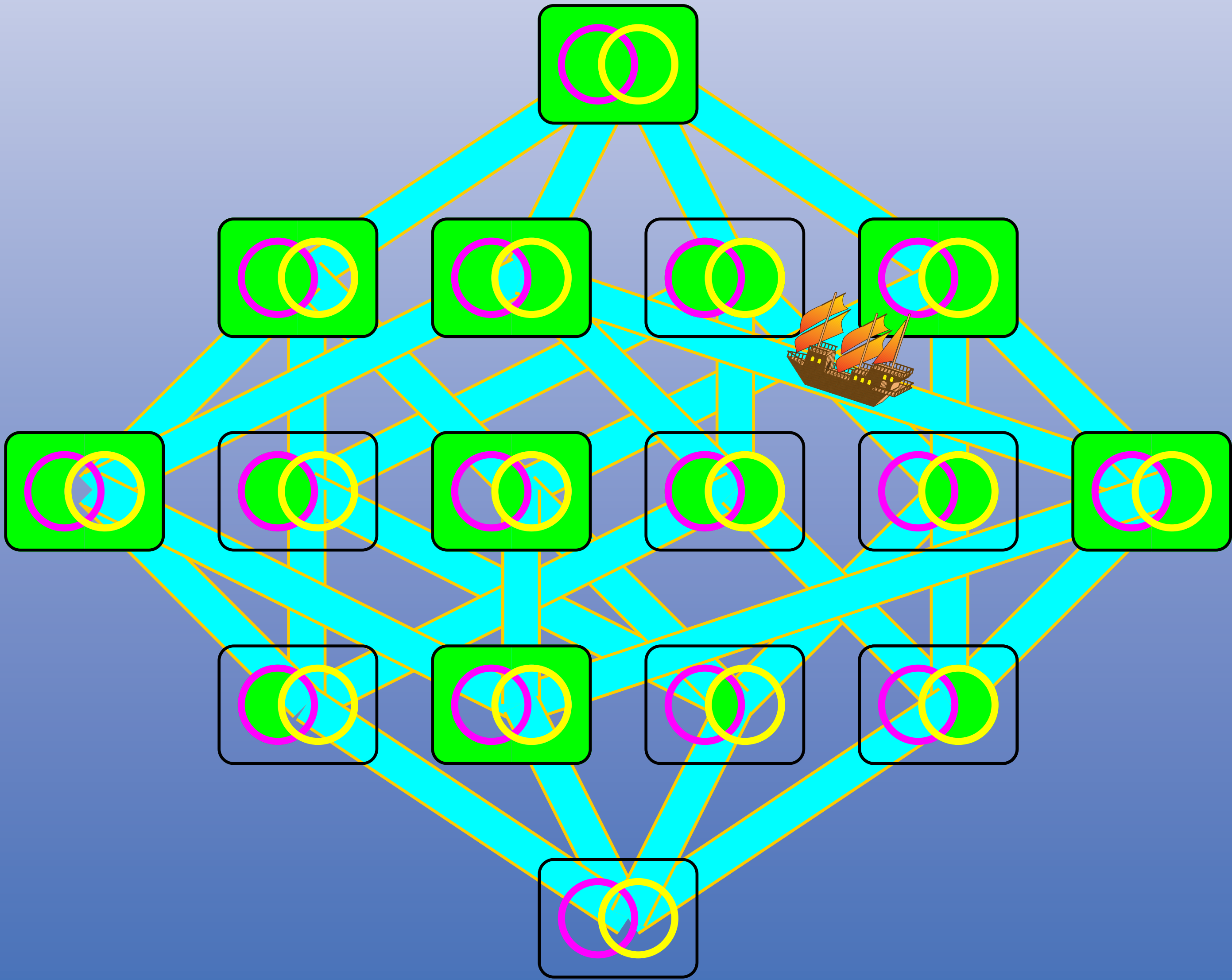
Sets, Relations, and Order Structures



Daniel J. Greenhoe

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1

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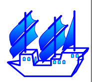


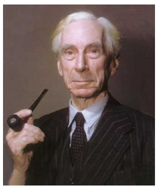
❝Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night? ❞

❝Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine. ❞

Alfred Edward Housman, English poet (1859–1936) [1]

❝The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning. ❞

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [2]



❝As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.❞

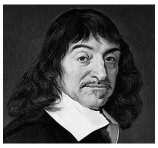
Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [3]

[1] quote: Housman (1936), page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[2] quote: Ewen (1961), page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[3] quote: Heijenoort (1967), page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS



❝*rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.*❞

❝*Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.*❞

René Descartes (1596–1650), French philosopher and mathematician [4]

❝*In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.*❞

Gottfried Leibniz (1646–1716), German mathematician, [5]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \dots$
$\mathbb{N}$	natural numbers	$1, 2, 3, \dots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\dots, -3, -2, -1, 0$

…continued on next page…

[4] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[5] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{Z}^-$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_o$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_e$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^+$	positive real numbers	$(0, \infty)$
$\mathbb{R}^-$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^*$	extended real numbers	$\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \triangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\not\subseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^c	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	
0	"false" condition	
$\neg$	logical NOT operation	

...continued on next page...

symbol	description	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\implies$	"implies";	"only if"
$\impliedby$	"implied by";	"if"
$\iff$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\le$	reflexive ordering relation	"less than or equal to"
$\ge$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle\triangle \mid \nabla\rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_j(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_j(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_j(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		
$\tilde{\mathbf{F}}$	*Fourier Transform* operator	
$\hat{\mathbf{F}}$	*Fourier Series* operator	

...continued on next page...

symbol	description
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator
$\mathbf{Z}$	*Z-Transform* operator
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $\left(\!\left(x_n \in \mathbb{C} \right)\!\right)_{n\in\mathbb{Z}}$
$\breve{\mathsf{x}}(z)$	*Z-Transform* of a sequence $\left(\!\left(x_n \in \mathbb{C} \right)\!\right)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

Part I

Foundational Structure

CHAPTER 1

FOUNDATIONAL CONSTRUCTION

Fundamental mathematical structures. This chapter presents six of the most fundamental structures in mathematics:

①	*sets*	(Definition 1.1	page 4)
②	*relations*	(Definition 1.8	page 7)
③	*equivalence relations*	(Definition 1.9	page 7)
④	*functions*	(Definition 1.11	page 8)
⑤	*logic operators*	(Definition 1.14	page 9)
⑥	*set operations*	(Definition 1.17	page 10)

These structures are "fundamental" in the sense that much of the remaining concepts of mathematics is built on them. We construct these structures as outlined next:

- We start with the concept of a *set*: a collection of objects that is uniquely defined by the objects that constitute the collection and that can be specified by any intelligible statement (Definition 1.1 page 4).
- Using sets, we can construct an *ordered pair* (Definition 1.5 page 6).
- The collection of all ordered pairs on a pair of sets is a *Cartesian product* (Definition 1.7 page 6).
- Any subset of a Cartesian product is a *relation* (Definition 1.8 page 7).
- An *equivalence relation* is a relation that is *reflexive*, *symmetric*, and *transitive* (Definition 1.9 page 7).
- A relation is a *function* if no value from the primary set is mapped to two different values in the secondary set (Definition 1.11 page 8).
- Using the concept of the function, we define the the logical operators *not* $\neg$, *logical and* $\wedge$, and *logical or* $\vee$,...which are also functions (Definition 1.14 page 9).
- Using logic operators, we can easily define the set operators *complement* c, *union* $\cup$, *intersection* $\cap$, *set difference* $\setminus$, and *symmetric difference* $\triangle$ (Definition 1.17 page 10).

1.1 Sets

We accept without definition the concepts of *true* and *false* and that any given statement is either true or false, but never both, and never anything inbetween (next axiom).

Axiom 1.1 (axiom of the excluded middle). [1]

AX
There exists a pair of properties 1 *and* 0,
such that any given statement has exactly one of these properties.
The symbol 1 *is called* ***true*** *and* 0 *is called* ***false***.

❝ *I call a manifold (an aggregate, a set) of elements, which belong to any conceptual sphere, well-defined, if on the basis of its definition and in consequence of the logical principle of excluded middle, it must be recognized that it is internally determined whether an arbitrary object of this conceptual sphere belongs to the manifold or not, and also, whether two objects in the set , in spite of formal differences in the manner in which they are given, are equal or not.*❞

Georg Cantor (1845–1918), German set theory pioneer [2]

There is more than one definition of a *set*. The most basic form of set theory is called *naive set theory* (Definition 1.1—next). However, in general, Naive set theory is flawed as shown by *Russell's Paradox* and illustrated by the *Barber of Seville* anecdote. The most commonly used replacement for the flawed Naive set theory is *Zermelo-Fraenkel (ZF) set theory*.[3]

Definition 1.1 (Naive set theory). [4]

DEF
A ***set*** *is any collection of objects that satisfies the following axioms:*
1. *Any two sets are equal if and only if their elements are the same.* (AXIOM OF EXTENSION)
2. *For every set* X *and statement* $s(x)$, *there exists a set* A *consisting exactly of elements* x *from set* X *such that* $s(x)$ *is true.* (AXIOM OF SPECIFICATION)

The *axiom of extension* states that the collection completely defines the set, and there is no other characteristic (no *extension*) which further defines the set. So if two sets have exactly the same elements, then the two sets are equal. The *axiom of specification* states that, you can form any subset A of a set X as long as you can *specify* which elements of X you want included in A by any possible intelligible statement $\mathsf{s}(x)$:

$$A = \{x \in X \,|\, \mathsf{s}(x) \text{ is true}\}$$

An element x that is included in a set X is denoted $x \in X$ (next definition). A set that has no elements is called the *empty set* and is denoted by the symbol $\varnothing$ (Definition 1.3 page 5). (Proposition 1.1 page 5) shows that, under the axioms of *naive set theory*, the emptyset exists and that there is only one set that is the emptyset.

Definition 1.2. [5]

DEF
An element x *is a member of a set* X *is denoted by* $x \in X$.
An element x *is* NOT *a member of a set* X *is denoted by* $x \notin X$.

[1] Cantor (1882) pages 114–115
[2] quote: Cantor (1882) pages 114–115
translation: Tait (2000), page 271
image: http://en.wikipedia.org/wiki/Image:Georg_Cantor.jpg
[3] Zermelo (1908a) pages 263–267 (7 axioms), Zermelo (1908b) (English translation), Fraenkel (1922), Wolf (1998), page 139
[4] Halmos (1960) pages 1–6
[5] Halmos (1960) page 2, Hausdorff (1937) page 12, Kuratowski (1921) page 161

Definition 1.3. [6]

DEF *The **empty set** $\varnothing$ is defined as*

$$\varnothing \triangleq \{x \in X | x \neq x\}$$

Proposition 1.1. [7] *Let $\varnothing$ be the empty set.*

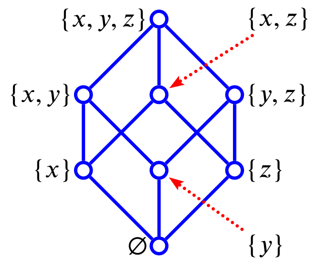
PRP
1. *$\varnothing$ exists.*
2. *$\varnothing$ is unique.*

✎PROOF:

1. $\varnothing$ exists by the *axiom of specification* (Definition 1.1 page 4) $\varnothing \triangleq \{x \in X | x \neq x\}$.
2. $\varnothing$ is unique by the *axiom of extension* (Definition 1.1 page 4).

⇨

Definition 1.4 (next) defines the *power set* 2^X of a set X. The power set is simply the set of all subsets of X. The notation 2^X is meaningful because the number of elements (subsets of X) in a finite set X is $2^{|X|}$ (Proposition 1.2 page 5). The tuple $\left(2^X, \cap, \cup; \subseteq\right)$ forms a *lattice* (Definition 5.3 page 73). The lattice $\left(2^{\{x,y,z\}}, \cap, \cup; \subseteq\right)$ is illustrated to the right.



Definition 1.4.

DEF *The **power set** 2^X on a set X is defined as*

$$2^X \triangleq \{A | A \subseteq X\} \qquad \text{(the set of all subsets of } X\text{)}$$

Proposition 1.2. *Let $|X|$ represent the number of elements in a finite set X.*

PRP

$$\left|2^X\right| = 2^{|X|}$$

✎PROOF:

1. Let $n = |X|$ be the number of elements in a set.
2. Let $X \triangleq \{x_1, x_2, x_3, \ldots, x_n\}$.
3. Each subset of X (each element in 2^X) has 2 possibilities for each of the n elements of X: Either the subset *has* a given element of X (represent by 1), or does *not* have it (represented by a 0).
4. Therefore, the number of possible subsets (the number of elements in 2^X) is

$$\begin{aligned}
\left|2^X\right| &= \underbrace{\left|\{\text{has } x_1, \text{ not have } x_1\}\right| \cdot \left|\{\text{has } x_2, \text{ not have } x_2\}\right| \cdots \left|\{\text{has } x_n, \text{ not have } x_n\}\right|}_{n \text{ times}} \\
&= \underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} \\
&= 2^n \\
&= 2^{|X|}
\end{aligned}$$

⇨

[6] Halmos (1960) page 8, Kelley (1955) page 3, Kuratowski (1961), page 26

[7] Halmos (1960) page 8, Hausdorff (1937) page 13

1.2 Relations

Ordered pair. In the set $\{a, b\}$, the order in which the elements are listed does *not* matter. That is, $\{a, b\}$ is equivalent to $\{b, a\}$. However, in some applications the order does (very much) matter. To help with this, we have the concept of the *ordered pair*. In an ordered pair (a, b), the order in which the elements are listed *is* significant. That is, (a, b) is equivalent to (b, a) if and only if $a = b$. The ordered pair can be defined in different ways. One of the most common definitions is due to mathematician Kazimierz Kuratowski in 1921 and is presented next:[8]

Definition 1.5. [9]

DEF *The **ordered pair** (a, b) is defined as* $\qquad (a, b) \triangleq \{\{a\}, \{a, b\}\}$

This definition may seem a little strange—it even has the strange consequence that $\{a, b\} \in (a, b)$. But the crucial test of validity is if

$$(a, b) = (c, d) \qquad \iff \qquad a = c \text{ and } b = d.$$

This statement is true under Definition 1.5 but is most easily proved once we have defined the *set intersection operator* $\cap$ and *set symmetric difference operator* $\triangle$. These are not defined until Definition 1.17 (page 10). And subsequently the above ordered pair equality is proved in Corollary 1.1 (page 10).

Cartesian product.

Definition 1.6. *Let X and Y be sets, 1 denote the logical property of "true", and 0 the logical property of "false"* (Axiom 1.1 page 3).

DEF $\owedge$ *is the "**relational and**" defined as*

$$\owedge \triangleq \{((0,0),0),\ ((0,1),0),\ ((1,0),0),\ ((1,1),1)\}$$

The "relational and" of Definition 1.6 can be illustrated using the *truth table* to the right →. Later, the relational and will be replaced by the *logical and*; but the *logical and* is a *function*, and functions are not defined until Definition 1.11 (page 8).

x	y	$x \owedge y$
0	0	0
0	1	0
1	0	0
1	1	1

Definition 1.7. [10] *Let X and Y be sets, and let $\owedge$ be the "relational and" relation of Definition 1.6 (page 6).*

DEF *The **Cartesian product** $X \times Y$ is defined as*

$$X \times Y \triangleq \{(x, y) \,|\, (x \in X) \owedge (y \in Y)\}$$

[8] *Alternative ordered pair definition*: As an alternative to the Kuratowski definition, the ordered pair can also be taken as an *axiom*. References:
Bourbaki (1968), page 72, Munkres (2000), page 13

[9] Suppes (1972) page 32, Halmos (1960) page 23, Kuratowski (1961), page 39, Kuratowski (1921) ⟨Def. V, page 171⟩, Wiener (1914)

[10] Halmos (1960) page 24, G. Frege, 2007 August 25, http://groups.google.com/group/sci.logic/msg/3b3294f5ac3a76f0

Relations. A set of ordered pairs represents a *relationship* between the set formed by the first elements of the ordered pairs and the set formed by the second elements of the ordered pairs. Any such set is called a *relation* (next definition). The set of all relations in $X \times Y$ is denoted 2^{XY}. This notation is meaningful because the number of relations in 2^{XY} is $2^{|X||Y|}$ (Proposition 17.1 page 252).

Definition 1.8. [11] *Let X and Y be sets.*

DEF

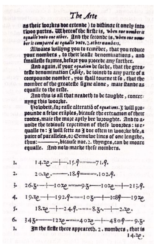
*A **relation** $® : X \to Y$ is any subset of $X \times Y$. That is,*

$$® \subseteq X \times Y$$

The set of all relations that are subsets of $X \times Y$ is denoted 2^{XY}. That is,

$$2^{XY} \triangleq \{® | ® \subseteq (X \times Y)\}$$

1.3 Equivalence relations



❝*To avoide the tedious repetition of these woordes: is equal to: I will sette as I doe often in woorke use, a paire of parralles, or Gemowe lines of one lengthe, thus: ═══, bicause noe 2 thynges, can be moare equalle.*❞

Robert Recorde (1510–1558), Welsh physician and mathematician [12]

Definition 1.9. [13]

DEF

*A relation $\bumpeq \in 2^{XX}$ is an **equivalence relation** on a set X if*

1. $x \bumpeq x$	$\forall x \in X$	(REFLEXIVE)	*and*
2. $x \bumpeq y \implies y \bumpeq x$	$\forall x, y \in X$	(SYMMETRIC)	*and*
3. $x \bumpeq y$ *and* $y \bumpeq z \implies x \bumpeq z$	$\forall x, y, z \in X$	(TRANSITIVE)	

Example 1.1. Examples of equivalence relations include

EX

1. The equality relation $=$ on the set of integers $\mathbb{Z}$.
2. The set equality relation $=$ $(A = B \implies A \subsetneq B$ and $B \subsetneq A)$ on the set of all sets.
3. The similarity relation $\sim$ (all angles equal) on the set of triangles.
4. The modulo relation $\sim$ of all whole numbers that divide 60 $(60|\cdot)$.

Definition 1.10.

DEF

*A set E is an **equivalence class** with respect to the equivalence relation $=$ if*

$$x = y \qquad \forall x, y \in E$$

Equivalence relations occur, of course, in the context of relations. In the context of sets, a structure that is "equivalent" to the equivalence relation is the *partition* (Definition 16.11 page 231). In particular, an equivalence relation on a set generates a partition, and a partition on a set defines an equivalence relation.

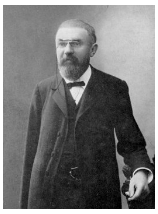
[11] Halmos (1960) page 26

[12] quote: Potts (1860) page 109
Recorde (1557)
image: http://nsm1.nsm.iup.edu/gsstoudt/history/images/witte.jpg
http://www-gap.dcs.st-and.ac.uk/~history/PictDisplay/Recorde.html
http://members.aol.com/jeff94100/witte.jpg

[13] Aliprantis and Burkinshaw (1998), page 7, Peano (1889b), page 91

1.4 Functions

❝*La logique parfois engendre des monstres. Depuis un demi-siècle on a vu surgir une foule de fonctions bizarres qui semblent s'efforcer de ressembler aussi peu que possible aux honnêtes fonctions qui servent à quelque chose.*❞



❝*Logic sometimes breeds monsters. For half a century there has been springing up a host of bizarre functions, which seem to strive to have as little resemblance as possible to honest functions that are of some use.*❞

Jules Henri Poincaré (1854–1912), French physicist and mathematician [14]

A *relation* is a kind of mapping between two sets. A relation $® \subseteq X \times Y$ permits a single point in X to map to two or more points in Y. That is, it is possible that (x, y_1) and (x, y_2) with $y_1 \neq y_2$ are both members of some relation $® \subseteq X \times Y$. An example of this is the relation $\leq$ in the set $\mathbb{Z} \times \mathbb{Z}$ where $1 \leq 2$ **and** $1 \leq 3$; so in this example the element $1 \in X$ has a relationship with both $2 \in Y$ and $3 \in Y$. Alternatively, we can say that both $(1, 2)$ and $(1, 3)$ are elements of the relation $\leq$. However, there is a special case of relations where this is not allowed; *functions* (next definition) are a kind of relation where the only way that both (x, y_1) and (x, y_2) can be elements of the *relation* is if $y_1 = y_2$. The set of all functions in $X \times Y$ is denoted Y^X. This notation is meaningful because the number of functions in Y^X is $|Y|^{|X|}$ (Proposition 17.5 page 264).

Definition 1.11. [15] *Let X and Y be sets. Let $\oslash$ be the "relational and"* (Definition 1.6 page 6).

DEF

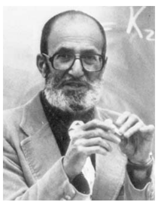
*A relation $\mathsf{f} \in 2^{XY}$ is a **function** if*

$$(x, y_1) \in \mathsf{f} \;\oslash\; (x, y_2) \in \mathsf{f} \implies y_1 = y_2 \qquad \text{(for each } x\text{, there is only one } \mathsf{f}(x)\text{)}$$

The set of all functions in 2^{XY} is denoted

$$Y^X \triangleq \left\{\mathsf{f} \in 2^{XY} | \mathsf{f} \text{ is a function}\right\}.$$

1.5 Logic



❝*My most nearly immortal contributions are an abbreviation and a typographical symbol. I invented "iff", for "if and only if"—but I could never believe that I was really its first inventor.... The symbol is definitely not my invention—it appeared in popular magazines (not mathematical ones) before I adopted it,... It is the symbol that sometimes looks like □, and is used to indicate an end, usually the end of a proof.*❞

Paul R. Halmos (1916–2006), Hungarian-born Jewish-American mathematician [16]

Definition 1.12 (logic order relation). *Let 1 and 0 represent the logical properties of "TRUE" and "FALSE", respectively* (Axiom 1.1 page 3).

DEF

*The **implies**, or **only if**, relation $\implies$ is defined as*

$$\implies \;\triangleq\; \{((0,0),1),\ ((0,1),1),\ ((1,0),0),\ ((1,1),1)\}$$

[14] quote: Poincaré (1908a) ⟨book 2, chap. 2, sec. 5, par. 3⟩, Poincaré (1908b) page 125
image: `http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Poincare.html`

[15] Suppes (1972) page 86, Kelley (1955) page 10, Bourbaki (1939)

[16] quote: Halmos (1985) page 403
image: `http://en.wikipedia.org/wiki/Image:Paul_Halmos.jpeg`

Definition 1.13. *Let* 1 *and* 0 *represent the logical properties of "*TRUE*" and "*FALSE*", respectively.*

DEF
$$x \Longleftarrow y \quad \text{if} \quad y \Longrightarrow x \qquad \forall x,y \in \{0,1\}$$
$$x \Longleftrightarrow y \quad \text{if} \quad x \Longrightarrow y \quad \text{and} \quad y \Longrightarrow x \qquad \forall x,y \in \{0,1\}$$

Alternatively, the relations presented in (Definition 1.12 page 8) and (Definition 1.13 page 9) can be represented in the form of truth tables:

"implies" or "only if"		
$\Longrightarrow : \{0,1\}^2 \to \{0,1\}$		
x	y	$x \Longrightarrow y$
0	0	1
0	1	1
1	0	0
1	1	1

"implied by" or "if"		
$\Longleftarrow : \{0,1\}^2 \to \{0,1\}$		
x	y	$x \Longleftarrow y$
0	0	1
0	1	0
1	0	1
1	1	1

"if and only if"		
$\Longleftrightarrow : \{0,1\}^2 \to \{0,1\}$		
x	y	$x \Longleftrightarrow y$
0	0	1
0	1	0
1	0	0
1	1	1

Definition 1.14 (Propositional logic). *Let* 1 *and* 0 *represent the logical properties of "*TRUE*" and "*FALSE*", respectively* (Axiom 1.1 page 3). *The following binary operators are defined according to the expressions below:*

DEF
$$\neg \triangleq \{(0,1), (1,0)\} \qquad \text{(LOGICAL NOT)}$$
$$\vee \triangleq \{((0,0),0), ((0,1),1), ((1,0),1), ((1,1),1)\} \qquad \text{(LOGICAL OR)}$$
$$\wedge \triangleq \{((0,0),0), ((0,1),0), ((1,0),0), ((1,1),1)\} \qquad \text{(LOGICAL AND)}$$
$$\oplus \triangleq \{((0,0),0), ((0,1),1), ((1,0),1), ((1,1),0)\} \qquad \text{(LOGICAL EXCLUSIVE OR)}$$

Alternatively, the relations presented in Definition 1.14 can be represented in the form of truth tables:

logical not	
x	$\neg x$
0	1
1	0

logical or		
x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

logical and		
x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

logical exclusive-or		
x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Note that the "logical and" *function* $\wedge$ as defined in Definition 1.14 (previous) and the "*relational and*" *relation* $\owedge$ as defined in Definition 1.7 (page 6) are equivalent relations.

More information about logic structure is presented in CHAPTER 14 (page 187). Key concepts concerning the structure of logic is more conveniently demonstrated once the concepts of *order* (CHAPTER 4 page 57), *lattices* (CHAPTER 5 page 71), and *Boolean algebras* (CHAPTER 10 page 127) have been introduced.

Definition 1.15 (predicate logic).

DEF
$$\exists i \in I \quad \text{such that} \quad p(x_i) \text{ is true} \iff \bigvee_{i \in I} [p(x_i) \text{ is true}] \qquad \textit{(existensial quantifier)}$$
$$\forall i \in I,\ p(x_i) \text{ is true} \iff \bigwedge_{i \in I} [p(x_i) \text{ is true}] \qquad \textit{(universal quantifier)}$$

Theorem 1.1. *(Wolf, 1998)62 Let* x *be a condition and* $P(x)$ *be a logical statement dependent on* x.

THM
$$\neg[\forall x,\ P(x)] \quad \textit{is equivalent to} \quad \exists x \text{ such that } \neg P(x)$$
$$\neg[\exists x, \text{ such that } P(x)] \quad \textit{is equivalent to} \quad \forall x,\ \neg P(x)$$

Theorem 1.2. [17]

[17] Gödel (1930a), Gödel (1930b), Henkin (1949), Takeuti (1975) page 43 ⟨Theorem 8.2⟩

THM *Any statement constructed using first order predicate logic is* COMPLETE.

1.6 Set operations

Equipped with the concept of the *relation*, we are now in a position to construct the following fundamental relationships between sets:

- The *set inclusion* relation $A \subseteq B$ means $x \in A \implies x \in B$.
- The *equality* relation $A = B$ means $A \subseteq B$ and $B \subseteq A$.
- The *non-equality* relation $A \neq B$ means $A = B$ is false (0).
- The *proper subset* relation $A \subsetneq B$ means $A \subseteq B$ and $A \neq B$.

These informal set relation descriptions are reinforced with their formal definitions next.

Definition 1.16. [18] *Let A and B be sets.*

DEF
*The relation "**subset**"* $\subseteq$	*is defined as the set*	$\{(A,B)\,\vert\, x \in A \implies x \in B\}$
*The relation "**set equality**"* $=$	*is defined as the set*	$\{(A,B)\,\vert\,(A \subseteq B) \land (B \subseteq A)\}$
*The relation "**set non-equality**"* $\neq$	*is defined as the set*	$\{(A,B)\,\vert\, A = B \text{ is false } (0)\}$
*The relation "**proper subset**"* $\subsetneq$	*is defined as the set*	$\{(A,B)\,\vert\,(A \subseteq B) \land (A \neq B)\}$

Definition 1.17. [19] [20] *Let* 2^X *be the* POWER SET *(Definition 1.4 page 5) on a set* X.

DEF
A^c	$\triangleq \{x \in X \,\vert\, \neg(x \in A)\}$	$\forall A \in 2^X$	*(**complement**)*
$A \cup B$	$\triangleq \{x \in X \,\vert\, (x \in A) \lor (x \in B)\}$	$\forall A,B \in 2^X$	*(**union**)*
$A \cap B$	$\triangleq \{x \in X \,\vert\, (x \in A) \land (x \in B)\}$	$\forall A,B \in 2^X$	*(**intersection**)*
$A \setminus B$	$\triangleq \{x \in X \,\vert\, (x \in A) \land \neg(x \in B)\}$	$\forall A,B \in 2^X$	*(**difference**)*
$A \triangle B$	$\triangleq \{x \in X \,\vert\, (x \in A) \oplus (x \in B)\}$	$\forall A,B \in 2^X$	*(**symmetric difference**)*

Definition 1.5 (page 6) defined the *ordered pair* using Kuratowski's somewhat cryptic expression $(a,b) \triangleq \{\{a\},\{a,b\}\}$. Theorem 1.3 (next) helps show that this expression is reasonable in that a and b can be extracted from the set $\{\{a\},\{a,b\}\}$ by simple set operations on its elements. Also, a key test to the usefulness of the definition of ordered pairs is whether or not $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$. In fact, this statement is true and demonstrated by Corollary 1.1 (page 10).

Theorem 1.3.

THM
$$\begin{aligned}
\{a\} &= \bigcap (a,b) &&= \bigcap \{\{a\},\{a,b\}\} &&= \{a\} \cap \{a,b\} \\
\{b\} &= \bigtriangleup (a,b) &&= \bigtriangleup \{\{a\},\{a,b\}\} &&= \{a\} \triangle \{a,b\}
\end{aligned}$$

Corollary 1.1. [21]

COR
$$(a,b) = (c,d) \iff a = c \text{ and } b = d$$

✎PROOF:

[18] Kelley (1955) page 2
[19] Vereščagin and Shen (2002) pages 1–2, Aliprantis and Burkinshaw (1998), pages 2–4
[20] Origin of ∪ and ∩: Peano (1888a), Peano (1888b)
[21] Apostol (1975) page 33, Hausdorff (1937) page 15

1. Proof that $(a,b) = (c,d) \implies a = c$ and $b = d$:

$$\begin{aligned}
\{a\} &= \bigcap (a,b) && \text{by Theorem 1.3 page 10}\\
&= \bigcap (c,d) && \text{by left hypothesis}\\
&= \{c\} && \text{by Theorem 1.3 page 10}\\
\{b\} &= \triangle (a,b) && \text{by Theorem 1.3 page 10}\\
&= \triangle (c,d) && \text{by left hypothesis}\\
&= \{d\} && \text{by Theorem 1.3 page 10}
\end{aligned}$$

2. Proof that $(a,b) = (c,d) \impliedby a = c$ and $b = d$:

$$(a,b) = (c,d) \qquad \text{by right hypothesis}$$

⇨

1.7 Abstract Mathematical Spaces

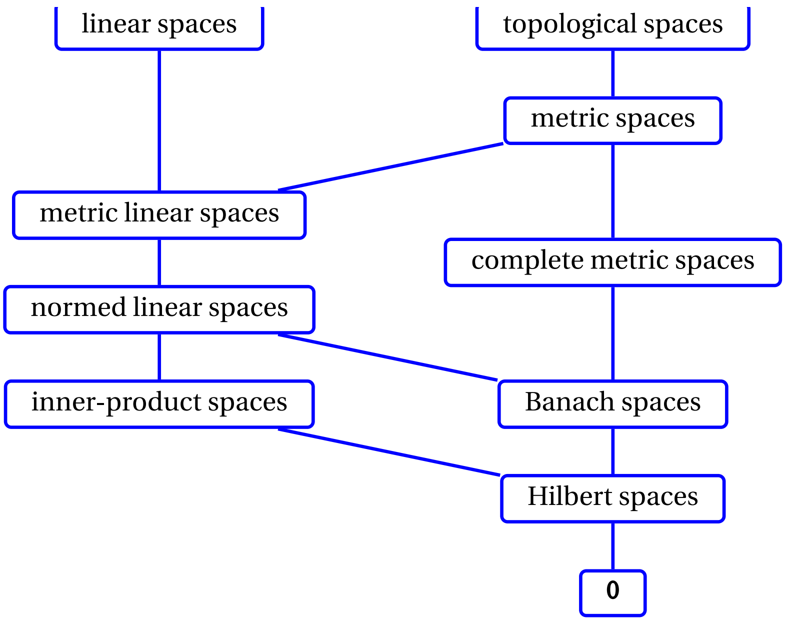


Figure 1.1: Lattice of mathematical spaces

The abstract space was introduced by Maurice Fréchet in his 1906 Ph.D. thesis.[22] An abstract **space** in mathematics does not really have a rigorous definition; but in general it is a set together with some other unifying structure.

❝...*A collection of these abstract elements will be called an abstract set. If to this set there is added some rule of association of these elements, or some relation between them, the set will be called an abstract space.*❞

Marice René Fréchet (1878–1973), French mathematician who in his 1906 Ph.D. dissertation introduced the concept of the *metric space* [23]

[22] Fréchet (1906), Fréchet (1928)

Examples of common abstract spaces include the following:

1. *linear space*: a set of "vectors" over a field with a vector-vector addition operator $+$ such vectors can be added together and a scalar-vector multiplication operator $\times$ such that field elements can be multiplied with vectors.
   Peano (1888a)

2. *metric space*: A set of elements together with a metric d such giving the "distance" between any two elements.
   Fréchet (1906)

3. *measure space*: A set of sets and a measure μ that measures the "size" of a set.

4. *normed vector space*: A vector space together with a norm $\|\cdot\|$ that measures the "length" of a vector.
   Banach (1922)

5. *inner-product space*: A vector space together with an inner-product $\langle \triangle \mid \nabla \rangle$.

6. *Banach space*: a normed vector space that is "complete" with respect to the norm $\|\cdot\|$.
   Banach (1922)

7. *Hilbert space*: an inner-product space that is "complete" with respect to the norm induced by the inner-product $\langle \triangle \mid \nabla \rangle$.
   von Neumann (1929)

[23] quote: Fréchet (1950) page 147
image: http://en.wikipedia.org/wiki/Frechet

CHAPTER 2

NUMBER SYSTEMS

> *Wherever there is number, there is beauty.*
>
> Proclus Lycaeus (412 – 485 AD), Greek philosopher [1]

The most common number systems are the following:

The set of *natural numbers*	$\mathbb{N}$	$\triangleq$	$\{1, 2, 3, \ldots\}$	Definition 2.1	page 14
The set of *whole numbers*	$\mathbb{W}$	$\triangleq$	$\{0, 1, 2, 3, \ldots\}$	Definition 2.5	page 22
The set of *integers*	$\mathbb{Z}$	$\triangleq$	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$	Definition 2.7	page 22
The set of *rational numbers*	$\mathbb{Q}$	$\triangleq$	$\left\{\frac{n}{m} \middle| n \in \mathbb{Z},\ m \in \mathbb{N}\right\}$	Definition 2.10	page 23
The set of *real numbers*	$\mathbb{R}$	$\triangleq$	completion of $\mathbb{Q}$	Definition 2.13	page 24
The set of *complex numbers*	$\mathbb{C}$	$\triangleq$	$\{(x, y) \mid x, y \in \mathbb{R}\}$	Definition 2.17	page 31

Generally there are two ways to construct these numbers systems:[2]

1. **ordinal approach**: Start by defining the natural numbers axiomatically using the *Peano axioms*. From there construct the other number systems. The ordinal approach presents the natural numbers as having an inherent *order* such that each natural number has a unique *successor* that is also a natural number.

2. **cardinal approach**: Start by defining the real numbers axiomatically using their *field properties* as axioms. From there define the natural numbers, whole numbers, integers, and rational numbers as subsets of the reals, and define the complex numbers as ordered pairs of the reals. The cardinal approach presents the real numbers as having inherent *size* such that any number except zero can be sliced into smaller and smaller pieces.

Either approach is perfectly acceptable. However, under the assumption that, say, four thousand years ago most every culture on earth had some kind of counting system, but relatively few, if any, were familiar with the real number system, one might argue that the ordinal approach is a more "natural" choice for number system construction. And indeed it is the ordinal approach that is presented in the following material.

[1] quote: Kline (1990) page 131

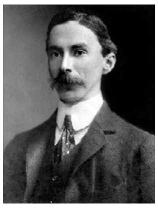
[2] Bunt et al. (1988) pages 3–4

2.1 Natural numbers

“I regard the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetic act, that of counting, and counting itself as nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already-formed individual to the consecutive new one to be formed.”

Richard Dedekind (1831–1915), German mathematician [3]

2.1.1 Definitions



“The Congress was a turning point in my intellectual life, because I there met Peano.... In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked, As the days went by, I decided that this must be owing to his mathematical logic.... It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring a new and powerful technique for the work that I had long wanted to do.”

Bertrand Russell (1872–1970), British mathematician, [4]

Axiom 2.1 (Peano's axioms). [5]

A X

*The set $\mathbb{N}$ of **natural numbers** satisfies the following axioms:*

1. $1 \in \mathbb{N}$ ($\mathbb{N}$ *is not empty*)
2. $\mathsf{s}(n) \in \mathbb{N}$ $\quad \forall n\in\mathbb{N}$ (*every element n in $\mathbb{N}$ has a* SUCCESSOR $\mathsf{s}(n)$ *in* $\mathbb{N}$)
3. $\mathsf{s}(n) = \mathsf{s}(m) \iff n = m$ $\quad \forall n,m\in\mathbb{N}$ (s *is a* BIJECTION)
4. $\mathsf{s}(n) \neq 1$ $\quad \forall n\in\mathbb{N}$ (*1 is not a successor of any element in* $\mathbb{N}$)
5. $\forall M \subseteq \mathbb{N}$ (AXIOM OF INDUCTION)

$$\left.\begin{array}{lll} \text{(5a)} & 1 \in M & \text{AND} \\ \text{(5b)} & m \in M \implies \mathsf{s}(m) \in M & \end{array}\right\} \implies M = \mathbb{N}$$

*The set $\mathbb{N}$ is called the set of **natural numbers**. The value $\mathsf{s}(n)$ is called the **successor** of n. The element 1 is called “**one**”.*

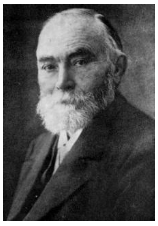
---

[3] quote: Dedekind (1872a)
translation: Dedekind (1872b), page 2, Dedekind (1872c), page 2?, Dedekind (1872d) page 768
image: http://turnbull.mcs.st-and.ac.uk/history/PictDisplay/Dedekind.html

[4] quote: Russell (1951), pages 216–217
image: http://en.wikipedia.org/wiki/File:Russell1907-2.jpg, public domain

[5] Landau (1966) page 2, Halmos (1960) page 46, Thurston (1956) page 51, Peano (1889a), Peano (1889b) page 94, Dedekind (1888a), Dedekind (1888b) page 67, Cori and Lascar (2001) pages 8–15 ⟨recursion theory⟩

2.1.2 Addition



❝And even now I do not see how arithmetic can be scientifically established; how numbers can be apprehended as logical objects, and brought under review; unless we are permitted – at least conditionally – to pass from a concept to its extension.❞

Gottlob Frege (1848–1925), German mathematician, logician, and philosopher [6]

Definition 2.1 (Addition over $\mathbb{N}$). [7] *Let* $\mathsf{s}(n)$ *be the* SUCCESSOR *of* n (Definition 2.1 page 14).

DEF

Let $+ : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ *be an operator that satisfies the following conditions:*

1. $\mathsf{s}(n) \;=\; n+1 \qquad \forall n\in\mathbb{N}$ *and*
2. $\mathsf{s}(n+m) \;=\; n+\mathsf{s}(m) \qquad \forall n,m\in\mathbb{N}.$

The quantity $n+m$ *is called the* ***sum*** *of* n *and* m.

Lemma 2.1. [8] *Let* $n+m$ *represent the* SUM *of* n *and* m.

LEM

$$\begin{array}{rcll} \mathsf{s}(m)+n &=& m+\mathsf{s}(n) & \forall n,m,k\in\mathbb{N}\\ \mathsf{s}(m+n) &=& \mathsf{s}(m)+n & \forall n,m\in\mathbb{N}\\ \mathsf{s}(n) &=& 1+n & \forall n\in\mathbb{N} \end{array}$$

✎PROOF:

1. Proof that $\mathsf{s}(m)+n = m+\mathsf{s}(n)$:

   (a) Let M be the set of values for which $\mathsf{s}(m)+n = m+\mathsf{s}(n)$.

   (b) Proof that $1 \in M$:

$$\begin{aligned} \mathsf{s}(m)+1 &= \mathsf{s}[\mathsf{s}(m)] && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\ &= \mathsf{s}[m+1] && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\ &= m+\mathsf{s}(1) && \text{by definition of addition over } \mathbb{N} \text{ page 15} \end{aligned}$$

   (c) Proof that $m\in M \implies \mathsf{s}(m)\in M$:

$$\begin{aligned} \mathsf{s}(m)+\mathsf{s}(n) &= \mathsf{s}(m)+[n+1] && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\ &= [\mathsf{s}(m)+n]+1 && \text{by associative property}\\ &= [m+\mathsf{s}(n)]+1 && \text{by left hypothesis}\\ &= m+[\mathsf{s}(n)+1] && \text{by associative property}\\ &= m+\mathsf{s}(\mathsf{s}(n)) && \text{by associative property} \end{aligned}$$

   (d) Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$ (all of $\mathbb{N}$ has the property of M).

2. Proof that $\mathsf{s}(m)+n = \mathsf{s}(m+n)$:

$$\begin{aligned} \mathsf{s}(m)+n &= m+\mathsf{s}(n) && \text{by previous lemma (2)}\\ &= m+[n+1] && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\ &= [m+n]+1 && \text{by associative property}\\ &= \mathsf{s}(m+n) && \text{by definition of addition over } \mathbb{N} \text{ page 15} \end{aligned}$$

[6] quote: Frege (1903)
translation: Frege (1952) page 234, Frege (1997) page 280
image: http://en.wikipedia.org/wiki/Image:Frege.jpg
[7] Dedekind (1888a), Dedekind (1888b) page 97, Landau (1966) page 4
[8] Dedekind (1888b) pages 97–99

3. Proof that $1 + n = \mathsf{s}(n)$:
   (a) Let M be the set of values for which $1 + n = \mathsf{s}(n)$.
   (b) Proof that $1 \in M$:

$$1 + 1 = \mathsf{s}(1) \qquad \text{by definition of addition over } \mathbb{N} \text{ page 15}$$

   (c) Proof that $m \in M \implies \mathsf{s}(m) \in M$:

$$\begin{aligned}
1 + \mathsf{s}(n) &= 1 + [n + 1] && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\
&= [1 + n] + 1 && \text{by associative property}\\
&= \mathsf{s}(n) + 1 && \text{by left hypothesis}\\
&= \mathsf{s}(\mathsf{s}(n)) && \text{by definition of addition over } \mathbb{N} \text{ page 15}
\end{aligned}$$

   (d) Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$ (all of $\mathbb{N}$ has the property of M).

⇰

Theorem 2.1 (Associative property of addition). [9] *Let $n + m$ represent the* SUM *of n and m.*

THM					
	$(n + m) + k$	$=$	$n + (m + k)$	$\forall n,m,k \in \mathbb{N}$	(ASSOCIATIVE)

✎PROOF:

1. Let M be the set of values for which $(\mathbb{N}, +)$ is associative.
2. Proof that $1 \in M$:

$$\begin{aligned}
(n + m) + 1 &= \mathsf{s}(n + m) && \text{by Definition 2.1 page 15}\\
&= n + \mathsf{s}(m) && \text{by Definition 2.1 page 15}\\
&= n + (m + 1) && \text{by Definition 2.1 page 15}
\end{aligned}$$

3. Proof that $k \in M \implies \mathsf{s}(k) \in M$:

$$\begin{aligned}
[n + m] + \mathsf{s}(k) &= \mathsf{s}([n + m] + k) && \text{by Definition 2.1 page 15}\\
&= \mathsf{s}(n + [m + k]) && \text{by left hypothesis}\\
&= n + [\mathsf{s}(m + k)] && \text{by Definition 2.1 page 15}\\
&= n + [m + \mathsf{s}(k)] && \text{by Definition 2.1 page 15}
\end{aligned}$$

4. Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$ (all of $\mathbb{N}$ is associative under $+$).

⇰

Theorem 2.2 (Commutative property of addition). [10] *Let $n + m$ represent the* SUM *of n and m.*

THM					
	$n + m$	$=$	$m + n$	$\forall n,m \in \mathbb{N}$	(COMMUTATIVE)

✎PROOF:

[9] Dedekind (1888b) pages 97–99, Landau (1966) pages 4–5, Thurston (1956) page 53
[10] Dedekind (1888b), pages 97–99, Landau (1966) pages 4–5, Thurston (1956) page 53

1. Let M be the set of values for which $(\mathbb{N}, +)$ is commutative.

2. Proof that $1 \in M$:

$$\begin{aligned} 1 + n &= \mathsf{s}(n) && \text{by Lemma 2.1 page 15} \\ &= n + 1 && \text{by definition of addition over } \mathbb{N} \text{ page 15} \end{aligned}$$

3. Proof that $m \in M \implies \mathsf{s}(m) \in M$:

$$\begin{aligned} n + \mathsf{s}(m) &= n + [m+1] && \text{by definition of addition over } \mathbb{N} \text{ page 15} \\ &= [n+m] + 1 && \text{by associative property} \\ &= [m+n] + 1 && \text{by left hypothesis} \\ &= m + [n+1] && \text{by associative property} \\ &= m + [1+n] && \text{by Lemma 2.1 page 15} \\ &= [m+1] + n && \text{by associative property} \\ &= \mathsf{s}(m) + n && \text{by definition of addition over } \mathbb{N} \text{ page 15} \end{aligned}$$

4. Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$ (all of $\mathbb{N}$ is commutative under $+$).

⇒

2.1.3 Multiplication

Definition 2.2 (Multiplication over $\mathbb{N}$). [11] *Let* $\mathsf{s}(n)$ *be the* SUCCESSOR *of* n.

DEF

Let $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *be an operator that satisfies*

1. $n \cdot 1 = n \quad \forall n \in \mathbb{N}$ *and*
2. $n \cdot \mathsf{s}(m) = n \cdot m + n \quad \forall n,m \in \mathbb{N}$

The quantity $n \cdot m$ *is called the* ***product*** *of* n *and* m.
The operation $+$ *is called* ***addition***.

Lemma 2.2. [12]

LEM

$$\begin{aligned} \mathsf{s}(n)m &= nm + m && \forall n,m \in \mathbb{N} \\ 1 \cdot n &= n && \forall n \in \mathbb{N} \end{aligned}$$

✎PROOF:

1. Proof that $\mathsf{s}(n)m = nm + m$:
   (a) Let M be the set of values for which $\mathsf{s}(n)m = nm + m$.
   (b) Proof that $1 \in M$:

$$\begin{aligned} \mathsf{s}(n) \cdot 1 &= \mathsf{s}(n) && \text{by definition of multiplication over } \mathbb{N} \text{ page 17} \\ &= n + 1 && \text{by definition of addition over } \mathbb{N} \text{ page 15} \\ &= n \cdot 1 + 1 && \text{by definition of multiplication over } \mathbb{N} \text{ page 17} \end{aligned}$$

[11] Dedekind (1888b) page 101, Landau (1966) page 14
[12] Dedekind (1888b) pages 101–102

(c) Proof that $m \in M \implies \mathsf{s}(n)\mathsf{s}(m) = n\mathsf{s}(m) + \mathsf{s}(m)$:

$$\begin{aligned}
\mathsf{s}(n)\mathsf{s}(m) &= \mathsf{s}(n)m + \mathsf{s}(n) && \text{by definition of multiplication over } \mathbb{N} \text{ page 17}\\
&= [nm + m] + \mathsf{s}(n) && \text{by left hypothesis}\\
&= nm + [m + \mathsf{s}(n)] && \text{by associative property of + over } \mathbb{N} \text{ page 16}\\
&= nm + [\mathsf{s}(n) + m] && \text{by commutative property of + over } \mathbb{N} \text{ page 16}\\
&= nm + [n + \mathsf{s}(m)] && \text{by Lemma 2.1 page 15}\\
&= [nm + n] + \mathsf{s}(m) && \text{by commutative property of + over } \mathbb{N} \text{ page 16}\\
&= n\mathsf{s}(m) + \mathsf{s}(m) && \text{by definition of multiplication over } \mathbb{N} \text{ page 17}
\end{aligned}$$

(d) Therefore, by the *axiom of induction* (page 14), $\mathsf{s}(n)m = nm + m$.

2. Proof that $1 \cdot n = n$:

(a) Let M be the set of values for which $1 \cdot n = n$.

(b) Proof that $1 \in M$:

$$1 \cdot 1 = 1 \qquad \text{by definition of multiplication over } \mathbb{N} \text{ page 17}$$

(c) Proof that $n \in M \implies 1 \cdot \mathsf{s}(n) = \mathsf{s}(n)$:

$$\begin{aligned}
1 \cdot \mathsf{s}(n) &= 1 \cdot n + 1 && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\
&= n + 1 && \text{by left hypothesis}\\
&= \mathsf{s}(n) && \text{by definition of addition over } \mathbb{N} \text{ page 15}
\end{aligned}$$

(d) Therefore, by the *axiom of induction* (page 14), $1 \cdot n = n$.

✎

Theorem 2.3 (Commutative property of multiplication). [13] *Let $n \cdot m$ represent the* PRODUCT *of n and m.*

THM				
nm	$=$	mn	$\forall n,m \in \mathbb{N}$	(COMMUTATIVE)

✎PROOF:

1. Let M be the set of values for which $nm = mn$.

2. Proof that $1 \in M$:

$$\begin{aligned}
n \cdot 1 &= n && \text{by definition of multiplication over } \mathbb{N} \text{ page 17}\\
&= 1 \cdot n && \text{by Lemma 2.2 page 17}
\end{aligned}$$

3. Proof that $m \in M \implies \mathsf{s}(m) \in M$:

$$\begin{aligned}
n \cdot \mathsf{s}(m) &= nm + n && \text{by definition of multiplication over } \mathbb{N} \text{ page 17}\\
&= mn + n && \text{by left hypothesis}\\
&= \mathsf{s}(m)n && \text{by Lemma 2.2 page 17}
\end{aligned}$$

4. Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$ ($nm = mn \quad \forall m \in \mathbb{N}$).

[13] Dedekind (1888b) page 102, Landau (1966) page 15

Theorem 2.4 (Distributive propterties). [14] *Let $n+m$ represent the* SUM *of n and m and $n \cdot m$ represent the* PRODUCT *of n and m.*

THM
$$\begin{array}{rclll} n(m+k) &=& nm+nk & \forall n,m,k \in \mathbb{N} & \text{(LEFT DISTRIBUTIVE)} \\ (n+m)k &=& nk+mk & \forall n,m,k \in \mathbb{N} & \text{(RIGHT DISTRIBUTIVE)} \end{array}$$

✎PROOF:

1. Proof that $\mathbb{N}$ is left distributive:
   (a) Let M be the set of values for which $(\mathbb{N}, +, \cdot)$ is distributive.
   (b) Proof that $1 \in M$:

$$\begin{aligned} n(m+1) &= n\mathsf{s}(m) && \text{by definition of } + \text{ (Definition 2.1 page 15)} \\ &= nm + n && \text{by definition of } \cdot \text{ (Definition 2.2 page 17)} \\ &= nm + n \cdot 1 && \text{by definition of } \cdot \text{ (Definition 2.2 page 17)} \end{aligned}$$

   (c) Proof that $k \in M \implies \mathsf{s}(k) \in M$:

$$\begin{aligned} n(m+\mathsf{s}(k)) &= n\mathsf{s}(m+k) && \text{by definition of } + \text{ (Definition 2.1 page 15)} \\ &= n(m+k)+n && \text{by definition of } \cdot \text{ (Definition 2.2 page 17)} \\ &= (nm+nk)+n && \text{by left hypothesis} \\ &= nm+(nk+n) && \text{by associative property of } (\mathbb{N},+) \text{ (Theorem 2.1 page 16)} \\ &= nm+n\mathsf{s}(k) && \text{by definition of } \cdot \text{ (Definition 2.2 page 17)} \end{aligned}$$

   (d) Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$ (all of $(\mathbb{N}, +, \cdot)$ is distributive).

2. Proof that $\mathbb{N}$ is right distributive:

$$\begin{aligned} (n+m)k &= k(n+m) && \text{by commutative property of multiplication over } \mathbb{N} \text{ (page 18)} \\ &= kn+km && \text{by left distributive property of } \mathbb{N} \\ &= nk+mk && \text{by commutative property of multiplication over } \mathbb{N} \text{ (page 18)} \end{aligned}$$

Theorem 2.5 (Associative propterty of multiplication). [15] *Let $n \cdot m$ represent the* PRODUCT *of n and m.*

THM
$$\begin{array}{rclll} (nm)k &=& n(mk) & \forall n,m,k \in \mathbb{N} & \text{(ASSOCIATIVE)} \end{array}$$

✎PROOF:

1. Let M be the set of values for which $n(mk) = (nm)k$.
2. Proof that $1 \in M$:

$$\begin{aligned} n(m \cdot 1) &= nm && \text{by definition of multiplication over } \mathbb{N} \text{ page 17} \\ &= (nm) \cdot 1 && \text{by definition of multiplication over } \mathbb{N} \text{ page 17} \end{aligned}$$

[14] Dedekind (1888b) pages 102–103, Landau (1966) page 16
[15] Dedekind (1888b) page 103, Landau (1966) page 16

3. Proof that $k \in M \implies \mathsf{s}(k) \in M$:

$$\begin{aligned}
n[m\mathsf{s}(k)] &= n[m(k+1)] && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\
&= n[mk+m)] && \text{by left distributive property page 19}\\
&= n(mk)+nm && \text{by left distributive property page 19}\\
&= (nm)k+nm && \text{by left hypothesis}\\
&= (nm)k+(nm)\cdot 1 && \text{by definition of multiplication over } \mathbb{N} \text{ page 17}\\
&= (nm)(k+1) && \text{by left distributive property page 19}\\
&= (nm)\mathsf{s}(k) && \text{by definition of addition over } \mathbb{N} \text{ page 15}
\end{aligned}$$

4. Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$ $(n(mk) = (nm)k \quad \forall k \in \mathbb{N})$.

Definition 2.3 (exponents on $\mathbb{N}$).[16] *Let* $\mathsf{s}(n)$ *be the* SUCCESSOR *of* n.

DEF

Let $^{\cdot}\mathbb{N} \to \mathbb{N}$ *be an operator that satisfies*

1. $a^1 = a \quad \forall a,n \in \mathbb{N}$ *and*
2. $a^{\mathsf{s}(n)} = a^n a \quad \forall a,n \in \mathbb{N}$

The quantity n *is called the* ***exponent*** *of* a^n. *The quantity* a *is called the* ***base*** *of* a^n.

Theorem 2.6.[17]

THM

$$\begin{aligned}
a^{(n+m)} &= \left(a^n\right)\left(a^m\right) && \forall n,m,a \in \mathbb{N}\\
\left(a^n\right)^m &= a^{(nm)} && \forall n,m,a \in \mathbb{N}\\
(ab)^n &= \left(a^n\right)\left(b^n\right) && \forall n,a,b \in \mathbb{N}
\end{aligned}$$

PROOF:

1. Proof that $a^{n+m} = a^n a^m$:
   (a) Let M be the set of values for which $a^{n+m} = a^n a^m$.
   (b) Proof that $1 \in M$:

$$\begin{aligned}
a^{n+1} &= a^{\mathsf{s}(n)} && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\
&= a^n\, a && \text{by definition of exponents over } \mathbb{N} \text{ page 20}\\
&= a^n\, a^1 && \text{by definition of exponents over } \mathbb{N} \text{ page 20}
\end{aligned}$$

   (c) Proof that $m \in M \implies \mathsf{s}(m) \in M$:

$$\begin{aligned}
a^{n+\mathsf{s}(m)} &= a^{n+[m+1]} && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\
&= a^{[n+m]+1} && \text{by Theorem 2.2 page 16}\\
&= a^{\mathsf{s}(n+m)} && \text{by definition of addition over } \mathbb{N} \text{ page 15}\\
&= a^{n+m}\, a && \text{by definition of exponents over } \mathbb{N} \text{ page 20}\\
&= \left[a^n a^m\right] a && \text{by left hypothesis}\\
&= a^n \left[a^m a\right] && \text{by Theorem 2.5 page 19}\\
&= a^n a^{\mathsf{s}(m)} && \text{by definition of exponents over } \mathbb{N} \text{ page 20}
\end{aligned}$$

   (d) Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$.

[16] Dedekind (1888b) page 104
[17] Dedekind (1888b) pages 104–105

2. Proof that $(a^n)^m = a^{nm}$:
   (a) Let M be the set of values for which $(a^n)^m = a^{nm}$.
   (b) Proof that $1 \in M$:

$$\begin{aligned}
(a^n)^1 &= (a^n)^1 \\
&= a^{\mathsf{s}(n)} && \text{by definition of addition over } \mathbb{N} \text{ page 15} \\
&= a^n\, a && \text{by definition of exponents over } \mathbb{N} \text{ page 20} \\
&= a^n\, a^1 && \text{by definition of exponents over } \mathbb{N} \text{ page 20}
\end{aligned}$$

   (c) Proof that $m \in M \implies \mathsf{s}(m) \in M$:

$$\begin{aligned}
(a^n)^{\mathsf{s}(m)} &= (a^n)^m\,(a^n) \\
&= a^{nm}\, a^n && \text{by left hypothesis} \\
&= a^{nm+n} && \text{by previous property} \\
&= a^{n\mathsf{s}(m)} && \text{by definition of multiplication over } \mathbb{N} \text{ page 17}
\end{aligned}$$

   (d) Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$.

3. Proof that $(ab)^n = a^n b^n$:
   (a) Let M be the set of values for which $(a^n)^m = a^{nm}$.
   (b) Proof that $1 \in M$:

$$\begin{aligned}
(ab)^1 &= ab && \text{by definition of exponents over } \mathbb{N} \text{ page 20} \\
&= \left(a^1\right)\left(b^1\right) && \text{by definition of exponents over } \mathbb{N} \text{ page 20}
\end{aligned}$$

   (c) Proof that $m \in M \implies \mathsf{s}(m) \in M$:

$$\begin{aligned}
(ab)^{\mathsf{s}(n)} &= (ab)^n\,(ab) && \text{by definition of exponents over } \mathbb{N} \text{ page 20} \\
&= (a^n b^n)\,(ab) && \text{by left hypothesis} \\
&= a^n(b^n a)b && \text{by associative property of } \cdot \text{ over } \mathbb{N} \text{ (page 19)} \\
&= a^n(ab^n)b && \text{by commutative property of } \cdot \text{ over } \mathbb{N} \text{ (page 18)} \\
&= (a^n a)\,(b^n b) && \text{by associative property of } \cdot \text{ over } \mathbb{N} \text{ (page 19)} \\
&= a^{\mathsf{s}(n)}\, b^{\mathsf{s}(n)} && \text{by definition of exponents over } \mathbb{N} \text{ page 20}
\end{aligned}$$

   (d) Therefore, by the *axiom of induction* (page 14), $M = \mathbb{N}$.

✎

2.1.4 Order

Definition 2.4. [18] *Let $\mathbb{N}$ be the set of natural numbers.*

DEF: $m \le n$ *if there exists* $k \in \mathbb{N}$ *such that* $n = m + k$ *or* $m = n$

Theorem 2.7.

THM: $(\mathbb{N}, \le)$ *is a* TOTALLY ORDERED SET.

[18] Landau (1966) page 9

2.2 Whole Numbers

Definition 2.5. *Let $\mathbb{N}$ be the set of natural numbers.*

DEF *The set of **whole numbers** $\mathbb{W}$ is defined as $\mathbb{W} \triangleq \mathbb{N} \cup 0$ where 0 is the element that satisfies the condition $\mathsf{s}(0) = 1$.*

Theorem 2.8 (Division). [19]

THM $\forall n, m \in \mathbb{W} \quad \exists r, q \in \mathbb{W} \quad$ such that $\quad n = qm + r$ *and* $r < qm$

✎PROOF:

1. Let M be defined as

$$M \triangleq \{n \in \mathbb{Z} | \forall m \in \mathbb{Z},\ \exists q, r \in \mathbb{W} \quad \text{such that} \quad n = qm + r\}.$$

2. Proof that $0 \in M$: $0 = 0 \cdot m + r$

3. Proof that $n \in M \implies \mathsf{s}(n) \in M$:

$$\begin{aligned}
\mathsf{s}(n) &= \mathsf{s}(qm + r) && \text{left hypothesis}\\
&= qm + \mathsf{s}(r) && \text{by Definition 2.1 page 15}\\
&= \begin{cases} qm + \mathsf{s}(r) & \text{for } \mathsf{s}(r) < qm \\ qm + m & \text{for } \mathsf{s}(r) = m \end{cases}\\
&= \begin{cases} qm + \mathsf{s}(r) & \text{for } \mathsf{s}(r) < qm \\ \mathsf{s}(q)m & \text{for } \mathsf{s}(r) = m \end{cases} && \text{by Lemma 2.2 page 17}
\end{aligned}$$

✏

2.3 Integers

Definition 2.6.

DEF *The **negative** number $-n$ is the number that satisfies $n + (-n) = 0$ forall $n \in \mathbb{N}$.*

Definition 2.7.

DEF *The set of integers $\mathbb{Z}$ is defined as* [20]
$$\mathbb{Z} \triangleq \mathbb{W} \cup \{-n | n \in \mathbb{N}\}$$

[19] Amann and Escher (2005) page 35

[20] note about $\mathbb{Z}$: "Z" stands for the German word *zahlen* which means *numbers*.
Reference: Stillwell (2002) page 404

2.4 Rational Numbers

Definition 2.8.

DEF

The fractions $\frac{m}{n}$ *and* $\frac{p}{q}$ *are* ***equivalent*** *if* $mq = np$.

The equivalence class reprented by the member $\frac{m}{n}$ *is denoted* $\left[\frac{m}{n}\right] \triangleq \left\{\frac{p}{q} \middle| \frac{p}{q} = \frac{m}{n}\right\}$.

Definition 2.9.

DEF

$$\frac{m}{n} \leq \frac{p}{q} \qquad \text{if} \qquad mq \lesssim np$$

A *rational number* is any number q that is a ratio of two integers, the denominator not being 0. This is formally defined next.

Definition 2.10.

DEF

The set of rational numbers $\mathbb{Q}$ *is defined as*[21]

$$\mathbb{Q} \triangleq \left\{\left[\frac{n}{m}\right] \middle| n \in \mathbb{Z} \text{ and } m \in \mathbb{N}\right\}.$$

A ***rational number*** *is any element of* $\mathbb{Q}$.

Definition 2.11. [22] *Let* $\mathbb{Q}$ *be the set of rational numbers (Definition 2.10 page 23). Let* $\oplus$ *be the operation of addition on the set of intergers* $\mathbb{Z}$. *Let* $\otimes$ *be the operation of multiplication on* $\mathbb{Z}$.

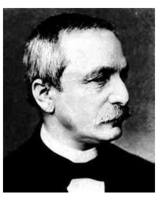
DEF

Let the ***addition operator*** $+$ *and* ***multiplication operator*** $\times$ *be defined as follows:*

$$\left.\begin{array}{rcll} r & \triangleq & \dfrac{n_r}{d_r} & {}_{n_r, d_r \in \mathbb{Z}} \quad \text{and} \\ s & \triangleq & \dfrac{n_s}{d_s} & {}_{n_s, d_s \in \mathbb{Z}} \end{array}\right\} \implies \left\{\begin{array}{rcl} r + s & \triangleq & \dfrac{(n_r \otimes d_s) \oplus (n_s \otimes d_r)}{d_r \otimes d_s} \\ r \times s & \triangleq & \dfrac{n_r \otimes n_s}{d_r \otimes d_s} \end{array}\right.$$

2.5 Real numbers

❝Sp ter ging Kronecker noch weiter, indem er die Existenz irrationaler Zahlen leugnete; so sagte er mir in seiner lebhaften und zu Paradoxen geneigten Art einmal: "Was n utzt uns Ihre schöne Untersuchung über die Zahl ? Wozu das Nachdenken über solche Probleme, wenne es doch gar keine irrationalen Zahlen gibt?"❞



❝Later, Kronecker went even further and denied the existence of irrational numbers; thus, he once said to me in his lively and paradoxical way, "Of what use to us are your beautiful researches about the number π? Why consider such problems when in fact there are no irrational numbers?"❞

Lindemann's recollection, published more than ten years after Leopold Kronecker's (pictured) death, of an encounter with Kronecker (1823–1891), German mathematician, after Lindemann had proved that π is a transcendental number. [23]

[21] note about $\mathbb{Q}$: "Q" stands for the English word "quotient".
Reference: Pugh (2002) page 2

[22] Strichartz (1995) page 19, Hobson (1926), pages 15–17

[23] quote: Poincaré et al. (1904) page 246 ⟨note 3⟩
translation: Edwards (2005) page 203
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Kronecker.html

❝*I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic.... For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.... It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. I succeeded Nov. 24, 1858,...*❞

Richard Dedekind (1831–1915), German mathematician [24]

2.5.1 Construction

Definition 2.12 (Dedekind cuts). [25]

DEF

*A set L is a **cut** if*

1. $L \subsetneq \mathbb{Q}$ *(L is a proper subset of $\mathbb{Q}$)* *and*
2. $L \neq \varnothing$ *(L is non-empty)* *and*
3. $r \in L$ *and* $q \in \mathbb{Q}$ *and* $q < r \implies q \in L$ *and*
4. $\forall r \in L,\ \exists q \in L$ such that $r < q$ *(L is open)*

Lemma 2.3. [26] *Let $(\mathbb{Q}, \leq)$ be the ordered set of rationals. Let L be a cut on $\mathbb{Q}$.*

LEM

$$\begin{array}{llll} p \in L & \text{and} & q \notin L & \implies p < q \\ q \notin L & \text{and} & q < r & \implies r \notin L \end{array}$$

✎PROOF:

$$\begin{aligned} p \geq q &\implies q \in L && \text{by (3) in Definition 2.12 page 24} \\ &\implies \text{contradiction of left hypothesis} \end{aligned}$$

$$\begin{aligned} r \in L &\implies q \in L && \text{by (3) in Definition 2.12 page 24} \\ &\implies \text{contradiction of left hypothesis} \end{aligned}$$

✏

Definition 2.13. [27]

DEF

The set of real numbers $\mathbb{R}$ is the set of all cuts on $\mathbb{Q}$. That is,

$$\mathbb{R} \triangleq \{L | L \text{ is a cut}\}$$

*Any element of $\mathbb{R}$ (any cut) is a **real number**.*

2.5.2 Order structure

Definition 2.14. [28] *Let $(\mathbb{Q}, \precsim)$ be the ordered set of rational numbers.*

DEF

$x \leq y$ *if* $x \subseteq y$ *for $x, y \in \mathbb{R}$ (x and y are cuts on $\mathbb{Q}$)*

---

[24] quote: Dedekind (1872b) page 1
image: http://turnbull.mcs.st-and.ac.uk/history/PictDisplay/Dedekind.html

[25] Landau (1966) page 43, Rudin (1976) page 17

[26] Rudin (1976) page 17

[27] Pugh (2002) page 12

[28] Pugh (2002) page 13

Theorem 2.9. *Let $\leq$ be the relation defined in Definition 2.14 (page 24).*

THM

$(\mathbb{R}, \leq)$ *is a* TOTALLY ORDERED SET. *In particular,*

1. $x \leq x$ $\quad \forall x \in \mathbb{R}$ (reflexive) and
2. $x \leq y$ and $y \leq z \implies x \leq z$ $\quad \forall x,y,z \in \mathbb{R}$ (transitive) and
3. $x \leq y$ and $y \leq x \implies x = y$ $\quad \forall x,y \in \mathbb{R}$ (antisymmetric) and
4. $x, y \in \mathbb{R} \implies x \leq y$ or $y \leq x$ $\quad \forall x,y \in \mathbb{R}$ (comparable).

(1–3: partial order)

Theorem 2.10. [29]

THM

The set $\mathbb{R}$ has the least upper bound property. [30]

2.5.3 Arithmetic

Definition 2.15 (real addition)**.**

DEF

$$x + y = \{a + b | a \in x,\ b \in y\} \qquad \forall x, y \in \mathbb{R}$$

2.5.4 Least upper bound properties

Theorem 2.11. [31] *Let $a, b, \epsilon \in \mathbb{R}$.*

THM

$$a \leq b + \epsilon \quad \forall \epsilon > 0 \implies a \leq b$$
$$a < b + \epsilon \quad \forall \epsilon > 0 \implies a \leq b$$

✎PROOF:

1. Choose $\epsilon = \frac{a-b}{2}$.

2.

$$\begin{aligned}
b < a \implies b + \epsilon &= b + \frac{a-b}{2} && \text{by choice of } \epsilon = \frac{a-b}{2}\\
&= \frac{a+b}{2}\\
&< \frac{a+a}{2} && \text{by } b < a \text{ assumption}\\
&= a\\
\implies a &> b + \epsilon\\
\implies &\text{contradiction of left hypotheses}\\
\implies a &\leq b
\end{aligned}$$

⇨

[29] Pugh (2002) page 13, Rudin (1976) page 18

[30] *least upper bound property*: Definition 4.23 (page 70)

[31] Apostol (1975) page 3

Theorem 2.12. [32] *Let $S \in 2^{\mathbb{R}}$ be a subset of the set of real numbers $\mathbb{R}$.*

THM

$$a < \sup S \implies \exists x \in S \text{ such that } a < x \leq \sup S$$

✎PROOF:

$$\begin{aligned}
\neg(\exists x \in S \text{ such that } a < x \leq \sup S) &\implies \forall x \in S,\ a < x \leq \sup S \\
&\implies \forall x \in S,\ x \leq a < \sup S \\
&\implies a \text{ is a least upper bound for } S \\
&\implies a = \sup S \\
&\implies \text{contradiction of left hypothesis} \\
&\implies \exists x \in S \text{ such that } a < x \leq \sup S
\end{aligned}$$

⇒

Theorem 2.13. [33] *Let $A \in 2^{\mathbb{R}}$ and $B \in 2^{\mathbb{R}}$ be subsets of the set of real numbers $\mathbb{R}$. Define $A + B \triangleq \{x + y | x \in A,\ y \in B\}$.*

THM

$$\left.\begin{array}{ll} C = A + B & \text{and} \\ \sup A \text{ exists} & \text{and} \\ \sup B \text{ exists} & \end{array}\right\} \implies \left\{\begin{array}{ll} \sup C \text{ exists} & \text{and} \\ \sup C = \sup A + \sup B & \end{array}\right.$$

✎PROOF:

1. Proof that $\sup C$ exists and $\sup C \leq \sup A + \sup B$:

$$z = x + y \leq \sup A + \sup B \implies \sup C \text{ exists and } \sup C \leq \sup A + \sup B$$

2. Proof that $\sup A + \sup B \leq \sup C$:

$$\begin{aligned}
&\exists x, y \text{ such that } (\sup A - \epsilon) + (\sup B - \epsilon) < x + y \leq \sup A + \sup B \\
&\implies \sup A + \sup B - 2\epsilon < x + y \leq \sup C \\
&\implies \sup A + \sup B \leq \sup C && \text{by Theorem 2.11 page 25}
\end{aligned}$$

⇒

Theorem 2.14. *Let $S \in 2^{\mathbb{R}}$ and $T \in 2^{\mathbb{R}}$ be subsets of the set of real numbers $\mathbb{R}$.*

THM

$$\left.\begin{array}{ll} s \leq t \quad \forall s \in S,\ t \in T & \text{and} \\ \sup T \text{ exists} & \end{array}\right\} \implies \left\{\begin{array}{ll} \sup S \text{ exists} & \text{and} \\ \sup S \leq \sup T & \end{array}\right.$$

✎PROOF:

$$s \leq t \leq \sup T \implies \sup S \text{ exists and } \sup S \leq \sup T$$

⇒

[32] Apostol (1975) page 9

[33] Apostol (1975) page 10

2.5.5 Completeness

Theorem 2.15. [34] *The set of rational numbers $\mathbb{Q}$ is not complete in the set of real numbers $\mathbb{R}$.*

✎PROOF: The proof is by an example of the number $\sqrt{2}$, which we will show is not a rational number.

1. Proof that $[n^2$ is even$] \implies [n$ is even$]$:

$$\begin{aligned}
n^2 \text{ is even} &\implies \exists m \in \mathbb{Z} \quad \text{such that} \quad n^2 = 2m^2 \\
&\implies n = \sqrt{2m^2} \\
&\implies \exists p \in \mathbb{Z} \quad \text{such that} \quad n = \sqrt{2 \cdot \underbrace{2 \cdot p^2}_{m^2}} \\
&\implies n = 2p \\
&\implies n \text{ is even}
\end{aligned}$$

2. Proof that $\sqrt{2} \notin \mathbb{Q}$:

$$\begin{aligned}
\sqrt{2} \in \mathbb{Q} &\implies \exists n \in \mathbb{Z}, m \in \mathbb{Z}\backslash\{0\}, \text{ not both even} \quad \text{such that} \quad \sqrt{2} = \frac{n}{m} \\
&\implies n^2 = 2m^2 \\
&\implies n^2 \text{ is even} \\
&\implies n \text{ is even} \\
&\implies n^2 \text{ is divisible by } 4 \\
&\implies 2m^2 \text{ is divisible by } 4 \\
&\implies m^2 \text{ is even} \\
&\implies m \text{ is even} \\
&\implies \text{both } n \text{ and } m \text{ are even} \\
&\implies \sqrt{2} \notin \mathbb{Q}
\end{aligned}$$

✏

Theorem 2.16 (The Archimedean Property / Eudoxus axiom). [35]

THM
$$\exists n \in \mathbb{N} \qquad \text{such that} \qquad nx > y \qquad \forall x, y \in \mathbb{R}^+$$

✎PROOF:

1. Proof that if $\sup A$ of a set A exists, then $\forall \epsilon > 0$, $\exists x$ such that $\sup A - \epsilon < x \leq \sup A$:

$$\begin{aligned}
\neg\left(\exists x \quad \text{such that} \quad \sup A - \epsilon < x \leq \sup A\right) &\implies \nexists x \quad \text{such that} \quad \sup A - \epsilon < x \\
&\implies x \leq \sup A - \epsilon \qquad \forall x \in A \\
&\implies (\sup A - \epsilon) \text{ is an upper bound of } A \\
&\implies \sup A \leq \sup A - \epsilon \\
&\implies \text{impossibility} \\
&\implies \exists x \quad \text{such that} \quad (\sup A - \epsilon) < x \leq \sup A
\end{aligned}$$

[34] 📖 Rudin (1976), page 2
[35] 📖 Aliprantis and Burkinshaw (1998), page 17
📖 Carothers (2000) page 5

2. Proof that $\mathbb{N}$ is unbounded above:

$$\begin{aligned}
\neg(\mathbb{N} \text{ is unbounded above}) &\implies \mathbb{N} \text{ is bounded above} \\
&\implies \exists x \in \mathbb{R} \quad \text{such that} \quad n \le x \quad \forall n \in \mathbb{N} \\
&\implies \exists s \in \mathbb{R} \quad \text{such that} \quad s = \sup \mathbb{N} && \text{by Theorem 2.10 page 25} \\
&\implies \exists n \quad \text{such that} \quad s - 1 < n && \text{by item (1)} \\
&\implies \exists n \quad \text{such that} \quad s < n + 1 \\
&\implies \exists n \quad \text{such that} \quad s < n + 1 \le s \\
&\implies s < s \\
&\implies \text{impossibility} \\
&\implies \mathbb{N} \text{ is unbounded above}
\end{aligned}$$

3. Proof that $\exists n \in \mathbb{N}$ such that $nx > y$:

$$\begin{aligned}
\neg[\exists n \in \mathbb{N} \quad \text{such that} \quad nx > y] &\implies \nexists n \in \mathbb{N} \quad \text{such that} \quad nx > y \\
&\implies nx \le y \quad \forall n \in \mathbb{N} \\
&\implies n \le \frac{y}{x} \quad \forall n \in \mathbb{N} \\
&\implies \mathbb{N} \text{ is bounded above by } \frac{y}{x} \\
&\implies \text{impossibility} && \text{by item (2)} \\
&\implies \exists n \in \mathbb{N} \quad \text{such that} \quad nx > y
\end{aligned}$$

✏

Theorem 2.17. [36] *Let $\mathbb{R}$ be the set of real numbers and $\mathbb{Q}$ the set of rational numbers.*

THM
$$\left.\begin{array}{l} x, y \in \mathbb{R} \quad \text{and} \\ x < y \end{array}\right\} \implies \exists r \in \mathbb{Q} \quad \text{such that} \quad x < r < y$$

✎PROOF:

$$\begin{aligned}
x, y \in \mathbb{R} \text{ and } x < y &\implies \forall x, y \in \mathbb{R} \quad \exists q \in \mathbb{N} \quad \text{such that} \quad q(y - x) > 1 && \text{by the Archimedean property page 27} \\
&\implies qy - qx > 1 \\
&\implies \exists p \in \mathbb{N} \quad \text{such that} \quad qy < p < qx \\
&\implies y < \frac{p}{q} < x \\
&\implies \exists r \in \mathbb{Q} \quad \text{such that} \quad x < r < y
\end{aligned}$$

✏

Theorem 2.18. [37]

THM
$$\left\{\begin{array}{ll} 1. & (\!(x_n)\!)_{n\in\mathbb{Z}} \textit{ is monotone} \quad \text{and} \\ 2. & (\!(x_n)\!)_{n\in\mathbb{Z}} \textit{ is bounded} \end{array}\right\} \implies \left\{\begin{array}{l} (\!(x_n)\!)_{n\in\mathbb{Z}} \\ \textit{converges} \end{array}\right\}$$

✎PROOF:

1. Proof for monotonic *increasing* sequences:

[36] Carothers (2000) page 5, Aliprantis and Burkinshaw (1998), page 17
[37] Carothers (2000) page

(a) By boundness hypothesis, $\sup (x_n)$ exists in $\mathbb{R}$ such that

$$x_m - \epsilon < \sup (x_n) \qquad \forall \epsilon > 0,\ m \in \mathbb{Z}$$

(b) By Theorem 2.17 (page 28), there exists $x \in \mathbb{R}$ between x_m and $\sup (x_n)$ such that

$$x_m - \epsilon < x \leq \sup (x_n) \qquad \forall \epsilon > 0,\ m \in \mathbb{Z}$$

(c) $x_m - x < \epsilon$

2. Proof for monotonic *decreasing* sequences:

⇨

2.5.6 Normed algebra structure

Definition 2.16. [38]

DEF *The **absolute value** $|\cdot| \in \mathbb{R}^{\mathbb{R}}$ is defined as*

$$|x| \triangleq \begin{cases} x & for \quad x \geq 0 \\ -x & for \quad x < 0 \end{cases}$$

Lemma 2.4. [39]

LEM $|x| \leq a \iff -a \leq x \leq a \qquad \forall x \in \mathbb{R}$

✎PROOF:

1. Proof that $|x| \leq a \implies -a \leq x \leq a$

$$\begin{aligned} |x| \leq a &\implies \left\{ \begin{array}{ccc} x & for & x \geq 0 \\ -x & for & x < 0 \end{array} \right\} \leq a && \text{by Definition 2.16 page 29} \\ &\implies \left\{ \begin{array}{ccc} x \leq a & for & x \geq 0 \\ (-x) \leq a & for & x < 0 \end{array} \right\} \\ &\implies \left\{ \begin{array}{ccc} x \leq a & for & x \geq 0 \\ (x \geq -a) & for & x < 0 \end{array} \right\} \\ &\implies -a \leq x \leq a \end{aligned}$$

2. Proof that $|x| \leq a \impliedby -a \leq x \leq a$

$$\begin{aligned} |x| &= \left\{ \begin{array}{ccc} x & for & x \geq 0 \\ -x & for & x < 0 \end{array} \right\} && \text{by Definition 2.16 page 29} \\ &\leq \left\{ \begin{array}{ccc} a & for & x \geq 0 \\ a & for & x < 0 \end{array} \right\} && \text{by left hypothesis} \\ &= a \end{aligned}$$

⇨

[38] Apostol (1975) page 13
[39] Apostol (1975) page 13

Theorem 2.19 (normed algebra properties). [40] *Let $|\cdot| \in \mathbb{R}^{\mathbb{R}}$ be the absolute value function.*

THM

The pair $(\mathbb{R}, |\cdot|)$ is a NORMED ALGEBRA. *In particular*

1.	$\lvert x\rvert \geq 0$	$\forall x \in \mathbb{R}$	*(non-negative)*
2.	$\lvert x\rvert = 0 \iff x = 0$	$\forall x \in \mathbb{R}$	*(nondegenerate)*
3.	$\lvert xy\rvert = \lvert x\rvert\,\lvert y\rvert$	$\forall x,y \in \mathbb{R}$	*(homogeneous / multiplicative condition)*
4.	$\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$	$\forall x,y \in \mathbb{R}$	*(subadditive / triangle inequality)*

✎PROOF:

1. Proof that $|x| \geq 0$: true by Definition 2.16 page 29.
2. Proof that $|x| = 0 \iff x = 0$: true by Definition 2.16 page 29.
3. Proof that $|xy| = |x||y|$:

$$\begin{aligned}
|xy| &= \begin{pmatrix} xy & for & xy \geq 0 \\ -xy & for & xy < 0 \end{pmatrix} && \text{by definition of } |\cdot| \text{ page 29} \\
&= \begin{pmatrix} x & for & x \geq 0 \\ -x & for & x < 0 \end{pmatrix} \begin{pmatrix} y & for & y \geq 0 \\ -y & for & y < 0 \end{pmatrix} \\
&= |x|\,|y| && \text{by definition of } |\cdot| \text{ page 29}
\end{aligned}$$

4. Proof that $|x+y| \leq |x| + |y|$:
   (a) Start with these inequalities:
   $$\begin{array}{rcccl} -|x| & \leq & x & \leq & |(|x) \\ -|y| & \leq & y & \leq & |(|y) \end{array}$$
   (b) Add the above two equations to get the following:
   $$-(|x| + |y|) \leq x + y \leq (|x| + |y|)$$
   (c) Then by Lemma 2.4 (page 29),
   $$|x+y| \leq |x| + |y|$$

⇨

2.6 Complex numbers

❝*Weil nun alle möglichen Zahlen, die man sich nur immer vorstellen mag, entweder größer oder kleiner als 0, oder etwa 0 selbst sind, so ist klar, daß die Quadratwurzeln von Negativzahlen nicht einmal zu den möglichen Zahlen gerechnet werden können. Folglich müssen wir sagen, daß dies unmögliche Zahlen sind. Und dieser Umstand leitet uns auf den Begriß von solchen Zahlen, welche ihrer Natur nach unmöglich sind, und gewöhnlich imaginäre oder eingebildete Zahlen genannt werden, weil sie bloßin der Einbildung vorhanden sind.*❞

❝*And, since all numbers which it is possible to conceive, are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers, which from their nature are impossible; and therefore they are usually called imaginary quantities, because they exist merely in the imagination.*❞

Leonhard Euler (1707–1783), mathematician [41]

[40] 📖 Apostol (1975) page 13

2.6.1 Definitions

Definition 2.17. [42]

DEF: *The set of* ***complex numbers*** $\mathbb{C}$ *is defined as*

$$\mathbb{C} \triangleq \{(a,b)\,|a,b \in \mathbb{R}\} \qquad \textit{(the set of all ordered pairs of real numbers)}$$

A ***complex number*** *is any element of* $\mathbb{C}$.

Addition and multiplication over $\mathbb{C}$ are defined next.

Definition 2.18. [43] *Let* $\mathbb{C}$ *be the set of complex numbers.*

DEF: ADDITION *and* MULTIPLICATION *on* $\mathbb{C}$ *are defined as follows:*

$$\begin{array}{rcl} (x_r, x_i) + (y_r, y_i) & \triangleq & (x_r + y_r, x_i + y_i) \qquad \forall (x_r,x_i),(y_r,y_i) \in \mathbb{C} \\ (x_r, x_i) \cdot (y_r, y_i) & \triangleq & (x_r y_r - x_i y_i, x_r y_i + x_i y_r) \qquad \forall (x_r,x_i),(y_r,y_i) \in \mathbb{C} \end{array}$$

Definition 2.19. [44] *Let* $\mathbb{C}$ *be the set of complex numbers.*

DEF: *The* ***imaginary number*** i *in* $\mathbb{C}$ *is defined as*

$$i \triangleq (0,1)$$

Proposition 2.1. [45] *Let* $\mathbb{C}$ *be the set of complex numbers and* i *the imaginary number.*

PRP:
$$i^2 = -1$$

✎PROOF:

$$\begin{aligned} i^2 &= (0,1)\,(0,1) && \text{by Definition 2.19 page 31} \\ &= (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) && \text{by Theorem 2.21 page 32} \\ &= (-1, 0) \\ &= -1 \end{aligned}$$

✏

Theorem 2.20. [46]

THM:
$$(a,b) = a + ib \qquad \forall (a,b) \in \mathbb{C} \qquad \text{(RECTANGULAR COORDINATES)}$$

✎PROOF:

$$\begin{aligned} (a,b) &= a(1,0) + b\,(0,1) \\ &= a1 + bi && \text{by Definition 2.19 page 31} \\ &= a + ib \end{aligned}$$

✏

[41] quote: Euler (1770a) page 60
translation: Euler (1770b) page 43
image: http://en.wikipedia.org/wiki/File:Leonhard_Euler.jpg, public domain

[42] Landau (1966) page 92, Rudin (1976) page 12

[43] Landau (1966) pages 93–96, Thurston (1956) page 121, Bottazzini (1986) page 180, Hamilton (1837) pages 8

[44] Landau (1966) page 133, Euler (1777) page 184, Cardano (1545a), Cardano (1545b)

[45] Landau (1966) page 133

[46] Landau (1966) page 133

2.6.2 Field structure

Theorem 2.21 (Additive abelian group). [47]

THM

$(\mathbb{C},+)$ *is a* COMMUTATIVE GROUP *under addition; in particular*

1. $(x+y)+z = x+(y+z)$ $\quad \forall x,y,z\in\mathbb{C}$ (ASSOCIATIVE)
2. $x+(0,0) = (0,0)+x = x$ $\quad \forall x\in\mathbb{C}$ ($(0,0)$ *is the* ADDITIVE IDENTITY ELEMENT)
3. $-(a,b) = (-a,-b)$ $\quad \forall (a,b)\in\mathbb{C}$ ($(-a,-b)$ *is the* ADDITIVE INVERSE *of* (a,b))
4. $x+y = y+x$ $\quad \forall x,y\in\mathbb{C}$ (COMMUTATIVE)

✎PROOF:

1. Proof that $(\mathbb{C},+)$ is *associative*:

$$\begin{aligned}
(x+y)+z &= \big[(x_r,x_i)+(y_r,y_i)\big]+(z_r,z_i) \\
&= (x_r+y_r, x_i+y_i)+(z_r,z_i) && \text{by definition of complex addition (page 31)} \\
&= ([x_r+y_r]+z_r, [x_i+y_i]+z_i) && \text{by definition of complex addition (page 31)} \\
&= (x_r+[y_r+z_r], x_i+[y_i+z_i]) \\
&= (x_r,x_i)+(y_r+z_r, y_i+z_i) && \text{by definition of complex addition (page 31)} \\
&= x+(y+z)
\end{aligned}$$

2. Proof that $(0,0)\in\mathbb{C}$ is the additive identity element:

$$\begin{aligned}
(a,b)+(0,0) &= (a+0, b+0) && \text{by definition of complex addition (page 31)} \\
&= (a,b) \\
(0,0)+(a,b) &= (0+a, 0+b) && \text{by definition of complex addition (page 31)} \\
&= (a,b)
\end{aligned}$$

3. Proof that $(-a,-b)\in\mathbb{C}$ is the inverse element of (a,b):

$$\begin{aligned}
(a,b)+(-a,-b) &= (a-a, b-b) && \text{by definition of complex addition (page 31)} \\
&= (0,0) \\
(-a,-b)+(a,b) &= (-a+a, -b+b) && \text{by definition of complex addition (page 31)} \\
&= (0,0)
\end{aligned}$$

4. Proof that $(\mathbb{C},+)$ is *commutative*:

$$\begin{aligned}
x+y &= (x_r,x_i)+(y_r,y_i) \\
&= (x_r+y_r, x_i+y_i) && \text{by definition of complex addition (page 31)} \\
&= (y_r+x_r, y_i+x_i) \\
&= (y_r,y_i)+(x_r,x_i) && \text{by definition of complex addition (page 31)} \\
&= y+x
\end{aligned}$$

✏

Theorem 2.22 (Multiplicative abelian group). [48]

THM

$(\mathbb{C},\cdot)$ *is a* COMMUTATIVE GROUP *under multiplication; in particular*

1. $(xy)z = x(yz)$ $\quad \forall x,y,z\in\mathbb{C}$ (ASSOCIATIVE)
2. $x\,(1,0) = (1,0)\,x = x$ $\quad \forall x\in\mathbb{C}$ ($(1,0)$ *is the* MULTIPLICATIVE IDENTITY ELEMENT)
3. $(a,b)^{-1} = \frac{1}{a^2+b^2}\,(a,-b)$ $\quad \forall (a,b)\in\mathbb{C}$ ($\frac{1}{a^2+b^2}\,(a,-b)$ *is the* MULTIPLICATIVE INVERSE *of* (a,b))
4. $xy = yx$ $\quad \forall x,y\in\mathbb{C}$ (COMMUTATIVE)

[47] Landau (1966) pages 93–94
[48] Landau (1966) pages 96–98

✎PROOF:

1. Proof that $(\mathbb{C}, \times)$ is *associative*:

$$\begin{aligned}
(xy)z &= \left[\left(x_r, x_i\right)\left(y_r, y_i\right)\right]\left(z_r, z_i\right) \\
&= \underbrace{\left(x_r y_r - x_i y_i, x_r y_i + x_i y_r\right)}_{xy}\underbrace{\left(z_r, z_i\right)}_{z} && \text{by Definition 2.18 page 31} \\
&= \left((x_r y_r - x_i y_i)z_r - (x_r y_i + x_i y_r)z_i, (x_r y_r - x_i y_i)z_i + (x_r y_i + x_i y_r)z_r\right) && \text{by Definition 2.18 page 31} \\
&= \left(x_r y_r z_r - x_i y_i z_r - x_r y_i z_i - x_i y_r z_i, x_r y_r z_i - x_i y_i z_i + x_r y_i z_r + x_i y_r z_r\right) \\
&= \left(x_r(y_r z_r - y_i z_i) - x_i(y_r z_i + y_i z_r), x_r(y_r z_i + y_i z_r) + x_i(y_r z_r - y_i z_i)\right) \\
&= \underbrace{\left(x_r, x_i\right)}_{x}\underbrace{\left(y_r z_r - y_i z_i, y_r z_i + y_i z_r\right)}_{yz} \\
&= \left(x_r, x_i\right)\left[\left(y_r, y_i\right)\left(z_r, z_i\right)\right] && \text{by Definition 2.18 page 31} \\
&= x(yz)
\end{aligned}$$

2. Proof that $(1,0) \in \mathbb{C}$ is the additive identity element:

$$\begin{aligned}
(a,b)\,(1,0) &= (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) && \text{by definition of complex multiplication (page 31)} \\
&= (a,b) \\
(1,0)\,(a,b) &= (1 \cdot a - 0 \cdot b, 1 \cdot b + 0 \cdot a) && \text{by definition of complex multiplication (page 31)} \\
&= (a,b)
\end{aligned}$$

3. Proof that $\frac{1}{a^2+b^2}\,(a,-b) \in \mathbb{C}$ is the inverse element of (a,b):

$$\begin{aligned}
(a,b)\left[\frac{1}{a^2+b^2}\,(a,-b)\right] &= \frac{1}{a^2+b^2}\,(aa+bb, -ab+ba) && \text{by definition of complex multiplication (page 31)} \\
&= \frac{1}{a^2+b^2}\left(a^2+b^2, 0\right) \\
&= \left(\frac{a^2+b^2}{a^2+b^2}, \frac{0}{a^2+b^2}\right) \\
&= (1,0) \\
\left[\frac{1}{a^2+b^2}\,(a,-b)\right](a,b) &= \frac{1}{a^2+b^2}\,(aa+bb, ab-ba) && \text{by definition of complex multiplication (page 31)} \\
&= \frac{1}{a^2+b^2}\left(a^2+b^2, 0\right) \\
&= \left(\frac{a^2+b^2}{a^2+b^2}, \frac{0}{a^2+b^2}\right) \\
&= (1,0)
\end{aligned}$$

4. Proof that $(\mathbb{C}, \times)$ is *commutative*:

$$\begin{aligned}
xy &= \left(x_r, x_i\right)\left(y_r, y_i\right) \\
&= \left(x_r y_r - x_i y_i, x_r y_i + x_i y_r\right) && \text{by definition of complex multiplication (page 31)} \\
&= \left(y_r x_r - y_i x_i, y_i x_r + y_r x_i\right) \\
&= \left(y_r, y_i\right)\left(x_r, x_i\right) \\
&= yx
\end{aligned}$$

5. Proof that $(\mathbb{C}, \times)$ forms a multiplicative group:

(a) Proof that $(1,0)$ is the multiplicative identity element:

$$\begin{aligned}(a,b)(1,0) &= (a1-b0, a0+b1) &&= (a,b)\\ (1,0)(a,b) &= (1a-0b, 0a+1b) &&= (a,b)\end{aligned}$$

(b) Proof that $\frac{1}{a^2+b^2}(a,-b)$ is the multiplicative inverse element of (a,b):

$$\begin{aligned}(a,b)\left[\frac{1}{a^2+b^2}(a,-b)\right] &= \frac{1}{a^2+b^2}(a^2+b^2, -ab+ba) &&= (1,0)\\ \left[\frac{1}{a^2+b^2}(a,-b)\right](a,b) &= \frac{1}{a^2+b^2}(a^2+b^2, ab-ba) &&= (1,0)\end{aligned}$$

(c) Proof that $(\mathbb{C},\times)$ is closed:

$$(a,b)(c,d) = (ac-bd, ad+bc) \in \mathbb{C}.$$

⇒

Proposition 2.2 (Distributive property). [49]

PRP

$z(x+y)$	$=$	zx	$+$	zy	$\forall x,y,z\in\mathbb{C}$	(LEFT DISTRIBUTIVE)
$(x+y)z$	$=$	xz	$+$	yz	$\forall x,y,z\in\mathbb{C}$	(RIGHT DISTRIBUTIVE)

✎PROOF:

$$\begin{aligned}
z(x+y) &= (z_r,z_i)\left[(x_r,x_i)+(y_r,y_i)\right]\\
&= (z_r,z_i)(x_r+y_r, x_i+y_i) && \text{by def. of complex add. (Definition 2.18 page 31)}\\
&= (z_r[x_r+y_r]-z_i[x_i+y_i], z_i[x_r+y_r]+z_r[x_i+y_i]) && \text{by def. of complex mult. (Definition 2.18 page 31)}\\
&= (z_rx_r - z_ix_i + z_ry_r - z_iy_i, z_ix_r + z_rx_i + z_iy_r + z_ry_i)\\
&= (z_rx_r - z_ix_i, z_ix_r + z_rx_i) + (z_ry_r - z_iy_i, z_iy_r + z_ry_i) && \text{by def. of complex add. (Definition 2.18 page 31)}\\
&= (z_r,z_i)(x_r,x_i) + (z_r,z_i)(y_r,y_i) && \text{by def. of complex mult. (Definition 2.18 page 31)}\\
&= zx + zy\\
(x+y)z &= z(x+y) && \text{by commutative prop. of Theorem 2.22 page 32}\\
&= zx+zy && \text{by previous result}\\
&= xz+yz && \text{by commutative prop. of Theorem 2.22 page 32}
\end{aligned}$$

⇒

Theorem 2.23.

THM

$(\mathbb{C},+,\times)$ *is a field.*

✎PROOF:

1. By Theorem 2.21 (page 32), $(\mathbb{C},+,\times)$ is an additive group.
2. By Theorem 2.22 (page 32), $(\mathbb{C},+,\times)$ is a multiplicative group.
3. By Proposition 2.2 (page 34), $(\mathbb{C},+,\times)$ is distributive.

49 Landau (1966) page 99

4. These properties collectively imply that $(\mathbb{C}, +, \times)$ is a field.

Proposition 2.3. [50]

PRP
$$(0,0)\,(a,b) = (a,b)\,(0,0) = (0,0) \qquad \forall (a,b)\in\mathbb{C}$$

✎PROOF:

$$\begin{aligned}
(0,0)\,(a,b) &= (0\cdot a - 0\cdot b, 0\cdot b + 0\cdot a) && \text{by Definition 2.18 page 31}\\
&= (0,0) && \text{by Definition 2.18 page 31}\\
(a,b)\,(0,0) &= (a\cdot 0 - b\cdot 0, b\cdot 0 + a\cdot 0) && \text{by Definition 2.18 page 31}\\
&= (0,0) && \text{by Definition 2.18 page 31}
\end{aligned}$$

⇨

2.6.3 Normed algebra structure

Definition 2.20. [51]

DEF
*The **absolute value** $|\cdot| \in \mathbb{R}^{\mathbb{C}}$ is defined as*
$$|(a,b)| \triangleq \sqrt{a^2+b^2} \qquad \forall (a,b)\in\mathbb{C}$$

Theorem 2.24 (normed algebra properties). [52] *Let $|\cdot| \in \mathbb{R}^{\vdash\mathbb{C}}$ be the absolute value function.*

THM
The pair $(\mathbb{C}, |\cdot|)$ is a NORMED ALGEBRA. *In particular*

1.	$\lvert x\rvert \geq 0$	$\forall x\in\mathbb{C}$	*(non-negative)*
2.	$\lvert x\rvert = 0 \iff x = 0$	$\forall x\in\mathbb{C}$	*(nondegenerate)*
3.	$\lvert xy\rvert = \lvert x\rvert\,\lvert y\rvert$	$\forall x,y\in\mathbb{C}$	*(homogeneous / multiplicative condition)*
4.	$\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$	$\forall x,y\in\mathbb{C}$	*(subadditive / triangle inequality)*

✎PROOF:

1. Proof that $|x| \geq 0$:

$$\begin{aligned}
|x|^2 &= a^2 + b^2 && \text{by definition of } |x|\\
&\geq 0
\end{aligned}$$

[50] Apostol (1975) page 16
[51] Landau (1966) page 108
[52] Landau (1966) pages 108–110,

2. Proof that $|x| = 0 \iff x = 0$:

$$\begin{aligned}
|x| = 0 &\implies |x|^2 = 0\\
&\implies |x| = 0\\
&\implies a^2 + b^2 = 0\\
&\implies a = b = 0\\
&\implies (a, b) = 0\\
&\implies x = 0\\
x = 0 &\implies (a, b) = 0\\
&\implies a = b = 0\\
&\implies a^2 + b^2 = 0\\
&\implies |x|^2 = 0\\
&\implies |x| = 0
\end{aligned}$$

3. Proof that $|xy| = |x||y|$:

$$\begin{aligned}
|xy|^2 &= |(a, b)\,(c, d)|^2 && \text{by definition of } |x|\\
&= |(ac - bd, ad + bc)|^2 && \text{by Definition 2.18 page 31}\\
&= (ac - bd)^2 + (ad + bc)^2 && \text{by definition of } |x|\\
&= \underbrace{(ac)^2 + (bd)^2 - 2(ac)(bd)}_{(ac-bd)^2} + \underbrace{(ad)^2 + (bc)^2 + 2(ad)(bc)}_{(ad+bc)^2}\\
&= (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2\\
&= (a^2 + b^2)(c^2 + d^2)\\
&= |x|^2|y|^2 && \text{by definition of } |x|
\end{aligned}$$

4. Proof that $|x + y| \le |x| + |y|$:

$$\begin{aligned}
|x + y|^2 &= |(a, b) + (c, d)|^2 && \text{by definition of } x, y\\
&= |(a + c, b + d)|^2 && \text{by Definition 2.18 page 31}\\
&= (a + c)^2 + (b + d)^2 && \text{by definition of } |x|\\
&= \underbrace{a^2 + c^2 + 2ac}_{(a+c)^2} + \underbrace{b^2 + d^2 + 2bd}_{(b+d)^2}\\
&= \underbrace{a^2 + b^2}_{|x|^2} + \underbrace{c^2 + d^2}_{|y|^2} + \underbrace{2ac + 2bd}_{2\Re(xy^*)}\\
&= |x|^2 + |y|^2 + 2\Re(xy^*)\\
&\le |x|^2 + |y|^2 + 2\left|xy^*\right| && \text{by Theorem 2.28 page 39}\\
&\le |x|^2 + |y|^2 && \text{by } |\cdot| \ge 0 \text{ property (1)}
\end{aligned}$$

Proposition 2.4. [53] *Let $|\cdot| \in \mathbb{C}^{\mathbb{R}^{\vdash}}$ be the absolute value function.*

PRP $\quad \left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|} \qquad \forall x, y \in \mathbb{C}$

[53] Apostol (1975) page 18

✎PROOF:

$$\begin{aligned}
\frac{|x|}{|y|} &= \frac{\left|y\frac{x}{y}\right|}{|y|} \\
&= \frac{|y|\left|\frac{x}{y}\right|}{|y|} && \text{by homogeneous property of Theorem 2.24 page 35} \\
&= \left|\frac{x}{y}\right|
\end{aligned}$$

✏

2.6.4 Star-algebra structure

Definition 2.21. [54] *Let $(a,b) \in \mathbb{C}$. The **conjugate operator** $* : \mathbb{C} \to \mathbb{C}$ is defined as*

DEF
$$(a,b)^* \triangleq (a,-b) \qquad \forall (a,b) \in \mathbb{C}$$

Theorem 2.25. [55]

THM
$$x = x^* \iff x \in \mathbb{R}$$

✎PROOF:

1. Proof that $x = x^* \implies x \in \mathbb{R}$

$$\begin{aligned}
x = (a,b) &\implies x^* = (a,b)^* && \text{by definition of } x \\
&\implies x^* = (a,-b) && \text{by Definition 2.21 page 37} \\
&\implies b = -b && \text{by left hypothesis} \\
&\implies b = 0 && \text{because } b \in \mathbb{R} \\
&\implies x = (a,0) \\
&\implies x \in \mathbb{R} && \text{by Definition 2.17 page 31}
\end{aligned}$$

2. Proof that $x = x^* \impliedby x \in \mathbb{R}$

$$\begin{aligned}
x &= (a,b) && \text{by definition of } x \\
&= (a,0) && \text{by right hypothesis} \\
&= (a,-0) \\
&= (a,0)^* && \text{by Definition 2.21 page 37} \\
&= (a,b)^* && \text{by right hypothesis} \\
&= x^* && \text{by definition of } x
\end{aligned}$$

✏

Theorem 2.26 ($*$-algebra properties)**.** [56] *Let $* : \mathbb{C} \to \mathbb{C}$ be the conjugate operator.*

THM
The pair $(\mathbb{C}, *)$ is a $*$-ALGEBRA[57]*. In particular,*

$$\begin{array}{rcllr}
x^{**} &=& x & \forall x \in \mathbb{C} & \textit{(involutary)} \\
(x+y)^* &=& x^* + y^* & \forall x,y \in \mathbb{C} & \textit{(distributive)} \\
(xy)^* &=& y^* x^* & \forall x,y \in \mathbb{C} & \textit{(antiautomorphic)}
\end{array}$$

[54] Landau (1966) page 106
[55] Berberian (1961) page 25
[56] Landau (1966) pages 106–107
[57] $*$-*algebra*: Definition B.3 page 288

✎PROOF:

$$\begin{aligned}
x^{**} &= (x^*)^* \\
&= \left[\left(x_r, x_i\right)^*\right]^* && \text{where } x \triangleq \left(x_r, x_i\right) \\
&= \left(x_r, -x_i\right)^* && \text{by definition of conjugate (page 37)} \\
&= \left(x_r, x_i\right) && \text{by definition of conjugate (page 37)} \\
&= x && \text{by the definition } x \triangleq \left(x_r, x_i\right)
\end{aligned}$$

$$\begin{aligned}
(x+y)^* &= [\left(x_r, x_i\right) + \left(y_r, y_i\right)]^* && \text{where } x \triangleq \left(x_r, x_i\right) \text{ and } y \triangleq \left(y_r, y_i\right) \\
&= (x_r + y_r, x_i + y_i)^* && \text{by definition of complex addition (page 31)} \\
&= (x_r + y_r, -x_i - y_i) && \text{by definition of complex conjugate (page 37)} \\
&= \left(x_r, -x_i\right) + \left(y_r, -y_i\right) && \text{by definition of complex addition (page 31)} \\
&= \left(x_r, x_i\right)^* + \left(y_r, y_i\right)^* && \text{by definition of complex conjugate (page 37)} \\
&= x^* + y^* && \text{by the definitions } x \triangleq \left(x_r, x_i\right) \text{ and } y \triangleq \left(y_r, y_i\right)
\end{aligned}$$

$$\begin{aligned}
(xy)^* &= [\left(x_r, x_i\right)\left(y_r, y_i\right)]^* && \text{where } x \triangleq \left(x_r, x_i\right) \text{ and } y \triangleq \left(y_r, y_i\right) \\
&= (x_r y_r - x_i y_i, x_r y_i + x_i y_r)^* && \text{by definition of complex multiplication (page 31)} \\
&= (x_r y_r - x_i y_i, -x_r y_i - x_i y_r) && \text{by definition of complex conjugate (page 37)} \\
&= \left(x_r, -x_i\right)\left(y_r, -y_i\right) && \text{by definition of complex multiplication (page 31)} \\
&= \left(y_r, -y_i\right)\left(x_r, -x_i\right) && \text{by Theorem 2.21 page 32} \\
&= \left(y_r, y_i\right)^*\left(x_r, x_i\right)^* && \text{by definition of complex conjugate (page 37)} \\
&= y^* x^*. && \text{by the definitions } x \triangleq \left(x_r, x_i\right) \text{ and } y \triangleq \left(y_r, y_i\right)
\end{aligned}$$

✏

Theorem 2.27. [58] *Let $* : \mathbb{C} \to \mathbb{C}$ be the conjugate operator.*

THM				
	$(-x)^* = -(x^*)$	$\forall x \in \mathbb{C}$	*(conjugate of additive inverse)*
	$\left(x^{-1}\right)^* = \left(x^*\right)^{-1}$	$\forall x,y \in \mathbb{C}$	*(conjugate of multiplicative inverse)*

✎PROOF:

$$\begin{aligned}
(-x)^* &= \left[-\left(x_r, x_i\right)\right]^* && \text{where } x \triangleq \left(x_r, x_i\right) \\
&= \left(-x_r, -x_i\right)^* \\
&= \left(-x_r, x_i\right) && \text{by definition of conjugate (page 37)} \\
&= -\left(x_r, -x_i\right) \\
&= -\left[\left(x_r, x_i\right)^*\right] && \text{by definition of conjugate (page 37)} \\
&= -\left(x^*\right) && \text{by the definition } x \triangleq \left(x_r, x_i\right)
\end{aligned}$$

$$\begin{aligned}
\left(x^{-1}\right)^* &= \left(\left(x_r, x_i\right)^{-1}\right)^* && \text{where } x \triangleq \left(x_r, x_i\right) \\
&= \left(\frac{1}{x_r^2 + x_i^2}\left(x_r, -x_i\right)\right)^* && \text{by Theorem 2.22 page 32} \\
&= \left(\frac{1}{x_r^2 + x_i^2}\right)^*\left(x_r, -x_i\right)^* && \text{by previous property}
\end{aligned}$$

[58] Landau (1966) page 107

$$
\begin{aligned}
&= \frac{1}{x_r^2 + x_i^2}(x_r, x_i) && \text{by definition of conjugate (page 37)}\\
&= (x_r, -x_i)^{-1} && \text{by Theorem 2.22 page 32}\\
&= \left((x_r, x_i)^*\right)^{-1} && \text{by definition of conjugate (page 37)}\\
&= (x^*)^{-1} && \text{by definition } x \triangleq (x_r, x_i)
\end{aligned}
$$

✏

Corollary 2.1. [59] *Let* $*$: $\mathbb{C} \to \mathbb{C}$ *be the conjugate operator.*

COR

$$\left(\frac{x}{y}\right)^* = \frac{x^*}{y^*} \qquad \forall x,y \in \mathbb{C}$$

✎PROOF:

$$
\begin{aligned}
\left(\frac{x}{y}\right)^* &= \left(x\frac{1}{y}\right)^*\\
&= [x^*]\left[\left(\frac{1}{y}\right)^*\right] && \text{by Theorem 2.26 page 37}\\
&= [x^*]\left[\left(y^{-1}\right)^*\right]\\
&= [x^*]\left[\left(y^*\right)^{-1}\right] && \text{by Theorem 2.26 page 37}\\
&= \frac{x^*}{y^*}
\end{aligned}
$$

✏

2.6.5 C-star algebraic structure

Theorem 2.28. [60] *Let* $|\cdot|$: $\mathbb{C} \to \mathbb{R}^{\vdash}$ *be the absolute value function on* $\mathbb{C}$.

