(Proposition C.3 page 125)}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) &\text{ is \textit{self-adjoint}} && \text{by definition of \textit{self-adjoint}} && \text{(Definition C.11 page 129)}
\end{aligned}$$

✏

3.6 Fourier transform properties

Proposition 3.11. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 3.3 page 40.*
Let $\mathbf{B}$ *be the* TWO-SIDED LAPLACE TRANSFORM *defined as* $[\mathbf{B}\mathsf{f}](s) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x$.

PRP

1.	$\mathbf{B}\mathbf{T}^n$	$=$	$e^{-sn}\mathbf{B}$			$\forall n\in\mathbb{Z}$
2.	$\mathbf{B}\mathbf{D}^j$	$=$	$\mathbf{D}^{-j}\mathbf{B}$			$\forall j\in\mathbb{Z}$
3.	$\mathbf{D}\mathbf{B}$	$=$	$\mathbf{B}\mathbf{D}^{-1}$			$\forall n\in\mathbb{Z}$
4.	$\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1}$	$=$	$\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B}$	$=$	$\mathbf{D}$	$\forall n\in\mathbb{Z}$ ($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5.	$\mathbf{D}\mathbf{B}\mathbf{D}$	$=$	$\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1}$	$=$	$\mathbf{B}$	$\forall n\in\mathbb{Z}$

✎PROOF:

$$\begin{aligned}
\mathbf{B}\mathbf{T}^n\mathsf{f}(x) &= \mathbf{B}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathrm{d}u && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right]\\
&= e^{-sn}\,\mathbf{B}\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}\mathbf{D}^j\mathsf{f}(x) &= \mathbf{B}\left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition 3.3 page 40)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right]e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathrm{d}u && \text{let } u \triangleq 2^j x \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathrm{d}u\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right] && \text{by Proposition 3.6 page 42 and} && \text{Proposition 3.7 page 43}\\
&= \mathbf{D}^{-j}\,\mathbf{B}\,\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{D}\mathbf{B}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x\right] && \text{by definition of } \mathbf{B}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathrm{d}x && \text{by definition of } \mathbf{D} && \text{(Definition 3.3 page 40)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{u}{2}\right)\right]e^{-su}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\,e^{-su}\,\mathrm{d}u && \text{by Proposition 3.6 page 42 and} && \text{Proposition 3.7 page 43}\\
&= \mathbf{B}\mathbf{D}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{B}\mathbf{D} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition C.3 page 112)}\\
\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} &= \mathbf{D}\mathbf{B}\mathbf{B}^{-1} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition C.3 page 112)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{DBD} &= \mathbf{DD}^{-1}\mathbf{B} && \text{by previous result} \\
&= \mathbf{B} && \text{by definition of operator inverse (Definition C.3 page 112)} \\
\mathbf{D}^{-1}\mathbf{BD}^{-1} &= \mathbf{D}^{-1}\mathbf{DB} && \text{by previous result} \\
&= \mathbf{B} && \text{by definition of operator inverse (Definition C.3 page 112)}
\end{aligned}$$

⇨

Corollary 3.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 3.3 page 40. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition H.2 page 192) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition D.1 page 141).*

COR

$$\begin{array}{llcll}
1. & \tilde{\mathbf{F}}\mathbf{T}^n & = & e^{-i\omega n}\tilde{\mathbf{F}} & \\
2. & \tilde{\mathbf{F}}\mathbf{D}^j & = & \mathbf{D}^{-j}\tilde{\mathbf{F}} & \\
3. & \mathbf{D}\tilde{\mathbf{F}} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1} & \\
4. & \mathbf{D} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} & = \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}} \\
5. & \tilde{\mathbf{F}} & = & \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} & = \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}
\end{array}$$

✎PROOF: These results follow directly from Proposition 3.11 page 46 with $\tilde{\mathbf{F}} = \mathbf{B}|_{s=i\omega}$. ⇨

Proposition 3.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 3.3 page 40. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition H.2 page 192) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition D.1 page 141).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = \frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{\mathsf{f}}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n\mathsf{f}(x) && \text{by Corollary 3.1 page 47 (3)} \\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}\mathsf{f}(x) && \text{by Corollary 3.1 page 47 (3)} \\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathsf{f}}(\omega) \\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{\mathsf{f}}\left(2^{-j}\omega\right) && \text{by Proposition 3.2 page 41}
\end{aligned}$$

⇨

Proposition 3.13. *Let* $\mathbf{T}$ *be the translation operator (Definition 3.3 page 40). Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition H.2 page 192) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$*. Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT *(Definition L.1 page 223) of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition I.2 page 203).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) \qquad \forall (\!(a_n)\!)\in\boldsymbol{\ell}^2_{\mathbb{R}}, \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\tilde{\mathbf{F}}\phi(x) && \text{by Corollary 3.1 page 47} \\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega) \\
&= \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) && \text{by definition of } \textit{DTFT} \text{ (Definition L.1 page 223)}
\end{aligned}$$

⇨

Definition 3.4. *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition D.1 page 141). Let $\boldsymbol{\ell}^2_{\mathbb{R}}$ be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition D.1 page 141).*

DEF

$\mathbf{S}$ *is the* ***sampling operator*** *in* $\boldsymbol{\ell}^{2\,\boldsymbol{L}^2_{\mathbb{R}}}_{\mathbb{R}}$ *if* $[\mathbf{S}\mathsf{f}(x)](n) \triangleq \mathsf{f}\left(\frac{2\pi}{\tau}n\right) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \tau\in\mathbb{R}^+$

Theorem 3.2 (Poisson Summation Formula—PSF). [6] *Let $\tilde{\mathsf{f}}(\omega)$ be the* FOURIER TRANSFORM *(Definition H.2 page 192) of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$. Let $\mathbf{S}$ be the* SAMPLING OPERATOR *(Definition 3.4 page 48).*

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_\tau \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)$ then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:

   Note that $\mathsf{h}(x)$ is *periodic* with period τ. Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of PSF (this theorem—Theorem 3.2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) && \text{by (1) lemma page 48}\\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \left(\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,dx\right]}_{\hat{\mathbf{F}}[\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition M.1 page 233)}\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau \mathsf{f}(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,dx\right]\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,du\right] && \text{where } u \triangleq x+n\tau \implies x = u-n\tau\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\cancel{e^{i2\pi kn}}^{\,1}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}ku}\,du\right]\\
&= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,du\right]}_{[\tilde{\mathbf{F}}\mathsf{f}]\left(\frac{2\pi}{\tau}k\right)} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem M.1 page 234)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}\mathsf{f}(x)]\left(\frac{2\pi}{\tau}k\right)\right] && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition H.2 page 192)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \mathbf{S} \quad \text{(Definition 3.4 page 48)}\\
&= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem M.1 page 234)}
\end{aligned}$$

✏

[6] Andrews et al. (2001) page 624, Knapp (2005b) page 389, Lasser (1996) page 254, Rudin (1987) pages 194–195, Folland (1992) page 337

Theorem 3.3 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition H.2 page 192) of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)}_{\text{summation in “frequency”}} \quad = \quad \underbrace{\frac{\tau}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\text{summation in “time”}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)$, then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:
   Note that $\mathsf{h}(\omega)$ is periodic with period $2\pi/T$:

   $$\mathsf{h}\left(\omega + \frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau} + \frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + (n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \triangleq \mathsf{h}(\omega)$$

   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of IPSF (this theorem—Theorem 3.3):

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \\
&= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) && \text{by (1) lemma page 49} \\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\sqrt{\frac{\tau}{2\pi}}\int_0^{\frac{2\pi}{\tau}} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\,\mathrm{d}\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)\right]} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition M.1 page 233)} \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{\tau}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega Tk}\,\mathrm{d}\omega\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-i\left(u-\frac{2\pi}{\tau}n\right)Tk}\,\mathrm{d}u\right] && \text{where } u \triangleq \omega + \frac{2\pi}{\tau}n \implies && \omega = u - \frac{2\pi}{\tau}n \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\cancel{e^{i2\pi nk}}^{\,1}\int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{d}u\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{d}u\right] \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{iu(-\tau k)}\,\mathrm{d}u\right]}_{[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\left[\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](-k\tau)\right] && \text{by value of } \tilde{\mathbf{F}}^{-1} && \text{(Theorem H.1 page 193)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\,\tilde{\mathsf{f}} && \text{by definition of } \mathbf{S} && \text{(Definition 3.4 page 48)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\mathsf{f}(x) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathsf{f}(-k\tau) && \text{by definition of } \mathbf{S} && \text{(Definition 3.4 page 48)} \\
&= \sqrt{\tau}\,\frac{1}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau)e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem M.1 page 234)}
\end{aligned}$$

[7] Gauss (1900) page 88

$$= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}} \sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau)e^{ik\tau\omega} \quad \text{by definition of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem M.1 page 234)}$$

$$= \frac{\tau}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \mathsf{f}(m\tau)e^{-i\omega m\tau} \quad \text{let } m \triangleq -k$$

✏

Remark 3.2. The left hand side of the *Poisson Summation Formula* (Theorem 3.2 page 48) is very similar to the *Zak Transform* **Z**: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark 3.3. A generalization of the *Poisson Summation Formula* (Theorem 3.2 page 48) is the **Selberg Trace Formula**.[9]

3.7 Examples

Example 3.2 (linear functions). [10] Let **T** be the *translation operator* (Definition 3.3 page 40). Let $\mathcal{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX
1. $\{x, \mathbf{T}x\}$ is a *basis* for $\mathcal{L}(\mathbb{C},\mathbb{C})$ and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x$ $\quad \forall \mathsf{f} \in \mathcal{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition 3.3 page 40}\\
&= (ax+b)|_{x=1}\, x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of f}\\
&= (a+b)x - b(x-1)\\
&= ax + bx - bx + b\\
&= ax + b\\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of f}
\end{aligned}$$

✏

Example 3.3 (Cardinal Series). Let **T** be the *translation operator* (Definition 3.3 page 40). The *Paley-Wiener* class of functions $\boldsymbol{PW}^2_\sigma$ are those functions which are "*bandlimited*" with respect to their Fourier transform (Definition H.2 page 192). The cardinal series forms an orthogonal basis for such a space. The *Fourier coefficients* (Definition 2.11 page 20) for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of $\mathsf{f}(x)$ taken at regular intervals. In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \,|\, \mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\,\mathrm{d}x \triangleq \mathsf{f}(n)$$

EX
1. $\left\{ \mathbf{T}^n \frac{\sin(\pi x)}{\pi x} \,\middle|\, n\in\mathbb{N} \right\}$ is a *basis* for $\boldsymbol{PW}^2_\sigma$ and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin(\pi x)}{\pi x}}_{\text{Cardinal series}}$ $\quad \forall \mathsf{f} \in \boldsymbol{PW}^2_\sigma,\ \sigma \le \frac{1}{2}$

8 Janssen (1988) page 24, Zayed (1996) page 482
9 Lax (2002) page 349, Selberg (1956), Terras (1999)
10 Higgins (1996) page 2

Example 3.4 (Fourier Series).

EX

1. $\left\{\mathbf{D}_n e^{ix} \middle| n\in\mathbb{Z}\right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and
2. $\mathsf{f}(x) = \dfrac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix} \quad \forall x\in(0:2\pi), \mathsf{f}\in\boldsymbol{L}(0:2\pi)$ where
3. $\alpha_n \triangleq \dfrac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x)\mathbf{D}_n e^{-ix}\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}(0:2\pi)$

✎PROOF: See Theorem M.1 page 234. ✏

Example 3.5 (Fourier Transform). [11]

EX

1. $\left\{\mathbf{D}_\omega e^{ix} \middle| \omega\in\mathbb{R}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \dfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)\mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\tilde{\mathsf{f}}(\omega) \triangleq \dfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}_\omega e^{-ix}\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example 3.6 (Gabor Transform). [12]

EX

1. $\left\{\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{ix}\right) \middle| \tau,\omega\in\mathbb{R}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \int_{\mathbb{R}} \mathsf{G}(\tau,\omega)\, \mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\mathsf{G}(\tau,\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{-ix}\right) \mathrm{d}x \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example 3.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX

1. $\left\{\mathbf{D}^k\mathbf{T}^n\psi(x) \middle| k,n\in\mathbb{Z}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha_{k,n}\mathbf{D}^k\mathbf{T}^n\psi(x) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\alpha_n \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}^k\mathbf{T}^n\psi^*(x)\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

[11] cross reference: Definition H.2 page 192

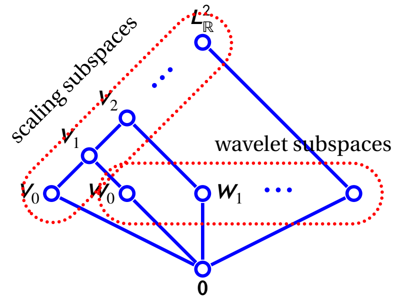
[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

CHAPTER 4

MRA STRUCTURES

4.1 Introduction

In 1989, Stéphane G. Mallat introduced the *Multiresolution Analysis* (MRA, Definition 4.1 page 54) method for wavelet construction. The MRA has become the dominate wavelet construction method. This text uses the MRA method extensively, and combines the MRA "scaling subspaces" (Definition 4.1 page 54) with "wavelet subspaces" (Definition 5.1 page 81) to form a subspace structure as represented by the *Hasse diagram* to the right. The *Fast Wavelet Transform* combines both sets of subspaces as well, providing the results of projections onto both wavelet and MRA subspaces. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA.[1]



The MRA is an **analysis** of the linear space $L^2_{\mathbb{R}}$. An analysis of a linear space X is any sequence $(\!(V_j)\!)_{j\in\mathbb{Z}}$ of linear subspaces of X. The partial or complete reconstruction of X from $(\!(V_j)\!)_{j\in\mathbb{Z}}$ is a **synthesis**.[2] An analysis is completely *characterized* by a *transform*. For example, a Fourier analysis is a sequence of subspaces with sinusoidal bases. Examples of subspaces in a Fourier analysis include $V_1 = \operatorname{span}\{e^{ix}\}$, $V_{2.3} = \operatorname{span}\{e^{i2.3x}\}$, $V_{\sqrt{2}} = \operatorname{span}\left\{e^{i\sqrt{2}x}\right\}$, etc. A **transform** is loosely defined as a function that maps a family of functions into an analysis. A very useful transform (a "*Fourier transform*") for Fourier Analysis is (Definition H.2 page 192)

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x$$

---

[1] Lemarié (1990), Mallat (1999) page 240

[2] The word *analysis* comes from the Greek word ἀνάλυσις, meaning "dissolution" (Perschbacher (1990) page 23 ⟨entry 359⟩), which in turn means "the resolution or separation into component parts" (Black et al. (2009), http://dictionary.reference.com/browse/dissolution)

4.2 Definition

A multiresolution analysis provides "coarse" approximations of a function in a linear space $\boldsymbol{L}^2_{\mathbb{R}}$ at multiple "scales" or "resolutions". Key to this process is a sequence of *scaling functions*. Most traditional transforms feature a single *scaling function* $\phi(x)$ set equal to one ($\phi(x) = 1$). This allows for convenient representation of the most basic functions, such as constants.[3] A multiresolution system, on the other hand, uses a generalized form of the scaling concept:

1. Instead of the scaling function simply being set *equal to unity* ($\phi(x) = 1$), a multiresolution system (Definition 4.3 page 63) is often constructed in such a way that the scaling function $\phi(x)$ forms a *partition of unity* such that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x) = 1$.
2. Instead of there being *just one* scaling function, there is an entire sequence of scaling functions $\left(\!\left(\mathbf{D}^j\phi(x)\right)\!\right)_{j\in\mathbb{Z}}$, each corresponding to a different "*resolution*".

Definition 4.1. [4] *Let $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ be a sequence of subspaces on $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141). Let A^- be the* CLOSURE *of a set A.*

DEF

*The sequence $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **multiresolution analysis** on $\boldsymbol{L}^2_{\mathbb{R}}$ if*

1.	$\boldsymbol{V}_j = \boldsymbol{V}_j^-$	$\forall j\in\mathbb{Z}$	(CLOSED)	and
2.	$\boldsymbol{V}_j \subset \boldsymbol{V}_{j+1}$	$\forall j\in\mathbb{Z}$	(LINEARLY ORDERED)	and
3.	$\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}$		(DENSE *in* $\boldsymbol{L}^2_{\mathbb{R}}$)	and
4.	$\mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{D}\mathsf{f} \in \boldsymbol{V}_{j+1}$	$\forall j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(SELF-SIMILAR)	and
5.	$\exists\phi$ such that $\left\{\mathbf{T}^n\phi \mid n\in\mathbb{Z}\right\}$ *is a* RIESZ BASIS *for* $\boldsymbol{V}_0$.			

A MULTIRESOLUTION ANALYSIS *is also called an **MRA**.*
*An element $\boldsymbol{V}_j$ of $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **scaling subspace** of the space $\boldsymbol{L}^2_{\mathbb{R}}$.*
*The pair $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right)\right)$ is a **multiresolution analysis space**, or **MRA space**.*
*The function ϕ is the **scaling function** of the* MRA SPACE.

The traditional definition of the *MRA* also includes the following:

1.	$\mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{T}^n\mathsf{f} \in \boldsymbol{V}_j$	$\forall n,j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(*translation invariant*)
2.	$\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \{\mathbb{0}\}$		(*greatest lower bound* is $\mathbf{0}$)

However, Proposition 4.1 (next) demonstrates that both of these follow from the *MRA* as defined in Definition 4.1.

Proposition 4.1. [5]

PRP

$$\left.\begin{array}{l}\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}} \text{ is an MRA} \\ \text{(Definition 4.1 page 54)}\end{array}\right\} \implies \left\{\begin{array}{llll} 1. & \mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{T}^n\mathsf{f} \in \boldsymbol{V}_j & \forall n,j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \text{(TRANSLATION INVARIANT) and} \\ 2. & \bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \{\mathbb{0}\} & & \text{(GREATEST LOWER BOUND is } \mathbf{0}) \end{array}\right.$$

[3] Jawerth and Sweldens (1994) page 8
[4] Hernández and Weiss (1996) page 44, Mallat (1999) page 221 ⟨Definition 7.1⟩, Mallat (1989) page 70, Meyer (1992) page 21 ⟨Definition 2.2.1⟩, Christensen (2003) page 284 ⟨Definition 13.1.1⟩, Bachman et al. (2000) pages 451–452 ⟨Definition 7.7.6⟩, Walnut (2002) pages 300–301 ⟨Definition 10.16⟩, Daubechies (1992) pages 129–140 ⟨Riesz basis: page 139⟩
[5] Hernández and Weiss (1996) page 45 ⟨Theorem 1.6⟩, Wojtaszczyk (1997) pages 19–28 ⟨Proposition 2.14⟩, Pinsky (2002) pages 313–314 ⟨Lemma 6.4.28⟩

✎PROOF: Proof for (1):

$$
\begin{aligned}
&\mathbf{T}^n \mathsf{f} \in \boldsymbol{V}_j \\
\iff\; & \mathbf{T}^n \mathsf{f} \in \operatorname{span}\left\{\mathbf{D}^j \mathbf{T}^m \phi |_{m\in\mathbb{Z}}\right\} && \text{by definition of } \{\phi\} && \text{(Definition 4.1 page 54)} \\
\iff\; & \exists (\!(\alpha_n)\!)_{n\in\mathbb{Z}} \text{ such that } \mathbf{T}^n \mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{D}^j \mathbf{T}^k \phi(x) && \text{by definition of } \{\phi\} && \text{(Definition 4.1 page 54)} \\
\iff\; & \exists (\!(\alpha_n)\!)_{n\in\mathbb{Z}} \text{ such that } \mathsf{f}(x) = \mathbf{T}^{-n} \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{D}^j \mathbf{T}^k \phi(x) && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)} \\
& = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{T}^{-n} \mathbf{D}^j \mathbf{T}^k \phi(x) \\
& = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{D}^j \mathbf{T}^{k-2n} \phi(x) && \text{by } \textit{commutator relation} && \text{(Proposition 3.5 page 42)} \\
& = \sum_{\ell\in\mathbb{Z}} \alpha_{\ell+2n} \mathbf{D}^j \mathbf{T}^{\ell} \phi(x) && \text{where } \ell \triangleq k-2n \implies && k = \ell + 2n \\
& = \sum_{\ell\in\mathbb{Z}} \beta_\ell \mathbf{D}^j \mathbf{T}^{\ell} \phi(x) && \text{where } \beta_\ell \triangleq \alpha_{\ell+2n} \\
& \iff \mathsf{f} \in \boldsymbol{V}_j && \text{by def. of } \{\mathbf{T}^n\phi\} && \text{(Definition 4.1 page 54)}
\end{aligned}
$$

Proof for (2):

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace $\boldsymbol{V}_j$ such that
   $\mathbf{V}_j = \left\{\mathbf{P}_j \mathsf{f} | \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}\right\}$
2. lemma: Functions with *compact support* are *dense* in $\boldsymbol{L}^2_{\mathbb{R}}$. Therefore, we only need to prove that the proposition is true for functions with support in $[-R : R]$, for all $R > 0$.
3. For some function $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$, let $(\!(\mathsf{f}_n)\!)_{n\in\mathbb{Z}}$ be a sequence of functions in $\boldsymbol{L}^2_{\mathbb{R}}$ with *compact support* such that
   $\operatorname{supp}\mathsf{f}_n \subseteq [-R : R]$ for some $R > 0$ and $\mathsf{f}(x) = \lim_{n\to\infty} (\!(\mathsf{f}_n(x))\!)$.
4. lemma: $\bigcap \boldsymbol{V}_j = \{\mathbb{0}\} \iff \lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\| = 0 \quad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. Proof:

$$
\begin{aligned}
\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j &= \bigcap_{j\in\mathbb{Z}} \left\{\mathbf{P}_j \mathsf{f} | \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}\right\} && \text{by definition of } \boldsymbol{V}_j && \text{(definition 1 page 55)} \\
&= \lim_{j\to-\infty} \left\{\mathbf{P}_j \mathsf{f} | \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}\right\} && \text{by definition of } \cap \\
&= \mathbb{0} \iff \lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\| = 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\| && \text{(Definition C.5 page 116)}
\end{aligned}
$$

5. lemma: $\lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\| = 0 \quad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. Proof:
   Let $\mathbb{1}_{A(x)}$ be the *set indicator function* (Definition 3.2 page 40)

$$
\begin{aligned}
&\lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\|^2 \\
&= \lim_{j\to-\infty} \left\|\mathbf{P}_j \lim_{n\to\infty} (\!(\mathsf{f}_n)\!)\right\|^2 && \text{by definition 3 page 55} \\
&\le \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbf{P}_j \lim_{n\to\infty} (\!(\mathsf{f}_n)\!) \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by } \textit{frame property} && \text{(Proposition 2.5 page 30)} \\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \lim_{n\to\infty} (\!(\mathsf{f}_n)\!) \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by definition of } \mathbf{P}_j && \text{(definition 1 page 55)} \\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbb{1}_{[-R:R]}(x) \lim_{n\to\infty} (\!(\mathsf{f}_n)\!) \,|\, \mathbf{D}^j \mathbf{T}^n \phi(x) \right\rangle\right|^2 && \text{by definition of } (\!(\mathsf{f}_n)\!) && \text{(definition 3 page 55)}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left| \left\langle \lim_{n\to\infty} (\mathsf{f}_n) \mid \mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x) \right\rangle \right|^2 && \text{prop. of } \langle \triangle \mid \triangledown \rangle \text{ in } L^2_{\mathbb{R}} && \text{(Definition D.1 page 141)}\\
&\le \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left\| \lim_{n\to\infty} (\mathsf{f}_n) \right\|^2 \left\| \mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x) \right\|^2 && \text{by } \textit{CS Inequality} \\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| \mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x) \right\|^2 && \text{by definition of } (\mathsf{f}_n) && \text{(definition 3 page 55)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| \left[\underbrace{\mathbf{D}^j\mathbf{D}^{-j}}_{\mathbf{I}} \mathbb{1}_{[-R:R]}(x) \right] \left[\mathbf{D}^j\mathbf{T}^n\phi(x) \right] \right\|^2 && \text{by property of } \mathbf{D} && \text{(Proposition 3.2 page 41)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| 2^{j/2}\mathbf{D}^j \left\{ \left[\mathbf{D}^{-j}\mathbb{1}_{[-R:R]}(x) \right] \left[\mathbf{T}^n\phi(x) \right] \right\} \right\|^2 && \text{by Proposition 3.4 page 41}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| \mathbf{D}^j \left\{ 2^{j/2}2^{-j/2}\mathbb{1}_{[-R:R]}(2^{-j}x) \left[\mathbf{T}^n\phi(x) \right] \right\} \right\|^2 && \text{by property of } \mathbf{D} && \text{(Proposition 3.2 page 41)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| \mathbf{D}^j \left\{ \left[\underbrace{\mathbf{T}^n\mathbf{T}^{-n}}_{\mathbf{I}} \mathbb{1}_{[-R:R]}(2^{-j}x) \right] \left[\mathbf{T}^n\phi(x) \right] \right\} \right\|^2 && \text{by property of } \mathbf{T} && \text{(Proposition 3.2 page 41)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| \mathbf{D}^j \left\{ \left[\mathbf{T}^n\mathbb{1}_{[-R:R]}(2^{-j}x+n) \right] \left[\mathbf{T}^n\phi(x) \right] \right\} \right\|^2 && \text{by property of } \mathbf{T} && \text{(Proposition 3.2 page 41)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| \mathbf{D}^j\mathbf{T}^n \left\{ \mathbb{1}_{[-R:R]}(2^{-j}x+n)\phi(x) \right\} \right\|^2 && \text{by property of } \mathbf{D} && \text{(Proposition 3.2 page 41)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\| \mathbb{1}_{[-R:R]}(2^{-j}x+n)\phi(x) \right\|^2 && \text{by } \textit{unitary} \text{ prop.} && \text{(Theorem 3.1 page 45)}\\
&= B\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \lim_{j\to-\infty} \left\| \mathbb{1}_{[-2^jR+n:2^jR+n]}(u)\phi(2^{-j}(u-n)) \right\|^2 && u \triangleq 2^jx+n \implies && x = 2^{-j}(u-n)\\
&= B\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \lim_{j\to-\infty} \int_{-2^jR+n}^{2^jR+n} \left| \phi(2^{-j}(u-n)) \right|^2 \,\mathrm{d}u \\
&= B\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \int_n^n |\phi(0)|^2 \,\mathrm{d}u \\
&= 0
\end{aligned}
$$

6. Final step in proof that $\bigcap \boldsymbol{V}_j = \{\mathbb{0}\}$: by (4) lemma page 55 and (5) lemma page 55

✏

Proposition 4.2. [6]

PRP

$$
\left\{ \begin{array}{lll} (1). & (\mathbf{T}^n\phi) \text{ is a } \textsc{Riesz sequence} & \text{and} \\ (2). & \tilde{\phi}(\omega) \text{ is } \textsc{continuous} \text{ at } 0 & \text{and} \\ (3). & \tilde{\phi}(0) \neq 0 & \end{array} \right\} \implies \left\{ \overline{\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right)} = L^2_{\mathbb{R}} \quad (\textsc{dense in } L^2_{\mathbb{R}}) \right\}
$$

✎PROOF:

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace $\boldsymbol{V}_j$ such that
   $\mathbf{V}_j = \{\mathbf{P}_j\mathsf{f} \,|\, \mathsf{f} \in \boldsymbol{H}\}$

2. definition: Choose $\mathsf{f} \in L^2_{\mathbb{R}}$ such that $\mathsf{f} \perp \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j$. Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier Transform* (Definition H.2 page 192) of $\mathsf{f}(x)$.

[6] Wojtaszczyk (1997) pages 28–31 ⟨Proposition 2.15⟩

3. lemma: The function f (definition 2 page 56) *exists* because the set of functions that can be chosen to be f at least contains 0 (it is not the emptyset). Proof:

$$\begin{aligned}
\mathsf{f}(x) = 0 \implies & \left\langle \mathsf{f} \,\middle|\, \left\{ \mathsf{h} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{h} \in \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right\} \right\rangle \\
&= \left\langle 0 \,\middle|\, \left\{ \mathsf{h} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{h} \in \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right\} \right\rangle \\
&= 0 \\
& \implies \quad \mathsf{f} \perp \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \\
& \implies \quad \mathsf{f} \text{ exists}
\end{aligned}$$

4. lemma: $\left\|\mathbf{P}_j \mathsf{f}\right\| = 0 \quad \forall j\in\mathbb{Z}$. Proof:

$$\begin{aligned}
\left\|\mathbf{P}_j \mathsf{f}\right\| &= \|0\| && \text{by definition of f} && \text{(definition 2 page 56)} \\
&= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|
\end{aligned}$$

5. definition: Choose some function $\mathsf{g} \in \boldsymbol{L}^2_{\mathbb{R}}$ such that $\tilde{\mathsf{g}}(\omega) = \tilde{\mathsf{f}}(\omega)\mathbb{1}_{[-R:R]}$ (Definition 3.2 page 40) for some $R > 0$ and such that $\|\mathsf{f} - \mathsf{g}\| < \varepsilon$. Let $\tilde{\mathsf{g}}(\omega)$ be the *Fourier Transform* (Definition H.2 page 192) of $\mathsf{g}(x)$.

6. lemma: The function g (definition 5 page 57) *exists*. Proof: For some (possibly very large) R,

$$\begin{aligned}
\varepsilon &> \left\|\tilde{\mathsf{f}}(\omega) - \tilde{\mathsf{g}}(\omega)\right\| && \text{by definition of g} && \text{(definition 5 page 57)} \\
&= \left\|\tilde{\mathbf{F}}\mathsf{f}(x) - \tilde{\mathbf{F}}\mathsf{g}(x)\right\| && \text{by definition of } \tilde{\mathsf{f}} \text{ and } \tilde{\mathsf{g}} && \text{(definition 2 page 56), (definition 5 page 57)} \\
&= \left\|\tilde{\mathbf{F}}\left[\mathsf{f}(x) - \mathsf{g}(x)\right]\right\| && \text{by } \textit{linearity} \text{ of } \tilde{\mathbf{F}} && \text{(Definition C.4 page 113)} \\
&= \|\mathsf{f}(x) - \mathsf{g}(x)\| && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem H.2 page 193)} \\
\implies & \quad \mathsf{g} \text{ exists} && \text{because it's possible to satisfy definition 5 page 57}
\end{aligned}$$

7. lemma: $\left\|\mathbf{P}_j \mathsf{g}\right\| < \varepsilon \quad \forall j\in\mathbb{Z}$ for sufficiently large R. Proof:

$$\begin{aligned}
\varepsilon &> \|\mathsf{f} - \mathsf{g}\| && \text{by definition of g} && \text{(definition 5 page 57)} \\
&\geq \left\|\mathbf{P}_j\left[\mathsf{f} - \mathsf{g}\right]\right\| && \text{by property of } \textit{projection operators} && \text{(Definition C.10 page 127)} \\
&= \left\|\mathbf{P}_j \mathsf{f} - \mathbf{P}_j \mathsf{g}\right\| && \text{by } \textit{additive} \text{ property of } \mathbf{P}_j && \text{(Definition C.4 page 113)} \\
&\geq \left|\left\|\mathbf{P}_j \mathsf{f}\right\| - \left\|\mathbf{P}_j \mathsf{g}\right\|\right| && \text{by } \textit{Reverse Triangle Inequality} \\
&= \left|0 - \left\|\mathbf{P}_j \mathsf{g}\right\|\right| && \text{by ((4) lemma page 57)} \\
&= \left\|\mathbf{P}_j \mathsf{g}\right\| && \text{by } \textit{strictly positive} \text{ property of } \|\cdot\| && \text{(Definition C.5 page 116)}
\end{aligned}$$

8. lemma: $\mathsf{g} = 0$. Proof:

$$\begin{aligned}
0 &= \lim_{j\to\infty} \left\|\mathbf{P}_j \mathsf{g}\right\|^2 && \text{by (7) lemma page 57} \\
&\geq \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbf{P}_j \mathsf{g} \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by } \textit{frame property} && \text{(Proposition 2.5 page 30)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \mathsf{g} \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by definition of } \mathbf{P}_j && \text{(item (1) page 56)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathbf{F}} \mathsf{g} \,|\, \tilde{\mathbf{F}} \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem H.2 page 193)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathsf{g}}(\omega) \,|\, 2^{-j/2} e^{-i2^{-j}\omega n} \tilde{\phi}(2^{-j}\omega) \right\rangle\right|^2 && \text{by Proposition 3.12 page 47}
\end{aligned}$$

$$= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{g}(\omega)\tilde{\phi}^*(2^{-j}\omega) \,|\, 2^{-j/2}e^{-i2^{-j}\omega n}\right\rangle\right|^2 \quad \text{by property of } \langle\triangle\,|\,\nabla\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}}$$

$$= \lim_{j\to\infty} A \left\|\tilde{g}(\omega)\tilde{\phi}^*(2^{-j}\omega)\right\|^2 \quad \text{by } \textit{Parseval's Identity} \quad \text{(Theorem 2.9 page 22)}$$

$$= A \left\|\tilde{g}(\omega)\tilde{\phi}^*(0)\right\|^2 \quad \text{by left hypothesis (2)}$$

$$= A\left|\tilde{\phi}^*(0)\right|^2 \|\tilde{g}(\omega)\|^2 \quad \text{by } \textit{homogeneous} \text{ property of } \|\cdot\|$$

$$= A\left|\tilde{\phi}(0)\right|^2 \|g\|^2 \quad \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} \quad \text{(Theorem H.2 page 193)}$$

$$\Longrightarrow \|g\| = 0 \quad \text{by left hypothesis (3)}$$

$$\Longleftrightarrow g = 0 \quad \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|$$

9. Final step in proof that $\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}$:

$$g = 0 \quad \text{by (8) lemma page 57}$$

$$\Longrightarrow f = 0 \quad \text{by definition of } g \quad \text{(definition 5 page 57)}$$

$$\Longrightarrow \left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}$$

⇨

Definition 4.1 defines an MRA on the space $\boldsymbol{L}^2_{\mathbb{R}}$, which is a special case of a *separable Hilbert space*. A Hilbert space is a *linear space* that is equipped with an *inner product*, is *complete* with respect to the *metric* induced by the inner product, and contains a subset that is *dense* in $\boldsymbol{L}^2_{\mathbb{R}}$.

An *inner product* on a linear space endows the linear space with a *topology*. The sum such as $\sum_{n=1}^{N} \alpha_n f_n$ is finite and thus suitable for a finite linear space only. An infinite space requires an infinite sum $\sum_{n=1}^{\infty} \alpha_n \phi_n$, and an infinite sum is defined in terms of a limit. The limit, in turn, is defined in terms of a *topology*. The *inner product* induces a *norm* (Definition C.5 page 116) which induces a *metric* which induces a topology.

Definition 4.1 defines each subspace $\boldsymbol{V}_j$ to be *closed* ($\boldsymbol{V}_j = \boldsymbol{V}_j^-$) in $\boldsymbol{L}^2_{\mathbb{R}}$. As one might imagine, the properties of *completeness* and *closure* are closely related. Moreover, Every *complete* sequence is also *bounded*, and so each subspace $\boldsymbol{V}_j$ is *bounded* as well.

Proposition 4.3. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!)\right)$ *be an* MRA SPACE.

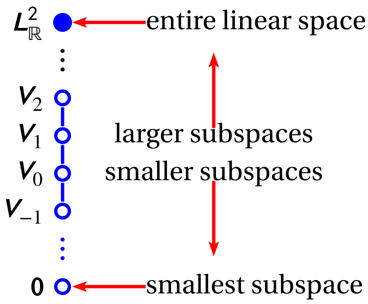
PRP *Each subspace* $\boldsymbol{V}_j$ *is* COMPLETE.

✎PROOF:

1. By definition Definition 4.1, $\boldsymbol{L}^2_{\mathbb{R}}$ is *complete*.
2. In any metric space, (which includes all inner product spaces such as $\boldsymbol{L}^2_{\mathbb{R}}$), a *closed* subspace of a *complete* metric space is itself also *complete*.
3. In any *complete* metric space $\boldsymbol{X}$ (which includes all Hilbert spaces such as $\boldsymbol{L}^2_{\mathbb{R}}$), the two properties coincide—that is, a subspace is complete *if and only if* it is closed in the space $\boldsymbol{X}$.
4. So because $\boldsymbol{L}^2_{\mathbb{R}}$ is *complete* and each $\boldsymbol{V}_j$ is *closed*, then each $\boldsymbol{V}_j$ is also *complete*.

⇨

4.3 Order structure



A *multiresolution analysis* (Definition 4.1 page 54) together with the set inclusion relation $\subseteq$ forms the *linearly ordered set* $(\langle\!\langle V_j \rangle\!\rangle, \subseteq)$, illustrated to the right by a *Hasse diagram*. Subspaces V_j increase in "size" with increasing j. That is, they contain more and more vectors (functions) for larger and larger j—with the upper limit of this sequence being $L^2_{\mathbb{R}}$. Alternatively, we can say that approximation within a subspace V_j yields greater "*resolution*" for increasing j.

The *least upper bound* (*l.u.b.*) of the linearly ordered set $(\langle\!\langle V_j \rangle\!\rangle, \subseteq)$ is $L^2_{\mathbb{R}}$ (Definition 4.1 page 54):

$$\left(\bigcup_{j\in\mathbb{Z}} V_j\right)^- = L^2_{\mathbb{R}}.$$

The *greatest lower bound* (*g.l.b.*) of the linearly ordered set $(\langle\!\langle V_j \rangle\!\rangle, \subseteq)$ is $\mathbf{0}$ (Proposition 4.1 page 54):

$$\bigcap_{j\in\mathbb{Z}} V_j = \mathbf{0}.$$

All linear subspaces contain the zero vector (Proposition B.3 page 99). So the intersection of any two subspaces must at least contain $\mathbb{0}$. If the intersection of any two linear subspaces X and Y is exactly $\{\mathbb{0}\}$, then for any vector in the sum of those subspaces ($u \in X \dot{+} Y$) there are **unique** vectors $\mathsf{f} \in X$ and $\mathsf{g} \in Y$ such that $u = \mathsf{f} + \mathsf{g}$. This is *not* necessarily true if the intersection contains more than just $\{\mathbb{0}\}$ (Theorem B.1 page 101).

4.4 Dilation equation

Several functions in mathematics exhibit a kind of *self-similar* or *recursive* property:

- If a function $\mathsf{f}(x)$ is *linear*, then (Example 3.2 page 50)
  $$\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x.$$
- If a function $\mathsf{f}(x)$ is sufficiently *bandlimited*, then the *Cardinal series* (Example 3.3 page 50) demonstrates
  $$\mathsf{f}(x) = \sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin[\pi(x)]}{\pi(x)}.$$
- *B-splines* are another example:
  $$\mathsf{N}_n(x) = \frac{1}{n}x\mathsf{N}_{n-1}(x) - \frac{1}{n}x\mathbf{T}\mathsf{N}_{n-1}(x) + \frac{n+1}{n}\mathbf{T}\mathsf{N}_{n-1}(x) \qquad \forall n\in\mathbb{W}\setminus\{1\}, \forall x\in\mathbb{R}.$$

The scaling function $\phi(x)$ (Definition 4.1 page 54) also exhibits a kind of *self-similar* property. By Definition 4.1 page 54, the dilation $\mathbf{D}\mathsf{f}$ of each vector f in V_0 is in V_1. If $\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for V_0, then $\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for V_1, $\{\mathbf{D}^2\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for V_2, ...; and in general $\{\mathbf{D}^j\mathbf{T}^m\phi | j \in \mathbb{Z}\}$ is a basis for V_j. Also, if ϕ is in V_0, then it is also in V_1 (because $V_0 \subset V_1$). And because ϕ is in V_1 and because $\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for V_1, ϕ is a linear combination of the elements in $\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$. That is, ϕ can be represented as a linear combination of translated and dilated versions of itself.

The resulting equation is called the *dilation equation* (Definition 4.2, next).[7]

Definition 4.2. [8] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!)\right)$ *be a* MULTIRESOLUTION ANALYSIS SPACE *with scaling function* ϕ (Definition 4.1 page 54). *Let* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE (Definition I.1 page 203) *in* $\ell^2_{\mathbb{R}}$ (Definition I.2 page 203).

DEF

The EQUATION $\left\{\phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) \qquad \forall x\in\mathbb{R}\right\}$ *is called the* ***dilation equation***.
It is also called the ***refinement equation***, ***two-scale difference equation***, *and* ***two-scale relation***.

Remark 4.1.

REM

The *dilation equation* under the defintions of $\mathbf{T}$ and $\mathbf{D}$ evaluates to

$$\phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n).$$

✎PROOF:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\phi(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n) && \text{by definition of } \mathbf{D} && \text{(Definition 3.3 page 40)}
\end{aligned}$$

⇰

Theorem 4.1 (dilation equation). *Let an* MRA SPACE *and* SCALING FUNCTION *be as defined in Definition 4.1 page 54.*

THM

$$\left\{\begin{array}{l}\left(L^2_{\mathbb{R}}, (\!(V_j)\!)\right) \text{ is an MRA SPACE} \\ \text{with SCALING FUNCTION } \phi\end{array}\right\} \implies \underbrace{\left\{\begin{array}{ll}\exists (\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}} \text{ such that} & \\ \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) & \forall x\in\mathbb{R}\end{array}\right\}}_{\text{DILATION EQUATION IN "TIME"}}$$

✎PROOF:

$$\begin{aligned}
\phi &\in V_0 && \text{by definition of MRA} && \text{(Definition 4.1 page 54)} \\
&\subseteq V_1 && \text{by definition of MRA} && \text{(Definition 4.1 page 54)} \\
&\triangleq \operatorname{span}\{\mathbf{D}\mathbf{T}^n\phi(x) \mid n\in\mathbb{Z}\} && \text{by definition of } V_j && \text{(Definition 4.1 page 54)} \\
&\implies \exists (\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}} \text{ such that } \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi(x) && \text{by definition of span} && \text{(Definition 2.2 page 9)}
\end{aligned}$$

⇰

Lemma 4.1. [9] *Let* $\phi(x)$ *be a function in* $L^2_{\mathbb{R}}$ (Definition D.1 page 141). *Let* $\tilde{\phi}(\omega)$ *be the* FOURIER TRANSFORM (Definition H.2 page 192) *of* $\phi(x)$. *Let* $\breve{\mathsf{h}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM (Definition L.1 page 223) *of a sequence* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$.

LEM

$$\begin{array}{llll}
(A)\quad \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi(x) \quad \forall x\in\mathbb{R} & \iff \tilde{\phi}(\omega) = \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) & \forall \omega\in\mathbb{R} & (1) \\
& \iff \tilde{\phi}(\omega) = \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \forall n\in\mathbb{N}, \omega\in\mathbb{R} & (2)
\end{array}$$

[7] The property of *translation invariance* is of particular significance in the theory of *normed linear spaces* (a Hilbert space is a complete normed linear space equipped with an inner product).

[8] Jawerth and Sweldens (1994) page 7

[9] Mallat (1999) page 228

✎PROOF:

1. Proof that (A) $\implies$ (1):

$$\begin{aligned}
\tilde{\phi}(\omega) &\triangleq \tilde{\mathbf{F}}\phi \\
&= \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) && \text{by (A)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \underbrace{\frac{\sqrt{2}}{2} e^{-i\frac{\omega}{2}n} \phi\left(\frac{\omega}{2}\right)}_{\tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n\phi(x)} && \text{by Proposition 3.12 page 47} \\
&= \underbrace{\frac{\sqrt{2}}{2}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\frac{\omega}{2}n}\right]}_{\breve{\mathsf{h}}(\omega/2)} \tilde{\phi}\left(\frac{\omega}{2}\right) \\
&= \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \textit{DTFT} \text{ (Definition L.1 page 223)}
\end{aligned}$$

2. Proof that (A) $\impliedby$ (1):

$$\begin{aligned}
\phi(x) &= \tilde{\mathbf{F}}^{-1}\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega) \\
&= \tilde{\mathbf{F}}^{-1} \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by (1)} \\
&= \tilde{\mathbf{F}}^{-1} \frac{\sqrt{2}}{2} \sum_{n\in\mathbb{Z}} h_n e^{-i\frac{\omega}{2}n} \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \textit{DTFT} \text{ (Definition L.1 page 223)} \\
&= \frac{\sqrt{2}}{2} \sum_{n\in\mathbb{Z}} h_n \tilde{\mathbf{F}}^{-1} e^{-i\frac{\omega}{2}n} \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by property of linear operators} \\
&= \frac{\sqrt{2}}{2} \sum_{n\in\mathbb{Z}} h_n \tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n \phi && \text{by Proposition 3.12 page 47} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) && \text{by definition of } \textit{operator inverse}
\end{aligned}$$

3. Proof that (1) $\implies$ (2):

(a) Proof for $N = 1$ case:

$$\begin{aligned}
\tilde{\phi}\left(\frac{\omega}{2^N}\right) \prod_{n=1}^{N} \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\Bigg|_{N=1} &= \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right) \\
&= \tilde{\phi}(\omega) && \text{by (1)}
\end{aligned}$$

(b) Proof that [N case] $\implies$ [$N+1$ case]:

$$\begin{aligned}
\tilde{\phi}\left(\frac{\omega}{2^{N+1}}\right) \prod_{n=1}^{N+1} \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) &= \left[\prod_{n=1}^{N} \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\right] \underbrace{\frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2^{N+1}}\right) \tilde{\phi}\left(\frac{\omega}{2^{N+1}}\right)}_{\tilde{\phi}(\omega/2^N)} \\
&= \tilde{\phi}(\omega/2^N) \prod_{n=1}^{N} \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) \\
&= \tilde{\phi}(\omega) && \text{by [} N \text{ case] hypothesis}
\end{aligned}$$

4. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\tilde{\phi}(\omega) &= \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\Bigg|_{N=1} && \text{by (2)}\\
&= \tilde{\phi}\left(\frac{\omega}{2}\right)\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)
\end{aligned}$$

✏

Lemma 4.2. *Let $\phi(x)$ be a function in $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141). Let $\tilde{\phi}(\omega)$ be the* FOURIER TRANSFORM *(Definition H.2 page 192) of $\phi(x)$. Let $\breve{\mathsf{h}}(\omega)$ be the* DISCRETE TIME FOURIER TRANSFORM *(Definition L.1 page 223) of $(\!(\mathsf{h}_n)\!)$. Let $\prod_{n=1}^{\infty} x_n \triangleq \lim_{N\to\infty}\prod_{n=1}^{N} x_n$, with respect to the standard norm in $\boldsymbol{L}^2_{\mathbb{R}}$.*

LEM

$$\left\{\begin{array}{ll}\tilde{\phi}(\omega) = C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \\ \forall C>0,\omega\in\mathbb{R} & (A)\end{array}\right\}
\begin{array}{llll}
\Longrightarrow & \phi(x) = \sum_{n\in\mathbb{Z}}\mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) & \forall x\in\mathbb{R} & (1)\\
\Longleftrightarrow & \tilde{\phi}(\omega) = \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) & \forall\omega\in\mathbb{R} & (2)\\
\Longleftrightarrow & \tilde{\phi}(\omega) = \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \forall n\in\mathbb{N},\omega\in\mathbb{R} & (3)
\end{array}$$

✎PROOF:

1. Proof that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3): by Lemma 4.1 page 60
2. Proof that (A) $\Longrightarrow$ (2):

$$\begin{aligned}
\tilde{\phi}(\omega) &= C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) && \text{by left hypothesis}\\
&= C\,\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^{n+1}}\right)\\
&= C\,\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega/2}{2^n}\right)\\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\left[C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega/2}{2^n}\right)\right]\\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by left hypothesis}
\end{aligned}$$

✏

Proposition 4.4. *Let $\phi(x)$ be a function in $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141). Let $\tilde{\phi}(\omega)$ be the* FOURIER TRANSFORM *(Definition H.2 page 192) of $\phi(x)$. Let $\breve{\mathsf{h}}(\omega)$ be the* DISCRETE TIME FOURIER TRANSFORM *(Definition L.1 page 223) of $(\!(\mathsf{h}_n)\!)$. Let $\prod_{n=1}^{\infty} x_n \triangleq \lim_{N\to\infty}\prod_{n=1}^{N} x_n$, with respect to the standard norm in $\boldsymbol{L}^2_{\mathbb{R}}$.*

PRP
$$\left\{\begin{array}{l}\tilde{\phi}(\omega) \text{ is} \\ \text{CONTINUOUS} \\ \text{at } \omega=0\end{array}\right\} \implies \left\{\begin{array}{rrcll}
 & \phi(x) & = & \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) & \forall x\in\mathbb{R} \quad (1)\\
\iff & \tilde{\phi}(\omega) & = & \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) & \forall \omega\in\mathbb{R} \quad (2)\\
\iff & \tilde{\phi}(\omega) & = & \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \forall n\in\mathbb{N}, \omega\in\mathbb{R} \quad (3)\\
\iff & \tilde{\phi}(\omega) & = & \tilde{\phi}(0)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \omega\in\mathbb{R} \quad (4)
\end{array}\right\}$$

✎PROOF:

1. Proof that (1) $\iff$ (2) $\iff$ (3): by Lemma 4.1 page 60

2. Proof that (3) $\implies$ (4):

$$\begin{aligned}
\tilde{\phi}(0)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) &= \lim_{N\to\infty}\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) && \text{by } \textit{continuity} \text{ and definition of } \prod_{n=1}^{\infty} x_n\\
&= \tilde{\phi}(\omega) && \text{by (3) and Lemma 4.1 page 60}
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4): by Lemma 4.2 page 62

✏

Definition 4.3 (next) formally defines the coefficients that appear in Theorem 4.1 (page 60).

Definition 4.3. *Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!)\right)$ be a multiresolution analysis space with scaling function ϕ. Let $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ be a sequence of coefficients such that $\phi = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi$.*

DEF *A **multiresolution system** is the tuple $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, (\!(h_n)\!)\right)$. The sequence $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ is the **scaling coefficient sequence**. A multiresolution system is also called an **MRA system**. An* MRA SYSTEM *is an **orthonormal MRA system** if $\left\{\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\}$ is* ORTHONORMAL.

Theorem 4.2. *Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, (\!(h_n)\!)\right)$ be an* MRA SYSTEM (Definition 4.3 page 63). *Let $\operatorname{span} A$ be the* LINEAR SPAN (Definition 2.2 page 9) *of a set A.*

THM
$$\underbrace{\operatorname{span}\left\{\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} = \boldsymbol{V}_0}_{\left\{\mathbf{T}^n\phi|\,n\in\mathbb{Z}\right\} \text{ is a BASIS for } \boldsymbol{V}_0} \quad\implies\quad \underbrace{\operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} = \boldsymbol{V}_j \quad \forall j\in\mathbb{W}}_{\left\{\mathbf{D}^j\mathbf{T}^n\phi|\,n\in\mathbb{Z}\right\} \text{ is a BASIS for } \boldsymbol{V}_j}$$

✎PROOF: Proof is by induction:[10]

1. induction basis (proof for $j=0$ case):

$$\begin{aligned}
\operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\}\Big|_{j=0} &= \operatorname{span}\left\{\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\}\\
&= \boldsymbol{V}_0 && \text{by left hypothesis}
\end{aligned}$$

[10] Smith (2011) page 4

2. induction step (proof that j case $\implies$ $j+1$ case):

$$
\begin{aligned}
&\operatorname{span}\left\{\mathbf{D}^{j+1}\mathbf{T}^n\phi \,\middle|\, n\in\mathbb{Z}\right\} \\
&= \left\{\mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\alpha_n) \quad \text{such that} \quad \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j+1}\mathbf{T}^n\phi\right\} && \text{by definition of span} && \text{(Definition 2.2 page 9)} \\
&= \left\{\mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\alpha_n) \quad \text{such that} \quad \mathsf{f}(x) = \mathbf{D}\sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \left\{\mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\alpha_n) \quad \text{such that} \quad \mathbf{D}^{-1}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \left\{[\mathbf{D}\mathsf{f}]\in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\alpha_n) \quad \text{such that} \quad \mathbf{D}^{-1}[\mathbf{D}\mathsf{f}(x)] = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \mathbf{D}\left\{\mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\alpha_n) \quad \text{such that} \quad \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \mathbf{D}\operatorname{span}\left\{\mathbf{D}^{j}\mathbf{T}^n\phi \,\middle|\, n\in\mathbb{Z}\right\} && \text{by definition of span} && \text{(Definition 2.2 page 9)} \\
&= \mathbf{D}\boldsymbol{V}_j && \text{by induction hypothesis} \\
&= \boldsymbol{V}_{j+1} && \text{by } \textit{self-similar} \text{ property} && \text{(Definition 4.1 page 54)}
\end{aligned}
$$

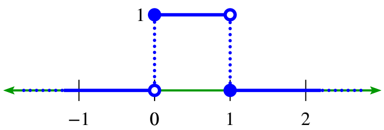
✎

Example 4.1.

EX

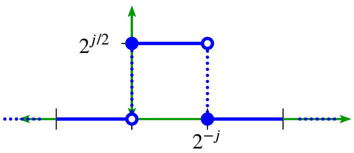
In the *Haar* MRA, the scaling function $\phi(x)$ is the *pulse function*

$$\phi(x) = \begin{cases} 1 & \text{for } x\in[0:1) \\ 0 & \text{otherwise.}\end{cases}$$



In the subspace $\boldsymbol{V}_j$ ($j\in\mathbb{Z}$) the scaling functions are

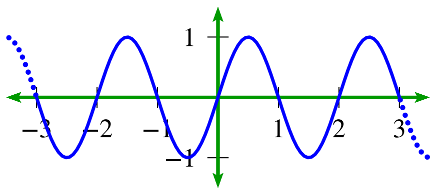
$$\mathbf{D}^j\phi(x) = \begin{cases} (2)^{j/2} & \text{for } x\in\left[0:\left(2^{-j}\right)\right) \\ 0 & \text{otherwise.}\end{cases}$$



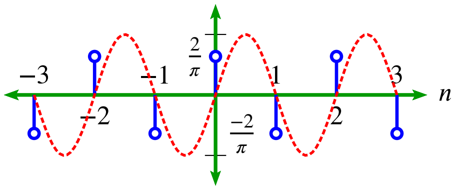
The scaling subspace $\boldsymbol{V}_0$ is the span $\boldsymbol{V}_0 \triangleq \operatorname{span}\left\{\mathbf{T}^n\phi | n\in\mathbb{Z}\right\}$. The scaling subspace $\boldsymbol{V}_j$ is the span $\boldsymbol{V}_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi | n\in\mathbb{Z}\right\}$. Note that $\left\|\mathbf{D}^j\mathbf{T}^n\phi\right\|$ for each resolution j and shift n is unity:

$$
\begin{aligned}
\left\|\mathbf{D}^j\mathbf{T}^n\phi\right\|^2 &= \|\phi\|^2 && \text{by } \textit{unitary} \text{ properties of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Theorem 3.1 page 45)} \\
&= \int_0^1 |1|^2 \,\mathrm{d}x && \text{by definition of } \|\cdot\| \text{ on } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition D.1 page 141)} \\
&= 1
\end{aligned}
$$

Let $\mathsf{f}(x) = \sin(\pi x)$. Suppose we want to project $\mathsf{f}(x)$ onto the subspaces $\boldsymbol{V}_0, \boldsymbol{V}_1, \boldsymbol{V}_2, \ldots$.



The values of the transform coefficients for the subspace $\boldsymbol{V}_j$ are given by



$$
\begin{aligned}
\left[\mathbf{R}_j \mathsf{f}(x)\right](n) &= \frac{1}{\|\mathbf{D}^j\mathbf{T}^n\phi\|^2}\left\langle \mathsf{f}(x) \,|\, \mathbf{D}^j\mathbf{T}^n\phi\right\rangle \\
&= \frac{1}{\cancel{\|\phi\|^2}^{\,1}}\left\langle \mathsf{f}(x) \,|\, 2^{j/2}\phi\left(2^j x - n\right)\right\rangle && \text{by Proposition 3.3 page 41} \\
&= 2^{j/2}\left\langle \mathsf{f}(x) \,|\, \phi\left(2^j x - n\right)\right\rangle \\
&= 2^{j/2}\int_{2^{-j}n}^{2^{-j}(n+1)} \mathsf{f}(x)\, \mathsf{d}x \\
&= 2^{j/2}\int_{2^{-j}n}^{2^{-j}(n+1)} \sin(\pi x)\, \mathsf{d}x \\
&= 2^{j/2}\left(-\frac{1}{\pi}\right)\cos(\pi x)\Big|_{2^{-j}n}^{2^{-j}(n+1)} \\
&= \frac{2^{j/2}}{\pi}\left[\cos\left(2^{-j}n\pi\right) - \cos\left(2^{-j}(n+1)\pi\right)\right]
\end{aligned}
$$

And the projection $\mathbf{A}_n\mathsf{f}(x)$ of the function $\mathsf{f}(x)$ onto the subspace $\boldsymbol{V}_j$ is

$$
\begin{aligned}
\mathbf{A}_j\mathsf{f}(x) &= \sum_{n\in\mathbb{Z}}\left\langle \mathsf{f}(x) \,|\, \mathbf{D}^j\mathbf{T}^n\phi\right\rangle \mathbf{D}^j\mathbf{T}^n\phi \\
&= \frac{2^{j/2}}{\pi}\sum_{n\in\mathbb{Z}}\left[\cos\left(2^{-j}n\pi\right) - \cos\left(2^{-j}(n+1)\pi\right)\right]2^{j/2}\phi\left(2^j x - n\right) \\
&= \frac{2^{j}}{\pi}\sum_{n\in\mathbb{Z}}\left[\cos\left(2^{-j}n\pi\right) - \cos\left(2^{-j}(n+1)\pi\right)\right]\phi\left(2^j x - n\right)
\end{aligned}
$$

The transforms of $\sin(\pi x)$ into the subspaces $\boldsymbol{V}_0$, $\boldsymbol{V}_1$, and $\boldsymbol{V}_2$, as well as the approximations in those subspaces are as illustrated in Figure 4.1 (page 66).

4.5 Necessary Conditions

Theorem 4.3 (admissibility condition). *Let* $\check{\mathsf{h}}(z)$ *be the* Z-TRANSFORM *(Definition I.4 page 204) and* $\check{\mathsf{h}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition L.1 page 223) of a sequence* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$.

THM

$\left\{\left(L^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)\right.$ *is an* MRA SYSTEM *(Definition 4.3 page 63)* $\left.\right\}$

$$
\begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array}
\underbrace{\left\{\sum_{n\in\mathbb{Z}}\mathsf{h}_n = \sqrt{2}\right\}}_{\text{(1) ADMISSIBILITY in "time"}}
\iff
\underbrace{\left\{\check{\mathsf{h}}(z)\Big|_{z=1} = \sqrt{2}\right\}}_{\text{(2) ADMISSIBILITY in "z domain"}}
\iff
\underbrace{\left\{\check{\mathsf{h}}(\omega)\Big|_{\omega=0} = \sqrt{2}\right\}}_{\text{(3) ADMISSIBILITY in "frequency"}}
$$

✎PROOF:

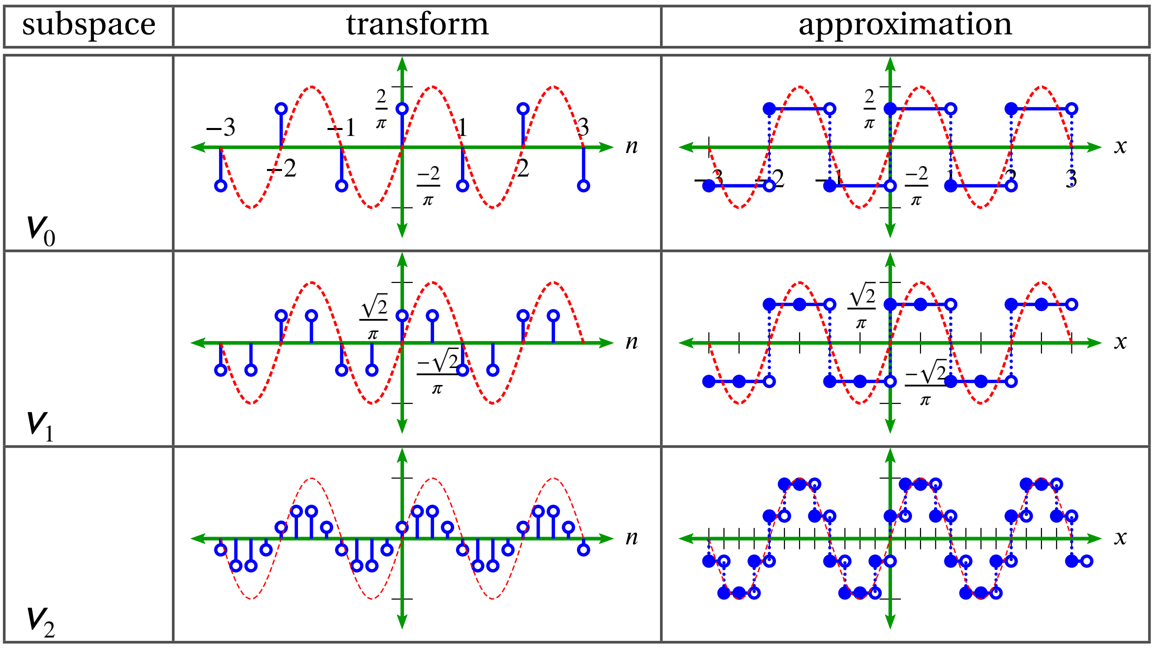


Figure 4.1: Projections of $\sin(\pi x)$ on Haar subspaces (Example 4.1 page 64)

1. Proof that MRA system $\implies$ (1):

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{h}_n &= \frac{\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x}{\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \\
&= \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(x)\,\mathrm{d}x \\
&= \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \frac{\sqrt{2}}{\sqrt{2}} \phi(2y-n)2\,\mathrm{d}y && \text{let } y \triangleq \frac{x+n}{2} \implies x = 2y-n \implies \mathrm{d}x = 2\,\mathrm{d}y \\
&= \frac{2}{\sqrt{2}} \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(y)\,\mathrm{d}y && \text{by definitions of } \mathbf{T} \text{ and } \mathbf{D} \text{ (Definition 3.3 page 40)} \\
&= \sqrt{2} \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x} \int_{\mathbb{R}} \phi(y)\,\mathrm{d}y && \text{by \emph{dilation equation} (Theorem 4.1 page 60)} \\
&= \sqrt{2}
\end{aligned}
$$

2. Alternate proof that MRA system $\implies$ (1):
Let $\mathsf{f}(x) \triangleq 1 \quad \forall x \in \mathbb{R}$.

$$
\begin{aligned}
\langle \phi \,|\, \mathsf{f} \rangle &= \left\langle \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi \,\middle|\, \mathsf{f} \right\rangle && \text{by \emph{dilation equation}} && \text{(Theorem 4.1 page 60)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \mathbf{D}\mathbf{T}^n \phi \,|\, \mathsf{f} \rangle && \text{by linearity of } \langle \triangle \,|\, \triangledown \rangle \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \phi \,|\, (\mathbf{D}\mathbf{T}^n)^* \mathsf{f} \rangle && \text{by definition of operator adjoint} && \text{(Theorem C.13 page 126)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \phi \,|\, (\mathbf{T}^*)^n \mathbf{D}^* \mathsf{f} \rangle && \text{by property of operator adjoint} && \text{(Theorem C.13 page 126)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,|\, (\mathbf{T}^{-1})^n \mathbf{D}^{-1} \mathsf{f} \right\rangle && \text{by unitary property of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Proposition 3.7 page 43)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,\middle|\, (\mathbf{T}^{-1})^n \frac{\sqrt{2}}{2} \mathsf{f} \right\rangle && \text{because } \mathsf{f} \text{ is a constant hypothesis} && \text{and by Proposition 3.2 page 41}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,|\, \tfrac{\sqrt{2}}{2}\mathsf{f} \right\rangle && \text{by } \mathsf{f}(x) = 1 \text{ definition} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \tfrac{\sqrt{2}}{2} \langle \phi \,|\, \mathsf{f} \rangle && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&= \tfrac{\sqrt{2}}{2} \langle \phi \,|\, \mathsf{f} \rangle \sum_{n\in\mathbb{Z}} \mathsf{h}_n \\
&\implies \sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2}
\end{aligned}
$$

3. Proof that (1) $\iff$ (2) $\iff$ (3): by Proposition L.2 page 225.

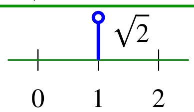
4. Proof for $\not\Leftarrow$ part: by Counterexample 4.1 page 67.

✏

Counterexample 4.1. Let $\left(\boldsymbol{L}^2_{\mathbb{R}},\, (\!(\boldsymbol{V}_j)\!),\, \phi,\, (\!(h_n)\!)\right)$ be an *MRA system* (Definition 4.3 page 63).

CNT

$$
\left\{ (\!(\mathsf{h}_n)\!) \triangleq \sqrt{2}\bar{\delta}_{n-1} \triangleq \begin{cases} \sqrt{2} & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases} \right\} \implies \{\phi(x) = 0\}
$$



which means

$$
\left\{ \sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2} \right\} \quad \not\Longrightarrow \quad \left\{ \left(\boldsymbol{L}^2_{\mathbb{R}},\, (\!(\boldsymbol{V}_j)\!),\, \phi,\, (\!(h_n)\!)\right) \text{ is an MRA system for } \boldsymbol{L}^2_{\mathbb{R}}. \right\}
$$

✎PROOF:

$$
\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) && \text{by } \textit{dilation equation} && \text{(Theorem 4.1 page 60)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x - n) && \text{by definitions of } \mathbf{D} \text{ and } \mathbf{T} && \text{(Definition 3.3 page 40)} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{\sqrt{2}\bar{\delta}_{n-1}}_{(\!(\mathsf{h}_n)\!)} \phi(2x - n) && \text{by definitions of } (\!(\mathsf{h}_n)\!) \\
&= \sqrt{2}\phi(2x - 1) && \text{by definition of } \phi(x) \\
\implies \phi(x) &= 0
\end{aligned}
$$

This implies $\phi(x) = 0$, which implies that $\left(\boldsymbol{L}^2_{\mathbb{R}},\, (\!(\boldsymbol{V}_j)\!),\, \phi,\, (\!(h_n)\!)\right)$ is *not* an *MRA system* for $\boldsymbol{L}^2_{\mathbb{R}}$ because

$$
\overline{\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right)} = \overline{\left(\bigcup_{j\in\mathbb{Z}} \operatorname{span}\left\{ \mathbf{D}^j \mathbf{T}^n \phi |_{n\in\mathbb{Z}} \right\} \right)} \neq \boldsymbol{L}^2_{\mathbb{R}}
$$

(the *least upper bound* is *not* $\boldsymbol{L}^2_{\mathbb{R}}$). ✏

Theorem 4.4 (Quadrature condition in "time"). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}},\, (\!(\boldsymbol{V}_j)\!),\, \phi,\, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 4.3 page 63).*

THM

$$
\sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}^*_k \left\langle \phi \,|\, \mathbf{T}^{2n-m+k} \phi \right\rangle = \langle \phi \,|\, \mathbf{T}^n \phi \rangle \qquad \forall n\in\mathbb{Z}
$$

✎PROOF:

$$
\begin{aligned}
\langle \phi \,|\, \mathbf{T}^n \phi \rangle &= \left\langle \sum_{m\in\mathbb{Z}} \mathsf{h}_m \mathbf{D}\mathbf{T}^m \phi \,\middle|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} \mathsf{h}_k \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by *dilation equation* (Theorem 4.1 page 60)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \mathbf{D}\mathbf{T}^m \phi \,|\, \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by properties of } \langle \triangle \,|\, \nabla \rangle \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \, \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by definition of operator adjoint (Proposition C.3 page 125)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \, \mathbf{D}\mathbf{T}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by Proposition 3.5 page 42} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, \mathbf{T}^{*m} \mathbf{D}^* \mathbf{D}\mathbf{T}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by operator star-algebra properties (Theorem C.13 page 126)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, \mathbf{T}^{-m} \mathbf{D}^{-1} \mathbf{D}\mathbf{T}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by Proposition 3.7 page 43} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k} \phi \right\rangle
\end{aligned}
$$

⇨

Theorem 4.5 (next) presents the *quadrature necessary conditions* of a *wavelet system*. These relations simplify dramatically in the special case of an *orthonormal wavelet system* (Theorem L.4 page 229).

Theorem 4.5 (Quadrature condition in "frequency").[11] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *be an* MRA SYSTEM *(Definition 4.3 page 63). Let* $\check{\mathsf{x}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition L.1 page 223) for a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$*. Let* $\tilde{\mathsf{S}}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM *(Definition O.3 page 241) of* ϕ*.*

THM
$$\left|\check{\mathsf{h}}(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) \;=\; 2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)$$

$\left(\begin{array}{c}\textit{Note: } \tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1 \\ \textit{for} \text{ ORTHONORMAL } \textit{MRA}\end{array}\right)$

✎PROOF:

$$
\begin{aligned}
&2\tilde{\mathsf{S}}_{\phi\phi}(2\omega) \\
&= 2(2\pi) \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(2\omega + 2\pi n)\right|^2 && \text{by Theorem O.1 page 241} \\
&= 2(2\pi) \sum_{n\in\mathbb{Z}} \left|\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right) \tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 && \text{by Lemma 4.1 page 60} \\
&= 2\pi \sum_{n\in\mathbb{Z}_e} \left|\check{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 + 2\pi \sum_{n\in\mathbb{Z}_o} \left|\check{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \\
&= 2\pi \sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega+2\pi n)\right|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi \sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega+2\pi n+\pi)\right|^2 \left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 \\
&= 2\pi \sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega)\right|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi \sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega+\pi)\right|^2 \left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 && \text{by Proposition L.1 page 223} \\
&= \left|\check{\mathsf{h}}(\omega)\right|^2 \left(2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega+2\pi n)\right|^2\right) + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2 \left(2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right) \\
&= \left|\check{\mathsf{h}}(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) && \text{by Theorem O.1 page 241}
\end{aligned}
$$

⇨

[11] Chui (1992) page 135, Goswami and Chan (1999) page 110

4.6 Sufficient conditions

Theorem 4.6 (next) gives a set of *sufficient* conditions on the *scaling function* (Definition 4.1 page 54) ϕ to generate an *MRA*.

Theorem 4.6. [12] *Let* $\boldsymbol{V}_j \triangleq \operatorname{span}\{\mathbf{T}\phi(x) | n\in\mathbb{Z}\}$ *(Definition 2.2 page 9).*

THM
$$\left\{\begin{array}{lll}(1). & (\!(\mathbf{T}^n\phi)\!) \text{ is a RIESZ SEQUENCE (Definition 2.14 page 27)} & \text{and}\\ (2). & \exists (\!(\mathsf{h}_n)\!) \text{ such that } \phi(x) = \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n\phi(x) & \text{and}\\ (3). & \tilde{\phi}(\omega) \text{ is CONTINUOUS at } 0 & \text{and}\\ (4). & \tilde{\phi}(0)\neq 0 & \end{array}\right\} \implies \left\{\begin{array}{c}(\!(\boldsymbol{V}_j)\!)_{j\in\mathbb{Z}} \text{ is an MRA}\\ \text{(Definition 4.1 page 54)}\end{array}\right\}$$

✎PROOF: For this to be true, each of the conditions in the definition of an *MRA* (Definition 4.1 page 54) must be satisfied:

1. Proof that each $\boldsymbol{V}_j$ is *closed*: by definition of span
2. Proof that $(\!(\boldsymbol{V}_j)\!)$ is *linearly ordered*:
$$\boldsymbol{V}_j \subseteq \boldsymbol{V}_{j+1} \iff \operatorname{span}\{\mathbf{D}^j\mathbf{T}^n\phi\} \subseteq \operatorname{span}\{\mathbf{D}^{j+1}\mathbf{T}^n\phi\} \qquad \Longleftarrow (2)$$
3. Proof that $\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j$ is *dense* in $\boldsymbol{L}^2_{\mathbb{R}}$: by Proposition 4.2 page 56
4. Proof of *self-similar* property:
$$\{\mathsf{f}\in\boldsymbol{V}_j \iff \mathbf{D}\mathsf{f}\in\boldsymbol{V}_{j+1}\} \iff \mathsf{f}\in\operatorname{span}\{\mathbf{T}^n\phi\} \iff \mathbf{D}\mathsf{f}\in\operatorname{span}\{\mathbf{D}\mathbf{T}^n\phi\} \qquad \Longleftarrow (2)$$
5. Proof for *Riesz basis*: by (1) and Proposition 4.2 page 56.

✏

4.7 Support size

The *support* of a function is what it's non-zero part "sits" on. If the support of the scaling coefficients $(\!(\mathsf{h}_n)\!)$ goes from say $[0,3]$ in $\mathbb{Z}$, what is the support of the scaling function $\phi(x)$? The answer is $[0,3]$ in $\mathbb{R}$—essentially the same as the support of $(\!(\mathsf{h}_n)\!)$ except that the two functions have different domains ($\mathbb{Z}$ versus $\mathbb{R}$). This concept is defined in Definition 4.4 (next definition), and proven in Theorem 4.7 (next theorem).

Definition 4.4. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 4.3 page 63). Let* X^- *represent the* CLOSURE *of a set* X *in* $\boldsymbol{L}^2_{\mathbb{R}}$, $\vee X$ *the* LEAST UPPER BOUND *of an* ORDERED SET $(X,\leq)$, $\wedge X$ *the* GREATEST LOWER BOUND *of* $(X,\leq)$, *and*

$$\begin{array}{rclll}\lfloor x\rfloor & \triangleq & \bigvee\{n\in\mathbb{Z} | n\leq x\} & \forall x\in\mathbb{R} & (\text{FLOOR of } x)\\ \lceil x\rceil & \triangleq & \bigwedge\{n\in\mathbb{Z} | n\geq x\} & \forall x\in\mathbb{R} & (\text{CEILING of } x).\end{array}$$

DEF
The set $\operatorname{supp}\mathsf{f}$ *of a function* $\mathsf{f}\in Y^X$ *is the* ***support*** *of* f *if*
$$\operatorname{supp}\mathsf{f} \triangleq \left\{\begin{array}{lll}\{x\in\mathbb{R} | \mathsf{f}(x)\neq 0\}^- & \text{for } X=\mathbb{R} & (\text{domain of } \mathsf{f} \text{ is } \mathbb{R}) \quad \text{and}\\ \{x\in\mathbb{R} | \mathsf{f}(\lfloor x\rfloor)\neq 0 \text{ and } \mathsf{f}(\lceil x\rceil)\neq 0\}^- & \text{for } X=\mathbb{Z} & (\text{domain of } \mathsf{f} \text{ is } \mathbb{Z}) \quad .\end{array}\right.$$

Theorem 4.7 (support size). [13] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 4.3 page 63).*

[12] Wojtaszczyk (1997) page 28 ⟨Theorem 2.13⟩, Pinsky (2002) page 313 ⟨Theorem 6.4.27⟩
[13] Mallat (1999) pages 243–244

Let $\operatorname{supp}\mathsf{f}$ *be the support of a function* f (Definition 4.4 page 69).

THM $\operatorname{supp}\phi = \operatorname{supp}\mathsf{h}$

✎PROOF:

1. Definitions: $\operatorname{supp}\phi \triangleq [a, b]$
   $\operatorname{supp}\mathsf{h} \triangleq [k, m].$

2. lemma: $\operatorname{supp}\phi(x) = [a, b] \iff \operatorname{supp}\phi(2x) = \left[\frac{a}{2}, \frac{b}{2}\right]$

3. lemma: $\operatorname{supp}[\lambda\phi(x)] = \operatorname{supp}[\phi(x)] \quad \forall\lambda \in \mathbb{R}\backslash 0$

4. Proof that $k = a$:

$$\begin{aligned}
a &= \bigwedge \operatorname{supp}\phi(x) && \text{by definition of } a && \text{(item (1) page 70)}\\
&\triangleq \bigwedge \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x)\right] && \text{by } \textit{dilation equation} && \text{(Theorem 4.1 page 60)}\\
&= \bigwedge \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 3.3 page 40)}\\
&= \bigwedge \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by (3) lemma}\\
&= \bigwedge \operatorname{supp}\left[\mathsf{h}_k \phi(2x-k)\right] && \text{because } n = k \text{ is the } \textit{least value} \text{ of } n \text{ for which } \mathsf{h}_n \neq 0\\
&= \bigwedge \operatorname{supp}[\phi(2x-k)] && \text{by (3) lemma}\\
&= \bigwedge \operatorname{supp}\left[\phi\left(2\left[x-\frac{k}{2}\right]\right)\right]\\
&= \bigwedge \left\{t | \phi\left(2\left[x-\frac{k}{2}\right]\right) \neq 0\right\} && \text{by definition of supp} && \text{(Definition 4.4 page 69)}\\
&= x \quad \text{such that} \quad x - \frac{k}{2} = \frac{a}{2} && \text{by (2) lemma}\\
&= \frac{k}{2} + \frac{a}{2}
\end{aligned}$$

$$\begin{aligned}
&\implies && \frac{k}{2} = a - \frac{a}{2}\\
&\iff && k = a
\end{aligned}$$

5. Proof that $m = b$:

$$\begin{aligned}
b &= \bigvee \operatorname{supp}\phi(x) && \text{by definition of } b && \text{(item (1) page 70)}\\
&\triangleq \bigvee \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x)\right] && \text{by } \textit{dilation equation} && \text{(Theorem 4.1 page 60)}\\
&= \bigvee \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 3.3 page 40)}\\
&= \bigvee \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by (3) lemma}\\
&= \bigvee \operatorname{supp}\left[\mathsf{h}_m \phi(2x-m)\right] && \text{because } n = m \text{ is the } \textit{greatest value} \text{ of } n \text{ for which } \mathsf{h}_n \neq 0\\
&= \bigvee \operatorname{supp}[\phi(2x-m)] && \text{by (3) lemma}
\end{aligned}$$

$$= \bigvee \operatorname{supp}\left[\phi\left(2\left[x - \frac{m}{2}\right]\right)\right]$$

$$= \bigvee \left\{t | \phi\left(2\left[x - \frac{m}{2}\right]\right) \neq 0\right\} \qquad \text{by definition of supp} \qquad \text{(Definition 4.4 page 69)}$$

$$= x \quad \text{such that} \quad x - \frac{m}{2} = \frac{b}{2} \qquad \text{by (2) lemma}$$

$$= \frac{m}{2} + \frac{b}{2}$$

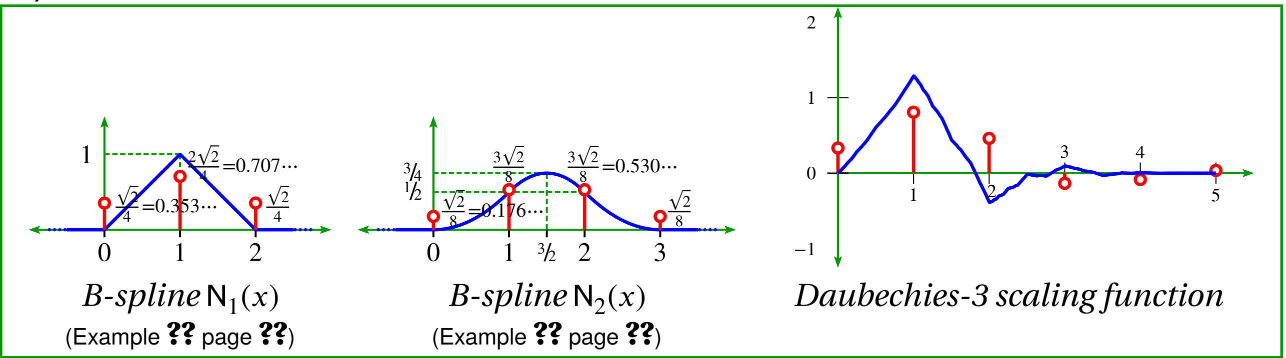
$$\implies \qquad \frac{m}{2} = b - \frac{b}{2}$$

$$\iff \qquad m = b$$

✏

Example 4.2.

EX



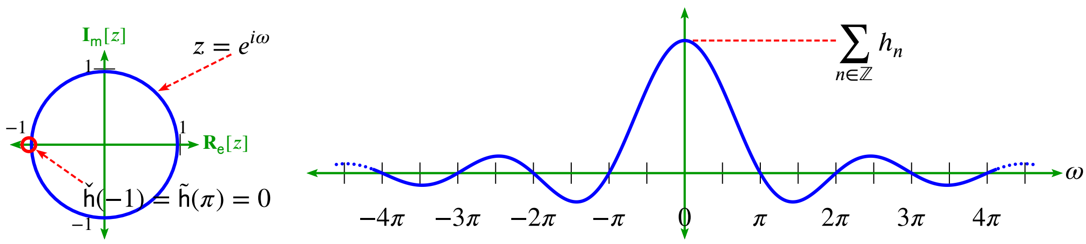
B-spline $\mathsf{N}_1(x)$ (Example ?? page ??)

B-spline $\mathsf{N}_2(x)$ (Example ?? page ??)

Daubechies-3 scaling function

4.8 Scaling functions with partition of unity

The Z transform (Definition I.4 page 204) of a sequence $(\!(h_n)\!)$ with sum $\sum_{n\in\mathbb{Z}}(-1)^n \mathsf{h}_n = 0$ has a zero at $z = -1$. Somewhat surprisingly, the *partition of unity* and *zero at* $z = -1$ properties are actually equivalent (next theorem).



Theorem 4.8. [14] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be a* MULTIRESOLUTION SYSTEM (Definition 4.3 page 63). *Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* FOURIER TRANSFORM (Definition H.2 page 192) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION (Definition 2.12 page 20). *Let* c *be some contant in* $\mathbb{R}\backslash 0$.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi = c}_{\text{(1) PARTITION OF UNITY}} \iff \underbrace{\sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n = 0}_{\text{(2) ZERO AT } z=-1} \iff \underbrace{\sum_{n\in\mathbb{Z}} \mathsf{h}_{2n} = \sum_{n\in\mathbb{Z}} \mathsf{h}_{2n+1} = \frac{\sqrt{2}}{2}}_{\text{(3) sum of even = sum of odd = } \frac{\sqrt{2}}{2}}$$

✎PROOF: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

[14] Jawerth and Sweldens (1994) page 8, Chui (1992) page 123

1. Proof that (1) $\Longleftarrow$ (2):

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \left[\sum_{m\in\mathbb{Z}} \mathsf{h}_m \mathbf{D}\mathbf{T}^m\phi\right] && \text{by } \textit{dilation equation} && \text{(Theorem 4.1 page 60)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathbf{D}\mathbf{T}^m\phi\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^m\phi && \text{by } \textit{commutator relation} && \text{(Proposition 3.5 page 42)}\\
&= \mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{T}^{2n}\mathbf{T}^m\phi\\
&= \mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \left[\sqrt{\frac{2\pi}{2}}\hat{\mathbf{F}}^{-1}\mathbf{S}_2\tilde{\mathbf{F}}(\mathbf{T}^m\phi)\right] && \text{by } \textit{PSF} && \text{(Theorem 3.2 page 48)}\\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1}\mathbf{S}_2 e^{-i\omega m}\tilde{\mathbf{F}}\phi && \text{by Corollary 3.1 page 47}\\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1} e^{-i\frac{2\pi}{2}km}\mathbf{S}_2\tilde{\mathbf{F}}\phi && \text{by definition of } \mathbf{S} && \text{(Definition 3.4 page 48)}\\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1}(-1)^{km}\mathbf{S}_2\tilde{\mathbf{F}}\phi\\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \left[\frac{\sqrt{2}}{2}\sum_{k\in\mathbb{Z}}(-1)^{km}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\frac{2\pi}{2}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem M.1 page 234)}\\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}\mathsf{h}_m\\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}\mathsf{h}_m + \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}\mathsf{h}_m\\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}}\mathsf{h}_m}_{\sqrt{2}} + \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}}(-1)^{m}\mathsf{h}_m}_{0}\\
&= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx} && \text{by Theorem 4.3 (page 65)} && \text{and right hypothesis}\\
&= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e}\tilde{\phi}\left(\frac{2\pi}{2}k\right)e^{i\pi kx} && \text{by definitions of } \tilde{\mathbf{F}} \text{ and } \mathbf{S}_2\\
&= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}}\tilde{\phi}(2\pi k)e^{i2\pi kx} && \text{by definition of } \mathbb{Z}_e\\
&= \frac{1}{\sqrt{2}}\mathbf{D}\left\{\sqrt{2\pi}\sum_{k\in\mathbb{Z}}\tilde{\phi}(2\pi k)e^{i2\pi kx}\right\}\\
&= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n\in\mathbb{Z}}\phi(x+n) && \text{by } \textit{PSF} && \text{(Theorem 3.2 page 48))}\\
&= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n}\mathbf{T}^n\phi && \text{by definition of } \mathbf{T} && \text{(Definition 3.3 page 40)}
\end{aligned}
$$

The above equation sequence demonstrates that

$$\mathbf{D}\sum_n \mathbf{T}^n\phi = \sqrt{2}\sum_n \mathbf{T}^n\phi$$

(essentially that $\sum_n \mathbf{T}^n\phi$ is equal to it's own dilation). This implies that $\sum_n \mathbf{T}^n\phi$ is a constant (Proposition 3.8 page 43).

2. Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi && \text{by left hypothesis} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\phi && \text{by } PSF \quad \text{(Theorem 3.2 page 48)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\underbrace{\sqrt{2}\left(\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\omega n}\right)\left(\mathbf{D}^{-1}\tilde{\mathbf{F}}\phi\right)}_{\tilde{\mathbf{F}}\phi} && \text{by Lemma 4.1 page 60} \\
&= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\omega n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) && \text{by Corollary 3.1 page 47} \\
&= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\frac{\omega}{2}n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) && \text{by evaluation of } \mathbf{D}^{-1} \quad \text{(Proposition 3.2 page 41)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\frac{2\pi k}{2}n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) && \text{by definition of } \mathbf{S} \quad \text{(Definition 3.4 page 48)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n (-1)^{kn}\right)\left(\mathbf{S}\mathbf{D}^{-1}\mathbf{F}\phi\right) \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n (-1)^{kn}\right)\left(\mathbf{S}\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{\omega}{2}\right)\right) && \text{by definition of } \mathbf{S} \quad \text{(Definition 3.4 page 48)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n (-1)^{kn}\right)\left(\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{2\pi k}{2}\right)\right) \\
&= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} && \text{by definition of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem M.1 page 234)} \\
&= \sqrt{\pi}\sum_{k \text{ even}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} + \sqrt{\pi}\sum_{k \text{ odd}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} \\
&= \sqrt{\pi}\sum_{k \text{ even}}\left(\cancelto{\sqrt{2}}{\sum_{n\in\mathbb{Z}} \mathsf{h}_n}\right)\tilde{\phi}(\pi k)e^{i2\pi kx} + \sqrt{\pi}\sum_{k \text{ odd}}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^n\right)\tilde{\phi}(\pi k)e^{i2\pi kx} \\
&= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sqrt{2}\,\tilde{\phi}(\pi 2k)e^{i2\pi 2kx} + \sqrt{\pi}\sum_{k\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^n\right)\tilde{\phi}(\pi[2k+1])e^{i2\pi[2k+1]x} && \text{by Theorem 4.3 page 65} \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\tilde{\phi}(0) + \sqrt{\pi}e^{i2\pi x}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^n\sum_{k\in\mathbb{Z}}\tilde{\phi}(\pi[2k+1])e^{i4\pi kx} && \text{by left hypothesis and Theorem ?? page ??} \\
\Longrightarrow &\qquad \left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^n\right) = 0 && \text{because the right side must equal } c
\end{aligned}$$

3. Proof that (2) $\Longrightarrow$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}_e} \mathsf{h}_n = \sum_{n\in\mathbb{Z}_o} \mathsf{h}_n &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n && \text{by (2) and Proposition L.4 page 226} \\
&= \frac{\sqrt{2}}{2} && \text{by } \textit{admissibility condition} \text{ (Theorem 4.3 page 65)}
\end{aligned}$$

4. Proof that (2) $\Longleftarrow$ (3):

$$\frac{\sqrt{2}}{2} = \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n\mathsf{h}_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n\mathsf{h}_n}_{\text{odd terms}} \qquad \text{by (3)}$$

$$\implies \sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n = 0 \qquad \text{by Proposition L.4 page 226}$$

✏

Not every function that forms a *partition of unity* is a *basis* for an *MRA*, as formerly stated next and demonstrated by Counterexample 4.2 (page 74) and Counterexample 4.3 (page 76).

Proposition 4.5.

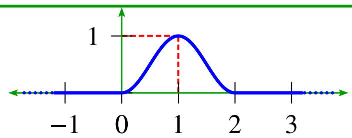
PRP | $\phi(x)$ *generates a* PARTITION OF UNITY $\quad\not\Longrightarrow\quad$ $\phi(x)$ *generates an* MRA *system.*

✎PROOF: By Counterexample 4.2 (page 74) and Counterexample 4.3 (page 76). ✏

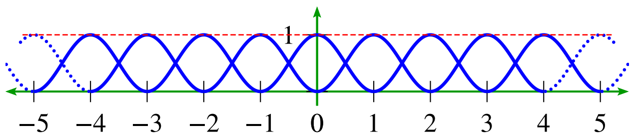
Counterexample 4.2. Let a function ϕ be defined in terms of the sine function (Definition F.2 page 153) as follows:

CNT

$$\phi(x) \triangleq \begin{cases} \sin^2\left(\frac{\pi}{2}x\right) & \text{for } x \in [0:2] \\ 0 & \text{otherwise} \end{cases}$$



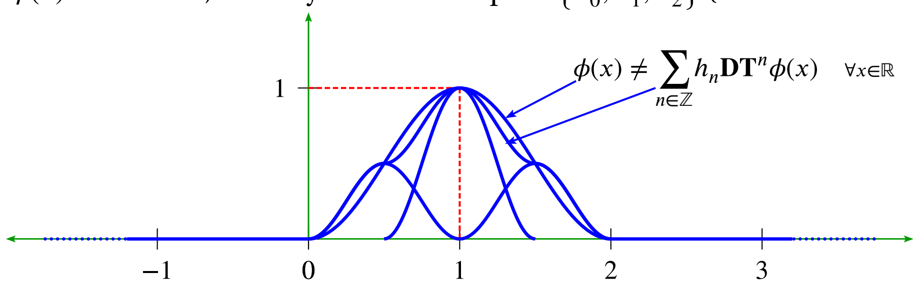
Then $\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x = 1$ and ϕ induces a *partition of unity*



but $\left\{\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\right\}$ does **not** generate an *MRA*.

✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 3.2 page 40) on a set A.

1. Proof that $\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x = 1$: by Example **??** (page **??**)
2. Proof that $\phi(x)$ forms a *partition of unity*: by Example **??** (page **??**)
3. Proof that $\phi(x) \notin \operatorname{span}\left\{\mathbf{DT}^n\phi(x) \,|\, n\in\mathbb{Z}\right\}$ (and so does not generate an *MRA*):
   (a) Note that the *support* (Definition 4.4 page 69) of ϕ is $\operatorname{supp}\phi = [0:2]$.
   (b) Therefore, the *support* of $(\!(h_n)\!)$ is $\operatorname{supp}(\!(h_n)\!) = \{0, 1, 2\}$ (Theorem 4.7 page 69).
   (c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\left\{h_0, h_1, h_2\right\}$ (the rest would be 0).



Here would be the values of $\{h_1, h_2, h_3\}$:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n) \\
&= \sum_{n=0}^{2} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 4.7 page 69}
\end{aligned}$$

(d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$ (see the diagram in item (3c) page 75):

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot\frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(1-n)\right)\mathbb{1}_{[0:2]}(1-n) \\
&= \mathsf{h}_0 \sin^2\left(\frac{\pi}{2}(1-0)\right)\mathbb{1}_{[0:2]}(1-0) + \mathsf{h}_1 \sin^2\left(\frac{\pi}{2}(1-1)\right)\mathbb{1}_{[0:2]}(1-1) \\
&\quad + \mathsf{h}_2 \sin^2\left(\frac{\pi}{2}(1-2)\right)\mathbb{1}_{[0:2]}(1-2) \\
&= \mathsf{h}_0\cdot 1\cdot 1 + \mathsf{h}_1\cdot 0\cdot 1 + \mathsf{h}_2(-1)\cdot 0 \\
&= \mathsf{h}_0
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot 1 &= \frac{\sqrt{2}}{2}(1)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{2}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi(1) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2-n)\right)\mathbb{1}_{[0:2]}(2-n) \\
&= \mathsf{h}_0 \sin^2\left(\frac{\pi}{2}(2-0)\right)\mathbb{1}_{[0:2]}(2-0) + \mathsf{h}_1 \sin^2\left(\frac{\pi}{2}(2-1)\right)\mathbb{1}_{[0:2]}(2-1) \\
&\quad + \mathsf{h}_2 \sin^2\left(\frac{\pi}{2}(2-2)\right)\mathbb{1}_{[0:2]}(2-2) \\
&= \mathsf{h}_0\cdot 0\cdot 1 + \mathsf{h}_1\cdot 1\cdot 1 + \mathsf{h}_2\cdot 0\cdot 1 \\
&= \mathsf{h}_1
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot\frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{-\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{3\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{3}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(3-n)\right)\mathbb{1}_{[0:2]}(3-n)
\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{h}_0 \sin^2\left(\frac{\pi}{2}(3-0)\right)\mathbb{1}_{[0:2]}(3-0) + \mathsf{h}_1 \sin^2\left(\frac{\pi}{2}(3-1)\right)\mathbb{1}_{[0:2]}(3-1) \\
&\quad + \mathsf{h}_2 \sin^2\left(\frac{\pi}{2}(3-2)\right)\mathbb{1}_{[0:2]}(3-2) \\
&= \mathsf{h}_0 \cdot (-1)\cdot 0 + \mathsf{h}_1 \cdot 0 \cdot 1 + \mathsf{h}_2 1 \cdot 1 \\
&= \mathsf{h}_2
\end{aligned}
$$

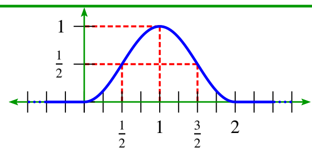
(e) These values for (h_0, h_1, h_2) are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 4.1 page 60). In particular,
$$\phi(x) \neq \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \phi(x)$$
(see the diagram in item (3c) page 75)

✏

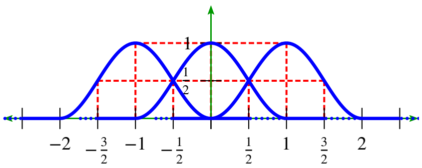
Counterexample 4.3 (raised sine). [15] Let a function f be defined in terms of a shifted cosine function (Definition F.1 page 153) as follows:

CNT

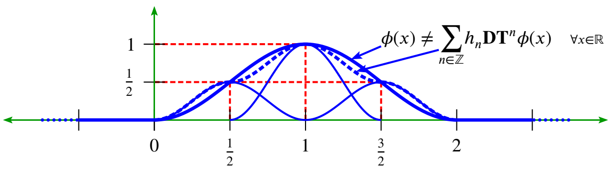
$$\phi(x) \triangleq \begin{cases} \frac{1}{2}\left\{1 + \cos[\pi(|x-1|)]\right\} & \text{for } 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$



Then ϕ forms a *partition of unity*:



but $\{\mathbf{T}^n\phi \mid n\in\mathbb{Z}\}$ does **not** generate an *MRA*.



✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 3.2 page 40) on a set A.

1. Proof that $\phi(x)$ forms a *partition of unity*:

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \phi(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \phi(x+1) && \text{by Proposition 3.1 page 40} \\
&= \sum_{n\in\mathbb{Z}} \phi(x+1-n) && \text{by Definition 3.3 page 40} \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2}\{1+\cos[\pi(|x-1+1-n|)]\}\mathbb{1}_{[0:2)}(x+1-n) && \text{by definition of } \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2}\{1+\cos[\pi(|x-n|)]\}\mathbb{1}_{[-1:1)}(x-n) && \text{by Definition 3.2 page 40} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{\frac{1}{2}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n| - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_{[-1:1)}(x-n)\Bigg|_{\beta=1}}_{\textit{raised cosine} \text{ (Example ?? page ??) with } \beta=1} \\
&= 1 && \text{by Example ?? page ??}
\end{aligned}
$$

2. Proof that $\phi(x) \notin \operatorname{span}\{\mathbf{DT}^n\phi(x) \mid n\in\mathbb{Z}\}$ (and so does not generate an *MRA*):

(a) Note that the *support* (Definition 4.4 page 69) of ϕ is $\operatorname{supp}\phi = [0:2]$.

[15] Proakis (2001) pages 560–561

(b) Therefore, the *support* of $(\!(h_n)\!)$ is $\operatorname{supp}(\!(h_n)\!) = \{0, 1, 2\}$ (Theorem 4.7 page 69).

(c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\{h_0, h_1, h_2\}$ (the rest would be 0). Here would be the values of $\{h_1, h_2, h_3\}$:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\ \frac{1}{2}\left\{\ 1 + \cos[\pi(|x-1|)]\right\}\mathbb{1}_{[0:2]}(x) && \text{by definition of } \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n) && \text{by Definition 3.3 page 40} \\
&= \sum_{n=0}^{2} \mathsf{h}_n\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 4.7 page 69}
\end{aligned}$$

(d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$ (see the diagram in item (3c) page 75):

$$\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} \mathsf{h}_n\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n)\Big|_{x=\frac{1}{2}} \\
&= \mathsf{h}_0\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[1-1-0]\right\} \\
&= \mathsf{h}_0\sqrt{2} \\
&\implies\ \mathsf{h}_0 = \frac{\sqrt{2}}{4}
\end{aligned}$$

$$\begin{aligned}
1 &= \sum_{n=0}^{2} \mathsf{h}_n\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n)\Big|_{x=1} \\
&= \mathsf{h}_1\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[2-1-1]\right\} \\
&= \mathsf{h}_1\sqrt{2} \\
&\implies\ \mathsf{h}_1 = \frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} \mathsf{h}_n\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n)\Big|_{x=\frac{3}{2}} \\
&= \mathsf{h}_2\ \frac{\sqrt{2}}{2}\left\{\ 1 + \cos[1-1-0]\right\} \\
&= \mathsf{h}_2\sqrt{2} \\
&\implies\ \mathsf{h}_2 = \frac{\sqrt{2}}{4}
\end{aligned}$$

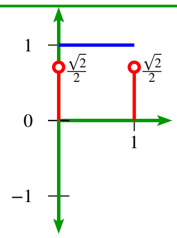
(e) These values for $(\!(h_0, h_1, h_2)\!)$ are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 4.1 page 60). In particular (see diagram),

$$\phi(x) \neq \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x)\,.$$

Example 4.3 (2 coefficient case/Haar wavelet system/order 0 B-spline wavelet system). [16]
Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be an *wavelet system*.

EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:1] & \text{(Theorem 4.7 page 69) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 4.3 page 65) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.8 page 71)} \end{array}\right\} \implies \left\{\begin{array}{r|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{2} \\ 1 & \frac{\sqrt{2}}{2} \\ \text{other} & 0 \end{array}\right\}$$



✎PROOF:

1. Proof that (1) $\implies$ that only h_0 and h_1 are non-zero: by Theorem 4.7 page 69.
2. Proof for values of h_0 and h_1:
   (a) Method 1: Under the constraint of two non-zero scaling coefficients, a scaling function design is fully constrained using the *admissibility equation* (Theorem 4.3 page 65) and the *partition of unity* constraint. The partition of unity formed by $\phi(x)$ is illustrated in Example ?? (page ??).

   Here are the equations:

$$\begin{array}{rcllll} \mathsf{h}_0 + \mathsf{h}_1 & = & \sqrt{2} & \text{(admissibility equation} & \text{Theorem 4.3} & \text{page 65)} \\ \mathsf{h}_0 - \mathsf{h}_1 & = & 0 & \text{(partition of unity/zero at } -1 & \text{Theorem 4.8} & \text{page 71)} \end{array}$$

   Here are the calculations for the coefficients:

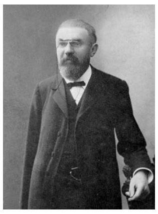
$$\begin{aligned} (h_0 + h_1) + (h_0 - h_1) = 2h_0 &= \sqrt{2} && \text{(add two equations together)} \\ (h_0 + h_1) - (h_0 - h_1) = 2h_1 &= \sqrt{2} && \text{(subtract second from first)} \end{aligned}$$

✏

[16] Haar (1910), Wojtaszczyk (1997) pages 14–15 ⟨"Sources and comments"⟩

CHAPTER 5

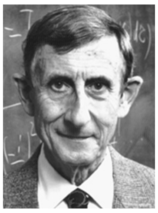
WAVELET STRUCTURES



❝…on fait la science avec des faits comme une maison avec des pierres ; mais une accumulation de faits n'est pas plus une science qu'un tas de pierres n'est une maison.❞

❝Science is built up of facts, as a house is built of stones; but an accumulation of facts is no more a science than a heap of stones is a house.❞

Jules Henri Poincaré (1854-1912), physicist and mathematician [1]



❝The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit. The problem is the overall design.❞

Freeman Dyson (1923–), physicist and mathematician [2]

5.1 Introduction

5.1.1 What are wavelets?

In Fourier analysis, *continuous* dilations (Definition 3.3 page 40) of the *complex exponential* (Definition F.4 page 158) form a *basis* (Definition 2.7 page 14) for the *space of square integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141) such that

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}_\omega e^{ix} |_{\omega\in\mathbb{R}}\right\}.$$

[1] quote: Poincaré (1902a) ⟨Chapter IX, paragraph 7⟩
translation: Poincaré (1902b) page 141
image: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Poincare.html

[2] quote: Albers and Dyson (1994) page 20
image: http://www.isepp.org/Media/Speaker%20Images/95-96%20Images/dyson.jpg

In Fourier series analysis (Theorem M.1 page 234), *discrete* dilations of the complex exponential form a basis for $\boldsymbol{L}^2_{\mathbb{R}}(0 : 2\pi)$ such that

$$\boldsymbol{L}^2_{\mathbb{R}}(0 : 2\pi) = \operatorname{span}\left\{ \mathbf{D}_j e^{ix} \middle| j\in\mathbb{Z} \right\}.$$

In Wavelet analysis, for some *mother wavelet* (Definition 5.1 page 81) $\psi(x)$,

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{ \mathbf{D}_\omega \mathbf{T}_\tau \psi(x) | \omega, \tau \in \mathbb{R} \right\}.$$

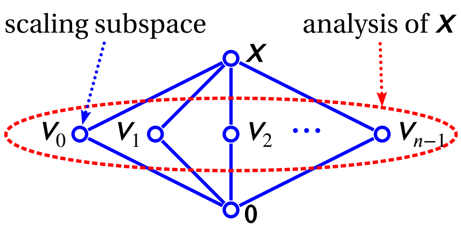
However, the ranges of parameters ω and τ can be much reduced to the countable set $\mathbb{Z}$ resulting in a *dyadic* wavelet basis such that for some mother wavelet $\psi(x)$,

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{ \mathbf{D}^j \mathbf{T}^n \psi(x) | j, n \in \mathbb{Z} \right\}.$$

This text deals almost exclusively with dyadic wavelets. Wavelets that are both *dyadic* and *compactly supported* have the attractive feature that they can be easily implemented in hardware or software by use of the *Fast Wavelet Transform* (Figure N.1 page 239).

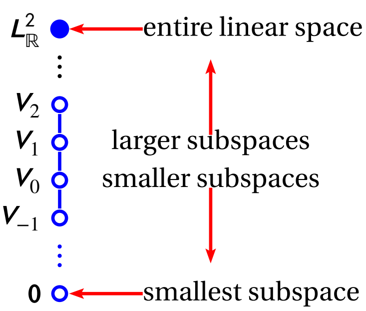
5.1.2 Analyses

An analysis can be partially characterized by its order structure with respect to an order relation such as the set inclusion relation $\subseteq$. Most transforms have a very simple M-n order structure, as illustrated to the right. The M-n lattices for $n \geq 3$ are *modular* but not *distributive*. Analyses typically have one subspace that is a *scaling* subspace; and this subspace is often simply a family of constants (as is the case with Fourier Analysis).

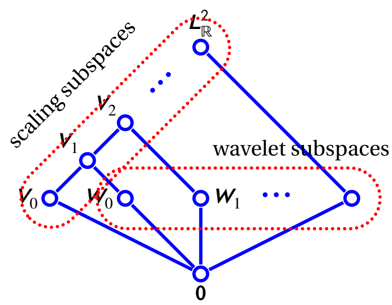


A special characteristic of wavelet analysis is that there is not just one scaling subspace, but an entire sequence of scaling subspaces. These scaling subspaces are *linearly ordered* with respect to the ordering relation $\subseteq$. In wavelet theory, this structure is called a *multiresolution analysis*, or *MRA* (Definition 4.1 page 54).

The MRA was introduced by Stéphane G. Mallat in 1989. The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan, and later as the *Gaussian Pyramid* by Burt and Adelson in the 1980s in the West.[3]

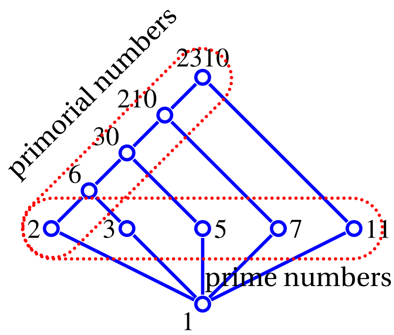


A second special characteristic of wavelet analysis is that it's order structure with respect to the $\subseteq$ relation is not a simple M-n lattice (as is with the case of Fourier and other analyses). Rather, it is a lattice of the form illustrated to the right. This lattice is *non-complemented*, *non-distributive*, *non-modular*, and *non-Boolean* (Proposition 5.1 page 83).



[3] Mallat (1989) page 70, Iijima (1959), Burt and Adelson (1983), Adelson and Burt (1981), Lindeberg (1993), Alvarez et al. (1993), Guichard et al. (2012), Weickert (1999) ⟨historical survey⟩

The wavelet subspace structure is similar in form to that of the *Primorial numbers*,[4] illustrated to the right by a *Hasse diagram*.



An analysis can be represented using three different structures:

① sequence of subspaces
② sequence of basis coefficients
③ sequence of basis vectors

These structures are isomorphic to each other, and can therefore be used interchangeably.

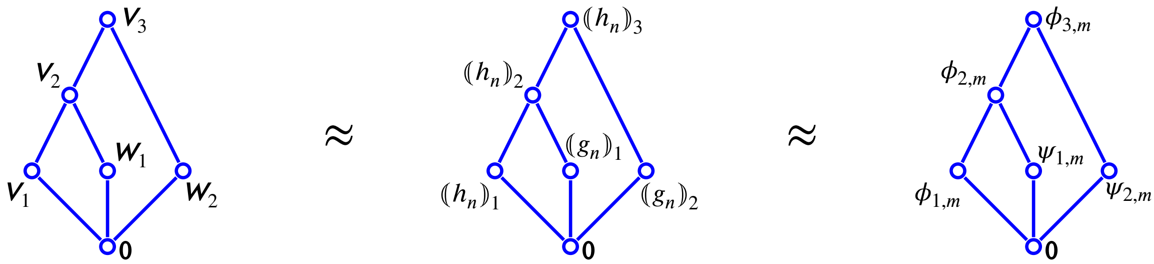


Figure 5.1 (page 91) illustrate the order structures of some analyses, including two wavelet analyses:

5.2 Definition

The term "wavelet" comes from the French word "*ondelette*", meaning "small wave". And in essence, wavelets are "small waves" (as opposed to the "long waves" of Fourier analysis) that form a basis for the Hilbert space $\boldsymbol{L}^2_{\mathbb{R}}$.[6]

Definition 5.1. [7] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 3.3 page 40.*

DEF

*A function $\psi(x)$ in $\boldsymbol{L}^2_{\mathbb{R}}$ is a **wavelet function** for $\boldsymbol{L}^2_{\mathbb{R}}$ if*

$$\left\{\mathbf{D}^j\mathbf{T}^n\psi|_{j,n\in\mathbb{Z}}\right\} \text{ is a RIESZ BASIS for } \boldsymbol{L}^2_{\mathbb{R}}.$$

*In this case, ψ is also called the **mother wavelet** of the basis $\left\{\mathbf{D}^j\mathbf{T}^n\psi|_{j,n\in\mathbb{Z}}\right\}$. The sequence of subspaces $(\!(\boldsymbol{W}_j)\!)_{j\in\mathbb{Z}}$ is the **wavelet analysis** induced by ψ, where each subspace $\boldsymbol{W}_j$ is defined as*

$$\boldsymbol{W}_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\psi\,\middle|\,_{n\in\mathbb{Z}}\right\}.$$

A *wavelet analysis* $(\!(\boldsymbol{W}_j)\!)$ is often constructed from a *multiresolution anaysis* (Definition 4.1 page 54) $(\!(\boldsymbol{V}_j)\!)$ under the relationship

$$\boldsymbol{V}_{j+1} = \boldsymbol{V}_j \dotplus \boldsymbol{W}_j, \qquad \text{where } \dotplus \text{ is subspace addition (\textit{Minkowski addition}).}$$

By this relationship alone, $(\!(\boldsymbol{W}_j)\!)$ is in no way uniquely defined in terms of a multiresolution analysis

[4] Sloane (2014) ⟨http://oeis.org/A002110⟩
[6] Strang and Nguyen (1996) page ix, Atkinson and Han (2009) page 191
[7] Wojtaszczyk (1997) page 17 ⟨Definition 2.1⟩

$\left(\!\left(V_j \right)\!\right)$. In general there are many possible complements of a subspace V_j. To uniquely define such a wavelet subspace, one or more additional constraints are required. One of the most common additional constraints is *orthogonality*, such that V_j and W_j are orthogonal to each other.

5.3 Dilation equation

Suppose $(\mathbf{T}^n\psi)_{n\in\mathbb{Z}}$ is a basis for W_0. By Definition 5.1 page 81, the wavelet subspace W_0 is contained in the scaling subspace V_1. By Definition 4.1 page 54, the sequence $(\mathbf{D}\mathbf{T}^n\phi)_{n\in\mathbb{Z}}$ is a basis for V_1. Because W_0 is contained in V_1, the sequence $(\mathbf{D}\mathbf{T}^n\phi)_{n\in\mathbb{Z}}$ is also a basis for W_0.

Theorem 5.1 (wavelet dilation equation). *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \phi, (h_n)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 4.3 page 63) and* $\left(\!\left(W_j \right)\!\right)_{j\in\mathbb{Z}}$ *be a* WAVELET ANALYSIS *(Definition 5.1 page 81) with respect to* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \phi, (h_n)\right)$ *and with* WAVELET FUNCTION ψ *(Definition 5.1 page 81).*

THM
$$\exists (g_n)_{n\in\mathbb{Z}} \quad \text{such that} \quad \psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{D}\mathbf{T}^n\phi$$

✎PROOF:

$$\begin{aligned}
\psi &\in W_0 && \text{by Definition 5.1 page 81}\\
&\subseteq V_1 && \text{by Definition 5.1 page 81}\\
&= \operatorname{span}(\mathbf{D}\mathbf{T}^n\phi(x))_{n\in\mathbb{Z}} && \text{by Definition 4.1 page 54 (MRA)}\\
&\implies \exists (g_n)_{n\in\mathbb{Z}} \quad \text{such that} \quad \psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{D}\mathbf{T}^n\phi
\end{aligned}$$

✏

A *wavelet system* (next definition) consists of two subspace sequences:

- A **multiresolution analysis** $\left(\!\left(V_j \right)\!\right)$ (Definition 4.1 page 54) provides "coarse" approximations of a function in $L^2_{\mathbb{R}}$ at different "scales" or resolutions.
- A **wavelet analysis** $\left(\!\left(W_j \right)\!\right)$ provides the "detail" of the function missing from the approximation provided by a given scaling subspace (Definition 5.1 page 81).

Definition 5.2. *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \phi, (h_n)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 4.1 page 54) and* $\left(\!\left(W_j \right)\!\right)_{j\in\mathbb{Z}}$ *a wavelet analysis (Definition 5.1 page 81) with respect to* $\left(\!\left(V_j \right)\!\right)_{j\in\mathbb{Z}}$. *Let* $(g_n)_{n\in\mathbb{Z}}$ *be a sequence of coefficients.*

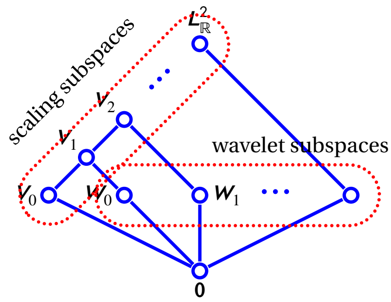
DEF
*A **wavelet system** is the tuple* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \left(\!\left(W_j \right)\!\right), \phi, \psi, (h_n), (g_n)\right)$
and the sequence $(g_n)_{n\in\mathbb{Z}}$ *that satisfies the equation* $\psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{D}\mathbf{T}^n\phi$
*is the **wavelet coefficient sequence**.*

Remark 5.1. The pair of coefficient sequences $\left((h_n), (g_n)\right)$ generates the scaling function $\phi(x)$ (Definition 4.1 page 54) and the wavelet function $\psi(x)$ (Definition 5.1 page 81). These functions in turn generate the multiresolution analysis $\left(\!\left(V_j \right)\!\right)$ (Definition 4.1 page 54) and the wavelet analysis $\left(\!\left(W_j \right)\!\right)$ (Definition 5.1 page 81). Therefore, the coefficient sequence pair $\left((h_n), (g_n)\right)$ totally defines a wavelet system $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \left(\!\left(W_j \right)\!\right), \phi, \psi, (h_n), (g_n)\right)$ (Definition 5.2 page 82).

Furthermore, especially in the case of orthonormal wavelets, the wavelet coefficient sequence $\left(\!\left(g_n\right)\!\right)_{n\in\mathbb{Z}}$ is often defined in terms of the scaling coefficient sequence $\left(\!\left(h_n\right)\!\right)_{n\in\mathbb{Z}}$ in a very simple and straightforward manner. Therefore, in the case of an orthonormal wavelet system, the coefficient scaling sequence $\left(\!\left(h_n\right)\!\right)_{n\in\mathbb{Z}}$ often totally defines the entire wavelet system. And in this case, designing a wavelet system is only a matter of finding a handful of scaling coefficients $\left(\!\left(h_1, h_2, \ldots, h_n\right)\!\right)$...because once you have these, you can generate everything else.

5.4 Order structure

The *wavelet system* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ (Definition 5.2 page 82) together with the set inclusion relation $\subseteq$ forms an *ordered set*, illustrated to the right by a *Hasse diagram*.



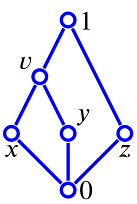
Proposition 5.1. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system with order relation* $\subseteq$. *The lattice* $\boldsymbol{L} \triangleq \left(\left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \vee, \wedge; \subseteq\right)$ *has the following properties:*

PRP
1. $\boldsymbol{L}$ *is* NONDISTRIBUTIVE.
2. $\boldsymbol{L}$ *is* NONMODULAR.
3. $\boldsymbol{L}$ *is* NONCOMPLEMENTED.
4. $\boldsymbol{L}$ *is* NONBOOLEAN.

✎PROOF:

1. Proof that $\boldsymbol{L}$ is *nondistributive*:
   (a) $\boldsymbol{L}$ contains the $N5$ lattice.
   (b) Because $\boldsymbol{L}$ contains the $N5$ lattice, $\boldsymbol{L}$ is *nondistributive*.
2. Proof that $\boldsymbol{L}$ is *nonmodular* and *nondistributive*:
   (a) $\boldsymbol{L}$ contains the $N5$ lattice.
   (b) Because $\boldsymbol{L}$ contains the $N5$ lattice, $\boldsymbol{L}$ is *nonmodular*.
3. Proof that $\boldsymbol{L}$ is *noncomplemented*:

$$\begin{aligned} x' &= y' = v' = z \\ z' &= \{x, y, v\} \\ x'' &= \left(x'\right)' \\ &= z' \\ &= \{x, y, v\} \\ &\neq x \end{aligned}$$



4. Proof that $\boldsymbol{L}$ is *nonBoolean*:
   (a) $\boldsymbol{L}$ is *nondistributive* (item (1)).
   (b) Because $\boldsymbol{L}$ is *nondistributive*, it is *nonBoolean*.

5.5 Subspace algebraic structure

Theorem 5.2. *Let* $\left(L^2_{\mathbb{R}}, ⦅V_j⦆, ⦅W_j⦆, \phi, \psi, ⦅h_n⦆, ⦅g_n⦆\right)$ *be a* WAVELET SYSTEM (Definition 5.2 page 82). *Let* $V_1 \dotplus V_2$ *represent* MINKOWSKI ADDITION *of two subspaces* V_1 *and* V_2 *of a Hilbert space* H.

T H M

$$\begin{aligned} L^2_{\mathbb{R}} &= \lim_{j\to\infty} V_j && (L^2_{\mathbb{R}} \text{ is equivalent to one very large scaling subspace}) \\ &= V_j \dotplus W_j \dotplus W_{j+1} \dotplus W_{j+2} \dotplus \cdots && \left(\begin{array}{l} L^2_{\mathbb{R}} \text{ is equivalent to one scaling space} \\ \text{and a sequence of wavelet subspaces} \end{array}\right) \\ &= \cdots \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus \cdots && (L^2_{\mathbb{R}} \text{ is equivalent to a sequence of wavelet subspaces}) \end{aligned}$$

✎PROOF:

1. Proof for (1):

$$L^2_{\mathbb{R}} = \lim_{j\to\infty} V_j \qquad \text{by Definition 4.1 page 54}$$

2. Proof for (2):

$$\begin{aligned} \underbrace{V_j \dotplus W_j}_{V_{j+1}} \dotplus W_{j+1} \dotplus W_{j+2} \dotplus \cdots &= \underbrace{V_{j+1} \dotplus W_{j+1}}_{V_{j+2}} \dotplus W_{j+2} \dotplus W_{j+3} \dotplus \cdots \\ &= \underbrace{V_{j+2} \dotplus W_{j+2}}_{V_{j+3}} \dotplus W_{j+3} \dotplus W_{j+4} \dotplus \cdots \\ &= \underbrace{V_{j+3} \dotplus W_{j+3}}_{V_{j+4}} \dotplus W_{j+4} \dotplus W_{j+5} \dotplus \cdots \\ &= \underbrace{V_{j+5} \dotplus W_{j+5}}_{V_{j+5}} \dotplus W_{j+6} \dotplus W_{j+6} \dotplus \cdots \\ &= \lim_{j\to\infty} V_{j+5} \dotplus W_{j+5} \dotplus W_{j+6} \dotplus W_{j+6} \dotplus \cdots \\ &= L^2_{\mathbb{R}} \end{aligned}$$

3. Proof for (3):

$$\begin{aligned} L^2_{\mathbb{R}} &= \underbrace{V_0}_{V_{-1} \dotplus W_{-1}} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots && \text{by (2)} \\ &= \underbrace{V_{-1}}_{V_{-2} \dotplus W_{-2}} W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\ &= \underbrace{V_{-2}}_{V_{-3} \dotplus W_{-3}} W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\ &= \underbrace{V_{-3}}_{V_{-4} \dotplus W_{-4}} W_{-3} \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\ &\vdots \\ &= \cdots \dotplus W_{-3} \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \end{aligned}$$

⇨

Remark 5.2. In the special case that two subspaces $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are *orthogonal* to each other, then the *subspace addition* operation $\boldsymbol{W}_1 \dotplus \boldsymbol{W}_2$ is frequently expressed as $\boldsymbol{W}_1 \oplus \boldsymbol{W}_2$. In the case of an *orthonormal wavelet system*, the expressions in Theorem 5.2 (page 84) could be expressed as

$$\begin{aligned} \boldsymbol{L}^2_{\mathbb{R}} &= \lim_{j\to\infty} \boldsymbol{V}_j \\ &= \boldsymbol{V}_j \oplus \boldsymbol{W}_j \oplus \boldsymbol{W}_{j+1} \oplus \boldsymbol{W}_{j+2} \oplus \cdots \\ &= \cdots \oplus \boldsymbol{W}_{-2} \oplus \boldsymbol{W}_{-1} \oplus \boldsymbol{W}_0 \oplus \boldsymbol{W}_1 \oplus \boldsymbol{W}_2 \oplus \cdots . \end{aligned}$$

.

5.6 Necessary conditions

Theorem 5.3 (quadrature conditions in "time"). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system* (Definition 5.2 page 82).

THM

1. $\displaystyle\sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \phi \,|\, \mathbf{T}^n \phi \right\rangle \quad \forall n\in\mathbb{Z}$
2. $\displaystyle\sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \psi \,|\, \mathbf{T}^n \psi \right\rangle \quad \forall n\in\mathbb{Z}$
3. $\displaystyle\sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \phi \,|\, \mathbf{T}^n \psi \right\rangle \quad \forall n\in\mathbb{Z}$

✎PROOF:

1. Proof for (1): by Theorem 4.4 page 67.
2. Proof for (2):

$$\begin{aligned}
\langle \psi \,|\, \mathbf{T}^n\psi \rangle &= \left\langle \sum_{m\in\mathbb{Z}} g_m \mathbf{D}\mathbf{T}^m\phi \,\middle|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} g_k \mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by } \textit{wavelet dilation equation} \quad \text{(Theorem 5.1 page 82)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by properties of } \langle \triangle \,|\, \triangledown \rangle \quad \text{(Definition C.9 page 124)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \,\mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by def. of operator adjoint} \quad \text{(Proposition C.3 page 125)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \,\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by Proposition 3.5 page 42} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{*m}\mathbf{D}^*\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by operator star-algebra prop.} \quad \text{(Theorem C.13 page 126)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{-m}\mathbf{D}^{-1}\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by Proposition 3.7 page 43} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
&\langle \phi \,|\, \mathbf{T}^n\psi \rangle \\
&= \left\langle \sum_{m\in\mathbb{Z}} h_m \mathbf{D}\mathbf{T}^m\phi \,\middle|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} g_k \mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by Theorem 4.1 page 60} \quad \text{and Theorem 5.1 page 82} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by properties of } \langle \triangle \,|\, \triangledown \rangle \quad \text{(Definition C.9 page 124)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{DT}^m)^* \, \mathbf{T}^n \mathbf{DT}^k \phi \right\rangle && \text{by definition of operator adjoint} && \text{(Proposition C.3 page 125)} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{DT}^m)^* \, \mathbf{DT}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by Proposition 3.5 page 42} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{*m} \mathbf{D}^* \mathbf{DT}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by operator star-algebra properties} && \text{(Theorem C.13 page 126)} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{-m} \mathbf{D}^{-1} \mathbf{DT}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by Proposition 3.7 page 43} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k} \phi \right\rangle
\end{aligned}
$$

⇨

Proposition 5.2. *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \left(\!\left(W_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system. Let* $\tilde{\phi}(\omega)$ *and* $\tilde{\psi}(\omega)$ *be the* FOURIER TRANSFORMS *(Definition H.2 page 192) of* $\phi(x)$ *and* $\psi(x)$*, respectively. Let* $\breve{g}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition L.1 page 223) of* $\left(\!\left(g_n\right)\!\right)$.

PRP
$$\tilde{\psi}(\omega) = \frac{\sqrt{2}}{2}\, \breve{g}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right)$$

✎PROOF:

$$
\begin{aligned}
\tilde{\psi}(\omega) &\triangleq \tilde{\mathbf{F}}\psi \\
&= \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} g_n \mathbf{DT}^n \phi && \text{by } \textit{wavelet dilation equation} && \text{(Theorem 5.1 page 82)} \\
&= \sum_{n\in\mathbb{Z}} g_n \tilde{\mathbf{F}} \mathbf{DT}^n \phi \\
&= \sum_{n\in\mathbb{Z}} g_n \mathbf{D}^{-1} \tilde{\mathbf{F}} \mathbf{T}^n \phi && \text{by Corollary 3.1 page 47} \\
&= \sum_{n\in\mathbb{Z}} g_n \mathbf{D}^{-1} e^{-i\omega n} \tilde{\mathbf{F}} \phi && \text{by Corollary 3.1 page 47} \\
&= \sum_{n\in\mathbb{Z}} g_n \sqrt{2} \left(\mathbf{D}^{-1} e^{-i\omega n}\right) \left(\mathbf{D}^{-1} \tilde{\mathbf{F}} \phi\right) && \text{by Proposition 3.4 page 41} \\
&= \sqrt{2} \left(\mathbf{D}^{-1} \sum_{n\in\mathbb{Z}} g_n e^{-i\omega n} \right) \left(\mathbf{D}^{-1} \tilde{\mathbf{F}} \phi\right) \\
&= \sqrt{2} \left(\mathbf{D}^{-1} \breve{\mathbf{F}} \left(\!\left(g_n\right)\!\right)\right) \left(\mathbf{D}^{-1} \tilde{\mathbf{F}} \phi\right) && \text{by definition of } \breve{\mathbf{F}} && \text{(Definition L.1 page 223)} \\
&= \sqrt{2} \frac{\sqrt{2}}{2}\, \breve{g}\left(\frac{\omega}{2}\right) \frac{\sqrt{2}}{2} \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by property of } \mathbf{D} && \text{(Proposition 3.2 page 41)} \\
&= \frac{\sqrt{2}}{2}\, \breve{g}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right)
\end{aligned}
$$

⇨

Theorem 5.4 (next) presents the *quadrature* necessary conditions of a wavelet system. These relations simplify dramatically in the special case of an *orthonormal wavelet system* (Theorem L.4 page 229).

Theorem 5.4 (Quadrature conditions in "frequency"). [8] *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \left(\!\left(W_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system. Let* $\breve{x}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition L.1 page 223) for a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$*. Let* $\tilde{\mathrm{S}}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM *(Definition O.3 page 241) of* ϕ*,* $\tilde{\mathrm{S}}_{\psi\psi}(\omega)$ *be*

[8] Chui (1992) page 135, Goswami and Chan (1999) page 110

the AUTO-POWER SPECTRUM *of* ψ, *and* $\tilde{\mathsf{S}}_{\phi\psi}(\omega)$ *be the* CROSS-POWER SPECTRUM *of* ϕ *and* ψ.

THM

1.	$\left\|\breve{\mathsf{h}}(\omega)\right\|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left\|\breve{\mathsf{h}}(\omega+\pi)\right\|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi)$	$=$	$2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)$
2.	$\left\|\breve{\mathsf{g}}(\omega)\right\|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left\|\breve{\mathsf{g}}(\omega+\pi)\right\|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi)$	$=$	$2\tilde{\mathsf{S}}_{\psi\psi}(2\omega)$
3.	$\breve{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega)\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \breve{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi)$	$=$	$2\tilde{\mathsf{S}}_{\phi\psi}(2\omega)$

✎PROOF:

1. Proof for (1): by Theorem 4.5 page 68.
2. Proof for (2):

$$\begin{aligned}
2\tilde{\mathsf{S}}_{\psi\psi}(2\omega) &\triangleq 2(2\pi)\sum_{n\in\mathbb{Z}}\left|\tilde{\psi}(2\omega+2\pi n)\right|^2 \\
&= 2(2\pi)\sum_{n\in\mathbb{Z}}\left|\frac{\sqrt{2}}{2}\breve{\mathsf{g}}\left(\frac{2\omega+2\pi n}{2}\right)\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 && \text{by Lemma 4.1 page 60} \\
&= 2\pi\sum_{n\in\mathbb{Z}_e}\left|\breve{\mathsf{g}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 + \\
&\qquad 2\pi\sum_{n\in\mathbb{Z}_o}\left|\breve{\mathsf{g}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\breve{\mathsf{g}}(\omega+2\pi n)\right|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}\left|\breve{\mathsf{g}}(\omega+2\pi n+\pi)\right|^2\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\breve{\mathsf{g}}(\omega)\right|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}\left|\breve{\mathsf{g}}(\omega+\pi)\right|^2\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 \\
&= \left|\breve{\mathsf{g}}(\omega)\right|^2\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2+\right)\left|\breve{\mathsf{g}}(\omega+\pi)\right|^2\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right) \\
&= \left|\breve{\mathsf{g}}(\omega)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\breve{\mathsf{g}}(\omega+\pi)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) && \text{by Theorem O.1 page 241}
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
2\tilde{\mathsf{S}}_{\phi\psi}(2\omega) &= 2(2\pi)\sum_{n\in\mathbb{Z}}\tilde{\phi}(2\omega+2\pi n)\tilde{\psi}^*(2\omega+2\pi n) \\
&= 2(2\pi)\sum_{n\in\mathbb{Z}}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}(\omega+\pi n)\,\tilde{\phi}(\omega+\pi n)\,\frac{\sqrt{2}}{2}\breve{\mathsf{g}}^*(\omega+\pi n)\,\tilde{\phi}^*(\omega+\pi n) && \text{by Lemma 4.1 page 60} \\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega+\pi n)\,\breve{\mathsf{g}}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}_o}\breve{\mathsf{h}}(\omega+\pi n)\,\breve{\mathsf{g}}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2 \\
&\qquad + 2\pi\sum_{n\in\mathbb{Z}_e}\breve{\mathsf{h}}(\omega+\pi n)\,\breve{\mathsf{g}}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega+2\pi n+\pi)\,\breve{\mathsf{g}}^*(\omega+2\pi n+\pi)\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 \\
&\qquad + 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega+2\pi n)\,\breve{\mathsf{g}}^*(\omega+2\pi n)\left|\tilde{\phi}(\omega+2\pi n)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega+\pi)\,\breve{\mathsf{g}}^*(\omega+\pi)\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega)\,\breve{\mathsf{g}}^*(\omega)\left|\tilde{\phi}(\omega+2\pi n)\right|^2 \\
&= \breve{\mathsf{h}}(\omega)\,\breve{\mathsf{g}}^*(\omega)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2\right)
\end{aligned}$$

$$+\breve{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right)$$

$$=\breve{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2\right)+\breve{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right)$$

$$=\breve{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega)\tilde{\mathsf{S}}_{\phi\phi}(\omega)+\breve{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) \qquad \text{by Theorem O.1 page 241}$$

⇨

5.7 Sufficient condition

In this text, an often used sufficient condition for designing the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 5.2 page 82) is the *conjugate quadrature filter condition* (Definition I.9 page 211). It expresses the sequence $(\!(g_n)\!)$ in terms of the *scaling coefficient sequence* (Definition 4.3 page 63) and a "shift" integer N as $g_n = \pm(-1)^n h^*_{N-n}$. The *CQF condition* has the following "nice" properties:

1. Given a *scaling coefficient sequence* $(\!(h_n)\!)$ (Definition 4.3 page 63), it is extremely simple to compute the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 5.2 page 82).
2. If $\{\mathbf{T}\phi\}$ of a *wavelet system* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ (Definition 5.2 page 82) is *orthonormal* and $\left((\!(g_n)\!), (\!(h_n)\!), N\right)$ satisfies the *CQF condition*, then $\left\{\mathbf{T}^n\psi\right\}$ is also *orthnormal*.
3. If $\{\mathbf{T}\phi\}$ of a *wavelet system* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ (Definition 5.2 page 82) is *orthonormal* and $\left((\!(g_n)\!), (\!(h_n)\!), N\right)$ satisfies the *CQF condition*, then the *wavelet subspace* $\boldsymbol{W}_0$ is *orthnormal* to the *scaling subspace* $\boldsymbol{V}_0$ ($\boldsymbol{W}_0 \perp \boldsymbol{V}_0$).

Theorem 5.5. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 5.2 page 82). Let* $\breve{\mathsf{g}}(\omega)$ *be the* DTFT *(Definition L.1 page 223) and* $\breve{\mathsf{g}}(z)$ *the* Z-TRANSFORM *(Definition I.4 page 204) of* $(\!(g_n)\!)$.

THM

$$\underbrace{g_n = \pm(-1)^n h^*_{N-n},\ N\in\mathbb{Z}}_{\text{CONJUGATE QUADRATURE FILTER}} \iff \breve{\mathsf{g}}(\omega) = \pm(-1)^N e^{-i\omega N}\breve{\mathsf{h}}^*(\omega+\pi)\Big|_{\omega=\pi} \qquad (1)$$

$$\implies \sum_{n\in\mathbb{Z}}(-1)^n g_n = \sqrt{2} \qquad (2)$$

$$\iff \breve{\mathsf{g}}(z)\Big|_{z=-1} = \sqrt{2} \qquad (3)$$

$$\iff \breve{\mathsf{g}}(\omega)\Big|_{\omega=\pi} = \sqrt{2} \qquad (4)$$

✎PROOF:

1. Proof that CQF $\iff$ (1): by Theorem I.5 page 211
2. Proof that CQF $\implies$ (4):

$$\breve{\mathsf{g}}(\pi) = \breve{\mathsf{g}}(\omega)\Big|_{\omega=\pi}$$

$$= \pm(-1)^N e^{-i\omega N}\breve{\mathsf{h}}^*(\omega+\pi)\Big|_{\omega=\pi} \qquad \text{by } \textit{CQF theorem} \qquad \text{(Theorem I.5 page 211)}$$

$$= \pm(-1)^N e^{-i\pi N}\breve{\mathsf{h}}^*(2\pi)$$

$$= \pm(-1)^N(-1)^N\breve{\mathsf{h}}^*(0) \qquad \text{by } \textit{DTFT periodicity} \qquad \text{(Proposition L.1 page 223)}$$

$$= \sqrt{2} \qquad \text{by } \textit{admissibility condition} \qquad \text{(Theorem 4.3 page 65)}$$

3. Proof that (2) $\iff$ (3) $\iff$ (4): by Proposition L.4 page 226

✏

5.8 Support size

Theorem 5.6 (support size). [9] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 5.2 page 82) induced by the* CQF CONDITIONS *(Theorem 5.5 page 88). Let* $\operatorname{supp} f$ *be the support of a function* f *(Definition 4.4 page 69).*

THM
$$\begin{aligned} \operatorname{supp}\phi &= \operatorname{supp} h \\ \operatorname{supp}\psi &= \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right] \end{aligned}$$

✎PROOF:

1. Proof that $\operatorname{supp}\phi = \operatorname{supp} h$: by Theorem 4.7 (page 69)

2. Proof that $\operatorname{supp}\psi = \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right]$:

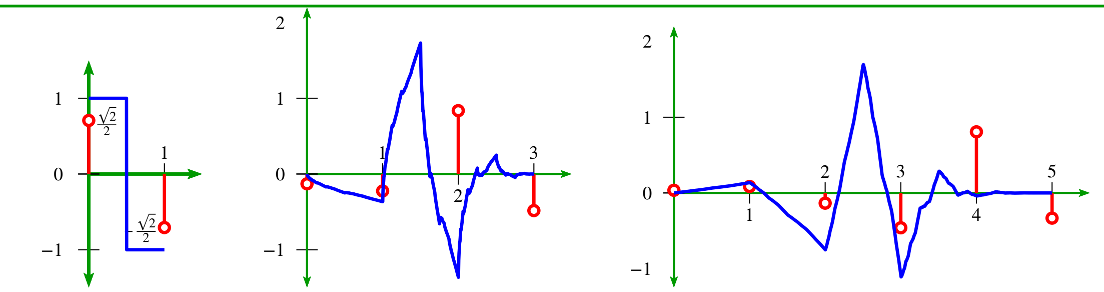
$$\begin{aligned}
\operatorname{supp}\psi(x) &= \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} g_n \mathbf{D}\mathbf{T}^n \phi(x)\right] && \text{by \textit{wavelet dilation equation}} && \text{(Theorem 5.1 page 82)} \\
&= \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} g_n \phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 3.3 page 40)} \\
&= \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \pm(-1)^N h(N-n)\phi(2x-n)\right] && \text{by \textit{CQF conditions}} && \text{(Theorem 5.5 page 88)} \\
&= \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} h(N-n)\phi(2x-n)\right] && \text{by (3) lemma (page 70)} \\
&= \left\{x\in\mathbb{R} \,\middle|\, \sum_{n\in\mathbb{Z}} h(N-n)\phi(2x-n) \neq 0\right\}^- && \text{by definition of supp} && \text{(Definition 4.4 page 69)} \\
&= \left[\frac{n_1}{2} + \frac{N-n_2}{2} : \frac{n_2}{2} + \frac{N-n_1}{2}\right] \\
&= \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right]
\end{aligned}$$

✏

Example 5.1. Here are some examples using *Daubechies wavelet functions*.

[9] Mallat (1999) pages 243–244

EX



Daubechies-1 *Daubechies-2* *Daubechies-3*

5.9 Examples

No further examples of wavelets are presented in this section. Examples begin in the next chapter which is about a property called the *partition of unity*. Other design constraints leading to wavelets with more "powerful" properties include *vanishing moments* (CHAPTER **??** page **??**), *orthonormality*, *compact support*, and *minimum phase* (Definition I.5 page 207).

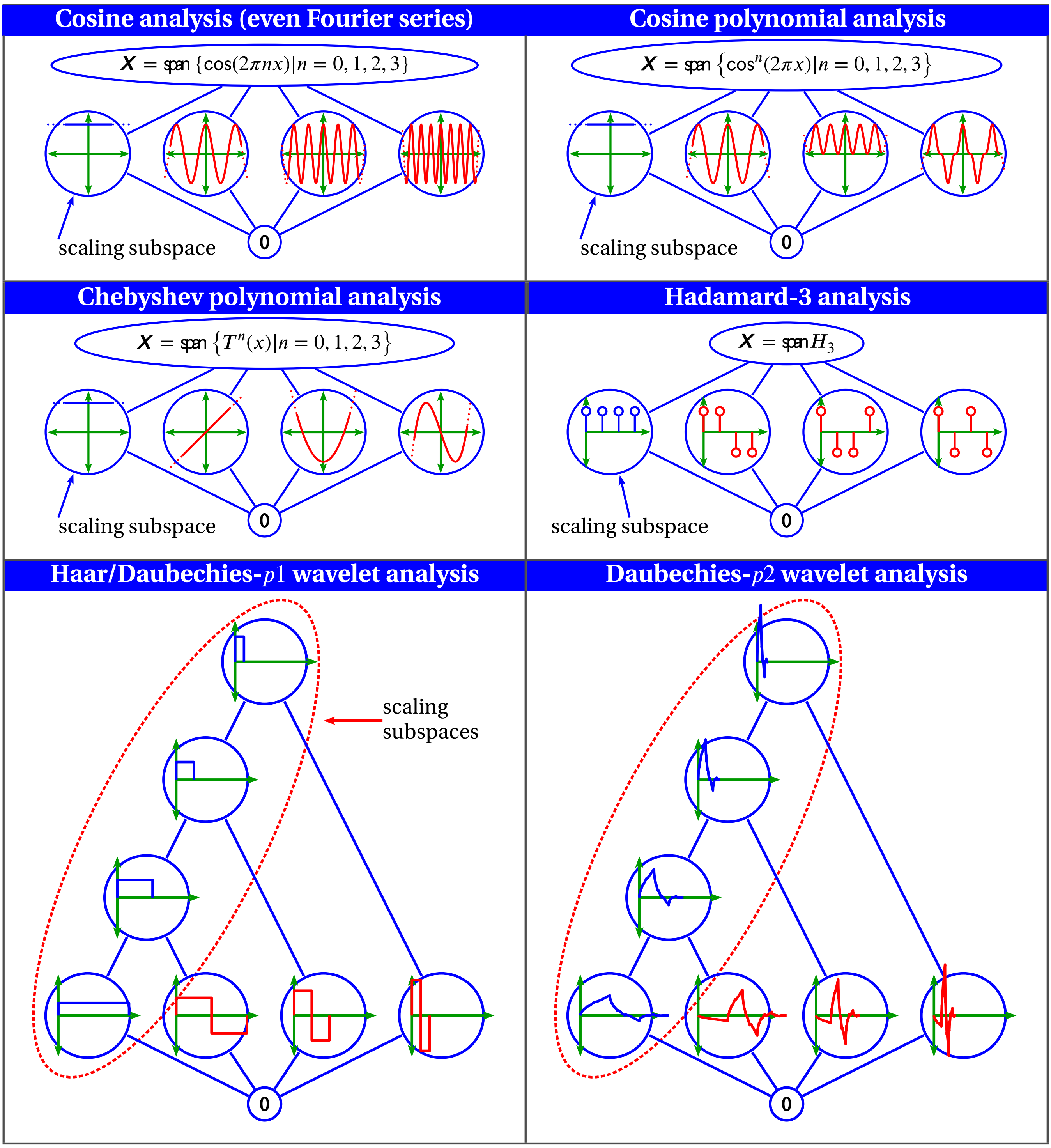


Figure 5.1: examples of the order structures of some analyses

APPENDIX A

ALGEBRAIC STRUCTURES

❝*In this book, learned reader, you have the rules of algebra...It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one.... this most abstruse and unsurpassed treasury of the entire arithmetic being brought to light and, as in a theater, exposed to the sight of all...*❞

Gerolamo Cardano (1501–1576), Italian mathematician, physician, and astrologer [1]

A set together with one or more operations forms several standard mathematical structures:

group $\supseteq$ *ring* $\supseteq$ *commutative ring* $\supseteq$ *integral domain* $\supseteq$ *field*

Definition A.1. [2] *Let X be a set and $\diamond : X \times X \to X$ be an operation on X.*

DEF

*The pair $(X, \diamond)$ is a **group** if*

1.	$\exists e \in X$ such that	$e \diamond x = x \diamond e = x$	$\forall x \in X$	(IDENTITY *element*)	*and*
2.	$\exists (-x) \in X$ such that	$(-x) \diamond x = x \diamond (-x) = e$	$\forall x \in X$	(INVERSE *element*)	*and*
3.		$x \diamond (y \diamond z) = (x \diamond y) \diamond z$	$\forall x,y,z \in X$	(ASSOCIATIVE)	

Definition A.2. [3] *Let $+ : X \times X \to X$ and $* : X \times X \to X$ be operations on a set X. Furthermore, let the operation $*$ also be represented by juxtapostion as in $a * b \equiv ab$.*

DEF

*The triple $(X, +, *)$ is a **ring** if*

1.	$(X, +)$ is a group.		(additive group)	*and*
2.	$x(yz) = (xy)z$	$\forall x,y,z \in X$	(ASSOCIATIVE *with respect to* $*$)	*and*
3.	$x(y+z) = (xy) + (xz)$	$\forall x,y,z \in X$	($*$ *is* LEFT DISTRIBUTIVE *over* $+$)	*and*
4.	$(x+y)z = (xz) + (yz)$	$\forall x,y,z \in X$	($*$ *is* RIGHT DISTRIBUTIVE *over* $+$).	

Definition A.3. [4]

---

[1] quote: Cardano (1545) page 1
image: http://en.wikipedia.org/wiki/Image:Cardano.jpg

[2] Durbin (2000) page 29

[3] Durbin (2000) pages 114–115

[4] Durbin (2000) page 118

DEF *A triple* $(X, +, *)$ *is a* ***commutative ring*** *if*

1. $(X, +, *)$ *is a* RING			*and*
2. $xy = yx$	$\forall x,y \in X$	(COMMUTATIVE).	

Definition A.4. [5] *Let* R *be a* COMMUTATIVE RING *(Definition A.3 page 95).*

DEF *A function* $|\cdot|$ *in* $\mathbb{R}^{\mathbb{R}}$ *is an* ***absolute value*** *(or* ***modulus****) if*

1.	$\lvert x\rvert \geq 0$	$x\in\mathbb{R}$	(NON-NEGATIVE)	*and*
2.	$\lvert x\rvert = 0 \iff x = 0$	$x\in\mathbb{R}$	(NONDEGENERATE)	*and*
3.	$\lvert xy\rvert = \lvert x\rvert \cdot \lvert y\rvert$	$x,y\in\mathbb{R}$	(HOMOGENEOUS / SUBMULTIPLICATIVE)	*and*
4.	$\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$	$x,y\in\mathbb{R}$	(SUBADDITIVE / TRIANGLE INEQUALITY)	

Definition A.5. [6]

DEF *The structure* $F \triangleq (X, +, \cdot, 0, 1)$ *is a* ***field*** *if*

1.	$(X, +, *)$ *is a ring*		*(ring)*	*and*
2.	$xy = yx$	$\forall x,y\in X$	*(commutative with respect to* $*$*)*	*and*
3.	$(X\setminus\{0\}, *)$ *is a group*		*(group with respect to* $*$*).*	

Definition A.6. [7] *Let* $\boldsymbol{V} = (F, +, \cdot)$ *be a vector space and* $\otimes : \boldsymbol{V}\times\boldsymbol{V}\to\boldsymbol{V}$ *be a vector-vector multiplication operator.*
An ***algebra*** *is any pair* $(\boldsymbol{V}, \otimes)$ *that satisfies (*$\otimes$ *is represented by juxtaposition)*

DEF

1.	$(\boldsymbol{ux})\boldsymbol{y} = \boldsymbol{u}(\boldsymbol{xy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V}$	(ASSOCIATIVE)	*and*
2.	$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}) = (\boldsymbol{ux}) + (\boldsymbol{uy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V}$	(LEFT DISTRIBUTIVE)	*and*
3.	$(\boldsymbol{u}+\boldsymbol{x})\boldsymbol{y} = (\boldsymbol{uy}) + (\boldsymbol{xy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V}$	(RIGHT DISTRIBUTIVE)	*and*
4.	$\alpha(\boldsymbol{xy}) = (\alpha\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x}(\alpha\boldsymbol{y})$	$\forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V}$ *and* $\alpha\in F$	(SCALAR COMMUTATIVE)	.