THM

1. $|x^*| = |x| \qquad \forall x \in \mathbb{C}$
2. $|x|^2 = xx^* \qquad \forall x \in \mathbb{C}$

✎PROOF:

$$
\begin{aligned}
|x^*| &= |(a,b)^*| && \text{by definition of } x\\
&= |(a,-b)| && \text{by Definition 2.21 page 37}\\
&= \sqrt{a^2 + (-b)^2} && \text{by definition of } |x|\\
&= \sqrt{a^2 + b^2}\\
&= |(a,b)| && \text{by definition of } |x|\\
&= |x| && \text{by definition of } x
\end{aligned}
$$

$$
\begin{aligned}
|x|^2 &= |(a,b)|^2 && \text{by definition of } x\\
&= a^2 + b^2 && \text{by Definition 2.20 page 35}
\end{aligned}
$$

[59] Landau (1966) page 107
[60] Rudin (1976) page 14

$$\begin{aligned}
&= (a^2 + b^2, 0) \\
&= (a^2 + b^2, ba - ab) \\
&= (a, b)\,(a, -b) \\
&= (a, b)\,(a, b)^* && \text{by Definition 2.20 page 35} \\
&= xx^* && \text{by definition of } x
\end{aligned}$$

⇨

2.6.6 Hermitian structure

The pair $(\mathbb{C}, *)$ is a special case of a *star-algebra* (*$*$-algebra*). In a star-algebra, the real and imaginary components of an element x are defined as follows:[61]

$$\Re x \triangleq \frac{1}{2}\left(x + x^*\right) \qquad\qquad \Im x \triangleq \frac{1}{2i}\left(x - x^*\right).$$

Theorem 2.29. [62] *Let $x \triangleq (a, b) \in \mathbb{C}$. Then*

THM
$$\begin{array}{rcll}
\Re\,(a,b) &=& a & \forall (a,b)\in\mathbb{C} \\
\Im\,(a,b) &=& b & \forall (a,b)\in\mathbb{C}
\end{array}$$

✎PROOF:

$$\begin{aligned}
\Re\,(a, b) &= \frac{1}{2}\left((a, b) + (a, b)^*\right) && \text{by Definition B.5 page 289} \\
&= \frac{1}{2}((a, b) + (a, -b)) && \text{by Definition 2.21 page 37} \\
&= \frac{1}{2}\,(a + a, b - b) && \text{by Definition 2.18 page 31} \\
&= \frac{1}{2}\,(2a, 0) \\
&= a
\end{aligned}$$

$$\begin{aligned}
\Im\,(a, b) &= \frac{1}{2}\left((a, b) - (a, b)^*\right) && \text{by Definition B.5 page 289} \\
&= \frac{1}{2}((a, b) - (a, -b)) && \text{by Definition 2.21 page 37} \\
&= \frac{1}{2}\,(a - a, b + b) && \text{by Definition 2.18 page 31} \\
&= \frac{1}{2}\,(0, 2b) \\
&= b
\end{aligned}$$

⇨

Theorem 2.30. [63] *Let $|\cdot| : \mathbb{C} \to \mathbb{R}^{\vdash}$ be the absolute value function on $\mathbb{C}$.*

THM
$$\begin{array}{llcll}
1. & |\Re x| &\leq& |x| & \forall x\in\mathbb{C} \\
2. & |\Im x| &\leq& |x| & \forall x\in\mathbb{C}
\end{array}$$

[61] $\Re$ and $\Im$: Definition B.5 (page 289)
[62] Munkres (2000), page 87
[63] Rudin (1976), page 14

✎PROOF:

$$\begin{aligned}
|\Re(x)| &= |\Re\,(a,b)| && \text{by definition of } x \\
&= |a| && \text{by Theorem 2.29 page 40} \\
&= \sqrt{a^2} \\
&\le \sqrt{a^2+b^2} \\
&= |(a,b)| && \text{by definition of } |x| \\
&= |x| && \text{by definition of } x
\end{aligned}$$

$$\begin{aligned}
|\Im(x)| &= |\Im\,(a,b)| && \text{by definition of } x \\
&= |b| && \text{by Theorem 2.29 page 40} \\
&= \sqrt{b^2} \\
&\le \sqrt{a^2+b^2} \\
&= |(a,b)| && \text{by definition of } |x| \\
&= |x| && \text{by definition of } x
\end{aligned}$$

2.7 Literature

Literature survey:

1. Classic/standard texts:
   - Dedekind (1872a)
   - Dedekind (1888a)
   - Landau (1966)
   - Thurston (1956)
   - Birkhoff and MacLane (1996)
2. English translations and reprints of Literature item 1:
   - Dedekind (1872b)
   - Dedekind (1872c)
   - Dedekind (1872d)
   - Dedekind (1888b)
   - Dedekind (1888c)
   - Thurston (2007)

CHAPTER 3

INFORMATION THEORY

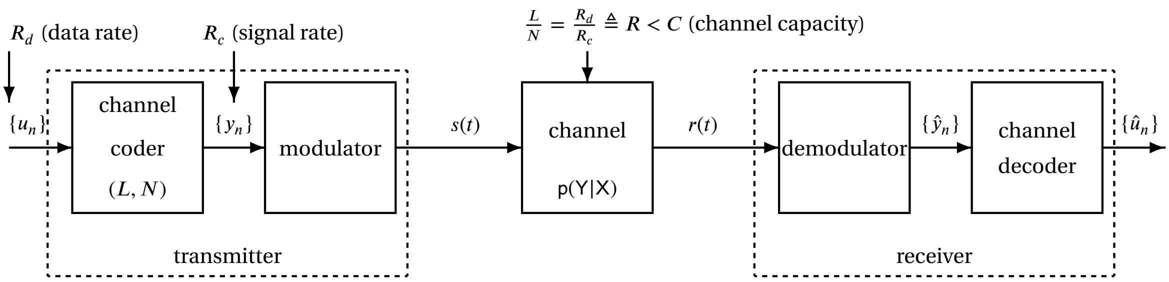


Figure 3.1: Memoryless modulation system model

3.1 Information Theory

3.1.1 Definitions

The *Kullback Leibler distance* $\mathsf{D}(\mathsf{p}_1, \mathsf{p}_2)$ (Definition 3.1 page 43) is a measure between two probability density functions p_1 and p_2. It is not a true distance measure[1] but it behaves in a similar manner. If $\mathsf{p}_1 = \mathsf{p}_2$, then the *KL distance* is 0. If p_1 is very different from p_2, then $|\mathsf{D}(\mathsf{p}_1, \mathsf{p}_2)|$ will be much larger.

Definition 3.1. [2] *Let* p_1 *and* p_2 *be probability density functions. Then the* ***Kullback Leibler distance*** *(the* KL DISTANCE, *also called the* ***relative entropy****) of* p_1 *and* p_2 *is*

DEF $$\mathsf{D}(\mathsf{p}_1, \mathsf{p}_2) \triangleq \mathsf{E} \log_2 \frac{\mathsf{p}_1(\mathsf{X})}{\mathsf{p}_2(\mathsf{X})} \quad \textit{bits}$$

If the base of logarithm is e *(the "natural logarithm") rather than* 2, *then the units are* NATS *rather than* BITS.

The *mutual information* $\mathsf{I}(\mathsf{X}; \mathsf{Y})$ of random variable X and Y is the *KL distance* between their *joint distribution* $\mathsf{p}(\mathsf{X}, \mathsf{Y})$ and the product of their *marginal distributions* $\mathsf{p}(\mathsf{X})$ and $\mathsf{p}(\mathsf{Y})$. If X and Y are independent, then the *KL distance* between joint and marginal product is $\log 1 = 0$ and they have no *mutual information* ($\mathsf{I}(\mathsf{X}; \mathsf{Y}) = 0$). If X and Y are highly correlated, then the *joint distribution* is

[1] *Distance measure*: Definition 12.2 (page 167)

[2] Kullback and Leibler (1951), Csiszar (1961), ichi Amari (2012), Cover and Thomas (1991) page 18

much different than the product of the marginals making the *KL distance* greater and along with it the *mutual information* greater as well.

Definition 3.2 (Mutual information). [3]

DEF $\mathsf{I}(\mathsf{X};\mathsf{Y}) \triangleq \mathsf{D}\left(\mathsf{p}(\mathsf{X},\mathsf{Y}), \mathsf{p}(\mathsf{X})\mathsf{p}(\mathsf{Y})\right) \triangleq \mathsf{E}_{\mathsf{xy}} \log_2 \frac{\mathsf{p}(\mathsf{X},\mathsf{Y})}{\mathsf{p}(\mathsf{X})\mathsf{p}(\mathsf{Y})}$ *bits*

The *self information* $\mathsf{I}(\mathsf{X};\mathsf{X})$ of random variable X is the *mutual information* between X and itself. That is, it is a measure of the information contained in X. Self information $\mathsf{I}(\mathsf{X};\mathsf{X})$ can also be viewed as the *KL distance* between the constant 1 (no information because 1 is completely known) and $\mathsf{p}(\mathsf{X})$.

Definition 3.3 (Self information). [4]

DEF $\mathsf{I}(\mathsf{X};\mathsf{X}) \triangleq \mathsf{D}\left(1, \mathsf{p}(\mathsf{X})\right) \triangleq \mathsf{E}_{\mathsf{x}} \log_2 \frac{1}{\mathsf{p}(\mathsf{X})}$ *bits*

The *entropy* $\mathsf{H}(\mathsf{X})$ of a random variable X is equivalent to the self information $\mathsf{I}(\mathsf{X};\mathsf{X})$ of X. That is, the entropy of X is a measure of the information contained in X.

Likewise, the *conditional entropy* $\mathsf{H}(\mathsf{X}|\mathsf{Y})$ of X given Y is the information contained in X given Y has occurred. If X and Y are independent, then X does not care about the occurrence of Y. Thus in this case, the occurrence of $\mathsf{Y} = y$ does not change the amount of information provided by X and $\mathsf{H}(\mathsf{X}|\mathsf{Y}) = \mathsf{H}(\mathsf{X})$. If X and Y are highly correlated, the occurrence of $\mathsf{Y} = y$ tells us a lot about what the value of X might turn out to be. Thus in this case, the information provided by X given Y is greatly reduced and $\mathsf{H}(\mathsf{X}|\mathsf{Y}) << \mathsf{H}(\mathsf{X})$.

The *joint entropy* $\mathsf{H}(\mathsf{X},\mathsf{Y})$ of X and Y is the amount of information contained in the ordered pair (X,Y).

Definition 3.4 (Entropy). [5]

DEF

entropy of X :	$\mathsf{H}(\mathsf{X})$	$\triangleq$	$\mathsf{E}_{\mathsf{x}} \log_2 \frac{1}{\mathsf{p}(\mathsf{X})}$	*bits*
joint entropy of X,Y :	$\mathsf{H}(\mathsf{X},\mathsf{Y})$	$\triangleq$	$\mathsf{E}_{\mathsf{xy}} \log_2 \frac{1}{\mathsf{p}(\mathsf{X},\mathsf{Y})}$	*bits*
conditional entropy of X *given* Y :	$\mathsf{H}(\mathsf{X}|\mathsf{Y})$	$\triangleq$	$\mathsf{E}_{\mathsf{xy}} \log_2 \frac{1}{\mathsf{p}(\mathsf{X}|\mathsf{Y})}$	*bits*

3.1.2 Relations

Theorem 3.1.

THM $\mathsf{H}(\mathsf{X},\mathsf{Y}) = \mathsf{H}(\mathsf{Y},\mathsf{X})$

✎PROOF:

$$\begin{aligned}
\mathsf{H}(\mathsf{X},\mathsf{Y}) &\triangleq \mathsf{E}_{\mathsf{xy}} \log \frac{1}{\mathsf{p}_{xy}(\mathsf{X},\mathsf{Y})} \\
&= \mathsf{E}_{\mathsf{yx}} \log \frac{1}{\mathsf{p}_{yx}(\mathsf{Y},\mathsf{X})} \\
&\triangleq \mathsf{H}(\mathsf{Y},\mathsf{X})
\end{aligned}$$

[3] Kullback (1959), Cover and Thomas (1991), pages 18–19
[4] Hartley (1928), Fano (1949), Cover and Thomas (1991), pages 18–19
[5] Cover and Thomas (1991), pages 15–17

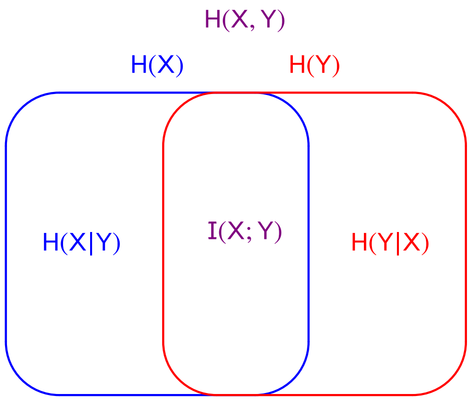


Figure 3.2: Relationship between information and entropy

Theorem 3.2 (Entropy chain rule).

THM

$$\begin{aligned}
\mathsf{H}(X,Y) &= \mathsf{H}(X|Y) + \mathsf{H}(Y) \\
&= \mathsf{H}(Y|X) + \mathsf{H}(X). \\
\mathsf{H}(X_1, X_2, \ldots, X_N) &= \sum_{n=1}^{N-1} \mathsf{H}(X_n | X_{n+1}, \ldots, X_N) + \mathsf{H}(X_N)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{H}(X,Y) &\triangleq \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(X,Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(X|Y)\mathsf{p}(Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(X|Y)} + \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(X|Y)} + \mathsf{E}_{y} \log \frac{1}{\mathsf{p}(Y)} \\
&= \mathsf{H}(X|Y) + \mathsf{H}(Y)
\end{aligned}$$

$$\begin{aligned}
\mathsf{H}(X,Y) &\triangleq \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(X,Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(Y|X)\mathsf{p}(X)} \\
&= \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(Y|X)} + \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(X)} \\
&= \mathsf{E}_{xy} \log \frac{1}{\mathsf{p}(Y|X)} + \mathsf{E}_{y} \log \frac{1}{\mathsf{p}(X)} \\
&= \mathsf{H}(Y|X) + \mathsf{H}(X)
\end{aligned}$$

$$\begin{aligned}
\mathsf{H}(X_1, X_2, \ldots, X_N) &= \mathsf{H}(X_1|X_2, \ldots, X_N) + \mathsf{H}(X_2, \ldots, X_N) \\
&= \mathsf{H}(X_1|X_2, \ldots, X_N) + \mathsf{H}(X_2|X_3, \ldots, X_N) + \mathsf{H}(X_3, \ldots, X_N) \\
&= \mathsf{H}(X_1|X_2, \ldots, X_N) + \mathsf{H}(X_2|X_3, \ldots, X_N) + \mathsf{H}(X_3|X_4, \ldots, X_N) + \mathsf{H}(X_4, \ldots, X_N)
\end{aligned}$$

$$= \sum_{n=1}^{N-1} \mathsf{H}(\mathsf{X}_n|\mathsf{X}_{n+1},\ldots,\mathsf{X}_n) + \mathsf{H}(\mathsf{X}_N)$$

✏

Theorem 3.3.

THM

$$\begin{aligned}
\mathrm{I}(\mathsf{X};\mathsf{Y}) &= \mathsf{H}(\mathsf{X}) - \mathsf{H}(\mathsf{X}|\mathsf{Y})\\
\mathrm{I}(\mathsf{X};\mathsf{Y}) &= \mathsf{H}(\mathsf{Y}) - \mathsf{H}(\mathsf{Y}|\mathsf{X})\\
\mathrm{I}(\mathsf{X};\mathsf{Y}) &= \mathsf{H}(\mathsf{X}) + \mathsf{H}(\mathsf{Y}) - \mathsf{H}(\mathsf{X},\mathsf{Y})\\
\mathrm{I}(\mathsf{X};\mathsf{Y}) &= \mathrm{I}(\mathsf{Y};\mathsf{X})\\
\mathrm{I}(\mathsf{X};\mathsf{X}) &= \mathsf{H}(\mathsf{X})
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathrm{I}(\mathsf{X};\mathsf{Y}) &\triangleq \mathsf{E}_{xy}\log_2\frac{\mathsf{p}(\mathsf{X},\mathsf{Y})}{\mathsf{p}(\mathsf{X})\mathsf{p}(\mathsf{Y})}\\
&= \mathsf{E}_{xy}\log_2\frac{\mathsf{p}(\mathsf{X}|\mathsf{Y})}{\mathsf{p}(\mathsf{X})}\\
&= \mathsf{E}_{xy}\log_2\frac{1}{\mathsf{p}(\mathsf{X})} + \mathsf{E}_{xy}\log_2\mathsf{p}(\mathsf{X}|\mathsf{Y})\\
&= \mathsf{E}_{xy}\log_2\frac{1}{\mathsf{p}(\mathsf{X})} - \mathsf{E}_{xy}\log_2\frac{1}{\mathsf{p}(\mathsf{X}|\mathsf{Y})}\\
&\triangleq \mathsf{H}(\mathsf{X}) - \mathsf{H}(\mathsf{X}|\mathsf{Y})
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(\mathsf{X};\mathsf{Y}) &\triangleq \mathsf{E}_{xy}\log_2\frac{\mathsf{p}(\mathsf{X},\mathsf{Y})}{\mathsf{p}(\mathsf{X})\mathsf{p}(\mathsf{Y})}\\
&= \mathsf{E}_{xy}\log_2\frac{\mathsf{p}(\mathsf{Y}|\mathsf{X})}{\mathsf{p}(\mathsf{Y})}\\
&= \mathsf{E}_{xy}\log_2\frac{1}{\mathsf{p}(\mathsf{Y})} + \mathsf{E}_{xy}\log_2\mathsf{p}(\mathsf{Y}|\mathsf{X})\\
&= \mathsf{E}_{xy}\log_2\frac{1}{\mathsf{p}(\mathsf{Y})} - \mathsf{E}_{xy}\log_2\frac{1}{\mathsf{p}(\mathsf{Y}|\mathsf{X})}\\
&\triangleq \mathsf{H}(\mathsf{Y}) - \mathsf{H}(\mathsf{Y}|\mathsf{X})
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(\mathsf{X};\mathsf{Y}) &= \mathsf{H}(\mathsf{Y}) - \mathsf{H}(\mathsf{Y}|\mathsf{X})\\
&= \mathrm{I}(\mathsf{Y};\mathsf{X})
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(\mathsf{X};\mathsf{X}) &\triangleq \mathsf{E}_{xy}\log_2\frac{\mathsf{p}(\mathsf{X},\mathsf{X})}{\mathsf{p}(\mathsf{X})\mathsf{p}(\mathsf{X})}\\
&= \mathsf{E}_{xy}\log_2\frac{\mathsf{p}(\mathsf{X})}{\mathsf{p}(\mathsf{X})\mathsf{p}(\mathsf{X})}\\
&= \mathsf{E}_{xy}\log_2\frac{1}{\mathsf{p}(\mathsf{X})}\\
&\triangleq \mathsf{H}(\mathsf{X})
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(\mathsf{X};\mathsf{Y}) &\triangleq \mathsf{H}(\mathsf{X}) - \mathsf{H}(\mathsf{X}|\mathsf{Y})\\
&= \mathsf{H}(\mathsf{X}) - [\mathsf{H}(\mathsf{X},\mathsf{Y}) - \mathsf{H}(\mathsf{Y})]\\
&= \mathsf{H}(\mathsf{X}) + \mathsf{H}(\mathsf{Y}) - \mathsf{H}(\mathsf{X},\mathsf{Y})
\end{aligned}$$

✏

Theorem 3.4 (Information chain rule).

THM
$$\mathrm{I}(X_1, X_2, \ldots, X_N; Y) = \sum_{n=1}^{N-1} \mathrm{I}(X_n | X_{n+1}, \ldots, X_N) + \mathrm{I}(X_N)$$

✎PROOF:

$$\begin{aligned}
\mathrm{I}(X_1, X_2, \ldots, X_N; Y) &= \mathrm{H}(X_1, X_2, \ldots, X_N) - \mathrm{H}(X_1, X_2, \ldots, X_N | Y) \\
&= \sum_{n=1}^{N-1} \mathrm{H}(X_n | X_{n+1}, \ldots, X_N) + \mathrm{H}(X_N) - \sum_{n=1}^{N-1} \mathrm{H}(X_n | X_{n+1}, \ldots, X_N, Y) - \mathrm{H}(X_N | Y) \\
&= \sum_{n=1}^{N-1} \left[\mathrm{H}(X_n | X_{n+1}, \ldots, X_N) - \mathrm{H}(X_n | X_{n+1}, \ldots, X_N, Y)\right] + \left[\mathrm{H}(X_N) - \mathrm{H}(X_N | Y)\right] \\
&= \sum_{n=1}^{N-1} \mathrm{I}(X_n | X_{n+1}, \ldots, X_N) + \mathrm{I}(X_N)
\end{aligned}$$

⇨

3.1.3 Properties

Theorem 3.5. [6]

THM
$$\begin{aligned}
\mathrm{D}(p_1, p_2) &\geq 0 \\
\mathrm{I}(X; Y) &\geq 0
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathrm{D}(p_1, p_2) &\triangleq \mathrm{E}_x \log \frac{p_1(X)}{p_2(X)} \\
&= \mathrm{E}_x \left[-\log \frac{p_2(X)}{p_1(X)}\right] \\
&\geq -\log \mathrm{E}_x \left[\frac{p_2(X)}{p_1(X)}\right] && \text{by *Jensen's Inequality* (Theorem ?? page ??)} \\
&= -\log \int_x p_1(x) \frac{p_2(x)}{p_1(x)}\, dx \\
&= -\log \int_x p_2(x)\, dx \\
&= -\log(1) \\
&= 0
\end{aligned}$$

⇨

[6] Cover and Thomas (1991), page 26

3.2 Channel Capacity

Definition 3.5. *Let* (L, N) *be a block coder with* N *output bits for each* L *input bits.*

$$
\begin{array}{rcll}
R & \triangleq & \frac{L}{N} & \textit{coding rate} \\
\mathsf{C} & \triangleq & \max \mathsf{I}(\mathsf{X};\mathsf{Y}) & \textit{channel capacity} \\
\mathsf{E}(R) & \triangleq & \max_\rho \max_Q [\mathsf{E}_0(\rho, Q) - \rho R] & \textit{random coding exponent}
\end{array}
$$

Theorem 3.6 (noisy channel coding theorem). [7]

THM

If $R < \mathsf{C}$ *then it is possible to construct an encoder and decoder such that the probability of error* P_e *is arbitrarily small. Specifically*

$$P_e \le e^{-N\mathsf{E}(R)}$$

For $0 \le R \ge \mathsf{C}$*, the function* $\mathsf{E}(R)$ *is* POSITIVE, DECREASING, *and* CONVEX.

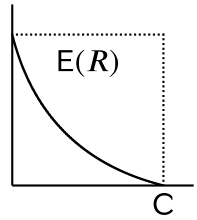


Figure 3.3: Typical $\mathsf{E}(R)$

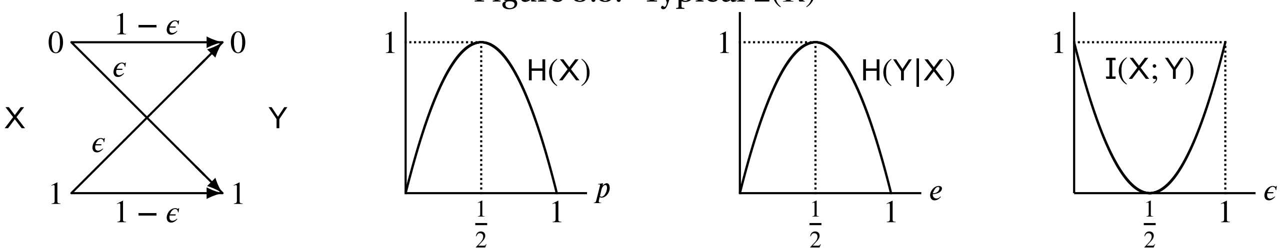


Figure 3.4: Binary symmetric channel (BSC)

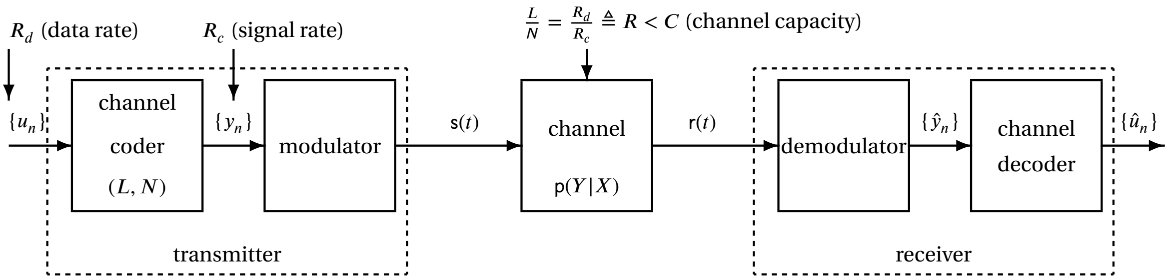


Figure 3.5: Memoryless modulation system model

How much information can be reliably sent through the channel? The answer depends on the *channel capacity* C. As proven by the *Noisy Channel Coding Theorem* (NCCT), each transmitted symbol can carry up to C bits for any arbitrarily small probability of error greater than zero. The price for decreasing error is increasing the block code size.

Note that the NCCT does not say at what rate (in bits/second) you can send data through the AWGN channel. The AWGN channel knows nothing of time (and is therefore not a realistic channel). The NCCT channel merely gives a *coding rate*. That is, the number of information bits each symbol can carry. Channels that limit the rate (in bits/second) that can be sent through it are obviously aware of time and are often referred to as *bandlimited channels*.

[7] Gallager (1968), page 143

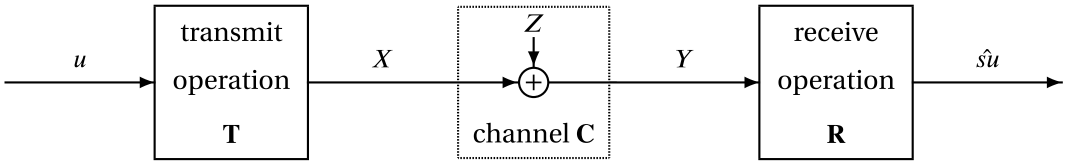


Figure 3.6: Additive noise system model

Theorem 3.7. *Let* $Z \sim \mathsf{N}\left(0, \sigma^2\right)$. *Then*

THM
$$\mathsf{H}(Z) = \frac{1}{2}\log_2 2\pi e\sigma^2$$

✎PROOF:

$$\begin{aligned}
\mathsf{H}(Z) &= \mathsf{E}_z \log \frac{1}{\mathsf{p}(Z)} \\
&= -\mathsf{E}_z \log \mathsf{p}(z) \\
&= -\mathsf{E}_z \log\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-z^2}{2\sigma^2}}\right] \\
&= -\mathsf{E}_z\left[-\frac{1}{2}\log(2\pi\sigma^2) + \frac{-z^2}{2\sigma^2}\log e\right] \\
&= \frac{1}{2}\mathsf{E}_z\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2} z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}\mathsf{E}_z z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}(\sigma^2 + 0)\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \log e\right] \\
&= \frac{1}{2}\log(2\pi e\sigma^2)
\end{aligned}$$

✏

Theorem 3.8. *Let* $Y = X + Z$ *be a Gaussian channel with* $\mathsf{E}X^2 = P$ *and* $Z \sim \mathsf{N}\left(0, \sigma^2\right)$. *Then*

THM
$$\mathrm{I}(X;Y) \le \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right) = \mathsf{C}$$

✎PROOF: No proof at this time.

Reference: (Cover and Thomas, 1991, page 241) ✏

Example 3.1. 1. If there is no transmitted energy ($P = 0$), then the capacity of the channel to pass information is

$$\begin{aligned}
\mathsf{C} &= \frac{1}{2}\log_2\left(1 + \frac{P}{\sigma^2}\right) \\
&= \frac{1}{2}\log_2\left(1 + \frac{0}{\sigma^2}\right) \\
&= 0
\end{aligned}$$

That is, the symbols cannot carry any information.

2. If there is finite symbol energy and no noise ($\sigma^2 = 0$), then the capacity of the channel to pass information is

$$\begin{aligned} \mathsf{C} &= \frac{1}{2}\log_2\left(1+\frac{P}{0}\right) \\ &= \infty \end{aligned}$$

That is, each symbol can carry an infinite amount of information. That is, we can use a modulation scheme with an infinite number of of signaling waveforms (analog modulation) and thus each symbol can be represented by one of an infinite number of waveforms.

3. If the transmitted energy is ($P = 15\sigma^2$), then the capacity of the channel to pass information is

$$\begin{aligned} \mathsf{C} &= \frac{1}{2}\log_2\left(1+\frac{15\sigma^2}{\sigma^2}\right) \\ &= \frac{1}{2}\log_2(1+15) \\ &= \frac{1}{2}4 \\ &= 2 \end{aligned}$$

This means

$$2 = \mathsf{C} > \mathsf{R} \triangleq \frac{\text{information bits}}{\text{symbol}} = \frac{\text{information bits}}{\text{coded bits}} \times \frac{\text{coded bits}}{\text{symbol}} = \mathsf{r_c r_s}$$

This means that if the coding rate is $\mathsf{r_c} = 1/4$, then we must use a modulation with 256 ($\mathsf{r_s} = 8$ bits/symbol) or fewer waveforms.

Conversely, if the modulation scheme uses 4 waveforms, then $\mathsf{r_s} = 2$ bits/symbol and so the code rate $\mathsf{r_c}$ can be up to 1 (almost no coding redundancy is needed).

4. If there is the transmitted energy ($P = \sigma^2$), then the capacity of the channel to pass information is

$$\begin{aligned} \mathsf{C} &= \frac{1}{2}\log_2\left(1+\frac{\sigma^2}{\sigma^2}\right) \\ &= \frac{1}{2}\log_2(1+1) \\ &= \frac{1}{2} \end{aligned}$$

That is, each symbol can carry just under 1/2 bits of information. This means

$$\frac{1}{2} = \mathsf{C} > \mathsf{R} \triangleq \frac{\text{information bits}}{\text{symbol}} = \frac{\text{information bits}}{\text{coded bits}} \times \frac{\text{coded bits}}{\text{symbol}} = \mathsf{r_c r_s}$$

This means that if the coding rate is $\mathsf{r_c} = 1/4$, then we must use a modulation with 4 ($\mathsf{r_s} = 2$ bits/symbol) or fewer waveforms.

Conversely, if the modulation scheme uses 16 waveforms, then $\mathsf{r_s} = 4$ bits/symbol and so the code rate $\mathsf{r_c}$ must be less than 1/8.

3.3 Specific channels

3.3.1 Binary Symmetric Channel (BSC)

The properties of the *binary symmetric channel (BSC)* are illustrated in Figure 3.4 (page 48) and stated in Theorem 17.8 (next).

Theorem 3.9 (Binary symmetric channel). *Let* $\mathbf{C} : \mathsf{X} \to \mathsf{Y}$ *be a channel operation with* $\mathsf{X}, \mathsf{Y} \in \{0, 1\}$ *and*

$$\begin{aligned} p &\triangleq \mathsf{P}\{\mathsf{X}=1\} \\ \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\} &= \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\} \triangleq \epsilon \end{aligned}$$

Then

THM

$$\begin{aligned} \mathsf{P}\{\mathsf{Y}=1\} &= \epsilon + p - 2\epsilon p \\ \mathsf{P}\{\mathsf{Y}=0\} &= 1 - p - \epsilon + 2\epsilon p \\ \mathsf{H}(\mathsf{X}) &= p\log_2\tfrac{1}{p} + (1-p)\log_2\tfrac{1}{(1-p)} \\ \mathsf{H}(\mathsf{Y}) &= (1-p-\epsilon+2\epsilon p)\log_2\tfrac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p)\log_2\tfrac{1}{\epsilon+p-2\epsilon p} \\ \mathsf{H}(\mathsf{Y}|\mathsf{X}) &= (1-\epsilon)\log_2\tfrac{1}{1-\epsilon} + \epsilon\log_2\tfrac{1}{\epsilon} \\ \mathsf{I}(\mathsf{X};\mathsf{Y}) &= (1-p-\epsilon+2\epsilon p)\log_2\tfrac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p)\log_2\tfrac{1}{\epsilon+p-2\epsilon p} \\ &\quad -(1-\epsilon)\log_2\tfrac{1}{1-\epsilon} + -\epsilon\log_2\tfrac{1}{\epsilon} \\ \mathsf{C} &= 1 + \epsilon\log_2\epsilon + (1-\epsilon)\log_2(1-\epsilon) \end{aligned}$$

✎PROOF:

$$\begin{aligned} \mathsf{P}\{\mathsf{X}=1\} &\triangleq p \\ \mathsf{P}\{\mathsf{X}=0\} &= 1-p \\ \mathsf{P}\{\mathsf{Y}=1\} &= \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\}\mathsf{P}\{\mathsf{X}=0\} + \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=1\}\mathsf{P}\{\mathsf{X}=1\} \\ &= \epsilon(1-p) + (1-\epsilon)p \\ &= \epsilon - \epsilon p + p - \epsilon p \\ &= \epsilon + p - 2\epsilon p \\ \mathsf{P}\{\mathsf{Y}=0\} &= \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=0\}\mathsf{P}\{\mathsf{X}=0\} + \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\}\mathsf{P}\{\mathsf{X}=1\} \\ &= (1-\epsilon)(1-p) + \epsilon p \\ &= 1 - p - \epsilon + \epsilon p + \epsilon p \\ &= 1 - p - \epsilon + 2\epsilon p \end{aligned}$$

$$\begin{aligned} \mathsf{H}(\mathsf{X}) &\triangleq \mathsf{E}_\mathsf{x}\log_2\frac{1}{\mathsf{p}(\mathsf{X})} \\ &= \sum_{n=0}^{1}\mathsf{P}\{\mathsf{X}=n\}\log_2\frac{1}{\mathsf{P}\{\mathsf{X}=n\}} \\ &= \mathsf{P}\{\mathsf{X}=0\}\log_2\frac{1}{\mathsf{P}\{\mathsf{X}=0\}} + \mathsf{P}\{\mathsf{X}=1\}\log_2\frac{1}{\mathsf{P}\{\mathsf{X}=1\}} \\ &= p\log_2\frac{1}{p} + (1-p)\log_2\frac{1}{(1-p)} \end{aligned}$$

$$\mathsf{H}(\mathsf{Y}) \triangleq \mathsf{E}_\mathsf{y}\log_2\frac{1}{\mathsf{p}(\mathsf{Y})}$$

$$
\begin{aligned}
&= \sum_{n=0}^{1} \mathsf{P}\{\mathsf{Y}=n\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=n\}} \\
&= \mathsf{P}\{\mathsf{Y}=0\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=0\}} + \mathsf{P}\{\mathsf{Y}=1\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=1\}} \\
&= (1-p-\epsilon+2\epsilon p)\log_2 \frac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p)\log_2 \frac{1}{\epsilon+p-2\epsilon p}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{H}(\mathsf{Y}|\mathsf{X}) &\triangleq \mathsf{E}_{xy}\log_2 \frac{1}{\mathsf{p}(\mathsf{Y}|\mathsf{X})} \\
&= \sum_{m=0}^{1}\sum_{n=0}^{1} \mathsf{P}\{\mathsf{X}=m,\mathsf{Y}=n\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=n|\mathsf{X}=m\}} \\
&= \sum_{m=0}^{1}\sum_{n=0}^{1} \mathsf{P}\{\mathsf{Y}=n|\mathsf{X}=m\}\,\mathsf{P}\{\mathsf{X}=m\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=n|\mathsf{X}=m\}} \\
&= \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=0\}\,\mathsf{P}\{\mathsf{X}=0\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=0\}} + \\
&\quad\ \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\}\,\mathsf{P}\{\mathsf{X}=1\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\}} + \\
&\quad\ \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\}\,\mathsf{P}\{\mathsf{X}=0\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\}} + \\
&\quad\ \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=1\}\,\mathsf{P}\{\mathsf{X}=1\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=1\}} \\
&= (1-\epsilon)(1-p)\log_2 \frac{1}{1-\epsilon} + \epsilon p\log_2 \frac{1}{\epsilon} + \epsilon(1-p)\log_2 \frac{1}{\epsilon} + (1-\epsilon)p\log_2 \frac{1}{1-\epsilon} \\
&= (1-p-\epsilon+\epsilon p+p-\epsilon p)\log_2 \frac{1}{1-\epsilon} + (\epsilon p+\epsilon-\epsilon p)\log_2 \frac{1}{\epsilon} \\
&= (1-\epsilon)\log_2 \frac{1}{1-\epsilon} + \epsilon\log_2 \frac{1}{\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{I}(\mathsf{X};\mathsf{Y}) &= \mathsf{H}(\mathsf{Y}) - \mathsf{H}(\mathsf{Y}|\mathsf{X}) \\
&= (1-p-\epsilon+2\epsilon p)\log_2 \frac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p)\log_2 \frac{1}{\epsilon+p-2\epsilon p} - (1-\epsilon)\log_2 \frac{1}{1-\epsilon} + -\epsilon\log_2 \frac{1}{\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{C} &\triangleq \max_p \mathrm{I}(\mathsf{X};\mathsf{Y}) \\
&= \mathrm{I}(\mathsf{X};\mathsf{Y})|_{p=\frac{1}{2}} \\
&= \frac{1}{2}\log_2 \frac{1}{\frac{1}{2}} + \frac{1}{2}\log_2 \frac{1}{\frac{1}{2}} - (1-\epsilon)\log_2 \frac{1}{1-\epsilon} + -\epsilon\log_2 \frac{1}{\epsilon} \\
&= 1 + \epsilon\log_2\epsilon + (1-\epsilon)\log_2(1-\epsilon)
\end{aligned}
$$

✏

Remark 3.1.

REM

When $\epsilon = 0$	(noiseless channel), the channel capacity is 1 bit (maximum capacity).
When $\epsilon = 1$	(inverting channel), the channel capacity is still 1 bit.
When $\epsilon = 1/2$	(totally random channel), the channel capacity is 0.
When $p = 1$ (1	is always transmitted), the entropy of X is 0.
When $p = 0$ (0	is always transmitted), the entropy of X is 0.
When $p = 1/2$	(totally random transmission), the entropy of X is 1 bit (maximum entropy).

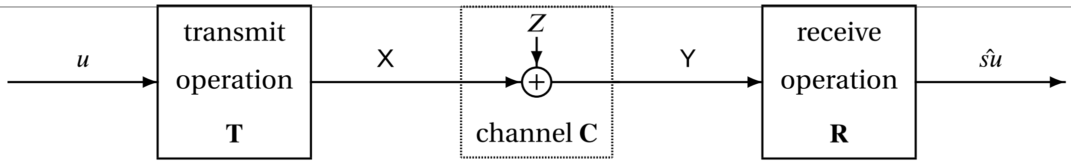


Figure 3.7: Additive noise system model

3.3.2 Gaussian Noise Channel

Theorem 3.10. *Let* $Z \sim \mathsf{N}(0, \sigma^2)$. *Then*

THM
$$\mathsf{H}(Z) = \frac{1}{2}\log_2 2\pi e \sigma^2$$

✎PROOF:

$$\begin{aligned}
\mathsf{H}(Z) &= \mathsf{E}_z \log \frac{1}{\mathsf{p}(Z)} \\
&= -\mathsf{E}_z \log \mathsf{p}(z) \\
&= -\mathsf{E}_z \log\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-z^2}{2\sigma^2}}\right] \\
&= -\mathsf{E}_z \left[-\frac{1}{2}\log(2\pi\sigma^2) + \frac{-z^2}{2\sigma^2}\log e\right] \\
&= \frac{1}{2}\mathsf{E}_z \left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2} z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}\mathsf{E}_z z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}(\sigma^2 + 0)\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \log e\right] \\
&= \frac{1}{2}\log(2\pi e \sigma^2)
\end{aligned}$$

✏

Theorem 3.11. [8] *Let* $\mathsf{Y} = \mathsf{X} + Z$ *be a Gaussian channel with* $\mathsf{E}\mathsf{X}^2 = P$ *and* $Z \sim \mathsf{N}(0, \sigma^2)$. *Then*

THM
$$\mathsf{I}(\mathsf{X}; \mathsf{Y}) \le \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right) = \mathsf{C} \qquad \textit{bits per usage}$$

Theorem 3.12. [9] *Let* $\mathsf{Y} = \mathsf{X} + Z$ *be a bandlimited Gaussian channel with* $\mathsf{E}\mathsf{X}^2 = P$ *and* $Z \sim \mathsf{N}(0, \sigma^2)$ *and bandwidth* W. *Then*

THM
$$\mathsf{C} = W \log\left(1 + \frac{P}{\sigma^2 W}\right) \qquad \textit{bits per second}$$

[8] Cover and Thomas (1991), page 241
[9] Cover and Thomas (1991), page 250

Part II

Order Structures

CHAPTER 4

ORDER

Equivalence relations (Definition 1.9 page 7) require *symmetry* ($x \doteq y \iff y \doteq x$). However another very important type of relation, the *order relation*, actually requires *anti-symmetry*. This chapter presents some useful structures regarding order relations. Ordering relations on a set allow us to *compare* some pairs of elements in a set and determine whether or not one element is *less than* another. In this case, we say that those two elements are *comparable*; otherwise, they are *incomparable*. A set together with an order relation is called an *ordered set*, a *partially ordered set*, or a *poset* (Definition 4.2 page 58).

4.1 Preordered sets

Definition 4.1. [1] *Let X be a set.*

DEF

*A relation $\sqsubseteq$ is a **preorder relation** on X if*

1. $x \sqsubseteq x$ $\quad\forall x \in X\quad$ (REFLEXIVE) *and*
2. $x \sqsubseteq y$ *and* $y \sqsubseteq z \implies x \sqsubseteq z$ $\quad\forall x,y,z \in X\quad$ (TRANSITIVE)

*A **preordered set** is the pair* $(X, \sqsubseteq)$.

Example 4.1. [2]

EX

$\sqsubseteq$ is a *preorder relation* on the set of *positive integers* $\mathbb{N}$ if

$$n \sqsubseteq m \iff (p \text{ is a prime factor of } n \implies p \text{ is a prime factor of } m)$$

[1] Schröder (2003) page 115, Brown and Watson (1991), page 317
[2] Shen and Vereshchagin (2002) page 43

4.2 Order relations

Definition 4.2. [3] *Let X be a set. Let 2^{XX} be the set of all relations on X.*

DEF

*A relation $\le$ is an **order relation** in 2^{XX} if*

1. $x \le x$ $\forall x \in X$ (REFLEXIVE)   *and*   ⎤ *preorder*
2. $x \le y$ *and* $y \le z \implies x \le z$ $\forall x,y,z \in X$ (TRANSITIVE)   *and*   ⎦
3. $x \le y$ *and* $y \le x \implies x = y$ $\forall x,y \in X$ (ANTI-SYMMETRIC)

*An **ordered set** is the pair $(X, \le)$. The set X is called the **base set** of $(X, \le)$. If $x \le y$ or $y \le x$, then elements x and y are said to be **comparable**, denoted $x \sim y$. Otherwise they are **incomparable**, denoted $x \| y$. The relation $\lneq$ is the relation $\le \setminus =$ ("less than but not equal to"), where $\setminus$ is the* SET DIFFERENCE *operator, and $=$ is the equality relation. An order relation is also called a **partial order relation**. An ordered set is also called a **partially ordered set** or **poset**.*

The familiar relations $\ge$, $<$, and $>$ (next) can be defined in terms of the order relation $\le$ (Definition 4.2—previous).

Definition 4.3. [4] *Let $(X, \le)$ be an ordered set.*

DEF

The relations $\ge$, $<$, $> \in 2^{XX}$ are defined as follows:

$$
\begin{array}{rclll}
x \ge y & \overset{\text{def}}{\iff} & y \le x & & \forall x,y \in X \\
x \lneq y & \overset{\text{def}}{\iff} & x \le y \text{ and } x \ne y & & \forall x,y \in X \\
x \gneq y & \overset{\text{def}}{\iff} & x \ge y \text{ and } x \ne y & & \forall x,y \in X
\end{array}
$$

*The relation $\ge$ is called the **dual** of $\le$.*

Theorem 4.1. [5] *Let X be a set.*

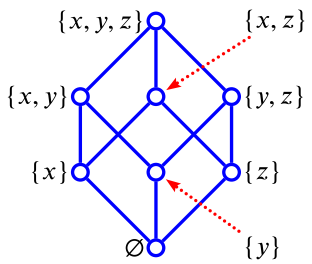
THM

$$(X, \le) \text{ is an ordered set} \iff (X, \ge) \text{ is an ordered set}$$

Example 4.2.

EX

	order relation		dual order relation
$\le$	(integer less than or equal to)	$\ge$	(integer greater than or equal to)
$\subseteq$	(subset)	$\supseteq$	(super set)
$\mid$	(divides)		(divided by)
$\implies$	(implies)	$\impliedby$	(implied by)

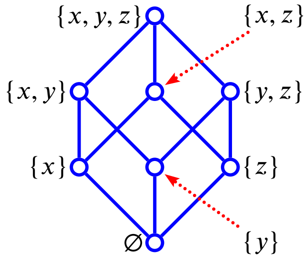


Example 4.3. The Hasse diagram to the left illustrates the ordered set $\left(2^{\{x,y,z\}}, \subseteq\right)$ and the Hasse diagram to the right illustrates its dual $\left(2^{\{x,y,z\}}, \supseteq\right)$.

[3] MacLane and Birkhoff (1999) page 470, Beran (1985) page 1, Korselt (1894) page 156 ⟨I, II, (1)⟩, Dedekind (1900) page 373 ⟨I–III⟩

[4] Peirce (1880b) page 2

[5] Grätzer (1998), page 3



4.3 Linearly ordered sets

In an ordered set we can say that some element is less than or equal to some other element. That is, we can say that these two elements are *comparable*—we can *compare* them to see which one is lesser or equal to the other. But it is very possible that there are two elements that are not comparable, or *incomparable*. That is, we cannot say that one element is less than the other—it is simply not possible to compare them because their ordered pair is not an element of the order relation.

For example, in the ordered set $(2^{\{x,y,z\}}, \subseteq)$ of Example 4.9, we can say that $\{x\} \subseteq \{x,z\}$ (we can compare these two sets with respect to the order relation $\subseteq$), but we cannot say $\{y\} \subseteq \{x,z\}$, nor can we say $\{x,z\} \subseteq \{y\}$. Rather, these two elements $\{y\}$ and $\{x,z\}$ are simply *incomparable*.

However, there are some ordered sets in which every element is comparable with every other element; and in this special case we say that this ordered set is a *totally ordered* set or is *linearly ordered* (next definition).

Definition 4.4. [6]

DEF
*A relation $\le$ is a **linear order relation** on X if*
1. $\le$ *is an* ORDER RELATION *(Definition 4.2 page 58) and*
2. $x \le y$ or $y \le x \quad \forall x,y \in X$ (COMPARABLE).

*A **linearly ordered set** is the pair $(X, \le)$.*
*A linearly ordered set is also called a **totally ordered set**, a **fully ordered set**, and a **chain**.*

Definition 4.5 (poset product). [7]

DEF
*The **product** $P \times Q$ of ordered pairs $P \triangleq (X, \precsim)$ and $Q \triangleq (Y, \trianglelefteq)$ is the ordered pair $(X \times Y, \le)$ where*

$$(x_1, y_1) \le (x_2, y_2) \quad \stackrel{\text{def}}{\iff} \quad x_1 \precsim x_2 \text{ and } y_1 \trianglelefteq y_2 \qquad \forall x_1,x_2 \in X;\ y_1,y_2 \in Y$$

4.4 Representation

Definition 4.6. [8]

DEF
*y **covers** x in the ordered set $(X, \le)$ if*
1. $x \le y$ *(y is greater than x)* and
2. $(x \le z \le y) \implies (z = x \text{ or } z = y)$ *(there is no element between x and y).*

The case in which y covers x is denoted
$x \prec y$.

[6] MacLane and Birkhoff (1999) page 470, Ore (1935) page 410
[7] Birkhoff (1948) page 7, MacLane and Birkhoff (1967), page 489
[8] Birkhoff (1933a) page 445

Example 4.4. Let $(\{x, y, z\}, \leq)$ be an ordered set with cover relation $\prec$.

EX

$$\{x < y < z\} \implies \left\{\begin{array}{lll} y & \text{covers} & x \\ z & \text{covers} & y \\ z & \text{does **not** cover} & x \end{array}\right\}$$

An ordered set can be represented in four ways:

1. Hasse diagram
2. tables
3. set of ordered pairs of order relations
4. set of ordered pairs of cover relations

Definition 4.7. *Let $(X, \leq)$ be an ordered pair.*

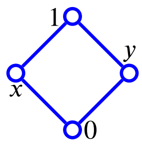
DEF

*A diagram is a **Hasse diagram** of $(X, \leq)$ if it satisfies the following criteria:*

- *Each element in X is represented by a dot or small circle.*
- *For each $x, y \in X$, if $x \prec y$, then y appears at a higher position than x and a line connects x and y.*

Example 4.5. Here are three ways of representing the ordered set $\left(2^{\{x,y\}}, \subseteq\right)$;

1. **Hasse diagrams**: If two elements are comparable, then the lesser of the two is drawn lower on the page than the other with a line connecting them.

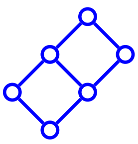


2. Sets of ordered pairs specifying *order relations* (Definition 4.2 page 58):

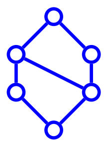
$$\subseteq = \left\{\begin{array}{cccc} (\varnothing, \varnothing), & (\{x\}, \{x\}), & (\{y\}, \{y\}), & (\{x,y\}, \{x,y\}), \\ (\varnothing, \{x\}), & (\varnothing, \{y\}), & (\varnothing, \{x,y\}), & (\{x\}, \{x,y\}), (\{y\}, \{x,y\}) \end{array}\right\}$$

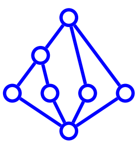
3. Sets of ordered pairs specifying *covering relations*:

$$\prec = \left\{ (\varnothing, \{x\}), \ (\varnothing, \{y\}), \ (\{x\}, \{x,y\}), (\{y\}, \{x,y\}) \right\}$$

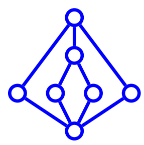


Example 4.6. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.

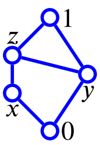




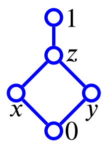
Example 4.7. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.



Example 4.8. The Hasse diagrams to the left and right represent *equivalent* ordered sets.



In particular, the line extending from 1 to y in the diagram to the left is redundant because other lines already indicate that $z \leq 1$ and $y \leq z$; and thus by the *transitive* property (Definition 4.2 page 58), these two relations imply $1 \leq y$. A more concise explanation is that both have the same convering relation:

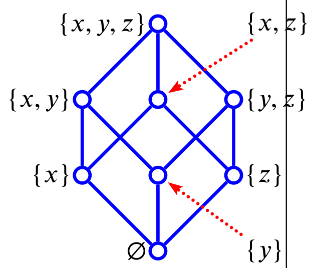


$$\prec = \{(z, 1), (x, z), (0, x), (y, z), (0, y)\}$$

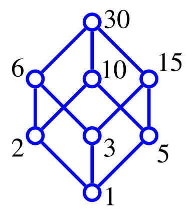
4.5 Examples

Examples of order relations include the following:

- set inclusion order relation: Example 4.9 page 61
- integer divides order relation: Example 4.10 page 61
- linear operator order relation: Example 4.11 page 61
- projection operator order relation: Example 4.12 page 61
- integer order relation: Example 4.13 page 62
- metric order relation: Example 4.14 page 62
- coordinatewise order relation Example 4.15 page 62
- lexicographical order relation Example 4.16 page 62



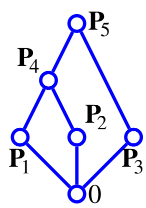
Example 4.9 (Set inclusion order relation). [9]Let X be a set, 2^X the power set of X, and $\subseteq$ the set inclusion relation. Then, $\subseteq$ is an *order relation* on the set 2^X and the pair $(2^X, \subseteq)$ is an ordered set. The ordered set $(2^{\{x,y,z\}}, \subseteq)$ is illustrated to the right by its *Hasse diagram*.



Example 4.10 (Integer divides order relation). [10]Let $|$ be the "divides" relation on the set $\mathbb{N}$ of positive integers such that $n|m$ represents m divides n. Then $|$ is an *order relation* on $\mathbb{N}$ and the pair $(\mathbb{N}, |)$ is an *ordered set*. The ordered set $(\{n \in \mathbb{N} | n|2 \text{ or } n|3 \text{ or } n|5\}, |)$ is illustrated by a *Hasse diagram* to the right.

Example 4.11 (Operator order relation). [11] Let $\boldsymbol{X}$ be an inner-product space. We can define the order relation $\lessapprox$ on the linear operators $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3 \ldots \in X^X$ as follows:

$$\mathbf{L}_1 \lessapprox \mathbf{L}_2 \quad \overset{\text{def}}{\iff} \quad \langle \mathbf{L}_2 \boldsymbol{x} - \mathbf{L}_1 \boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X}$$



Example 4.12 (Projection operator order relation). [12]Let $(\boldsymbol{V}_n)$ be a sequence of subspaces in a Hilbert space $\boldsymbol{X}$. We can define a projection operator $\mathbf{P}_n$ for every subspace $\boldsymbol{V}_n \subseteq \boldsymbol{X}$ in a subspace lattice such that

$$\boldsymbol{V}_n = \mathbf{P}_n \boldsymbol{X} \qquad \forall n \in \mathbb{Z}.$$

Each projection operator $\mathbf{P}_n$ in the lattice "projects" the range space $\boldsymbol{X}$ onto a subspace $\boldsymbol{V}_n$. We can define an order relation on the projection operators as follows:

$$\mathbf{P}_1 \leq \mathbf{P}_2 \quad \overset{\text{def}}{\iff} \quad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1$$

[9] Menini and Oystaeyen (2004) pages 56–57
[10] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩
[11] Michel and Herget (1993) page 429, Pedersen (2000) page 87
[12] Isham (1999) pages 21–22, Dunford and Schwartz (1957), page 481, Svozil (1994) page 72

Example 4.13 (Integer order relation). Let $\le$ be the standard order relation on the set of integers $\mathbb{Z}$. Then the ordered pair $(\mathbb{Z}, \le)$ is a totally ordered set. The totally ordered set $(\{1,2,3,4\}, \le)$ is illustrated to the right. Other familiar examples of totally ordered sets include the pair $(\mathbb{Q}, \le)$ (where $\mathbb{Q}$ is the set of rational numbers) and $(\mathbb{R}, \le)$ (where $\mathbb{R}$ is the set of real numbers).

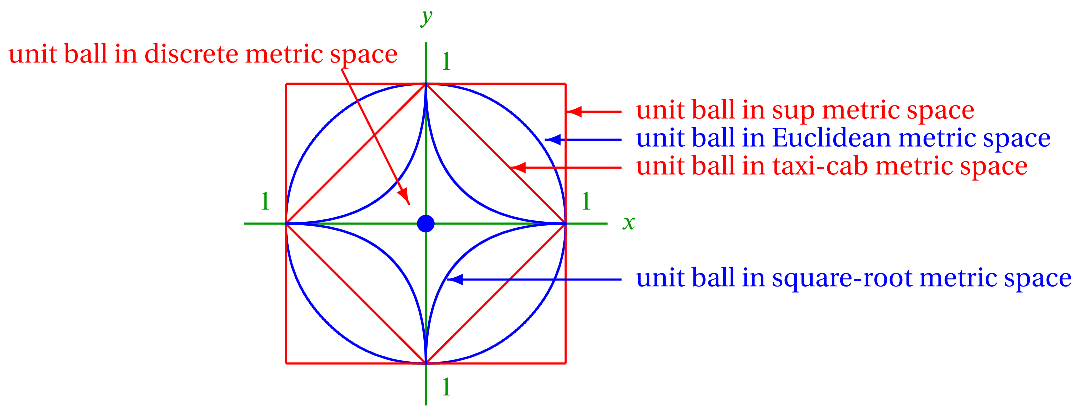
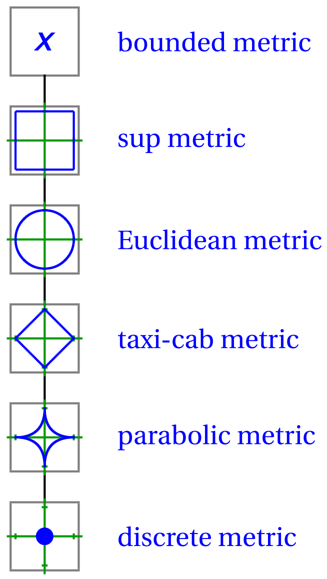


Figure 4.1: Balls on the set $\mathbb{R}^2$ using different metrics



Example 4.14 (Metric order relation). [13]Let d_n be a metric on the set X and B_n be the unit ball centered at "0" in the metric space (X, d_n). Define an order relation $\le$ on the set of metric spaces $\{(X, \mathsf{d}_n) \mid n = 1, 2, \ldots\}$ such that

$$(X, \mathsf{d}_n) \le (X, \mathsf{d}_m) \qquad \Longleftrightarrow \qquad \mathsf{B}_n \subseteq \mathsf{B}_m.$$

The the tuple $(\{(X, \mathsf{d}_n) \mid n = 1, 2, \ldots\}, \le)$ is an ordered set. The ordered set of several common metric spaces is a *totally ordered* set, as illustrated to the right and with associated unit balls illustrated in Figure 4.1 (page 62).

Example 4.15 (Coordinatewise order relation). [14] Let $(X, \le)$ be an ordered set. Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

EX

The **coordinatewise order relation** $\precsim$ on the Cartesian product X^n is defined for all $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \precsim \boldsymbol{y} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \{x_1 \le y_1 \text{ and } x_2 \le y_2 \text{ and } \ldots \text{ and } x_n \le y_n\}$$

Example 4.16 (Lexicographical order relation). [15] Let $(X, \le)$ be an ordered set. Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

[13] Michel and Herget (1993) page 354, Giles (1987) page 29
[14] Shen and Vereshchagin (2002) page 43
[15] Shen and Vereshchagin (2002) page 44, Halmos (1960) page 58, Hausdorff (1937) page 54

EX

The **lexicographical order relation** $\precsim$ on the Cartesian product X^n is defined forall $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \precsim \boldsymbol{y} \overset{\text{def}}{\iff} \left\{ \begin{array}{lllll} (\, x_1 < y_1 & & &) & \text{or} \\ (\, x_2 < y_2 & \text{and} & x_1 = y_1 &) & \text{or} \\ (\, x_3 < y_3 & \text{and} & (x_1, x_2) = (y_1, y_2) &) & \text{or} \\ (\, \ldots & \ldots & \ldots &) & \text{or} \\ (\, x_{n-1} < y_{n-1} & \text{and} & (x_1, x_2, \ldots, x_{n-2}) = (y_1, y_2, \ldots, y_{n-2}) &) & \text{or} \\ (\, x_n \leq y_n & \text{and} & (x_1, x_2, \ldots, x_{n-1}) = (y_1, y_2, \ldots, y_{n-1}) &) & \end{array} \right\}$$

The lexicographical order relation is also called the **dictionary order relation** or **alphabetic order relation**.

Definition 4.8.

DEF

An ordered set is ***labeled*** *if the labels on the elements are significant.*
An ordered set is ***unlabeled*** *if the labels on the elements are not significant.*

Proposition 4.1. [16] *Let X_n be a finite set with order $n = |X_n|$. Let P_n be the number of labeled ordered sets on X_n and p_n the number of unlabeled ordered sets.*

PRP

n	0	1	2	3	4	5	6	7	8	9
P_n	1	1	3	19	219	4231	130,023	6,129,859	431,723,379	44,511,042,511
p_n	1	1	2	5	16	63	318	2045	16,999	183,231

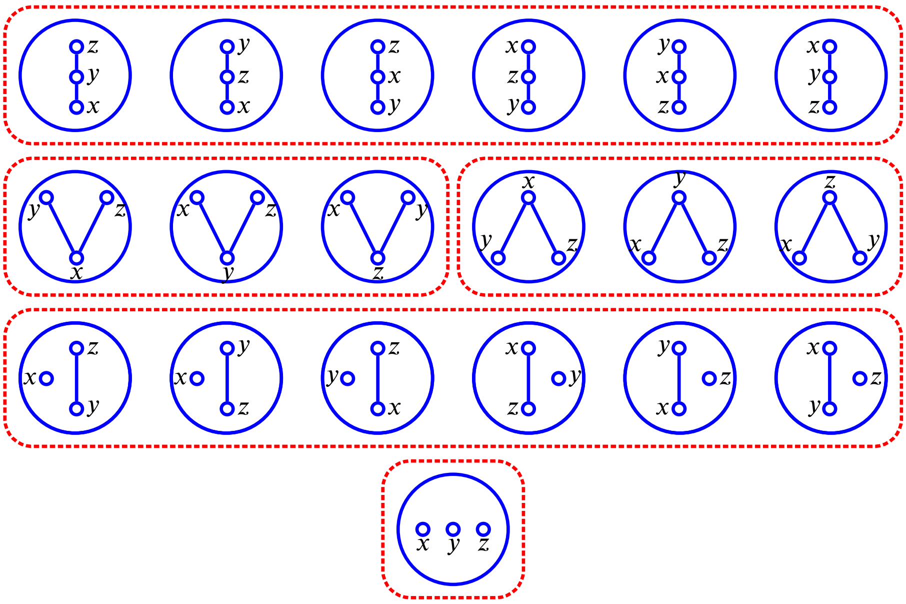


Figure 4.2: All possible orderings of the set $\{x, y, z\}$ (Example 4.17 page 63).

Example 4.17. Proposition 4.1 (page 63) indicates that there are exactly 19 labeled order relations on the set $\{x, y, z\}$ and 5 unlabeled order relations.

EX

The 19 labeled order relations on $\{x, y, z,\}$ are represented here using three methods:

1. Hasse diagrams: Figure 4.2 page 63
2. order relations: Table 4.2 page 64
3. covering relations: Table 4.3 page 64

In each of these three methods, the 19 *labeled* order relations are arranged into 5 groups, each group representing one of the 5 *unlabeled* order relations.

[16] Sloane (2014) ⟨http://oeis.org/A001035⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Comtet (1974) page 60, Brinkmann and McKay (2002)

labeled order relations on $\{x, y, z\}$						
$\leq_1$	$=$	$\{$	$(x,x), (y,y), (z,z)$			$\}$
$\leq_2$	$=$	$\{$	$(x,x), (y,y), (z,z),$	(y,z)		$\}$
$\leq_3$	$=$	$\{$	$(x,x), (y,y), (z,z),$	(z,y)		$\}$
$\leq_4$	$=$	$\{$	$(x,x), (y,y), (z,z),$	(x,z)		$\}$
$\leq_5$	$=$	$\{$	$(x,x), (y,y), (z,z),$	(z,x)		$\}$
$\leq_6$	$=$	$\{$	$(x,x), (y,y), (z,z),$	(x,y)		$\}$
$\leq_7$	$=$	$\{$	$(x,x), (y,y), (z,z),$	(y,x)		$\}$
$\leq_8$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(x,y), (x,z)$		$\}$
$\leq_9$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(x,y), (y,z)$		$\}$
$\leq_{10}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(z,x), (z,y)$		$\}$
$\leq_{11}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(y,x), (z,x)$		$\}$
$\leq_{12}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(x,y), (z,y)$		$\}$
$\leq_{13}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(x,z), (y,z)$		$\}$
$\leq_{14}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(x,y), (y,z),$	(x,z)	$\}$
$\leq_{15}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(x,z), (x,y),$	(z,y)	$\}$
$\leq_{16}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(y,x), (y,z),$	(x,z)	$\}$
$\leq_{17}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(y,z), (y,x),$	(z,x)	$\}$
$\leq_{18}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(z,x), (z,y),$	(x,y)	$\}$
$\leq_{19}$	$=$	$\{$	$(x,x), (y,y), (z,z),$	$(z,y), (z,x),$	(y,x)	$\}$

Table 4.2: labeled order relations on $\{x, y, z\}$

labeled cover relations on $\{x, y, z\}$		
$\prec_1 = \varnothing$		$\prec_{11} = \{(y,x), (z,x)\}$
$\prec_2 = \{(y,z)\}$		$\prec_{12} = \{(x,y), (z,y)\}$
$\prec_3 = \{(z,y)\}$		$\prec_{13} = \{(x,z), (y,z)\}$
$\prec_4 = \{(x,z)\}$		$\prec_{14} = \{(x,y), (y,z)\}$
$\prec_5 = \{(z,x)\}$		$\prec_{15} = \{(x,z), (x,y)\}$
$\prec_6 = \{(x,y)\}$		$\prec_{16} = \{(y,x), (y,z)\}$
$\prec_7 = \{(y,x)\}$		$\prec_{17} = \{(y,z), (y,x)\}$
$\prec_8 = \{(x,y), (x,z)\}$		$\prec_{18} = \{(z,x), (z,y)\}$
$\prec_9 = \{(x,y), (y,z)\}$		$\prec_{19} = \{(z,y), (z,x)\}$
$\prec_{10} = \{(z,x), (z,y)\}$		

Table 4.3: labeled cover relations on $\{x, y, z\}$

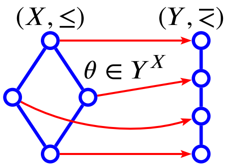
4.6 Functions on ordered sets

Definition 4.9. [17] *Let $(X, \leq)$ and $(Y, \precsim)$ be ordered sets.*

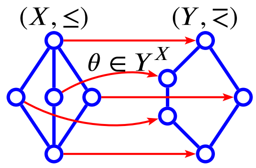
DEF *A function $\theta \in Y^X$ is **order preserving** with respect to $\leq$ and $\precsim$ if*

$$x \leq y \quad \Longrightarrow \quad \theta(x) \precsim \theta(y) \qquad \forall x, y \in X.$$

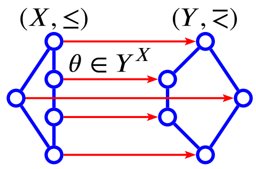
Example 4.18. [18] In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} is *not* order preserving. This example also illustrates the fact that that order preserving does not imply *isomorphic*.



Example 4.19. In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} is *not* order preserving. Like Example 4.18 (page 65), this example also illustrates the fact that that order preserving does not imply *isomorphic*.



Example 4.20. In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} *is also* order preserving. In this case, θ is an *isomorphism* and the ordered sets $(X, \leq)$ and $(Y, \precsim)$ are *isomorphic*.



Example 4.21. [19]

EX The function $\mathsf{f}(x) \triangleq \dfrac{x}{1 - x^2}$ in $\mathbb{R}^{(-1:1)}$ is *bijective* and *order preserving*.

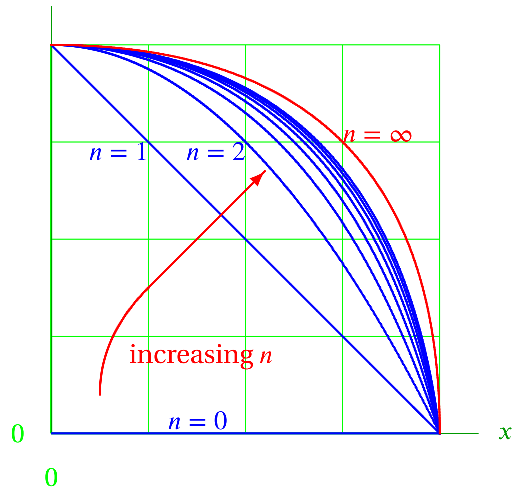
Theorem 4.2 (Pointwise ordering relation). [20] *Let X be a set, $(Y, \leq)$ an ordered set, and $\mathsf{f}, \mathsf{g} \in Y^X$.*

THM

$$\mathsf{f}(x) \leq \mathsf{g}(x) \forall x \in X \quad \Longrightarrow \quad \left(Y^X, \precsim\right) \text{ is an ordered set.}$$

In this case we say f is "dominated by" g in X, or we say g "dominates" f in X.

Example 4.22 (Pointwise ordering relation). [21] Let $\mathsf{f} \precsim \mathsf{g}$ represent that $\mathsf{f}(x) \leq \mathsf{g}(x)$ for all $0 \leq x \leq 1$ (we say f is "dominated by" g in the region $[0, 1]$, or we say g "dominates" f in the region $[0, 1]$). The pair $(\{\mathsf{f}_n(x) = 1 - x^n | n \in \mathbb{N}\}, \precsim)$ is a totally ordered set.



[17] Burris and Sankappanavar (2000), page 10
[18] Burris and Sankappanavar (2000), page 10
[19] Munkres (2000) page 25 ⟨Example 1§3.9⟩
[20] Shen and Vereshchagin (2002), page 43, Giles (2000), page 252
[21] Shen and Vereshchagin (2002), page 43, Giles (2000), page 252, Aliprantis and Burkinshaw (2006) page 2

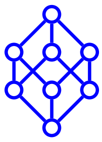
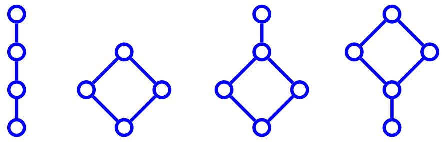
4.7 Decomposition

4.7.1 Subposets

Definition 4.10. [22]

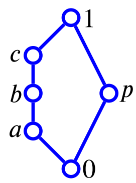
DEF *The tupple $(Y, \lessapprox)$ is a **subposet** of the ordered set $(X, \le)$ if*
1. $Y \subseteq X$ $\quad$ *(Y is a subset of X)* $\quad$ *and*
2. $\lessapprox = \le \cap Y^2$ $\quad$ *($\lessapprox$ is the relation $\le$ restricted to $Y \times Y$)*

Example 4.23.

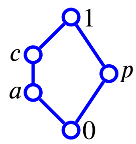
Subposets of  include 

Example 4.24. Let

$$\begin{aligned}(X, \le) \triangleq \Big(\{0, a, b, c, p, 1\}, \quad & \{ (0,0), (a,a), (b,b), (c,c), (p,p), (1,1), \\ & (0,a), (0,b), (0,c), (0,p), (0,1), \\ & (a,b), (a,c), (a,1), (p,1), \\ & (b,c), (b,1), (c,1), (p,1) \}\Big)\end{aligned}$$



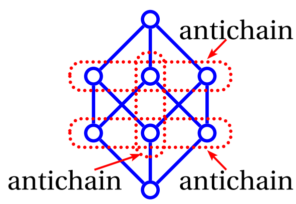
$$\begin{aligned}(Y, \lessapprox) \triangleq \Big(\{0, a, c, p, 1\}, \quad & \{ (0,0), (a,a), (c,c), (p,p), (1,1), \\ & (0,a), (0,c), (0,p), (0,1), \\ & (a,c), (a,1), (p,1), (c,1), (p,1) \}\Big).\end{aligned}$$



Then $(Y, \lessapprox)$ is a subposet of $(X, \le)$ because $Y \subseteq X$ and $\lessapprox = (\le \cap Y^2)$.

A *chain* is an ordered set in which every pair of elements is *comparable* (Definition 4.4 page 59). An *antichain* is just the opposite—it is an ordered set in which *no* pair of elements is comparable (next definition).

Definition 4.11. [23]



DEF *The subposet $(A, \le)$ in the ordered set $(X, \le)$ is an **antichain** if*
$\quad a \| b \qquad \forall a, b \in A$
(all elements in A are INCOMPARABLE*).*

Definition 4.12. [24]

DEF
- *The **length** of a chain $(C, \le)$ equals $|C| - 1$.*
- *The **length** of a poset $(X, \le)$ is the length of the longest chain in the ordered set.*
- *The **width** of a poset $(X, \le)$ is number of elements in the largest antichain in the ordered set.*

Theorem 4.3 (Dilworth's theorem). [25] *Let $(X, \le)$ be an ordered set with width n.*

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[22] Grätzer (2003) page 2
[23] Grätzer (2003) page 2
[24] Grätzer (2003) page 2, Birkhoff (1967) page 5
[25] Dilworth (1950a) page 161, Dilworth (1950b), Farley (1997) page 4

THM $\left\{\begin{array}{l}\text{WIDTH } n \text{ of } (X,\le)\\ \text{is FINITE}\end{array}\right\} \implies \left\{\begin{array}{ll}1. & \text{there exists a PARTITION of } (X,\le) \text{ into } n \text{ chains} \quad \text{and}\\ 2. & \text{there does not exist any PARTITION}\\ & \text{of } (X,\le) \text{ into less than } n \text{ chains}\end{array}\right\}$

4.7.2 Operations on posets

Definition 4.13. [26] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **direct sum** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} + \boldsymbol{Q} \triangleq (X \cup Y, \le)$$

where $x \le y$ if

1. $x, y \in X$ and $x \precsim y$ or
2. $x, y \in Y$ and $x \trianglelefteq y$

*The direct sum operation is also called the **disjoint union**. The notation $n\boldsymbol{P}$ is defined as*

$$n\boldsymbol{P} \triangleq \underbrace{\boldsymbol{P} + \boldsymbol{P} + \cdots + \boldsymbol{P}}_{n-1 \text{ “+” operations}}.$$

Definition 4.14. [27] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **direct product** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} \times \boldsymbol{Q} \triangleq (X \times Y, \le)$$

where $(x_1, y_1) \le (x_2, y_2)$ if $x_1 \precsim x_2$ and $y_1 \precsim y_2$.
*The direct product operation is also called the **cartesian product**. The order relation $\le$ is called a **coordinate wise order relation**. The notation $\boldsymbol{P}^n$ is defined as*

$$\boldsymbol{P}^n \triangleq \underbrace{\boldsymbol{P} \times \boldsymbol{P} \times \cdots \times \boldsymbol{P}}_{n-1 \text{ “}\times\text{” operations}}.$$

Definition 4.15. [28] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **ordinal sum** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} \oplus \boldsymbol{Q} \triangleq (X \cup Y, \le)$$

where $x \le y$ if

1. $x, y \in X$ and $x \precsim y$ or
2. $x, y \in Y$ and $x \trianglelefteq y$ or
3. $x \in X$ and $y \in Y$.

Definition 4.16. [29] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **ordinal product** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*

$$\boldsymbol{P} \otimes \boldsymbol{Q} \triangleq (X \times Y, \le)$$

where $(x_1, y_1) \le (x_2, y_2)$ if

1. $x_1 \ne x_2$ and $x_1 \precsim x_2$ or
2. $x_1 = x_2$ and $y_1 \trianglelefteq y_2$

*The order relation $\le$ is called a **lexicographical** order relation, **dictionary order relation**, or **alphabetic order relation**.*

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[26] Stanley (1997) page 100
[27] Stanley (1997) pages 100–101, Shen and Vereshchagin (2002) page 43
[28] Stanley (1997) page 100
[29] Stanley (1997) page 101, Shen and Vereshchagin (2002) page 44, Halmos (1960) page 58, Hausdorff (1937) page 54

Definition 4.17. [30] *Let $\mathbf{P} \triangleq (X, \leq)$ be an ordered set. Let $\geq$ be the dual order relation of $\leq$.*

DEF: *The **dual** of $\mathbf{P}$ is defined as*
$$\mathbf{P}^* \triangleq (X, \geq)$$

Definition 4.18. [31] *Let X and Y be disjoint sets. Let $\mathbf{P} \triangleq (X, \precsim)$ and $\mathbf{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF: *The **ordinal product** of $\mathbf{P}$ and $\mathbf{Q}$ is defined as*
$$\mathbf{Q}^{\mathbf{P}} \triangleq \left(\left\{ \mathsf{f} \in Y^X | \mathsf{f} \text{ is ORDER PRESERVING} \right\}, \leq \right)$$
where $\mathsf{f} \leq \mathsf{g}$ iff $\mathsf{f}(x) \leq \mathsf{g}(x) \quad \forall x \in X$.
*The order relation $\leq$ is called a **pointwise order relation*** (Example 4.22 page 65).

Theorem 4.4 (cardinal arithmetic). [32] *Let $\mathbf{P} \triangleq (X, \leq)$ be an ordered set.*

THM:

1.	$\mathbf{P} + \mathbf{Q}$	$=$	$\mathbf{Q} + \mathbf{P}$	*commutative*
2.	$\mathbf{P} \times \mathbf{Q}$	$=$	$\mathbf{Q} \times \mathbf{P}$	*commutative*
3.	$(\mathbf{P} + \mathbf{Q}) + (\mathbb{R}, \leq)$	$=$	$\mathbf{P} + (\mathbf{Q} + (\mathbb{R}, \leq))$	*associative*
4.	$(\mathbf{P} \times \mathbf{Q}) \times (\mathbb{R}, \leq)$	$=$	$\mathbf{P} \times (\mathbf{Q} \times (\mathbb{R}, \leq))$	*associative*
5.	$\mathbf{P} \times (\mathbf{Q} + (\mathbb{R}, \leq))$	$=$	$(\mathbf{P} \times \mathbf{Q}) + (\mathbf{P} \times (\mathbb{R}, \leq))$	*distributive*
6.	$(\mathbb{R}, \leq)^{\mathbf{P}+\mathbf{Q}}$	$=$	$(\mathbb{R}, \leq)^{\mathbf{P}} \times (\mathbb{R}, \leq)^{\mathbf{Q}}$	
7.	$\left(\mathbf{P}^{\mathbf{Q}}\right)^{(\mathbb{R}, \leq)}$	$=$	$\mathbf{P}^{\mathbf{Q} \times (\mathbb{R}, \leq)}$	

4.7.3 Primitive subposets

Definition 4.19.

DEF: *The ordered set $\mathbf{L}_1$ is defined as $(\{x\}, \leq)$, for some value x.*

The $\mathbf{L}_1$ ordered set is illustrated by the Hasse diagram to the right.

Definition 4.20.

DEF: *The ordered set $\mathbb{2}$ is defined as $\mathbb{2} \triangleq \mathbb{1}^2$.*

The $\mathbb{2}$ ordered set is illustrated by the Hasse diagram to the right.

4.7.4 Decomposition examples

Example 4.25. Figure 4.3 (page 69) illustrates the four ordered set operations $+$, $\times$, $\oplus$, and $\otimes$.

Example 4.26. [33] The ordered set $n\mathbb{1}$ is the *anti-chain* with n elements. The ordered set $4\mathbb{1}$ is illustrated to the right.

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[30] Stanley (1997) page 101
[31] Stanley (1997) page 101
[32] Stanley (1997) page 102
[33] Stanley (1997) page 100

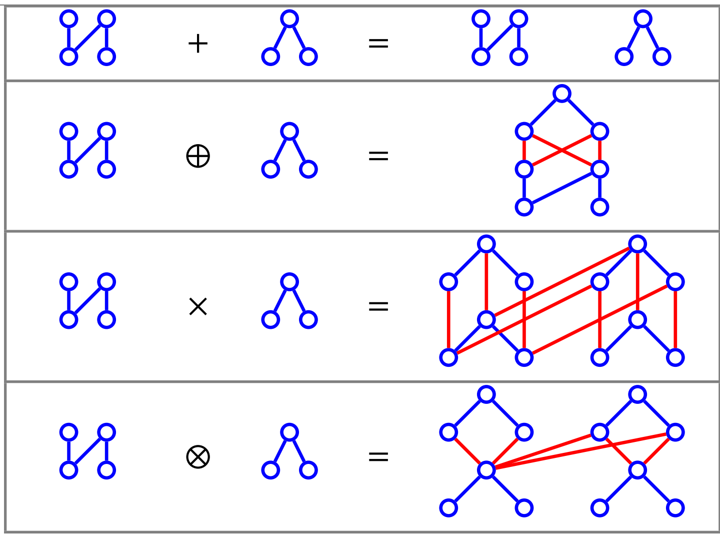
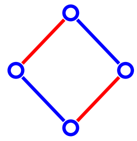


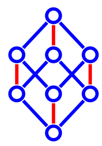
Figure 4.3: Operations on ordered sets (Example 4.25 page 68)

Example 4.27. The ordered set $\mathbb{1}^n$ is the *chain* with n elements. The ordered set $\mathbb{1}^4$ is illustrated to the right.

Example 4.28. The ordered set $\mathbb{2}^2$ is the 4 element *Boolean algebra* illustrated to the right.

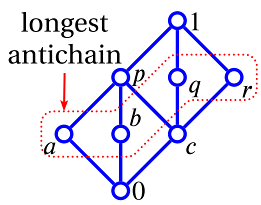


Example 4.29. The ordered set $\mathbb{2}^3$ is the 8 element *Boolean algebra* illustrated to the right.



Example 4.30. [34]The longest *antichain* (Definition 4.11 page 66) in the figure to the right has 4 elements giving this ordered set a *width* (Definition 4.12 page 66) of 4. The longest chain also has 4 elements, giving the ordered set a *length* (Definition 4.12 page 66) of 3. By *Dilworth's theorem* (Theorem 4.3 page 66), the smallest *partition* consists of four *chains* (Definition 4.4 page 59). One such *partition* is

$$\{\{0, a, p, 1\}, \{b\}, \{c, q\}, \{r\}\}.$$



4.8 Bounds on ordered sets

In an *ordered set* (Definition 4.2 page 58), a pair of elements $\{x, y\}$ may not be *comparable*. Despite this, we may still be able to find elements that *are* comparable to both x and y and are "*greater*" than both of them. Such a greater element is called an *upper bound* of x and y. There may be many elements that are upper bounds of x and y. But if one of these upper bounds is comparable with and is smaller than all the other upper bounds, than this "smallest" of the "greater" elements is called the *least upper bound* (*lub*) of x and y, and is denoted $x \vee y$ (Definition 4.21 page 70). Likewise, we may also be

[34] Farley (1997) page 4

able to find elements that are comparable to $\{x, y\}$ and are "*lesser*" than both of them. Such a lesser element is called a *lower bound* of x and y. If one of these lower bounds is comparable with and is larger than all the other lower bounds, than this "largest" of the "lesser" elements is called the *greatest lower bound* (*glb*) of $\{x, y\}$ and is denoted $x \wedge y$ (Definition 4.22 page 70). If every pair of elements in an ordered set has both a least upper bound and a greatest lower bound in the ordered set, then that ordered set is a *lattice* (Definition 5.3 page 73).

Definition 4.21. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

*For any set $A \in 2^X$, c is an **upper bound** of A in $(X, \leq)$ if*

1. $x \leq c \quad \forall x \in A$.

*An element b is the **least upper bound**, or **lub**, of A in $(X, \leq)$ if*

2. b *and* c *are* UPPER BOUNDS *of* $A \implies b \leq c$.

*The least upper bound of the set A is denoted $\bigvee A$. It is also called the **supremum** of A, which is denoted $\sup A$. The **join** $x \vee y$ of x and y is defined as $x \vee y \triangleq \bigvee \{x, y\}$.*

Definition 4.22. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

*For any set $A \in 2^X$, p is a **lower bound** of A in $(X, \leq)$ if*

1. $p \leq x \quad \forall x \in A$.

*An element a is the **greatest lower bound**, or **glb**, of A in $(X, \leq)$ if*

2. a *and* p *are* LOWER BOUNDS *of* $A \implies p \leq a$.

*The greatest lower bound of the set A is denoted $\bigwedge A$. It is also called the **infimum** of A, which is denoted $\inf A$. The **meet** $x \wedge y$ of x and y is defined as $x \wedge y \triangleq \bigwedge \{x, y\}$.*

Definition 4.23 (least upper bound property). [35] *Let X be a set. Let $\sup A$ be the supremum (least upper bound) of a set A.*

DEF

*A set X satisfies the **least upper bound property** if*

$$\left.\begin{array}{lll} 1. & A \subseteq X & \textit{and} \\ 2. & A \neq \varnothing & \textit{and} \\ 3. & \exists b \in X \quad \text{such that} \quad \forall a \in A,\ a \leq b & (A \text{ is bounded above in } X) \end{array}\right\} \implies \exists \sup A \in X$$

*A set X that satisfies the least upper bound property is also said to be **complete**.*

Proposition 4.2. *Let $(X, \vee, \wedge; \leq)$ be an* ORDERED SET *(Definition 4.2 page 58).*

PRP

$$x \leq y \iff \left\{\begin{array}{llll} 1. & x \wedge y = x & \textit{and} \\ 2. & x \vee y = y & \end{array}\right\} \quad \forall x,y \in X$$

Proposition 4.3. *Let 2^X be the* POWER SET *of a set X.*

PRP

$$A \subseteq B \implies \left\{\begin{array}{ll} 1. & \bigvee A \leq \bigvee B \quad \textit{and} \\ 2. & \bigwedge A \leq \bigwedge B \end{array}\right\} \quad \forall A,B \in 2^X$$

[35] Pugh (2002) page 13, Rudin (1976) page 4

CHAPTER 5

LATTICES

5.1 Semi-lattices

Definition 4.21 (page 70) defined the least upper bound $\vee$ of pairs of elements in terms of an ordering relation $\le$. However, the converse development is also possible— we can first define a binary operation $\circledvee$ with a handful of "least upper bound like properties", and then define an ordering relation $\lessapprox$ in terms of $\circledvee$ (Definition 5.1 page 71). In fact, Theorem 5.1 (page 71) shows that under Definition 5.1, $(X, \lessapprox)$ *is* a partially ordered set and $\circledvee$ is a least upper bound on that ordered set.

The same development is performed with regards to a greatest lower bound $\circledwedge$ with the result that $(X, \lessapprox)$ *is* a partially ordered set and $\circledwedge$ is a greatest lower bound on that ordered set (Theorem 5.2 page 72).

Definition 5.1. [1] *Let* $\circledvee, \lessapprox : X^2 \to X$ *be binary operators on a set* X.

DEF *The algebraic structure* $(X, \lessapprox, \circledvee)$ *is a* ***join semilattice*** *if*

1. $x \circledvee x = x$	$\forall x \in X$	(IDEMPOTENT)	and
2. $x \circledvee y = y \circledvee x$	$\forall x, y \in X$	(COMMUTATIVE)	and
3. $(x \circledvee y) \circledvee z = x \circledvee (y \circledvee z)$	$\forall x, y, z \in X$	(ASSOCIATIVE).	

Definition 5.2. [2] *Let* $\circledwedge, \lessapprox : X^2 \to X$ *be binary operators on a set* X.

DEF *The algebraic structure* $(X, \lessapprox, \circledwedge)$ *is a* ***meet semilattice*** *if*

1. $x \circledwedge x = x$	$\forall x \in X$	(IDEMPOTENT)	and
2. $x \circledwedge y = y \circledwedge x$	$\forall x, y \in X$	(COMMUTATIVE)	and
3. $(x \circledwedge y) \circledwedge z = x \circledwedge (y \circledwedge z)$	$\forall x, y, z \in X$	(ASSOCIATIVE).	

Theorem 5.1. [3] *Let* $\circledvee, \lessapprox : X^2 \to X$ *be binary operators over a set* X.

THM $\left\{ \begin{array}{l} (X, \lessapprox, \circledvee) \text{ is a} \\ \text{JOIN SEMILATTICE} \end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & (X, \lessapprox) \text{ is a PARTIALLY ORDERED SET} & \text{and} \\ 2. & x \circledvee y \text{ is a LEAST UPPER BOUND of } x \text{ and } y & \forall x, y \in X. \end{array} \right\}$

✎PROOF: In order for $(X, \le)$ to be an ordered set, $\le$ must be, according to Definition 4.2 (page 58), *reflexive*, *antisymmetric*, and *transitive*;

[1] MacLane and Birkhoff (1999) page 475, Birkhoff (1967) page 22
[2] MacLane and Birkhoff (1999) page 475
[3] MacLane and Birkhoff (1999) page 475

Proof that $\le$ is reflexive:

$$\begin{aligned}
& x = x \ovee x && \text{by idempotent hypothesis}\\
\iff\ & x \le x && \text{by definition of } \le\\
\implies\ & \le \text{ is reflexive}
\end{aligned}$$

Proof that $\le$ is antisymmetric:

$$\begin{aligned}
x \le y \text{ and } y \le x &\iff x \ovee y = y \text{ and } y \ovee x = x && \text{by definition of } \le\\
&\implies x \ovee y = y \text{ and } x \ovee y = x && \text{by commutative hypothesis}\\
&\implies x = y\\
&\implies \le \text{ is antisymmetric}
\end{aligned}$$

Proof that $\le$ is transitive:

$$\begin{aligned}
x \le y \text{ and } y \le z &\iff x \ovee y = y \text{ and } y \ovee z = z && \text{by definition of } \le\\
&\implies (x \ovee y) \ovee z = z\\
&\iff x \ovee (y \ovee z) = z && \text{by associative hypothesis}\\
&\implies x \ovee z = z\\
&\iff x \le z\\
&\iff \le \text{ is transitive}
\end{aligned}$$

Proof that $x \ovee y$ is a lub of x and y:

$$\begin{aligned}
x \ovee y = y &\iff x \le y && \text{by definition of } \le\\
&\iff x \vee y = y && \text{by definition of } \vee\\
&\implies x \ovee y = x \vee y\\
&\implies x \ovee y \text{ is the lub of } x \text{ and } y
\end{aligned}$$

Theorem 5.2. [4] *Let* $\owedge, \eqslantless \colon X^2 \to X$ *be binary operators over a set* X.

THM $\left\{ (X, \eqslantless, \owedge) \text{ is a MEET SEMILATTICE} \right\} \implies \left\{ \begin{array}{lll} 1. & (X, \eqslantless) \text{ is a PARTIALLY ORDERED SET} & \text{and}\\ 2. & x \owedge y \text{ is a GREATEST LOWER BOUND of } x \text{ and } y & \forall x, y \in X. \end{array} \right\}$

✎PROOF: In order for $(X, \le)$ to be an ordered set, $\le$ must be, according to Definition 4.2 (page 58), *reflexive*, *antisymmetric*, and *transitive*;

Proof that $\le$ is reflexive:

$$\begin{aligned}
& x = x \owedge x && \text{by idempotent hypothesis}\\
\iff\ & x \le x && \text{by definition of } \le\\
\implies\ & \le \text{ is reflexive}
\end{aligned}$$

Proof that $\le$ is antisymmetric:

$$\begin{aligned}
x \le y \text{ and } y \le x &\iff x \owedge y = x \text{ and } y \owedge x = y && \text{by definition of } \le\\
&\implies x \owedge y = x \text{ and } x \owedge y = y && \text{by commutative hypothesis}\\
&\implies x = y\\
&\implies \le \text{ is antisymmetric}
\end{aligned}$$

[4] MacLane and Birkhoff (1999) page 475

Proof that $\leq$ is transitive:

$$\begin{aligned}
x \leq y \text{ and } y \leq z &\iff x \oslash y = x \text{ and } y \oslash z = y && \text{by definition of } \leq\\
&\implies x \oslash (y \oslash z) = x\\
&\iff (x \oslash y) \oslash z = x && \text{by associative hypothesis}\\
&\implies x \oslash z = x\\
&\iff x \leq z\\
&\iff \leq \text{ is transitive}
\end{aligned}$$

Proof that $x \oslash y$ is a glb of x and y:

$$\begin{aligned}
x \oslash y = x &\iff x \leq y && \text{by definition of } \leq\\
&\iff x \wedge y = x && \text{by definition of } \wedge\\
&\implies x \oslash y = x \wedge y\\
&\implies x \oslash y \text{ is the glb of } x \text{ and } y
\end{aligned}$$

5.2 Lattices

An *ordered set* is a set (Definition 1.1 page 4) together with the additional structure of an ordering relation (Definition 4.2 page 58). However, this amount of structure tends to be insufficient to ensure "well-behaved" mathematical systems. This situation is greatly remedied if every pair of elements in an ordered set (partially or linearly ordered) has both a *least upper bound* and a *greatest lower bound* (Definition 4.22 page 70) in the ordered set; in this case, that ordered set is a *lattice* (next definition). Gian-Carlo Rota (1932–1999) illustrates the advantage of lattices over simple ordered sets by pointing out that the *ordered set* of partitions of an integer "is fraught with pathological properties", while the *lattice* of partitions of a set "remains to this day rich in pleasant surprises".[5] Further examples of lattices follow in Section 5.3 (page 78).

Definition 5.3. [6]

DEF

An algebraic structure $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *is a **lattice** if*

1. $(X, \leq)$ *is an ordered set* *and*
2. $x, y \in X \implies x \vee y \in X$ *and*
3. $x, y \in X \implies x \wedge y \in X$

The algebraic structure $\mathbf{L}^* \triangleq (X, \varovee, \owedge; \geq)$ *is the **dual** lattice of* $\mathbf{L}$*, where* $\varovee$ *and* $\owedge$ *are determined by* $\geq$*. The* LATTICE $\mathbf{L}$ *is **linear** if* $(X, \leq)$ *is a* CHAIN *(Definition 4.4 page 59).*

Definition 5.3 (previous) characterizes lattices in terms of *order properties*. Under this definition, lattices have an equivalent characterization in terms of *algebraic properties*. In particular, all lattices have four basic algebraic properties: all lattices are *idempotent*, *commutative*, *associative*, and *absorptive*. Conversely, any structure that possesses these four properties *is* a lattice. These results are demonstrated by Theorem 5.3 (next). However, note that the four properties are not *independent*, as it is possible to prove that any structure $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ that is *commutative*, *associative*, and *absorptive*, is also *idempotent* (Theorem 5.8 page 82). Thus, when proving that $\mathbf{L}$ is a lattice, it is only necessary to prove that it is *commutative*, *associative*, and *absorptive*.

[5] Rota (1997) page 1440 ⟨Introduction⟩, Rota (1964) page 498 ⟨partitions of a set⟩

[6] MacLane and Birkhoff (1999) page 473, Birkhoff (1948) page 16, Ore (1935), Birkhoff (1933a) page 442, Maeda and Maeda (1970), page 1

Theorem 5.3. [7]

THM

$(X, \vee, \wedge; \leq)$ *is a* LATTICE $\iff$

$$\left\{\begin{array}{rcl|rcl|lll} x \vee x &=& x & x \wedge x &=& x & \forall x \in X & \text{(IDEMPOTENT)} & \text{and} \\ x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \forall x,y \in X & \text{(COMMUTATIVE)} & \text{and} \\ (x \vee y) \vee z &=& x \vee (y \vee z) & (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & \forall x,y,z \in X & \text{(ASSOCIATIVE)} & \text{and} \\ x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \forall x,y \in X & \text{(ABSORPTIVE)}. & \end{array}\right\}$$

✎PROOF:

1. Proof that $(X, \vee, \wedge; \leq)$ is a lattice $\implies$ 4 properties:
   These follow directly from the definitions of least upper bound $\vee$ and greatest lower bound $\wedge$. For the absorptive property,

$$\begin{array}{rcl} x \leq y &\implies& x \vee (x \wedge y) = x \vee x = x \\ y \leq x &\implies& x \vee (x \wedge y) = x \vee y = x \\ x \leq y &\implies& x \wedge (x \vee y) = x \wedge y = x \\ y \leq x &\implies& x \wedge (x \vee y) = x \wedge x = x \end{array}$$

2. Proof that $(X, \vee, \wedge; \leq)$ is a lattice $\impliedby$ 4 properties:

   According to Definition 5.3 (page 73), in order for $(X, \vee, \wedge; \leq)$ to be a lattice, $(X, \vee, \wedge; \leq)$ must be an ordered set, $x \vee y$ must be the least upper bound for any $x, y \in X$ and $x \wedge y$ must be the greatest lower bound for any $x, y \in X$.

   (a) By Theorem 5.1 (page 71), $(X, \vee, \wedge; \leq)$ is an ordered set.

   (b) By Theorem 5.1 (page 71), $x \vee y$ is the least upper bound for any $x, y \in X$.

   (c) Proof that $x \wedge y$ is the greatest lower bound for any $x, y \in X$: To prove this, we must show that $x \leq y \iff x \wedge y = x$.

   Proof that $x \leq y \implies x \wedge y = x$:

$$\begin{array}{rll} x &= x \wedge (x \vee y) & \text{by absorptive hypothesis} \\ &= x \wedge y & \text{by } x \leq y \text{ hypothesis and definition of } \leq \end{array}$$

   Proof that $x \leq y \impliedby x \wedge y = x$:

$$\begin{array}{rll} y &= y \vee (y \wedge x) & \text{by absorptive hypothesis} \\ &= y \vee (x \wedge y) & \text{by commutative hypothesis} \\ &= y \vee x & \text{by } x \wedge y = x \text{ hypothesis} \\ &= x \vee y & \text{by commutative hypothesis} \\ \implies & x \leq y & \text{by definition of } \leq \end{array}$$

⇰

[7] MacLane and Birkhoff (1999) pages 473–475 ⟨LEMMA 1, THEOREM 4⟩, Burris and Sankappanavar (1981) pages 4–7, Birkhoff (1938), pages 795–796, Ore (1935) page 409 ⟨(α)⟩, Birkhoff (1933a) page 442, Dedekind (1900) pages 371–372 ⟨(1)–(4)⟩. Peirce (1880b) credits Boole and Jevons with the *commutative* property: Peirce (1880b), page 33 ⟨"(5)"⟩. Peirce (1880b) credits Boole and Jevons with the *associative* property. Peirce (1880b) credits Jevons (1864) with the *idempotent* property: Jevons (1864), page 41

$$\begin{array}{rcll} A + A &=& A & \text{"Law of Unity"} \\ AA &=& A & \text{"Law of Simplicity"} \end{array}$$

Lemma 5.1. [8] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition 5.3 page 73).*

LEM

$$x \leq y \iff x = x \wedge y \qquad \forall x,y \in \boldsymbol{L}$$

✎PROOF:

1. Proof for $\Longrightarrow$ case: by left hypothesis and definition of $\wedge$ (Definition 4.22 page 70).
2. Proof for $\Longleftarrow$ case: by right hypothesis and definition of $\wedge$ (Definition 4.22 page 70).

⇒

The identities of Theorem 5.3 (page 74) occur in pairs that are *duals* of each other. That is, for each identity, if you swap the join and meet operations, you will have the other identity in the pair. Thus, the characterization of lattices provided by Theorem 5.3 (page 74) is called *self-dual*. And because of this, lattices support the *principle of duality* (next theorem).

Theorem 5.4 (Principle of duality). [9] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

THM

$$\left\{\begin{array}{l}\phi \text{ is an identity on } \boldsymbol{L} \text{ in terms} \\ \text{of the operations } \vee \text{ and } \wedge\end{array}\right\} \Longrightarrow \mathbf{T}\phi \text{ is also an identity on } \boldsymbol{L}$$

where the operator $\mathbf{T}$ *performs the following mapping on the operations of* ϕ*:*

$$\vee \to \wedge, \qquad \wedge \to \vee$$

✎PROOF: For each of the identities in Theorem 5.3 (page 74), the operator $\mathbf{T}$ produces another identity that is also in the set of identities:

$$\begin{array}{lclclclclcl}
\mathbf{T}(1a) & = & \mathbf{T}[x \vee y & = & y \vee x] & = & [x \wedge y & = & y \wedge x] & = & (1b) \\
\mathbf{T}(1b) & = & \mathbf{T}[x \wedge y & = & y \wedge x] & = & [x \vee y & = & y \vee x] & = & (1a) \\
\mathbf{T}(2a) & = & \mathbf{T}[x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z)] & = & [x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z)] & = & (2b) \\
\mathbf{T}(2b) & = & \mathbf{T}[x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z)] & = & [x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z)] & = & (2a)
\end{array}$$

Therefore, if the statement ϕ is consistent with regards to the lattice $\boldsymbol{L}$, then $\mathbf{T}\phi$ is also consistent with regards to the lattice $\boldsymbol{L}$. ⇒

Proposition 5.1 (Monotony laws). [10] *Let* $(X, \vee, \wedge; \leq)$ *be a lattice.*

PRP

$$\left.\begin{array}{lcll} a & \leq & b & \text{and} \\ x & \leq & y. & \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcll} a \wedge x & \leq & b \wedge y & \text{and} \\ a \vee x & \leq & b \vee y. & \end{array}\right.$$

[8] Holland (1970), page ???
[9] Padmanabhan and Rudeanu (2008) pages 7–8, Beran (1985) pages 29–30
[10] Givant and Halmos (2009) page 39, Doner and Tarski (1969) pages 97–99

✎PROOF:

$$\begin{array}{rll}
1.(a \wedge x) & \le a & \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition 4.22 page 70} \\
& \le b & \text{by left hypothesis} \\
2.(a \wedge x) & \le x & \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition 4.22 page 70} \\
& \le y & \text{by left hypothesis} \\
3.(a \wedge x) & = \underbrace{(a \wedge x)}_{\le b} \wedge \underbrace{(a \wedge x)}_{\le y} & \text{by } \textit{idempotent} \text{ property Theorem 5.3 page 74} \\
& \le b \wedge y & \text{by 1 and 2} \\
4.(a \vee x) & = \underbrace{(a \vee x)}_{\le b} \vee \underbrace{(a \vee x)}_{\le y} & \text{by } \textit{idempotent} \text{ property Theorem 5.3 page 74} \\
& \le b \vee y & \text{by 1 and 2}
\end{array}$$

✏

Minimax inequality. Suppose we arrange a finite sequence of values into m groups of n elements per group. This could be represented as an $m \times n$ matrix. Suppose now we find the minimum value in each row, and the maximum value in each column. We can call the maximum of all the minimum row values the *maximin*, and the minimum of all the maximum column values the *minimax*. Now, which is greater, the maximin or the minimax? The *minimax inequality* demonstrates that the maximin is always less than or equal to the minimax. The minimax inequality is illustrated below and stated formerly in Theorem 5.5 (page 76).

$$\underbrace{\bigvee_1^m \left\{ \begin{array}{l}
\bigwedge_1^n \{\; x_{11} \quad x_{12} \quad \cdots \quad x_{1n} \;\} \\ \hline
\bigwedge_1^n \{\; x_{21} \quad x_{22} \quad \cdots \quad x_{2n} \;\} \\ \hline
\bigwedge_1^n \{\; \vdots \quad \ddots \quad \ddots \quad \vdots \;\} \\ \hline
\bigwedge_1^n \{\; x_{m1} \quad x_{m2} \quad \cdots \quad x_{mn} \;\}
\end{array} \right\}}_{\text{maximin}}
\le
\underbrace{\bigwedge_1^n \left\{ \begin{array}{c|c|c|c}
\bigvee_1^m & \bigvee_1^m & \bigvee_1^m & \bigvee_1^m \\
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{array} \right\}}_{\text{minimax}}$$

Theorem 5.5 (Minimax inequality). [11] *Let* $(X, \vee, \wedge; \le)$ *be a lattice.*

THM

$$\underbrace{\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} x_{ij}}_{\textit{maxmini: largest of the smallest}} \le \underbrace{\bigwedge_{j=1}^{n} \bigvee_{i=1}^{m} x_{ij}}_{\textit{minimax: smallest of the largest}} \qquad \forall x_{ij} \in X$$

[11] Birkhoff (1948) pages 19–20

✎PROOF:

$$
\begin{array}{rcccl}
& \underbrace{\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{smallest for any given } i} \leq x_{ij} \leq \underbrace{\left(\bigvee_{k=1}^{n} x_{kj}\right)}_{\text{largest for any given } j} & & & \forall i,j \\
\implies & \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{largest amoung all } i\text{s of the smallest values}} \leq \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{k=1}^{m} x_{kj}\right)}_{\text{smallest amoung all } j\text{s of the largest values}} & & & \\
\implies & \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{j=1}^{n} x_{ij}\right)}_{\text{maxmini}} \leq \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{i=1}^{m} x_{ij}\right)}_{\text{minimax}} & & & \text{(change of variables)}
\end{array}
$$

✏

Distributive inequalities. Special cases of the minimax inequality include three distributive *inequalities* (next theorem). If for some lattice any *one* of these inequalities is an *equality*, then *all three* are *equalities* (Theorem 8.1 page 102); and in this case, the lattice is a called a *distributive* lattice (Definition 8.2 page 102).

Theorem 5.6 (distributive inequalities). [12]

THM $(X, \vee, \wedge; \leq)$ *is a lattice* $\implies$ *for all* $x,y,z \in X$

$$
\begin{array}{rcll}
x \wedge (y \vee z) & \geq & (x \wedge y) \vee (x \wedge z) & \text{(JOIN SUPER-DISTRIBUTIVE)} \quad \textit{and} \\
x \vee (y \wedge z) & \leq & (x \vee y) \wedge (x \vee z) & \text{(MEET SUB-DISTRIBUTIVE)} \quad \textit{and} \\
(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \leq & (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \text{(MEDIAN INEQUALITY).}
\end{array}
$$

✎PROOF:

1. Proof that $\wedge$ sub-distributes over $\vee$:

$$
\begin{aligned}
(x \wedge y) \vee (x \wedge z) &\leq (x \vee x) \wedge (y \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem 5.5 page 76)} \\
&= x \wedge (y \vee z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem 5.3 page 74)}
\end{aligned}
$$

$$
\bigvee\left\{\begin{array}{c} \bigwedge\left\{\begin{array}{cc} x & y \end{array}\right\} \\ \hline \bigwedge\left\{\begin{array}{cc} x & z \end{array}\right\} \end{array}\right\} \quad \leq \quad \bigwedge\left\{\begin{array}{c|c} \bigvee & \bigvee \\ x & y \\ x & z \end{array}\right\}
$$

2. Proof that $\vee$ super-distributes over $\wedge$:

$$
\begin{aligned}
x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem 5.3 page 74)} \\
&\leq (x \vee y) \wedge (x \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem 5.5 page 76)}
\end{aligned}
$$

$$
\bigvee\left\{\begin{array}{c} \bigwedge\left\{\begin{array}{cc} x & x \end{array}\right\} \\ \hline \bigwedge\left\{\begin{array}{cc} y & z \end{array}\right\} \end{array}\right\} \quad \leq \quad \bigwedge\left\{\begin{array}{c|c} \bigvee & \bigvee \\ x & x \\ y & z \end{array}\right\}
$$

3. Proof that of median inequality: by *minimax inequality* (Theorem 5.5 page 76)

✏

[12] Davey and Priestley (2002) page 85, Grätzer (2003) page 38, Birkhoff (1933a) page 444, Korselt (1894) page 157, Müller-Olm (1997) page 13 ⟨terminology⟩

Modular inequalities. Besides the distributive property, another consequence of the minimax inequality is the *modularity inequality* (next theorem). A lattice in which this inequality becomes equality is said to be *modular* (Definition 7.3 page 92).

Theorem 5.7 (Modular inequality). [13] *Let* $(X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition 5.3 page 73).*

THM
$$x \leq y \qquad \Longrightarrow \qquad x \vee (y \wedge z) \leq y \wedge (x \vee z)$$

✎PROOF:

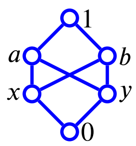
$$\begin{aligned} x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by } \textit{absorptive} \text{ property (Theorem 5.3 page 74)}\\ &\leq (x \vee y) \wedge (x \vee z) && \text{by the } \textit{minimax inequality} \text{ (Theorem 5.5 page 76)}\\ &= y \wedge (x \vee z) && \text{by left hypothesis} \end{aligned}$$

$$\bigvee\left\{\begin{array}{c}\bigwedge\{\; x \quad x \;\}\\ \hline \bigwedge\{\; y \quad z \;\}\end{array}\right\} \quad \leq \quad \bigwedge\left\{\begin{array}{c|c}\bigvee & \bigvee\\ x & x\\ y & z\end{array}\right\}$$

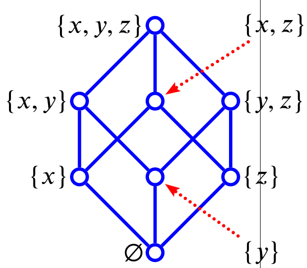
✏

5.3 Examples

Example 5.1. [14] the ordered set illustrated to the right is **not** a lattice because, for example, while x and y have *upper bounds* a, b, and 1, x and y have no *least upper bound*. Obviously 1 is not the least upper bound because $a \leq 1$ and $b \leq 1$. And neither a nor b is a least upper bound because $a \not\leq b$ and $b \not\leq a$; rather, a and b are incomparable $(a \| b)$. Note that if we remove either or both of the two lines crossing the center, the ordered set becomes a lattice.



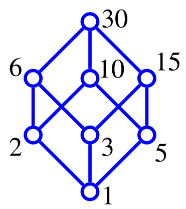
Example 5.2 (Discrete lattice). Let 2^A be the power set of a set A, $\subseteq$ the set inclusion relation, $\cup$ the set union operation, and $\cap$ the set intersection operation. Then the tupple $\left(2^{\{x,y,z\}}, \cup, \cap; \subseteq\right)$ is a lattice.



Examples of least upper bounds	Examples of greatest lower bounds
$\{x\} \cup \{z\} = \{x, z\}$	$\{x\} \cap \{z\} = \varnothing$
$\{x, y\} \cup \{y\} = \{x, y\}$	$\{x, y\} \cap \{y\} = \{y\}$
$\{x, z\} \cup \{y, z\} = \{x, y, z\}$	$\{x, z\} \cap \{y, z\} = \{z\}$

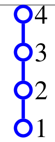
Example 5.3 (Integer factor lattice). [15] For any pair of natural numbers $n, m \in \mathbb{N}$, let $n|m$ represent the relation "m divides n", $\operatorname{lcm}(n, m)$ the *least common multiple* of n and m, and $\gcd(n, m)$ the *greatest common divisor* of n and m.

EX
$$\left(\{1, 2, 3, 5, 6, 10, 15, 30\}, \gcd, \operatorname{lcm}; |\right) \text{ is a lattice.}$$

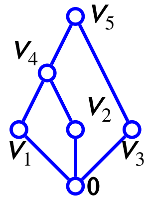


[13] Birkhoff (1948) page 19, Burris and Sankappanavar (1981) page 11, Dedekind (1900) page 374
[14] Oxley (2006) page 54, Farley (1997), page 3, Farley (1996), page 5, Birkhoff (1967) pages 15–16
[15] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩

Example 5.4 (Linear lattice). Let $\le$ be the standard counting ordering relation on the set of integers; and for any pair of integers $n, m \in \mathbb{N}$, let $\max(n, m)$ be the maximum of n and m, and $\min(n, m)$ be the minimum of n and m. Then the tupple $(\{1, 2, 3, 4\}, \max, \min; \le)$ is a lattice.



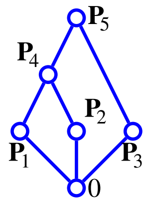
Example 5.5 (Subspace lattices). [16]Let $(\boldsymbol{V}_n)$ be a sequence of subspaces, $\subseteq$ be the set inclusion relation, $+$ the subspace addition operator, and $\cap$ the set intersection operator. Then the tuple $(\{\boldsymbol{V}_n\}, +, \cap; \subseteq)$ is a lattice.



Example 5.6 (Projection operator lattices). [17]Let $(\mathbf{P}_n)$ be a sequence of projection operators in a Hilbert space X.

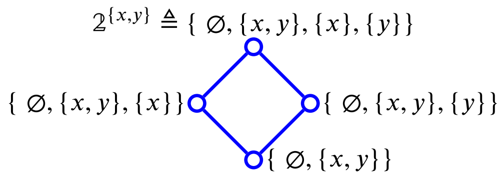
EX $(\{\mathbf{P}_n\}, \vee, \wedge; \le)$ is a lattice

$$\begin{array}{lrcl} \text{where} & \mathbf{P}_1 \le \mathbf{P}_2 & \overset{\text{def}}{\iff} & \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1 \\ & \mathbf{P}_1 \vee \mathbf{P}_2 & = & \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 \\ & \mathbf{P}_1 \wedge \mathbf{P}_2 & = & \mathbf{P}_1\mathbf{P}_2 \end{array}$$



Example 5.7 (Lattice of a single topology). [18] Let X be a set, τ a topology on X, $\subseteq$ the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $(\tau, \cup, \cap; \subseteq)$ is a lattice.

Example 5.8 (Lattice of topologies). [19]Let X be a set and $\{\tau_1, \tau_2, \tau_3, \ldots\}$ all the possible topologies on X. Let $\subseteq$ be the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $(\{(X, \tau_n)\}, \cup, \cap; \subseteq)$ is a lattice.



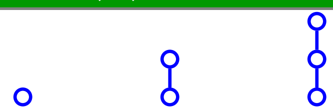
Proposition 5.2. [20] *Let X_n be a finite set with order $n = |X_n|$. Let L_n be the number of labeled lattices on X_n, l_n the number of unlabeled lattices, and p_n the number of unlabeled posets.*

PRP

n	0	1	2	3	4	5	6	7	8	9	10
L_n	1	1	2	6	36	380	6390	157962	5396888	243, 179, 064	13, 938, 711, 210
l_n	1	1	1	1	2	5	15	53	222	1078	5994
p_n	1	1	2	5	16	63	318	2045	16, 999	183, 231	2, 567, 284

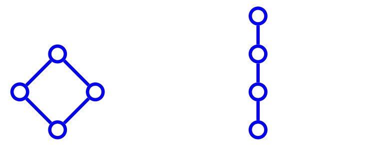
Example 5.9 (lattices on 1–3 element sets). [21]There is only one unlabeled lattice for finite sets with 3 or fewer elements (Proposition 5.2 page 79). Thus, these lattices are all linearly ordered. These 3 lattices are illustrated to the right.

lattices on 1, 2, and 3 element sets



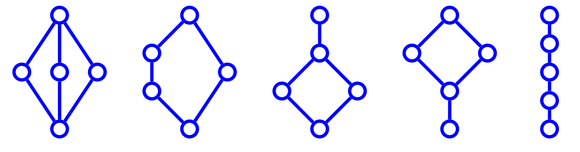
[16] Isham (1999) pages 21–22
[17] Isham (1999) pages 21–22, Dunford and Schwartz (1957), pages 481–482
[18] Burris and Sankappanavar (1981) page 9, Birkhoff (1936a) page 161
[19] Isham (1999) page 44, Isham (1989), page 1515
[20] Sloane (2014) ⟨http://oeis.org/A055512⟩, Sloane (2014) ⟨http://oeis.org/A006966⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Heitzig and Reinhold (2002)
[21] Kyuno (1979), page 412, Stanley (1997), page 102

lattices on 4 element sets



Example 5.10 (lattices on a 4 element set). [22]There are 2 unlabeled lattices on a 4 element set (Proposition 5.2 page 79). These are illustrated to the right.

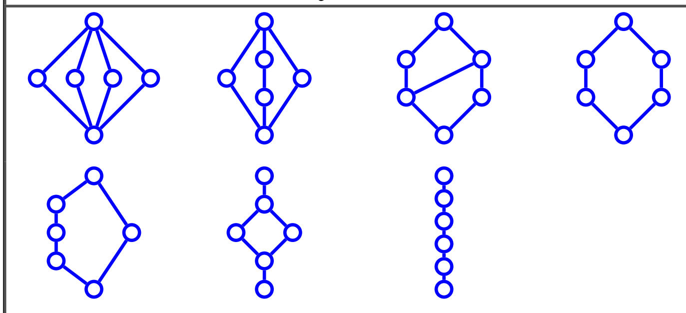
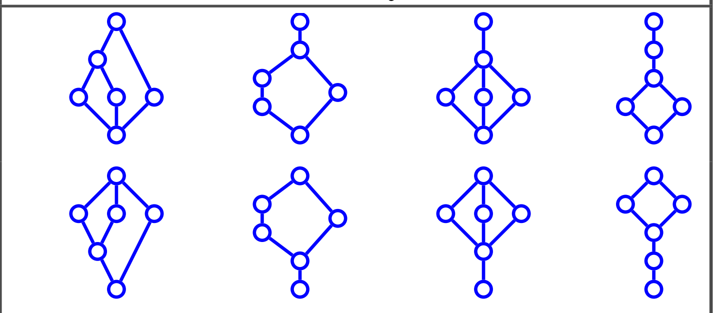
lattices on 5 element sets



Example 5.11 (lattices on a 5 element set). [23]There are 5 unlabeled lattices on a 5 element set (Proposition 5.2 page 79). These are illustrated to the right.

Example 5.12 (lattices on a 6 element set). [24] There are 15 *unlabeled lattices* on a 6 element set (Proposition 5.2 page 79). These are illustrated in the following table. Notice that the lattices in the second row are simply generated from the 5 element lattices (Example 5.11 page 80) with a "head" or "tail" added to each one.

lattices on 6 element sets

self-dual	*non-self dual*
	

Example 5.13 (lattices on a 7 element set). [25] There are 53 unlabeled lattices on a 7 element set (Proposition 5.2 page 79). These are illustrated in Figure 5.1 (page 81).

Example 5.14 (lattices on 8 element sets). There are 222 unlabeled lattices on a 8 element set (Proposition 5.2 page 79). See Kyuno's paper[26] for Hasse diagrams of all 222 lattices.

5.4 Characterizations

Theorem 5.3 (page 74)gave eight equations in three variables and two operators that are true of all lattices. But the converse is also true: that is, if the eight equations of Theorem 5.3 are true for all values of the underlying set, then that set together with the two operators are a lattice.

That is, the eight equations in three variables of Theorem 5.3 *characterize* lattices, or serve as an *equational basis* for lattices.[27] And this is not the only system of equations that characterize a lattice. There are other systems that use fewer equations in more variables. Here are some examples of lattice characterizations:

[22] Kyuno (1979), page 412, Stanley (1997), page 102
[23] Kyuno (1979), page 413, Stanley (1997), page 102
[24] Kyuno (1979), page 413, Stanley (1997), page 102
[25] Kyuno (1979), pages 413–414
[26] Kyuno (1979), pages 415–421
[27] McKenzie (1970) page 24, Tarski (1966)

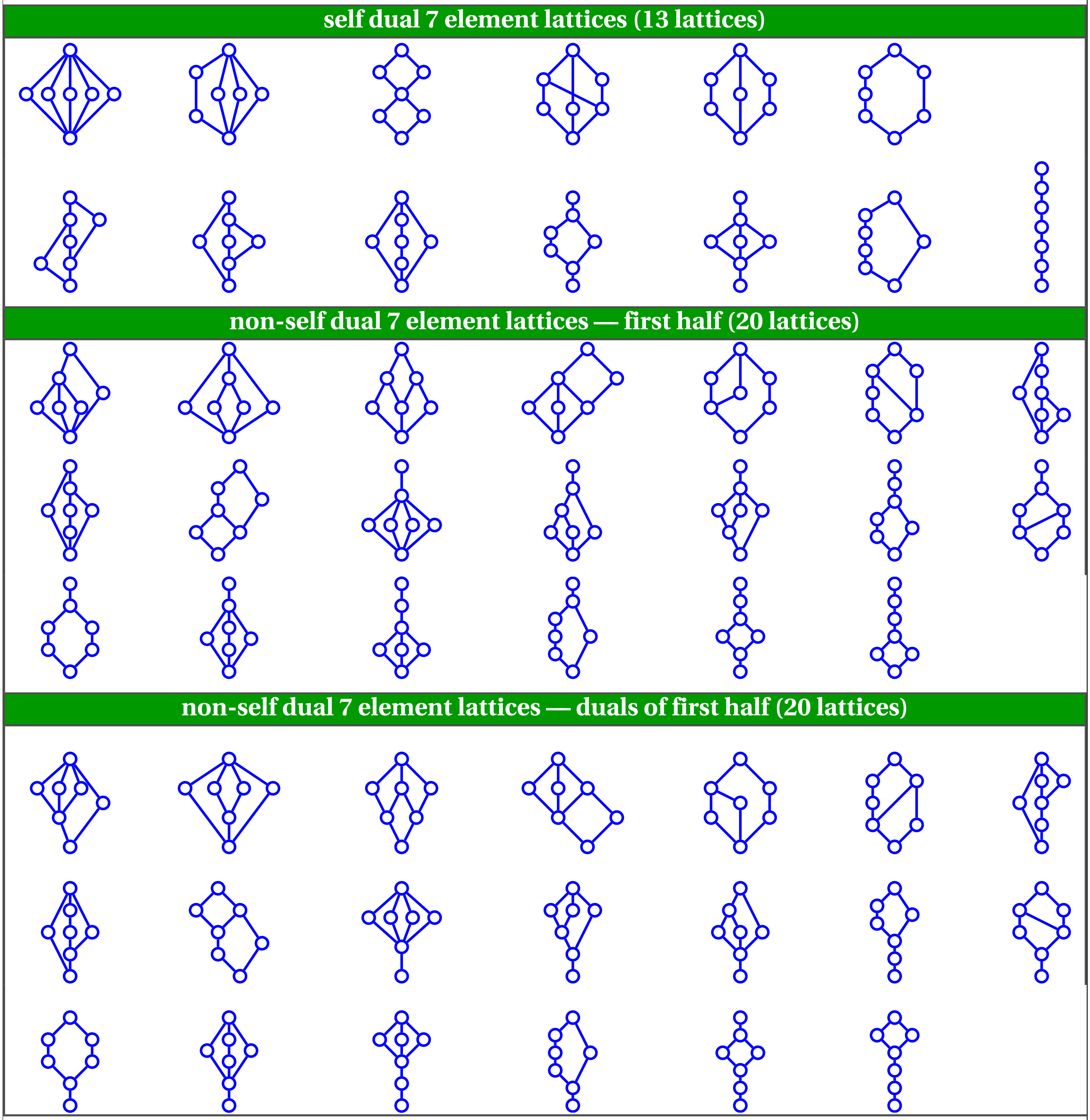


Figure 5.1: The 53 unlabeled lattices on a 7 element set (Example 5.13 page 80)

Since these characterizations are equivalent to the definition of the lattice, we could in fact change things around and essentially make any of these characterizations into the definition, and make the definition into a theorem.[28]

Theorem 5.3 (page 74) gave 4 necessary and sufficient pairs of properties for a structure $(X, \vee, \wedge; \le)$ to be a *lattice*. However, these 4 pairs are actually *overly* sufficient (they are not *independent*), as demonstrated next.

Theorem 5.8. [29]

THM

$(X, \vee, \wedge; \le)$ *is a lattice* $\iff$

$$\left\{\begin{array}{rcl|rcl|lll} x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \forall x,y \in X & (\text{COMMUTATIVE}) & \textit{and} \\ (x \vee y) \vee z &=& x \vee (y \vee z) & (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & \forall x,y,z \in X & (\text{ASSOCIATIVE}) & \textit{and} \\ x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \forall x,y \in X & (\text{ABSORPTIVE}) & \end{array}\right\}$$

✎PROOF: Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$.

1. Proof that $\boldsymbol{L}$ is a *lattice* $\implies$ 3 properties: by Theorem 5.3 page 74
2. Proof that $\boldsymbol{L}$ is a *lattice* $\impliedby$ 3 properties:
   (a) Proof that 3 properties $\implies$ $\boldsymbol{L}$ is *idempotent*:

$$\begin{aligned} x \vee x &= x \vee [x \wedge (x \vee y)] && \text{by } \textit{absorptive} \text{ property} \\ &= x \vee [x \wedge z] && \text{where } z \triangleq x \vee y \\ &= x && \text{by } \textit{absorptive} \text{ property} \\ x \wedge x &= x \wedge [x \vee (x \wedge y)] && \text{by } \textit{absorptive} \text{ property} \\ &= x \wedge [x \vee z] && \text{where } z \triangleq x \wedge y \\ &= x && \text{by } \textit{absorptive} \text{ property} \end{aligned}$$

   (b) By Theorem 5.3 page 74 and because $\boldsymbol{L}$ is *commutative*, *associative*, *absorptive*, and *idempotent* with respect to $\vee$ and $\wedge$, $\boldsymbol{L}$ is a *lattice*.

⇰

Theorem 5.9 (Lattice characterization in 2 equations and 5 variables). [30] *Let X be a set and $\vee$ and $\wedge$ be two binary operators on X.*

THM

$(X, \le, \vee, \wedge)$ *is a lattice* ***if and only if***

$$\begin{array}{rcll} x &=& (x \wedge y) \vee x & \forall x,y \in X \quad \textit{and} \\ [(x \wedge y) \wedge z \vee u] \vee w &=& [(y \wedge z) \wedge x \vee w] \vee (y \vee u) \wedge u & \forall x,y,z,u,w \in X \end{array}$$

Theorem 5.10 (Lattice characterizations in 1 equation). [31] *Let X be a set and $\vee$ and $\wedge$ be two binary*

[28] Burris and Sankappanavar (1981) pages 6–7,
[29] Padmanabhan and Rudeanu (2008) page 8, Beran (1985) page 5, McKenzie (1970) page 24
[30] Tamura (1975) page 137
[31] McCune et al. (2003b) page 2, McCune et al. (2003a), McCune and Padmanabhan (1996) page 144, `http://www.cs.unm.edu/%7Everoff/LT/`

operators on X.

THM *The following four statements are all equivalent:*

1. $(X, \vee, \wedge; \leq)$ *is a* ***lattice***
2. $(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x$ $\quad \forall x,y,z,u,v,w,s,t \in X$ *(1 equation, 8 variables, length 29)*
3. $(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (((w \vee x) \wedge (s \vee x)) \vee t) = x$ $\quad \forall x,y,z,u,v,w,s,t \in X$ *(1 equation, 8 variables, length 29)*
4. $(((x \wedge y) \vee (y \wedge (x \vee y))) \wedge z) \vee (((x \wedge (((x_1 \wedge y) \vee (y \wedge x_2)) \vee y)) \vee (((y \wedge (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)) \vee (u \wedge (y \vee (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)))) \wedge (x \vee (((x_1 \wedge y) \vee (y \wedge x_2)) \vee y)))) \wedge (((x \wedge y) \vee (y \wedge (x \vee y))) \vee z)) = y$ $\quad \forall x,y,z,x_1,x_2,x_3,u \in X$ *(1 equation, 7 variables, length 79)*

5.5 Functions on lattices

5.5.1 Isomorphisms

Lattices and *ordered set* (Definition 4.2 page 58) are examples of mathematical *order structures*. Often we are interested in similarities between two lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ with respect to order. Similarities between lattices can be described by defining a function θ that maps from the first lattice to the second. The degree of similarity can be roughly described in terms of the mapping θ as follows:

1. If there exists a mapping that is *bijective* (Definition 17.11 page 266) then the number of elements in $\mathbf{L}_1$ and $\mathbf{L}_2$ is the same. However, their order structure may still be very different.
2. Lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are more similar if there exists a mapping that is *bijective* and *order preserving* (Definition 4.9 page 65). Despite having this property however, their order structure may still be remarkably different, as illustrated by Example 4.18 (page 65) and Example 4.19 (page 65).
3. Lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are essentially identical (except possibly for their labeling) if there exists a mapping θ that is not only *bijective* and *order preserving*, but whose *inverse* (Definition 17.2 page 255) is *also bijective* (Theorem 5.11 page 83). In this case, the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are *isomorphic* and the mapping θ is an *isomorphism*. An isomorphism between $\mathbf{L}_1$ and $\mathbf{L}_2$ implies that the two lattices have an identical order structure. In particular, the isomorphism θ preserves joins and meets (next definition).

Definition 5.4. *Let $\mathbf{L}_1 \triangleq (X, \vee, \wedge; \leq)$ and $\mathbf{L}_2 \triangleq (Y, \varovee, \varowedge; \precsim)$ be lattices.*

DEF *$\mathbf{L}_1$ and $\mathbf{L}_2$ are* ***algebraically isomorphic****, or simply* ***isomorphic****, if there exists a function $\theta \in Y^X$ such that*

1. $\theta(x \vee y) = \theta(x) \varovee \theta(y) \quad \forall x, y \in X$ (PRESERVES JOINS) *and*
2. $\theta(x \wedge y) = \theta(x) \varowedge \theta(y) \quad \forall x, y \in X$ (PRESERVES MEETS).

In this case, the function θ is said to be an ***isomorphism*** *from $\mathbf{L}_1$ to $\mathbf{L}_2$, and the isomorphic relationship between $\mathbf{L}_1$ and $\mathbf{L}_2$ is denoted as*

$\mathbf{L}_1 \equiv \mathbf{L}_2$.

Theorem 5.11. [32] *Let $(X, \vee, \wedge; \leq)$ and $(Y, \varovee, \varowedge; \precsim)$ be lattices and $\theta \in Y^X$ be a* BIJECTIVE *function with inverse $\theta^{-1} \in X^Y$. Let $(X, \vee, \wedge; \leq) \equiv (Y, \varovee, \varowedge; \precsim)$ represent the condition that the two lattices*

[32] Burris and Sankappanavar (2000), page 10

are ISOMORPHIC.

THM

$$\underbrace{\left.\begin{array}{lll} x_1 \le x_2 & \implies \theta(x_1) \precsim \theta(x_2) & \forall x_1,x_2 \in X \\ y_1 \precsim y_2 & \implies \theta^{-1}(y_1) \precsim \theta^{-1}(y_2) & \forall y_1,y_2 \in Y \end{array}\right\}}_{\theta \text{ and } \theta^{-1} \text{ are ORDER PRESERVING with respect to } \le \text{ and } \precsim} \iff \underbrace{(X, \vee, \wedge; \le) \equiv (Y, \varovee, \varowedge; \precsim)}_{\text{isomorphic}}$$

(footnote [33] on "$\precsim$")

✎PROOF: Let $\theta \in Y^X$ be the isomorphism between lattices $(X, \vee, \wedge; \le)$ and $(Y, \varovee, \varowedge; \precsim)$.

1. Proof that *order preserving* $\implies$ *preserves joins*:
   (a) Proof that $\theta(x_1 \vee x_2) \succsim \theta(x_1) \varovee \theta(x_2)$:
      i. Note that
$$x_1 \le x_1 \vee x_2$$
$$x_2 \le x_1 \vee x_2.$$
      ii. Because θ is *order preserving*
$$\theta(x_1) \precsim \theta(x_1 \vee x_2)$$
$$\theta(x_2) \precsim \theta(x_1 \vee x_2).$$
      iii. We can then finish the proof of item (1a):
$$\begin{aligned} \theta(x_1) \varovee \theta(x_2) &\precsim \underbrace{\theta(x_1 \vee x_2)}_{x_1 \le x_1 \vee x_2} \varovee \underbrace{\theta(x_1 \vee x_2)}_{x_2 \le x_1 \vee x_2} && \text{by \textit{order preserving} hypothesis} \\ &= \theta(x_1 \vee x_2) && \text{by \textit{idempotent} property page 74} \end{aligned}$$
   (b) Proof that $\theta(x_1 \vee x_2) \precsim \theta(x_1) \varovee \theta(x_2)$:
      i. Just as in item (1a), note that $\theta^{-1}(y_1) \vee \theta^{-1}(y_2) \le \theta^{-1}(y_1 \varovee y_2)$:
$$\begin{aligned} \theta^{-1}(y_1) \vee \theta^{-1}(y_2) &\le \underbrace{\theta^{-1}(y_1 \varovee y_2)}_{y_1 \precsim y_1 \varovee y_2} \vee \underbrace{\theta^{-1}(y_1 \varovee y_2)}_{y_2 \precsim y_1 \varovee y_2} && \text{by \textit{order preserving} hypothesis} \\ &= \theta^{-1}(y_1 \varovee y_2) && \text{by \textit{idempotent} property page 74} \end{aligned}$$
      ii. Because θ is *order preserving*
$$\begin{aligned} \theta\left[\theta^{-1}(y_1) \vee \theta^{-1}(y_2)\right] &\precsim \theta\theta^{-1}(y_1 \varovee y_2) && \text{by item (1(b)i) page 84} \\ &= y_1 \varovee y_2 && \text{by definition of inverse function } \theta^{-1} \end{aligned}$$
      iii. Let $u_1 \triangleq \theta(x_1)$ and $u_2 \triangleq \theta(x_2)$.
      iv. We can then finish the proof of item (1b):
$$\begin{aligned} \theta(x_1 \vee x_2) &= \theta\left[\theta^{-1}\theta(x_1) \vee \theta^{-1}\theta(x_2)\right] && \text{by definition of inverse function } \theta^{-1} \\ &= \theta\left[\theta^{-1}(u_1) \vee \theta^{-1}(u_2)\right] && \text{by definition of } u_1, u_2 \text{, item (1(b)iii)} \\ &\precsim u_1 \varovee u_2 && \text{by item (1(b)ii)} \\ &= \theta(x_1) \varovee \theta(x_2) && \text{by definition of } u_1, u_2 \text{, item (1(b)iii)} \end{aligned}$$
   (c) And so, combining item (1a) and item (1b), we have
$$\left.\begin{array}{llll} \theta(x_1 \vee x_2) & \succsim & \theta(x_1) \varovee \theta(x_2) & \text{(item (1a) page 84)} \quad \text{and} \\ \theta(x_1 \vee x_2) & \precsim & \theta(x_1) \varovee \theta(x_2) & \text{(item (1b) page 84)} \end{array}\right\} \implies \theta(x_1 \vee x_2) = \theta(x_1) \varovee \theta(x_2)$$

[33] *order preserving*: Definition 4.9 page 65

2. Proof that *order preserving* $\implies$ *preserves meets*:

   (a) Proof that $\theta(x_1 \wedge x_2) \precsim \theta(x_1) \oslash \theta(x_2)$:

$$\begin{aligned}
\theta(x_1) \oslash \theta(x_2) &\otimes \underbrace{\theta(x_1 \wedge x_2)}_{x_1 \ge x_1 \wedge x_2} \oslash \underbrace{\theta(x_1 \wedge x_2)}_{x_2 \ge x_1 \wedge x_2} && \text{by \textit{order preserving} hypothesis}\\
&= \theta(x_1 \wedge x_2) && \text{by \textit{idempotent} property page 74}
\end{aligned}$$

   (b) Proof that $\theta(x_1 \wedge x_2) \otimes \theta(x_1) \oslash \theta(x_2)$:

      i. Just as in item (2a), note that $\theta^{-1}(y_1) \wedge \theta^{-1}(y_2) \ge \theta^{-1}(y_1 \oslash y_2)$:

$$\begin{aligned}
\theta^{-1}(y_1) \wedge \theta^{-1}(y_2) &\ge \underbrace{\theta^{-1}(y_1 \oslash y_2)}_{y_1 \otimes y_1 \oslash y_2} \oslash \underbrace{\theta^{-1}(y_1 \oslash y_2)}_{y_2 \otimes y_1 \oslash y_2} && \text{by \textit{order preserving} hypothesis}\\
&= \theta^{-1}(y_1 \oslash y_2) && \text{by \textit{idempotent} property page 74}
\end{aligned}$$

      ii. Because θ is *order preserving*

$$\begin{aligned}
\theta\left[\theta^{-1}(y_1) \wedge \theta^{-1}(y_2)\right] &\otimes \theta\theta^{-1}(y_1 \oslash y_2) && \text{by item (2(b)i)}\\
&= y_1 \oslash y_2
\end{aligned}$$

      iii. Let $v_1 \triangleq \theta(x_1)$ and $v_2 \triangleq \theta(x_2)$.

      iv. We can then finish the proof of item (2a):

$$\begin{aligned}
\theta(x_1 \wedge x_2) &= \theta\left[\theta^{-1}\theta(x_1) \wedge \theta^{-1}\theta(x_2)\right]\\
&= \theta\left[\theta^{-1}(v_1) \wedge \theta^{-1}(v_2)\right] && \text{by item (2(b)iii)}\\
&\otimes v_1 \oslash v_2 && \text{by item (2(b)ii)}\\
&= \theta(x_1) \oslash \theta(x_2) && \text{by item (2(b)iii)}
\end{aligned}$$

   (c) And so, combining item (2a) and item (2b), we have

$$\left.\begin{array}{lcll}
\theta(x_1 \wedge x_2) & \precsim & \theta(x_1) \oslash \theta(x_2) & \text{(item (2a) page 85)} \quad \text{and}\\
\theta(x_1 \wedge x_2) & \otimes & \theta(x_1) \oslash \theta(x_2) & \text{(item (2b) page 85)}
\end{array}\right\} \implies \theta(x_1 \wedge x_2) = \theta(x_1) \oslash \theta(x_2)$$

3. Proof that *order preserving* $\impliedby$ *isomorphic*:

$$\begin{aligned}
x \le y &\implies \theta(y) = \theta(x \vee y) = \theta(x) \varovee \theta(y) && \text{by right hypothesis}\\
&\implies \theta(x) \precsim \theta(y)\\
x \le y &\implies \theta(x) = \theta(x \wedge y) = \theta(x) \oslash \theta(y) && \text{by right hypothesis}\\
&\implies \theta(x) \precsim \theta(y)
\end{aligned}$$

✏

Example 5.15. Let $\boldsymbol{L} \equiv \boldsymbol{M}$ represent the condition that a lattice $\boldsymbol{L}$ and a lattice $\boldsymbol{M}$ are *isomorphic*.

EX
$\left(2^{\{x,y,z\}}, \cup, \cap; \subseteq\right) \equiv \left(\{1,2,3,5,6,10,15,30\}, \text{lcm}, \text{gcd}; |\right)$
with isomorphism
$\theta(A) = 5^{\mathbb{1}_A(z)} \cdot 3^{\mathbb{1}_A(y)} \cdot 2^{\mathbb{1}_A(x)} \qquad \forall A \in 2^{\{a,b,c\}}$

Explicit cases are listed below and illustrated in Example 4.9 (page 61) and Example 4.10 (page 61).

$$\begin{aligned}
\theta(\emptyset) &= 5^0 \cdot 3^0 \cdot 2^0 &&= 1 & \theta(\{z\}) &= 5^1 \cdot 3^0 \cdot 2^0 &&= 5\\
\theta(\{x\}) &= 5^0 \cdot 3^0 \cdot 2^1 &&= 2 & \theta(\{x,z\}) &= 5^1 \cdot 3^0 \cdot 2^1 &&= 10\\
\theta(\{y\}) &= 5^0 \cdot 3^1 \cdot 2^0 &&= 3 & \theta(\{y,z\}) &= 5^1 \cdot 3^1 \cdot 2^0 &&= 15\\
\theta(\{x,y\}) &= 5^0 \cdot 3^1 \cdot 2^1 &&= 6 & \theta(\{x,y,z\}) &= 5^1 \cdot 3^1 \cdot 2^1 &&= 30
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\theta(A\cup B) &= 5^{\mathbb{1}_{A\cup B}(a)}\cdot 3^{\mathbb{1}_{A\cup B}(b)}\cdot 2^{\mathbb{1}_{A\cup B}(c)}\\
&= 5^{\mathbb{1}_A(a)\vee\mathbb{1}_B(a)}\cdot 3^{\mathbb{1}_A(b)\vee\mathbb{1}_B(b)}\cdot 2^{\mathbb{1}_A(c)\vee\mathbb{1}_B(c)} && \text{by Theorem 17.11 page 272}\\
&= \operatorname{lcm}\left(5^{\mathbb{1}_A(a)},\ 5^{\mathbb{1}_B(a)}\right)\cdot\operatorname{lcm}\left(3^{\mathbb{1}_A(b)},\ 3^{\mathbb{1}_B(b)}\right)\cdot\operatorname{lcm}\left(2^{\mathbb{1}_A(c)},\ 2^{\mathbb{1}_B(c)}\right)\\
&= \operatorname{lcm}\left(5^{\mathbb{1}_A(a)}\cdot 3^{\mathbb{1}_A(b)}\cdot 2^{\mathbb{1}_A(c)},\ 5^{\mathbb{1}_B(a)}\cdot 3^{\mathbb{1}_B(b)}\cdot 2^{\mathbb{1}_B(c)}\right)\\
&= \operatorname{lcm}(\theta(A),\ \theta(B))
\end{aligned}$$

$$\begin{aligned}
\theta(A\cap B) &= 5^{\mathbb{1}_{A\cap B}(a)}\cdot 3^{\mathbb{1}_{A\cap B}(b)}\cdot 2^{\mathbb{1}_{A\cap B}(c)}\\
&= 5^{\mathbb{1}_A(a)\wedge\mathbb{1}_B(a)}\cdot 3^{\mathbb{1}_A(b)\wedge\mathbb{1}_B(b)}\cdot 2^{\mathbb{1}_A(c)\wedge\mathbb{1}_B(c)} && \text{by Theorem 17.11 page 272}\\
&= \gcd\left(5^{\mathbb{1}_A(a)},\ 5^{\mathbb{1}_B(a)}\right)\cdot\gcd\left(3^{\mathbb{1}_A(b)},\ 3^{\mathbb{1}_B(b)}\right)\cdot\gcd\left(2^{\mathbb{1}_A(c)},\ 2^{\mathbb{1}_B(c)}\right)\\
&= \gcd\left(5^{\mathbb{1}_A(a)}\cdot 3^{\mathbb{1}_A(b)}\cdot 2^{\mathbb{1}_A(c)},\ 5^{\mathbb{1}_B(a)}\cdot 3^{\mathbb{1}_B(b)}\cdot 2^{\mathbb{1}_B(c)}\right)\\
&= \gcd(\theta(A),\ \theta(B))
\end{aligned}$$

✏

5.5.2 Metrics

Definition 5.5. [34] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

DEF

A function $\mathsf{v} \in \mathbb{R}^X$ *is a* ***subvaluation*** *if*

1. $\mathsf{v}(x) \geq 0 \quad \forall x \in X$ *and*
2. $\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) \leq \mathsf{v}(x) + \mathsf{v}(y) \quad \forall x,y \in X$

A subvaluation v *is* ***isotone*** *if* $x \leq y \implies \mathsf{v}(x) \leq \mathsf{v}(y)$.
A subvaluation v *is* ***positive*** *if* $x < y \implies \mathsf{v}(x) < \mathsf{v}(y)$.

Definition 5.6. [35] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

DEF

A function $\mathsf{v} \in \mathbb{R}^X$ *is a* ***valuation*** *if*

1. $\mathsf{v}(x) \geq 0 \quad \forall x \in X$ *and*
2. $\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) = \mathsf{v}(x) + \mathsf{v}(y) \quad \forall x,y \in X$ *and*
3. $x \leq y \implies \mathsf{v}(x) \leq \mathsf{v}(y) \quad \forall x,y \in X$ (ISOTONE).

Proposition 5.3 (lattice subvaluation metric). [36] *Let* $\mathbf{L}$ *be a lattice.*

PRP

$\left\{\mathsf{v} \text{ is a positive SUBVALUATION on } \mathbf{L}\right\} \implies \left\{\mathsf{d}(x,y) = 2\mathsf{v}(x\vee y) - \mathsf{v}(x) - \mathsf{v}(y) \text{ is a metric.}\right\}$

Proposition 5.4 (lattice valuation metric). [37] *Let* $\mathbf{L}$ *be a lattice.*

PRP

$\left\{\mathsf{v} \text{ is a positive VALUATION on } \mathbf{L}\right\} \implies \left\{\mathsf{d}(x,y) = \mathsf{v}(x) + \mathsf{v}(y) - 2\mathsf{v}(x\wedge y) \text{ is a metric.}\right\}$

---

34 Deza and Deza (2006) page 143
35 Deza and Deza (2006) page 143, Istrăţescu (1987) page 127 ⟨differs from Deza⟩, Birkhoff (1948) page 74 ⟨not compatible with Deza⟩
36 Deza and Deza (2006) page 143
37 Deza and Deza (2006) page 143

5.5.3 Lattice products

Theorem 5.12 (lattice product). [38] *Let* $(X \times Y, \leq)$ *be the* POSET PRODUCT[39] *of* $(X, \precsim)$ *and* $(Y, \trianglelefteq)$.

THM
$$\left.\begin{array}{lll} (X, \varovee, \varowedge; \precsim) & \text{is a lattice} & \text{and} \\ (Y, \veebar, \barwedge; \trianglelefteq) & \text{is a lattice} & \end{array}\right\} \implies (X \times Y, \vee, \wedge; \leq) \text{ is also a lattice}$$

5.6 Literature

Literature survey:

1. Early lattice theory concepts:
   Dedekind (1900)
   Ore (1935)
2. Garrett Birkhoff's contribution:
   (a) The modern concept of the lattice was introduced by Garrett Birkhoff in 1933:
   Birkhoff (1933a)
   Birkhoff (1933b)
   (b) However, Birkhoff came to realize that the concept of the lattice had actually already been published in 1900 by Richard Dedekind. Birkhoff later remarked in an interview "My ideas about lattice theory developed gradually ...It was my father who, when he told Ore at Yale about what I was doing some time in 1933, found out from Ore that my lattices coincided with Dedekind's Dualgruppen ...I was lucky to have gone beyond Dedekind before I discovered his work. It would have been quite discouraging if I had discovered all my results anticipated by Dedekind."[40]
   (c) Birkhoff wrote a book in 1940 called *Lattice Theory*. There are basically three editions:
   Birkhoff (1940)
   Birkhoff (1948)
   Birkhoff (1967) With regards to his *Lattice Theory* book and another book entitled *A Survey of Modern Algebra* written with Saunders MacLane, Birkhoff remarked, "Morse had told me that no one under 30 should write a book. So I thought it over and wrote two!"[41]
3. Standard text books of lattice theory:
   Birkhoff (1967)
   Grätzer (1998)
   Crawley and Dilworth (1973)
4. Characterizations / equational bases:
   (a) General discussion:
   Tarski (1966)
   Baker (1969)
   McKenzie (1970)
   McKenzie (1972)
   Pigozzi (1975)
   Taylor (1979)
      Taylor (2008)
   Jipsen and Rose (1992) pages 115–127 ⟨Chapter 5⟩
   Padmanabhan and Rudeanu (2008)
   (b) Characterizations for lattices:
   Kalman (1968)
   Tamura (1975)
   Sobociński (1979)

[38] MacLane and Birkhoff (1967), page 489
[39] *poset product*: Definition 4.5 page 59
[40] Albers and Alexanderson (1985), page 4
[41] Albers and Alexanderson (1985), page 4

(c) Specific characterizations:
 Padmanabhan (1969) ⟨2 equations in 7 variables⟩
 McCune and Padmanabhan (1996), page 144 ⟨1 equation, 7 variables, length 79⟩
 McCune et al. (2003a) ⟨1 equation, 8 variables, length 29⟩
 McCune et al. (2003b) ⟨1 equation, 8 variables, length 29⟩

5. Lattice drawing program:
Ralph Freese, http://www.math.hawaii.edu/~ralph/LatDraw/

CHAPTER 6

BOUNDED LATTICES

Let $\mathbf{L} \triangleq (X, \vee, \wedge ; \leq)$ be a lattice. By the definition of a *lattice* (Definition 5.3 page 73), the *upper bound* $(x \vee y)$ and *lower bound* $(x \wedge y)$ of any two elements in X is also in X. But what about the upper and lower bounds of the entire set X ($\bigvee X$ and $\bigwedge X$)[1]? If both of these are in X, then the lattice $\mathbf{L}$ is said to be *bounded* (next definition). All *finite* lattices are bounded (next proposition). However, not all lattices are bounded—for example, the lattice $(\mathbb{Z}, \leq)$ (the lattice of integers with the standard integer ordering relation) is *unbounded*. Proposition 6.2 (page 89) gives two properties of bounded lattices. Boundedness is one of the "*classic 10*" properties (Theorem 10.2 page 132) of *Boolean algebras* (Definition 10.1 page 127). Conversely, a bounded and complemented lattice that satisfies the conditions $1' = 0$ and *Elkan's law is* a *Boolean algebra* (Proposition 10.4 page 143).

Definition 6.1. *Let $\mathbf{L} \triangleq (X, \vee, \wedge ; \leq)$ be a lattice. Let $\bigvee X$ be the least upper bound of $(X, \leq)$ and let $\bigwedge X$ be the greatest lower bound of $(X, \leq)$.*

DEF

$\mathbf{L}$ *is* ***upper bounded***	*if* $\left(\bigvee X\right) \in X$.
$\mathbf{L}$ *is* ***lower bounded***	*if* $\left(\bigwedge X\right) \in X$.
$\mathbf{L}$ *is* ***bounded***	*if* $\mathbf{L}$ *is both upper and lower bounded.*

A BOUNDED *lattice is optionally denoted* $(X, \vee, \wedge, 0, 1 ; \leq)$, *where* $0 \triangleq \bigwedge X$ *and* $1 \triangleq \bigvee X$.

Proposition 6.1. *Let $\mathbf{L} \triangleq (X, \vee, \wedge ; \leq)$ be a lattice.*

PRP

$$\mathbf{L} \text{ is FINITE} \implies \mathbf{L} \text{ is BOUNDED}$$

Proposition 6.2. *Let $\mathbf{L} \triangleq (X, \vee, \wedge ; \leq)$ be a lattice with $\bigvee X \triangleq 1$ and $\bigwedge X \triangleq 0$.*

PRP

$$\left\{\begin{array}{l}\mathbf{L} \text{ is BOUNDED} \\ \text{(Definition 6.1 page 89)}\end{array}\right\} \implies \left\{\begin{array}{lclll} x \vee 1 & = & 1 & \forall x \in X & \text{(UPPER BOUNDED)} \quad \text{and} \\ x \wedge 0 & = & 0 & \forall x \in X & \text{(LOWER BOUNDED)} \quad \text{and} \\ x \vee 0 & = & x & \forall x \in X & \text{(JOIN-IDENTITY)} \quad \text{and} \\ x \wedge 1 & = & x & \forall x \in X & \text{(MEET-IDENTITY)} \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} x \vee 1 &= x \vee \left(\bigvee X\right) && \text{by definition of } 1 \text{ (Definition 6.1 page 89)} \\ &= \bigvee X && \text{because } x \in X \end{aligned}$$

[1] $\bigvee X$: Definition 4.21 page 70, $\bigwedge X$:Definition 4.22 (page 70)

$$\begin{aligned}
&= 1 && \text{by definition of } 1 \text{ (Definition 6.1 page 89)} \\
x \wedge 0 &= x \wedge \left(\bigwedge X\right) && \text{by definition of } 0 \text{ (Definition 6.1 page 89)} \\
&= \bigwedge X && \text{because } x \in X \\
&= 0 && \text{by definition of } 0 \text{ (Definition 6.1 page 89)} \\
\boxed{x} &= \bigvee \{x\} \\
&\le \bigvee \{x, 0\} && \text{because } \{x\} \subseteq \{0, x\} \text{ and } \textit{isotone} \text{ property (Proposition 4.3 page 70)} \\
&= \boxed{x \vee 0} && \text{by definition of } \vee \text{ (Definition 4.21 page 70)} \\
&= x \vee \left(\bigwedge X\right) && \text{by definition of } 0 \text{ (Definition 6.1 page 89)} \\
&\le x \vee \left(\bigwedge \{x\}\right) && \text{because } \{x\} \subseteq X \text{ and } \textit{isotone} \text{ property (Proposition 4.3 page 70)} \\
&\le x \vee \left(\bigwedge \{x, x\}\right) && \text{by definition of } \{\cdot\} \\
&= x \vee (x \wedge x) && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)} \\
&= \boxed{x} && \text{by } \textit{absorptive} \text{ property of lattices (Theorem 5.3 page 74)} \\
&= x \wedge (x \vee x) && \text{by } \textit{absorptive} \text{ property of lattices (Theorem 5.3 page 74)} \\
&\triangleq x \wedge \left(\bigvee \{x, x\}\right) && \text{by definition of } \vee \text{ (Definition 4.21 page 70)} \\
&\triangleq x \wedge \left(\bigvee \{x\}\right) && \text{by definition of set } \{\cdot\} \\
&\le x \wedge \left(\bigvee X\right) && \text{because } \{x\} \subseteq \{x, 1\} \text{ and by } \textit{isotone} \text{ property of } \bigwedge \text{ (Proposition 4.3 page 70)} \\
&= \boxed{x \wedge 1} && \text{by definition of } 1 \text{ (Definition 6.1 page 89)} \\
&= \bigwedge \{x, 1\} && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)} \\
&\le \bigwedge \{x\} && \text{because } \{x\} \subseteq \{x, 1\} \text{ and by } \textit{isotone} \text{ property of } \bigwedge \text{ (Proposition 4.3 page 70)} \\
&= \boxed{x}
\end{aligned}$$

⇨

Definition 6.2. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89).*

DEF

A set $\{x_1, x_2, \ldots\}$ *is a* ***partition*** *of an element* $y \in X$ *if*

1.	$x_n \neq 0$	$\forall n$	NON-EMPTY	*and*
2.	$x_n \wedge x_m = 0$	$\forall n \neq m$	MUTUALLY EXCLUSIVE	*and*
3.	$\bigvee_n x_n = 1$			

Definition 6.3. [2] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89).*

DEF

The ***height*** $\mathsf{h}(x)$ *of a point* $x \in \boldsymbol{L}$ *is the* LEAST UPPER BOUND *of the* LENGTHS *(Definition 4.12 page 66) of all the* CHAINS *that have* 0 *and in which* x *is the* LEAST UPPER BOUND. *The* ***height*** $\mathsf{h}(\boldsymbol{L})$ *of the lattice* $\boldsymbol{L}$ *is defined as*

$$\mathsf{h}(\boldsymbol{L}) \triangleq \mathsf{h}(1).$$

[2] Birkhoff (1967) page 5

CHAPTER 7

MODULAR LATTICES

7.1 Modular relation

Definition 7.1. [1] *Let $(X, \vee, \wedge; \leq)$ be a lattice. Let 2^{XX} be the set of all* RELATIONS *in X^2.*

DEF

*The **modularity** relation $Ⓜ \in 2^{XX}$ and the **dual modularity** relation $Ⓜ^* \in 2^{XX}$ are defined as*

$$x Ⓜ y \overset{\text{def}}{\iff} \left\{(x,y) \in X^2 | a \leq y \implies y \wedge (x \vee a) = (y \wedge x) \vee a \quad \forall a \in X\right\}$$

$$x Ⓜ^* y \overset{\text{def}}{\iff} \left\{(x,y) \in X^2 | a \geq y \implies y \vee (x \wedge a) = (y \vee x) \wedge a \quad \forall a \in X\right\}.$$

*A pair $(x,y) \in Ⓜ$ is alternatively denoted as $(x,y)Ⓜ$, and is called a **modular** pair. A pair $(x,y) \in Ⓜ^*$ is alternatively denoted as $(x,y)Ⓜ^*$, and is called a **dual modular** pair. A pair (x,y) that is* NOT *a modular pair ($(x,y) \notin Ⓜ$) is denoted $x \not{Ⓜ} y$. A pair (x,y) that is* NOT *a dual modular pair is denoted $x \not{Ⓜ}^* y$.*

Proposition 7.1. [2] *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ be a lattice.*

PRP

$$\left\{x Ⓜ y \iff x Ⓜ^* y\right\} \quad \forall x,y \in X$$

✎PROOF:

$$\begin{aligned}
x Ⓜ y &\iff \{a \leq y \implies y \wedge (x \vee a) = (y \wedge x) \vee a \quad \forall a \in X\} && \text{by definition of } Ⓜ \text{ (Definition 7.1 page 91)}\\
&\iff \{a \geq y \implies a \wedge (x \vee y) = (a \wedge x) \vee y \quad \forall a \in X\} && \text{by definition of } \geq \text{ (Definition 4.3 page 58)}\\
&\iff \{a \geq y \implies (a \wedge x) \vee y = a \wedge (x \vee y) \quad \forall a \in X\} && \text{by \textit{symmetric} property of } = \text{ (Definition 1.9 page 7)}\\
&\iff \{a \geq y \implies y \vee (x \wedge a) = (y \vee x) \wedge a \quad \forall a \in X\} && \text{by \textit{commutative} prop. of lat. (Theorem 5.3 page 74)}\\
&\iff x Ⓜ^* y && \text{by definition of } Ⓜ^* \text{ (Definition 7.1 page 91)}
\end{aligned}$$

✏

Proposition 7.2. [3] *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ be a lattice.*

---

[1] Stern (1999) page 11, Maeda and Maeda (1970), page 1 ⟨Definition (1.1)⟩, Maeda (1966) page 248
[2] Maeda and Maeda (1970), page 1 ⟨Lemma (1.2)⟩
[3] Maeda and Maeda (1970), page 1

PRP

$$\underbrace{\left.\begin{array}{l} x \le y \quad or \\ y \le x \end{array}\right\}}_{x,\,y \text{ are COMPARABLE}} \implies \left\{\begin{array}{llll} x & Ⓜ & y & and \\ y & Ⓜ & x & and \\ x & Ⓜ^* & y & and \\ y & Ⓜ^* & x. & \end{array}\right.$$

✎PROOF:

$$\begin{aligned}
x \le y &\implies \{a \le y \implies y \wedge (x \vee a) = x \vee a = (y \wedge x) \vee a \quad \forall a \in X\} \\
&\iff xⓂy \qquad \text{by definition of Ⓜ (Definition 7.1 page 91)} \\
x \le y &\implies \{a \le x \implies x \wedge (y \vee a) = x = x \vee a = (x \wedge y) \vee a \quad \forall a \in X\} \\
&\iff yⓂx \qquad \text{by definition of Ⓜ (Definition 7.1 page 91)} \\
x \le y &\implies xⓂ^*y \qquad \text{because } x \le y \implies xⓂy \text{ and by Proposition 7.1 page 91} \\
x \le y &\implies yⓂ^*x \qquad \text{because } x \le y \implies yⓂx \text{ and by Proposition 7.1 page 91}
\end{aligned}$$

✏

Proposition 7.3. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ *be a lattice.*

PRP

x	Ⓜ	x	$\forall x \in X$	(Ⓜ *is* REFLEXIVE)
x	$Ⓜ^*$	x	$\forall x \in X$	($Ⓜ^*$ *is* REFLEXIVE)

✎PROOF: Because $x \le x$ and by Proposition 7.2 (page 91). ✏

7.2 Semimodular lattices

Definition 7.2. [4]

DEF

A lattice $(X, \vee, \wedge; \le)$ *is* ***semimodular*** *if*

$$xⓂy \implies yⓂx$$

A semimodular lattice is also called ***M-symmetric****.*

7.3 Modular lattices

Modular lattices are a generalization of the distributive lattice in the sense that all distributive lattices are modular, but not equivalent because not all modular lattices are distributive (Theorem 8.5 page 117).

Definition 7.3. [5]

DEF

A lattice $(X, \vee, \wedge; \le)$ *is* ***modular*** *if*

$$xⓂy \qquad \forall x, y \in X.$$

7.3.1 Characterizations

This section describes some characterizations of modular lattices— that is, sets of properties that are equivalent to the definition of modular lattices (Definition 7.3 page 92):

[4] Maeda and Maeda (1970), page 3 ⟨Definition (1.7)⟩

[5] Birkhoff (1967) page 82, Maeda and Maeda (1970), page 3 ⟨Definition (1.7)⟩

Ore 1935	(order characterization)	Theorem 7.1	page 93
N5 lattice	(order characterization)	Theorem 7.2	page 94
Riecan 1957	(algebraic characterization)	Theorem 7.3	page 96

Alternatively, any of the sets of properties listed in this section could be used as the definition of modular lattices and the definition would in turn become a theorem/proposition.

Order characterizations

Theorem 7.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge ; \leq)$ *be a lattice.*

THM

$$\begin{array}{lclcll} \boldsymbol{L} \text{ is MODULAR} & \iff & \{x \leq y & \implies & x \vee (z \wedge y) = (x \vee z) \wedge y\} & \forall x,y,z \in X \\ & \iff & x \vee [(x \vee y) \wedge z] & = & (x \vee y) \wedge (x \vee z) & \forall x,y,z \in X \\ & \iff & x \wedge [(x \wedge y) \vee z] & = & (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X \end{array}$$

✎PROOF:

1. Proof that $\boldsymbol{L}$ is *modular* $\iff \{x \leq y \implies x \vee (z \wedge y) = (x \vee z) \wedge y\}$:

$$\begin{array}{rlll} \{\boldsymbol{L} \text{ is } modular\} & \iff \{x \leq y \implies y \wedge (z \vee x) = (y \wedge z) \vee x & \forall x, y, z \in X\} & \text{by Definition 7.3 page 92} \\ & \iff \{a \leq y \implies y \wedge (x \vee a) = (y \wedge x) \vee a & \forall x, y, a \in X\} & \text{by change of variables} \\ & \iff \{x \text{Ⓜ} y \quad \forall x,y \in X\} & & \text{by Definition 7.1 page 91} \end{array}$$

2. Proof that $\boldsymbol{L}$ is *modular* $\iff x \vee [(x \vee y) \wedge z] = (x \vee y) \wedge (x \vee z)$:

   (a) Proof that $\boldsymbol{L}$ is *modular* $\implies x \vee [(x \vee y) \wedge z] = (x \vee y) \wedge (x \vee z)$:
   First note that $x \leq x \vee y$.

$$\begin{aligned} x \vee [(x \vee y) \wedge z] &= x \vee (u \wedge z)|_{u \triangleq x \vee y} && \text{by substitution } u \triangleq x \vee y \\ &= u \wedge (x \vee z)|_{u \triangleq x \vee y} && \text{by modularity hypothesis} \\ &= (x \vee y) \wedge (x \vee z) && \text{because } u \triangleq x \vee y \end{aligned}$$

   (b) Proof that $\boldsymbol{L}$ is *modular* $\impliedby x \vee [(x \vee y) \wedge z] = (x \vee y) \wedge (x \vee z)$:

$$\begin{aligned} x \leq y \implies x \vee (y \wedge z) &= x \vee (y \wedge z) && \text{by right hypothesis and } x \leq y \\ &= x \vee (z \wedge y) && \text{by } commutative \text{ property Theorem 5.3 page 74} \\ &= x \vee [z \wedge (x \vee y)] && \text{because } x \leq y \\ &= x \vee [(x \vee y) \wedge z] && \text{by } commutative \text{ property Theorem 5.3 page 74} \\ &= (x \vee y) \wedge (x \vee z) && \text{by right hypothesis} \\ &= y \wedge (x \vee z) && \text{because } x \leq y \end{aligned}$$

3. Proof that $\boldsymbol{L}$ is modular $\iff \{y \leq x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\}$:

$$\begin{aligned} \boldsymbol{L} \text{ is modular} &\iff \underbrace{\{x \leq y \implies x \vee (y \wedge z) = y \wedge (x \vee z)\}}_{\text{modularity definition (Definition 7.3 page 92)}} && \text{by definition of modular page 92} \\ &\iff \{y \leq x \implies y \vee (x \wedge z) = x \wedge (y \vee z)\} && \text{by change of variables: } x \leftrightarrow y \\ &\iff \underbrace{\{y \leq x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\}}_{\text{dual of Definition 7.3}} && \text{by } reflexive \text{ property of } = \text{ (Definition 1.9 page 7)} \end{aligned}$$

[6] Padmanabhan and Rudeanu (2008) page 39, Ore (1935) page 413 ⟨(2)⟩

4. Proof that $\{y \le x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\} \iff \{x \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee (x \wedge z)\}$:

   (a) Proof that $\{y \le x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\} \implies \{x \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee (x \wedge z)\}$:
   First note that $x \wedge y \le x$.

$$\begin{aligned} x \wedge [(x \wedge y) \vee z] &= x \wedge (u \vee z)|_{u \triangleq x \wedge y} && \text{by substitution } u \triangleq x \wedge y \\ &= u \vee (x \wedge z)|_{u \triangleq x \wedge y} && \text{by left hypothesis} \\ &= (x \wedge y) \vee (x \wedge z) && \text{because } u \triangleq x \wedge y \end{aligned}$$

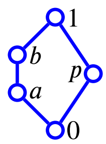
   (b) Proof that $\{y \le x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\} \impliedby \{x \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee (x \wedge z)\}$:

$$\begin{aligned} y \le x \implies x \wedge (y \vee z) &= x \wedge (z \vee y) && \text{by } \textit{commutative} \text{ property Theorem 5.3 page 74} \\ &= x \wedge [z \vee (x \wedge y)] && \text{because } y \le x \\ &= x \wedge [(x \wedge y) \vee z] && \text{by } \textit{commutative} \text{ property Theorem 5.3 page 74} \\ &= (x \wedge y) \vee (x \wedge z) && \text{by right hypothesis} \\ &= y \vee (x \wedge z) && \text{because } y \le x \end{aligned}$$

⇨

Definition 7.4 (N5 lattice/pentagon). [7]

DEF *The* ***N5 lattice*** *is the ordered set* $(\{0, a, b, p, 1\}, \le)$ *with cover relation* $\lessdot = \{(0, a), (a, b), (b, 1), (p, 1), (0, p)\}$.
The N5 lattice is also called the ***pentagon***.



Lemma 7.1. [8]

LEM *The $N5$ lattice (pentagon lattice) is* NON-MODULAR.

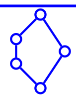
✎PROOF:

$$\begin{aligned} x \le y \implies y \wedge (z \vee x) &= y \wedge b && \text{by Definition 4.21 page 70 (lub)} \\ &= y && \text{by Definition 4.22 page 70 (glb)} \\ &\ne x \\ &= x \vee a && \text{by Definition 4.21 page 70 (lub)} \\ &= (y \wedge z) \vee x && \text{by Definition 4.21 page 70 (lub)} \end{aligned}$$

⇨

Theorem 7.2. [9] *Let* $\boldsymbol{L}$ *be a* LATTICE *(Definition 5.3 page 73).*

THM $\boldsymbol{L}$ *is* MODULAR $\iff$ $\boldsymbol{L}$ *does* NOT *contain $N5$ as a sublattice.*



✎PROOF:

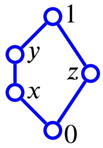
1. Proof that $\boldsymbol{L}$ is modular $\implies$ $\boldsymbol{L}$ does *not* contain $N5$:
   This is because $N5$ is a non-modular lattice. Proof: Lemma 7.1 page 94

[7] Beran (1985) pages 12–13, Dedekind (1900) pages 391–392 ⟨(44) and (45)⟩
[8] Burris and Sankappanavar (1981) page 11
[9] Burris and Sankappanavar (1981) page 11, Grätzer (1971) page 70, Dedekind (1900) ⟨cf Stern 1999 page 10⟩

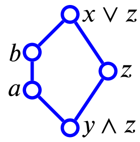
2. Proof that $\boldsymbol{L}$ does not contain $N5 \implies \boldsymbol{L}$ is modular:

   (a) In what follows, we will prove the equivalent contrapositive statement:
   $N5 \in L \qquad \Longleftarrow \qquad L$ is not modular
   (every non-modular lattice *must* contain $N5$).

   (b) We will show that for any choice of $x, y \in L$ such that $x \le y$ and under the following definitions, all non-modular lattices contain the $N5$ lattice illustrated below:



$$a \triangleq x \vee (y \wedge z)$$
$$b \triangleq y \wedge (x \vee z)$$



   (c) Proofs for comparable elements:

$$\begin{aligned}
b &= y \wedge (x \vee z) && \text{by definition of } b \text{ in item (2b)}\\
&\le x \vee z && \text{by definition of } \wedge \text{ page 70}
\end{aligned}$$

$$\begin{aligned}
a &= x \vee (y \wedge z) && \text{by definition of } a \text{ in item (2b)}\\
&\le y \wedge (x \vee z) && \text{by modularity inequality Theorem 5.7}\\
&= b && \text{by definition of } b \text{ in item (2b)}
\end{aligned}$$

$$\begin{aligned}
y \wedge z &\le x \vee (y \wedge z) && \text{by definition of } \vee \text{ page 70}\\
&= a && \text{by definition of } a \text{ in item (2b)}
\end{aligned}$$

$$z \le x \vee z \qquad \text{by definition of } \wedge \text{ page 70}$$

$$y \wedge z \le z \qquad \text{by definition of } \wedge \text{ page 70}$$

   (d) Proofs for noncomparable elements:

$$\begin{aligned}
a \vee z &= [x \vee (y \wedge z)] \vee z && \text{by definition of } a\\
&= z \vee [x \vee (y \wedge z)] && \text{by \textit{commutative property} of lattices (page 74)}\\
&= [z \vee x] \vee (y \wedge z) && \text{by \textit{associative property} of lattices (page 74)}\\
&= [x \vee z] \vee (y \wedge z) && \text{by \textit{commutative property} of lattices (page 74)}\\
&= x \vee [z \vee (y \wedge z)] && \text{by \textit{associative property} of lattices (page 74)}\\
&= x \vee z && \text{by \textit{absorptive property} of lattices (page 74)}
\end{aligned}$$

$$\begin{aligned}
b \vee z &= (b \vee a) \vee z && \text{by previous result}\\
&= b \vee (a \vee z) && \text{by \textit{associative property} of lattices (page 74)}\\
&= b \vee (x \vee z) && \text{by previous result}\\
&= x \vee z && \text{by previous result}
\end{aligned}$$

$$\begin{aligned}
a \wedge z &= (a \wedge b) \wedge z && \text{by previous result}\\
&= a \wedge (b \wedge z) && \text{by \textit{associative property} of lattices (page 74)}\\
&= a \wedge (y \wedge z) && \text{by previous result}\\
&= y \wedge z && \text{by previous result}
\end{aligned}$$

$$\begin{aligned}
b \wedge z &= [y \wedge (x \vee z)] \vee z && \text{by definition of } a\\
&= z \wedge [y \wedge (x \vee z)] && \text{by \textit{commutative property} of lattices (page 74)}
\end{aligned}$$

$$
\begin{aligned}
&= [z \wedge y] \wedge (x \vee z) && \text{by } \textit{associative property} \text{ of lattices (page 74)}\\
&= [y \wedge z] \wedge (x \vee z) && \text{by } \textit{commutative property} \text{ of lattices (page 74)}\\
&= y \wedge [z \wedge (x \vee z)] && \text{by } \textit{associative property} \text{ of lattices (page 74)}\\
&= y \wedge z && \text{by } \textit{absorptive property} \text{ of lattices (page 74)}
\end{aligned}
$$

(e) Thus, *all* non-modular lattices *must* contain an $N5$ sublattice. That is,

$$\textbf{\textit{L}} \text{ is a non-modular lattice} \quad\Longrightarrow\quad \textbf{\textit{L}} \text{ contains an } N5 \text{ sublattice.}$$

And this implies (by the contrapostive of the statement)

$$\textbf{\textit{L}} \text{ does } \textit{not} \text{ contain an } N5 \text{ sublattice} \quad\Longrightarrow\quad \textbf{\textit{L}} \text{ is modular lattice.}$$

Algebraic characterizations

Theorem 7.3. [10] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge ; \leq)$ *be an algebraic structure.*

$$
\text{THM}\quad
\left\{\begin{array}{lcll}
(x \wedge y) \vee (x \wedge z) & = & [(z \wedge x) \vee y] \wedge x & \forall x,y,z \in X \quad \textit{and}\\
{[x \vee (y \vee z)] \wedge z} & = & z & \forall x,y,z \in X
\end{array}\right\}
\iff
\left\{\begin{array}{l}
\textbf{\textit{A}} \textit{ is a}\\
\textbf{\textit{modular lattice}}
\end{array}\right\}
$$

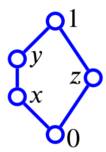
7.3.2 Special cases

Theorem 7.4. [11] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *be a bounded lattice.*

$$
\text{THM}\quad
\left\{\begin{array}{lll}
1. & \textbf{\textit{L}} \textit{ is } \text{COMPLEMENTED} & \textit{and}\\
2. & \textbf{\textit{L}} \textit{ is } \text{ATOMIC} & \textit{and}\\
3. & \textbf{\textit{L}} \textit{ does } \text{NOT} \textit{ contain an N5 lattice} & \\
 & \textit{with elements } 0 \textit{ and } 1 &
\end{array}\right\}
\implies
\left\{\begin{array}{lll}
1. & \textbf{\textit{L}} \textit{ does not contain} & \\
 & \textit{any N5 sublattice} & \textit{and}\\
2. & \textbf{\textit{L}} \textit{ is } \text{MODULAR} &
\end{array}\right\}
$$

7.4 Examples

Example 7.1. The lattice illustrated to the right is the *N5 lattice* (Definition 7.4 page 94). The N5 lattice has a total of $5 \times 5 = 25$ pairs of elements of the form (x, y) where $x, y \in X$. Of these 25, *all* are modular pairs *except* for the pair (z, y). That is, $z \not{\text{Ⓜ}} y$. Therefore, the N5 lattice is *non-semimodular* (and *non-modular*).



✎PROOF:

1. Five are of the form (x, x) and are therefore modular pairs by the *reflexive* property and Proposition 7.3 page 92:
$1 \text{Ⓜ} 1$, $y \text{Ⓜ} y$, $x \text{Ⓜ} x$, $z \text{Ⓜ} z$, $0 \text{Ⓜ} 0$.

[10] Padmanabhan and Rudeanu (2008) pages 42–43, Riečan (1957)

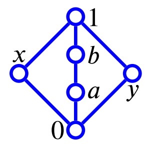
[11] Saliĭ (1988) page 27, Dilworth (1982), pages 333–353 ⟨cf Stern 1999⟩, Stern (1999) page 11, McLaughlin (1956)

2. Of the remaining 20, 16 more are modular pairs simply because they are *comparable* and by Proposition 7.2 (page 91):
$$\begin{array}{llllllll} 1\,\text{Ⓜ}\,y & 1\,\text{Ⓜ}\,x & 1\,\text{Ⓜ}\,0 & y\,\text{Ⓜ}\,x & y\,\text{Ⓜ}\,0 & x\,\text{Ⓜ}\,0 & 1\,\text{Ⓜ}\,z & z\,\text{Ⓜ}\,0 \\ y\,\text{Ⓜ}\,1 & x\,\text{Ⓜ}\,1 & 0\,\text{Ⓜ}\,1 & x\,\text{Ⓜ}\,y & 0\,\text{Ⓜ}\,y & 0\,\text{Ⓜ}\,x & z\,\text{Ⓜ}\,1 & 0\,\text{Ⓜ}\,z \end{array}$$

3. Of the remaining 4, 3 are modular pairs and 1 is a nonmodular pair:
$$\begin{array}{ll} y\,\text{Ⓜ}\,z & x\,\text{Ⓜ}\,z \\ z\not\text{Ⓜ}\,y & z\,\text{Ⓜ}\,x \end{array}$$

$$\begin{array}{rlllllll} x \le y & \implies y \wedge (z \vee x) = y \wedge 1 & = y & \ne x & = 0 \vee x & = (y \wedge z) \vee x & \implies & z\not\text{Ⓜ}\,y \\ 0 \le z & \implies z \wedge (y \vee 0) = z \wedge y & = 0 & & = 0 \vee 0 & = (z \wedge y) \vee 0 & \implies & y\,\text{Ⓜ}\,z \\ 0 \le z & \implies z \wedge (x \vee 0) = z \wedge x & = 0 & & = 0 \vee 0 & = (z \wedge x) \vee 0 & \implies & x\,\text{Ⓜ}\,z \\ 0 \le x & \implies x \wedge (z \vee 0) = x \wedge z & = 0 & & = 0 \vee 0 & = (x \wedge z) \vee 0 & \implies & z\,\text{Ⓜ}\,x \end{array}$$

⇨



Example 7.2. Of the non-comparable pairs in the lattice illustrated to the right, the following are *modular* pairs:

$x\,\text{Ⓜ}\,y$, $y\,\text{Ⓜ}\,x$, $x\,\text{Ⓜ}\,a$, $a\,\text{Ⓜ}\,x$, $y\,\text{Ⓜ}\,a$, $a\,\text{Ⓜ}\,y$, $b\,\text{Ⓜ}\,x$, $b\,\text{Ⓜ}\,y$

and the remaining non-comparable pairs are *non-modular*:

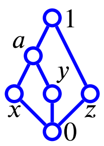
$x\not\text{Ⓜ}\,b$, $y\not\text{Ⓜ}\,b$.

Therefore, although the Hasse diagram shown is horizontally and vertically symmetric, the lattice itself is *not M-symmetric* (not semimodular), and thus also not modular and not distributive.

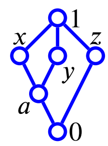
✎Proof:

$$\begin{array}{rlllll} y(x+0) = yx & = yx + 0 & & & & \implies x\,\text{Ⓜ}\,y \\ x(y+0) = xy & = xy + 0 & & & & \implies y\,\text{Ⓜ}\,x \\ a(x+0) = ax & = ax + 0 & & & & \implies x\,\text{Ⓜ}\,a \\ x(a+0) = xa & = xa + 0 & & & & \implies a\,\text{Ⓜ}\,x \\ a(y+0) = ay & = ay + 0 & & & & \implies y\,\text{Ⓜ}\,a \\ y(a+0) = ya & = ya + 0 & & & & \implies a\,\text{Ⓜ}\,y \\ b(x+a) = b1 & = b & \ne a & = 0 + a & = bx + a & \implies x\not\text{Ⓜ}\,b \\ x(b+0) = xb & = xb + 0 & & & & \implies b\,\text{Ⓜ}\,x \\ b(y+a) = b1 & = b & \ne a & = 0 + a & = by + a & \implies y\not\text{Ⓜ}\,b \\ y(b+0) = yb & = yb + 0 & & & & \implies b\,\text{Ⓜ}\,y \end{array}$$

⇨



Example 7.3. The lattices illustrated to the right and left are duals of each other. Both are *non-modular* and both are *non-semimodular*.



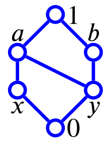
✎Proof:

Left hand side lattice:

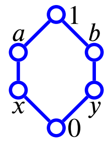
$$\begin{array}{rlllll} a(z+x) = a1 & = a & \ne x & = 0 + x & = az + x & \implies z\not\text{Ⓜ}\,a \\ z(a+0) = za & = za + 0 & & & & \implies a\,\text{Ⓜ}\,z \end{array}$$

Right hand side lattice:

$$\begin{array}{rlllll} z(x+0) = zx & = zx + 0 & & & & \implies x\,\text{Ⓜ}\,z \\ x(z+a) = x1 & = x & \ne a & = 0 + a & = xz + a & \implies z\not\text{Ⓜ}\,x \end{array}$$



Example 7.4. The lattice illustrated to the left is *modular*. The lattice illustrated to the right is *non-modular* and *non-semimodular*.



✎PROOF:

1. Proof that the left hand side is *modular*: because it does not contain the N5 lattice and by Theorem 7.2 (page 94).

2. Proof that the right hand side is *non-modular* and *non-semimodular*:

$$\begin{array}{llllll}
x(b+y) = xb & = 0 & & = 0 + y & = xb + y & \implies b\,\textcircled{M}\,x \\
b(x+y) = b1 & = b & \neq y & = 0 + y & = bx + y & \implies x\,\cancel{\textcircled{M}}\,b \\
y(a+x) = ya & = 0 & & = 0 + x & = ya + x & \implies a\,\textcircled{M}\,y \\
a(y+x) = a1 & = a & \neq x & = 0 + x & = ay + x & \implies y\,\cancel{\textcircled{M}}\,a
\end{array}$$

⇨

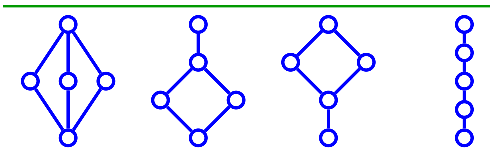
Proposition 7.4. [12] *Let X_n be a finite set with order $n = |X_n|$. Let l_n be the number of unlabeled lattices on X_n, d_n the number of unlabeled distributive lattices on X_n, and m_n the number of unlabeled modular lattices on X_n.*

PRP	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
	l_n	1	1	1	1	2	5	15	53	222	1078	5994	37622	262,776	2,018,305
	m_n	1	1	1	1	2	4	8	16	34	72	157	343		

Example 7.5 (modularity in 5 element sets). There are a total of five unlabeled lattices on a five element set (Proposition 5.2 page 79); and of these five, four are modular, and three of the five are *distributive* (Example 8.2 page 119).

EX

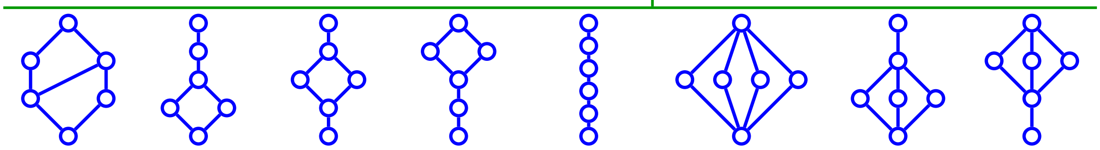
modular lattices on 5 element sets



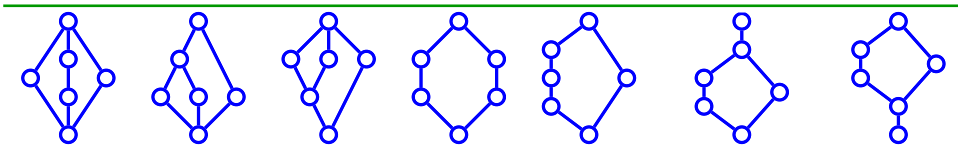
Example 7.6 (modularity in 6 element sets). There are a total of 15 unlabeled lattices on a six element set (Proposition 5.2 page 79 and Example 5.12 page 80); and of these 15, eight are modular, and five of the eight are distributive (Proposition 8.3 page 119). There are no six element non-modular lattices that are also *semimodular*.

EX

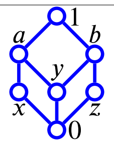
modular and distributive lattices | modular but not distributive



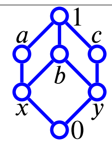
non-semimodular lattices (and non-modular and non-distributive)



[12] l_n: 💻 Sloane (2014) ⟨http://oeis.org/A006966⟩ | m_n: 💻 Sloane (2014) ⟨http://oeis.org/A006981⟩ | l_n: 📖 Heitzig and Reinhold (2002)



Example 7.7. The lattices illustrated to the left and right are duals of each other. Both are *non-modular*. The left hand side lattice is also *non-semimodular*, however the right hand side lattice is *semimodular*.[13]



✎PROOF:

Proof for lattice on left hand side:

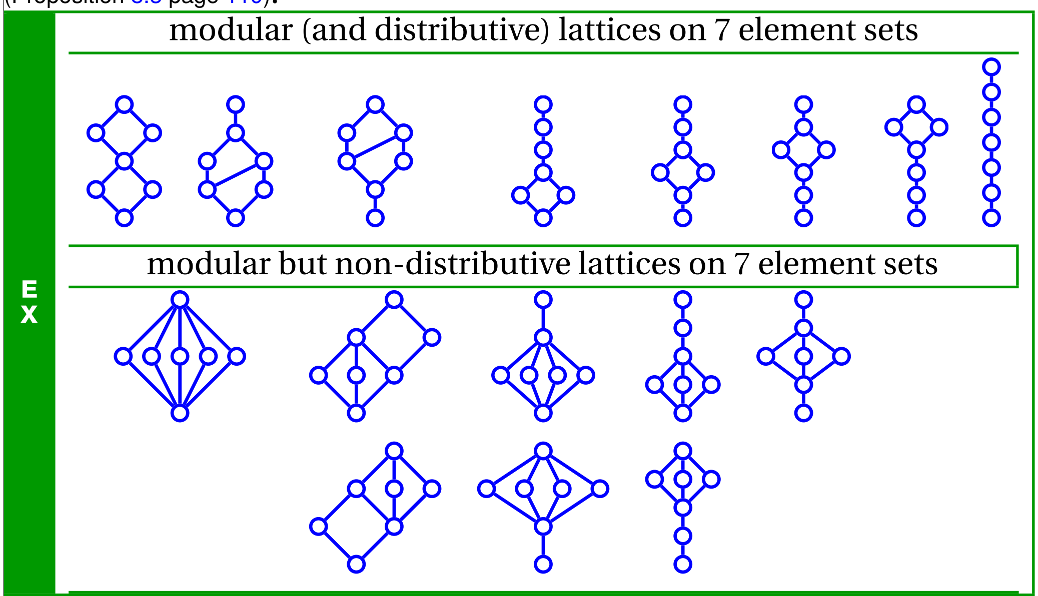
$$\begin{array}{llllll}
y(a+0)=ya & & & =ya+0 & & \implies a\textcircled{M}y\\
a(y+x)=aa & =a & =y+x & =ay+x & & \implies y\textcircled{M}a\\
b(a+z)=b1 & =b & =y+z & =ba+z & & \implies a\textcircled{M}b\\
a(b+x)=a1 & =a & =y+x & =ab+x & & \implies b\textcircled{M}a\\
b(x+z)=b1 & =b & \neq z & =0+z & bx+z & \implies x\not\textcircled{M}b\\
x(b+0)=xb & & = & =xb+0 & & \implies b\textcircled{M}x
\end{array}$$

Proof for lattice on right hand side:

$$\begin{array}{llllll}
c(x+y)=cb & =y & =0+y & =cx+y & & \implies x\textcircled{M}c\\
x(c+0)=xc & =xc+0 & & & & \implies c\textcircled{M}x\\
b(a+x)=ba & =x & =x+x & =ba+x & \text{and} & \\
b(a+y)=b1 & =b & =x+y & =ba+y & & \implies a\textcircled{M}b\\
a(b+x)=ab & =1 & =1+x & =ab+x & & \implies b\textcircled{M}a\\
c(a+y)=c1 & =c & \neq y & =0+y & =ca+y & \implies a\not\textcircled{M}c\\
a(c+x)=a1 & =a & \neq x & =0+x & =ac+x & \implies c\not\textcircled{M}a\\
c(x+y)=cb & =y & & =0+y & =cx+y & \implies x\textcircled{M}c\\
x(c+0)=xc & =xc+0 & & & & \implies c\textcircled{M}x\\
\vdots & & & & &
\end{array}$$

⇨

Example 7.8 (modular lattices on 7 element sets). There are a total of 53 unlabeled lattices on a seven element set (Example 5.13 page 80). Of these 53, 16 are modular, and 8 of these 16 are distributive (Proposition 8.3 page 119).



13 Maeda and Maeda (1970), page 5 ⟨Exercise 1.1⟩

CHAPTER 8

DISTRIBUTIVE LATTICES

8.1 Distributivity relation

Definition 8.1. [1] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE (Definition 5.3 page 73). *Let* 2^{XXX} *be the set of all* RELATIONS *in* X^3.

DEF

The ***distributivity*** *relation* $\textcircled{D} \in 2^{XXX}$ *and the* ***dual distributivity*** *relation* $\textcircled{D}^* \in 2^{XXX}$ *are defined as*

$$\textcircled{D} \triangleq \left\{(x,y,z) \in X^3 | x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\right\} \quad \text{(each } (x,y,z) \text{ is DISJUNCTIVE DISTRIBUTIVE) and}$$
$$\textcircled{D}^* \triangleq \left\{(x,y,z) \in X^3 | x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\right\} \quad \text{(each } (x,y,z) \text{ is CONJUNCTIVE DISTRIBUTIVE).}$$

A triple $(x, y, z) \in \textcircled{D}$ *is alternatively denoted as* $(x, y, z)\textcircled{D}$*, and is called a* ***distributive*** *triple. A triple* $(x, y, z) \in \textcircled{D}^*$ *is alternatively denoted as* $(x, y, z)\textcircled{D}^*$*, and is called a* ***dual distributive*** *triple. A set* $\{x, y, z\} \subseteq X$ *is* ***distributive*** *in* $\mathbf{L}$ *if each of the possible* $3! = 6$ *triples* $[(x, y, z), (z, x, y), \ldots]$ *constructed from the set is* DISTRIBUTIVE *in* $\mathbf{L}$.

8.2 Distributive Lattices

8.2.1 Definition

This section introduces *distributive lattices*. Theorem 5.6 (page 77) demonstrates that *all* lattices $(X, \vee, \wedge; \leq)$ satisfy the following *distributive inequalities*:

$$\begin{array}{rcllll} x \wedge (y \vee z) & \geq & (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X & \text{(join super-distributive)} & \text{and} \\ x \vee (y \wedge z) & \leq & (x \vee y) \wedge (x \vee z) & \forall x,y,z \in X & \text{(meet sub distributive).} & \text{and} \\ (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \leq & (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \forall x,y,z \in X & \text{(median inequality).} & \end{array}$$

Theorem 8.1 (page 102) demonstrates that when *one* of these inequalities is equality, then *all three* of them are equalities. And in this case, the lattice is defined to be *distributive* (next definition).

[1] Maeda and Maeda (1970), page 15 ⟨Definition 4.1⟩, Foulis (1962) page 67, von Neumann (1960), page 32 ⟨Definition 5.1⟩, Davis (1955) page 314 ⟨*disjunctive distributive* and *conjunctive distributive* f.⟩

Definition 8.2. [2]

DEF *A lattice* $(X, \vee, \wedge; \leq)$ *is* ***distributive*** *if*

$$(x, y, z) \in \textcircled{D} \quad \forall x, y, z \in X$$

Are all lattices *distributive*? The answer is "no". Lemma 8.1 (page 104) and Lemma 8.2 (page 105) demonstrate two lattices that are *not* distributive: the N5 lattice (Definition 7.4 page 94) and the M3 lattice (Definition 8.3 page 105).

8.2.2 Characterizations

This section describes some characterizations (equational bases) of distributive lattices both in terms of lattices (order characterizations) and in terms of abstract algebraic structures (algebraic characterizations).

- Order characterizations (first assuming a structure is a lattice):
  - Median property 1894 Theorem 8.1 page 102
  - Birkhoff distributivity criterion 1934 Theorem 8.2 page 106
  - Cancellation property 1934 Theorem 8.3 page 109
- Algebraic characterizations (first assuming nothing):
  - Birkhoff 1946 Proposition 8.1 page 112
  - Birkhoff 1948 Proposition 8.2 page 112
  - Sholander 1951 Theorem 8.4 page 112

Alternatively, any of the sets of properties listed in this section could be used as the definition of distributive lattices and the definition would in turn become a theorem/proposition.

In addition, if a lattice is *uniquely complemented* and satisfies any one of a number of *Huntington properties*, then it is also *distributive* (Theorem 9.2 page 123), and hence also a *Boolean algebra* (Definition 10.1 page 127).

Order characterizations

By the definition given in Definition 8.2 (page 102), a lattice is *distributive* if the meet operation $\wedge$ distributes over the join operation $\vee$. And in view that the properties of lattices are self-dual, it is perhaps not surprising that the dual of the identity of Definition 8.2 is also true for any distributive lattice— that is, the join operation $\vee$ distributes over the meet operation $\wedge$ (next theorem). But besides these two identities that are duals of each other, there is another identity that is not only equivalent to the first two, but is a dual of itself. This is called the *median property*,[3] and is given by (3) in Theorem 8.1 (next theorem).

Theorem 8.1. [4] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition 5.3 page 73).*

[2] Burris and Sankappanavar (1981) page 10, Birkhoff (1948) page 133, Ore (1935) page 414 ⟨*arithmetic axiom*⟩, Birkhoff (1933a) page 453, Balbes and Dwinger (1975) page 48 ⟨Definition II.5.1⟩

[3] *median property*: see also Literature item 5 page 124

[4] Dilworth (1984) page 237, Burris and Sankappanavar (1981) page 10, Ore (1935) page 416 ⟨(7),(8), Theorem 3⟩, Ore (1940) ⟨cf Gratzer 2003 page 159⟩, Schröder (1890) page 286 ⟨cf Birkhoff(1948)p.133⟩, Korselt (1894) ⟨cf Birkhoff(1948)p.133⟩

THM ***L* is DISTRIBUTIVE** (Definition 8.2 page 102)

$$\iff x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x,y,z \in X \qquad \text{(DISJUNCTIVE DISTRIBUTIVE)}$$
$$\iff x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x,y,z \in X \qquad \text{(CONJUNCTIVE DISTRIBUTIVE)}$$
$$\iff (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \quad \forall x,y,z \in X \qquad \text{(MEDIAN PROPERTY)}$$

✎PROOF: Let the join operation $\vee$ be represented by $+$, the meet operation $\wedge$ be represented by juxtaposition, and let meet take algebraic precedence over join $(+)$.

1. Proof that *distributive* $\iff$ *disjunctive distributive*:

$$\begin{aligned}
\{\boldsymbol{L} \text{ is distributive}\} &\iff \{x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in X\} && \text{by Definition 8.2 page 102}\\
&\iff \{(x, y, z) \in \mathbb{D} \quad \forall x, y, z \in X\} && \text{by Definition 8.1 page 101}
\end{aligned}$$

2. Proof that *disjunctive distributive* $\implies$ *conjunctive distributive*:

$$\begin{aligned}
x + (yz) &= \underbrace{[x + (xy)]}_{\text{expand } x \text{ wrt } y} + (yz) && \text{by \textit{absorptive} property of lattices page 74}\\
&= x + [(xy) + (yz)] && \text{by \textit{associative} property of lattices page 74}\\
&= x + [(yx) + (yz)] && \text{by \textit{commutative} property of lattices page 74}\\
&= x + [y(x + z)] && \text{by left hypothesis}\\
&= \underbrace{[x(x + z)]}_{\text{expand } x \text{ wrt } z} + [y(x + z)] && \text{by \textit{absorptive} property of lattices page 74}\\
&= [(x + z)x] + [(x + z)y] && \text{by \textit{commutative} property of lattices page 74}\\
&= (x + z)(x + y) && \text{by left hypothesis}\\
&= (x + y)(x + z) && \text{by \textit{commutative} property of lattices page 74}
\end{aligned}$$

3. Proof that *conjunctive distributive* $\implies$ *disjunctive distributive*:

$$\begin{aligned}
x(y + z) &= \underbrace{[x(x + y)]}_{\text{expand } x \text{ wrt } y} (y + z) && \text{by \textit{absorptive} property of lattices page 74}\\
&= x[(x + y)(y + z)] && \text{by \textit{associative} property of lattices page 74}\\
&= x[(y + x)(y + z)] && \text{by \textit{commutative} property of lattices page 74}\\
&= x[y + (xz)] && \text{by right hypothesis}\\
&= \underbrace{[x + (xz)]}_{\text{expand } x \text{ wrt } z} [y + (xz)] && \text{by \textit{absorptive} property of lattices page 74}\\
&= [(xz) + x][(xz) + y] && \text{by \textit{commutative} property of lattices page 74}\\
&= (xz) + (xy) && \text{by left hypothesis}\\
&= (xy) + (xz) && \text{by \textit{commutative} property of lattices page 74}
\end{aligned}$$

4. Proof that *disjunctive distributive* $\implies$ *median property*:

$$\begin{aligned}
&(x + y)(x + z)(y + z)\\
&= (x + y)[(x + z)y + (x + z)z] && \text{by \textit{disjunctive distributive} hypothesis}\\
&= (x + y)[y(x + z) + z(x + z)] && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&= (x + y)(yx + yz + zx + zz) && \text{by \textit{disjunctive distributive} hypothesis}\\
&= (x + y)(xy + xz + yz + z) && \text{by Theorem 5.3 page 74}\\
&= (x + y)xy + (x + y)xz + (x + y)yz + (x + y)z && \text{by \textit{disjunctive distributive} hypothesis}
\end{aligned}$$

$$\begin{aligned}
&= xy(x+y) + xz(x+y) + yz(x+y) + z(x+y) && \text{by } \textit{commutative} \text{ property (Theorem 5.3 page 74)}\\
&= xyx + xyy + xzx + xzy + yzx + yzy + zx + zy && \text{by } \textit{disjunctive distributive} \text{ hypothesis}\\
&= xy + xy + xz + xyz + xyz + yz + xz + yz && \text{by Theorem 5.3 page 74}\\
&= xy + xyz + xz + yz && \text{by } \textit{idempotent} \text{ property (Theorem 5.3 page 74)}\\
&= (xy)(xy) + xyz + xz + yz && \text{by } \textit{idempotent} \text{ property (Theorem 5.3 page 74)}\\
&= (xy)(xy + z) + xz + yz && \text{by } \textit{disjunctive distributive} \text{ hypothesis}\\
&= xy + xz + yz && \text{by } \textit{absorptive} \text{ property (Theorem 5.3 page 74)}
\end{aligned}$$

5. Proof that *median property* $\implies$ *disjunctive distributive*:

   (a) Proof that $\boldsymbol{L}$ is *modular*:

$$\begin{aligned}
y \le x \implies x(y+z) &= x(x+z)(y+z) && \text{by } \textit{absorptive} \text{ property (Theorem 5.3 page 74)}\\
&= (x+y)(x+z)(y+z) && \text{by } y \le x \text{ hypothesis}\\
&= xy + xz + yz && \text{by } \textit{median property} \text{ hypothesis}\\
&= y + xz + yz && \text{by } y \le x \text{ hypothesis}\\
&= y + xz && \text{by } \textit{absorptive} \text{ property (Theorem 5.3 page 74)}\\
&\implies \boldsymbol{L} \text{ is } \textit{modular}
\end{aligned}$$

   (b) Proof that $a + ab = a$:

$$\begin{aligned}
& ab \le a && \text{by definition of } \wedge \text{ Definition 4.22 page 70}\\
& \implies a + ab = a && \text{by definition of } \vee \text{ Definition 4.21 page 70}
\end{aligned}$$

   (c) Proof that *median property* $\implies$ *disjunctive distributive*:

$$\begin{aligned}
x(y+z) &= xx(y+z) && \text{by } \textit{idempotent} \text{ property (Theorem 5.3 page 74)}\\
&= \underbrace{x(x+y)}_{x}\underbrace{x(x+z)}_{x}(y+z) && \text{by } \textit{absorptive} \text{ property (Theorem 5.3 page 74)}\\
&= x[(x+y)(x+z)(y+z)] && \text{by Theorem 5.3 page 74}\\
&= x(xy + \underbrace{xz + yz}_{z'}) && \text{by } \textit{median property} \text{ hypothesis}\\
&= x(xy) + x(xz + \underbrace{yz}_{z''}) && \text{by item (5a) and by Theorem 7.1 page 93}\\
&= x(xy) + x(xz) + x(yz) && \text{by item (5a) and by Theorem 7.1 page 93}\\
&= xy + xz + xyz && \text{by Theorem 5.3 page 74}\\
&= xy + xz && \text{by item (5b)}
\end{aligned}$$

✏

Lemma 8.1. [5]

LEM *The $N5$ lattice is* NON-DISTRIBUTIVE

5 Burris and Sankappanavar (1981) page 11

✎PROOF:

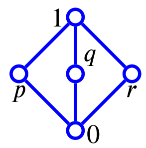
$$\begin{aligned}
y \wedge (x \vee z) &= y \wedge b && \text{by Definition 4.21 page 70 (lub)}\\
&= y && \text{by Definition 4.22 page 70 (glb)}\\
&= y \vee a && \text{by Definition 4.21 page 70 (lub)}\\
&= y \vee (y \wedge z) && \text{by Definition 4.22 page 70 (glb)}\\
&\neq x \vee (y \wedge z) && \text{because } x \neq y\\
&= (y \wedge x) \vee (y \wedge z) && \text{by Definition 4.22 page 70 (glb)}
\end{aligned}$$

✏

Definition 8.3 (M3 lattice/diamond). [6]

DEF *The **M3 lattice** is the ordered set* $(\{0, p, q, r, 1\}, \leq)$ *with covering relation*
$\prec = \{(p, 1), (q, 1), (r, 1), (0, p), (0, q), (0, r)\}$.
*The M3 lattice is also called the **diamond**,*

and is illustrated by the Hasse diagram to the right.



Remark 8.1. The M3 lattice is isomorphic to the lattices

- $(\mathcal{P}(\{x, y, z\}), \leq)$ [7] where $\mathcal{P}(\{x, y, z\})$ is the set of *partitions* on $\{x, y, z\}$ and with $\leq$ defined as in Proposition 16.8 (page 231)
- $(\mathcal{R}(\{x, y\}), \subseteq)$ where $\mathcal{R}(\{x, y\})$ is the set of *rings of sets* on $\{x, y\}$
- $(\mathcal{A}(\{x, y, z\}), \subseteq)$ where $\mathcal{A}(\{x, y, z\})$ is the set of *algebras of sets* on $\{x, y, z\}$.

See Example 16.11 (page 231), Example 16.7 (page 229), Example 16.16 (page 243), and Figure 16.8 (page 245).

Lemma 8.2. [8]

LEM $\left\{ \begin{array}{l} \text{L is an \textbf{M3 lattice}} \\ \text{(Definition 8.3 page 105)} \end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & \text{L is NOT distributive} & \text{(Definition 8.2 page 102) and} \\ 2. & \text{L IS modular} & \text{(Definition 7.3 page 92)} \end{array} \right\}$

✎PROOF:

1. Proof that $M3$ is non-distributive:

$$\begin{aligned}
x \wedge (a \vee c) &= x \wedge y && \text{by def. of l.u.b. page 70}\\
&= x && \text{by def. of g.l.b. page 70}\\
&\neq b\\
&= b \vee b && \text{by Theorem 5.3 page 74 (idempotent property)}\\
&= \underbrace{(x \wedge a)}_{b} \vee \underbrace{(x \wedge c)}_{b} && \text{by def. of g.l.b. page 70}
\end{aligned}$$

2. Proof that $M3$ is modular: (proof by exhaustion)

$$\begin{aligned}
x \vee (y \wedge a) &= x \vee a\\
&= y\\
&= y \wedge y
\end{aligned}
\qquad
\begin{aligned}
&= y \wedge (x \vee a)\\
x \vee (y \wedge b) &= x \vee b\\
&= x\\
&= y \wedge x
\end{aligned}
\qquad
\begin{aligned}
&= y \wedge (x \vee b)\\
x \vee (y \wedge c) &= x \vee c\\
&= y\\
&= y \wedge y
\end{aligned}$$

[6] Beran (1985) pages 12–13, Korselt (1894) page 157 $\langle p_1 \equiv x,\ p_2 \equiv y,\ p_3 \equiv z,\ g \equiv 1,\ 0 \equiv 0\rangle$
[7] Saliĭ (1988) page 22
[8] Birkhoff (1948) page 6, Burris and Sankappanavar (1981) page 11, Korselt (1894) page 157 ⟨cf Salii1988 p. 37⟩

$$\begin{aligned}
&= y \wedge (x \vee c)\\
a \vee (y \wedge x) &= a \vee x\\
&= y\\
&= y \wedge y\\
&= y \wedge (a \vee x)\\
a \vee (y \wedge b) &= a \vee b\\
&= a\\
&= y \wedge a\\
&= y \wedge (a \vee b)\\
a \vee (y \wedge c) &= a \vee c\\
&= y\\
&= y \wedge y\\
&= y \wedge (a \vee c)\\
c \vee (y \wedge a) &= c \vee a\\
&= y\\
&= y \wedge y\\
&= y \wedge (c \vee a)\\
c \vee (y \wedge x) &= c \vee x\\
&= y\\
&= y \wedge y\\
&= y \wedge (c \vee x)\\
c \vee (y \wedge b) &= c \vee b\\
&= c\\
&= y \wedge c\\
&= y \wedge (c \vee b)
\end{aligned}$$

$$\begin{aligned}
b \vee (y \wedge a) &= b \vee a\\
&= a\\
&= y \wedge a\\
&= y \wedge (b \vee a)\\
b \vee (y \wedge x) &= b \vee x\\
&= x\\
&= y \wedge x\\
&= y \wedge (b \vee x)\\
b \vee (y \wedge c) &= b \vee c\\
&= c\\
&= y \wedge c\\
&= y \wedge (b \vee c)\\
b \vee (a \wedge x) &= b \vee b\\
&= b\\
&= a \wedge x\\
&= a \wedge (b \vee x)\\
b \vee (a \wedge y) &= b \vee a\\
&= a\\
&= a \wedge y\\
&= a \wedge (b \vee y)\\
b \vee (a \wedge c) &= b \vee b\\
&= b\\
&= a \wedge c\\
&= a \wedge (b \vee c)
\end{aligned}$$

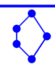
$$\begin{aligned}
b \vee (x \wedge a) &= b \vee b\\
&= b\\
&= x \wedge a\\
&= x \wedge (b \vee a)\\
b \vee (x \wedge c) &= b \vee b\\
&= b\\
&= x \wedge c\\
&= x \wedge (b \vee c)\\
b \vee (x \wedge y) &= b \vee x\\
&= x\\
&= x \wedge y\\
&= x \wedge (b \vee y)\\
b \vee (c \wedge x) &= b \vee b\\
&= b\\
&= c \wedge x\\
&= c \wedge (b \vee x)\\
b \vee (c \wedge y) &= b \vee c\\
&= c\\
&= c \wedge y\\
&= c \wedge (b \vee y)\\
b \vee (c \wedge a) &= b \vee b\\
&= b\\
&= c \wedge a\\
&= c \wedge (b \vee a)
\end{aligned}$$

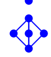
⇒

The *Birkhoff distributivity criterion* (next) demonstrates that a lattice is distributive *if and only if* it does not contain either the N5 or M3 lattices. If a lattice does contain either of these, it is *not* distributive. If a lattice is distributive, it does *not* contain either the N5 or M3 lattices. There was a similar theorem for *modular* lattices and the N5 lattice (Theorem 7.2 page 94).

Theorem 8.2 (Birkhoff distributivity criterion). [9] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE.

THM $\boldsymbol{L}$ *is* DISTRIBUTIVE $\iff \begin{cases} \boldsymbol{L} \textit{ does } \textbf{not} \textit{ contain } N5 \textit{ as a sublattice} & \text{and} \\ \boldsymbol{L} \textit{ does } \textbf{not} \textit{ contain } M3 \textit{ as a sublattice} \end{cases}$





✎PROOF:

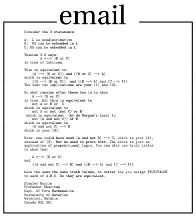
1. Proof that $\boldsymbol{L}$ is distributive $\implies$ $\boldsymbol{L}$ does *not* contain $N5$:
   This follows directly from Lemma 8.1 (page 104).

[9] Burris and Sankappanavar (1981) page 12, Birkhoff (1948) page 134, Birkhoff and Hall (1934)

2. Proof that $\boldsymbol{L}$ is distributive $\implies$ $\boldsymbol{L}$ does *not* contain $M3$:
   This follows directly from Lemma 8.2 (page 105).