[5] Cohn (2002) page 312
[6] Durbin (2000) page 123, Weber (1893)
[7] Abramovich and Aliprantis (2002) page 3, Michel and Herget (1993) page 56

APPENDIX B

LINEAR SUBSPACES

B.1 Subspaces of a linear space

Linear spaces (Definition C.1 page 111) can be decomposed into a collection of *linear subspaces* (Definition B.1 page 98). Often such a collection along with an *order relation* forms a *lattice*.

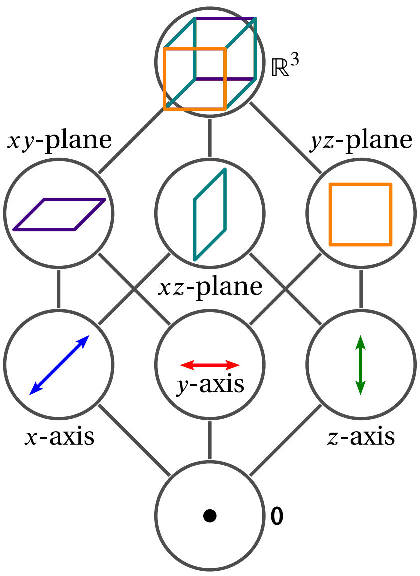
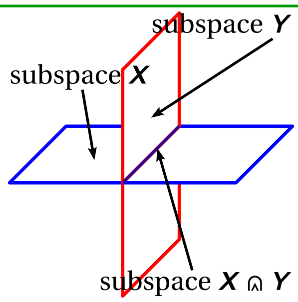


Figure B.1: lattice of subspaces of $\mathbb{R}^3$ (Example B.1 page 97)

EX

Example B.1. The 3-dimensional Euclidean space $\mathbb{R}^3$ contains the 2-dimensional xy-plane and xz-plane subspaces, which in turn both contain the 1-dimensional x-axis subspace. These subspaces are illustrated in the figure to the right and in Figure B.1 (page 97).



Definition B.1. [1] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition C.1 page 111)*.

DEF

A tupple $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *is a* ***linear subspace*** *of* $\boldsymbol{\Omega}$ *if*

1. $Y \neq \varnothing$ *(Y must contain at least one element)* and
2. $Y \subseteq X$ *(Y is a subset of X)* and
3. $\boldsymbol{x}, \boldsymbol{y} \in Y \implies \boldsymbol{x} + \boldsymbol{y} \in Y$ *(closed under vector addition)* and
4. $\boldsymbol{x} \in Y$ *and* $\alpha \in \mathbb{F} \implies \alpha\boldsymbol{x} \in Y$ *(closed under scalar-vector multiplication)*.

A linear subspace is also called a ***linear manifold***.

Every *linear space* (Definition C.1 page 111) $\boldsymbol{X}$ has at least two *linear subspaces*—itself and $\mathbf{0}$ (Proposition B.1 page 98), called the *trivial linear space*. The *linear span* (Definition 2.2 page 9) of every subset of a linear linear space is a subspace (Proposition B.2 page 99). Every *linear subspace* contains the "zero" vector $\mathbb{0}$, and is *convex*.

Proposition B.1. [2] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\mathbf{0} \triangleq (\{\mathbb{0}\}, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

PRP

$$\left\{ \boldsymbol{X} \text{ is a LINEAR SPACE (Definition C.1 page 111)} \right\} \implies \left\{ \begin{array}{lll} 1. & \mathbf{0} \text{ is a LINEAR SUBSPACE of } \boldsymbol{X} & \text{and} \\ 2. & \boldsymbol{X} \text{ is a LINEAR SUBSPACE of } \boldsymbol{X} & \end{array} \right\}$$

✎PROOF: For a structure to be a linear subspace of $\boldsymbol{X}$, it must satisfy the requirements of Definition B.1 (page 98).

1. Proof that $\{\mathbb{0}\}$ is a linear subspace:

   (a) Note that $\{\mathbb{0}\} \neq \varnothing$.

   (b) Proof that $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbb{0}\} \implies \boldsymbol{x} + \boldsymbol{y} \in \{\mathbb{0}\}$:

   $$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &= \mathbb{0} + \mathbb{0} && \text{by } \boldsymbol{x}, \boldsymbol{y} \in \{\mathbb{0}\} \text{ hypothesis} \\ &= \mathbb{0} \\ &\in \{\mathbb{0}\} \end{aligned}$$

   (c) Proof that $\boldsymbol{x} \in \{\mathbb{0}\}, \alpha \in \mathbb{F} \implies \alpha\boldsymbol{x} \in \{\mathbb{0}\}$:

   $$\begin{aligned} \alpha\boldsymbol{x} &= \alpha\mathbb{0} && \text{by } \boldsymbol{x} \in \{\mathbb{0}\} \text{ hypothesis} \\ &= \mathbb{0} && \text{by definition of } \mathbb{0} \\ &\in \{\mathbb{0}\} \end{aligned}$$

2. Proof that $\boldsymbol{\Omega}$ is a linear subspace of itself:

   (a) Proof that $X \neq \varnothing$:

   $$X \neq \varnothing$$

   (b) Proof that $\boldsymbol{x}, \boldsymbol{y} \in X \implies \boldsymbol{x} + \boldsymbol{y} \in X$:

   $\boldsymbol{x} + \boldsymbol{y} \in \{\mathbb{0}\}$ because $+ : X \times X \to X$ (X is closed under vector addition)

   (c) Proof that $\boldsymbol{x} \in X, \alpha \in \mathbb{F} \implies \alpha\boldsymbol{x} \in X$:

   $\alpha\boldsymbol{x} \in X$ because $\cdot : \mathbb{F} \times X \to X$ (X is closed under scalar-vector multiplication)

[1] Michel and Herget (1993) page 81 ⟨Definition 3.2.1⟩, Berberian (1961) page 13 ⟨Definition I.5.1⟩, Halmos (1958) page 16
[2] Michel and Herget (1993) pages 81–83, Haaser and Sullivan (1991) page 43

Proposition B.2. [3] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition C.1 page 111). *Let* span *be the* LINEAR SPAN *of a set* Y *in* $\boldsymbol{X}$.

$$\text{PRP}\quad \left\{\begin{array}{l} Y \text{ is a SUBSET of the set } X \\ (Y \subseteq X) \end{array}\right\} \implies \left\{ \operatorname{span} Y \quad \text{is a LINEAR SUBSPACE of } \boldsymbol{X}. \right\}$$

Proposition B.3. [4] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *and* $\mathbb{0}$ *the zero vector of* $\boldsymbol{X}$.

$$\text{PRP}\quad \left\{ \boldsymbol{Y} \text{ is a LINEAR SUBSPACE of } \boldsymbol{X} \right\} \implies \left\{\begin{array}{lll} 1. & \mathbb{0} \in \boldsymbol{Y} & \text{and} \\ 2. & \boldsymbol{Y} \text{ is CONVEX in } \boldsymbol{X} & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} \boldsymbol{Y} \text{ is a } \textit{subspace} &\implies \exists(\alpha \boldsymbol{y}) \in \boldsymbol{Y} \quad \forall \alpha \in \mathbb{F} && \text{by Definition B.1 page 98} \\ &\implies \exists \mathbb{0} \in \boldsymbol{Y} && \text{because } \alpha = 0 \in \mathbb{F} \end{aligned}$$

$$\begin{aligned} \boldsymbol{Y} \text{ is a linear subspace} &\implies \boldsymbol{x} + \boldsymbol{y} \in Y \ \forall \boldsymbol{x}, \boldsymbol{y} \in Y \\ &\implies \lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in Y \ \forall \boldsymbol{x}, \boldsymbol{y} \in Y \\ &\implies \boldsymbol{Y} \text{ is } \textit{convex} \end{aligned}$$

Definition B.2. [5] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be* LINEAR SUBSPACES (Definition B.1 page 98) *of a* LINEAR SPACE (Definition C.1 page 111) $\boldsymbol{\Omega} \triangleq (\Omega, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

$$\text{DEF}\quad \begin{array}{rcll} \boldsymbol{X} \hat{+} \boldsymbol{Y} & \triangleq & (\{\boldsymbol{x}+\boldsymbol{y} | \boldsymbol{x} \in X \text{ and } \boldsymbol{y} \in Y\}, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})) & \text{(Minkowski addition)} \\ \boldsymbol{X} \uplus \boldsymbol{Y} & \triangleq & (X \cup Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})) & \text{(subspace union)} \\ \boldsymbol{X} \Cap \boldsymbol{Y} & \triangleq & (X \cap Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})) & \text{(subspace intersection)} \end{array}$$

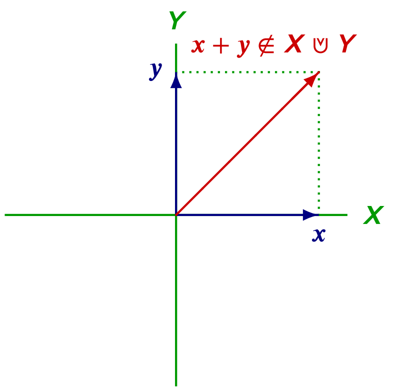
Example B.2. Some examples of operations on subspaces in $\mathbb{R}^3$ are illustrated next:

Remark B.1.

Notice the similarities between the properties of linear subspaces in a linear space (Proposition B.4 page 100) and the properties of closed sets in a topological space:

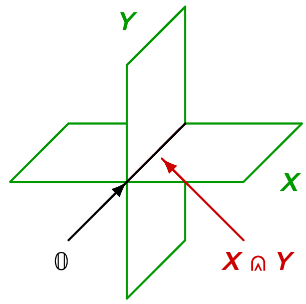
linear subspaces	*closed sets*
$\mathbb{0}$	$\emptyset$
$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$
$\boldsymbol{X} \hat{+} \boldsymbol{Y}$	$X \cup Y$
$\bigcap_{n=1}^{N} \boldsymbol{X}_n$	$\bigcap_{\gamma \in \Gamma} X_\gamma$

One key difference is that the union of two linear subspaces is not in general a linear subspace. For example, if $\boldsymbol{x}$ is the vector $[1\ 0]$ in the x direction linear subspace of $\mathbb{R}^2$ and $\boldsymbol{y}$ is the vector $[0\ 1]$ in the y direction linear subspace, then $\boldsymbol{x} + \boldsymbol{y}$ is not in the union of the two linear subspaces (it is not on the x axis or y axis but rather at $(1, 1)$).[6]



[3] Michel and Herget (1993) page 86
[4] Michel and Herget (1993) page 81
[5] Wedderburn (1907) page 79
[6] Michel and Herget (1993) page 82

In general, the set of all linear subspaces of a linear space $\boldsymbol{\Omega}$ is *not* closed under the subspace union ($\uplus$) operation; that is, the union of of two linear subspaces is *not* necessarily a linear subspace. However the set *is* closed under Minkowski sum ($\dotplus$) and subspace intersection ($\Cap$). Proposition B.4 (next) shows four useful objects are always subspaces. Some of these in Euclidean space $\mathbb{R}^3$ are illustrated to the right.



Proposition B.4. [7] *Let* $\boldsymbol{X}$ *be a* LINEAR SPACE *(Definition C.1 page 111)*.

PRP
$$\left.\begin{array}{l}\left\{\boldsymbol{X}_n \middle| {\scriptstyle n=1,2,\ldots,N}\right\} \textit{ are}\\ \text{LINEAR SUBSPACES } \textit{of } \boldsymbol{X}\end{array}\right\} \implies \left\{\begin{array}{lll} 1. & \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_N & \textit{is a } \text{LINEAR SUBSPACE } \textit{of } \boldsymbol{X}\\ & \textit{and} & \\ 2. & \boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_N & \textit{is a } \text{LINEAR SUBSPACE } \textit{of } \boldsymbol{X}\end{array}\right.$$

✎PROOF: For a structure to be a linear subspace of $\boldsymbol{X}$, it must satisfy the requirements of Definition B.1 (page 98).

1. Proof that $\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_N$ is a *linear subspace* (proof by induction):

   (a) proof for $N = 1$ case: by left hypothesis.

   (b) proof for $N = 2$ case:

      i. proof that $\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \neq \varnothing$:

$$\begin{aligned}
\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 &= \left\{\boldsymbol{v}+\boldsymbol{w} \middle| \boldsymbol{v} \in \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in Y\right\} && \text{by Definition B.2 page 99}\\
&\supseteq \left\{\boldsymbol{v}+\boldsymbol{w} \middle| \boldsymbol{v} \in \{\mathbb{0}\} \subseteq \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in \{\mathbb{0}\} \subseteq \boldsymbol{X}_2\right\}\\
&= \{\mathbb{0}+\mathbb{0}\}\\
&= \{\mathbb{0}\}\\
&\neq \varnothing
\end{aligned}$$

      ii. proof that $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \implies \boldsymbol{x}+\boldsymbol{y} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2$:

$$\begin{aligned}
\boldsymbol{x}+\boldsymbol{y} &= (\boldsymbol{v}_1+\boldsymbol{w}_1)+(\boldsymbol{v}_2+\boldsymbol{w}_2) && \text{by } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \text{ hypothesis}\\
&= \underbrace{(\boldsymbol{v}_1+\boldsymbol{v}_2)}_{\text{in } \boldsymbol{X}_1} + \underbrace{(\boldsymbol{w}_1+\boldsymbol{w}_2)}_{\text{in } \boldsymbol{X}_2 \text{ because } \boldsymbol{X}_2 \text{ is a linear subspace}}\\
&\in \left\{\boldsymbol{v}+\boldsymbol{w} \middle| \boldsymbol{v} \in \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in Y\right\}\\
&= \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 && \text{by Definition B.2 page 99}
\end{aligned}$$

      iii. proof that $\boldsymbol{v} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2, \alpha \in F \implies \alpha\boldsymbol{v} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2$:

$$\begin{aligned}
\alpha\boldsymbol{x} &= \alpha(\boldsymbol{v}_1+\boldsymbol{w}_1) && \text{by } \boldsymbol{x} \in \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \text{ hypothesis}\\
&= \underbrace{\alpha\boldsymbol{v}_1}_{\text{in } \boldsymbol{X}_1} + \underbrace{\alpha\boldsymbol{w}_1}_{\text{in } \boldsymbol{X}_2 \text{ because } \boldsymbol{X}_2 \text{ is a linear subspace}}\\
&\in \left\{\boldsymbol{v}+\boldsymbol{w} \middle| \boldsymbol{v} \in \boldsymbol{X}_1 \text{ and } \boldsymbol{w} \in Y\right\}\\
&= \boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 && \text{by Definition B.2 page 99}
\end{aligned}$$

   (c) Proof that [N case] $\implies$ [$N+1$ case]:

$$\begin{aligned}
\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_{N+1} &= \underbrace{\left(\boldsymbol{X}_1 \dotplus \boldsymbol{X}_2 \dotplus \cdots \dotplus \boldsymbol{X}_N\right)}_{\text{linear subspace by } N \text{ case hypothis}} \dotplus \boldsymbol{X}_{N+1}\\
&\implies \text{linear subspace by } N = 2 \text{ case} && \text{(item (1b) page 100)}
\end{aligned}$$

[7] Michel and Herget (1993) pages 81–83

2. Proof that $\boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_N$ is a *linear subspace* (proof by induction):
   (a) proof for $N = 1$ case: $\boldsymbol{X}_1$ is a linear subspace by left hypothesis.
   (b) Proof for $N = 2$ case:
      i. proof that $\boldsymbol{X} \Cap \boldsymbol{Y} \neq \varnothing$:

$$\begin{aligned}\boldsymbol{X} \Cap \boldsymbol{Y} &= \left\{\boldsymbol{x} \in X | \boldsymbol{x} \in X \text{ and } \boldsymbol{w} \in Y\right\}\\ &\supseteq \left\{\boldsymbol{x} \in X | \boldsymbol{x} \in \{\mathbb{0}\} \subseteq \boldsymbol{X} \text{ and } \boldsymbol{x} \in \{\mathbb{0}\} \subseteq \boldsymbol{Y}\right\}\\ &= \{\mathbb{0} + \mathbb{0}\}\\ &= \{\mathbb{0}\}\\ &\neq \varnothing\end{aligned}$$

      ii. proof that $\boldsymbol{x}, \boldsymbol{y} \in X \Cap \boldsymbol{Y} \implies \boldsymbol{x} + \boldsymbol{y} \in X \Cap \boldsymbol{Y}$:

$$\begin{aligned}\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X} \Cap \boldsymbol{Y} &\implies \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X} \text{ and } \boldsymbol{x}, \boldsymbol{y} \in Y\\ &\implies \boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{X} \text{ and } \boldsymbol{x} + \boldsymbol{y} \in Y && \text{because } \boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ are linear subspaces}\\ &\implies \boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{X} \Cap \boldsymbol{Y}\end{aligned}$$

      iii. proof that $\boldsymbol{v} \in X \Cap \boldsymbol{Y}, \alpha \in F \implies \alpha \boldsymbol{v} \in X \Cap \boldsymbol{Y}$:

$$\begin{aligned}\boldsymbol{x} \in \boldsymbol{X} \Cap \boldsymbol{Y} &\implies \boldsymbol{x} \in \boldsymbol{X} \text{ and } \boldsymbol{x} \in Y\\ &\implies \alpha\boldsymbol{x} \in \boldsymbol{X} \text{ and } \alpha\boldsymbol{x} \in Y && \text{because } \boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ are linear subspaces}\\ &\implies \alpha\boldsymbol{x} \in \boldsymbol{X} \Cap \boldsymbol{Y}\end{aligned}$$

   (c) Proof that $[N \text{ case}] \implies [N+1 \text{ case}]$:

$$\begin{aligned}\boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_{N+1} &= \underbrace{\left(\boldsymbol{X}_1 \Cap \boldsymbol{X}_2 \Cap \cdots \Cap \boldsymbol{X}_N\right)}_{\text{linear subspace by } N \text{ case hypothesis}} \Cap \boldsymbol{X}_{N+1}\\ &\implies \text{linear subspace by } N = 2 \text{ case (item (2b) page 101)}\end{aligned}$$

✏

Every linear subspace contains the zero vector $\mathbb{0}$ (Proposition B.3 page 99). But if a pair of linear subspaces of a linear space $\boldsymbol{X}$ *only* have $\mathbb{0}$ in commmon, then any vector in $\boldsymbol{X}$ can be *uniquely* represented by a single vector from each of the two subspaces (next).

Theorem B.1. [8] *Let* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *and* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be* LINEAR SUBSPACES *(Definition B.1 page 98) of a* LINEAR SPACE *(Definition C.1 page 111)* $\boldsymbol{\Omega} \triangleq \left(\Omega, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

THM

$$X \cap Y = \{\mathbb{0}\} \iff \left\{\begin{array}{l}\text{for every } \boldsymbol{u} \in \boldsymbol{X} \hat{+} \boldsymbol{Y} \text{ there exist } \boldsymbol{x} \in X \text{ and } \boldsymbol{y} \in Y \text{ such that}\\ \quad 1.\ \boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y} \qquad \text{and}\\ \quad 2.\ \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are UNIQUE.}\end{array}\right\}$$

✎PROOF:

1. Proof that $X \cap Y = \{\mathbb{0}\} \implies$ *unique* $\boldsymbol{x}, \boldsymbol{y}$:
   Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are not unique, but rather $\boldsymbol{u} = \boldsymbol{x}_1 + \boldsymbol{y}_1 = \boldsymbol{x}_2 + \boldsymbol{y}_2$ where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in Y$.

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{x}_1 + \boldsymbol{y}_1 = \boldsymbol{x}_2 + \boldsymbol{y}_2 &\implies \underbrace{\boldsymbol{x}_1 - \boldsymbol{x}_2}_{\in X} = \underbrace{\boldsymbol{y}_2 - \boldsymbol{y}_1}_{\in Y}\\ &\implies \boldsymbol{x}_1 - \boldsymbol{x}_2,\ \boldsymbol{y}_2 - \boldsymbol{y}_1 \in X \cap Y\\ &\implies \boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{y}_2 - \boldsymbol{y}_1 = \mathbb{0} && \text{by left hypothesis}\\ &\implies \boldsymbol{x}_1 = \boldsymbol{x}_2 \quad \text{and} \quad \boldsymbol{y}_2 = \boldsymbol{y}_1\\ &\implies \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are } \textit{unique}\end{aligned}$$

[8] Michel and Herget (1993) page 83 ⟨Theorem 3.2.12⟩, Kubrusly (2001) page 67 ⟨Theorem 2.14⟩

2. Proof that $X \cap Y = \{\mathbb{0}\} \impliedby$ *unique* $\boldsymbol{x}, \boldsymbol{y}$:

$$
\begin{aligned}
\boldsymbol{u} &= \boldsymbol{x} + \boldsymbol{y} \\
&= \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{y} && \text{for some vector } \boldsymbol{y} \in X \cap Y \\
&= \underbrace{(\boldsymbol{x} + \boldsymbol{y})}_{\in X} + \underbrace{(\boldsymbol{y} - \boldsymbol{y})}_{\in Y} && \text{because } \boldsymbol{x} \in X \text{ and } \boldsymbol{y} \in X \cap Y \ldots \\
&\implies \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are } \textit{not unique} \text{ if } \boldsymbol{y} \neq \mathbb{0} \\
&\implies \boldsymbol{y} = \mathbb{0} && \text{by right hypothesis} \\
&\implies X \cap Y = \{\mathbb{0}\}
\end{aligned}
$$

✏

Theorem B.2. [9] *Let $\boldsymbol{\Omega}$ be a linear subspace and $2^{\boldsymbol{\Omega}}$ the set of closed linear subspaces of $\boldsymbol{\Omega}$.*

THM

$\left(2^{\boldsymbol{\Omega}},\ \hat{+},\ \Cap,\ \mathbf{0},\ \boldsymbol{\Omega};\ \subseteq\right)$ *is a* LATTICE. *In particular*

$$
\begin{array}{rclrcll}
\boldsymbol{X} \hat{+} \boldsymbol{X} &=& \boldsymbol{X} & \boldsymbol{X} \Cap \boldsymbol{X} &=& \boldsymbol{X} & \forall \boldsymbol{X} \in 2^{\boldsymbol{\Omega}} \\
\boldsymbol{X} \hat{+} \boldsymbol{Y} &=& \boldsymbol{Y} \hat{+} \boldsymbol{X} & \boldsymbol{X} \Cap \boldsymbol{Y} &=& \boldsymbol{Y} \Cap \boldsymbol{X} & \forall \boldsymbol{X},\boldsymbol{Y} \in 2^{\boldsymbol{\Omega}} \\
(\boldsymbol{X} \hat{+} \boldsymbol{Y}) \hat{+} \boldsymbol{Z} &=& \boldsymbol{X} \hat{+} (\boldsymbol{Y} \hat{+} \boldsymbol{Z}) & (\boldsymbol{X} \Cap \boldsymbol{Y}) \Cap \boldsymbol{Z} &=& \boldsymbol{X} \Cap (\boldsymbol{Y} \Cap \boldsymbol{Z}) & \forall \boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z} \in 2^{\boldsymbol{\Omega}} \\
\boldsymbol{X} \hat{+} (\boldsymbol{X} \Cap \boldsymbol{Y}) &=& \boldsymbol{X} & \boldsymbol{X} \Cap (\boldsymbol{X} \hat{+} \boldsymbol{Y}) &=& \boldsymbol{X} & \forall \boldsymbol{X},\boldsymbol{Y} \in 2^{\boldsymbol{\Omega}}
\end{array}
$$

✎PROOF: These results follow directly from the properties of lattices. ✏

B.2 Subspaces of an inner product space

Definition B.3. [10] *Let $\boldsymbol{\Omega} \triangleq \left(X,\ +,\ \cdot,\ (\mathbb{F},\ \dot{+},\ \dot{\times}),\ \langle \triangle \mid \nabla \rangle\right)$ be an* INNER PRODUCT SPACE *(Definition C.9 page 124).*

DEF

*The **orthogonal complement** $A^{\perp}$ in $\boldsymbol{\Omega}$ of a set $A \subseteq X$ is*

$$A^{\perp} \triangleq \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A\}.$$

The expression $A^{\perp\perp}$ is defined as $\left(A^{\perp}\right)^{\perp}$.

Proposition B.5. [11] *Let $\left(X,\ +,\ \cdot,\ (\mathbb{F},\ \dot{+},\ \dot{\times}),\ \langle \triangle \mid \nabla \rangle\right)$ be an* INNER PRODUCT SPACE *(Definition C.9 page 124).*

PRP

$$A \subseteq B \implies B^{\perp} \subseteq A^{\perp} \quad \forall A,B \in 2^X \qquad \text{(ANTITONE)}$$

✎PROOF:

$$
\begin{aligned}
B^{\perp} &\triangleq \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in B\} && \text{by definition of } B^{\perp} \text{ (Definition B.3 page 102)} \\
&\subseteq \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A\} && \text{by } A \subseteq B \text{ hypothesis} \\
&= A^{\perp} && \text{by definition of } A^{\perp} \text{ (Definition B.3 page 102)}
\end{aligned}
$$

✏

Every *linear space* $\boldsymbol{X}$ contains $\mathbf{0}$ and $\boldsymbol{X}$ as *linear subspaces* (Proposition B.1 page 98). If $\boldsymbol{X}$ is also an *inner product space*, then $\mathbf{0}$ and $\boldsymbol{X}$ are *orthogonal complements* of each other (next proposition).

[9] Iturrioz (1985) pages 56–57

[10] Berberian (1961) page 59 ⟨Definition III.2.1⟩, Michel and Herget (1993) page 382, Kubrusly (2001) page 328

[11] Berberian (1961) page 60 ⟨Theorem III.2.2⟩, Kubrusly (2011) page 326

Proposition B.6. [12] *Let $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ be an* INNER PRODUCT SPACE *(Definition C.9 page 124) and $\mathbb{0}$ the* VECTOR ADDITIVE IDENTITY ELEMENT *(Definition C.1 page 111) in $\boldsymbol{\Omega}$.*

PRP
1. $\{\mathbb{0}\}^{\perp} = X$
2. $X^{\perp} = \{\mathbb{0}\}$

✎PROOF:

$$\begin{aligned}
\{\mathbb{0}\}^{\perp} &= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in \{\mathbb{0}\}\} && \text{by definition of } \perp \text{ (Definition B.3 page 102)}\\
&= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \mathbb{0} \rangle = 0\}\\
&= X\\
X^{\perp} &= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in X\} && \text{by definition of } \perp \text{ Definition B.3 page 102}\\
&= \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle = 0\}\\
&= \{\mathbb{0}\}
\end{aligned}$$

✏

For any set A contained in a linear space $\boldsymbol{X}$, $A^{\perp\perp}$ is a *linear subspace*, and it is the smallest linear subspace containing the set A ($A^{\perp\perp} = \operatorname{span} A$, next theorem). In the case that A is a *linear subspace* rather than just a subset, results simplify significantly (next corollary).

Theorem B.3. [13] *Let $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ be an* INNER PRODUCT SPACE *(Definition C.9 page 124). Let $\operatorname{span} A$ be the span of a set A (Definition 2.2 page 9).*

THM
$$\left\{\begin{array}{l} A \text{ is a } \textbf{subset} \text{ of } X \\ (A \subseteq X) \end{array}\right\} \implies \left\{\begin{array}{llcll}
1. & A \cap A^{\perp} & = & \left\{\begin{array}{ll} \{\mathbb{0}\} & \text{if } \mathbb{0} \in A \\ \varnothing & \text{if } \mathbb{0} \notin A \end{array}\right. & \text{and}\\
2. & A & \subseteq & A^{\perp\perp} = \operatorname{span} A & \text{and}\\
3. & A^{\perp} & = & A^{\perp\perp\perp} = A^{\perp -} = A^{-\perp} = (\operatorname{span} A)^{\perp} & \text{and}\\
4. & A^{\perp} \text{ is a } \textbf{subspace} \text{ of } \boldsymbol{\Omega} & & &
\end{array}\right\}$$

✎PROOF:

1. Proof that $A \cap A^{\perp} = \cdots$:

$$\begin{aligned}
A \cap A^{\perp} &= \{\boldsymbol{x} \in X \mid \boldsymbol{x} \in A\} \cap \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle \quad \forall \boldsymbol{y} \in A\} && \text{by definition of } A^{\perp}\\
&= \left\{\boldsymbol{x} \in X \mid \boldsymbol{x} \in A \quad \text{and} \quad \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle \quad \forall \boldsymbol{y} \in A\right\}\\
&= \left\{\begin{array}{ll} \{\mathbb{0}\} & \text{if } \mathbb{0} \in A \\ \varnothing & \text{if } \mathbb{0} \notin A \end{array}\right.
\end{aligned}$$

2. Proof that $A \subseteq A^{\perp\perp} = \operatorname{span} A$:

$$\begin{aligned}
\boldsymbol{x} \in A &\implies \{\boldsymbol{x}\}^{\perp\perp} \subseteq A^{\perp\perp}\\
&\implies \boldsymbol{x} \in \{\boldsymbol{x}\}^{\perp\perp} \subseteq A^{\perp\perp}\\
&\implies \boldsymbol{x} \in A^{\perp\perp}
\end{aligned}$$

but

$$\boldsymbol{x} \in A^{\perp\perp} \not\Longrightarrow \boldsymbol{x} \in A$$

Here is an example for the $\not\Longrightarrow$ part using the linear space $\mathbb{R}^3$:

[12] Kubrusly (2011) page 326, Michel and Herget (1993) page 383
[13] Michel and Herget (1993) page 383, Kubrusly (2011) page 326

(a) Let $A \triangleq \{i\}$, where i is the unit vector on the x-axis.

(b) Then $A^\perp = \left\{\boldsymbol{x} \in X | \boldsymbol{x} \in \text{yz plane}\right\}$.

(c) Then $A^{\perp\perp} = \{\boldsymbol{x} \in X | \boldsymbol{x} \in \text{x axis}\}$.

(d) Therefore, $A \subsetneq A^{\perp\perp}$

3. Proof for $A^\perp$ equivalent expressions:

(a) Proof that $A^\perp = A^{\perp\perp\perp}$:

$$\begin{aligned}
A^\perp &\subseteq \left(A^\perp\right)^{\perp\perp} && \text{by item (2)} \\
&= \left(A^{\perp\perp}\right)^\perp \\
&= A^{\perp\perp\perp} && \text{by Definition B.3 page 102} \\
A^{\perp\perp\perp} &= \left(A^{\perp\perp}\right)^\perp && \text{by Definition B.3 page 102} \\
&\subseteq A^\perp && \text{by item (2) and Proposition B.5 (page 102)}
\end{aligned}$$

(b) Proof that $A^{\perp\perp\perp} = (\operatorname{span} A)^\perp$: follows directly from item (2) ($A^{\perp\perp} = \operatorname{span} A$).

(c) Proof that $A^\perp = A^{\perp -}$:

i. Let $(\!(\boldsymbol{x}_n)\!)$ be an $A^\perp$-valued sequence that converges to the limit $\boldsymbol{x}$ in X.

ii. The limit point $\boldsymbol{x}$ must be in $A^\perp$ because for all $\boldsymbol{y} \in A$

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle &= \langle \lim \boldsymbol{x}_n \,|\, \boldsymbol{y} \rangle && \text{by definition of the sequence } (\!(\boldsymbol{x}_n)\!) \\
&= \lim \langle \boldsymbol{x}_n \,|\, \boldsymbol{y} \rangle \\
&= 0 && \text{because } (\!(\boldsymbol{x}_n)\!) \text{ is } A^\perp\text{-valued}
\end{aligned}$$

iii. Because $\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A$, $\boldsymbol{x}$ is in $A^\perp$.

iv. Because $A^\perp$ contains all its limit points, and by the *Closed Set Theorem* (Theorem ?? page ??), it must be *closed* ($A^\perp = A^{\perp -}$)

(d) Proof that $A^\perp = A^{-\perp}$:

i. Let $x \in A^\perp$ and $y \in A^-$.

ii. Let $(\!(\boldsymbol{y}_n)\!)$ be an $A^\perp$-valued sequence that converges in X to y.

iii. Thus $A^\perp \perp A^-$ because

$$\begin{aligned}
\langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle &= \langle \lim \boldsymbol{y}_n \,|\, \boldsymbol{x} \rangle && \text{by definition of } (\!(\boldsymbol{y}_n)\!) \\
&= \lim \langle \boldsymbol{y}_n \,|\, \boldsymbol{x} \rangle \\
&= 0 && \text{because } (\!(\boldsymbol{y}_n)\!) \text{ is } A^\perp\text{-valued}
\end{aligned}$$

iv. Because $A^\perp \perp A^-$, so $A^\perp \subseteq A^{\perp -}$.

v. But $A^{\perp -} \subseteq A^\perp$ because

$$A \subseteq A^- \implies A^{\perp -} \subseteq A^\perp \qquad \text{by \textit{antitone} property (Proposition B.5 page 102)}$$

vi. And so $A^\perp = A^{\perp -}$.

4. Proof that $A^\perp$ is a **subspace** of $\boldsymbol{\Omega}$ (must satisfy the conditions of Definition B.1 page 98):

(a) Proof that $A^\perp \neq \varnothing$: $A^\perp$ has at least one element, the element $\mathbb{0}$...

$$\begin{aligned}
\langle \mathbb{0} \,|\, \boldsymbol{y} \rangle = 0 \quad \forall \boldsymbol{y} \in A & && \text{by definition of } \mathbb{0} \\
\implies \mathbb{0} \in A^\perp & && \text{by definition of } A^\perp \text{ (Definition B.3 page 102)}
\end{aligned}$$

(b) Proof that $A^\perp \subseteq X$:

$$\begin{aligned}
\boldsymbol{u} \in A^\perp &\implies \boldsymbol{u} \in \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle = 0 \quad \forall \boldsymbol{y} \in A\} && \text{by definition of } A^\perp \text{ (Definition B.3 page 102)}\\
&\implies \boldsymbol{u} \in X && \text{by definition of sets}
\end{aligned}$$

(c) Proof that $\boldsymbol{u}, \boldsymbol{v} \in A^\perp \implies (\boldsymbol{u}+\boldsymbol{v}) \in A^\perp$:

$$\begin{aligned}
\boldsymbol{u}, \boldsymbol{v} \in A^\perp &\implies \langle \boldsymbol{u} \mid \boldsymbol{y}\rangle = \langle \boldsymbol{v} \mid \boldsymbol{y}\rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by definition of } A^\perp \text{ (Definition B.3 page 102)}\\
&\implies \langle \boldsymbol{u} \mid \boldsymbol{y}\rangle + \langle \boldsymbol{v} \mid \boldsymbol{y}\rangle = 0 \quad \forall \boldsymbol{y} \in A\\
&\implies \langle \boldsymbol{u}+\boldsymbol{v} \mid \boldsymbol{y}\rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by \textit{additive} property of } \langle \triangle \mid \nabla \rangle \text{ (Definition C.9 page 124)}\\
&\implies \boldsymbol{u}+\boldsymbol{v} \in A^\perp && \text{by definition of } A^\perp \text{ (Definition B.3 page 102)}
\end{aligned}$$

(d) Proof that $\boldsymbol{v} \in \Omega \implies \alpha\boldsymbol{v} \in A^\perp$:

$$\begin{aligned}
\boldsymbol{v} \in A^\perp &\implies \langle \boldsymbol{v} \mid \boldsymbol{y}\rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by definition of } A^\perp \text{ (Definition B.3 page 102)}\\
&\implies \alpha \langle \boldsymbol{v} \mid \boldsymbol{y}\rangle = \alpha \cdot 0 \quad \forall \boldsymbol{y} \in A\\
&\implies \langle \alpha\boldsymbol{v} \mid \boldsymbol{y}\rangle = 0 \quad \forall \boldsymbol{y} \in A && \text{by \textit{homogeneous} property of } \langle \triangle \mid \nabla \rangle \text{ (Definition C.9 page 124)}\\
&\implies \alpha\boldsymbol{v} \in A^\perp && \text{by definition of } A^\perp \text{ (Definition B.3 page 102)}
\end{aligned}$$

⇨

Corollary B.1. *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be* INNER PRODUCT SPACE*s. Let* $\operatorname{span} Y$ *be the span of the set* Y (Definition 2.2 page 9).

COR

$$\left\{ \boldsymbol{Y} \text{ is a \textbf{linear subspace} of } \boldsymbol{X} \right\} \implies \left\{ \begin{array}{llcll} 1. & Y \cap Y^\perp & = & \{\mathbb{0}\} & \text{and}\\ 2. & Y & = & Y^{\perp\perp} = \operatorname{span} Y & \text{and}\\ 3. & Y^\perp & = & Y^{\perp\perp\perp} & \text{and}\\ 4. & \multicolumn{4}{l}{Y^\perp \text{ is a \textbf{subspace} of } \boldsymbol{X}} \end{array} \right\}$$

✎PROOF:

1. Proof that $\boldsymbol{Y} \cap Y^\perp = \{\mathbb{0}\}$: This follows from Theorem B.3 (page 103) and the fact that all subspaces contain the zero vector $\mathbb{0}$ (Proposition B.3 page 99).
2. Proof that $\boldsymbol{Y} = \boldsymbol{Y}^{\perp\perp} = \operatorname{span} \boldsymbol{Y}$: This follows directly from Theorem B.3 (page 103).
3. Proof that $\boldsymbol{Y}^\perp = \boldsymbol{Y}^{\perp\perp\perp}$: This follows directly from Theorem B.3 (page 103).
4. Proof that $Y^\perp$ is a **subspace** of $\boldsymbol{X}$: This follows directly from Theorem B.3 (page 103).

⇨

Theorem B.4. [14] *Let* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and* $\boldsymbol{Z} \triangleq (Z, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be* LINEAR SUBSPACE*s of an* INNER PRODUCT SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM

$$\boldsymbol{Y} \perp \boldsymbol{Z} \implies Y \cap Z = \{\mathbb{0}\}$$

✎PROOF:

$$\begin{aligned}
\boldsymbol{x} \in \boldsymbol{Y} \cap \boldsymbol{Z} &\implies \boldsymbol{x} \in \boldsymbol{Y} \text{ and } \boldsymbol{x} \in \boldsymbol{Z} && \text{by definition of } \cap\\
&\implies \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 && \text{by hypothesis } \boldsymbol{Y} \perp \boldsymbol{Z}\\
&\implies \boldsymbol{x} = \mathbb{0} && \text{by \textit{non-isotropic}property of } \langle \triangle \mid \nabla \rangle \text{ (Definition C.9 page 124)}
\end{aligned}$$

⇨

[14] Kubrusly (2001) page 324

Theorem B.5. [15] *Let* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle)$ *and* $\boldsymbol{Z} \triangleq (Z, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle)$ *be linear subspaces of an* INNER PRODUCT SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle)$.

THM
$$\left\{\begin{array}{ll} 1. & \boldsymbol{Y} \perp \boldsymbol{Z} \text{ and} \\ 2. & \boldsymbol{x} \in \boldsymbol{Y} \hat{+} \boldsymbol{Z} \end{array}\right\} \implies \left\{\begin{array}{ll} 1. & \text{There exists } \boldsymbol{y} \in \boldsymbol{Y} \text{ and } \boldsymbol{z} \in \boldsymbol{Z} \text{ such that } \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z} \text{ and} \\ 2. & \boldsymbol{y} \text{ and } \boldsymbol{z} \text{ are UNIQUE.} \end{array}\right\}$$

✎PROOF:

1. Proof that $\boldsymbol{y}$ and $\boldsymbol{z}$ exist: by definition of Minkowski addition operator $\hat{+}$ (Definition B.2 page 99).
2. Proof that $\boldsymbol{y}$ and $\boldsymbol{z}$ are *unique*:
   (a) Suppose $\boldsymbol{x} = \boldsymbol{y}_1 + \boldsymbol{z}_1 = \boldsymbol{y}_1 + \boldsymbol{z}_2$ for $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{Y}$ and $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \boldsymbol{Z}$.
   (b) This implies
$$\begin{aligned} \mathbb{0} &= \boldsymbol{x} - \boldsymbol{x} \\ &= (\boldsymbol{y}_1 + \boldsymbol{z}_1) - (\boldsymbol{y}_1 + \boldsymbol{z}_2) \\ &= \underbrace{(\boldsymbol{y}_1 - \boldsymbol{y}_2)}_{\text{in } \boldsymbol{Y}} + \underbrace{(\boldsymbol{z}_1 - \boldsymbol{z}_2)}_{\text{in } \boldsymbol{Z}} \end{aligned}$$
   (c) Because $\boldsymbol{y}_1 - \boldsymbol{y}_2 \in \boldsymbol{Y}$, $\boldsymbol{z}_1 - \boldsymbol{z}_2 \in \boldsymbol{Z}$, $(\boldsymbol{y}_1 - \boldsymbol{y}_2) + (\boldsymbol{z}_1 + \boldsymbol{z}_2) = \mathbb{0}$, and $\langle \boldsymbol{y}_1 - \boldsymbol{y}_2 \,|\, \boldsymbol{z}_1 - \boldsymbol{z}_2 \rangle = 0$, then by Theorem **??** (page **??**), $\boldsymbol{y}_1 - \boldsymbol{y}_2 = \mathbb{0}$ and $\boldsymbol{z}_1 - \boldsymbol{z}_2 = \mathbb{0}$.
   (d) This implies $\boldsymbol{y}_1 = \boldsymbol{y}_2$ and $\boldsymbol{z}_1 = \boldsymbol{z}_2$.
   (e) This implies $\boldsymbol{y}$ and $\boldsymbol{z}$ are *unique*.

✏

B.3 Subspaces of a Hilbert Space

Theorem B.6. [16] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle)$ *be a* HILBERT SPACE *(Definition **??** page **??**). Let* Y *be a* SUBSET *of* X*, and let* $\mathsf{d}(\boldsymbol{x}, Y) \triangleq \inf_{\boldsymbol{y} \in Y} \|\boldsymbol{x} - \boldsymbol{y}\|$.

THM
$$\left\{\begin{array}{lll} 1. & Y \neq \varnothing & \text{and} \\ 2. & Y \text{ is CLOSED} & \text{and} \\ 3. & Y \text{ is CONVEX} & \end{array}\right\} \implies \left\{\begin{array}{lll} \multicolumn{3}{l}{\text{There exists } \boldsymbol{p} \in Y \text{ such that}} \\ 1. & \mathsf{d}(\boldsymbol{x}, Y) = \|\boldsymbol{x} - \boldsymbol{p}\| & \text{and} \\ 2. & \boldsymbol{p} \text{ is UNIQUE.} & \end{array}\right\}$$

✎PROOF:

1. Let $\delta \triangleq \inf\{\boldsymbol{x} - \boldsymbol{y} | \boldsymbol{y} \in Y\}$.
2. Let $(\boldsymbol{y}_n)_{n\in\mathbb{Z}}$ be a sequence such that $\|\boldsymbol{x} - \boldsymbol{y}_n\| \to \delta$.

[15] Berberian (1961) page 61 ⟨Theorem III.2.3⟩
[16] Kubrusly (2001) page 330 ⟨Theorem 5.13⟩, Aliprantis and Burkinshaw (1998) page 290 ⟨Theorem 33.6⟩, Berberian (1961) page 68 ⟨Theorem III.5.1⟩

3. Proof that $(\!(\boldsymbol{y}_n)\!)$ is *Cauchy*:

$$\begin{aligned}
&\lim_{m,n\to\infty} \|\boldsymbol{y}_n - \boldsymbol{y}_m\|^2 \\
&= \lim_{m,n\to\infty} \|(\boldsymbol{y}_n - x) + (x - \boldsymbol{y}_m)\|^2 \\
&= \lim_{m,n\to\infty} \left\{ -\|(\boldsymbol{y}_n - x) - (x - \boldsymbol{y}_m)\|^2 + 2\|\boldsymbol{y}_n - x\|^2 + 2\|x - \boldsymbol{y}_m\|^2 \right\} && \text{by \textit{parallelogram law} (page ??)} \\
&= \lim_{m,n\to\infty} \left\{ -4 \left\| \underbrace{\left(\frac{1}{2}\boldsymbol{y}_n + \frac{1}{2}\boldsymbol{y}_m\right)}_{\text{in } Y \text{ by \textit{convexity}}} - x \right\|^2 + 2\|\boldsymbol{y}_n - x\|^2 + 2\|x - \boldsymbol{y}_m\|^2 \right\} \\
&\le \lim_{m,n\to\infty} \left\{ -4\delta^2 + 2\|\boldsymbol{y}_n - x\|^2 + 2\|x - \boldsymbol{y}_m\|^2 \right\} && \text{by definition of } \delta \text{ (item (1))} \\
&= -4\delta^2 + \lim_{m,n\to\infty} \left\{ 2\|\boldsymbol{y}_n - x\|^2 \right\} + \lim_{m,n\to\infty} \left\{ 2\|x - \boldsymbol{y}_m\|^2 \right\} \\
&= -4\delta^2 + 2\delta^2 + 2\delta^2 && \text{by definition of } \delta \text{ (item (1))} \\
&= 0
\end{aligned}$$

4. Proof that $\mathsf{d}(\boldsymbol{x}, Y) = \|\boldsymbol{x} - \boldsymbol{y}\|$: because $(\!(\boldsymbol{y}_n)\!)$ is *Cauchy* (item (1)) and by the *closed* hypothesis.

5. Proof that $\boldsymbol{y}$ is *unique*: Because in a metric space, the limit of a convergent sequence is *unique*.

✏

Theorem B.7. [17] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a* HILBERT SPACE *(Definition ?? page ??). Let* $\mathsf{d}(\boldsymbol{x}, Y) \triangleq \inf_{\boldsymbol{y}\in Y} \|\boldsymbol{x} - \boldsymbol{y}\|$*. Let* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *and* $Y^\perp$ *the* ORTHOGONAL COMPLEMENT *of* Y.

THM

$$\left\{ \boldsymbol{Y} \text{ is a SUBSPACE of } \boldsymbol{H} \right\} \implies \left\{ \begin{array}{lll} \multicolumn{3}{l}{\text{There exists } \boldsymbol{p} \in Y \text{ such that}} \\ 1. & \mathsf{d}(\boldsymbol{x}, Y) = \|\boldsymbol{x} - \boldsymbol{p}\| & \text{and} \\ 2. & \boldsymbol{p} \text{ is UNIQUE} & \text{and} \\ 3. & \boldsymbol{x} - \boldsymbol{p} \in Y^\perp. & \end{array} \right\}$$

Theorem B.8 (Projection Theorem). [18] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a Hilbert space.*

THM

$$\left\{ \boldsymbol{Y} \text{ is a SUBSPACE of } \boldsymbol{H} \right\} \implies \left\{ \boldsymbol{Y} \hat{+} \boldsymbol{Y}^\perp = \boldsymbol{H} \right\}$$

✎PROOF:

$$\begin{aligned}
\boldsymbol{Y} \hat{+} \boldsymbol{Y}^\perp &= \left[\boldsymbol{Y} \hat{+} \boldsymbol{Y}^\perp\right]^{\perp\perp} && \text{by Corollary B.1 page 105} \\
&= \left[\boldsymbol{Y}^\perp \cap \boldsymbol{Y}^{\perp\perp}\right]^\perp && \text{by Proposition B.5 (page 102)} \\
&= \{\mathbb{0}\}^\perp && \text{by Corollary B.1 page 105} \\
&= \boldsymbol{H} && \text{by Proposition B.6 page 103}
\end{aligned}$$

✏

The inclusion relation $\subseteq$ is an order relation on the set of subspaces of a linear space $\boldsymbol{\Omega}$.

[17] Kubrusly (2001) page 330 ⟨Theorem 5.13⟩
[18] Bachman and Narici (1966) page 172 ⟨Theorem 10.8⟩, Kubrusly (2001) page 339 ⟨Theorem 5.20⟩

Proposition B.7. *Let $\boldsymbol{S}$ be the set of subspaces of a linear space $\boldsymbol{\Omega}$. Let $\subseteq$ be the inclusion relation.*

PRP $(S, \subseteq)$ *is an* ***ordered set***

✎PROOF: $(S, \subseteq)$ is an *ordered set* and because

1.	$\boldsymbol{X} \subseteq \boldsymbol{X}$	$\forall \boldsymbol{X} \in S$	(reflexive)	and	preorder
2.	$\boldsymbol{X} \subseteq \boldsymbol{Y}$ and $\boldsymbol{Y} \subseteq \boldsymbol{Z} \implies \boldsymbol{X} \subseteq \boldsymbol{Z}$	$\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in S$	(transitive)	and	
3.	$\boldsymbol{X} \subseteq \boldsymbol{Y}$ and $\boldsymbol{Y} \subseteq \boldsymbol{X} \implies \boldsymbol{X} = \boldsymbol{Y}$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in S$	(anti-symmetric)		

✏

Theorem B.9. [19] *Let $\boldsymbol{H}$ be a Hilbert space and $2^{\boldsymbol{H}}$ the set of closed linear subspaces of $\boldsymbol{H}$.*

THM $\left(2^{\boldsymbol{H}}, \hat{+}, \Cap, \boldsymbol{0}, \boldsymbol{H}; \subseteq\right)$ *is an* ORTHOMODULAR LATTICE. *In particular*

1.	$\boldsymbol{X} \hat{+} \boldsymbol{X}^\perp$	$=$	$\boldsymbol{H}$	$\forall \boldsymbol{X} \in \boldsymbol{H}$	(COMPLEMENTED)
2.	$\boldsymbol{X} \Cap \boldsymbol{X}^\perp$	$=$	$\boldsymbol{0}$	$\forall \boldsymbol{X} \in \boldsymbol{H}$	(COMPLEMENTED)
3.	$\left(\boldsymbol{X}^\perp\right)^\perp$	$=$	$\boldsymbol{X}$	$\forall \boldsymbol{X} \in \boldsymbol{H}$	(INVOLUTORY)
4.	$\boldsymbol{X} \leq \boldsymbol{Y}$	$\implies$	$\boldsymbol{Y}^\perp \leq \boldsymbol{X}^\perp$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{H}$	(ANTITONE)
5.	$\boldsymbol{X} \leq \boldsymbol{Y}$	$\implies$	$\boldsymbol{X} \hat{+} \left(\boldsymbol{X}^\perp \Cap \boldsymbol{Y}\right) = \boldsymbol{Y}$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in X$	(ORTHOMODULAR IDENTITY)

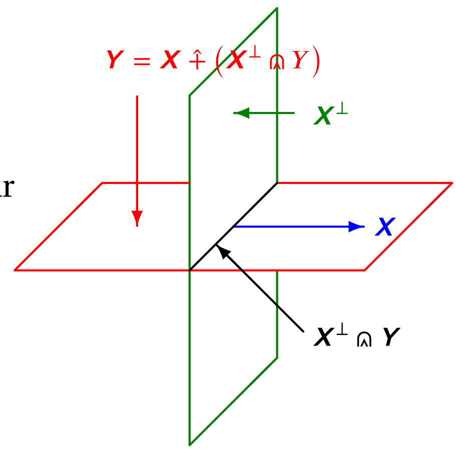
✎PROOF:

1. Proof for *complemented* (1) property: by *Projection Theorem* (Theorem B.8 page 107).
2. Proof for *complemented* (2) property: by Corollary B.1 (page 105).
3. Proof for *involutory* property: by Corollary B.1 (page 105).
4. Proof for *antitone* property: by Proposition B.5 (page 102).
5. Proof for *orthomodular identity* property:
6. Proof that lattice is *orthomodular*: by 5 properties and definition of *orthomodular lattice*.

✏

This concept is illustrated to the right where $\boldsymbol{X}, \boldsymbol{Y} \in 2^{\boldsymbol{H}}$ are linear subspaces of the linear space $\boldsymbol{H}$ and

$$\boldsymbol{X} \subseteq \boldsymbol{Y} \implies \boldsymbol{Y} = \boldsymbol{X} \hat{+} \left(\boldsymbol{X}^\perp \Cap Y\right).$$



Corollary B.2. *Let $\boldsymbol{H}$ be a Hilbert space with orthogonality operation $\perp$. Let $\left(2^{\boldsymbol{H}}, \hat{+}, \Cap, \boldsymbol{0}, \boldsymbol{H}; \subseteq\right)$ be the lattice of subspaces of $\boldsymbol{H}$.*

COR

$(\boldsymbol{X} \hat{+} \boldsymbol{Y})^\perp$	$=$	$\boldsymbol{X}^\perp \Cap \boldsymbol{Y}^\perp$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in 2^{\boldsymbol{H}}$	(DE MORGAN)	*and*
$(\boldsymbol{X} \Cap \boldsymbol{Y})^\perp$	$=$	$\boldsymbol{X}^\perp \hat{+} \boldsymbol{Y}^\perp$	$\forall \boldsymbol{X}, \boldsymbol{Y} \in 2^{\boldsymbol{H}}$	(DE MORGAN)	

✎PROOF: By properties of *orthocomplemented lattices* . ✏

[19] Iturrioz (1985) pages 56–57

B.4 Subspace Metrics

Definition B.4 (Hilbert space gap metric). [20] *Let $\boldsymbol{X}$ be a **Hilbert space** and $\boldsymbol{S}$ the set of subspaces of $\boldsymbol{X}$. Then we define the following metric between subspaces of $\boldsymbol{X}$.*

DEF

$$\begin{aligned} \mathsf{d}(\boldsymbol{V},\boldsymbol{W}) &\triangleq \|\mathbf{P}-\mathbf{Q}\| \quad \forall \boldsymbol{V},\boldsymbol{W}\in\boldsymbol{S} && \text{(the distance between subspaces } \boldsymbol{V} \text{ and } \boldsymbol{W} \text{ is the size of the difference of their projection operators)}\\ \text{where } \boldsymbol{V} &\triangleq \mathbf{P}\boldsymbol{X} && (\mathbf{P} \text{ is the projection operator that generates the subspace } \boldsymbol{V})\\ \text{and } \boldsymbol{W} &\triangleq \mathbf{Q}\boldsymbol{X} && (\mathbf{Q} \text{ is the projection operator that generates the subspace } \boldsymbol{W}). \end{aligned}$$

Definition B.5 (Banach space gap metric). [21] *Let $\boldsymbol{X}$ be a **Banach space** and $\boldsymbol{S}$ the set of subspaces of $\boldsymbol{X}$. Then we define the following metric between subspaces of $\boldsymbol{X}$.*

DEF

$$\begin{aligned} \mathsf{d}(\boldsymbol{V},\boldsymbol{W}) &\triangleq \max\left\{ \sup_{\boldsymbol{v}\in\boldsymbol{V},\|\boldsymbol{v}\|=1} \mathsf{p}(\boldsymbol{v},\boldsymbol{W}),\; \sup_{\boldsymbol{w}\in\boldsymbol{W},\|\boldsymbol{w}\|=1} \mathsf{p}(\boldsymbol{w},\boldsymbol{V}) \right\} \quad \forall \boldsymbol{V},\boldsymbol{W}\in\boldsymbol{S}\\ \text{where } \mathsf{p}(\boldsymbol{v},\boldsymbol{W}) &\triangleq \inf_{\boldsymbol{w}\in\boldsymbol{W}} \|\boldsymbol{v}-\boldsymbol{w}\| && \text{(metric from the point } \boldsymbol{v} \text{ to the subspace } \boldsymbol{W}) \end{aligned}$$

Definition B.6 (Schäffer's metric). [22]

DEF

$$\begin{aligned} \mathsf{d}(\boldsymbol{V},\boldsymbol{W}) &= \log\left(1+\max\{\mathsf{r}(\boldsymbol{V},\boldsymbol{W}),\mathsf{r}(\boldsymbol{W},\boldsymbol{V})\}\right) \quad \text{where}\\ \mathsf{r}(\boldsymbol{V},\boldsymbol{W}) &\triangleq \begin{cases} \inf\{\|\mathbf{A}-\mathbf{I}\| \,|\, \mathbf{AV}=\mathbf{W}\} & \text{if } \mathbf{A} \text{ and } \mathbf{A}^{-1} \text{ both exist}\\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

B.5 Literature

Literature survey:

1. Lattice of subspaces
   - Birkhoff and Neumann (1936)
   - Husimi (1937)
   - Sasaki (1954)
   - Loomis (1955)
   - von Neumann (1960)
   - Holland (1970)
   - Halmos (1998b)
   - Amemiya and Araki (1966)
   - Gudder (1979)
   - Gudder (2005)
2. Characterizations of lattice of Hilbert subspaces (cf Iturrioz (1985) page 60):
   - Kakutani and Mackey (1946) ⟨using Banach spaces⟩
   - Piron (1964a) ⟨using pre-Hilbert spaces⟩
     - Piron (1964b) ⟨using pre-Hilbert spaces⟩
   - Amemiya and Araki (1966) ⟨using pre-Hilbert spaces⟩
   - Wilbur (1975) ⟨using locally convex spaces⟩
3. Metrics on subspaces:
   - Burago et al. (2001)

[20] Deza and Deza (2006) page 235, Akhiezer and Glazman (1993) page 69, Berkson (1963) page 8, Krein and Krasnoselski (1947)

[21] Akhiezer and Glazman (1993) page 70, Berkson (1963) page 8, Krein et al. (1948)

[22] Massera and Schäffer (1958) pages 562–563, Berkson (1963) pages 7–8

APPENDIX C

OPERATORS ON LINEAR SPACES

❝*And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.*❞
Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

C.1 Operators on linear spaces

C.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition C.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD (Definition A.5 page 96). *Let* X *be a set, let* $+$ *be an* OPERATOR (Definition C.2 page 112) *in* X^{X^2}*, and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

---

[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991) page 41, Halmos (1948) pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

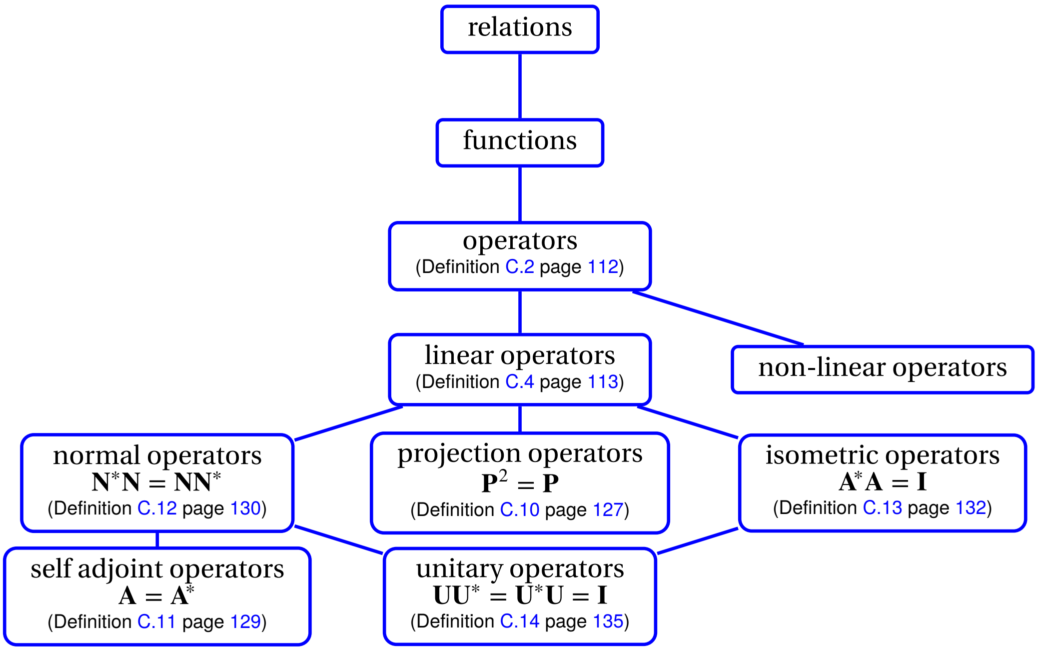


Figure C.1: Some operator types

DEF *The structure* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1.	$\exists \mathbb{0} \in X$ such that	$x + \mathbb{0} = x$	$\forall x \in X$	(+ IDENTITY)	*
2.	$\exists y \in X$ such that	$x + y = \mathbb{0}$	$\forall x \in X$	(+ INVERSE)	
3.		$(x + y) + z = x + (y + z)$	$\forall x,y,z \in X$	(+ *is* ASSOCIATIVE)	
4.		$x + y = y + x$	$\forall x,y \in X$	(+ *is* COMMUTATIVE)	
5.		$1 \cdot x = x$	$\forall x \in X$	($\cdot$ IDENTITY)	
6.		$\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$	$\forall \alpha,\beta \in S$ *and* $x \in X$	($\cdot$ ASSOCIATES *with* $\cdot$)	
7.		$\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$	$\forall \alpha \in S$ *and* $x,y \in X$	($\cdot$ DISTRIBUTES *over* +)	
8.		$(\alpha + \beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x)$	$\forall \alpha,\beta \in S$ *and* $x \in X$	($\cdot$ PSEUDO-DISTRIBUTES *over* +)	

The set X is called the ***underlying set****. The elements of X are called* ***vectors****. The elements of $\mathbb{F}$ are called* ***scalars****. A linear space is also called a* ***vector space****. If $\mathbb{F} \triangleq \mathbb{R}$, then $\boldsymbol{\Omega}$ is a* ***real linear space****. If $\mathbb{F} \triangleq \mathbb{C}$, then $\boldsymbol{\Omega}$ is a* ***complex linear space****.*

Definition C.2. [3]

DEF *A function* $\mathbf{A}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *is an* ***operator*** *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *if* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both* LINEAR SPACES *(Definition C.1 page 111).*

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}x = \mathbf{B}x$ for all $x \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{X}\boldsymbol{Y}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}x = \mathbf{I}$ for all $x \in \boldsymbol{X}$.

Definition C.3. [4] *Let $\boldsymbol{X}^{\boldsymbol{X}}$ be the set of all operators with from a* LINEAR SPACE $\boldsymbol{X}$ *to* $\boldsymbol{X}$*. Let* $\mathbf{I}$ *be an operator in* $\boldsymbol{X}^{\boldsymbol{X}}$*. Let* $\mathbb{1}(\boldsymbol{X})$ *be the* IDENTITY ELEMENT *in* $\boldsymbol{X}^{\boldsymbol{X}}$*.*

DEF $\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\quad \mathbf{I} = \mathbb{1}(\boldsymbol{X})$.

[3] Heil (2011) page 42
[4] Michel and Herget (1993) page 411

C.1.2 Linear operators

Definition C.4. [5] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be linear spaces.*

DEF

An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *is* ***linear*** *if*

1. $\mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) = \mathbf{L}\boldsymbol{x}+\mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$ (ADDITIVE) and
2. $\mathbf{L}(\alpha\boldsymbol{x}) = \alpha\mathbf{L}\boldsymbol{x} \quad \forall \boldsymbol{x}\in\boldsymbol{X}, \quad \forall\alpha\in\mathbb{F}$ (HOMOGENEOUS).

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ *such that*

$$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \left\{\mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}} | \mathbf{L} \text{ is linear}\right\}.$$

Theorem C.1. [6] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$.

THM

$$\{\mathbf{L} \text{ is LINEAR}\} \implies \left\{\begin{array}{llcll}
1. & \mathbf{L}\mathbb{0} & = & \mathbb{0} & & \text{and}\\
2. & \mathbf{L}(-\boldsymbol{x}) & = & -(\mathbf{L}\boldsymbol{x}) & \forall\boldsymbol{x}\in\boldsymbol{X} & \text{and}\\
3. & \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} & \forall\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X} & \text{and}\\
4. & \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) & = & \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}\boldsymbol{x}_n\right) & \boldsymbol{x}_n\in\boldsymbol{X}, \alpha_n\in\mathbb{F} &
\end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned}
\mathbf{L}\mathbb{0} &= \mathbf{L}(0\cdot\mathbb{0}) && \text{by additive identity property}\\
&= 0\cdot(\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition C.4 page 113)}\\
&= \mathbb{0} && \text{by } \textit{additive identity} \text{ property}
\end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned}
\mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1\cdot\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property}\\
&= -1\cdot(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition C.4 page 113)}\\
&= -(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property}
\end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}$:

$$\begin{aligned}
\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x}+(-\boldsymbol{y})) && \text{by } \textit{additive inverse} \text{ property}\\
&= \mathbf{L}(\boldsymbol{x})+\mathbf{L}(-\boldsymbol{y}) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} && \text{(Definition C.4 page 113)}\\
&= \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} && \text{by item (2)}
\end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) = \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}\boldsymbol{x}_n\right)$:

   (a) Proof for $N=1$:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_1\boldsymbol{x}_1\right) && \text{by } N=1 \text{ hypothesis}\\
&= \alpha_1\left(\mathbf{L}\boldsymbol{x}_1\right) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition C.4 page 113)}
\end{aligned}$$

[5] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[6] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

(b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1}\alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1} + \sum_{n=1}^{N}\alpha_n \boldsymbol{x}_n\right) \\
&= \mathbf{L}(\alpha_{N+1}\boldsymbol{x}_{N+1}) + \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n \boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \quad \text{(Definition C.4 page 113)} \\
&= \alpha_{N+1}\mathbf{L}(\boldsymbol{x}_{N+1}) + \sum_{n=1}^{N}\mathbf{L}(\alpha_n \boldsymbol{x}_n) && \text{by left } N+1 \text{ hypothesis} \\
&= \sum_{n=1}^{N+1}\mathbf{L}(\alpha_n \boldsymbol{x}_n)
\end{aligned}$$

✏

Theorem C.2. [7] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathcal{N}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{X}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathcal{I}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{Y}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	

✎PROOF:

1. Proof that $\mathcal{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0}\in\mathcal{N}(\mathbf{L}) \implies \mathcal{N}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{N}(\mathbf{L}) \triangleq \{\boldsymbol{x}\in\boldsymbol{X} | \mathbf{L}\boldsymbol{x}=\mathbb{0}\} \subseteq \boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{N}(\mathbf{L}) \implies \mathbb{0}=\mathbf{L}(\boldsymbol{x}+\boldsymbol{y})=\mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\boldsymbol{X} \implies \mathbb{0}=\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
2. Proof that $\mathcal{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0}\in\mathcal{I}(\mathbf{L}) \implies \mathcal{I}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{I}(\mathbf{L}) \triangleq \{\boldsymbol{y}\in\boldsymbol{Y} | \exists\boldsymbol{x}\in\boldsymbol{X} \text{ such that } \boldsymbol{y}=\mathbf{L}\boldsymbol{x}\} \subseteq \boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{v}\in\boldsymbol{X} \text{ such that } \mathbf{L}\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{x}\in\boldsymbol{X} \text{ such that } \boldsymbol{y}=\mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y}=\alpha\mathbf{L}\boldsymbol{x}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$

✏

Example C.1. [8] Let $C([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX
$C([a:b],\mathbb{R})$ is a linear space.

Theorem C.3. [9] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of a linear operator $\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM		
$\mathbf{L}\boldsymbol{x}=\mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y}\in\mathcal{N}(\mathbf{L})$
$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathcal{N}(\mathbf{L})=\{\mathbb{0}\}$

[7] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩
[8] Eidelman et al. (2004) page 3
[9] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

✎PROOF:

1. Proof that $\mathbf{L}x = \mathbf{L}y \implies x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}(x-y) &= \mathbf{L}x - \mathbf{L}y && \text{by Theorem C.1 page 113}\\
&= \mathbb{0} && \text{by left hypothesis}\\
\implies\quad & x - y \in \mathcal{N}(\mathbf{L}) && \text{by definition of } \textit{null space}
\end{aligned}$$

2. Proof that $\mathbf{L}x = \mathbf{L}y \impliedby x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}y &= \mathbf{L}y + \mathbb{0} && \text{by definition of linear space (Definition C.1 page 111)}\\
&= \mathbf{L}y + \mathbf{L}(x-y) && \text{by right hypothesis}\\
&= \mathbf{L}y + (\mathbf{L}x - \mathbf{L}y) && \text{by Theorem C.1 page 113}\\
&= (\mathbf{L}y - \mathbf{L}y) + \mathbf{L}x && \text{by } \textit{associative} \text{ and } \textit{commutative} \text{ properties (Definition C.1 page 111)}\\
&= \mathbf{L}x
\end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned}
\mathbf{L} \text{ is } \textit{injective} &\iff \{(\mathbf{L}x = \mathbf{L}y \iff x = y) \quad \forall x, y \in X\}\\
&\iff \left\{\left[\mathbf{L}x - \mathbf{L}y = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \left\{\left[\mathbf{L}(x-y) = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}
\end{aligned}$$

✏

Theorem C.4. [10] *Let W, X, Y, and Z be linear spaces over a field $\mathbb{F}$.*

THM

1.	$\mathbf{L}(\mathbf{MN})$	$=$	$(\mathbf{LM})\mathbf{N}$	$\forall \mathbf{L}\in\mathcal{L}(Z,W), \mathbf{M}\in\mathcal{L}(Y,Z), \mathbf{N}\in\mathcal{L}(X,Y)$	(ASSOCIATIVE)
2.	$\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})$	$=$	$(\mathbf{LM}) \dotplus (\mathbf{LN})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(X,Y), \mathbf{N}\in\mathcal{L}(X,Y)$	(LEFT DISTRIBUTIVE)
3.	$(\mathbf{L} \dotplus \mathbf{M})\mathbf{N}$	$=$	$(\mathbf{LN}) \dotplus (\mathbf{MN})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(Y,Z), \mathbf{N}\in\mathcal{L}(X,Y)$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{LM})$	$=$	$(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(X,Y), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{MN}) = (\mathbf{LM})\mathbf{N}$: Follows directly from property of *associative* operators.
2. Proof that $\mathbf{L}(\mathbf{M} \dotplus \mathbf{N}) = (\mathbf{LM}) \dotplus (\mathbf{LN})$:

$$\begin{aligned}
\left[\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})\right]x &= \mathbf{L}\left[(\mathbf{M} \dotplus \mathbf{N})x\right]\\
&= \mathbf{L}\left[(\mathbf{M}x) \dotplus (\mathbf{N}x)\right]\\
&= [\mathbf{L}(\mathbf{M}x)] \dotplus [\mathbf{L}(\mathbf{N}x)] && \text{by } \textit{additive} \text{ property Definition C.4 page 113}\\
&= [(\mathbf{LM})x] \dotplus [(\mathbf{LN})x]
\end{aligned}$$

3. Proof that $(\mathbf{L} \dotplus \mathbf{M})\mathbf{N} = (\mathbf{LN}) \dotplus (\mathbf{MN})$: Follows directly from property of *associative* operators.
4. Proof that $\alpha(\mathbf{LM}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.
5. Proof that $\alpha(\mathbf{LM}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned}
[\alpha(\mathbf{LM})]x &= \alpha[(\mathbf{LM})x]\\
&= \mathbf{L}[\alpha(\mathbf{M}x)] && \text{by } \textit{homogeneous} \text{ property Definition C.4 page 113}\\
&= \mathbf{L}[(\alpha\mathbf{M})x]\\
&= [\mathbf{L}(\alpha\mathbf{M})]x
\end{aligned}$$

[10] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

Theorem C.5 (Fundamental theorem of linear equations). *Michel and Herget (1993) page 99 Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ (Definition ?? page ??).*

THM

$$\dim \mathcal{I}(\mathbf{L}) + \dim \mathcal{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1, 2, \ldots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for $\mathcal{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{ \boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x} \} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \mathbb{0} \right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ⇨

C.2 Operators on Normed linear spaces

C.2.1 Operator norm

Definition C.5. [11] *Let $\boldsymbol{V} = (X, \mathbb{F}, \dot{+}, \cdot)$ be a linear space and $\mathbb{F}$ be a field with absolute value function $|\cdot| \in \mathbb{R}^{\mathbb{F}}$ (Definition A.4 page 96).*

DEF

*A **norm** is any functional $\|\cdot\|$ in $\mathbb{R}^X$ that satisfies*

1.	$\|\boldsymbol{x}\| \geq 0$	$\forall \boldsymbol{x} \in X$	(STRICTLY POSITIVE)	and
2.	$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	(NONDEGENERATE)	and
3.	$\|a\boldsymbol{x}\| = \|a\| \, \|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X,\ a \in \mathbb{C}$	(HOMOGENEOUS)	and
4.	$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(SUBADDITIVE/*triangle inquality*).	

*A **normed linear space** is the pair $(\boldsymbol{V}, \|\cdot\|)$.*

[11] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

Definition C.6. [12] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [13]

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} \qquad \forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition C.1 (next) shows that the functional defined in Definition C.6 (previous) is a *norm* (Definition C.5 page 116).