3. Proof that $\boldsymbol{L}$ is distributive $\impliedby$ $N5 \notin L$ and $M3 \notin L$:

   (a) Proof that this statement is equivalent to email



```
Consider the 3 statements:

A:  L is nondistributive
B:  M3 can be embedded in L
C: N5 can be embedded in L

Theorem 3.6 says:
        A <--> (B or C)
is true of lattices.

This is equivalent to:
    (A --> (B or C)) and ((B or C) --> A)
which is equivalent to
    ((A --> (B or C))  and ((B --> A) and (C --> A))
The last two implications are your (1) and (2).

So what remains after these two is to show
    A --> (B or C)
is true. But this is equivalent to
    not A or B or  C
which is equivalent to
    not A or not (not C) or B
 which is equivalent  (by de Morgan's Laws) to
    not (A and not C)) or B
which is equivalent to
    (A and not C) --> B
which is your (3).

Note: one could have used (A and not B) --> C, which is your (4),
instead of (3). But no need to prove both. The above is just an
application of propositional logic. You can also use truth tables
to show that

    A <--> (B or C)
and
    ((A and not C) --> B) and ((B --> A) and (C --> A))

have the same the same truth values, no matter how you assign TRUE/FALSE
to each of A,B,C. So they are equivalent.

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Many many thanks to University of Waterloo I
his brilliant help with the logical structure of th
as a pdf file, zoom in on the figure to the left t
October 9 email.)

$$(\boldsymbol{L} \text{ is nondistributive}) \wedge (N5 \notin L) \implies (M3 \in L):$$

Let $P \equiv Q$ denote that statement P is equivalent to statement Q. Then …

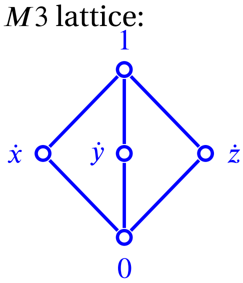
$$\begin{array}{lll}
& (\boldsymbol{L} \text{ is distributive}) \impliedby (N5 \notin L) \wedge (M3 \notin L) & \\
\equiv & (\boldsymbol{L} \text{ is nondistributive}) \implies (N5 \in L) \vee (M3 \in L) & \text{contrapositive} \\
\equiv & \neg(\boldsymbol{L} \text{ is nondistributive}) \vee [(N5 \in L) \vee (M3 \in L)] & \text{by definition of } \implies \text{ (Definition 14.1 page 199)} \\
\equiv & \big[\neg(\boldsymbol{L} \text{ is nondistributive}) \vee (N5 \in L)\big] \vee (M3 \in L) & \text{by associative property (Theorem 14.2 page 199)} \\
\equiv & \neg\neg\big[\neg(\boldsymbol{L} \text{ is nondistributive}) \vee \neg(N5 \notin L)\big] \vee (M3 \in L) & \text{by involutary property (Theorem 14.2 page 199)} \\
\equiv & \neg\big[(\boldsymbol{L} \text{ is nondistributive}) \wedge (N5 \notin L)\big] \vee (M3 \in L) & \text{by de Morgan's law (Theorem 14.2 page 199)} \\
\equiv & (\boldsymbol{L} \text{ is nondistributive}) \wedge (N5 \notin L) \implies (M3 \in L) & \text{by definition of } \implies \text{ (Definition 14.1 page 199)}
\end{array}$$

   (b) Proof that $\boldsymbol{L}$ is *not* distributive and $N5 \notin L \implies M3 \in L$:

      i. Because $N5 \notin L$ and by Theorem 7.2 (page 94), $\boldsymbol{L}$ is modular (so we can use the modularity property of Definition 7.3 page 92).

      ii. We will show that the five values defined below form an $M3$ lattice:

$$\begin{aligned}
b &\triangleq (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \\
a &\triangleq (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \\
\dot{x} &\triangleq (x \wedge b) \vee a \\
\dot{y} &\triangleq (y \wedge b) \vee a \\
\dot{z} &\triangleq (z \wedge b) \vee a
\end{aligned}$$



      iii. Proof that $a \leq b$:

$$\begin{array}{rll}
a &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \text{by definition of } a \text{ (item (3(b)ii))} \\
&= (x \wedge y \wedge x) \vee (x \wedge z \wedge z) \vee (y \wedge z \wedge z) & \text{by \textit{idempotent property} of lattices (page 74)} \\
&\leq (x \vee x \vee y) \wedge (y \vee z \vee z) \wedge (x \vee z \vee z) & \text{by minimax inequality Theorem 5.5 page 76} \\
&= (x \vee y) \wedge (y \vee z) \wedge (x \vee z) & \text{by \textit{idempotent property} of lattices (page 74)} \\
&= (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \text{by \textit{commutative property} of lattices (page 74)} \\
&= b & \text{by definition of } b \text{ (item (3(b)ii))}
\end{array}$$

$$\bigvee \left\{ \begin{array}{c} \bigwedge \left\{ \begin{array}{ccc} x & y & x \end{array} \right\} \\ \hline \bigwedge \left\{ \begin{array}{ccc} x & z & z \end{array} \right\} \\ \hline \bigwedge \left\{ \begin{array}{ccc} y & z & z \end{array} \right\} \end{array} \right\} \leq \bigwedge \left\{ \begin{array}{c|c|c} \bigvee & \bigvee & \bigvee \\ x & y & x \\ x & z & z \\ y & z & z \end{array} \right\}$$

      iv. Proof that $a \leq \dot{x} \leq \dot{y} \leq \dot{z} \leq b$:

         A. By item (3(b)iii), $a \leq b$.

         B. By definition of $\wedge$, $(x \wedge b)$ must be less than or equal to b.

C. By definition of $\vee$, $(x \wedge b) \vee a$ must be greater than or equal to a.

D. By definition of $\dot{x}$ (item (3(b)ii)), $a \le \dot{x} \le b$.

E. The proofs for $a \le \dot{y} \le b$ and $a \le \dot{z} \le b$ are essentially identical to that of $a \le \dot{x} \le b$.

v. Proof that $\dot{x} \wedge \dot{y} = \dot{x} \wedge \dot{z} = \dot{y} \wedge \dot{z} = a$:

$$\begin{aligned}
\dot{x} \wedge \dot{y} &= [\underbrace{(x \wedge b) \vee a}_{\dot{x}}] \wedge \dot{y} && \text{by definition of } \dot{x} \text{ item (3(b)ii)} \\
&= [(x \wedge b) \underline{\wedge \dot{y}}] \underline{\vee a} && \text{by modularity page 92} \\
&= [(x \wedge b) \wedge \underbrace{((y \wedge b) \vee a)}_{\dot{y}}] \vee a && \text{by definition of } \dot{y} \text{ item (3(b)ii)} \\
&= [(x \wedge b) \wedge (y \underline{\vee a}) \underline{\wedge b}] \vee a && \text{by modularity page 92} \\
&= [(x \wedge b) \wedge (y \vee a)] \vee a && \text{by idempotent property page 74} \\
&= \left[\left(x \wedge \underbrace{[(x \vee y) \wedge (x \vee z) \wedge (y \vee z)]}_{b}\right) \wedge \right. \\
&\qquad \left.\left(y \vee \underbrace{[(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)]}_{a}\right)\right] \vee a && \text{by definitions of } a \text{ and } b \text{ item (3(b)ii)} \\
&= [(x \wedge (y \vee z)) \wedge (y \vee (x \wedge z))] \vee a && \text{by absorption property page 74} \\
&= \left[x \wedge \left(\underline{y \vee}\left(\underline{(y \vee z) \wedge} (x \wedge z)\right)\right)\right] \vee a && \text{by modularity page 92} \\
&= [x \wedge (y \vee (x \wedge z))] \vee a && \text{because } (x \wedge z) \le (y \vee z) \\
&= \left[(x \wedge z) \underline{\vee} (\underline{x \wedge} y)\right] \vee a && \text{by modularity page 92} \\
&= [(x \wedge z) \vee (x \wedge y)] \vee \underbrace{[(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)]}_{a} && \text{by definition of } a \text{ item (3(b)ii)} \\
&= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) && \text{by idempotent property page 74} \\
&= a && \text{by definition of } a \text{ item (3(b)ii)}
\end{aligned}$$

vi. To prove that $\dot{x} \wedge \dot{z} = a$, simply replace $\dot{y}$ with $\dot{z}$ and y with z in item (3(b)v).

vii. To prove that $\dot{y} \wedge \dot{z} = a$, simply replace $\dot{x}$ with $\dot{z}$ and x with z in item (3(b)v).

viii. Proof that $\dot{x} \vee \dot{y} = b$:

$$\begin{aligned}
\dot{x} \vee \dot{y} &= [\underbrace{(x \wedge b) \vee a}_{\dot{x}}] \vee \dot{y} && \text{by definition of } \dot{x} \text{ item (3(b)ii)} \\
&= [(x \underline{\vee a}) \underline{\wedge b}] \vee \dot{y} && \text{by modularity page 92} \\
&= [(x \vee a) \underline{\vee \dot{y}}] \underline{\wedge b} && \text{by modularity page 92} \\
&= [(x \vee a) \vee \underbrace{((y \wedge b) \vee a)}_{\dot{y}}] \wedge b && \text{by definition of } \dot{y} \text{ item (3(b)ii)} \\
&= [(x \vee a) \vee (y \wedge b)] \wedge b && \text{by idempotent property page 74} \\
&= \left[\left(x \vee \underbrace{[(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)]}_{a}\right) \vee \right. \\
&\qquad \left.\left(y \wedge \underbrace{[(x \vee y) \wedge (x \vee z) \wedge (y \vee z)]}_{b}\right)\right] \wedge b && \text{by definitions of } a \text{ and } b \text{ item (3(b)ii)} \\
&= [(x \vee (y \wedge z)) \vee (y \wedge (x \vee z))] \wedge b && \text{by absorption property page 74}
\end{aligned}$$

$$
\begin{aligned}
&= [x \vee (y \wedge z) \vee (y \wedge (x \vee z))] \wedge b && \text{by associative property page 74}\\
&= \left[x \vee \left(\underline{y \wedge}\left[\underline{(y \wedge z) \vee (x \vee z)}\right]\right)\right] \wedge b && \text{by modularity page 92}\\
&= [x \vee (y \wedge (x \vee z))] \wedge b && \text{by Definition 4.21 and Definition 4.22}\\
&= \left[\underline{(x \vee z) \wedge}\left(\underline{x \vee} y\right)\right] \wedge b && \text{by modularity page 92}\\
&= [(x \vee z) \wedge (x \vee y)] \wedge \underbrace{[(x \vee z) \wedge (x \vee y) \wedge (y \vee z)]}_{b} && \text{by definition of } b \text{ item (3(b)ii)}\\
&= (x \vee z) \wedge (x \vee y) \wedge (y \vee z) && \text{by idempotent property page 74}\\
&= b && \text{by definition of } b \text{ item (3(b)ii)}
\end{aligned}
$$

ix. To prove that $\dot{x} \vee \dot{z} = b$, simply replace $\dot{y}$ with $\dot{z}$ and y with z in item (3(b)viii).

x. To prove that $\dot{y} \vee \dot{z} = b$, simply replace $\dot{x}$ with $\dot{z}$ and x with z in item (3(b)viii).

✏

Theorem 8.3 (cancellation criterion). [10] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE.

THM

$$
\boldsymbol{L} \text{ is DISTRIBUTIVE} \iff \underbrace{\left\{\left\{\begin{array}{lcll} x \vee z &=& y \vee z & \forall x,y,z \in X \quad \text{and} \quad (1)\\ x \wedge z &=& y \wedge z & \forall x,y,z \in X \quad\quad\quad\;\; (2)\end{array}\right\} \implies x = y\right\}}_{\text{CANCELLATION } property}
$$

✎PROOF:

1. Proof that *distributive* property $\implies$ *cancellation* property:

$$
\begin{aligned}
x &= x(x+z) && \text{by } absorbtive \text{ property (Theorem 5.3 page 74)}\\
&= x(y+z) && \text{by (1)}\\
&= xy + xz && \text{by } distributive \text{ hypothesis}\\
&= xy + yz && \text{by (2)}\\
&= yx + yz && \text{by } commutative \text{ property (Theorem 5.3 page 74)}\\
&= y(x+z) && \text{by } distributive \text{ hypothesis}\\
&= y(y+z) && \text{by (1)}\\
&= y && \text{by } absorbtive \text{ property (Theorem 5.3 page 74)}
\end{aligned}
$$

2. Proof that *distributive* property $\impliedby$ *cancellation* property:

   (a) Define

$$
\begin{aligned}
a &\triangleq x(y+z)\\
b &\triangleq y(x+z)\\
c &\triangleq z(x+y)\\
d &\triangleq (x+y)(x+z)(y+z)
\end{aligned}
$$

[10] Blyth (2005) pages 67–68, Birkhoff and Hall (1934)

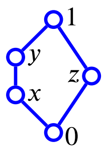
(b) Proof that $ab = xy$, $ac = xz$, and $bc = yz$:

$$\begin{aligned}
ab &= [x(y+z)][y(x+z)] && \text{by item (2a)}\\
&= [x(x+z)][y(y+z)] && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&= xy && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}\\
ac &= [x(y+z)][z(x+y)] && \text{by item (2a)}\\
&= [x(x+y)][z(z+y)] && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&= xz && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}\\
bc &= [y(x+z)][z(x+y)] && \text{by item (2a)}\\
&= [y(y+x)][z(z+x)] && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&= yz && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}
\end{aligned}$$

(c) Proof of some inequalities:

$$\begin{aligned}
a &= x(y+z) && \text{by item (2a)}\\
&\le (x+y)(y+z) && \text{by definition of } \vee\\
&\le (x+y)[(x+y)+z] && \text{by definition of } \vee\\
&= x+y && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}\\
a &= x(y+z) && \text{by item (2a)}\\
&= x(z+y) && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&\le (x+z)(z+y) && \text{by definition of } \vee\\
&\le (x+z)[(x+z)+y] && \text{by definition of } \vee\\
&= x+z && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}\\
b &= y(x+z) && \text{by item (2a)}\\
&\le (x+y)(x+z) && \text{by definition of } \vee\\
&\le (x+y)[(x+y)+z] && \text{by definition of } \vee\\
&= x+y && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}\\
c &= z(x+y) && \text{by item (2a)}\\
&\le (x+z)(x+y) && \text{by definition of } \vee\\
&\le (x+z)[(x+z)+y] && \text{by definition of } \vee\\
&= x+z && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}
\end{aligned}$$

(d) Proof that $\boldsymbol{L}$ is *modular*:

i. Consider the following $N5$ lattice: 

ii. For the $N5$ lattice, the *cancellation* property does not hold because
$$\begin{aligned} 1 &= x+z &= y+z &= 1 \quad\text{and}\\ 0 &= xz &= yz &= 0, \end{aligned}$$
but yet $x \neq y$.

iii. Because $N5$ does *not* support the *cancellation* property and by the hypothesis that $\boldsymbol{L}$ *does* support the cancellation property, $\boldsymbol{L}$ therefore does *not* contain $N5$.

iv. Because $\boldsymbol{L}$ does not contain $N5$ and by Theorem 7.2 (page 94), $\boldsymbol{L}$ is *modular*.

(e) Proof that $a+b=a+c=b+c=d$:

$$\begin{aligned}
a+b &= a+y(x+z) && \text{by definition of } c \text{ (item (2a) page 109)}\\
&= (a+y)(x+z) && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= [x(y+z)+y](x+z) && \text{by definition of } a \text{ (item (2a) page 109)}\\
&= [y+x(y+z)](x+z) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= (y+x)(y+z)(x+z) && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= (x+y)(x+z)(y+z) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= d && \text{by definition of } d \text{ (item (2a) page 109)}\\
a+c &= a+z(x+y) && \text{by definition of } c \text{ (item (2a) page 109)}\\
&= (a+z)(x+y) && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= [x(y+z)+z](x+y) && \text{by definition of } a \text{ (item (2a) page 109)}\\
&= [z+x(y+z)](x+y) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= (z+x)(y+z)(x+y) && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= (x+y)(x+z)(y+z) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= d && \text{by definition of } d \text{ (item (2a) page 109)}\\
b+c &= b+z(x+y) && \text{by definition of } c \text{ (item (2a) page 109)}\\
&= (b+z)(x+y) && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= [y(x+z)+z](x+y) && \text{by definition of } a \text{ (item (2a) page 109)}\\
&= [z+y(x+z)](x+y) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= (z+y)(x+z)(x+y) && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= (x+y)(x+z)(y+z) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= d && \text{by definition of } d \text{ (item (2a) page 109)}
\end{aligned}$$

(f) Proof that $(a+yz)+c=(b+xz)+c$ and $(a+yz)c=(b+xz)c$:

$$\begin{aligned}
(a+yz)+c &= (a+bc)+c && \text{by item (2b)}\\
&= a+(c+cb) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= a+c && \text{by \emph{absorptive} property (Theorem 5.3 page 74)}\\
&= d && \text{by item (2e)}\\
&= b+c && \text{by item (2e)}\\
&= b+(c+ca) && \text{by \emph{absorptive} property (Theorem 5.3 page 74)}\\
&= (b+ac)+c && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= (b+xz)+c && \text{by item (2b)}\\
(a+yz)c &= c(a+yz) && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= c(a+bc) && \text{by item (2b)}\\
&= (bc+a)c && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= bc+ac && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= ac+bc && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= (ac+b)c && \text{by \emph{modularity}: item (2c) and item (2d)}\\
&= (b+ac)c && \text{by \emph{commutative} property (Theorem 5.3 page 74)}\\
&= (b+xz)c && \text{by item (2b)}
\end{aligned}$$

(g) Proof that $a+yz=b+xz$: by item (2f) and *cancellation hypothesis*.

(h) Proof that $a + yz = d$:

$$\begin{aligned}
a + yz &= (a + yz) + (a + yz) && \text{by \textit{idempotent} property (Theorem 5.3 page 74)}\\
&= (a + yz) + (b + xz) && \text{by item (2g)}\\
&= (a + bc) + (b + ac) && \text{by item (2b)}\\
&= (a + ac) + (b + bc) && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&= a + b && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}\\
&= d && \text{by item (2e)}
\end{aligned}$$

(i) Proof that $z(x + y) = zx + zy$ (*distributivity*):

$$\begin{aligned}
z(x + y) &= c && \text{by item (2a)}\\
&= c(c + a) && \text{by \textit{absorptive} property (Theorem 5.3 page 74)}\\
&= c(a + c) && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&= cd && \text{by item (2e)}\\
&= c(a + yz) && \text{by item (2h)}\\
&= c(a + bc) && \text{by item (2b)}\\
&= (bc + a)c && \text{by \textit{commutative} property (Theorem 5.3 page 74)}\\
&= bc + ac && \text{by \textit{modularity}: item (2c) and item (2d)}\\
&= yz + xz && \text{by item (2b)}\\
&= zx + zy && \text{by \textit{commutative} property (Theorem 5.3 page 74)}
\end{aligned}$$

✏

Algebraic characterizations

Proposition 8.1. [11] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\left\{ \begin{array}{l} \boldsymbol{A} \text{ is a} \\ \textbf{distributive lattice} \end{array} \right\} \iff \left\{ \begin{array}{llcll}
1. & x \wedge x & = & x & \forall x \in X \quad \text{and}\\
2. & x \vee 1 & = & 1 \vee x = 1 & \forall x \in X \quad \text{and}\\
3. & x \wedge 1 & = & 1 \wedge x = x & \forall x \in X \quad \text{and}\\
4. & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) & \forall x, y, z \in X \quad \text{and}\\
5. & (y \vee z) \wedge x & = & (y \wedge x) \vee (z \wedge x) & \forall x, y, z \in X
\end{array} \right\}$$

Proposition 8.2. [12] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\left\{ \begin{array}{l} \boldsymbol{A} \text{ is a} \\ \textbf{distributive lattice} \end{array} \right\} \iff \left\{ \begin{array}{llcll}
1. & x \wedge x & = & x & \forall x \in X \quad \text{and}\\
2. & x \vee y & = & y \vee x & \forall x, y \in X \quad \text{and}\\
3. & x \wedge y & = & y \wedge x & \forall x, y \in X \quad \text{and}\\
4. & x \wedge (y \wedge z) & = & (x \wedge y) \wedge z & \forall x, y, z \in X \quad \text{and}\\
5. & x \wedge (x \vee y) & = & x & \forall x, y \in X \quad \text{and}\\
6. & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) & \forall x, y, z \in X.
\end{array} \right\}$$

Theorem 8.4. [13] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

THM

$$\left\{ \begin{array}{l} \boldsymbol{A} \text{ is a} \\ \textbf{distributive lattice} \end{array} \right\} \iff \left\{ \begin{array}{llcll}
1. & x \wedge (x \vee y) & = & x & \forall x, y \in X \quad \text{and}\\
2. & x \wedge (y \vee z) & = & (z \wedge x) \vee (y \wedge x) & \forall x, y, z \in X
\end{array} \right\}$$

[11] Birkhoff (1948) pages 135–136, Birkhoff and Birkhoff (1946) ⟨???⟩
[12] Padmanabhan and Rudeanu (2008) page 58, Birkhoff (1948) pages 134–135 ⟨Ex.6⟩
[13] Padmanabhan and Rudeanu (2008) page 59, Sholander (1951) page 28 ⟨P1, P2⟩

✎PROOF:

1. Proof that $xx = x$ (*meet idempotent* property):

$$\begin{aligned}
xx &= x[x(x+x)] && \text{by 1}\\
&= x(xx+xx) && \text{by 2}\\
&= xxx+xxx && \text{by 2}\\
&= xxx(x+x)+xxx(x+x) && \text{by 1}\\
&= xx(xx+xx)+xx(xx+xx) && \text{by 2}\\
&= xx+xx && \text{by 1}\\
&= x(x+x) && \text{by 2}\\
&= x && \text{by 1}
\end{aligned}$$

2. Proof that $x + x = x$ (*join idempotent* property):

$$\begin{aligned}
x+x &= xx+xx && \text{by \textit{meet idempotent} property (item (1) page 113)}\\
&= x(x+x) && \text{by 2}\\
&= x && \text{by 1}
\end{aligned}$$

3. Proof that $xy = yx$ (*meet commutative* property):

$$\begin{aligned}
xy &= xy+xy && \text{by \textit{join idempotent} property (item (2) page 113)}\\
&= y(x+x) && \text{by 2}\\
&= yx && \text{by \textit{join idempotent} property (item (2) page 113)}
\end{aligned}$$

4. Proof that $x(y + z) = xy + xz$ (*conjunctive distributive* property):

$$\begin{aligned}
x(y+z) &= yx+zx && \text{by 2}\\
&= xy+xz && \text{by \textit{meet commutative} property (item (3) page 113)}
\end{aligned}$$

5. Proof that $x + xy = x$ (*join absorptive* property):

$$\begin{aligned}
x &= x(x+y) && \text{by 1}\\
&= yx+xx && \text{by 2}\\
&= yx+x && \text{by \textit{meet idempotent} property (item (1) page 113)}\\
&= (yx+x)(yx+x) && \text{by \textit{meet idempotent} property (item (1) page 113)}\\
&= x(yx+x)+yx(yx+x) && \text{by 2}\\
&= x(yx+x)+yx && \text{by 1}\\
&= [xx+(yx)x]+yx && \text{by 2}\\
&= x(yx+x)+yx && \text{by 2}\\
&= x(yx+xx)+yx && \text{by \textit{meet idempotent} property (item (1) page 113)}\\
&= x[x(x+y)]+yx && \text{by 2}\\
&= xx+yx && \text{by 1}\\
&= x+yx && \text{by \textit{meet idempotent} property (item (1) page 113)}\\
&= x+xy && \text{by \textit{meet commutative} property (item (3) page 113)}
\end{aligned}$$

6. Proof that $x+y=y+x$ (*join commutative* property):

$$\begin{aligned}
x+y &= (x+y)(x+y) && \text{by } \textit{meet idempotent} \text{ property (item (2) page 113)}\\
&= y(x+y)+x(x+y) && \text{by 2}\\
&= y(x+y)+x && \text{by 1}\\
&= (yy+xy)+x && \text{by 2}\\
&= (y+xy)+x && \text{by } \textit{meet idempotent} \text{ property (item (2) page 113)}\\
&= (y+yx)+x && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= y+x && \text{by } \textit{join absorptive} \text{ property (item (5) page 113)}
\end{aligned}$$

7. Proof that $(x+y)+z=x+(y+z)$ (*join associative* property):

   (a) Let $P \triangleq (x+y)+z$ and $Q \triangleq x+(y+z)$

   (b) Proof that $Px=x$, $Py=y$, and $Pz=z$:

$$\begin{aligned}
Px &= [(x+y)+z]x && \text{by definition of } P \text{ (item (7a) page 114)}\\
&= x[(x+y)+z] && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= x(x+y)+xz && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 113)}\\
&= x+xz && \text{by 1}\\
&= x && \text{by } \textit{join absorptive} \text{ property (item (5) page 113)}\\
Py &= [(x+y)+z]y && \text{by definition of } P \text{ (item (7a) page 114)}\\
&= y[(x+y)+z] && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= y(x+y)+yz && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 113)}\\
&= y(y+x)+yz && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= y+yz && \text{by 1}\\
&= y && \text{by } \textit{join absorptive} \text{ property (item (5) page 113)}\\
Pz &= [(x+y)+z]z && \text{by definition of } P \text{ (item (7a) page 114)}\\
&= z[(x+y)+z] && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= z[z+(x+y)] && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= z && \text{by 1}
\end{aligned}$$

   (c) Proof that $Qx=x$, $Qy=y$, and $Qz=z$:

$$\begin{aligned}
Qx &= [x+(y+z)]x && \text{by definition of } Q \text{ (item (7a) page 114)}\\
&= x[x+(y+z)] && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= x && \text{by 1}\\
Qy &= [x+(y+z)]y && \text{by definition of } Q \text{ (item (7a) page 114)}\\
&= y[x+(y+z)] && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= yx+y(y+z) && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 113)}\\
&= yx+y && \text{by 2}\\
&= y+yx && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= y && \text{by } \textit{join absorptive} \text{ property (item (5) page 113)}\\
Qz &= [x+(y+z)]z && \text{by definition of } Q \text{ (item (7a) page 114)}\\
&= z[x+(y+z)] && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= zx+z(y+z) && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 113)}\\
&= z(z+y)+zx && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= z+zx && \text{by 1}\\
&= z+zx && \text{by 1}\\
&= z && \text{by } \textit{join absorptive} \text{ property (item (5) page 113)}
\end{aligned}$$

(d) Proof that $(x+y)+z = x+(y+z)$:

$$\begin{aligned}
(x+y)+z &= Qx + (Qy + Qz) && \text{by item (7c)}\\
&= Qx + Q(y+z) && \text{by \textit{conjunctive distributive} property (item (4) page 113)}\\
&= Q[x+(y+z)] && \text{by \textit{conjunctive distributive} property (item (4) page 113)}\\
&= QP && \text{by definition of } Q \text{ (item (7a) page 114)}\\
&= PQ && \text{by \textit{meet commutative} property (item (3) page 113)}\\
&= PQ && \text{by \textit{meet commutative} property (item (3) page 113)}\\
&= P[x+(y+z)] && \text{by definition of } Q \text{ (item (7a) page 114)}\\
&= Px + P(y+z) && \text{by \textit{conjunctive distributive} property (item (4) page 113)}\\
&= Px + (Py + Pz) && \text{by \textit{conjunctive distributive} property (item (4) page 113)}\\
&= x + (y+z) && \text{by item (7b)}
\end{aligned}$$

8. Proof that $x + yz = (x+y)(x+z)$ (*disjunctive distributive* property):

$$\begin{aligned}
(x+y)(x+z) &= (x+y)x + (x+y)z && \text{by \textit{conjunctive distributive} property (item (4) page 113)}\\
&= x(x+y) + z(x+y) && \text{by \textit{meet commutative} property (item (3) page 113)}\\
&= x + z(x+y) && \text{by 1}\\
&= x + (zx + zy) && \text{by \textit{conjunctive distributive} property (item (4) page 113)}\\
&= x + (xz + yz) && \text{by \textit{meet commutative} property (item (3) page 113)}\\
&= (x + xz) + yz && \text{by \textit{join associatiave} property (item (7) page 114)}\\
&= x + yz && \text{by \textit{join absorptive} property (item (5) page 113)}
\end{aligned}$$

9. Proof that $(xy)z = x(yz)$ (*meet associative* property):

(a) Let $P \triangleq (xy)z$ and $Q \triangleq x(yz)$

(b) Proof that $P + x = x$, $P + y = y$, and $P + z = z$:

$$\begin{aligned}
P + x &= (xy)z + x && \text{by definition of } P \text{ (item (9a) page 115)}\\
&= x + (xy)z && \text{by \textit{join commutative} property (item (6) page 114)}\\
&= [x + (xy)][x + z] && \text{by \textit{disjunctive distributive} property (item (8) page 115)}\\
&= x[x+z] && \text{by 1}\\
&= x && \text{by 1}\\
P + y &= (xy)z + y && \text{by definition of } P \text{ (item (9a) page 115)}\\
&= y + (xy)z && \text{by \textit{join commutative} property (item (6) page 114)}\\
&= y + (yx)z && \text{by \textit{meet commutative} property (item (3) page 113)}\\
&= [y + (yx)][y + z] && \text{by \textit{disjunctive distributive} property (item (8) page 115)}\\
&= y[y+z] && \text{by 1}\\
&= y && \text{by 1}\\
P + z &= (xy)z + z && \text{by definition of } P \text{ (item (9a) page 115)}\\
&= z + (xy)z && \text{by \textit{join commutative} property (item (6) page 114)}\\
&= z + z(yx) && \text{by \textit{meet commutative} property (item (3) page 113)}\\
&= z && \text{by 1}
\end{aligned}$$

(c) Proof that $Q + x = x$, $Q + y = y$, and $Q + z = z$:

$$\begin{aligned}
Q + x &= x(yz) + x && \text{by definition of } Q \text{ (item (9a) page 115)}\\
&= x + x(yz) && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= x && \text{by 1}\\
Q + y &= x(yz) + y && \text{by definition of } Q \text{ (item (9a) page 115)}\\
&= y + x(yz) && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= (y + x)(y + yz) && \text{by } \textit{disjunctive distributive} \text{ property (item (8) page 115)}\\
&= (y + x)y && \text{by 1}\\
&= y(y + x) && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= y && \text{by 1}\\
Q + z &= x(yz) + z && \text{by definition of } Q \text{ (item (9a) page 115)}\\
&= z + x(yz) && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= (z + x)(z + yz) && \text{by } \textit{disjunctive distributive} \text{ property (item (8) page 115)}\\
&= (z + x)(z + zy) && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= (z + x)z && \text{by 1}\\
&= z(z + x) && \text{by } \textit{meet commutative} \text{ property (item (3) page 113)}\\
&= z && \text{by 1}
\end{aligned}$$

(d) Proof that $(xy)z = x(yz)$:

$$\begin{aligned}
(xy)z &= [(Q + x)(Q + y)](Q + z) && \text{by item (9c)}\\
&= (Q + xy)(Q + z) && \text{by } \textit{disjunctive distributive} \text{ property (item (8) page 115)}\\
&= Q + (xy)z && \text{by } \textit{disjunctive distributive} \text{ property (item (8) page 115)}\\
&= Q + P && \text{by definition of } P \text{ (item (9a) page 115)}\\
&= P + Q && \text{by } \textit{join commutative} \text{ property (item (6) page 114)}\\
&= P + x(yz) && \text{by definition of } Q \text{ (item (9a) page 115)}\\
&= (P + x)(P + yz) && \text{by } \textit{disjunctive distributive} \text{ property (item (8) page 115)}\\
&= (P + x)[(P + y)(P + z)] && \text{by } \textit{disjunctive distributive} \text{ property (item (8) page 115)}\\
&= x(yz) && \text{by item (9b)}
\end{aligned}$$

10. Proof that $\boldsymbol{A}$ is a *distributive* lattice:

(a) Proof that $\boldsymbol{A}$ is a lattice:

i. $\boldsymbol{A}$ is *idempotent* by item (1) and item (2).

ii. $\boldsymbol{A}$ is *commutative* by item (3) and item (6).

iii. $\boldsymbol{A}$ is *associative* by item (9) and item (7).

iv. $\boldsymbol{A}$ is *absorptive* by 1 and item (5).

v. Because $\boldsymbol{A}$ is *idempotent*, *commutative*, *associative*, and *absorptive*, then by Theorem 5.3 (page 74), $\boldsymbol{A}$ is a *lattice*.

(b) Proof that $\boldsymbol{A}$ is *distributive*: by item (4) and Definition 8.2 (page 102).

✎

8.2.3 Properties

Distributive lattices are a special case of modular lattices. That is, all distributive lattices are modular, but not all modular lattices are distributive (next theorem). An example is the M3 lattice— it is modular, but yet it is not *distributive* (Lemma 8.2 page 105).

Theorem 8.5. [14] *Let* $(X, \vee, \wedge; \leq)$ *be a lattice.*

THM $(X, \vee, \wedge; \leq)$ *is* DISTRIBUTIVE $\underset{\not\Leftarrow}{\Longrightarrow}$ $(X, \vee, \wedge; \leq)$ *is* MODULAR.

✎PROOF:

1. Proof that distributivity $\Longrightarrow$ modularity:

$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) && \text{by distributive hypothesis} \\ &= y \wedge (x \vee z) && \text{by } x \leq y \text{ hypothesis} \end{aligned}$$

2. Proof that distributivity $\not\Longleftarrow$ modularity:
   By Lemma 8.2 page 105, the $M3$ lattice is modular, but yet it is *non-distributive*.

✏

Theorem 8.6 (Birkhoff's Theorem). [15] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice. Let* 2^X *be the power set of some set* X.

THM $\left\{ \begin{array}{l} \mathbf{L} \textit{ is} \\ \text{DISTRIBUTIVE} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} \mathbf{L} \textit{ is isomorphic to a sublattice of } (2^X, \cup, \cap; \subseteq) \\ \textit{for some set } X. \end{array} \right\}$

Theorem 8.7. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

THM $\left\{ \begin{array}{l} \mathbf{L} \textit{ is} \\ \text{DISTRIBUTIVE} \end{array} \right\} \Longrightarrow \left\{ \overbrace{\left(\bigwedge_{n=1}^{N} x_n \right) \vee y = \bigvee_{n=1}^{N} (x_n \vee y)}^{\textit{tautology}} \quad \overbrace{\left(\bigvee_{n=1}^{N} x_n \right) \wedge y = \bigwedge_{n=1}^{N} (x_n \wedge y)}^{\textit{dual}} \right\}$

✎PROOF:

1. Proof that $\left(\bigwedge_{n=1}^{N} x_n \right) \vee y = \bigvee_{n=1}^{N} (x_n \vee y)$ (by induction):

   Proof for $N = 1$ case:

$$\begin{aligned} \left(\bigwedge_{n=1}^{N=1} x_n \right) \vee y &= x_1 \vee y && \text{by definition of } \wedge \\ &= \bigwedge_{n=1}^{N=1} (x_n \vee y) && \text{by definition of } \wedge \end{aligned}$$

   Proof for $N = 2$ case:

$$\begin{aligned} \left(\bigwedge_{n=1}^{N=2} x_n \right) \vee y &= (x_1 \vee y) \wedge (x_2 \vee y) && \text{by Theorem 8.1 page 102} \\ &= \bigwedge_{n=1}^{N=2} (x_n \vee y) && \text{by definition of } \wedge \end{aligned}$$

[14] Birkhoff (1948) page 134, Burris and Sankappanavar (1981) page 11
[15] Saliĭ (1988) page 24

Proof that (N case) $\implies$ ($N+1$ case):

$$\begin{aligned}
\left(\bigwedge_{n=1}^{N+1} x_n\right) \vee y &= \left[\left(\bigwedge_{n=1}^{N} x_n\right) \wedge x_{N+1}\right] \vee y && \text{by definition of } \wedge\\
&= \left[\left(\bigwedge_{n=1}^{N} x_n\right) \vee y\right] \wedge (x_{N+1} \vee y) && \text{by Theorem 8.1 page 102}\\
&= \left[\bigwedge_{n=1}^{N} (x_n \vee y)\right] \wedge (x_{N+1} \vee y) && \text{by left hypothesis}\\
&= \bigwedge_{n=1}^{N+1} (x_n \vee y) && \text{by definition of } \wedge
\end{aligned}$$

2. Proof that $\left(\bigvee_{n=1}^{N} x_n\right) \wedge y = \bigwedge_{n=1}^{N}(x_n \wedge y)$: by *principle of duality* (Theorem 5.4 page 75).

Theorem 8.8. [16] *Let* $(X, \vee, \wedge; \leq)$ *be a lattice.*

THM

$$\underbrace{(X, \leq)}_{\text{ordered set}} \text{ is LINEARLY ORDERED} \implies \underbrace{(X, \vee, \wedge; \leq)}_{\text{lattice}} \text{ is DISTRIBUTIVE}$$

✎PROOF:

$$\begin{array}{llllll}
x \leq y \leq z \implies & x \wedge (y \vee z) & = x \wedge z & = x & = x \vee x & = (x \wedge y) \vee (x \wedge z)\\
x \leq z \leq y \implies & x \wedge (y \vee z) & = x \wedge y & = x & = x \vee x & = (x \wedge y) \vee (x \wedge z)\\
z \leq x \leq y \implies & x \wedge (y \vee z) & = x \wedge y & = x & = x \vee z & = (x \wedge y) \vee (x \wedge z)\\
y \leq z \leq x \implies & x \wedge (y \vee z) & = x \wedge z & = z & = y \vee z & = (x \wedge y) \vee (x \wedge z)\\
y \leq x \leq z \implies & x \wedge (y \vee z) & = x \wedge z & = x & = y \vee x & = (x \wedge y) \vee (x \wedge z)\\
z \leq y \leq x \implies & x \wedge (y \vee z) & = x \wedge y & = y & = y \vee z & = (x \wedge y) \vee (x \wedge z)
\end{array}$$

Theorem 8.9. [17] *Let* $Y^X \triangleq \{\mathsf{f} : X \to Y\}$ *(the set of all functions from the set* X *to the set* Y*).*

THM

$(Y, \varovee, \varowedge; \precapprox)$ *is a distributive lattice* $\implies$ $\left(Y^X, \vee, \wedge; \leq\right)$ *is a distributive lattice*
where $\mathsf{f} \leq \mathsf{g} \iff \mathsf{f}(x) \precapprox \mathsf{g}(x) \quad \forall x \in X$

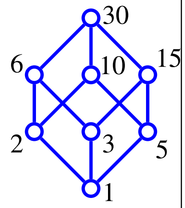
✎PROOF:

$$\begin{aligned}
[\mathsf{f} \wedge (\mathsf{g} \vee \mathsf{h})](x) &= \mathsf{f}(x) \varowedge (\mathsf{g}(x) \varovee \mathsf{h}(x))\\
&= (\mathsf{f}(x) \varowedge \mathsf{g}(x)) \varovee (\mathsf{f}(x) \varowedge \mathsf{h}(x)) && \text{because } (Y, \varovee, \varowedge; \precapprox) \text{ is distributive}\\
&= [\mathsf{f} \wedge \mathsf{g}](x) \vee [\mathsf{f} \wedge \mathsf{h}](x) && \text{because } (Y, \varovee, \varowedge; \precapprox) \text{ is distributive}
\end{aligned}$$

[16] MacLane and Birkhoff (1999) page 484

[17] MacLane and Birkhoff (1999) page 484

8.2.4 Examples



Example 8.1. [18] For any pair of natural numbers $n, m \in \mathbb{N}$, let $n|m$ represent the relation "m divides n", $\operatorname{lcm}(n, m)$ the least common multiple of n and m, and $\gcd(n, m)$ the greatest common divisor of n and m.

EX $(\mathbb{N}, \gcd, \operatorname{lcm}\,;\, |)$ is a *distributive* lattice.

✎PROOF:

1. For all $m \in \mathbb{N}$, m can be analyzed as a product of prime factors such that
$$m = 2^{\mathsf{e}(1)} 3^{\mathsf{e}(2)} 5^{\mathsf{e}(3)} 7^{\mathsf{e}(4)} \cdots p_k^{\mathsf{e}(k)}$$
where $\mathsf{e}(n)$ is a function $\mathsf{e} : \mathbb{N} \to \mathbb{W}$ expressing the number of prime factors p_n in m. For example,
$$84 = 2^2 3^1 7^1 \qquad \Longrightarrow \qquad \mathsf{e}(1) = 2,\ \mathsf{e}(2) = 1,\ \mathsf{e}(3) = 0,\ \mathsf{e}(4) = 1,\ \mathsf{e}(5) = 0,\ \mathsf{e}(6) = 0, \ldots$$
2. Because $\mathbb{W}$ is a chain and by Theorem 8.8 page 118, $(\mathbb{W}, \vee, \wedge\,;\, \leq)$ is a distributive lattice where $\leq$ is the standard ordering on $\mathbb{W}$ and $\vee$ and $\wedge$ are defined in terms of $\leq$.
3. Let $\mathbb{W}^{\mathbb{N}}$ represent the set of all functions $\mathsf{e} : \mathbb{N} \to \mathbb{W}$. By Theorem 8.9 page 118, $(\mathbb{W}^{\mathbb{N}}, \otimes, \oslash\,;\, \precsim)$ is also a distributive lattice where $\precsim$ is defined in terms of $\leq$ as
$$\mathsf{e} \precsim \mathsf{f} \qquad \Longleftrightarrow \qquad \mathsf{e}(n) \leq \mathsf{f}(n) \quad \forall n \in \mathbb{N}.$$
4. Again by Theorem 8.9 page 118, $(\mathbb{N}, \gcd, \operatorname{lcm}\,;\, |)$ is a distributive lattice because $m|k$ if $e_m(n) \precsim e_k(n)$.

✏

Proposition 8.3. [19] *Let X_n be a finite set with order $n = |X_n|$. Let l_n be the number of unlabeled lattices on X_n, m_n the number of unlabeled modular lattices on X_n, and d_n the number of unlabeled distributive lattices on X_n.*

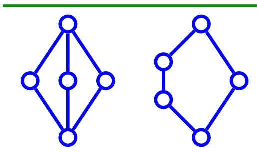
THM

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
l_n	1	1	1	1	2	5	15	53	222	1078	5994	37622			
m_n	1	1	1	1	2	4	8	16	34	72	157	343			
d_n	1	1	1	1	2	3	5	8	15	26	47	82	151	269	494

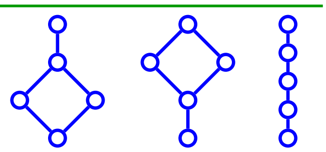
Example 8.2. [20] There are a total of five unlabeled lattices on a five element set; and of these five, three are distributive (Proposition 8.3 page 119). Example 5.11 (page 80) illustrated all five of the unlabeled lattices, Example 7.5 (page 98) illustrated the 4 modular lattices, and the following table illustrates the 3 distributive lattices. Note that none of these lattices are *complemented* (none are *Boolean* (Definition 10.1 page 127)).

EX

non-distributive



distributive



[18] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩

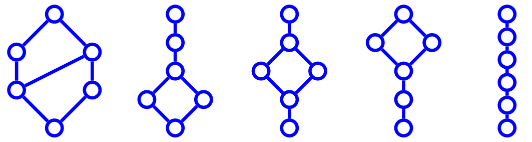
[19] l_n: Sloane (2014) ⟨http://oeis.org/A006966⟩ | m_n: Sloane (2014) ⟨http://oeis.org/A006981⟩ | d_n: Sloane (2014) ⟨http://oeis.org/A006982⟩ | l_n: Heitzig and Reinhold (2002) | m_n: Thakare et al. (2002)? | d_n: Erné et al. (2002), page 17

[20] Erné et al. (2002), pages 4–5

Example 8.3. [21] There are a total of 15 unlabeled lattices on a six element set; and of these 15, five are distributive (Proposition 8.3 page 119). Example 5.12 (page 80) illustrated all 15 of the unlabeled lattices, Example 7.6 (page 98) illustrated the 8 modular lattices, and the following illustrates the 5 distributive lattices.
Note that none of these lattices are *complemented* (none are *Boolean* (Definition 10.1 page 127)).

EX

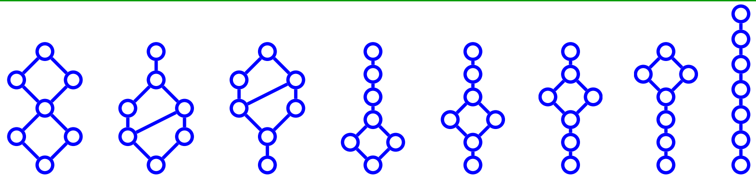
distributive lattices on 6 element sets



Example 8.4. [22] There are a total of 53 unlabeled lattices on a seven element set; and of these, 8 are *distributive* (Proposition 8.3 page 119). Example 5.13 (page 80) illustrated all 53 of the unlabeled lattices, Example 7.8 (page 99) illustrated the 16 *modular* lattices, and the following illustrates the 8 distributive lattices. Note that none of these lattices are *complemented* (none are *Boolean* (Definition 10.1 page 127)).

EX

distributive lattices on 7 element sets



[21] Erné et al. (2002), pages 4–5
[22] Erné et al. (2002), pages 4–5

CHAPTER 9

COMPLEMENTED LATTICES

9.1 Definitions

Definition 9.1. [1] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89).*

DEF

An element $x' \in X$ *is a* ***complement*** *of an element* x *in* $\mathbf{L}$ *if*

1. $x \wedge x' = 0$ (NON-CONTRADICTION) *and*
2. $x \vee x' = 1$ (EXCLUDED MIDDLE).

An element x' *in* $\mathbf{L}$ *is the* UNIQUE COMPLEMENT *of* x *in* $\mathbf{L}$ *if* x' *is a* COMPLEMENT *of* x *and* y' *is a* COMPLEMENT *of* $x \implies x' = y'$. $\mathbf{L}$ *is* ***complemented*** *if every element in* X *has a complement in* X. $\mathbf{L}$ *is* ***uniquely complemented*** *if every element in* X *has a unique complement in* X. *A complemented lattice that is* NOT *uniquely complemented is* ***multiply complemented***. *A* ***complemented lattice*** *is optionally denoted* $(X, \vee, \wedge, 0, 1\,;\, \leq)$.

Definition 9.1 (previous) introduced the concept of a *complement* of a lattice. Definition 9.2 (next) introduces the concept of a *relative complement* in an *interval* (Definition A.2 page 279).

Definition 9.2. [2] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge\,;\, \leq)$ *be a lattice.*

DEF

An element $y \in X$ *is a* ***relative complement*** *of* x *in* $[a, b]$ *with respect to* $\mathbf{L}$ *if*

1. $x \vee y = b$ *and*
2. $x \wedge y = a$.

A lattice $\mathbf{L}$ *is* ***relatively complemented*** *if every element in every closed interval* $[a, b]$ *in* $\mathbf{L}$ *has a complement in* $[a, b]$.

9.2 Examples

Example 9.1. [3] The lattice $\left(2^{\{x,y,z\}}, \cup, \cap\,;\, \subseteq\right)$ of Example 5.2 page 78 is a complemented lattice. The "lattice complement" of each element A is simply the "set complement" $A^c \triangleq 2^{\{x,y,z\}} \setminus A$:

---

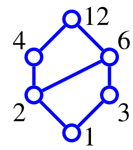
[1] Stern (1999) page 9, Birkhoff (1948) page 23
[2] Birkhoff (1948) page 23
[3] Svozil (1994) page 72

EX

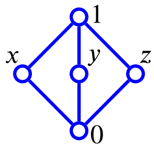
A^c	$A \cup A^c$	$A \cap A^c$
$\mathsf{c}\varnothing = \{x, y, z\}$	$\varnothing \cup \{x, y, z\} = \{x, y, z\}$	$\varnothing \cap \{x, y, z\} = \varnothing$
$\mathsf{c}\{x\} = \{y, z\}$	$\{x\} \cup \{y, z\} = \{x, y, z\}$	$\{x\} \cap \{y, z\} = \varnothing$
$\mathsf{c}\{y\} = \{x, z\}$	$\{y\} \cup \{x, z\} = \{x, y, z\}$	$\{y\} \cap \{x, z\} = \varnothing$
$\mathsf{c}\{x, y\} = \{z\}$	$\{x, y\} \cup \{z\} = \{x, y, z\}$	$\{x, y\} \cap \{z\} = \varnothing$
$\mathsf{c}\{z\} = \{x, y\}$	$\{z\} \cup \{x, y\} = \{x, y, z\}$	$\{z\} \cap \{x, y\} = \varnothing$
$\mathsf{c}\{x, z\} = \{y\}$	$\{x, z\} \cup \{y\} = \{x, y, z\}$	$\{x, z\} \cap \{y\} = \varnothing$
$\mathsf{c}\{y, z\} = \{x\}$	$\{y, z\} \cup \{x\} = \{x, y, z\}$	$\{y, z\} \cap \{x\} = \varnothing$
$\mathsf{c}\{x, y, z\} = \varnothing$	$\{x, y, z\} \cup \varnothing = \{x, y, z\}$	$\{x, y, z\} \cap \varnothing = \varnothing$

Example 9.2 (factors of 12). [4] The lattice $\boldsymbol{L} \triangleq (\{1, 2, 3, 4, 6, 12\}, \text{lcm}, \text{gcd}; |)$ (illustrated to the right) is *non-complemented*. In particular, the elements 2 and 6 have no complements in $\boldsymbol{L}$:

$$\begin{array}{rclll} \text{lcm}(2,3) &=& 6 & \neq & 12 \\ \text{lcm}(2,4) &=& 4 & \neq & 12 \\ \text{lcm}(2,6) &=& 6 & \neq & 12 \\ \text{lcm}(6,3) &=& 6 & \neq & 12 \\ \text{lcm}(6,4) &=& 12 & & \end{array} \qquad \begin{array}{rclll} \text{gcd}(2,3) &=& 1 & & \\ \text{gcd}(2,4) &=& 2 & \neq & 1 \\ \text{gcd}(2,2) &=& 2 & \neq & 1 \\ \text{gcd}(6,3) &=& 3 & \neq & 1 \\ \text{gcd}(6,4) &=& 2 & \neq & 1 \end{array}$$

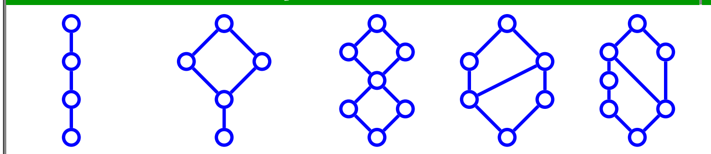


Example 9.3. [5] The lattice illustrated in the figure to the right is *complemented*. In this complemented lattice, complements are *not unique*. For example, the complement of x is both y and z, the complement of y is both x and z, and the complement of z is both x and y.

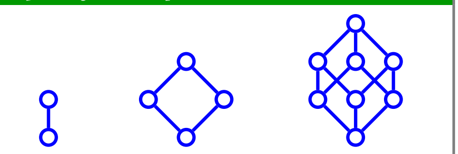


Example 9.4. Here are some more examples:

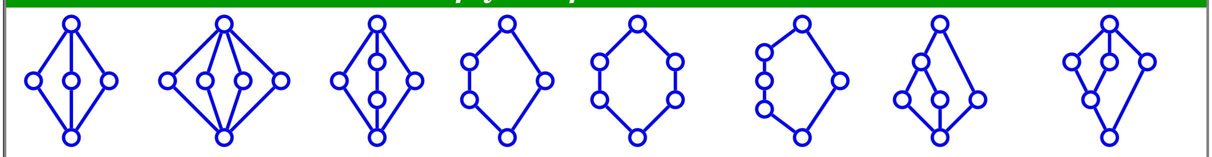
***non-complemented* lattices**



***uniquely complemented* lattices**



***multiply complemented* lattices**



Example 9.5.

EX

Of the 53 unlabeled lattices on a 7 element set (Example 5.13 page 80),
- 0 are complemented with unique complements,
- 17 are complemented with multiple complements, and
- 36 are non-complemented.

9.3 Properties

Theorem 9.1 (next) is a landmark theorem in mathematics.

Theorem 9.1. [6]

---

[4] Durbin (2000) page 271, Saliĭ (1988) pages 26–27

[5] Durbin (2000) page 271

[6] Dilworth (1945) page 123, Saliĭ (1988) page 51, Grätzer (2003) page 378 ⟨Corollary 3.8⟩

THM

For every lattice $\boldsymbol{L}$, *there exists a lattice* $\boldsymbol{U}$ *such that*
1. $\boldsymbol{L} \subseteq \boldsymbol{U}$ ($\boldsymbol{L}$ *is a sublattice of* $\boldsymbol{U}$) *and*
2. $\boldsymbol{U}$ *is* UNIQUELY COMPLEMENTED.

"I therefore propose the following problem...". With these words, Edward Huntington in a 1904 paper introduced one of the most famous problems in mathematical history;[7] a question that took some 40 years to answer, and that in the end had a very surprising solution. Huntington's problem was essentially this: *Are all uniquely complemented lattices also distributive?*[8] This question is significant because if a lattice is both complemented and distributive, then it is *uniquely complemented* (Corollary 9.1—next) and, more importantly, is a *Boolean algebra* (Definition 10.1 page 127). Being a Boolean algebra is very significant in that it implies the lattice has several powerful properties including that it satisfies *de Morgan's laws* (Theorem 5.3 page 74) and that it is isomorphic to an *algebra of sets* (Theorem 16.4 page 228).

A uniquely complemented lattice that satifies any one of a number of other conditions is distributive (Theorem 9.2 page 123, Literature item 3 page 124). So there was ample evidence that the answer to Huntington's question is "yes". But the final answer to Huntington's problem is actually "no"—an answer that took the mathematical community 40 years to find. The resulting effort had a profound impact on lattice theory in general. In fact, George Grätzer, in a 2007 paper, identified uniquely complemented lattices as one of the "two problems that shaped a century of lattice theory".[9]

This final solution to Huntington's problem was found by Robert Dilworth and published in a 1945 paper.[10] And the answer is this: *Every lattice is a sublattice of a uniquely complemented lattice* (Theorem 9.1 page 122). To understand why this answers the question, consider either the *M3 lattice* (Definition 8.3 page 105) or the *N5 lattice* (Definition 7.4 page 94). Neither of these lattices are *distributive* (Theorem 8.2 page 106), but yet either of them can be a sublattice in a uniquely complemented lattice (by *Dilworth's theorem*). That is, it is therefore possible to have a lattice that is both *uniquely complemented* and *non-distributive*.

Corollary 9.1. [11] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

COR

$$\left\{\begin{array}{lll} 1. & \boldsymbol{L} \text{ is DISTRIBUTIVE} & \text{and} \\ 2. & \boldsymbol{L} \text{ is COMPLEMENTED} & \end{array}\right\} \implies \{\boldsymbol{L} \text{ is UNIQUELY COMPLEMENTED}\}$$

✎PROOF:

$\boldsymbol{L}$ is complemented

$$\begin{aligned}
&\iff \forall x \in L \exists a, b \text{ such that } a, b \text{ are complements of } x \text{ in } \boldsymbol{L} && \text{by definition of complement page 121}\\
&\iff x \vee a = 1,\ x \vee b = 1,\ x \wedge a = 0,\ x \wedge b = 0 && \text{by definition of complement page 121}\\
&\implies a = b && \text{by Theorem 8.3 page 109}\\
&\implies \boldsymbol{L} \text{ is uniquely complemented}
\end{aligned}$$

⇰

Theorem 9.2 (Huntington properties). [12] *Let* $\boldsymbol{L}$ *be a lattice.*

[7] For more discussion, see Literature item 7 page 125
[8] Huntington (1904) page 305
[9] Grätzer (2007) page 696
[10] Dilworth (1945) page 123
[11] MacLane and Birkhoff (1999) page 488, Saliĭ (1988) page 30 ⟨Theorem 10⟩
[12] Roman (2008) page 103, Adams (1990) page 79, Saliĭ (1988) page 40, Dilworth (1945) page 123, Grätzer (2007), page 698

THM
$$\left\{\begin{array}{l} \boldsymbol{L} \textit{ is} \\ \text{UNIQUELY} \\ \text{COMPLEMENTED} \end{array}\right\} \textit{ and } \underbrace{\left\{\begin{array}{ll} \boldsymbol{L} \textit{ is } \text{MODULAR} & \textit{or} \\ \boldsymbol{L} \textit{ is } \text{ATOMIC} & \textit{or} \\ \boldsymbol{L} \textit{ is } \text{ORTHO-COMPLEMENTED} & \textit{or} \\ \boldsymbol{L} \textit{ has } \text{FINITE WIDTH} & \textit{or} \\ \boldsymbol{L} \textit{ has } \text{DE MORGAN } \textit{properties} & \end{array}\right\}}_{\text{HUNTINGTON PROPERTIES}} \implies \left\{\begin{array}{l} \boldsymbol{L} \textit{ is} \\ \text{DISTRIBUTIVE} \end{array}\right\}$$

Theorem 9.3 (Peirce's Theorem). [13] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a bounded lattice. Let* $\complement y \triangleq \{y' \in X | y' \text{ is a complement of } y\}$.

THM
$$\{\forall y' \in \complement y,\ x \nleq y' \implies x \wedge y \neq 0\} \implies \left\{\begin{array}{lll} 1. & \boldsymbol{L} \textit{ is } \text{UNIQUELY COMPLEMENTED} & \textit{and} \\ 2. & \boldsymbol{L} \textit{ is } \text{DISTRIBUTIVE} & \end{array}\right\}$$

9.4 Literature

Literature survey:

1. General treatment of lattice varieties:
   Jipsen and Rose (1992)
2. Distributive lattices:
   Grätzer (1971)
   Balbes and Dwinger (1975)
   Dilworth (1984)
3. Uniquely complemented lattices:
   Dilworth (1945) ⟨"Every lattice is a sublattice of a lattice with unique complements."⟩
   Saliĭ (1988)
   Adams (1990) pages 79–84
   Grätzer (2007)
   Roman (2008) page 103
   Bergman (1929) ⟨uniquely complemented + *modular* = distributive⟩
   Birkhoff (1940) ⟨uniquely complemented + *ortho-complemented* = distributive⟩
   Birkhoff and Ward (1939a) ⟨uniquely complemented + *atomic* = distr.⟩
   Birkhoff and Ward (1939b) ⟨uniquely complemented + *atomic* = distributive⟩
4. Projective distributive lattices:
   Balbes (1967)
   Balbes and Horn (1970)
5. Median property:
   Birkhoff and Kiss (1947a)
   Birkhoff and Kiss (1947b)
   Grau (1947)
   Evans (1977)
   Isbell (1980)
   Bandelt and Hedlíková (1983)
   Birkhoff and Ward (1987) pages 1–8
   Artamonov (2000) page 554 ⟨median algebras⟩
   Grätzer (2008) page 356
6. Properties of lattices
   (a) The fact that lattices are not in general *distributive* was not always universally accepted. In a famous 1880 paper, Charles S. Peirce(Peirce, 1880b)33 presents distributivity as a property of all lattices but says that "the proof is too tedius to give".

[13] Saliĭ (1988) pages 38–39 ⟨"Peirce's Theorem"⟩, Peirce ⟨1902 January 31 entry⟩, Peirce (1903) ⟨letter to Huntington⟩, Peirce (1904) ⟨letter to Huntington⟩, Huntington (1904)

7. Note about *Huntington's problem* concerning uniquely complemented lattices:

   (a) Salii[14] suggests that Huntington's problem is actually motivated by and a simple extension of *Peirce's Theorem* (Theorem 9.3 page 124). That is, Huntington's problem is equivalent to asking if the uniquely complemented property is equivalent to the left hypothesis in Peirce's Theorem.

   (b) George Grätzer in a 2007 paper seems to indicate that Huntington's 1904 paper[15] is *not* the original source of "Huntington's problem". In particular, Grätzer says "...Neither gives any references as to the origin of the problem. G. Birkhoff and M. Ward, 1933, reference E. V. Huntington, 1904, for the lattice axioms, which Huntington stated as being due to E. Schröder, but not for the problem. If the reader is surprised, I suggest he try to read the original paper of E. V. Huntington, and there he may find the clue. In my earlier papers on the subject, I reference only R. P. Dilworth, 1945, but in my lattice books (e.g., [7]) I give the correct reference. But I have no recollection of reading E. V. Huntington, 1904, until the preparation for this article." (Grätzer (2007), page 699) The reference [7] is Grätzer (2003). In this reference, Dilworth's 1945 theorem is presented on page 378, and its historical background is discussed on page 392. However, this discussion does not seem to give credit for Huntington's problem to anyone other than Huntington (1904). Perhaps it is Peirce that Grätzer has in mind with these comments—but so far the person referred to by Grätzer is unclear (to me). See also `http://groups.google.com/group/sci.math/browse_thread/thread/b7790be1efe8946e#`

8. General treatment of lattice varieties:
   Jipsen and Rose (1992)

9. Atomic lattices:
   Birkhoff (1938), page 800 ⟨see footnote ‡⟩

[14] Saliĭ (1988) pages 38–39 ⟨"Peirce's Theorem"⟩

[15] Huntington (1904) page 305

CHAPTER 10

BOOLEAN LATTICES

❝*That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved....by Mr. Boole. The unity of the forms of thought in all the applications of reason, however remotely separated, will one day be matter of notoriety and common wonder: and Boole's name will be remembered in connection with one of the most important steps towards the attainment of knowledge.*❞
Augustus de Morgan (1806–1871), British mathematician and logician, [1]

10.1 Definition and properties

A *Boolean algebra* (next definition) is a *bounded* (Definition 6.1 page 89), *distributive* (Definition 8.2 page 102), and *complemented* (Definition 9.1 page 121), *lattice* (Definition 5.3 page 73).

Definition 10.1. [2]

DEF

The BOUNDED LATTICE *(Definition 6.1 page 89)* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *is* ***Boolean*** *if*

1. $\mathbf{L}$ *is* COMPLEMENTED *(Definition 9.1 page 121)* *and*
2. $\mathbf{L}$ *is* DISTRIBUTIVE *(Definition 8.2 page 102)* .

A BOUNDED LATTICE $\mathbf{L}$ *that is* BOOLEAN *is a* ***Boolean algebra*** *or a* ***Boolean lattice***.
A BOOLEAN LATTICE *with* 2^N *elements is denoted* $\mathbf{L}_2^N$.

Several examples of *Boolean lattices* are illustrated in Example 11.2 (page 152).

Proposition 10.1.

PRP

The algebraic structure $\mathbf{A} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *is a* ***Boolean algebra*** *(Definition 10.1 page 127) if*

1. $(X, \vee, \wedge, 0, 1 ; \leq)$ *is a* BOUNDED LATTICE *(Definition 6.1 page 89)* *and*
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x,y,z \in X$ (DISTRIBUTIVE) *and*
3. $x \wedge x' = 0 \quad \forall x \in X$ (NON-CONTRADICTION) *and*
4. $x \vee x' = 1 \quad \forall x \in X$ (EXCLUDED MIDDLE).

[1] quote: DeMorgan (1872) page 80
image: http://en.wikipedia.org/wiki/Augustus_De_Morgan
[2] MacLane and Birkhoff (1999) page 488, Jevons (1864)

✎PROOF: This follows directly from Definition 10.1 (page 127). ⇨

Boolean algebras support the *principle of duality* (next theorem).

Theorem 10.1 (Principle of duality). [3] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a Boolean algebra.*

THM

$$\left\{\begin{array}{l}\phi \text{ is an identity on } \boldsymbol{B} \text{ in terms}\\ \text{of the operations}\\ \vee, \wedge, ', 0, \text{ and } 1\end{array}\right\} \implies \mathbf{T}\phi \text{ is also an identity on } \boldsymbol{B}$$

where the operator $\mathbf{T}$ *performs the following mapping on the operations in* X^X:

$$0 \to 1, \qquad 1 \to 0, \qquad \vee \to \wedge, \qquad \wedge \to \vee$$

✎PROOF: For each of the identities in the definition of Boolean algebras (Proposition 10.5 page 143), the operator $\mathbf{T}$ produces another identity that is also in the definition:

$$\begin{array}{lllllllllll}
\mathbf{T}(1a) &=& \mathbf{T}[x \vee y &=& y \vee x] &=& [x \wedge y &=& y \wedge x] &=& (1b)\\
\mathbf{T}(1b) &=& \mathbf{T}[x \wedge y &=& y \wedge x] &=& [x \vee y &=& y \vee x] &=& (1a)\\
\mathbf{T}(2a) &=& \mathbf{T}[x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& [x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& (2b)\\
\mathbf{T}(2b) &=& \mathbf{T}[x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& [x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& (2a)\\
\mathbf{T}(3a) &=& \mathbf{T}[x \vee 0 &=& x] &=& [x \wedge 1 &=& x] &=& (3b)\\
\mathbf{T}(3b) &=& \mathbf{T}[x \wedge 1 &=& x] &=& [x \vee 0 &=& x] &=& (3a)\\
\mathbf{T}(4a) &=& \mathbf{T}[x \vee x' &=& 1] &=& [x \wedge x' &=& 0] &=& (4b)\\
\mathbf{T}(4b) &=& \mathbf{T}[x \wedge x' &=& 0] &=& [x \vee x' &=& 1] &=& (4a)
\end{array}$$

Therefore, if the statement ϕ is consistent with regards to the Boolean algebra $\boldsymbol{B}$, then $\mathbf{T}\phi$ is also consistent with regards to the Boolean algebra $\boldsymbol{B}$. ⇨

10.2 Order properties

The definition of Boolean algebras given by Definition 10.1 is a set of postulates known as *Huntington's* FIRST SET. Lemma 10.1 (next) gives a link between *Huntington's* FIRST SET of Boolean algebra postulates and the *classic 10* set of Boolean algebra postulates (Theorem 10.2 page 132).

Lemma 10.1. [4] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a bounded lattice.*

LEM

If $\forall x, y, z \in X$

$$\left\{\begin{array}{l|lll|lll|ll}
① & x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \text{(COMMUTATIVE)} & \text{and}\\
② & x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z) & x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z) & \text{(DISTRIBUTIVE)} & \text{and}\\
③ & x \vee 0 &=& x & x \wedge 1 &=& x & \text{(IDENTITY)} & \text{and}\\
④ & x \vee x' &=& 1 & x \wedge x' &=& 0 & \text{(COMPLEMENTED)} &
\end{array}\right\}$$

then $\forall x, y, z \in X$

$$\left\{\begin{array}{l|lll|lll|ll}
1. & x \vee x &=& x & x \wedge x &=& x & \text{(IDEMPOTENT)} & \text{and}\\
2. & x \vee (y \vee z) &=& (x \vee y) \vee z & x \wedge (y \wedge z) &=& (x \wedge y) \wedge z & \text{(ASSOCIATIVE)}^5 & \text{and}\\
3. & x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \text{(ABSORPTIVE)} & \text{and}\\
4. & x \vee 1 &=& 1 & x \wedge 0 &=& 0 & \text{(BOUNDED)} & \text{and}\\
5. & (x \vee y)' &=& x' \wedge y' & (x \wedge y)' &=& x' \vee y' & \text{(DE MORGAN'S LAWS).} &
\end{array}\right\}$$

✎PROOF: For each pair of properties, it is only necessary to prove one of them, as the other follows by the *principle of duality* (Theorem 10.1 page 128). Let the *join* $\vee$ be represented by $+$, the operation *meet* $\wedge$ represented by $\cdot$ or juxtaposition, and let $\wedge$ have algebraic precedence over $\vee$.

[3] Givant and Halmos (2009) pages 20–22 ⟨Chapter 4⟩, Sikorski (1969), page 8

[4] Huntington (1904) pages 292–296 ⟨"1st set"⟩, Joshi (1989) pages 224–227

[5] K.D. Joshi comments that having the *associative* property as a result of an axiom rather than as an axiom, is a very unusual and "remarkable property" in the world of algebras. Joshi (1989) pages 225–226

1. Proof that $x+x=x$ and $xx=x$ (*idempotent* properties):

$$\begin{aligned}
x+x &= (x+x)\cdot 1 && \text{by \textit{identity} property,} && \text{③b}\\
&= (x+x)(x+x') && \text{by \textit{complemented} property,} && \text{④a}\\
&= x+(xx') && \text{by \textit{distributive} property,} && \text{②a}\\
&= x+0 && \text{by \textit{complemented} property,} && \text{④b}\\
&= x && \text{by \textit{identity} property,} && \text{③a}
\end{aligned}$$

2. Proof that $x+1=1$ and $x\cdot 0=0$ (*bounded* properties):

$$\begin{aligned}
x+1 &= (x+1)\cdot 1 && \text{by \textit{identity} property,} && \text{③b}\\
&= 1\cdot(x+1) && \text{by \textit{commutative} property,} && \text{①b}\\
&= (x+x')(x+1) && \text{by \textit{complemented} property,} && \text{④a}\\
&= x+(x'\cdot 1) && \text{by \textit{distributive} property,} && \text{②a}\\
&= x+x' && \text{by \textit{identity} property,} && \text{③b}\\
&= 1 && \text{by \textit{complemented} property,} && \text{④a}
\end{aligned}$$

3. Proof that $x+(xy)=x$ and $x(x+y)=x$: (*absorptive* properties)

$$\begin{aligned}
x+(x\cdot y) &= (x\cdot 1)+(xy) && \text{by \textit{identity} property,} && \text{③b}\\
&= x\cdot(1+y) && \text{by \textit{distributive} property,} && \text{②b}\\
&= x\cdot(y+1) && \text{by \textit{commutative} property,} && \text{①a}\\
&= x\cdot 1 && \text{by item (2)} && \\
&= x && \text{by \textit{identity} property,} && \text{③b}
\end{aligned}$$

4. Proof that $(x+y)+z=x+(y+z)$ and $(xy)z=x(yz)$ (*associative* properties):
   Let $a \triangleq x(yz)$ and $b \triangleq (xy)z$.

   (a) Proof that $a+x=b+x$:

$$\begin{aligned}
a+x &= x(yz)+x && \text{by definition of } a && \\
&= x(yz)+x1 && \text{by \textit{identity} property,} && \text{③b}\\
&= x(yz+1) && \text{by \textit{distributive} property,} && \text{②a}\\
&= x(1) && \text{by \textit{bounded} property,} && \text{item (2)}\\
&= x && \text{by \textit{identity} property,} && \text{③b}\\
&= x(x+z) && \text{by \textit{absorptive} property,} && \text{item (3)}\\
&= (x+xy)(x+z) && \text{by \textit{absorptive} property,} && \text{item (3)}\\
&= x+(xy)z && \text{by \textit{distributive} property,} && \text{②b}\\
&= (xy)z+x && \text{by \textit{commutative} property,} && \text{①a,b}\\
&= b+x && \text{by definition of } b &&
\end{aligned}$$

(b) Proof that $a + x' = b + x'$:

$$
\begin{aligned}
a + x' &= x(yz) + x' && \text{by definition of } a \\
&= x' + x(yz) && \text{by \textit{commutative} property, } ①\text{a,b} \\
&= (x' + x)(x' + yz) && \text{by \textit{distributive} property, } ②\text{b} \\
&= 1 \cdot (x' + yz) && \text{by \textit{complemented} property, } ④\text{a} \\
&= x' + yz && \text{by \textit{identity} property, } ③\text{b} \\
&= (x' + y)(x' + z) && \text{by \textit{distributive} property, } ②\text{b} \\
&= \left[(x' + y) \cdot 1\right](x' + z) && \text{by \textit{identity} property, } ③\text{b} \\
&= \left[1 \cdot (x' + y)\right](x' + z) && \text{by \textit{commutative} property, } ①\text{b} \\
&= \left[(x + x')(x' + y)\right](x' + z) && \text{by \textit{complemented} property, } ④\text{a} \\
&= (x' + xy)(x' + z) && \text{by \textit{distributive} property, } ②\text{b} \\
&= x' + (xy)z && \text{by \textit{distributive} property, } ②\text{b} \\
&= (xy)z + x' && \text{by \textit{commutative} property, } ①\text{a} \\
&= b + x' && \text{by definition of } b
\end{aligned}
$$

(c) Proof that $x(yz) = (xy)z$:

$$
\begin{aligned}
x(yz) &\triangleq a && \text{by definition of } a \\
&= a + a && \text{by \textit{idempotent} property, item (1)} \\
&= a + a1 + 0 && \text{by \textit{identity} property, } ③\text{a,b} \\
&= a + a(x + x') + xx' && \text{by \textit{complemented} property, } ④\text{a,b} \\
&= a + ax + ax' + xx' && \text{by \textit{distributive} property, } ②\text{a} \\
&= a + ax' + xa + xx' && \text{by \textit{commutative} property, } ①\text{a,b} \\
&= aa + ax' + xa + xx' && \text{by \textit{idempotent} property, item (1)} \\
&= a(a + x') + x(a + x') && \text{by \textit{distributive} property, } ②\text{a} \\
&= (a + x)(a + x') && \text{by \textit{distributive} property, } ②\text{a} \\
&= (b + x)(a + x') && \text{by item (4a)} \\
&= (b + x)(b + x') && \text{by item (4b)} \\
&= (b + x)b + (b + x)x' && \text{by \textit{distributive} property, } ②\text{a} \\
&= b(b + x) + x'(b + x) && \text{by \textit{commutative} property, } ①\text{b} \\
&= bb + bx + x'b + x'x && \text{by \textit{distributive} property, } ②\text{a} \\
&= b + bx + x'b + x'x && \text{by \textit{idempotent} property, item (1)} \\
&= b + bx + bx' + x'x && \text{by \textit{commutative} property, } ①\text{b} \\
&= b + b(x + x') + x'x && \text{by \textit{distributive} property, } ②\text{a} \\
&= b + b \cdot 1 + 0 && \text{by \textit{complemented} property, } ④\text{a,b} \\
&= b + b && \text{by \textit{identity} property, } ③\text{a,b} \\
&= b && \text{by \textit{idempotent} property, item (1)} \\
&\triangleq (xy)z && \text{by definition of } b
\end{aligned}
$$

5. Proof that $(x + y)' = x'y'$ and $(xy)' = x' + y'$: (*de Morgan* properties)

(a) Proof that $(x + y) + (x'y') = 1$:

$$
\begin{aligned}
&(x + y) + (x'y') \\
&= \left[(x + y) + x'\right]\left[(x + y) + y'\right] && \text{by \textit{distributive} property, } ②\text{a} \\
&= \left[x' + (x + y)\right]\left[y' + (x + y)\right] && \text{by \textit{commutative} property, } ①\text{a} \\
&= \left[(x' + (x + y))1\right]\left[(y' + (x + y))1\right] && \text{by \textit{identity} property, } ③\text{b}
\end{aligned}
$$

$$\begin{aligned}
&= \left[1\left(x' + (x+y)\right)\right]\left[1\left(y' + (y+x)\right)\right] && \text{by \textit{distributive} property, ②b}\\
&= \left[\left(x'+x\right)\left(x' + (x+y)\right)\right]\left[\left(y'+y\right)\left(y' + (y+x)\right)\right] && \text{by \textit{complemented} property, ④a}\\
&= \left[x' + (x(x+y))\right]\left[y' + (y(y+x))\right] && \text{by \textit{distributive} property, ②a}\\
&= \left[x'+x\right]\left[y'+y\right] && \text{by \textit{absorptive} property, item (3)}\\
&= [1][1] && \text{by \textit{complemented} property, ④a}\\
&= 1 && \text{by \textit{bounded} property, item (2)}
\end{aligned}$$

(b) Proof that $(x+y)(x'y') = 0$:

$$\begin{aligned}
(x+y)(x'y') &= \left[x(x'y')\right] + \left[y(x'y')\right] && \text{by \textit{distributive} property, ②b}\\
&= \left[0 + x(x'y')\right] + \left[0 + y(x'y')\right] && \text{by \textit{identity} property, ③a}\\
&= \left[(xx') + x(x'y')\right] + \left[(yy') + y(x'y')\right] && \text{by \textit{complemented} property, ④b}\\
&= \left[x\left(x' + x'y'\right)\right] + \left[y\left(y' + x'y'\right)\right] && \text{by \textit{distributive} property, ②b}\\
&= \left[xx'\right] + \left[yy'\right] && \text{by \textit{absorptive} property, item (3)}\\
&= [0] + [0] && \text{by \textit{complemented} property, ④b}\\
&= 0 && \text{by \textit{bounded} property, item (2)}
\end{aligned}$$

(c) Proof that $(x+y)' = x'y'$:
The quanities $(x+y)$ and $x'y'$ are *complements* of each other as demonstrated by item (5a) ($(x+y)+(x'y') = 1$) and item (5b) ($(x+y)(x'y') = 0$). Therefore, $(x+y)' = x'y'$.

⇨

Proposition 10.2. [6] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a Boolean algebra.*

PRP

The pair $(X, \leq)$ *is an* ORDERED SET. *In particular,*

1. $x \leq x$	$\forall x \in X$	(REFLEXIVE)	*and*
2. $x \leq y$ *and* $y \leq z \implies x \leq z$	$\forall x,y,z \in X$	(TRANSITIVE)	*and*
3. $x \leq y$ *and* $y \leq x \implies x = y$	$\forall x,y \in X$	(ANTI-SYMMETRIC).	

✎PROOF:

1. Proof that $\leq$ is *reflexive* in $(X, \leq)$:

$$\begin{aligned}
x \leq x &\iff x \vee x = x && \text{by definition of } \leq \text{ (Definition 10.1 page 127)}\\
&\iff \text{true} && \text{by Lemma 10.1 page 128}
\end{aligned}$$

2. Proof that $\leq$ is *transitive* in $(X, \leq)$:

$$\begin{aligned}
\{(x \leq y) \text{ and } (y \leq z)\} &\iff \{(x \vee y = y) \text{ and } (y \vee z = z)\} && \text{by definition of } \leq \text{ (Definition 10.1 page 127)}\\
&\implies (x \vee z)\\
&= x \vee (y \vee z)\\
&= (x \vee y) \vee z && \text{by \textit{associative} property of Lemma 10.1 page 128}\\
&= y \vee z\\
&= z
\end{aligned}$$

3. Proof that $\leq$ is *anti-symmetric* in $(X, \leq)$:

$$\begin{aligned}
\{(x \leq y) \text{ and } (y \leq x)\} &\iff \{(x \vee y = y) \text{ and } (y \vee x = x)\} && \text{by definition of } \leq \text{ (Definition 10.1 page 127)}\\
&\iff \{(x \vee y = y) \text{ and } (x \vee y = x)\} && \text{by \textit{commutative} property of Definition 10.1 page 12}\\
&\iff x = x \vee y = y\\
&\implies x = y
\end{aligned}$$

[6] Sikorski (1969), page 7

Proposition 10.3. *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be a Boolean algebra.*

PRP

$x \vee y$ *is the* LEAST UPPER BOUND *of* x *and* y *in* $(X, \leq)$.
$x \wedge y$ *is the* GREATEST LOWER BOUND *of* x *and* y *in* $(X, \leq)$.

Theorem 10.2 (classic 10 Boolean properties). [7]

THM

$\boldsymbol{A} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *is a* ***Boolean algebra*** $\iff$ $\forall x, y, z \in X$

property with emphasis on $\vee$	dual property with emphasis on $\wedge$	property name	
$x \vee x = x$	$x \wedge x = x$	(IDEMPOTENT)	*and*
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	(COMMUTATIVE)	*and*
$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(ASSOCIATIVE)	*and*
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	(ABSORPTIVE)	*and*
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(DISTRIBUTIVE)	*and*
$x \vee 0 = x$	$x \wedge 1 = x$	(IDENTITY)	*and*
$x \vee 1 = 1$	$x \wedge 0 = 0$	(BOUNDED)	*and*
$x \vee x' = 1$	$x \wedge x' = 0$	(COMPLEMENTED)	*and*
$(x \vee y)' = x' \wedge y'$	$(x \wedge y)' = x' \vee y'$	(DE MORGAN)	*and*
$(x')' = x$		(INVOLUTORY).	

✎PROOF:

1. Proof that Proposition 10.5 (page 143) $\Longrightarrow$ Theorem 10.2 (page 132):

1.	Proof that $\boldsymbol{A}$ is *idempotent*:	by 1	of Lemma 10.1	page 128
2.	Proof that $\boldsymbol{A}$ is *commutative*:	by 1	of Proposition 10.5	page 143
3.	Proof that $\boldsymbol{A}$ is *associative*:	by 2	of Lemma 10.1	page 128
4.	Proof that $\boldsymbol{A}$ is *absorptive*:	by 3	of Lemma 10.1	page 128
5.	Proof that $\boldsymbol{A}$ is *distributive*:	by 2	of Proposition 10.5	page 143
6.	Proof that $\boldsymbol{A}$ is *identity*:	by 3	of Proposition 10.5	page 143
7.	Proof that $\boldsymbol{A}$ is *bounded*:	by 4	of Lemma 10.1	page 128
8.	Proof that $\boldsymbol{A}$ is *complemented*:	by 4	of Proposition 10.5	page 143
9.	Proof that $\boldsymbol{A}$ is *involutory*:	by	Corollary 9.1	page 123
10.	Proof that $\boldsymbol{A}$ is *de Morgan*:	by 5	of Lemma 10.1	page 128

2. Proof that Proposition 10.5 (page 143) $\Longleftarrow$ Theorem 10.2 (page 132):

1.	Proof that $\boldsymbol{A}$ is *commutative*:	by 2	of Theorem 10.2	page 132
2.	Proof that $\boldsymbol{A}$ is *distributive*:	by 5	of Theorem 10.2	page 132
3.	Proof that $\boldsymbol{A}$ is *identity*:	by 6	of Theorem 10.2	page 132
4.	Proof that $\boldsymbol{A}$ is *complemented*:	by 8	of Theorem 10.2	page 132

Lemma 10.2.

LEM

$$\left.\begin{array}{l}(X, \vee, \wedge, 0, 1; \leq)\\ \text{is a BOOLEAN ALGEBRA}\end{array}\right\} \Longrightarrow \left\{\begin{array}{llllll}1. & x' \vee (x \wedge y) & = & x' \vee y & \forall x,y \in X & \text{(SASAKI HOOK) and}\\ 2. & x \vee (x' \wedge y) & = & x \vee y & \forall x,y \in X & \end{array}\right.$$

[7] Huntington (1904) pages 292–293 ⟨"1st set"⟩, Huntington (1933) page 280 ⟨"4th set"⟩, MacLane and Birkhoff (1999) page 488, Givant and Halmos (2009) page 10, Müller (1909) pages 20–21, Schröder (1890), Whitehead (1898) pages 35–37

✎PROOF:

$$\begin{aligned}
x' \vee (x \wedge y) &= \underbrace{x' \vee (x' \wedge y)}_{x'} \vee (x \wedge y) && \text{by } \textit{absorption} \text{ property (Theorem 10.2 page 132)}\\
&= x' \vee \left[(x' \vee x) \wedge y\right] && \text{by } \textit{associative} \text{ and } \textit{distributive} \text{ properties (Theorem 10.2 page 132)}\\
&= x' \vee [1 \wedge y] && \text{by } \textit{excluded middle} \text{ property (Theorem 10.2 page 132)}\\
&= x' \vee y && \text{by definition of } 1 \text{ (Definition 4.21 page 70)}\\
x \vee (x' \wedge y) &= \underbrace{x \vee (x \wedge y)}_{x} \vee (x \wedge y) && \text{by } \textit{absorption} \text{ property (Theorem 10.2 page 132)}\\
&= x \vee \left[(x \vee x') \wedge y\right] && \text{by } \textit{associative} \text{ and } \textit{distributive} \text{ properties (Theorem 10.2 page 132)}\\
&= x \vee [1 \wedge y] && \text{by } \textit{excluded middle} \text{ property (Theorem 10.2 page 132)}\\
&= x \vee y && \text{by definition of } 1 \text{ (Definition 4.21 page 70)}
\end{aligned}$$

⇰

Theorem 10.3. [8] *Let $|X|$ be the number of elements in a finite set X.*

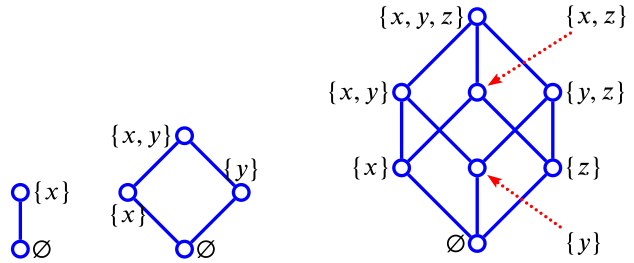
THM

$\boldsymbol{A}$ *is a* BOOLEAN ALGEBRA $\implies$ $|\boldsymbol{A}| = 2^n$ *for some* $n \in \mathbb{N}$.

Example 10.1. Here are some lattices that are Boolean algebras.

EX

Boolean algebras as algebras of sets



Theorem 10.4.

THM

If $(X, \vee, \wedge, 0, 1; \leq)$ *is a* BOOLEAN ALGEBRA *then*

$$\left\{\begin{array}{rclrcl}
\multicolumn{3}{c}{\textit{tautology}} & \multicolumn{3}{c}{\textit{dual}}\\
\hline
\neg\left(\bigwedge_{n=1}^{N} x_n\right) &=& \bigvee_{n=1}^{N}(\neg x_n) & \neg\left(\bigvee_{n=1}^{N} x_n\right) &=& \bigwedge_{n=1}^{N}(\neg x_n) \quad \forall x_n \in X, N \in \mathbb{N}\\
\left(\bigwedge_{n=1}^{N} x_n\right) \vee y &=& \bigwedge_{n=1}^{N}(x_n \vee y) & \left(\bigvee_{n=1}^{N} x_n\right) \wedge y &=& \bigvee_{n=1}^{N}(x_n \wedge y)
\end{array}\right\}$$

✎PROOF:

1. Proof that $\neg\left(\bigwedge_{n=1}^{N} x_n\right) = \bigvee_{n=1}^{N}(\neg x_n)$ (by induction):

Proof for $N = 1$ case:

$$\begin{aligned}
\neg\left(\bigwedge_{n=1}^{N=1} x_n\right) &= \neg x_n && \text{by definition of } \wedge\\
&= \bigvee_{n=1}^{N=1}(\neg x_n) && \text{by definition of } \vee
\end{aligned}$$

[8] Joshi (1989) page 237

Proof for $N = 2$ case:

$$\begin{aligned}
\neg\left(\bigwedge_{n=1}^{N=2} x_n\right) &= (\neg x_1) \vee (\neg x_2) && \text{by Theorem 10.2 page 132}\\
&= \bigvee_{n=1}^{N=2} (\neg x_n) && \text{by definition of } \vee
\end{aligned}$$

Proof that (N case) $\implies$ ($N+1$ case):

$$\begin{aligned}
\neg\left(\bigwedge_{n=1}^{N+1} x_n\right) &= \neg\left[\left(\bigwedge_{n=1}^{N} x_n\right) \wedge x_N\right] && \text{by definition of } \wedge\\
&= \left(\neg\bigwedge_{n=1}^{N} x_n\right) \vee (\neg x_{N+1}) && \text{by Theorem 10.2 page 132}\\
&= \left[\bigvee_{n=1}^{N} (\neg x_n)\right] \vee (\neg x_{N+1}) && \text{by left hypothesis}\\
&= \bigvee_{n=1}^{N+1} (\neg x_n) && \text{by definition of } \vee
\end{aligned}$$

2. Proof that $\neg\left(\bigvee_{n=1}^{N} x_n\right) = \bigwedge_{n=1}^{N} (\neg x_n)$:

$$\begin{aligned}
\neg\left(\bigvee_{n=1}^{N} x_n\right) &= \neg\left(\bigvee_{n=1}^{N} (\neg\neg x_n)\right) && \text{by Theorem 10.2 page 132}\\
&= \neg\neg\left(\bigwedge_{n=1}^{N} (\neg x_n)\right) && \text{by previous result 1.}\\
&= \bigwedge_{n=1}^{N} (\neg x_n) && \text{by Theorem 10.2 page 132}
\end{aligned}$$

3. Proof that $\left(\bigwedge_{n=1}^{N} x_n\right) \vee y = \bigvee_{n=1}^{N} (x_n \vee y)$ (by induction):

Proof for $N = 1$ case:

$$\begin{aligned}
\left(\bigwedge_{n=1}^{N=1} x_n\right) \vee y &= x_1 \vee y && \text{by definition of } \wedge\\
&= \bigwedge_{n=1}^{N=1} (x_n \vee y) && \text{by definition of } \wedge
\end{aligned}$$

Proof for $N = 2$ case:

$$\begin{aligned}
\left(\bigwedge_{n=1}^{N=2} x_n\right) \vee y &= (x_1 \vee y) \wedge (x_2 \vee y) && \text{by Theorem 10.2 page 132}\\
&= \bigwedge_{n=1}^{N=2} (x_n \vee y) && \text{by definition of } \wedge
\end{aligned}$$

Proof that (N case) $\implies$ ($N+1$ case):

$$\left(\bigwedge_{n=1}^{N+1} x_n\right) \vee y = \left[\left(\bigwedge_{n=1}^{N} x_n\right) \wedge x_{N+1}\right] \vee y \qquad \text{by definition of } \wedge$$

$$= \left[\left(\bigwedge_{n=1}^{N} x_n\right) \vee y\right] \wedge (x_{N+1} \vee y) \qquad \text{by Theorem 10.2 page 132}$$

$$= \left[\bigwedge_{n=1}^{N} (x_n \vee y)\right] \wedge (x_{N+1} \vee y) \qquad \text{by left hypothesis}$$

$$= \bigwedge_{n=1}^{N+1} (x_n \vee y) \qquad \text{by definition of } \wedge$$

4. Proof that $\left(\bigvee_{n=1}^{N} x_n\right) \wedge y = \bigwedge_{n=1}^{N} (x_n \wedge y)$:

$$\left(\bigvee_{n=1}^{N} x_n\right) \wedge y = \neg\neg\left[\left(\bigvee_{n=1}^{N} x_n\right) \wedge y\right] \qquad \text{by Theorem 10.2 page 132}$$

$$= \neg\left[\neg\left(\bigvee_{n=1}^{N} x_n\right) \vee (\neg y)\right] \qquad \text{by Theorem 10.2 page 132}$$

$$= \neg\left[\left(\bigwedge_{n=1}^{N} (\neg x_n)\right) \vee (\neg y)\right] \qquad \text{by previous result 2.}$$

$$= \neg\left(\bigwedge_{n=1}^{N} [(\neg x_n) \vee (\neg y)]\right) \qquad \text{by previous result 3.}$$

$$= \left(\bigvee_{n=1}^{N} \neg[(\neg x_n) \vee (\neg y)]\right) \qquad \text{by previous result 1.}$$

$$= \bigvee_{n=1}^{N} (x_n \wedge y) \qquad \text{by Theorem 10.2 page 132}$$

10.3 Additional operations

Propositional logic has a total of $2^4 = 16$ operations in the class of functions $\{0, 1\}^{\{0,1\}^2}$ (see page 201). The 16 logic operations of propositional logic can all be represented using the logic operations of *disjunction* $\vee$, *conjunction* $\wedge$, and *negation* $\neg$. Using these representations, all 16 operations can be generalized to *Boolean algebras* using the equivalent Boolean algebra/lattice operations of *join*, *meet*, and *complement*.[9] Several of these additional operations for Boolean algebras are defined in Definition 10.2 (next).

Definition 10.2 (additional Boolean algebra operations). [10] *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be a Boolean algebra. The following table defines additional operations in* $X^{X\times X}$ *in terms of* $\vee$, $\wedge$, *and* $'$. *Let* $x' \triangleq {}'x$

[9] Givant and Halmos (2009), page 32

[10] Givant and Halmos (2009) pages 32–33, Bernstein (1934) page 876 ⟨implication ⊃⟩, Huntington (1933) page 276, Taylor (1920) page 243, Bernstein (1914) page 93, Sheffer (1913) pages 487–488, Peirce (1902) page 216, Peirce (1880a) pages 218–221

and $y' \triangleq {}'y$.

name	symbol	definition	
rejection	$\downarrow$	$x \downarrow y \triangleq x' \wedge y'$	$\forall x,y \in X$
exception	$-$	$x - y \triangleq x \wedge y'$	$\forall x,y \in X$
adjunction	$\div$	$x \div y \triangleq x \vee y'$	$\forall x,y \in X$
Sheffer stroke	$\mid$	$x \mid y \triangleq x' \vee y'$	$\forall x,y \in X$
Boolean addition	$\triangle$	$x \triangle y \triangleq (x' \wedge y) \vee (x \wedge y')$	$\forall x,y \in X$
inhibit x	$\ominus$	$x \ominus y \triangleq x' \wedge y$	$\forall x,y \in X$
implication	$\Rightarrow$	$x \Rightarrow y \triangleq x' \vee y$	$\forall x,y \in X$
biconditional	$\Leftrightarrow$	$x \Leftrightarrow y \triangleq (x \wedge y) \vee (x' \wedge y')$	$\forall x,y \in X$

Theorem 10.5. [11]

T H M

$\vee$	*(join)*	*is the dual of*	$\downarrow$	*(rejection)*
$\wedge$	*(meet)*	*is the dual of*	$\mid$	*(Sheffer stroke)*
$\triangle$	*(Boolean addition)*	*is the dual of*	$\Leftrightarrow$	*(biconditional)*
$-$	*(exception)*	*is the dual of*	$\Rightarrow$	*(implication)*
$\div$	*(adjunction)*	*is the dual of*	$\ominus$	*(inhibit x)*

✎PROOF:

$$
\begin{aligned}
&(\textit{join}) & (x \vee y)' &= x' \wedge y' && \text{by \textit{de Morgan's law} property (Theorem 10.2 page 132)}\\
& & &= x \downarrow y \quad (\textit{rejection}) && \text{by definition of \textit{rejection} } \downarrow \text{ (Definition 10.2 page 136)}\\
&(\textit{meet}) & (x \wedge y)' &= x' \vee y' && \text{by \textit{de Morgan's law} property (Theorem 10.2 page 132)}\\
& & &= x \mid y \quad (\textit{Sheffer stroke}) && \text{by definition of \textit{Sheffer stroke} } \mid \text{ (Definition 10.2 page 136)}\\
&(\textit{Boolean addition}) & (x \triangle y)' &= (x'y \vee xy')' && \text{by def. of \textit{Boolean addition} } \triangle \text{ (Definition 10.2 page 136)}\\
& & &= (x \vee y')(x' \vee y) && \text{by \textit{de Morgan's law} property (Theorem 10.2 page 132)}\\
& & &= xx' \vee xy \vee y'x' \vee y'y && \text{by \textit{distributive} property (Theorem 10.2 page 132)}\\
& & &= xy \vee x'y' && \\
& & &= x \Leftrightarrow y \quad (\textit{biconditional}) && \text{by def. of \textit{biconditional} } \Leftrightarrow \text{ (Definition 10.2 page 136)}\\
&(\textit{exception}) & (x - y)' &= (xy')' && \text{by definition of \textit{exception} } - \text{ (Definition 10.2 page 136)}\\
& & &= x' \vee y && \text{by \textit{de Morgan's law} property (Theorem 10.2 page 132)}\\
& & &= x \Rightarrow y \quad (\textit{implication}) && \text{by definition of \textit{implication} } \Rightarrow \text{ (Definition 10.2 page 136)}\\
&(\textit{adjunction}) & (x \div y)' &= (x \vee y')' && \text{by definition of \textit{adjunction} } \div \text{ (Definition 10.2 page 136)}\\
& & &= x'y && \text{by \textit{de Morgan's law} property (Theorem 10.2 page 132)}\\
& & &= x \ominus y \quad (\textit{inhibit } x) && \text{by definition of \textit{inhibit} } x \ominus \text{ (Definition 10.2 page 136)}\\
&(\textit{complement } x) & (x \oplus y)' &= (x')' && \text{by definition of \textit{complement} } x \oplus\\
& & &= x && \text{by \textit{involutory} property (Theorem 10.2 page 132)}\\
& & &= x \DashV y \quad (\textit{transfer } x) && \text{by definition of \textit{transfer} } x \DashV\\
&(\textit{complement } y) & (x \oplus y)' &= (y')' && \text{by definition of \textit{complement} } y \oplus\\
& & &= y && \text{by \textit{involutory} property (Theorem 10.2 page 132)}\\
& & &= x \VDash y \quad (\textit{transfer } y) && \text{by definition of \textit{transfer} } y \VDash
\end{aligned}
$$

⇰

Theorem 10.6. [12] *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

[11] Givant and Halmos (2009) page 33

[12] Givant and Halmos (2009) page 39

THM

$$\begin{array}{lllll} x \le y & \iff & y' \le x' & & \forall x,y \in X \\ x \le y & \iff & x - y = 0 & & \forall x,y \in X \\ x \le y & \iff & x \Rightarrow y = 1 & & \forall x,y \in X \end{array}$$

✎PROOF:

1. Proof that $x \le y \iff y' \le x'$:

$$\begin{aligned} x \le y &\iff x \wedge y = x && \text{by definition of } \textit{meet } \wedge, && \text{Definition 4.22 page 70} \\ &\iff (x \wedge y)' = x' && \text{by } \textit{de Morgan's law} \text{ property,} && \text{Theorem 10.2 page 132} \\ &\iff x' \vee y' = x' && \text{by } \textit{de Morgan's law} \text{ property,} && \text{Theorem 10.2 page 132} \\ &\iff y' \le x' && \text{by definition of } \textit{join } \vee, && \text{Definition 4.21 page 70} \end{aligned}$$

2. Proof that $x \le y \implies x - y = 0$:

$$\begin{aligned} x - y &= x \wedge y' && \text{by definition of } \textit{exception } -, && \text{Definition 10.2 page 136} \\ &\le y \wedge y' && \text{by left hypothesis} \\ &= 0 && \text{by definition of } \textit{complement}, && \text{Definition 9.1 page 121} \end{aligned}$$

3. Proof that $x \le y \impliedby x - y = 0$:

$$\begin{aligned} x - y = 0 &\iff x \wedge y' = 0 && \text{by definition of } \textit{exception } -, && \text{Definition 10.2 page 136} \\ &\iff \end{aligned}$$

✏

10.4 Representation

A Boolean algebra $(X, \vee, \wedge, 0, 1; \le)$ can be represented in terms of five operators (see Theorem 10.2 page 132):

- the binary operators join $\vee$ and meet $\wedge$,
- the unary operator complement $'$, and
- the nullary opeartors 0 and 1.

However, it is also possible to represent a Boolean algebra with fewer operators— in fact, as few as one operator. When a set of operators can completely represent all the operators of a Boolean algebra, then that set is called *functionally complete* (next definition).

Definition 10.3. [13] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a Boolean algebra.*

DEF
A set of operators Φ is ***functionally complete*** *in $\boldsymbol{B}$ if*
$\vee, \wedge, ', 0,$ *and* 1
can all be expressed in terms of Φ.

[13] Whitesitt (1995) page 69

Here are some examples of functionally complete sets:

	$\{\downarrow\}$	(*rejection*)	Theorem 10.9	page 138
	$\{\mid\}$	(*Sheffer stroke*)	Theorem 10.10	page 138
	$\{\div, 0\}$	(*adjunction* and 0)	Theorem 10.12	page 140
	$\{-, 1\}$	(*exception* and 1)	Theorem 10.13	page 140
	$\{\vee, '\}$	(*join* and *complement*)	Theorem 10.7	page 138
	$\{\wedge, '\}$	(*meet* and *complement*)	Theorem 10.8	page 138
	$\{\triangle, \wedge, 1\}$	(*Boolean addition*, *meet*, and 1)	Theorem 10.14	page 141
	$\{\triangle, \vee, 1\}$	(*Boolean addition*, *join*, and 1)	Theorem 10.15	page 142
	$\{\triangle, -, '\}$	(*Boolean addition*, *exception*, and *complement*)	Theorem 10.16	page 142

Theorem 10.7. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ *be a* BOOLEAN ALGEBRA.

THM

The set $\{\vee, '\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned} x \wedge y &= (x' \vee y)' && \forall x,y \in X \\ 0 &= (x \vee x')' && \forall x \in X \\ 1 &= x \vee x' && \forall x \in X \end{aligned}$$

✎PROOF:

$$\begin{aligned} x \wedge y &= (x \wedge y)'' && \text{by } \textit{involutory} \text{ property Theorem 10.2 page 132} \\ &= (x' \vee y')' && \text{by de Morgan's Law property Theorem 10.2 page 132} \\ 1 &= x \vee x' && \text{by complement property Theorem 10.2 page 132} \\ 0 &= 1' \\ &= (x \vee x')' && \text{by complement property Theorem 10.2 page 132} \end{aligned}$$

✏

Theorem 10.8. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ *be a* BOOLEAN ALGEBRA.