Proposition C.1. [14] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ and $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1.	$|||\mathbf{A}||| \ge 0$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(NON-NEGATIVE)	and
2.	$|||\mathbf{A}||| = 0 \iff \mathbf{A} \doteq \mathbb{0}$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(NONDEGENERATE)	and
3.	$|||\alpha\mathbf{A}||| = |\alpha|\ |||\mathbf{A}|||$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)	and
4.	$\left|\left|\left|\mathbf{A} \mathring{+} \mathbf{B}\right|\right|\right| \le |||\mathbf{A}||| + |||\mathbf{B}|||$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space.***

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \ne \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)}\\
&> 0
\end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \doteq \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)}\\
&= \sup_{x\in\boldsymbol{X}} \{\|\mathbb{0}x\| \mid \|x\| \le 1\}\\
&= 0
\end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha|\ |||\mathbf{A}|||$:

$$\begin{aligned}
|||\alpha\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\alpha\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)}\\
&= \sup_{x\in\boldsymbol{X}} \{|\alpha|\ \|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)}\\
&= |\alpha| \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of sup}\\
&= |\alpha|\ \ |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)}
\end{aligned}$$

[12] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225
[13] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990) page 290
[14] Rudin (1991) page 93

4. Proof that $|||\mathbf{A} \mathbin{\dot{+}} \mathbf{B}||| \le |||\mathbf{A}||| + |||\mathbf{B}|||$:

$$\begin{aligned}
|||\mathbf{A} \mathbin{\dot{+}} \mathbf{B}||| &\triangleq \sup_{x \in \boldsymbol{X}} \left\{ \left\|(\mathbf{A} \mathbin{\dot{+}} \mathbf{B})\boldsymbol{x}\right\| \mid \|\boldsymbol{x}\| \le 1 \right\} && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)} \\
&= \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \\
&\le \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x}\| + \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)} \\
&\le \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} + \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \\
&\triangleq |||\mathbf{A}||| + |||\mathbf{B}||| && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)}
\end{aligned}$$

⇨

Lemma C.1. *Let* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *and* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.

LEM

$$|||\mathbf{L}||| = \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} \qquad \forall \boldsymbol{x} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

✎Proof: [15]

1. Proof that $\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \ge \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \}$:

$$\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \ge \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

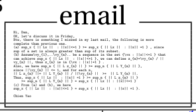
2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ \boldsymbol{y} \in X \,\middle|\, \begin{array}{ll} 1. & \|\mathbf{L}\boldsymbol{y}\| = \sup_{x \in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \text{ and} \\ 2. & 0 < \|\boldsymbol{y}\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \le \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \}$:

$$\begin{aligned}
\sup_{x \in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} &= \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of set } Y \\
&= \frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \|\mathbf{L}\boldsymbol{y}\| \\
&= \|\boldsymbol{y}\| \left\| \frac{1}{\|\boldsymbol{y}\|} \mathbf{L}\boldsymbol{y} \right\| && \text{by homogeneous property (page 116)} \\
&= \|\boldsymbol{y}\| \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| && \text{by homogeneous property (page 113)} \\
&\le \|\boldsymbol{y}\| \sup_{\boldsymbol{y} \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \right\} && \text{by definition of supremum} \\
&= \|\boldsymbol{y}\| \sup_{\boldsymbol{y} \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \text{ for all } \boldsymbol{y} \in Y \\
&\le \sup_{\boldsymbol{y} \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } 0 < \|\boldsymbol{y}\| \le 1 \\
&\le \sup_{x \in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} && \text{because } \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in X \quad \forall \boldsymbol{y} \in Y
\end{aligned}$$

[15] email



Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

4. By (1) and (3),

$$\sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\|\le 1\} = \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| = 1\}$$

✏

Proposition C.2. [16] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{X}), |||\cdot|||)$.

PRP: $|||\mathbf{I}||| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}x\| \mid \|x\|\le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition C.6 page 117)}\\
&= \sup\{\|x\| \mid \|x\|\le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition C.3 page 112)}\\
&= 1
\end{aligned}$$

✏

Theorem C.6. [17] *Let* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM:

$$\begin{array}{rcll}
\|\mathbf{L}x\| & \le & |||\mathbf{L}|||\ \|x\| & \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}),\ x\in\boldsymbol{X}\\
|||\mathbf{K}\mathbf{L}||| & \le & |||\mathbf{K}|||\ |||\mathbf{L}||| & \forall \mathbf{K},\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})
\end{array}$$

✎PROOF:

1. Proof that $\|\mathbf{L}x\| \le |||\mathbf{L}|||\ \|x\|$:

$$\begin{aligned}
\|\mathbf{L}x\| &= \frac{\|x\|}{\|x\|}\|\mathbf{L}x\|\\
&= \|x\|\left\|\frac{1}{\|x\|}\mathbf{L}x\right\| && \text{by property of norms}\\
&= \|x\|\left\|\mathbf{L}\frac{x}{\|x\|}\right\| && \text{by property of linear operators}\\
&\triangleq \|x\|\,\|\mathbf{L}y\| && \text{where } y \triangleq \frac{x}{\|x\|}\\
&\le \|x\|\sup_y \|\mathbf{L}y\| && \text{by definition of supremum}\\
&= \|x\|\sup_y\{\|\mathbf{L}y\| \mid \|y\| = 1\} && \text{because } \|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1\\
&\triangleq \|x\|\,\|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

[16] Michel and Herget (1993) page 410
[17] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

2. Proof that $|||\mathbf{KL}||| \le |||\mathbf{K}|||\ |||\mathbf{L}|||$:

$$\begin{aligned}
|||\mathbf{KL}||| &\triangleq \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{\|(\mathbf{KL})\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by Definition C.6 page 117 } (|||\cdot|||)\\
&= \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{\|\mathbf{K}(\mathbf{L}\boldsymbol{x})\| \mid \|\boldsymbol{x}\| \le 1\}\\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{|||\mathbf{K}|||\ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by 1.}\\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{|||\mathbf{K}|||\ |||\mathbf{L}|||\ \|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by 1.}\\
&= \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{|||\mathbf{K}|||\ |||\mathbf{L}|||\ 1 \mid \|\boldsymbol{x}\| \le 1\} && \text{by definition of sup}\\
&= |||\mathbf{K}|||\ |||\mathbf{L}||| && \text{by definition of sup}
\end{aligned}$$

✏

C.2.2 Bounded linear operators

Definition C.7. [18] *Let* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ *be a normed space of linear operators.*

DEF

An operator $\mathbf{B}$ *is* ***bounded*** *if* $|||\mathbf{B}||| < \infty$.
The quantity $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ *is the set of all* ***bounded linear operators*** *on* $(\boldsymbol{X},\boldsymbol{Y})$ *such that*

$$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \{\mathbf{L} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \mid |||\mathbf{L}||| < \infty\}.$$

Theorem C.7. [19] *Let* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ *be the set of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

THM

The following conditions are all EQUIVALENT:

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $\boldsymbol{x}_0 \in X$ $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ $\iff$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point* $\boldsymbol{x} \in X$*)* $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ $\iff$
3. $|||\mathbf{L}||| < \infty$ *(*$\mathbf{L}$ *is* BOUNDED*)* $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ $\iff$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}\boldsymbol{x}\| \le M\,\|\boldsymbol{x}\|$ $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \boldsymbol{x}\in X$

✎PROOF:

1. Proof that $1 \implies 2$:

$$\begin{aligned}
\epsilon &> \|\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{x}_0\| && \text{by hypothesis 1}\\
&= \|\mathbf{L}(\boldsymbol{x} - \boldsymbol{x}_0)\| && \text{by linearity (Definition C.4 page 113)}\\
&= \|\mathbf{L}(\boldsymbol{x} + \boldsymbol{y} - \boldsymbol{x}_0 - \boldsymbol{y})\|\\
&= \|\mathbf{L}(\boldsymbol{x} + \boldsymbol{y}) - \mathbf{L}(\boldsymbol{x}_0 + \boldsymbol{y})\| && \text{by linearity (Definition C.4 page 113)}\\
&\implies \mathbf{L} \text{ is continuous at point } \boldsymbol{x} + \boldsymbol{y}\\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}$$

2. Proof that $2 \implies 1$: obvious:

[18] Rudin (1991) pages 92–93
[19] Aliprantis and Burkinshaw (1998) page 227

3. Proof that 4 $\implies$ 2:[20]

$$\begin{aligned}
|||\mathbf{L}\boldsymbol{x}||| \le M\,\|\boldsymbol{x}\| &\implies |||\mathbf{L}(\boldsymbol{x}-\boldsymbol{y})||| \le M\,\|\boldsymbol{x}-\boldsymbol{y}\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le M\,\|\boldsymbol{x}-\boldsymbol{y}\| && \text{by linearity of } \mathbf{L} \text{ (Definition C.4 page 113)}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } M\,\|\boldsymbol{x}-\boldsymbol{y}\| < \epsilon\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } \|\boldsymbol{x}-\boldsymbol{y}\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that 3 $\implies$ 4:

$$\begin{aligned}
\|\mathbf{L}\boldsymbol{x}\| &\le \underbrace{|||\mathbf{L}|||}_{M}\ \|\boldsymbol{x}\| && \text{by Theorem C.6 page 119}\\
&= M\ \|\boldsymbol{x}\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that 1 $\implies$ 3:[21]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} = \infty\\
&\implies \exists\, (\boldsymbol{x}_n) \quad \text{such that} \quad \|\boldsymbol{x}_n\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\|\mathbf{L}\boldsymbol{x}_n\| \mid \|\boldsymbol{x}_n\| \le 1\right\} = \infty\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \|\mathbf{L}\boldsymbol{x}_n\|\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \|\mathbf{L}\boldsymbol{x}_n\| \ge n\\
&\implies \frac{1}{n}\|\boldsymbol{x}_n\| = \frac{1}{n} \text{ and } \frac{1}{n}\|\mathbf{L}\boldsymbol{x}_n\| \ge 1\\
&\implies \left\|\frac{\boldsymbol{x}_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{\boldsymbol{x}_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

✐

C.2.3 Adjoints on normed linear spaces

Definition C.8. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *of $\boldsymbol{X}$.*

DEF: $\mathbf{B}^*$ *is the **adjoint** of an operator $\mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ if*

$$\mathsf{f}(\mathbf{B}\boldsymbol{x}) = \left[\mathbf{B}^*\mathsf{f}\right](\boldsymbol{x}) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ \boldsymbol{x} \in \boldsymbol{X}$$

Theorem C.8. [22] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *$\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM:
$$\begin{aligned}
(\mathbf{A} \dotplus \mathbf{B})^* &= \mathbf{A}^* \dotplus \mathbf{B}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\lambda\mathbf{A})^* &= \lambda\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\mathbf{A}\mathbf{B})^* &= \mathbf{B}^*\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})
\end{aligned}$$

[20] Bollobás (1999) page 29
[21] Aliprantis and Burkinshaw (1998) page 227
[22] Bollobás (1999) page 156

✎PROOF:

$$
\begin{aligned}
\left[\mathbf{A} \mathring{+} \mathbf{B}\right]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \mathring{+} \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition C.8 page 121)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition C.4 page 113)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of \textit{linear functional}} &&\\
&= \mathbf{A}^{*}\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^{*}\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition C.8 page 121)}\\
&= \left[\mathbf{A}^{*} + \mathbf{B}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{linear functional}} &&
\end{aligned}
$$

$$
\begin{aligned}
[\lambda\mathbf{A}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition C.8 page 121)}\\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of \textit{linear functional}} &&\\
&= \left[\lambda\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition C.8 page 121)}
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{A}\mathbf{B}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition C.8 page 121)}\\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of \textit{linear operators}} && \text{(Definition C.4 page 113)}\\
&= \left[\mathbf{A}^{*}\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition C.8 page 121)}\\
&= \mathbf{B}^{*}\left[\mathbf{A}^{*}\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition C.8 page 121)}\\
&= \left[\mathbf{B}^{*}\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{adjoint}} && \text{(Definition C.8 page 121)}
\end{aligned}
$$

⇰

Theorem C.9. [23] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^{*}$ be the adjoint of an operator $\mathbf{B}$.*

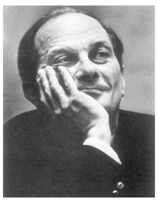
THM
$$|||\mathbf{B}||| = \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| \qquad \forall \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF:

$$
\begin{aligned}
|||\mathbf{B}||| &\triangleq \sup\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by Definition C.6 page 117}\\
&\stackrel{?}{=} \sup\left\{|\mathsf{g}(\mathbf{B}\boldsymbol{x};\boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x};\mathbf{B}^{*}\boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&\triangleq \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&= \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&\triangleq \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| && \text{by Definition C.6 page 117}
\end{aligned}
$$

⇰

C.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞

Stanislaus M. Ulam (1909–1984), Polish mathematician [24]

[23] Rudin (1991) page 98

Theorem C.10 (Mazur-Ulam theorem). [25] *Let* $\phi \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *be a function on normed linear spaces* $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ *and* $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. *Let* $\mathbf{I} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$ *be the identity operator on* $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.

THM

$$\left.\begin{array}{ll} 1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} \quad \text{and} \\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}}_{\text{isometric}} \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$
   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$
2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$
3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned}
\left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\
&= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis} \\
&= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1}
\end{aligned}$$

4. Proof that $\mathrm{g}\boldsymbol{z} = \boldsymbol{z}$:

$$\begin{aligned}
2\lambda &= 2\sup\{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\} && \text{by definition of } \lambda \text{ item (2)} \\
&\leq 2\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\| && \text{by definition of sup} \\
&= \|2\boldsymbol{z} - 2\mathrm{g}\boldsymbol{z}\| \\
&= \|\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}\boldsymbol{z}\| && \text{by definition of } \psi \text{ item (1)} \\
&= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}^{-1}\mathrm{g}\boldsymbol{z}\right\| && \text{by item (3)} \\
&= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| && \text{by definition of } \mathrm{g}^{-1} \\
&= \left\|\psi\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| \\
&= \left\|\mathrm{g}^{*}\boldsymbol{z} - \boldsymbol{z}\right\| \\
&\leq \lambda && \text{by definition of } \lambda \text{ item (2)} \\
&\implies 2\lambda \leq \lambda \\
&\implies \lambda = 0 \\
&\implies \mathrm{g}\boldsymbol{z} = \boldsymbol{z}
\end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) = \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}$:

$$\begin{aligned}
\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) &= \\
&= \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}
\end{aligned}$$

[24] quote: Ulam (1991) page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[25] Oikhberg and Rosenthal (2007) page 598, Väisälä (2003) page 634, Giles (2000) page 11, Dunford and Schwartz (1957) page 91, Mazur and Ulam (1932)

6. Proof that $\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}$:

$$\begin{aligned}\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) &= \\ &= [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}\end{aligned}$$

⇰

Theorem C.11 (Neumann Expansion Theorem). [26] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{lll} 1. & \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}\text{ is bounded}) \\ 2. & |||\mathbf{A}||| < 1 & \end{array}\right\} \implies \left\{\begin{array}{lrcl} 1. & (\mathbf{I}-\mathbf{A})^{-1} & & \text{exists} \\ 2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \le & \dfrac{1}{1-|||\mathbf{A}|||} \\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \displaystyle\sum_{n=0}^{\infty}\mathbf{A}^n \\ & \multicolumn{3}{c}{\text{with uniform convergence}} \end{array}\right.$$

C.3 Operators on Inner product spaces

C.3.1 General Results

Definition C.9. [27] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space.*

DEF

A function $\langle \triangle \mid \triangledown \rangle \in \mathbb{F}^{X\times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

1.	$\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle \ge 0$	$\forall \boldsymbol{x}\in X$	(non-negative)	and
2.	$\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x}\in X$	(nondegenerate)	and
3.	$\langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y}\in X,\ \forall\alpha\in\mathbb{C}$	(homogeneous)	and
4.	$\langle \boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{u}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{u}\rangle + \langle \boldsymbol{y} \mid \boldsymbol{u}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X$	(additive)	and
5.	$\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle = \langle \boldsymbol{y} \mid \boldsymbol{x}\rangle^*$	$\forall \boldsymbol{x},\boldsymbol{y}\in X$	(conjugate symmetric).	

An inner product is also called a ***scalar product****.*
An ***inner product space*** *is the pair* $(\boldsymbol{\Omega}, \langle \triangle \mid \triangledown \rangle)$.

Theorem C.12. [28] *Let* $\mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM

$$\begin{array}{rclclrcll} \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle & = & 0 & \forall \boldsymbol{x}\in X & \iff & \mathbf{B}\boldsymbol{x} & = & \mathbb{0} & \forall \boldsymbol{x}\in X \\ \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle & = & \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle & \forall \boldsymbol{x}\in X & \iff & \mathbf{A} & = & \mathbf{B} & \end{array}$$

✎PROOF:

[26] Michel and Herget (1993) page 415
[27] Haaser and Sullivan (1991) page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72
[28] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

1. Proof that $\langle \mathbf{B}x \,|\, x\rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x+\mathbf{B}x) \,|\, (x+\mathbf{B}x)\rangle + i\,\langle \mathbf{B}(x+i\mathbf{B}x) \,|\, (x+i\mathbf{B}x)\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}x+\mathbf{B}^2x) \,|\, x+\mathbf{B}x\right\rangle\right\} + i\left\{\left\langle \mathbf{B}x+i\mathbf{B}^2x) \,|\, x+i\mathbf{B}x\right\rangle\right\} && \text{by Definition C.4 page 113}\\
&= \left\{\langle \mathbf{B}x \,|\, x\rangle + \langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \,|\, x\right\rangle + \left\langle \mathbf{B}^2x \,|\, \mathbf{B}x\right\rangle\right\} && \text{by Definition C.9 page 124}\\
&\quad + i\left\{\langle \mathbf{B}x \,|\, x\rangle - i\,\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \,|\, x\right\rangle - i^2\left\langle \mathbf{B}^2x \,|\, \mathbf{B}x\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \,|\, x\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \,|\, x\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \,|\, x\right\rangle\right\} + \left\{\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle - \left\langle \mathbf{B}^2x \,|\, x\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}x \,|\, \mathbf{B}x\rangle\\
&= 2\,\|\mathbf{B}x\|^2\\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition C.5 page 116}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \,|\, x\rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products.

3. Proof that $\langle \mathbf{A}x \,|\, x\rangle = \langle \mathbf{B}x \,|\, x\rangle \implies \mathbf{A} \doteq \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \,|\, x\rangle - \langle \mathbf{B}x \,|\, x\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}x - \mathbf{B}x \,|\, x\rangle && \text{by } \textit{additivity} \text{ property of } \langle \triangle \,|\, \triangledown\rangle \text{ (Definition C.9 page 124)}\\
&= \langle (\mathbf{A}-\mathbf{B})x \,|\, x\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A}-\mathbf{B})x &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \,|\, x\rangle = \langle \mathbf{B}x \,|\, x\rangle \impliedby \mathbf{A} \doteq \mathbf{B}$:

$$\langle \mathbf{A}x \,|\, x\rangle = \langle \mathbf{B}x \,|\, x\rangle \qquad \text{by } \mathbf{A} \doteq \mathbf{B} \text{ hypothesis}$$

✏

C.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition C.3 page 125). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem C.13 page 126).
- Both support decomposition into "real" and "imaginary" parts (Theorem E.3 page 148).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *null space* of an operator (Theorem C.14 page 127).

Proposition C.3. [29] *Let* $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be the space of* BOUNDED LINEAR OPERATORS *(Definition C.7 page 120) on a* HILBERT SPACE $\boldsymbol{H}$.

PRP: *An operator* $\mathbf{B}^*$ *is the* ***adjoint*** *of* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *if*

$$\langle \mathbf{B}x \,|\, y\rangle = \langle x \,|\, \mathbf{B}^* y\rangle \qquad \forall x, y \in \boldsymbol{H}.$$

✎PROOF:

[29] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000) page 182, von Neumann (1929) page 49, Stone (1932) page 41

1. For fixed y, $\mathsf{f}(\boldsymbol{x}) \triangleq \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle$ is a *functional* in $\mathbb{F}^X$.
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}
\langle \mathbf{B}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle &\triangleq \mathsf{f}(\mathbf{B}\boldsymbol{x}) \\
&\triangleq \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{operator adjoint} && \text{(Definition C.8 page 121)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{B}^*\boldsymbol{y} \rangle
\end{aligned}$$

✏

Example C.2.

EX

In matrix algebra ("linear algebra")

- The inner product operation $\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle$ is represented by $\boldsymbol{y}^H\boldsymbol{x}$.
- The linear operator is represented as a matrix $\boldsymbol{A}$.
- The operation of $\boldsymbol{A}$ on a vector $\boldsymbol{x}$ is represented as $\boldsymbol{A}\boldsymbol{x}$.
- The adjoint of matrix $\boldsymbol{A}$ is the Hermitian matrix $\boldsymbol{A}^H$.

✎PROOF:

$$\langle \boldsymbol{A}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H\boldsymbol{A}\boldsymbol{x} = [(\boldsymbol{A}\boldsymbol{x})^H\boldsymbol{y}]^H = [\boldsymbol{x}^H\boldsymbol{A}^H\boldsymbol{y}]^H = (\boldsymbol{A}^H\boldsymbol{y})^H\boldsymbol{x} \triangleq \left\langle \boldsymbol{x} \,|\, \boldsymbol{A}^H\boldsymbol{y} \right\rangle$$

✏

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition E.3 page 146). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[30]

Theorem C.13 (operator star-algebra). [31] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM

The pair $(\boldsymbol{H}, *)$ *is a* $*$-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \dotplus \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(INVOLUTARY)	

✎PROOF:

$$\begin{aligned}
\left\langle \boldsymbol{x} \,|\, (\mathbf{A} \dotplus \mathbf{B})^*\boldsymbol{y} \right\rangle &= \left\langle (\mathbf{A} \dotplus \mathbf{B})\boldsymbol{x} \,|\, \boldsymbol{y} \right\rangle && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)} \\
&= \langle \mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle + \langle \mathbf{B}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition C.9 page 124)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A}^*\boldsymbol{y} \rangle + \langle \boldsymbol{x} \,|\, \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of operator addition} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A}^*\boldsymbol{y} + \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition C.9 page 124)} \\
&= \langle \boldsymbol{x} \,|\, (\mathbf{A}^* + \mathbf{B}^*)\boldsymbol{y} \rangle && \text{by definition of operator addition}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, (\alpha\mathbf{A})^*\boldsymbol{y} \rangle &= \langle (\alpha\mathbf{A})\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)} \\
&= \langle \alpha(\mathbf{A}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle && \text{by definition of scalar multiplication} \\
&= \alpha \langle \mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition C.9 page 124)} \\
&= \alpha \langle \boldsymbol{x} \,|\, \mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)} \\
&= \langle \boldsymbol{x} \,|\, \alpha^*\mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition C.9 page 124)}
\end{aligned}$$

[30] Sakai (1998) page 1
[31] Halmos (1998a) pages 39–40, Rudin (1991) page 311

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, (\mathbf{AB})^* \boldsymbol{y} \rangle &= \langle (\mathbf{AB})\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)} \\
&= \langle \mathbf{A}(\mathbf{B}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle && \text{by definition of operator multiplication} \\
&= \langle (\mathbf{B}\boldsymbol{x}) \,|\, \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{B}^* \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{A}^{**} \boldsymbol{y} \rangle &= \langle \mathbf{A}^* \boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)} \\
&= \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle^* && \text{by definition of inner product} && \text{(Definition C.9 page 124)} \\
&= \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* && \text{by definition of adjoint} && \text{(Proposition C.3 page 125)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A} \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition C.9 page 124)}
\end{aligned}$$

⇨

Theorem C.14. [32] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator* $\mathbf{L}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *and* $\mathcal{I}(\mathbf{L})$ *the* IMAGE SET *of* $\mathbf{L}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$.

THM
$$\begin{aligned}
\mathcal{N}(\mathbf{A}) &= \mathcal{I}(\mathbf{A}^*)^{\perp} \\
\mathcal{N}(\mathbf{A}^*) &= \mathcal{I}(\mathbf{A})^{\perp}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathcal{I}(\mathbf{A}^*)^{\perp} &= \left\{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}^*) \right\} \\
&= \left\{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} \\
&= \{ y \in \boldsymbol{H} | \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathbf{A}^* && \text{(Proposition C.3 page 125)} \\
&= \{ y \in \boldsymbol{H} | \mathbf{A} \boldsymbol{y} = 0 \} \\
&= \mathcal{N}(\mathbf{A}) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\mathbf{A})^{\perp} &= \{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}) \} \\
&= \{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \mathbf{A} \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathcal{I} \\
&= \left\{ y \in \boldsymbol{H} | \langle \mathbf{A}^* \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition C.3 page 125)} \\
&= \left\{ y \in \boldsymbol{H} | \mathbf{A}^* \boldsymbol{y} = 0 \right\} \\
&= \mathcal{N}(\mathbf{A}^*) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

⇨

C.4 Special Classes of Operators

C.4.1 Projection operators

Definition C.10. [33] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF
$\mathbf{P}$ *is a* ***projection*** *operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[32] Rudin (1991) page 312

[33] Rudin (1991) page 133 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 6, Halmos (1958) page 73 ⟨§41. Projections⟩

Theorem C.15. [34] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ with* NULL SPACE $\mathcal{N}(\mathbf{P})$ *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

$$\left.\begin{array}{llcll} 1. & \mathbf{P}^2 & = & \mathbf{P} & (\mathbf{P} \textit{ is a projection operator}) \quad \textit{and} \\ 2. & \boldsymbol{\Omega} & = & \boldsymbol{X} \dotplus \boldsymbol{Y} & (\boldsymbol{Y} \textit{ compliments } \boldsymbol{X} \textit{ in } \boldsymbol{\Omega}) \quad \textit{and} \\ 3. & \mathbf{P}\boldsymbol{\Omega} & = & \boldsymbol{X} & (\mathbf{P} \textit{ projects onto } \boldsymbol{X}) \end{array}\right\} \implies \left\{\begin{array}{llcll} 1. & \mathcal{I}(\mathbf{P}) & = & \boldsymbol{X} & \textit{and} \\ 2. & \mathcal{N}(\mathbf{P}) & = & \boldsymbol{Y} & \textit{and} \\ 3. & \boldsymbol{\Omega} & = & \mathcal{I}(\mathbf{P}) \dotplus \mathcal{N}(\mathbf{P}) & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\ &= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\ &= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\ &= \boldsymbol{\Omega}_1 \end{aligned}$$

$$\begin{aligned} \mathcal{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\} \\ &= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_2 \end{aligned}$$

✏

Theorem C.16. [35] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

THM

$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P} \textit{ is a projection operator}} \iff \underbrace{(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})}_{(\mathbf{I} - \mathbf{P}) \textit{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} (\mathbf{I} - \mathbf{P})^2 &= (\mathbf{I} - \mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I}(\mathbf{I} - \mathbf{P}) + (-\mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2 \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} && \text{by left hypothesis} \\ &= \mathbf{I} - \mathbf{P} \end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} \mathbf{P}^2 &= \underbrace{\mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P}^2}_{(\mathbf{I} - \mathbf{P})^2} - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P})^2 - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P}) - (\mathbf{I} - \mathbf{P} - \mathbf{P}) && \text{by right hypothesis} \\ &= \mathbf{P} \end{aligned}$$

✏

[34] Michel and Herget (1993) pages 120–121
[35] Michel and Herget (1993) page 121

Theorem C.17. [36] *Let $\boldsymbol{H}$ be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathcal{N}(\mathbf{P})$, *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

$$\begin{array}{llcll}
1. & \mathbf{P}^* & = & \mathbf{P} & (\mathbf{P} \text{ is SELF-ADJOINT}) \iff \\
2. & \mathbf{P}^*\mathbf{P} & = & \mathbf{P}\mathbf{P}^* & (\mathbf{P} \text{ is NORMAL}) \iff \\
3. & \mathcal{I}(\mathbf{P}) & = & \mathcal{N}(\mathbf{P})^\perp & \iff \\
4. & \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle & = & \|\mathbf{P}\boldsymbol{x}\|^2 \quad \forall x \in X &
\end{array}$$

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\implies$ (2):

$$\begin{aligned}
\mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)} \\
&= \mathbf{P}\mathbf{P}^* && \text{by Theorem C.13 page 126}
\end{aligned}$$

Proof that (1) $\implies$ (3):

$$\begin{aligned}
\mathcal{I}(\mathbf{P}) &= \mathcal{N}(\mathbf{P}^*)^\perp && \text{by Theorem C.14 page 127} \\
&= \mathcal{N}(\mathbf{P})^\perp && \text{by (1)}
\end{aligned}$$

Proof that (3) $\implies$ (4):

Proof that (4) $\implies$ (1):

✏

C.4.2 Self Adjoint Operators

Definition C.11. [37] *Let* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be* ***self-adjoint*** *or* ***hermitian*** *if* $\mathbf{B} \stackrel{\circ}{=} \mathbf{B}^*$.

Example C.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}.$$

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \displaystyle\int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}}\mathsf{f}(u)\,\mathsf{d}u$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

Theorem C.18. [38] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.

THM

$$\left\{\underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S}\text{ is self adjoint}}\right\} \implies \left\{\begin{array}{lll}
1. & \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle \in \mathbb{R} & (\text{the hermitian quadratic form of } \mathbf{S} \text{ is REAL-VALUED}) \\
2. & \lambda_n \in \mathbb{R} & (\text{eigenvalues of } \mathbf{S} \text{ are REAL-VALUED}) \\
3. & \lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0 & (\text{eigenvectors are ORTHOGONAL})
\end{array}\right\}$$

---

[36] Rudin (1991) page 314

[37] Historical works regarding self-adjoint operators: von Neumann (1929) page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932) page 50 ⟨"self-adjoint transformations"⟩

[38] Lax (2002) pages 315–316, Keener (1988) pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩, Bertero and Boccacci (1998) page 225 ⟨§"9.2 SVD of a matrix ...If all eigenvectors are normalized..."⟩

✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \in \mathbb{R}$:

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition C.9 page 124}
\end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned}
\lambda_n \left\|\psi_n\right\|^2 &= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition} \\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition C.9 page 124} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_n \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \lambda_n^* \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition C.9 page 124} \\
&= \lambda_n^* \left\|\psi_n\right\|^2 && \text{by definition}
\end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition C.9 page 124} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_m \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \lambda_m^* \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition C.9 page 124} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

C.4.3 Normal Operators

Definition C.12. [39] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{N}^*$ be the adjoint of an operator $\mathbf{N} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF: $\mathbf{N}$ *is* ***normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

Theorem C.19. [40] *Let $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$.*

THM: $$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \text{ is normal}} \iff \left\|\mathbf{N}^*\boldsymbol{x}\right\| = \left\|\mathbf{N}\boldsymbol{x}\right\| \qquad \forall \boldsymbol{x} \in \boldsymbol{H}$$

[39] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969) page 167, Frobenius (1878), Frobenius (1968) page 391

[40] Rudin (1991) pages 312–313

✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition C.3 page 125 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (}\mathbf{N}\text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition C.3 page 125 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition C.3 page 125 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem C.13 page 126 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis } (\|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by Proposition C.3 page 125 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem C.20. [41] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ *and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same null space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \left\{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{(definition of } \mathcal{N})\\
&= \left\{\boldsymbol{x} |\, \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition C.5 page 116)}\\
&= \left\{\boldsymbol{x} |\, \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \\
&= \left\{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x}\in\boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition C.5 page 116)}\\
&= \mathcal{N}(\mathbf{N}) && \text{(definition of } \mathcal{N})
\end{aligned}$$

✏

Theorem C.21. [42] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ *and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m\rangle = 0}_{\text{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. [Rudin] claims both to be true.(Rudin, 1991)313

[41] Rudin (1991) pages 312–313
[42] Rudin (1991) pages 312–313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\quad\; = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem C.13 page 126} \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem C.13 page 126} \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\Longrightarrow \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\iff \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n\psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition C.9 page 124} \\
&= \langle \mathbf{N}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{N}^*\psi_m \rangle && \text{by Proposition C.3 page 125 (definition of adjoint)} \\
&= \langle \psi_n \,|\, \lambda_m^*\psi_m \rangle && \text{by (4.)} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition C.9 page 124}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

⇰

C.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition C.13. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES *(Definition C.5 page 116).*

DEF *An operator* $\mathbf{M} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is* ***isometric*** *if*
$$\|\mathbf{M}x\| = \|x\| \qquad \forall x \in X.$$

Theorem C.22. [43] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM
$$\underbrace{\|\mathbf{M}x\| = \|x\| \quad \forall x \in X}_{\text{isometric in length}} \iff \underbrace{\|\mathbf{M}x - \mathbf{M}y\| = \|x - y\| \quad \forall x, y \in X}_{\text{isometric in distance}}$$

✎PROOF:

1. Proof that $\|\mathbf{M}x\| = \|x\| \implies \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$:

$$\begin{aligned}
\|\mathbf{M}x - \mathbf{M}y\| &= \|\mathbf{M}(x - y)\| && \text{by definition of linear operators (Definition C.4 page 113)} \\
&= \|\mathbf{M}u\| && \text{let } u \triangleq x - y \\
&= \|x - y\| && \text{by left hypothesis}
\end{aligned}$$

[43] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

2. Proof that $\|\mathbf{M}x\| = \|x\| \impliedby \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$:

$$\begin{aligned}
\|\mathbf{M}x\| &= \|\mathbf{M}(x - \mathbb{0})\| \\
&= \|\mathbf{M}x - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition C.4 page 113)} \\
&= \|x - \mathbb{0}\| && \text{by right hypothesis} \\
&= \|x\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition C.13 page 132) in the more general normed linear spaces, while Theorem C.22 (page 132) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}x\| = \|x\| \iff \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$ for all $x, y \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem C.23 (next) demonstrates two additional equivalent properties.

Theorem C.23. [44] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$.*

THM *The following conditions are all **equivalent**:*

1.	$\mathbf{M}^*\mathbf{M}$	$=$	$\mathbf{I}$			$\iff$
2.	$\langle \mathbf{M}x \mid \mathbf{M}y \rangle$	$=$	$\langle x \mid y \rangle$	$\forall x, y \in X$	($\mathbf{M}$ *is surjective*)	$\iff$
3.	$\|\mathbf{M}x - \mathbf{M}y\|$	$=$	$\|x - y\|$	$\forall x, y \in X$	(*isometric in distance*)	$\iff$
4.	$\|\mathbf{M}x\|$	$=$	$\|x\|$	$\forall x \in X$	(*isometric in length*)	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}x \mid \mathbf{M}y \rangle &= \langle x \mid \mathbf{M}^*\mathbf{M}y \rangle && \text{by Proposition C.3 page 125 (definition of adjoint)} \\
&= \langle x \mid \mathbf{I}y \rangle && \text{by (1)} \\
&= \langle x \mid y \rangle && \text{by Definition C.3 page 112 (definition of } \mathbf{I}\text{)}
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}x\| &= \sqrt{\langle \mathbf{M}x \mid \mathbf{M}x \rangle} && \text{by definition of } \|\cdot\| \\
&= \sqrt{\langle x \mid x \rangle} && \text{by right hypothesis} \\
&= \|x\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}x \mid \mathbf{M}y \rangle &= \|\mathbf{M}x + \mathbf{M}y\|^2 - \|\mathbf{M}x - \mathbf{M}y\|^2 + i\,\|\mathbf{M}x + i\mathbf{M}y\|^2 - i\,\|\mathbf{M}x - i\mathbf{M}y\|^2 && \text{by polarization id.} \\
&= \|\mathbf{M}(x + y)\|^2 - \|\mathbf{M}(x - y)\|^2 + i\,\|\mathbf{M}(x + iy)\|^2 - i\,\|\mathbf{M}(x - iy)\|^2 && \text{by Definition C.4} \\
&= \|x + y\|^2 - \|x - y\|^2 + i\,\|x + iy\|^2 - i\,\|x - iy\|^2 && \text{by left hypothesis}
\end{aligned}$$

4. Proof that (3) $\iff$ (4): by Theorem C.22 page 132

[44] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

5. Proof that (4) $\implies$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition C.3 page 125 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem C.13 page 126 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y = 0\\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Definition C.3 page 112 (definition of } \mathbf{I})\\
\implies \quad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}$$

✎

Theorem C.24. [45] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M} \text{ is isometric}} \implies \left\{\begin{array}{rcll} |||\mathbf{M}||| &=& 1 & \text{(UNIT LENGTH)} \quad \text{and}\\ |\lambda| &=& 1 & \forall \lambda \in \Lambda \end{array}\right.$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \implies |||\mathbf{M}||| = 1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{\boldsymbol{x}\in X}\{\|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Definition C.6 page 117}\\
&= \sup_{\boldsymbol{x}\in X}\{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Theorem C.23 page 133}\\
&= \sup_{\boldsymbol{x}\in X}\{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x}, \lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|}\ \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem C.23 page 133}\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|}\ |\lambda|\ \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

✎

Example C.4 (One sided shift operator). [46] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ $(0, 1, 2, \ldots)$ and shift operators defined as

1. $\mathbf{S}_r\,(x_0, x_1, x_2, \ldots) \triangleq (0, x_0, x_1, x_2, \ldots)$ (right shift operator)
2. $\mathbf{S}_l\,(x_0, x_1, x_2, \ldots) \triangleq (x_1, x_2, x_3, \ldots)$ (left shift operator)

EX

1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

[45] Michel and Herget (1993) page 432
[46] Michel and Herget (1993) page 441

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r (\!(x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_0, y_1, y_2, \ldots)\!) \rangle &= \langle (\!(0, x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_0, y_1, y_2, \ldots)\!) \rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\, y_n^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \langle (\!(x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_1, y_2, y_3, \ldots)\!) \rangle \\
&= \left\langle (\!(x_0, x_1, x_2, \ldots)\!) \,|\, \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} (\!(y_0, y_1, y_2, \ldots)\!) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✏

C.4.5 Unitary operators

Definition C.14. [47] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF: *The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition C.4. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP: $\left.\begin{array}{l}\mathbf{U} \text{ is UNITARY and} \\ \mathbf{V} \text{ is UNITARY}\end{array}\right\} \implies (\mathbf{UV}) \text{ is UNITARY.}$

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\left(\mathbf{V}^*\mathbf{U}^*\right) && \text{by Theorem C.8 page 121} \\
&= \mathbf{U}\left(\mathbf{VV}^*\right)\mathbf{U}^* && \text{by associative property} \\
&= \mathbf{UIU}^* && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition C.14 page 135)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition C.14 page 135)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \left(\mathbf{V}^*\mathbf{U}^*\right)(\mathbf{UV}) && \text{by Theorem C.8 page 121} \\
&= \mathbf{V}^*\left(\mathbf{U}^*\mathbf{U}\right)\mathbf{V} && \text{by associative property} \\
&= \mathbf{V}^*\mathbf{IV} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition C.14 page 135)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition C.14 page 135)}
\end{aligned}$$

[47] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

Theorem C.25. [48] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{I}(\mathbf{U})$ be the* IMAGE SET *of* $\mathbf{U}$.

THM: *If $\mathbf{U}$ is a **bounded linear operator** ($\mathbf{U}\in\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$), then the following conditions are **equivalent**:*

1. $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ (UNITARY) $\iff$
2. $\langle \mathbf{U}x \,|\, \mathbf{U}y\rangle = \langle \mathbf{U}^*x \,|\, \mathbf{U}^*y\rangle = \langle x \,|\, y\rangle$ *and* $\mathcal{I}(\mathbf{U}) = X$ (SURJECTIVE) $\iff$
3. $\|\mathbf{U}x - \mathbf{U}y\| = \|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x-y\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN DISTANCE) $\iff$
4. $\|\mathbf{U}x\| = \|x\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN LENGTH)

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) $\langle \mathbf{U}x \,|\, \mathbf{U}y\rangle = \langle \mathbf{U}^*x \,|\, \mathbf{U}^*y\rangle = \langle x \,|\, y\rangle$ by Theorem C.23 (page 133).
   (b) Proof that $\mathcal{I}(\mathbf{U}) = X$:

$$\begin{aligned}
X &\supseteq \mathcal{I}(\mathbf{U}) && \text{because } \mathbf{U}\in X^X\\
&\supseteq \mathcal{I}(\mathbf{U}\mathbf{U}^*)\\
&= \mathcal{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I})\\
&= X && \text{by Definition C.3 page 112 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\iff$ (3) $\iff$ (4): by Theorem C.23 page 133.
3. Proof that (3) $\implies$ (1):
   (a) Proof that $\|\mathbf{U}x - \mathbf{U}y\| = \|x-y\| \implies \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem C.23 page 133
   (b) Proof that $\|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x-y\| \implies \mathbf{U}\mathbf{U}^* = \mathbf{I}$:

$$\begin{aligned}
\|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x-y\| \implies\quad & \mathbf{U}^{**}\mathbf{U}^* = \mathbf{I} && \text{by Theorem C.23 page 133}\\
& \mathbf{U}\mathbf{U}^* = \mathbf{I} && \text{by Theorem C.13 page 126}
\end{aligned}$$

Theorem C.26. *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathcal{N}(\mathbf{U})$ the* NULL SPACE *of* $\mathbf{U}$, *and $\mathcal{I}(\mathbf{U})$ the* IMAGE SET *of* $\mathbf{U}$.

THM:

$$\underbrace{\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U}\text{ is unitary}} \implies \left\{\begin{array}{lllll}
\mathbf{U}^{-1} &=& \mathbf{U}^* & & \textit{and}\\
\mathcal{I}(\mathbf{U}) &=& \mathcal{I}(\mathbf{U}^*) &= X & \textit{and}\\
\mathcal{N}(\mathbf{U}) &=& \mathcal{N}(\mathbf{U}^*) &= \{\mathbb{0}\} & \textit{and}\\
|||\mathbf{U}||| &=& |||\mathbf{U}^*||| &= 1 & \text{(UNIT LENGTH)}
\end{array}\right\}$$

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both *isometric* and *normal*:

$$\begin{array}{lllll}
\mathbf{U}^*\mathbf{U} & & &= \mathbf{I} & \implies \mathbf{U}\text{ is } \textit{isometric}\\
\mathbf{U}\mathbf{U}^* &=& \mathbf{U}^*\mathbf{U} &= \mathbf{I} & \implies \mathbf{U}^*\text{ is } \textit{isometric}\\
\mathbf{U}^{-1} &=& \mathbf{U}^* & & \implies \mathbf{U}^{-1}\text{ is } \textit{isometric}\\
\\
\mathbf{U}^*\mathbf{U} &=& \mathbf{U}\mathbf{U}^* &= & \implies \mathbf{U}\text{ is } \textit{normal}\\
\mathbf{U}\mathbf{U}^* &=& \mathbf{U}^*\mathbf{U} &= & \implies \mathbf{U}^*\text{ is } \textit{normal}\\
\mathbf{U}^{-1} &=& \mathbf{U}^* & & \implies \mathbf{U}^{-1}\text{ is } \textit{normal}
\end{array}$$

[48] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem C.25 page 136.

3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}^{-1})$:

$$\begin{aligned}
\mathcal{N}(\mathbf{U}^*) &= \mathcal{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both } \textit{normal} \text{ and by Theorem C.21 page 131}\\
&= \mathcal{I}(\mathbf{U})^\perp && \text{by Theorem C.14 page 127}\\
&= X^\perp && \text{by above result}\\
&= \{\mathbb{0}\}
\end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem C.24 page 134.

✏

Example C.5 (Rotation matrix). [49]

EX
$$\underbrace{\left\{ \mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}}_{\text{rotation matrix } \mathbf{R}_\theta \,:\, \mathbb{R}^2 \to \mathbb{R}^2} \implies \left\{ \begin{array}{llcll} (1). & \mathbf{R}^{-1}{}_\theta & = & \mathbf{R}_{-\theta} & \text{and} \\ (2). & \mathbf{R}^*{}_\theta & = & \mathbf{R}^{-1}{}_\theta & (\mathbf{R} \text{ is unitary}) \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H\\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R}\\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of } \textit{Hermetian transpose} \text{ operator } H\\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{by Theorem F.2 page 155}\\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R}\\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✏

Example C.6. [50] Let $\mathbf{A}$ and $\mathbf{B}$ be matrix operators.

EX
$$\mathbf{A} \triangleq \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} \triangleq \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\underbrace{\mathbf{A} \text{ is a } \textit{rotation operator}. \quad \mathbf{B} \text{ is a } \textit{reflection operator}.}_{\text{Both } \mathbf{A} \text{ and } \mathbf{B} \text{ are } \textbf{unitary}.}$

Example C.7. Examples of *Fredholm integral operators* include

EX

1.	**Fourier Transform**	$[\tilde{\mathbf{F}}\mathsf{x}](f) = \int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-i2\pi ft}\,\mathrm{d}t$	$\kappa(t,f) = e^{-i2\pi ft}$
2.	**Inverse Fourier Transform**	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t) = \int_{f\in\mathbb{R}} \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathrm{d}f$	$\kappa(f,t) = e^{i2\pi ft}$
3.	**Laplace operator**	$[\mathbf{L}\mathsf{x}](s) = \int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t,s) = e^{-st}$

Example C.8 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathsf{f}(x) \triangleq \mathsf{f}(x-1) \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

[49] Noble and Daniel (1988) page 311
[50] Gel'fand (1963) page 4, Gelfand et al. (2018) page 4

EX

1. $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ (inverse translation operator)
2. $\mathbf{T}^* = \mathbf{T}^{-1}$ ($\mathbf{T}$ is invertible)
3. $\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^* = \mathbf{I}$ ($\mathbf{T}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$
$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned}
\langle \mathbf{T}\mathsf{f}(x) \mid \mathsf{g}(x)\rangle &= \langle \mathsf{f}(x-1) \mid \mathsf{g}(x)\rangle && \text{by definition of } \mathbf{T}\\
&= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathrm{d}x\\
&= \langle \mathsf{f}(x) \mid \mathsf{g}(x+1)\rangle\\
&= \left\langle \mathsf{f}(x) \mid \underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example C.9 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX

1. $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ (inverse dilation operator)
2. $\mathbf{D}^* = \mathbf{D}^{-1}$ ($\mathbf{D}$ is invertible)
3. $\mathbf{D}^*\mathbf{D} = \mathbf{D}\mathbf{D}^* = \mathbf{I}$ ($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x) \mid \mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x) \mid \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\,\mathrm{d}u\\
&= \left\langle \mathsf{f}(x) \mid \frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x) \mid \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

Example C.10 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq (\!(x_{n-1})\!)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq \left(\!\!\left(x_{n+1}\right)\!\!\right)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\,(\!(x_n)\!) \,|\, (\!(y_n)\!)\rangle &= \langle (\!(x_{n-1})\!) \,|\, (\!(y_n)\!)\rangle && \text{by definition of } \mathbf{D}\\
&= \sum_n \boldsymbol{x}_{n-1}\; \boldsymbol{y}_n^*\\
&= \sum_n \boldsymbol{x}_n\; \boldsymbol{y}_{n+1}^*\\
&= \left\langle (\!(x_n)\!) \,|\, \left(\!\!\left(y_{n+1}\right)\!\!\right)\right\rangle\\
&= \left\langle (\!(x_n)\!) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\,(\!(y_n)\!)\right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary. ✏

Example C.11 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator (Theorem M.1 page 234)

$$[\tilde{\mathbf{F}}\mathsf{x}](f) \triangleq \int_t \mathsf{x}(t)\underbrace{e^{-i2\pi ft}}_{\kappa(t,f)}\,\mathrm{d}t \qquad\qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](t) \triangleq \int_f \tilde{\mathsf{x}}(f)\underbrace{e^{i2\pi ft}}_{\kappa^*(t,f)}\,\mathrm{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{x} \,|\, \tilde{\mathsf{y}}\right\rangle &= \left\langle \int_t \mathsf{x}(t)e^{-i2\pi ft}\,\mathrm{d}t \,|\, \tilde{\mathsf{y}}(f)\right\rangle\\
&= \int_t \mathsf{x}(t)\left\langle e^{-i2\pi ft} \,|\, \tilde{\mathsf{y}}(f)\right\rangle\,\mathrm{d}t\\
&= \int_t \mathsf{x}(t)\int_f e^{-i2\pi ft}\tilde{\mathsf{y}}^*(f)\,\mathrm{d}f\,\mathrm{d}t\\
&= \int_t \mathsf{x}(t)\left[\int_f e^{i2\pi ft}\tilde{\mathsf{y}}(f)\,\mathrm{d}f\right]^*\,\mathrm{d}t\\
&= \left\langle \mathsf{x}(t) \,|\, \int_f \tilde{\mathsf{y}}(f)e^{i2\pi ft}\,\mathrm{d}f\right\rangle\\
&= \left\langle \mathsf{x} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathsf{y}}\right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✏

C.5 Operator order

Definition C.15. [51] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF

$\mathbf{P}$ *is* ***positive*** *if* $\langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \;\forall \boldsymbol{x} \in \boldsymbol{X}$.
This condition is denoted $\mathbf{P} \geq 0$.

Theorem C.27. [52]

THM

$$\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \implies \left\{ \begin{array}{lcllr} (\mathbf{P}+\mathbf{Q}) & \geq & 0 & & ((\mathbf{P}+\mathbf{Q}) \text{ is positive}) \\ \mathbf{A}^*\mathbf{P}\mathbf{A} & \geq & 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive}) \\ \mathbf{A}^*\mathbf{A} & \geq & 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{A} \text{ is positive}) \end{array} \right.$$

✎PROOF:

$$\begin{aligned}
\langle (\mathbf{P}+\mathbf{Q})\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle + \langle \mathbf{Q}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by additive property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition C.9 page 124)} \\
&\geq \langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \mathbf{A}\boldsymbol{x} \rangle && \text{by definition of adjoint (Proposition C.3 page 125)} \\
&= \langle \mathbf{P}\boldsymbol{y} \,|\, \boldsymbol{y} \rangle && \text{where } \boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{I}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition C.3 page 112)} \\
&\geq 0 && \text{by non-negative property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition C.9 page 124)} \\
&\implies \mathbf{I} \text{ is positive} \\
\langle \mathbf{A}^*\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition C.3 page 112)} \\
&\geq 0 && \text{by two previous results}
\end{aligned}$$

✏

Definition C.16. [53] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be* BOUNDED *operators.*

DEF

$\mathbf{A} \geq \mathbf{B}$ *("*$\mathbf{A}$ *is greater than or equal to* $\mathbf{B}$*") if*
$\mathbf{A} - \mathbf{B} \geq 0$ *("(*$\mathbf{A} - \mathbf{B}$*) is positive")*

[51] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩
[52] Michel and Herget (1993) page 429
[53] Michel and Herget (1993) page 429

APPENDIX D

CALCULUS

Definition D.1. *Let* $\mathbb{R}$ *be the set of real numbers,* $\mathscr{B}$ *the set of* BOREL SETS *on* $\mathbb{R}$*, and* μ *the standard* BOREL MEASURE *on* $\mathscr{B}$*. Let* $\mathbb{R}^{\mathbb{R}}$ *be as in Definition 3.1 page 39.*

DEF

The ***space of Lebesgue square-integrable functions*** $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *(or* $\boldsymbol{L}^2_{\mathbb{R}}$*) is defined as*

$$\boldsymbol{L}^2_{\mathbb{R}} \triangleq \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \, d\mu < \infty \right\}.$$

The ***standard inner product*** $\langle \triangle \mid \nabla \rangle$ *on* $\boldsymbol{L}^2_{\mathbb{R}}$ *is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, dx.$$

The ***standard norm*** $\|\cdot\|$ *on* $\boldsymbol{L}^2_{\mathbb{R}}$ *is defined as* $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$

Definition D.2. *Let* $\mathsf{f}(x)$ *be a* FUNCTION *in* $\mathbb{R}^{\mathbb{R}}$.

DEF

$$\frac{d}{dx}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon \to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition D.1.

PRP

$$\left\{ \begin{array}{lll} (1). & \mathsf{f}(x) \text{ is CONTINUOUS} & \text{and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC about a point } a} & \end{array} \right\} \implies \left\{ \begin{array}{lrcll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) & \text{(ANTI-SYMMETRIC about } a\text{)} \\ (2). & \mathsf{f}'(a) & = & 0 & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned} \mathsf{f}'(a+x) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\ &= -\mathsf{f}(a-x) \end{aligned}$$

$$\begin{aligned} \mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\ &= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result} \end{aligned}$$

$$= 0$$

✎

Lemma D.1.

LEM $\mathsf{f}(x)$ *is* INVERTIBLE $\implies \left\{ \frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) = \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to0}\frac{\mathsf{f}^{-1}(y+\varepsilon)-\mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}y} && \text{(Definition D.2 page 141)}\\
&= \lim_{\delta\to0}\left.\frac{1}{\left[\frac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1}\\
&\triangleq \left.\frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x)}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition D.2 page 141)}\\
&= \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✎

Theorem D.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f*.*

THM $$\int_{[0:1)^n}\mathsf{f}^{(n)}\left(\sum_{k=1}^{n}x_k\right)\mathsf{d}x_1\,\mathsf{d}x_2\cdots\,\mathsf{d}x_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n\in\mathbb{N}$$

✎PROOF: Proof by induction:

1. Base case …proof for $n=1$ case:

$$\begin{aligned}
\int_{[0:1)}\mathsf{f}^{(1)}(x)\,\mathsf{d}x &= \mathsf{f}(1)-\mathsf{f}(0) && \text{by } \textit{Fundamental theorem of calculus}\\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1) + (-1)^{1+0}\binom{1}{0}\mathsf{f}(0)\\
&= \sum_{k=0}^{1}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

2. Induction step …proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_{n+1}\\
&= \int_{[0:1)^n}\left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1}+\sum_{k=1}^{n} x_k\right)\mathrm{d}x_{n+1}\right]\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n\\
&= \int_{[0:1)^n}\left[\mathsf{f}^{(n)}\left(x_{n+1}+\sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right]\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n && \text{by \textit{Fundamental theorem of calculus}}\\
&= \int_{[0:1)^n}\left[\mathsf{f}^{(n)}\left(1+\sum_{k=1}^{n} x_k\right)-\mathsf{f}^{(n)}\left(0+\sum_{k=1}^{n} x_k\right)\right]\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n\\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1)-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by induction hypothesis}\\
&= \sum_{m=1}^{m=n+1}(-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m)+\sum_{k=0}^{n}(-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{where } m \triangleq k+1 \implies k=m-1\\
&= \left[\mathsf{f}(n+1)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right]+\left[(-1)^{n+1}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right]\\
&= \mathsf{f}(n+1)+(-1)^{n+1}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1}+\binom{n}{k}\right]}_{\text{use \textit{Stifel formula}}}\mathsf{f}(k)\\
&= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1)+(-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by \textit{Stifel formula}}\\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)
\end{aligned}$$

✏

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma D.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots,$ *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example D.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\mathsf{f}(x)\mathsf{g}(x)\right] = \mathsf{f}'''(x)\mathsf{g}(x)+3\mathsf{f}''(x)\mathsf{g}'(x)+3\mathsf{f}'(x)\mathsf{g}''(x)+\mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem D.2 (Leibniz integration rule). [3]

THM
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathsf{a}(x)}^{\mathsf{b}(x)} \mathsf{g}(t)\,\mathrm{d}t = \mathsf{g}[\mathsf{b}(x)]\mathsf{b}'(x)-\mathsf{g}[\mathsf{a}(x)]\mathsf{a}'(x)$$

---

[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)

[3] Flanders (1973) page 615 ⟨(1.1)⟩ Talvila (2001), Knapp (2005b) page 389 ⟨Chapter VII⟩, ? page 422 ⟨Leibniz Rule. Theorem 1.⟩, http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

APPENDIX E

NORMED ALGEBRAS

E.1 Algebras

All *linear spaces* are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition E.1. [3] *Let* $\boldsymbol{A}$ *be an* ALGEBRA.

DEF *An algebra* $\boldsymbol{A}$ *is* ***unital*** *if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition E.2. [4] *Let* $\boldsymbol{A}$ *be an* UNITAL ALGEBRA *(Definition E.1 page 145) with unit* e.

DEF

$$\begin{array}{llcll}
\textit{The } \textbf{\textit{spectrum}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \sigma(x) & \triangleq & \left\{\lambda \in \mathbb{C} | \lambda e - x \textit{ is not invertible}\right\}. & \\
\textit{The } \textbf{\textit{resolvent}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \rho_x(\lambda) & \triangleq & (\lambda e - x)^{-1} & \forall \lambda \notin \sigma(x). \\
\textit{The } \textbf{\textit{spectral radius}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & r(x) & \triangleq & \sup\{|\lambda| | \lambda \in \sigma(x)\}. &
\end{array}$$

[1] Fuchs (1995) page 2
[2] Hazewinkel (2000) page v
[3] Folland (1995) page 1
[4] Folland (1995) pages 3–4

E.2 Star-Algebras

Definition E.3. [5] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF

*The pair $(\boldsymbol{A}, *)$ is a **$*$-algebra**, or **star-algebra**, if*

1.	$(x+y)^* = x^* + y^*$	$\forall x,y \in \boldsymbol{A}$	(DISTRIBUTIVE)	and
2.	$(\alpha x)^* = \bar{\alpha} x^*$	$\forall x \in \boldsymbol{A}, \alpha \in \mathbb{C}$	(CONJUGATE LINEAR)	and
3.	$(xy)^* = y^* x^*$	$\forall x,y \in \boldsymbol{A}$	(ANTIAUTOMORPHIC)	and
4.	$x^{**} = x$	$\forall x \in \boldsymbol{A}$	(INVOLUTORY)	

*The operator $*$ is called an **involution** on the algebra $\boldsymbol{A}$.*

Proposition E.1. [6] *Let $(\boldsymbol{A}, *)$ be an* UNITAL $*$-ALGEBRA.

PRP

x is invertible $\implies$ $\begin{cases} 1. \quad x^* \text{ is INVERTIBLE} & \forall x \in \boldsymbol{A} \quad \text{and} \\ 2. \quad (x^*)^{-1} = (x^{-1})^* & \forall x \in \boldsymbol{A} \end{cases}$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= (x\,e^*)^{**} && \text{by *involutory* property of } * && \text{(Definition E.3 page 146)}\\
&= (x^*\,e^{**})^* && \text{by *antiautomorphic* property of } * && \text{(Definition E.3 page 146)}\\
&= (x^*\,e)^* && \text{by *involutory* property of } * && \text{(Definition E.3 page 146)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by *involutory* property of } * && \text{(Definition E.3 page 146)}\\
e^*\,x &= (e^*\,x)^{**} && \text{by *involutory* property of } * && \text{(Definition E.3 page 146)}\\
&= (e^{**}\,x^*)^* && \text{by *antiautomorphic* property of } * && \text{(Definition E.3 page 146)}\\
&= (e\,x^*)^* && \text{by *involutory* property of } * && \text{(Definition E.3 page 146)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by *involutory* property of } * && \text{(Definition E.3 page 146)}
\end{aligned}$$

2. Proof that $(x^*)^{-1} = (x^{-1})^*$:

$$\begin{aligned}
(x^{-1})^*\,(x^*) &= \left[x\,(x^{-1})\right]^* && \text{by *antiautomorphic* and *involution* properties of } * && \text{(Definition E.3 page 146)}\\
&= e^*\\
&= e && \text{by item (1) page 146}\\
(x^*)\,(x^{-1})^* &= \left[x^{-1}\,x\right]^* && \text{by *antiautomorphic* and *involution* properties of } * && \text{(Definition E.3 page 146)}\\
&= e^*\\
&= e && \text{by item (1) page 146}
\end{aligned}$$

✎

Definition E.4. [7] *Let $(\boldsymbol{A}, \|\cdot\|)$ be a* $*$-ALGEBRA *(Definition E.3 page 146).*

DEF

- *An element $x \in \boldsymbol{A}$ is **hermitian** or **self-adjoint** if $x^* = x$.*
- *An element $x \in \boldsymbol{A}$ is **normal** if $xx^* = x^*x$.*
- *An element $x \in \boldsymbol{A}$ is a **projection** if $xx = x$ (INVOLUTORY) and $x^* = x$ (HERMITIAN).*

[5] Rickart (1960) page 178, Gelfand and Naimark (1964), page 241
[6] Folland (1995) page 5
[7] Rickart (1960) page 178, Gelfand and Naimark (1964), page 242

Theorem E.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition E.3 page 146)*.

THM

$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are HERMITIAN}} \implies \left\{\begin{array}{lll} x+y &=& (x+y)^* \quad (x+y \text{ is self adjoint}) \\ x^* &=& (x^*)^* \quad (x^* \text{ is self adjoint}) \\ \underbrace{xy = (xy)^*}_{(xy) \text{ is HERMITIAN}} & \iff & \underbrace{xy = yx}_{\text{commutative}} \end{array}\right.$$

✎PROOF:

$$\begin{aligned} (x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition E.3 page 146)} \\ &= x + y && \text{by left hypothesis} \end{aligned}$$

$$\left(x^*\right)^* = x \qquad \text{by } \textit{involutory} \text{ property of } * \qquad \text{(Definition E.3 page 146)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned} xy &= (xy)^* && \text{by left hypothesis} \\ &= y^* x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition E.3 page 146)} \\ &= yx && \text{by left hypothesis} \end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned} (xy)^* &= (yx)^* && \text{by left hypothesis} \\ &= x^* y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition E.3 page 146)} \\ &= xy && \text{by left hypothesis} \end{aligned}$$

✏

Definition E.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition E.3 page 146)*.

DEF

The ***real part*** *of* x *is defined as* $\mathbf{R}_{\mathrm{e}} x \triangleq \frac{1}{2}\left(x + x^*\right)$

The ***imaginary part*** *of* x *is defined as* $\mathbf{I}_{\mathrm{m}} x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem E.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition E.3 page 146)*.

THM

$$\begin{array}{llll} \mathbf{R}_{\mathrm{e}} x = \left(\mathbf{R}_{\mathrm{e}} x\right)^* & \forall x \in \boldsymbol{A} & (\mathbf{R}_{\mathrm{e}} x \text{ is HERMITIAN}) \\ \mathbf{I}_{\mathrm{m}} x = \left(\mathbf{I}_{\mathrm{m}} x\right)^* & \forall x \in \boldsymbol{A} & (\mathbf{I}_{\mathrm{m}} x \text{ is HERMITIAN}) \end{array}$$

✎PROOF:

$$\begin{aligned} \left(\mathbf{R}_{\mathrm{e}} x\right)^* &= \left(\frac{1}{2}(x + x^*)\right)^* && \text{by definition of } \mathfrak{R} && \text{(Definition E.5 page 147)} \\ &= \frac{1}{2}(x^* + x^{**}) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition E.3 page 146)} \\ &= \frac{1}{2}(x^* + x) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 146)} \\ &= \mathbf{R}_{\mathrm{e}} x && \text{by definition of } \mathfrak{R} && \text{(Definition E.5 page 147)} \\ \left(\mathbf{I}_{\mathrm{m}} x\right)^* &= \left(\frac{1}{2i}(x - x^*)\right)^* && \text{by definition of } \mathfrak{I} && \text{(Definition E.5 page 147)} \end{aligned}$$

[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998a) page 42

$$= \frac{1}{2i}\left(x^* - x^{**}\right)$$ by *distributive* property of $*$ (Definition E.3 page 146)

$$= \frac{1}{2i}\left(x^* - x\right)$$ by *involutory* property of $*$ (Definition E.3 page 146)

$$= \mathbf{I}_{\mathrm{m}} x$$ by definition of $\mathfrak{I}$ (Definition E.5 page 147)

✎

Theorem E.3 (Hermitian representation). [11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition E.3 page 146).*

THM

$$a = x + iy \iff x = \mathbf{R}_{\mathrm{e}} a \quad \textit{and} \quad y = \mathbf{I}_{\mathrm{m}} a$$

✎PROOF:

Proof that $a = x + iy \implies x = \mathbf{R}_{\mathrm{e}} a$ and $y = \mathbf{I}_{\mathrm{m}} a$:

$a = x + iy$	by left hypothesis	
$\implies a^* = (x+iy)^*$	by definition of *adjoint*	(Definition E.4 page 146)
$= x^* - iy^*$	by *distributive* property of $*$	(Definition E.3 page 146)
$= x - iy$	by Theorem E.2 page 147	
$\implies x = a - iy$	by solving for x in $a = x + iy$ equation	
$x = a^* + iy$	by solving for x in $a^* = x - iy$ equation	
$\implies x + x = a + a^*$	by adding previous 2 equations	
$\implies 2x = a + a^*$	by solving for x in previous equation	
$\implies x = \frac{1}{2}\left(a + a^*\right)$		
$= \mathbf{R}_{\mathrm{e}} a$	by definition of $\mathfrak{R}$	(Definition E.5 page 147)
$iy = a - x$	by solving for iy in $a = x + iy$ equation	
$iy = -a^* + x$	by solving for iy in $a = x + iy$ equation	
$\implies iy + iy = a - a^*$	by adding previous 2 equations	
$\implies y = \frac{1}{2i}\left(a - a^*\right)$	by solving for iy in previous equations	
$= \mathbf{I}_{\mathrm{m}} a$	by definition of $\mathfrak{I}$	(Definition E.5 page 147)

Proof that $a = x + iy \impliedby x = \mathbf{R}_{\mathrm{e}} a$ and $y = \mathbf{I}_{\mathrm{m}} a$:

$$x + iy = \mathbf{R}_{\mathrm{e}} a + i\,\mathbf{I}_{\mathrm{m}} a$$ by right hypothesis

$$= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\mathbf{R}_{\mathrm{e}} a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\mathbf{I}_{\mathrm{m}} a}$$ by definition of $\mathfrak{R}$ and $\mathfrak{I}$ (Definition E.5 page 147)

$$= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \cancelto{0}{\left(\frac{1}{2}a^* - \frac{1}{2}a^*\right)}$$

$$= a$$

✎

[11] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Neumark (1943b) page 7

E.3 Normed Algebras

Definition E.6. [12] *Let $\boldsymbol{A}$ be an algebra.*

DEF

*The pair $(\boldsymbol{A}, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in \boldsymbol{A} \qquad \text{(multiplicative condition)}$$

*A normed algebra $(\boldsymbol{A}, \|\cdot\|)$ is a **Banach algebra** if $(\boldsymbol{A}, \|\cdot\|)$ is also a Banach space.*

Proposition E.2.

PRP

$(\boldsymbol{A}, \|\cdot\|)$ *is a normed algebra* $\implies$ *multiplication is **continuous** in* $(\boldsymbol{A}, \|\cdot\|)$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x-y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(\boldsymbol{A}, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 149)}\\
&= \|z(x-y)\| \\
&\le \|z\|\,\|x-y\| && \text{by definition of \textit{normed algebra}} && \text{(Definition E.6 page 149)}\\
&\triangleq \|z\|\;\delta && \text{by definition of } \delta && \text{(item (2) page 149)}\\
&\le \epsilon && \text{for some value of } \delta > 0
\end{aligned}$$

✏

Theorem E.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l}(\boldsymbol{A}, \|\cdot\|) \text{ is a Banach algebra}\\ \text{every nonzero } x \in \boldsymbol{A} \text{ is invertible}\end{array}\right\} \implies \boldsymbol{A} \equiv \mathbb{C} \quad (\boldsymbol{A} \text{ is isomorphic to } \mathbb{C})$$

E.4 C* Algebras

Definition E.7. [14]

DEF

*The triple $(\boldsymbol{A}, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(\boldsymbol{A}, \|\cdot\|)$ *is a Banach algebra* and
2. $(\boldsymbol{A}, *)$ *is a $*$-algebra* and
3. $\|x^*x\| = \|x\|^2 \quad \forall x \in \boldsymbol{A}$.

*A C^* **algebra** $(\boldsymbol{A}, \|\cdot\|, *)$ is also called a **C star algebra**.*

[12] Rickart (1960) page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩

[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩

[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem E.5. [15] *Let A be an algebra.*

THM: $(A, \|\cdot\|, *)$ *is a* C^* ***algebra*** $\implies$ $\|x^*\| = \|x\|$

✎PROOF:

$$\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\ \|x\|^2 \\
&= \frac{1}{\|x\|}\ \|x^* x\| && \text{by definition of } C^*\textit{-algebra} && \text{(Definition E.7 page 149)} \\
&\le \frac{1}{\|x\|}\ \|x^*\|\ \|x\| && \text{by definition of \textit{normed algebra}} && \text{(Definition E.6 page 149)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by \textit{involution} property of } * && \text{(Definition E.3 page 146)}
\end{aligned}$$

✏

[15] Folland (1995) page 1, Gelfand and Neumark (1943b) page 4, Gelfand and Neumark (1943a)

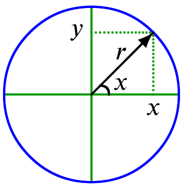
APPENDIX F

TRIGONOMETRIC FUNCTIONS

F.1 Definition Candidates

There are several ways of defining the sine and cosine functions, including the following:[1]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[2]



$$\begin{aligned}\cos x &\triangleq \frac{x}{r}\\ \sin x &\triangleq \frac{y}{r}\end{aligned}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[3]

$$\cos x \triangleq \mathbf{R}_\mathrm{e} e^{ix} \qquad \sin x \triangleq \mathbf{I}_\mathrm{m}\left(e^{ix}\right)$$

3. **Polynomial:** Let $\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of *Taylor expansions* such that[4]

$$\begin{aligned}\cos(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\ \sin(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

[1] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means "bay" or "inlet"—thus resulting in the Latin translation *sinus*, which also means "bay" or "inlet". Reference: Boyer and Merzbach (1991) page 252

[2] Abramowitz and Stegun (1972) page 78

[3] Euler (1748)

[4] Rosenlicht (1968) page 157, Abramowitz and Stegun (1972) page 74

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \prod_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of a product of factors such that[5]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1-\left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[6]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-(n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-(n\pi)^2}\right)}_{\cot(x)}\sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ such that

$$\begin{array}{llllll}
\cos(x) & \triangleq & \mathsf{f}(x) & \text{where} & \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}+\mathsf{f}=0}_{\text{differential equation}} & \underbrace{\mathsf{f}(0)=1}_{\text{1st initial condition}} & \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)=0}_{\text{2nd initial condition}}\\
\sin(x) & \triangleq & \mathsf{g}(x) & \text{where} & \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{g}+\mathsf{g}=0}_{\text{differential equation}} & \underbrace{\mathsf{g}(0)=0}_{\text{1st initial condition}} & \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{g}\right](0)=1}_{\text{2nd initial condition}}
\end{array}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[7]

$$\begin{array}{llllll}
\cos(x) & \triangleq & \mathsf{f}^{-1}(x) & \text{where} & \mathsf{f}(x) \triangleq & \underbrace{\int_x^1 \sqrt{\frac{1}{1-y^2}}\,\mathrm{d}y}_{\arccos(x)}\\
\sin(x) & \triangleq & \mathsf{g}^{-1}(x) & \text{where} & \mathsf{g}(x) \triangleq & \underbrace{\int_0^x \sqrt{\frac{1}{1-y^2}}\,\mathrm{d}y}_{\arcsin(x)}
\end{array}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ (Definition F.1 page 153). Support for such an approach includes the following:

- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ (Theorem F.1 page 154).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem F.3 page 156).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem F.4 page 157).

[5] Abramowitz and Stegun (1972) page 75
[6] Abramowitz and Stegun (1972) page 75
[7] Abramowitz and Stegun (1972) page 79

- The complex exponential function is a solution of a second order homogeneous differential equation (Definition F.4 page 158).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section F.6 page 166).

F.2 Definitions

Definition F.1. [8] *Let* $\boldsymbol{C}$ *be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and* $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ *the differentiation operator.*

DEF *The function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***cosine*** *function* $\cos(x) \triangleq \mathsf{f}(x)$ *if*

$$\begin{array}{llcll} 1. & \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} & = & 0 & \text{(second order homogeneous differential equation)} \quad \text{and} \\ 2. & \mathsf{f}(0) & = & 1 & \text{(first initial condition)} \quad \text{and} \\ 3. & \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) & = & 0 & \text{(second initial condition).} \end{array}$$

Definition F.2. [9] *Let* $\boldsymbol{C}$ *and* $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ *be defined as in definition of* $\cos(x)$ (Definition F.1 page 153).