THM

The set $\{\wedge, '\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned} x \vee y &= (x' \wedge y)' && \forall x,y \in X \\ 0 &= x \wedge x' && \forall x \in X \\ 1 &= (x \wedge x')' && \forall x \in X \end{aligned}$$

✎PROOF:

$$\begin{aligned} x \vee y &= (x \vee y)'' && \text{by } \textit{involutory} \text{ property Theorem 10.2 page 132} \\ &= (x' \wedge y')' && \text{by de Morgan's Law property Theorem 10.2 page 132} \\ 0 &= x \wedge x' && \text{by complement property Theorem 10.2 page 132} \\ 1 &= 0' \\ &= (x \wedge x')' && \text{by complement property Theorem 10.2 page 132} \end{aligned}$$

✏

Theorem 10.9. [14] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ *be a* BOOLEAN ALGEBRA. *Let* $\downarrow$ *represent the* REJECTION *operator (Definition 10.2 page 136).*

THM

The set $\{\downarrow\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned} x \vee y &= (x \downarrow y) \downarrow (x \downarrow y) && \forall x,y \in X \\ x \wedge y &= (x \downarrow x) \downarrow (y \downarrow y) && \forall x,y \in X \\ x' &= x \downarrow x && \forall x \in X \\ 0 &= x \downarrow (x \downarrow x) && \forall x \in X \\ 1 &= [x \downarrow (x \downarrow x)] \downarrow [x \downarrow (x \downarrow x)] && \forall x \in X \end{aligned}$$

[14] Givant and Halmos (2009) page 33

✎PROOF:

$$
\begin{aligned}
x' &= (x \vee x)' && \text{by Theorem 10.2 page 132}\\
&= x \downarrow x && \text{by definition of } \downarrow \text{ page 136}\\
x \vee y &= (x \vee y)'' && \text{by Theorem 10.2 page 132}\\
&= (x \downarrow y)' && \text{by definition of } \downarrow \text{ page 136}\\
&= (x \downarrow y) \downarrow (x \downarrow y) && \text{by previous result}\\
x \wedge y &= (x \wedge y)'' && \text{by Theorem 10.2 page 132}\\
&= \left(x' \vee y'\right)' && \text{by de Morgan's Law page 132}\\
&= x' \downarrow y' && \text{by definition of } \downarrow \text{ page 136}\\
&= (x \downarrow x) \downarrow (y \downarrow y) && \text{by previous result}\\
0 &= 1' &&\\
&= \left(x \vee x'\right)' && \text{by Theorem 10.2 page 132}\\
&= x \downarrow \left(x'\right) && \text{by definition of } \downarrow \text{ page 136}\\
&= x \downarrow (x \downarrow x) &&\\
1 &= \left(x \vee x'\right) && \text{by Theorem 10.2 page 132}\\
&= \left(x \vee x'\right)'' && \text{by Theorem 10.2 page 132}\\
&= \left(x \vee x'\right)' \downarrow \left(x \vee x'\right)' && \text{by definition of } \downarrow \text{ page 136}\\
&= \left[x \downarrow \left(x'\right)\right] \downarrow \left[x \downarrow \left(x'\right)\right] &&\\
&= [x \downarrow (x \downarrow x)] \downarrow [x \downarrow (x \downarrow x)] &&
\end{aligned}
$$

✏

Theorem 10.10. *Let $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ be a* BOOLEAN ALGEBRA. *Let $|$ represent the* SHEFFER STROKE *operator (Definition 10.2 page 136).*

THM *The set $\{|\}$ is* FUNCTIONALLY COMPLETE *with respect to $\boldsymbol{B}$. In particular,*

$$
\begin{array}{rcll}
x \vee y &=& (x|x)|(y|y) & \forall x,y \in X\\
x \wedge y &=& (x|y)|(x|y) & \forall x,y \in X\\
x' &=& x|x & \forall x \in X\\
0 &=& [x|(x|x)]|[x|(x|x)] & \forall x \in X\\
1 &=& x|(x|x) & \forall x \in X
\end{array}
$$

✎PROOF:

$$
\begin{aligned}
x' &= (x \wedge x)' && \text{by Theorem 10.2 page 132}\\
&= x|x && \text{by definition of } | \text{ page 136}\\
x \vee y &= (x \vee y)'' && \text{by Theorem 10.2 page 132}\\
&= \left(x' \wedge y'\right)' && \text{by de Morgan's Law page 132}\\
&= x'|y' && \text{by definition of } | \text{ page 136}\\
&= (x|x)|(y|y) && \text{by first result}\\
x \wedge y &= (x \wedge y)'' && \text{by Theorem 10.2 page 132}\\
&= (x|y)' && \text{by definition of } | \text{ page 136}\\
&= (x|y)|(x|y) && \text{by first result}\\
1 &= 0' &&\\
&= \left(x \wedge x'\right)' && \text{by Theorem 10.2 page 132}\\
&= x|\left(x'\right) && \text{by definition of } | \text{ page 136}\\
&= x|(x|x) &&
\end{aligned}
$$

$$\begin{aligned}
0 &= (x \wedge x') && \text{by Theorem 10.2 page 132}\\
&= (x \wedge x')'' && \text{by Theorem 10.2 page 132}\\
&= (x \wedge x')' | (x \wedge x')' && \text{by definition of } | \text{ page 136}\\
&= [x|(x')] | [x|(x')]\\
&= [x|(x|x)] | [x|(x|x)]
\end{aligned}$$

✏

Besides the *rejection* singleton $\{\downarrow\}$ and the Sheffer stroke singleton $\{|\}$, there are no single opertor sets that are *functionally complete* (next theorem).

Theorem 10.11. [15] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a Boolean algebra. Let* $\downarrow$ *be the* REJECTION *operator and* $|$ *be the* SHEFFER STROKE *operator.*

THM: $\{+\}$ *is* FUNCTIONALLY COMPLETE *in* $\boldsymbol{B}$ $\implies$ $+ = \downarrow$ *or* $+ = |$

Theorem 10.12. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA. *Let* $\div$ *represent the* ADJUNCTION *operator (Definition 10.2 page 136).*

THM: *The set* $\{\div, 0\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned}
x \vee y &= x \div (0 \div y) && \forall x,y \in X\\
x \wedge y &= 0 \div [(0 \div x) \div y] && \forall x,y \in X\\
x' &= 0 \div x && \forall x \in X\\
1 &= x \div x && \forall x \in X
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
x' &= 0 \vee x' && \text{by Theorem 10.2 page 132}\\
&= 0 \div x && \text{by definition of } \div \text{ (Definition 10.2 page 136)}\\
x \vee y &= x \vee y'' && \text{by Theorem 10.2 page 132}\\
&= x \div (y') && \text{by definition of } \div \text{ (Definition 10.2 page 136)}\\
&= x \div (0 \div y) && \text{by previous result}\\
x \wedge y &= (x' \vee y')' && \text{by \textit{de Morgan's law} property Theorem 10.2 page 132}\\
&= (x' \div y)' && \text{by definition of } \div \text{ (Definition 10.2 page 136)}\\
&= [(0 \div x) \div y]' && \text{by previous result}\\
&= 0 \div [(0 \div x) \div y] && \text{by previous result}\\
1 &= x \vee x' && \text{by \textit{complement} property Theorem 10.2 page 132}\\
&= x \div x && \text{by definition of } \div \text{ (Definition 10.2 page 136)}
\end{aligned}$$

✏

Theorem 10.13. [16] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA. *Let* $-$ *represent the* EXCEPTION *operator (Definition 10.2 page 136).*

THM: *The set* $\{-, 1\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned}
x \vee y &= 1 - [(1 - x) - y] && \forall x,y \in X\\
x \wedge y &= x - (1 - y) && \forall x,y \in X\\
x' &= 1 - x && \forall x \in X\\
0 &= x - x && \forall x \in X
\end{aligned}$$

15 Quine (1979) page 49, Żyliński (1925) page 208 $\langle \downarrow = \phi_{15}, | = \phi_2 \rangle$
16 Bernstein (1914) pages 89–91

✎PROOF:

$$\begin{aligned}
x' &= 1 \wedge x' && \text{by Theorem 10.2 page 132}\\
&= 1 - x && \text{by definition of } - \text{ (Definition 10.2 page 136)}\\
x \wedge y &= x \wedge y'' && \text{by Theorem 10.2 page 132}\\
&= x - (y') && \text{by definition of } - \text{ (Definition 10.2 page 136)}\\
&= x - (1 - y) && \text{by previous result}\\
x \vee y &= (x' \wedge y')' && \text{by \textit{de Morgan's law} property Theorem 10.2 page 132}\\
&= (x' - y)' && \text{by definition of } - \text{ (Definition 10.2 page 136)}\\
&= [(1 - x) - y]' && \text{by previous result}\\
&= 1 - [(1 - x) - y] && \text{by previous result}\\
0 &= x \wedge x' && \text{by \textit{complement} property Theorem 10.2 page 132}\\
&= x - x && \text{by definition of } - \text{ (Definition 10.2 page 136)}
\end{aligned}$$

✏

Theorem 10.14. [17] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM

The set $\{\triangle, \wedge, 1\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned}
x \vee y &= xy \triangle x \triangle y && \forall x,y \in X\\
x' &= x \triangle 1 && \forall x \in X\\
0 &= x \triangle x && \forall x \in X
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
x' &= x' \vee 0 && \text{by Theorem 10.2 page 132}\\
&= (x' \wedge 1) \vee (x \wedge 0) && \text{by Theorem 10.2 page 132}\\
&= (x' \wedge 1) \vee (x \wedge 1')\\
&= x \triangle 1 && \text{by definition of } \triangle \text{ (Definition 10.2 page 136)}\\
0 &= 0 \vee 0 && \text{by Theorem 10.2 page 132}\\
&= (x' \wedge x) \vee (x \wedge x') && \text{by Theorem 10.2 page 132}\\
&= x \triangle x && \text{by definition of } \triangle \text{ (Definition 10.2 page 136)}\\
xy \oplus x \oplus y &= (xy) \triangle (x \triangle y) && \text{by \textit{associative} property Theorem 10.2 page 132}\\
&= (xy) \oplus (x'y \vee xy') && \text{by definition of } \triangle \text{ (Definition 10.2 page 136)}\\
&= (xy)'(x'y \vee xy') \vee (xy)(x'y \vee xy')' && \text{by definition of } \triangle \text{ (Definition 10.2 page 136)}\\
&= (x' \vee y')(x'y \vee xy') \vee (xy)\left[(x'y)'(xy')'\right] && \text{by \textit{de Morgan's law} Theorem 10.2 page 132}\\
&= (x' \vee y')(x'y \vee xy') \vee (xy)\left[(x'' \vee y')(x' \vee y'')\right] && \text{by \textit{de Morgan's law} Theorem 10.2 page 132}\\
&= (x' \vee y')(x'y \vee xy') \vee (xy)\left[(x \vee y')(x' \vee y)\right]\\
&= (x'y \vee xy') \vee (xy)\left[xy \vee x'y'\right]\\
&= (x'y \vee xy') \vee xy\\
&= (x'y \vee xy') \vee (xy \vee xy) && \text{by \textit{idempotent} property Theorem 10.2}\\
&= (xy \vee x'y) \vee (xy \vee xy') && \text{by Theorem 10.2 page 132}\\
&= (x \vee x')y \vee x(y \vee y') && \text{by \textit{distributive} property Theorem 10.2}\\
&= (1)y \vee x(1)\\
&= x \vee y
\end{aligned}$$

✏

[17] Roth (2006) page 42

Theorem 10.15. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM *The set* $\{\triangle, \vee, 1\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned} x \wedge y &= [(x \triangle 1) \vee (y \triangle 1)] \triangle 1 && \forall x,y \in X \\ x' &= x \triangle 1 && \forall x \in X \\ 0 &= x \triangle x && \forall x \in X \end{aligned}$$

✎PROOF:

$$\begin{aligned} 0 &= x \triangle x \\ x' &= x \triangle 1 \\ x \wedge y &= (x' \vee y')' \\ &= [(x \triangle 1) \vee (y \triangle 1)] \triangle 1 \end{aligned}$$

⇨

Theorem 10.16. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM *The set* $\{\triangle, -, '\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned} x \vee y &= (x - y) \triangle y && \forall x,y \in X \\ x \wedge y &= x - (x - y) && \forall x,y \in X \\ 0 &= x \triangle x && \forall x \in X \end{aligned}$$

✎PROOF:

$$\begin{aligned} x \vee y &= x(y \vee y') \vee y \\ &= xy \vee xy' \vee y && \text{by \textit{distributive} property (Theorem 10.2 page 132)} \\ &= (y \vee xy) \vee xy' && \text{by \textit{associative} property (Theorem 10.2 page 132)} \\ &= y \vee xy' && \text{by \textit{absorptive} property (Theorem 10.2 page 132)} \\ &= (y \vee x'y) \vee xy' && \text{by \textit{absorptive} property (Theorem 10.2 page 132)} \\ &= (y \vee x')y \vee (xy')y' && \text{by \textit{distributive} and \textit{idempotent} properties (Theorem 10.2 page 132)} \\ &= (xy')'y \vee (xy')y' && \text{by \textit{de Morgan's law} property (Theorem 10.2 page 132)} \\ &= (xy') \triangle y && \text{by definition of } \triangle \text{ (Definition 10.2 page 136)} \\ &= (x - y) \triangle y && \text{by definition of } - \text{ (Definition 10.2 page 136)} \\ x \wedge y &= xx' \vee xy \\ &= x(x' \vee y) && \text{by \textit{distributive} and \textit{idempotent} properties (Theorem 10.2 page 132)} \\ &= x(x''y')' && \text{by \textit{de Morgan's law} property (Theorem 10.2 page 132)} \\ &= x(xy')' && \text{by \textit{involutory} property (Theorem 10.2 page 132)} \\ &= x(x - y)' && \text{by definition of } - \text{ (Definition 10.2 page 136)} \\ &= x - (x - y) && \text{by definition of } - \text{ (Definition 10.2 page 136)} \\ 0 &= xx' \\ &= x - (x - x') && \text{by previous result} \end{aligned}$$

⇨

10.5 Characterizations

❝*The algebra of symbolic logic…has recently assumed some importance as an independent calculus; it may therefore be not without interest to consider it from a purely mathematical or abstract point of view…*❞
Edward V. Huntington (1874–1952), American mathematician[18]

Order characterizations

An order characterization of Boolean algebras has already been given by Definition 10.1 (page 127): A lattice is a Boolean algebra if and only if it is *distributive* and *complemented*.

Proposition 10.4. [19] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ *be a* BOUNDED *and* COMPLEMENTED LATTICE.

PRP
$$\left\{\begin{array}{l}\boldsymbol{A} \text{ is a}\\ \textbf{Boolean algebra}\end{array}\right\} \iff \left\{\begin{array}{llll}1. & 1' = 0 & & \text{and}\\ 2. & (x \wedge y')' = y \vee (x' \wedge y') \quad \forall x,y \in X & \text{(ELKAN'S LAW)} & \end{array}\right\}$$

Algebraic characterizations

This section presents several algebraic characterizations. One such characterization has already been provided by Theorem 10.2 (page 132)— the standard properties of Boolean algebras characterized by 19 identities. If a system satisfies these 19 identities, then that system *is* a Boolean algebra. However, the set of 19 identities is very much an *over*-specification. It is possible to characterize Boolean algebras using much fewer relationships, from which all of the 19 identities of Theorem 10.2 can be derived. Here are some of these reduced characterizations:

- *Huntington's first set*: (1904) 8 relationships, Proposition 10.5 page 143
- *Huntington's fourth set*: (1933) 4 relationships, Proposition 10.6 page 145
- *Huntington's fifth set*: (1933) 3 relationships, Proposition 10.7 page 145
- *Stone*: (1935) 7 relationships, Proposition 10.8 page 146
- *Byrne's Formulation A*: (1946) 3 relationships, Proposition 10.9 page 146
- *Byrne's Formulation B*: (1946) 2 relationships, Proposition 10.10 page 148

All of these characterizations use 3 variables. It might be reasonable to ask if there exists a characterization that uses only two variables. The answer is "No", as demonstrated by the next theorem.

Theorem 10.17. [20]

THM *There does* NOT *exist a characterization of Boolean algebras consisting of only 2 variables.*

Proposition 10.5 (Huntington's first set). [21] *Let* X *be a set,* $\leq$ *a relation in* 2^{XX}, $\vee$ *and* $\wedge$ *binary*

[18] quote: Huntington (1904) page 288
image: http://en.wikipedia.org/wiki/Edward_V._Huntington

[19] Kondo and Dudek (2008) page 1035, Elkan et al. (1994), page 3 ⟨*Elkan's law*⟩

[20] Sikorski (1969), page 3, Diamond and McKinsey (1947) page 961, Gerrish (1978), page 36

[21] Gerrish (1978), page 35, Saliĭ (1988) page 33 ⟨"Huntington's Theorem"⟩, Joshi (1989) page 222 ⟨(B1)–(B4)⟩, Huntington (1904) pages 292–293 ⟨"1st set"⟩, Huntington (1933) page 277 ⟨"1st set"⟩, Givant and Halmos (2009) page 10

operations in $X^{X\times X}$, $'$ *an unary operation in* X^X, *and* 0 *and* 1 *nullary operations on* X.

PRP $(X, \vee, \wedge, 0, 1; \leq)$ *is a* ***Boolean algebra*** *if for all* $x, y, z \in X$

1.	$x \vee y$	$=$	$y \vee x$	$x \wedge y$	$=$	$y \wedge x$	(COMMUTATIVE)
2.	$x \vee (y \wedge z)$	$=$	$(x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z)$	$=$	$(x \wedge y) \vee (x \wedge z)$	(DISTRIBUTIVE)
3.	$x \vee 0$	$=$	x	$x \wedge 1$	$=$	x	(IDENTITY)
4.	$x \vee x'$	$=$	1	$x \wedge x'$	$=$	0	(COMPLEMENTED)

and where the relation $\leq$ *is defined as* $x \leq y \iff x \vee y = y \quad \forall x, y \in X$.

The property $x \vee x' = 1$ *is referred to as "the law of the* EXCLUDED MIDDLE*". The property* $x \wedge x' = 0$ *is referred to as "the law of* NON-CONTRADICTION*".*

✎PROOF:

1. Proof that $\boldsymbol{A}$ is a Boolean algebra $\implies$ $\boldsymbol{A}$ is a *distributive complemented lattice*:
   (a) Proof that $\boldsymbol{A}$ is *distributive*: by Definition 10.1 page 127
   (b) Proof that $\boldsymbol{A}$ is *complemented*: by Definition 10.1 page 127
   (c) Proof that $\boldsymbol{A}$ is *bounded*: by Lemma 10.1 page 128
   (d) Proof that $\boldsymbol{A}$ is a *lattice*:
      i. Proof that $\boldsymbol{A}$ is *idempotent*: by Lemma 10.1 page 128
      ii. Proof that $\boldsymbol{A}$ is *commutative*: by Definition 10.1 page 127
      iii. Proof that $\boldsymbol{A}$ is *associative*: by Lemma 10.1 page 128
      iv. Proof that $\boldsymbol{A}$ is *absorptive*: by Lemma 10.1 page 128
      v. Therefore, by Theorem 5.3 (page 74), $\boldsymbol{A}$ is a *lattice*

2. Proof that $\boldsymbol{A}$ is a Boolean algebra $\impliedby$ $\boldsymbol{A}$ is a *distributive complemented lattice*:
   (a) Proof that $\boldsymbol{A}$ is *commutative*: by property of lattices, Theorem 5.3 page 74
   (b) Proof that $\boldsymbol{A}$ is *distributive*: by right hypothesis
   (c) Proof that $\boldsymbol{A}$ has *identity*:

$$\begin{aligned}
x \vee 0 &= x \vee (x \wedge x') && \text{by \textit{complemented} property in right hypothesis}\\
&= x && \text{by \textit{absorptive} property of lattices Theorem 5.3 page 74}\\
x \wedge 1 &= x \wedge (x \vee x') && \text{by \textit{complemented} property in right hypothesis}\\
&= x && \text{by \textit{absorptive} property of lattices Theorem 5.3 page 74}
\end{aligned}$$

   (d) Proof that $\boldsymbol{A}$ is *complemented*: by right hypothesis

✏

Huntington's fourth set (next) characterizes Boolean algebras in terms of the standard properties of *idempotent*, *commutative*, and *associative* (see Theorem 10.2 page 132), and also in terms of an additional property called *Huntington's axiom*,[22] or (in terms of x and y), x *commutes* y. Huntington's axiom is significant in the context of *orthomodular* lattices in that an orthomodular lattice that satisfies Huntington's axiom is a Boolean algebra.[23]

[22] Givant and Halmos (2009) page 13 ⟨problem 7⟩
[23] Renedo et al. (2003), page 72 ⟨Definition 3⟩, Beran (1985) page 52, Beran (1982)

Proposition 10.6 (Huntington's fourth set).[24] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$\boldsymbol{A}$ *is a* ***Boolean algebra*** $\iff$

$$\left\{\begin{array}{llllll}
1. & x \vee x & = x & \forall x \in X & \text{(IDEMPOTENT)} & \text{and} \\
2. & x \vee y & = y \vee x & \forall x,y \in X & \text{(COMMUTATIVE)} & \text{and} \\
3. & (x \vee y) \vee z & = x \vee (y \vee z) & \forall x,y,z \in X & \text{(ASSOCIATIVE)} & \text{and} \\
4. & \left(x' \vee y'\right)' \vee \left(x' \vee y\right)' & = x & \forall x,y \in X. & \text{(HUNTINGTON'S AXIOM)} &
\end{array}\right\}$$

✎PROOF:

1. Proof that [$\boldsymbol{A}$ is a *Boolean algebra*] $\implies$ [$\boldsymbol{A}$ satisfies the 4 pairs of properties]:

   (a) Proof that $x \vee x = x$ (*idempotent* property with respect to $\vee$):
   by 1a of Lemma 10.1 (page 128).

   (b) Proof that $x \vee y = y \vee x$ (*commutative* property with respect to $\vee$):
   by 1a of this proposition.

   (c) Proof that $(x \vee y) \vee z = x \vee (y \vee z)$ (*associative* property with respect to $\vee$):
   by 2a of Lemma 10.1 (page 128).

   (d) Proof that $(x \wedge y) \vee \left(x \wedge y'\right) = x$ (*Huntington's axiom*):

$$\begin{aligned}
(x \wedge y) \vee \left(x \wedge y'\right) &= x \wedge \left(y \vee y'\right) && \text{by 2a} && (\textit{distributive} \text{ property wrt } \vee) \\
&= x \wedge 1 && \text{by 3a} && (\textit{complemented} \text{ property wrt } \vee) \\
&= x && \text{by 4b} && (\textit{identity} \text{ property wrt } \wedge)
\end{aligned}$$

2. Proof that [$\boldsymbol{A}$ is a *Boolean algebra*] $\impliedby$ [$\boldsymbol{A}$ satisfies the 4 pairs of properties]:

   (a) Proof that $x \vee y = y \vee x$: by 2 of Definition 10.1 page 127.

   (b) Proof that $x \wedge y = y \wedge x$:

   (c) Proof that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$:

   (d) Proof that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$:

   (e) Proof that $x \vee x' = 1$:

   (f) Proof that $x \wedge x' = 0$:

   (g) Proof that $x \vee 0 = x$:

   (h) Proof that $x \wedge 1 = x$:

✏

Proposition 10.7 (Huntington's fifth set).[25] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$\boldsymbol{A}$ *is a* ***Boolean algebra*** $\iff$

$$\left\{\begin{array}{llll}
1. & x'' = x & \forall x,y,z \in X & \text{and} \\
2. & x \vee \left(y \vee y'\right)' = x & \forall x,y \in X & \text{and} \\
3. & x \vee (y \vee z)' = \left[\left(y' \vee x\right)' \vee \left(z' \vee x\right)'\right]' & \forall x,y,z \in X. &
\end{array}\right\}$$

[24] Huntington (1933) page 280 ⟨"4th set"⟩

[25] Givant and Halmos (2009) page 13, Huntington (1933) page 286 ⟨"5th set"⟩

Proposition 10.8 (Stone). [26] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\boldsymbol{A} \textit{ is a Boolean algebra} \iff \left\{\begin{array}{llllll} 1. & x \vee y = y \vee x & \forall x,y \in X & \text{(JOIN COMMUTATIVE)} & \text{and} \\ 2. & x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X & \text{(LEFT DISTRIBUTIVE)} & \text{and} \\ 3. & (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) & \forall x,y,z \in X & \text{(RIGHT DISTRIBUTIVE)} & \text{and} \\ 4. & x \vee 0 = x & \forall x \in X & \text{(JOIN IDENTITY)} & \text{and} \\ 5. & \exists x' \text{ such that } x \vee x' = 1 \textit{ and } x \wedge x' = 0 & \forall x \in X & \text{(COMPLEMENTED)} & \text{and} \\ 6. & x \vee x = x & \forall x \in X & \text{(IDEMPOTENT)} & \text{and} \\ 7. & x \wedge x = x & \forall x \in X & & \end{array}\right\}$$

Proposition 10.9 (Byrne's FORMULATION A). [27] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\boldsymbol{A} \textit{ is a Boolean algebra} \iff \left\{\begin{array}{lllll} 1. & x \vee y = y \vee x & \forall x,y \in X & \text{(COMMUTATIVE)} & \text{and} \\ 2. & (x \vee y) \vee z = x \vee (y \vee z) & \forall x,y,z \in X & \text{(ASSOCIATIVE)} & \text{and} \\ 3. & x \vee y' = z \vee z' \iff x \vee y = x & \forall x,y,z \in X. & & \end{array}\right\}$$

✎PROOF:

1. Proof that $\boldsymbol{A}$ is a Boolean algebra $\implies$ 3 identities:
   (a) *commutative* property: By Theorem 10.2 (page 132), all Boolean algebras are *commutative*.
   (b) *associative* property: By Theorem 10.2 (page 132), all Boolean algebras are *associative*.
   (c) Proof that $x \vee y' = y \vee y' \implies x \vee y = x$:

$$\begin{aligned} x \vee y &= y \vee x && \text{by Boolean hypothesis and Theorem 10.2 page 132} \\ &= y \vee (x')' && \text{by Boolean hypothesis and Theorem 10.2 page 132} \\ &= y \vee (x')' && \text{by Boolean hypothesis and Theorem 10.2 page 132} \\ &= x' \vee (x')' && \text{by } x \vee y' = y \vee y' \text{ hypothesis} \\ &= x' \vee x && \text{by Boolean hypothesis and Theorem 10.2 page 132} \\ &= x && \text{by Boolean hypothesis and Theorem 10.2 page 132} \end{aligned}$$

   (d) Proof that $x \vee y' = y \vee y' \impliedby x \vee y = x$:

$$\begin{aligned} x \vee y' &= (x \vee y) \vee y' && \text{by } x \vee y = x \text{ hypothesis} \\ &= x \vee (y \vee y') && \text{by Boolean hypothesis and Theorem 10.2 page 132} \\ &= x \vee 1 && \text{by Boolean hypothesis and Theorem 10.2 page 132} \\ &= x && \text{by Boolean hypothesis and Theorem 10.2 page 132} \end{aligned}$$

2. Proof that $\boldsymbol{A}$ is a Boolean algebra $\impliedby$ 3 identities:
   (a) Proof that $x \vee x = x$ (*idempotent* property): because $x \vee x' = x \vee x'$ and by identity 3
   (b) Proof that $x \vee x' = y \vee y'$: by item (2a) and identity 3
   (c) Proof that $x \vee y = x$ and $y \vee z = y \implies x \vee z = x$:

$$\begin{aligned} x \vee z &= (x \vee y) \vee z && \text{by } x \vee y = x \text{ hypothesis} \\ &= x \vee (y \vee z) && \text{by identity 2 (\textit{associative} property)} \\ &= x \vee y && \text{by } y \vee z = y \text{ hypothesis} \\ &= x && \text{by } x \vee y = x \text{ hypothesis} \end{aligned}$$

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[26] Stone (1935) page 705
[27] Givant and Halmos (2009) page 13, Byrne (1946) page 270 ⟨"FORMULATION A"⟩

(d) Proof that $x'' = x$ (*involutory* property):

$$\begin{aligned}
x'' \lor x' &= x' \lor x'' && \text{by identity 1 (\textit{commutative} property)} && (10.1)\\
&= z \lor z' && \text{by item (2b)}\\
x'' \lor x &= x'' && \text{by equation (10.1) and identity 3} && (10.2)\\
x''' \lor x' &= x''' && \text{by equation (10.2)} && (10.3)\\
x'''' \lor x'' &= x'''' && \text{by equation (10.2)} && (10.4)\\
x'''' \lor x &= x'''' && \text{by equation (10.4), equation (10.5), and item (2c)} && (10.5)\\
x'''' \lor x' &= z \lor z' && \text{by equation (10.5) and identity 3} && (10.6)\\
x' \lor x''' &= x' && \text{by equation (10.6) and identity 3} && (10.7)\\
x''' &= x''' \lor x' && \text{by equation (10.3)} && (10.8)\\
&= x' \lor x''' && \text{by identity 1 (\textit{commutative} property)}\\
&= x' && \text{by equation (10.7)}\\
x \lor x''' &= x \lor x' && \text{by equation (10.8)} && (10.9)\\
&= z \lor z' && \text{by item (2b)}\\
x \lor x'' &= x && \text{by equation (10.9) and identity 3} && (10.10)\\
x'' &= x'' \lor x && \text{by equation (10.2)}\\
&= x \lor x'' && \text{by identity 1 (\textit{commutative} property)}\\
&= x && \text{by equation (10.10)}
\end{aligned}$$

(e) Proof that $x \lor (x' \lor y)'' = z \lor z'$:

$$\begin{aligned}
x \lor (x' \lor y)'' &= x \lor (x' \lor y) && \text{by item (2d) (\textit{involutory} property)}\\
&= (x \lor x') \lor y && \text{by identity 2 (\textit{associative} property)}\\
&= y \lor (x \lor x') && \text{by identity 1 (\textit{commutative} property)}\\
&= y \lor (y \lor y') && \text{by item (2b)}\\
&= (y \lor y) \lor y' && \text{by identity 2 (\textit{associative} property)}\\
&= y \lor y' && \text{by item (2a)}\\
&= z \lor z' && \text{by item (2b)}
\end{aligned}$$

(f) Proof that $x \lor (x' \lor y)' = x$: by item (2e) and identity 3

(g) Proof that $x \lor y'' \lor (x \lor y)' = z \lor z'$:

$$\begin{aligned}
x \lor y'' \lor (x \lor y)' &= x \lor y \lor (x \lor y)' && \text{by item (2d)}\\
&= z \lor z' && \text{by item (2b)}
\end{aligned}$$

(h) Proof that $x \lor (x \lor y)' = x \lor y'$:

$$\begin{aligned}
x \lor (x \lor y)' &= x \lor (x \lor y)' \lor y' && \text{by item (2g) and identity 3}\\
&= x \lor y' \lor (x \lor y)' && \text{by identity 1 (\textit{commutative} property)}\\
&= x \lor y' \lor \left[(x \lor y')' z\right] && \text{by item (2f)}\\
&= x \lor y' && \text{by item (2f)}
\end{aligned}$$

(i) Proof that $\left[\left(x' \lor y'\right)' \lor \left(x' \lor y\right)'\right] \lor x' = z \lor z'$:

$$\begin{aligned}
\left[\left(x' \lor y'\right)' \lor \left(x' \lor y\right)'\right] \lor x' &= x' \lor \left[\left(x' \lor y'\right)' \lor \left(x' \lor y\right)'\right] && \text{by identity 1 (\textit{commutative} property)}\\
&= \left[x' \lor \left(x' \lor y'\right)'\right] \lor \left(x' \lor y\right)' && \text{by identity 2 (\textit{associative} property)}\\
&= \left(x' \lor y''\right) \lor \left(x' \lor y\right)' && \text{by item (2h)}\\
&= \left(x' \lor y\right) \lor \left(x' \lor y\right)' && \text{by item (2d) (\textit{involutory})}\\
&= z \lor z' && \text{by item (2b)}
\end{aligned}$$

(j) Proof that $\left(x' \vee y'\right)' \vee \left(x' \vee y\right)' = x$ (*Huntington's axiom*):

$$\begin{aligned}
\underbrace{\left(x' \vee y'\right)' \vee \left(x' \vee y\right)'}_{\text{“}x\text{” in identity 3}} &= \underbrace{(x' \vee y')' \vee (x' \vee y)'}_{\text{“}x\text{” in identity 3}} \vee \underbrace{x}_{\text{“}y\text{”}} && \text{by item (2i) and identity 3}\\
&= x \vee \underbrace{(x' \vee y)' \vee (x' \vee y')'}_{x \text{ by item (2f)}} && \text{by identity 1 (\textit{commutative} property)}\\
&= \underbrace{x \vee (x' \vee y')'}_{x \text{ by item (2f)}} && \text{by item (2f)}\\
&= x && \text{by item (2f)}
\end{aligned}$$

(k) The three identities therefore imply that $\boldsymbol{A}$

i. is *idempotent* (item (2a)),

ii. is *commutative* (identity 1),

iii. is *associative* (identity 2), and

iv. satisfies *Huntington's axiom* (item (2j)).

Therefore, by Proposition 10.6 page 145 (*Huntington's Fourth Set*), $\boldsymbol{A}$ is a *Boolean algebra*.

✏

Proposition 10.10 (Byrne's Formulation B). [28] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$\boldsymbol{A}$ *is a* ***Boolean algebra*** $\iff$
$$\left\{\begin{array}{llllll}
1. & x \vee y' = z \vee z' & \iff & x \vee y = x & \forall x,y,z \in X & \text{and}\\
2. & (x \vee y) \vee z & = & (y \vee z) \vee x & \forall x,y,z \in X. &
\end{array}\right\}$$

Theorem 10.18. [29] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

THM

$\boldsymbol{A}$ *is a* ***Boolean algebra*** $\iff$
$$\left\{\begin{array}{lllll}
1. & x \wedge (x \vee y) & = & x & \forall x,y \in X \quad \text{and}\\
2. & x \wedge (y \vee z) & = & (z \wedge x) \vee (y \wedge x) & \forall x,y,z \in X \quad \text{and}\\
3. & \exists y' \quad \text{such that} \quad x \wedge (y \vee y') & = & x \vee (y \wedge y') & \forall x,y \in X.
\end{array}\right\}$$

✎PROOF:

1. Proof that $\boldsymbol{A}$ is a *distributive lattice*: by 1 and 2 and by Theorem 8.4 (page 112).

2. Define $0 \triangleq x \wedge x'$ and $1 \triangleq x \vee x'$.

3. Proof that 0 is the *join-identity* element and that 1 is the *meet-identity* element:

$$\begin{aligned}
x \vee 0 &= x \vee \left(y \wedge y'\right) && \text{by definition of 0 (item (2) page 148)}\\
&= (x \vee x) \vee \left(y \wedge y'\right) && \text{by \textit{idempotent} property of lattices (Theorem 5.3 page 74)}\\
&= x \vee \left[x \vee \left(y \wedge y'\right)\right] && \text{by \textit{associative} property of lattices (Theorem 5.3 page 74)}\\
&= x \vee \left[x \wedge \left(y \vee y'\right)\right] && \text{by 3}\\
&= x && \text{by \textit{absorptive} property of lattices (Theorem 5.3 page 74)}
\end{aligned}$$

$$\begin{aligned}
x \wedge 1 &= x \wedge \left(y \vee y'\right) && \text{by definition of 1 (item (2) page 148)}\\
&= (x \wedge x) \wedge \left(y \vee y'\right) && \text{by \textit{idempotent} property of lattices (Theorem 5.3 page 74)}\\
&= x \wedge \left[x \wedge \left(y \vee y'\right)\right] && \text{by \textit{associative} property of lattices (Theorem 5.3 page 74)}\\
&= x \wedge \left[x \vee \left(y \wedge y'\right)\right] && \text{by 3}\\
&= x && \text{by \textit{absorptive} property of lattices (Theorem 5.3 page 74)}
\end{aligned}$$

[28] Byrne (1946) page 271 ⟨“Formulation B”⟩
[29] Sholander (1951) pages 28–29, $P1$, $P2$, $P3^*$

4. Proof that $\boldsymbol{A}$ is *bounded* with 0 being the *greatest lower bound* and 1 being the *least upper bound*:

$$\begin{aligned}
x \wedge 0 &= (x \vee 0) \wedge 0 && \text{by } \textit{identity} \text{ property (item (3) page 148)} \\
&= 0 \wedge (0 \vee x) && \text{by } \textit{commutative} \text{ property of lattices (Theorem 5.3 page 74)} \\
&= 0 && \text{by } \textit{absorptive} \text{ property of lattices (Theorem 5.3 page 74)}
\end{aligned}$$

$$\begin{aligned}
x \vee 1 &= (x \wedge 1) \vee 1 && \text{by } \textit{identity} \text{ property (item (3) page 148)} \\
&= 1 \vee (1 \wedge x) && \text{by } \textit{commutative} \text{ property of lattices (Theorem 5.3 page 74)} \\
&= 1 && \text{by } \textit{absorptive} \text{ property of lattices (Theorem 5.3 page 74)}
\end{aligned}$$

5. Proof that $\boldsymbol{A}$ is *complemented*: Because $\boldsymbol{A}$ is *bounded* with greatest lower bound 0 and least upper bound 1 (item (4)) and because $x \wedge x' = 0$ and $x \vee x' = 1$ (definition of 0 and 1 (item (2) page 148)).

6. Proof that $\boldsymbol{A}$ is a *Boolean algebra*: Because $\boldsymbol{A}$ is *distributive* (item (1)) and *complemented* (item (5)), and by Definition 10.1 (page 127).

✏

10.6 Literature

Literature survey:

1. General information about Boolean algebras:
   Sikorski (1969)
   Dwinger (1971)
   Dwinger (1961)
   Halmos (1972)
   Monk (1989)
   Givant and Halmos (2009)

2. Characterizations:

   (a) Survey of characterizations:
   Padmanabhan and Rudeanu (2008)

   (b) Characterizations in terms of traditional *binary* operations *join* $\vee$, *meet* $\wedge$, and *complement* $'$:
   Huntington (1904) ⟨
   ⟩ Huntington (1933) ⟨
   ⟩ Diamond (1933)
   Diamond (1934)
   Stone (1935)
   Hoberman and McKinsey (1937)
   Frink (1941) ⟨4 identities involving $\vee, \wedge, '$⟩
   Newman (1941)
   Braithwaite (1942)
   Byrne (1946) ⟨Form. A and B⟩
   Gerrish (1978) ⟨independence of Huntington's characterizations⟩

   (c) Characterizations in terms of non-traditional *binary* operations:
   Sheffer (1913) ⟨rejection $\downarrow$⟩
   Bernstein (1914) ⟨exception $-$⟩
   Bernstein (1916) ⟨rejection $\downarrow$⟩
   Bernstein (1933) ⟨rejection $\downarrow$⟩
   Bernstein (1934) ⟨implication $\Rightarrow$⟩
   Bernstein (1936) ⟨complete disjuction $\triangle$⟩
   Byrne (1948) ⟨inclusion⟩

Byrne (1951) ⟨ring operations⟩
Miller (1952) ⟨ring operations⟩

(d) Characterizations in terms of *ternary* operations:
Whiteman (1937) *ternary rejection*

(e) Characterizations involving *Elkan's law*:
Kondo and Dudek (2008) ⟨for bounded lattices⟩
Renedo et al. (2003) ⟨for orthomodular lattices⟩
Trillas et al. (2004) ⟨for orthocomplemented lattices⟩

3. Analytic properties:
Vladimirov (2002)

4. Miscellaneous:
Montague and Tarski (1954)
Rudeanu (1961) ⟨referenced by Sikorski (1969)⟩

5. Actually, "Boolean algebras" are not really "algebras". Rather, they are "a commutative ring with unit, without nilpotents, and having idempotents which stood for classes"
Hailperin (1981), page 184

6. Pioneering works related to Boolean algebras:
Boole (1847)
Boole (1854)
Jevons (1864) ⟨join and meet operations⟩
Peirce (1870a) ⟨order concepts⟩
Huntington (1904) ⟨axiomization⟩

7. History of development of Boolean algebra:
Burris (2000)

CHAPTER 11

ORTHOCOMPLEMENTED LATTICES

Orthocomplemented lattices (Definition 11.1 page 152) are a kind of generalization of *Boolean algebras*. The relationship between lattices of several types, including orthocomplemented and Boolean lattices, is stated in Theorem 11.7 (page 163) and illustrated in Figure 11.1 (page 151).

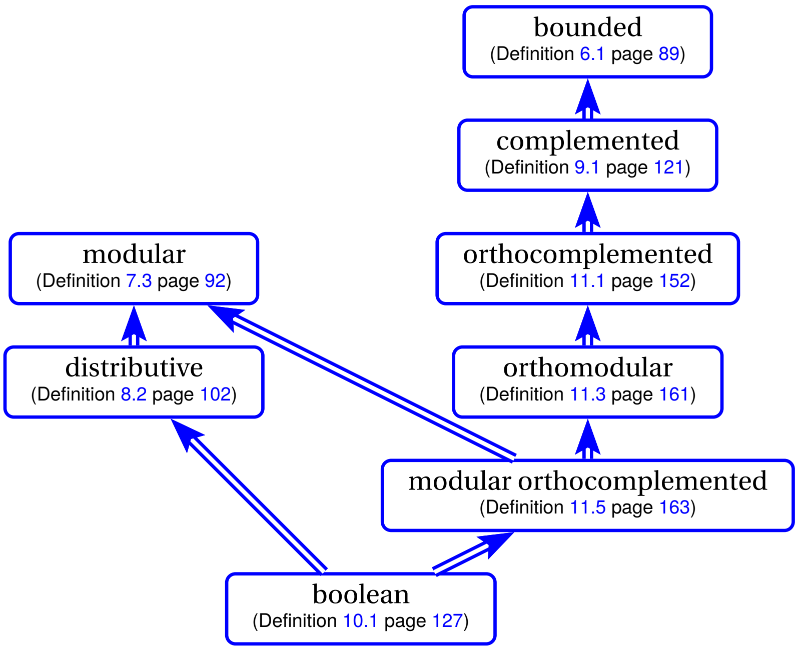


Figure 11.1: lattice of orthocomplemented lattices

11.1 Orthocomplemented Lattices

11.1.1 Definition

Definition 11.1. [1] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89).*

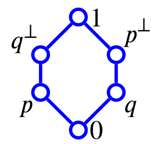
DEF

An element $x^{\perp} \in X$ *is an* ***orthocomplement*** *of an element* $x \in X$ *if*

1. $x^{\perp\perp} = x$ (INVOLUTORY) *and*
2. $x \wedge x^{\perp} = 0$ (NON-CONTRADICTION) *and*
3. $x \leq y \implies y^{\perp} \leq x^{\perp} \quad \forall y \in X$ (ANTITONE).

The LATTICE $\mathbf{L}$ *is* ***orthocomplemented*** *(*$\mathbf{L}$ *is an* ***orthocomplemented lattice****) if every element* x *in* X *has an* ORTHOCOMPLEMENT $x^{\perp}$ *in* X.

Definition 11.2. [2]



DEF

The $\mathbf{O_6}$ ***lattice*** *is the ordered set* $(\{0, p, q, p^{\perp}, q^{\perp}, 1\}, \leq)$ *with cover relation* $\lessdot = \{(0,p), (0,q), (p,q^{\perp}), (q,p^{\perp}), (p^{\perp},1), (q^{\perp},1)\}$.

The O_6 *lattice is illustrated by the Hasse diagram to the right.*

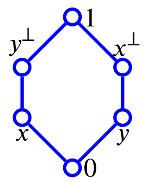
Example 11.1. [3]

EX

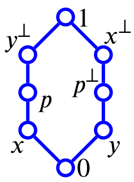
The O_6 **lattice** (Definition 11.2 page 152) is an **orthocomplemented lattice** (Definition 11.1 page 152).

Example 11.2. [4]There are a total of 10 **orthocomplemented lattices** with 8 elements or less. These 10, along with 3 other orthocomplemented lattices with 10 elements, are illustrated next:

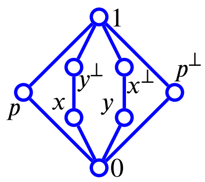
Lattices that are **orthocomplemented** but *non-orthomodular* and hence also *not modular orthocomplemented* and *non-Boolean*:



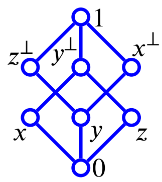
1. O_6 *lattice*



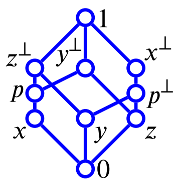
2. O_8 *lattice*



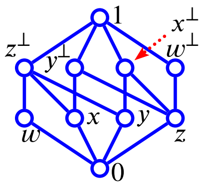
3.



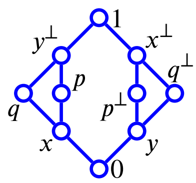
4.



5.



6.



7.

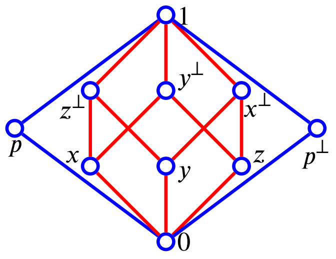
Lattices that are **orthocomplemented** and **orthomodular** but *not modular orthocomplemented* and hence also *non-Boolean*:

[1] Stern (1999) page 11, Beran (1985) page 28, Kalmbach (1983) page 16, Gudder (1988) page 76, Loomis (1955) page 3, Birkhoff and Neumann (1936) page 830 ⟨L71–L73⟩

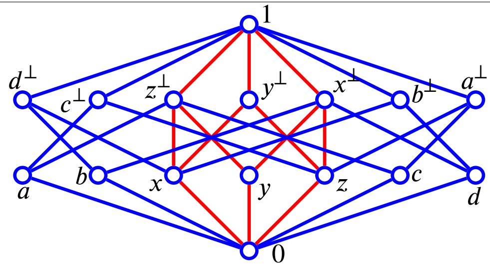
[2] Kalmbach (1983) page 22, Holland (1970), page 50, Beran (1985) page 33, Stern (1999) page 12, The O_6 *lattice* is also called the **Benzene ring** or the **hexagon**.

[3] Holland (1963), page 50

[4] Beran (1985) pages 33–42, Maeda (1966) page 250, Kalmbach (1983) page 24 ⟨Figure 3.2⟩, Stern (1999) page 12, Holland (1970), page 50

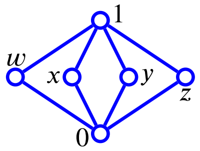


8.

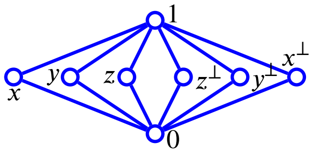


9.

Lattices that are **orthocomplemented**, **orthomodular**, and **modular orthocomplemented** but *non-Boolean*:



10. M_4 *lattice*

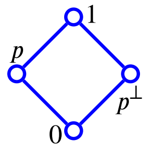


11. M_6 *lattice*

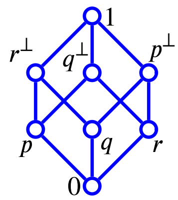
Lattices that are **orthocomplemented**, **orthomodular**, **modular orthocomplemented** and **Boolean**:

12. L_1 *lattice*

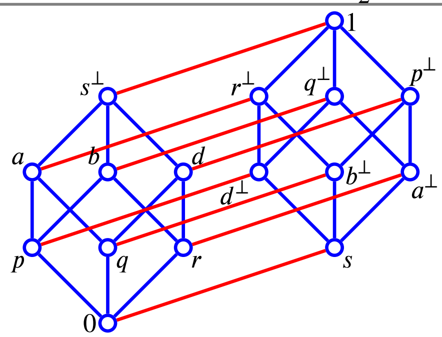
13. L_2 *lattice*



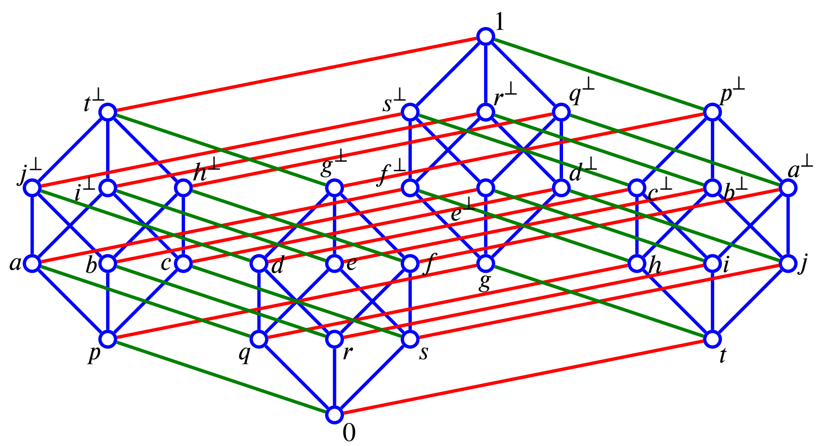
14. L_2^2 *lattice*



15. L_2^3 *lattice*



16. L_2^4 *lattice*

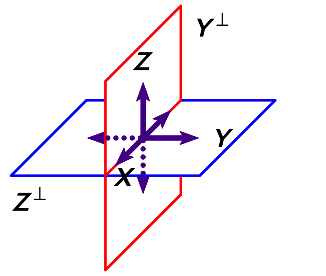


17. L_2^5 *lattice*

Example 11.3.

EX

The structure $\left(2^{\mathbb{R}^N}, +, \cap, \emptyset, \boldsymbol{H}; \subseteq\right)$ is an **orthocomplemented lattice** where
- $\mathbb{R}^N$ is an **Euclidean space** with dimension N
- $2^{\mathbb{R}^N}$ is the set of all subspaces of $\mathbb{R}^N$
- $\boldsymbol{V} + \boldsymbol{W}$ is the *Minkowski sum* of subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$
- $\boldsymbol{V} \cap \boldsymbol{W}$ is the *intersection* of subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$



Example 11.4.

EX

The structure $\left(2^{\boldsymbol{H}}, \oplus, \cap, \emptyset, \boldsymbol{H}; \subseteq\right)$ is an **orthocomplemented lattice** where
- $\boldsymbol{H}$ is a **Hilbert space**
- $2^{\boldsymbol{H}}$ is the set of all closed subspaces of $\boldsymbol{H}$
- $\boldsymbol{X} + \boldsymbol{Y}$ is the *Minkowski sum* of subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$
- $\boldsymbol{X} \oplus \boldsymbol{Y} \triangleq (\boldsymbol{X} + \boldsymbol{Y})^-$ is the *closure* of $\boldsymbol{X} + \boldsymbol{Y}$
- $\boldsymbol{X} \cap \boldsymbol{Y}$ is the *intersection* of subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$

11.1.2 Properties

Theorem 11.1. [5] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE.

THM

$$\left\{\begin{array}{l}\boldsymbol{L} \textit{ is}\\ \text{ORTHOCOMPLEMENTED}\\ \text{(Definition 11.1 page 152)}\end{array}\right\} \implies \left\{\begin{array}{lrcll}(1). & 0^\perp &=& 1 & & \text{(BOUNDARY CONDITION)} \quad \textit{and}\\ (2). & 1^\perp &=& 0 & & \text{(BOUNDARY CONDITION)} \quad \textit{and}\\ (3). & (x \vee y)^\perp &=& x^\perp \wedge y^\perp & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN)} \quad \textit{and}\\ (4). & (x \wedge y)^\perp &=& x^\perp \vee y^\perp & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)} \quad \textit{and}\\ (5). & x \vee x^\perp &=& 1 & \forall x \in X & \text{(EXCLUDED MIDDLE)}.\end{array}\right.$$

✎PROOF: Let $x^\perp \triangleq \neg x$, where $\neg$ is an *ortho negation* function (Definition 13.3 page 172). Then, this theorem follows directly from Theorem 13.5 (page 176). ⇨

Corollary 11.1. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89)*.

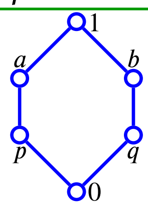
COR

$$\left\{\begin{array}{l}\boldsymbol{L} \textit{ is orthocomplemented}\\ \text{(Definition 11.1 page 152)}\end{array}\right\} \implies \left\{\begin{array}{l}\boldsymbol{L} \textit{ is complemented}\\ \text{(Definition 9.1 page 121)}\end{array}\right\}$$

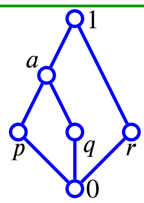
✎PROOF: This follows directly from the definition of *orthocomplemented lattices* (Definition 11.1 page 152) and *complemented lattices* (Definition 9.1 page 121). ⇨

Example 11.5.

EX



The O_6 *lattice* (Definition 11.2 page 152) illustrated to the left is both **ortho-complemented** (Definition 11.1 page 152) and **multiply complemented** (Definition 9.1 page 121). The lattice illustrated to the right is **multiply complemented**, but is **non-orthocomplemented**.



✎PROOF:

1. Proof that O_6 *lattice* is multiply complemented: b and q are both *complements* of p.

[5] Beran (1985) pages 30–31, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Cohen (1989) page 37 ⟨3B.13. Theorem⟩

2. Proof that the right side lattice is multiply complemented: a, p, and q are all *complements* of r.

⇨

Lemma 11.1 (next) is useful in proving that *de Morgan*'s laws (Theorem 16.8 page 236) hold in orthocomplemented lattices (Theorem 11.1 page 154) and in proving the characterization of Theorem 11.2 (page 156).

Lemma 11.1. [6] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

LEM

$$\underbrace{x \leq y \implies y^\perp \leq x^\perp}_{\text{ANTITONE}} \iff \underbrace{\left\{\begin{array}{rcll} (x \vee y)^\perp &=& x^\perp \wedge y^\perp & {\scriptstyle x,y \in X} \quad \textit{and} \\ (x \wedge y)^\perp &=& x^\perp \vee y^\perp & {\scriptstyle x,y \in X} \end{array}\right.}_{\text{DE MORGAN}}$$

✎PROOF: This follows directly from Lemma 13.2 (page 174). ⇨

Lemma 11.2. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

LEM

The set $\{0, x, x^\perp\}$ *is* DISTRIBUTIVE *(Definition 8.1 page 101) for all* $x \in X$.

✎PROOF:

$0 \wedge (x \vee x^\perp) = 0$	by *lower bounded* property	(Proposition 6.2 page 89)
$= 0 \vee 0$	by *join identity*	(Proposition 6.2 page 89)
$= (0 \wedge x) \vee (0 \wedge x^\perp)$	by *lower bounded* property	(Proposition 6.2 page 89)
$0 \wedge (x^\perp \vee x) = 0$	by *lower bounded* property	(Proposition 6.2 page 89)
$= 0 \vee 0$	by *join identity*	(Proposition 6.2 page 89)
$= (0 \wedge x^\perp) \vee (0 \wedge x)$	by *lower bounded* property	(Proposition 6.2 page 89)
$x \wedge (x^\perp \vee 0) = x \wedge x^\perp$	by *join identity*	(Proposition 6.2 page 89)
$= 0$	by *non-contradiction* property	(Definition 11.1 page 152)
$= 0 \vee 0$	by *join identity*	(Proposition 6.2 page 89)
$= (x \wedge x^\perp) \vee 0$	by *non-contradiction* property	(Definition 11.1 page 152)
$= (x \wedge x^\perp) \vee (x \wedge 0)$	by *lower bounded* property	(Proposition 6.2 page 89)
$x \wedge (0 \vee x^\perp) = x \wedge (x^\perp \vee 0)$	by *commutative* property of *lattices*	(Theorem 5.3 page 74)
$= (x \wedge x^\perp) \vee (x \wedge 0)$	by previous result	
$= (x \wedge 0) \vee (x \wedge x^\perp)$	by *commutative* property of *lattices*	(Theorem 5.3 page 74)
$x^\perp \wedge (x \vee 0) = (x^\perp \wedge x) \vee (x^\perp \wedge 0)$	by $x \wedge (x^\perp \vee 0)$ result	
$x^\perp \wedge (0 \vee x) = (x^\perp \wedge 0) \vee (x^\perp \wedge x)$	by $x \wedge (0 \vee x^\perp)$ result	

⇨

[6] Beran (1985) pages 30–31, Fáy (1967) ⟨cf Beran 1985 page 30⟩, Nakano and Romberger (1971) ⟨cf Beran 1985⟩

11.1.3 Characterization

Theorem 11.2. [7] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an algebraic structure.*

THM

$$\left\{ \begin{array}{l} \boldsymbol{L} \text{ is an} \\ \text{orthocomplemented lattice} \end{array} \right\} \iff \left\{ \begin{array}{lllll} 1. & (z^\perp \wedge y^\perp)^\perp \vee x & = & (x \vee y) \vee z & \forall x,y,z \in X \quad \text{and} \\ 2. & x \wedge (x \vee y) & = & x & \forall x,y \in X \quad \text{and} \\ 3. & x \vee (y \wedge y^\perp) & = & x & \forall x,y \in X. \end{array} \right.$$

✎PROOF:

1. Proof that orthocomplemented lattice $\implies$ 3 properties:

$$\begin{aligned} (z^\perp \wedge y^\perp)^\perp \vee x &= \left[(z^\perp)^\perp \vee (y^\perp)^\perp\right] \vee x && \text{by } \textit{de Morgan} \text{ property (Theorem 11.1 page 154)} \\ &= (z \vee y) \vee x && \text{by } \textit{involutory} \text{ property (Definition 11.1 page 152)} \\ &= x \vee (z \vee y) && \text{by } \textit{commutative} \text{ property (Theorem 5.3 page 74)} \\ &= x \vee (y \vee z) && \text{by } \textit{commutative} \text{ property (Theorem 5.3 page 74)} \\ &= (x \vee y) \vee z && \text{by } \textit{associative} \text{ property (Theorem 5.3 page 74)} \end{aligned}$$

$$x \wedge (x \vee y) = x \qquad \text{by } \textit{absorptive} \text{ property (Theorem 5.3 page 74)}$$

$$\begin{aligned} x \vee (y \wedge y^\perp) &= x \vee 0 && \text{by } \textit{complemented} \text{ property (Definition 11.1 page 152)} \\ &= x \end{aligned}$$

2. Proof that orthocomplemented lattice $\impliedby$ 3 properties:

   (a) Proof that $\boldsymbol{L}$ is *meet-idempotent*:

$$\begin{aligned} x \wedge x &= x \wedge \left[x \vee (y \wedge y^\perp)\right] && \text{by (3)} \\ &= x \wedge \left[x \vee (y \wedge y^\perp)\right] && \text{by (3)} \\ &= x && \text{by (2)} \end{aligned}$$

   (b) Define $0 \triangleq xx^\perp$ for some $x \in X$. Proof that 0 is the *greatest lower bound* of $\boldsymbol{L}$: The element 0 is the greatest lower bound if and only if $xx^\perp = yy^\perp \quad \forall x, y \in X\ldots$

   i. Proof that $(xx^\perp)^{\perp\perp} = (xx^\perp) \quad \forall x \in X$:

$$\begin{aligned} (xx^\perp)^{\perp\perp} &= (xx^\perp)^{\perp\perp} + (xx^\perp) && \text{by (3)} \\ &= \left[(xx^\perp)^\perp (xx^\perp)^\perp\right]^\perp + (xx^\perp) && \text{by item (2a)} \\ &= \left[(xx^\perp) + (xx^\perp)\right] + (xx^\perp) && \text{by (1)} \\ &= \left[(xx^\perp)\right] + (xx^\perp) && \text{by (3)} \\ &= (xx^\perp) && \text{by (3)} \end{aligned}$$

   ii. Proof that $a = (xx^\perp) + a \quad \forall a, x \in X$:

$$\begin{aligned} a &= a + (xx^\perp) && \text{by (3)} \\ &= \left[a + (xx^\perp)\right] + (xx^\perp) && \text{by (3)} \\ &= \left[(xx^\perp)^\perp (xx^\perp)^\perp\right]^\perp + a && \text{by (1)} \\ &= \left[(xx^\perp)^\perp\right]^\perp + a && \text{by item (2a)} \\ &= (xx^\perp) + a && \text{by item (2(b)i)} \end{aligned}$$

[7] Beran (1985) pages 31–33, Beran (1976) pages 251–252

iii. Proof that $(xx^{\perp}) = (yy^{\perp}) \quad \forall x, y \in X$:

$$\begin{aligned}
(xx^{\perp}) &= (xx^{\perp}) + (yy^{\perp}) && \text{by (3)}\\
&= (yy^{\perp}) && \text{by item (2(b)ii)}
\end{aligned}$$

(c) Proof that $x + 0 = 0 + x = x \quad \forall x \in X$ (*join identity*):

$$\begin{aligned}
x + 0 &= x + (yy^{\perp}) && \text{by item (2(b)iii)}\\
&= x && \text{by (3)}\\
0 + x &= (uu^{\perp}) + x && \text{by item (2(b)iii)}\\
&= x && \text{by item (2(b)ii)}
\end{aligned}$$

(d) Proof that $x + y = (y^{\perp}x^{\perp})^{\perp} \quad \forall x, y \in X$:

$$\begin{aligned}
(y^{\perp}x^{\perp})^{\perp} &= (y^{\perp}x^{\perp})^{\perp} + 0 && \text{by item (2c)}\\
&= (0 + x) + y && \text{by (1)}\\
&= x + y && \text{by item (2c)}
\end{aligned}$$

(e) Proof that $x + x = x^{\perp\perp} \quad \forall x \in X$:

$$\begin{aligned}
x + x &= (x^{\perp}x^{\perp})^{\perp} && \text{by item (2d)}\\
&= (x^{\perp})^{\perp} && \text{by item (2a)}
\end{aligned}$$

(f) Proof that $x + y = y + x \quad \forall x, y \in X$ (*join-commutative*):

$$\begin{aligned}
x + y &= (x + 0) + y && \text{by item (2c)}\\
&= (0^{\perp}x^{\perp})^{\perp} + y && \text{by item (2d)}\\
&= (y + x) + 0 && \text{by (1)}\\
&= y + x && \text{by item (2c)}
\end{aligned}$$

(g) Proof that $(x + y) + z = x + (y + z) \quad \forall x, y, z \in X$ (*join-associative*):

$$\begin{aligned}
(x + y) + z &= (z^{\perp}y^{\perp})^{\perp} + x && \text{by (1)}\\
&= (y + z) + x && \text{by item (2d)}\\
&= x + (y + z) && \text{by item (2f)}
\end{aligned}$$

(h) Proof that $x^{\perp\perp} = x \quad \forall x \in X$ (*involutory*):

$$\begin{aligned}
x^{\perp\perp} &= \left(x^{\perp}\right)\perp && \text{by definition of } x^{\perp\perp}\\
&= \left[x^{\perp}\left(x^{\perp} + x\right)\right]\perp && \text{by (2)}\\
&= \left[x^{\perp}\left(x^{\perp}x^{\perp\perp}\right)^{\perp}\right]\perp && \text{by item (2d)}\\
&= \left(x^{\perp}x^{\perp\perp}\right) + x && \text{by item (2d)}\\
&= (0) + x && \text{by item (2b)}\\
&= x && \text{by item (2c)}
\end{aligned}$$

(i) Proof of *de Morgan*'s laws:

$$\begin{aligned}
(x + y)^{\perp} &= (y + x)^{\perp} && \text{by item (2g)}\\
&= \left[\left(x^{\perp}y^{\perp}\right)^{\perp}\right]^{\perp} && \text{by item (2d)}\\
&= x^{\perp}y^{\perp} && \text{by item (2h)}
\end{aligned}$$

$$\begin{aligned}
(xy)^{\perp} &= \left(x^{\perp\perp}y^{\perp\perp}\right)^{\perp} && \text{by item (2h)}\\
&= y^{\perp} + x^{\perp} && \text{by item (2d)}\\
&= x^{\perp} + y^{\perp} && \text{by item (2g)}
\end{aligned}$$

(j) Proof that $(xy)z = x(yz) \quad \forall x,y,z \in X \quad$ (*meet-commutative*):

$$\begin{aligned}
xy &= (xy)^{\perp\perp} && \text{by item (2h)}\\
&= \left(x^{\perp} + y^{\perp}\right)^{\perp} && \text{by item (2i)}\\
&= \left(y^{\perp} + x^{\perp}\right)^{\perp} && \text{by item (2g)}\\
&= y^{\perp\perp}x^{\perp\perp} && \text{by item (2i)}\\
&= yx && \text{by item (2i)}
\end{aligned}$$

(k) Proof that $(xy)z = x(yz) \quad \forall x,y,z \in X \quad$ (*meet-associative*):

$$\begin{aligned}
(xy)z &= [(xy)z]^{\perp\ \perp} && \text{by item (2h)}\\
&= \left[(xy)^{\perp} + z^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[\left(x^{\perp} + y^{\perp}\right) + z^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp} + \left(y^{\perp} + z^{\perp}\right)\right]^{\perp} && \text{by item (2g)}\\
&= x^{\perp\perp}\left(y^{\perp} + z^{\perp}\right)^{\perp} && \text{by item (2i)}\\
&= x^{\perp\perp}\left(y^{\perp\perp}z^{\perp\perp}\right) && \text{by item (2i)}\\
&= x(yz) && \text{by item (2h)}
\end{aligned}$$

(l) Proof that $x + (xz) = x$ (*join-meet-absorptive*):

$$\begin{aligned}
x \vee (xz) &= [x + (xz)]^{\perp\perp} && \text{by item (2h)}\\
&= \left[x^{\perp}(xz)^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp}\left(x^{\perp} + z^{\perp}\right)\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp}\right]^{\perp} && \text{by (2)}\\
&= x && \text{by item (2h)}
\end{aligned}$$

(m) Because $\boldsymbol{L}$ is *commutative* (item (2f) and item (2j)), *associative* (item (2g) and item (2k)), and *absorptive* ((2) and item (2l)), and by Theorem 5.8 (page 82), $\boldsymbol{L}$ is a *lattice*.

(n) Define $1 \triangleq x + x^{\perp}$ for some $x \in X$. Proof that 1 is the *least upper bound* of $\boldsymbol{L}$: The element 1 is the least upper bound if and only if $x + x^{\perp} = y + y^{\perp} \quad \forall x,y \in X\ldots$

$$\begin{aligned}
1 &= \left(x + x^{\perp}\right) && \text{by definition of 1}\\
&= \left(x + x^{\perp}\right)^{\perp\perp} && \text{by item (2h)}\\
&= \left(x^{\perp}x\right)^{\perp} && \text{by item (2h)}\\
&= \left(xx^{\perp}\right)^{\perp} && \text{by item (2j)}\\
&= \left(yy^{\perp}\right)^{\perp} && \text{by item (2(b)iii)}\\
&= y^{\perp} + y^{\perp\perp} && \text{by item (2i)}\\
&= y^{\perp} + y && \text{by item (2h)}\\
&= y + y^{\perp} && \text{by item (2f)}
\end{aligned}$$

(o) Proof that $\boldsymbol{L}$ is *antitone*: by Theorem 13.4 (page 176).

(p) Proof that $\boldsymbol{L}$ is *complemented*: by item (2(b)iii) and item (2n).

(q) Because $\boldsymbol{L}$ is a *bounded* (item (2b) and item (2n)) lattice (item (2m)), and because $\boldsymbol{L}$ is *complemented* (item (2p)), is *involutory* (item (2h)), and is *antitone* (item (2o)), and by Definition 11.1 (page 152), $\boldsymbol{L}$ is an *orthocomplemented lattice*.

11.1.4 Restrictions resulting in Boolean algebras

Proposition 11.1. [8] *Let* $\mathbf{L} = (X, \vee, \wedge, 0, 1; \leq)$ *be a* LATTICE *(Definition 5.3 page 73).*

PRP: $\left\{\begin{array}{lll} 1. & \mathbf{L} \text{ is } \textbf{\textit{orthocomplemented}} & \text{(Definition 11.1 page 152) and} \\ 2. & \mathbf{L} \text{ is } \textbf{\textit{distributive}} & \text{(Definition 8.2 page 102)} \end{array}\right\} \implies \left\{\begin{array}{l} \mathbf{L} \text{ is } \textbf{\textit{Boolean}} \\ \text{(Definition 10.1 page 127)} \end{array}\right\}$

✎PROOF: To be a *Boolean algebra*, $\mathbf{L}$ must satisfy the 8 requirements of *boolean algebras* (Definition 10.1 page 127):

1. Proof for *commutative* properties: These are true for *all* lattices (Definition 5.3 page 73).
2. Proof for *join-distributive* property: by hypothesis (2).
3. Proof for *meet-distributive* property: by *join-distributive* property and the *Principle of duality* (Theorem 5.4 page 75) for lattices.
4. Proof for *identity* properties: because $\mathbf{L}$ is a *bounded lattice* and by definitions of 1 (*least upper bound*), 0 (*greatest lower bound*), $\vee$, and $\wedge$.
5. Proof for *complemented* properties: by hypothesis (1) and definition of *orthocomplemented lattices* (Definition 11.1 page 152).

⇒

Proposition 11.2. *Let* $\mathbf{L} = (X, \vee, \wedge, 0, 1; \leq)$ *be a* LATTICE *(Definition 5.3 page 73).*

PRP: $\left\{\begin{array}{lll} 1. & \mathbf{L} \text{ is } \textbf{\textit{orthocomplemented}} & \text{(Definition 11.1 page 152) and} \\ 2. & \text{Every } x \in \mathbf{L} \text{ is in the } \textbf{\textit{center}} \text{ of } \mathbf{L} & \text{(Definition 15.4 page 211)} \end{array}\right\} \iff \left\{\begin{array}{l} \mathbf{L} \text{ is} \\ \textbf{\textit{Boolean}} \end{array}\right\}$

✎PROOF:

1. Proof that (1,2) $\implies$ *Boolean*: $\mathbf{L}$ is *Boolean* because it satisfies *Huntington's Fourth Set* (Proposition 10.6 page 145), as demonstrated by the following …
   (a) Proof that $x \vee x = x$ (*idempotent*): $\mathbf{L}$ is a *lattice* (by definition of $\mathbf{L}$), and all lattices are *idempotent* (Definition 5.3 page 73).
   (b) Proof that $x \vee y = y \vee x$ (*commutative*): $\mathbf{L}$ is a *lattice* (by definition of $\mathbf{L}$), and all lattices are *commutative* (Definition 5.3 page 73).
   (c) Proof that $(x \vee y) \vee z = x \vee (y \vee z)$ (*associative*): $\mathbf{L}$ is a *lattice* (by definition of $\mathbf{L}$), and all lattices are *associative* (Definition 5.3 page 73).
   (d) Proof that $(x^\perp \vee y^\perp)^\perp \vee (x^\perp \vee y)^\perp = x$ (*Huntington's axiom*):

$$\begin{aligned}
(x^\perp \vee y^\perp)^\perp \vee (x^\perp \vee y)^\perp &= (x^\perp \perp \wedge y^\perp \perp) \vee (x^\perp \perp \wedge y^\perp) && \text{by } \textit{de Morgan} \text{ property (Theorem 11.1 page 154)} \\
&= (x \wedge y) \vee (x \wedge y^\perp) && \text{by } \textit{involution} \text{ property (Definition 11.1 page 152)} \\
&= x && \text{by definition of } \textit{center} \text{ (Definition 15.4 page 211)}
\end{aligned}$$

2. Proof that (1) $\impliedby$ *Boolean*:
   (a) Proof that $x \vee x^\perp = 1$: by definition of *Boolean algebras* (Definition 10.1 page 127).
   (b) Proof that $x \wedge x^\perp = 0$: by definition of *Boolean algebras* (Definition 10.1 page 127).

[8] Kalmbach (1983) page 22

(c) Proof that $x^{\perp\perp} = x$: by *involutory* property of *Boolean algebra* (Theorem 10.2 page 132).

(d) Proof that $x \le y \implies y^{\perp} \le x^{\perp}$:

$$\begin{aligned}
y^{\perp} \le x^{\perp} &\iff y^{\perp} = y^{\perp} \wedge x^{\perp} && \text{by Lemma 5.1 page 75}\\
&\iff y^{\perp\perp} = \left(y^{\perp} \wedge x^{\perp}\right)^{\perp}\\
&\iff y^{\perp\perp} = y^{\perp\perp} \vee x^{\perp\perp} && \text{by } \textit{de Morgan} \text{ property (Theorem 10.2 page 132)}\\
&\iff y = y \vee x && \text{by } \textit{involutory} \text{ property (Theorem 10.2 page 132)}\\
&\iff y = y && \text{by } x \le y \text{ hypothesis}
\end{aligned}$$

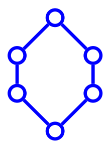
3. Proof that (2) $\impliedby$ *Boolean*: for all $x, y \in \boldsymbol{L}$

$$\begin{aligned}
(x \wedge y) \vee \left(x \wedge y^{\perp}\right) &= [(x \wedge y) \vee x] \wedge \left[(x \wedge y) \vee y^{\perp}\right] && \text{by } \textit{distributive} \text{ property (Theorem 10.2 page 132)}\\
&= x \wedge \left[(x \wedge y) \vee y^{\perp}\right] && \text{by } \textit{absorptive} \text{ property (Theorem 10.2 page 132)}\\
&= x \wedge \left[\left(x \vee y^{\perp}\right) \wedge \left(y \vee y^{\perp}\right)\right] && \text{by } \textit{distributive} \text{ property (Theorem 10.2 page 132)}\\
&= x \wedge \left(x \vee y^{\perp}\right) \wedge 1 && \text{by } \textit{complement} \text{ property (Theorem 10.2 page 132)}\\
&= x && \text{by } \textit{absorptive} \text{ property (Theorem 10.2 page 132)}\\
&\implies x \copyright y \quad \forall x, y \in \boldsymbol{L} && \text{by Definition 15.2 page 207}\\
&\implies x \text{ is in the } \textit{center} \text{ of } \boldsymbol{L} \text{ for all } x \in \boldsymbol{L} && \text{by Definition 15.4 page 211}
\end{aligned}$$

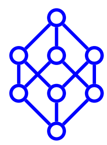
✎

Example 11.6.

E X



The O_6 *lattice* (Definition 11.2 page 152) illustrated to the left is **orthocomplemented** (Definition 11.1 page 152) but **non-join-distributive** (Definition 8.2 page 102),and hence *non-Boolean*. The lattice illustrated to the right is **orthocomplemented** *and* **distributive** and hence also **Boolean** (Proposition 11.1 page 159). Alternatively, the right side lattice is **orthocomplemented** *and* every element is in the *center*, and hence also **Boolean** (Proposition 11.2 page 159).



Note that of the 5 lattices on 5 element sets (Example 5.11 page 80), the 15 lattices on 6 element sets (Example 5.12 page 80), and 53 lattices on 7 element sets (Example 5.13 page 80), **none** are **uniquely complemented**.

✎PROOF:

1. Proof that the O_6 *lattice* is *non-join-distributive*:

$$\begin{aligned}
x \vee (x^{\perp} \wedge z^{\perp}) &= x \vee 0\\
&= x\\
&\neq z^{\perp}\\
&= 1 \wedge z^{\perp}\\
&= (x \vee x^{\perp}) \wedge (x \vee z^{\perp})
\end{aligned}$$

2. Proof that the O_6 *lattice* is also *non-meet-distributive*:

$$\begin{aligned}
z^{\perp} \wedge (x \vee z) &= z^{\perp} \wedge 1\\
&= z^{\perp}\\
&\neq x\\
&= x \vee 1\\
&= (z^{\perp} \wedge x) \vee (z^{\perp} \wedge z)
\end{aligned}$$

✎

11.2 Orthomodular lattices

11.2.1 Properties

Definition 11.3. [9] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an algebraic structure.*

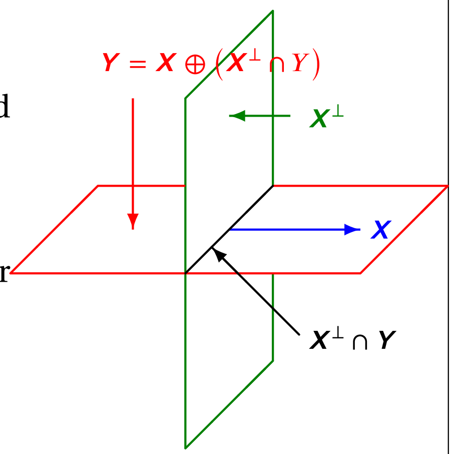
DEF
*$\boldsymbol{L}$ is an **orthomodular lattice** if*
1. $\boldsymbol{L}$ *is an* ORTHOCOMPLEMENTED LATTICE *and*
2. $x \leq y \implies x \vee (x^{\perp} \wedge y) = y \qquad \forall x,y \in X$ (ORTHOMODULAR IDENTITY)

Example 11.7.

EX
The O_6 *lattice* (Definition 11.2 page 152) is *orthocomplemented*, but *non-orthomodular* (and hence, *non-modular* and *non-Boolean*).

Example 11.8. [10]Let $\boldsymbol{H}$ be a Hilbert space and $2^{\boldsymbol{H}}$ the set of closed linear subspaces of $\boldsymbol{H}$.

EX
$(2^{\boldsymbol{H}}, \oplus, \cap, \varnothing, \boldsymbol{H}\,;\, \subseteq)$
is an orthomodular lattice.



This concept is illustrated to the right where $\boldsymbol{X}, \boldsymbol{Y} \in 2^{\boldsymbol{H}}$ are linear subspaces of the linear space $\boldsymbol{H}$ and

$$\boldsymbol{X} \subseteq \boldsymbol{Y} \implies \boldsymbol{Y} = \boldsymbol{X} \oplus (\boldsymbol{X}^{\perp} \cap Y).$$

Theorem 11.3. [11] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a lattice.*

THM
$$\left.\begin{array}{ll} 1. & \boldsymbol{L} \text{ is ORTHOMODULAR } \text{ and} \\ 2. & y \copyright x \text{ and } z \copyright x \end{array}\right\} \implies (x, y, z) \in Ⓓ$$

11.2.2 Characterizations

Theorem 11.4. [12] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE (Definition 11.1 page 152). *Let* Ⓜ *and* Ⓜ* *be the modularity relation and dual modularity relation, respectively* (Definition 7.1 page 91), $\perp$ *the orthogonality relation (Definition 15.1 page 205), and* © *the commutes relation (Definition 15.2 page 207).*

[9] Kalmbach (1983) page 22, Lidl and Pilz (1998) page 90, Husimi (1937)
[10] Iturrioz (1985) pages 56–57
[11] Kalmbach (1983) page 25, Holland (1963) pages 69–70 ⟨THEOREM 3⟩, Foulis (1962) page 68 ⟨THEOREM 5⟩
[12] Kalmbach (1983) page 22, Stern (1999) page 12, Nakamura (1957), Holland (1963), Foulis (1962), Maeda and Maeda (1970), page 132 ⟨Theorem 29.13⟩

THM

The following statements are EQUIVALENT:

	1.	$\boldsymbol{L}$ *is* ORTHOMODULAR
$\iff$	2.	$x \le y$ *and* $y \wedge x^{\perp} = 0 \implies x = y$
$\iff$	3.	$\boldsymbol{L}$ *does* NOT *contain the* O_6 *lattice*
$\iff$	4.	$x \circledC y \iff y \circledC x$ ($\circledC$ *is* SYMMETRIC)
$\iff$	5.	$x \circledM x^{\perp} \qquad \forall x \in X$
$\iff$	6.	$x \circledM^{*} x^{\perp} \qquad \forall x \in X$
$\iff$	7.	$x \vee \left[x^{\perp} \wedge (x \vee y)\right] = x \vee y \qquad \forall x, y \in X$
$\iff$	8.	$x \le y \implies \exists p \in X$ such that $x \perp p$ *and* $x \vee p = y$

✎PROOF:

1. Proof that *orthomodular* $\iff$ *symmetric*: by Proposition 15.3 (page 208).

✏

11.2.3 Restrictions resulting in Boolean algebras

Theorem 11.5. [13] *Let* $\boldsymbol{L} = (X, \vee, \wedge, 0, 1; \le)$ *be an algebraic structure.*

THM

$$\left\{\begin{array}{ll} \boldsymbol{L} \text{ is an } \textbf{orthomodular lattice} & \text{and} \\ \underbrace{\left(x \wedge y^{\perp}\right)^{\perp} = y \vee \left(x^{\perp} \wedge y^{\perp}\right)}_{\text{ELKAN'S LAW}} & \forall x, y \in X \end{array}\right\} \implies \left\{\begin{array}{l} \boldsymbol{L} \text{ is a} \\ \textbf{Boolean algebra} \\ \text{(Definition 10.1 page 127)} \end{array}\right\}$$

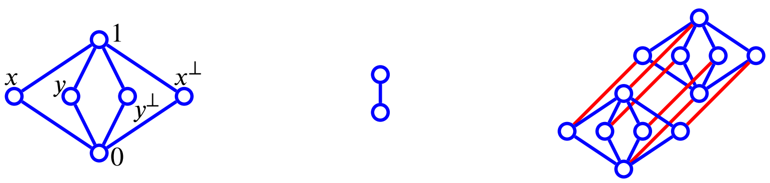
Definition 11.4. [14]

DEF

The $\textbf{MO}_2$ ***lattice*** *is the ordered set* $\left(\{0, x, y, x^{\perp}, y^{\perp}, 1\}, \le\right)$ *with cover relation*

$$\lessdot = \left\{(0,x),\ (0,y),\ \left(0,x^{\perp}\right),\ \left(0,y^{\perp}\right),\ (x,1),\ (y,1),\ \left(x^{\perp},1\right),\ \left(y^{\perp},1\right)\right\}$$

This lattice is also called the ***Chinese lantern***.



MO_2 $\qquad$ L_2 $\qquad$ $MO_2 \times L_2$

Theorem 11.6. [15] *Let* $\boldsymbol{M} = (X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOMODULAR *lattice.*

THM

$$\left\{\begin{array}{l} \boldsymbol{M} \text{ is} \\ \text{BOOLEAN} \end{array}\right\} \iff \left\{\begin{array}{ll} 1. & \boldsymbol{M} \text{ does not contain the } MO_2 \text{ lattice (Definition 11.4 page 162)} \quad \text{and} \\ 2. & \boldsymbol{M} \text{ does not contain the } MO_2 \times L_2 \text{ lattice.} \end{array}\right\}$$

[13] Renedo et al. (2003) page 72

[14] Iturrioz (1985) page 57, Davey and Priestley (2002) pages 18–19 ⟨1.25 Products⟩

[15] Iturrioz (1985) page 57, Carrega (1982) ⟨cf Iturrioz 1985 page 57⟩

11.3 Modular orthocomplemented lattices

Definition 11.5. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE (Definition 6.1 page 89).

DEF

$\mathbf{L}$ *is a* ***modular orthocomplemeted lattice*** *if*

1. $\mathbf{L}$ is ***orthocomplemented*** (Definition 11.1 page 152) *and*
2. $\mathbf{L}$ is ***modular*** (Definition 7.3 page 92)

11.4 Relationships between orthocomplemented lattices

Theorem 11.7. [16] *Let* $\mathbf{L}$ *be a lattice.*

THM

$$\left\{\begin{array}{l}\mathbf{L} \text{ is}\\ \text{BOOLEAN}\end{array}\right\} \implies \left\{\begin{array}{l}\mathbf{L} \text{ is}\\ \text{MODULAR}\\ \text{ORTHOCOM-}\\ \text{PLEMENTED}\end{array}\right\} \implies \left\{\begin{array}{l}\mathbf{L} \text{ is ORTHO-}\\ \text{MODULAR}\end{array}\right\} \implies \left\{\begin{array}{l}\mathbf{L} \text{ is}\\ \text{ORTHOCOM-}\\ \text{PLEMENTED}\end{array}\right\}$$

Remark 11.1. [17] Lattice number 8 in Example 11.2 (page 152) was originally introduced by Dilworth as a counterexample to *Husimi's conjecture* (1937). Kalmbach(1983) points out that this lattice was the first example of a *finite orthomodular* lattice.

[16] Kalmbach (1983) page 32 ⟨20.⟩, Iturrioz (1985) page 57

[17] Dilworth (1940), Dilworth (1990), Kalmbach (1983) page 9

Part III

Functions on Lattices

CHAPTER 12

VALUATIONS ON LATTICES

Definition 12.1. [1] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition 5.3 page 73).*

DEF: *A function* $\mathsf{v} \in \mathbb{R}^X$ *is a* ***valuation*** *on* $\boldsymbol{L}$ *if*

$$\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) = \mathsf{v}(x) + \mathsf{v}(y) \quad \forall x,y \in X$$

Proposition 12.1. *Let* $\mathsf{v} \in \mathbb{R}^X$ *be a* FUNCTION *on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *(Definition 5.3 page 73).*

PRP: $\left\{ \boldsymbol{L} \text{ is LINEAR (Definition 5.3 page 73)} \right\} \implies \left\{ \mathsf{v} \text{ is a VALUATION (Definition 12.1 page 167)} \right\}$

✎PROOF: Let $x, y \in X$ such that $x \leq y$ or $y \lesssim x$.

$$\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) = \mathsf{v}(x) + \mathsf{v}(y) \qquad \text{because } \boldsymbol{L} \text{ is } linear$$

⇰

Example 12.1. [2] Consider the *real valued lattice* $\boldsymbol{L} \triangleq (\mathbb{R}, \vee, \wedge; \leq)$. The *absolute value* function $|\cdot|$ is a *valuation* on $\boldsymbol{L}$.

✎PROOF: $\boldsymbol{L}$ is *linear* (Definition 5.3 page 73), so v is a *valuation* by Proposition 12.1 (page 167). ⇰

Definition 12.2. [3] *Let* X *be a set and* $\mathbb{R}^{\vdash}$ *the set of non-negative real numbers.*

DEF: *A function* $\mathsf{d} \in \mathbb{R}^{\vdash X \times X}$ *is a* ***metric*** *on* X *if*

1. $\mathsf{d}(x, y) \geq 0 \quad \forall x,y \in X$ (NON-NEGATIVE) *and*
2. $\mathsf{d}(x, y) = 0 \iff x = y \quad \forall x,y \in X$ (NONDEGENERATE) *and*
3. $\mathsf{d}(x, y) = \mathsf{d}(y, x) \quad \forall x,y \in X$ (SYMMETRIC) *and*
4. $\mathsf{d}(x, y) \leq \mathsf{d}(x, z) + \mathsf{d}(z, y) \quad \forall x,y,z \in X$ (SUBADDITIVE/TRIANGLE INEQUALITY).[4]

A ***metric space*** *is the pair* (X, d). *A metric is also called a* ***distance function***.

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[1] Istrăţescu (1987) page 127, Birkhoff (1967) page 230 ⟨Definition X.1 (V1)⟩, Blyth (2005) page 58 ⟨Exercise 4.25⟩, Deza and Laurent (1997) page 105 ⟨(8.1.1)⟩, Deza and Deza (2006) page 143 ⟨§10.3⟩, Deza and Deza (2009) page 193 ⟨§10.3⟩

[2] Khamsi and Kirk (2001) page 119 ⟨§5.7⟩

[3] Dieudonné (1969), page 28, Copson (1968), page 21, Hausdorff (1937) page 109, Fréchet (1928), Fréchet (1906) page 30

[4] Euclid (circa 300BC) ⟨Book I Proposition 20⟩

Actually, it is possible to significantly simplify the definition of a metric to an equivalent statement requiring only half as many conditions. These equivalent conditions (a "*characterization*") are stated in Theorem 12.1 (next).

Theorem 12.1 (metric characterization). [5] *Let* d *be a function in* $(\mathbb{R}^{\vdash})^{X\times X}$.

THM

$$\mathsf{d}(x,y) \text{ is a metric} \iff \left\{\begin{array}{llllll} 1. & \mathsf{d}(x,y) & = & 0 \iff x=y & \forall x,y\in X & \text{and} \\ 2. & \mathsf{d}(x,y) & \le & \mathsf{d}(z,x)+\mathsf{d}(z,y) & \forall x,y,z\in X & \end{array}\right.$$

Definition 12.3 (next) defines the *open ball*. In a *metric space* (Definition 12.2 page 167), sets are often specified in terms of an *open ball*; and an open ball is specified in terms of a metric.

Definition 12.3. [6] *Let* (X,d) *be a* METRIC SPACE *(Definition 12.2 page 167).*

DEF

*An **open ball** centered at* x *with radius* r	*is the set* $\mathsf{B}(x,r) \triangleq \{y\in X \mid \mathsf{d}(x,y) < r\}$.
*A **closed ball** centered at* x *with radius* r	*is the set* $\overline{\mathsf{B}}(x,r) \triangleq \{y\in X \mid \mathsf{d}(x,y) \le r\}$.
*A **unit ball** centered at* x	*is the set* $\mathsf{B}(x,1)$.
*A **closed unit ball** centered at* x	*is the set* $\overline{\mathsf{B}}(x,1)$.

Theorem 12.2. [7] *Let* $\mathsf{v}\in\mathbb{R}^{\boldsymbol{X}}$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X,\vee,\wedge;\le)$ *(Definition 5.3 page 73).*

THM

$$\left.\begin{array}{llll} 1. & \mathsf{v}(x\vee y)+\mathsf{v}(x\wedge y)=\mathsf{v}(x)+\mathsf{v}(y) & \forall x,y\in X & (\text{VALUATION}) \text{ and} \\ 2. & x\le y \implies \mathsf{v}(x)\le\mathsf{v}(y) & \forall x,y\in X & (\text{ISOTONE}) \end{array}\right\} \implies \left\{\begin{array}{l} \mathsf{d}(x,y)\triangleq \\ \mathsf{v}(x\vee y)-\mathsf{v}(x\wedge y) \\ \text{is a METRIC on } \boldsymbol{L} \end{array}\right.$$

Definition 12.4. [8] *Let* v *be a* VALUATION *(Definition 12.1 page 167) on a* LATTICE $\boldsymbol{L}\triangleq(X,\vee,\wedge;\le)$ *(Definition 5.3 page 73). Let* $\mathsf{d}(x,y)$ *be the* METRIC *defined in Theorem 12.2 (page 168).*

DEF

The pair $(\boldsymbol{L},\mathsf{d})$ *is called a* METRIC LATTICE.

For *finite modular* lattices, the *height* function $\mathsf{h}(x)$ (Definition 6.3 page 90) can serve as the isotone valuation that induces a metric (next proposition). Such a height function actually satisfies the stronger condition of being *positive* (rather than just being *isotone*)—all *positive* functions are also *isotone*.

Proposition 12.2. [9] *Let* $\mathsf{h}(x)$ *be the* HEIGHT *(Definition 6.3 page 90) of a point* x *in a* BOUNDED LATTICE *(Definition 6.1 page 89)* $\boldsymbol{L}\triangleq(X,\vee,\wedge,0,1;\le)$.

PRP

$$\left\{\begin{array}{ll} 1. & \boldsymbol{L} \text{ is MODULAR and} \\ 2. & \boldsymbol{L} \text{ is FINITE} \end{array}\right\}$$
$$\implies \left\{\begin{array}{llll} 1. & \mathsf{h}(x\vee y)+\mathsf{h}(x\wedge y)=\mathsf{h}(x)+\mathsf{h}(y) & \forall x,y\in X & (\text{VALUATION}) \text{ and} \\ 2. & x\lessapprox y \implies \mathsf{h}(x)\lessapprox\mathsf{h}(y) & \forall x,y\in X & (\text{POSITIVE}) \end{array}\right\}$$
$$\implies \left\{\begin{array}{llll} 1. & \mathsf{h}(x\vee y)+\mathsf{h}(x\wedge y)=\mathsf{h}(x)+\mathsf{h}(y) & \forall x,y\in X & (\text{VALUATION}) \text{ and} \\ 2. & x\le y \implies \mathsf{h}(x)\le\mathsf{h}(y) & \forall x,y\in X & (\text{ISOTONE}) \end{array}\right\}$$

Theorem 12.3. [10] *Let* v *be a* VALUATION *(Definition 12.1 page 167) on a* LATTICE $\boldsymbol{L}\triangleq(X,\vee,\wedge;\le)$ *(Definition 5.3 page 73). Let* $\mathsf{d}(x,y)$ *be the* METRIC *defined in Theorem 12.2 (page 168).*

THM

$$\left\{\begin{array}{c} (\boldsymbol{L},\mathsf{d}) \text{ is a METRIC LATTICE} \\ \text{(Definition 12.4 page 168)} \end{array}\right\} \implies \left\{\begin{array}{c} \boldsymbol{L} \text{ is MODULAR} \\ \text{(Definition 7.3 page 92)} \end{array}\right\}$$

[5] Michel and Herget (1993), page 264, Giles (1987), page 18
[6] Aliprantis and Burkinshaw (1998), page 35
[7] Deza and Laurent (1997) page 105 ⟨(8.1.2)⟩, Birkhoff (1967) pages 230–231
[8] Deza and Laurent (1997) page 105, Birkhoff (1967) page 231 ⟨§X.2⟩
[9] Birkhoff (1967) page 230
[10] Birkhoff (1967) page 232Theorem X.2, Deza and Laurent (1997) pages 105–106, Blyth (2005) page 58 ⟨Exercise 4.25⟩

Example 12.2. The function h on the *Boolean* (and thus also *modular*) lattice $\boldsymbol{L}_2^3$ illustrated to the right is a *valuation* (Definition 12.1 page 167) that is *positive* (and thus also *isotone*, Example 12.2 page 168). Therefore

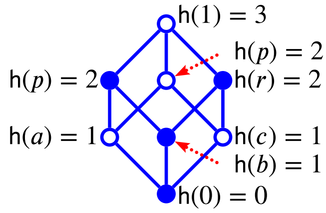
$$\mathsf{d}(x, y) \triangleq \mathsf{h}(x \vee y) - \mathsf{h}(x \wedge y) \qquad \forall x,y \in X$$

is a *metric* (Definition 12.4 page 168) on $\boldsymbol{L}_2^3$. For example,

$$\mathsf{d}(b, q) \triangleq \mathsf{h}(b \vee q) - \mathsf{h}(b \wedge q) = \mathsf{h}(1) - \mathsf{h}(0) = 3 - 0 = 3\,.$$

The *closed unit ball* centered at b (Definition 12.3 page 168) and illustrated with solid dots to the right is

$$\mathsf{B}(b, 1) \triangleq \{x \in X | \mathsf{d}(b, x) \leq 1\} = \{b, p, r, 0\}$$

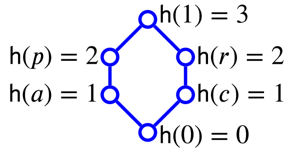


Example 12.3. The *height* function h (Definition 6.3 page 90) on the *orthocomplemented* but *non-modular* lattice O_6 illustrated to the right is *not* a *valuation* because for example

$$\mathsf{h}(a \vee c) + \mathsf{h}(a \wedge c) = \mathsf{h}(1) + \mathsf{h}(0) = 3 + 0 = 3 \neq 2 = 1 + 1 = \mathsf{h}(a) + \mathsf{h}(b).$$

Moreover, we might expect the "distance" from a to c to be 2. However, if we attempt to use $\mathsf{h}(x)$ to define a metric on O_6, then we get

$$\mathsf{d}(a, c) \triangleq \mathsf{h}(a \vee c) - \mathsf{h}(a \wedge c) = \mathsf{h}(1) - \mathsf{h}(0) = 3 - 0 = 3 \neq 2.$$



CHAPTER 13

NEGATION

> ❝ *When we say not-being, we speak, I think, not of something that is the opposite of being, but only of something different. …Then when we are told that the negative signifies the opposite, we shall not admit it; we shall admit only that the particle "not" indicates something different from the words to which it is prefixed, or rather from the things denoted by the words that follow the negative.* ❞
>
> Plato's the *Sophist* (circa 360 B.C.) [1]

> ❝ *Clearly, then, it is a principle of this kind that is the most certain of all principles.…Let us next state what this principle is. "It is impossible for the same attribute at once to belong and not to belong to the same thing and in the same relation"; …This is the most certain of all principles,…for it is impossible for anyone to suppose that the same thing is and is not,…it is by nature the starting-point of all the other axioms as well.* ❞
>
> Aristotle (384BC–322BC), Greek philosopher [2]

13.1 Definitions

Definition 13.1. [3] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89)*.

DEF: *A* FUNCTION $\neg \in X^X$ *is a* ***subminimal negation*** *on* $\mathbf{L}$ *if*

$$x \leq y \implies \neg y \leq \neg x \quad \forall x, y \in X \qquad \text{(ANTITONE)}^{4}$$

Remark 13.1. [5] In the context of natural language, D. Devidi argues that, *subminimal negation* (Definition 13.1 page 171) is "difficult to take seriously as" a negation. He essentially gives this example: Let $x \triangleq$"p is a fish" and $y \triangleq$"p has gills". Suppose "p is a fish" implies "p has gills" ($x \leq y$). Now let $p \triangleq$ "many dogs". Then the *antitone* property and $x \leq y$ tells us ($\implies$) that "Not many dogs have gills" implies that "Not many dogs are fish".

[1] Plato (circa 360 B.C.) ⟨257b–257c⟩, Horn (2001), page 5
[2] Aristotle page 4.1005b
[3] Dunn (1996) pages 4–6, Dunn (1999) pages 24–26 ⟨2 THE KITE OF NEGATIONS⟩
[4] The *antitone* property may also be referred to as *antitonic*, *order-reversing*, or *contrapositive*.
[5] Devidi (2010) page 511, Devidi (2006) page 568

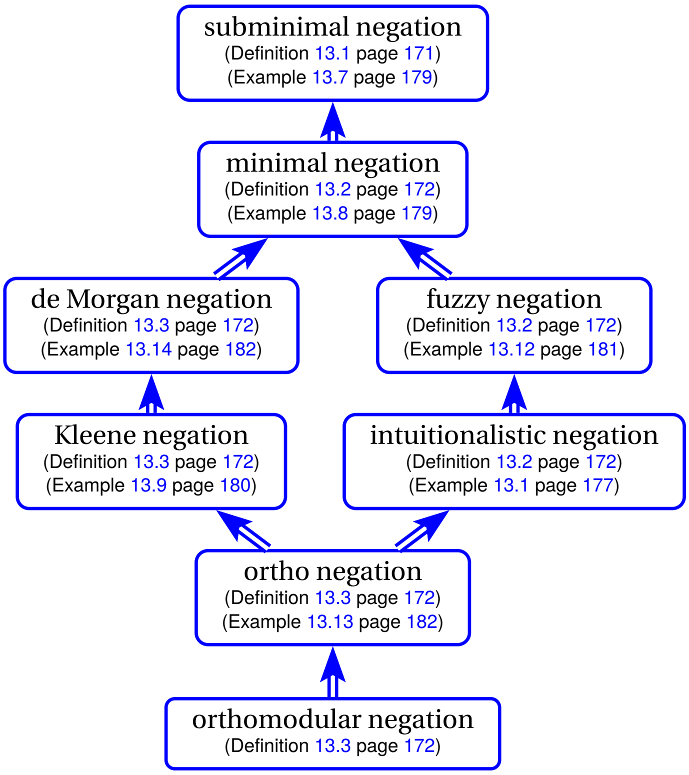


Figure 13.1: lattice of negations

Definition 13.2. [6] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89).*

DEF

A FUNCTION $\neg \in X^X$ *is a* ***negation****, or* ***minimal negation****, on* $\mathbf{L}$ *if*

1. $x \leq y \implies \neg y \leq \neg x \quad \forall x,y \in X$ (ANTITONE) *and*
2. $x \leq \neg\neg x \quad \forall x \in X$ (WEAK DOUBLE NEGATION).

A MINIMAL NEGATION $\neg$ *is an* ***intuitionistic negation*** *if*

3. $x \wedge \neg x = 0 \quad \forall x,y \in X$ (NON-CONTRADICTION).

A MINIMAL NEGATION $\neg$ *is a* ***fuzzy negation*** *if*

4. $\neg 1 = 0$ (BOUNDARY CONDITION).

Definition 13.3. [7] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89).*

DEF

A MINIMAL NEGATION $\neg$ *is a* ***de Morgan negation*** *if*

5. $x = \neg\neg x \quad \forall x \in X$ (INVOLUTORY).

A DE MORGAN NEGATION $\neg$ *is a* ***Kleene negation*** *if*

6. $x \wedge \neg x \leq y \vee \neg y \quad \forall x,y \in X$ (KLEENE CONDITION).

A DE MORGAN NEGATION $\neg$ *is an* ***ortho negation*** *if*

7. $x \wedge \neg x = 0 \quad \forall x,y \in X$ (NON-CONTRADICTION).

A DE MORGAN NEGATION $\neg$ *is an* ***orthomodular negation*** *if*

8. $x \wedge \neg x = 0 \quad \forall x,y \in X$ (NON-CONTRADICTION) *and*
9. $x \leq y \implies x \vee (y \wedge \neg x) = y \quad \forall x,y \in X$ (ORTHOMODULAR).

---

[6] Dunn (1996) pages 4–6, Dunn (1999) pages 24–26 ⟨2 The Kite of Negations⟩, Troelstra and van Dalen (1988) page 4 ⟨1.6 Intuitionism. (b)⟩, de Vries (2007) page 11 ⟨Definition 16⟩, Gottwald (1999) page 21 ⟨Definition 3.3⟩, Novák et al. (1999) page 50 ⟨Definition 2.26⟩, Nguyen and Walker (2006) pages 98–99 ⟨5.4 Negations⟩, Höhle (1978) ⟨???⟩, Bellman and Giertz (1973) pages 155–156 ⟨(N1) $\neg 0 = 1$ and $\neg 1 = 0$, (N3) $\neg\neg x = x$⟩

[7] Dunn (1999) pages 24–26 ⟨2 The Kite of Negations⟩, Jenei (2003) page 283, Kalmbach (1983) page 22, Lidl and Pilz (1998) page 90, Husimi (1937)

Remark 13.2. [8] The *Kleene condition* is basically a weakened form of the *non-contradiction* and *excluded middle* properties because $\underbrace{x \wedge \neg x = 0}_{\text{non-contradiction}} \leq \underbrace{1 = y \vee \neg y}_{\text{excluded middle}}$.

Definition 13.4. [9]

D E F

A MINIMAL NEGATION $\neg \in X^X$ *is* ***strict*** *($\neg$ is a* ***strict negation****) if*

1. $x \lneq y \implies \neg y \lneq \neg x \quad \forall x,y \in X$ (STRICTLY ANTITONE) *and*
2. $\neg$ *is* CONTINUOUS

A STRICT NEGATION $\neg$ *is* ***strong*** *($\neg$ is a* ***strong negation****) if*

3. $\neg\neg x = x \quad \forall x \in X$ (INVOLUTORY).

Definition 13.5. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1\,;\, \leq)$ *be a* BOUNDED LATTICE (Definition 6.1 page 89) *with a function* $\neg$ *in* X^X.

D E F

If $\neg$ *is a* MINIMAL NEGATION, *then* $\boldsymbol{L}$ *is a* ***lattice with negation***.

13.2 Properties of negations

Lemma 13.1. [10] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ (Definition 6.1 page 89).

L E M

$$\underbrace{x \leq y \implies \neg y \leq \neg x\}}_{\text{ANTITONE}} \implies \left\{\begin{array}{lllll} \neg x \vee \neg y & \leq & \neg(x \wedge y) & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN INEQ.) and} \\ \neg(x \vee y) & \leq & \neg x \wedge \neg y & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN INEQ.) and} \end{array}\right.$$

✎PROOF:

1. Proof that *antitone* $\implies$ *conjunctive de Morgan*:

$$\begin{array}{ll} x \wedge y \leq x \text{ and } x \wedge y \leq y & \text{by definition of } \wedge \\ \implies \neg(x \wedge y) \geq \neg x \text{ and } \neg(x \wedge y) \geq \neg y & \text{by } \textit{antitone} \\ \implies \neg(x \wedge y) \geq \neg x \vee \neg y & \text{by definition of } \vee \end{array}$$

2. Proof that *antitone* $\implies$ *disjunctive de Morgan*:

$$\begin{array}{ll} x \leq x \vee y \text{ and } y \leq x \vee y & \text{by definition of } \vee \\ \implies \neg x \geq \neg(x \vee y) \text{ and } \neg y \geq \neg(x \vee y) & \text{by } \textit{antitone} \\ \implies \neg x \wedge \neg y \geq \neg(x \vee y) & \text{by definition of } \wedge \\ \implies \neg(x \vee y) \leq \neg x \wedge \neg y & \end{array}$$

⇰

[8] Cattaneo and Ciucci (2009) page 78

[9] Fodor and Yager (2000), pages 127–128, Bellman and Giertz (1973)

[10] Beran (1985) page 31 ⟨Theorem 1.2 Proof⟩, Fáy (1967) page 268 ⟨Lemma 1 Proof⟩, de Vries (2007) page 12 ⟨Theorem 18⟩

Lemma 13.2. [11] *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ (Definition 5.3 page 73).

LEM

If $x = (\neg\neg x)$ *for all* $x \in X$ (INVOLUTORY), *then*

$$\underbrace{\{x \leq y \implies \neg y \leq \neg x\}}_{\text{ANTITONE}} \iff \underbrace{\left\{\begin{array}{lcll} \neg(x \vee y) & = & \neg x \wedge \neg y \quad \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN) and} \\ \neg(x \wedge y) & = & \neg x \vee \neg y \quad \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)} \end{array}\right.}_{\text{DE MORGAN}}$$

✎PROOF:

1. Proof that *antitone* $\implies$ *de Morgan* equalities:
   (a) Proof that $\neg(\neg x \wedge \neg y) \geq x \vee y$:

   $$\begin{aligned} \neg(\neg x \wedge \neg y) &\geq \neg\neg x \vee \neg\neg y && \text{by Lemma 13.1} \\ &= x \vee y && \text{by } \textit{involutory} \text{ property (Definition 13.5 page 173)} \end{aligned}$$

   (b) Proof that $\neg(\neg x \vee \neg y) \leq x \wedge y$:

   $$\begin{aligned} \neg(\neg x \vee \neg y) &\leq \neg\neg x \wedge \neg\neg y && \text{by Lemma 13.1} \\ &= x \wedge y && \text{by } \textit{involutory} \text{ property (Definition 13.5 page 173)} \end{aligned}$$

   (c) Proof that $\neg(x \wedge y) = \neg x \vee \neg y$:

   $$\begin{aligned} \neg(x \wedge y) &\geq \neg x \vee \neg y && \text{by Lemma 13.1} \\ \neg(x \wedge y) &= \neg[\neg\neg x \wedge \neg\neg y] && \text{by } \textit{involutory} \text{ property (Definition 13.5 page 173)} \\ &\leq \neg x \vee \neg y && \text{by item (1b)} \end{aligned}$$

   (d) Proof that $\neg(x \vee y) = \neg x \wedge \neg y$:

   $$\begin{aligned} \neg(x \vee y) &\geq \neg x \wedge \neg y && \text{by Lemma 13.1} \\ \neg(x \vee y) &= \neg[\neg\neg x \vee \neg\neg y] && \text{by } \textit{involutory} \text{ property (Definition 13.5 page 173)} \\ &\leq \neg x \wedge \neg y && \text{by item (1a)} \end{aligned}$$

2. Proof that *antitone* $\impliedby$ *de Morgan*:

   $$\begin{aligned} x \leq y \implies \neg y &= \neg(x \vee y) && \text{because } x \leq y \\ &= \neg x \wedge \neg y && \text{by } \textit{de Morgan} \\ &\leq \neg x && \text{by definition of } \wedge \end{aligned}$$

⇨

Lemma 13.3. *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ (Definition 5.3 page 73).

LEM

$$\left\{\begin{array}{lll} 1. & x \leq \neg\neg x \quad \forall x \in X & \text{(WEAK DOUBLE NEGATION) and} \\ 2. & \neg 1 = 0 & \text{(BOUNDARY CONDITION)} \end{array}\right\} \implies \{\ \neg 0 = 1 \quad \text{(BOUNDARY CONDITION)}\ \}$$

✎PROOF:

$$\begin{aligned} \neg 0 &= \neg\neg 1 && \text{by } \textit{boundary condition} \text{ hypothesis (2)} \\ &\geq 1 && \text{by } \textit{weak double negation} \text{ hypothesis (1)} \\ &\implies \neg 0 = 1 && \text{by } \textit{upper bound} \text{ property (Definition 6.1 page 89)} \end{aligned}$$

⇨

[11] Beran (1985) pages 30–31 ⟨Theorem 1.2⟩, Fáy (1967) page 268 ⟨Lemma 1⟩, Nakano and Romberger (1971) ⟨cf Beran 1985⟩

Lemma 13.4. *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *(Definition 5.3 page 73).*

LEM
$$\left\{ (x \wedge \neg x = 0 \quad \forall x \in X \quad \text{(NON-CONTRADICTION)} \right\} \implies \left\{ \neg 1 = 0 \quad \text{(BOUNDARY CONDITION)} \right\}$$

✎PROOF:

$$\begin{aligned} 0 &= 1 \wedge \neg 1 && \text{by } \textit{non-contradiction} \text{ hypothesis} \\ &= \neg 1 && \text{by definition of g.u.b. } 1 \text{ and } \wedge \end{aligned}$$

✏

Lemma 13.5. [12] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition 6.1 page 89).*

LEM
$$\left\{ \begin{array}{lll} \text{(A).} & \neg \text{ is BIJECTIVE} & \textit{and} \\ \text{(B).} & x \leq y \implies \neg y \leq \neg x \quad \forall x,y \in X & \text{(ANTITONE)} \end{array} \right\} \implies \underbrace{\left\{ \begin{array}{lll} \text{(1).} & \neg 0 = 1 & \textit{and} \\ \text{(2).} & \neg 1 = 0 & \end{array} \right\}}_{\text{BOUNDARY CONDITIONS}}$$

✎PROOF:

1. Proof that $\neg 0 = 1$:

$$\begin{aligned} & x \leq 1 && \forall x \in X && \text{by definition of l.u.b. } 1 \\ \implies & \neg 1 \leq \neg x && \forall x \in X && \text{by } \textit{antitone} \text{ hypothesis} \\ \implies & \neg 1 \leq y && \forall y \in X && \text{by } \textit{bijective} \text{ hypothesis} \\ \implies & \neg 1 = 0 && && \text{by definition of g.l.b. } 0 \end{aligned}$$

2. Proof that $\neg 0 = 1$:

$$\begin{aligned} & 0 \leq x && \forall x \in X && \text{by definition of g.l.b. } 0 \\ \implies & \neg x \leq \neg 0 && \forall x \in X && \text{by } \textit{antitone} \text{ hypothesis} \\ \implies & \neg x \leq y && \forall y \in X && \text{by } \textit{bijective} \text{ hypothesis} \\ \implies & \neg 0 = 1 && && \text{by definition of l.u.b. } 1 \end{aligned}$$

✏

Theorem 13.1. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition 6.1 page 89).*

THM
$$\left\{ \begin{array}{l} \neg \textit{ is an} \\ \text{INTUITIONISTIC NEGATION} \end{array} \right\} \implies \left\{ \neg 1 = 0 \quad \text{(BOUNDARY CONDITION)} \right\}$$

✎PROOF: This follows directly from Definition 13.5 (page 173) and Lemma 13.4 (page 175). ✏

Theorem 13.2. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition 6.1 page 89).*

THM
$$\left\{ \begin{array}{l} \neg \textit{ is a} \\ \text{FUZZY NEGATION} \end{array} \right\} \implies \left\{ \neg 0 = 1 \quad \text{(BOUNDARY CONDITION)} \right\}$$

[12] Varadarajan (1985) page 42

✎PROOF: This follows directly from Definition 13.2 (page 172) and Lemma 13.3 (page 174). ✑

Theorem 13.3. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition 6.1 page 89).*

$$\text{THM}\quad \left\{\begin{array}{l}\neg \textit{ is a} \\ \textit{minimal} \\ \textit{negation}\end{array}\right\} \implies \left\{\begin{array}{lllll}\neg x \vee \neg y & \leq & \neg(x \wedge y) & \forall x,y\in X & (\text{CONJUNCTIVE DE MORGAN INEQUALITY}) \text{ and} \\ \neg(x \vee y) & \leq & \neg x \wedge \neg y & \forall x,y\in X & (\text{DISJUNCTIVE DE MORGAN INEQUALITY})\end{array}\right\}$$

✎PROOF: This follows directly from Definition 13.5 (page 173) and Lemma 13.1 (page 173). ✑

Theorem 13.4. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition 6.1 page 89).*

$$\text{THM}\quad \left\{\begin{array}{l}\neg \textit{ is a} \\ \textit{de Morgan negation}\end{array}\right\} \implies \left\{\begin{array}{lllll}\neg(x \vee y) & = & \neg x \wedge \neg y & \forall x,y\in X & (\text{DISJUNCTIVE DE MORGAN}) \text{ and} \\ \neg(x \wedge y) & = & \neg x \vee \neg y & \forall x,y\in X & (\text{CONJUNCTIVE DE MORGAN})\end{array}\right.$$

✎PROOF: This follows directly from Definition 13.5 (page 173) and Lemma 13.2 (page 174). ✑

Theorem 13.5. [13] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition 6.1 page 89).*

$$\text{THM}\quad \left\{\begin{array}{l}\neg \textit{ is an} \\ \textit{ortho negation}\end{array}\right\} \implies \left\{\begin{array}{llllll}1. & \neg 0 & = & 1 & & (\text{BOUNDARY CONDITION}) \text{ and} \\ 2. & \neg 1 & = & 0 & & (\text{BOUNDARY CONDITION}) \text{ and} \\ 3. & \neg(x \vee y) & = & \neg x \wedge \neg y & \forall x,y\in X & (\text{DISJUNCTIVE DE MORGAN}) \text{ and} \\ 4. & \neg(x \wedge y) & = & \neg x \vee \neg y & \forall x,y\in X & (\text{CONJUNCTIVE DE MORGAN}) \text{ and} \\ 5. & x \vee \neg x & = & 1 & \forall x\in X & (\text{EXCLUDED MIDDLE}) \text{ and} \\ 6. & x \wedge \neg x & \leq & y \vee \neg y & \forall x,y\in X & (\text{KLEENE CONDITION}).\end{array}\right\}$$

✎PROOF:

1. Proof for $0 = \neg 1$ boundary condition: by Lemma 13.4 (page 175)
2. Proof for boundary conditions:

$$\begin{aligned} 1 &= \neg\neg 1 && \text{by } \textit{involutory} \text{ property} \\ &= \neg 0 && \text{by previous result} \end{aligned}$$

3. Proof for *de Morgan* properties:
   (a) By Definition 13.5 (page 173), *ortho negation* is *involutory* and *antitone*.
   (b) Therefore by Lemma 13.2 (page 174), *de Morgan* properties hold.
4. Proof for *excluded middle* property:

$$\begin{aligned} x \vee \neg x &= (x \vee \neg x)^{\neg\neg} && \text{by } \textit{involutory} \text{ property of } \textit{ortho negation} \text{ (Definition 13.5 page 173)} \\ &= \neg(x\neg \wedge x^{\neg\neg}) && \text{by } \textit{disjunctive de Morgan} \text{ property} \\ &= \neg(\neg x \wedge x) && \text{by } \textit{involutory} \text{ property of } \textit{ortho negation} \text{ (Definition 13.5 page 173)} \\ &= \neg(x \wedge \neg x) && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Definition 5.3 page 73)} \\ &= \neg 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{ortho negation} \text{ (Definition 13.5 page 173)} \\ &= 1 && \text{by } \textit{boundary condition} \text{ (item (2) page 176) of } \textit{minimal negation} \end{aligned}$$

[13] Beran (1985) pages 30–31, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Cohen (1989) page 37 ⟨3B.13. Theorem⟩

5. Proof for *Kleene condition*:

$$\begin{aligned}
x \wedge \neg x &= 0 && \text{by \textit{non-contradiction} property (Definition 13.5 page 173)}\\
&\le 1 && \text{by definition of 0 and 1}\\
&= y \vee \neg y && \text{by \textit{excluded middle} property (item (4) page 176)}
\end{aligned}$$

⇨

13.3 Examples

Example 13.1 (discrete negation). [14] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ be a *bounded lattice* (Definition 6.1 page 89) with a function $\neg \in X^X$.

EX The function $\neg x$ defined as

$$\neg x \triangleq \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

is an *intuitionistic negation* (Definition 13.2 page 172) and a *fuzzy negation* (Definition 13.2 page 172).

✎PROOF: To be an *intuitionistic negation*, $\neg x$ must be *antitone*, have *weak double negation*, and have the *non-contradiction* property (Definition 13.2 page 172). To be a *fuzzy negation*, $\neg x$ must be *antitone*, have *weak double negation*, and have the *boundary condition* $\neg 1 = 0$.

$$\left\{\begin{array}{lcll} \neg y \le \neg x & \Longleftrightarrow & 1 \le 1 & \text{for } 0 = x = y \\ \neg y \le \neg x & \Longleftrightarrow & 0 \le 1 & \text{for } 0 = x \lneq y \\ \neg y \le \neg x & \Longleftrightarrow & 0 \le 0 & \text{for } 0 \ne x \le y \end{array}\right\} \implies \neg x \text{ is \textit{antitone}}$$

$$\left\{\begin{array}{ccccccccl} \neg\neg x & = & \neg 1 & = & 0 & \ge & 0 & = & x \quad \text{for } x = 0 \\ \neg\neg x & = & \neg 0 & = & 1 & \ge & x & = & x \quad \text{for } x \ne 0 \end{array}\right\} \implies \neg x \text{ has \textit{weak double negation}}$$

$$\left\{\begin{array}{cccccccl} x \wedge \neg x & = & x \wedge 1 & = & 0 \wedge 0 & = & 0 & \text{for } x = 0 \\ x \wedge \neg x & = & x \wedge 0 & = & x \wedge 0 & = & 0 & \text{for } x \ne 0 \end{array}\right\} \implies \neg x \text{ has \textit{non-contradiction} property}$$

$$\neg 1 = 0 \implies \neg x \text{ has the \textit{boundary condition} property}$$

⇨

Example 13.2 (dual discrete negation). [15] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ be a *bounded lattice* (Definition 6.1 page 89) with a function $\neg \in X^X$.

EX The function $\neg x$ defined as

$$\neg x \triangleq \begin{cases} 0 & \text{for } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

is a *subminimal negation* (Definition 13.1 page 171) but it is *not* a *minimal negation* (Definition 13.2 page 172) (and not any other negation defined here).

✎PROOF: To be an *subminimal negation*, $\neg x$ must be *antitone* (Definition 13.1 page 171). To be a *minimal negation*, $\neg x$ must be *antitone* and have *weak double negation* (Definition 13.2 page 172).

$$\left\{\begin{array}{lcll} \neg y \le \neg x & \Longleftrightarrow & 0 \le 0 & \text{for } x = y = 1 \\ \neg y \le \neg x & \Longleftrightarrow & 0 \le 1 & \text{for } x \lneq y = 1 \\ \neg y \le \neg x & \Longleftrightarrow & 1 \le 1 & \text{for } x \le y \ne 1 \end{array}\right\} \implies \neg x \text{ is \textit{antitone}}$$

$$\left\{\begin{array}{ccccccl} \neg\neg x & = & \neg 0 & = & 1 & \ge & x \quad \text{for } x = 1 \\ \neg\neg x & = & \neg 1 & = & 0 & \le & x \quad \text{for } x \ne 1 \end{array}\right\} \implies \neg x \text{ does \textit{not} have \textit{weak double negation}}$$

⇨

[14] Fodor and Yager (2000) page 128, Yager (1980) pages 256–257, Yager (1979) ⟨cf Fodor⟩
[15] Fodor and Yager (2000) page 128, Ovchinnikov (1983) page 235 ⟨Example 4⟩

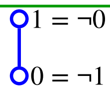
Example 13.3. [16] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ be a *bounded lattice*

EX The function $\neg x$ is an *intuitionistic negation* (Definition 13.2 page 172) if

$$\neg x \triangleq \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 13.4.

EX The function $\neg$ illustrated to the right is an *ortho negation* (Definition 13.3 page 172).



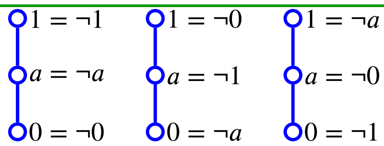
✎PROOF:

1. Proof that $\neg$ is *antitone*: $0 \le 1 \implies \neg 1 = 0 \le x = \neg 0 \implies \neg$ is *antitone* over $(0, 1)$
2. Proof that $\neg$ is *involutory*: $1 = \neg 0 = \neg\neg 1$
3. Proof that $\neg$ has the *non-contradiction* property: $\begin{array}{ccccccccc} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 \\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 \end{array}$

⇰

Example 13.5.

EX The functions $\neg$ illustrated to the right are *not* any negation defined here. In particular, they are *not antitone*.



(a) (b) (c)

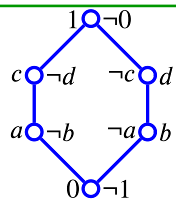
✎PROOF:

1. Proof that (a) is *not antitone*: $a \le 1 \implies \neg 1 = 1 \not\le a = \neg a$
2. Proof that (b) is *not antitone*: $a \le 1 \implies \neg 1 = a \not\le 0 = \neg a$
3. Proof that (c) is *not antitone*: $0 \le a \implies \neg a = 1 \not\le a = \neg 0$

⇰

Example 13.6.

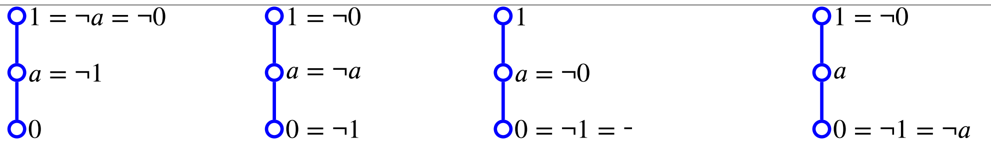
EX The function $\neg$ as illustrated to the right is *not* a *subminimal negation* (it is *not antitone*) and so is *not* any negation defined here. Note however that the problem is *not* the O_6 *lattice*—it is possible to define a negation on an O_6 *lattice* (Example 13.16 page 183).



✎PROOF: Proof that $\neg$ is *not antitone*: $a \le c \implies \neg c = d \not\le b = \neg a$ ⇰

Remark 13.3. The concept of a *complement* (Definition 9.1 page 121) and the concept of a *negation* are fundamentally different. A *complement* is a *relation* (Definition 17.1 page 251) on a lattice $\boldsymbol{L}$ and a *negation* is a *function* (Definition 17.8 page 263). In Example 13.6 (page 178), b and d are both complements of a, but yet $\neg$ is *not* a negation. In the right side lattice of Example 13.16 (page 183), both b and d are complements of a (and so the lattice is *multipy complemented*), but yet only d is equal to the negation of a ($d = \neg a$). It can also be said that complementation is a property *of* a lattice, whereas negation is a function defined *on* a lattice.

[16] Fodor and Yager (2000) page 128

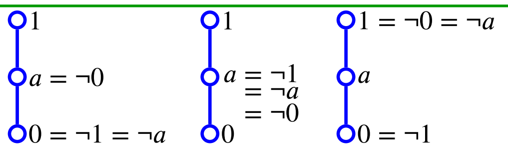


(A) *minimal negation* (Example 13.8 page 179) (B) *Kleene negation* (Example 13.9 page 180) (C) *intuitionistic negation* (Example 13.10 page 180) (D) *fuzzy* and *intuitionistic* negation (Example 13.11 page 181)

Figure 13.2: negations on $\boldsymbol{L}_3$

Example 13.7.

EX Each of the functions $\neg$ illustrated to the right is a *subminimal negation* (Definition 13.1 page 171); *none* of them is a *minimal negation* (each fails to have *weak double negation*).



(a) (b) (c)

✎PROOF:

1. Proof that (a) $\neg$ is *antitone*:
$$\begin{array}{ccccccccccccl} a & \le & 1 & \implies & \neg 1 & = & 0 & \le & 0 & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ over } (a,1)\\ 0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,1)\\ 0 & \le & a & \implies & \neg a & = & 0 & \le & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,a) \end{array}$$

2. Proof that (a) $\neg$ *fails* to have *weak double negation*:
$1 \not\le a = \neg 0 = \neg\neg 1$

3. Proof that (b) $\neg$ is *antitone*:
$$\begin{array}{ccccccccccccl} a & \le & 1 & \implies & \neg 1 & = & a & \le & a & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ over } (a,1)\\ 0 & \le & 1 & \implies & \neg 1 & = & a & \le & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,1)\\ 0 & \le & a & \implies & \neg a & = & a & \le & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,a) \end{array}$$

4. Proof that (b) $\neg$ *fails* to have *weak double negation*: $1 \not\le a = \neg a = \neg\neg 1$

5. (c) is a special case of the *dual discrete negation* (Example 13.2 page 177).

✏

Example 13.8. The function $\neg$ illustrated in Figure 13.2 page 179 (A) is a **minimal negation** (Definition 13.2 page 172); it is *not* an *intuitionistic negation* (it does not have the *non-contradiction* property), it is *not* a *de Morgan negation* (it is not *involutory*), and it is *not* a *fuzzy negation* ($\neg 1 \neq 0$).

✎PROOF:

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{ccccccccccccl} a & \le & 1 & \implies & \neg 1 & = & a & \le & 1 & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ over } (a,1)\\ 0 & \le & 1 & \implies & \neg 1 & = & a & \le & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,1)\\ 0 & \le & a & \implies & \neg a & = & 1 & \le & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,a) \end{array}$$

2. Proof that $\neg$ is a *weak double negation* (and so is a *minimal negation*, but is *not* a *de Morgan negation*):
$$\begin{array}{cccccccccl} 1 & = & 1 & = & \neg a & = & \neg\neg 1 & \implies & \neg \text{ is } \textit{involutory} \text{ at } 1\\ a & = & a & = & \neg 1 & = & \neg\neg a & \implies & \neg \text{ is } \textit{involutory} \text{ at } a\\ 0 & \le & a & = & \neg 1 & = & 0^{\neg\neg} & \implies & \neg \text{ is a } \textit{weak double negation} \text{ at } 0 \end{array}$$

3. Proof that $\neg$ does *not* have the *non-contradiction* property (and so is not an *intuitionistic negation*):
$1 \wedge \neg 1 = 1 \wedge a = a \neq 0$

4. Proof that $\neg$ is not a *fuzzy negation*: $\neg 1 = a \neq 0$

Example 13.9 (Łukasiewicz 3-valued logic/Kleene 3-valued logic/RM_3 logic). [17] The function $\neg$ illustrated in Figure 13.2 page 179 (B) is a **Kleene negation** (Definition 13.3 page 172), and is also a *fuzzy negation* (Definition 13.2 page 172); but it is *not* an *ortho negation* and is *not* an *intuitionistic negation* (it does not have the *non-contradiction* property).

✎PROOF:

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{lllllllllll} a & \le & 1 & \implies & \neg 1 & = & 0 & \le & a & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ over } (a,1)\\ 0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,1)\\ 0 & \le & a & \implies & \neg a & = & a & \le & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ over } (0,a)\end{array}$$

2. Proof that $\neg$ is *involutory* (and so is a *de Morgan negation*):
$$\begin{array}{llllllll} 1 & = & \neg 0 & = & \neg\neg 1 & \implies & \neg \text{ is } \textit{involutory} \text{ at } 1\\ a & = & \neg a & = & \neg\neg a & \implies & \neg \text{ is } \textit{involutory} \text{ at } a\\ 0 & = & \neg 0 & = & 0^{\neg\neg} & \implies & \neg \text{ is } \textit{involutory} \text{ at } 0\end{array}$$

3. Proof that $\neg$ does *not* have the *non-contradiction* property (and so is not an *ortho negation*):
$x \wedge \neg x = x \wedge x = x \neq 0$

4. Proof that $\neg$ satisfies the *Kleene condition* (and so is a *Kleene negation*):
$$\begin{array}{lllllllllllllllllll} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 & \le & a & = & a & \vee & a & = & a & \vee & \neg a\\ 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 & \le & 1 & = & 0 & \vee & 1 & = & 0 & \vee & \neg 0\\ a & \wedge & \neg a & = & 1 & \wedge & a & = & a & \le & 1 & = & 1 & \vee & 0 & = & 1 & \vee & \neg 1\\ a & \wedge & \neg a & = & 1 & \wedge & a & = & a & \le & 1 & = & 0 & \vee & 1 & = & 0 & \vee & \neg 0\\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 & \le & 1 & = & 1 & \vee & 0 & = & 1 & \vee & \neg 1\\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 & \le & a & = & a & \vee & a & = & a & \vee & \neg a\end{array}$$

✎

Example 13.10. The function $\neg$ illustrated in Figure 13.2 page 179 (C) an **intuitionistic negation** (Definition 13.2 page 172); but it is *not* a *fuzzy negation* ($1 \neq \neg 0$), and it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF:

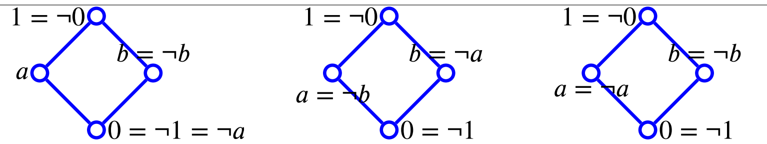
1. Proof that $\neg$ is *antitone*:
$$\begin{array}{lllllllllll} a & \le & 1 & \implies & \neg 1 & = & 0 & \le & 0 & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ at } (a,1)\\ 0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ at } (0,1)\\ 0 & \le & a & \implies & \neg a & = & 0 & \le & a & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ at } (0,a)\end{array}$$

2. Proof that $\neg$ has *weak double negation* property (and so is a *minimal negation*, but *not* a *de Morgan negation*):
$$\begin{array}{llllllllll} 1 & \le & a & = & \neg 0 & = & \neg\neg 1 & \implies & \neg \text{ has } \textit{weak double negation} \text{ at } 1\\ a & = & \neg 0 & = & \neg\neg a & & & \implies & \neg \text{ has } \textit{weak double negation} \text{ at } a\\ 0 & = & \neg a & = & 0^{\neg\neg} & & & \implies & \neg \text{ is } \textit{involutory} \text{ at } 0\end{array}$$

3. Proof that $\neg$ has the *non-contradiction* property (and so is an *intuitionistic negation*):
$$\begin{array}{lllllll} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0\\ a & \wedge & \neg a & = & a & \wedge & 0 & = & 0\\ 0 & \wedge & \neg 0 & = & 0 & \wedge & a & = & 0\end{array}$$

4. Proof that $\neg$ is *not* a *fuzzy negation*: $\neg 1 \neq 0$

[17] Łukasiewicz (1920), Avron (1991) pages 277–278, Kleene (1938) page 153, Kleene (1952), pages 332–339 (§64. The 3-valued logic), Sobociński (1952)



(A) *fuzzy negation* (Example 13.12 page 181) (B) *ortho negation* (Example 13.13 page 182) (C) *de Morgan negation* (Example 13.14 page 182)

Figure 13.3: negations on $\boldsymbol{M}_2$

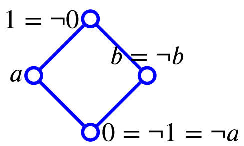
Example 13.11 (Heyting 3-valued logic/Jaśkowski's first matrix). [18] The function ¬ illustrated in Figure 13.2 page 179 (D) is an **intuitionistic negation** (Definition 13.2 page 172), and is also a **fuzzy negation** (Definition 13.2 page 172), but it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF: This is simply a special case of the *discrete negation* (Example 13.1 page 177). ✏

Remark 13.4. There is only one linearly ordered (Definition 4.4 page 59) 3-element lattice ($\boldsymbol{L}_3$) that is a *fuzzy negation* (Example 13.11 page 181). However, this lattice is also an *intuitionistic negation*. There are no $\boldsymbol{L}_3$ lattices that are *fuzzy* but yet not *intuitionistic*. In fact, there are only three linearly ordered 3-element lattices with with $1 = \neg 0$ and $0 = \neg 1$. Of these three, only one is both *fuzzy* and *intuitionistic* (Example 13.11 page 181), one is *Kleene* but not *fuzzy* (Example 13.9 page 180), and one is *subminimal* but not *fuzzy* (Example 13.7 page 179). It can be claimed that the "simplist" *fuzzy negation* that is not *de Morgan* and *not intuitionistic* is the $\boldsymbol{M}_2$ lattice of Example 13.12 (next).

Example 13.12. The function ¬ illustrated in Figure 13.3 page 181 (A) is a **fuzzy negation** (Definition 13.2 page 172). It is not an *intuitionistic negation* (it does not have the *non-contradiction* property) and it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF: Note that



(Example 13.12 page 181)

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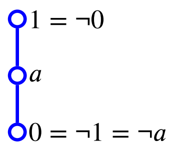
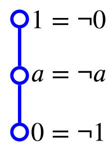


fuzzy and *intuitionistic* (Example 13.11 page 181)

+



Kleene negation (Example 13.9 page 180)

1. Proof that ¬ is *antitone*:
$$\begin{array}{ccccccccccccl}
a & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & 0 & = & \neg a & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (a,1)\\
0 & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & 1 & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,1)\\
0 & \le & a & \Longrightarrow & \neg a & = & 0 & \le & 1 & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,a)\\
b & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & b & = & \neg b & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (b,1)\\
0 & \le & b & \Longrightarrow & \neg b & = & b & \le & 1 & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,b)
\end{array}$$

2. Proof that ¬ has *weak double negation* property (and so is a *minimal negation*, but *not* a *de Morgan negation*):
$$\begin{array}{cccccccl}
1 & = & \neg 0 & = & \neg\neg 1 & & \Longrightarrow & \neg \text{ is } \textit{involutory} \text{ at } 1\\
a & \le & 1 & = & \neg 0 & = \neg\neg a & \Longrightarrow & \neg \text{ has } \textit{weak double negation} \text{ at } a\\
0 & = & \neg 1 & = & 0^{\neg\neg} & & \Longrightarrow & \neg \text{ is } \textit{involutory} \text{ at } 0\\
b & = & \neg b & = & \neg\neg b & = & \Longrightarrow & \neg \text{ is } \textit{involutory} \text{ at } b
\end{array}$$

3. Proof that ¬ does *not* have the *non-contradiction* property (and so is *not* an *intuitionistic negation*):
$b \wedge \neg b = b \wedge b = b \ne 0$

4. Proof that ¬ is has *boundary conditions* (and so is a *fuzzy negation*): $\neg 1 = 0, \quad \neg 0 = 1$

[18] Karpenko (2006) page 45, Johnstone (1982) page 9 ⟨§1.12⟩, Heyting (1930a), Heyting (1930b), Heyting (1930c), Heyting (1930d), Jaskowski (1936), Mancosu (1998)

Example 13.13. [19] The function ¬ illustrated in Figure 13.3 page 181 (B) is an *ortho negation* (Definition 13.3 page 172).

✎PROOF:

1. Proof that ¬ is *antitone*:
$$\begin{array}{ccccccccc} a & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & b & = & \neg a \\ 0 & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & 1 & = & \neg 0 \\ 0 & \le & a & \Longrightarrow & \neg a & = & b & \le & 1 & = & \neg 0 \\ b & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & a & = & \neg b \\ 0 & \le & b & \Longrightarrow & \neg b & = & a & \le & 1 & = & \neg 0 \end{array}$$

2. Proof that ¬ is *involutory* (and so is a *de Morgan negation*):
$$\begin{array}{ccccc} 1 & = & \neg 0 & = & \neg\neg 1 \\ a & = & \neg a & = & \neg\neg a \\ b & = & \neg b & = & \neg\neg b \\ 0 & = & \neg 0 & = & 0^{\neg\neg} \end{array}$$

3. Proof that ¬ is has the *non-contradiction* property (and so is an *ortho negation*):
$$\begin{array}{ccccccccc} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 \\ a & \wedge & \neg a & = & a & \wedge & b & = & 0 \\ b & \wedge & \neg b & = & b & \wedge & a & = & 0 \\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 \end{array}$$

Example 13.14 (BN_4). [20] The function ¬ illustrated in Figure 13.3 page 181 (C) is a **de Morgan negation** (Definition 13.3 page 172), but it is *not* a *Kleene negation* and not an *ortho negation* (it does *not* satisfy the *Kleene condition*).

✎PROOF:

1. Proof that ¬ is *antitone*:
$$\begin{array}{ccccccccc} a & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & b & = & \neg a \\ 0 & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & 1 & = & \neg 0 \\ 0 & \le & a & \Longrightarrow & \neg a & = & a & \le & 1 & = & \neg 0 \\ b & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & b & = & \neg b \\ 0 & \le & b & \Longrightarrow & \neg b & = & b & \le & 1 & = & \neg 0 \end{array}$$

2. Proof that ¬ is *involutory* (and so is a *de Morgan negation*):
$$\begin{array}{ccccc} 1 & = & \neg 0 & = & \neg\neg 1 \\ a & = & \neg a & = & \neg\neg a \\ b & = & \neg b & = & \neg\neg b \\ 0 & = & \neg 0 & = & 0^{\neg\neg} \end{array}$$

3. Proof that ¬ does *not* have the *non-contradiction* property (and so is *not* an *ortho negation*):
$$\begin{array}{ccccccccccc} a & \wedge & \neg a & = & a & \wedge & a & = & a & \ne & 0 \\ b & \wedge & \neg b & = & b & \wedge & b & = & b & \ne & 0 \end{array}$$

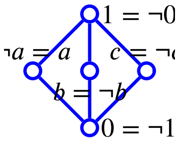
4. Proof that ¬ does *not* satisfy the *Kleene condition* (and so is a *de Morgan negation*):
$$a \wedge \neg a = a \wedge a = a \not\le b \wedge \neg b = b$$

[19] Belnap (1977) page 13, Restall (2000) page 177 ⟨Example 8.44⟩, Pavičić and Megill (2008) page 28 ⟨Definition 2, *classical implication*⟩

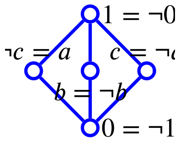
[20] Cignoli (1975) page 270, Restall (2000) page 171 ⟨Example 8.39⟩, de Vries (2007) pages 15–16 ⟨Example 26⟩, Dunn (1976), Belnap (1977)

Example 13.15.

EX

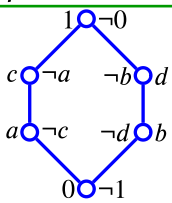


The function ¬ illustrated to the left is a *de Morgan negation* (Definition 13.3 page 172), but it is *not* a *Kleene negation* and not an *ortho negation* (it does *not* satisfy the *Kleene condition*). The *negation* illustrated to the right is a *Kleene negation* (Definition 13.3 page 172), but it is *not* an *ortho negation* (it does *not* have the *non-contradiction* property).

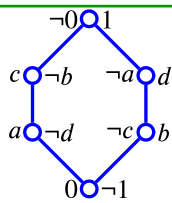


Example 13.16.

EX

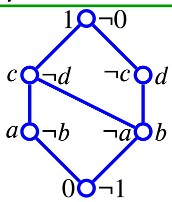


The function ¬ illustrated to the left is a *de Morgan negation* (Definition 13.3 page 172); it is *not* a *Kleene negation* (it does not satisfy the Kleene condition). The *negation* illustrated to the right is an *ortho negation* (Definition 13.3 page 172).

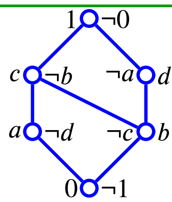


Example 13.17.

EX



The function ¬ illustrated to the left is *not antitone* and therefore is not a *negation* (Definition 13.2 page 172). The function ¬ illustrated to the right is a *Kleene negation* (Definition 13.3 page 172); it is *not* an *ortho negation* (it does not have the *non-contradiction* property).



✎PROOF:

1. Proof that left ¬ is *not antitone*: $a \le c$ but $\neg c \nleq \neg a$.

2. Proof that right ¬ satisfies the *Kleene condition*:

$$x \wedge \neg x = \begin{cases} b & \text{for } x = b \\ 0 & \text{otherwise} \end{cases} \quad \forall x \in X \qquad \text{and} \qquad y \wedge \neg y = \begin{cases} c & \text{for } y = c \\ 0 & \text{otherwise} \end{cases} \quad \forall y \in X$$

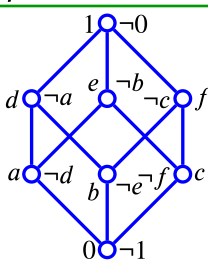
$$\implies \quad x \wedge \neg x \le y \vee \neg y \quad \forall x, y \in X$$

3. Proof that right ¬ does not have the *non-contradiction* property: $b \wedge \neg b = b \wedge c = b \neq 0$

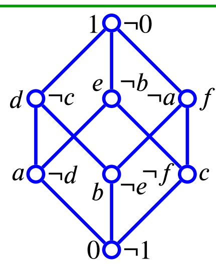
✏

Example 13.18.

EX



The lattices illustrated to the left and right are *Boolean* (Definition 10.1 page 127). The function ¬ illustrated to the left is a *Kleene negation* (Definition 13.3 page 172), but it is *not* an *ortho negation* (it does *not* have the *non-contradiction* property). The *negation* illustrated to the right is an *ortho negation* (Definition 13.3 page 172).



✎PROOF:

1. Proof that left side negation does *not* have *non-contradiction* property (and so is *not* an *ortho negation*):
$a \wedge \neg a = a \wedge d = a \neq 0$

2. Proof that left side negation does *not* satisfy *Kleene condition* (and so is *not* a *Kleene negation*):
$a \wedge \neg a = a \wedge d = a \not\leq f = c \vee f = c \vee \neg c$

13.4 Projections

Definition 13.6. [21] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

DEF

A function $\phi_x \in X^X$ *is a* ***Sasaki projection*** *on* $x \in X$ *if*

$$\phi_x(y) \triangleq \left(y \vee x^\perp\right) \wedge x.$$

The SASAKI PROJECTIONS ϕ_x *and* ϕ_y *are* ***permutable*** *if*

$$\phi_x \circ \phi_y(u) = \phi_y \circ \phi_x(u) \quad \forall u \in X.$$

Proposition 13.1. *Let* $\phi_x(y)$ *be the* SASAKI PROJECTION OF y ONTO x *(Definition 13.6 page 185) in an* ORTHOCOMPLEMENTED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$.

PRP

$$\begin{array}{llcrcl l}
(1). & x \leq y & \Longrightarrow & & \phi_x(y) = & x & \forall x,y \in X \\
(2). & y \leq x & \Longrightarrow & y \leq & \phi_x(y) \leq & x & \forall x,y \in X \\
(3). & y \leq x \text{ and } \boldsymbol{L} \text{ is BOOLEAN} & \Longrightarrow & & \phi_x(y) = & y & \forall x,y \in X
\end{array}$$

✎PROOF:

$$\begin{aligned}
x \leq y \implies \phi_x(y) &\triangleq \left(y \vee x^\perp\right) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition 13.6 page 185)} \\
&= 1 \wedge x && \text{by } x \leq y \text{ hypothesis and Proposition 15.1 page 205} \\
&= x && \text{by property of bounded lattices (Proposition 6.2 page 89)} \\
y \leq x \implies \boxed{y} &= y \wedge x && \text{by } y \leq x \text{ hypothesis} \\
&\leq (y \vee x^\perp) \wedge x && \text{by definition of } \vee \text{ (Definition 4.21 page 70)} \\
&= \boxed{\phi_x(y)} && \text{by definition of } \textit{Sasaki projection} \text{ (Definition 13.6 page 185)} \\
&\leq (y \vee x^\perp) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition 13.6 page 185)} \\
&\leq \boxed{x} && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)} \\
y \leq x \text{ and } \textit{Boolean} \implies \phi_x(y) &= (y \vee x^\perp) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition 13.6 page 185)} \\
&= (y \wedge x) \vee (x^\perp \wedge x) && \text{by } \textit{distributive} \text{ prop. of } \textit{Boolean lattices} \text{ (Theorem 10.2 page 132)} \\
&= (y \wedge x) \vee 0 && \text{by } \textit{non-contradiction} \text{ of Boolean lat. (Theorem 10.2 page 132)} \\
&= (y \wedge x) && \text{by } \textit{boundary} \text{ prop. of } \textit{bounded lattices} \text{ (Proposition 6.2 page 89)} \\
&= y && \text{by } y \leq x \text{ hypothesis and definition of } \wedge \text{ (Definition 4.22 page 70)}
\end{aligned}$$

✏

Proposition 13.2. *Let* $\phi_x(y)$ *be the* SASAKI PROJECTION OF y ONTO x *(Definition 13.6 page 185) in an* ORTHOCOMPLEMENTED LATTICE $(X, \vee, \wedge, 0, 1; \leq)$.

PRP

$$\begin{array}{lrcll}
(1). & \phi_0(y) & = & 0 & \forall y \in X \\
(2). & \phi_x(0) & = & 0 & \forall x \in X \\
(3). & \phi_1(y) & = & 1 & \forall y \in X \\
(4). & \phi_x(1) & = & x & \forall x \in X \\
(5). & \phi_x\left(x^\perp\right) & = & 0 & \forall x \in X
\end{array}$$

[21] Nakamura (1957) pages 158–159 ⟨equation (S)⟩
Sasaki (1954) page 300 ⟨Def.5.1, cf Foulis 1962⟩
Kalmbach (1983) page 117

✎PROOF:

$\phi_0(y) = 0$	because $0 \le y$ and by Proposition 13.1 page 185
$\phi_x(0) \triangleq (0 \vee x^\perp) \wedge x$	by definition of *Sasaki projection* (Definition 13.6 page 185)
$= x^\perp \wedge x$	by property of bounded lattices (Proposition 6.2 page 89)
$= 0$	by definition of *orthocomplemented* (Definition 11.1 page 152)
$\phi_1(y) \triangleq (y \vee 1^\perp) \wedge 1$	by definition of *Sasaki projection* (Definition 13.6 page 185)
$= (y \vee 0) \wedge 1$	by *boundary condition* *(Theorem 13.5 page 176)*
$= y \wedge 1$	by property of bounded lattices (Proposition 6.2 page 89)
$= 1$	by property of bounded lattices (Proposition 6.2 page 89)
$\phi_x(1) = x$	because $x \le 1$ and by Proposition 13.1 page 185
$\phi_x(x^\perp) \triangleq (x^\perp \vee x^\perp) \wedge x$	by definition of *Sasaki projection* (Definition 13.6 page 185)
$= x^\perp \wedge x$	by *idempotency* of lattices (Theorem 5.3 page 74)
$= 0$	by *non-contradiction* property of *orthocomplemented lattice* (Definition 11.1 page 152)

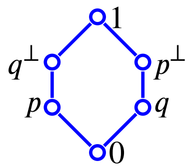
✏

Example 13.19.

EX

Here are some examples of projections in the O_6 *lattice* onto the element x:

$$\begin{array}{lllllllllll}
\phi_p(q) & \triangleq & (q \vee p^\perp) \wedge p & = & p^\perp \wedge p & = & 0 & \text{(because } p \perp q\text{)}\\
\phi_p(p^\perp) & \triangleq & (p^\perp \vee p^\perp) \wedge p & = & p^\perp \wedge p & = & 0 & \text{(because } p \perp p^\perp\text{)}\\
\phi_p(q^\perp) & \triangleq & (q^\perp \vee p^\perp) \wedge p & = & 1 \wedge p & = & p & \text{(because } p \le q^\perp\text{)}\\
\phi_{q^\perp}(p) & \triangleq & (p \vee q) \wedge q^\perp & = & 1 \wedge q^\perp & = & q^\perp & \text{(because } q^\perp \le 1\text{)}\\
\phi_p(1) & \triangleq & (1 \vee p^\perp) \wedge p & = & 1 \wedge p & = & p & \text{(because } p \le 1\text{)}\\
\phi_p(0) & \triangleq & (0 \vee p^\perp) \wedge p & = & p^\perp \wedge p & = & 0 & \text{(because } p \perp 0\text{)}
\end{array}$$



Example 13.20.

EX

Here are some examples of projections in lattice 5 of Example 11.2 (page 152):

$$\begin{array}{lllllllllll}
\phi_x(p) & \triangleq & (p \vee x^\perp) \wedge x & = & 1 \wedge x & = & x & \\
\phi_x(y) & \triangleq & (y \vee x^\perp) \wedge x & = & x^\perp \wedge x & = & 0 & \text{(because } x \perp y\text{)}\\
\phi_x(z) & \triangleq & (z \vee x^\perp) \wedge x & = & x^\perp \wedge x & = & 0 & \text{(because } x \perp z\text{)}\\
\phi_x(p^\perp) & \triangleq & (p^\perp \vee x^\perp) \wedge x & = & p^\perp \wedge x & = & 0 & \\
\phi_x(x^\perp) & \triangleq & (x^\perp \vee x^\perp) \wedge x & = & x^\perp \wedge x & = & 0 & \text{(because } x \perp x^\perp\text{)}\\
\phi_x(y^\perp) & \triangleq & (y^\perp \vee x^\perp) \wedge x & = & 1 \wedge x & = & x & \text{(because } x \le y^\perp\text{)}\\
\phi_x(z^\perp) & \triangleq & (z^\perp \vee x^\perp) \wedge x & = & 1 \wedge x & = & x & \text{(because } x \le z^\perp\text{)}\\
\phi_x(1) & \triangleq & (1 \vee x^\perp) \wedge x & = & 1 \wedge x & = & x & \text{(because } x \le 1\text{)}\\
\phi_x(0) & \triangleq & (0 \vee x^\perp) \wedge x & = & x^\perp \wedge x & = & 0 & \text{(because } x \perp 0\text{)}
\end{array}$$

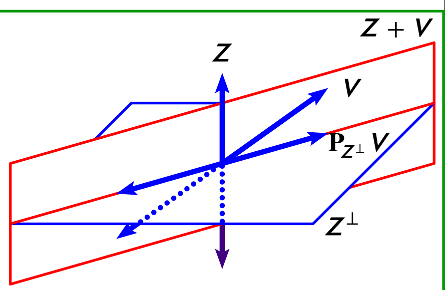
Example 13.21.

EX

Let $\mathbb{R}^3$ be the *3-dimensional Euclidean space* (Example 11.3 page 153) with subspaces $\boldsymbol{Z}$ and $\boldsymbol{V}$. Then the projection operator $P_{\boldsymbol{Z}^\perp}$ onto $\boldsymbol{Z}^\perp$ is a *sasaki projection* $\phi_{\boldsymbol{Z}^\perp}$. In particular

$$\begin{aligned}
\mathbf{P}_{\boldsymbol{Z}^\perp}\boldsymbol{V} &\triangleq \phi_{\boldsymbol{Z}^\perp}(\boldsymbol{V})\\
&\triangleq (\boldsymbol{V} + \boldsymbol{Z}^{\perp\perp}) \cap \boldsymbol{Z}^\perp\\
&= (\boldsymbol{V} + \boldsymbol{Z}) \cap \boldsymbol{Z}^\perp
\end{aligned}$$

as illustrated to the right.



CHAPTER 14

LOGIC

❝ *I dare say that this is the last effort of the human mind, and when this project shall have been carried out, all that men will have to do will be to be happy, since they will have an instrument that will serve to exalt the intellect not less than the telescope serves to perfect their vision.*❞

Gottfried Leibniz (1646–1716), German mathematician, sharing his thoughts regarding *mathematical logic.* [1]

❝ *I cannot forget or omit to record this day last week. I was sleeping as usual for the night at St. Michael's Hamlet. As I awoke in the morning, the sun was shining brightly into my room. There was a consciousness on my mind that I was the discoverer of the true logic of the future. For a few minutes I felt a delight such as one can seldom hope to feel. But it would not last long— I remembered only too soon how unworthy and weak an instrument I was for accomplishing so great a work, and how hardly could I expect to do it.*❞

William Stanley Jevons (1835–1882), English economist and logician [2]

14.1 Implications

Arguably a logic is not a logic without the inclusion of an *implication* function $\to$. The mathematical structure *logic* is formally defined in Definition 14.2 (page 193). But before defining a logic, this text offers a very general definition (a "weak" definition) of implication that can be used in defining a very wide class of logics—including *non-Boolean* ones. For *Boolean* logics, the *classical implication* function $x \overset{c}{\to} y$ (Example 14.1 page 189) is arguably adequate. Two key properties of *classical implication* on a *Boolean* logic are *entailment* and *modus ponens*. The following definition exploits weakened versions of these two properties to define implication. Note that the definition

[1] quote: Padoa (1912), page 21
Cajori (1993) ⟨paragraph 541⟩
image: http://en.wikipedia.org/wiki/Gottfried_Leibniz, public domain

[2] image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Jevons.html
quote: Jevons (1886), page 219 ⟨1866 March 28 entry⟩

is at this time probably not standard in the literature. But without it, it is difficult to offer a complete definition of a logic.

Definition 14.1. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE *(Definition 6.1 page 89).*

DEF *The function* $\to$ *in* X^X *is an* ***implication*** *on* $\mathbf{L}$ *if*

1. $\{x \leq y\} \implies x \to y \;\geq\; x \vee y \quad \forall x,y \in X$ (WEAK ENTAILMENT) *and*
2. $x \wedge (x \to y) \;\leq\; \neg x \vee y \quad \forall x,y \in X$ (WEAK MODUS PONENS)

Proposition 14.1. *Let* $\to$ *be an* IMPLICATION *(Definition 14.1 page 188) on a* BOUNDED LATTICE $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition 6.1 page 89).*

PRP $$\{x \leq y\} \iff \{x \to y \geq x \vee y\} \qquad \forall x,y \in X$$

✎PROOF:

1. Proof for $\implies$ case: by *weak entailment* property of *implications* (Definition 14.1 page 188).
2. Proof for $\impliedby$ case:

$$\begin{aligned} y &\geq x \wedge (x \to y) && \text{by right hypothesis} \\ &\geq x \wedge (x \vee y) && \text{by } \textit{modus ponens} \text{ property of } \to \text{ (Definition 14.1 page 188)} \\ &= x && \text{by } \textit{absorptive} \text{ property of } \textit{lattices} \text{ (Definition 5.3 page 73)} \end{aligned}$$

✏

Remark 14.1. [3] Let $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ be a *bounded lattice* (Definition 6.1 page 89). In the context of *ortho lattices*, a more common (and stronger) definition of *implication* $\to$ might be

1. $x \leq y \implies x \to y \;=\; 1 \quad \forall x,y \in X$ (*entailment / strong entailment*) and
2. $x \wedge (x \to y) \;\leq\; y \quad \forall x,y \in X$ (*modus ponens / strong modus ponens*)

This definition yields a result stronger than that of Proposition 14.1 (page 188):

$$\{x \leq y\} \iff \{x \to y = 1\} \qquad \forall x,y \in X$$

The *Heyting 3-valued logic* (Example 14.6 page 196) and *Sasaki hook logic* (Example 14.9 page 197) have both *strong entailment* and *strong modus ponens*. However, for non-ortho logics in general, these two properties seem inappropriate to serve as a definition for *implication*. For example, the *Kleene 3-valued logic* (Example 14.3 page 194), RM_3 *logic* (Example 14.5 page 195), and BN_4 *logic* (Example 14.10 page 197) do not have the *strong entailment* property; and the *Kleene 3-valued logic*, *Łukasiewicz 3-valued logic* (Example 14.4 page 195), and BN_4 *logic* do not have the *strong modus ponens* property.

✎PROOF:

1. Proof for $\implies$ case: by *entailment* property of *implications* (Definition 14.1 page 188).
2. Proof for $\impliedby$ case:

$$\begin{aligned} x \to y = 1 &\implies x \wedge 1 \leq y && \text{by } \textit{modus ponens} \text{ property (Definition 14.1 page 188)} \\ &\implies x \leq y && \text{by definition of } 1 \text{ (} \textit{least upper bound} \text{) (Definition 4.21 page 70)} \end{aligned}$$

✏

[3] Hardegree (1979) page 59 ⟨(E),(MP),(E*)⟩, Kalmbach (1973) page 498, Kalmbach (1983) pages 238–239 ⟨Chapter 4 §15⟩, Pavičić and Megill (2008) page 24, Xu et al. (2003) page 27 ⟨Definition 2.1.1⟩, Xu (1999) page 25, Jun et al. (1998) page 54

Example 14.1. [4] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1\, ; \leq)$ be a *lattice with negation* (Definition 13.5 page 173).

If $\boldsymbol{L}$ is an **orthomodular lattice** (Definition 13.3 page 172), then the functions listed below are all examples of valid *implication* functions (Definition 14.1 page 188) on $\boldsymbol{L}$. If $\boldsymbol{L}$ is an **ortho lattice**, then 1–5 are *implication* relations.

1. $x \overset{c}{\to} y \triangleq \neg x \vee y \quad \forall x,y \in X$ (*classical implication/ material implication/ horseshoe*)
2. $x \overset{s}{\to} y \triangleq \neg x \vee (x \wedge y) \quad \forall x,y \in X$ (*Sasaki hook / quantum implication*)
3. $x \overset{d}{\to} y \triangleq y \vee (\neg x \wedge \neg y) \quad \forall x,y \in X$ (*Dishkant implication*)
4. $x \overset{k}{\to} y \triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (x \wedge (\neg x \vee y)) \quad \forall x,y \in X$ (*Kalmbach implication*)
5. $x \overset{n}{\to} y \triangleq (\neg x \wedge y) \vee (x \wedge y) \vee ((\neg x \vee y) \wedge \neg y) \quad \forall x,y \in X$ (*non-tollens implication*)
6. $x \overset{r}{\to} y \triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) \quad \forall x,y \in X$ (*relevance implication*)

Moreover, if $\boldsymbol{L}$ is a **Boolean lattice**, then all of these implications are equivalent to $\overset{c}{\to}$, and all of them have *strong entailment* and *strong modus ponens*.

Note that $\forall x,y \in X$, $\quad x \overset{d}{\to} y = \neg y \overset{s}{\to} \neg x \quad$ and $\quad x \overset{n}{\to} y = \neg y \overset{k}{\to} \neg x$. The values for the 6 implications on an *orthocomplemented O_6 lattice* (Definition 11.2 page 152) are listed in Example 14.11 (page 197).

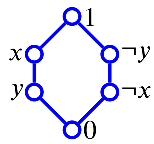
✎PROOF:

1. Proofs for the *classical implication* $\overset{c}{\to}$:

 (a) Proof that on an *ortho lattice*, $\overset{c}{\to}$ is an *implication*:

$$
\begin{aligned}
x \leq y \implies x \overset{c}{\to} y &\triangleq \neg x \vee y && \text{by definition of } \overset{c}{\to} \\
&\geq \neg y \vee y && \text{by } x \leq y \text{ and } \textit{antitone} \text{ prop. of } \neg \text{ (Definition 13.3 page 172)} \\
&= 1 && \text{by } \textit{excluded middle} \text{ prop. of } \neg \text{ (Theorem 13.5 page 176)} \\
&\implies \textit{strong entailment} && \text{by definition of } \textit{strong entailment} \\
x \wedge (\neg x \vee y) &\leq \neg x \vee y && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)} \\
&\implies \textit{weak modus ponens} && \text{by definition of } \textit{weak modus ponens}
\end{aligned}
$$

 Note that in general for an *ortho lattice*, the bound cannot be tightened to *strong modus ponens* because, for example in the O_6 *lattice* (Definition 11.2 page 152) illustrated to the right



$$x \wedge (\neg x \vee y) = x \wedge 1 = x \not\leq y \implies \textit{not strong modus ponens}$$

 (b) Proof that on a *Boolean lattice*, $\overset{c}{\to}$ is an *implication*:

$$
\begin{aligned}
x \wedge (\neg x \vee y) &= (x \wedge \neg x) \vee (x \wedge y) && \text{by } \textit{distributive} \text{ prop. of Boolean lat. (Definition 10.1 page 127)} \\
&= 1 \vee (x \wedge y) && \text{by } \textit{excluded middle} \text{ property of } \textit{Boolean lattices} \\
&= x \wedge y && \text{by definition of 1} \\
&\leq y && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)} \\
&\implies \textit{strong modus ponens} && \text{by definition of } \textit{strong modus ponens}
\end{aligned}
$$

2. Proofs for *Sasaki implication* $\overset{s}{\to}$:

[4] Kalmbach (1973) page 499, Kalmbach (1974), Mittelstaedt (1970) ⟨Sasaki hook⟩, Finch (1970) page 102 ⟨Sasaki hook (1.1)⟩, Kalmbach (1983) page 239 ⟨Chapter 4 §15, 3. THEOREM⟩

(a) Proof that on an *ortho lattice*, $\overset{s}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{s}{\to} y &\triangleq \neg x \vee (x \wedge y) && \text{by definition of } \overset{k}{\to}\\
&= \neg x \vee x && \text{by } x \le y \text{ hypothesis}\\
&= 1 && \text{by \textit{excluded middle} prop. of ortho neg. (Theorem 13.5 page 176)}\\
&\implies \textit{strong entailment} && \text{by definition of \textit{strong entailment}}\\
x \wedge (x \overset{s}{\to} y) &\triangleq x \wedge [\neg x \vee (x \wedge y)] && \text{by definition of } \overset{s}{\to}\\
&\le [\neg x \vee (x \wedge y)] && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\le \neg x \vee y && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(b) Proof that on a *Boolean lattice*, $\overset{s}{\to} = \overset{c}{\to}$:

$$\begin{aligned}
x \overset{s}{\to} y &\triangleq \neg x \vee (x \wedge y) && \text{by definition of } \overset{s}{\to}\\
&= \neg x \vee y && \text{by Lemma 10.2 (page 132)}\\
&= x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}$$

3. Proofs for *Dishkant implication* $\overset{d}{\to}$:

(a) Proof that $x \overset{d}{\to} y \equiv \neg y \overset{s}{\to} \neg x$:

$$\begin{aligned}
x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= y \vee (\neg y \wedge \neg x) && \text{by \textit{commutative} property of \textit{lattices} (Theorem 5.3 page 74)}\\
&= \neg\neg y \vee (\neg y \wedge \neg x) && \text{by \textit{involutory} property of \textit{ortho negations} (Definition 13.3 page 172)}\\
&\triangleq \neg y \overset{s}{\to} \neg x && \text{by definition of } \overset{s}{\to}
\end{aligned}$$

(b) Proof that on an *ortho lattice*, $\overset{d}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= y \vee \neg y && \text{by } x \le y \text{ hypoth. and \textit{antitone} prop. (Definition 13.3 page 172)}\\
&= 1 && \text{by \textit{excluded middle} prop. of ortho neg. (Theorem 13.5 page 176)}\\
&\implies \textit{strong entailment} && \text{by definition of \textit{strong entailment}}\\
x \wedge (x \overset{d}{\to} y) &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= y \vee \neg x && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(c) Proof that on a *Boolean lattice*, $\overset{d}{\to} = \overset{c}{\to}$:

$$\begin{aligned}
x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= \neg x \vee y && \text{by Lemma 10.2 (page 132)}\\
&= x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}$$

4. Proofs for the *Kalmbach implication* $\overset{k}{\to}$:

(a) Proof that on an *ortho lattice*, $\overset{k}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{k}{\to} y
&\triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \overset{k}{\to}\\
&= (\neg x \wedge y) \vee (\neg y) \vee [x \wedge (\neg x \vee y)] && \text{by \textit{antitone} property (Definition 13.3 page 172)}\\
&= (\neg x \wedge y) \vee \neg y \vee [x \wedge (1)] &&\\
&= (\neg x \wedge y) \vee (x \vee \neg y) && \text{by definition of 1 (Definition 4.21 page 70)}\\
&= \neg\neg(\neg x \wedge y) \vee (x \vee \neg y) && \text{by \textit{involutory} property (Definition 13.3 page 172)}\\
&= \neg(\neg\neg x \vee \neg y) \vee (x \vee \neg y) && \text{by \textit{de Morgan} property (Theorem 13.5 page 176)}\\
&= \neg(x \vee \neg y) \vee (x \vee \neg y) && \text{by \textit{involutory} property (Definition 13.3 page 172)}\\
&= 1 && \text{by \textit{excluded middle} property (Theorem 13.5 page 176)}\\
&\implies \textit{strong entailment}
\end{aligned}$$

$$\begin{aligned}
x \wedge (x \overset{k}{\to} y) &\triangleq x \wedge [(\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)]] && \text{by definition of } \overset{k}{\to}\\
&\le (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\le (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (\neg x \vee y) && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\le y \vee (\neg x \wedge \neg y) \vee \neg x \vee y && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&= y \vee \neg x \vee (\neg x \wedge \neg y) && \text{by \textit{idempotent} p. (Theorem 5.3 page 74)}\\
&\le y \vee \neg x \vee \neg x && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&= \neg x \vee y && \text{by \textit{idempotent} p. (Theorem 5.3 page 74)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(b) Proof that on a *Boolean lattice*, $\overset{k}{\to} = \overset{c}{\to}$:

$$\begin{aligned}
x \overset{k}{\to} y &\triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \overset{k}{\to}\\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [(x \wedge \neg x) \vee (x \wedge y)] && \text{by \textit{distributive} property (Definition 10.1 page 127)}\\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [(0) \vee (x \wedge y)] && \text{by \textit{non-contradiction} property}\\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (x \wedge y) && \text{by \textit{bounded} property (Definition 6.1 page 89)}\\
&= \neg x \wedge (y \vee \neg y) \vee (x \wedge y) && \text{by \textit{distributive} property (Definition 10.1 page 127)}\\
&= \neg x \wedge 1 \vee (x \wedge y) && \text{by \textit{excluded middle} property}\\
&= \neg x \vee (x \wedge y) && \text{by definition of 1 (Definition 4.21 page 70)}\\
&= \neg x \vee y && \text{by Lemma 10.2 (page 132)}\\
&\triangleq x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}$$

5. Proofs for the *non-tollens implication* $\overset{n}{\to}$:

(a) Proof that $x \overset{n}{\to} y \equiv \neg y \overset{k}{\to} \neg x$:

$$\begin{aligned}
x \overset{n}{\to} y &\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee [(\neg x \vee y) \wedge \neg y] && \text{by definition of } \overset{n}{\to}\\
&= (y \wedge \neg x) \vee (y \wedge x) \vee [\neg y \wedge (y \vee \neg x)] &&\\
&= (\neg\neg y \wedge \neg x) \vee (\neg\neg y \wedge \neg\neg x) \vee [\neg y \wedge (\neg\neg y \vee \neg x)] &&\\
&\triangleq \neg y \overset{k}{\to} \neg x && \text{by definition of } \overset{k}{\to}
\end{aligned}$$

(b) Proof that on an *ortho lattice*, $\stackrel{n}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \stackrel{n}{\to} y \\
&\equiv \neg y \stackrel{k}{\to} \neg x && \text{by item (5a) page 191}\\
&= 1 && \text{by item (4a) page 191}\\
&\implies \textit{strong entailment}
\end{aligned}$$

$$\begin{aligned}
x \wedge (x \stackrel{n}{\to} y) &= x \wedge (\neg y \stackrel{k}{\to} \neg x) && \text{by item (5a) page 191}\\
&\le \neg\neg y \vee \neg x && \text{by item (4a) page 191}\\
&= y \vee \neg x && \text{by \textit{involutory} property of } \neg \text{ (Definition 13.3 page 172)}\\
&= \neg x \vee y && \text{by \textit{commutative} property of \textit{lattices} (Definition 5.3 page 73)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

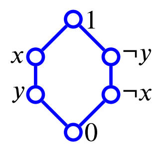
(c) Proof that on a *Boolean lattice*, $\stackrel{n}{\to} = \stackrel{c}{\to}$:

$$\begin{aligned}
x \stackrel{n}{\to} y &= \neg y \stackrel{k}{\to} \neg x && \text{by item (5a) page 191}\\
&= \neg\neg y \vee \neg x && \text{by item (4b) page 191}\\
&= y \vee \neg x && \text{by \textit{involutory} property of } \neg \text{ (Definition 13.3 page 172)}\\
&= \neg x \vee y && \text{by \textit{commutative} property of \textit{lattices} (Definition 5.3 page 73)}\\
&\triangleq x \stackrel{c}{\to} y && \text{by definition of } \stackrel{c}{\to}
\end{aligned}$$

6. Proofs for the *relevance implication* $\stackrel{r}{\to}$:

(a) Proof that on an *ortho lattice*, $\stackrel{r}{\to}$ does *not* have *weak entailment*:
In the *ortho lattice* to the right...

$$\begin{aligned}
x \le y \implies x \stackrel{r}{\to} y \\
&\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \stackrel{r}{\to}\\
&= 0 \vee x \vee \neg y\\
&= x \vee \neg y\\
&\ne x \vee y
\end{aligned}$$



(b) Proof that on an *orthomodular lattice*, $\stackrel{r}{\to}$ *does* have *strong entailment*:

$$\begin{aligned}
x \le y \implies x \stackrel{r}{\to} y \\
&\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \stackrel{r}{\to}\\
&= (\neg x \wedge y) \vee x \vee (\neg x \wedge \neg y) && \text{by } x \le y \text{ hypothesis}\\
&= (\neg x \wedge y) \vee x \vee \neg y && \text{by } x \le y \text{ and \textit{antitone} property (Definition 13.3 page 172)}\\
&= y \vee \neg y && \text{by \textit{orthomodular identity} (Definition 11.3 page 161)}\\
&= 1 && \text{by \textit{excluded middle} property of } \neg \text{ (Theorem 13.5 page 176)}
\end{aligned}$$

(c) Proof that on an *ortho lattice*, $\stackrel{r}{\to}$ *does* have *weak modus ponens*:

$$\begin{aligned}
x \wedge (x \stackrel{r}{\to} y) &\triangleq x \wedge [(\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y)] && \text{by definition of } \stackrel{r}{\to}\\
&\le [(\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y)] && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\le \neg x \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\le \neg x \vee y \vee (\neg x \wedge \neg y) && \text{by definition of } \wedge \text{ (Definition 4.22 page 70)}\\
&\le \neg x \vee y && \text{by \textit{absorption} property (Theorem 5.3 page 74)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(d) Proof that on a *Boolean lattice*, $\overset{r}{\to} = \overset{\varsigma}{\to}$:

$x \overset{r}{\to} y \triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y)$	by definition of $\overset{r}{\to}$
$= [\neg x \wedge (y \vee \neg y)] \vee (x \wedge y)$	by *distributive* property (Definition 10.1 page 127)
$= [\neg x \wedge 1] \vee (x \wedge y)$	by *excluded middle* property of $\neg$ (Theorem 13.5 page 176)
$= \neg x \vee (x \wedge y)$	by definition of 1 and $\wedge$ (Definition 4.22 page 70)
$= \neg x \vee y$	by property of *Boolean lattices* (Lemma 10.2 page 132)
$\triangleq x \overset{\varsigma}{\to} y$	by definition of $\overset{\varsigma}{\to}$

⇨

14.2 Logics

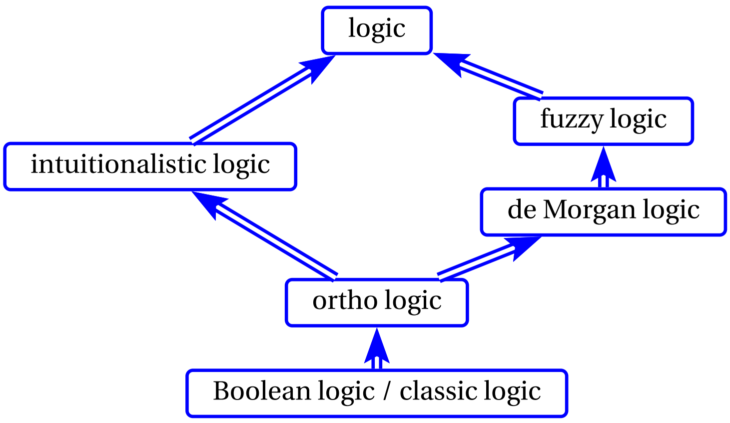


Figure 14.1: lattice of logics

Definition 14.2. [5] *Let* $\to$ *be an* IMPLICATION (Definition 14.1 page 188) *defined on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1 ; \leq)$ (Definition 13.5 page 173).

DEF	$(X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *is a* ***logic***	*if* $\neg$ *is a* MINIMAL NEGATION.
	$(X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *is a* ***fuzzy logic***	*if* $\neg$ *is a* FUZZY NEGATION.
	$(X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *is an* ***intuitionalistic logic***	*if* $\neg$ *is an* INTUITIONALISTIC NEGATION.
	$(X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *is a* ***de Morgan logic***	*if* $\neg$ *is a* DE MORGAN NEGATION.
	$(X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *is a* ***Kleene logic***	*if* $\neg$ *is a* KLEENE NEGATION.
	$(X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *is an* ***ortho logic***	*if* $\neg$ *is an* ORTHO NEGATION.
	$(X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *is a* ***Boolean logic***	*if* $\neg$ *is an* ORTHO NEGATION *and* $\boldsymbol{L}$ *is* BOOLEAN.

Definition 14.3. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1 ; \leq, \to)$ *be a* LOGIC (Definition 14.2 page 193).

DEF *The function* $\leftrightarrow$ *in* X^X *is an* ***equivalence*** *on* $\boldsymbol{L}$ *if*

$$x \leftrightarrow y \quad \triangleq \quad (x \to y) \wedge (y \to x) \qquad \forall x,y \in X$$

Example 14.2 (Aristotelian logic/classical logic). [7]

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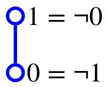
[5] Straßburger (2005) page 136 ⟨Definition 2.1⟩, de Vries (2007) page 11 ⟨Definition 16⟩
[6] Novák et al. (1999) page 18
[7] Novák et al. (1999) pages 17–18 ⟨Example 2.1⟩

EX The *classical bi-variate logic* is defined below. It is a 2 element *Boolean logic* (Definition 14.2 page 193). with $\boldsymbol{L} \triangleq (\{1,0\}, \wedge, \neg, 0, 1, \leq; \vee)$ and a *classical implication* $\rightarrow$ with *strong entailment* and *strong modus ponens*. The value 1 represents "*true*" and 0 represents "*false*".



$$x \rightarrow y \triangleq \left\{ \begin{array}{ll} 1 & \forall x \leq y \\ y & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{c|cc} \rightarrow & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 1 \end{array} \qquad \forall x,y \in X \right\} = \neg x \vee y$$

✎PROOF:

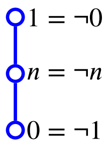
1. Proof that $\neg$ is an *ortho negation*: by Definition 13.3 (page 172)
2. Proof that $\rightarrow$ is an *implication* with *strong entailment* and *strong modus ponens*:
   (a) $\boldsymbol{L}$ is *Boolean* and therefore is *orthocomplemented*.
   (b) $\rightarrow$ is equivalent to the *classical implication* $\overset{c}{\rightarrow}$ (Example 14.1 page 189).
   (c) By Example 14.1 (page 189), $\rightarrow$ has *strong entailment* and *strong modus ponens*.

✏

The *classical logic* (previous example) can be generalized in several ways. Arguably one of the simplest of these is the 3-valued logic due to Kleene (next example).

Example 14.3 (Kleene 3-valued logic). [8]

EX The *Kleene 3-valued logic* $(X, \vee, \wedge, \neg, 0, 1; \leq, \rightarrow)$ is defined below. The function $\neg$ is a *Kleene negation* (Definition 13.3 page 172, Example 13.9 page 180) defined on a 3 element *linearly ordered lattice* (Definition 4.4 page 59). The function $\rightarrow$ is the *classical implication* $x \rightarrow y \triangleq \neg x \vee y$. The values 1 represents "*true*", 0 represents "*false*", and n represents "*neutral*" or "*undecided*".



$$x \rightarrow y \triangleq \left\{ \neg x \vee y \quad \forall x \in X \right\} = \left\{ \begin{array}{c|ccc} \rightarrow & 1 & n & 0 \\ \hline 1 & 1 & n & 0 \\ n & 1 & n & n \\ 0 & 1 & 1 & 1 \end{array} \qquad \forall x,y \in X \right\}$$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 13.9 (page 180)
2. Proof that $\rightarrow$ is an *implication*: This follows directly from the definition of $\rightarrow$ and the definition of an *implication* (Definition 14.1 page 188).
3. Proof that $\rightarrow$ does not have *strong entailment*: $n \rightarrow n = n = n \vee n \neq 1$.
4. Proof that $\rightarrow$ does not have *strong modus ponens*: $n \rightarrow 0 = n = \neg n \vee 0 \not\leq 0$.

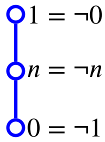
✏

A lattice and negation alone do not uniquely define a logic. Łukasiewicz also introduced a 3-valued logic with identical lattice structure to Kleene, but with a different implication relation (next example). Historically, Łukasiewicz's logic was introduced before Kleene's.

[8] Kleene (1938) page 153, Kleene (1952), pages 332–339 ⟨§64. The 3-valued logic⟩, Avron (1991) page 277

Example 14.4 (Łukasiewicz 3-valued logic). [9]

The *Łukasiewicz 3-valued logic* $(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ is defined to the right and below. The function $\neg$ is a *Kleene negation* (Definition 13.3 page 172) defined on a 3 element *linearly ordered lattice* (Definition 4.4 page 59). The implication has *strong entailment* but *weak modus ponens*. In the implication table below, values that differ from the classical $x \to y \triangleq \neg x \vee y$ are shaded.



$$x \to y \triangleq \begin{cases} 1 & \forall x \le y \\ \neg x \vee y & \text{otherwise} \end{cases} = \left\{ \begin{array}{c|ccc} \to & 1 & n & 0 \\ \hline 1 & 1 & n & 0 \\ n & 1 & \boxed{1} & n \\ 0 & 1 & 1 & 1 \end{array} \quad \forall x,y \in X \right\} = \begin{cases} 1 & \text{for } x = y = n \\ \neg x \vee y & \text{otherwise} \end{cases}$$

✎PROOF:

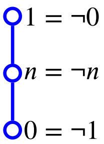
1. Proof that $\neg$ is a *Kleene negation*: see Example 13.9 (page 180)
2. Proof that $\to$ is an *implication*: This follows directly from the definition of $\to$ and the definition of an *implication* (Definition 14.1 page 188).
3. Proof that $\to$ does not have *strong modus ponens*: $n \to 0 = n = \neg n \vee 0 \not\le 0$.

⇒

Example 14.5 (RM_3 logic). [10]

EX

The RM_3 *logic* $(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ is defined below. The function $\neg$ is a *Kleene negation* (Definition 13.3 page 172) defined on a 3 element *linearly ordered lattice* (Definition 4.4 page 59). The implication function has *weak entailment* by *strong modus ponens*. In the implication table below, values that differ from the classical $x \to y \triangleq \neg x \vee y$ are shaded.



$$x \to y \triangleq \begin{cases} 1 & \forall x < y \\ n & \forall x = y \\ 0 & \forall x > y \end{cases} = \left\{ \begin{array}{c|ccc} \to & 1 & n & 0 \\ \hline 1 & 1 & \boxed{0} & 0 \\ n & 1 & n & \boxed{0} \\ 0 & 1 & 1 & 1 \end{array} \quad \forall x,y \in X \right\}$$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 13.9 (page 180)
2. Proof that $\to$ is an *implication*: This follows directly from the definition of $\to$ and the definition of an *implication* (Definition 14.1 page 188).
3. Proof that $\to$ does not have *strong entailment*: $n \to n = n = n \vee n \neq 1$.

⇒

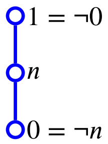
In a 3-valued logic, the negation does not necessarily have to be as in the previous three examples. The next example offers a different negation.

[9] Łukasiewicz (1920) page 17 ⟨II. The principles of consequence⟩, Avron (1991) page 277 ⟨Łukasiewicz.⟩
[10] Avron (1991) pages 277–278
Sobociński (1952)

Example 14.6 (Heyting 3-valued logic/Jaśkowski's first matrix). [11]

EX

The *Heyting 3-valued logic* $(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ is defined below. The negation $\neg$ is both *intuitionistic* and *fuzzy* (Definition 13.2 page 172), and is defined on a 3 element *linearly ordered lattice* (Definition 4.4 page 59). The implication function has both *strong entailment* and *strong modus ponens*. In the implication table below, values that differ from the classical $x \to y \triangleq \neg x \vee y$ are shaded.



$$x \to y \triangleq \begin{cases} 1 & \forall x \le y \\ y & \text{otherwise} \end{cases} = \left\{ \begin{array}{c|ccc} \to & 1 & n & 0 \\ \hline 1 & 1 & n & 0 \\ n & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array} \qquad \forall x,y \in X \right\}$$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 13.11 (page 181)

2. Proof that $\to$ is an *implication*: by definition of *implication* (Definition 14.1 page 188)

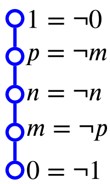
✏

Of course it is possible to generalize to more than 3 values (next example).

Example 14.7 (Łukasiewicz 5-valued logic). [12]

EX

The *Łukasiewicz 5-valued logic* $(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ is defined below. The implication function has *strong entailment* but *weak modus ponens*. In the implication table below, values that differ from the classical $x \to y \triangleq \neg x \vee y$ are shaded.



$$x \to y \triangleq \left\{ \begin{array}{c|ccccc} \to & 1 & p & n & m & 0 \\ \hline 1 & 1 & p & n & m & 0 \\ p & 1 & 1 & n & m & m \\ n & 1 & 1 & 1 & m & n \\ m & 1 & 1 & 1 & 1 & p \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right. \qquad \forall x,y \in X$$

✎PROOF:

✏

All the previous examples in this section are *linearly ordered*. The following examples employ logics that are not.

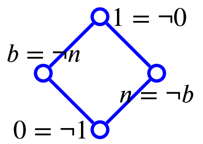
Example 14.8 (Boolean 4-valued logic). [13]

[11] Karpenko (2006) page 45, Johnstone (1982) page 9 ⟨§1.12⟩, Heyting (1930a), Heyting (1930b), Heyting (1930c), Heyting (1930d), Jaskowski (1936), Mancosu (1998)

[12] Xu et al. (2003) page 29 ⟨Example 2.1.3⟩
Jun et al. (1998) page 54 ⟨Example 2.2⟩

[13] Belnap (1977) page 13, Restall (2000) page 177 ⟨Example 8.44⟩, Pavičić and Megill (2008) page 28 ⟨Definition 2, *classical implication*⟩, Mittelstaedt (1970), Finch (1970) page 102 ⟨(1.1)⟩, Smets (2006) page 270

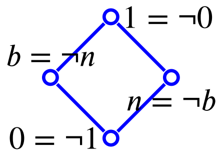
EX The *Boolean 4-valued logic* is defined below. The negation function ¬ is an *ortho negation* (Example 13.13 page 182) defined on an M_2 *lattice*. The value 1 represents "*true*", 0 represents "*false*", and m and n represent some intermediate values.



$$x \to y \triangleq \neg x \vee y = \left\{ \begin{array}{c|cccc} \to & 1 & b & n & 0 \\ \hline 1 & 1 & b & n & 0 \\ b & 1 & 1 & n & n \\ n & 1 & b & 1 & b \\ 0 & 1 & 1 & 1 & 1 \end{array} \right. \qquad \forall x,y \in X$$

Example 14.9 (Sasaki hook / quantum implication). [14]

EX The *Sasaki hook logic* $(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ is defined below. The order structure and negation are the same as in Example 14.8 (page 196).

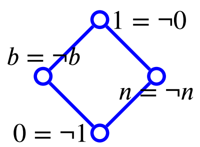


$$x \to y \triangleq \neg x \vee (x \wedge y) = \left\{ \begin{array}{c|cccc} \to & 1 & b & n & 0 \\ \hline 1 & 1 & b & n & 0 \\ b & 1 & 1 & n & n \\ n & 1 & b & 1 & b \\ 0 & 1 & 1 & 1 & 1 \end{array} \right. \qquad \forall x,y \in X$$

All the previous examples in this section are *distributive*; the previous example was *Boolean*. The next example is *non-distributive*, and *de Morgan* (but *non-Boolean*). Note for a given order structure, the method of negation may not be unique; in the previous and following examples both have identical lattices, but are negated differently.

Example 14.10 (BN_4 logic). [15]

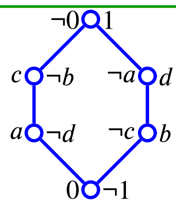
EX The BN_4 *logic* is defined below. The function ¬ is a *de Morgan negation* (Example 13.14 page 182) defined on a 4 element M_2 *lattice*. The value 1 represents "*true*", 0 represents "*false*", b represents "*both*" (both true and false), and n represents "*neither*". In the implication table below, the values that differ from those of the *classical implication* $\overset{c}{\to}$ are shaded.



$$x \to y \triangleq \left\{ \begin{array}{c|cccc} \to & 1 & n & b & 0 \\ \hline 1 & 1 & n & 0 & 0 \\ n & 1 & 1 & n & n \\ b & 1 & n & b & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array} \right. \qquad \forall x,y \in X$$

Example 14.11.

EX The tables that follow are the 6 implications defined in Example 14.1 (page 189) on the O_6 *lattice with ortho negation* (Definition 13.3 page 172), or the O_6 *orthocomplemented lattice* (Definition 11.2 page 152), illustrated to the right. In the tables, the values that differ from those of the *classical implication* $\overset{c}{\to}$ are shaded.



$\overset{c}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	c	1	a	a
c	1	d	1	b	1	b
b	1	1	c	1	c	c
a	1	d	1	d	1	d
0	1	1	1	1	1	1

$\overset{s}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	c	c
a	1	1	1	d	1	d
0	1	1	1	1	1	1

$\overset{d}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	c	1	a	a
c	1	d	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

[14] Pavičić and Megill (2008) page 28 ⟨Definition 2⟩, Mittelstaedt (1970), Finch (1970) page 102 ⟨(1.1)⟩, Smets (2006) page 270

[15] Restall (2000) page 171 ⟨Example 8.39⟩

$\overset{k}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

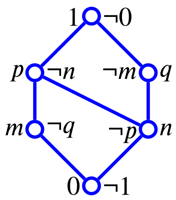
$\overset{n}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

$\overset{r}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

Example 14.12. [16]

EX

A 6 element logic is defined below. The function $\neg$ is a *Kleene negation* (Example 13.17 page 183). The implication has *strong entailment* but *weak modus ponens*. In the implication table below, the values that differ from those of the *classical implication* $\overset{c}{\to}$ are shaded.



$$x \to y \triangleq \left\{ \begin{array}{c|cccccc} \to & 1 & p & q & m & n & 0 \\ \hline 1 & 1 & p & q & m & n & 0 \\ p & 1 & \boxed{1} & q & p & \boxed{q} & n \\ q & 1 & p & 1 & m & p & m \\ m & 1 & 1 & q & 1 & q & q \\ n & 1 & \boxed{1} & 1 & p & \boxed{1} & p \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right. \qquad \forall x,y \in X$$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 13.17 (page 183)
2. Proof that $\to$ is an *implication*: This follows directly from the definition of $\to$ and the definition of an *implication* (Definition 14.1 page 188).
3. Proof that $\to$ does not have *strong modus ponens*:

$$\begin{array}{lllllllllll}
\neg p \wedge (p \to m) & = & n \wedge p & = & n & \leq & p & = & \neg p \vee m & \not\leq & m \\
\neg n \wedge (n \to m) & = & n \wedge p & = & n & \leq & p & = & \neg p \vee m & \not\leq & m \\
\neg p \wedge (p \to 0) & = & n \wedge n & = & n & \leq & n & = & \neg p \vee 0 & \not\leq & 0 \\
\neg n \wedge (n \to 0) & = & p \wedge n & = & n & \leq & p & = & \neg n \vee 0 & \not\leq & 0
\end{array}$$

✏

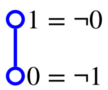
For an example of an 8-valued logic, see Kamide (2013). For examples of 16-valued logics, see Shramko and Wansing (2005).

14.3 Classical two-valued logic

Definition 14.4 (Aristotelian logic/classical logic). [17]

DEF

The ***classical 2-value logic*** *is a 2 element* LATTICE WITH ORTHO NEGATION (Definition 13.3 page 172) $(\{1,0\}, \vee, \wedge, \neg, 0, 1; \leq, \overset{c}{\to})$ *as illustrated below with values* 1 *representing* "TRUE", 0 *representing* "FALSE", *and with an implication connective* $\Longrightarrow$ *as specified below:*



$$x \Longrightarrow y \triangleq \left\{ \begin{array}{ll} 1 & \forall x \leq y \\ y & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{c|cc} \Longrightarrow & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 1 \end{array} \qquad \forall x,y \in X \right\} = \neg x \vee y$$

[16] Xu et al. (2003) pages 29–30 ⟨Example 2.1.4⟩
[17] Novák et al. (1999) pages 17–18 ⟨EXAMPLE 2.1⟩

Theorem 14.1.

THM

If $(\{1,0\}, \vee, \wedge, \neg, 0, 1; \leq, \overset{\neg}{\Rightarrow})$ *is the* CLASSICAL 2-VALUE LOGIC (Definition 14.4 page 198), *then the* ***logical OR*** $\vee$, ***logical AND*** $\wedge$, *and* ***logical equivalence*** $\Longleftrightarrow$ *operations are defined as follows:*

$$\begin{array}{c|cc} \vee & 1 & 0 \\ \hline 1 & 1 & 1 \\ 0 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \wedge & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \qquad \begin{array}{c|cc} \Longleftrightarrow & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

✎PROOF:

1. Proof for *logical OR* operation $\vee$: This follows from the *lattice* (Definition 5.3 page 73) properties of $\boldsymbol{L}_2$.
2. Proof for *logical AND* operation $\wedge$: This follows from the *lattice* (Definition 5.3 page 73) properties of $\boldsymbol{L}_2$.
3. Proof for *logical if and only if* operation $\Longleftrightarrow$: This follows from the definition of $\Longrightarrow$ (Definition 14.4 page 198) and Definition 14.3 (page 193).

⇨

One of the most useful facts concerning propositional logic systems is that they form a *Boolean algebra* (next theorem). Because they are a Boolean algebra, a number of useful properties automatically follow (next theorem) from the properties of Boolean algebras (Theorem 10.2 page 132).

Theorem 14.2 (Boolean algebra properties). [18] *Let* $\{0, 1\}$ *be the set of logical properties* FALSE *and* TRUE (Axiom 1.1 page 3). *Let* $\vee$ *be the* LOGICAL OR *and* $\wedge$ *the* LOGICAL AND *operations* (Definition 14.1 page 199). *Let* $\Longrightarrow$ *be the* LOGICAL IMPLIES *relation* (Definition 1.12 page 8).

THM

$(\{0, 1\}, \vee, \wedge; \Longrightarrow)$ *is a* BOOLEAN ALGEBRA. *In particular for all* $x, y, z \in \{0, 1\}$,

$x \vee x = x$	$x \wedge x = x$	(IDEMPOTENT)
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	(COMMUTATIVE)
$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(ASSOCIATIVE)
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	(ABSORPTIVE)
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(DISTRIBUTIVE)
$x \vee 0 = x$	$x \wedge 1 = x$	(IDENTITY)
$x \vee 1 = 1$	$x \wedge 0 = 0$	(BOUNDED)
$x \vee x' = 1$	$x \wedge x' = 0$	(COMPLEMENTED)[19]
$(x')' = x$		(UNIQUELY COMP.)
$(x \vee y)' = x' \wedge y'$	$(x \wedge y)' = x' \vee y'$	(DE MORGAN'S LAWS)
property with emphasizing $\vee$	*dual property emphasizing* $\wedge$	*property name*

✎PROOF: This follows directly from the fact that the *classical 2-valued logic* (Definition 14.4 page 198) is a *Boolean algebra* (Definition 10.1 page 127) and from Theorem 10.2 (page 132). ⇨

Definition 14.5 (additional logic operations). [20] *Let* $(\{0, 1\}, \Longrightarrow, \vee, \wedge, \neg, 0, 1)$ *be a propositional logic system. Let* $x' \triangleq \neg x$ *and* $y' \triangleq \neg y$*. The following table defines additional operations on* $\{0, 1\}$ *in*

[18] MacLane and Birkhoff (1999) page 488, Givant and Halmos (2009) page 10, Müller (1909), pages 20–21, Schröder (1890), Whitehead (1898) pages 35–37, Peano (1889b), page 88

[19] The property $x \vee x' = 1$ is also called the *law of the excluded middle.*
The property $x \wedge x' = 0$ is also called *non-contradiction* or *explosion.*
References: Renedo et al. (2003), page 71
Restall (2004) pages 73–75
Restall (2001), pages 1–3

[20] Givant and Halmos (2009) page 32 ⟨disjunction, conjunction, negation⟩, Shiva (1998) page 83 ⟨inhibit, transfer⟩, Whitesitt (1995) pages 68–69 ⟨*Sheffer stroke functions* $\downarrow = \uparrow, | = \downarrow$⟩, Quine (1979) pages 45–48 ⟨joint denial $\downarrow$, alternate denial $|$⟩, Bernstein (1934) page 876 ⟨implication $\supset$⟩

terms of $\vee$, $\wedge$, *and* $\neg$.

name	symbol	definition			
joint denial	$\downarrow$	$x \downarrow y$	$\triangleq$	$x' \wedge y'$	$\forall x,y \in \{0,1\}$
inhibit x	$\ominus$	$x \ominus y$	$\triangleq$	$x' \wedge y$	$\forall x,y \in \{0,1\}$
inhibit y	$-$	$x - y$	$\triangleq$	$x \wedge y'$	$\forall x,y \in \{0,1\}$
complete disjunction	$\oplus$	$x \oplus y$	$\triangleq$	$(x' \wedge y) \vee (x \wedge y')$	$\forall x,y \in \{0,1\}$
alternative denial	$\mid$	$x \mid y$	$\triangleq$	$x' \vee y'$	$\forall x,y \in \{0,1\}$

There are a total of $2^4 = 16$ possible binary operations on the set of relations $\{0,\ 1\} \times \{0,\ 1\} \to \{0,\ 1\}$. The following table summarizes these 16 operations.[21]

logic operations									
name and symbol		$(x,y) =$ 11	10	01	00	operation in terms of $\vee$, $\wedge$, and $\neg$			
zero	0	0	0	0	0	0	$=$	$x \wedge x'$	$\forall x \in \{0,1\}$
joint denial	$\downarrow$	0	0	0	1	$x \downarrow y$	$=$	$x' \wedge y'$	$\forall x,y \in \{0,1\}$
inhibit x	$\ominus$	0	0	1	0	$x \ominus y$	$=$	$x' \wedge y$	$\forall x,y \in \{0,1\}$
complement x	$\oplus$	0	0	1	1	$x \oplus y$	$=$	x'	$\forall x,y \in \{0,1\}$
inhibit y	$-$	0	1	0	0	$x - y$	$=$	$x \wedge y'$	$\forall x,y \in \{0,1\}$
complement y	$\oplus$	0	1	0	1	$x \oplus y$	$=$	y'	$\forall x,y \in \{0,1\}$
complete disjunction	$\oplus$	0	1	1	0	$x \oplus y$	$=$	$(x' \wedge y) \vee (x \wedge y')$	$\forall x,y \in \{0,1\}$
alternative denial	$\mid$	0	1	1	1	$x \mid y$	$=$	$x' \vee y'$	$\forall x,y \in \{0,1\}$
conjunction	$\wedge$	1	0	0	0	$x \wedge y$	$=$	$x \wedge y$	$\forall x,y \in \{0,1\}$
equivalence	$\Leftrightarrow$	1	0	0	1	$x \Leftrightarrow y$	$=$	$(x \wedge y) \vee (x' \wedge y')$	$\forall x,y \in \{0,1\}$
transfer y	$\Vdash$	1	0	1	0	$x \Vdash y$	$=$	y	$\forall x,y \in \{0,1\}$
implication	$\Rightarrow$	1	0	1	1	$x \Rightarrow y$	$=$	$x' \vee y$	$\forall x,y \in \{0,1\}$
transfer x	$\Dashv$	1	1	0	0	$x \Dashv y$	$=$	x	$\forall x,y \in \{0,1\}$
implied by	$\Leftarrow$	1	1	0	1	$x \Leftarrow y$	$=$	$x \vee y'$	$\forall x,y \in \{0,1\}$
disjunction	$\vee$	1	1	1	0	$x \vee y$	$=$	$x \vee y$	$\forall x,y \in \{0,1\}$
identity	1	1	1	1	1	1	$=$	$x \vee x'$	$\forall x \in \{0,1\}$

The 16 logic operations of propositional logic can all be represented using the logic operations of *disjunction* $\vee$, *conjunction* $\wedge$, and *negation* $\neg$. Using these representations, all 16 operations can be generalized to *Boolean algebras* using the equivalent Boolean algebra/lattice operations of *join*, *meet*, and *complement*.[22]

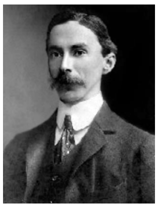
In addition to Boolean algebras, the 16 operations can also have equivalent operations on *algebra of sets* where the logic operations essentially define the set operations as in

$$\begin{aligned}
A \cup B &= \{x \in X \mid (x \in A) \vee (x \in B)\} \\
A \cap B &= \{x \in X \mid (x \in A) \wedge (x \in B)\} \\
A \setminus B &= \{x \in X \mid (x \in A) \ominus (x \in B)\} \\
A \triangle B &= \{x \in X \mid (x \in A) \oplus (x \in B)\} \\
A^c &= \{x \in X \mid \neg(x \in A)\}
\end{aligned}$$

[21] Shiva (1998) page 83
[22] Givant and Halmos (2009), page 32

Computer science also makes use of some of the 16 logic operations, where *disjunction* becomes *OR*, and *conjunction* becomes *AND*. So, there are four fields (Boolean algebra, logic, set theory, computer science) that all use essentially the same operations, but sometimes call them by different names. The following table attempts to identify to these terms across the four fields:[23]

terminology								
	Boolean algebra		logic		algebra of sets		computer science	
0000	0	**bottom**	0	*false*	$\varnothing$	**empty set**	0	**zero**
0001	$\downarrow$	**rejection**	$\downarrow$	*joint denial*	$\downarrow$	**rejection**	$\downarrow$	**nor**
0010	$\ominus$	**inhibit** x	$\ominus$	*inhibit* x	$\ominus$	**inhibit** x	$\ominus$	**inhibit** x
0011	⦶	**complement** x	⦶	*negation* x	c_x	**complement** x	⦶	**not** x
0100	$-$	**exception**	$-$	*inhibit* y	$\setminus$	**difference**	$-$	**difference**
0101	⦶	**complement** y	⦶	*negation* y	c_y	**complement** y	⦶	**not** y
0110	$\triangle$	**Boolean addition**	$\oplus$	*complete disjunction*	$\triangle$	**symmetric difference**	$\oplus$	**exclusive-or**
0111	$\mid$	**Sheffer stroke**	$\mid$	*alternate denial*	$\mid$	**Sheffer stroke**	$\mid$	**nand**
1000	$\wedge$	**meet**	$\wedge$	*conjuction*	$\cap$	**intersection**	$\wedge$	**and**
1001	$\Leftrightarrow$	**biconditional**	$\Leftrightarrow$	*equivalence*	$\Leftrightarrow$	**equivalence**	$\Leftrightarrow$	**equivalence**
1010	⊩	**projection** y	⊩	*transfer* y	⊩	**projection** y	⊩	**projection** y
1011	$\Rightarrow$	**implication**	$\Rightarrow$	*implication*	$\Rightarrow$	**implication**	$\Rightarrow$	**implication**
1100	⫣	**projection** x	⫣	*transfer* x	⫣	**projection** x	⫣	**projection** x
1101	$\div$	**adjunction**	$\Leftarrow$	*implied by*	$\div$	**adjunction**	$\div$	**adjunction**
1110	$\vee$	**join**	$\vee$	*disjunction*	$\cup$	**union**	$\vee$	**or**
1111	1	**top**	1	*true*	X	**universal set**	1	**one**



❝*I spent September in extending his [Peano's] methods to the logic of relations.... The time was one of intellectual intoxication. My sensations resembled those one has after climbing a mountain in a mist, when, on reaching the summit, the mist suddenly clears, and the country becomes visible for forty miles in every direction.... Suddenly, in the space of a few weeks, I discovered what appeared to be definitive answers to the problems which had baffled me for years. And in the course of discovering these answers, I was introducing a new mathematical technique, by which regions formerly abandoned to the vaguenesses of philosophers were conquered for the precision of exact formulae. Intellectually, the month of September 1900 was the highest point of my life. I went about saying to myself that now at last I had done something worth doing, and I had the feeling that I must be careful not to be run over in the street before I had written it down.*❞

Bertrand Russell (1872–1970), British mathematician, [24]

[23] http://groups.google.com/group/sci.math/browse_thread/thread/c1e9a7beb9a82311

[24] quote: Russell (1951), pages 217–218
image: http://en.wikipedia.org/wiki/File:Russell1907-2.jpg, public domain

Part IV

Relations

CHAPTER 15

RELATIONS ON LATTICES WITH NEGATION

The relations in this chapter are typically defined on an *orthocomplemented lattice* (Definition 11.1 page 152). Here, some relations are generalized to a *lattice with negation* (Definition 13.5 page 173). A *lattice* (Definition 5.3 page 73) with an *ortho negation* negation successfully defined on it is an *orthocomplemented lattice* (Definition 11.1 page 152). In many cases, these relations only work well on an *orthocomplemented lattice*, and thus many results are restricted to orthocomplemented lattices.

15.1 Orthogonality

Proposition 15.1. *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

PRP

$$x \leq y \implies \left\{ \begin{array}{ccccc} x^{\perp} & \vee & y & = & 1 \quad \textit{and} \\ x & \wedge & y^{\perp} & = & 0 \end{array} \right\} \quad \forall x,y \in X$$

✎PROOF:

$$\begin{aligned}
x \leq y &\implies x \vee x^{\perp} \leq y \vee x^{\perp} && \text{by \textit{monotone} property of \textit{lattices} (Proposition 5.1 page 75)}\\
&\implies 1 \leq y \vee x^{\perp} && \text{by \textit{excluded middle} property of \textit{ortho lattices} (Definition 11.1 page 152)}\\
&\implies x^{\perp} \vee y = 1 && \text{by \textit{upper bounded} property of \textit{bounded lattices} (Definition 6.1 page 89)}\\
x \leq y &\implies x \wedge y^{\perp} \leq y \wedge y^{\perp} && \text{by \textit{monotone} property of \textit{lattices} (Proposition 5.1 page 75)}\\
&\implies x \wedge y^{\perp} \leq 0 && \text{by \textit{non-contradiction} property of \textit{ortho lattices} (Definition 11.1 page 152)}\\
&\implies x \wedge y^{\perp} = 0 && \text{by \textit{lower bounded} property of \textit{bounded lattices} (Definition 6.1 page 89)}
\end{aligned}$$

⇰

Definition 15.1. [1] *Let* $(X, \vee, \wedge, \neg, 0, 1; \leq)$ *be a* LATTICE WITH NEGATION *(Definition 13.5 page 173).*

DEF

*The **orthogonality** relation* $\perp \in 2^{XX}$ *is defined as*

$$x \perp y \quad \overset{\text{def}}{\iff} \quad x \leq \neg y$$

If $x \perp y$*, we say that* x *is **orthogonal** to* y*.*

[1] Stern (1999) page 12, Loomis (1955) page 3

Lemma 15.1. *Let* $(X, \vee, \wedge, \neg, 0, 1 ; \le)$ *be a* LATTICE WITH NEGATION *(Definition 13.5 page 173).*

LEM
$$\left\{ x \perp y \quad (\text{ORTHOGONAL Definition 15.1 page 205}) \right\} \implies \left\{ y \perp x \quad (\text{SYMMETRIC}) \right\}$$

✎PROOF:

$x \perp y$	$\implies x \le \neg y$	by definition of $\perp$ (Definition 15.1 page 205)
	$\implies (\neg\neg y) \le \neg x$	by *antitone* property (Definition 11.1 page 152)
	$\implies y \le \neg x$	by *weak double negation* property of *negation* (Definition 13.2 page 172)
	$\implies y \perp x$	by definition of $\perp$ (Definition 15.1 page 205)

✏

Lemma 15.2. [2] *Let* $(X, \vee, \wedge, 0, 1 ; \le)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

LEM
$$\underbrace{x \perp y}_{\text{ORTHOGONAL (Definition 15.1 page 205)}} \implies \left\{ \begin{array}{llll} 1. & x \wedge y & = & 0 \quad \text{and} \\ 2. & x^\perp \vee y^\perp & = & 1 \end{array} \right\}$$