DEF *The function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***sine*** *function* $\sin(x) \triangleq \mathsf{f}(x)$ *if*

$$\begin{array}{llcll} 1. & \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} & = & 0 & \text{(second order homogeneous differential equation)} \quad \text{and} \\ 2. & \mathsf{f}(0) & = & 0 & \text{(first initial condition)} \quad \text{and} \\ 3. & \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) & = & 1 & \text{(second initial condition).} \end{array}$$

Definition F.3. [10]

DEF *Let* π *("pi") be defined as the element in* $\mathbb{R}$ *such that*

$$\begin{array}{llcll} (1). & \cos\left(\frac{\pi}{2}\right) & = & 0 & \text{and} \\ (2). & \pi & > & 0 & \text{and} \end{array}$$

(3). π *is the* ***smallest*** *of all elements in* $\mathbb{R}$ *that satisfies (1) and (2).*

F.3 Basic properties

Lemma F.1. [11] *Let* $\boldsymbol{C}$ *be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and* $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ *the differentiation operator.*

LEM

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\begin{array}{rcl} \mathsf{f}(x) & = & \underbrace{[\mathsf{f}](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \\ & = & \left(\mathsf{f}(0) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)x\right) - \left(\frac{\mathsf{f}(0)}{2!}x^2 + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{3!}x^3\right) + \left(\frac{\mathsf{f}(0)}{4!}x^4 + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{5!}x^5\right)\cdots \end{array}\right\}$$

[8] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[10] Rosenlicht (1968) page 158
[11] Rosenlicht (1968) page 156, Liouville (1839)

✎PROOF: Let $\mathsf{f}'(x) \triangleq \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x)$.

$$\begin{aligned}
\mathsf{f}'''(x) &= -\left[\tfrac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](x) \\
\mathsf{f}^{(4)}(x) &= -\left[\tfrac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](x) && = -\left[\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) = \mathsf{f}(x)
\end{aligned}$$

1. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) + \mathsf{f}(x) = 0 \implies \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\mathsf{f}(x) &= \sum_{n=0}^{\infty}\frac{\mathsf{f}^{(n)}(0)}{n!}x^n && \text{by Taylor expansion} \\
&= \mathsf{f}(0) + \left[\tfrac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)x - \frac{\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](0)}{2!}x^2 - \frac{\mathsf{f}^3(0)}{3!}x^3 + \frac{\mathsf{f}^4(0)}{4!}x^4 + \frac{\mathsf{f}^5(0)}{5!}x^5 - \cdots \\
&= \mathsf{f}(0) + \left[\tfrac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)x - \frac{\mathsf{f}(0)}{2!}x^2 - \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{3!}x^3 + \frac{\mathsf{f}(0)}{4!}x^4 + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{5!}x^5 - \cdots \\
&= \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]
\end{aligned}$$

2. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) + \mathsf{f}(x) = 0 \impliedby \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\left[\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) &= \tfrac{\mathrm{d}}{\mathrm{d}x}\tfrac{\mathrm{d}}{\mathrm{d}x}[\mathsf{f}(x)] \\
&= \tfrac{\mathrm{d}}{\mathrm{d}x}\tfrac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right] && \text{by right hypothesis} \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)\mathsf{f}(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n-1}\right] \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n-1)!}x^{2n-1}\right] \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right] \\
&= -\mathsf{f}(x) && \text{by right hypothesis}
\end{aligned}$$

⇒

Theorem F.1 (Taylor series for cosine/sine). [12]

THM

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && \forall x\in\mathbb{R} \\
\sin(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \forall x\in\mathbb{R}
\end{aligned}$$

[12] Rosenlicht (1968) page 157

✎PROOF:

$$\begin{aligned}
\cos(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma F.1 page 153}\\
&= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by cos initial conditions (Definition F.1 page 153)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
\sin(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma F.1 page 153}\\
&= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by sin initial conditions (Definition F.2 page 153)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

✏

Theorem F.2. [13]

THM

$$\begin{array}{rcl|rcll}
\cos(0) &=& 1 & \cos(-x) &=& \cos(x) & \forall x\in\mathbb{R}\\
\sin(0) &=& 0 & \sin(-x) &=& -\sin(x) & \forall x\in\mathbb{R}
\end{array}$$

✎PROOF:

$$\begin{aligned}
\cos(0) &= 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\Big|_{x=0} && \text{by } \textit{Taylor series for cosine} && \text{(Theorem F.1 page 154)}\\
&= 1\\
\sin(0) &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\Big|_{x=0} && \text{by } \textit{Taylor series for sine} && \text{(Theorem F.1 page 154)}\\
&= 0\\
\cos(-x) &= 1-\frac{(-x)^2}{2}+\frac{(-x)^4}{4!}-\frac{(-x)^6}{6!}+\cdots && \text{by } \textit{Taylor series for cosine} && \text{(Theorem F.1 page 154)}\\
&= 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\\
&= \cos(x) && \text{by } \textit{Taylor series for cosine} && \text{(Theorem F.1 page 154)}\\
\sin(-x) &= (-x)-\frac{(-x)^3}{3!}+\frac{(-x)^5}{5!}-\frac{(-x)^7}{7!}+\cdots && \text{by } \textit{Taylor series for sine} && \text{(Theorem F.1 page 154)}\\
&= -\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right]\\
&= \sin(x) && \text{by } \textit{Taylor series for sine} && \text{(Theorem F.1 page 154)}
\end{aligned}$$

✏

Lemma F.2. [14]

LEM

$$\begin{array}{rcl|l}
\cos(1) &>& 0 & x\in(0:2) \implies \sin(x)>0\\
\cos(2) &<& 0 &
\end{array}$$

[13] Rosenlicht (1968) page 157
[14] Rosenlicht (1968) page 158

✎PROOF:

$$\begin{aligned}
\cos(1) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=1} && \text{by } \textit{Taylor series for cosine} && \text{(Theorem F.1 page 154)}\\
&= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots \\
&> 0 \\
\cos(2) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=2} && \text{by } \textit{Taylor series for cosine} && \text{(Theorem F.1 page 154)}\\
&= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots \\
&< 0
\end{aligned}$$

$$\begin{aligned}
x \in (0:2) &\implies \text{each term in the sequence } \left(\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots\right) \text{ is } > 0\\
&\implies \sin(x) > 0
\end{aligned}$$

✏

Proposition F.1. *Let π be defined as in Definition F.3 (page 153).*

PRP
(A). *The value π **exists** in $\mathbb{R}$.*
(B). $2 < \pi < 4$.

✎PROOF:

$$\begin{aligned}
&\cos(1) > 0 && \text{by Lemma F.2 page 155}\\
&\cos(2) < 0 && \text{by Lemma F.2 page 155}\\
&\implies 1 < \frac{\pi}{2} < 2\\
&\implies 2 < \pi < 4
\end{aligned}$$

✏

Theorem F.3. [15] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $\mathrm{f}'(0) \triangleq \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)$.*

THM
$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f} + \mathrm{f} = 0\right\} \iff \left\{\mathrm{f}(x) = \mathrm{f}(0)\cos(x) + \mathrm{f}'(0)\sin(x)\right\} \qquad \forall \mathrm{f} \in \boldsymbol{C}, \forall x \in \mathbb{R}$$

✎PROOF:

1. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}\right](x) = -\mathrm{f}(x) \implies \mathrm{f}(x) = \mathrm{f}(0)\cos(x) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)\sin(x)$:

$$\begin{aligned}
\mathrm{f}(x) &= \mathrm{f}(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by left hypothesis and Lemma F.1 page 153}\\
&= \mathrm{f}(0)\cos x + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)\sin x && \text{by definitions of cos and sin (Definition F.1 page 153, Definition F.2 page 153)}
\end{aligned}$$

---

[15] Rosenlicht (1968) page 157. The general solution for the *non-homogeneous* equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = \mathrm{g}(x)$ with initial conditions $\mathrm{f}(a) = 1$ and $\mathrm{f}'(a) = \rho$ is $\mathrm{f}(x) = \cos(x) + \rho\sin(x) + \int_a^x \mathrm{g}(y)\sin(x-y)\,\mathrm{d}y$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992) page 371, Liouville (1839). Volterra equation references: Pedersen (2000) page 99, Lalescu (1908), Lalescu (1911)

2. Proof that $\frac{d^2}{dx^2}f = -f \impliedby f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$\begin{aligned}
f(x) &= f(0)\cos x + \left[\tfrac{d}{dx}f\right](0)\sin x && \text{by right hypothesis}\\
&= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\tfrac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)}\\
\implies\quad & \frac{d^2}{dx^2}f + f = 0 && \text{by Lemma F.1 page 153}
\end{aligned}$$

Theorem F.4. [16] *Let* $\frac{d}{dx} \in \mathbf{C}^{\mathbf{C}}$ *be the differentiation operator.*

THM

$$\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x\in\mathbb{R} \qquad \frac{d}{dx}\sin(x) = \cos(x) \quad \forall x\in\mathbb{R} \qquad \cos^2(x) + \sin^2(x) = 1 \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} && \text{by *Taylor series* (Theorem F.1 page 154)}\\
&= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}\\
&= -\sin(x) && \text{by *Taylor series* (Theorem F.1 page 154)}\\
\frac{d}{dx}\sin(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by *Taylor series* (Theorem F.1 page 154)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= \cos(x) && \text{by *Taylor series* (Theorem F.1 page 154)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left[\cos^2(x) + \sin^2(x)\right] &= -2\cos(x)\sin(x) + 2\sin(x)\cos(x)\\
&= 0\\
&\implies \cos^2(x) + \sin^2(x) \text{ is } \textit{constant}\\
&\implies \cos^2(x) + \sin^2(x)\\
&= \cos^2(0) + \sin^2(0)\\
&= 1 + 0 = 1 && \text{by Theorem F.2 page 155}
\end{aligned}$$

Proposition F.2.

PRP

$$\sin\left(\frac{\pi}{2}\right) = 1$$

[16] Rosenlicht (1968) page 157

✎PROOF:

$$\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2) + 0} \\
&= \pm\sqrt{\sin^2(\pi/2) + \cos^2(\pi/2)} && \text{by definition of } \pi \quad \text{(Definition F.3 page 153)} \\
&= \pm\sqrt{1} && \text{by Theorem F.4 page 157} \\
&= \pm 1 \\
&= 1 && \text{by Lemma F.2 page 155}
\end{aligned}$$

⇒

F.4 The complex exponential

Definition F.4.

DEF *The function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq \mathsf{f}(x)$ *if*

1. $\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f} + \mathsf{f} = 0$ *(second order homogeneous differential equation)* *and*
2. $\mathsf{f}(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0) = i$ *(second initial condition).*

Theorem F.5 (Euler's identity). [17]

THM $$e^{ix} = \cos(x) + i\sin(x) \qquad \forall x \in \mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\exp(ix) &= \mathsf{f}(0)\cos(x) + \left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sin(x) && \text{by Theorem F.3 page 156} \\
&= \cos(x) + i\sin(x) && \text{by Definition F.4 page 158}
\end{aligned}$$

⇒

Proposition F.3.

PRP $$e^{-i\pi/2} = -i \quad \Big| \quad e^{i\pi/2} = i$$

✎PROOF:

$$\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by \textit{Euler's identity} (Theorem F.5 page 158)} \\
&= 0 + i && \text{by Theorem F.2 (page 155) and Proposition F.2 (page 157)} \\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by \textit{Euler's identity} (Theorem F.5 page 158)} \\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem F.2 page 155} \\
&= 0 - i && \text{by Theorem F.2 (page 155) and Proposition F.2 (page 157)}
\end{aligned}$$

⇒

Corollary F.1.

COR $$e^{ix} = \sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x \in \mathbb{R}$$

[17] Euler (1748), Bottazzini (1986) page 12

✎PROOF:

$$\begin{aligned}
\boxed{e^{ix}} &= \cos(x) + i\sin(x) && \text{by } \textit{Euler's identity} && \text{(Theorem F.5 page 158)}\\
&= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by } \textit{Taylor series} && \text{(Theorem F.1 page 154)}\\
&= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} && = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!}\\
&= \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} && = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}
\end{aligned}$$

✏

Corollary F.2 (Euler formulas). [18]

COR

$$\cos(x) = \mathbf{R_e}\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \sin(x) = \mathbf{I_m}\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{R_e}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}+\left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} && \text{by definition of } \Re && \text{(Definition E.5 page 147)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} && \text{by } \textit{Euler's identity} && \text{(Theorem F.5 page 158)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} && = \frac{\cos(x)}{2} + \frac{\cos(x)}{2} && = \boxed{\cos(x)}\\
\boxed{\mathbf{I_m}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}-\left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} && \text{by definition of } \Im && \text{(Definition E.5 page 147)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} && \text{by } \textit{Euler's identity} && \text{(Theorem F.5 page 158)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} && = \frac{i\sin(x)}{2i} + \frac{i\sin(x)}{2i} && = \boxed{\sin(x)}
\end{aligned}$$

✏

Theorem F.6. [19]

THM

$$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

✎PROOF:

$$\begin{aligned}
e^{\alpha}\, e^{\beta} &= \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) && \text{by Corollary F.1 page 158}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{n!\,\alpha^k}{n!\,k!} \frac{\beta^{n-k}}{(n-k)!}
\end{aligned}$$

[18] Euler (1748), Bottazzini (1986) page 12
[19] Rudin (1987) page 1

$$\begin{aligned}
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\,\alpha^k\,\beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}\alpha^k\,\beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the \textit{Binomial Theorem}} \\
&= e^{\alpha+\beta} && \text{by Corollary F.1 page 158}
\end{aligned}$$

✎

F.5 Trigonometric Identities

Theorem F.7 (shift identities).

THM								
	$\cos\left(x+\frac{\pi}{2}\right)$	$=$	$-\sin x$	$\forall x\in\mathbb{R}$	$\sin\left(x+\frac{\pi}{2}\right)$	$=$	$\cos x$	$\forall x\in\mathbb{R}$
	$\cos\left(x-\frac{\pi}{2}\right)$	$=$	$\sin x$	$\forall x\in\mathbb{R}$	$\sin\left(x-\frac{\pi}{2}\right)$	$=$	$-\cos x$	$\forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)}+e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)} \\
&= \frac{e^{ix}e^{i\frac{\pi}{2}}+e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta}=e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem F.6 page 159)} \\
&= \frac{e^{ix}(i)+e^{-ix}(-i)}{2} && \text{by Proposition F.3 page 158} \\
&= \frac{e^{ix}-e^{-ix}}{-2i} \\
&= -\sin x && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)} \\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)}+e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)} \\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}}+e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta}=e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem F.6 page 159)} \\
&= \frac{e^{ix}(-i)+e^{-ix}(i)}{2} && \text{by Proposition F.3 page 158} \\
&= \frac{e^{ix}-e^{-ix}}{2i} \\
&= \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)} \\
\sin\left(x+\frac{\pi}{2}\right) &= \cos\left(\left[x+\frac{\pi}{2}\right]-\frac{\pi}{2}\right) && \text{by previous result} \\
&= \cos(x) \\
\sin\left(x-\frac{\pi}{2}\right) &= -\cos\left(\left[x-\frac{\pi}{2}\right]+\frac{\pi}{2}\right) && \text{by previous result} \\
&= -\cos(x)
\end{aligned}$$

✎

Theorem F.8 (product identities).

THM

(A).	$\cos x \cos y$	$=$	$\frac{1}{2}\cos(x-y)$	$+$	$\frac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$
(B).	$\cos x \sin y$	$=$	$-\frac{1}{2}\sin(x-y)$	$+$	$\frac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
(C).	$\sin x \cos y$	$=$	$\frac{1}{2}\sin(x-y)$	$+$	$\frac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
(D).	$\sin x \sin y$	$=$	$\frac{1}{2}\cos(x-y)$	$-$	$\frac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary F.2 page 159)
(algebraic method requiring *complex number system* $\mathbb{C}$):

$$\begin{aligned}
\cos x \cos y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right) && \text{by } \textit{Euler formulas} \quad \text{(Corollary F.2 page 159)}\\
&= \frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(-x+y)}+e^{i(-x-y)}}{4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{4}+\frac{e^{i(x-y)}+e^{-i(x-y)}}{4}\\
&= \frac{2\cos(x+y)}{4}+\frac{2\cos(x-y)}{4} && \text{by } \textit{Euler formulas} \quad \text{(Corollary F.2 page 159)}\\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem F.3 page 156)
(differential equation method requiring only *real number system* $\mathbb{R}$):

$$\begin{aligned}
&\mathsf{f}(x) \triangleq \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)\\
&\implies \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) = -\tfrac{1}{2}\sin(x-y)-\tfrac{1}{2}\sin(x+y) && \text{by Theorem F.4 page 157}\\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x) = -\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y) && \text{by Theorem F.4 page 157}\\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x)+\mathsf{f}(x) = 0 && \text{by } \textit{additive inverse} \text{ property}\\
&\implies \underbrace{\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)}_{\mathsf{f}(x)} = \underbrace{[\tfrac{1}{2}\cos(0-y)+\tfrac{1}{2}\cos(0+y)]}_{\mathsf{f}''(0)}\cos(x)+\underbrace{[-\tfrac{1}{2}\sin(0-y)-\tfrac{1}{2}\sin(0+y)]}_{\mathsf{f}'(0)}\sin(x)\\
&\implies \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y) = \cos y\cos x+0\sin(x)\\
&\implies \cos x\cos y = \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)
\end{aligned}$$

3. Proof for (B) using *Euler formulas* (Corollary F.2 page 159):

$$\begin{aligned}
\sin x \sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary F.2 page 159}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary F.2 page 159}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x \sin y &= \cos(x)\cos\left(y - \frac{\pi}{2}\right) && \text{by } \textit{shift identities} && \text{(Theorem F.7 page 160)}\\
&= \frac{1}{2}\cos\left(x + y - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(x - y + \frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y) - \frac{1}{2}\sin(x-y) && \text{by } \textit{shift identities} && \text{(Theorem F.7 page 160)}\\
\sin x \cos y &= \cos y \sin x\\
&= \frac{1}{2}\sin(y+x) - \frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y) && \text{by Theorem F.2 page 155}
\end{aligned}$$

✏

Proposition F.4.

PRP	(A). $\cos(\pi) = -1$	(C). $\cos(2\pi) = 1$	(E). $e^{i\pi} = -1$
	(B). $\sin(\pi) = 0$	(D). $\sin(2\pi) = 0$	(F). $e^{i2\pi} = 0$

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1 + 1 + \cos(\pi)\\
&= -1 + 2[{}^1\!/\!_2\cos({}^\pi\!/\!_2 - {}^\pi\!/\!_2) + {}^1\!/\!_2\cos({}^\pi\!/\!_2 + {}^\pi\!/\!_2)] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem F.2 page 155)}\\
&= -1 + 2\cos({}^\pi\!/\!_2)\cos({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem F.8 page 160)}\\
&= -1 + 2(0)(0) && \text{by definition of } \pi && \text{(Definition F.3 page 153)}\\
&= -1\\
\sin(\pi) &= 0 + \sin(\pi)\\
&= 2[-{}^1\!/\!_2\sin({}^\pi\!/\!_2 - {}^\pi\!/\!_2) + {}^1\!/\!_2\sin({}^\pi\!/\!_2 + {}^\pi\!/\!_2)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem F.2 page 155)}\\
&= 2\cos({}^\pi\!/\!_2)\sin({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem F.8 page 160)}\\
&= 2(0)\sin({}^\pi\!/\!_2) && \text{by definition of } \pi && \text{(Definition F.3 page 153)}\\
&= 0\\
\cos(2\pi) &= 1 + \cos(2\pi) - 1\\
&= 2[{}^1\!/\!_2\cos(\pi - \pi) + {}^1\!/\!_2\cos(\pi + \pi)] - 1 && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem F.2 page 155)}\\
&= 2\cos(\pi)\cos(\pi) - 1 && \text{by } \textit{product identities} && \text{(Theorem F.8 page 160)}\\
&= 2(-1)(-1) - 1 && \text{by (A)}\\
&= 1\\
\sin(2\pi) &= 0 + \sin(2\pi)\\
&= 2[{}^1\!/\!_2\sin(\pi - \pi) + {}^1\!/\!_2\sin(\pi + \pi)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem F.2 page 155)}\\
&= 2\sin(\pi)\cos(\pi) && \text{by } \textit{product identities} && \text{(Theorem F.8 page 160)}\\
&= 2(0)(-1) && \text{by (A) and (B)}\\
&= 0\\
e^{i\pi} &= \cos(\pi) + i\sin(\pi) && \text{by } \textit{Euler's identity} && \text{(Theorem F.5 page 158)}\\
&= -1 + 0 && \text{by (A) and (B)}\\
&= -1\\
e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) && \text{by } \textit{Euler's identity} && \text{(Theorem F.5 page 158)}\\
&= 1 + 0 && \text{by (C) and (D)}\\
&= 1
\end{aligned}$$

✏

Theorem F.9 (double angle formulas). [20]

THM

(A).	$\cos(x+y)$	$=$	$\cos x \cos y - \sin x \sin y$	$\forall x,y \in \mathbb{R}$
(B).	$\sin(x+y)$	$=$	$\sin x \cos y + \cos x \sin y$	$\forall x,y \in \mathbb{R}$
(C).	$\tan(x+y)$	$=$	$\dfrac{\tan x + \tan y}{1 - \tan x \tan y}$	$\forall x,y \in \mathbb{R}$

✎PROOF:

1. Proof for (A) using *product identities* (Theorem F.8 page 160).

$$\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x+y)}_{\cos(x+y)} + \underbrace{\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x-y)}_{0} \\
&= \left[\frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\right] - \left[\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\right] \\
&= \cos x \cos y - \sin x \sin y && \text{by Theorem F.8 page 160}
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem F.3 page 156):

$$\begin{aligned}
\mathsf{f}(x) \triangleq \cos(x+y) &\implies \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) = -\sin(x+y) && \text{by Theorem F.4 page 157} \\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x) = -\cos(x+y) && \text{by Theorem F.4 page 157} \\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x) + \mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property} \\
&\implies \cos(x+y) = \cos y \cos x - \sin y \sin x && \text{by Theorem F.3 page 156} \\
&\implies \cos(x+y) = \cos x \cos y - \sin x \sin y && \text{by \textit{commutative} property}
\end{aligned}$$

3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x - \frac{\pi}{2} + y\right) && \text{by \textit{shift identities} (Theorem F.7 page 160)} \\
&= \cos\left(x - \frac{\pi}{2}\right)\cos(y) - \sin\left(x - \frac{\pi}{2}\right)\sin(y) && \text{by (A)} \\
&= \sin(x)\cos(y) + \cos(x)\sin(y) && \text{by \textit{shift identities} (Theorem F.7 page 160)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\
&= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} && \text{by (A)} \\
&= \left(\frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}\right)\left(\frac{\cos x \cos y}{\cos x \cos y}\right) \\
&= \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}
\end{aligned}$$

✏

Theorem F.10 (trigonometric periodicity).

THM

(A).	$\cos(x+M\pi)$	$=$	$(-1)^M\cos(x)$	$\forall x \in \mathbb{R},\ M \in \mathbb{Z}$		(D).	$\cos(x+2M\pi)$	$=$	$\cos(x)$	$\forall x \in \mathbb{R},\ M \in \mathbb{Z}$
(B).	$\sin(x+M\pi)$	$=$	$(-1)^M\sin(x)$	$\forall x \in \mathbb{R},\ M \in \mathbb{Z}$		(E).	$\sin(x+2M\pi)$	$=$	$\sin(x)$	$\forall x \in \mathbb{R},\ M \in \mathbb{Z}$
(C).	$e^{i(x+M\pi)}$	$=$	$(-1)^M e^{ix}$	$\forall x \in \mathbb{R},\ M \in \mathbb{Z}$		(F).	$e^{i(x+2M\pi)}$	$=$	e^{ix}	$\forall x \in \mathbb{R},\ M \in \mathbb{Z}$

[20] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

✎PROOF:

1. Proof for (A):

(a) $M = 0$ case: $\cos(x + 0\pi) \quad = \quad \cos(x) \quad = \quad (-1)^0 \cos(x)$

(b) Proof for $M > 0$ cases (by induction):

i. Base case $M = 1$:

$$\begin{aligned}
\cos(x + \pi) &= \cos x \cos \pi - \sin x \sin \pi && \text{by \textit{double angle formulas}} && \text{(Theorem F.9 page 163)}\\
&= \cos x(-1) - \sin x(0) && \text{by } \cos\pi = -1 \text{ result} && \text{(Proposition F.4 page 162)}\\
&= (-1)^1 \cos x
\end{aligned}$$

ii. Inductive step...Proof that M case $\implies$ $M + 1$ case:

$$\begin{aligned}
\cos(x + [M + 1]\pi) &= \cos([x + \pi] + M\pi)\\
&= (-1)^M \cos(x + \pi) && \text{by induction hypothesis } (M \text{ case})\\
&= (-1)^M (-1)\cos(x) && \text{by base case (item (1(b)i) page 164)}\\
&= (-1)^{M+1} \cos(x)\\
&\implies M + 1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\cos(x + M\pi) &\triangleq \cos(x - N\pi) && \text{by definition of } N\\
&= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem F.9 page 163)}\\
&= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem F.2 page 155}\\
&= \cos(x)\cos(0 + N\pi) + \sin(x)\sin(0 + N\pi)\\
&= \cos(x)(-1)^N \cos(0) + \sin(x)(-1)^N \sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (1b) page 164)}\\
&= (-1)^N \cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem F.2 page 155)}\\
&\triangleq (-1)^{-M} \cos(x) && \text{by definition of } N\\
&= (-1)^M \cos(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\cos(x + M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)}\\
&= e^{iM\pi}\left[\frac{e^{ix} + e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem F.6 page 159)}\\
&= \left(e^{i\pi}\right)^M \cos x && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)}\\
&= (-1)^M \cos x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition F.4 page 162)}
\end{aligned}$$

2. Proof for (B):

(a) $M = 0$ case: $\sin(x + 0\pi) \quad = \quad \sin(x) \quad = \quad (-1)^0 \sin(x)$

(b) Proof for $M > 0$ cases (by induction):

i. Base case $M = 1$:

$$\begin{aligned}
\sin(x + \pi) &= \sin x \cos \pi + \cos x \sin \pi && \text{by \textit{double angle formulas}} && \text{(Theorem F.9 page 163)}\\
&= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} && \text{(Proposition F.4 page 162)}\\
&= (-1)^1 \sin x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\sin(x+[M+1]\pi) &= \sin([x+\pi]+M\pi) \\
&= (-1)^M \sin(x+\pi) && \text{by induction hypothesis (M case)} \\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 164)} \\
&= (-1)^{M+1}\sin(x) \\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M<0$ cases: Let $N \triangleq -M \ldots \implies N>0$.

$$\begin{aligned}
\sin(x+M\pi) &\triangleq \sin(x-N\pi) && \text{by definition of } N \\
&= \sin(x)\sin(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem F.9 page 163)} \\
&= \sin(x)\sin(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem F.2 page 155} \\
&= \sin(x)\sin(0+N\pi) + \sin(x)\sin(0+N\pi) \\
&= \sin(x)(-1)^N\sin(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (2b) page 164)} \\
&= (-1)^N\sin(x) && \text{by } \sin(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem F.2 page 155)} \\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N \\
&= (-1)^M\sin(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)} - e^{-i(x+M\pi)}}{2i} && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)} \\
&= e^{iM\pi}\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem F.6 page 159)} \\
&= \left(e^{i\pi}\right)^M \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)} \\
&= (-1)^M \sin x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition F.4 page 162)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem F.6 page 159)} \\
&= \left(e^{i\pi}\right)^M\left(e^{ix}\right) \\
&= (-1)^M e^{ix} && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition F.4 page 162)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):

$$\begin{aligned}
\cos(i(x+2M\pi)) &= (-1)^{2M}\cos(ix) &&= \cos(ix) && \text{by (A)} \\
\sin(i(x+2M\pi)) &= (-1)^{2M}\sin(ix) &&= \sin(ix) && \text{by (B)} \\
e^{i(x+2M\pi)} &= (-1)^{2M}e^{ix} &&= e^{ix} && \text{by (C)}
\end{aligned}$$

✏

Theorem F.11 (half-angle formulas/squared identities).

THM

(A). $\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$

(B). $\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$

(C). $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x) + \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem F.8 page 160)} \\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem F.2 page 155)} \\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x) - \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem F.8 page 160)} \\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem F.2 page 155)} \\
\cos^2 x + \sin^2 x &= \tfrac{1}{2}[1+\cos(2x)] + \tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)} \\
& && \text{note: see also} && \text{Theorem F.4 page 157}
\end{aligned}$$

✏

F.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

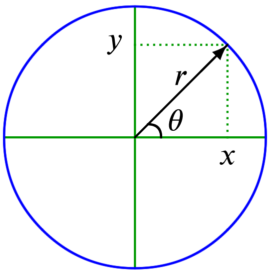
$$\begin{aligned}
\langle \cos(x) \,|\, \sin(x) \rangle &= \int_{-\pi}^{+\pi} \cos(x)\sin(x)\;\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x-x)\;\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x+x)\;\mathrm{d}x && \text{by Theorem F.8 page 160} \\
&= \cancelto{0}{\frac{1}{2}\int_{-\pi}^{+\pi} \sin(0)\;\mathrm{d}x} + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(2x)\;\mathrm{d}x \\
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi}\cos(2x) \\
&= -\frac{1}{4}[\cos(2\pi) - \cos(-2\pi)] \\
&= 0
\end{aligned}$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

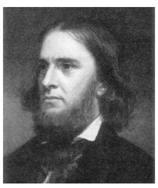
We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$



$$\begin{aligned}
\cos\theta &\triangleq \frac{x}{r} & \qquad \sec\theta &\triangleq \frac{r}{x} \\
\sin\theta &\triangleq \frac{y}{r} & \csc\theta &\triangleq \frac{r}{y} \\
\tan\theta &\triangleq \frac{y}{x} & \cot\theta &\triangleq \frac{x}{y}
\end{aligned}$$

F.7 The power of the exponential



“Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.”

Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [21]

[21] quote: Kasner and Newman (1940) page 104
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

"Young man, in mathematics you don't understand things. You just get used to them."
John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [22]

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary F.3. [23]

COR
$$e^{i\pi} + 1 = 0$$

✎PROOF:

$$\begin{aligned}
e^{ix}\Big|_{x=\pi} &= [\cos x + i \sin x]_{x=\pi} && \text{by Euler's identity (Theorem F.5 page 158)}\\
&= -1 + i \cdot 0 && \text{by Proposition F.4 page 162}\\
&= -1
\end{aligned}$$

✏

There are many transforms available, several of them integral transforms $[\mathbf{A}\mathsf{f}](s) \triangleq \int_t \mathsf{f}(s)\kappa(t,s)\,\mathrm{d}s$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:

① The *Laplace Transform* with kernel $\kappa(t,\ s) \triangleq e^{st}$
② The *Fourier Transform* with kernel $\kappa(t,\ \omega) \triangleq e^{i\omega t}$.

Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem F.12 page 167).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem F.12. [24] *Let* $\mathbf{L}$ *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*
$\check{\mathsf{h}}(s) \triangleq \left\langle \mathsf{h}(t,\omega) \,|\, e^{st} \right\rangle$ (LAPLACE TRANSFORM).

---

[22] quote: Zukav (1980) page 208
image: http://en.wikipedia.org/wiki/John_von_Neumann
The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."

[23] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

[24] Mallat (1999) page 2, ...page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

THM $\left\{\begin{array}{ll} 1. & \mathbf{L} \textit{ is } \text{LINEAR} \textit{ and} \\ 2. & \mathbf{L} \textit{ is } \text{TIME-INVARIANT} \end{array}\right\} \implies \left\{ \mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{\textit{eigenvalue}} \; \underbrace{e^{st}}_{\textit{eigenvector}} \right\}$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \,|\, \mathsf{h}((t;u),s)\rangle && \text{by linear hypothesis} \\
&= \langle e^{su} \,|\, \mathsf{h}((t-u),s)\rangle && \text{by time-invariance hypothesis} \\
&= \left\langle e^{s(t-v)} \,|\, \mathsf{h}(v,s)\right\rangle && \text{let } v = t-u \implies u = t-v \\
&= e^{st} \; \langle e^{-sv} \,|\, \mathsf{h}(v,s)\rangle && \text{by additivity of } \langle \triangle \,|\, \triangledown \rangle \\
&= \langle \mathsf{h}(v,s) \,|\, e^{-sv}\rangle^* \; e^{st} && \text{by conjugate symmetry of } \langle \triangle \,|\, \triangledown \rangle \\
&= \left\langle \mathsf{h}(v,s) \,|\, e^{(-s)v}\right\rangle^* \; e^{st} \\
&= \check{\mathsf{h}}^*(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}$$

✏

APPENDIX G

TRIGONOMETRIC POLYNOMIALS

❝*I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives.*❞
Charles Hermite (1822 – 1901), French mathematician, in an 1893 letter to Stieltjes, in response to the "pathological" everywhere continuous but nowhere differentiable *Weierstrass functions* $\mathsf{f}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$.[1]

G.1 Trigonometric expansion

Theorem G.1 (DeMoivre's Theorem)**.**

THM
$$\left(re^{ix}\right)^n = r^n(\cos nx + i\sin nx) \qquad \forall r,x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\left(re^{ix}\right)^n &= r^n e^{inx} \\
&= r^n\left(\cos nx + i\sin nx\right) && \text{by Euler's identity (Theorem F.5 page 158)}
\end{aligned}$$

(Theorem F.5 page 158)

✏

The cosine with argument nx can be expanded as a polynomial in $\cos(x)$ (next).

Theorem G.2 (trigonometric expansion)**.** [2]

---

[1] quote: Hermite (1893)
translation: Lakatos (1976) page 19
image: http://www-groups.dcs.sx-and.ac.uk/~history/PictDisplay/Hermite.html

[2] Rivlin (1974) page 3 ⟨(1.8)⟩

THM

$$\cos(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

$$\sin(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\sin x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(nx) &= \Re\left(\cos nx + i\sin nx\right) \\
&= \Re\left(e^{inx}\right) \\
&= \Re\left[\left(e^{ix}\right)^n\right] \\
&= \Re\left[(\cos x + i\sin x)^n\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n}\binom{n}{k}(\cos x)^{n-k}(i\sin x)^k\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n} i^k\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \Re\left[\sum_{k\in\{0,4,\dots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + i\sum_{k\in\{1,5,\dots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right. \\
&\qquad \left. - \sum_{k\in\{2,6,\dots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + -i\sum_{k\in\{3,7,\dots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \sum_{k\in\{0,4,\dots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x - \sum_{k\in\{2,6,\dots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x \\
&= \sum_{k\in\{0,2,\dots,n\}}\binom{n}{k}(-1)^{\frac{k}{2}}\cos^{n-k}x\sin^k x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\sin^{2k}x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x(1-\cos^2 x)^k \\
&= \left[\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\right]\left[\sum_{m=0}^{k}\binom{k}{m}(-1)^m\cos^{2m}x\right] \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}x
\end{aligned}$$

$$\begin{aligned}
\sin(nx) &= \cos\left(nx - \frac{\pi}{2}\right) \\
&= \cos\left(n\left[x - \frac{\pi}{2n}\right]\right) \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}\left(n\left[x - \frac{\pi}{2n}\right]\right)
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(nx - \frac{\pi}{2} \right)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \sin^{n-2(k-m)} (nx)$$

✏

Example G.1.

EX	$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ $\sin 5x = 16\sin^5 x - 20\sin^3 x + 5\sin x.$

✎PROOF:

1. Proof using *DeMoivre's Theorem* (Theorem G.1 page 169):

$$\begin{aligned}
&\cos 5x + i\sin 5x \\
&= e^{i5x} \\
&= \left(e^{ix}\right)^5 \\
&= (\cos x + i\sin x)^5 \\
&= \sum_{k=0}^{5} \binom{5}{k} [\cos x]^{5-k} [i\sin x]^k \\
&= \binom{5}{0}[\cos x]^{5-0}[i\sin x]^0 + \binom{5}{1}[\cos x]^{5-1}[i\sin x]^1 + \binom{5}{2}[\cos x]^{5-2}[i\sin x]^2 + \\
&\binom{5}{3}[\cos x]^{5-3}[i\sin x]^3 + \binom{5}{4}[\cos x]^{5-4}[i\sin x]^4 + \binom{5}{5}[\cos x]^{5-5}[i\sin x]^5 \\
&= 1\cos^5 x + i5\cos^4 x\sin x - 10\cos^3 x\sin^2 x - i10\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i1\sin^5 x \\
&= \left[\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\right] + i\left[5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)(1-\cos^2 x)\right] + \\
&i\left[5(1-\sin^2 x)(1-\sin^2 x)\sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)\right] + \\
&i\left[5(1 - 2\sin^2 x + \sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5(\cos x - 2\cos^3 x + \cos^5 x)\right] + \\
&i\left[5(\sin x - 2\sin^3 x + \sin^5 x) - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \underbrace{\left[16\cos^5 x - 20\cos^3 x + 5\cos x\right]}_{\cos 5x} + i\underbrace{\left[16\sin^5 x - 20\sin^3 x + 5\sin x\right]}_{\sin 5x}
\end{aligned}$$

2. Proof using trigonometric expansion (Theorem G.2 page 169):

$$\begin{aligned}
\cos 5x &= \sum_{k=0}^{\lfloor \frac{5}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{2} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{5-2(k-m)} \\
&= (-1)^0 \binom{5}{0}\binom{0}{0}\cos^5 x + (-1)^1 \binom{5}{2}\binom{1}{0}\cos^3 x + (-1)^2 \binom{5}{2}\binom{1}{1}\cos^5 x + \\
&(-1)^2 \binom{5}{4}\binom{2}{0}\cos^1 x + (-1)^3 \binom{5}{4}\binom{2}{1}\cos^3 x + (-1)^4 \binom{5}{4}\binom{2}{2}\cos^5 x
\end{aligned}$$

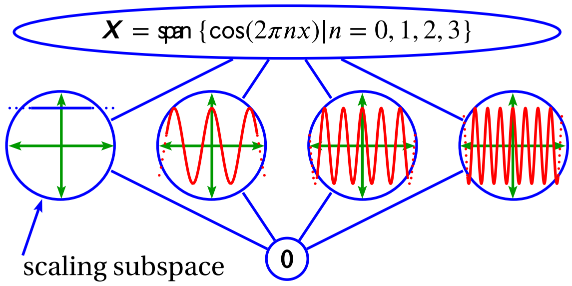


Figure G.1: Lattice of harmonic cosines $\{\cos(nx)|n = 0, 1, 2, \dots\}$

$$
\begin{aligned}
&= +(1)(1)\cos^5 x - (10)(1)\cos^3 x + (10)(1)\cos^5 x + (5)(1)\cos x - (5)(2)\cos^3 x + (5)(1)\cos^5 x \\
&= +(1 + 10 + 5)\cos^5 x + (-10 - 10)\cos^3 x + 5\cos x \\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}
$$

✏

Example G.2. [3]

EX	n	$\cos nx$		polynomial in $\cos x$	n	$\cos nx$		polynomial in $\cos x$
	0	$\cos 0x$	$=$	1	4	$\cos 4x$	$=$	$8\cos^4 x - 8\cos^2 x + 1$
	1	$\cos 1x$	$=$	$\cos^1 x$	5	$\cos 5x$	$=$	$16\cos^5 x - 20\cos^3 x + 5\cos x$
	2	$\cos 2x$	$=$	$2\cos^2 x - 1$	6	$\cos 6x$	$=$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
	3	$\cos 3x$	$=$	$4\cos^3 x - 3\cos x$	7	$\cos 7x$	$=$	$64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

✎PROOF:

$$
\begin{aligned}
\cos 2x &= \sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{2-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0}\cos^2 x + (-1)^1 \binom{3}{2}\binom{1}{0}\cos^0 x + (-1)^2 \binom{3}{2}\binom{1}{1}\cos^2 x \\
&= +(1)(1)\cos^2 x - (1)(1) + (1)(1)\cos^2 x \\
&= 2\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 3x &= \sum_{k=0}^{\lfloor \frac{3}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{3-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0}\cos^3 x + (-1)^1 \binom{3}{2}\binom{1}{0}\cos^1 x + (-1)^2 \binom{3}{2}\binom{1}{1}\cos^3 x \\
&= +\binom{3}{0}\binom{0}{0}\cos^3 x - \binom{3}{2}\binom{1}{0}\cos^1 x + \binom{3}{2}\binom{1}{1}\cos^3 x \\
&= +(1)(1)\cos^3 x - (3)(1)\cos^1 x + (3)(1)\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}
$$

$$
\cos 4x = \sum_{k=0}^{\lfloor \frac{4}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{4}{2k} \binom{k}{m} (\cos x)^{4-2(k-m)}
$$

[3] Abramowitz and Stegun (1972) page 795, Guillemin (1957) page 593 ⟨(21)⟩, Sloane (2014) ⟨http://oeis.org/A039991⟩, Sloane (2014) ⟨http://oeis.org/A028297⟩

$$\begin{aligned}
&= \sum_{k=0}^{2}\sum_{m=0}^{k}(-1)^{k+m}\binom{4}{2k}\binom{k}{m}(\cos x)^{4-2(k-m)} \\
&= (-1)^{0+0}\binom{4}{2\cdot 0}\binom{0}{0}(\cos x)^{4-2(0-0)} + (-1)^{1+0}\binom{4}{2\cdot 1}\binom{1}{0}(\cos x)^{4-2(1-0)} \\
&\quad + (-1)^{1+1}\binom{4}{2\cdot 1}\binom{1}{1}(\cos x)^{4-2(1-1)} + (-1)^{2+0}\binom{4}{2\cdot 2}\binom{2}{0}(\cos x)^{4-2(2-0)} \\
&\quad + (-1)^{2+1}\binom{4}{2\cdot 2}\binom{2}{1}(\cos x)^{4-2(2-1)} + (-1)^{2+2}\binom{4}{2\cdot 2}\binom{2}{2}(\cos x)^{4-2(2-2)} \\
&= (1)(1)\cos^4 x - (6)(1)\cos^2 x + (6)(1)\cos^4 x + (1)(1)\cos^0 x - (1)(2)\cos^2 x + (1)(1)\cos^4 x \\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}$$

$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ see Example G.1 page 171

$$\begin{aligned}
\cos 6x &= \sum_{k=0}^{\left\lfloor\frac{6}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{6}{2k}\binom{k}{m}(\cos x)^{6-2(k-m)} \\
&= (-1)^0\binom{6}{0}\binom{0}{0}\cos^6 x + (-1)^1\binom{6}{2}\binom{1}{0}\cos^4 x + (-1)^2\binom{6}{2}\binom{1}{1}\cos^6 x + (-1)^2\binom{6}{4}\binom{2}{0}\cos^2 x + \\
&(-1)^3\binom{6}{4}\binom{2}{1}\cos^4 x + (-1)^4\binom{6}{4}\binom{2}{2}\cos^6 x + (-1)^3\binom{6}{6}\binom{3}{0}\cos^0 x + (-1)^4\binom{6}{6}\binom{3}{1}\cos^2 x + \\
&(-1)^5\binom{6}{6}\binom{3}{2}\cos^4 x + (-1)^6\binom{6}{6}\binom{3}{3}\cos^6 x \\
&= +(1)(1)\cos^6 x - (15)(1)\cos^4 x + (15)(1)\cos^6 x + (15)(1)\cos^2 x - (15)(2)\cos^4 x + (15)(1)\cos^6 x \\
&\quad - (1)(1)\cos^0 x + (1)(3)\cos^2 x - (1)(3)\cos^4 x + (1)(1)\cos^6 x \\
&= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1
\end{aligned}$$

$$\begin{aligned}
\cos 7x &= \sum_{k=0}^{\left\lfloor\frac{7}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{3}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{7-2(k-m)} \\
&= (-1)^0\binom{7}{0}\binom{0}{0}\cos^7 x + (-1)^1\binom{7}{2}\binom{1}{0}\cos^5 x + (-1)^2\binom{7}{2}\binom{1}{1}\cos^7 x + (-1)^2\binom{7}{4}\binom{2}{0}\cos^3 x \\
&\quad + (-1)^3\binom{7}{4}\binom{2}{1}\cos^5 x + (-1)^4\binom{7}{4}\binom{2}{2}\cos^7 x + (-1)^3\binom{7}{6}\binom{3}{0}\cos^1 x + (-1)^4\binom{7}{6}\binom{3}{1}\cos^3 x \\
&\quad + (-1)^5\binom{7}{6}\binom{3}{2}\cos^5 x + (-1)^6\binom{7}{6}\binom{3}{3}\cos^7 x \\
&= (1)(1)\cos^7 x - (21)(1)\cos^5 x + (21)(1)\cos^7 x + (35)(1)\cos^3 x \\
&\quad - (35)(2)\cos^5 x + (35)(1)\cos^7 x - (7)(1)\cos^1 x + (7)(3)\cos^3 x \\
&\quad - (7)(3)\cos^5 x + (7)(1)\cos^7 x \\
&= (1 + 21 + 35 + 7)\cos^7 x - (21 + 70 + 21)\cos^5 x + (35 + 21)\cos^3 x - (7)\cos^1 x \\
&= 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x
\end{aligned}$$

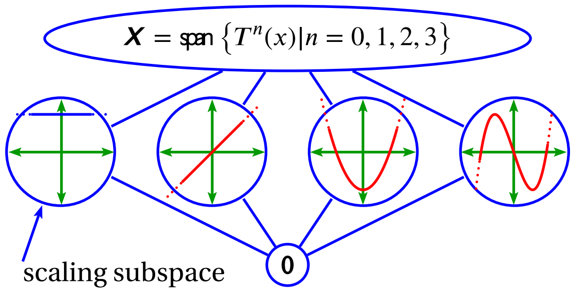


Figure G.2: Lattice of Chebyshev polynomials $\left\{T_n(x)|n = 0, 1, 2, 3\right\}$

Note: Trigonometric expansion of $\cos(nx)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[4]

```
trigexpand(cos(2*x));
trigexpand(cos(3*x));
trigexpand(cos(4*x));
trigexpand(cos(5*x));
trigexpand(cos(6*x));
trigexpand(cos(7*x));
```

✏

Definition G.1.

DEF *The nth **Chebyshev polynomial of the first kind** is defined as*

$$T_n(x) \triangleq \cos nx \qquad \textit{where} \qquad \cos x \triangleq x$$

Theorem G.3. [5] *Let $T_n(x)$ be a* CHEBYSHEV POLYNOMIAL *with $n \in \mathbb{W}$.*

THM

$$\begin{array}{lcl} n \text{ is EVEN} & \implies & T_n(x) \text{ is EVEN.} \\ n \text{ is ODD} & \implies & T_n(x) \text{ is ODD.} \end{array}$$

Example G.3. Let $T_n(x)$ be a *Chebyshev polynomial* with $n \in \mathbb{W}$.

EX

$T_0(x) = 1$	$T_4(x) = 8x^4 - 8x^2 + 1$
$T_1(x) = x$	$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_2(x) = 2x^2 - 1$	$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
$T_3(x) = 4x^3 - 3x$	

✎PROOF: Proof of these equations follows directly from Example G.2 (page 172). ✏

G.2 Trigonometric reduction

Theorem G.2 (page 169) showed that $\cos nx$ can be expressed as a polynomial in $\cos x$. Conversely, Theorem G.4 (next) shows that a polynomial in $\cos x$ can be expressed as a linear combination of $(\!(\cos nx)\!)_{n\in\mathbb{Z}}$.

Theorem G.4 (trigonometric reduction)**.**

[4] maxima pages 157–158 ⟨10.5 Trigonometric Functions⟩
[5] Rivlin (1974) page 5 ⟨(1.13)⟩, Süli and Mayers (2003) page 242 ⟨Lemma 8.2⟩, Davidson and Donsig (2010) page 222 ⟨exercise 10.7.A(a)⟩

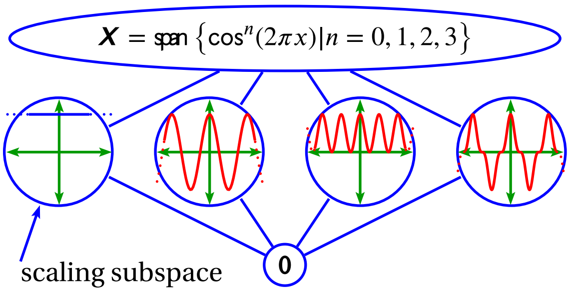


Figure G.3: Lattice of exponential cosines $\{\cos^n x | n = 0, 1, 2, 3\}$

THM

$$\begin{aligned}\cos^n x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\ &= \begin{cases} \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ even} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ odd}\end{cases}\end{aligned}$$

✎PROOF:

$$\begin{aligned}\cos^n x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n \\ &= \mathbf{R_e}\left[\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\right] \\ &= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)x}e^{-ikx}\right] \\ &= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}\right] \\ &= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\cos[(n-2k)x]+i\sin[(n-2k)x]\right)\right] \\ &= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x]+i\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sin[(n-2k)x]\right] \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\ &= \begin{cases} \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ even} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ odd}\end{cases}\end{aligned}$$

✏

Example G.4. [6]

[6] Abramowitz and Stegun (1972) page 795, Sloane (2014) ⟨http://oeis.org/A100257⟩, Sloane (2014) ⟨http://oeis.org/A008314⟩

EX

n	$\cos^n x$	trigonometric reduction	n	$\cos^n x$	trigonometric reduction
0	$\cos^0 x$	$= 1$	4	$\cos^4 x$	$= \dfrac{\cos 4x + 4\cos 2x + 3}{2^3}$
1	$\cos^1 x$	$= \cos x$	5	$\cos^5 x$	$= \dfrac{\cos 5x + 5\cos 3x + 10\cos x}{2^4}$
2	$\cos^2 x$	$= \dfrac{\cos 2x + 1}{2}$	6	$\cos^6 x$	$= \dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{2^5}$
3	$\cos^3 x$	$= \dfrac{\cos 3x + 3\cos x}{2^2}$	7	$\cos^7 x$	$= \dfrac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{2^6}$

✎PROOF:

$$\begin{aligned}
\cos^0 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=0} \\
&= \frac{1}{2^0}\sum_{k=0}^{0}\binom{0}{k}\cos[(0-2k)x] \\
&= \binom{0}{0}\cos[(0-2\cdot 0)x] \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\cos^1 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=1} \\
&= \frac{1}{2^1}\sum_{k=0}^{1}\binom{1}{k}\cos[(1-2k)x] \\
&= \frac{1}{2}\left[\binom{1}{0}\cos[(1-2\cdot 0)x] + \binom{1}{1}\cos[(1-2\cdot 1)x]\right] \\
&= \frac{1}{2}\left[1\cos x + 1\cos(-x)\right] \\
&= \frac{1}{2}(\cos x + \cos x) \\
&= \cos x
\end{aligned}$$

$$\begin{aligned}
\cos^2 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=2} \\
&= \frac{1}{2^2}\sum_{k=0}^{2}\binom{2}{k}\cos([2-2k]x) \\
&= \frac{1}{2^2}\left[\binom{2}{0}\cos([2-2\cdot 0]x) + \binom{2}{1}\cos([2-2\cdot 1]x) + \binom{2}{2}\cos([2-2\cdot 2]x) + \right] \\
&= \frac{1}{2^2}\left[1\cos(2x) + 2\cos(0x) + 1\cos(-2x)\right] \\
&= \frac{1}{2^2}\left[\cos(2x) + 2 + \cos(2x)\right] \\
&= \frac{1}{2}\left[\cos(2x) + 1\right]
\end{aligned}$$

$$\begin{aligned}
\cos^3 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=3} \\
&= \frac{1}{2^3}\sum_{k=0}^{3}\binom{3}{k}\cos([3-2k]x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^3}\left[1\cos(3x) + 3\cos(1x) + 3\cos(-1x) + 1\cos(-3x)\right] \\
&= \frac{1}{2^3}\left[\cos(3x) + 3\cos(x) + 3\cos(x) + \cos(3x)\right] \\
&= \frac{1}{2^2}\left[\cos(3x) + 3\cos(x)\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^4 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=4} \\
&= \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}\cos([4-2k]x) \\
&= \frac{1}{2^4}\left[1\cos(4x) + 4\cos(2x) + 6\cos(0x) + 4\cos(-2x) + 1\cos(-4x)\right] \\
&= \frac{1}{2^3}\left[\cos(4x) + 4\cos(2x) + 3\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^5 x &= \frac{1}{2^{5-1}}\sum_{k=0}^{\left\lfloor\frac{5}{2}\right\rfloor}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\sum_{k=0}^{2}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\left[\binom{5}{0}\cos 5x + \binom{5}{1}\cos 3x + \binom{5}{2}\cos x\right] \\
&= \frac{1}{16}\left[\cos 5x + 5\cos 3x + 10\cos x\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^6 x &= \frac{1}{2^6}\binom{6}{\frac{6}{2}} + \frac{1}{2^{6-1}}\sum_{k=0}^{\frac{6}{2}-1}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{2^6}\binom{6}{3} + \frac{1}{2^5}\sum_{k=0}^{2}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{64}20 + \frac{1}{32}\left[\binom{6}{0}\cos 6x + \binom{6}{1}\cos 4x \binom{6}{2}\cos 2x\right] \\
&= \frac{1}{32}\left[\cos 6x + 6\cos 4x + 15\cos 2x + 10\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^7 x &= \frac{1}{2^{7-1}}\sum_{k=0}^{\left\lfloor\frac{7}{2}\right\rfloor}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\sum_{k=0}^{2}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\left[\binom{7}{0}\cos 7x + \binom{7}{1}\cos 5x + \binom{7}{2}\cos 3x + \binom{7}{3}\cos x\right] \\
&= \frac{1}{64}\left[\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right]
\end{aligned}
$$

Note: Trigonometric reduction of $\cos^n(x)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[7]

```
trigreduce((cos(x))^2);
trigreduce((cos(x))^3);
trigreduce((cos(x))^4);
trigreduce((cos(x))^5);
trigreduce((cos(x))^6);
trigreduce((cos(x))^7);
```

[7] http://maxima.sourceforge.net/docs/manual/en/maxima_15.html
maxima page 158 ⟨10.5 Trigonometric Functions⟩

✏

G.3 Spectral Factorization

Theorem G.5 (Fejér-Riesz spectral factorization). [8] *Let* $[0, \infty) \subsetneq \mathbb{R}$ *and*

$$\mathsf{p}\left(e^{ix}\right) \triangleq \sum_{n=-N}^{N} a_n e^{inx} \qquad \text{(Laurent trigonometric polynomial order } 2N\text{)}$$

$$\mathsf{q}\left(e^{ix}\right) \triangleq \sum_{n=1}^{N} b_n e^{inx} \qquad \text{(standard trigonometric polynomial order } N\text{)}$$

THM $\mathsf{p}\left(e^{ix}\right) \in [0, \infty) \quad \forall x \in [0, 2\pi] \implies \left\{ \begin{array}{ll} \exists \left(\!\left(b_n\right)\!\right)_{n\in\mathbb{Z}} & \text{such that} \\ \mathsf{p}\left(e^{ix}\right) = \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right) & \forall x \in \mathbb{R} \end{array}\right.$

✎PROOF:

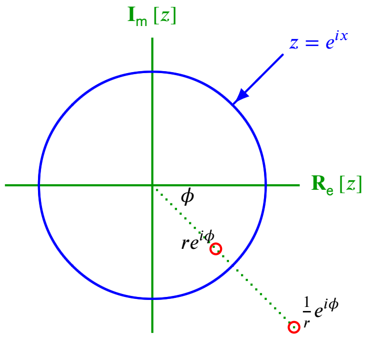
1. Proof that $a_n = a^*_{-n}$ ($\left(\!\left(a_n\right)\!\right)_{n\in\mathbb{Z}}$ is *Hermitian symmetric*):
   Let $a_n \triangleq r_n e^{i\phi_n}$, $r_n, \phi_n \in \mathbb{R}$. Then

$$\begin{aligned}
\mathsf{p}\left(e^{inx}\right) &\triangleq \sum_{n=-N}^{N} a_n e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{i\phi_n} e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{inx+\phi_n} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \sum_{n=-N}^{N} r_n \sin(nx+\phi_n) \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(0x+\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) + \sum_{n=1}^{N} r_{-n}\sin(-nx+\phi_{-n}) \right]}_{\text{imaginary part must equal 0 because } p(x)\in\mathbb{R}} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) - \sum_{n=1}^{N} r_{-n}\sin(nx-\phi_{-n}) \right]}_{\implies r_n = r_{-n},\ \phi_n = -\phi_{-n} \implies a_n = a^*_{-n},\ a_0 \in \mathbb{R}}
\end{aligned}$$

2. Because the coefficients $\left(\!\left(c_n\right)\!\right)_{n\in\mathbb{Z}}$ are *Hermitian symmetric*, the zeros of $P(z)$ occur in *conjugate reciprocol pairs*. This means that if $\sigma \in \mathbb{C}$ is a zero of $P(z)$ ($P(\sigma) = 0$), then $\frac{1}{\sigma^*}$ is also a zero of $P(z)$ ($P\left(\frac{1}{\sigma^*}\right) = 0$). In the complex z plane, this relationship means zeros are reflected across the unit circle such that

$$\frac{1}{\sigma^*} = \frac{1}{(re^{i\phi})^*} = \frac{1}{r}\frac{1}{e^{-i\phi}} = \frac{1}{r}e^{i\phi}$$

[8] Pinsky (2002) pages 330–331



3. Because the zeros of $\mathsf{p}(z)$ occur in conjugate recipricol pairs, $\mathsf{p}\left(e^{ix}\right)$ can be factored:

$$\begin{aligned}
\mathsf{p}\left(e^{ix}\right) &= \mathsf{p}(z)|_{z=e^{ix}} \\
&= z^{-N} C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N}\left(z-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N} z^{-1}\left(z-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N}\left(1-\frac{1}{\sigma_n^*}z^{-1}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n)\prod_{n=1}^{N}\left(z^{-1}-\sigma_n^*\right)\left(-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= \left[C\prod_{n=1}^{N}\left(-\frac{1}{\sigma_n^*}\right)\right]\left[\prod_{n=1}^{N}(z-\sigma_n)\right]\left[\prod_{n=1}^{N}\left(\frac{1}{z^*}-\sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \left[C_2\prod_{n=1}^{N}(z-\sigma_n)\right]\left[C_2\prod_{n=1}^{N}\left(\frac{1}{z^*}-\sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}(z)\mathsf{q}^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right)
\end{aligned}$$

✏

G.4 Dirichlet Kernel

❝*Dirichlet alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous proof is. Rather we learn it first from him. When Gauss says he has proved something it is clear; when Cauchy says it, one can wager as much pro as con; when Dirichlet says it, it is certain.*❞

Carl Gustav Jacob Jacobi (1804–1851), Jewish-German mathematician [9]

[9] quote: Schubring (2005) page 558
image: http://en.wikipedia.org/wiki/File:Carl_Jacobi.jpg, public domain

The *Dirichlet Kernel* is critical in proving what is not immediately obvious in examining the Fourier Series—that for a broad class of periodic functions, a function can be recovered from (with uniform convergence) its Fourier Series analysis.

Definition G.2. [10]

DEF *The **Dirichlet Kernel** $\mathsf{D}_n \in \mathbb{R}^{\mathbb{W}}$ with period τ is defined as*

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}kx}$$

Proposition G.1. [11] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition G.2 page 180).

PRP
$$\mathsf{D}_n(x) = \frac{1}{\tau}\frac{\sin\left(\frac{\pi}{\tau}[2n+1]x\right)}{\sin\left(\frac{\pi}{\tau}x\right)}$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &\triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \mathsf{D}_n \quad \text{(Definition G.2 page 180)}\\
&= \frac{1}{\tau}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}(k-n)x} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}kx} = \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\sum_{k=0}^{2n}\left(e^{i\frac{2\pi}{\tau}x}\right)^k\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\frac{1-\left(e^{i\frac{2\pi}{\tau}x}\right)^{2n+1}}{1-e^{i\frac{2\pi}{\tau}x}} && \text{by } \textit{geometric series}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\frac{1-e^{i\frac{2\pi}{\tau}(2n+1)x}}{1-e^{i\frac{2\pi}{\tau}x}} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(\frac{e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{i\frac{\pi}{\tau}x}}\right)\frac{e^{-i\frac{\pi}{\tau}(2n+1)x}-e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{-i\frac{\pi}{\tau}x}-e^{i\frac{\pi}{\tau}x}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(e^{i\frac{2\pi n}{\tau}x}\right)\frac{-2i\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{-2i\sin\left[\frac{\pi}{\tau}x\right]} = \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}x\right]}
\end{aligned}$$

✏

Proposition G.2. [12] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition G.2 page 180).

PRP
$$\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x = 1$$

✎PROOF:

$$\begin{aligned}
\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x &\triangleq \int_0^\tau \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \mathsf{D}_n \text{ (Definition G.2 page 180)}\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right) + i\sin\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x
\end{aligned}$$

[10] Katznelson (2004) page 14, Heil (2011) pages 443–444, Folland (1992) pages 33–34
[11] Katznelson (2004) page 14, Heil (2011) page 444, Folland (1992) page 34
[12] Bruckner et al. (1997) pages 620–621

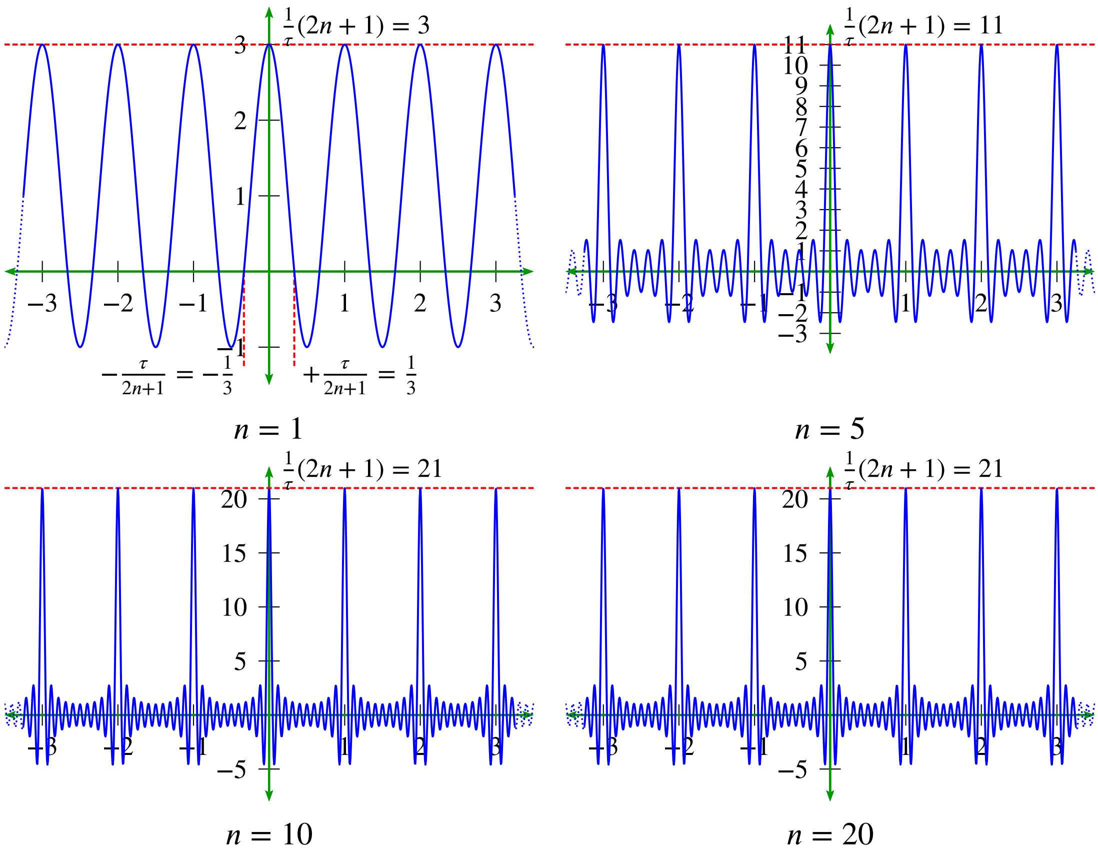


Figure G.4: D_n function for $N = 1, 5, 10, 20$. $\mathsf{D}_n \to$ comb. (See Proposition G.1 page 180).

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x \\
&= \frac{1}{\tau}\sum_{k=-n}^{n} \left.\frac{\sin\left(\frac{2\pi}{\tau}nx\right)}{\frac{2\pi}{\tau}n}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left[\frac{\sin\left(\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n} - \frac{\sin\left(-\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n}\right] \\
&= \frac{1}{\tau}\frac{\tau}{2}\sum_{k=-n}^{n}\left[\frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi n)}{\pi n}\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n)}{\pi n}\right]_{k=0} \\
&= 1
\end{aligned}
$$

✏

Proposition G.3. *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition G.2 page 180). *Let* w_N *(the "*WIDTH*" of* $\mathsf{D}_n(x)$*) be the distance between the two points where the center pulse of* $\mathsf{D}_n(x)$ *intersects the* x *axis.*

PRP

$$
\begin{aligned}
\mathsf{D}_n(0) &= \frac{1}{\tau}(2n+1) \\
w_n &= \frac{2\tau}{2n+1}
\end{aligned}
$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(0) &= \mathsf{D}_n(x)\Big|_{t=0} \\
&= \frac{1}{\tau}\left.\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by Proposition G.1 page 180} \\
&= \frac{1}{\tau}\left.\frac{\frac{d}{dx}\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\frac{d}{dx}\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \frac{1}{\tau}\left.\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{\cos\left[\frac{\pi}{\tau}(2n+1)x\right]}{\cos\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{1}{1} \\
&= \frac{1}{\tau}(2n+1)
\end{aligned}$$

The center pulse of kernel $\mathsf{D}_n(x)$ intersects the x axis at

$$t = \pm\frac{\tau}{(2n+1)}$$

which implies

$$w_n = \frac{\tau}{2n+1} + \frac{\tau}{2n+1} = \frac{2\tau}{(2n+1)}.$$

✏

Proposition G.4. [13] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ *(Definition G.2 page 180).*

PRP: $\mathsf{D}_n(x) = \mathsf{D}_n(-x)$ (D_n *is an* EVEN *function*)

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]} && \text{by Proposition G.1 page 180} \\
&= \frac{1}{\tau}\frac{-\sin\left[-\frac{\pi}{\tau}(2n+1)x\right]}{-\sin\left[-\frac{\pi}{\tau}t\right]} && \text{because } \sin x \text{ is an } \textit{odd} \text{ function} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)(-x)\right]}{\sin\left[\frac{\pi}{\tau}(-x)\right]} \\
&= \mathsf{D}_n(-x) && \text{by Proposition G.1 page 180}
\end{aligned}$$

✏

[13] Bruckner et al. (1997) pages 620–621

G.5 Trigonometric summations

Theorem G.6 (Lagrange trigonometric identities). [14]

THM

$$\sum_{n=0}^{N-1}\cos(nx) = \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)+\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)+\cos\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=0}^{N-1}\Re e^{inx} = \Re\sum_{n=0}^{N-1}e^{inx} = \Re\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Re\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \Re\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-i\frac{1}{2}x}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Re\left[\left(e^{i\frac{1}{2}(N-1)x}\right)\left(\frac{-i\frac{1}{2}\sin\left(\frac{1}{2}Nx\right)}{-i\frac{1}{2}\sin\left(\frac{1}{2}x\right)}\right)\right] \\
&= \cos\left(\frac{1}{2}(N-1)x\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{-\frac{1}{2}\sin\left(-\frac{1}{2}x\right)+\frac{1}{2}\sin\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by } \textit{product identities} \quad \text{(Theorem F.8 page 160)} \\
&= \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=0}^{N-1}\Im e^{inx} = \Im\sum_{n=0}^{N-1}e^{inx} = \Im\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Im\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \Im\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-ix/2}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Im\left[\left(e^{i(N-1)x/2}\right)\left(\frac{-\frac{1}{2}i\sin\left(\frac{1}{2}Nx\right)}{-\frac{1}{2}i\sin\left(\frac{1}{2}x\right)}\right)\right]
\end{aligned}$$

[14] Muniz (1953) page 140 ⟨"Lagrange's Trigonometric Identities"⟩, Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (14), (13)⟩

$$= \sin\left(\frac{(N-1)x}{2}\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right)$$

$$= \frac{\frac{1}{2}\cos\left(-\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} \qquad \text{by } \textit{product identities} \qquad \text{(Theorem F.8 page 160)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}$$

Note that these results (summed with indices from $n=0$ to $n=N-1$) are compatible with Muniz (1953) page 140 (summed with indices from $n=1$ to $n=N$) as demonstrated next:

$$\sum_{n=0}^{N-1}\cos(nx) = \sum_{n=1}^{N}\cos(nx) + [\cos(0x) - \cos(Nx)]$$

$$= \left[-\frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + [\cos(0x) - \cos(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \left(1-\frac{1}{2}\right) + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\cos(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\left[\sin\left(\left[\frac{1}{2}-N\right]x\right) + \sin\left[\left(\frac{1}{2}+N\right)x\right]\right]}{2\sin\left(\frac{1}{2}x\right)} \qquad \text{by Theorem F.8 page 160}$$

$$= \frac{1}{2} + \frac{\sin\left(\frac{1}{2}[2N-1]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \sum_{n=1}^{N}\sin(nx) + [\sin(0x) - \sin(Nx)]$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} + [0 - \sin(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\sin(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - \left[\cos\left(\left[\frac{1}{2}-N\right]x\right) - \cos\left(\left[\frac{1}{2}+N\right]x\right)\right]}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

✎

Theorem G.7. [15]

[15] Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (16) and (17)⟩

THM

$$\sum_{n=0}^{N-1}\cos(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] - \sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + \sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)-\sin(nx)\sin(y)] && \text{by \textit{double angle formulas} (Theorem F.9 page 163)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx)-\sin(y)\sum_{n=0}^{N-1}\sin(nx)\\
\sum_{n=0}^{N-1}\sin(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)+\sin(nx)\sin(y)] && \text{by \textit{double angle formulas} (Theorem F.9 page 163)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx)+\sin(y)\sum_{n=0}^{N-1}\sin(nx)
\end{aligned}$$

✏

Corollary G.1 (Summation around unit circle).

THM

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right) = 0 \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

$$\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right) = \frac{N}{2} \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

✎PROOF:

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\\
&= \cos(\theta)\sum_{n=0}^{N-1}\cos\left(\frac{2nM\pi}{N}\right)-\sin(\theta)\sum_{n=0}^{N-1}\sin\left(\frac{2nM\pi}{N}\right) && \text{by Theorem F.9 page 163}\\
&= \cos(\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{2M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] && \text{by Theorem G.6 page 183}\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{\cos\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{M\pi}{N}\right)}{\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity} (Theorem F.10 page 163)}\\
&= \cos(\theta)[0]-\sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1}\cos\left(\theta-\frac{\pi}{2}+\frac{2nM\pi}{N}\right) && \text{by \textit{shift identities}} && \text{(Theorem F.7 page 160)}\\
&= \sum_{n=0}^{N-1}\cos\left(\phi+\frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta-\frac{\pi}{2}\\
&= 0 && \text{by previous result}
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)\\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(\left[\theta+\frac{2nM\pi}{N}\right]-\left[\theta+\frac{2nM\pi}{N}\right]\right)+\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(\left[\theta+\frac{2nM\pi}{N}\right]+\left[\theta+\frac{2nM\pi}{N}\right]\right) && \text{by Theorem F.8 page 160}\\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\cancelto{0}{\sin(0)}+\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(2\theta+\frac{4nM\pi}{N}\right)\\
&= \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)+\frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right) && \text{by Theorem F.9 page 163}\\
&= \cos(2\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{4M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{4M\pi}{N}\right)}\right] && \text{by Theorem G.6 page 183}\\
&= \cos(2\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{2M\pi}{N}-4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]-\sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)-\frac{\cos\left(\frac{2M\pi}{N}-4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{2M\pi}{N}\right)}{\sin\left(\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity}} \\
& && \text{(Theorem F.10 page 163)}\\
&= \cos(\theta)[0]-\sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left[1+\cos\left(2\theta+\frac{4nM\pi}{N}\right)\right] && \text{by Theorem F.11 page 165}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}\left[1+\cos(2\theta)\cos\left(\frac{4nM\pi}{N}\right)-\sin(2\theta)\sin\left(\frac{4nM\pi}{N}\right)\right] && \text{by Theorem F.9 page 163}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}1+\frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\cos\left(\frac{4nM\pi}{N}\right)-\frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)\\
&= \left[\frac{1}{2}\sum_{n=0}^{N-1}1\right]+\frac{1}{2}\cos(2\theta)0-\frac{1}{2}\sin(2\theta)0 && \text{by previous results}\\
&= \frac{N}{2}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos^2\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by } \textit{shift identities} \text{ (Theorem F.7 page 160)}\\
&= \sum_{n=0}^{N-1} \cos^2\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2}\\
&= \frac{N}{2} && \text{by previous result}
\end{aligned}$$

✎

G.6 Summability Kernels

Definition G.3. [16] *Let $(\!(\kappa_n)\!)_{n\in\mathbb{Z}}$ be a sequence of* CONTINUOUS 2π PERIODIC *functions.*

DEF

*The sequence $(\!(\kappa_n)\!)_{n\in\mathbb{Z}}$ is a **summability kernel** if*

1. $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \kappa_n(x)\,dx = 1 \quad \forall n\in\mathbb{Z}$ *and*
2. $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} |\kappa_n(x)|\,dx \in \mathbb{R} \quad \forall n\in\mathbb{Z}$ *and*
3. $\displaystyle\lim_{n\to\infty}\int_\delta^{2\pi-\delta} |\kappa_n(x)|\,dx = 0 \quad \forall n\in\mathbb{Z}, 0<\delta<\pi$

Theorem G.8. [17] *Let $(\!(\kappa_n)\!)_{n\in\mathbb{Z}}$ be a sequence. Let $\mathbb{T}$ be the quotient $\mathbb{R}/2\pi\mathbb{Z}$.*

THM

$$\left.\begin{array}{ll} 1. & \mathsf{f} \in L^1(\mathbb{T}) \quad \textit{and} \\ 2. & (\!(\kappa_n)\!) \textit{ is a summability kernel} \end{array}\right\} \implies \mathsf{f}(x) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbb{T}} \kappa_n(x)\mathsf{f}(x-x)\,dx$$

The *Dirichlet kernel* (Definition G.2 page 180) is *not* a *summability kernel*. Examples of kernels that *are* summability kernels include

1. *Fejér's kernel* (Definition G.4 page 187)
2. *de la Vallée Poussin kernel* (Definition G.5 page 189)
3. *Jackson kernel* (Definition G.6 page 189)
4. *Poisson kernel* (Definition G.7 page 189 .)