✎PROOF:

$x \perp y$	$\implies x \le y^\perp$	by definition of $\perp$ (Definition 15.1 page 205)
	$\implies x \wedge y \le y^\perp \wedge y$	by *monotone* property of *lattices* (Proposition 5.1 page 75)
	$\implies x \wedge y \le y \wedge y^\perp$	by *commutative* property of lattices (Theorem 5.3 page 74)
	$\implies x \wedge y \le 0$	by *non-contradiction* property of *ortho negation* (Definition 13.3 page 172)
	$\implies x \wedge y = 0$	by *lower bound* property of *bounded lattices* (Definition 6.1 page 89)

$x \perp y$	$\implies x \le y^\perp$	by definition of $\perp$ (Definition 15.1 page 205)
	$\implies x^\perp \vee x \le x^\perp \vee y^\perp$	by *monotone* property of *lattices* (Proposition 5.1 page 75)
	$\implies x \vee x^\perp \le x^\perp \vee y^\perp$	by *commutative* property of *lattices* (Theorem 5.3 page 74)
	$\implies 1 \le x^\perp \vee y^\perp$	by *excluded middle* property of *ortho lattices* (Theorem 13.5 page 176)
	$\implies x^\perp \vee y^\perp$	by *upper bound* property of *bounded lattices* (Definition 6.1 page 89)

✏

Remark 15.1. In an *orthocomplemented lattice* $\boldsymbol{L}$, the *orthogonality* relation $\perp$ is in general *non-associative*. That is
$$\left\{ \begin{array}{ccc} x & \perp & y \quad \text{and} \\ y & \perp & z \end{array} \right\} \not\Longrightarrow x \perp z$$

✎PROOF: Consider the $\boldsymbol{L}_2^4$ *Boolean lattice* in Example 11.2 (page 152).

- $a^\perp \perp p$ because $a^\perp \le p^\perp$.
- $p \perp r$ because $p \le r^\perp$.
- But yet $a^\perp$ is *not* orthogonal to r because $a^\perp \not\le r^\perp$.

✏

Example 15.1.

EX
In the O_6 *lattice* (Definition 11.2 page 152), there are a total of $\binom{6}{2} = \frac{6!}{(6-2)!2!} = \frac{6\times 5}{2} = 15$ distinct unordered (the $\perp$ relation is *symmetric* by Lemma 15.1 page 206 so the order doesn't matter) pairs of elements.

Of these 15 pairs, 8 are orthogonal to each other, and 0 is orthogonal to itself, making a total of 9 orthogonal pairs:

$x \perp y$	$x \perp 0$	$y^\perp \perp 0$
$x \perp x^\perp$	$y \perp 0$	$1 \perp 0$
$y \perp y^\perp$	$x^\perp \perp 0$	$0 \perp 0$

[2] Holland (1963), page 67

Example 15.2.

EX

In lattice 5 of Example 11.2 (page 152), there are a total of $\binom{10}{2} = \frac{10!}{(10-2)!2!} = \frac{10\times 9}{2} = 45$ distinct unordered pairs of elements.

Of these 45 pairs, 18 are orthogonal to each other, and 0 is orthogonal to itself, making a total of 19 orthogonal pairs:

$p \perp p^\perp$	$x \perp x^\perp$	$y \perp z$	$x^\perp \perp 0$
$p \perp x^\perp$	$x \perp y$	$y \perp 0$	$y^\perp \perp 0$
$p \perp y$	$x \perp z$	$z \perp z^\perp$	$z^\perp \perp 0$
$p \perp z$	$x \perp 0$	$z \perp 0$	$0 \perp 0$
$p \perp 0$	$y \perp y^\perp$	$p^\perp \perp 0$	

Example 15.3.

EX

In the $\mathbb{R}^3$ **Euclidean space** illustrated in Example 11.3 (page 153),

$$\begin{array}{llll} \boldsymbol{X} \subseteq \boldsymbol{Y}^\perp & \implies \boldsymbol{X} \perp \boldsymbol{Y} & \boldsymbol{Y} \subseteq \boldsymbol{X}^\perp & \implies \boldsymbol{Y} \perp \boldsymbol{X} \\ \boldsymbol{X} \subseteq \boldsymbol{Z}^\perp & \implies \boldsymbol{X} \perp \boldsymbol{Z} & \boldsymbol{Y} \subseteq \boldsymbol{Z}^\perp & \implies \boldsymbol{Y} \perp \boldsymbol{Z} \\ \boldsymbol{X} \wedge \boldsymbol{Y} = \boldsymbol{X} \wedge \boldsymbol{Z} = \boldsymbol{Y} \wedge \boldsymbol{Z} = \boldsymbol{0} \end{array}$$

15.2 Commutativity

The *commutes* relation is defined next. Motivation for the name "commutes" is provided by Proposition 15.4 (page 210) which shows that if x commutes with y in a lattice $\boldsymbol{L}$, then x and y commute in the *Sasaki projection* $\phi_x(y)$ on $\boldsymbol{L}$.

Definition 15.2. [3] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ *be a* LATTICE WITH NEGATION (Definition 13.5 page 173).

DEF

*The **commutes** relation* © *is defined as*

$$x © y \quad \overset{\text{def}}{\iff} \quad x = (x \wedge y) \vee (x \wedge \neg y) \qquad \forall x,y \in X,$$

*in which case we say, "x **commutes** with y in* $\boldsymbol{L}$*".*
That is, © *is a relation in* 2^{XX} *such that*

$$© \triangleq \{(x,y) \in X^2 | x = (x \wedge y) \vee (x \wedge \neg y)\}$$

Proposition 15.2. [4] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOCOMPLEMENTED LATTICE.

PRP

$$\begin{array}{llll} x©0 & \text{and} & 0©x & \forall x \in X \\ x©1 & \text{and} & 1©x & \forall x \in X \\ x©x & & & \forall x \in X \end{array} \qquad \begin{array}{llll} x©y & \iff & x©y^\perp & \forall x,y \in X \\ x \le y & \implies & x©y & \forall x,y \in X \\ x \perp y & \implies & x©y & \forall x,y \in X \end{array}$$

✎PROOF:

$$\begin{aligned} (x \wedge 0) \vee (x \wedge 0^\perp) &= 0 \vee (x \wedge 0^\perp) && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition 6.1 page 89)} \\ &= 0 \vee (x \wedge 1) && \text{by } \textit{boundary condition} \text{ of } \textit{ortho negation} \text{ (Theorem 13.5 page 176)} \\ &= 0 \vee (x) && \text{by } \textit{upper bound} \text{ property of } \textit{bounded lattices} \text{ (Definition 6.1 page 89)} \\ &= x && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition 6.1 page 89)} \\ &\implies x©0 && \text{by definition of © relation (Definition 15.2 page 207)} \\ (0 \wedge x) \vee (0 \wedge x^\perp) &= 0 \vee (0) && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition 6.1 page 89)} \\ &= 0 && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition 6.1 page 89)} \\ &\implies 0©x && \text{by definition of © relation (Definition 15.2 page 207)} \end{aligned}$$

[3] Kalmbach (1983) page 20, Holland (1970), page 79 ⟨A. Commutativity⟩, Maeda (1958), page 227 ⟨Hilfssatz (Lemma) XII.1.2⟩, Sasaki (1954) page 301 ⟨Def.5.2, cf Foulis 1962⟩, Birkhoff (1936b) page 833 ⟨"$a = (a \cap x) \cup (a \cap x')$"⟩

[4] Holland (1963), page 67

$(x \wedge 1) \vee (x \wedge 1^{\perp}) = x \vee (x \wedge 1^{\perp})$	by *lower bound* property of *bounded lattices* (Definition 6.1 page 89)
$= x \vee (x \wedge 0)$	by *boundary condition* of *ortho negation* (Theorem 13.5 page 176)
$= (x) \vee (0)$	by *lower bound* property of *bounded lattices* (Definition 6.1 page 89)
$= x$	by *lower bound* property of *bounded lattices* (Definition 6.1 page 89)
$\implies x \copyright 1$	by definition of © relation (Definition 15.2 page 207)
$(1 \wedge x) \vee (1 \wedge x^{\perp}) = (x) \vee (x^{\perp})$	by *non-contradiction* prop. of *ortho negation* (Definition 13.3 page 172)
$= 1$	by *excluded middle* property of *ortho negation* (Theorem 13.5 page 176)
$\implies 1 \copyright x$	by definition of © relation (Definition 15.2 page 207)
$(x \wedge x) \vee (x \wedge x^{\perp}) = x \vee (x \wedge x^{\perp})$	by *idempotent* property of *lattices* (Theorem 5.3 page 74)
$= x \vee (0)$	by *non-contradiction* prop. of *ortho negation* (Definition 13.3 page 172)
$= x$	by *lower bound* property of *bounded lattices* (Definition 6.1 page 89)
$\implies x \copyright x$	by definition of © relation (Definition 15.2 page 207)
$x \copyright y \implies (x \wedge y^{\perp}) \vee (x \wedge y^{\perp\perp})$	by definition of © (Definition 15.2 page 207)
$= (x \wedge y^{\perp}) \vee (x \wedge y)$	by *involution* property of $\perp$ (Definition 11.1 page 152)
$= (x \wedge y) \vee (x \wedge y^{\perp})$	by *commutative* property of *lattices* (Definition 5.3 page 73)
$= x$	by $x \copyright y$ hypothesis and Definition 15.2 page 207
$\implies x \copyright y^{\perp}$	by definition of © relation (Definition 15.2 page 207)
$x \copyright y^{\perp} \implies (x \wedge y) \vee (x \wedge y^{\perp})$	by definition of © (Definition 15.2 page 207)
$= (x \wedge y^{\perp\perp}) \vee (x \wedge y^{\perp})$	by *involution* property of $\perp$ (Definition 11.1 page 152)
$= (x \wedge y^{\perp}) \vee (x \wedge y^{\perp\perp})$	by *commutative* property of *lattices* (Definition 5.3 page 73)
$= x$	by $x \copyright y^{\perp}$ hypothesis and Definition 15.2 page 207
$\implies x \copyright y$	by definition of © relation (Definition 15.2 page 207)
$x \leq y \implies (x \wedge y) \vee (x \wedge y^{\perp})$	by definition of © (Definition 15.2 page 207)
$= x \vee (x \wedge y^{\perp})$	by $x \leq y$ hypothesis
$= x$	by *absorptive* property (Theorem 5.3 page 74)
$\implies x \copyright y$	by definition of © (Definition 15.2 page 207)
$x \perp y \implies (x \wedge y) \vee (x \wedge y^{\perp})$	by definition of © (Definition 15.2 page 207)
$= 0 \vee (x \wedge y^{\perp})$	by Lemma 15.2 page 206
$= 0 \vee x$	by $x \perp y$ hypothesis ($x \perp y \implies x \leq y^{\perp}$)
$= x \vee 0$	by *commutative* property (Theorem 5.3 page 74)
$= x$	by *identity* property of *bounded lattices*
$\implies x \copyright y$	by definition of © (Definition 15.2 page 207)

⇨

Definition 15.3. *Let © be the* COMMUTES *relation (Definition 15.2 page 207) on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \leq)$ *(Definition 13.5 page 173).*

DEF

$\boldsymbol{L}$ *is* ***symmetric*** *if*

$$x \copyright y \implies y \copyright x \qquad \forall x,y \in X$$

In general, the commutes relation is not *symmetric*. But Proposition 15.3 (next) describes some conditions under which it *is* symmetric.

Proposition 15.3. [5] *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

[5] Holland (1963) page 68, Nakamura (1957) page 158

PRP

$$\underbrace{\{x © y \implies y © x\}}_{© \text{ is SYMMETRIC at } (x,y) \ (1)} \iff \left\{x \le y \implies y = x \vee \left(x^{\perp} \wedge y\right)\right\} \quad \text{(ORTHOMODULAR IDENTITY)} \quad (2)$$
$$\iff \left\{x \le y \implies x = y \wedge \left(x \vee y^{\perp}\right)\right\} \quad (x = \phi_y(x) \ \text{(SASAKI PROJECTION)}) \quad (3)$$
$$\iff \left\{y = (x \wedge y) \vee \left[y \wedge (x \wedge y)^{\perp}\right]\right\} \quad (4)$$
$$\iff \left\{x = (x \vee y) \wedge \left[x \vee (x \vee y)^{\perp}\right]\right\} \quad (5)$$

✎PROOF:

1. Proof that (2) $\iff$ (3):

$$\begin{aligned}
x \le y &\implies y^{\perp} \le x^{\perp} && \text{by } \textit{antitone} \text{ property (Definition 11.1 page 152)}\\
&\implies x^{\perp} = y^{\perp} \vee \left(y^{\perp\perp} \wedge x^{\perp}\right) && \text{by left hypothesis}\\
&\implies \left(x^{\perp}\right)^{\perp} = \left[y^{\perp} \vee \left(y^{\perp\perp} \wedge x^{\perp}\right)\right]^{\perp} &&\\
&\implies x = \left[y^{\perp} \vee \left(y^{\perp\perp} \wedge x^{\perp}\right)\right]^{\perp} && \text{by } \textit{involutory} \text{ property (Definition 11.1 page 152)}\\
&= y^{\perp\perp} \wedge \left(y^{\perp\perp} \wedge x^{\perp}\right)^{\perp} && \text{by } \textit{de Morgan} \text{ property (Theorem 11.1 page 154)}\\
&= y \wedge \left(y \wedge x^{\perp}\right)^{\perp} && \text{by } \textit{involutory} \text{ property (Definition 11.1 page 152)}\\
&= y \wedge \left(y^{\perp} \vee x^{\perp\perp}\right) && \text{by } \textit{de Morgan} \text{ property (Theorem 11.1 page 154)}\\
&= y \wedge \left(y^{\perp} \vee x\right) && \text{by } \textit{involutory} \text{ property (Definition 11.1 page 152)}\\
&= y \wedge \left(x \vee y^{\perp}\right) && \text{by } \textit{commutative} \text{ property (Theorem 5.3 page 74)}
\end{aligned}$$

$$\begin{aligned}
x \le y &\implies y^{\perp} \le x^{\perp} && \text{by } \textit{antitone} \text{ property (Definition 11.1 page 152)}\\
&\implies y^{\perp} = x^{\perp} \wedge \left(y^{\perp} \vee x^{\perp\perp}\right) && \text{by right hypothesis}\\
&\implies \left(y^{\perp}\right)^{\perp} = \left[x^{\perp} \wedge \left(y^{\perp} \vee x^{\perp\perp}\right)\right]^{\perp} &&\\
&\implies y = \left[x^{\perp} \wedge \left(y^{\perp} \vee x^{\perp\perp}\right)\right]^{\perp} && \text{by } \textit{involutory} \text{ property (Definition 11.1 page 152)}\\
&= x^{\perp\perp} \vee \left(y^{\perp} \vee x^{\perp\perp}\right)^{\perp} && \text{by } \textit{de Morgan} \text{ property (Theorem 11.1 page 154)}\\
&= x \vee \left(y^{\perp} \vee x\right)^{\perp} && \text{by } \textit{involutory} \text{ property (Definition 11.1 page 152)}\\
&= x \vee \left(y^{\perp\perp} \wedge x^{\perp}\right) && \text{by } \textit{de Morgan} \text{ property (Theorem 11.1 page 154)}\\
&= x \vee \left(y \wedge x^{\perp}\right) && \text{by } \textit{involutory} \text{ property (Definition 11.1 page 152)}\\
&= x \vee \left(x^{\perp} \wedge y\right) && \text{by } \textit{commutative} \text{ property (Theorem 5.3 page 74)}
\end{aligned}$$

2. Proof that (2) $\iff$ (4):

$$\begin{aligned}
(xy) \vee \left[y(xy)^{\perp}\right] &= u \vee \left[yu^{\perp}\right] && \text{where } u \triangleq xy \le y\\
&= u \vee \left[u^{\perp}y\right] && \text{by } \textit{commutative} \text{ property of lattices (Theorem 5.3 page 74)}\\
&= y && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies x \vee \left(x^{\perp}y\right) &= xy \vee \left[(xy)^{\perp}y\right] && \text{by } x \le y \text{ hypothesis}\\
&= xy \vee \left[y(xy)^{\perp}\right] && \text{by } \textit{commutative} \text{ property of lattices (Theorem 5.3 page 74)}\\
&= y && \text{by right hypothesis}
\end{aligned}$$

3. Proof that (3) $\iff$ (5):

$$\begin{aligned}
(x \vee y)\left[x \vee (x \vee y)^{\perp}\right] &= u\left[x \vee u^{\perp}\right] && \text{where } x \le u \triangleq x \vee y\\
&= x && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies y\left(x \vee y^{\perp}\right) &= (x \vee y)\left[x \vee (x \vee y)^{\perp}\right] && \text{by } x \le y \text{ hypothesis}\\
&= x && \text{by right hypothesis}
\end{aligned}$$

4. Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}
x \le y &\Longrightarrow x\copyright y && \text{by Proposition 15.2 page 207}\\
&\Longrightarrow y\copyright x && \text{by } \textit{symmetry} \text{ hypothesis (left hypothesis)}\\
&\Longrightarrow y = (y \wedge x) \vee \left(y \wedge x^{\perp}\right) && \text{by definition of } \copyright \text{ (Definition 15.2 page 207)}\\
&\Longrightarrow y = x \vee \left(y \wedge x^{\perp}\right) && \text{by } x \le y \text{ hypothesis}\\
&\Longrightarrow y = x \vee \left(x^{\perp} \wedge y\right) && \text{by } \textit{commutative} \text{ property of lattices (Theorem 5.3 page 74)}
\end{aligned}$$

5. Proof that (2) $\Longrightarrow$ (4):

(a) lemma: proof that $x\copyright y \Longrightarrow x^{\perp}y = (xy)^{\perp}y$:

$$\begin{aligned}
x\copyright y \Longrightarrow x^{\perp}y &= \left(xy \vee xy^{\perp}\right)^{\perp}y && \text{by definition of } \copyright \text{ (Definition 15.2 page 207)}\\
&= (xy)^{\perp}\left(xy^{\perp}\right)^{\perp}y && \text{by } \textit{de Morgan}\text{'s law (Theorem 13.4 page 176)}\\
&= (xy)^{\perp}\left[\left(x^{\perp} \vee y^{\perp\perp}\right)y\right] && \text{by } \textit{de Morgan}\text{'s law (Theorem 13.4 page 176)}\\
&= (xy)^{\perp}\left[\left(x^{\perp} \vee y\right)y\right] && \text{by } \textit{involutory}\text{'s property (Definition 11.1 page 152)}\\
&= (xy)^{\perp}y && \text{by } \textit{absorptive} \text{ property of lattices (Theorem 5.3 page 74)}
\end{aligned}$$

(b) Completion of proof for (2) $\Longrightarrow$ (4):

$$\begin{aligned}
x\copyright y \Longrightarrow xy \vee y(xy)^{\perp} &= xy \vee (xy)^{\perp}y && \text{by } \textit{commutative} \text{ property (Theorem 5.3 page 74)}\\
&= xy \vee x^{\perp}y && \text{by } x\copyright y \text{ hypothesis and item (5a)}\\
&= (yx) \vee \left[yx^{\perp}\right] && \text{by } \textit{commutative} \text{ property (Theorem 5.3 page 74)}\\
&\Longrightarrow y\copyright x && \text{by definition of } \copyright \text{ (Definition 15.2 page 207)}
\end{aligned}$$

⇰

Theorem 15.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

THM

$$\{x\copyright c \quad \forall x \in X\} \iff \left\{\boldsymbol{L} \text{ is ISOMORPHIC to } [0:c] \times \left[0:c^{\perp}\right]\right\}$$
with isomorphism $\theta(x) \triangleq \left([0:c], \left[0:c^{\perp}\right]\right)$.

Proposition 15.4. [7] *Let* $(X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOMODULAR *lattice.*

PRP

$$x\copyright y \iff \phi_x(y) = \phi_y(x) = x \wedge y \qquad \forall x,y \in X$$

15.3 Center

An element in an *orthocomplemented lattice* (Definition 11.1 page 152) is in the *center* of the lattice if that element *commutes* (Definition 15.2 page 207) with every other element in the lattice (next definition). *All* the elements of an *orthocomplemented lattice* are in the *center* if and only if that lattice is *Boolean* (Proposition 11.2 page 159).

[6] Kalmbach (1983) page 20, MacLaren (1964)
[7] Foulis (1962) page 66, Sasaki (1954) ⟨cf Foulis 1962⟩

Definition 15.4. [8] *Let* $\copyright$ *be the* COMMUTES *relation (Definition 15.2 page 207) on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1 ; \leq)$ *(Definition 13.5 page 173).*

DEF | *The* ***center*** *of* $\boldsymbol{L}$ *is defined as*
$$\{x \in X | x \copyright y \quad \forall y \in X\}$$

Proposition 15.5. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

PRP | 0 *and* 1 *are in the* ***center*** *of* $\boldsymbol{L}$.

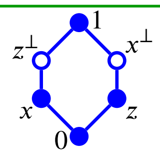
✎PROOF: This follows directly from Definition 15.2 (page 207) and Proposition 15.2 (page 207). ⇨

Theorem 15.2. [9] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition 11.1 page 152).*

THM | *The* CENTER *of* $\boldsymbol{L}$ *is* BOOLEAN *(Definition 10.1 page 127).*

Example 15.4.

EX | The **center** of the O_6 **lattice** (Definition 11.2 page 152) is the set $\{0, x, z, 1\}$. The elements $x^\perp$ and $z^\perp$ are **not** in the center of $\boldsymbol{L}$. The O_6 lattice is illustrated to the right, with the center elements as solid dots. Note that the center is the *Boolean* lattice $\boldsymbol{L}_2^2$ (Proposition 11.2 page 159).



✎PROOF:

1. Proof that 0 and 1 are in the *center* of $\boldsymbol{L}$: by Proposition 15.5 (page 211).

2. Proof that x is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(x \wedge x) \vee (x \wedge x^\perp) &= x \vee 0 &&= x &&\implies x \copyright x \\
(x \wedge z) \vee (x \wedge z^\perp) &= 0 \vee x &&= x &&\implies x \copyright z
\end{aligned}$$

$x \copyright x$, $x \copyright x^\perp$, $x \copyright z^\perp$, $x \copyright 0$, and $x \copyright 1$ by Proposition 15.2 (page 207).

3. Proof that z is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(z \wedge z) \vee (z \wedge z^\perp) &= z \vee 0 &&= z &&\implies z \copyright z \\
(z \wedge x) \vee (z \wedge x^\perp) &= 0 \vee z &&= z &&\implies z \copyright x
\end{aligned}$$

$z \copyright z$, $z \copyright x^\perp$, $z \copyright z^\perp$, $z \copyright 0$, and $z \copyright 1$ by Proposition 15.2 (page 207).

4. Proof that $x^\perp$ and $z^\perp$ are *not* in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(x^\perp \wedge y) \vee (x^\perp \wedge y^\perp) &= y \vee 0 &&= y &&\implies x^\perp \not\copyright y \\
(z^\perp \wedge x) \vee (z^\perp \wedge x^\perp) &= x \vee 0 &&= x &&\implies z^\perp \not\copyright x
\end{aligned}$$

⇨

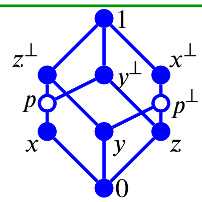
[8] Holland (1970), page 80

[9] Jeffcott (1972) page 645 ⟨§5. Main theorem⟩

Example 15.5.

EX

The **center** the lattice illustrated to the right (Example 11.2 page 152), with center elements as solid dots, is the set $\{0, 1, p, y, z, x^\perp, y^\perp, z^\perp,\}$. The elements x and $p^\perp$ are *not* in the *center* of $\boldsymbol{L}$. Note that the center is the *Boolean* lattice $\boldsymbol{L}_2^3$ (Proposition 11.2 page 159).



✎PROOF:

1. Proof that 0 and 1 are in the *center* of $\boldsymbol{L}$: by Proposition 15.5 (page 211).

2. Proof that x is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{rll}
(x \wedge p) \vee (x \wedge p^\perp) = x \vee 0 & = x & \implies x \copyright p \\
(x \wedge y) \vee (x \wedge y^\perp) = 0 \vee x & = x & \implies x \copyright y \\
(x \wedge z) \vee (x \wedge z^\perp) = 0 \vee x & = x & \implies x \copyright z
\end{array}$$

$x \copyright x$, $x \copyright x^\perp$, $x \copyright p^\perp$, $x \copyright y^\perp$, $x \copyright z^\perp$, $x \copyright 0$, and $x \copyright 1$ by Proposition 15.2 (page 207).

3. Proof that y is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{rll}
(y \wedge x) \vee (y \wedge x^\perp) = 0 \vee y & = y & \implies y \copyright x \\
(y \wedge p) \vee (y \wedge p^\perp) = 0 \vee y & = y & \implies y \copyright p \\
(y \wedge z) \vee (y \wedge z^\perp) = 0 \vee y & = y & \implies y \copyright z
\end{array}$$

$y \copyright y$, $y \copyright x^\perp$, $y \copyright p^\perp$, $y \copyright y^\perp$, $y \copyright z^\perp$, $y \copyright 0$, and $y \copyright 1$ by Proposition 15.2 (page 207).

4. Proof that z is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{rll}
(z \wedge x) \vee (z \wedge x^\perp) = 0 \vee z & = z & \implies z \copyright x \\
(z \wedge p) \vee (z \wedge p^\perp) = 0 \vee z & = z & \implies z \copyright p \\
(z \wedge y) \vee (z \wedge y^\perp) = 0 \vee z & = z & \implies z \copyright y
\end{array}$$

$z \copyright z$, $z \copyright x^\perp$, $z \copyright p^\perp$, $z \copyright y^\perp$, $z \copyright z^\perp$, $z \copyright 0$, and $z \copyright 1$ by Proposition 15.2 (page 207).

5. Proof that $x^\perp$ is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{rll}
(p^\perp \wedge x) \vee (p^\perp \wedge x^\perp) = 0 \vee p^\perp & = p^\perp & \implies p^\perp \copyright x \\
(p^\perp \wedge y) \vee (p^\perp \wedge y^\perp) = y \vee z & = p^\perp & \implies p^\perp \copyright y \\
(p^\perp \wedge z) \vee (p^\perp \wedge z^\perp) = z \vee y & = p^\perp & \implies p^\perp \copyright z
\end{array}$$

$p^\perp \copyright x^\perp$, $p^\perp \copyright p^\perp$, $p^\perp \copyright y^\perp$, $p^\perp \copyright z^\perp$, $p^\perp \copyright 0$, and $p^\perp \copyright 1$ by Proposition 15.2 (page 207).

6. Proof that $y^\perp$ is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{rll}
(y^\perp \wedge x) \vee (y^\perp \wedge x^\perp) = x \vee z & = y^\perp & \implies y^\perp \copyright x \\
(y^\perp \wedge p) \vee (y^\perp \wedge p^\perp) = p \vee z & = y^\perp & \implies y^\perp \not{\copyright} p \\
(y^\perp \wedge z) \vee (y^\perp \wedge z^\perp) = z \vee p & = y^\perp & \implies y^\perp \copyright z
\end{array}$$

$p^\perp \copyright x^\perp$, $p^\perp \copyright p^\perp$, $p^\perp \copyright y^\perp$, $p^\perp \copyright z^\perp$, $p^\perp \copyright 0$, and $p^\perp \copyright 1$ by Proposition 15.2 (page 207).

7. Proof that $z^\perp$ is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(z^\perp \wedge x) \vee (z^\perp \wedge x^\perp) = x \vee y &\quad= z^\perp &&\implies z^\perp \copyright x\\
(z^\perp \wedge p) \vee (z^\perp \wedge p^\perp) = p \vee y &\quad= z^\perp &&\implies y^\perp \not\copyright p\\
(z^\perp \wedge y) \vee (z^\perp \wedge y^\perp) = z \vee p &\quad= z^\perp &&\implies y^\perp \copyright z
\end{aligned}$$

$z^\perp \copyright x^\perp$, $z^\perp \copyright p^\perp$, $z^\perp \copyright y^\perp$, $z^\perp \copyright z^\perp$, $z^\perp \copyright 0$, and $z^\perp \copyright 1$ by Proposition 15.2 (page 207).

8. Proof that p and $x^\perp$ are *not* in the *center* of $\boldsymbol{L}$:

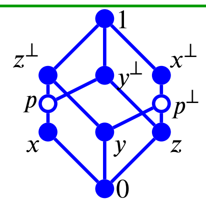
$$\begin{aligned}
(p \wedge x) \vee (p \wedge x^\perp) = x \vee 0 &\quad= x &&\implies p \not\copyright x\\
(x^\perp \wedge p) \vee (x^\perp \wedge p^\perp) = 0 \vee p^\perp &\quad= p^\perp &&\implies x^\perp \not\copyright p
\end{aligned}$$

✏

Example 15.6.

EX

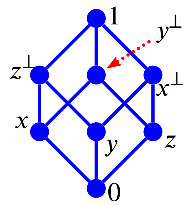
The **center** of the lattice illustrated to the right is illustrated with solid dots. Note that the center is the *Boolean* lattice $\boldsymbol{L}_2^2$ (Proposition 11.2 page 159).



Example 15.7.

EX

In a *Boolean* lattice, such as the one illustrated to the right, every element is in the center (Proposition 11.2 page 159).



CHAPTER 16

SET STRUCTURES

16.1 General set structures

Similar to the definition of a *relation* on a set X as being any subset of the *Cartesian product* $X \times X$ (Definition 17.1 page 251), a *set structure* on a set X is simply any subset of the *power set* 2^X (next) of the set X.

Definition 16.1.

DEF *The **power set** 2^X on a set X is defined as*

$$2^X \triangleq \{A | A \subseteq X\} \qquad \text{(the set of all subsets of } X\text{)}$$

Definition 16.2. [1] *Let 2^X be the* POWER SET *(Definition 16.1 page 215) of a set X.*

DEF *A set $S(X)$ is a **set structure** on X if* $S(X) \subseteq 2^X$.
A SET STRUCTURE $Q(X)$ *is a **paving** on X if* $\varnothing \in Q(X)$.

Definition 16.3. [2] *Let $Q(X)$ be a* PAVING *(Definition 16.2 page 215) on a set X. Let Y be a set containing the element* 0.

DEF *A function $\mathsf{m} \in Y^{Q(X)}$ is a **set function** if*

$$\mathsf{m}(\varnothing) = 0.$$

16.2 Operations on the power set

16.2.1 Standard operations

Definition 16.4. [3] *Let 2^X be a set. Let $|X|$ be a function in the function space $[0 : +\infty]^X$ (Definition 17.8 page 263).*

---

[1] Molchanov (2005) page 389, Pap (1995) page 7, Hahn and Rosenthal (1948) page 254

[2] Pap (1995) page 8 ⟨Definition 2.3: extended real-valued set function⟩, Halmos (1950) page 30 ⟨§7. MEASURE ON RINGS⟩, Hahn and Rosenthal (1948), Choquet (1954)

[3] Tao (2011) page 12 ⟨Example 3.6⟩, Tao (2010) page 7 ⟨Example 1.1.14⟩

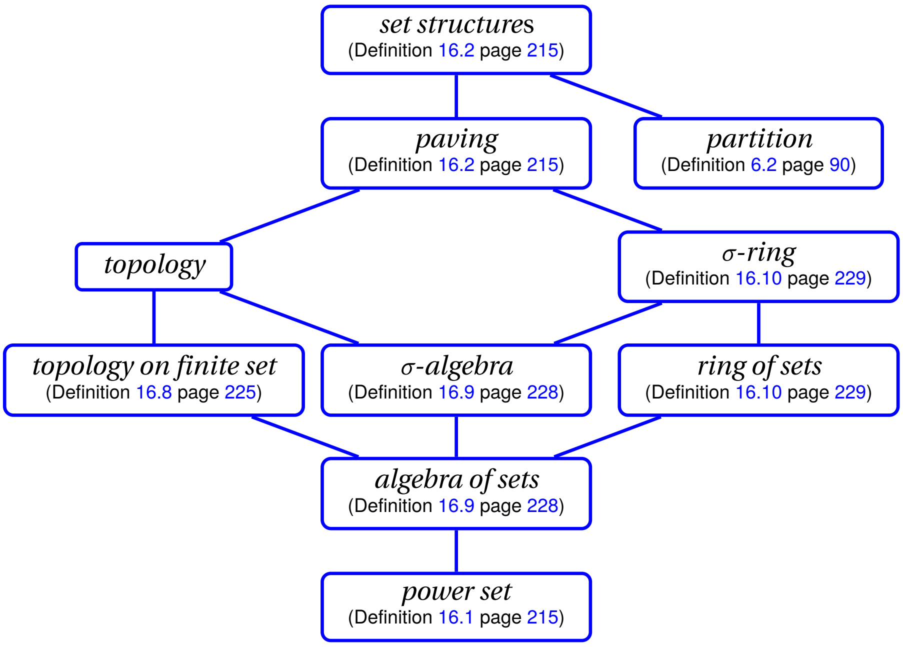


Figure 16.1: some standard set structures

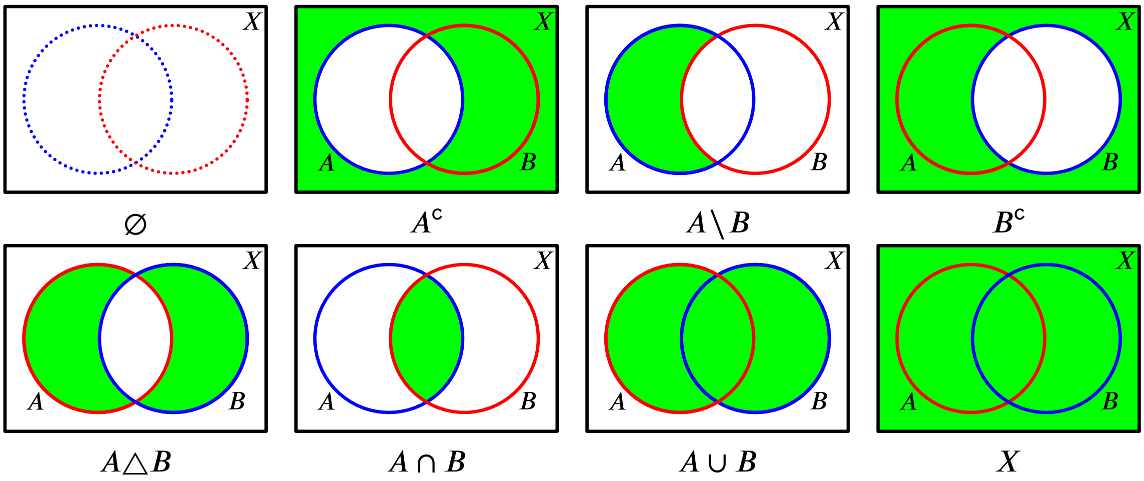


Figure 16.2: Venn diagrams for standard set operations (Definition 16.5 page 216)

DEF $|X|$ *is the* ***cardinality*** *or* ***order*** *of* X *if*

$$|X| \triangleq \begin{cases} \textit{number of elements in } X & \textit{if } X \textit{ is } \textsc{finite} \\ +\infty & \textit{otherwise} \end{cases}$$

Definition 16.5 (next) introduces seven standard set operations: two *nullary* operations, one *unary* operation, and four *binary operations* (Definition 17.9 page 264).

Definition 16.5. [4] *Let* 2^X *be the* POWER SET *(Definition 16.1 page 215) on a set* X. *Let* $\neg$ *represent the* LOGICAL NOT *operation,* $\vee$ *represent the* LOGICAL OR *operation,* $\wedge$ *represent the* LOGICAL AND *operation (Definition 14.2 page 193), and* $\oplus$ *represent the* LOGICAL EXCLUSIVE-OR *operation (Definition 14.5 page 199).*

[4] Aliprantis and Burkinshaw (1998) pages 2–4

DEF

name/symbol		arity	definition	domain
emptyset	$\varnothing$	0	$\varnothing \triangleq \{x \in X \mid x \neq x\}$	
universal set	X	0	$X \triangleq \{x \in X \mid x = x\}$	
complement	c	1	$A^c \triangleq \{x \in X \mid \neg(x \in A)\}$	$\forall A \in 2^X$
union	$\cup$	2	$A \cup B \triangleq \{x \in X \mid (x \in A) \vee (x \in B)\}$	$\forall A,B \in 2^X$
intersection	$\cap$	2	$A \cap B \triangleq \{x \in X \mid (x \in A) \wedge (x \in B)\}$	$\forall A,B \in 2^X$
difference	$\setminus$	2	$A \setminus B \triangleq \{x \in X \mid (x \in A) \wedge \neg(x \in B)\}$	$\forall A,B \in 2^X$
symmetric difference	$\triangle$	2	$A \triangle B \triangleq \{x \in X \mid (x \in A) \oplus (x \in B)\}$	$\forall A,B \in 2^X$

With regards to the standard seven set operations only, Theorem 16.1 (next) expresses each of the set operations in terms of pairs of other operations.

Theorem 16.1.

THM

$$\begin{array}{rclclclcl}
X &=& \varnothing^c \\
\varnothing &=& X^c = \left(A \cup A^c\right)^c &=& A \cap A^c &=& A \setminus A &=& A \triangle A \\
X &=& A \cup A^c &=& \left(A \cap A^c\right)^c \\
A^c &=& X \setminus A &=& X \triangle A \\
A \cup B &=& \left(A^c \cap B^c\right)^c &=& (A \triangle B) \triangle (A \cap B) &=& (A \setminus B) \triangle B \\
A \cap B &=& \left(A^c \cup B^c\right)^c &=& (A \cup B) \triangle A \triangle B &=& A \setminus (A \setminus B) \\
A \setminus B &=& \left(A^c \cup B\right)^c &=& A \cap B^c &=& (A \cup B) \triangle B &=& (A \triangle B) \cap A \\
A \triangle B &=& \left[\left(A^c \cup B\right)^c\right] \cup \left[\left(A \cup B^c\right)^c\right] & & &=& \left[\left(A^c \cap B^c\right)^c\right] \cap (A \cap B)^c \\
&=& (A \setminus B) \cup (B \setminus A)
\end{array}$$

Proposition 16.1. *Let X be a set and 2^X the power set of X. Let $\boldsymbol{R} \subseteq X$ such that $\boldsymbol{R}$ is closed with respect to the set symmetric difference operator $\triangle$.*

PRP

$(\boldsymbol{R}, \triangle)$ *is a* GROUP. *In particular,*

1. $\varnothing \triangle A = A \triangle \varnothing = A$ $\quad \forall A \in \boldsymbol{R}$ $\quad$ *($\varnothing$ is the* IDENTITY *element)*
2. $A \triangle A = \varnothing$ $\quad \forall A \in \boldsymbol{R}$ $\quad$ *(A is the* INVERSE *of A)*
3. $A \triangle (B \triangle C) = (A \triangle B) \triangle C$ $\quad \forall A,B,C \in \boldsymbol{R}$ $\quad$ *(*ASSOCIATIVE*)*

✎PROOF:

Proof that $\varnothing$ is the *identity* element:

1a. Proof that $\varnothing \in \boldsymbol{R}$:

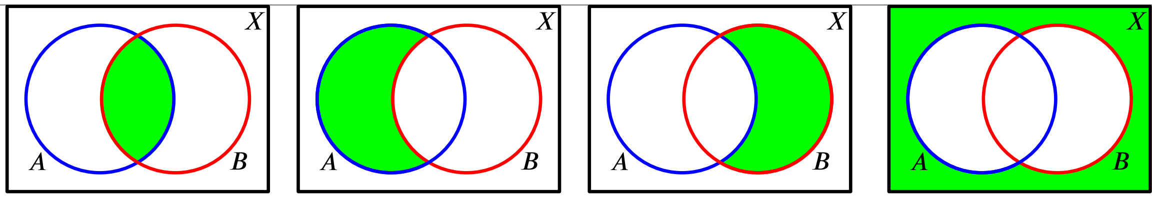
$$\begin{aligned}
\varnothing &= A \triangle A && \triangle \text{ closed with respect to } \boldsymbol{R} \\
&\in \boldsymbol{R}
\end{aligned}$$

1b. Proof that $\varnothing \triangle A = A$:

$$\begin{aligned}
\varnothing \triangle A &= \{x \in X \mid (x \in \varnothing) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 216} \\
&= \{x \in X \mid (x \in \{x \in X \mid x \neq x\}) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 216} \\
&= \{x \in X \mid 0 \oplus (x \in A)\} \\
&= \{x \in X \mid (x \in A)\} && \text{by definition of } \oplus \text{ (Definition 14.1 page 199)} \\
&= A
\end{aligned}$$

1c. Proof that $A \triangle \varnothing = A$:

$$\begin{aligned}
A \triangle \varnothing &= \{x \in X \mid (x \in A) \oplus (x \in \varnothing)\} && \text{by definition of } \triangle \text{ page 216} \\
&= \{x \in X \mid (x \in A) \oplus (x \in \{x \in X \mid x \neq x\})\} && \text{by definition of } \triangle \text{ page 216} \\
&= \{x \in X \mid (x \in A) \oplus 0\} \\
&= \{x \in X \mid (x \in A)\} && \text{by definition of } \oplus \text{ (Definition 14.1 page 199)} \\
&= A
\end{aligned}$$



$A \cap B$ (in both A and B) $A \cap B^c$ (in A but not in B) $A^c \cap B$ (not in A but in B) $A^c \cap B^c$ (neither in A nor B)

Figure 16.3: The partition of a set X into 4 regions by subsets A and B

2. Proof that $A \triangle A$:

$$
\begin{aligned}
A \triangle A &= \{x \in X | (x \in A) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 216} \\
&= \{x \in X | 0\} && \text{by definition of } \triangle \text{ page 216} \\
&= \varnothing && \text{by definition of } \triangle \text{ page 216}
\end{aligned}
$$

3. Proof that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$:

$$
\begin{aligned}
A \triangle (B \triangle C) &= \{x \in X | (x \in A) \oplus [x \in (B \triangle C)]\} && \text{by definition of } \triangle \text{ page 216} \\
&= \{x \in X | (x \in A) \oplus [(x \in B) \oplus (x \in C)]\} && \text{by definition of } \triangle \text{ page 216} \\
&= \{x \in X | [(x \in A) \oplus (x \in B)] \oplus (x \in C)\} \\
&= (A \triangle B) \triangle C
\end{aligned}
$$

✏

16.2.2 Non-standard operations

Two subsets A and B of a set X that are intersecting but yet one is not contained in the other, partition the set X into four regions, as illustrated in Figure 16.3 (page 218). Because there are four regions, the number of ways we can select one or more of them is $2^4 = 16$. Therefore, a binary operator on sets A and B can likewise result in one of $2^4 = 16$ possibilities. Definition 16.6 (page 218) presents 7 set operations. Therefore, there should be an additional $16 - 7 = 9$ operations. Definition 16.6 (next definition) attempts to define these additional operations. Some definitions are adapted from logic (Table 14.3 page 200). But in general these definitions are non-standard definitions with respect to set theory. The 16 set operations under the inclusion relation $\subseteq$ form a lattice; this lattice is illustrated by a *Hasse diagram* in Figure 16.4 (page 219).

Definition 16.6. [5] *Let 2^X be the power set on a set X. For any sets $A, B \in 2^X$, let $AB \triangleq (A \cap B)$.*

[5] standard ops: Aliprantis and Burkinshaw (1998) pages 2–4

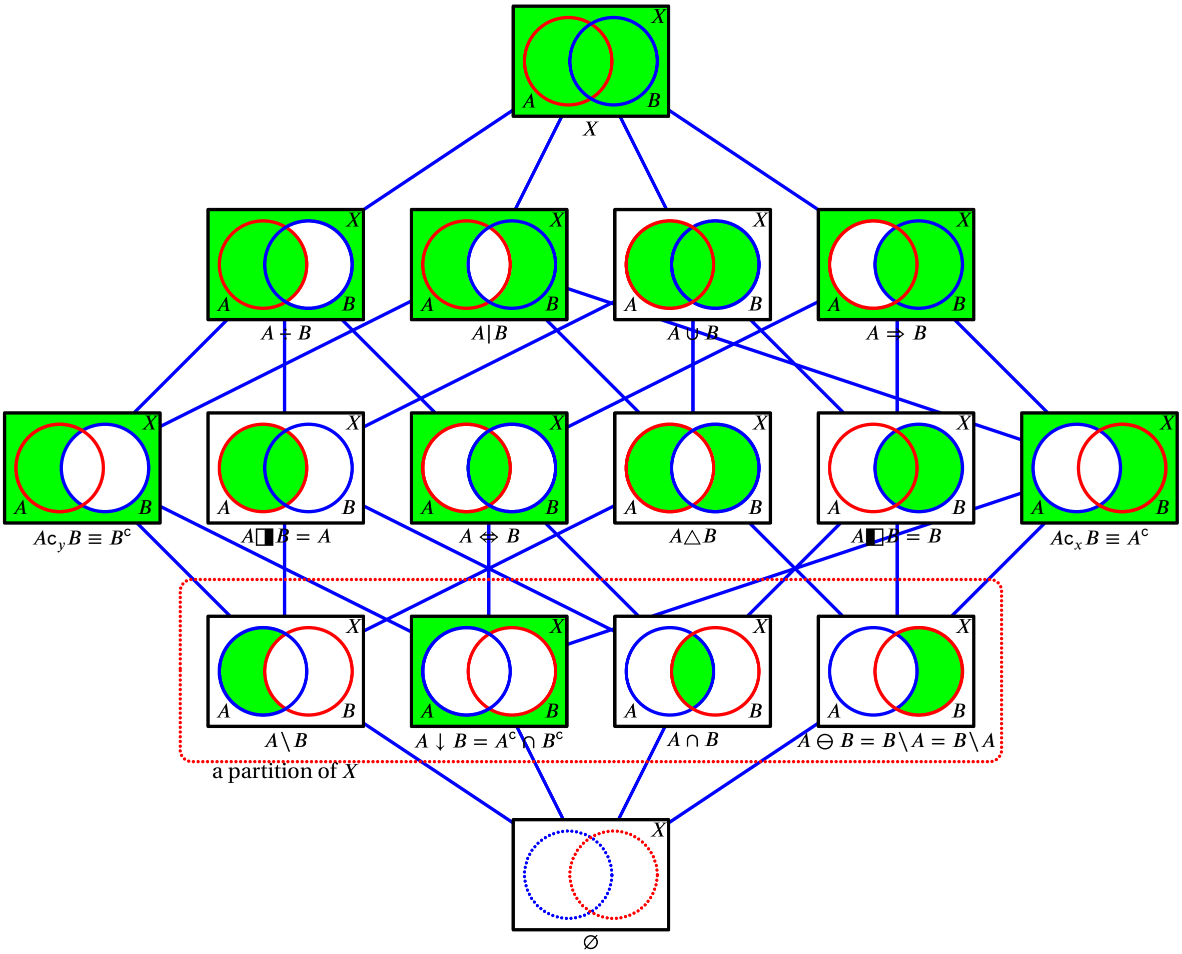


Figure 16.4: lattice of set operations

DEF

name/symbol		*arity*	*definition*	*domain*
empty set	$\oslash$	2	$A \oslash B \triangleq \varnothing$	$\forall A,B\in 2^X$
rejection	$\downarrow$	2	$A \downarrow B \triangleq A^c B^c$	$\forall A,B\in 2^X$
inhibit *x*	$\ominus$	2	$A \ominus B \triangleq A^c B$	$\forall A,B\in 2^X$
complement *x*	c_x	2	$A\, c_x\, B \triangleq A^c B \cup A^c B^c$	$\forall A,B\in 2^X$
difference	$\setminus$	2	$A \setminus B \triangleq A B^c$	$\forall A,B\in 2^X$
complement *y*	c_y	2	$A\, c_y\, B \triangleq A B^c \cup A^c B^c$	$\forall A,B\in 2^X$
symmetric difference	$\triangle$	2	$A \triangle B \triangleq A B^c \cup A^c B$	$\forall A,B\in 2^X$
Sheffer stroke	$\mid$	2	$A \mid B \triangleq A B^c \cup A^c B \cup A^c B^c$	$\forall A,B\in 2^X$
intersection	$\cap$	2	$A \cap B \triangleq AB \cup$	$\forall A,B\in 2^X$
equivalence	$\Leftrightarrow$	2	$A \Leftrightarrow B \triangleq AB \cup A^c B^c$	$\forall A,B\in 2^X$
projection *y*	$\Vdash$	2	$A \Vdash B \triangleq AB \cup A^c B$	$\forall A,B\in 2^X$
implication	$\Rightarrow$	2	$A \Rightarrow B \triangleq AB \cup A^c B \cup A^c B^c$	$\forall A,B\in 2^X$
projection *x*	$\dashv\mid$	2	$A \dashv\mid B \triangleq AB \cup AB^c$	$\forall A,B\in 2^X$
adjunction	$\div$	2	$A \div B \triangleq AB \cup AB^c \cup A^c B^c$	$\forall A,B\in 2^X$
union	$\cup$	2	$A \cup B \triangleq AB \cup AB^c \cup A^c B$	$\forall A,B\in 2^X$
universal set	$\otimes$	2	$A \otimes B \triangleq AB \cup AB^c \cup A^c B \cup A^c B^c$	$\forall A,B\in 2^X$

16.2.3 Generated operations

Definition 16.5 (page 216) defines set operations in terms of logical operations. However, it is also possible to express set operations in terms of two or more other set operations. When all the set operations can be expressed in terms of a set of operations, then that set of operations is *functionally complete* (next definition, but see also Definition 10.3 page 137).

Definition 16.7. [6] *Let S be a set structure.*

DEF

*A set of operations Φ is **functionally complete** in S if*

$\cup, \cap, c, \varnothing$, *and* X

can all be expressed in terms of elements of Φ.

Example 16.1. Here are some examples of *functionally complete* sets:

EX

- $\{\downarrow\}$ (*rejection*)
- $\{\mid\}$ (*Sheffer stroke*)
- $\{\div, \varnothing\}$ (*adjunction* and $\varnothing$)
- $\{\setminus, X\}$ (*set difference* and X)
- $\{\cup, c\}$ (*union* and *complement*)
- $\{\cap, c\}$ (*intersection* and *complement*)
- $\{\triangle, \cap, X\}$ (*symmetric difference*, *intersection*, and X)
- $\{\triangle, \cup, X\}$ (*symmetric difference*, *union*, and X)
- $\{\triangle, \setminus, c\}$ (*symmetric difference*, *set difference*, and *complement*)

The five theorems that follow demonstrate which operations can be generated by sets of generating operations:

- 2 generators, $\binom{7}{2} = 21$ possibilities, Proposition 16.2 page 221
- 3 generators, $\binom{7}{3} = 35$ possibilities, Proposition 16.3 page 221
- 4 generators, $\binom{7}{4} = 35$ possibilities, Proposition 16.4 page 222
- 5 generators, $\binom{7}{5} = 21$ possibilities, Proposition 16.5 page 223
- 6 generators, $\binom{7}{6} = 7$ possibilities, Proposition 16.6 page 223

[6] Whitesitt (1995) page 69

Starting with any two subsets A and B and using all the operations of a *functionally complete* set of operations, an *algebra of sets* (Definition 16.9 page 228) is produced. Thus, a *functionally complete* set of set operations induces an *algebra of sets*. Other less powerful sets of operations generate fewer operations and induce only a *ring of sets* (Definition 16.10 page 229). And some sets of operations, such as $\{\cup, \cap\}$, generate no set operations but themselves.

Proposition 16.2 (2 generators)**.** *The following table demonstrates the "standard" operations generated by sets of 2 operations.*

	generators		generated operations							induced set structure
1.	$\varnothing$	X	$\varnothing$	X						
2.	$\varnothing$	c	$\varnothing$	X	c					
3.	$\varnothing$	$\cup$	$\varnothing$			$\cup$				
4.	$\varnothing$	$\cap$	$\varnothing$				$\cap$			
5.	$\varnothing$	$\setminus$	$\varnothing$					$\setminus$		
6.	$\varnothing$	$\triangle$	$\varnothing$						$\triangle$	
7.	X	c	$\varnothing$	X	c					
8.	X	$\cup$		X		$\cup$				
9.	X	$\cap$		X			$\cap$			
10.	X	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
11.	X	$\triangle$	$\varnothing$	X	c				$\triangle$	
12.	c	$\cup$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
13.	c	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
14.	c	$\setminus$	$\varnothing$	X	c			$\setminus$		
15.	c	$\triangle$	$\varnothing$	X	c				$\triangle$	
16.	$\cup$	$\cap$				$\cup$	$\cap$			
17.	$\cup$	$\setminus$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
18.	$\cup$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
19.	$\cap$	$\setminus$	$\varnothing$				$\cap$	$\setminus$		
20.	$\cap$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
21.	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*

Proposition 16.3 (3 generators)**.** *The following table demonstrates the "standard" operations generated by sets of 3 operations.*

	generators			generated operations							induced set structure
1.	$\varnothing$	X	c	$\varnothing$	X	c					
2.	$\varnothing$	X	$\cup$	$\varnothing$	X		$\cup$				
3.	$\varnothing$	X	$\cap$	$\varnothing$	X			$\cap$			
4.	$\varnothing$	X	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
5.	$\varnothing$	X	$\triangle$	$\varnothing$	X	c				$\triangle$	
6.	$\varnothing$	c	$\cup$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
7.	$\varnothing$	c	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
8.	$\varnothing$	c	$\setminus$	$\varnothing$	X	c			$\setminus$		
9.	$\varnothing$	c	$\triangle$	$\varnothing$	X	c				$\triangle$	
10.	$\varnothing$	$\cup$	$\cap$	$\varnothing$			$\cup$	$\cap$			
11.	$\varnothing$	$\cup$	$\setminus$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
12.	$\varnothing$	$\cup$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
13.	$\varnothing$	$\cap$	$\setminus$	$\varnothing$				$\cap$	$\setminus$		
14.	$\varnothing$	$\cap$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
15.	$\varnothing$	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
16.	X	c	$\cup$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*

17.	X	c	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
18.	X	c	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
19.	X	c	$\triangle$	$\varnothing$	X	c				$\triangle$	
20.	X	$\cup$	$\cap$		X		$\cup$	$\cap$			
21.	X	$\cup$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
22.	X	$\cup$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
23.	X	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
24.	X	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
25.	X	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
26.	c	$\cup$	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
27.	c	$\cup$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
28.	c	$\cup$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
29.	c	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
30.	c	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
31.	c	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
32.	$\cup$	$\cap$	$\setminus$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
33.	$\cup$	$\cap$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
34.	$\cup$	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
35.	$\cap$	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*

Proposition 16.4 (4 generators). *The following table demonstrates the "standard" operations generated by sets of 4 operations.*

	generators				generated operations							induced set structure
1.	$\varnothing$	X	c	$\cup$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
2.	$\varnothing$	X	c	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
3.	$\varnothing$	X	c	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
4.	$\varnothing$	X	c	$\triangle$	$\varnothing$	X	c				$\triangle$	
5.	$\varnothing$	X	$\cup$	$\cap$	$\varnothing$	X		$\cup$	$\cap$			*pre-topology*
6.	$\varnothing$	X	$\cup$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
7.	$\varnothing$	X	$\cup$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
8.	$\varnothing$	X	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
9.	$\varnothing$	X	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
10.	$\varnothing$	X	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
11.	$\varnothing$	c	$\cup$	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
12.	$\varnothing$	c	$\cup$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
13.	$\varnothing$	c	$\cup$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
14.	$\varnothing$	c	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
15.	$\varnothing$	c	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
16.	$\varnothing$	c	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
17.	$\varnothing$	$\cup$	$\cap$	$\setminus$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
18.	$\varnothing$	$\cup$	$\cap$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
19.	$\varnothing$	$\cup$	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
20.	$\varnothing$	$\cap$	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
21.	X	c	$\cup$	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
22.	X	c	$\cup$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
23.	X	c	$\cup$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
24.	X	c	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
25.	X	c	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
26.	X	c	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
27.	X	$\cup$	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*

28.	X	$\cup$	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
29.	X	$\cup$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
30.	X	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
31.	c	$\cup$	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
32.	c	$\cup$	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
33.	c	$\cup$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
34.	c	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
35.	$\cup$	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*

Proposition 16.5 (5 generators)**.** *The following table demonstrates the "standard" operations generated by sets of 5 operations.*

	generators					generated operations							induced set structure
1.	$\varnothing$	X	c	$\cup$	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
2.	$\varnothing$	X	c	$\cup$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
3.	$\varnothing$	X	c	$\cup$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
4.	$\varnothing$	X	c	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
5.	$\varnothing$	X	c	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
6.	$\varnothing$	X	c	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
7.	$\varnothing$	X	$\cup$	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
8.	$\varnothing$	X	$\cup$	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
9.	$\varnothing$	X	$\cup$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
10.	$\varnothing$	X	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
11.	$\varnothing$	c	$\cup$	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
12.	$\varnothing$	c	$\cup$	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
13.	$\varnothing$	c	$\cup$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
14.	$\varnothing$	c	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
15.	$\varnothing$	$\cup$	$\cap$	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
16.	X	c	$\cup$	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
17.	X	c	$\cup$	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
18.	X	c	$\cup$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
19.	X	c	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
20.	X	$\cup$	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
21.	c	$\cup$	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*

Proposition 16.6 (6 genererators)**.** *The following table demonstrates the "standard" operations generated by sets of 6 operations.*

	generators						generated operations							induced set structure
1.	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
2.	$\varnothing$	X	c	$\cup$	$\cap$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
3.	$\varnothing$	X	c	$\cup$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
4.	$\varnothing$	X	c	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
5.	$\varnothing$	X	$\cup$	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
6.	$\varnothing$	c	$\cup$	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
7.	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*

16.2.4 Set multiplication

The *Cartesian product* operation $\times$ (Definition 1.7 page 6) is a kind of *set multiplication* operation. Theorem 16.2 (next theorem) demonstrates how this set operation interacts with certain other set operations. The Cartesian product is of critical importance in general because, for example, relations (Definition 17.1 page 251) and functions (Definition 17.8 page 263) are subsets of Cartesian products.

Theorem 16.2. [7] *Let X, Y, Z be sets.*

THM

$$\begin{array}{rcll} X\times(Y\cup Z) &=& (X\times Y)\cup(X\times Z) & (\times \text{ distributes over } \cup)\\ X\times(Y\cap Z) &=& (X\times Y)\cap(X\times Z) & (\times \text{ distributes over } \cap)\\ X\times(Y\setminus Z) &=& (X\times Y)\setminus(X\times Z) & (\times \text{ distributes over } \setminus)\\ (X\times Y)\cap(Y\times X) &=& (X\cap Y)\times(Y\cap X) & \\ (X\times X)\cap(Y\times Y) &=& (X\cap Y)\times(X\cap Y) & \end{array}$$

✎PROOF:

$$\begin{aligned} X\times(Y\cup Z) &= \{(a,b)\,|(a\in X)\wedge(b\in Y\cup Z)\} && \text{by Definition 1.5}\\ &= \{(a,b)\,|(a\in X)\wedge[(b\in Y)\vee(b\cup Z)]\} && \text{by Definition 16.5}\\ &= \{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\vee[(a\in X)\wedge(b\in Z)]\} && \text{by Theorem 14.2}\\ &= \underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\}}_{X\times Y}\cup\underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Z)]\}}_{X\times Z} && \text{by Definition 16.5}\\ &= (X\times Y)\cup(X\times Z) && \text{by Definition 1.5} \end{aligned}$$

$$\begin{aligned} X\times(Y\cap Z) &= \{(a,b)\,|(a\in X)\wedge(b\in Y\cap Z)\} && \text{by Definition 1.5}\\ &= \{(a,b)\,|(a\in X)\wedge[(b\in Y)\wedge(b\cup Z)]\} && \text{by Definition 16.5}\\ &= \{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\wedge[(a\in X)\wedge(b\in Z)]\} && \\ &= \underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\}}_{X\times Y}\cap\underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Z)]\}}_{X\times Z} && \text{by Definition 16.5}\\ &= (X\times Y)\cap(X\times Z) && \text{by Definition 1.5} \end{aligned}$$

$$\begin{aligned} X\times(Y\setminus Z) &= \{(a,b)\,|(a\in X)\wedge(b\in Y\setminus Z)\} && \text{by Definition 1.5}\\ &= \left\{(a,b)\,|(a\in X)\wedge(b\in Y\cap Z^{c})\right\} && \text{by Theorem 16.1}\\ &= \left\{(a,b)\,|(a\in X)\wedge\left[(b\in Y)\wedge(b\in Z^{c})\right]\right\} && \text{by Definition 16.5}\\ &= \left\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\wedge\left[(a\in X)\wedge(b\in Z^{c})\right]\right\} && \\ &= \underbrace{\{(a,b)\,|\,[(a\in X)\wedge(b\in Y)]\}}_{X\times Y}\cap\underbrace{\left\{(a,b)\,|\left[(a\in X)\wedge(b\in Z^{c})\right]\right\}}_{X\times Z^{c}} && \text{by Definition 16.5}\\ &= (X\times Y)\cap(X\times Z^{c}) && \text{by Definition 1.5}\\ &\neq (X\times Y)\setminus(X\times Z) && \end{aligned}$$

$$\begin{aligned} (X\times Y)\cap(Y\times X) &= \{(a,b)\,|(a\in X)\wedge(b\in Y)\}\cap\{(a,b)\,|(a\in Y)\wedge(b\in X)\} && \text{by Definition 1.5}\\ &= \{(a,b)\,|[(a\in X)\wedge(b\in Y)]\wedge[(a\in Y)\wedge(b\in X)]\} && \text{by Definition 16.5}\\ &= \{(a,b)\,|[(a\in X)\wedge(a\in Y)]\wedge[(b\in Y)\wedge(b\in X)]\} && \\ &= \{(a,b)\,|(a\in X\cap Y)\wedge(b\in Y\cap X)\} && \\ &= (X\cap Y)\times(Y\cap X) && \text{by Definition 1.5} \end{aligned}$$

[7] Menini and Oystaeyen (2004), page 50, Halmos (1960) page 25

$$
\begin{aligned}
(X\times X)\cap(Y\times Y) &= \{(a,b)\,|(a\in X)\wedge(b\in X)\}\cap\{(a,b)\,|(a\in Y)\wedge(b\in Y)\} && \text{by Definition 1.5}\\
&= \{(a,b)\,|[(a\in X)\wedge(b\in X)]\ \wedge\ [(a\in Y)\wedge(b\in Y)]\} && \text{by Definition 16.5}\\
&= \{(a,b)\,|[(a\in X)\wedge(a\in Y)]\ \wedge\ [(b\in X)\wedge(b\in Y)]\}\\
&= \{(a,b)\,|(a\in X\cap Y)\ \wedge\ (b\in X\cap Y)\}\\
&= (X\cap Y)\times(X\cap Y) && \text{by Definition 1.5}
\end{aligned}
$$

✏

16.3 Standard set structures

Set structures are typically designed to satisfy some special properties— such as being closed with respect to certain set operations. Examples of commonly occurring set structures include

- *power set* (Definition 16.1 page 215)
- *topologies* (Definition 16.8 page 225)
- *algebra of sets* (Definition 16.9 page 228)
- *ring of sets* (Definition 16.10 page 229)
- *partitions* (Definition 16.11 page 231)

16.3.1 Topologies

Definition 16.8. [8] *Let Γ be a set with an arbitrary (possibly uncountable) number of elements. Let 2^X be the* POWER SET *of a set X.*

DEF

*A family of sets $\mathbf{T}\subseteq 2^X$ is a **topology** on a set X if*

1. $\varnothing\in\mathbf{T}$ — ($\varnothing$ is in $\mathbf{T}$) and
2. $X\in\mathbf{T}$ — (X is in $\mathbf{T}$) and
3. $U,V\in\mathbf{T} \implies U\cap V\in\mathbf{T}$ — (the intersection of a finite number of open sets is open) and
4. $\{U_\gamma|\gamma\in\Gamma\}\subseteq\mathbf{T} \implies \bigcup_{\gamma\in\Gamma}U_\gamma\in\mathbf{T}$ — (the union of an arbitrary number of open sets is open).

*A **topological space** is the pair $(X,\mathbf{T})$. An **open set** is any member of $\mathbf{T}$.*
*A **closed set** is any set D such that D^c is* OPEN.
The set of topologies on a set X is denoted $\mathcal{T}(X)$. That is,

$$\mathcal{T}(X)\triangleq\left\{\mathbf{T}\subseteq 2^X|\mathbf{T}\text{ is a \textbf{topology}}\right\}.$$

If X is FINITE, *then $\mathbf{T}$ is a **topology on a finite set**, and (4.) can be replaced by*

$$U,V\in\mathbf{T}\implies U\cup V\in\mathbf{T}.$$

Example 16.2. [9] Let $\mathcal{T}(X)$ be the set of topologies on a set X and 2^X the *power set* (Definition 16.1 page 215) on X.

EX

$\{\varnothing,\ X\}$	is a *topology* in	$\mathcal{T}(X)$	(*indiscrete topology* or *trivial topology*)
2^X	is a *topology* in	$\mathcal{T}(X)$	(*discrete topology*)

Example 16.3. [10] There are four topologies on the set $X\triangleq\{x,y\}$:

[8] Munkres (2000) page 76, Riesz (1909), Hausdorff (1914), Tietze (1923) ⟨cited by Thron page 18⟩, Hausdorff (1937) page 258
[9] Munkres (2000), page 77, Kubrusly (2011) page 107 ⟨Example 3.J⟩, Steen and Seebach (1978) pages 42–43 ⟨II.4⟩, DiBenedetto (2002) page 18
[10] Isham (1999), page 44, Isham (1989), page 1515

EX

topologies on $\{x, y\}$	corresponding closed sets
$T_0 = \{\varnothing, X\}$	$\{\varnothing, X\}$
$T_1 = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y\}, X\}$
$T_2 = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x\}, X\}$
$T_3 = \{\varnothing, \{x\}, \{y\}, X\}$	$\{\varnothing, \{x\}, \{y\}, X\}$

The topologies (X, T_1) and (X, T_2), as well as their corresponding closed set topological spaces, are all *Serpiński spaces*.

Example 16.4. There are a total of 29 *topologies* (Definition 16.8 page 225) on the set $X \triangleq \{x, y, z\}$:

topologies on $\{x, y, z\}$	corresponding closed sets
$T_{00} = \{\varnothing, X\}$	$\{\varnothing, X\}$
$T_{01} = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y,z\}, X\}$
$T_{02} = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x,z\} X\}$
$T_{04} = \{\varnothing, \{z\}, X\}$	$\{\varnothing, \{x,y\}, X\}$
$T_{10} = \{\varnothing, \{x,y\}, X\}$	$\{\varnothing, \{z\}, X\}$
$T_{20} = \{\varnothing, \{x,z\}, X\}$	$\{\varnothing, \{y\}, X\}$
$T_{40} = \{\varnothing, \{y,z\}, X\}$	$\{\varnothing, \{x\}, X\}$
$T_{11} = \{\varnothing, \{x\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{y,z\}, X\}$
$T_{21} = \{\varnothing, \{x\}, \{x,z\}, X\}$	$\{\varnothing, \{y\} \{y,z\}, X\}$
$T_{41} = \{\varnothing, \{x\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y,z\}, X\}$
$T_{12} = \{\varnothing, \{y\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,z\} X\}$
$T_{22} = \{\varnothing, \{y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,z\}, X\}$
$T_{42} = \{\varnothing, \{y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,z\}, X\}$
$T_{14} = \{\varnothing, \{z\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,y\}, X\}$
$T_{24} = \{\varnothing, \{z\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,y\}, X\}$
$T_{44} = \{\varnothing, \{z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,y\}, X\}$
$T_{31} = \{\varnothing, \{x\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$
$T_{52} = \{\varnothing, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$
$T_{64} = \{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$
$T_{13} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{25} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,y\}, \{y,z\}, X\}$
$T_{46} = \{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,y\}, \{x,z\}, X\}$
$T_{33} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{53} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{35} = \{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$
$T_{65} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$
$T_{56} = \{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$
$T_{66} = \{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$
$T_{77} = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$

Theorem 16.3. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE.

THM

T *is a* TOPOLOGY $\Longrightarrow$ $(T, \cup, \cap; \subseteq)$ *is a* DISTRIBUTIVE LATTICE

✎PROOF:

1. By Proposition 16.15 (page 238), $(S, \subseteq)$ is an *ordered set*.
2. By Proposition 16.16 (page 239), $\cup$ is *least upper bound* operation on $(S, \subseteq)$. and $\cap$ is *greatest lower bound* operation on $(S, \subseteq)$.
3. Therefore, by Definition 5.3 (page 73), $(S, \cup, \cap; \subseteq)$ is a lattice.
4. By Theorem 5.3 (page 74), $(S, \cup, \cap; \subseteq)$ is *idempotent*, *commutative*, *associative*, and *absorptive*.
5. Proof that $(S, \cup, \cap; \subseteq)$ is *distributive*:

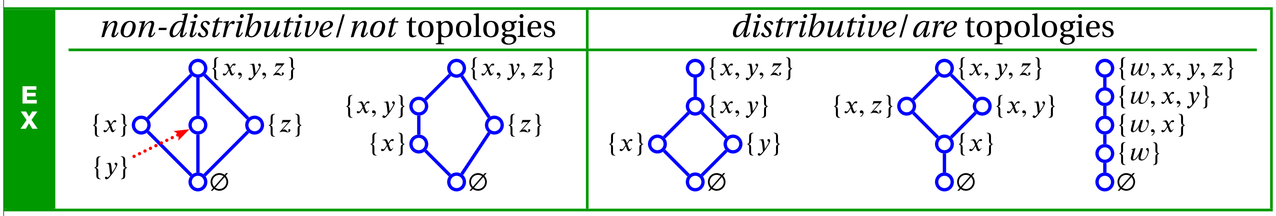
(a) Proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:

$$\begin{aligned}
&A \cap (B \cup C) \\
&= \{x \in X | x \in A \land x \in (B \cup C)\} && \text{by definition of } \cap \text{ (Definition 16.5 page 216)} \\
&= \{x \in X | x \in A \land x \in \{x \in X | x \in B \lor x \in C\}\} && \text{by definition of } \cup \text{ (Definition 16.5 page 216)} \\
&= \{x \in X | x \in A \land (x \in B \lor x \in C)\} \\
&= \{x \in X | (x \in A \land x \in B) \lor (x \in A \land x \in C)\} && \text{by Theorem 14.2 page 199} \\
&= \{x \in X | x \in A \land x \in B\} \cup \{x \in X | x \in A \land x \in C\} && \text{by definition of } \cup \text{ (Definition 16.5 page 216)} \\
&= (A \cap B) \cup (A \cap C) && \text{by definition of } \cap \text{ (Definition 16.5 page 216)}
\end{aligned}$$

(b) Proof that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:
This follows from the fact that $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is a lattice (item (3) page 226), that $\cap$ distributes over $\cup$ (item (5) page 226), and by Theorem 8.1 (page 102).

⇒

Example 16.5. There are five unlabeled lattices on a five element set (Proposition 5.2 page 79). Of these five, three are *distributive* (Proposition 8.3 page 119). The following illustrates that the distributive lattices are isomorphic to topologies, while the non-distributive lattices are not.



✎PROOF:

1. The first two lattices are non-distributive by *Birkhoff distributivity criterion* (Theorem 8.2 page 106).

   (a) This lattice is not a topology because, for example,
   $\{x\} \lor \{y\} = \{x,y,z\} \neq \{x,y\} = \{x\} \cup \{y\}$.
   That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\lor$.

   (b) This lattice is not a topology because, for example,
   $\{x\} \lor \{y\} = \{y\} \neq \{x,y\} = \{x\} \cup \{y\}$

2. The last three lattices are distributive by *Birkhoff distributivity criterion* (Theorem 8.2 page 106).

   (a) This lattice is the topology $\boldsymbol{T}_{13}$ of Example 16.4 (page 226). On the set $\{x,y,z\}$, there are a total of three topologies that have this order structure (see Example 16.4):

   $$\begin{aligned}
   \boldsymbol{T}_{13} &= \{ \emptyset, \{x\}, \{y\}, \{x,y\}, \{x,y,z\} \} \\
   \boldsymbol{T}_{25} &= \{ \emptyset, \{x\}, \{z\}, \{x,z\}, \{x,y,z\} \} \\
   \boldsymbol{T}_{46} &= \{ \emptyset, \{y\}, \{z\}, \{y,z\}, \{x,y,z\} \}
   \end{aligned}$$

   (b) This lattice is the topology $\boldsymbol{T}_{31}$ of Example 16.4 (page 226). On the set $\{x,y,z\}$, there are a total of three topologies that have this order structure (see Example 16.4):

   $$\begin{aligned}
   \boldsymbol{T}_{31} &= \{ \emptyset, \{x\}, \{x,y\}, \{x,z\}, \{x,y,z\} \} \\
   \boldsymbol{T}_{52} &= \{ \emptyset, \{y\}, \{x,y\}, \{y,z\}, \{x,y,z\} \} \\
   \boldsymbol{T}_{64} &= \{ \emptyset, \{z\}, \{x,z\}, \{y,z\}, \{x,y,z\} \}
   \end{aligned}$$

   (c) This lattice is a topology by Definition 16.8 (page 225).

⇒

16.3.2 Algebras of sets

Definition 16.9. [11] *Let X be a set with* POWER SET 2^X (Definition 16.1 page 215).

DEF

$\boldsymbol{A} \subseteq 2^X$ *is an **algebra of sets** on X if*

1. $A \in \boldsymbol{A} \implies A^c \in \boldsymbol{A}$ (closed under complement operation) and
2. $A, B \in \boldsymbol{A} \implies A \cap B \in \boldsymbol{A}$ (closed under $\cap$)

The set of all algebra of sets on a set X is denoted $\mathcal{A}(X)$ such that

$$\mathcal{A}(X) \triangleq \{\boldsymbol{A} \subseteq 2^X | \boldsymbol{A} \text{ is an algebra of sets}\}.$$

An ALGEBRA OF SETS $\boldsymbol{A}$ *on X is a **σ-algebra** on X if*

3. $\{A_n | n \in \mathbb{Z}\} \subseteq \boldsymbol{A} \implies \bigcup_{n\in\mathbb{Z}} A_n \in \boldsymbol{A}$ (closed under countable union operations).

On every set X with at least 2 elements, there are always two particular algebras of sets: the *smallest algebra* and the *largest algebra*, as demonstrated by Example 16.6 (next).

Example 16.6. [12] Let $\mathcal{A}(X)$ be the set of *algebras of sets* (Definition 16.9 page 228) on a set X and 2^X the *power set* (Definition 16.1 page 215) on X.

EX

$$\begin{array}{lcll} \{\varnothing, X\} & \in & \mathcal{A}(X) & \text{(smallest algebra)} \\ 2^X & \in & \mathcal{A}(X) & \text{(largest algebra)} \end{array}$$

Isomorphically, all *algebras of sets* are *boolean algebras* (Definition 10.1 page 127) and all boolean algebras are algebras of sets (next theorem).

Theorem 16.4 (Stone Representation Theorem). [13] *Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ be a* LATTICE.

THM

$$\boldsymbol{L} \text{ is BOOLEAN} \iff \left\{ \begin{array}{l} \boldsymbol{L} \text{ is isomorphic to } (\boldsymbol{A}, \cup, \cap, \varnothing, X; \subseteq) \\ \text{for some ALGEBRA OF SETS (Definition 16.9 page 228) } \boldsymbol{A} \end{array} \right\}$$

✎PROOF:

1. Proof that *algebra of sets* $\implies$ *Boolean algebra*:
   (a) Proof that $\boldsymbol{S}$ is closed under $\cup$ and $\cap$: by hypothesis.
   (b) By item (1b) and by Theorem 16.6 (page 235), $\boldsymbol{L}$ is a *distributive* lattice.
   (c) By item (1b) and properties of *lattices* (Theorem 5.3 page 74), $\boldsymbol{L}$ is *idempotent*, *commutative*, *associative*, and *absorptive*.
   (d) Proof that $\boldsymbol{L}$ has *identity*:

$$\begin{aligned} A \cup \varnothing &= \{x \in X | (x \in A) \vee (x \in \varnothing)\} && \text{by definition of } \cup \text{ Definition 16.5 page 216} \\ &= \{x \in X | x \in A\} && \text{by definition of } \varnothing \text{ Definition 16.5 page 216} \\ &= A \\ A \cap X &= \{x \in X | (x \in A) \wedge (x \in X)\} && \text{by definition of } \cap \text{ Definition 16.5 page 216} \\ &= \{x \in X | x \in A\} && \text{by definition of } \varnothing \text{ Definition 16.5 page 216} \\ &= A \end{aligned}$$

   (e) Proof that $\boldsymbol{L}$ is *complemented*: by hypothesis.

[11] Aliprantis and Burkinshaw (1998) page 95, Aliprantis and Burkinshaw (1998) page 151, Halmos (1950) page 21, Hausdorff (1937) page 91

[12] Stroock (1999) page 33, Aliprantis and Burkinshaw (1998) pages 95–96

[13] Levy (2002) page 257, Grätzer (2003) page 85, Joshi (1989) page 224, Saliĭ (1988) page 32 ⟨"Stone's Theorem"⟩, Stone (1936)

(f) Because $\boldsymbol{L}$ is *commutative* (item (1c) page 228), *distributive* (item (1b) page 228), has *identity* (item (1d) page 228), and is *complemented* (item (1e) page 228), and by the definition of *Boolean algebras* (Definition 10.1 page 127), $\boldsymbol{L}$ is a *Boolean algebra*.

2. Proof that *Boolean algebra* $\Longrightarrow$ *algebra of sets*: not included at this time.

⇰

16.3.3 Rings of sets

A *ring of sets* (next definition) is a family of subsets that is closed under an "addition-like" set union operator $\cup$ and "subtraction-like" set difference operator $\setminus$. Using these two operations, it is not difficult to show that a ring of sets is also closed under a "multiplication-like" set intersection operator $\cap$. Because of this, a ring of sets behaves like an *algebraic ring*. Note however that a ring of sets is not necessarily a *topology* (Definition 16.8 page 225) because it does not necessarily include X itself.

Definition 16.10. [14] *Let X be a set with* POWER SET 2^X (Definition 16.1 page 215).

DEF

*$\boldsymbol{R} \subseteq 2^X$ is a **ring of sets** on X if*

1. $A, B \in \boldsymbol{R} \implies A \cup B$ *(closed under $\cup$)* *and*
2. $A, B \in \boldsymbol{R} \implies A \setminus B \in \boldsymbol{R}$ *(closed under $\setminus$)*

The set of all rings of sets on a set X is denoted $\mathcal{R}(X)$ such that

$$\mathcal{R}(X) \triangleq \left\{\boldsymbol{R} \subseteq 2^X \,|\, \boldsymbol{R} \text{ is a ring of sets}\right\}.$$

A RING OF SETS *$\boldsymbol{R}$ on X is a **σ-ring** on X if*

3. $\left\{A_n \middle| n\in\mathbb{Z}\right\} \subseteq \boldsymbol{R} \implies \bigcup_{n\in\mathbb{Z}} A_n \in \boldsymbol{R}$ *(closed under countable union operations).*

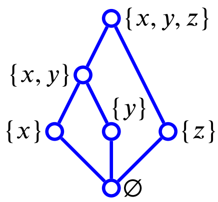
Example 16.7. Table 16.7 (page 230) lists some *rings of sets* on a finite set X.

Example 16.8. Let $X \triangleq \{x, y, z\}$ be a set and $\boldsymbol{R}$ be the family of sets

$\boldsymbol{R} \triangleq \{\varnothing,\ X,\ \{x\},\ \{y\},\ \{z\},\ \{x, y\}\}$.

Note that $(\boldsymbol{R},\ \subseteq,\ \cup,\ \cap)$ is a lattice as illustrated in the figure to the right. However, $\boldsymbol{R}$ is *not* a ring of sets on X because, for example,

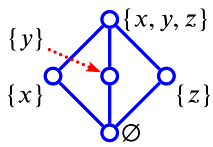
$\{x, y, z\}\setminus\{x\} = \{y, z\} \notin \boldsymbol{R}$.



Example 16.9. Let $X \triangleq \{x, y, z\}$ be a set and $\boldsymbol{R}$ be the family of sets

$\boldsymbol{R} \triangleq \{\varnothing,\ X,\ \{x\},\ \{y\},\ \{z\}\}$. Note that $(\boldsymbol{T}, \subseteq) \cup \cap$ is a lattice as illustrated in the figure to the right. However, $\boldsymbol{R}$ is *not* a ring of sets on X because, for example,

$\{x, y, z\}\setminus\{x\} = \{y, z\} \notin \boldsymbol{R}$.



Proposition 16.7. [15] *Let $\mathcal{R}(X)$ be the set of* RINGS OF SETS (Definition 16.10 page 229) *on a set X.*

PRP

$$\left\{\begin{array}{l}\boldsymbol{R}_1 \text{ and } \boldsymbol{R}_2 \\ \text{are rings of sets}\end{array}\right\} \implies \left\{\begin{array}{l}(\boldsymbol{R}_1 \cap \boldsymbol{R}_2) \\ \text{is a ring of sets}\end{array}\right\}$$

16.3.4 Partitions

The following definition is a special case of *partition* defined on lattices (Definition 6.2 page 90).

[14] Berezansky et al. (1996) page 4, Halmos (1950) page 19, Hausdorff (1937) page 90

[15] Kolmogorov and Fomin (1975) page 32, Bartle (2001) page 318

rings $\mathcal{R}(X)$ on a set X
$\mathcal{R}(\varnothing) = \left\{ \boldsymbol{R}_1 = \{\varnothing\} \right\}$
$\mathcal{R}(\{x\}) = \left\{ \begin{array}{llllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\} & \} \end{array} \right\}$
$\mathcal{R}(\{x,y\}) = \left\{ \begin{array}{llllllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\}, & & & \} \\ \boldsymbol{R}_3 & = & \{ & \varnothing, & & \{y\}, & & \} \\ \boldsymbol{R}_4 & = & \{ & \varnothing, & & & \{x,y\} & \} \\ \boldsymbol{R}_5 & = & \{ & \varnothing, & \{x\}, & \{y\}, & \{x,y\} & \} \end{array} \right\}$
$\mathcal{R}(\{x,y,z\}) = \left\{ \begin{array}{lllllllllll} \boldsymbol{R}_1 & =\{ & \varnothing, & & & & & & & & \} \\ \boldsymbol{R}_2 & =\{ & \varnothing, & \{x\}, & & & & & & & \} \\ \boldsymbol{R}_3 & =\{ & \varnothing, & & \{y\}, & & & & & & \} \\ \boldsymbol{R}_4 & =\{ & \varnothing, & & & \{z\}, & & & & & \} \\ \boldsymbol{R}_5 & =\{ & \varnothing, & & & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_6 & =\{ & \varnothing, & & & & & \{x,z\}, & & & \} \\ \boldsymbol{R}_7 & =\{ & \varnothing, & & & & & & \{y,z\}, & & \} \\ \boldsymbol{R}_8 & =\{ & \varnothing, & \{x\}, & \{y\}, & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_9 & =\{ & \varnothing, & \{x\}, & & \{z\}, & & \{x,z\}, & & & \} \\ \boldsymbol{R}_{10} & =\{ & \varnothing, & & \{y\}, & \{z\}, & & & \{y,z\}, & & \} \\ \boldsymbol{R}_{11} & =\{ & \varnothing, & & & & & & & X & \} \\ \boldsymbol{R}_{12} & =\{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X & \} \\ \boldsymbol{R}_{13} & =\{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X & \} \\ \boldsymbol{R}_{14} & =\{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X & \} \\ \boldsymbol{R}_{15} & =\{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X & \} \end{array} \right\}$

Table 16.7: some *rings of sets* on a finite set X (Example 16.7 page 229)

Definition 16.11. [16]

DEF

A SET STRUCTURE $\left\{ P_n \in 2^X \middle| n=1,2,\ldots,N \right\}$ *is a* ***partition*** *of the set* X *if*

1. $P_n \neq \emptyset \quad \forall n \in \{1,2,\ldots,N\}$ NON-EMPTY *and*
2. $P_n \cap P_m = \emptyset \quad \forall n \neq m$ MUTUALLY EXCLUSIVE *and*
3. $\bigcup_{n\in\mathbb{Z}} P_n = X$

Example 16.10. Let $A, B \subseteq X$, as illustrated in Figure 16.3 (page 218). There are a total of 15 partitions of X induced by A and B (Proposition 16.11 page 233). Here are 5 of these partitions:

EX

1. $\{X\}$ (1 region)
2. $\{A,\ A^c\}$ (2 regions)
3. $\{A \cup B,\ A^c \cap B^c\}$ (2 regions)
4. $\{A \cap B,\ A \triangle B,\ A^c \cap B^c\}$ (3 regions)
5. $\{A \cap B,\ A \cap B^c,\ A^c \cap B,\ A^c \cap B^c\}$ (4 regions) [see also Figure 16.3 page 218 and Figure 16.4 page 219]

Proposition 16.8. [17] *Let* $\mathcal{P}(X)$ *be the set of partitions on a set* X.

PRP

The relation $\trianglelefteq \in 2^{\mathbb{PP}}$ *defined as*

$$\boldsymbol{P} \trianglelefteq \boldsymbol{Q} \quad \overset{def}{\iff} \quad \forall B \in \boldsymbol{Q},\ \exists A \in \boldsymbol{P} \quad \text{such that} \quad B \subseteq A$$

is an ordering relation on $\mathcal{P}(X)$.

Example 16.11. Table 16.8 (page 232) lists some partitions $\boldsymbol{P}(X)$ on a finite set X.

16.4 Numbers of set structures

Proposition 16.9. [18]

PRP

The ***number of topologies*** t_n *on a finite set* X_n *with* n *elements is*

n	0	1	2	3	4	5	6	7	8
t_n	1	1	4	29	355	6942	209,527	9,535,241	642,779,354

n	9	10
t_n	63,260,289,423	8,977,053,873,043

Proposition 16.10. [19] *Let* t_n *be the number of topologies on a finite set with* n *elements.*

PRP

$$\lim_{n\to\infty} \frac{t_n}{2^{\frac{n^2}{4}}} = \infty \qquad \textit{(lower bound)}$$

$$\lim_{n\to\infty} \frac{t_n}{2^{\left(\frac{1}{2}+\epsilon\right)n^2}} = 0 \qquad \forall \epsilon>0 \qquad \textit{(upper bound)}$$

$$t_n > n t_{n-1} \qquad \textit{(rate of growth)}$$

Similar to the amazing relationship between e, π, i, 1, and 0 given by $e^{i\pi}+1=0$, we find another relationship between e and the number of partitions, rings of sets, and algebras of sets (Theorem 16.5 page 234).

[16] Munkres (2000), page 23, Rota (1964), page 498, Halmos (1950) page 31

[17] Roman (2008) page 111, Comtet (1974) page 220, Grätzer (2007), page 697

[18] Sloane (2014) ⟨http://oeis.org/A000798⟩, Brown and Watson (1996), page 31, Comtet (1974) page 229, Comtet (1966), Chatterji (1967), page 7, Evans et al. (1967), Krishnamurthy (1966), page 157

[19] Chatterji (1967), pages 6–7, Kleitman and Rothschild (1970)

partitions $\mathcal{P}(X)$ on a set X		
$\mathcal{P}(\varnothing)$	$=$	$\{\, \boldsymbol{P}_1 = \varnothing \,\}$
$\mathcal{P}(\{x\})$	$=$	$\{\, \boldsymbol{P}_1 = \{ \; \{x\} \; \} \,\}$
$\mathcal{P}(\{x,y\})$	$=$	$\boldsymbol{P}_1 = \{ \; \{x\}, \; \{y\}, \; \}$
		$\boldsymbol{P}_2 = \{ \; \{x,y\} \; \}$
$\mathcal{P}(\{x,y,z\})$	$=$	$\boldsymbol{P}_1 = \{ \; \{x,y,z\} \; \}$
		$\boldsymbol{P}_2 = \{ \; \{x\}, \; \{y,z\}, \; \}$
		$\boldsymbol{P}_3 = \{ \; \{y\}, \; \{x,z\}, \; \}$
		$\boldsymbol{P}_4 = \{ \; \{z\}, \; \{x,y\} \; \}$
		$\boldsymbol{P}_5 = \{ \; \{x\}, \; \{y\}, \; \{z\} \; \}$
$\mathcal{P}(\{w,x,y,z\})$	$=$	$\boldsymbol{P}_1 = \{ \; X \; \}$
		$\boldsymbol{P}_2 = \{ \; \{w\}, \; \{x,y,z\} \; \}$
		$\boldsymbol{P}_3 = \{ \; \{x\}, \; \{w,y,z\} \; \}$
		$\boldsymbol{P}_4 = \{ \; \{y\}, \; \{w,x,z\} \; \}$
		$\boldsymbol{P}_5 = \{ \; \{z\}, \; \{w,x,y\} \; \}$
		$\boldsymbol{P}_6 = \{ \; \{w,x\}, \{y,z\} \; \}$
		$\boldsymbol{P}_7 = \{ \; \{w,y\}, \{x,z\} \; \}$
		$\boldsymbol{P}_8 = \{ \; \{w,z\}, \{x,y\}, \; \}$
		$\boldsymbol{P}_9 = \{ \; \{w\}, \{x\} \; \{y,z\} \; \}$
		$\boldsymbol{P}_{10} = \{ \; \{w\}, \{y\}, \; \{x,z\} \; \}$
		$\boldsymbol{P}_{11} = \{ \; \{w\}, \{z\}, \; \{x,y\} \; \}$
		$\boldsymbol{P}_{12} = \{ \; \{x\}, \{y\}, \; \{w,z\} \; \}$
		$\boldsymbol{P}_{13} = \{ \; \{x\}, \{z\}, \; \{w,y\} \; \}$
		$\boldsymbol{P}_{14} = \{ \; \{y\}, \{z\}, \; \{w,x\} \; \}$
		$\boldsymbol{P}_{15} = \{ \; \{w\}, \{x\}, \{y\}, \{z\}, \; \}$

Table 16.8: some partitions $\boldsymbol{P}(X)$ on a finite set X (Example 16.11 page 231)

Definition 16.12. [20]

DEF

*The **Bell numbers** are the elements of the sequence* $(\!(B_n)\!)_{n\in\mathbb{W}}$ *defined as the solution to the following equation:*

$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

*The Bell numbers are also called the **exponential numbers**.*

Proposition 16.11. [21] *Let* $(\!(B_n)\!)_{n\in\mathbb{W}}$ *be the sequence of Bell numbers. Then* $(\!(B_n)\!)$ *has the following values:*

PRP

n	0	1	2	3	4	5	6	7	8	9	10	11
B_n	1	1	2	5	15	52	203	877	4140	21,147	115,975	678,570

✎PROOF: By Definition 16.12 (page 233), the sequence $(\!(B_n)\!)$ is the solution to

$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Let $\mathsf{f}^{(n)}(x)$ be the nth derivative of a function $\mathsf{f} : \mathbb{R} \to \mathbb{R}$. The Maclaurin expansion of $\mathsf{f}(x)$ is

$$\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\mathsf{f}^n(0)}{n!} x^n$$

Let $\mathsf{f}(x) \triangleq e^{e^x}$. Then

$$\begin{aligned}
\mathsf{f}^{(0)}(0) &= \mathsf{f}^{(0)}(x)\Big|_{x=0} \\
&= e^{e^0} \\
&= e \\
\mathsf{f}^{(1)}(0) &= \mathsf{f}^{(1)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\Big|_{x=0} \\
&= e^{e^x} e^x\Big|_{x=0} \\
&= e \\
\mathsf{f}^{(2)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(1)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x} e^x\Big|_{x=0} \\
&= \left(e^{e^x} e^x\right) e^x + e^{e^x} e^x\Big|_{x=0} \\
&= e^{e^x}\left(e^{2x} + e^x\right)\Big|_{x=0} \\
&= 2e \\
\mathsf{f}^{(3)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(2)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x} e^x \left(e^{2x} + e^x\right) + e^{e^x}\left(2e^{2x} + e^x\right)\Big|_{x=0}
\end{aligned}$$

[20] Comtet (1974) pages 210–211, Rota (1964), page 499, Bell (1934) page 417, d'Ocagne (1887) page 371
[21] Sloane (2014) ⟨http://oeis.org/A000110⟩

$$
\begin{aligned}
&= e^{e^x}\left(e^{3x}+3e^{2x}+e^x\right)\Big|_{x=0}\\
&= 5e\\
f^{(4)}(0) &= \frac{d}{dx}f^{(3)}(x)\Big|_{x=0}\\
&= \frac{d}{dx}e^{e^x}\left(e^{3x}+3e^{2x}+e^x\right)\Big|_{x=0}\\
&= \left(e^{e^x}e^x\right)\left(e^{3x}+3e^{2x}+e^x\right)+e^{e^x}\left(3e^{3x}+6e^{2x}+e^x\right)\Big|_{x=0}\\
&= e^{e^x}\left(e^{4x}+6e^{3x}+7e^{2x}+e^x\right)\Big|_{x=0}\\
&= 15e\\
f^{(5)}(0) &= \frac{d}{dx}f^{(4)}(x)\Big|_{x=0}\\
&= \frac{d}{dx}e^{e^x}\left(e^{4x}+6e^{3x}+7e^{2x}+e^x\right)\Big|_{x=0}\\
&= \frac{d}{dx}\left(e^{e^x}e^x\right)\left(e^{4x}+6e^{3x}+7e^{2x}+e^x\right)+e^{e^x}\left(4e^{4x}+18e^{3x}+14e^{2x}+e^x\right)\Big|_{x=0}\\
&= \frac{d}{dx}e^{e^x}\left(e^{5x}+10e^{4x}+25e^{3x}+15e^{2x}+e^x\right)\Big|_{x=0}\\
&= 52e\\
f^{(6)}(0) &= \frac{d}{dx}f^{(5)}(x)\Big|_{x=0}\\
&= \frac{d}{dx}e^{e^x}\left(e^{5x}+10e^{4x}+25e^{3x}+15e^{2x}+e^x\right)\Big|_{x=0}\\
&= \left(e^{e^x}e^x\right)\left(e^{5x}+10e^{4x}+25e^{3x}+15e^{2x}+e^x\right)+e^{e^x}\left(5e^{5x}+40e^{4x}+75e^{3x}+30e^{2x}+e^x\right)\Big|_{x=0}\\
&= e^{e^x}\left(e^{6x}+15e^{5x}+65e^{4x}+90e^{3x}+31e^{2x}+e^x\right)\Big|_{x=0}\\
&= 203e
\end{aligned}
$$

Thus, e^{e^x} has Maclaurin expansion

$$
e^{e^x} = e\left(1+x+\frac{2}{2}x^2+\frac{5}{3!}x^3+\frac{15}{4!}x^4+\frac{52}{5!}x^5+\frac{203}{6!}x^6+\cdots\right) = e\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n
$$

✎

Theorem 16.5. [22] *Let X_n be a finite set with n elements. Let $(\!(B_n)\!)_{n\in\mathbb{W}}$ be the sequence of Bell numbers.*

T H M

The number of	PARTITIONS	*on X_n is B_n.*
The number of	RINGS OF SETS	*on X_n is B_{n+1}.*
The number of	ALGEBRAS OF SETS	*on X_n is B_n.*

[22] http://groups.google.com/group/sci.math/browse_thread/thread/70a73e734b69a6ec/

16.5 Operations on set structures

Proposition 16.12.

PRP

closed under	*partition*	*ring of sets*	*algebra of sets*	*topology*
$\varnothing$		✓	✓	✓
X	✓		✓	✓
c			✓	
$\cup$		✓	✓	✓
$\cap$		✓	✓	✓
$\triangle$		✓	✓	
$\setminus$		✓	✓	

✎PROOF:

1. Proof for closure in a *topology*: Definition 16.8 (page 225)
2. Proof for closure in a *ring of sets*: Definition 16.10 (page 229) and Theorem 16.14 (page 237)
3. Proof for closure in an *algebra of sets*: Definition 16.9 (page 228) and Theorem 16.13 (page 235)

✏

Theorem 16.6. *Let T be a* SET STRUCTURE (Definition 16.2 page 215) *on a set X.*

THM

T *is a* ***topology*** $\implies$ $\forall A, B, C \in T$

property with emphasis on $\cup$	dual property with emphasis on $\cap$	property name
$A \cup A = A$	$A \cap A = A$	(IDEMPOTENT)
$A \cup B = B \cup A$	$A \cap B = B \cap A$	(COMMUTATIVE)
$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	(ASSOCIATIVE)
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	(ABSORPTIVE)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(DISTRIBUTIVE)

✎PROOF:

1. By Definition 16.8 (page 225), T is a *topology*.
2. By Theorem 16.4 (page 228), $(T, \cup, \cap; \subseteq)$ is a *distributive lattice*.
3. The properties listed are all properties of *distributive lattices*, as provided by Theorem 5.3 (page 74), Definition 8.2 (page 102), and Theorem 8.1 (page 102).

✏

Proposition 16.13. *Let $\boldsymbol{A}$ be a* SET STRUCTURE (Definition 16.2 page 215) *on a set X.*

PRP

$$\left\{ \begin{array}{l}\boldsymbol{A} \text{ is an} \\ \textbf{algebra of sets}\end{array} \right\} \implies \left\{ \begin{array}{llllll}
1. & \varnothing & \in \boldsymbol{A} & & (\boldsymbol{A} \text{ includes the } \cup \text{ identity element}) \\
2. & X & \in \boldsymbol{A} & & (\boldsymbol{A} \text{ includes the } \cap \text{ identity element}) \\
3. & A^{c} & \in \boldsymbol{A} & \forall A \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } c) \\
4. & A \cup B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \cup) \\
5. & A \cap B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \cap) \\
6. & A \setminus B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \setminus) \\
7. & A \triangle B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \triangle)
\end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\varnothing &= A \cap A^c \\
X &= c\varnothing \\
A \cup B &= c(A^c \cap B^c) && \text{by } \textit{de Morgan's Law} \text{ (Theorem 16.8 page 236)} \\
A \setminus B &= A \cap B^c \\
A \triangle B &= (A \setminus B^c) \cup (B \setminus A)
\end{aligned}$$

$(\boldsymbol{A}, \cup, \setminus)$ is a ring of sets because $\cup$ and $\setminus$ are closed in $\boldsymbol{A}$ (as shown above). ✏

Theorem 16.7. [23] *Let $\boldsymbol{A}$ be a* SET STRUCTURE *(Definition 16.2 page 215) on a set X.*

THM: *$\boldsymbol{A}$ is an **algebra of sets*** $\implies$ $\forall A, B, C \in \boldsymbol{A}$

property emphasizing $\cup$	dual property emphasizing $\cap$	property name
$A \cup A = A$	$A \cap A = A$	(IDEMPOTENT)
$A \cup B = B \cup A$	$A \cap B = B \cap A$	(COMMUTATIVE)
$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	(ASSOCIATIVE)
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	(ABSORPTIVE)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(DISTRIBUTIVE)
$A \cup \varnothing = A$	$A \cap X = A$	(IDENTITY)
$A \cup X = X$	$A \cap \varnothing = \varnothing$	(BOUNDED)
$A \cup A^c = X$	$A \cap A^c = \varnothing$	(COMPLEMENTED)
$(A^c)^c = A$		(UNIQUELY COMPLEMENTED)
$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$	(DE MORGAN)

✎PROOF:

1. By Definition 16.9 (page 228), $\boldsymbol{S}$ is an *algebra of sets.*
2. By the *Stone Representation Theorem* (Theorem 16.4 page 228), $(\boldsymbol{S}, \cup, \cap, \varnothing, X; \subseteq)$ is a *Boolean algebra.*
3. The properties listed are all properties of *Boolean algebras* (Theorem 10.2 page 132).

✏

Theorem 16.8. [24] *Let $\boldsymbol{A}$ be an* ALGEBRA OF SETS *(Definition 16.9 page 228) on a set X.*

THM: *$\boldsymbol{A}$ is an **algebra of sets*** $\implies$ $\forall A_1, A_2, \ldots, A_N, B \in \boldsymbol{A}$ *and* $\forall N \in \mathbb{N}$

property emphasizing $\cup$	dual property emphasizing $\cap$	property name
$\left(\bigcup_{n=1}^{N} A_n\right)^c = \bigcap_{n=1}^{N} A^c{}_n$	$\left(\bigcap_{n=1}^{N} A_n\right)^c = \bigcup_{n=1}^{N} A^c{}_n$	(DE MORGAN)
$\left(\bigcup_{n=1}^{N} A_n\right) \cap B = \bigcup_{n=1}^{N} (A_n \cap B)$	$\left(\bigcap_{n=1}^{N} A_n\right) \cup B = \bigcap_{n=1}^{N} (A_n \cup B)$	(DISTRIBUTIVE *with respect to* $\cup$ *and* $\cap$)
$\left(\bigcup_{n=1}^{N} A_n\right) \setminus B = \bigcup_{n=1}^{N} (A_n \setminus B)$	$\left(\bigcap_{n=1}^{N} A_n\right) \setminus B = \bigcap_{n=1}^{N} (A_n \setminus B)$	(DISTRIBUTIVE *with respect to* $\setminus$ *and* $\cap$)

✎PROOF:

1. By Theorem 16.4 (page 228), the lattice $(X, \cup, \cap; \subseteq)$ is *Boolean.*
2. The first four properties are true any Boolean system Theorem 10.4 (page 133).

[23] Dieudonné (1969) pages 3–4, Copson (1968) page 9
[24] Michel and Herget (1993) page 12, Aliprantis and Burkinshaw (1998) page 4, Vaidyanathaswamy (1960) pages 3–4

3. Proof for the remaining two:

$$\begin{aligned}
\left(\bigcap_{n=1}^{N} A_n\right)\setminus B &= \left(\bigcap_{n=1}^{N} A_n\right)\cap B^c && \text{by Theorem 16.1 page 217}\\
&= \bigcap_{n=1}^{N}(A_n \cap B^c) && \text{by previous result}\\
&= \bigcap_{n=1}^{N}(A_n \setminus B) && \text{by Theorem 16.1 page 217}
\end{aligned}$$

$$\begin{aligned}
\left(\bigcup_{n=1}^{N} A_n\right)\setminus B &= \left(\bigcup_{n=1}^{N} A_n\right)\cap B^c && \text{by Theorem 16.1 page 217}\\
&= \bigcup_{n=1}^{N}(A_n \cap B^c) && \text{by previous result}\\
&= \bigcup_{n=1}^{N}(A_n \setminus B) && \text{by Theorem 16.1 page 217}
\end{aligned}$$

⇨

Proposition 16.14. [25] *Let $\boldsymbol{R}$ be a* SET STRUCTURE (Definition 16.2 page 215) *on a set X.*

PRP

$$\left\{\begin{array}{l}\boldsymbol{R} \text{ is a}\\ \textbf{ring of sets}\\ \text{on } X\end{array}\right\} \implies \left\{\begin{array}{llllll}
1. & \varnothing & \in \boldsymbol{R} & & (\boldsymbol{R} \text{ includes the} \cup \text{ identity element}) & \text{and}\\
2. & A\cup B & \in \boldsymbol{R} & \forall A,B\in\boldsymbol{R} & (\boldsymbol{R} \text{ is closed under} \cup) & \text{and}\\
3. & A\cap B & \in \boldsymbol{R} & \forall A,B\in\boldsymbol{R} & (\boldsymbol{R} \text{ is closed under} \cap) & \text{and}\\
4. & A\setminus B & \in \boldsymbol{R} & \forall A,B\in\boldsymbol{R} & (\boldsymbol{R} \text{ is closed under} \setminus) & \text{and}\\
5. & A\triangle B & \in \boldsymbol{R} & \forall A,B\in\boldsymbol{R} & (\boldsymbol{R} \text{ is closed under} \triangle) &
\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
A\triangle B &= (A\setminus B)\cup(B\setminus A)\\
A\cap B &= (A\cup B)\setminus(A\triangle B)\\
A\setminus A &= \varnothing
\end{aligned}$$

⇨

Theorem 16.9. [26] *Let $\boldsymbol{R}$ be a* SET STRUCTURE (Definition 16.2 page 215) *on a set X.*

THM

*If $\boldsymbol{R}$ is an **ring of sets** on X, then $(\boldsymbol{R},\triangle,\cap)$ is an* ALGEBRAIC RING; *in particular,*

$$\begin{array}{llll|llll}
A\triangle\varnothing &=& A & \forall A\in\boldsymbol{R} & A\cap\varnothing &=& \varnothing & \forall A\in\boldsymbol{R}\\
A\triangle X &=& A^c & \forall A\in\boldsymbol{R} & A\cap X &=& A & \forall A\in\boldsymbol{R}\\
A\triangle\varnothing &=& A & \forall A\in\boldsymbol{R} & A\cap A &=& A & \forall A\in\boldsymbol{R}
\end{array}$$

$$A\cap(B\triangle C) = (A\cap B)\triangle(A\cap C) \qquad \forall A,B,C\in\boldsymbol{R}$$

properties emphasizing $\triangle$ | properties emphasizing $\cap$

✎PROOF:

[25] Berezansky et al. (1996) page 4, Halmos (1950) pages 19–20

[26] Vaidyanathaswamy (1960) pages 17–18, Kelley and Srinivasan (1988) page 22, Wilker (1982), page 211, Vaidyanathaswamy (1960) page 19