Definition G.4. [18]

DEF

Fejér's kernel K_n *is defined as*

$$\mathsf{K}_n(x) \triangleq \sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx}$$

Proposition G.5. [19] *Let K_n be Fejér's kernel* (Definition G.4 page 187).

PRP

$$\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$$

[16] Cerdà (2010) page 56, Katznelson (2004) page 10, de Reyna (2002) page 21, Walnut (2002) pages 40–41, Heil (2011) page 440, Istrăţescu (1987) page 309
[17] Katznelson (2004) page 11
[18] Katznelson (2004) page 12
[19] Katznelson (2004) page 12, Heil (2011) page 448

✎PROOF:

1. Lemma: Proof that $\sin^2\frac{x}{2} \equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right)$:

$$\begin{aligned}
\sin^2\frac{x}{2} &\equiv \left(\frac{e^{-i\frac{x}{2}} - e^{+i\frac{x}{2}}}{2i}\right)^2 && \text{by Euler Formulas (Corollary F.2 page 159)}\\
&\equiv \frac{-1}{4}\left(e^{-2i\frac{x}{2}} - 2e^{-i\frac{x}{2}}e^{i\frac{x}{2}} + e^{2i\frac{x}{2}}\right)\\
&\equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right) \;:
\end{aligned}$$

2. Lemma:

$$2|k| - |k+1| - |k-1| = \begin{cases} -2 & \text{for } k = 0\\ 0 & \text{for } k \in \mathbb{Z}\backslash 0 \end{cases}$$

3. Proof that $\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$:

$$\begin{aligned}
&-4(n+1)\left(\sin\frac{1}{2}x\right)^2 \mathsf{K}_n(x)\\
&= -4(n+1)\left(\frac{-1}{4}\right)\left(e^{-ix} - 2 + e^{ix}\right)\mathsf{K}_n(x) && \text{by item (1)}\\
&= (n+1)\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx} && \text{by Definition G.4}\\
&= (n+1)\frac{1}{n+1}\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= e^{-ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}e^{ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= \sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k-1)x} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k+1)x}\\
&= \sum_{k=-n-1}^{k=n-1}(n+1-|k+1|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n+1}^{k=n+1}(n+1-|k-1|)e^{ikx}\\
&= \underbrace{e^{-i(n+1)x}}_{k=-n-1} + \underbrace{2e^{-inx}}_{k=-n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad \underbrace{-2e^{-inx}}_{k=-n} + \underbrace{-2e^{inx}}_{k=n} - 2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad \underbrace{e^{i(n+1)x}}_{k=n+1} + \underbrace{2e^{inx}}_{k=n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}\\
&= e^{-i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad -2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}
\end{aligned}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} [(n+1-|k+1|) - 2(n+1-|k|) + (n+1-|k-1|)]\, e^{ikx}$$
$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} (2|k| - |k+1| - |k-1|)\, e^{ikx}$$
$$= e^{-i(n+1)x} + e^{i(n+1)x} - 2 \qquad \text{by item (2)}$$
$$= -4\left(\sin\frac{n+1}{2}x\right)^2 \qquad \text{by item (1)}$$

✏

Definition G.5. [20] *Let* K_n *be* FEJÉR'S KERNEL (Definition G.4 page 187).

DEF *The* ***de la Vallée Poussin kernel*** V_n *is defined as*

$$\mathsf{V}_n(x) \triangleq 2K_{2n+1}(x) - K_n(x)$$

Definition G.6. [21] *Let* K_n *be* FEJÉR'S KERNEL (Definition G.4 page 187).

DEF *The* ***Jackson kernel*** J_n *is defined as*

$$\mathsf{J}_n(x) \triangleq \left\|\mathsf{K}_n\right\|^{-2} \mathsf{K}_n^2(x)$$

Definition G.7. [22]

DEF *The* ***Poisson kernel*** P *is defined as*

$$P(r,x) \triangleq \sum_{k\in\mathbb{Z}} r^{|k|} e^{ikx}$$

[20] Katznelson (2004) page 16
[21] Katznelson (2004) page 17
[22] Katznelson (2004) page 16

APPENDIX H

FOURIER TRANSFORM

❝ *Up to this point we have supposed that the function whose development is required in a series of sines of multiple arcs can be developed in a series arranged according to powers of the variable x, …We can extend the same results to any functions, even to those which are discontinuous and entirely arbitrary. …even entirely arbitrary functions may be developed in series of sines of multiple arcs.* ❞

Joseph Fourier (1768–1830) [1]

H.1 Introduction

Historically, before the Fourier Transform was the Taylor Expansion (transform). The Taylor Expansion demonstrates that for **analytic** functions knowledge of the derivatives of a function at a location $x = a$ allows you to determine (predict) arbitrarily closely all the points $\mathsf{f}(x)$ in the vicinity of $x = a$ (Chapter J page 217). But analytic functions are by definition functions for which all their derivatives exist. Thus, if a function is *discontinuous*, it is simply not a candidate for a Taylor Expansion. And some 300 years ago, mathematician giants of the day were fairly content with this.

But then in came an engineer named Joseph Fourier whose day job was working as a governor of lower Egypt under Napolean. He claimed that, rather than expansion based on derivatives, one could expand based on integrals over sinusoids, and that this would work not just for analytic functions, but for **discontinuous** ones as well![2]

Needless to say, this did not go over too well initially in the mathematical community. But over time (on the order of 200 or so years), the Fourier Transform has in many ways won the day.

[3]

---

[1] quote: Fourier (1878) page 184,186 ⟨§219,220⟩
image: `http://en.wikipedia.org/wiki/File:Fourier2.jpg`, public domain

[2] Robinson (1982) page 886

[3] Caricature of Legendre (left) and Fourier (right), 1820, by Julien-Léopold Boilly (1796–1874). "*Album de 73*

H.2 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on ⊠, and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \middle| \int_{\mathbb{R}} |\mathsf{f}|^2 \,\mathrm{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \,|\, \nabla \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \,\mathrm{d}\mu$ such that

$$\langle \mathsf{f} \,|\, \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\,\mathrm{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \,|\, \nabla \rangle \right)$ is a *Hilbert space*.

Definition H.1. *Let κ be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF *The function κ is the **Fourier kernel** if* $\quad \kappa(x,\omega) \triangleq e^{i\omega x} \qquad \forall x,\omega \in \mathbb{R}$

Definition H.2. [4] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF *The **Fourier Transform** operator $\tilde{\mathbf{F}}$ is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \,\mathrm{d}x \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

*This definition of the Fourier Transform is also called the **unitary Fourier Transform**.*

Remark H.1 (**Fourier transform scaling factor**). [5] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B \int_{\mathbb{R}} \mathsf{F}(\omega) e^{i\omega x}\,\mathrm{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\,\mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x}\,\mathrm{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x}\,\mathrm{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int \cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\,\mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem H.2 page 193)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

Portraits-Charge Aquarelle's des Membres de I'Institute (watercolor portrait #29). Biliotheque de l'Institut de France." Public domain. https://en.wikipedia.org/wiki/File:Legendre_and_Fourier_(1820).jpg

[4] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965) page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

[5] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005b) pages 374–375

H.3 Operator properties

Theorem H.1 (Inverse Fourier transform). [6] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition H.2 page 192). *The inverse* $\tilde{\mathbf{F}}^{-1}$ *of* $\tilde{\mathbf{F}}$ *is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\,\mathrm{d}\omega \qquad \forall \tilde{\mathsf{f}} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem H.2. *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with inverse* $\tilde{\mathbf{F}}^{-1}$ *and adjoint* $\tilde{\mathbf{F}}^{*}$.

THM
$$\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \mathsf{g} \right\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathrm{d}x \,|\, \mathsf{g}(\omega) \right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 192}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,\left\langle e^{-i\omega x} \,|\, \mathsf{g}(\omega) \right\rangle\,\mathrm{d}x && \text{by \textit{additive property} of } \langle \triangle \,|\, \nabla \rangle\\
&= \int_{\mathbb{R}} \mathsf{f}(x)\,\frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle^{*}\,\mathrm{d}x && \text{by \textit{conjugate symmetric property} of } \langle \triangle \,|\, \nabla \rangle\\
&= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle \right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle\\
&= \left\langle \mathsf{f} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^{*}}\mathsf{g} \right\rangle && \text{by Theorem H.1 page 193}
\end{aligned}$$

✏

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* [7] (Corollary H.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem H.3 page 193).

Corollary H.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^{*}$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

COR
$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{*}\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{-1}$ (Theorem H.2 page 193). ✏

Theorem H.3. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^{*}$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $\left(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \nabla \rangle\right)$. *Let* $\mathcal{R}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

THM
$$\begin{array}{rclcll}
\mathcal{R}(\mathbf{F}\tau) &=& \mathcal{R}(\tilde{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| &=& |||\tilde{\mathbf{F}}^{-1}||| &=& 1 & \text{(UNITARY)}\\
\left\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g} \right\rangle &=& \left\langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g} \right\rangle &=& \langle \mathsf{f} \,|\, \mathsf{g} \rangle & \text{(PARSEVAL'S EQUATION)}\\
\|\tilde{\mathbf{F}}\mathsf{f}\| &=& \|\tilde{\mathbf{F}}^{-1}\mathsf{f}\| &=& \|\mathsf{f}\| & \text{(PLANCHEREL'S FORMULA)}\\
\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\| &=& \|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\| &=& \|\mathsf{f} - \mathsf{g}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary H.1 page 193) and from the properties of unitary operators (Theorem C.26 page 136). ✏

[6] Chorin and Hald (2009) page 13

[7] *unitary operators*: Definition C.14 page 135

H.4 Shift relations

Theorem H.4 (Shift relations). *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition H.2 page 192).

T H M
$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega)\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r)
\end{aligned}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition K.1 page 219).

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= \mathbf{L}[\mathsf{f}(x-y)](s)|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition K.1 page 219)}\\
&= e^{-sy}\,[\mathbf{L}\mathsf{f}(x)](s)|_{s=i\omega} && \text{by \textit{Laplace shift relation}} && \text{(Theorem K.1 page 219)}\\
&= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)}\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\mathbf{L}\left(e^{irx}\mathsf{g}(x)\right)\right](s)\Big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition K.1 page 219)}\\
&= \left[\left[\mathbf{L}\mathsf{g}(x)\right](s-r)\right]\Big|_{s=i\omega} && \text{by \textit{Laplace shift relation}} && \text{(Theorem K.1 page 219)}\\
&= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)}
\end{aligned}$$

✏

Theorem H.5 (Complex conjugate). *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator and* $*$ *represent the complex conjugate operation on the set of complex numbers.*

T H M
$$\begin{aligned}
& \tilde{\mathbf{F}}\mathsf{f}^*(-x) &= -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \forall \mathsf{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}\\
\mathsf{f} \text{ is real} \implies & \tilde{\mathsf{f}}(-\omega) &= \left[\tilde{\mathsf{f}}(\omega)\right]^* && \forall \omega\in\mathbb{R} && \text{REALITY CONDITION}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(-x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)}\\
&= \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(u)e^{i\omega u}(-1)\,\mathrm{d}u && \text{where } u\triangleq -x \implies \mathrm{d}x = -\,\mathrm{d}u\\
&= -\left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right]^*\\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)}\\
\tilde{\mathsf{f}}(-\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i(-\omega)x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)}\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(x)e^{-i\omega x}\,\mathrm{d}x\right]^*\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]^* && \text{by } \mathsf{f} \text{ is real hypothesis}\\
&\triangleq \tilde{\mathsf{f}}^*(\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)}
\end{aligned}$$

✏

H.5 Convolution relations

Definition H.3. [8]

DEF *The **convolution operation** is defined as*

$$[f \star g](x) \triangleq f(x) \star g(x) \triangleq \int_{u\in\mathbb{R}} f(u)g(x-u)\,du \qquad \forall f,g\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem I.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem H.6 (convolution theorem). [9] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition H.2 page 192) *and $\star$ the convolution operator* (Definition K.2 page 220).

THM

$$\underbrace{\tilde{\mathbf{F}}[f(x)\star g(x)](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\,[\tilde{\mathbf{F}}f](\omega)\,[\tilde{\mathbf{F}}g](\omega)}_{\text{multiplication in "frequency domain"}} \qquad \forall f,g\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\tilde{\mathbf{F}}[f(x)g(x)](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}[\tilde{\mathbf{F}}f](\omega)\star[\tilde{\mathbf{F}}g](\omega)}_{\text{convolution in "frequency domain"}} \qquad \forall f,g\in L^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition K.1 page 219).

$$\begin{aligned}
\tilde{\mathbf{F}}[f(x)\star g(x)](\omega) &= \mathbf{L}[f(x)\star g(x)](s)\big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition K.1 page 219)}\\
&= \sqrt{2\pi}[\mathbf{L}f](s)\,[\mathbf{L}g](s)\Big|_{s=i\omega} && \text{by \textit{Laplace convolution} result} && \text{(Theorem K.2 page 220)}\\
&= \sqrt{2\pi}\,[\tilde{\mathbf{F}}f](\omega)\,[\tilde{\mathbf{F}}g](\omega)\\
\tilde{\mathbf{F}}[f(x)g(x)](\omega) &= \mathbf{L}[f(x)g(x)](s)|_{s=i\omega}\\
&= \frac{1}{\sqrt{2\pi}}[\mathbf{L}f](s)\star[\mathbf{L}g](s)\Big|_{s=i\omega}\\
&= \frac{1}{\sqrt{2\pi}}[\tilde{\mathbf{F}}f](\omega)\star[\tilde{\mathbf{F}}g](\omega)
\end{aligned}$$

✏

H.6 Calculus relations

Theorem H.7. *Let $\tilde{\mathbf{F}}$ be the* FOURIER TRANSFORM *operator* (Definition H.2 page 192).

THM

$$\left\{\lim_{t\to-\infty} x(t) = 0\right\} \implies \left\{\tilde{\mathbf{F}}\left[\frac{d}{dt}x(t)\right] = i\omega[\tilde{\mathbf{F}}x](\omega)\right\}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition K.1 page 219).

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\frac{d}{dt}x(t)\right] &\triangleq \mathbf{L}\left[\frac{d}{dt}x(t)\right](s)\Big|_{s=i\omega} && \text{by definitions of } \mathbf{L} \text{ and } \tilde{\mathbf{F}} && \text{(Definition K.1 page 219)}\\
&= s[\mathbf{L}x(t)](s)|_{s=i\omega} && \text{by Theorem K.3 page 221}\\
&= i\omega[\tilde{\mathbf{F}}x](\omega)
\end{aligned}$$

✏

[8] Bachman (1964) page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩
[9] Bracewell (1978) page 110

Theorem H.8. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* (Definition H.2 page 192).

THM
$$\tilde{\mathbf{F}} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u = \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)$$

Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition K.1 page 219). ✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u &\triangleq \mathbf{L} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u\bigg|_{s=i\omega} \\
&= \frac{1}{s}[\mathbf{L}\mathsf{x}(t)](s)\bigg|_{s=i\omega} && \text{by Theorem K.4 page 221} \\
&= \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}(t)\right](\omega)
\end{aligned}$$

✏

H.7 Real valued functions

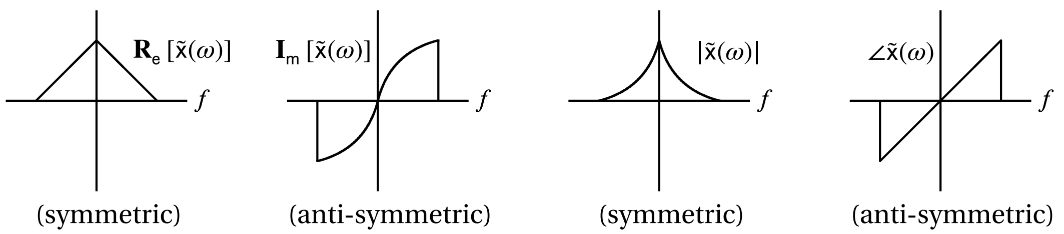


Figure H.1: Fourier transform components of real-valued signal

Theorem H.9. *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM
$$\left\{\begin{array}{l} \mathsf{f}(x) \textit{ is } \text{REAL-VALUED} \\ (\mathsf{f} \in \mathbb{R}^{\mathbb{R}}) \end{array}\right\} \implies \left\{\begin{array}{rcll}
\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & (\text{HERMITIAN SYMMETRIC}) \\
\mathbf{R}_e\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R}_e\left[\tilde{\mathsf{f}}(-\omega)\right] & (\text{SYMMETRIC}) \\
\mathbf{I}_m\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I}_m\left[\tilde{\mathsf{f}}(-\omega)\right] & (\text{ANTI-SYMMETRIC}) \\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| & (\text{SYMMETRIC}) \\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & (\text{ANTI-SYMMETRIC}).
\end{array}\right\}$$

✎PROOF:

$$\begin{array}{lclcl}
\tilde{\mathsf{f}}(\omega) &\triangleq& [\tilde{\mathbf{F}}\mathsf{f}(x)](\omega) &\triangleq& \left\langle \mathsf{f}(x) \,|\, e^{i\omega x}\right\rangle = \left\langle \mathsf{f}(x) \,|\, e^{i(-\omega)x}\right\rangle^* \triangleq \tilde{\mathsf{f}}^*(-\omega) \\
\mathbf{R}_e\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R}_e\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& \mathbf{R}_e\left[\tilde{\mathsf{f}}(-\omega)\right] \\
\mathbf{I}_m\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{I}_m\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& -\mathbf{I}_m\left[\tilde{\mathsf{f}}(-\omega)\right] \\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}^*(-\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| \\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}^*(-\omega) &=& -\angle\tilde{\mathsf{f}}(-\omega)
\end{array}$$

✏

H.8 Moment properties

Definition H.4. [10]

DEF
The quantity M_n *is the* n***th moment*** *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\,\mathsf{d}x \qquad \textit{for } n \in \mathbb{W}.$$

[10] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83

Lemma H.1. [11] *Let* M_n *be the* *n*TH MOMENT (Definition H.4 page 196) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition H.2 page 192) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141).

LEM

$$\begin{aligned}
\mathsf{M}_n &= \sqrt{2\pi}(i)^n \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}\\
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} &= \frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition H.2 page 192)}\\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\right] \mathrm{d}x\bigg|_{\omega=0}\\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathrm{d}x\bigg|_{\omega=0}\\
&= (-i^2)^n \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x\\
&= \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x\\
&\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \quad \text{(Definition H.4 page 196)}
\end{aligned}$$

✏

Lemma H.2. [12] *Let* M_n *be the* *n*TH MOMENT (Definition H.4 page 196) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition H.2 page 192) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141).

LEM

$$\mathsf{M}_n = 0 \quad\iff\quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
0 &= \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by left hypothesis}\\
&= \sqrt{2\pi}(-i)^{-n} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by Lemma H.1 page 197}\\
&\Longrightarrow \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} = 0
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by right hypothesis}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\bigg|_{\omega=0} && \text{by definition of } \tilde{\mathsf{f}}(\omega)\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\,\mathrm{d}x\bigg|_{\omega=0}\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathrm{d}x\bigg|_{\omega=0}\\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x\\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by definition of } \langle \cdot \,|\, \cdot \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{(Definition D.1 page 141)}
\end{aligned}$$

[11] Goswami and Chan (1999) pages 38–39
[12] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

Lemma H.3 (Strang-Fix condition). [13] *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* M_n *the* NTH MOMENT (Definition H.4 page 196) *of* $\mathsf{f}(x)$. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR (Definition 3.3 page 40).

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "time"}} \iff \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "frequency"}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by definition of } \tilde{\mathsf{f}}(\omega) \quad \text{(Definition H.2 page 192)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k)\bar{\delta}_k && \text{by } \textit{PSF} \quad \text{(Theorem 3.2 page 48)} \\
&= \frac{1}{\sqrt{2\pi}} (-i)^n \bar{\delta}_k \mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[(-i)^n \bar{\delta}_k \mathsf{M}_n\right] e^{-i2\pi kx} && \text{by definition of } \bar{\delta} \quad \text{(Definition 2.12 page 20)} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } \textit{PSF} \quad \text{(Theorem 3.2 page 48)}
\end{aligned}$$

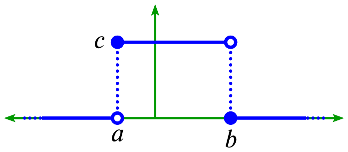
[13] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83, Mallat (1999) pages 241–243, Fix and Strang (1969)

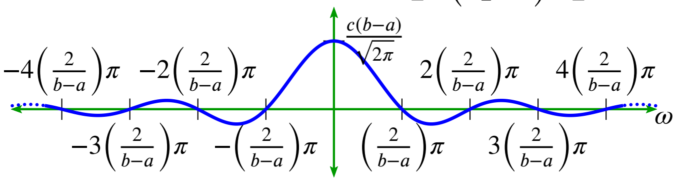
H.9 Examples

Example H.1 (rectangular pulse). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right]$$





✎PROOF:

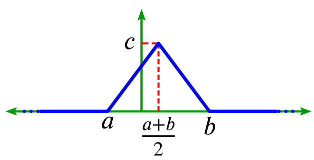
$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x-\frac{a+b}{2}\right)\right](\omega) && \text{by \textit{shift relation}} && \text{(Theorem H.4 page 194)} \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x-\frac{a+b}{2}\right)\right](\omega) && \text{by definition of } \mathsf{f}(x) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} && \text{(Definition 3.2 page 40)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition H.2 page 192)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} ce^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} && \text{(Definition 3.2 page 40)} \\
&= \frac{c}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\frac{1}{-i\omega}e^{-i\omega x}\Big|_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \\
&= \frac{2c}{\sqrt{2\pi}\omega}e^{-i\left(\frac{a+b}{2}\right)\omega}\left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)}-e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i}\right] \\
&= \frac{c(b-a)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right] && \text{by \textit{Euler formulas}} && \text{(Corollary F.2 page 159)}
\end{aligned}$$

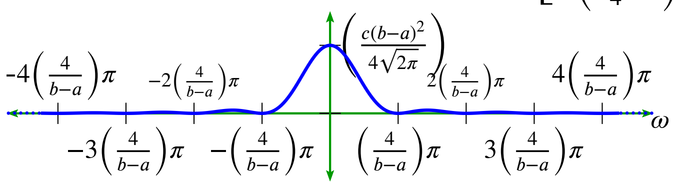
✏

Example H.2 (triangle). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \left\{\begin{array}{ll} c\left[1-\frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{array}\right\} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$





✎PROOF:

$$\tilde{\mathsf{f}}(\omega) = \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) \qquad \text{by definition of } \tilde{\mathsf{f}}(\omega)$$

$$= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x-\frac{a+b}{2}\right)\right](\omega) \qquad \text{by } \textit{shift relation} \qquad \text{(Theorem H.4 page 194)}$$

$$= \tilde{\mathbf{F}}\left[c\left(1-\frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) \qquad \text{by definition of } \mathsf{f}(x)$$

$$= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x) \star \mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega)$$

$$= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] \qquad \text{by } \textit{convolution theorem} \qquad \text{(Theorem I.2 page 206)}$$

$$= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2$$

$$= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 \qquad \text{by } \textit{Rectangular pulse} \text{ ex.} \qquad \text{Example H.1 page 199}$$

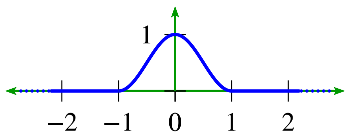
$$= \frac{c(b-a)^2}{4\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$

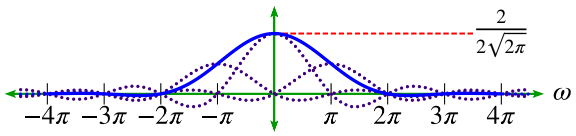
✏

Example H.3. Let a function f be defined in terms of the cosine function (Definition F.1 page 153) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$





✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 3.2 page 40) on a set A.

$$\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x \qquad \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition H.2)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,\mathrm{d}x \qquad \text{by definition of } \mathsf{f}(x)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right)e^{-i\omega x}\,\mathrm{d}x \qquad \text{by definition of } \mathbb{1} \text{ (Definition 3.2)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left[\frac{e^{i\frac{\pi}{2}x}+e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\,\mathrm{d}x \qquad \text{by Corollary F.2 page 159}$$

$$= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1}\left[2+e^{i\pi x}+e^{-i\pi x}\right]e^{-i\omega x}\,\mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1} 2e^{-i\omega x}+e^{-i(\omega+\pi)x}+e^{-i(\omega-\pi)x}\,\mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}}\left[2\frac{e^{-i\omega x}}{-i\omega}+\frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2\sqrt{2\pi}}\left[2\frac{e^{-i\omega}-e^{+i\omega}}{-2i\omega}+\frac{e^{-i(\omega+\pi)}-e^{+i(\omega+\pi)}}{-2i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)}-e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$

✏

APPENDIX I

Z TRANSFORM

I.1 Convolution operator

Definition I.1. [1] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF

A function f *in* X^Y *is a* ***sequence*** *over X if $Y = \mathbb{Z}$.*
A sequence may be denoted in the form $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ or simply as $\left(\!\left(x_n\right)\!\right)$.

Definition I.2. [2] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a* FIELD (Definition A.5 page 96).

DEF

The ***space of all absolutely square summable sequences*** $\boldsymbol{\ell}^2_{\mathbb{F}}$ *over $\mathbb{F}$ is defined as*

$$\boldsymbol{\ell}^2_{\mathbb{F}} \triangleq \left\{ \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} \left|x_n\right|^2 < \infty \right\}$$

The space $\boldsymbol{\ell}^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is isomorphic to $\boldsymbol{L}^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition I.3.

DEF

The ***convolution*** *operation $\star$ is defined as*

$$\left(\!\left(x_n\right)\!\right) \star \left(\!\left(y_n\right)\!\right) \triangleq \left(\!\!\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right)\!\!\right)_{n\in\mathbb{Z}} \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}, \left(\!\left(y_n\right)\!\right)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$$

Proposition I.1. *Let $\star$ be the* CONVOLUTION OPERATOR (Definition I.3 page 203).

PRP

$$\left(\!\left(x_n\right)\!\right) \star \left(\!\left(y_n\right)\!\right) = \left(\!\left(y_n\right)\!\right) \star \left(\!\left(x_n\right)\!\right) \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}, \left(\!\left(y_n\right)\!\right)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

[1] Bromwich (1908) page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[2] Kubrusly (2011) page 347 ⟨Example 5.K⟩

✎PROOF:

$$\begin{aligned}
[x \star y](n) &\triangleq \sum_{m\in\mathbb{Z}} x_m y_{n-m} && \text{by Definition I.3 page 203}\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{where } k \triangleq n-m \implies m = n-k\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{by commutativity of addition}\\
&= \sum_{m\in\mathbb{Z}} x_{n-m} y_m && \text{by change of variables}\\
&= \sum_{m\in\mathbb{Z}} y_m x_{n-m} && \text{by commutative property of the field over } \mathbb{C}\\
&\triangleq (y \star x)_n && \text{by Definition I.3 page 203}
\end{aligned}$$

✏

Proposition I.2. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition I.3 page 203). Let $\ell^2_{\mathbb{R}}$ be the set of* ABSOLUTELY SUMMABLE *sequences (Definition I.2 page 203).*

PRP
$$\left\{\begin{array}{lllll} \text{(A).} & \mathsf{x}(n) & \in & \ell^2_{\mathbb{R}} & \text{and}\\ \text{(B).} & \mathsf{y}(n) & \in & \ell^2_{\mathbb{R}} & \end{array}\right\} \implies \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] = \mathsf{x}[-n] \star \mathsf{y}(n) \right\}$$

✎PROOF:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \implies k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by absolutely summable hypothesis} && \text{(Definition I.2 page 203)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \implies \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of convolution } \star && \text{(Definition I.3 page 203)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n]
\end{aligned}$$

✏

I.2 Z-transform

Definition I.4. [3]

DEF
*The **z-transform** $\mathbf{Z}$ of $(x_n)_{n\in\mathbb{Z}}$ is defined as*
$$[\mathbf{Z}(x_n)](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\text{Laurent series}} \qquad \forall (x_n)\in\ell^2_{\mathbb{R}}$$

Theorem I.1. *Let $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ be the* Z-TRANSFORM *of $\mathsf{x}[n]$.*

THM
$$\left\{ \check{\mathsf{x}}(z) \triangleq \mathbf{Z}(\mathsf{x}[n]) \right\} \implies \left\{\begin{array}{lrcll}
(1). & \mathbf{Z}(\alpha\mathsf{x}[n]) & = & \alpha\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
(2). & \mathbf{Z}(\mathsf{x}[n-k]) & = & z^{-k}\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
(3). & \mathbf{Z}(\mathsf{x}[-n]) & = & \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
(4). & \mathbf{Z}(\mathsf{x}^*[n]) & = & \check{\mathsf{x}}^*(z^*) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
(5). & \mathbf{Z}(\mathsf{x}^*[-n]) & = & \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}}
\end{array}\right\}$$

[3] *Laurent series*: Abramovich and Aliprantis (2002) page 49

✎PROOF:

$$
\begin{aligned}
\alpha\mathbb{Z}\check{x}(z) &\triangleq \alpha\mathbf{Z}\left(\!\left(x[n] \right)\!\right) && \text{by definition of } \check{x}(z)\\
&\triangleq \alpha\sum_{n\in\mathbb{Z}} x[n]z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator}\\
&\triangleq \sum_{n\in\mathbb{Z}} (\alpha x[n])z^{-n} && \text{by } \textit{distributive} \text{ property}\\
&\triangleq \mathbf{Z}\left(\!\left(\alpha x[n] \right)\!\right) && \text{by definition of } \mathbf{Z} \text{ operator}\\
z^{-k}\check{x}(z) &= z^{-k}\mathbf{Z}\left(\!\left(x[n] \right)\!\right) && \text{by definition of } \check{x}(z) && \text{(left hypothesis)}\\
&\triangleq z^{-k}\sum_{n=-\infty}^{n=+\infty} x[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
&= \sum_{n=-\infty}^{n=+\infty} x[n]z^{-n-k}\\
&= \sum_{m-k=-\infty}^{m-k=+\infty} x[m-k]z^{-m} && \text{where } m \triangleq n+k && \implies n = m-k\\
&= \sum_{m=-\infty}^{m=+\infty} x[m-k]z^{-m}\\
&= \sum_{n=-\infty}^{n=+\infty} x[n-k]z^{-n} && \text{where } n \triangleq m\\
&\triangleq \mathbf{Z}\left(\!\left(x[n-k] \right)\!\right) && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
\mathbf{Z}\left(\!\left(x^*[n] \right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
&\triangleq \left(\sum_{n\in\mathbb{Z}} x[n](z^*)^{-n}\right)^* && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
&\triangleq \check{x}^*(z^*) && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
\mathbf{Z}\left(\!\left(x[-n] \right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
&= \sum_{-m\in\mathbb{Z}} x[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]z^{m} && \text{by } \textit{absolutely summable} \text{ property} && \text{(Definition I.2 page 203)}\\
&= \sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z}\right)^{-m} && \text{by } \textit{absolutely summable} \text{ property} && \text{(Definition I.2 page 203)}\\
&\triangleq \check{x}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
\mathbf{Z}\left(\!\left(x^*[-n] \right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
&= \sum_{-m\in\mathbb{Z}} x^*[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x^*[m]z^{m} && \text{by } \textit{absolutely summable} \text{ property} && \text{(Definition I.2 page 203)}\\
&= \sum_{m\in\mathbb{Z}} x^*[m]\left(\frac{1}{z}\right)^{-m} && \text{by } \textit{absolutely summable} \text{ property} && \text{(Definition I.2 page 203)}\\
&= \left(\sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z^*}\right)^{-m}\right)^* && \text{by } \textit{absolutely summable} \text{ property} && \text{(Definition I.2 page 203)}
\end{aligned}
$$

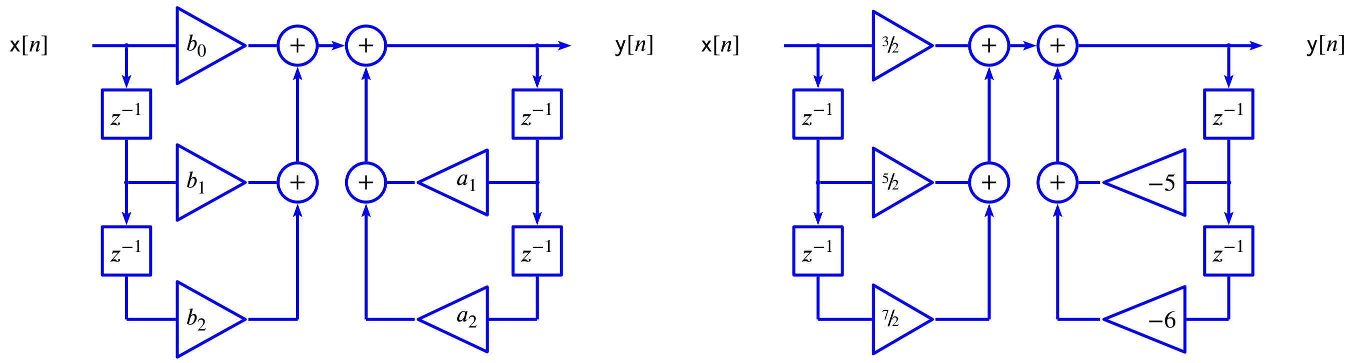


Figure I.1: Direct form 1 order 2 IIR filters

$$\triangleq \check{x}^*\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition I.4 page 204)}$$

Theorem I.2 (convolution theorem). *Let $\star$ be the convolution operator* (Definition I.3 page 203).

THM

$$\underbrace{\mathbf{Z}\left(\langle\!\langle x_n\rangle\!\rangle \star \langle\!\langle y_n\rangle\!\rangle\right)}_{\text{sequence convolution}} = \underbrace{\left(\mathbf{Z}\langle\!\langle x_n\rangle\!\rangle\right)\left(\mathbf{Z}\langle\!\langle y_n\rangle\!\rangle\right)}_{\text{series multiplication}} \qquad \forall \langle\!\langle x_n\rangle\!\rangle_{n\in\mathbb{Z}}, \langle\!\langle y_n\rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x \star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by definition of } \star && \text{(Definition I.3 page 203)}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k \triangleq n-m && \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right]\\
&\triangleq \left[\mathbf{Z}\langle\!\langle x_n\rangle\!\rangle\right]\left[\mathbf{Z}\langle\!\langle y_n\rangle\!\rangle\right] && \text{by definition of } \mathbf{Z} && \text{(Definition I.4 page 204)}
\end{aligned}$$

I.3 From z-domain back to time-domain

$$\check{y}(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) - a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Example I.1. See Figure I.1 (page 206)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2\left(z^2+5z+6\right)} = \frac{\left(\tfrac{3}{2}z^2+\tfrac{5}{2}z+\tfrac{7}{2}\right)}{z^2+5z+6} = \frac{\left(\tfrac{3}{2}+\tfrac{5}{2}z^{-1}+\tfrac{7}{2}z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

I.4 Zero locations

The system property of *minimum phase* is defined in Definition I.5 (next) and illustrated in Figure I.2 (page 207).

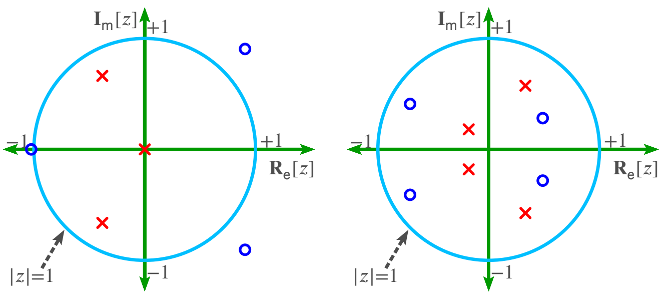


Figure I.2: Minimum Phase filter

Definition I.5. [4] *Let* $\check{x}(z) \triangleq \mathbf{Z}\left(\!\left(x_n\right)\!\right)$ *be the* Z TRANSFORM (Definition I.4 page 204) *of a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\left(\!\left(z_n\right)\!\right)_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\check{x}(z)$.

DEF

The sequence $\left(\!\left(x_n\right)\!\right)$ *is* ***minimum phase*** *if*

$$\underbrace{\left|z_n\right| < 1 \qquad \forall n \in \mathbb{Z}}_{\check{x}(z)\text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem I.3 (Robinson's Energy Delay Theorem). [5] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM

$$\left\{\begin{array}{lll} \mathsf{p} & \text{is MINIMUM PHASE} & \text{and} \\ \mathsf{q} & \text{is NOT } \textit{minimum phase} & \end{array}\right\} \implies \underbrace{\sum_{n=0}^{m-1}\left|a_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{p}(z)} \geq \underbrace{\sum_{n=0}^{m-1}\left|b_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{q}(z)} \qquad \forall 0 \leq m \leq N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle (Figure I.3 page 208).

Example I.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure I.3 (page 208).

[4] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36
[5] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976) pages 52–53

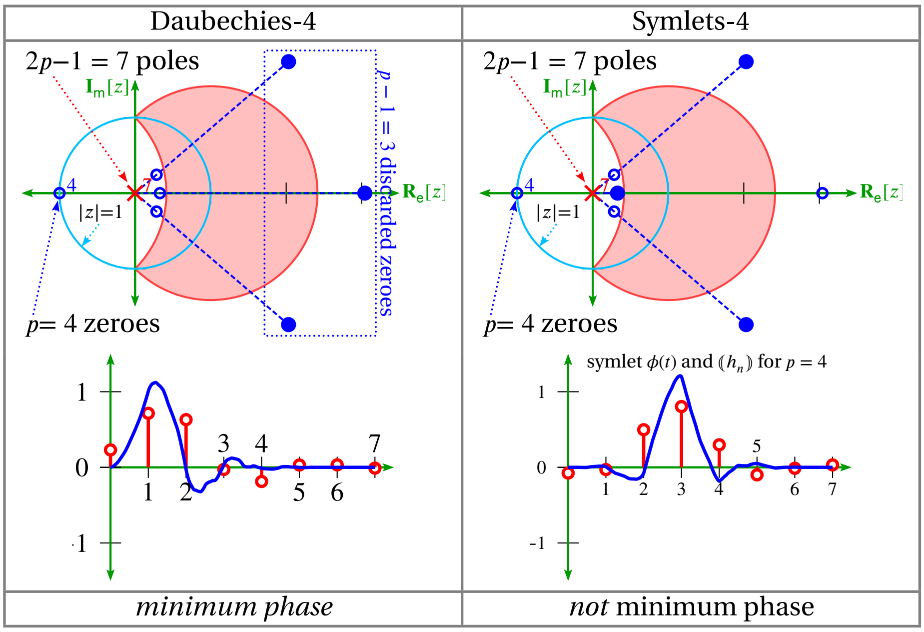


Figure I.3: Daubechies-4 and Symlet-4 scaling functions pole-zero plots

I.5 Pole locations

Definition I.6.

> DEF *A filter (or system or operator)* **H** *is* ***causal***
> *if its current output does not depend on future inputs.*

Definition I.7.

> DEF *A filter (or system or operator)* **H** *is* ***time-invariant***
> *if the mapping it performs does not change with time.*

Definition I.8.

> DEF *An operation* **H** *is* ***linear*** *if any output y_n can be described as a linear combination of inputs x_n as in*
> $$\mathsf{y}_n = \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\mathsf{x}(n-m)\ .$$

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem I.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Example I.3. Stable/unstable filters are illustrated in Figure I.4 (page 209).

True or False? This filter has no poles:

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

$$\mathsf{H}(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$

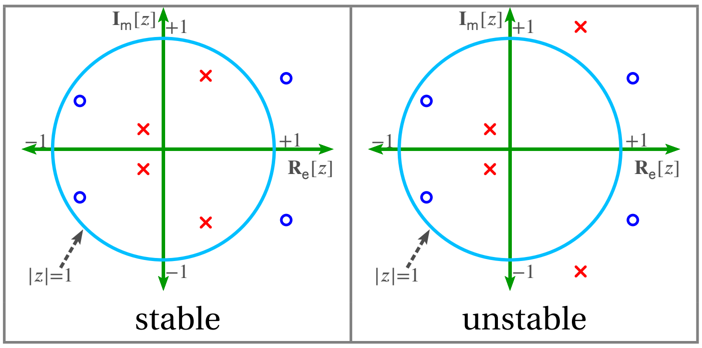


Figure I.4: Pole-zero plot stable/unstable causal LTI filters (Example I.3 page 208)

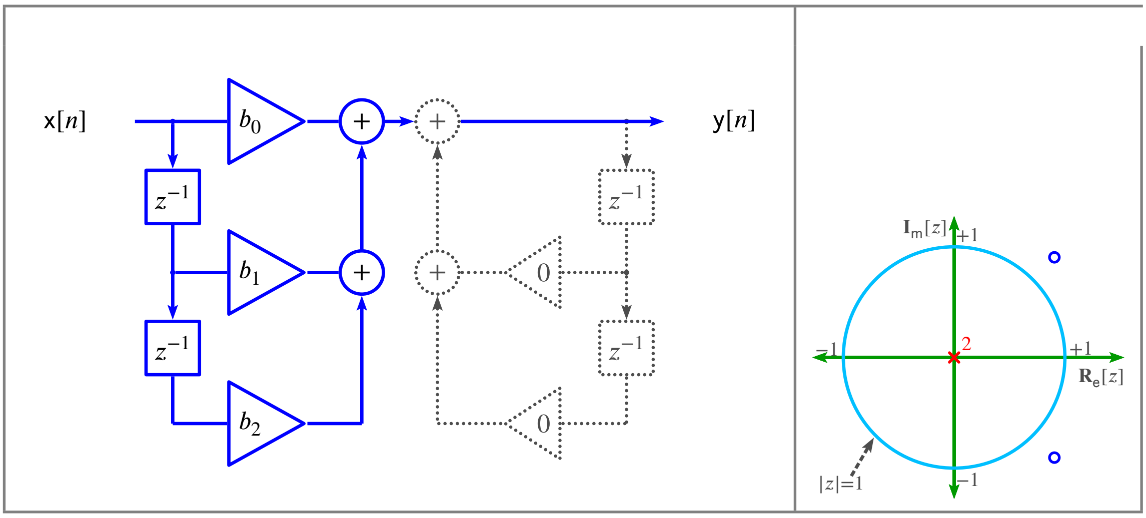


Figure I.5: FIR filters

I.6 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

Proposition I.3.

PRP $\left\{\begin{array}{l}\text{ZEROS } \textit{and} \text{ POLES}\\ \textit{occur in} \text{ CONJUGATE PAIRS}\end{array}\right\} \implies \left\{\begin{array}{l}\text{COEFFICIENTS}\\ \textit{are} \text{ REAL.}\end{array}\right\}$

✎PROOF:

$$\begin{aligned}\left(z - p_1\right)\left(z - p_1^*\right) &= [z - (a + ib)][z - (a - ib)]\\ &= z^2 + [-a + ib - ib - a]z - [ib]^2\end{aligned}$$

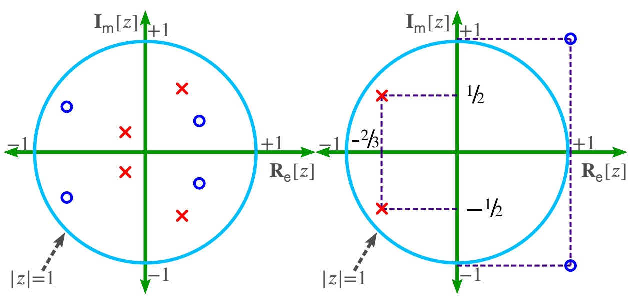


Figure I.6: Conjugate pair structure yielding real coefficients

$$= z^2 - 2az + b^2$$

✏

Example I.4. See Figure I.6 (page 209).

$$\begin{aligned}
H(z) &= G\frac{\left[z-z_1\right]\left[z-z_2\right]}{\left[z-p_1\right]\left[z-p_2\right]} = G\frac{[z-(1+i)][z-(1-i)]}{[z-(-2/3+i1/2)][z-(-2/3-i1/2)]}\\
&= G\frac{z^2-z[(1-i)+(1+i)]+(1-i)(1+i)}{z^2-z[(-2/3+i1/2)+(-2/3+i1/2)]+(-2/3+i1/2)(-2/3+i1/2)}\\
&= G\frac{z^2-2z+2}{z^2-4/3z+(4/3+1/4)} = G\frac{z^2-2z+2}{z^2-4/3z+19/12}
\end{aligned}$$

I.7 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma I.1. *A causal LTI operator* **H** *can be expressed as a rational expression* $\check{h}(z)$.

$$\begin{aligned}
\check{h}(z) &= \frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Nz^{-N}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}\\
&= \frac{\sum\limits_{n=0}^{N} b_nz^{-n}}{1+\sum\limits_{n=1}^{N} a_nz^{-n}}
\end{aligned}$$

A filter operation $\check{h}(z)$ can be expressed as a product of its roots (poles and zeros).

$$\begin{aligned}
\check{h}(z) &= \frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Nz^{-N}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}\\
&= \alpha\frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}
\end{aligned}$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with $\times$ and zeros with $\bigcirc$. An example is shown in Figure I.7 page 211. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

I.8 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{x}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{y}(z)$ is a *high-pass filter*.

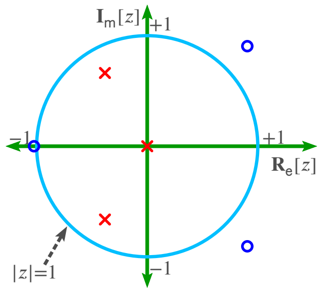


Figure I.7: Pole-zero plot for rational expression with real coefficients

Definition I.9. [6] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES (Definition I.1 page 203) *in* $\ell^2_{\mathbb{R}}$ (Definition I.2 page 203).

DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter***.
Any triple $\left((\!(x_n)\!), (\!(y_n)\!), N\right)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition***.

Theorem I.5 (CQF theorem). [7] *Let* $\breve{y}(\omega)$ *and* $\breve{x}(\omega)$ *be the* DTFTs (Definition L.1 page 223) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$, *respectively, in* $\ell^2_{\mathbb{R}}$ (Definition I.2 page 203).

THM

$$\begin{array}{llclll}
\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{\text{(1) CQF in "time"}} & \iff & \breve{y}(z) & = & \pm(-1)^N z^{-N}\breve{x}^*\left(\dfrac{-1}{z^*}\right) & \text{(2) CQF in "z-domain"}\\
& \iff & \breve{y}(\omega) & = & \pm(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi) & \text{(3) CQF in "frequency"}\\
& \iff & x_n & = & \pm(-1)^N(-1)^n y^*_{N-n} & \text{(4) "reversed" CQF in "time"}\\
& \iff & \breve{x}(z) & = & \pm z^{-N}\breve{y}^*\left(\dfrac{-1}{z^*}\right) & \text{(5) "reversed" CQF in "z-domain"}\\
& \iff & \breve{x}(\omega) & = & \pm e^{-i\omega N}\breve{y}^*(\omega+\pi) & \text{(6) "reversed" CQF in "frequency"}
\end{array}$$

$\forall N\in\mathbb{Z}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{y}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-transform} \quad \text{(Definition I.4 page 204)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm\sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies \quad n = N-m\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*
\end{aligned}$$

[6] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

[7] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

$$= \pm(-1)^N z^{-N} \breve{x}^*\left(\frac{-1}{z^*}\right) \qquad \text{by definition of } z\text{-transform} \quad \text{(Definition I.4 page 204)}$$

2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\breve{y}(z) &= \pm(-1)^N z^{-N} \breve{x}^*\left(\frac{-1}{z^*}\right) && \text{by (2)}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-transform} \quad \text{(Definition I.4 page 204)}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m^* \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-transform} \quad \text{(Definition I.4 page 204)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x_m^* z^{-(N-m)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^n x_{N-n}^* z^{-n} && \text{where } n = N-m \implies \quad m \triangleq N-n\\
&\implies x_n = \pm(-1)^n x_{N-n}^*
\end{aligned}$$

3. Proof that (1) $\implies$ (3):

$$\begin{aligned}
\breve{y}(\omega) &\triangleq \breve{x}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition L.1 page 223)}\\
&= \left[\pm(-1)^N z^{-N} \breve{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)}\\
&= \pm(-1)^N e^{-i\omega N} \breve{x}\left(e^{i\pi} e^{i\omega}\right)\\
&= \pm(-1)^N e^{-i\omega N} \breve{x}\left(e^{i(\omega+\pi)}\right)\\
&= \pm(-1)^N e^{-i\omega N} \breve{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition L.1 page 223)}
\end{aligned}$$

4. Proof that (1) $\implies$ (6):

$$\begin{aligned}
\breve{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT \quad \text{(Definition L.1 page 223)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x_{N-n}^*}_{CQF} e^{-i\omega n} && \text{by (1)}\\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x_m^* e^{-i\omega(N-m)} && \text{where } m \triangleq N-n \implies \quad n = N-m\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} (-1)^m x_m^* e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} e^{i\pi m} x_m^* e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} x_m^* e^{i(\omega+\pi)m}\\
&= \pm(-1)^N e^{-i\omega N}\left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^*\\
&= \pm(-1)^N e^{-i\omega N} \breve{x}^*(\omega+\pi) && \text{by definition of } DTFT \quad \text{(Definition L.1 page 223)}
\end{aligned}$$

5. Proof that (1) $\Longleftarrow$ (3):

$$\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{\mathsf{y}}(\omega)e^{i\omega n}\,\mathrm{d}\omega && \text{by } \textit{inverse DTFT} && \text{(Theorem L.3 page 229)}\\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\breve{\mathsf{x}}^*(\omega+\pi)}_{\text{right hypothesis}}e^{i\omega n}\,\mathrm{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{\mathsf{x}}^*(\omega+\pi)e^{i\omega(n-N)}\,\mathrm{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{0}^{2\pi} \breve{\mathsf{x}}^*(v)e^{i(v-\pi)(n-N)}\,\mathrm{d}v && \text{where } v \triangleq \omega+\pi \implies && \omega = v-\pi\\
&= \pm(-1)^N e^{-i\pi(n-N)}\frac{1}{2\pi}\int_{0}^{2\pi} \breve{\mathsf{x}}^*(v)e^{iv(n-N)}\,\mathrm{d}v\\
&= \pm(-1)^N\underbrace{(-1)^N}_{e^{i\pi N}}\underbrace{(-1)^n}_{e^{-i\pi n}}\left[\frac{1}{2\pi}\int_{0}^{2\pi} \breve{\mathsf{x}}(v)e^{iv(N-n)}\,\mathrm{d}v\right]^*\\
&= \pm(-1)^n x^*_{N-n} && \text{by } \textit{inverse DTFT} && \text{(Theorem L.3 page 229)}
\end{aligned}$$

6. Proof that (1) $\Longleftrightarrow$ (4):

$$\begin{aligned}
y_n = \pm(-1)^n x^*_{N-n} &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x^*_{N-n}\\
&\iff \pm(-1)^n y_n = x^*_{N-n}\\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x^*_{N-n}\right)^*\\
&\iff \pm(-1)^n y^*_n = x_{N-n}\\
&\iff x_{N-n} = \pm(-1)^n y^*_n\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m} && \text{where } m \triangleq N-n \implies && n = N-m\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m}\\
&\iff x_m = \pm(-1)^{N}(-1)^m y^*_{N-m}\\
&\iff x_n = \pm(-1)^{N}(-1)^n y^*_{N-n} && \text{by change of free variables}
\end{aligned}$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

⇛

Theorem I.6. [8] *Let $\breve{\mathsf{y}}(\omega)$ and $\breve{\mathsf{x}}(\omega)$ be the* DTFTs (Definition L.1 page 223) *of the sequences $(\!(y_n)\!)_{n\in\mathbb{Z}}$ and $(\!(x_n)\!)_{n\in\mathbb{Z}}$, respectively, in $\ell^2_{\mathbb{R}}$* (Definition I.2 page 203).

THM *Let $y_n = \pm(-1)^n x^*_{N-n}$* (CQF CONDITION, *Definition I.9 page 211*). *Then*

$$\left\{\begin{array}{llcllll}
\text{(A)} & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{y}}(\omega)\Big|_{\omega=0} = 0 & \iff & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\Big|_{\omega=\pi} & = & 0 & \text{(B)}\\
 & & \iff & \sum_{k\in\mathbb{Z}}(-1)^k k^n x_k & = & 0 & \text{(C)}\\
 & & \iff & \sum_{k\in\mathbb{Z}} k^n y_k & = & 0 & \text{(D)}
\end{array}\right\} \quad \forall n\in\mathbb{W}$$

✎PROOF:

[8] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

1. Proof that (A) $\implies$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{y}(\omega)\bigg|_{\omega=0} && \text{by (A)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi)\bigg|_{\omega=0} && \text{by } \textit{CQF theorem} \quad \text{(Theorem I.5 page 211)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^\ell \left[e^{-i\omega N}\right]\cdot\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\bigg|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma D.2 page 143)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell e^{-i\omega N}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\bigg|_{\omega=0}\\
&= (\pm)(-1)^N \cancel{e^{-i0N}} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\bigg|_{\omega=0}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{x}^{(0)}(\pi) &= 0\\
&\implies & \breve{x}^{(1)}(\pi) &= 0\\
&\implies & \breve{x}^{(2)}(\pi) &= 0\\
&\implies & \breve{x}^{(3)}(\pi) &= 0\\
&\implies & \breve{x}^{(4)}(\pi) &= 0\\
&\vdots & &\vdots\\
&\implies & \breve{x}^{(n)}(\pi) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

2. Proof that (A) $\impliedby$ (B):

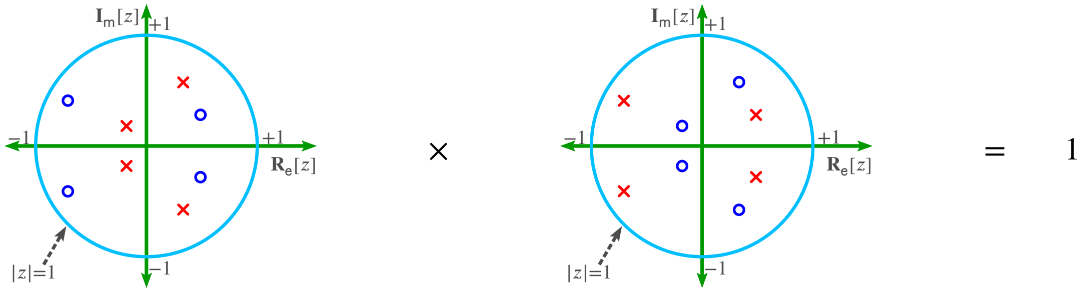
$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=\pi} && \text{by (B)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)e^{-i\omega N}\breve{y}^*(\omega+\pi)\bigg|_{\omega=\pi} && \text{by } \textit{CQF theorem} \quad \text{(Theorem I.5 page 211)}\\
&= (\pm)\sum_{\ell=0}^{n} \binom{n}{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^\ell \left[e^{-i\omega N}\right]\cdot\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma D.2 page 143)}\\
&= (\pm)\sum_{\ell=0}^{n} \binom{n}{\ell}(-iN)^\ell e^{-i\omega N}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}\\
&= (\pm)\cancel{e^{-i\pi N}}\sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}\\
&= (\pm)(-1)^N\sum_{\ell=0}^{n} \binom{n}{\ell} -iN^\ell \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{y}^{(0)}(0) &= 0\\
&\implies & \breve{y}^{(1)}(0) &= 0\\
&\implies & \breve{y}^{(2)}(0) &= 0\\
&\implies & \breve{y}^{(3)}(0) &= 0\\
&\implies & \breve{y}^{(4)}(0) &= 0\\
&\vdots & &\vdots\\
&\implies & \breve{y}^{(n)}(0) &= 0\\
&\implies & \breve{y}^{(n)}(0) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

3. Proof that (B) $\iff$ (C): by Theorem L.5 page 231

4. Proof that (A) $\iff$ (D): by Theorem L.5 page 231

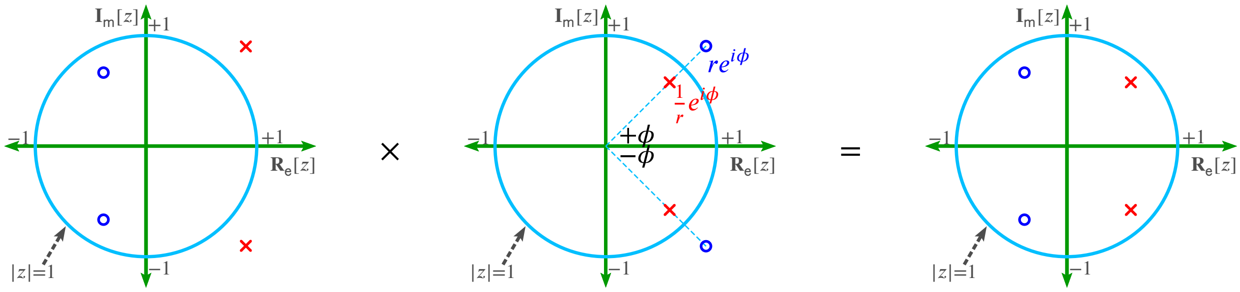
5. Proof that (CQF) $\not\Leftarrow$ (A): Here is a counterexample: $\breve{y}(\omega) = 0$.



$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

I.9 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.



unstable — all-pass — stable!

$$\begin{aligned}
|A(z)|_{z=e^{i\omega}} &= \frac{1}{r}\left|\frac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}} = \left|\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|e^{i\phi}\left(\frac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} = \left|z\left(\frac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|-z\left(\frac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} = \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\frac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
&= \left|\frac{1}{e^{-iv}}\left(\frac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right| = \left|\frac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right| \\
&= 1
\end{aligned}$$

APPENDIX J

TAYLOR EXPANSIONS (TRANSFORMS)

J.1 Introduction

For modeling real-world processes above the quantum level, measurements are *continuous* in time—that is, the first derivative of a function over time representing the measurement *exists*.

But even for "simple" physical systems, it is not just the first derivative that matters. For example, the classical "vibrating string" vertical displacement $\mathsf{u}(x,t)$ wave equation can be described as

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Not only do physical systems demonstrate heavy dependence on the derivatives of their measurement functions, but also commonly exhibit *oscillation*, as demonstrated by sunspot activity over the last 300 years or earthquake activity (Figure J.1 page 218).

In fact, derivatives and oscillations are fundamentally linked as demonstrated by the fact that all solutions of homogeneous second order differential equations are linear combinations of sine and cosine functions (Theorem F.3 page 156):

$$\left\{\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \mathsf{f}'(0)\sin(x)\right\} \qquad \forall \mathsf{f}\in\boldsymbol{C}, \forall x\in\mathbb{R}$$

Derivatives are calculated *locally* about a point. Oscillations are observed *globally* over a range, and analyzed (decomposed) by projecting the function onto a sequence of basis functions—sinusoids in the case of Fourier Transform family. Projection is accomplished using inner products, and often these are calculated using *integration*. Note that derivatives and integrals are also fundamentally linked as demonstrated by the *Fundamental Theorem of Calculus*...which shows that integration can be calculated using anti-differentiation:

$$\int_a^b \mathsf{f}(x)\,\mathsf{d}x = \mathsf{F}(b) - \mathsf{F}(a) \qquad \text{where } \mathsf{F}(x) \text{ is the } \textit{antiderivative} \text{ of } \mathsf{f}(x).$$

Brook Taylor showed that for *analytic* functions,[1] knowledge of the derivatives of a function at a location $x = a$ allows you to determine (predict) arbitrarily closely all the points $\mathsf{f}(x)$ in the vicinity

[1] *analytic* functions: Functions for which all their derivatives exist.

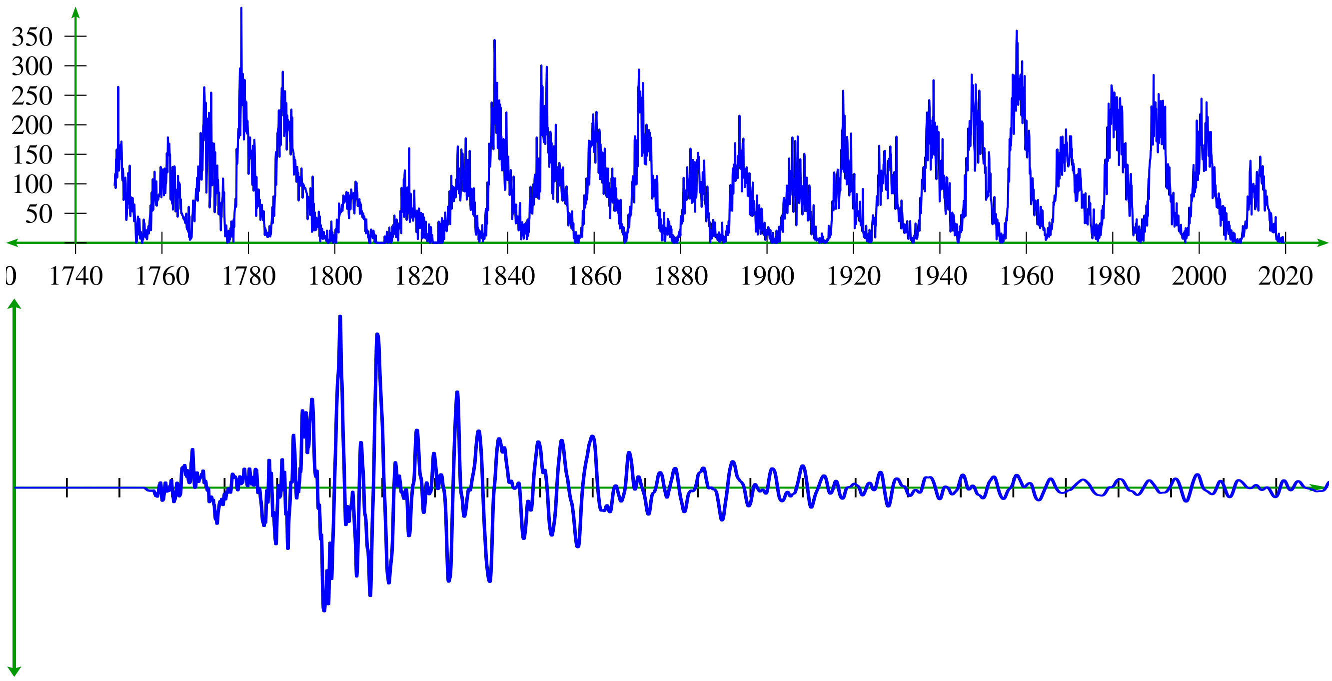


Figure J.1: Sunspot and earthquake measurements

of $x = a$:[2]

$$\mathsf{f}(x) = \mathsf{f}(a) + \frac{1}{1!}\mathsf{f}'(a)[x-a] + \frac{1}{2!}\mathsf{f}''(x)[x-a]^2 + \frac{1}{3!}\mathsf{f}'''(x)[x-a]^3 + \cdots$$

On the other hand, the *Fourier Transform* is a kind of counter-part of the Taylor expansion:[3]

	Taylor coefficients	Fourier coefficients
	Depend on derivatives $\frac{\mathsf{d}^n}{\mathsf{d}x^n}\mathsf{f}(x)$	Depend on integrals $\int_{x\in\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x$
	Behavior in the vicinity of a point.	Behavior over the entire function.
	Demonstrate trends locally.	Demonstrate trends globally, such as oscillations.
	Admits *analytic* functions only.	Admits *non-analytic* functions as well.
	Function must be *continuous*.	Function can be *discontinuous*.

J.2 Taylor Expansion

Theorem J.1 (Taylor Series). [4] *Let* $\boldsymbol{C}$ *be the space of all* ANALYTIC *functions and* $\frac{\mathsf{d}}{\mathsf{d}x}$ *in* $\boldsymbol{C}^{\boldsymbol{C}}$ *the* DIFFERENTIATION OPERATOR.

THM

A ***Taylor Series*** *about the point* $x = a$ *of a function* $\mathsf{f}(x) \in \boldsymbol{C}^{\boldsymbol{C}}$ *is*

$$\mathsf{f}(x) \quad = \quad \sum_{n=0}^{\infty} \underbrace{\frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}^n \mathsf{f}\right](a)}{n!}}_{\text{coefficient}} \underbrace{(x-a)^n}_{\text{basis function}} \qquad \forall a\in\mathbb{R}, \mathsf{f}\in\boldsymbol{C}$$

A ***Maclaurin Series*** *is a* TAYLOR SERIES *about the point* $a = 0$.

[2] Robinson (1982) page 886
[3] Robinson (1982) page 886
[4] Flanigan (1983) page 221 ⟨Theorem 15⟩, Strichartz (1995) page 281, Sohrab (2003) page 317 ⟨Theorem 8.4.9⟩, Taylor (1715), Maclaurin (1742)

APPENDIX K

LAPLACE TRANSFORM

K.1 Definition

Definition K.1. *Let* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF *The **Laplace Transform** operator* $\mathbf{L}$ *is here defined as*

$$[\mathbf{L}\mathsf{f}](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x\in\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

In Definition K.1, the scaling factor $\frac{1}{\sqrt{2\pi}}$ is not normally found in most definitions of the Laplace Transform. However it is included here to make the operator $\mathbf{L}$ more directly compatible with the *Unitary Fourier Transform* operator $\tilde{\mathbf{F}}$ (Definition H.2 page 192).

K.2 Shift relations

Theorem K.1 (Shift relations). *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition K.1 page 219).

THM

$$\begin{aligned}
\mathbf{L}[\mathsf{f}(x-y)](s) &= e^{-sy}\,[\mathbf{L}\mathsf{f}(x)](s) \\
\left[\mathbf{L}\left(e^{rx}\mathsf{g}(x)\right)\right](s) &= \left[\mathbf{L}\mathsf{g}(x)\right](s-r)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{L}[\mathsf{f}(x-y)](s) &= \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}} \mathsf{f}(x-y)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{L} && \text{(Definition K.1 page 219)} \\
&= \frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-s(y+u)}\,\mathrm{d}u && \text{where } u \triangleq x-y && \implies x = y+u \\
&= e^{-sy}\,\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u && \text{by property of exponents} && a^{x+y} = a^x a^y \\
&= e^{-sy}\,\frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}u && \text{by change of variable} && u \to x \\
&= e^{-sy}\,[\mathbf{L}\mathsf{f}(x)](s) && \text{by definition of } \mathbf{L} && \text{(Definition K.1 page 219)}
\end{aligned}$$

$$\begin{aligned}
\left[\mathbf{L}(e^{rx}\mathsf{g}(x))\right](s) &= \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}} e^{rx}\mathsf{g}(x)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{L} && \text{(Definition K.1 page 219)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}} \mathsf{g}(x)e^{-(s-r)x}\,\mathrm{d}x && \text{by property of exponents} && a^{x+y}=a^x a^y\\
&= \left[\mathbf{L}\mathsf{g}(x)\right](s-r) && \text{by definition of } \mathbf{L} && \text{(Definition K.1 page 219)}
\end{aligned}$$

⇨

K.3 Convolution relations

Definition K.2. [1]

DEF *The **convolution operation** is defined as*

$$\left[\mathsf{f}\star\mathsf{g}\right](x) \triangleq \mathsf{f}(x)\star\mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem K.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "s domain" and vice-versa.

Theorem K.2 (convolution theorem). *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition K.1 page 219) *and* $\star$ *the convolution operator* (Definition K.2 page 220).

THM

$$\underbrace{\mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s)}_{\text{multiplication in "s domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\mathbf{L}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s)\star\left[\mathbf{L}\mathsf{g}\right](s)}_{\text{convolution in "s domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](s) &= \mathbf{L}\left[\int_{u\in\mathbb{R}}\mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u\right](s) && \text{by definition of } \star && \text{(Definition K.2 page 220)}\\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)\left[\mathbf{L}\mathsf{g}(x-u)\right](s)\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-su}\left[\mathbf{L}\mathsf{g}(x)\right](s)\,\mathrm{d}u && \text{by \textit{Fourier shift theorem}} && \text{(Theorem H.4 page 194)}\\
&= \sqrt{2\pi}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-su}\,\mathrm{d}u\right)}_{[\mathbf{L}\mathsf{f}](s)}\left[\mathbf{L}\mathsf{g}\right](s)\\
&= \sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s) && \text{by definition of } \mathbf{L} && \text{(Definition H.2 page 192)}\\
\mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s) &= \mathbf{L}\left[\left(\mathbf{L}^{-1}\mathbf{L}\mathsf{f}(x)\right)\mathsf{g}(x)\right](s) && \text{by def. of operator inverse} && \text{(Definition C.3 page 112)}\\
&= \mathbf{L}\left[\left(\frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}[\mathbf{L}\mathsf{f}(x)](v)e^{sxv}\,\mathrm{d}v\right)\mathsf{g}(x)\right](s) && \text{by Theorem H.1 page 193}\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}[\mathbf{L}\mathsf{f}(x)](v)\left[\mathbf{L}\left(e^{sxv}\,\mathsf{g}(x)\right)\right](s,v)\,\mathrm{d}v\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}[\mathbf{L}\mathsf{f}(x)](v)\left[\mathbf{L}\mathsf{g}(x)\right](s-v)\,\mathrm{d}v && \text{by Theorem H.4 page 194}\\
&= \frac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s)\star\left[\mathbf{L}\mathsf{g}\right](s) && \text{by definition of } \star && \text{(Definition K.2 page 220)}
\end{aligned}$$

[1] Bachman (1964) page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩

K.4 Calculus relations

Theorem K.3. *Let* **L** *be the* LAPLACE TRANSFORM *operator* (Definition K.1 page 219).

$$\text{THM}\quad \left\{\lim_{t\to-\infty} \mathsf{x}(t) = 0\right\} \implies \left\{\mathbf{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t)\right] = s[\mathbf{L}\mathsf{x}](s)\right\}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t)\right]} &\triangleq \int_{t\in\mathbb{R}} \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t)\right]}_{\mathrm{d}v}\underbrace{e^{-st}}_{u}\,\mathrm{d}t && \text{by definition of } \mathbf{L}\\
&= \underbrace{e^{-st}}_{u}\underbrace{\mathsf{x}(t)}_{v}\Big|_{t=-\infty}^{t=+\infty} - \int_{t\in\mathbb{R}} \underbrace{\mathsf{x}(t)}_{v}\underbrace{(-s)e^{-st}\,\mathrm{d}t}_{\mathrm{d}u} && \text{by \textit{Integration by Parts}}\\
&= \cancelto{0}{e^{-s\infty}}\mathsf{x}(\infty) - e^{s\infty}\cancelto{0}{\mathsf{x}(-\infty)} - (-s)\underbrace{\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathrm{d}t}_{\text{\textit{Laplace Transform} of } \mathsf{x}(t)} && \text{by left hypothesis}\\
&= \boxed{s[\mathbf{L}\mathsf{x}](s)}
\end{aligned}$$

Theorem K.4. *Let* **L** *be the* LAPLACE TRANSFORM *operator* (Definition K.1 page 219).

$$\text{THM}\quad \mathbf{L}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathrm{d}u = \frac{1}{s}[\mathbf{L}\mathsf{x}](s)$$

✎PROOF:

1. Define the *Heaviside function* $\mathsf{h}(t)$ as $\mathsf{h}(t) \triangleq \begin{cases} 1 & \text{for } t \ge 0\\ 0 & \text{otherwise}\end{cases}$

2. Remainder of proof…

$$\begin{aligned}
\boxed{\mathbf{L}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathrm{d}u} &\triangleq \int_{t=-\infty}^{t=+\infty}\left[\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathrm{d}u\right]e^{-st}\,\mathrm{d}t && \text{by definition of } \mathbf{L}\\
&= \int_{t=-\infty}^{t=+\infty}\left[\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)\mathsf{h}(t-u)\,\mathrm{d}u\right]e^{-st}\,\mathrm{d}t && \left(\begin{array}{l}\text{by definition of \textit{Heaviside function}}\\ \text{definition 1}\end{array}\right)\\
&= \int_{v=-\infty}^{v=+\infty}\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)\mathsf{h}(v)e^{-s(u+v)}\,\mathrm{d}u\,\mathrm{d}v && \left(\begin{array}{ll}\text{where} & v \triangleq t-u\\ \implies & t = u+v\end{array}\right)\\
&= \left[\int_{v=-\infty}^{v=+\infty} \mathsf{h}(v)e^{-sv}\,\mathrm{d}v\right]\underbrace{\left[\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)e^{-su}\,\mathrm{d}u\right]}_{\text{\textit{Laplace Transform} of } \mathsf{x}(t)}\\
&= \left[\int_{v=0}^{v=\infty} e^{-sv}\,\mathrm{d}v\right][\mathbf{L}\mathsf{x}](s) && \left(\begin{array}{l}\text{by definition of \textit{Heaviside function}}\\ \text{definition 1}\end{array}\right)
\end{aligned}$$

$$= \frac{1}{-s} e^{-sv} \Big|_{v=0}^{v=\infty} [\mathbf{L}\mathrm{x}](s) \qquad \text{by } \textit{Fundamental Theorem of Calculus}$$

$$= \boxed{\frac{1}{s}[\mathbf{L}\mathrm{x}](s)}$$

⇰

APPENDIX L

DISCRETE TIME FOURIER TRANSFORM

L.1 Definition

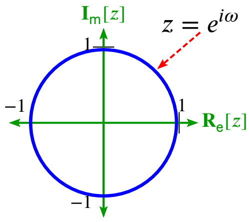
Definition L.1.

DEF

The ***discrete-time Fourier transform*** $\breve{\mathbf{F}}$ *of* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *is defined as*

$$\left[\breve{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition L.1 page 223) to the definition of the Z-transform (Definition I.4 page 204), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.



L.2 Properties

Proposition L.1 (DTFT periodicity). *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition L.1 page 223) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP

$$\underbrace{\breve{x}(\omega) = \breve{x}(\omega + 2\pi n)}_{\text{PERIODIC } \textit{with period } 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\breve{x}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} & &= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi nm}} \\
&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} & &= \breve{x}(\omega)
\end{aligned}$$

✏

Theorem L.1. *Let* $\tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition L.1 page 223) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

T H M
$$\left\{ \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x[n])\!) \right\} \implies \left\{ \begin{array}{llcll} (1). & \breve{\mathbf{F}}(\!(x[-n])\!) & = & \tilde{x}(-\omega) & \text{and} \\ (2). & \breve{\mathbf{F}}(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition L.1 page 223)} \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m} \\
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x^*[n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition L.1 page 223)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\mathbf{algebras} && \text{(Definition E.3 page 146)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{-i(-\omega)n}\right)^* \\
&\triangleq \tilde{x}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x^*[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition L.1 page 223)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[-n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\mathbf{algebras} && \text{(Definition E.3 page 146)} \\
&= \left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n = -m \\
&\triangleq \tilde{x}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

⊳

Theorem L.2. *Let* $\tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition L.1 page 223) of a sequence* $(\!(x[n])\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

T H M
$$\left\{ \begin{array}{lll} (1). & \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x[n])\!) & \text{and} \\ (2). & (\!(x[n])\!) \text{ is REAL-VALUED} & \end{array} \right\} \implies \left\{ \begin{array}{llcllll} (1). & \breve{\mathbf{F}}(\!(x[-n])\!) & = & \tilde{x}(-\omega) & & & \text{and} \\ (2). & \breve{\mathbf{F}}(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & = & \tilde{x}(\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & = & \tilde{x}(-\omega) & \end{array} \right\}$$

✎PROOF:

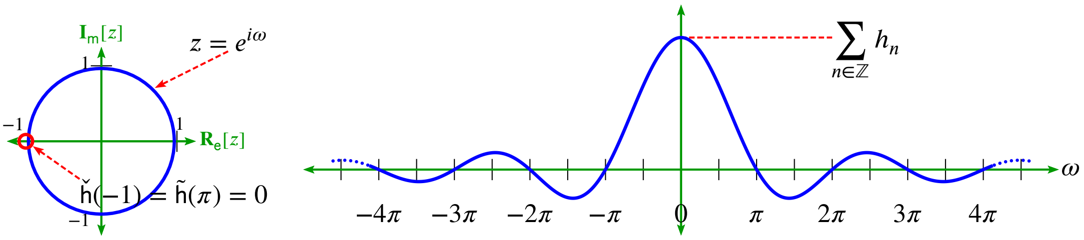
$$\begin{aligned}
\breve{\mathbf{F}}(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition L.1 page 223)} \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}
\end{aligned}$$

$$\triangleq \breve{\mathsf{x}}(-\omega) \qquad \text{by left hypothesis}$$

$$\boxed{\tilde{\mathsf{x}}^*(-\omega)} = \boxed{\breve{\mathbf{F}}\,(\!(\mathsf{x}^*[n])\!)} \qquad \text{by Theorem L.1 page 224}$$
$$= \breve{\mathbf{F}}\,(\!(\mathsf{x}[n])\!) \qquad \text{by } \textit{real-valued} \text{ hypothesis}$$
$$= \boxed{\tilde{\mathsf{x}}(\omega)} \qquad \text{by definition of } \tilde{\mathsf{x}}(\omega) \quad \text{(Definition L.1 page 223)}$$

$$\boxed{\tilde{\mathsf{x}}^*(\omega)} = \boxed{\breve{\mathbf{F}}\,(\!(\mathsf{x}^*[-n])\!)} \qquad \text{by Theorem L.1 page 224}$$
$$= \breve{\mathbf{F}}\,(\!(\mathsf{x}[-n])\!) \qquad \text{by } \textit{real-valued} \text{ hypothesis}$$
$$= \boxed{\tilde{\mathsf{x}}(-\omega)} \qquad \text{by result (1)}$$

✏



Proposition L.2. *Let* $\breve{\mathsf{x}}(z)$ *be the* Z-TRANSFORM *(Definition I.4 page 204) and* $\breve{\mathsf{x}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition L.1 page 223) of* $(\!(x_n)\!)$.

PRP

$$\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}, c\in\mathbb{R}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\breve{\mathsf{x}}(z)\Big|_{z=1} = \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} \qquad \text{by definition of } \breve{\mathsf{x}}(z) \text{ (Definition I.4 page 204)}$$
$$= \sum_{n\in\mathbb{Z}} x_n \qquad \text{because } z^n = 1 \text{ for all } n\in\mathbb{Z}$$
$$= c \qquad \text{by hypothesis (1)}$$

2. Proof that (2) $\implies$ (3):

$$\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} = \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} \qquad \text{by definition of } \breve{\mathsf{x}}(\omega) \quad \text{(Definition L.1 page 223)}$$
$$= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1}$$
$$= \breve{\mathsf{x}}(z)\Big|_{z=1} \qquad \text{by definition of } \breve{\mathsf{x}}(z) \quad \text{(Definition I.4 page 204)}$$
$$= c \qquad \text{by hypothesis (2)}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=0} \\
&= \breve{\mathsf{x}}(\omega) && \text{by definition of } \breve{\mathsf{x}}(\omega) && \text{(Definition L.1 page 223)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

✎

Proposition L.3. *If the coefficients are* ***real****, then the magnitude response (MR) is* ***symmetric****.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathsf{h}}(-\omega)\right| &\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{i\omega m}\right| && = \left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}^*[m]e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } real} && = \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right| \\
&\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\tilde{\mathsf{h}}(\omega)\right|
\end{aligned}$$

✎

Proposition L.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}}(-1)^n x_n = c}_{\text{(1) in "time"}} \iff \underbrace{\breve{\mathsf{x}}(z)|_{z=-1} = c}_{\text{(2) in "z domain"}} \iff \underbrace{\breve{\mathsf{x}}(\omega)|_{\omega=\pi} = c}_{\text{(3) in "frequency"}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{\mathsf{x}}(z)|_{z=-1} &= \left.\sum_{n\in\mathbb{Z}} x_n z^{-n}\right|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}}(-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

2. Proof that (2) $\Longrightarrow$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\bigg|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \sum_{n\in\mathbb{Z}} z^{-n} x_n\bigg|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\Longrightarrow$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n\bigg|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\Longrightarrow$ (4):

(a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

(b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

(e) Solutions to these equations give

$$\sum_{n\in\mathbb{Z}} x_{2n} \triangleq A = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)$$
$$\sum_{n\in\mathbb{Z}} x_{2n+1} \triangleq B = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1} \\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1} \\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)} \\
&= c
\end{aligned}$$

✏

Lemma L.1. *Let* $\tilde{f}(\omega)$ *be the* DTFT (Definition L.1 page 223) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

LEM

$$\underbrace{(x_n \in \mathbb{R})_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{x}(\omega)|^2 = |\breve{x}(-\omega)|^2}_{\text{EVEN}} \qquad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
|\breve{x}(\omega)|^2 &= |\breve{x}(z)|^2\Big|_{z=e^{i\omega}} \\
&= \breve{x}(z)\breve{x}^*(z)\Big|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^*(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right]$$
$$= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2\sum_{n\in\mathbb{Z}}\sum_{m>n} x_n x_m \cos[\omega(m-n)]$$

Since cos is real and even, then $|\breve{x}(\omega)|^2$ must also be real and even. ✏

Theorem L.3 (inverse DTFT). [2] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition L.1 page 223) of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\tilde{x}^{-1}$ be the inverse of $\breve{x}$.*

THM
$$\underbrace{\left\{ \breve{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\breve{x}(\omega)\triangleq\breve{\mathbf{F}}(x_n)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega)e^{i\omega n}\,d\omega \quad \forall\alpha\in\mathbb{R} \right\}}_{(x_n)=\breve{\mathbf{F}}^{-1}\breve{\mathbf{F}}(x_n)} \quad \forall (x_n)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega)e^{i\omega n}\,d\omega &= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m}\right]}_{\breve{x}(\omega)} e^{i\omega n}\,d\omega && \text{by definition of } \breve{x}(\omega)\\
&= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\,d\omega\\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\,d\omega\\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \left[2\pi\bar{\delta}_{m-n}\right]\\
&= x_n
\end{aligned}$$

✏

Theorem L.4 (orthonormal quadrature conditions). [3] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition L.1 page 223) of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\bar{\delta}_n$ be the* KRONECKER DELTA FUNCTION *at n (Definition 2.12 page 20).*

THM
$$\begin{array}{rcll}
\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 & \iff & \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0 & \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}}\\
\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n & \iff & |\breve{x}(\omega)|^2 + |\breve{x}(\omega+\pi)|^2 = 2 & \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}}
\end{array}$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩
[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

1. Proof that $2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)$:

$$\begin{aligned}
&2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n}\left(1+e^{i\pi n}\right)\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_{m} z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_{m} e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n\\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_{m} z^{m} + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_{m} e^{+i(\omega+\pi)m}\\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^*\\
&\triangleq \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)
\end{aligned}$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 0 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n}\\
&= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 2 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

✏

L.3 Derivatives

Theorem L.5. [4] *Let $\breve{x}(\omega)$ be the* DTFT (Definition L.1 page 223) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$.

THM

$$\text{(A)}\quad \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} = 0 \iff \sum_{k\in\mathbb{Z}} k^n x_k = 0 \quad \text{(B)} \quad \forall n\in\mathbb{W}$$

$$\text{(C)}\quad \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=\pi} = 0 \iff \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 \quad \text{(D)} \quad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} && \text{by hypothesis (A)}\\
&= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \breve{x}(\omega) \text{ (Definition L.1 page 223)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} &= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \tilde{g}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k\\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

[4] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

3. Proof that $(C) \implies (D)$:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} && \text{by hypothesis (C)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=\pi} && \text{by definition of } \breve{x} \text{ (Definition L.1 page 223)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k
\end{aligned}$$

4. Proof that $(C) \impliedby (D)$:

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=\pi} && \text{by definition of } \breve{x} \text{ (Definition L.1 page 223)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k\\
&= 0 && \text{by hypothesis (D)}
\end{aligned}$$

✏

APPENDIX M

FOURIER SERIES

❝ *...et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur.*❞

MÉMOIRE

SUR LA

PROPAGATION DE LA CHALEUR

DANS LES CORPS SOLIDES.

Par M. FOURIER (¹).

Présenté le 21 décembre 1807 à l'Institut national.

Nouveau Bulletin des Sciences par la Société philomathique de Paris, t. I, p. 112-116, n° 6; mars 1808. Paris, Bernard.

L'auteur de ce Mémoire s'est proposé de soumettre la Théorie de la chaleur à l'Analyse mathématique et de vérifier, par l'expérience, les résultats du calcul. Pour exposer l'état de la question, supposons une barre de métal, cylindrique et d'une longueur indéterminée, plongée par une de ses extrémités dans un fluide entretenu à une température constante : la chaleur se répandra successivement dans la barre; et, sans la perte qui a lieu à sa surface et à son autre extrémité, elle prendrait dans toute son étendue la température constante du foyer; mais, à cause de cette perte, la chaleur ne s'étendra d'une manière sensible que jusqu'à une distance du foyer dépendante de la grosseur de la barre, de la conductibilité du métal et de son degré de poli, qui influe sur le rayonnement; de sorte que des thermomètres placés dans l'éten-

(¹) Cet Article, que nous avons déjà signalé dans l'Avant-Propos du Tome I, n'est pas de Fourier. Signé de l'initiale P, il a été écrit par Poisson, qui était un des rédacteurs du *Bulletin des Sciences* pour la partie mathématique. A raison de l'intérêt historique qu'il présente comme étant le premier écrit où l'on ait fait connaître la théorie de Fourier, nous avons cru devoir le reproduire intégralement. G. D.

❝ *...and the innovation of the subject, together with its importance, convinced the committee to crown this work. By observing however that the way in which the author arrives at his equations is not free from difficulties, and the analysis of which, to integrate them, still leaves something to be desired, either relative to generality, or even on the side of rigour.*❞

A competition awards committee consisting of the mathematical giants Lagrange, Laplace, Legendre, and others, commenting on Fourier's 1807 landmark paper *Dissertation on the propagation of heat in solid bodies* that introduced the *Fourier Series*. [1]

M.1 Definition

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition M.1. [2]

DEF *The **Fourier Series operator** $\hat{\mathbf{F}} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{R}}$ is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{f}(x) e^{-i\frac{2\pi}{\tau}nx}\, \mathsf{d}x \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{f} \text{ is periodic with period } \tau\right\}$$

[1] quote: Lagrange et al. (1812b) page 374, Lagrange et al. (1812a) page 112, Kahane (2008) page 199
translation: assisted by Google Translate, Castanedo (2005) ⟨chapter 2 footnote 5⟩
paper: Fourier (1807)

[2] Katznelson (2004) page 3

M.2 Inverse Fourier Series operator

Theorem M.1. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

THM

*The **inverse Fourier Series** operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{x}}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall \left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right) \in \ell^2_{\mathbb{R}}$$

✎PROOF: The proof of the pointwise convergence of the Fourier Series is notoriously difficult. It was conjectured in 1913 by Nokolai Luzin that the Fourier Series for all square summable periodic functions are pointwise convergent: Luzin (1913)

Fifty-three years later (1966) at a conference in Moscow, Lennart Axel Edvard Carleson presented one of the most spectacular results ever in mathematics; he demonstrated that the Luzin conjecture is indeed correct. Carleson formally published his result that same year: Carleson (1966)

Carleson's proof is expounded upon in Reyna's (2002) 175 page book: de Reyna (2002)

Interestingly enough, Carleson started out trying to disprove Luzin's conjecture. Carleson said this in an interview published in 2001:[3] ❝*Well, the problem of course presents itself already when you are a student and I was thinking of the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I suddenly realized why there should be a counter-example and how you should produce it. I thought I really understood what was the background and then to my amazement I could prove that this "correct" counter-example couldn't exist and therefore I suddenly realized that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. The most important aspect in solving a mathematical problem is the conviction of what is the true result! Then it took like 2 or 3 years using the technique that had been developed during the past 20 years or so. It is actually a problem related to analytic functions basically even though it doesn't look that way.*❞

For now, if you just want some intuitive justification for the Fourier Series, and you can somehow imagine that the Dirichlet kernel generates a *comb function* of *Dirac delta* functions, then perhaps what follows may help (or not). It is certainly not mathematically rigorous and is by no means a real proof (but at least it is less than 175 pages).

$$\begin{aligned}
\left[\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{x}\right](x) &= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}\mathsf{x}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition M.1 page 233)}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}\,\mathrm{d}u\right]e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \hat{\mathbf{F}}^{-1}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{i\frac{2\pi}{\tau}n(x-u)}\,\mathrm{d}u
\end{aligned}$$

[3] Carleson and Engquist (2001)

$$
\begin{aligned}
&= \int_0^\tau \mathsf{x}(u) \underbrace{\frac{1}{\tau}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi}{\tau}n(x-u)}}_{\lim_{N\to\infty}\mathsf{D}_n(x)} \,\mathrm{d}u \\
&= \int_0^\tau \mathsf{x}(u) \left[\sum_{n\in\mathbb{Z}} \delta(x-u-n\tau)\right] \mathrm{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{u=0}^{u=\tau} \mathsf{x}(u)\delta(x-u-n\tau)\,\mathrm{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{v-n\tau=0}^{v-n\tau=\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathrm{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathrm{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v)\delta(x-v)\,\mathrm{d}v && \text{because } \mathsf{x} \text{ is periodic with period } \tau \\
&= \int_{\mathbb{R}} \mathsf{x}(v)\delta(x-v)\,\mathrm{d}v \\
&= \mathsf{x}(x) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} && \text{(Definition C.3 page 112)}
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](n) &= \hat{\mathbf{F}}\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) e^{i\frac{2\pi}{\tau}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau \left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) e^{i\frac{2\pi}{\tau}kx}\right] e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition M.1 page 233)} \\
&= \frac{1}{\tau}\int_0^\tau \left[\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) e^{i\frac{2\pi}{\tau}(k-n)x}\right] \mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) \left[\frac{1}{\tau}\int_0^\tau e^{i\frac{2\pi}{\tau}(k-n)x}\,\mathrm{d}x\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) \frac{1}{\tau}\left[\frac{1}{i\frac{2\pi}{\tau}(k-n)} e^{i\frac{2\pi}{\tau}(k-n)x}\right]_0^\tau \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) \frac{1}{i2\pi(k-n)}\left[e^{i2\pi(k-n)} - 1\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\,\bar{\delta}(k-n) \lim_{x\to 0}\left[\frac{e^{i2\pi x}-1}{i2\pi x}\right] \\
&= \tilde{\mathsf{x}}(n) \left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{i2\pi x}-1\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(i2\pi x)}\right|_{x=0} && \text{by \textit{l'Hôpital's rule}} \\
&= \tilde{\mathsf{x}}(n) \left.\frac{i2\pi e^{i2\pi x}}{i2\pi}\right|_{x=0} \\
&= \tilde{\mathsf{x}}(n) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} && \text{(Definition C.3 page 112)}
\end{aligned}
$$

✏

Theorem M.2.

THM *The **Fourier Series adjoint** operator* $\hat{\mathbf{F}}^*$ *is given by*

$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathrm{x}(x) \,|\, \tilde{\mathrm{y}}(n) \right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}} \int_0^\tau \mathrm{x}(x) e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x \,|\, \tilde{\mathrm{y}}(n) \right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition M.1 page 233)}\\
&= \frac{1}{\sqrt{\tau}} \int_0^\tau \mathrm{x}(x) \left\langle e^{-i\frac{2\pi}{\tau}nx} \,|\, \tilde{\mathrm{y}}(n) \right\rangle_{\mathbb{Z}} \mathrm{d}x && \text{by additivity property of } \langle \triangle \,|\, \nabla \rangle\\
&= \int_0^\tau \mathrm{x}(x) \frac{1}{\sqrt{\tau}} \left\langle \tilde{\mathrm{y}}(n) \,|\, e^{-i\frac{2\pi}{\tau}nx} \right\rangle_{\mathbb{Z}}^{*} \mathrm{d}x && \text{by property of } \langle \triangle \,|\, \nabla \rangle\\
&= \int_0^\tau \mathrm{x}(x) \left[\hat{\mathbf{F}}^{-1}\tilde{\mathrm{y}}(n)\right]^{*} \mathrm{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem M.1 page 234)}\\
&= \left\langle \mathrm{x}(x) \,|\, \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^{*}} \tilde{\mathrm{y}}(n) \right\rangle_{\mathbb{R}}
\end{aligned}$$

✑

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary* [4] (Corollary M.1 page 236).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary M.2 page 236).

Corollary M.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\hat{\mathbf{F}}$ *be the Fourier Series operator with adjoint* $\hat{\mathbf{F}}^{*}$.

COR: $\left\{\hat{\mathbf{F}}\hat{\mathbf{F}}^{*} = \hat{\mathbf{F}}^{*}\hat{\mathbf{F}} = \mathbf{I}\right\}$ ($\hat{\mathbf{F}}$ *is* ***unitary*** *...and thus also* NORMAL *and* ISOMETRIC)

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^{*} = \hat{\mathbf{F}}^{-1}$ (Theorem M.2 page 235). ✑

Corollary M.2. *Let* $\hat{\mathbf{F}}$ *be the Fourier series operator with adjoint* $\hat{\mathbf{F}}^{*}$ *and inverse* $\hat{\mathbf{F}}^{-1}$.

COR:

$$\begin{array}{rclcll}
\mathcal{R}(\hat{\mathbf{F}}) &=& \mathcal{R}(\hat{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
\left\|\hat{\mathbf{F}}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\right\| &=& 1 & \text{(UNITARY)}\\
\left\langle \hat{\mathbf{F}}\boldsymbol{x} \,|\, \hat{\mathbf{F}}\boldsymbol{y} \right\rangle &=& \left\langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \,|\, \hat{\mathbf{F}}^{-1}\boldsymbol{y} \right\rangle &=& \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle & \text{(PARSEVAL'S EQUATION)}\\
\left\|\hat{\mathbf{F}}\boldsymbol{x}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\right\| &=& \|\boldsymbol{x}\| & \text{(PLANCHEREL'S FORMULA)}\\
\left\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\right\| &=& \|\boldsymbol{x} - \boldsymbol{y}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary M.1 page 236) and from the properties of unitary operators (Theorem C.26 page 136). ✑

M.3 Fourier series for compactly supported functions

Theorem M.3.

THM: *The set*

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i\frac{2\pi}{\tau}nx} \,\middle|\, n\in\mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathrm{f}(x)$ *with support in* $[0 : \tau]$.

[4] *unitary operators*: Definition C.14 page 135

APPENDIX N

FAST WAVELET TRANSFORM (FWT)

The Fast Wavelet Transform can be computed using simple discrete filter operations (as a conjugate mirror filter).

Definition N.1 (Wavelet Transform)**.** *Let the wavelet transform* $\mathbf{W} : \{f : \mathbb{R} \to \mathbb{C}\} \to \{w : \mathbb{Z}^2 \to \mathbb{C}\}$ *be defined as*[1]

DEF
$$[\mathbf{W}\mathsf{f}](j,n) \triangleq \langle \mathsf{f}(x) \,|\, \psi_{k,n}(x) \rangle$$

Definition N.2. *The following relations are defined as described below:*

DEF

scaling coefficients	$v_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$v_j(n)$	$\triangleq$	$\langle f(x) \,|\, \phi_{j,n}(x) \rangle$
wavelet coefficients	$w_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$w_j(n)$	$\triangleq$	$\langle f(x) \,|\, \psi_{j,n}(x) \rangle$
scaling filter coefficients	$\bar{\mathsf{h}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{h}}(n)$	$\triangleq$	$\mathsf{h}(-n)$
wavelet filter coefficients	$\bar{\mathsf{g}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{g}}(n)$	$\triangleq$	$\mathsf{g}(-n)$

The scaling and wavelet filter coefficients at scale j are equal to the filtered and downsampled (Theorem **??** page **??**) scaling filter coefficients at scale $j+1$:[2]

- The convolution (Definition I.3 page 203) of $v_{j+1}(n)$ with $\bar{\mathsf{h}}(n)$ and then downsampling by 2 produces $v_j(n)$.
- The convolution of $v_{j+1}(n)$ with $\bar{\mathsf{g}}(n)$ and then downsampling by 2 produces $w_j(n)$.

This is formally stated and proved in the next theorem.

---

[1] Notice that this definition is similar to the definition of transforms of other analysis systems:

Laplace Transform	$\mathcal{L}f(s)$	$\triangleq$	$\langle \mathsf{f}(x) \,|\, e^{sx} \rangle$	$\triangleq$	$\int_x \mathsf{f}(x)e^{-sx}\,\mathrm{d}x$
Continuous Fourier Transform	$\mathcal{F}f(\omega)$	$\triangleq$	$\langle \mathsf{f}(x) \,|\, e^{i\omega x} \rangle$	$\triangleq$	$\int_x \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x$
Fourier Series Transform	$\mathcal{F}_s f(k)$	$\triangleq$	$\left\langle \mathsf{f}(x) \,\middle|\, e^{i\frac{2\pi}{T}kx} \right\rangle$	$\triangleq$	$\int_x \mathsf{f}(x)e^{-i\frac{2\pi}{T}kx}\,\mathrm{d}x$
Z-Transform	$\mathcal{Z}f(z)$	$\triangleq$	$\langle \mathsf{f}(x) \,|\, z^n \rangle$	$\triangleq$	$\sum_n \mathsf{f}(x)z^{-n}$
Discrete Fourier Transform	$\mathcal{F}_d f(k)$	$\triangleq$	$\left\langle f(n) \,\middle|\, e^{i\frac{2\pi}{N}kn} \right\rangle$	$\triangleq$	$\sum_n \mathsf{f}(x)e^{-i\frac{2\pi}{N}kn}$

[2] Mallat (1999) page 257, Burrus et al. (1998) page 35

Theorem N.1.

THM
$$\begin{aligned} v_j(n) &= [\bar{\mathsf{h}} \star v_{j+1}](2n) \\ w_j(n) &= [\bar{\mathsf{g}} \star v_{j+1}](2n) \end{aligned}$$

✎PROOF:

$$\begin{aligned}
v_j(n) &= \left\langle \mathsf{f}(x) \,|\, \phi_{j,n}(x) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\phi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\sqrt{2}\sum_m h(m)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sum_m h(j)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \,|\, \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \,|\, \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m h(m) v_{j+1}(2n+m) \\
&= \sum_p h(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p - 2n \\
&= \sum_p \bar{\mathsf{h}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{h}} \star v_{j+1}](2n)
\end{aligned}$$

$$\begin{aligned}
w_j(n) &= \left\langle \mathsf{f}(x) \,|\, \psi_{j,n}(x) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\psi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\sqrt{2}\sum_m \mathsf{g}(j)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sum_m \mathsf{g}(m)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \,|\, \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \,|\, \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m \mathsf{g}(m) v_{j+1}(2n+m) \\
&= \sum_p \mathsf{g}(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p - 2n \\
&= \sum_p \bar{\mathsf{g}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{g}} \star v_{j+1}](2n)
\end{aligned}$$

✏

These filtering and downsampling operations are equivalent to the operations performed by a filter bank. Therefore, a filter bank can be used to implement a *Fast Wavelet Transform* (*FWT*), as illustrated in Figure N.1 (page 239).

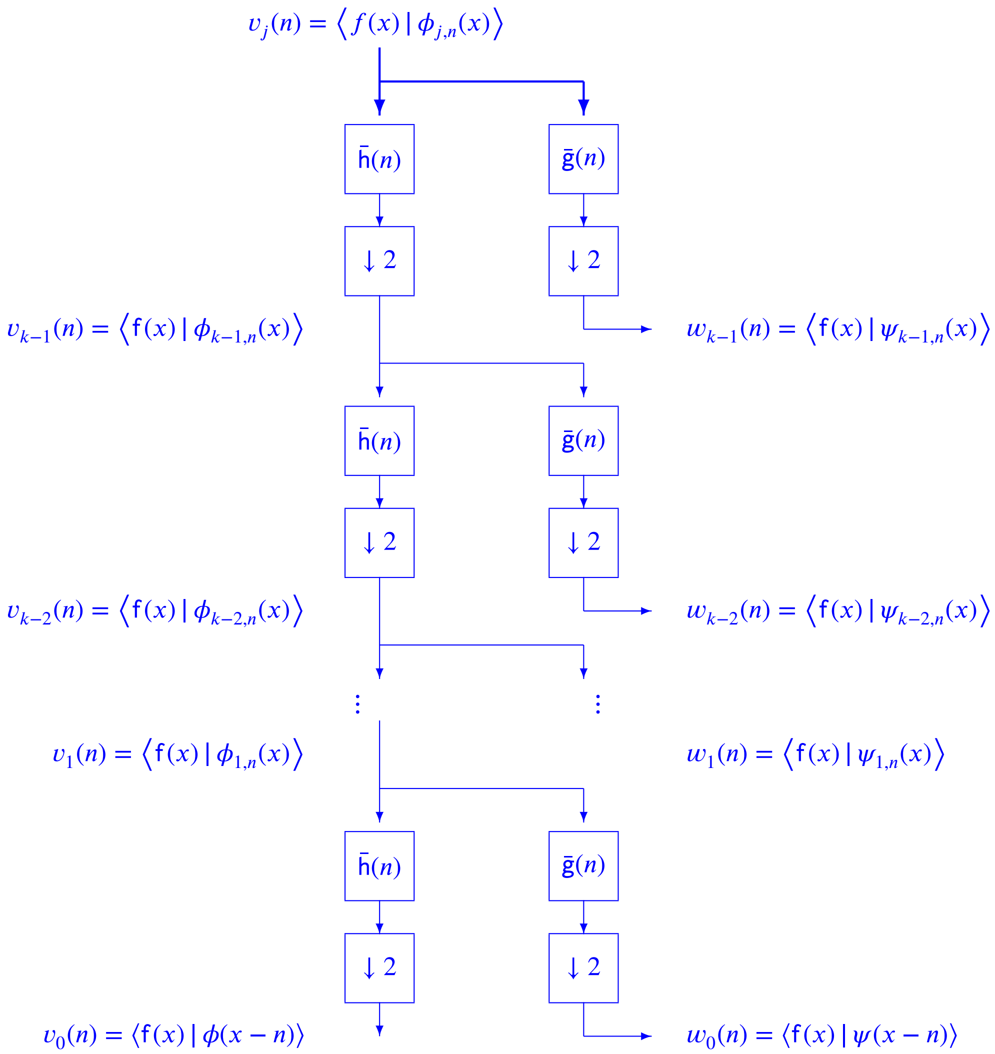


Figure N.1: k-Stage Fast Wavelet Transform

APPENDIX O

POWER SPECTRUM FUNCTIONS

O.1 Correlation

Definition O.1 and Definition O.2 define four quantities. In this document, the quantities' notation and terminology are similar to those used in the study of *random processes*.

Definition O.1. [1] *Let* $\langle \triangle \mid \triangledown \rangle$ *be the* STANDARD INNER PRODUCT *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition D.1 page 141).

DEF
$$\begin{array}{rcllll} \mathsf{R}_{\mathsf{fg}}(n) & \triangleq & \langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{g}(x) \rangle, & n \in \mathbb{Z}; & \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{cross-correlation function} \text{ of } \mathsf{f} \text{ and } \mathsf{g}. \\ \mathsf{R}_{\mathsf{ff}}(n) & \triangleq & \langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{f}(x) \rangle, & n \in \mathbb{Z}; & \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{autocorrelation function} \text{ of } \mathsf{f}. \end{array}$$

Definition O.2. [2] *Let* $\mathsf{R}_{\mathsf{fg}}(n)$ *and* $\mathsf{R}_{\mathsf{ff}}(n)$ *be the sequences defined in Definition O.1 page 241. Let* $\mathbf{Z}\,(\!(x_n)\!)$ *be the* Z-TRANSFORM (Definition I.4 page 204) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$.

DEF
$$\begin{array}{rclll} \check{\mathsf{S}}_{\mathsf{fg}}(z) & \triangleq & \mathbf{Z}\left[\mathsf{R}_{\mathsf{fg}}(n)\right], & \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{complex cross-power spectrum} \text{ of } \mathsf{f} \text{ and } \mathsf{g}. \\ \check{\mathsf{S}}_{\mathsf{ff}}(z) & \triangleq & \mathbf{Z}\left[\mathsf{R}_{\mathsf{ff}}(n)\right], & \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{complex auto-power spectrum} \text{ of } \mathsf{f}. \end{array}$$

O.2 Power Spectrum

Definition O.3. [3] *Let* $\check{\mathsf{S}}_{\mathsf{fg}}(z)$ *and* $\check{\mathsf{S}}_{\mathsf{ff}}(z)$ *be the functions defined in Definition O.2 page 241.*

DEF
$$\begin{array}{rclll} \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) & \triangleq & \check{\mathsf{S}}_{\mathsf{fg}}\left(e^{i\omega}\right), & \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{cross-power spectrum} \text{ of } \mathsf{f} \text{ and } \mathsf{g}. \\ \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) & \triangleq & \check{\mathsf{S}}_{\mathsf{ff}}\left(e^{i\omega}\right), & \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{auto-power spectrum} \text{ of } \mathsf{f}. \end{array}$$

Theorem O.1. [4] *Let* $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ *and* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be defined as in Definition O.3 (page 241).*
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition H.2 page 192) *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{F}}$.

THM
$$\begin{array}{rcll} \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) & = & 2\pi \displaystyle\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega + 2\pi n)\tilde{\mathsf{g}}^*(\omega + 2\pi n) & \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}} \\ \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) & = & 2\pi \displaystyle\sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega + 2\pi n)\right|^2 & \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{F}} \end{array}$$

---

[1] Chui (1992) page 134, Papoulis (1991) pages 294–332 ⟨(10-29), (10-169)⟩
[2] Chui (1992) page 134, Papoulis (1991) page 334 ⟨(10-178)⟩
[3] Chui (1992) page 134, Papoulis (1991) page 333 ⟨(10-179)⟩
[4] Chui (1992) page 135

✎PROOF: Let $z \triangleq e^{i\omega}$.

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) &\triangleq \check{\mathsf{S}}_{\mathsf{fg}}(z) && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition O.3 page 241)}\\
&= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n) z^{-n} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition O.2 page 241)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\, z^{-n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition O.3 page 241)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathbf{F}}[\mathsf{f}(x)] \,|\, \tilde{\mathbf{F}}\left[\mathsf{g}(x-n)\right] \right\rangle z^{-n} && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} && \text{(Theorem H.3 page 193)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathsf{f}}(v) \,|\, e^{-ivn}\tilde{\mathsf{g}}(v) \right\rangle z^{-n} && \text{by \textit{shift relation}} && \text{(Theorem H.4 page 194)}\\
&= \sum_{n\in\mathbb{Z}} \sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v) e^{ivu}\,\mathrm{d}v\right]_{u=n} z^{-n} && \text{by definition of } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition D.1 page 141)}\\
&= \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \left[\tilde{\mathbf{F}}^{-1}\left(\sqrt{2\pi}\tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v)\right)\right]_{u=n} e^{-i\omega n} && \text{by Theorem H.1 page 193}\\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n) && \text{by \textit{IPSF} with } \tau = 1 && \text{(Theorem 3.3 page 49)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)\Big|_{\mathsf{g}=\mathsf{f}} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) && \text{(Definition O.3 page 241)}\\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n)\Big|_{\mathsf{g}\triangleq\mathsf{f}} && \text{by previous result}\\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{f}}^*(\omega+2\pi n)\\
&= 2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{because } |z|^2 \triangleq zz^* \quad \forall z\in\mathbb{C}
\end{aligned}$$

✏

Proposition O.1. *Let $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition O.3 (page 241).*

PRP: $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) \;\geq\; 0$ (NON-NEGATIVE)

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= 2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem O.1 page 241}\\
&\geq 0 && \text{because } |z| \geq 0 \quad \forall z\in\mathbb{C}
\end{aligned}$$

✏

Proposition O.2. *Let $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ and $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition O.3 (page 241).*

PRP:
$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) && \text{(PERIODIC \textit{with period} } 2\pi)\\
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) && \text{(PERIODIC \textit{with period} } 2\pi)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(\omega+2\pi) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\omega+2\pi+2\pi n)\tilde{g}^*(\omega+2\pi+2\pi n) && \text{by Theorem O.1 page 241}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}[\omega+2\pi(n+1)]\tilde{g}^*[\omega+2\pi(n+1)]\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{f}[\omega+2\pi m]\tilde{g}^*[\omega+2\pi m] && \text{where } m\triangleq n+1\\
&= \tilde{S}_{fg}(\omega) && \text{by Theorem O.1 page 241}\\
\tilde{S}_{ff}(\omega+2\pi) &= \tilde{S}_{fg}(\omega+2\pi)\Big|_{g=f}\\
&= \tilde{S}_{fg}(\omega)\Big|_{g=f} && \text{by previous result}\\
&= \tilde{S}_{ff}(\omega)
\end{aligned}$$

✑

Proposition O.3. *Let $\tilde{S}_{fg}(\omega)$ and $\tilde{S}_{ff}(\omega)$ be defined as in Definition O.3 (page 241).*

PRP

$$\begin{array}{llrcll}
f,g \text{ are real} & \Longrightarrow & \tilde{S}_{fg}(-\omega) & = & \tilde{S}_{gf}(\omega) & \\
f \text{ is real} & \Longrightarrow & \tilde{S}_{ff}(-\omega) & = & \tilde{S}_{ff}(\omega) & (\text{SYMMETRIC about } 0)\\
f,g \text{ are real} & \Longrightarrow & \tilde{S}_{fg}(\pi-\omega) & = & \tilde{S}_{gf}(\pi+\omega) & \\
f \text{ is real} & \Longrightarrow & \tilde{S}_{ff}(\pi-\omega) & = & \tilde{S}_{ff}(\pi+\omega) & (\text{SYMMETRIC about } \pi)
\end{array}$$

✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(-\omega+2\pi n)\tilde{g}^*(-\omega+2\pi n) && \text{by Theorem O.1 page 241}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\omega-2\pi n)\tilde{g}(\omega-2\pi n) && \text{by hypothesis and Theorem H.5 page 194}\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\omega+2\pi m)\tilde{f}^*(\omega+2\pi m) && \text{where } m\triangleq -n\\
&= \tilde{S}_{gf}(\omega) && \text{by Theorem O.1 page 241}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{fg}(\pi-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\pi-\omega+2\pi n)\tilde{g}^*(\pi-\omega+2\pi n) && \text{by Theorem O.1 page 241}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(-\pi+\omega-2\pi n)\tilde{g}(-\pi+\omega-2\pi n) && \text{by hypothesis and Theorem H.5 page 194}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega-2\pi-2\pi n)\tilde{g}(\pi+\omega-2\pi-2\pi n)\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega+2\pi(-n-1))\tilde{g}(\pi+\omega+2\pi(-n-1))\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\pi+\omega+2\pi m)\tilde{f}^*(\pi+\omega+2\pi m) && \text{where } m\triangleq -n-1\\
&= \tilde{S}_{gf}(\pi+\omega) && \text{by Theorem O.1 page 241}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{ff}(-\omega) &= \tilde{S}_{fg}(-\omega)\Big|_{g\triangleq f}\\
&= \tilde{S}_{gf}(+\omega)\Big|_{g\triangleq f} && \text{by previous result}\\
&= \tilde{S}_{ff}(+\omega) && \text{by definition of g } (g\triangleq f)
\end{aligned}$$

$$\tilde{S}_{ff}(\pi-\omega) = \tilde{S}_{fg}(\pi-\omega)\Big|_{g\triangleq f}$$

$$= \tilde{S}_{gf}(\pi+\omega)\big|_{g \triangleq f} \qquad \text{by previous result}$$
$$= \tilde{S}_{ff}(\pi+\omega) \qquad \text{by definition of } g\ (g \triangleq f)$$

⇨

Proposition O.4. *Let* $\tilde{S}_{ff}(\omega)$ *be the* AUTO-POWER SPECTRUM (Definition O.3 page 241) *of a function* $f(x) \in L^2_{\mathbb{R}}$ *and* $\tilde{S}'_{ff}(\omega) \triangleq \frac{d}{d\omega}\tilde{S}_{ff}(\omega)$ (Definition D.2 page 141).

PRP

$$\left\{\begin{array}{ll}(a). & f \text{ is REAL and}\\ (b). & \tilde{S}_{ff}(\omega) \text{ is CONTINUOUS at } \omega=0\end{array}\right\} \implies \left\{\begin{array}{ll}(1). & \tilde{S}'_{ff}(0)=0 \text{ and}\\ (2). & \underbrace{\tilde{S}'_{ff}(\omega) = -\tilde{S}'_{ff}(-\omega) \quad \forall\omega\in\mathbb{R}}_{\text{ANTI-SYMMETRIC about } 0}\end{array}\right\}$$

$$\left\{\begin{array}{ll}(c). & f \text{ is REAL and}\\ (d). & \tilde{S}_{ff}(\omega) \text{ is CONTINUOUS at } \omega=\pi\end{array}\right\} \implies \left\{\begin{array}{ll}(3). & \tilde{S}'_{ff}(\pi)=0 \text{ and}\\ (4). & \underbrace{\tilde{S}'_{ff}(\pi+\omega) = -\tilde{S}'_{ff}(\pi-\omega) \quad \forall\omega\in\mathbb{R}}_{\text{ANTI-SYMMETRIC about } \pi}\end{array}\right\}$$

✎PROOF: This follows from Proposition O.3 (page 243) and Proposition D.1 (page 141). ⇨

Theorem O.2 (next) is a major result and provides strong motivation for bothering with *power spectrum* functions in the first place. In particular, the *auto-power spectrum* being *bounded* provides a necessary and sufficient condition for a sequence of functions $\langle\!\langle \phi(x-n)\rangle\!\rangle_{n\in\mathbb{Z}}$ to be a *Riesz basis* (Definition 2.13 page 27) for the *span* $\operatorname{span}\langle\!\langle \phi(x-n)\rangle\!\rangle$ of the sequence.

Theorem O.2. [5] *Let* $\tilde{S}_{ff}(\omega)$ *be defined as in Definition O.3 (page 241). Let* $\|\cdot\|$ *be defined as in Definition D.1 (page 141). Let* $0 < A < B$.

THM

$$\underbrace{\left\{A\sum_{n\in\mathbb{N}}|a_n|^2 \le \left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 \le B\sum_{n\in\mathbb{N}}|\alpha_n|^2 \quad \forall (a_n)\in\ell^2_{\mathbb{F}}\right\}}_{\langle\!\langle\phi(x-n)\rangle\!\rangle \text{ is a RIESZ BASIS for } \operatorname{span}\langle\!\langle\phi(x-n)\rangle\!\rangle \text{ (Theorem 2.13 page 28)}} \iff \left\{A \le \tilde{S}_{\phi\phi}(\omega) \le B\right\}$$

✎PROOF:

1. lemma:

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 &= \left\|\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 && \text{because } \tilde{\mathbf{F}} \text{ is } \textit{unitary} \text{ (Theorem H.2 page 193)}\\
&= \left\|\breve{a}(\omega)\tilde{\phi}(\omega)\right\|^2 && \text{by Proposition 3.13 page 47}\\
&= \int_{\mathbb{R}} \left|\breve{a}(\omega)\tilde{\phi}(\omega)\right|^2 d\omega && \text{by definition of } \|\cdot\|\\
&= \sum_{n\in\mathbb{Z}}\int_0^{2\pi} \left|\breve{a}(\omega+2\pi n)\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega\\
&= \int_0^{2\pi}\sum_{n\in\mathbb{Z}} |\breve{a}(\omega+2\pi n)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega\\
&= \int_0^{2\pi}\sum_{n\in\mathbb{Z}} |\breve{a}(\omega)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega && \text{by Proposition L.1 page 223}\\
&= \int_0^{2\pi} |\breve{a}(\omega)|^2\frac{1}{2\pi}2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega
\end{aligned}$$

[5] Wojtaszczyk (1997) pages 22–23 ⟨Proposition 2.8⟩, Igari (1996) page 219 ⟨Lemma 9.6⟩, Pinsky (2002) page 306 ⟨Theorem 6.4.8⟩

$$= \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega \qquad \text{by definition of } \tilde{\mathsf{S}}_{\phi\phi}(\omega) \text{ (Theorem O.1 page 241)}$$

2. lemma:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2\, \mathrm{d}\omega &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right|^2 \mathrm{d}\omega && \text{by def. of } \mathit{DTFT} \text{ (Definition L.1 page 223)}\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\left[\sum_{m\in\mathbb{Z}} a_m e^{-i\omega m}\right]^* \mathrm{d}\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\left[\sum_{m\in\mathbb{Z}} a_m^* e^{i\omega m}\right] \mathrm{d}\omega\\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n a_m^* \int_0^{2\pi} e^{-i\omega(n-m)}\, \mathrm{d}\omega\\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n a_m^* 2\pi \bar{\delta}_{nm}\\
&= \sum_{n\in\mathbb{Z}} |a_n|^2 && \text{by definition of } \bar{\delta} \text{ (Definition 2.12 page 20)}
\end{aligned}$$

3. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\boxed{A\sum_{n\in\mathbb{Z}} |a_n|^2} &= \frac{A}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2\, \mathrm{d}\omega && \text{by (2) lemma page 245}\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 A\, \mathrm{d}\omega\\
&\boxed{\leq} \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by right hypothesis}\\
&= \boxed{\left\|\sum_{n\in\mathbb{Z}} a_n \phi(x-n)\right\|^2} && \text{by (1) lemma page 244}\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by (1) lemma page 244}\\
&\boxed{\leq} \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 B\, \mathrm{d}\omega && \text{by right hypothesis}\\
&= \frac{B}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2\, \mathrm{d}\omega\\
&= \boxed{B\sum_{n\in\mathbb{Z}} |a_n|^2} && \text{by (2) lemma page 245}
\end{aligned}$$

4. Proof for ($\Longrightarrow$) case:

(a) Let $Y \triangleq \left\{\omega \in [0:2\pi] | \tilde{\mathsf{S}}_{\phi\phi}(\omega) > \alpha\right\}$
and $X \triangleq \left\{\omega \in [0:2\pi] | \tilde{\mathsf{S}}_{\phi\phi}(\omega) < \alpha\right\}$

(b) Let $\mathbb{1}_{A(x)}$ be the *set indicator* (Definition 3.2 page 40) of a set A.
Let $(\!(b_n)\!)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem L.3 page 229) of $\mathbb{1}_Y(\omega)$ such that

$$\mathbb{1}_Y(\omega) \triangleq \sum_{n\in\mathbb{N}} b_n e^{-i\omega n} \triangleq \tilde{\mathsf{b}}(\omega).$$

Let $(\!(a_n)\!)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem L.3 page 229) of $\mathbb{1}_X(\omega)$ such that

$$\mathbb{1}_X(\omega) \triangleq \sum_{n\in\mathbb{N}} a_n e^{-i\omega n} \triangleq \breve{\mathsf{a}}(\omega).$$

(c) Proof that $\alpha \le B$:
Let $\mu(A)$ be the *measure* of a set A.

$$\begin{aligned}
\boxed{B} \sum_{n\in\mathbb{Z}} |b_n|^2 &\ge \left\| \sum_{n\in\mathbb{Z}} b_n \phi(x-n) \right\|^2 && \text{by left hypothesis} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\tilde{\mathsf{b}}(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by (1) lemma page 244} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\mathbb{1}_Y(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 245)} \\
&= \frac{1}{2\pi}\int_Y |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 245)} \\
&\boxed{\ge} \frac{\alpha}{2\pi}\mu(Y) && \text{by definition of } Y && \text{(item (4a) page 245)} \\
&= \int_0^{2\pi} \left|\mathbb{1}_Y(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 245)} \\
&= \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}} b_n e^{-i\omega n} \right|^2 \mathrm{d}\omega && \text{by definition of } (\!(b_n)\!) && \text{(item (4b) page 245)} \\
&= \int_0^{2\pi} \left|\tilde{\mathsf{b}}(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \tilde{\mathsf{b}}(\omega) && \text{(item (4b) page 245)} \\
&= \boxed{\alpha} \sum_{n\in\mathbb{Z}} |b_n|^2 && \text{by (2) lemma page 245}
\end{aligned}$$

(d) Proof that $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$:
  (i). $\tilde{\mathsf{S}}_{\phi\phi}(\omega) > \alpha$ whenever $\omega \in Y$ (item (4a) page 245).
  (ii). But even then, $\alpha \le B$ (item (4c) page 246).
  (iii). So, $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$.

(e) Proof that $A \le \alpha$:
Let $\mu(A)$ be the *measure* of a set A.

$$\begin{aligned}
\boxed{A} \sum_{n\in\mathbb{Z}} |a_n|^2 &\le \left\| \sum_{n\in\mathbb{Z}} a_n \phi(x-n) \right\|^2 && \text{by left hypothesis} \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by (1) lemma page 244} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\mathbb{1}_X(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition 3.2 page 40)} \\
&= \frac{1}{2\pi}\int_X |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition 3.2 page 40)} \\
&\boxed{\le} \frac{\alpha}{2\pi}\mu(X) && \text{by definition of } X && \text{(item (4a) page 245)} \\
&= \int_0^{2\pi} \left|\mathbb{1}_X(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition 3.2 page 40)} \\
&= \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \right|^2 \mathrm{d}\omega && \text{by definition of } (\!(a_n)\!) && \text{((2) lemma page 245)} \\
&= \int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \mathrm{d}\omega && \text{by definition of } \breve{\mathsf{a}}(\omega) && \text{((2) lemma page 245)} \\
&= \boxed{\alpha} \sum_{n\in\mathbb{Z}} |a_n|^2 && \text{by (2) lemma page 245}
\end{aligned}$$

(f) Proof that $A \le \tilde{S}_{\phi\phi}(\omega)$:
  (i). $\tilde{S}_{\phi\phi}(\omega) < \alpha$ whenever $\omega \in X$ (item (4a) page 245).
  (ii). But even then, $A \le \alpha$ (item (4e) page 246).
  (iii). So, $A \le \tilde{S}_{\phi\phi}(\omega)$.

✏

In the case that f and g are *orthonormal*, the spectral density relations simplify considerably (next).

Theorem O.3. [6] *Let* $\tilde{S}_{ff}$ *and* $\tilde{S}_{fg}$ *be the* SPECTRAL DENSITY FUNCTIONS *(Definition O.3 page 241).*

THM

$$\begin{array}{lllllll}
\langle f(x) \,|\, f(x-n)\rangle &=& \bar{\delta}_n & (\langle f(x-n)\rangle \text{ is ORTHONORMAL}) & \iff & \tilde{S}_{ff}(\omega) = 1 & \forall f \in L^2_{\mathbb{F}} \\
\langle f(x) \,|\, g(x-n)\rangle &=& 0 & (f(x) \text{ is ORTHOGONAL to } \langle g(x-n)\rangle) & \iff & \tilde{S}_{fg}(\omega) = 0 & \forall f,g \in L^2_{\mathbb{F}}
\end{array}$$

✎PROOF:

1. Proof that $\langle f(x) \,|\, f(x-n)\rangle = \bar{\delta}_n \iff \tilde{S}_{ff}(\omega) = 1$: This follows directly from Theorem O.2 (page 244) with $A = B = 1$ (by *Parseval's Identity* Theorem 2.9 page 22 since $\{\mathbf{T}^n f\}$ is *orthonormal*)

2. Alternate proof that $\langle f(x) \,|\, f(x-n)\rangle = \bar{\delta}_n \implies \tilde{S}_{ff}(\omega) = 1$:

$$\begin{aligned}
\tilde{S}_{ff}(\omega) &= \sum_{n\in\mathbb{Z}} R_{ff}(n) e^{-i\omega n} && \text{by definition of } \tilde{S}_{fg} && \text{(Definition O.3 page 241)} \\
&= \sum_{n\in\mathbb{Z}} \langle f(x) \,|\, f(x-n)\rangle\, e^{-i\omega n} && \text{by definition of } R_{ff} && \text{(Definition O.1 page 241)} \\
&= \sum_{n\in\mathbb{Z}} \bar{\delta}_n e^{-i\omega n} && \text{by left hypothesis} \\
&= 1 && \text{by definition of } \bar{\delta} && \text{(Definition 2.12 page 20)}
\end{aligned}$$

3. Alternate proof that $\langle f(x) \,|\, f(x-n)\rangle = \bar{\delta}_n \impliedby \tilde{S}_{ff}(\omega) = 1$:

$$\begin{aligned}
&\langle f(x) \,|\, f(x-n)\rangle \\
&= \left\langle \tilde{\mathbf{F}} f(x) \,\middle|\, \tilde{\mathbf{F}} f(x-n)\right\rangle && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem H.3 page 193)} \\
&= \left\langle \tilde{f}(\omega) \,\middle|\, e^{-i\omega n}\tilde{f}(\omega)\right\rangle && \text{by } \textit{shift property} \text{ of } \tilde{\mathbf{F}} && \text{(Theorem H.4 page 194)} \\
&= \int_{\mathbb{R}} \tilde{f}(\omega) e^{i\omega n} \tilde{f}^*(\omega)\, d\omega && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition D.1 page 141)} \\
&= \int_{\mathbb{R}} \left|\tilde{f}(\omega)\right|^2 e^{i\omega n}\, d\omega \\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \left|\tilde{f}(\omega)\right|^2 e^{i\omega n}\, d\omega \\
&= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \left|\tilde{f}(u+2\pi n)\right|^2 e^{i(u+2\pi n)n}\, du && \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{f}(u+2\pi n)\right|^2 \right] e^{iun} \overbrace{\cancel{e^{i2\pi nn}}}^{1}\, du \\
&= \frac{1}{2\pi} \int_0^{2\pi} \tilde{S}_{ff}(\omega) e^{iun}\, du && \text{by Theorem O.1 page 241} \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{iun}\, du && \text{by right hypothesis} \\
&= \bar{\delta}_n && \text{by definition of } \bar{\delta} && \text{(Definition 2.12 page 20)}
\end{aligned}$$

[6] Hernández and Weiss (1996) page 50 ⟨PROPOSITION 2.1.11⟩, Wojtaszczyk (1997) page 23 ⟨Corollary 2.9⟩, Igari (1996) pages 214–215 ⟨Lemma 9.2⟩, Pinsky (2002) page 306 ⟨Corollary 6.4.9⟩

4. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle = 0 \implies \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

$$
\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) &= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n) e^{-i\omega n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition O.3 page 241)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle \, e^{-i\omega n} && \text{by definition of } \mathsf{R}_{\mathsf{fg}} && \text{(Definition O.1 page 241)}\\
&= \sum_{n\in\mathbb{Z}} 0 e^{-i\omega n} && \text{by left hypothesis}\\
&= 0
\end{aligned}
$$

5. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle = 0 \impliedby \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

$$
\begin{aligned}
&\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\\
&= \left\langle \tilde{\mathbf{F}}\mathsf{f}(x) \,\middle|\, \tilde{\mathbf{F}}\mathsf{g}(x-n)\right\rangle && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem H.3 page 193)}\\
&= \left\langle \tilde{\mathsf{f}}(\omega) \,\middle|\, e^{-i\omega n}\tilde{\mathsf{g}}(\omega)\right\rangle && \text{by } \textit{shift property} \text{ of } \tilde{\mathbf{F}} && \text{(Theorem H.4 page 194)}\\
&= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) e^{i\omega n} \tilde{\mathsf{g}}^*(\omega) \, \mathsf{d}\omega && \text{by definition of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition D.1 page 141)}\\
&= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) \tilde{\mathsf{g}}^*(\omega) e^{i\omega n} \, \mathsf{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \tilde{\mathsf{f}}(\omega) \tilde{\mathsf{g}}^*(\omega) e^{i\omega n} \, \mathsf{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{0}^{2\pi} \tilde{\mathsf{f}}(u+2\pi n) \tilde{\mathsf{g}}^*(u+2\pi n) e^{i(u+2\pi n)n} \, \mathsf{d}u && \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n\\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(u+2\pi n) \tilde{\mathsf{g}}^*(u+2\pi n) \right] e^{iun} \cancelto{1}{e^{i2\pi nn}} \, \mathsf{d}u\\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{\mathsf{S}}_{\mathsf{fg}}(u) e^{iun} \, \mathsf{d}u && \text{by Theorem O.1 page 241}\\
&= \frac{1}{2\pi} \int_{0}^{2\pi} 0 \cdot e^{iun} \, \mathsf{d}u && \text{by right hypothesis}\\
&= 0
\end{aligned}
$$

✏

APPENDIX P

CONTINUOUS RANDOM PROCESSES

P.1 Definitions

Definition P.1. [1] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE.

DEF	*The function* $\mathsf{x} : \Omega \to \mathbb{R}$ *is a* ***random variable***. *The function* $\mathsf{y} : \mathbb{R} \times \Omega \to \mathbb{R}$ *is a* ***random process***.

The random process $\mathsf{x}(t,\omega)$, where t commonly represents time and $\omega \in \Omega$ is an outcome of an experiment, can take on more specialized forms depending on whether t and ω are fixed or allowed to vary. These forms are illustrated in Figure P.1 page 249[2] and Figure P.2 page 250.

$\mathsf{x}(t,\omega)$	fixed t	variable t
fixed ω	number	time function
variable ω	random variable	random process

Figure P.1: Specialized forms of a random process $\mathsf{x}(t,\omega)$

Definition P.2. [3] *Let* $\mathsf{x}(t)$ *and* $\mathsf{y}(t)$ *be random processes.*

DEF	*The* ***mean***	$\mu_{\mathsf{x}}(t)$	*of* $\mathsf{x}(t)$ *is*	$\mu_{\mathsf{x}}(t)$	$\triangleq$	$\mathsf{E}[\mathsf{x}(t)]$
	The ***cross-correlation***	$\mathsf{R}_{\mathsf{xy}}(t)$	*of* $\mathsf{x}(t)$ *and* $\mathsf{y}(t)$ *is*	$\mathsf{R}_{\mathsf{xy}}(t,u)$	$\triangleq$	$\mathsf{E}\left[\mathsf{x}(t)\mathsf{y}^*(u)\right]$
	The ***auto-correlation function***	$\mathsf{R}_{\mathsf{xx}}(t)$	*of* $\mathsf{x}(t)$ *is*	$\mathsf{R}_{\mathsf{xx}}(t,u)$	$\triangleq$	$\mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right]$

Remark P.1. [4] The equation $\int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{f}(u)\,\mathsf{d}u$ is a *Fredholm integral equation of the first kind* and $\mathsf{R}_{\mathsf{xx}}(t,u)$ is the *kernel* of the equation.

---

[1] Papoulis (1991) page 63, Papoulis (1991) page 285
[2] Papoulis (1991) pages 285–286
[3] Papoulis (1984) page 216 ⟨$R_{xy}(t_1,t_2) = E\left\{\mathsf{x}(t_1)\mathsf{y}^*(t_2)\right\}$ (9-35)⟩,
[4] Fredholm (1900), Fredholm (1903) page 365, Michel and Herget (1993) page 97, Keener (1988) page 101

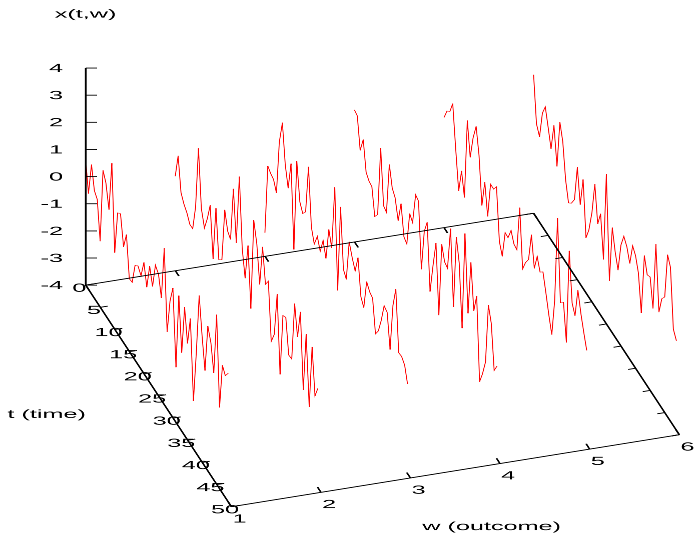


Figure P.2: Example of a random process $\mathsf{x}(t,\omega)$

P.2 Properties

Theorem P.1. *Let* $\mathsf{x}(t)$ *and* $\mathsf{y}(t)$ *be random processes with cross-correlation* $\mathsf{R}_{\mathsf{xy}}(t,u)$ *and let* $\mathsf{R}_{\mathsf{xx}}(t,u)$ *be the auto-correlation of* $\mathsf{x}(t)$.

THM

$$\begin{aligned}
\mathsf{R}_{\mathsf{xx}}(t,u) &= \mathsf{R}^*_{\mathsf{xx}}(u,t) \quad \text{(conjugate symmetric)}\\
\mathsf{R}_{\mathsf{xy}}(t,u) &= \mathsf{R}^*_{\mathsf{yx}}(u,t)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{\mathsf{xx}}(t,u) &\triangleq \mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right] &&= \mathsf{E}\left[\mathsf{x}^*(u)\mathsf{x}(t)\right] = \left(\mathsf{E}\left[\mathsf{x}(u)\mathsf{x}^*(t)\right]\right)^* &&\triangleq \mathsf{R}^*_{\mathsf{xx}}(u,t)\\
\mathsf{R}_{\mathsf{xy}}(t,u) &\triangleq \mathsf{E}\left[\mathsf{x}(t)\mathsf{y}^*(u)\right] &&= \mathsf{E}\left[\mathsf{y}^*(u)\mathsf{x}(t)\right] = \left(\mathsf{E}\left[\mathsf{y}(u)\mathsf{x}^*(t)\right]\right)^* &&\triangleq \mathsf{R}^*_{\mathsf{yx}}(u,t)
\end{aligned}$$

✏

APPENDIX Q

SPECTRAL THEORY

Q.1 Operator Spectrum

Definition Q.1. [1] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator over the linear spaces* $\boldsymbol{X} = (X, F, \oplus, \otimes)$ *and* $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$. *Let* $\mathcal{N}(\mathbf{A})$ *be the* NULL SPACE *of* $\mathbf{A}$.

DEF

An ***eigenvalue*** *of* $\mathbf{A}$ *is any value* λ *such that there exists* $\mathbf{x}$ *such that* $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.
The ***eigenspace*** $\boldsymbol{H}_\lambda$ *of* $\mathbf{A}$ *at eigenvalue* λ *is* $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$.
An ***eigenvector*** *of* $\mathbf{A}$ *associated with eigenvalue* λ *is any element of* $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$.

Example Q.1. [2] Let $\mathbf{D}$ be the differntial operator.

EX

The set $\left\{e^{\lambda x} | \lambda \in \mathbb{C}\right\}$ are the eigenvectors of $\mathbf{D}$.

$$\begin{array}{rcll}
\rho(\mathbf{D}) &=& \emptyset & \text{($\mathbf{D}$ has no non-spectral points whatsoever)}\\
\sigma_{\mathsf{p}}(\mathbf{D}) &=& \sigma(\mathbf{D}) & \text{(the spectrum of $\mathbf{D}$ is all eigenvalues)}\\
\sigma_{\mathsf{c}}(\mathbf{D}) &=& \emptyset & \text{($\mathbf{D}$ has no continuous spectrum)}\\
\sigma_{\mathsf{r}}(\mathbf{D}) &=& \emptyset & \text{($\mathbf{D}$ has no resolvent spectrum)}
\end{array}$$

✎PROOF:

$$\begin{aligned}
(\mathbf{D} - \lambda\mathbf{I})e^{\lambda x} &= \mathbf{D}e^{\lambda x} - \lambda\mathbf{I}e^{\lambda x}\\
&= \lambda e^{\lambda x} - \lambda e^{\lambda x}\\
&= 0 && \forall \lambda \in \mathbb{C}
\end{aligned}$$

This theorem and proof needs more work and investigation to prove/disprove its claims. ✏

Definition Q.2. [3] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator over the linear spaces* $\boldsymbol{X} = (X, F, \oplus, \otimes)$ *and* $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$.

[1] Bollobás (1999) page 168, Descartes (1637a), Descartes (1954), Cayley (1858), Hilbert (1904) page 67, Hilbert (1912),
[2] Pedersen (2000) page 79
[3] Michel and Herget (1993) page 439

quantity		$\mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\}$ ($x = \mathbb{0}$ is the only solution)	$\overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X}$ (dense)	$(\mathbf{A}-\lambda\mathbf{I})^{-1} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ (continuous/bounded)
$\rho(\mathbf{A})$	(resolvent set)	1	1	1
$\sigma_{\mathrm{p}}(\mathbf{A})$	(point spectrum)	0		
$\sigma_{\mathrm{r}}(\mathbf{A})$	(residual spectrum)	1	0	
$\sigma_{\mathrm{c}}(\mathbf{A})$	(continuous spectrum)	1	1	0

Table Q.1: Spectrum of an operator $\mathbf{A}$

DEF

*The **resolvent set** $\rho(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\rho(\mathbf{A}) \triangleq \left\{ \lambda \in F \,\middle|\, \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \textit{(no non-zero eigenvectors)} \quad \textit{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X} & \textit{(the range is dense in } \boldsymbol{X}\textit{).} \quad \textit{and} \\ 3. & (\mathbf{A}-\lambda\mathbf{I})^{-1} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) & \textit{(inverse is continuous/bounded).} \end{array} \right\}$$

*The **spectrum** $\sigma(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma(\mathbf{A}) \triangleq F \setminus \rho(\mathbf{A}).$$

Definition Q.3. [4] *Let $\mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ be an operator over the linear spaces $\boldsymbol{X} = (X, F, \oplus, \otimes)$ and $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$.*

DEF

*The **point spectrum** $\sigma_{\mathrm{p}}(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

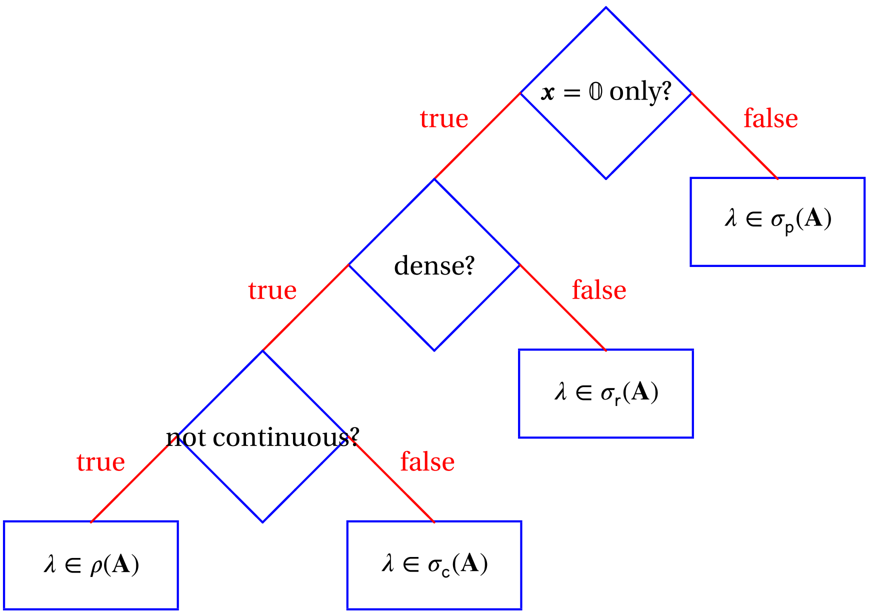
$$\sigma_{\mathrm{p}}(\mathbf{A}) \triangleq \left\{ \lambda \in F \,\middle|\, 1.\ \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) \supsetneq \{\mathbb{0}\} \quad \textit{(has non-zero eigenvector)} \right\}$$

*The **residual spectrum** $\sigma_{\mathrm{r}}(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma_{\mathrm{r}}(\mathbf{A}) \triangleq \left\{ \lambda \in F \,\middle|\, \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \textit{(no non-zero eigenvectors)} \quad \textit{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} \neq \boldsymbol{X} & \textit{(not dense in } \boldsymbol{X}\textit{—has gaps).} \end{array} \right\}$$

*The **continuous spectrum** $\sigma_{\mathrm{c}}(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma_{\mathrm{c}}(\mathbf{A}) \triangleq \left\{ \lambda \in F \,\middle|\, \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \textit{(no non-zero eigenvectors)} \quad \textit{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X} & \textit{(dense in } \boldsymbol{X}\textit{).} \quad \textit{and} \\ 3. & (\mathbf{A}-\lambda\mathbf{I})^{-1} \notin \mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) & \textit{(not continuous / not bounded)} \end{array} \right\}$$



The spectral components' definitions are illustrated in the figure to the left and summarized in Table Q.1 (page 252). Let a family of operators $\mathbf{B}(\lambda)$ be defined with respect to an operator $\mathbf{A}$ such that $\mathbf{B}(\lambda) \triangleq (\mathbf{A} - \lambda\mathbf{I})$. Normally, we might expect a "normal" or "regular" or even "mundane" operator $\mathbf{B}(\lambda)$ to have the properties

1. $\mathbf{B}(\lambda)\boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \mathbb{0}$
2. $\mathbf{B}(\lambda)\boldsymbol{x}$ spans virtually all of $\boldsymbol{X}$ as we vary $\boldsymbol{x}$
3. $\mathbf{B}^{-1}(\lambda)$ is continuous.

After all, these are the properties that we would have if $\mathbf{B}(\lambda)$ were simply an affine operator in the

[4] Bollobás (1999) page 168, Hilbert (1906) pages 169–172

field of real numbers— such as $[\mathbf{B}(\lambda)](x) \triangleq [\lambda](x) = \lambda x$ which is 0 if and only if $x = 0$, has range $\boldsymbol{R}(\lambda) = \mathbb{R}$, and its inverse $\lambda^{-1}x$ is continuous.

If for some λ the operator $\mathbf{B}(\lambda)$ does have all these "regular" properties, then that λ part of the *resolvent set* of $\mathbf{A}$ and λ is called *regular*. However if for some λ the operator $\mathbf{B}(\lambda)$ fails any of these conditions, then that λ part of the *spectrum* of $\mathbf{A}$. And which conditions it fails determines which component of the spectrum it is in.

Theorem Q.1. [5] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator.*

THM
$$\sigma(\mathbf{A}) = \sigma_{\mathsf{p}}(\mathbf{A}) \cup \sigma_{\mathsf{c}}(\mathbf{A}) \cup \sigma_{\mathsf{r}}(\mathbf{A})$$

Theorem Q.2 (Spectral Theorem)**.** [6] *Let* $\mathbf{N} \in Y^X$ *be an operator.*

THM
$$\left.\begin{array}{ll} (A). & \underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is NORMAL}} \\ (B). & \mathbf{N}\text{ is COMPACT} \end{array}\right\} \implies \left\{\begin{array}{lrcl} (1). & \mathbf{N} &=& \sum_n \lambda_n \mathbf{P}_n \\ (2). & \sum_n \mathbf{P}_n &=& \mathbf{I} \\ (3). & \mathbf{P}_n\mathbf{P}_m &=& \bar{\delta}_{n-m}\mathbf{P}_n \\ (4). & \dim(\boldsymbol{H}_n) &<& \infty \\ (5). & \left|\left\{\lambda_n | \lambda_n \neq 0\right\}\right| & & \text{is COUNTABLY INFINITE} \end{array}\right.$$

where

$$\begin{array}{rcll} (\lambda_n)_{n\in\mathbb{Z}} & \triangleq & \sigma_{\mathsf{p}}(\mathbf{N}) & \text{(eigenvalues of } \mathbf{N}) \\ \boldsymbol{H}_n & \triangleq & \mathcal{N}(\mathbf{N} - \lambda_n\mathbf{I}) & (\lambda_n \text{ is the eigenspace of } \mathbf{N} \text{ at } \lambda_n \text{ in } \boldsymbol{Y}) \\ \boldsymbol{H}_n & = & \mathbf{P}_n\boldsymbol{Y} & (\mathbf{P}_n \text{ is the projection operator that generates } \boldsymbol{H}_n) \end{array}$$

Q.2 Fredholm kernels

Definition Q.4. [7]

DEF
*A **Fredholm operator*** $\mathbf{K}$ *is defined as*

$$\underbrace{[\mathbf{K}\mathsf{f}](t) \triangleq \int_a^b \underbrace{\kappa(t,s)}_{\text{kernel}}\mathsf{f}(s)\,\mathsf{d}s}_{\text{Fredholm integral equation of the first kind}^8} \qquad \forall \mathsf{f} \in \boldsymbol{L}_2([a,b])$$

Example Q.2. Examples of Fredholm operators include

$$\begin{array}{lllcllcl} 1. & \text{Fourier Transform} & [\tilde{\mathbf{F}}\mathsf{x}](f) &=& \int_t \mathsf{x}(t)e^{-i2\pi ft}\,\mathsf{d}t & \kappa(t,f) &=& e^{-i2\pi ft} \\ 2. & \text{Inverse Fourier Transform} & [\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t) &=& \int_f \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathsf{d}f & \kappa(f,t) &=& e^{i2\pi ft} \\ 3. & \text{Laplace operator} & [\mathbf{L}\mathsf{x}](s) &=& \int_t \mathsf{x}(t)e^{-st}\,\mathsf{d}t & \kappa(t,s) &=& e^{-st} \\ 4. & \text{autocorrelation operator} & [\mathbf{R}\mathsf{x}](t) &=& \int_s R(t,s)\mathsf{x}(s)\,\mathsf{d}s & \kappa(t,s) &=& R(t,s) \end{array}$$

Theorem Q.3. *Let* $\mathbf{K}$ *be a Fredholm operator with kernel* $\kappa(t,s)$ *and adjoint* $\mathbf{K}^*$.

THM
$$[\mathbf{K}\mathsf{f}](t) = \int_A \kappa(t,s)\mathsf{f}(s)\,\mathsf{d}s \iff \left[\mathbf{K}^*\mathsf{f}\right](t) = \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathsf{d}s$$

[5] Michel and Herget (1993) page 440

[6] Michel and Herget (1993) page 457, Bollobás (1999) page 200, Hilbert (1906), Hilbert (1912), von Neumann (1929), de Witt (1659)

[7] Michel and Herget (1993) page 425

[8] The equation $\int_u \kappa(t,s)\mathsf{f}(s)\,\mathsf{d}s$ is a **Fredholm integral equation of the first kind** and $\kappa(t,u)$ is the **kernel** of the equation. References: Fredholm (1900), Fredholm (1903) page 365, Michel and Herget (1993) page 97, Keener (1988) page 101

✎PROOF:

$$\begin{aligned}
[\mathbf{K}\mathsf{f}](t) &= \int_A \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s \\
\iff \langle[\mathbf{K}\mathsf{f}](t)\,|\,\mathsf{g}(t)\rangle &= \left\langle \int_s \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s\,\middle|\,\mathsf{g}(t)\right\rangle && \text{by left hypothesis} \\
&= \int_s \mathsf{f}(s)\,\langle\kappa(t,s)\,|\,\mathsf{g}(t)\rangle\,\mathrm{d}s && \text{by additivity property of } \langle\triangle\,|\,\nabla\rangle \\
&= \int_s \mathsf{f}(s)\,\langle\mathsf{g}(t)\,|\,\kappa(t,s)\rangle^*\,\mathrm{d}s && \text{by conjugate symmetry property of } \langle\triangle\,|\,\nabla\rangle \\
&= \langle\mathsf{f}(s)\,|\,\langle\mathsf{g}(t)\,|\,\kappa(t,s)\rangle\rangle && \text{by local definition of } \langle\triangle\,|\,\nabla\rangle \\
&= \left\langle \mathsf{f}(s)\,\middle|\,\underbrace{\int_t \kappa^*(t,s)\mathsf{g}(t)\,\mathrm{d}t}_{[\mathbf{K}^*\mathsf{g}](s)}\right\rangle && \text{by local definition of } \langle\triangle\,|\,\nabla\rangle \\
\iff [\mathbf{K}^*\mathsf{g}](s) &= \int_A \kappa^*(t,s)\mathsf{g}(t)\,\mathrm{d}t && \text{by right hypothesis} \\
\iff [\mathbf{K}^*\mathsf{g}](\sigma) &= \int_A \kappa^*(\tau,\sigma)\mathsf{g}(\tau)\,\mathrm{d}\tau && \text{by change of variable: } \tau = t,\ \sigma = s \\
\iff [\mathbf{K}^*\mathsf{f}](t) &= \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathrm{d}s && \text{by change of variable: } t=\sigma,\ s=\tau,\ \mathsf{f}=\mathsf{g}
\end{aligned}$$

✏

Corollary Q.1. [9] *Let* $\mathbf{K}$ *be an Fredholm operator with kernel* $\kappa(t,s)$ *and adjoint* $\mathbf{K}^*$.

COR

$$\underbrace{\mathbf{K} = \mathbf{K}^*}_{\mathbf{K}\text{ is self-adjoint}} \iff \underbrace{\kappa(t,s) = \kappa^*(s,t)}_{\text{kernel is conjugate symmetric}}$$

✎PROOF:

$$\begin{aligned}
\mathbf{K} = \mathbf{K}^* &\iff \int_A \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s = \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathrm{d}s && \text{by Theorem Q.3 page 253} \\
&\iff \kappa(t,s) = \kappa^*(s,t)
\end{aligned}$$

✏

Theorem Q.4 (Mercer's Theorem). [10] *Let* $\mathbf{K}$ *be an Fredholm operator with kernel* $\kappa(t,s)$ *and eigensystem* $(\!(\lambda_n,\phi_n(t))\!)_{n\in\mathbb{Z}}$.

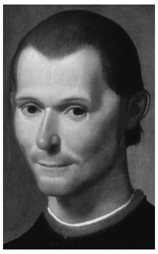
THM

$$\left\{\begin{array}{ll}
(A). & \underbrace{\int_a^b\int_a^b \kappa(t,s)\mathsf{f}(t)\mathsf{f}^*(s)\,\mathrm{d}t \geq 0}_{\text{positive}} \quad \textit{and} \\
(B). & \kappa(t,s)\ \textit{is}\ \text{CONTINUOUS}\ \textit{on}\ [a:b]\times[a:b]
\end{array}\right\}
\implies
\left\{\begin{array}{ll}
(1). & \kappa(t,s) = \sum_n \lambda_n\phi_n(t)\phi_n^*(s) \quad \textit{and} \\
(2). & \kappa(t,s)\ \text{CONVERGES ABSOLUTELY}\ \textit{and}\ \text{UNIFORMLY}\ \textit{on}\ [a:b]\times[a:b]
\end{array}\right\}$$

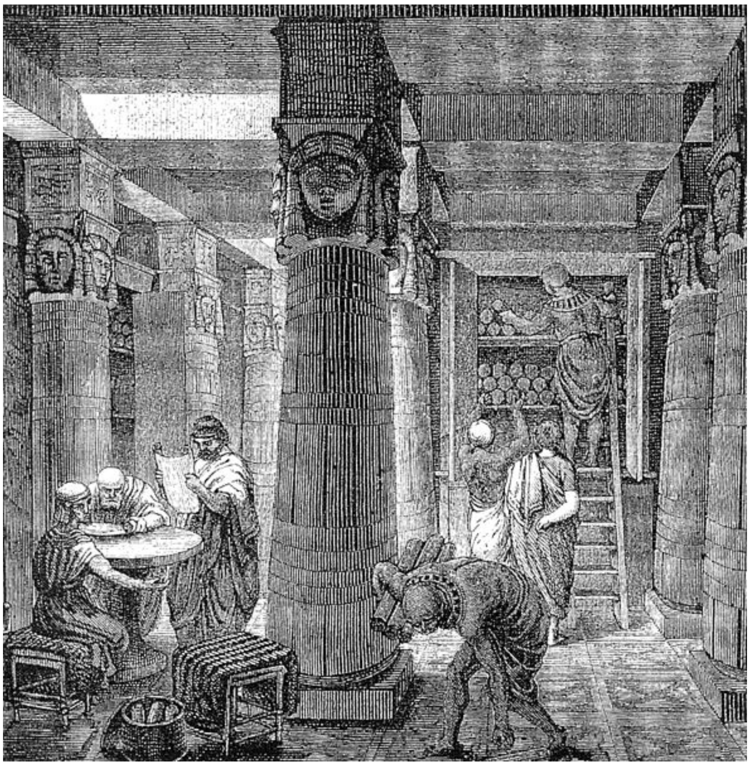
[9] Michel and Herget (1993) page 430
[10] Gohberg et al. (2003) page 198, Courant and Hilbert (1930) pages 138–140, Mercer (1909) page 439

Back Matter

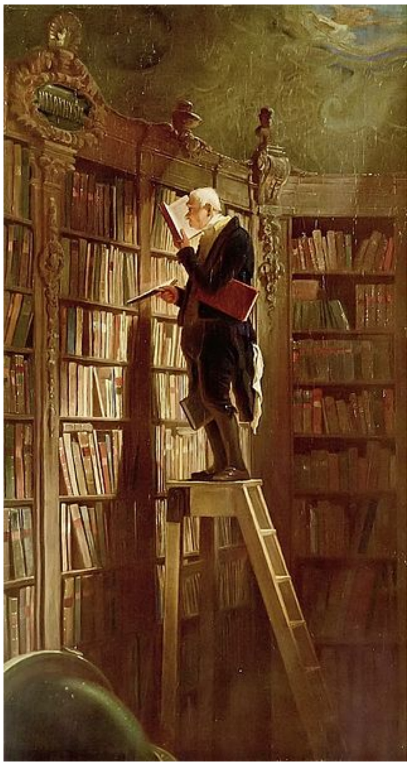
❝*It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.*❞
Niels Henrik Abel (1802–1829), Norwegian mathematician [11]



❝*When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.*❞
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [12]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850

[13]

[11] quote: Simmons (2007) page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

[12] quote: Machiavelli (1961) page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

[13] http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

❝*To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.*❞
Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [14]

[14] quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

BIBLIOGRAPHY

A prelude to sampling, wavelets, and tomography. In John Benedetto and Ahmed I. Zayed, editors, *Sampling, Wavelets, and Tomography*, Applied and Numerical Harmonic Analysis, pages 1–32. Springer, 2004. ISBN 9780817643041. URL http://books.google.com/books?vid=ISBN0817643044.

Yuri A. Abramovich and Charalambos D. Aliprantis. *An Invitation to Operator Theory*. American Mathematical Society, Providence, Rhode Island, 2002. ISBN 0-8218-2146-6. URL http://books.google.com/books?vid=ISBN0821821466.

Milton Abramowitz and Irene A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards, 1972. URL http://www.cs.bham.ac.uk/~aps/research/projects/as/book.php.

Edward H. Adelson and Peter J. Burt. Image data compression with the laplacian pyramid. In *Proceedings of the Pattern Recognition and Information Processing Conference*, pages 218–223, Dallas Texas, 1981. IEEE Computer Society Press. URL citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.50.791.

N. I. Akhiezer and I. M. Glazman. *Theory of Linear Operators in Hilbert Spaces*, volume 1. Dover, New York, 1993. URL http://books.google.com/books?vid=ISBN0486677486. Translated from the original Russian text *Teoriia lineinykh operatorov v Gil'bertovom prostranstve*.

Donald J. Albers and Freeman Dyson. Freeman dyson: Mathematician, physicist, and writer. *The College Mathematics Journal*, 25(1):3–21, January 1994. doi: 10.2307/2687079. URL http://www.jstor.org/stable/2687079.

Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis*. Acedemic Press, London, 3 edition, 1998. ISBN 9780120502578. URL http://www.amazon.com/dp/0120502577.

Luis Alvarez, Frédéric Guichard, Pierre-Louis Lions, and Jean Michel Morel. Axioms and fundamental equations of image processing. *Archive for Rational Mechanics and Analysis*, 123(3):199–257, 1993. URL http://link.springer.com/article/10.1007/BF00375127.

Ichiro Amemiya and Huzihiro Araki. A remark on piron's paper. *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, 2(3):423–427, 1966. ISSN 0034-5318. doi: 10.2977/prims/1195195769. URL http://projecteuclid.org/euclid.prims/1195195769.

George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*, volume 71 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, Cambridge, U.K., new edition, February 15 2001. ISBN 0521789885. URL http://books.google.com/books?vid=ISBN0521789885.

Kendall E. Atkinson and Weimin Han. *Theoretical Numerical Analysis: A Functional Analysis Framework*, volume 39 of *Texts in Applied Mathematics*. Springer, 3 edition, 2009. ISBN 9781441904584. URL http://books.google.com/books?vid=ISBN1441904581.

Léon Autonne. Sur l'hermitien (on the hermitian). In *Comptes Rendus Des SéAnces De L'AcadéMie Des Sciences*, volume 133, pages 209–268. De L'Académie des sciences (Academy of Sciences), Paris, 1901. URL http://visualiseur.bnf.fr/Visualiseur?O=NUMM-3089. Comptes Rendus Des SéAnces De L'AcadéMie Des Sciences (Reports Of the Meetings Of the Academy of Science).

Léon Autonne. Sur l'hermitien (on the hermitian). *Rendiconti del Circolo Matematico di Palermo*, 16:104–128, 1902. Rendiconti del Circolo Matematico di Palermo (Statements of the Mathematical Circle of Palermo).

George Bachman. *Elements of Abstract Harmonic Analysis*. Academic paperbacks. Academic Press, New York, 1964. URL http://books.google.com/books?id=ZP8-AAAAIAAJ.

George Bachman and Lawrence Narici. *Functional Analysis*. Academic Press textbooks in mathematics; Pure and Applied Mathematics Series. Academic Press, 1 edition, 1966. ISBN 9780486402512. URL http://books.google.com/books?vid=ISBN0486402517. “unabridged republication” available from Dover (isbn 0486402517).

George Bachman, Lawrence Narici, and Edward Beckenstein. *Fourier and Wavelet Analysis*. Universitext Series. Springer, 2000. ISBN 9780387988993. URL http://books.google.com/books?vid=ISBN0387988998.

Stefan Banach. Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales (on abstract operations and their applications to the integral equations). *Fundamenta Mathematicae*, 3:133–181, 1922. URL http://matwbn.icm.edu.pl/ksiazki/fm/fm3/fm3120.pdf.

Stefan Banach. *Théorie des opérations linéaires*. Monografje Matematyczne, Warsaw, Poland, 1932a. URL http://matwbn.icm.edu.pl/kstresc.php?tom=1&wyd=10. (Theory of linear operations).

Stefan Banach. *Theory of Linear Operations*, volume 38 of *North-Holland mathematical library*. North-Holland, Amsterdam, 1932b. ISBN 0444701842. URL http://www.amazon.com/dp/0444701842/. English translation of 1932 French edition, published in 1987.

Adi Ben-Israel and Robert P. Gilbert. *Computer-supported calculus*. Texts and monographs in symbolic computation. Springer, 2002. ISBN 3-211-82924-5. URL http://books.google.com/books?vid=ISBN3211829245.

Ladislav Beran. Three identities for ortholattices. *Notre Dame Journal of Formal Logic*, 17(2):251–252, 1976. doi: 10.1305/ndjfl/1093887530. URL http://projecteuclid.org/euclid.ndjfl/1093887530.

Ladislav Beran. *Orthomodular Lattices: Algebraic Approach*. Mathematics and Its Applications (East European Series). D. Reidel Publishing Company, Dordrecht, 1985. ISBN 90-277-1715-X. URL http://books.google.com/books?vid=ISBN902771715X.

Sterling Khazag Berberian. *Introduction to Hilbert Space*. Oxford University Press, New York, 1961. URL http://books.google.com/books?vid=ISBN0821819127.

Earl Berkson. Some metrics on the subspaces of a banach space. *Pacific Journal of Mathematics*, 13(1):7–22, 1963. URL http://projecteuclid.org/euclid.pjm/1103035953.

M. Bertero and P. Boccacci. *Introduction to Inverse Problems in Imaging*. CRC Press, 1998. ISBN 9781439822067. URL http://books.google.com/books?vid=ISBN9781439822067.

Garrett Birkhoff and John Von Neumann. The logic of quantum mechanics. *The Annals of Mathematics*, 37(4):823–843, October 1936. URL http://www.jstor.org/stable/1968621.

Duncan Black, Ian Brooks, and et al. Robert Groves, editors. *Collins English Dictionary—Complete and Unabridged*. HarperCollins Publishers, 10 edition, 2009. ISBN 978-0-00-732119-3. URL http://books.google.com/books?vid=ISBN9780007321193.

Béla Bollobás. *Linear Analysis; an introductory course*. Cambridge mathematical textbooks. Cambridge University Press, Cambridge, 2 edition, March 1 1999. ISBN 978-0521655774. URL http://books.google.com/books?vid=ISBN0521655773.

Umberto Bottazzini. *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*. Springer-Verlag, New York, 1986. ISBN 0-387-96302-2. URL http://books.google.com/books?vid=ISBN0387963022.

Carl Benjamin Boyer and Uta C. Merzbach. *A History of Mathematics*. Wiley, New York, 2 edition, 1991. ISBN 0471543977. URL http://books.google.com/books?vid=ISBN0471543977.

Ronald Newbold Bracewell. *The Fourier transform and its applications*. McGraw-Hill electrical and electronic engineering series. McGraw-Hill, 2, illustrated, international student edition edition, 1978. ISBN 9780070070134. URL http://books.google.com/books?vid=ISBN007007013X.

Thomas John I'Anson Bromwich. *An Introduction to the Theory of Infinite Series*. Macmillan and Company, 1 edition, 1908. ISBN 9780821839768. URL http://www.archive.org/details/anintroductiont00bromgoog.

Andrew M. Bruckner, Judith B. Bruckner, and Brian S. Thomson. *Real Analysis*. Prentice-Hall, Upper Saddle River, N.J., 1997. ISBN 9780134588865. URL http://books.google.com/books?vid=ISBN013458886X.

Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A Course in Metric Geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001. ISBN 978-0821821299. URL http://books.google.com/books?vid=ISBN0821821296.

C. Sidney Burrus, Ramesh A. Gopinath, and Haitao Guo. *Introduction to Wavelets and Wavelet Transforms; A Primer*. Prentice Hall, Upper Saddle River, New Jersey, USA, 1998. ISBN 0-13-489600-9. URL http://www.dsp.rice.edu/software/wavebook.shtml.

Peter J. Burt and Edward H. Adelson. The laplacian pyramid as a compact image code. *IEEE Transactions On Communications*, COM-3L(4):532–540, April 1983. URL http://citeseer.ist.psu.edu/burt83laplacian.html.

Charles L. Byrne. *Signal Processing: A Mathematical Approach*. AK Peters Series. A K Peters, 2005. ISBN 1568812426.

Florian Cajori. A history of mathematical notations; notations mainly in higher mathematics. In *A History of Mathematical Notations; Two Volumes Bound as One*, volume 2. Dover, Mineola, New York, USA, 1993. ISBN 0-486-67766-4. URL http://books.google.com/books?vid=ISBN0486677664. reprint of 1929 edition by *The Open Court Publishing Company*.

Gerolamo Cardano. *Ars Magna or the Rules of Algebra*. Dover Publications, Mineola, New York, 1545. ISBN 0486458733. URL http://www.amazon.com/dp/0486458733. English translation of the Latin *Ars Magna* edition, published in 2007.

Lennart Axel Edvard Carleson. Convergence and growth of partial sums of fourier series. *Acta Mathematica*, 116(1):135–157, 1966. doi: 10.1007/BF02392815. URL http://dx.doi.org/10.1007/BF02392815.

Lennart Axel Edvard Carleson and Björn Engquist. After the 'golden age': what next? lennart carleson interviewed by björn engquist. In Björn Engquist and Wilfried Schmid, editors, *Mathematics Unlimited—2001 and beyond*, pages 455–461. Springer, Berlin, 2001. ISBN 978-3-540-66913-5. URL https://link.springer.com/chapter/10.1007/978-3-642-56478-9_22.

N. L. Carothers. *A Short Course on Banach Space Theory*. Number 64 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2005. ISBN 0521842832. URL http://www.comm.musashi-tech.ac.jp/~arimoto/Drarimoto/preprints/calothers/Assemble.ps.

Peter G. Casazza and Mark C. Lammers. Bracket products for weyl-heisenberg frames. In Hans G. Feichtinger and Thomas Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 71–98. Birkhäuser, 1998. ISBN 9780817639594.

Clémente Ibarra Castanedo. *Quantitative subsurface defect evaluation by pulsed phase thermography: depth retrieval with the phase*. PhD thesis, Université Laval, October 2005. URL http://archimede.bibl.ulaval.ca/archimede/fichiers/23016/23016.html. Faculte' Des Sciences Et De Génie.

Arthur Cayley. A memoir on the theory of matrices. *Philosophical Transactions of the Royal Society of London*, 148:17–37, 1858. ISSN 1364-503X. URL http://www.jstor.org/view/02610523/ap000059/00a00020/0.

Joan Cerdà. *Linear functional analysis*, volume 116 of *Graduate studies in mathematics*. American Mathematical Society, July 16 2010. ISBN 0821851152. URL http://books.google.com/books?vid=ISBN0821851152.

Alexandre J. Chorin and Ole H. Hald. *Stochastic Tools in Mathematics and Science*, volume 1 of *Surveys and Tutorials in the Applied Mathematical Sciences*. Springer, New York, 2 edition, 2009. ISBN 978-1-4419-1001-1. URL http://books.google.com/books?vid=ISBN9781441910011.

Ole Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston/Basel/Berlin, 2003. ISBN 0-8176-4295-1. URL http://books.google.com/books?vid=ISBN0817642951.

Ole Christensen. *Frames and bases: An Introductory Course*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston/Basel/Berlin, 2008. ISBN 9780817646776. URL http://books.google.com/books?vid=ISBN0817646779.

Charles K. Chui. *An Introduction to Wavelets*. Academic Press, San Diego, California, USA, January 3 1992. ISBN 9780121745844. URL http://books.google.com/books?vid=ISBN0121745848.

Jon F. Claerbout. *Fundamentals of Geophysical Data Processing with Applications to Petroleum Prospecting*. International series in the earth and planetary sciences, Tab Mastering Electronics Series. McGraw-Hill, New York, 1976. ISBN 9780070111172. URL http://sep.stanford.edu/sep/prof/.

Paul M. Cohn. *Basic Algebra; Groups, Rings and Fields*. Springer, December 6 2002. ISBN 1852335874. URL http://books.google.com/books?vid=isbn1852335874.

R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume 1. Interscience Publishers, New York, 1930. URL http://www.worldcat.org/isbn/0471504475.

Xingde Dai and David R. Larson. *Wandering vectors for unitary systems and orthogonal wavelets*. Number 640 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence R.I., July 1998. ISBN 0821808001. URL http://books.google.com/books?vid=ISBN0821808001.

Xingde Dai and Shijie Lu. Wavelets in subspaces. *Michigan Math. J.*, 43(1):81–98, 1996. doi: 10.1307/mmj/1029005391. URL http://projecteuclid.org/euclid.mmj/1029005391.

Ingrid Daubechies. *Ten Lectures on Wavelets*. Society for Industrial and Applied Mathematics, Philadelphia, 1992. ISBN 0-89871-274-2. URL http://www.amazon.com/dp/0898712742.

Ingrid Daubechies, A. Grossman, and Y. Meyer. Painless nonorthogonal expansions. *Journal of Mathematical Physics*, 27(5):1271–1283, May 1986. ISSN 0022-2488. URL link.aip.org/link/?jmp/27/1271.

Kenneth R. Davidson and Allan P. Donsig. *Real Analysis and Applications*. Springer, 2010. ISBN 9781441900050. URL http://books.google.com/books?vid=ISBN1441900055.

Charles Jean de la Vallée-Poussin. Sur l'intégrale de lebesgue. *Transactions of the American Mathematical Society*, 16(4):435–501, October 1915. URL http://www.jstor.org/stable/1988879.

Juan-Arias de Reyna. *Pointwise Convergence of Fourier Series*, volume 1785 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin/Heidelberg/New York, 2002. ISBN 3540432701. URL http://books.google.com/books?vid=ISBN3540432701.

Johan de Witt. Elementa curvarum linearum (elements of curves). In *La Géométrie*. 1659. Elementa curvarum linearum was an appendix to de Witt's translation of of Descartes' *La Géométrie*.

René Descartes. *La géométrie*. 1637a. URL http://historical.library.cornell.edu/math/math_D.html.

René Descartes. *Discours de la méthode pour bien conduire sa raison, et chercher la verite' dans les sciences*. Jan Maire, Leiden, 1637b. URL http://www.gutenberg.org/etext/13846.

René Descartes. *Discourse on the Method of Rightly Conducting the Reason in the Search for Truth in the Sciences*. 1637c. URL http://www.gutenberg.org/etext/59.

René Descartes. *Regulae ad directionem ingenii*. 1684a. URL http://www.fh-augsburg.de/~harsch/Chronologia/Lspost17/Descartes/des_re00.html.

René Descartes. *Rules for Direction of the Mind*. 1684b. URL http://en.wikisource.org/wiki/Rules_for_the_Direction_of_the_Mind.

René Descartes. *The Geometry of Rene Descartes*. Courier Dover Publications, June 1 1954. ISBN 0486600688. URL http://books.google.com/books?vid=isbn0486600688. orginally published by Open Court Publishing, Chicago, 1925; translation of La géométrie.

Elena Deza and Michel-Marie Deza. *Dictionary of Distances*. Elsevier Science, Amsterdam, 2006. ISBN 0444520872. URL http://books.google.com/books?vid=ISBN0444520872.

Jean Alexandre Dieudonné. *Foundations of Modern Analysis.* Academic Press, New York, 1969. ISBN 1406727911. URL http://books.google.com/books?vid=ISBN1406727911.

R. J. Duffin and A. C. Schaeffer. A class of nonharmonic fourier series. *Transactions of the American Mathematical Society*, 72(2):341–366, March 1952. ISSN 1088-6850. URL http://www.jstor.org/stable/1990760.

Bogdan Dumitrescu. *Positive Trigonometric Polynomials and Signal Processing Applications.* Signals and Communication Technology. Springer, 2007. ISBN 978-1-4020-5124-1. URL gen.lib.rus.ec/get?md5=5346e169091b2d928d8333cd053300f9.

Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part 1, General Theory*, volume 7 of *Pure and applied mathematics.* Interscience Publishers, New York, 1957. ISBN 0471226394. URL http://www.amazon.com/dp/0471608483. with the assistance of William G. Bade and Robert G. Bartle.

John R. Durbin. *Modern Algebra; An Introduction.* John Wiley & Sons, Inc., 4 edition, 2000. ISBN 0-471-32147-8. URL http://www.worldcat.org/isbn/0471321478.

Robert E. Edwards. *Functional Analysis: Theory and Applications.* Dover books on mathematics. Dover, New York, 1995. ISBN 0-486-68143-2. URL http://books.google.com/books?vid=ISBN0486681432.

Yuli Eidelman, Vitali D. Milman, and Antonis Tsolomitis. *Functional Analysis: An Introduction*, volume 66 of *Graduate Studies in Mathematics.* American Mathematical Society, 2004. ISBN 0821836463. URL http://books.google.com/books?vid=ISBN0821836463.

Per Enflo. A counterexample to the approximation problem in banach spaces. *Acta Mathematica*, 130:309–317, 1973. URL http://link.springer.com/content/pdf/10.1007/BF02392270.

Leonhard Euler. *Introductio in analysin infinitorum*, volume 1. Marcum-Michaelem Bousquet & Socios, Lausannæ, 1748. URL http://www.math.dartmouth.edu/~euler/pages/E101.html. Introduction to the Analysis of the Infinite.

Leonhard Euler. *Introduction to the Analysis of the Infinite.* Springer, 1988. ISBN 0387968245. URL http://books.google.com/books?vid=ISBN0387968245. translation of 1748 Introductio in analysin infinitorum.

David Ewen. *The Book of Modern Composers.* Alfred A. Knopf, New York, 1950. URL http://books.google.com/books?id=yHw4AAAAIAAJ.

David Ewen. *The New Book of Modern Composers.* Alfred A. Knopf, New York, 3 edition, 1961. URL http://books.google.com/books?id=bZIaAAAAMAAJ.

Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos, Vaclav Zizler, and Václav Zizler. *Banach Space Theory: The Basis for Linear and Nonlinear Analysis.* CMS Books in Mathematics. Springer, 2010. ISBN 1441975144. URL http://books.google.com/books?vid=isbn1441975144.

Lorenzo Farina and Sergio Rinaldi. *Positive Linear Systems: Theory and Applications.* Pure and applied mathematics. John Wiley & Sons, 1 edition, July 3 2000. ISBN 9780471384564. URL http://books.google.com/books?vid=ISBN0471384569.

G.L. Fix and G. Strang. Fourier analysis of the finite element method in ritz-galerkin theory. *Studies in Applied Mathematics*, 48:265–273, 1969.

Harley Flanders. Differentiation under the integral sign. *The American Mathematical Monthly*, 80(6):615–627, June–July 1973. URL http://sgpwe.izt.uam.mx/files/users/uami/jdf/proyectos/Derivar_inetegral.pdf. http://www.jstor.org/pss/2319163.

Francis J. Flanigan. *Complex Variables; Harmonic and Analytic Functions*. Dover, New York, 1983. ISBN 9780486613888. URL http://books.google.com/books?vid=ISBN0486613887.

Gerald B. Folland. *Fourier Analysis and its Applications*. Wadsworth & Brooks / Cole Advanced Books & Software, Pacific Grove, California, USA, 1992. ISBN 0-534-17094-3. URL http://www.worldcat.org/isbn/0534170943.

Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1995. ISBN 0-8493-8490-7. URL http://books.google.com/books?vid=ISBN0849384907.

Brigitte Forster and Peter Massopust, editors. *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*. Applied and Numerical Harmonic Analysis. Springer, November 19 2009. ISBN 9780817648909. URL http://books.google.com/books?vid=ISBN0817648909.

Jean-Baptiste-Joseph Fourier. Mémoire sur la propagation de la chaleur dans les corps solides (dissertation on the propagation of heat in solid bodies). In M. Gaston Darboux, editor, *Œuvres De Fourier*, pages 215–221. Ministère de L'instruction Publique, Paris, France, 2 edition, December 21 1807. URL http://gallica.bnf.fr/ark:/12148/bpt6k33707/f220n7.

Jean-Baptiste-Joseph Fourier. *Théorie Analytique de la Chaleur (The Analytical Theory of Heat)*. Chez Firmin Didot, pere et fils, Paris, 1822. URL http://books.google.com/books?vid=04X2vlqZx7hydlQUWEq&id=TDQJAAAAIAAJ.

Jean-Baptiste-Joseph Fourier. *The Analytical Theory of Heat (Théorie Analytique de la Chaleur)*. Cambridge University Press, Cambridge, February 20 1878. URL http://www.archive.org/details/analyticaltheory00fourrich. 1878 English translation of the original 1822 French edition. A 2003 Dover edition is also available: isbn 0486495310.

Maurice René Fréchet. Sur quelques points du calcul fonctionnel (on some points of functional calculation). *Rendiconti del Circolo Matematico di Palermo*, 22:1–74, April 22 1906. URL https://www.lpsm.paris/pageperso/mazliak/Frechet_1906.pdf. Rendiconti del Circolo Matematico di Palermo (Statements of the Mathematical Circle of Palermo).

Maurice René Fréchet. *Les Espaces abstraits et leur théorie considérée comme introduction a l'analyse générale*. Borel series. Gauthier-Villars, Paris, 1928. URL http://books.google.com/books?id=9czoHQAACAAJ. Abstract spaces and their theory regarded as an introduction to general analysis.

Erik Ivar Fredholm. Sur une nouvelle méthode pour la résolution du problème de dirichlet (on a new method for the resolution of the problem of dirichlet). */:Oefversigt af Kongl. Sv. Vetenskaps-Academiens Förhandlingar*, 57:39–66, 1900.

Erik Ivar Fredholm. Sur une classe d'equations fonctionnelles (on a class of functional equations). *Acta Mathematica*, 27(1):365–390, December 1903. ISSN 0001-5962. doi: 10.1007/BF02421317. URL http://www.springerlink.com/content/c41371137837p252/.

Ferdinand Georg Frobenius. Uber lineare substitutionen und bilineare formen. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 84:1–63, 1878. ISSN 0075-4102. URL http://www.digizeitschriften.de/home/services/pdfterms/?ID=509796.

Ferdinand Georg Frobenius. Uber lineare substitutionen und bilineare formen. In Jean Pierre Serre, editor, *Gesammelte Abhandlungen (Collected Papers)*, volume I, pages 343–405. Springer, Berlin, 1968. URL http://www.worldcat.org/oclc/253015. reprint of Frobenius' 1878 paper.

Jürgen Fuchs. *Affine Lie Algebras and Quantum Groups: An Introduction, With Applications in Conformal Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1995. ISBN 052148412X. URL http://books.google.com/books?vid=ISBN052148412X.

Dennis Gabor. Theory of communication. *Journal of the Institution of Electrical Engineers*, 93(26): 429–457, November 1946. URL http://bigwww.epfl.ch/chaudhury/gabor.pdf.

Carl Friedrich Gauss. *Carl Friedrich Gauss Werke*, volume 8. Königlichen Gesellschaft der Wissenschaften, B.G. Teubneur In Leipzig, Göttingen, 1900. URL http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN236010751.

Israel M. Gelfand. Normierte ringe. *Mat. Sbornik*, 9(51):3–24, 1941.

Israel M. Gelfand and Mark A. Naimark. Normed rings with an involution and their representations. In *Commutative Normed Rings*, number 170 in AMS Chelsea Publishing Series, pages 240–274. Chelsea Publishing Company, Bronx, 1964. ISBN 9780821820223. URL http://books.google.com/books?vid=ISBN0821820222.

Israel M. Gelfand and Mark A. Neumark. On the imbedding of normed rings into the ring of operators in hilbert space. *Mat. Sbornik*, 12(54:2):197–217, 1943a.

Israel M. Gelfand and Mark A. Neumark. On the imbedding of normed rings into the ring of operators in hilbert space. In Robert S. Doran, editor, *C*-algebras: 1943–1993: a Fifty Year Celebration: Ams Special Session Commenorating the First Fifty Years of C*-Algebra Theory January 13–14, 1993*, pages 3–19. 1943b. ISBN 978-0821851753. URL http://books.google.com/books?vid=ISBN0821851756.

Israel M. Gelfand, R. A. Minlos, and Z. .Ya. Shapiro. *Representations of the rotation and Lorentz groups and their applications*. Courier Dover Publications, reprint edition, 2018. ISBN 9780486823850.

Izrail' Moiseevich Gel'fand. *Representations of the rotation and Lorentz groups and their applications*. Pergamon Press book, 1963. 2018 Dover edition available.

John Robilliard Giles. *Introduction to the Analysis of Normed Linear Spaces*. Number 13 in Australian Mathematical Society lecture series. Cambridge University Press, Cambridge, 2000. ISBN 0-521-65375-4. URL http://books.google.com/books?vid=ISBN0521653754.

Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek. *Basic Classes of Linear Operators*. Birkhäuser, Basel, 1 edition, 2003. ISBN 3764369302. URL http://books.google.com/books?vid=ISBN3764369302.

T. N. T. Goodman, S. L. Lee, and W. S. Tang. Wavelets in wandering subspaces. *Transactions of the A.M.S.*, 338(2):639–654, August 1993a. URL http://www.jstor.org/stable/2154421. Transactions of the American Mathematical Society.

T. N. T. Goodman, S. L. Lee, and W. S. Tang. Wavelets in wandering subspaces. *Advances in Computational Mathematics 1*, pages 109–126, February 1993b.

Jaideva C. Goswami and Andrew K. Chan. *Fundamentals of Wavelets; Theory, Algorithms, and Applications*. John Wiley & Sons, Inc., 1999. ISBN 0-471-19748-3. URL http://vadkudr.boom.ru/Collection/fundwave_contents.html.

Daniel J. Greenhoe. *Wavelet Structure and Design*, volume 3 of *Mathematical Structure and Design series*. Abstract Space Publishing, August 2013. ISBN 9780983801139. URL http://books.google.com/books?vid=ISBN0983801134. revised online version available at https://www.researchgate.net/publication/312529555.

Stanley Gudder. *Stochastic Methods in Quantum Mechanics*. North Holland, 1979. ISBN 0444002995. URL http://books.google.com/books?vid=ISBN0444002995.

Stanley Gudder. *Stochastic Methods in Quantum Mechanics*. Dover, Mineola NY, 2005. ISBN 0486445321. URL http://books.google.com/books?vid=ISBN0486445321.

Frédéric Guichard, Jean-Michel Morel, and Robert Ryan. Contrast invariant image analysis and pde's. 2012. URL http://dev.ipol.im/~morel/JMMBook2012.pdf.

Ernst Adolph Guillemin. *Synthesis of Passive Networks: Theory and Methods Appropriate to the Realization and Approximation Problems*. John Wiley & Sons, 1957. ISBN 9780882754819. URL http://books.google.com.tw/books?id=JQ4nAAAAMAAJ.

Amritava Gupta. *Real & Abstract Analysis*. Academic Publishers Calcutta, 1998. ISBN 9788186358443. URL http://books.google.com/books?vid=ISBN8186358447. first published in 1998, reprinted in 2000 and 2006.

Alfréd Haar. Zur theorie der orthogonalen funktionensysteme. *Mathematische Annalen*, 69:331–371, September 1910. ISSN 1432-1807. doi: 10.1007/BF01456326.

Norman B. Haaser and Joseph A. Sullivan. *Real Analysis*. Dover Publications, New York, 1991. ISBN 0-486-66509-7. URL http://books.google.com/books?vid=ISBN0486665097.

Paul R. Haddad and Ali N. Akansu. *Multiresolution Signal Decomposition: Transforms, Subbands, and Wavelets*. Acedemic Press, October 1 1992. ISBN 0323138365. URL http://books.google.com/books?vid=ISBN0323138365.

Paul R. Halmos. *Finite Dimensional Vector Spaces*. Princeton University Press, Princeton, 1 edition, 1948. ISBN 0691090955. URL http://books.google.com/books?vid=isbn0691090955.

Paul R. Halmos. *Finite Dimensional Vector Spaces*. Springer-Verlag, New York, 2 edition, 1958. ISBN 0-387-90093-4. URL http://books.google.com/books?vid=isbn0387900934.

Paul R. Halmos. *Intoduction to Hilbert Space and the Theory of Spectral Multiplicity*. Chelsea Publishing Company, New York, 2 edition, 1998a. ISBN 0821813781. URL http://books.google.com/books?vid=ISBN0821813781.

Paul R. Halmos. *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*. AMS Chelsea, Providence RI, 2 edition, 1998b. ISBN 0821813781. URL http://books.google.com/books?vid=ISBN0821813781.

Georg Hamel. Eine basis aller zahlen und die unstetigen lösungen der funktionalgleichung $f(x+y) = f(x) + f(y)$. *Mathematische Annalen*, 60(3):459–462, 1905. URL http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN002260395&IDDOC=28580.

Deguang Han, Keri Kornelson, David Larson, and Eric Weber. *Frames for Undergraduates*, volume 40 of *Student Mathematical Library*. American Mathematical Society, 2007. ISBN 0821842129. URL http://books.google.com/books?vid=ISBN0821842129. Deguang Han = ???

Godfrey H. Hardy. *A Mathematician's Apology*. Cambridge University Press, Cambridge, 1940. URL http://www.math.ualberta.ca/~mss/misc/A%20Mathematician's%20Apology.pdf.

Felix Hausdorff. *Set Theory*. Chelsea Publishing Company, New York, 3 edition, 1937. ISBN 0828401195. URL http://books.google.com/books?vid=ISBN0828401195. 1957 translation of the 1937 German *Grundzüge der Mengenlehre*.

Michiel Hazewinkel, editor. *Handbook of Algebras*, volume 2. North-Holland, Amsterdam, 1 edition, 2000. ISBN 044450396X. URL http://books.google.com/books?vid=ISBN044450396X.

Jean Van Heijenoort. *From Frege to Gödel : A Source Book in Mathematical Logic, 1879-1931*. Harvard University Press, Cambridge, Massachusetts, 1967. URL http://www.hup.harvard.edu/catalog/VANFGX.html.

Christopher Heil. *A Basis Theory Primer*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, expanded edition edition, 2011. ISBN 9780817646868. URL http://books.google.com/books?vid=ISBN9780817646868.

Christopher E. Heil and David F. Walnut. Continuous and discrete wavelet transforms. *Society for Industrial and Applied Mathematics*, 31(4), December 1989. URL http://citeseer.ist.psu.edu/viewdoc/download?doi=10.1.1.132.1241&rep=rep1&type=pdf.

Charles Hermite. Lettre à stieltjes. In B. Baillaud and H. Bourget, editors, *Correspondance d'Hermite et de Stieltjes*, volume 2, pages 317–319. Gauthier-Villars, Paris, May 20 1893. published in 1905.

Eugenio Hernández and Guido Weiss. *A First Course on Wavelets*. CRC Press, New York, 1996. ISBN 0849382742. URL http://books.google.com/books?vid=ISBN0849382742.

John Rowland Higgins. *Sampling Theory in Fourier and Signal Analysis: Foundations*. Oxford Science Publications. Oxford University Press, August 1 1996. ISBN 9780198596998. URL http://books.google.com/books?vid=ISBN0198596995.

David Hilbert. Grundzüge einer allgemeinen theorie der linearen integralgleichungen (fundamentals of a general theory of the linear integral equations). *Mathematisch-Physikalische Klasse (Mathematical-Physical Class)*, pages 49–91, March 1904. URL http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?did=D57552. Report 1 of 6.

David Hilbert. Grundzüge einer allgemeinen theorie der linearen integralgleichungen (fundamentals of a general theory of the linear integral equations). *Mathematisch-Physikalische Klasse (Mathematical-Physical Class)*, pages 157–228, March 1906. URL http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?did=D58133. Report 4 of 6.

David Hilbert. *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen (Fundamentals of a general theory of the linear integral equations)*, volume 26 of *Fortschritte der mathematischen Wissenschaften in Monographien (Progress of the mathematical sciences in Monographs)*. B.G. Teubner, Leipzig und Berlin, 1912. URL http://www.worldcat.org/oclc/13468199.

David Hilbert, Lothar Nordheim, and John von Neumann. über die grundlagen der quantenmechanik (on the bases of quantum mechanics). *Mathematische Annalen*, 98:1–30, 1927. ISSN 0025-5831 (print) 1432-1807 (online). URL http://dz-srv1.sub.uni-goettingen.de/cache/toc/D27776.html.

Samuel S. Holland, Jr. The current interest in orthomodular lattices. In James C. Abbott, editor, *Trends in Lattice Theory*, pages 41–126. Van Nostrand-Reinhold, New York, 1970. URL http://books.google.com/books?id=ZfA-AAAAIAAJ. from Preface: “The present volume contains written versions of four talks on lattice theory delivered to a symposium on Trends in Lattice Theory held at the United States Naval Academy in May of 1966.”.

Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990. ISBN 0-521-30586-1. URL http://books.google.com/books?vid=isbn0521305861. Library: QA188H66 1985.

Alfred Edward Housman. *More Poems*. Alfred A. Knopf, 1936. URL http://books.google.com/books?id=rTMiAAAAMAAJ.

K Husimi. Studies on the foundations of quantum mechanics i. *Proceedings of the Physico-Mathematical Society of Japan*, 19:766–789, 1937.

Satoru Igari. *Real Analysis—With an Introduction to Wavelet Theory*, volume 177 of *Translations of mathematical monographs*. American Mathematical Society, 1996. ISBN 9780821821046. URL http://books.google.com/books?vid=ISBN0821821040.

Taizo Iijima. Basic theory of pattern observation. *Papers of Technical Group on Automata and Automatic Control*, December 1959. see Weickert 1999 for historical information.

Chris J. Isham. *Modern Differential Geometry for Physicists*. World Scientific Publishing, New Jersey, 2 edition, 1999. ISBN 9810235623. URL http://books.google.com/books?vid=ISBN9810235623.

C.J. Isham. Quantum topology and quantisation on the lattice of topologies. *Classical and Quantum Gravity*, 6:1509–1534, November 1989. doi: 10.1088/0264-9381/6/11/007. URL http://www.iop.org/EJ/abstract/0264-9381/6/11/007.

Vasile I. Istrăţescu. *Inner Product Structures: Theory and Applications*. Mathematics and Its Applications. D. Reidel Publishing Company, 1987. ISBN 9789027721822. URL http://books.google.com/books?vid=ISBN9027721823.

Luisa Iturrioz. Ordered structures in the description of quantum systems: mathematical progress. In *Methods and applications of mathematical logic: proceedings of the VII Latin American Symposium on Mathematical Logic held July 29-August 2, 1985*, volume 69, pages 55–75, Providence Rhode Island, July 29–August 2 1985. Sociedade Brasileira de Lógica, Sociedade Brasileira de Matemática, and the Association for Symbolic Logic, AMS Bookstore (1988). ISBN 0821850768.

A. J. E. M. Janssen. The zak transform: A signal transform for sampled time-continuous signals. *Philips Journal of Research*, 43(1):23–69, 1988.

Bjorn Jawerth and Wim Sweldens. An overview of wavelet based multiresolutional analysis. *SIAM Review*, 36:377–412, September 1994. URL http://cm.bell-labs.com/who/wim/papers/papers.html#overview.

Alan Jeffrey and Hui Hui Dai. *Handbook of Mathematical Formulas and Integrals*. Handbook of Mathematical Formulas and Integrals Series. Academic Press, 4 edition, January 18 2008. ISBN 9780080556840. URL http://books.google.com/books?vid=ISBN0080556841.

Palle E. T. Jørgensen, Kathy Merrill, Judith Packer, and Lawrence W. Baggett. *Representations, Wavelets and Frames: A Celebration of the Mathematical Work of Lawrence Baggett*. Applied and Numerical Harmonic Analysis. Birkhäuser, 2008. ISBN 9780817646820. URL http://books.google.com/books?vid=ISBN0817646825.

K. D. Joshi. *Applied Discrete Structures*. New Age International, New Delhi, 1997. ISBN 8122408265. URL http://books.google.com/books?vid=ISBN8122408265.

J.S.Chitode. *Signals And Systems*. Technical Publications, 2009. ISBN 9788184316780. URL http://books.google.com/books?vid=ISBN818431678X.

Jean-Pierre Kahane. Partial differential equations, trigonometric series, and the concept of function around 1800: a brief story about lagrange and fourier. In Dorina Mitrea and V. G. Mazía, editors, *Perspectives In Partial Differential Equations, Harmonic Analysis And Applications: a Volume in Honor of Vladimir G. Maz'ya's 70th birthday*, volume 79 of *Proceedings of Symposia in Pure Mathematics*, pages 187–206. American Mathematical Society, 2008. ISBN 0821844245. URL http://books.google.com/books?vid=ISBN0821844245.

Shizuo Kakutani and George W. Mackey. Ring and lattice characterizations of complex hilbert space. *Bulletin of the American Mathematical Socieity*, 52:727–733, 1946. doi: 10.1090/S0002-9904-1946-08644-9. URL http://www.ams.org/bull/1946-52-08/S0002-9904-1946-08644-9/.

David W. Kammler. *A First Course in Fourier Analysis*. Cambridge University Press, 2 edition, 2008. ISBN 9780521883405. URL http://books.google.com/books?vid=ISBN0521883407.

Edward Kasner and James Roy Newman. *Mathematics and the Imagination*. Simon and Schuster, 1940. ISBN 0486417034. URL http://books.google.com/books?vid=ISBN0486417034. “unabridged and unaltered republication” available from Dover.

Yitzhak Katznelson. *An Introduction to Harmonic Analysis*. Cambridge mathematical library. Cambridge University Press, 3 edition, 2004. ISBN 0521543592. URL http://books.google.com/books?vid=ISBN0521543592.

James P. Keener. *Principles of Applied Mathematics; Transformation and Approximation*. Addison-Wesley Publishing Company, Reading, Massachusets, 1988. ISBN 0201156741. URL http://www.worldcat.org/isbn/0201156741.

Yoshida Kenko. *The Tsuredzure Gusa of Yoshida No Kaneyoshi. Being the meditations of a recluse in the 14th Century (Essays in Idleness)*. circa 1330. URL http://www.humanistictexts.org/kenko.htm. 1911 translation of circa 1330 text.

Anthony W Knapp. *Advanced Real Analysis*. Cornerstones. Birkhäuser, Boston, Massachusetts, USA, 1 edition, July 29 2005a. ISBN 0817643826. URL http://books.google.com/books?vid=ISBN0817643826.

Anthony W Knapp. *Basic Real Analysis*. Cornerstones. Birkhäuser, Boston, Massachusetts, USA, 1 edition, July 29 2005b. ISBN 0817632506. URL http://books.google.com/books?vid=ISBN0817632506.

M. G. Krein and M. A. Krasnoselski. Fundamental theorems concerning the extension of hermitian operators and some of their applications to the theory of orthogonal polynomials and the moment problem. *Uspekhi Matematicheskikh Nauk*, 2(3), 1947. URL http://www.turpion.org/php/homes/pa.phtml?jrnid=rm. (Russian Mathematical Surveys).

M. G. Krein, M. A. Krasnoselski, and D. P. Milman. Concerning the deficiency numbers of linear operators in banach space and some geometric questions. *Sbornik Trudov Inst. Matem. AN Ukrainian SSR*, 11:97–112, 1948.

Carlos S. Kubrusly. *The Elements of Operator Theory*. Springer, 1 edition, 2001. ISBN 9780817641740. URL http://books.google.com/books?vid=ISBN0817641742.

Carlos S. Kubrusly. *The Elements of Operator Theory*. Springer, 2 edition, 2011. ISBN 9780817649975. URL http://books.google.com/books?vid=ISBN0817649972.

Andrew Kurdila and Michael Zabarankin. *Convex Functional Analysis*. Systems & Control: Foundations & Applications. Birkhäuser, Boston, 2005. ISBN 9783764321987. URL http://books.google.com/books?vid=ISBN3764321989.

Joseph-Louis Lagrange, Pierre-Simon Laplace, Étienne Louis Malus, René Just Haüy, and Adrien-Marie Legendre. Proclamation des prix décernés dans la séance publique de 6 janvier 1812. *Esprit des Journaux, Francais et étrangers par Une Societe de Gens Delettres*, 2:111–112, January 6 1812a. URL http://books.google.com/books?id=QpUUAAAAQAAJ.

Joseph-Louis Lagrange, Pierre-Simon Laplace, Étienne Louis Malus, René Just Haüy, and Adrien-Marie Legendre. Proclamation des prix décernés dans la séance publique de 6 janvier 1812. *Mercure De France, Journal Littéraire et Politique*, 50:374–375, January 6 1812b. URL http://books.google.com/books?id=8HxBAAAAcAAJ.

Imre Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge University Press, Cambridge, 1976. ISBN 0521290384. URL http://books.google.com/books?vid=ISBN0521290384.

Traian Lalescu. *Sur les équations de Volterra*. PhD thesis, University of Paris, 1908. advisor was Émile Picard.

Traian Lalescu. *Introduction à la théorie des équations intégrales (Introduction to the Theory of Integral Equations)*. Librairie Scientifique A. Hermann, Paris, 1911. URL http://www.worldcat.org/oclc/1278521. first book about integral equations ever published.

Rupert Lasser. *Introduction to Fourier Series*, volume 199 of *Monographs and textbooks in pure and applied mathematics*. Marcel Dekker, New York, New York, USA, February 8 1996. ISBN 978-0824796105. URL http://books.google.com/books?vid=ISBN0824796101. QA404.L33 1996.

Peter D. Lax. *Functional Analysis*. John Wiley & Sons Inc., USA, 2002. ISBN 0-471-55604-1. URL http://www.worldcat.org/isbn/0471556041. QA320.L345 2002.

Gottfried W. Leibniz. Symbolismus memorabilis calculi algebraici et infinitesimalis, in comparatione potentiarum et differentiarum; et de lege homogeneorum transcendentali. *Miscellanea Berolinensia ad incrementum scientiarum, ex scriptis Societati Regiae scientarum*, pages 160–165, 1710. URL http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/anzeige/index_html?band=01-misc/1&seite:int=184.

Gottfried Wilhelm Leibniz. Letter to christian huygens, 1679. In Leroy E. Loemker, editor, *Philosophical Papers and Letters*, volume 2 of *The New Synthese Historical Library*, chapter 27, pages 248–249. Kluwer Academic Press, Dordrecht, 2 edition, September 8 1679. ISBN 902770693X. URL http://books.google.com/books?vid=ISBN902770693X.

Pierre Gilles Lemarié, editor. *Les Ondelettes en 1989*, volume 1438 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. ISBN 978-3-540-52932-3. URL http://link.springer.com/book/10.1007/BFb0083510/.

Tony Lindeberg. *Scale-Space Theory in Computer Vision*. The Springer International Series in Engineering and Computer Science. Springer, 1993. ISBN 9780792394181. URL http://books.google.com/books?vid=ISBN0792394186.

Joram Lindenstrauss and Lior Tzafriri. *Classical Banach Spaces I: Sequence Spaces*, volume 92 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1977. ISBN 978-3642665592. URL http://www.amazon.com/dp/3642665594.

J. Liouville. Sur l'integration d'une classe d'équations différentielles du second ordre en quantités finies explicites. *Journal De Mathematiques Pures Et Appliquees*, 4:423–456, 1839. URL http://gallica.bnf.fr/ark:/12148/bpt6k16383z.

Lynn H. Loomis. *The Lattice Theoretic Background of the Dimension Theory of Operator Algebras*, volume 18 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence RI, 1955. ISBN 0821812181. URL http://books.google.com/books?id=P3Vl_lXCFRkC.

Lynn H. Loomis and Ethan D. Bolker. *Harmonic analysis*. Mathematical Association of America, 1965. URL http://books.google.com/books?id=MEfvAAAAMAAJ.

Nikolai Luzin. Sur la convergence des séries trigonom etriers de fourier. *C. R. Acad. Sci.*, 156:1655–1658, 1913.

Niccolò Machiavelli. *The Literary Works of Machiavelli: Mandragola, Clizia, A Dialogue on Language, and Belfagor, with Selections from the Private Correspondence*. Oxford University Press, 1961. ISBN 0313212481. URL http://www.worldcat.org/isbn/0313212481.

Colin Maclaurin. *Treatise of Fluxions*. W. Baynes, 1742. URL http://www.amazon.com/dp/B000863E7M.

Stéphane G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $l^2(\mathbb{R})$. *Transactions of the American Mathematical Society*, 315(1):69–87, September 1989. URL http://blanche.polytechnique.fr/~mallat/papiers/math_multiresolution.pdf.

Stéphane G. Mallat. *A Wavelet Tour of Signal Processing*. Elsevier, 2 edition, September 15 1999. ISBN 9780124666061. URL http://books.google.com/books?vid=ISBN012466606X.

J. L. Massera and J. J. Schäffer. Linear differential equations and functional analysis, i. *The Annals of Mathematics, 2nd Series*, 67(3):517–573, May 1958. URL http://www.jstor.org/stable/1969871.

maxima. *Maxima Manual version 5.28.0*. 5.28.0 edition. URL http://maxima.sourceforge.net/documentation.html.

Stefan Mazur. Sur les anneaux linéaires. *Comptes rendus de l'Académie des sciences*, 207:1025–1027, 1938.

Stefan Mazur and Stanislaus M. Ulam. Sur les transformations isométriques d'espaces vectoriels normées. *Comptes rendus de l'Académie des sciences*, 194:946–948, 1932.

James Mercer. Functions of positive and negative type and their connection with the theory of integral equations. *Philosophical Transactions of the Royal Society of London*, 209:415–446, 1909. ISSN 02643952. URL http://www.jstor.org/stable/91043.

Yves Meyer. *Wavelets and Operators*, volume 37 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, January 12 1992. ISBN 0521458692. URL http://books.google.com/books?vid=ISBN0521458692.

Anthony N. Michel and Charles J. Herget. *Applied Algebra and Functional Analysis*. Dover Publications, Inc., 1993. ISBN 048667598X. URL http://books.google.com/books?vid=ISBN048667598X. original version published by Prentice-Hall in 1981.

Fred Mintzer. Filters for distortion-free two-band multi-rate filter banks. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 32, 1985.

Eddie Ortiz Muniz. A method for deriving various formulas in electrostatics and electromagnetism using lagrange's trigonometric identities. *American Journal of Physics*, 21(140), 1953. doi: 10.1119/1.1933371. URL http://dx.doi.org/10.1119/1.1933371.

Ben Noble and James W. Daniel. *Applied Linear Algebra*. Prentice-Hall, Englewood Cliffs, NJ, USA, 3 edition, 1988. ISBN 0130412600. URL http://www.worldcat.org/isbn/0130412600.

Timur Oikhberg and Haskell Rosenthal. A metric characterization of normed linear spaces. *Rocky Mountain Journal Of Mathematics*, 37(2):597–608, 2007. URL http://www.ma.utexas.edu/users/rosenthl/pdf-papers/95-oikh.pdf.

Judith Packer. Applications of the work of stone and von neumann to wavelets. In Robert S. Doran and Richard V. Kadison, editors, *Operator Algebras, Quantization, and Noncommutative Geometry: A Centennial Celebration Honoring John Von Neumann and Marshall H. Stone : AMS Special Session on Operator Algebras, Quantization, and Noncommutative Geometry, a Centennial Celebration Honoring John Von Neumann and Marshall H. Stone, January 15-16, 2003, Baltimore, Maryland*, volume 365 of *Contemporary mathematics—American Mathematical Society*, pages 253–280, Baltimore, Maryland, 2004. American Mathematical Society. ISBN 9780821834022. URL http://books.google.com/books?vid=isbn0821834029.

Lincoln P. Paine. *Warships of the World to 1900*. Ships of the World Series. Houghton Mifflin Harcourt, 2000. ISBN 9780395984147. URL http://books.google.com/books?vid=ISBN9780395984149.

Anthanasios Papoulis. *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill Series in Electrical Engineering. McGraw-Hill Book Company, New York, 2 edition, 1984. ISBN 9780070484689. URL http://books.google.com/books?vid=ISBN0070484686.

Anthanasios Papoulis. *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, New York, 3 edition, 1991. ISBN 0070484775. URL http://books.google.com/books?vid=ISBN0070484775.

Giuseppe Peano. *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva*. Fratelli Bocca Editori, Torino, 1888a. Geometric Calculus: According to the *Ausdehnungslehre* of H. Grassmann.

Giuseppe Peano. *Geometric Calculus: According to the Ausdehnungslehre of H. Grassmann*. Springer (2000), 1888b. ISBN 0817641262. URL http://books.google.com/books?vid=isbn0817641262. originally published in 1888 in Italian.

Michael Pedersen. *Functional Analysis in Applied Mathematics and Engineering*. Chapman & Hall/CRC, New York, 2000. ISBN 9780849371691. URL http://books.google.com/books?vid=ISBN0849371694. Library QA320.P394 1999.

Wesley J. Perschbacher, editor. *The New Analytical Greek Lexicon*. Hendrickson Publishers, Peabody, Mass., 1990. ISBN 978-0-943575-33-9. URL http://www.amazon.com/dp/0943575338.

Mark A. Pinsky. *Introduction to Fourier Analysis and Wavelets*. Brooks/Cole, Pacific Grove, 2002. ISBN 0-534-37660-6. URL http://www.amazon.com/dp/0534376606.

C Piron. Axiomatique quantique. *Helvetica Physica Acta*, 37:439–468, 1964a. ISSN 0018-0238. English translation completed by M. Cole.

C Piron. Qunatum axiomatics. *Helvetica Physica Acta*, 1964b. English translation of *Axiomatique quantique*, RB4 Technical memo 107/106/104, GPO Engineering Department (London).

Henri Poincaré. *La Science et l'hypothèse.* 1902a. URL http://fr.wikisource.org/wiki/La_Science_et_l%27hypoth%C3%A8se. (Science and Hypothesis).

Henri Poincaré. *Science and Hypothesis.* Dover Publications (1952), New York, 1902b. ISBN 0486602214. URL http://books.google.com/books?vid=isbn0486602214. translation of La Science et l'hypothèse.

Lakshman Prasad and Sundararaja S. Iyengar. *Wavelet Analysis with Applications to Image Processing.* CRC Press LLC, Boca Raton, 1997. ISBN 978-0849331695. URL http://books.google.com/books?vid=ISBN0849331692. Library TA1637.P7 1997.

John G. Proakis. *Digital Communications.* McGraw Hill, 4 edition, 2001. ISBN 0-07-232111-3. URL http://www.mhhe.com/.

Ptolemy. *Ptolemy's Almagest.* Springer-Verlag (1984), New York, circa 100AD. ISBN 0387912207. URL http://gallica.bnf.fr/ark:/12148/bpt6k3974x.

Shie Qian and Dapang Chen. *Joint time-frequency analysis: methods and applications.* PTR Prentice Hall, 1996. ISBN 9780132543842. URL http://books.google.com/books?vid=ISBN0132543842.

Charles Earl Rickart. *General Theory of Banach Algebras.* University series in higher mathematics. D. Van Nostrand Company, Yale University, 1960. URL http://books.google.com/books?id=PVrvAAAAMAAJ.

Theodore J. Rivlin. *The Chebyshev Polynomials.* Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts. John Wiley & Sons, New York, 1974. ISBN 0-471-72470-X. URL http://books.google.com/books?vid=ISBN047172470X.

Enders A. Robinson. *Random Wavelets and Cybernetic Systems*, volume 9 of *Griffins Statistical Monographs & Courses.* Lubrecht & Cramer Limited, London, June 1962. ISBN 0852640757. URL http://books.google.com/books?vid=ISBN0852640757.

Enders A. Robinson. Multichannel z-transforms and minimum delay. *Geophyics*, 31(3):482–500, June 1966. doi: 10.1190/1.1439788. URL http://dx.doi.org/10.1190/1.1439788.

Enders A. Robinson. A historical perspective of spectrum estimation. *Proceedings of the IEEE*, 70 (9):885–907, September 1982. doi: 10.1109/PROC.1982.12423. URL http://www.archive.ece.cmu.edu/~ece792/handouts/Robinson82.pdf.

Maxwell Rosenlicht. *Introduction to Analysis.* Dover Publications, New York, 1968. ISBN 0-486-65038-3. URL http://books.google.com/books?vid=ISBN0486650383.

Walter Rudin. *Real and Complex Analysis.* McGraw-Hill Book Company, New York, New York, USA, 3 edition, 1987. ISBN 9780070542341. URL http://www.amazon.com/dp/0070542341. Library QA300.R8 1976.

Walter Rudin. *Functional Analysis.* McGraw-Hill, New York, 2 edition, 1991. ISBN 0-07-118845-2. URL http://www.worldcat.org/isbn/0070542252. Library QA320.R83 1991.

Bryan P. Rynne and Martin A. Youngson. *Linear Functional Analysis.* Springer undergraduate mathematics series. Springer, 2 edition, January 1 2008. ISBN 9781848000056. URL http://books.google.com/books?vid=ISBN1848000057.

Shôichirô Sakai. *C*-Algebras and W*-Algebras*. Springer-Verlag, Berlin, 1 edition, 1998. ISBN 9783540636335. URL http://books.google.com/books?vid=ISBN3540636331. reprint of 1971 edition.

Usa Sasaki. Orthocomplemented lattices satisfying the exchange axiom. *Journal of Science of the Hiroshima University*, 17:293–302, 1954. ISSN 0386-3034. URL http://journalseek.net/cgi-bin/journalseek/journalsearch.cgi?field=issn&query=0386-3034.

Juljusz Schauder. Zur theorie stetiger abbildungen in funktionalräumen. *Mathematische Zeitschrift*, 26:47–65, 1927. URL http://eudml.org/doc/167911;jsessionid=156A34EBAB6C0E2DDAAC8C1232D23E8F.

Juljusz Schauder. Eine eigenschaft des haarschen orthogonalsystems. *Mathematische Zeitschrift*, 28:317–320, 1928.

Gert Schubring. *Conflicts Between Generalization, Rigor, and Intuition: Number Concepts Underlying the Development of Analysis in 17th–19th Century France and Germany*. Sources and studies in the history of mathematics and physical sciences. Springer, New York, 1 edition, June 2005. ISBN 0387228365. URL http://books.google.com/books?vid=ISBN0387228365.

Isaac Schur. Uber die charakterischen wurzeln einer linearen substitution mit enier anwendung auf die theorie der integralgleichungen (over the characteristic roots of one linear substitution with an application to the theory of the integral). *Mathematische Annalen*, 66:488–510, 1909. URL http://dz-srv1.sub.uni-goettingen.de/cache/toc/D38231.html.

Mícheál Ó Searcóid. *Elements of Abstract Analysis*. Springer Undergraduate Mathematics Series. Springer, 2002. ISBN 9781852334246. URL http://books.google.com/books?vid=ISBN185233424X.

Atle Selberg. Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to dirichlet series. *Journal of the Indian Mathematical Society*, 20:47–87, 1956.

George Finlay Simmons. *Calculus Gems: Brief Lives and Memorable Mathematicians*. Mathematical Association of America, Washington DC, 2007. ISBN 0883855615. URL http://books.google.com/books?vid=ISBN0883855615.

Iván Singer. *Bases in Banach Spaces I*, volume 154 of *Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete*. Springer-Verlag New York, New York, 1970. ISBN 9780387048338.

Neil J. A. Sloane. On-line encyclopedia of integer sequences. World Wide Web, 2014. URL http://oeis.org/.

Jonathan D. H. Smith. *Introduction to Abstract Algebra*. CRC Press, March 23 2011. ISBN 1420063723. URL http://books.google.com/books?vid=ISBN1420063723.

M.J.T. Smith and T.P. Barnwell. A procedure for designing exact reconstruction filter banks for tree-structured subband coders. *IEEE International Conference on Acoustics, Speech and Signal Processing*, 9:421–424, 1984a. T.P. Barnwell is T.P. Barnwell III.

M.J.T. Smith and T.P. Barnwell. The design of digital filters for exact reconstruction in subband coding. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 34(3):434–441, June 1984b. ISSN 0096-3518. doi: 10.1109/TASSP.1986.1164832. T.P. Barnwell is T.P. Barnwell III.

Houshang H. Sohrab. *Basic Real Analysis*. Birkhäuser, Boston, 1 edition, 2003. ISBN 978-0817642112. URL http://books.google.com/books?vid=ISBN0817642110.

Lynn Arthur Steen. Highlights in the history of spectral theory. *The American Mathematical Monthly*, 80(4):359–381, April 1973. ISSN 00029890. URL http://www.jstor.org/stable/2319079.

Anne K. Steiner. The lattice of topologies: Structure and complementation. *Transactions of the American Mathematical Society*, 122(2):379–398, April 1966. URL http://www.jstor.org/stable/1994555.

Marshall Harvey Stone. *Linear transformations in Hilbert space and their applications to analysis*, volume 15 of *American Mathematical Society. Colloquium publications*. American Mathematical Society, New York, 1932. URL http://books.google.com/books?vid=ISBN0821810154. 1990 reprint of the original 1932 edition.

Gilbert Strang and Truong Nguyen. *Wavelets and Filter Banks*. Wellesley-Cambridge Press, Wellesley, MA, 1996. ISBN 9780961408879. URL http://books.google.com/books?vid=ISBN0961408871.

Robert S. Strichartz. *The Way of Analysis*. Jones and Bartlett Publishers, Boston-London, 1995. ISBN 978-0867204711. URL http://books.google.com/books?vid=ISBN0867204710.

Endre Süli and David F. Mayers. *An Introduction to Numerical Analysis*. Cambridge University Press, August 28 2003. ISBN 9780521007948. URL http://books.google.com/books?vid=ISBN0521007941.

Wim Sweldens and Robert Piessens. Wavelet sampling techniques. In *1993 Proceedings of the Statistical Computing Section*, pages 20–29. American Statistical Association, August 1993. URL http://citeseer.ist.psu.edu/18531.html.

Erik Talvila. Necessary and sufficient conditions for differentiating under the integral sign. *The American Mathematical Monthly*, 108(6):544–548, June–July 2001. URL http://arxiv.org/abs/math/0101012.

Brook Taylor. *Methodus Incrementorum Directa et Inversa*. London, 1715.

Audrey Terras. *Fourier Analysis on Finite Groups and Applications*. Number 43 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1999. ISBN 0-521-45718-1. URL http://books.google.com/books?vid=ISBN0521457181.

Brian S. Thomson, Andrew M. Bruckner, and Judith B. Bruckner. *Elementary Real Analysis*. www.classicalrealanalysis.com, 2 edition, 2008. ISBN 9781434843678. URL http://classicalrealanalysis.info/com/Elementary-Real-Analysis.php.

Stanislaw Marcin Ulam. *Adventures of a Mathematician*. University of California Press, Berkeley, 1991. ISBN 0520071549. URL http://books.google.com/books?vid=ISBN0520071549.

P.P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall Signal Processing Series. Prentice Hall, 1993. ISBN 0136057187. URL http://books.google.com/books?vid=ISBN0136057187.

Jussi Väisälä. A proof of the mazur-ulam theorem. *The American Mathematical Monthly*, 110(7): 633–635, August–September 2003. URL http://www.helsinki.fi/~jvaisala/mazurulam.pdf.

Brani Vidakovic. *Statistical Modeling by Wavelets*. John Wiley & Sons, Inc, New York, 1999. ISBN 9780471293651. URL http://www.amazon.com/dp/0471293652.

John von Neumann. Allgemeine eigenwerttheorie hermitescher funktionaloperatoren. *Mathematische Annalen*, 102(1):49–131, 1929. ISSN 0025-5831 (print) 1432-1807 (online). URL http://resolver.sub.uni-goettingen.de/purl?GDZPPN002273535. General eigenvalue theory of Hermitian functional operators.

John von Neumann. *Continuous Geometry*. Princeton mathematical series. Princeton University Press, Princeton, 1960. URL http://books.google.com/books?id=3bjqOgAACAAJ.

David F. Walnut. *An Introduction to Wavelet Analysis*. Applied and numerical harmonic analysis. Springer, 2002. ISBN 0817639624. URL http://books.google.com/books?vid=ISBN0817639624.

Gilbert G. Walter and XiaoPing Shen. *Wavelets and Other Orthogonal Systems*. Chapman and Hall/CRC, New York, 2 edition, 2001. ISBN 9781584882275. URL http://books.google.com/books?vid=ISBN1584882271.

Heinrich Martin Weber. Die allgemeinen grundlagen der galois'schen gleichungstheorie. *Mathematische Annalen*, 43(4):521–549, December 1893. URL http://resolver.sub.uni-goettingen.de/purl?GDZPPN002254670. The general foundation of Galois' equation theory.

Joseph H. Maclagan Wedderburn. On hypercomplex numbers. *Proceedings of the London Mathematical Society*, 6:77–118, 1907. URL http://plms.oxfordjournals.org/cgi/reprint/s2-6/1/77.

Joachim Weickert. Linear scale-space has first been proposed in japan. *Journal of Mathematical Imaging and Vision*, 10:237–252, May 1999. URL http://dl.acm.org/citation.cfm?id=607668.

Hermann Weyl. The method of orthogonal projection in potential theory. *Duke Mathematical Journal*, 7(1):411–444, 1940. URL http://projecteuclid.org/euclid.dmj/1077492266.

W. John Wilbur. Quantum logic and the locally convex spaces. *Transactions of the American Mathematical Society*, 207:343–360, June 1975. URL http://www.jstor.org/stable/1997181.

P. Wojtaszczyk. *A Mathematical Introduction to Wavelets*, volume 37 of *London Mathematical Society student texts*. Cambridge University Press, February 13 1997. ISBN 9780521578943. URL http://books.google.com/books?vid=ISBN0521578949.

Robert M. Young. *An introduction to nonharmonic Fourier series*, volume 93 of *Pure and applied mathematics*. Academic Press, 1 edition, 1980. ISBN 0127728503. URL http://books.google.com/books?vid=ISBN0127728503.

Robert M. Young. *An introduction to nonharmonic Fourier series*, volume 93 of *Pure and applied mathematics*. Academic Press, revised first edition, May 16 2001. ISBN 0127729550. URL http://books.google.com/books?vid=ISBN0127729550.

Ahmed I. Zayed. *Handbook of Function and Generalized Function Transformations*. Mathematical Sciences Reference Series. CRC Press, Boca Raton, 1996. ISBN 0849378516. URL http://books.google.com/books?vid=ISBN0849378516.

Gary Zukav. *The Dancing Wu Li Masters : An Overview of the New Physics*. Bantam Books, New York, 1980. ISBN 055326382X. URL http://books.google.com/books?vid=ISBN055326382X.

REFERENCE INDEX

SUBJECT INDEX

License

…last page …please stop reading …