1. Proof that $(\boldsymbol{R}, \cup, \setminus)$ is an *algebraic ring*: by Theorem 16.9 (page 237)

2. Proof that a ring of sets is equivalent to $(\boldsymbol{R}, \cup, \setminus)$: This is proven simply by noting that $\cup$ and $\setminus$ (the two operations in a ring of sets $(\boldsymbol{R}, \cup, \setminus)$) can be expressed in terms of $\triangle$ and $\cap$ (the two operations in the algebraic ring $(\boldsymbol{R}, \triangle, \cap)$) and vice-versa. And this is demonstrated by Theorem 16.1 (page 217).

1. Proof that $(\boldsymbol{S}, \triangle)$ is a group: see Proposition 16.1 (page 217).

2. Proof that $A \cap (B \cap C) = (A \cap B) \cap C$:

$$\begin{aligned}
A \cap (B \cap C) &= \{x \in X | (x \in A) \wedge [(x \in B) \wedge (x \in C)]\} && \text{by definition of } \cap \text{ page 216}\\
&= \{x \in X | [(x \in A) \wedge (x \in B)] \wedge (x \in C)\}\\
&= (A \cap B) \cap C && \text{by definition of } \cap \text{ page 216}
\end{aligned}$$

3. Proof that $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$:

$$\begin{aligned}
A \cap (B \triangle C) &= \{x \in X | (x \in A) \wedge [(x \in B) \oplus (x \in C)]\} && \text{by definition of } \cap, \triangle \text{ page 216}\\
&= \{x \in X | [(x \in A) \wedge (x \in B)] \oplus [(x \in A) \wedge (x \in C)]\}\\
&= (A \cap B) \triangle (A \cap C) && \text{by definition of } \cap, \triangle \text{ page 216}
\end{aligned}$$

4. Proof that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$:

$$\begin{aligned}
(A \triangle B) \cap C &= \{x \in X | [(x \in A) \oplus (x \in B)] \wedge (x \in C)\} && \text{by definition of } \cap, \triangle \text{ page 216}\\
&= \{x \in X | [(x \in A) \wedge (x \in C)] \oplus [(x \in B) \wedge (x \in C)]\}\\
&= (A \cap C) \triangle (B \cap C) && \text{by definition of } \cap, \triangle \text{ page 216}
\end{aligned}$$

16.6 Lattices of set structures

16.6.1 Ordering relations

The *set inclusion* relation $\subseteq$ (Definition 16.13 page 238) is an *order relation* (Definition 4.2 page 58) on set structures, as demonstrated by Proposition 16.15 (next proposition).

Definition 16.13. *Let $\boldsymbol{S}$ be a* SET STRUCTURE *(Definition 16.2 page 215) on a set X.*

DEF
The relation $\subseteq \in 2^{\boldsymbol{SS}}$ is defined as

$$A \subseteq B \quad \text{if} \quad x \in A \implies x \in B \qquad \forall x \in X$$

Proposition 16.15 (order properties)**.** *Let $\boldsymbol{S}$ be a* SET STRUCTURE *(Definition 16.2 page 215) on a set X.*

PRP
The pair $(\boldsymbol{S}, \subseteq)$ is an ORDERED SET. *In particular,*

$$\begin{array}{lllllllll}
A \subseteq A & & & & \forall A \in \boldsymbol{S} & (\text{REFLEXIVE}) & \text{and}\\
A \subseteq B & \text{and} & B \subseteq C & \implies & A \subseteq C \quad \forall A,B,C \in \boldsymbol{S} & (\text{TRANSITIVE}) & \text{and}\\
A \subseteq B & \text{and} & B \subseteq A & \implies & A = B \quad \forall A,B \in \boldsymbol{S} & (\text{ANTI-SYMMETRIC}).
\end{array}$$

✎PROOF: By Definition 4.2 (page 58), a relation is an *order relation* if it is *reflexive*, *transitive*, and *anti-symmetric*.

1. Proof that $\subseteq$ is *reflexive* on 2^X:

$$\begin{aligned} x \in A &\implies x \in A \\ &\implies A \subseteq A \end{aligned}$$

2. Proof that $\subseteq$ is *transitive* on 2^X:

$$\begin{aligned} x \in A &\implies x \in B && \text{by first left hypothesis} \\ &\implies x \in C && \text{by second left hypothesis} \\ &\implies A \subseteq C && \text{by Definition 16.13 page 238} \end{aligned}$$

3. Proof that $\subseteq$ is *anti-symmetric* on 2^X:

$$\begin{aligned} A \subseteq B &\implies (x \in A \implies x \in B) \\ B \subseteq A &\implies (x \in B \implies x \in A) \\ A \subseteq B \text{ and } B \subseteq A &\implies (x \in A \iff x \in B) \\ &\implies A = B \end{aligned}$$

⇨

In a set structure that is *closed* under the *union* operation $\cup$ and *intersection* operation $\cap$, the *greatest lower bound* of any two elements A and B is simply $A \cap B$ and *least upper bound* is simply $A \cup B$ (Proposition 16.16 page 239). However, this may not be true for a set structure that is *not* closed under these operations (Example 16.12 page 240).

Proposition 16.16. *Let S be a* SET STRUCTURE *(Definition 16.2 page 215) on a set X.*

PRP

If** S is closed under $\cup$ and $\cap$ **then

$A \cup B$ *is the* LEAST UPPER BOUND *of A and B in $(S, \subseteq)$* $(\cup = \vee)$ *and*

$A \cap B$ *is the* GREATEST LOWER BOUND *of A and B in $(S, \subseteq)$* $(\cap = \wedge)$.

✎PROOF:

1. Proof that $A \cup B$ is the least upper bound:

$$\begin{aligned} A &= \{x \in X | x \in A\} \\ &\subseteq \{x \in X | x \in A \text{ or } x \in B\} \\ &= A \cup B && \text{by Definition 16.5 page 216} \\ B &= \{x \in X | x \in B\} \\ &\subseteq \{x \in X | x \in A \text{ or } x \in B\} \\ &= A \cup B && \text{by Definition 16.5 page 216} \\ A \subseteq C \text{ and } B \subseteq C &\implies \left\{x \in A \text{ and } y \in B \implies x, y \in C\right\} \\ &\implies \{x \in A \text{ or } x \in B \implies x \in C\} \\ &\implies \{x \in A \cup B \implies x \in C\} \\ &\implies A \cup B \subseteq C \end{aligned}$$

2. Proof that $A \cap B$ is the greatest lower bound:

$$\begin{aligned}
A \cap B &= \{x \in X | x \in A \text{ and } x \in B\} && \text{by Definition 16.5 page 216}\\
&\subsetneq \{x \in X | x \in A\}\\
&= A\\
A \cap B &= \{x \in X | x \in A \text{ and } x \in B\} && \text{by Definition 16.5 page 216}\\
&\subsetneq \{x \in X | x \in B\}\\
&= B\\
C \subseteq A \text{ and } C \subseteq B &\implies \{x \in C \implies x \in A \quad \text{and} \quad x \in C \implies x \in B\}\\
&\implies \{x \in C \implies x \in A \text{ or } x \in B\}\\
&\implies \{x \in C \implies x \in A \cap B\}\\
&\implies C \subseteq A \cap B
\end{aligned}$$

⇰

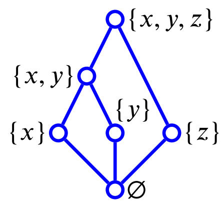
Example 16.12. The set structure

$\boldsymbol{S} \triangleq \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,y,z\}\}$

ordered by the set inclusion relation $\subseteq$ is illustrated by the Hasse diagram to the right. Note that

$\{x\} \vee \{z\} = \{x,y,z\} \neq \{x,z\} = \{x\} \cup \{z\}$.

That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.

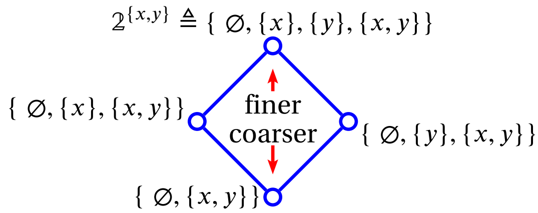


16.6.2 Lattices of topologies

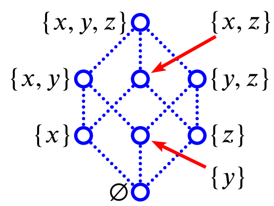
Example 16.13. [27] Example 16.3 (page 225) lists the four topologies on the set $X \triangleq \{x,y\}$. The lattice of these topologies

$\left(\{\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{T}_4\}, \cup, \cap; \subseteq\right)$

is illustrated by the *Hasse diagram* to the right.



Example 16.14. [28]Let a given topology in $\mathcal{T}(\{x,y,z\})$ be represented by a Hasse diagram as illustrated to the right, where a circle present means the indicated set is in the topology, and a circle absent means the indicated set is not in the topology. Example 16.4 (page 226) lists the 29 topologies $\mathcal{T}(\{x,y,z\})$. The lattice of these 29 topologies $(\mathcal{T}(\{x,y,z\}), \cup, \cap; \subseteq)$ is illustrated in Figure 16.5 (page 241). The five topologies $\boldsymbol{T}_1$, $\boldsymbol{T}_{41}$, $\boldsymbol{T}_{22}$, $\boldsymbol{T}_{14}$, and $\boldsymbol{T}_{77}$ are also *algebras of sets* (Definition 16.9 page 228); these five sets are shaded in Figure 16.5.



Theorem 16.10. [29] *Let $\mathcal{T}(X)$ be the **lattice of topologies** on a set X with $|X|$ elements.*

THM

$|X| \leq 2 \implies \mathcal{T}(X)$ *is* DISTRIBUTIVE

$|X| \geq 3 \implies \mathcal{T}(X)$ *is* NOT MODULAR *(and not distributive)*

---

[27] Isham (1999), page 44, Isham (1989), page 1515

[28] Isham (1999), page 44, Isham (1989), page 1516, Steiner (1966), page 386

[29] Steiner (1966), page 384

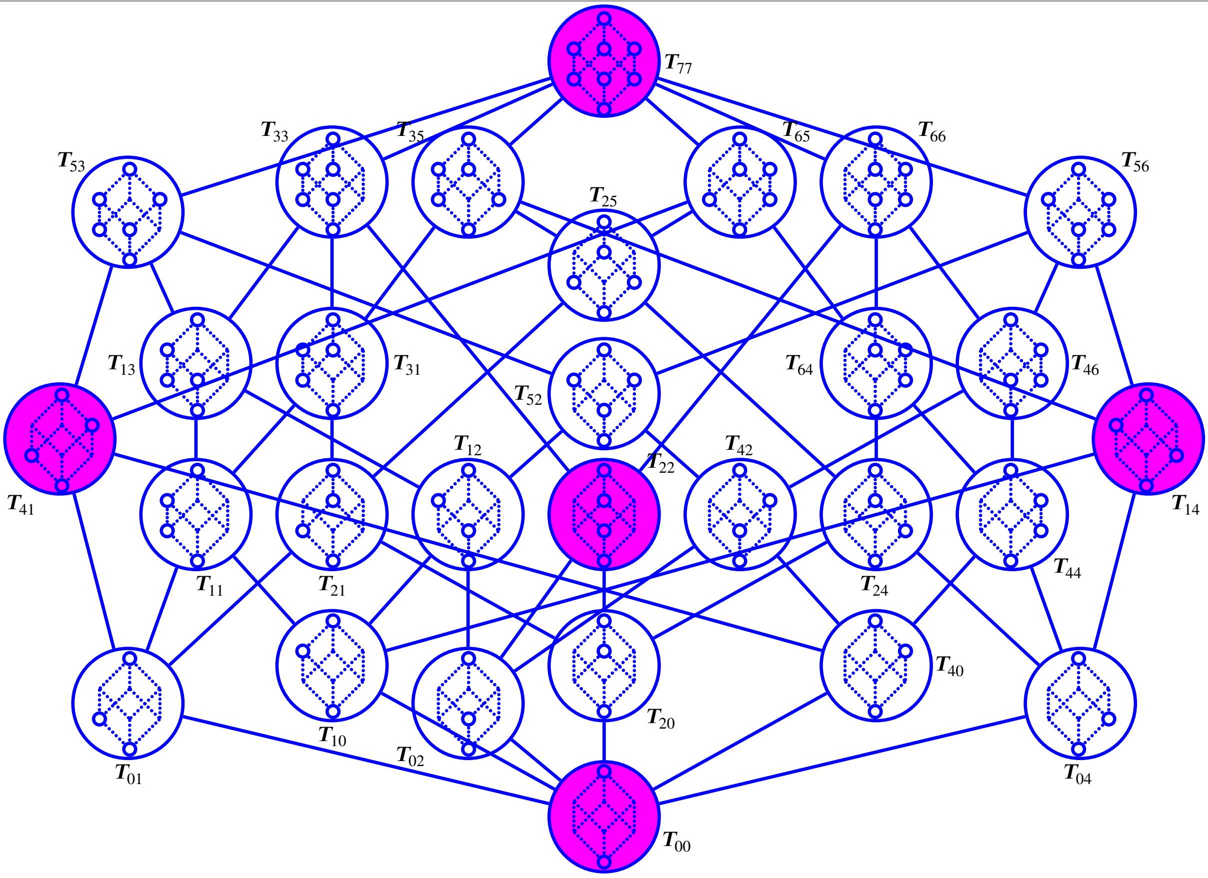


Figure 16.5: Lattice of *topologies* on $X \triangleq \{x, y, z\}$ (see Example 16.14 page 240)

Theorem 16.11. [30] *Let $\mathcal{T}(X)$ be the **lattice of topologies** on a set X.*

THM		
$\mathcal{T}(X)$ *is* SELF-DUAL	$\iff$	$\lvert X \rvert \le 3$

Theorem 16.12. [31]

THM: *Every lattice of topologies is complemented.*

Theorem 16.13. [32]

THM: *Every* TOPOLOGY (Definition 16.8 page 225) *except the* DISCRETE TOPOLOGY *and* INDISCRETE TOPOLOGY (Example 16.2 page 225) *in the **lattice of topologies** on a set X has at least $\lvert X \rvert - 1$* COMPLEMENTS.

Example 16.15. Example 16.4 (page 226) lists the 29 topologies on a set $X \triangleq \{x, y, z\}$. By Theorem 16.13 (page 241), with the exception of $\boldsymbol{T}_{00}$ (the indiscrete topology) and $\boldsymbol{T}_{77}$ (the discrete topology), each of those topologies has exactly $\lvert X \rvert - 1 = 3 - 1 = 2$ complements. Table 16.9 (page 242) lists the 29 topologies on $\{x, y, z\}$ along with their respective complements.

Theorem 16.14. [33]

THM		
$\mathcal{T}(X)$ *is a topology of sets*	$\implies$	$\begin{cases} \mathcal{T}(X) \text{ is atomic.} \\ \mathcal{T}(X) \text{ is anti-atomic.} \end{cases}$

[30] Steiner (1966), page 385
[31] van Rooij (1968), Steiner (1966), page 397, Gaifman (1961), Hartmanis (1958)
[32] Hartmanis (1958), Schnare (1968), page 56, Watson (1994), Brown and Watson (1996), page 32
[33] Larson and Andima (1975), page 179, Frölich (1964), Vaidyanathaswamy (1960), Vaidyanathaswamy (1947)

topologies on $\{x, y, z\}$	1st complement	2nd compl.
$T_{00} = \{\varnothing, X\}$	T_{77}	
$T_{01} = \{\varnothing, \{x\}, X\}$	T_{56}	T_{66}
$T_{02} = \{\varnothing, \{y\}, X\}$	T_{65}	T_{35}
$T_{04} = \{\varnothing, \{z\}, X\}$	T_{53}	T_{33}
$T_{10} = \{\varnothing, \{x,y\}, X\}$	T_{65}	T_{66}
$T_{20} = \{\varnothing, \{x,z\}, X\}$	T_{53}	T_{56}
$T_{40} = \{\varnothing, \{y,z\}, X\}$	T_{33}	T_{35}
$T_{11} = \{\varnothing, \{x\}, \{x,y\}, X\}$	T_{64}	T_{46}
$T_{21} = \{\varnothing, \{x\}, \{x,z\}, X\}$	T_{52}	T_{46}
$T_{41} = \{\varnothing, \{x\}, \{y,z\}, X\}$	T_{22}	T_{14}
$T_{12} = \{\varnothing, \{y\}, \{x,y\}, X\}$	T_{64}	T_{25}
$T_{22} = \{\varnothing, \{y\}, \{x,z\}, X\}$	T_{41}	T_{14}
$T_{42} = \{\varnothing, \{y\}, \{y,z\}, X\}$	T_{31}	T_{25}
$T_{14} = \{\varnothing, \{z\}, \{x,y\}, X\}$	T_{41}	T_{22}
$T_{24} = \{\varnothing, \{z\}, \{x,z\}, X\}$	T_{52}	T_{13}
$T_{44} = \{\varnothing, \{z\}, \{y,z\}, X\}$	T_{31}	T_{13}
$T_{31} = \{\varnothing, \{x\}, \{x,y\}, \{x,z\}, X\}$	T_{42}	T_{44}
$T_{52} = \{\varnothing, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{21}	T_{24}
$T_{64} = \{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{11}	T_{12}
$T_{13} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$	T_{24}	T_{44}
$T_{25} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$	T_{12}	T_{42}
$T_{46} = \{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$	T_{11}	T_{21}
$T_{33} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{04}	T_{40}
$T_{53} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	T_{04}	T_{20}
$T_{35} = \{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	T_{02}	T_{40}
$T_{65} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{02}	T_{10}
$T_{56} = \{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	T_{01}	T_{20}
$T_{66} = \{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{01}	T_{10}
$T_{77} = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	T_{00}	

Table 16.9: the 29 topologies on a set $\{x, y, z\}$ along with their respective complements (Example 16.15 page 241)

Theorem 16.15. [34] *Let $\mathcal{T}(X)$ be the lattice of topologies on a set X and let $n \triangleq |X|$.*

THM

$$\begin{array}{lll}
\mathcal{T}(X) \text{ contains } 2^n - 2 \text{ atoms} & \text{for finite } X. \\
\mathcal{T}(X) \text{ contains } 2^{|X|} \text{ atoms} & \text{for infinite } X. \\
\mathcal{T}(X) \text{ contains } n(n-1) \text{ anti-atoms} & \text{for finite } X. \\
\mathcal{T}(X) \text{ contains } 2^{2^{|X|}} \text{ anti-atoms} & \text{for infinite } X.
\end{array}$$

16.6.3 Lattices of algebra of sets

Example 16.16. The following table lists some algebras of sets on a finite set X. Lattices of algebras of sets are illustrated in Figure 16.8 (page 245) and Figure 16.6 (page 244).

algebra of sets $\mathcal{A}(X)$ on a set X

$$\mathcal{A}(\varnothing) = \left\{ \mathbf{A}_1 = \{\varnothing\} \right\}$$

$$\mathcal{A}(\{x\}) = \left\{ \mathbf{A}_1 = \{\varnothing, \{x\}\} \right\}$$

$$\mathcal{A}(\{x,y\}) = \left\{ \begin{array}{lcl}
\mathbf{A}_1 &=& \{\varnothing, X\} \\
\mathbf{A}_2 &=& \{\varnothing, \{x\}, \{y\}, X\}
\end{array} \right\}$$

$$\mathcal{A}(\{x,y,z\}) = \left\{ \begin{array}{lcllllllll}
\mathbf{A}_1 &=& \{ & \varnothing, & & & & & & X \;\} \\
\mathbf{A}_2 &=& \{ & \varnothing, & \{x\}, & & & & \{y,z\}, & X \;\} \\
\mathbf{A}_3 &=& \{ & \varnothing, & & \{y\}, & & \{x,z\}, & & X \;\} \\
\mathbf{A}_4 &=& \{ & \varnothing, & & & \{z\},\; \{x,y\}, & & & X \;\} \\
\mathbf{A}_5 &=& \{ & \varnothing, & \{x\}, & \{y\}, & \{z\},\; \{x,y\}, & \{x,z\}, & \{y,z\}, & X \;\}
\end{array} \right\}$$

$$\mathcal{A}(\{w,x,y,z\}) =$$

$$\left\{ \begin{array}{lcllllll}
\mathbf{A}_1 &=& \{ & \varnothing, & & & & X \;\} \\
\mathbf{A}_2 &=& \{ & \varnothing, & \{w\}, & & \{x,y,z\}, & X \;\} \\
\mathbf{A}_3 &=& \{ & \varnothing, & \{x\}, & & \{w,y,z\}, & X \;\} \\
\mathbf{A}_4 &=& \{ & \varnothing, & \{y\}, & & \{w,x,z\}, & X \;\} \\
\mathbf{A}_5 &=& \{ & \varnothing, & \{z\}, & & \{w,x,y\}, & X \;\} \\
\mathbf{A}_6 &=& \{ & \varnothing, & & \{w,x\}, \{y,z\}, & & X \;\} \\
\mathbf{A}_7 &=& \{ & \varnothing, & & \{w,y\}, \{x,z\}, & & X \;\} \\
\mathbf{A}_8 &=& \{ & \varnothing, & & \{w,z\}, \{x,y\}, & & X \;\} \\
\mathbf{A}_9 &=& \{ & \varnothing, & \{w\}, \{x\}, & \{w,x\}, \{y,z\}, & \{w,y,z\}, \{x,y,z\}, & X \;\} \\
\mathbf{A}_{10} &=& \{ & \varnothing, & \{w\}, \{y\}, & \{w,y\}, \{x,z\}, & \{w,x,z\}, \{x,y,z\}, & X \;\} \\
\mathbf{A}_{11} &=& \{ & \varnothing, & \{w\}, \{z\}, & \{w,z\}, \{x,y\}, & \{w,x,y\}, \{x,y,z\}, & X \;\} \\
\mathbf{A}_{12} &=& \{ & \varnothing, & \{x\}, \{y\}, & \{w,z\}, \{x,y\}, & \{w,x,z\}, \{w,y,z\}, & X \;\} \\
\mathbf{A}_{13} &=& \{ & \varnothing, & \{x\}, \{z\}, & \{w,y\}, \{x,z\}, & \{w,x,y\}, \{w,y,z\}, & X \;\} \\
\mathbf{A}_{14} &=& \{ & \varnothing, & \{y\}, \{z\}, & \{w,x\}, \{y,z\}, & \{w,x,y\}, \{w,x,z\}, & X \;\} \\
\mathbf{A}_{15} &=& 2^X & & & & &
\end{array} \right\}$$

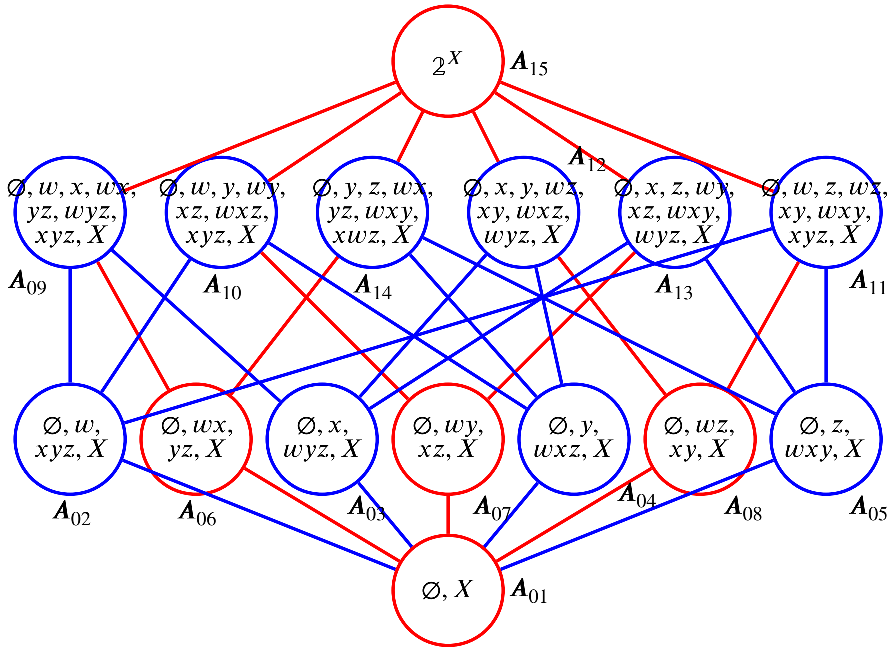


Figure 16.6: lattice of *algebras of sets* on $\{w, x, y, z\}$ (Example 16.16 page 243)

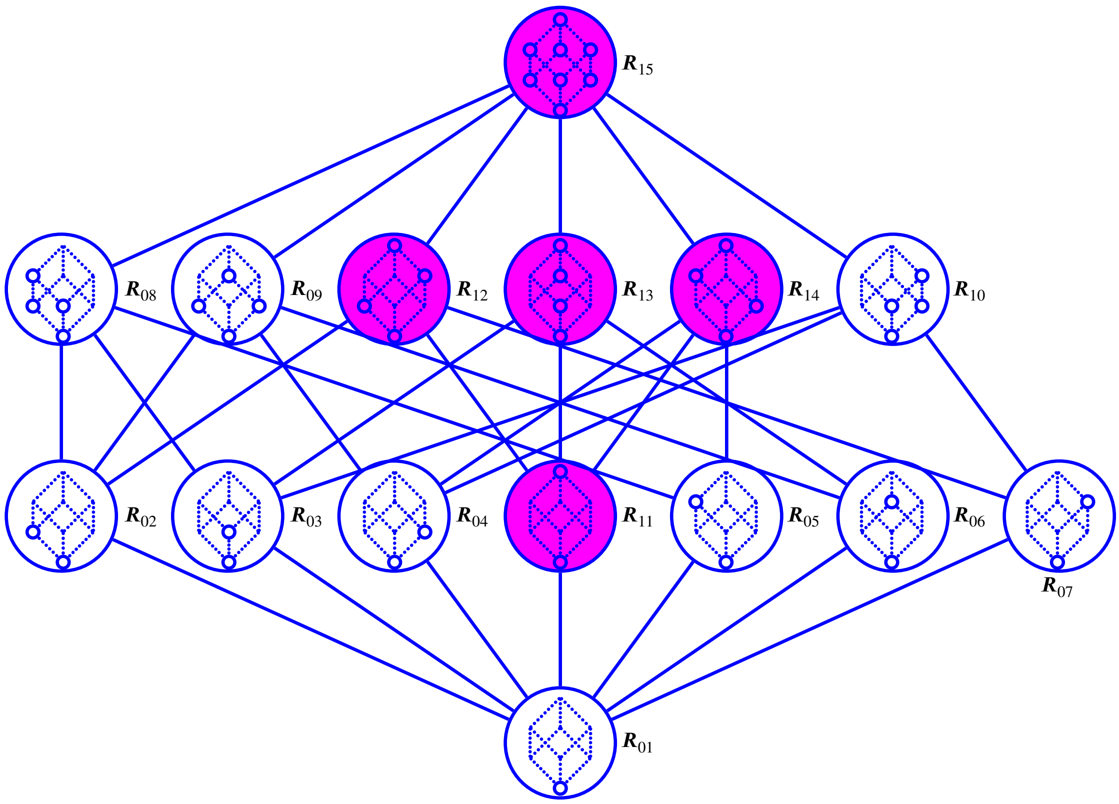


Figure 16.7: Lattice of rings of sets on $X \triangleq \{x, y, z\}$ (Example 16.17 page 245)

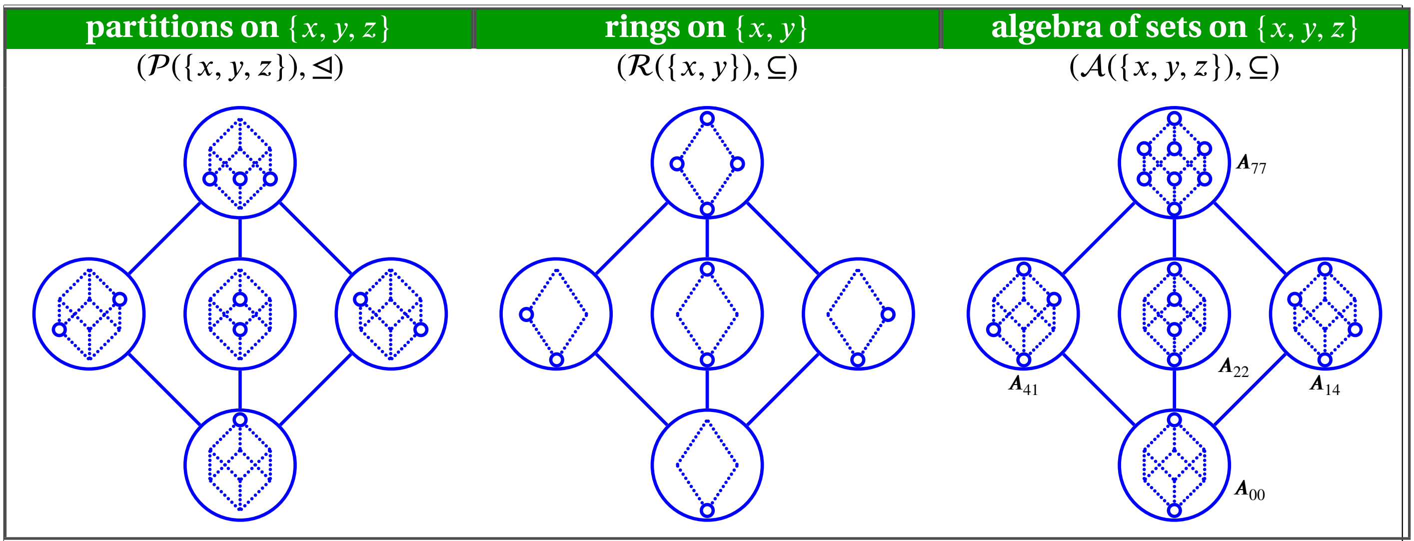


Figure 16.8: Lattices of set structures (see Example 16.18 (page 245), Example 16.7 (page 229), and Example 16.16 (page 243))

16.6.4 Lattices of rings of sets

Example 16.17. There are a total of **15** rings of sets on the set $X \triangleq \{x, y, z\}$. These rings of sets are listed in Example 16.7 (page 229) and illustrated in Figure 16.7 (page 244). The five rings containing X ($\boldsymbol{R}_{11}$–$\boldsymbol{R}_{15}$) are also *algebras of sets* (Proposition 16.18 page 247), and thus also *Boolean algebras* (Theorem 16.4 page 228). The five algebras of sets are shaded Figure 16.7.

16.6.5 Lattices of partitions of sets

Example 16.18. There are a total of **5** partitions of sets on the set $X \triangleq \{x, y, z\}$. These sets are listed in Example 16.11 (page 231) and illustrated in Figure 16.8 (page 245).

Example 16.19. There are a total of **15** partitions of sets on the set $X \triangleq \{w, x, y, z\}$. These sets are listed in Example 16.11 (page 231) and illustrated in Figure 16.9 (page 246).

In 1946, Philip Whitman proposed an amazing conjecture—that all finite lattices are isomorphic to a lattice of partitions. A proof for this was published some 30 years later by Pavel Pudlák and Jiří Tůma (next theorem).

Theorem 16.16. [35] *Let* $\boldsymbol{L}$ *be a lattice.*

THM			
THM	$\boldsymbol{L}$ *is* FINITE	$\implies$	$\boldsymbol{L}$ *is isomorphic to a* LATTICE OF PARTITIONS

Example 16.20. There are five unlabeled lattices on a five element set as stated in Proposition 5.2 (page 79) and illustrated in Example 5.11 (page 80). All of these lattices are isomorphic to a lattice of partitions (Theorem 16.16 page 245), as illustrated next.

[34] Larson and Andima (1975), page 179, Frölich (1964)

[35] Pudlák and Tůma (1980) ⟨improved proof⟩, Pudlák and Tůma (1977) ⟨proof⟩, Whitman (1946) ⟨conjecture⟩, Salii (1988) page vii ⟨list of lattice theory breakthroughs⟩

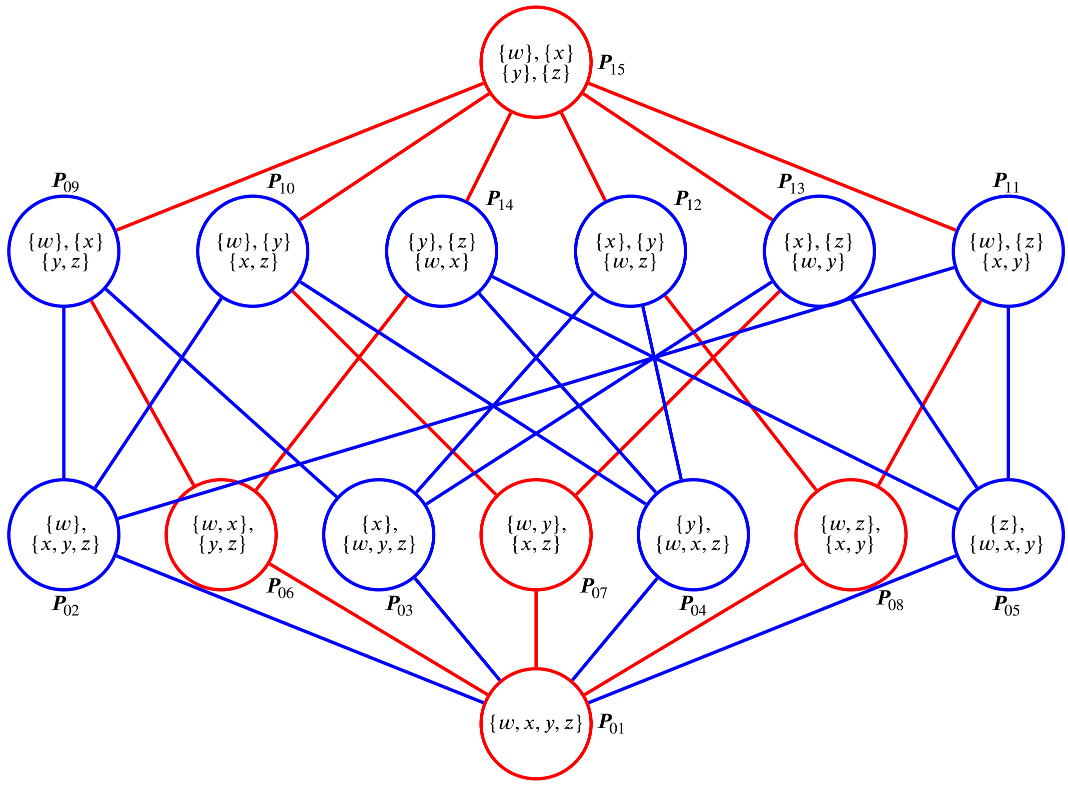
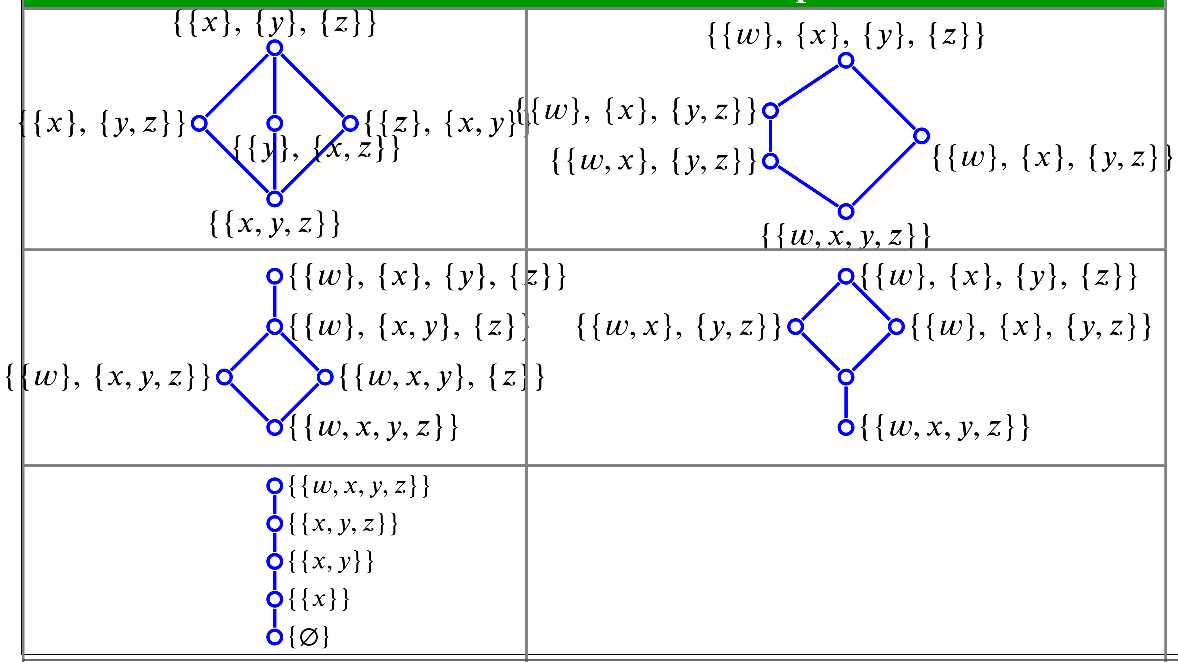


Figure 16.9: Lattice of partitions of sets on $X \triangleq \{w, x, y, z\}$ (Example 16.19 page 245)

lattices on 5 element set as lattices of partitions



16.7 Relationships between set structures

Proposition 16.17. [36]

PRP
$$\left\{\begin{array}{l} \boldsymbol{R} \text{ is a } \textbf{ring of sets} \\ \text{on a set } X \end{array}\right\} \implies \left\{\begin{array}{l} \boldsymbol{R} \cup X \text{ is an } \textbf{algebra of sets} \\ \text{on } X \end{array}\right\}$$

Theorem 16.17. *Let X be a set.*

THM
$$\left\{\begin{array}{l} \boldsymbol{A} \text{ is an } \textbf{algebra of sets} \\ \text{on } X \end{array}\right\} \iff \left\{\begin{array}{lll} 1. & \boldsymbol{A} \text{ is a } \textbf{topology} \text{ on } X & \text{and} \\ 2. & \boldsymbol{A} \text{ is a } \textbf{ring of sets} \text{ on } X & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} \boldsymbol{A} \text{ is an algebra of sets on } X &\implies \boldsymbol{A} \text{ is closed under } \cup, \cap, \mathsf{c}, \setminus, \varnothing, X && \text{by Theorem 16.12 page 235} \\ &\implies \left\{\begin{array}{ll} 1. & \boldsymbol{A} \text{ is a topology on } X \\ & \text{AND} \\ 2. & \boldsymbol{A} \text{ is a ring of sets on } X \end{array}\right\} \end{aligned}$$

$$\begin{aligned} \left\{\begin{array}{ll} 1. & \boldsymbol{A} \text{ is a topology on } X \\ & \text{AND} \\ 2. & \boldsymbol{A} \text{ is a ring of sets on } X \end{array}\right\} &\implies \boldsymbol{A} \text{ is closed under } \mathsf{c} \text{ and } \cap && \text{by Theorem 16.12 page 235} \\ &\implies \boldsymbol{A} \text{ is a ring of sets} \end{aligned}$$

✏

Corollary 16.1. *Let X be a set and 2^X the power set of X.*

COR
$$\left\{\boldsymbol{A} \subseteq 2^X \middle| \boldsymbol{A} \text{ is an algebra of sets on } X\right\} = \left\{\boldsymbol{T} \subseteq 2^X \middle| \boldsymbol{T} \text{ is a topology on } X\right\} \cap \left\{\boldsymbol{R} \subseteq 2^X \middle| \boldsymbol{R} \text{ is a ring of sets on } X\right\}$$

✎PROOF:

$$\begin{aligned} &\left\{\boldsymbol{T} \middle| \boldsymbol{T} \text{ is a topology}\right\} \cap \left\{\boldsymbol{R} \middle| \boldsymbol{R} \text{ is a ring of sets}\right\} \\ &= \left\{\boldsymbol{Y} \middle| \boldsymbol{Y} \text{ is a topology AND a ring of sets}\right\} && \text{by Definition 16.5 page 216} \\ &= \left\{\boldsymbol{Y} \middle| \boldsymbol{Y} \text{ is an algebra of sets}\right\} && \text{by Theorem 16.17 page 247} \\ &= \left\{\boldsymbol{A} \middle| \boldsymbol{A} \text{ is an algebra of sets}\right\} && \text{by change of variable} \end{aligned}$$

✏

Example 16.21. Note that the *intersection* of the lattice of topologies on $\{x, y, z\}$ (Figure 16.5 page 241) *and* the lattice of rings of sets on $\{x, y, z\}$ (Figure 16.7 page 244) is *equal to* the lattice of algebras of sets on $\{x, y, z\}$ (Figure 16.8 page 245).

Proposition 16.18. *Let $\mathcal{R}(X)$ be the set of* RINGS OF SETS *(Definition 16.10 page 229) and $\mathcal{A}(X)$ the set of* ALGEBRAS OF SETS *(Definition 16.9 page 228) on a set X.*

PRP
$$\left\{\begin{array}{lll} 1. & \boldsymbol{R} \text{ is a } \textbf{ring of sets} & \text{and} \\ 2. & X \in \boldsymbol{R} & \end{array}\right\} \iff \left\{ \boldsymbol{R} \text{ is an } \textbf{algebra of sets} \right\}$$

✎PROOF:

$$\begin{aligned} A^{\mathsf{c}} &= X \setminus A && \text{by Theorem 16.1 page 217} \\ A \cap B &= A \setminus (A \setminus B) && \text{by Theorem 16.1 page 217} \end{aligned}$$

[36] Berezansky et al. (1996) page 4, Halmos (1950) page 21

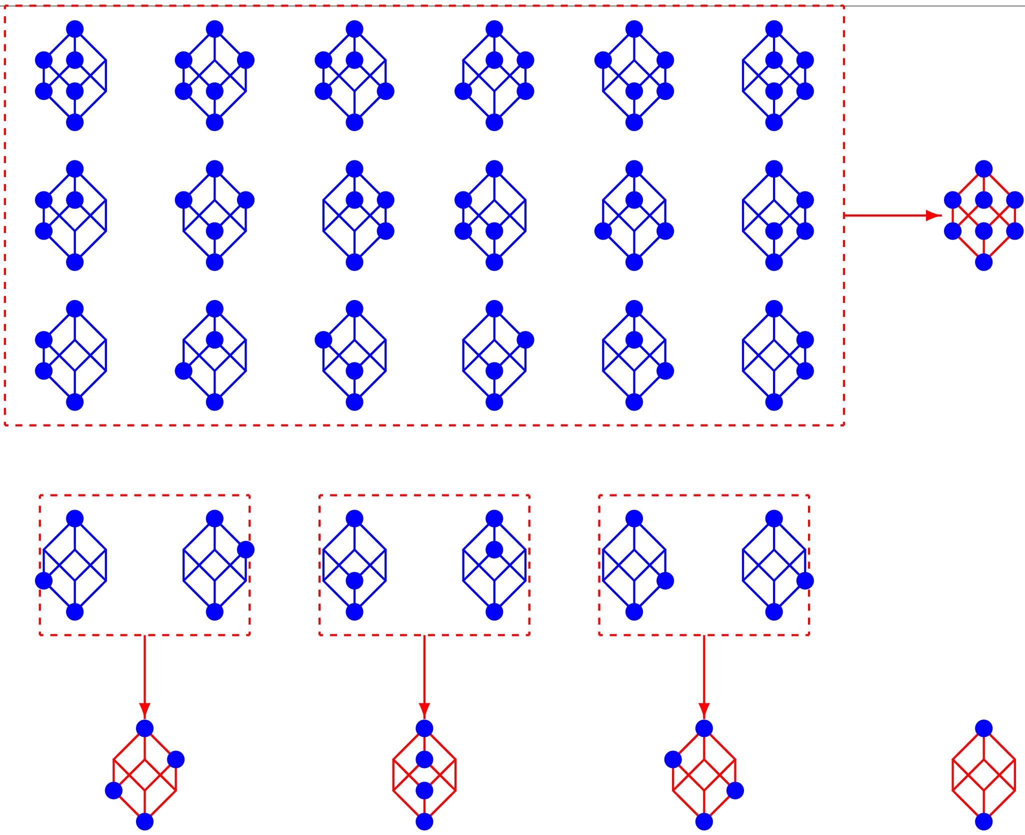


Figure 16.10: Algebras of sets generated by topologies on the set $X \triangleq \{x, y, z\}$ (see Example 16.23 page 248)

Therefore, $(\boldsymbol{R} \cup X)$ is closed under c and ∩, and thus by the definition of algebras of sets (Definition 16.9 page 228), $(\boldsymbol{R} \cup X)$ is an algebra of sets. ⇨

Definition 16.14. [37]

DEF *The **Borel set** $\mathrm{B}(X, \boldsymbol{T})$ generated by the topological space $(X, \boldsymbol{T})$ is the σ-algebra generated by the topology $\boldsymbol{T}$.*

Example 16.22. Suppose we have a dice with the standard six possible outcomes X. Suppose also we construct the following topology $\boldsymbol{T}$ on X, and this in turn generates the following Borel set (σ-algebra) B on X:

EX

$$X = \{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}$$

$$\boldsymbol{T} = \left\{ \underbrace{\{\;\}}_{\varnothing}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}}_{\Omega}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\}}_{\text{first four}}, \underbrace{\{\text{⚃},\text{⚄},\text{⚅}\}}_{\text{last three}}, \underbrace{\{\text{⚃}\}}_{\{1234\}\cap\{456\}}, \right\}$$

$$B = \left\{ \underbrace{\{\;\}}_{\varnothing}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}}_{\Omega}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\}}_{\text{first four}}, \underbrace{\{\text{⚃},\text{⚄},\text{⚅}\}}_{\text{last three}}, \underbrace{\{\text{⚃}\}}_{\{1234\}\cap\{456\}}, \right.$$
$$\left. \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚄},\text{⚅}\}}_{\{4\}}, \underbrace{\{\text{⚄},\text{⚅}\}}_{(\{4\})\cap\{456\}}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂}\}}_{\{1234\}\cap\{4\}}, \right\}$$

Example 16.23. There are a total of 29 *topologies* on the set $X \triangleq \{x, y, z\}$; and of these, 5 are also *algebras of sets*, 24 are not. Figure 16.10 (page 248) illustrates the 24 topologies on the set $\{x, y, z\}$

[37] Aliprantis and Burkinshaw (1998) page 97

that are *not* algebras of sets and the 5 algebras of sets that they generate.

16.8 Literature

Literature survey:

1. Origin of the symbols ∪ and ∩:
   Peano (1888a)
   Peano (1888b)
2. There is some difference in the definition of *ring of sets*:
   (a) *ring of sets* defined as closed under △, ∩:
   Stone (1936), page 38
   Kolmogorov and Fomin (1975) page 31
   Kolmogorov and Fomin (1999) page 20
   Constantinescu (1984) page 155
   (b) *ring of sets* defined as closed under ∪, \ (compatible definition):
   Wilker (1982), page 211
   Kelley and Srinivasan (1988) page 21
   Aliprantis and Burkinshaw (1998) page 96
   Haaser and Sullivan (1991) page 2
   Hewitt and Ross (1994) page 118
   (c) *ring of sets* defined as closed under ∪, \, ∅ (compatible definition):
   Rao (2004) page 15
   (d) *ring of sets* defined as closed under ∪, ∩ (incompatible definition):
   Hausdorff (1927) ⟨???,p.77?⟩
   Hausdorff (1937) page 90
   Birkhoff (1937), page 443
   Erdös and Tarski (1943), page 315
   MacLane and Birkhoff (1999) page 485
3. Relationship to lattices (order theory):
   Stone (1936)
4. More references dealing with set structures ...
   Vaidyanathaswamy (1947)
   Bagley (1955)
   Hartmanis (1958)
   Vaidyanathaswamy (1960)
   Gaifman (1961)
   Gaifman (1966)
   Steiner (1966)
   van Rooij (1968)
   Schnare (1968)
   Rayburn (1969)
   Larson and Andima (1975)
   Pudlák and Tůma (1980)
   Brown and Watson (1991)
   Watson (1994)
   Brown and Watson (1996)
5. Partitions
   Deza and Deza (2006) page 142
   Day (1981)
   Rota (1964)
6. Distributive and modular properties in lattice of topologies

(a) Remark that "It can be shewn easily that the lattice of topologies is not distributive."
Vaidyanathaswamy (1947)
Vaidyanathaswamy (1960) page 134

(b) Proof that the lattice of $T1$ topologies is not modular:
Bagley (1955)

(c) Proof that the lattice of topologies on any set with 3 or more elements is not modular (and thus also not distributive):
Steiner (1966), page 384

7. Complements in lattice of topologies:

(a) Proof that every lattice of topologies over a *finite* set is complemented:
Hartmanis (1958)

(b) Proof that every lattice of topologies over a *countably infinite* set is complemented:
Gaifman (1961)

(c) Proof that every lattice of topologies over a *any arbitrary* set is complemented:
Steiner (1966), page 397

(d) van Rooij (1968)

(e) Every topology in $\hat{\Sigma}(X)$ has at least 2 complements for $|X| \geq 3$:
Hartmanis (1958)

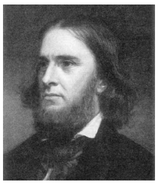
(f) Every topology in $\hat{\Sigma}(X)$ has at least $|X| - 1$ complements for $|X| \geq 2$:
Schnare (1968)

(g) A large number of topologies in $\hat{\Sigma}(X)$ have at least $2^{|X|}$ complements for $|X| \geq 4$:
Brown and Watson (1996)

CHAPTER 17

RELATIONS AND FUNCTIONS

17.1 Relations



❝*A dual relative term, such as "lover," "benefactor," "servant," is a common name signifying a pair of objects. Of the two members of the pair, a determinate one is generally the first, and the other the second; so that if the order is reversed, the pair is not considered as remaining the same.*❞

Charles Sanders Peirce (1839–1914), American mathematician and logician [1]

17.1.1 Definition and examples

A relation on the sets X and Y is any subset of the Cartesian product $X \times Y$ (next definition). Alternatively, a relation is a generalization of a *function* (Definition 17.8 page 263) in the sense that both are subsets of a Cartesian product, but the relation allows mapping from a single element in its domain to two different elements in its range, whereas functions do not— a single element in a function's domain may map to one and only one element in its range. The set of all relations in $X \times Y$ is denoted 2^{XY}, which is suitable since the number of relations in $X \times Y$ when X and Y are finite is $2^{|X|\cdot|Y|}$ (Proposition 17.1 page 252). Examples include the following:

- Example 17.2 page 252 Relations in the Cartesian product $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$
- Example 17.20 page 265 Functions in the Cartesian product $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$
- Example 17.21 page 265 Functions in the Cartesian product $\{x, y, z\} \times \{x, y, z\}$
- Example 17.18 page 264 discrete examples
- Example 17.19 page 264 continuous examples

Definition 17.1. [2] *Let X and Y be sets.*

---

[1] quote: Peirce (1883a), page 187
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

[2] Maddux (2006) page 4, Halmos (1960) page 26

DEF

*A **relation** $®: X \to Y$ is any subset of $X \times Y$. That is,*

$$® \subseteq X \times Y$$

A pair $(x, y) \in ®$ is alternatively denoted $x®y$.

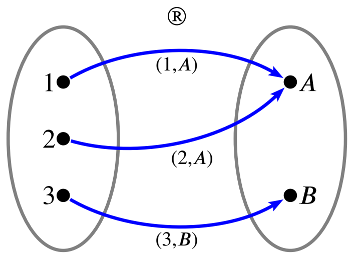
The set of all relations that are subsets of $X \times Y$ is denoted 2^{XY}; that is,

$$2^{XY} \triangleq \{® | ® \subseteq (X \times Y)\}.$$

Example 17.1.

Let $X \triangleq \{1, 2, 3\}$
$Y \triangleq \{A, B\}$
$® \triangleq \{(1, A), (2, A), (3, B)\}$

The sets X and Y and the relation ® are illustrated to the right.



Proposition 17.1. *Let 2^{XY} be the set of all relations from a set X to a set Y. Let $|\cdot|$ be the counting measure for sets.*

PRP

$$\underbrace{\left|2^{XY}\right|}_{\text{number of possible relations in } X\times Y} = 2^{|X \times Y|} = 2^{|X|\cdot|Y|}$$

✎PROOF:

1. Let X be a finite set with m elements.

2. Let Y be a finite set with n elements.

3. Then the number of elements in $X \times Y$ is mn.

4. A relation is any subset of $X \times Y$, which may (represent this with a 1) or may not (represent this with a 0) contain a given element of $X \times Y$.

5. Therefore, the number of possible relations is $2^{mn} = 2^{|X|\cdot|Y|}$.

✏

Example 17.2 (next) lists all of the 64 possible relations in the Cartesian product $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$. Eight of these 64 relations are also functions. These eight functions are listed in Example 17.20 (page 265). Of these eight functions, six are *surjective*. These six surjective functions are listed in Example 17.27 (page 268).

Example 17.2. Let $X \triangleq \{x_1, x_2, x_3\}$ and $Y \triangleq \{y_1, y_2\}$. Let 2^{XY} be the set of all relations in $X \times Y$. There are a total of $\left|2^{XY}\right| = 2^{|X|\cdot|Y|} = 2^{3\times2} = 64$ possible relations. These are listed below. Of these 64 relations, only 8 are *functions*, as listed in Example 17.20 (page 265).

relations in $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$

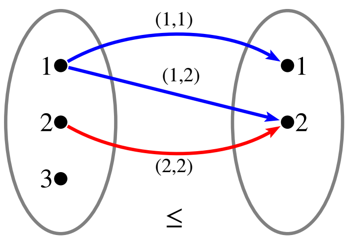
$®_1$	=	$\varnothing$					
$®_2$	=	{	(x_1, y_1),				}
$®_3$	=	{		(x_1, y_2)			}
$®_4$	=	{	(x_1, y_1),	(x_1, y_2)			}
$®_5$	=	{			(x_2, y_1)		}

$$
\begin{array}{lcllllllll}
\circledR_{6} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & & & \} \\
\circledR_{7} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & & & \} \\
\circledR_{8} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & & & & \} \\
\circledR_{9} & = & \{ & & & & (x_2,y_2) & & & \} \\
\circledR_{10} & = & \{ & (x_1,y_1), & & & (x_2,y_2) & & & \} \\
\circledR_{11} & = & \{ & & (x_1,y_2) & & (x_2,y_2) & & & \} \\
\circledR_{12} & = & \{ & (x_1,y_1), & (x_1,y_2) & & (x_2,y_2) & & & \} \\
\circledR_{13} & = & \{ & & & (x_2,y_1) & (x_2,y_2) & & & \} \\
\circledR_{14} & = & \{ & (x_1,y_1), & & (x_2,y_1) & (x_2,y_2) & & & \} \\
\circledR_{15} & = & \{ & & (x_1,y_2), & (x_2,y_1) & (x_2,y_2) & & & \} \\
\circledR_{16} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & (x_2,y_2) & & & \} \\
\circledR_{17} & = & \{ & & & & & (x_3,y_1) & & \} \\
\circledR_{18} & = & \{ & (x_1,y_1), & & & & (x_3,y_1) & & \} \\
\circledR_{19} & = & \{ & & (x_1,y_2) & & & (x_3,y_1) & & \} \\
\circledR_{20} & = & \{ & (x_1,y_1), & (x_1,y_2) & & & (x_3,y_1) & & \} \\
\circledR_{21} & = & \{ & & & (x_2,y_1) & & (x_3,y_1) & & \} \\
\circledR_{22} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & (x_3,y_1) & & \} \\
\circledR_{23} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & (x_3,y_1) & & \} \\
\circledR_{24} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & & (x_3,y_1) & & \} \\
\circledR_{25} & = & \{ & & & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{26} & = & \{ & (x_1,y_1), & & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{27} & = & \{ & & (x_1,y_2) & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{28} & = & \{ & (x_1,y_1), & (x_1,y_2) & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{29} & = & \{ & & & (x_2,y_1) & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{30} & = & \{ & (x_1,y_1), & & (x_2,y_1) & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{31} & = & \{ & & (x_1,y_2), & (x_2,y_1) & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{32} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{33} & = & \{ & & & & & & (x_3,y_2) & \} \\
\circledR_{34} & = & \{ & (x_1,y_1), & & & & & (x_3,y_2) & \} \\
\circledR_{35} & = & \{ & & (x_1,y_2) & & & & (x_3,y_2) & \} \\
\circledR_{36} & = & \{ & (x_1,y_1), & (x_1,y_2) & & & & (x_3,y_2) & \} \\
\circledR_{37} & = & \{ & & & (x_2,y_1) & & & (x_3,y_2) & \} \\
\circledR_{38} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & & (x_3,y_2) & \} \\
\circledR_{39} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & & (x_3,y_2) & \} \\
\circledR_{40} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & & & (x_3,y_2) & \} \\
\circledR_{41} & = & \{ & & & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{42} & = & \{ & (x_1,y_1), & & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{43} & = & \{ & & (x_1,y_2) & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{44} & = & \{ & (x_1,y_1), & (x_1,y_2) & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{45} & = & \{ & & & (x_2,y_1) & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{46} & = & \{ & (x_1,y_1), & & (x_2,y_1) & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{47} & = & \{ & & (x_1,y_2), & (x_2,y_1) & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{48} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{49} & = & \{ & & & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{50} & = & \{ & (x_1,y_1), & & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{51} & = & \{ & & (x_1,y_2) & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{52} & = & \{ & (x_1,y_1), & (x_1,y_2) & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{53} & = & \{ & & & (x_2,y_1) & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{54} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{55} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & (x_3,y_1) & (x_3,y_2) & \}
\end{array}
$$

$\circledR_{56}$	$=$	$\{$	(x_1, y_1),	(x_1, y_2),	(x_2, y_1),		(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{57}$	$=$	$\{$				(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{58}$	$=$	$\{$	(x_1, y_1),			(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{59}$	$=$	$\{$		(x_1, y_2)		(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{60}$	$=$	$\{$	(x_1, y_1),	(x_1, y_2)		(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{61}$	$=$	$\{$			(x_2, y_1)	(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{62}$	$=$	$\{$	(x_1, y_1),		(x_2, y_1)	(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{63}$	$=$	$\{$		(x_1, y_2),	(x_2, y_1)	(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$
$\circledR_{64}$	$=$	$\{$	(x_1, y_1),	(x_1, y_2),	(x_2, y_1),	(x_2, y_2)	(x_3, y_1)	(x_3, y_2) $\}$

Example 17.3.
Let $X \triangleq \{1, 2, 3\}$, $Y \triangleq \{1, 2\}$, and 2^{XY} the set of all of the $2^{3\times2} = 64$ relations in $X \times Y$. Furthermore, let $x_1 \triangleq 1$, $x_2 \triangleq 2$, $x_3 \triangleq 3$, $y_1 \triangleq 1$, and $y_2 \triangleq 2$. Then the following common relations are the relations of Example 17.2 (page 252):

$\le$	$\equiv$	$\{(1,1), (1,2), (2,2)\}$	$\equiv$	$\circledR_{12}$
$\ge$	$\equiv$	$\{(1,1), (2,1), (2,2), (3,1), (3,2)\}$	$\equiv$	$\circledR_{62}$
$<$	$\equiv$	$\{(1,2)\}$	$\equiv$	$\circledR_{3}$
$>$	$\equiv$	$\{(2,1), (3,1), (3,2)\}$	$\equiv$	$\circledR_{53}$
$=$	$\equiv$	$\{(1,1), (2,2)\}$	$\equiv$	$\circledR_{10}$



17.1.2 Calculus of Relations

Proposition 17.2. [3] *Let 2^{XY} be the set of all relations in $X \times Y$.*

$$\varnothing \in 2^{XY} \qquad (\varnothing \textit{ is a relation})$$

✎PROOF:

$$\begin{aligned} &\varnothing \subseteq X \times Y \\ &\implies \varnothing \text{ is a relation.} && \text{by definition of relation Definition 17.1 page 251} \end{aligned}$$

✑

Proposition 17.3. [4] *Let 2^{XY} be the set of all relations from the sets X to the set Y.*

PRP
$$\left.\begin{array}{lll} \circledR \in 2^{XY} & (\circledR \textit{ is a relation}) & \textit{and} \\ \circledS \subseteq \circledR & (\circledS \textit{ is a subset of } \circledR) & \end{array}\right\} \implies \circledS \in 2^{XY} \quad (\circledS \textit{ is a relation})$$

✎PROOF:

$$\begin{aligned} \circledS &\subseteq \circledR && \text{by right hypothesis} \\ &\subseteq X \times Y && \text{by definition of relation Definition 17.1 page 251} \\ &\implies \varnothing \text{ is a relation.} && \text{by definition of relation Definition 17.1 page 251} \end{aligned}$$

[3] Suppes (1972) page 58
[4] Suppes (1972) page 58

A function does not always have an inverse that is also a function. But unlike functions, *every* relation has an inverse that is also a relation. Note that since all functions are relations, every function *does* have an inverse that is at least a relation, and in some cases this inverse is also a function.

Definition 17.2. [5] *Let $\circledR$ be a relation in 2^{XY}.*

DEF

$\circledR^{-1}$ *is the **inverse** of relation* $\circledR$ *if*

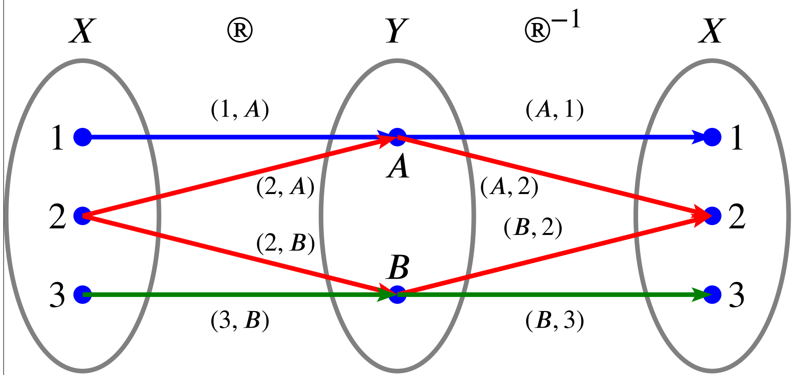
$$\circledR^{-1} \triangleq \{(y,x) \in Y \times X \mid (x,y) \in \circledR\}$$

*The inverse relation $\circledR^{-1}$ is also called the **converse** of $\circledR$.*

Example 17.4.

Let	X	$\triangleq$	$\{1,\ 2,\ 3\}$
and	Y	$\triangleq$	$\{A,\ B\}$
and	$\circledR$	$\triangleq$	$\{(1,A)\,,\,(2,A)\,,\,(2,B)\,,\,(3,B)\}$.
Then	$\circledR^{-1}$	$=$	$\{(A,1)\,,\,(A,2)\,,\,(B,2)\,,\,(B,3)\}$.

The sets X and Y and the relations $\circledR$ and $\circledR^{-1}$ are illustrated below.



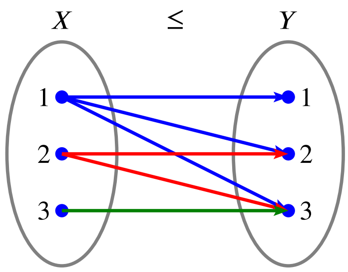
Example 17.5.

Let $X \triangleq \{1,2,3\}$. Then the "less than or equal to" relation $\leq$ in 2^{XX} is

$$\leq \equiv \{(1,1)\,,\,(1,2)\,,\,(1,3)\,,\,(2,2)\,,\,(2,3)\,,\,(3,3)\}$$

and it's inverse $\leq^{-1}$ is equivalent to the "greater than or equal to" relation $\geq$:

$$\leq^{-1} \equiv \{(1,1)\,,\,(2,1)\,,\,(3,1)\,,\,(2,2)\,,\,(3,2)\,,\,(3,3)\} \equiv \geq.$$



Example 17.6.

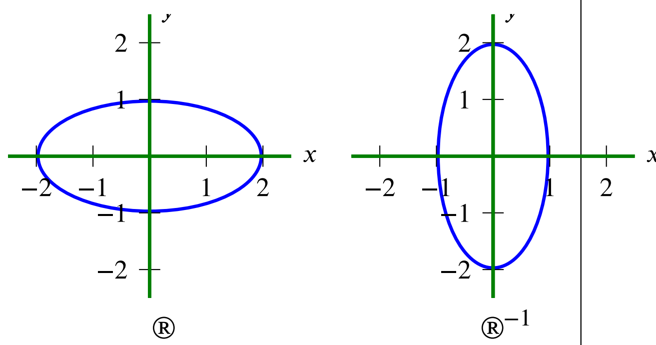
Let $\circledR$ be the *ellipse* relation in $2^{\mathbb{R}\mathbb{R}}$ such that

$$\circledR \triangleq \left\{(x,y) \in \mathbb{R}^2 \middle| \frac{x^2}{2^2} + \frac{y^2}{1^2} = 1\right\}.$$

Then the inverse relation $\circledR^{-1}$ is

$$\circledR^{-1} = \left\{(x,y) \in \mathbb{R}^2 \middle| \frac{x^2}{2^2} + \frac{2^2}{2^2} = 1\right\}.$$

Both of these relations are illustrated to the right.



Example 17.7. Let $\mathbf{I} \in X^X$ be an identity function, and $\mathsf{f}, \mathsf{f}^{-1} \in X^X$ be functions.

f^{-1} is the **inverse** of f if $\mathsf{f}\mathsf{f}^{-1} = \mathsf{f}^{-1}\mathsf{f} = \mathbf{I}$.

[5] Suppes (1972) page 61 ⟨Defintion 6, inverse="converse"⟩, Kelley (1955) page 7, Peirce (1883a) page 188 ⟨inverse="converse"⟩

Theorem 17.1. [6] *Let $\circledR$ be a relation with inverse $\circledR^{-1}$.*

THM
$$\left(\circledR^{-1}\right)^{-1} = \circledR$$

✎PROOF:

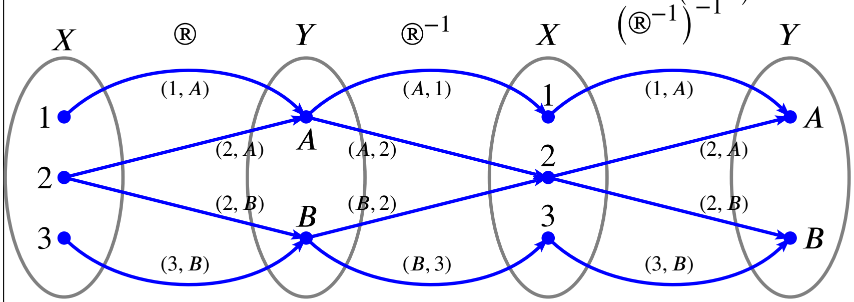
$$\begin{aligned}
\left(\circledR^{-1}\right)^{-1} &= \underbrace{\{(x,y) \mid (y,x) \in \circledR\}}_{\circledR^{-1}}{}^{-1} && \text{by definition of } \circledR^{-1} \text{ (Definition 17.2 page 255)}\\
&= \{(x,y) \mid (y,x) \in \{(y,x) \mid (x,y) \in \circledR\}\} && \text{by definition of } \circledR^{-1} \text{ (Definition 17.2 page 255)}\\
&= \{(x,y) \mid (x,y) \in \circledR\}\\
&= \circledR
\end{aligned}$$

⇰

Example 17.8.

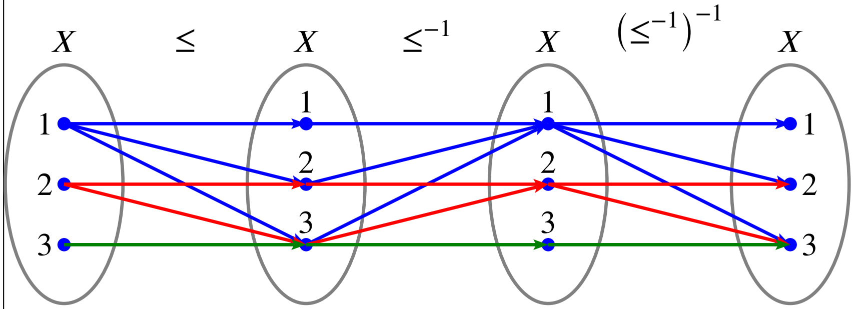
Let	X	$\triangleq$	$\{1,\ 2,\ 3\}$
and	Y	$\triangleq$	$\{A,\ B\}$
and	$\circledR$	$\triangleq$	$\{(1,A),\ (2,A),\ (2,B),\ (3,B)\}$.
Then	$\circledR^{-1}$	$=$	$\{(A,1),\ (A,2),\ (B,2),\ (B,3)\}$
and	$\left(\circledR^{-1}\right)^{-1}$	$=$	$\{(1,A),\ (2,A),\ (2,B),\ (3,B)\} = \circledR$.

The sets X and Y and the relations $\circledR$, $\circledR^{-1}$, and $\left(\circledR^{-1}\right)^{-1}$ are illustrated below.



Example 17.9. Let $X \triangleq \{1,2,3\}$. Let $\leq \in 2^{XX}$ be the "less than or equal to" relation in 2^{XX}.

$$\begin{aligned}
\left(\leq^{-1}\right)^{-1} &\triangleq \left(\{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3)\}^{-1}\right)^{-1}\\
&= (\{(1,1),\ (2,1),\ (3,1),\ (2,2),\ (3,2),\ (3,3)\})^{-1}\\
&= (\{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3)\})\\
&\triangleq \leq
\end{aligned}$$



Definition 17.3. [7] *Let $\circledR \in 2^{XY}$ and $\circledS \in 2^{YZ}$ be relations. Let $\wedge$ be the logical and function.*

DEF
The composition function $\circ$ on relations $\circledR$ and $\circledS$ is defined as
$$\circledS \circ \circledR \triangleq \{(x,z) \mid \exists y \quad \text{such that} \quad (x,y) \in \circledR \ \wedge \ (y,z) \in \circledS\}$$

[6] Kelley (1955) page 8, Peirce (1883a) page 188
[7] Kelley (1955) pages 7–8, Fuhrmann (2012) page 2

Theorem 17.2. [8] *Let X, Y, and Z be sets.*

THM	$(\circledR \circ \circledS)^{-1} = \left(\circledS^{-1}\right) \circ \left(\circledR^{-1}\right)$	$\forall \circledR \in 2^{WX},\ \circledS \in 2^{XY}$		(IDEMPOTENT)
	$\textcircled{Q} \circ (\circledS \circ \circledR) = (\textcircled{Q} \circ \circledS) \circ \circledR$	$\forall \circledR \in 2^{WX},\ \circledS \in 2^{XY},\ \textcircled{Q} \in 2^{YZ}$		(ASSOCIATIVE)

✎PROOF:

$$
\begin{aligned}
(\circledR \circ \circledS)^{-1} &= \left\{(x,z) \,|\, \exists y \quad \text{such that} \quad (x,y) \in \circledR \text{ and } (y,z) \in \circledS\right\}^{-1} && \text{by definition of } \circ \text{ (page 256)}\\
&= \left\{(z,x) \,|\, (x,z) \in \left\{(x,z) \,|\, \exists y \quad \text{such that} \quad (x,y) \in \circledR \text{ and } (y,z) \in \circledS\right\}\right\} && \text{by definition of } \circledR^{-1} \text{ (page 255)}\\
&= \left\{(z,x) \,|\, \exists y \quad \text{such that} \quad (x,y) \in \circledR \text{ and } (y,z) \in \circledS\right\}\\
&= \left\{(z,x) \,|\, \exists y \quad \text{such that} \quad (y,x) \in \circledR^{-1} \text{ and } (z,y) \in \circledS^{-1}\right\} && \text{by definition of } \circledR^{-1} \text{ (page 255)}\\
&= \left(\circledS^{-1}\right) \circ \left(\circledR^{-1}\right) && \text{by definition of } \circ \text{ (page 256)}
\end{aligned}
$$

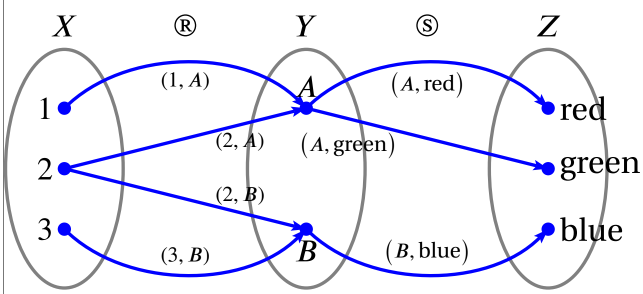
$$
\begin{aligned}
&\textcircled{Q} \circ (\circledS \circ \circledR)\\
&= \left\{(w,z) \,|\, \exists y \quad \text{such that} \quad (w,y) \in (\circledS \circ \circledR) \text{ and } (y,z) \in \textcircled{Q}\right\}\\
&\qquad \text{by definition of } \circ \text{ (page 256)}\\
&= \left\{(w,z) \,|\, \exists y \quad \text{such that} \quad (w,y) \in \left\{(w,y) \,|\, \exists x \quad \text{such that} \quad (w,x) \in \circledR \text{ and } (x,y) \in \circledS\right\} \text{ and } (y,z) \in \textcircled{Q}\right\}\\
&\qquad \text{by definition of } \circ \text{ (page 256)}\\
&= \left\{(w,z) \,|\, \exists x,y \quad \text{such that} \quad (w,x) \in \circledR \text{ and } (x,y) \in \circledS \text{ and } (y,z) \in \textcircled{Q}\right\}\\
&= \left\{(w,z) \,|\, \exists x \quad \text{such that} \quad (w,x) \in \circledR \text{ and } (x,z) \in \left\{(x,z) \,|\, \exists y \quad \text{such that} \quad (x,y) \in \circledS \text{ and } (y,z) \in \textcircled{Q}\right\}\right\}\\
&= \left\{(w,z) \,|\, \exists x \quad \text{such that} \quad (w,x) \in \circledR \text{ and } (x,z) \in (\circledS \circ \textcircled{Q})\right\}\\
&\qquad \text{by definition of } \circ \text{ (page 256)}\\
&= (\textcircled{Q} \circ \circledS) \circ \circledR\\
&\qquad \text{by definition of } \circ \text{ (page 256)}
\end{aligned}
$$

⇒

Example 17.10.

$$
\begin{array}{lll}
\text{Let} & X & \triangleq \{1,\ 2,\ 3\}\\
\text{and} & Y & \triangleq \{A,\ B\}\\
\text{and} & Z & \triangleq \{\text{red, green, blue}\}\\
\text{and} & \circledR & \triangleq \{(1,A),\ (2,A),\ (2,B),\ (3,B)\}.\\
\text{and} & \circledS & \triangleq \{(A,\text{red}),\ (A,\text{green}),\ (B,\text{blue})\}.\\
\text{Then} & \circledR \circ \circledS & = \{(1,\text{red}),\ (1,\text{green}),\ (2,\text{green}),\ (2,\text{blue}),\ (3,\text{blue})\}.\\
\text{and} & (\circledR \circ \circledS)^{-1} & = \{(\text{red},1),\ (\text{green},1),\ (\text{green},2),\ (\text{blue},2),\ (\text{blue},3)\}.\\
 & & = \circledS^{-1} \circ \circledR^{-1}
\end{array}
$$

The quanitities are illustrated below.



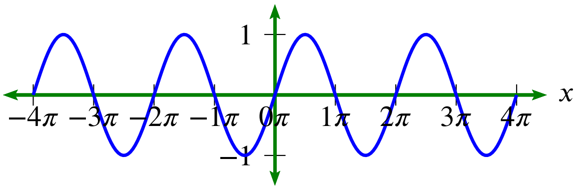
[8] Kelley (1955) page 8

Definition 17.4. [9] *Let* $\circledR \in 2^{XY}$ *be a relation.*

The ***domain*** *of* $\circledR$ *is*	$\mathcal{D}(\circledR)$	$\triangleq$	$\{x \in X \mid \exists y$ such that $(x, y) \in \circledR\}$.
The ***image set*** *of* $\circledR$ *is*	$\mathcal{I}(\circledR)$	$\triangleq$	$\{y \in Y \mid \exists x$ such that $(x, y) \in \circledR\}$.
The ***null space*** *of* $\circledR$ *is*	$\mathcal{N}(\circledR)$	$\triangleq$	$\{x \in X \mid (x, 0) \in \circledR\}$.
The ***range*** *of* $\circledR$ *is*	*any set* $\mathcal{R}(\circledR)$ *such that* $\mathcal{I}(\circledR) \subseteq \mathcal{R}(\circledR)$		

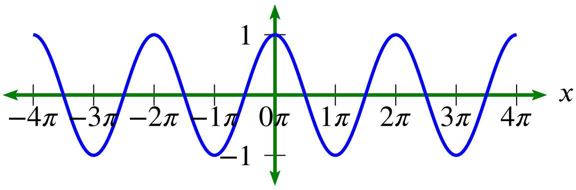
Example 17.11. Let $\circledR \triangleq \sin x$. Then ...

$$\begin{aligned}
\mathcal{D}(\circledR) &= \mathbb{R}\\
\mathcal{I}(\circledR) &= -1 \le y \le 1\\
\mathcal{N}(\circledR) &= \{n\pi \mid n \in \mathbb{Z}\}.\\
\mathcal{R}(\circledR) &= \mathbb{R}
\end{aligned}$$



Example 17.12. Let $\circledR \triangleq \cos x$. Then ...

$$\begin{aligned}
\mathcal{D}(\circledR) &= \mathbb{R}\\
\mathcal{I}(\circledR) &= -1 \le y \le 1\\
\mathcal{N}(\circledR) &= \left\{\left(n + \tfrac{1}{2}\right)\pi \mid n \in \mathbb{Z}\right\}.\\
\mathcal{R}(\circledR) &= \mathbb{R}
\end{aligned}$$



Example 17.13. (Rudin, 1991)99 Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be linear functions and Y^X be the set of all functions from $\boldsymbol{X}$ to $\boldsymbol{Y}$. Let f be an function in Y^X.

The **domain** of f is	$\mathcal{D}(\mathsf{f})$	$\triangleq$	$\boldsymbol{X}$
The **range** of f is	$\mathcal{I}(\mathsf{f})$	$\triangleq$	$\{\boldsymbol{y} \in \boldsymbol{Y} \mid \exists \boldsymbol{x} \in \boldsymbol{X}$ such that $\boldsymbol{y} = \mathsf{f}\boldsymbol{x}\}$
The **null space** of f is	$\mathcal{N}(\mathsf{f})$	$\triangleq$	$\{\boldsymbol{x} \in \boldsymbol{X} \mid \mathsf{f}\boldsymbol{x} = 0\}$

Theorem 17.3. [10] *Let* $\mathcal{D}(\circledR)$ *be the domain of a relation* $\circledR$ *and* $\mathcal{I}(\circledR)$ *the image of* $\circledR$.

$$\begin{aligned}
\mathcal{D}\left(\bigcup_{i\in I}\circledR_i\right) &= \bigcup_{i\in I}\mathcal{D}(\circledR_i) & \qquad \mathcal{I}\left(\bigcup_{i\in I}\circledR_i\right) &= \bigcup_{i\in I}\mathcal{I}(\circledR_i)\\
\mathcal{D}\left(\bigcap_{i\in I}\circledR_i\right) &\subseteq \bigcap_{i\in I}\mathcal{D}(\circledR_i) & \qquad \mathcal{I}\left(\bigcap_{i\in I}\circledR_i\right) &\subseteq \bigcap_{i\in I}\mathcal{I}(\circledR_i)\\
\mathcal{D}(\circledR\setminus ⓢ) &\supseteq \mathcal{D}(\circledR)\setminus\mathcal{D}(ⓢ) & \qquad \mathcal{I}(\circledR\setminus ⓢ) &\supseteq \mathcal{I}(\circledR)\setminus\mathcal{I}(ⓢ)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathcal{D}\left(\bigcup_{i\in I}\circledR_i\right) &= \left\{x \mid \exists y \text{ such that } (x, y) \in \bigcup_{i\in I}\circledR_i\right\} && \text{by Definition 17.4 page 258}\\
&= \left\{x \mid \exists y \text{ such that } (x, y) \in \left\{(x, y) \mid \bigvee_i (x, y) \in \circledR_i\right\}\right\} && \text{by Definition 16.5 page 216}\\
&= \left\{x \mid \exists y \text{ such that } \bigvee_i (x, y) \in \circledR_i\right\}\\
&= \left\{x \mid \bigvee_i [\exists y \text{ such that } (x, y) \in \circledR_i]\right\}\\
&= \bigcup_i \{x \mid \exists y \text{ such that } (x, y) \in \circledR_i\} && \text{by Definition 16.5 page 216}\\
&= \bigcup_i \mathcal{D}(\circledR_i) && \text{by Definition 17.4 page 258}
\end{aligned}$$

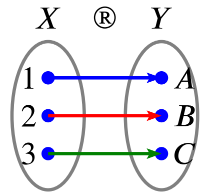
[9] Munkres (2000), page 16, Kelley (1955) page 7
[10] Suppes (1972) pages 60–61

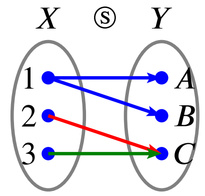
$$
\begin{aligned}
\mathcal{D}\left(\bigcap_{i\in I}\circledR_i\right) &= \left\{x|\exists y \quad \text{such that} \quad (x,y)\in\bigcap_{i\in I}\circledR_i\right\} && \text{by Definition 17.4 page 258}\\
&= \left\{x|\exists y \quad \text{such that} \quad (x,y)\in\left\{(x,y)\,|\bigwedge_i (x,y)\in\circledR_i\right\}\right\} && \text{by Definition 16.5 page 216}\\
&= \left\{x|\exists y \quad \text{such that} \quad \bigwedge_i (x,y)\in\circledR_i\right\}\\
&= \left\{x|\bigwedge_i \left[\exists y \quad \text{such that} \quad (x,y)\in\circledR_i\right]\right\}\\
&= \bigcap_i \left\{x|\exists y \quad \text{such that} \quad (x,y)\in\circledR_i\right\} && \text{by Definition 16.5 page 216}\\
&= \bigcap_i \mathcal{D}(\circledR_i) && \text{by Definition 17.4 page 258}
\end{aligned}
$$

✎

Example 17.14.

$$
\begin{array}{llll}
\text{Let} & X & \triangleq & \{1,2,3\}\\
\text{and} & Y & \triangleq & \{A,B,C\}\\
\text{and} & \circledR & \triangleq & \{(1,A),\,(2,B),\,(3,C)\}\\
\text{and} & \circledS & \triangleq & \{(1,A),\,(1,B),\,(2,C),\,(3,C)\}.
\end{array}
$$

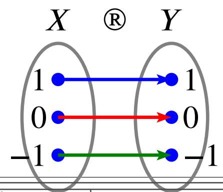


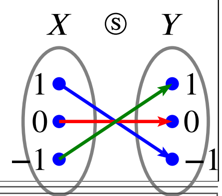


$$
\begin{aligned}
\mathcal{D}(\circledR\cup\circledS) &= \mathcal{D}(\{(1,A),\,(2,B),\,(3,C)\}\cup\{(1,A),\,(1,B),\,(2,C),\,(3,C)\}).\\
&= \mathcal{D}\{(1,A),\,(1,B),\,(2,B),\,(2,C),\,(3,C)\}\\
&= \{1,\ 2,\ 3\}\\
&= \{1,\ 2,\ 3\}\cup\{1,\ 2\,3\}\\
&= \mathcal{D}\circledR\cup\mathcal{D}\circledS\\
\mathcal{D}(\circledR\cap\circledS) &= \{(1,A),\,(3,C)\}\\
&= \{1,\ 3\}\\
&\subseteq \{1,\ 2,\ 3\}\cap\{1,\ 2\,3\}\\
&= \mathcal{D}\circledR\cap\mathcal{D}\circledS\\
\mathcal{I}(\circledR\cup\circledS) &= \mathcal{I}(\{(1,A),\,(2,B),\,(3,C)\}\cup\{(1,A),\,(1,B),\,(2,C),\,(3,C)\}).\\
&= \mathcal{I}\{(1,A),\,(1,B),\,(2,B),\,(2,C),\,(3,C)\}\\
&= \{A,\ B,\ C\}\\
&= \{A,\ B,\ C\}\cup\{A,\ B\,C\}\\
&= \mathcal{I}\circledR\cup\mathcal{I}\circledS\\
\mathcal{I}(\circledR\cap\circledS) &= \{(1,A),\,(3,C)\}\\
&= \{A,\ C\}\\
&\subseteq \{A,\ B,\ C\}\cap\{A,\ B\,C\}\\
&= \mathcal{I}\circledR\cap\mathcal{I}\circledS
\end{aligned}
$$

Example 17.15.

$$
\begin{array}{llll}
\text{Let} & X & \triangleq & \{-1,0,1\}\\
\text{and} & Y & \triangleq & \{-1,0,1\}\\
\text{and} & \circledR & \triangleq & \{(-1,-1),\,(0,0),\,(1,1)\}\\
\text{and} & \circledS & \triangleq & \{(-1,1),\,(0,0),\,(1,-1)\}.
\end{array}
$$

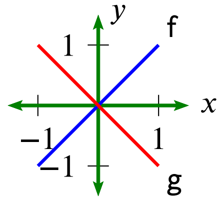




$$\begin{aligned}
\boldsymbol{\mathcal{D}}(\circledR \cup \circledS) &= \boldsymbol{\mathcal{D}}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cup \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \boldsymbol{\mathcal{D}}\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\,,\,(-1,1)\,,\,(1,-1)\} \\
&= \{-1,\ 0,\ 1\} \\
&= \{-1,\ 0,\ 1\} \cup \{-1,\ 0\,1\} \\
&= \boldsymbol{\mathcal{D}}\circledR \cup \boldsymbol{\mathcal{D}}\circledS \\
\boldsymbol{\mathcal{D}}(\circledR \cap \circledS) &= \boldsymbol{\mathcal{D}}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cap \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \boldsymbol{\mathcal{D}}\{(0,0)\} \\
&= \{0\} \\
&\subseteq \{-1,\ 0,\ 1\} \cap \{-1,\ 0\,1\} \\
&= \boldsymbol{\mathcal{D}}\circledR \cap \boldsymbol{\mathcal{D}}\circledS \\
\boldsymbol{\mathcal{I}}(\circledR \cup \circledS) &= \boldsymbol{\mathcal{I}}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cup \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \boldsymbol{\mathcal{I}}\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\,,\,(-1,1)\,,\,(1,-1)\} \\
&= \{-1,\ 0,\ 1\} \\
&= \{-1,\ 0,\ 1\} \cup \{-1,\ 0\,1\} \\
&= \boldsymbol{\mathcal{I}}\circledR \cup \boldsymbol{\mathcal{D}}\circledS \\
\boldsymbol{\mathcal{I}}(\circledR \cap \circledS) &= \boldsymbol{\mathcal{I}}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cap \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \boldsymbol{\mathcal{I}}\{(0,0)\} \\
&= \{0\} \\
&\subseteq \{-1,\ 0,\ 1\} \cap \{-1,\ 0\,1\} \\
&= \boldsymbol{\mathcal{I}}\circledR \cap \boldsymbol{\mathcal{I}}\circledS
\end{aligned}$$

Example 17.16.

$$\begin{array}{llll}
\text{Let} & \mathsf{f}(x) & \triangleq & x \\
\text{and} & \mathsf{g}(x) & \triangleq & -x.
\end{array}$$



$$\begin{aligned}
\boldsymbol{\mathcal{D}}(\mathsf{f} \cup \mathsf{g}) &= \boldsymbol{\mathcal{D}}\left(\left\{(x,y) \in \mathbb{R}^2 | y = x\right\} \cup \left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
&= \boldsymbol{\mathcal{D}}\left\{(x,y) \in \mathbb{R}^2 | y = x \text{ or } y = -x\right\} \\
&= \mathbb{R} \\
&= \mathbb{R} \cup \mathbb{R} \\
&= \left(\boldsymbol{\mathcal{D}}\left\{(x,y) \in \mathbb{R}^2 | y = x\right\}\right) \cup \left(\boldsymbol{\mathcal{D}}\left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
\boldsymbol{\mathcal{D}}(\mathsf{f} \cap \mathsf{g}) &= \boldsymbol{\mathcal{D}}\left(\left\{(x,y) \in \mathbb{R}^2 | y = x\right\} \cap \left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
&= \boldsymbol{\mathcal{D}}\left\{(x,y) \in \mathbb{R}^2 | y = x \text{ and } y = -x\right\} \\
&= \boldsymbol{\mathcal{D}}\{(0,0)\} \\
&= \{0\} \\
&\subseteq \mathbb{R} \\
&= \mathbb{R} \cap \mathbb{R} \\
&= \left(\boldsymbol{\mathcal{D}}\left\{(x,y) \in \mathbb{R}^2 | y = x\right\}\right) \cap \left(\boldsymbol{\mathcal{D}}\left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\mathsf{f} \cup \mathsf{g}) &= \mathcal{I}\left(\left\{(x,y) \in \mathbb{R}^2 | y = x\right\} \cup \left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
&= \mathcal{I}\left\{(x,y) \in \mathbb{R}^2 | y = x \text{ or } y = -x\right\} \\
&= \mathbb{R} \\
&= \mathbb{R} \cup \mathbb{R} \\
&= \left(\mathcal{I}\left\{(x,y) \in \mathbb{R}^2 | y = x\right\}\right) \cup \left(\mathcal{I}\left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
\mathcal{I}(\mathsf{f} \cap \mathsf{g}) &= \mathcal{I}\left(\left\{(x,y) \in \mathbb{R}^2 | y = x\right\} \cap \left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
&= \mathcal{I}\left\{(x,y) \in \mathbb{R}^2 | y = x \text{ and } y = -x\right\} \\
&= \mathcal{I}\{(0,0)\} \\
&= \{0\} \\
&\subseteq \mathbb{R} \\
&= \mathbb{R} \cap \mathbb{R} \\
&= \left(\mathcal{I}\left\{(x,y) \in \mathbb{R}^2 | y = x\right\}\right) \cap \left(\mathcal{I}\left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right)
\end{aligned}$$

Definition 17.5. [11] *Let ® be a relation in 2^{XY}.*

DEF

$$\begin{array}{rcllll}
®(A) & \triangleq & \{y \in Y | \exists x \in A & \text{such that} & (x,y) \in ®\} & \forall A \in 2^X & \text{(image of } A \text{ under } ®) \\
®^{-1}(B) & \triangleq & \{x \in X | \exists y \in B & \text{such that} & (x,y) \in ®\} & \forall B \in 2^Y & \text{(image of } B \text{ under } ®^{-1})
\end{array}$$

Theorem 17.4. [12]

THM

$$\begin{aligned}
®(\varnothing) &= \varnothing \\
®\left(\bigcup_{i \in I} A_i\right) &= \bigcup_{i \in I} ®(A_i) \\
®\left(\bigcap_{i \in I} A_i\right) &\subseteq \bigcap_{i \in I} ®(A_i)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
®(\varnothing) &= \{y \in Y | \exists x \in \varnothing \text{ such that } (x,y) \in ®\} && \text{by Definition 17.5 page 261} \\
&= \varnothing \\
®\left(\bigcup_{i \in I} A_i\right) &= \left\{y \in Y | \exists x \in \bigcup_{i \in I} A_i \text{ such that } (x,y) \in ®\right\} && \text{by Definition 17.5 page 261} \\
&= \left\{y \in Y | \exists x \in \left\{x \in X | \bigvee_{i \in I} x \in A_i\right\} \text{ such that } (x,y) \in ®\right\} && \text{by Definition 16.5 page 216} \\
&= \left\{y \in Y | \exists x \in X \text{ such that } \left[\bigvee_{i \in I} x \in A_i\right] \wedge (x,y) \in ®\right\} \\
&= \left\{y \in Y | \exists x \in X \text{ such that } \bigvee_{i \in I} \left[x \in A_i \wedge (x,y) \in ®\right]\right\} \\
&= \left\{y \in Y | \bigvee_{i \in I} \left[\exists x \in X \text{ such that } x \in A_i \wedge (x,y) \in ®\right]\right\} \\
&= \bigcup_{i \in I} \{y \in Y | \exists x \in X \text{ such that } x \in A_i \wedge (x,y) \in ®\} && \text{by Definition 16.5 page 216} \\
&= \bigcup_{i \in I} ®(A_i) && \text{by Definition 17.5 page 261}
\end{aligned}$$

[11] Kelley (1955) page 8

[12] Kelley (1955) page 8

$$\begin{aligned}
®\left(\bigcap_{i\in I} A_i\right) &= \left\{y\in Y \,|\, \exists x\in \bigcap_{i\in I} A_i \quad \text{such that} \quad (x,y)\in ®\right\} && \text{by Definition 17.5 page 261}\\
&= \left\{y\in Y \,|\, \exists x\in \left\{x\in X \,|\, \bigwedge_{i\in I} x\in A_i\right\} \quad \text{such that} \quad (x,y)\in ®\right\} && \text{by Definition 16.5 page 216}\\
&= \left\{y\in Y \,|\, \exists x\in X \quad \text{such that} \quad \left[\bigwedge_{i\in I} x\in A_i\right] \wedge (x,y)\in ®\right\}\\
&= \left\{y\in Y \,|\, \exists x\in X \quad \text{such that} \quad \bigwedge_{i\in I} \left[x\in A_i \wedge (x,y)\in ®\right]\right\}\\
&\subseteq \left\{y\in Y \,|\, \bigwedge_{i\in I} \left[\exists x\in X \quad \text{such that} \quad x\in A_i \wedge (x,y)\in ®\right]\right\}\\
&= \bigcap_{i\in I} \left\{y\in Y \,|\, \exists x\in X \quad \text{such that} \quad x\in A_i \wedge (x,y)\in ®\right\} && \text{by Definition 16.5 page 216}\\
&= \bigcap_{i\in I} ®(A_i) && \text{by Definition 17.5 page 261}
\end{aligned}$$

✏

Definition 17.6 (next) provides some properties associated with special types of relations. Relations can be defined based on their properties. For example, *equivalence relations* (Definition 1.9 page 7) are *reflexive, symmetric*, and *transitive*; whereas *order relations* are (Definition 4.2 page 58) are *reflexive, anti-symmetric*, and *transitive*.

Definition 17.6. [13] *Let X be a set and ® a relation in 2^{XX}.*

DEF

® is **reflexive**	if	$x®x$	$\forall x\in X$
® is **irreflexive**	if	$(x,x)\notin ®$	$\forall x\in X$
® is **symmetric**	if	$x®y \implies y®x$	$\forall x,y\in X$
® is **asymmetric**	if	$x®y \implies (y,x)\notin ®$	$\forall x,y\in X$
® is **anti-symmetric**	if	$x®y$ and $y®x \implies x=y$	$\forall x,y\in X$
® is **transitive**	if	$x®y$ and $y®z \implies x®z$	$\forall x,y,z\in X$
® is **connected**	if	$x\neq y \implies x®y$ or $y®x$	$\forall x,y,z\in X$
® is **strongly connected**	if	$x®y$ or $y®x$	$\forall x,y,z\in X$

Definition 17.7. [14]

DEF

*The **identity element** ①(X) with respect to $® \in 2^{XX}$ is defined as*

$$①(X) \triangleq \{(x,x)\,|\,(x,x)\in ®\}.$$

The identity element ①(X) may also be denoted as simply ①.

Proposition 17.4. *Let ① be the identity element in 2^{XX} with respect to the composition function $\circ$.*

PRP

$$① \circ ® = ® \circ ① = ® \qquad \forall ® \in 2^{XX}$$

Example 17.17. (Michel and Herget, 1993)411 Let $\boldsymbol{X}$ be a linear space and X^X the set of all functions from $\boldsymbol{X}$ to $\boldsymbol{X}$ (Definition 17.8 page 263). Let $\mathbf{I}$ be an function in X^X. $\mathbf{I}$ is an **identity function** in X^X if $\mathbf{I}x = x \qquad \forall x\in \boldsymbol{X}$.

Theorem 17.5. [15] *Let ® be a relation in 2^{XX}. Let ① be the identity element in 2^{XX} with respect to composition.*

---

[13] Suppes (1972) page 69 ⟨Defintion 10–Definition 17⟩, Kelley (1955) page 9

[14] Kelley (1955) page 9

[15] Kelley (1955) page 9

THM

$$\begin{array}{llrcl}
® \text{ is reflexive} & \iff & ① & \subseteq & ® \\
® \text{ is symmetric} & \iff & ® & = & ®^{-1} \\
® \text{ is anti-symmetric} & \iff & ® \cap ®^{-1} & = & \varnothing \\
® \text{ is transitive} & \iff & ® \circ ® & \subseteq & ® \\
® \text{ is transitive and reflexive} & \implies & ® \circ ® & = & ®
\end{array}$$

✎PROOF:

$$\begin{array}{rlll}
® \text{ is } \textit{reflexive} & \iff (x,x) \in ® \quad \forall x \in X & & \text{by Definition 17.6 page 262} \\
& \iff ① \subseteq ® & & \text{by Definition 17.7 page 262} \\
® \text{ is } \textit{symmetric} & \iff [(x,y) \in ® \implies (y,x) \in ®] & & \text{by Definition 17.6 page 262} \\
& \iff ® = ®^{-1} & & \text{by Definition 17.2 page 255} \\
® \text{ is } \textit{anti-symmetric} & \iff [(x,y) \in ® \implies (y,x) \notin ®] & & \text{by Definition 17.6 page 262} \\
& \iff ® \bigcap ®^{-1} = \varnothing & & \text{by Definition 17.2 page 255} \\
® \text{ is } \textit{transitive} & \iff [(x,y),(y,z) \in ® \implies (x,z) \in ®] & & \text{by Definition 17.6 page 262} \\
& \iff ® \circ ® \subseteq ® & & \text{by Definition 17.3 page 256} \\
® \text{ is } \textit{transitive} \text{ and } \textit{reflexive} & \iff \left[® \circ ® \subseteq ® \text{ and } ① \subseteq ®\right] & & \text{by previous results} \\
& \implies \left[® \circ ® \subseteq ® \text{ and } ® = ① \circ ® \subseteq ® \circ ®\right] & & \text{by definition of ① page 262} \\
& \iff \left[® \circ ® \subseteq ® \text{ and } ® \subseteq ® \circ ®\right] & & \\
& \implies ® \circ ® = ® & &
\end{array}$$

✉

17.2 Functions

The function is a special case of the relation in that while both are subsets of a Cartesian product, an element in the domain of a function can only map to *one* element in the range (Definition 17.8—next definition). The set of all functions in the Cartesian product $X \times Y$ is denoted Y^X; this is suitable because the number of functions in $X \times Y$ for finite X and Y is $|Y|^{|X|}$ (Proposition 17.5 page 264). The fact that not all functions are relations is demonstrated in Example 17.18 (page 264) (discrete cases) and Example 17.19 (page 264) (continuous cases).

17.2.1 Definition and examples

Definition 17.8. [16] *Let X and Y be sets. Let $\wedge$ be the "logical and" operation* (Definition 14.1 page 199).

DEF

*A relation $\mathsf{f} \in 2^{XY}$ is a **function** if*

$$\left(x, y_1\right) \in \mathsf{f} \;\wedge\; \left(x, y_2\right) \in \mathsf{f} \implies y_1 = y_2 \qquad \text{(for each } x \text{, there is only one } \mathsf{f}(x)\text{)}$$

The set of all functions in 2^{XY} is denoted

$$Y^X \triangleq \left\{\mathsf{f} \in 2^{XY} | \mathsf{f} \text{ is a function}\right\}.$$

*A function may also be referred to as a **correspondence**, **transformation**, or **map**.*

As indicated in Definition 17.8 (previous definition), functions customarily come disguised in different names depending on the context in which they are found. This is particularly true with respect to *vector spaces*, as illustrated next:

[16] Suppes (1972) page 86, Kelley (1955) page 10, Bourbaki (1939), Bottazzini (1986), page 7

① *function*: maps from a field to a field
② *functional*: maps from a vector space to a field
③ *function*: maps from a vector space to a vector space

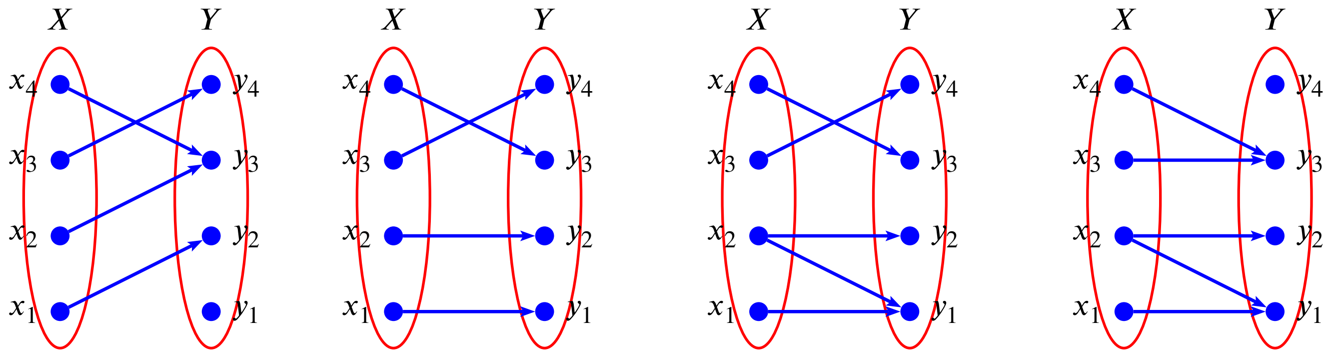
However, no matter what name is used, a function is still a function as long as it satisfies Definition 17.8.

Definition 17.9. [17]

DEF

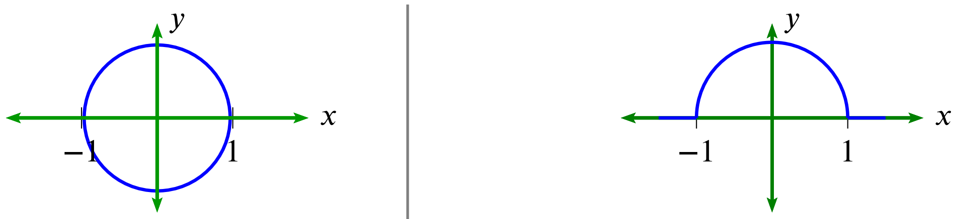
A function $\mathsf{f} \in Y^{X^n}$	*is said to have* ***arity*** n.
A function $\mathsf{f} \in Y^{X^3}$	*is said to be* ***ternary***.
A function $\mathsf{f} \in Y^{X^2}$	*is said to be* ***binary***.
A function $\mathsf{f} \in Y^{X^1} \triangleq Y^X$	*is said to be* ***unary***.
A function $\mathsf{f} \in Y^{X^0} \triangleq Y$	*is said to be* ***nullary***.

Example 17.18. The figure below illustrates two discrete examples of relations that *are* functions and two that are *not*.



two *relations* in 2^{XY} that *are functions* two *relations* in 2^{XY} that are *not functions*

Example 17.19. [18] The figures below illustrates one example of a continuous relation that is *not* a function and one that *is*.



$$\left\{(x,y) \in X \times Y \,|\, x^2 + y^2 = 1\right\}$$

(a relation that is *not* a function)

$$\left\{(x,y) \in X \times Y \,\middle|\, \begin{array}{ll} y = \sqrt{1-x^2} & \text{for } -1 < x < 1 \\ y = 0 & \text{otherwise} \end{array}\right\}$$

(a relation that *is* a function)

Proposition 17.5. [19] *Let* Y^X *be the set of all functions from a set* X *to a set* Y. *Let* $|\cdot|$ *be the counting measure for sets.*

PRP

$$\underbrace{\left|Y^X\right|}_{\text{number of possible functions in } X \times Y} = |Y|^{|X|}$$

[17] Burris and Sankappanavar (2000), pages 25–26
[18] Apostol (1975) page 34
[19] Comtet (1974) page 4

✎PROOF: Let $X \triangleq \{x_1, x_2, \ldots, x_m\}$.
Let $Y \triangleq \{y_1, y_2, \ldots, y_n\}$.
Then x_1 can map to exactly one of the n elements in set Y: y_1, y_2, ..., or y_n.
Likewise, x_2 can also map to one of the n elements in set Y.
So, the total number of possible functions in Y^X is

$$n^m = |Y|^{|X|}$$

✏

Example 17.20. Let $X \triangleq \{x_1, x_2, x_3\}$ and $Y \triangleq \{y_1, y_2\}$. There are a total of $|\mathbb{R}| = 2^{|X|\cdot|Y|} = 2^{3\times 2} = 64$ possible relations on $X \times Y$, as listed in Example 17.2 (page 252). Let $\mathbb{F} \triangleq (\!(\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \ldots)\!)$ be the set of all **functions** from X to Y. There are a total of $|\mathbb{F}| = |Y|^{|X|} = 2^3 = 8$ possible functions. These 8 functions are listed below. Of these 8 functions, 6 are *surjective*, as listed in Example 17.27 (page 268).

functions on $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$	
$\boldsymbol{F}_1 = \{(x_1, y_1), (x_2, y_1), (x_3, y_1)\}$	$\boldsymbol{F}_5 = \{(x_1, y_1), (x_2, y_1), (x_3, y_2)\}$
$\boldsymbol{F}_2 = \{(x_1, y_2), (x_2, y_1), (x_3, y_1)\}$	$\boldsymbol{F}_6 = \{(x_1, y_2), (x_2, y_1), (x_3, y_2)\}$
$\boldsymbol{F}_3 = \{(x_1, y_1), (x_2, y_2), (x_3, y_1)\}$	$\boldsymbol{F}_7 = \{(x_1, y_1), (x_2, y_2), (x_3, y_2)\}$
$\boldsymbol{F}_4 = \{(x_1, y_2), (x_2, y_2), (x_3, y_1)\}$	$\boldsymbol{F}_8 = \{(x_1, y_2), (x_2, y_2), (x_3, y_2)\}$

Example 17.21. Let $X \triangleq \{x, y, z\}$ There are a total of $|\mathbb{R}| = 2^{|X\times X|} = 2^{|X|\cdot|X|} = 2^{3\times 3} = 2^9 = 512$ possible relations on X^2. Of these 512 relations, only 27 are **functions**. These 27 functions are listed below. Of these 27 functions, only 7 are *surjective* functions, as listed in Example 17.28 (page 269).

functions on $\{x, y, z\} \times \{x, y, z\}$	
$\boldsymbol{F}_1 = \{(x,x), (y,x), (z,x)\}$	$\boldsymbol{F}_{15} = \{(x,z), (y,y), (z,y)\}$
$\boldsymbol{F}_2 = \{(x,y), (y,x), (z,x)\}$	$\boldsymbol{F}_{16} = \{(x,x), (y,z), (z,y)\}$
$\boldsymbol{F}_3 = \{(x,z), (y,x), (z,x)\}$	$\boldsymbol{F}_{17} = \{(x,y), (y,z), (z,y)\}$
$\boldsymbol{F}_4 = \{(x,x), (y,y), (z,x)\}$	$\boldsymbol{F}_{18} = \{(x,z), (y,z), (z,y)\}$
$\boldsymbol{F}_5 = \{(x,y), (y,y), (z,x)\}$	$\boldsymbol{F}_{19} = \{(x,x), (y,x), (z,z)\}$
$\boldsymbol{F}_6 = \{(x,z), (y,y), (z,x)\}$	$\boldsymbol{F}_{20} = \{(x,y), (y,x), (z,z)\}$
$\boldsymbol{F}_7 = \{(x,x), (y,z), (z,x)\}$	$\boldsymbol{F}_{21} = \{(x,z), (y,x), (z,z)\}$
$\boldsymbol{F}_8 = \{(x,y), (y,z), (z,x)\}$	$\boldsymbol{F}_{22} = \{(x,x), (y,y), (z,z)\}$
$\boldsymbol{F}_9 = \{(x,z), (y,z), (z,x)\}$	$\boldsymbol{F}_{23} = \{(x,y), (y,y), (z,z)\}$
$\boldsymbol{F}_{10} = \{(x,x), (y,x), (z,y)\}$	$\boldsymbol{F}_{24} = \{(x,z), (y,y), (z,z)\}$
$\boldsymbol{F}_{11} = \{(x,y), (y,x), (z,y)\}$	$\boldsymbol{F}_{25} = \{(x,x), (y,z), (z,z)\}$
$\boldsymbol{F}_{12} = \{(x,z), (y,x), (z,y)\}$	$\boldsymbol{F}_{26} = \{(x,y), (y,z), (z,z)\}$
$\boldsymbol{F}_{13} = \{(x,x), (y,y), (z,y)\}$	$\boldsymbol{F}_{27} = \{(x,z), (y,z), (z,z)\}$
$\boldsymbol{F}_{14} = \{(x,y), (y,y), (z,y)\}$	

Definition 17.10. [20] *Let Y^X be the set of functions from a set X to a set Y.*

DEF

Functions $\mathsf{f} \in Y^X$ *and* $\mathsf{g} \in Y^X$ *are* ***equal*** *if*

$$\mathsf{f}(x) = \mathsf{g}(x) \qquad \forall x \in X$$

This is denoted as $\mathsf{f} \doteq \mathsf{g}$.

[20] Berberian (1961) page 73

17.2.2 Properties of functions

Theorem 17.6. [21] *Let* f *be a* FUNCTION (Definition 17.8 page 263) *in* Y^X *with inverse relation* f^{-1} *in* 2^{XY}.

THM

1.	$\mathsf{f}(\varnothing) = \varnothing$	$\forall \mathsf{f}\in Y^X$	
2.	$\mathsf{f}^{-1}(\varnothing) = \varnothing$	$\forall \mathsf{f}\in Y^X$	
3.	$A \subseteq B \implies \mathsf{f}(A) \subseteq \mathsf{f}(B)$	$\forall \mathsf{f}\in Y^X, A,B\in 2^X$	(ISOTONE)
4.	$A \subseteq B \implies \mathsf{f}^{-1}(A) \subseteq \mathsf{f}^{-1}(B)$	$\forall \mathsf{f}\in Y^X, A,B\in 2^Y$	(ISOTONE)

✎PROOF:

1. Proof that $\mathsf{f}(\varnothing) = \varnothing$:

$$\begin{aligned}
\mathsf{f}(\varnothing) &= \{y \in Y | \exists x \in \varnothing \text{ such that } (x,y) \in \mathsf{f}\} && \text{by Definition 17.5 page 261}\\
&= \varnothing && \text{by definition of } \varnothing \text{ page 5}
\end{aligned}$$

2. Proof that $A \subseteq B \implies \mathsf{f}(A) \subseteq \mathsf{f}(B)$:

$$\begin{aligned}
\mathsf{f}(A) &= \{y \in Y | \exists x \in A \text{ such that } (x,y) \in \mathsf{f}\} && \text{by Definition 17.5 page 261}\\
&\subseteq \{y \in Y | \exists x \in B \text{ such that } (x,y) \in \mathsf{f}\} && \text{by left hypothesis}\\
&= \mathsf{f}(B) && \text{by Definition 17.5 page 261}
\end{aligned}$$

3. Proof that $\mathsf{f}^{-1}(\varnothing) = \varnothing$:

$$\begin{aligned}
\mathsf{f}^{-1}(\varnothing) &= \{x \in X | \exists y \in \varnothing \text{ such that } (x,y) \in \mathsf{f}\} && \text{by Definition 17.5 page 261}\\
&= \varnothing && \text{by definition of } \varnothing \text{ page 5}
\end{aligned}$$

4. Proof that $A \subseteq B \implies \mathsf{f}^{-1}(A) \subseteq \mathsf{f}^{-1}(B)$:

$$\begin{aligned}
\mathsf{f}^{-1}(A) &= \left\{x \in X | \exists y \in A \text{ such that } (x,y) \in \mathsf{f}^{-1}\right\} && \text{by Definition 17.5 page 261}\\
&\subseteq \{x \in X | \exists y \in B \text{ such that } (x,y) \in \mathsf{f}\} && \text{by left hypothesis}\\
&= \mathsf{f}^{-1}(B) && \text{by Definition 17.5 page 261}
\end{aligned}$$

✑

17.2.3 Types of functions

In general, a function $\mathsf{f} \in Y^X$ can be described as "*into*" because f maps each element of X *into* Y such that $\mathsf{f}(X) \subseteq Y$. However there are some common more restrictive special types of functions. These are defined in Definition 17.11 (next defintion).

Definition 17.11. [22] *Let* $\mathsf{f} \in Y^X$.

DEF

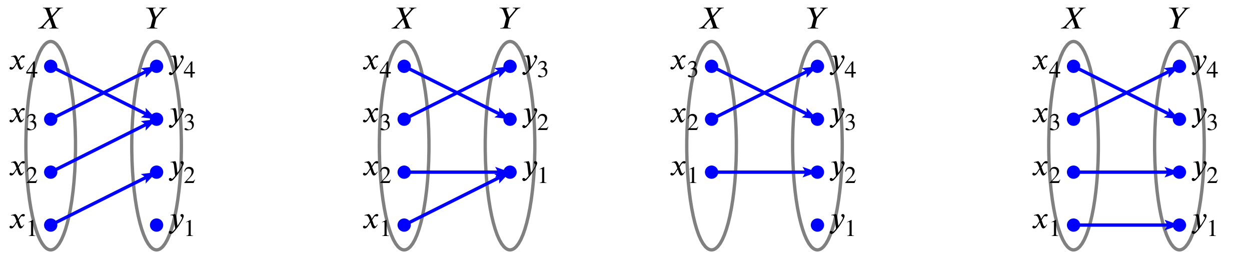
f *is* ***surjective***	*(also called* ***onto****)*	*iff* $\mathsf{f}(X) = Y$
f *is* ***injective***	*(also called* ***one-to-one****)*	*iff* $\mathsf{f}(x_n) = \mathsf{f}(x_m) \implies x_n = x_m$
f *is* ***bijective***	*(also called* ***one-to-one and onto****)*	*iff* f *is both surjective and injective.*

We also define the following sets of functions:

$$\begin{aligned}
\mathcal{S}_{\mathrm{j}}(X,Y) &\triangleq \left\{\mathsf{f} \in Y^X | \mathsf{f} \text{ is surjective}\right\} && \text{(the set of all surjective functions in } Y^X\text{)}\\
\mathcal{I}_{\mathrm{j}}(X,Y) &\triangleq \left\{\mathsf{f} \in Y^X | \mathsf{f} \text{ is injective}\right\} && \text{(the set of all injective functions in } Y^X\text{)}\\
\mathcal{B}_{\mathrm{j}}(X,Y) &\triangleq \left\{\mathsf{f} \in Y^X | \mathsf{f} \text{ is bijective}\right\} && \text{(the set of all bijective functions in } Y^X\text{)}
\end{aligned}$$

[21] Davis (2005) pages 6–8, Vaidyanathaswamy (1960) page 10
[22] Michel and Herget (1993), pages 14–15, Fuhrmann (2012) page 2, Comtet (1974) page 5

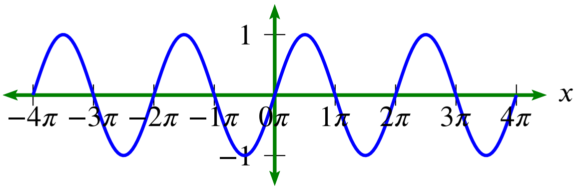
The types described in Definition 17.11 are illustrated below:



"into" (arbitrary function in Y^X)	*"onto"* *surjective*	*"one-to-one"* *injective*	*"one-to-one and onto"* *bijective*

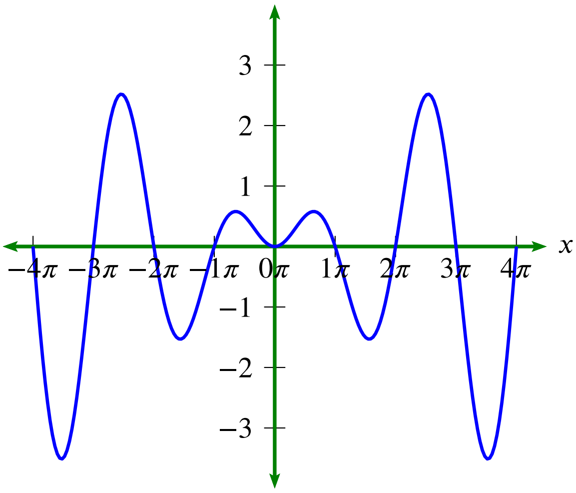
Example 17.22.

In the set $\mathbb{R}^{\mathbb{R}}$, the function $\sin x$ is *not injective, not surjective*, and *not bijective*.



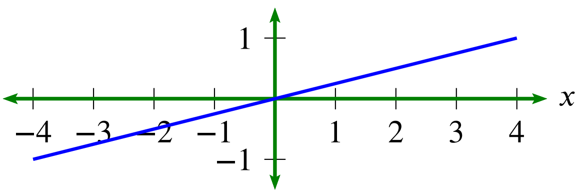
Example 17.23.

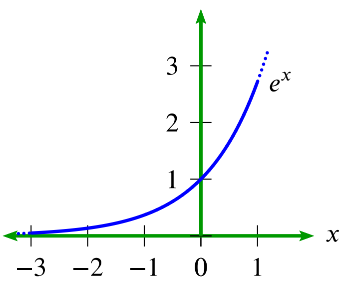
In the set $\mathbb{R}^{\mathbb{R}}$, the function $x \sin x$ is *surjective*, but *not injective* and *not bijective*.



Example 17.24.

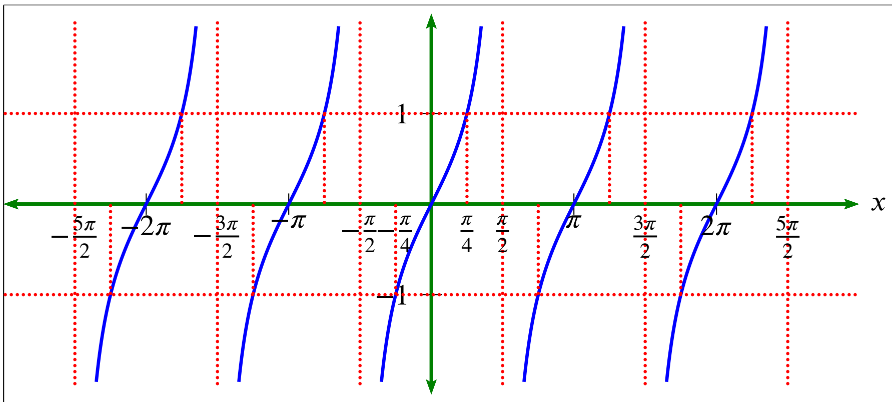
In the set $\mathbb{R}^{\mathbb{R}}$, the function $y = \frac{1}{4}x$ is *injective, surjective*, and *bijective*.





Example 17.25. In the set $\mathbb{R}^{\mathbb{R}}$, the function e^x is *injective*, but *not surjective* and *not bijective*.

Example 17.26. In the set $\mathbb{R}^{\mathbb{R}}$, the function $\tan x$ is *not injective, not surjective* (it's range does not include $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc.) and *not bijective*.



Theorem 17.7. [23]

THM

$$\begin{array}{llllll}
\mathsf{f} \textit{ and } \mathsf{g} & \textit{are} & \textit{surjective} & \Longrightarrow & \mathsf{g}\circ\mathsf{f} & \textit{is surjective} \\
\mathsf{g}\circ\mathsf{f} & \textit{is} & \textit{surjective} & \Longrightarrow & \mathsf{g} & \textit{is surjective} \\
\mathsf{f} \textit{ and } \mathsf{g} & \textit{are} & \textit{injective} & \Longrightarrow & \mathsf{g}\circ\mathsf{f} & \textit{is injective} \\
\mathsf{g}\circ\mathsf{f} & \textit{is} & \textit{injective} & \Longrightarrow & \mathsf{f} & \textit{is injective}
\end{array}$$

✎PROOF:

$$\begin{array}{rcll}
\mathsf{f},\mathsf{g} \text{ are surjective} & \Longrightarrow & \mathsf{f}(X)=Y, \text{ and } \mathsf{g}(Y)=Z & \text{by definition of surjective page 266} \\
& \Longrightarrow & \mathsf{g}\circ\mathsf{f}(X)=\mathsf{g}(Y)=Z & \\
& \Longrightarrow & \mathsf{g}\circ\mathsf{f} \text{ is surjective} & \text{by definition of surjective page 266}
\end{array}$$

$$\begin{array}{rcll}
\mathsf{g}\circ\mathsf{f} \text{ is surjective} & \Longrightarrow & \mathsf{g}\circ\mathsf{f}(X)=Z & \text{by definition of surjective page 266} \\
& \Longrightarrow & \mathsf{g}(\mathsf{f}(X))=Z & \\
& \Longrightarrow & \mathsf{g}(Y)=Z & \text{because } \mathsf{f}(X)\subseteq Y \text{ and by isotone property page 266} \\
& \Longrightarrow & \mathsf{g} \text{ is surjective} & \text{by definition of surjective page 266}
\end{array}$$

$$\begin{array}{rcll}
\mathsf{g}\circ\mathsf{f}(x_1)=\mathsf{g}\circ\mathsf{f}(x_2) & \Longrightarrow & \mathsf{g}(\mathsf{f}(x_1))=\mathsf{g}(\mathsf{f}(x_2)) & \\
& \Longrightarrow & \mathsf{f}(x_1)=\mathsf{f}(x_2) & \text{because } \mathsf{g} \text{ is injective} \\
& \Longrightarrow & x_1=x_2 & \text{because } \mathsf{f} \text{ is injective} \\
& \Longrightarrow & \mathsf{g}\circ\mathsf{f} \text{ is injective} &
\end{array}$$

$$\begin{array}{rcll}
\mathsf{f}(x_1)=\mathsf{f}(x_2) & \Longrightarrow & \mathsf{g}(\mathsf{f}(x_1))=\mathsf{g}(\mathsf{f}(x_2)) & \\
& \Longrightarrow & \mathsf{g}\circ\mathsf{f}(x_1)=\mathsf{g}\circ\mathsf{f}(x_2) & \\
& \Longrightarrow & x_1=x_2 & \text{because } \mathsf{g}\circ\mathsf{f} \text{ is injective} \\
& \Longrightarrow & \mathsf{f} \text{ is injective} &
\end{array}$$

✏

Theorem 17.8 (Bernstein-Cantor-Schröder Theorem). [24]

$$\left(\exists\mathsf{f}\in\mathcal{I}_{\mathsf{j}}(X,Y)\right) \textit{ and } \left(\exists\mathsf{g}\in\mathcal{I}_{\mathsf{j}}(Y,X)\right) \quad\Longrightarrow\quad \exists\mathsf{h}\in\mathcal{B}_{\mathsf{j}}(X,Y)$$

Example 17.27. Let $X \triangleq \{x_1,\ x_2,\ x_3\}$ and $Y \triangleq \{y_1,\ y_2\}$. There are a total of $|\,\mathbb{R}\,| = 2^{3\times 2} = 64$ possible relations, as listed in Example 17.2 (page 252). There are a total of $|\,\mathbb{F}\,| = 2^3 = 8$ possible functions, as listed in Example 17.20 (page 265). Let $\mathbb{S} \triangleq ⦇\boldsymbol{S}_1,\ \boldsymbol{S}_2,\ \boldsymbol{S}_3,\ \ldots⦈$ be the set of all **surjective** functions

[23] Durbin (2000), pages 16–17

[24] Schröder (2003), page 116, Nievergelt (2002), page 213, Suppes (1972) page 95, Fraenkel (1953), pages 102–103 ⟨???⟩

from X to Y. There are a total of $|\mathbb{S}| = 6$ possible surjective functions, as listed next:

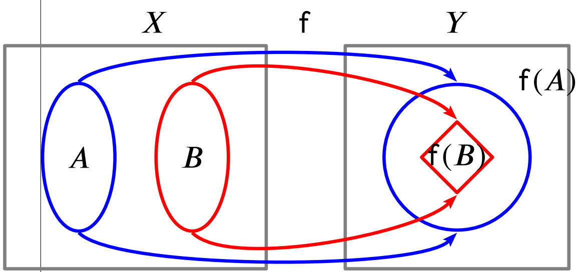
surjective functions on $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$	
$S_1 = \{(x_1, y_2), (x_2, y_1), (x_3, y_1)\}$	$S_4 = \{(x_1, y_1), (x_2, y_1), (x_3, y_2)\}$
$S_2 = \{(x_1, y_1), (x_2, y_2), (x_3, y_1)\}$	$S_5 = \{(x_1, y_2), (x_2, y_1), (x_3, y_2)\}$
$S_3 = \{(x_1, y_2), (x_2, y_2), (x_3, y_1)\}$	$S_6 = \{(x_1, y_1), (x_2, y_2), (x_3, y_2)\}$

Example 17.28. Let $X \triangleq \{x, y, z\}$ There are a total of $|\mathbb{R}| = 2^{|X\times X|} = 2^{|X|\cdot|X|} = 2^{3\times 3} = 2^9 = 512$ possible relations on $X \times X$. Of these 512 relations, only 27 are **functions**. These 27 functions are listed in Example 17.21 (page 265). Of these 27 functions, only 7 are *surjective* functions, as listed below. Actually, in the case of a function mapping from a finite set onto the same finite set, The set $\mathbb{S}$ of surjective functions is equal to the set of injective functions and the set of bijective functions.

surjective functions on $\{x, y, z\} \times \{x, y, z\}$	
$S_1 = \{(x, z), (y, x), (z, x)\}$	$S_5 = \{(x, x), (y, z), (z, y)\}$
$S_2 = \{(x, z), (y, y), (z, x)\}$	$S_6 = \{(x, y), (y, x), (z, z)\}$
$S_3 = \{(x, y), (y, z), (z, x)\}$	$S_7 = \{(x, x), (y, y), (z, z)\}$
$S_4 = \{(x, z), (y, x), (z, y)\}$	

17.2.4 Image relations

Consider two subsets A and B of the domain of a function f. What is the relationship between the image under f of their union and the union of their images under f? Are they equal? Is one a subset of the other? What is the relationship between the image of their intersection under f and the intersection of their images f? Theorem 17.9 (next theorem) answers these questions.



Theorem 17.9. [25] *Let* f *be a function in* Y^X.

THM

$$\mathsf{f}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathsf{f}(A_i) \qquad \forall \mathsf{f}\in Y^X, A_i\in 2^X \qquad \text{(additive)}$$

$$\mathsf{f}\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} \mathsf{f}(A_i) \qquad \forall \mathsf{f}\in Y^X, A_i\in 2^X$$

✎PROOF:

[25] Davis (2005) pages 6–7, Vaidyanathaswamy (1960) page 10

1. Proof that $\mathsf{f}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathsf{f}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}\left(\bigcup_{i\in I} A_i\right) &= \left\{ y \in Y | \exists x \in \bigcup_{i\in I} A_i \quad \text{such that} \quad (x,y) \in \mathsf{f} \right\} && \text{by Definition 17.5 page 261}\\
&= \bigcup_{i\in I} \left\{ y \in Y | \exists x \in A_i \quad \text{such that} \quad (x,y)\in \mathsf{f} \right\}\\
&= \bigcup_{i\in I} \mathsf{f}\left(A_i\right) && \text{by Definition 17.5 page 261}
\end{aligned}$$

2. Proof that $\mathsf{f}\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} \mathsf{f}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}\left(\bigcap_{i\in I} A_i\right) &= \left\{ y \in Y | \exists x \in \bigcap_{i\in I} A_i \quad \text{such that} \quad (x,y) \in \mathsf{f} \right\} && \text{by Definition 17.5 page 261}\\
&= \left\{ y \in Y | \exists x \quad \text{such that} \quad \bigwedge_{i\in I} [x \in A_i] \text{ and } (x,y)\in \mathsf{f}] \right\} && \text{by Definition 16.5 page 216}\\
&\subseteq \left\{ y \in Y | \bigwedge_{i\in I} [\exists x \in A_i \quad \text{such that} \quad (x,y)\in \mathsf{f}] \right\}\\
&= \bigcap_{i\in I} \left\{ y \in Y | \exists x \in A_i \quad \text{such that} \quad (x,y)\in \mathsf{f} \right\} && \text{by Definition 16.5 page 216}\\
&= \bigcap_{i\in I} \mathsf{f}\left(A_i\right) && \text{by Definition 17.5 page 261}
\end{aligned}$$

✏

Theorem 17.10. [26] *Let $\mathsf{f}^{-1} \in X^Y$ be the inverse of a function $\mathsf{f} \in Y^X$.*

THM

$$\begin{array}{rcll}
\mathsf{f}^{-1}(Y) &=& X & \forall \mathsf{f}\in Y^X\\
\mathsf{f}^{-1}(A^c) &=& \mathsf{c}\left[\mathsf{f}^{-1}(A)\right] & \forall \mathsf{f}\in Y^X, A\in 2^Y\\
\mathsf{f}^{-1}\left(\bigcup_{i\in I} A_i\right) &=& \bigcup_{i\in I} \mathsf{f}^{-1}\left(A_i\right) & \forall \mathsf{f}\in Y^X, A_i\in 2^Y\\
\mathsf{f}^{-1}\left(\bigcap_{i\in I} A_i\right) &=& \bigcap_{i\in I} \mathsf{f}^{-1}\left(A_i\right) & \forall \mathsf{f}\in Y^X, A_i\in 2^Y
\end{array}$$

✎PROOF:

1. Proof that $\mathsf{f}^{-1}(A^c) = \mathsf{c}\left[\mathsf{f}^{-1}(A)\right]$:

$$\begin{aligned}
\mathsf{c}\left[\mathsf{f}^{-1}(Y)\right] &= \mathsf{c}\left\{ x \in X | \exists y \in A \quad \text{such that} \quad (x,y)\in \mathsf{f} \right\} && \text{by Definition 17.5 page 261}\\
&= \left\{ x \in X | \neg\{\exists y \in A \quad \text{such that} \quad (x,y)\in \mathsf{f}\} \right\} && \text{by Definition 16.5 page 216}\\
&= \left\{ x \in X | \nexists y \in A \quad \text{such that} \quad (x,y)\in \mathsf{f} \right\} && \text{by Definition 16.5 page 216}\\
&= \left\{ x \in X | \exists y \in A^c \quad \text{such that} \quad (x,y)\in \mathsf{f} \right\}\\
&= \mathsf{f}^{-1}(A^c) && \text{by Definition 17.5 page 261}
\end{aligned}$$

[26] Davis (2005) pages 7–8, Vaidyanathaswamy (1960) page 10

2. Proof that $\mathsf{f}^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}\mathsf{f}^{-1}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}^{-1}\left(\bigcup_{i\in I} A_i\right) &= \left\{ x\in X | \exists y\in\bigcup_{i\in I} A_i \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\} && \text{by Definition 17.5 page 261}\\
&= \left\{ x\in X | \bigvee_{i\in I}\left\{\exists y\in A_i \quad \text{such that} \quad (x,y)\in\mathsf{f}\right\} \right\}\\
&= \bigcup_{i\in I}\left\{\exists x\in X | y\in A_i \quad \text{such that} \quad (x,y)\in\mathsf{f}\right\} && \text{by Definition 16.5 page 216}\\
&= \bigcup_{i\in I}\mathsf{f}^{-1}(A_i) && \text{by Definition 17.5 page 261}
\end{aligned}$$

3. Proof that $\mathsf{f}^{-1}(Y) = X$:

$$\begin{aligned}
\mathsf{f}^{-1}(Y) &= \mathsf{f}^{-1}(\mathcal{I}X \cup Y\setminus\mathcal{I}X)\\
&= \mathsf{f}^{-1}(\mathcal{I}X)\cup\mathsf{f}^{-1}(Y\setminus\mathcal{I}X) && \text{by item 4}\\
&= X\cup\varnothing && \text{by Definition 17.4 page 258}\\
&= X
\end{aligned}$$

4. Proof that $\mathsf{f}^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I}\mathsf{f}^{-1}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}^{-1}\left(\bigcap_{i\in I} A_i\right) &= \left\{ x\in X | \exists y\in\bigcap_{i\in I} A_i \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\} && \text{by Definition 17.5 page 261}\\
&= \left\{ x\in X | \exists y \quad \text{such that} \quad \left\{ y\in\bigwedge_{i\in I} A_i \text{ and } (x,y)\in\mathsf{f} \right\} \right\} && \text{by Definition 16.5 page 216}\\
&= \left\{ x\in X | \bigwedge_{i\in I}[\exists y\in A_i \quad \text{such that} \quad (x,y)\in\mathsf{f}] \right\} && \text{by definition of function page 263}\\
&= \bigcap_{i\in I}\left\{x\in X | \exists y\in A_i \quad \text{such that} \quad (x,y)\in\mathsf{f}\right\} && \text{by Definition 16.5 page 216}\\
&= \bigcap_{i\in I}\mathsf{f}^{-1}\left(A_i\right) && \text{by Definition 17.5 page 261}
\end{aligned}$$

5. Proof that $\mathsf{f}^{-1}(Y\setminus A) = X\setminus\mathsf{f}^{-1}(A)$:

$$\begin{aligned}
\mathsf{f}^{-1}(Y\setminus A) &= \mathsf{f}^{-1}(Y\cap A^{\mathsf{c}})\\
&= \mathsf{f}^{-1}(Y)\cap\mathsf{f}^{-1}(A^{\mathsf{c}}) && \text{by 6.}\\
&= X\cap\mathsf{f}^{-1}(A^{\mathsf{c}}) && \text{by 5.}\\
&= X\cap\mathsf{c}\left[\mathsf{f}^{-1}(A)\right] && \text{by 3.}\\
&= X\setminus\mathsf{f}^{-1}(A) && \text{by Definition 16.5 page 216}
\end{aligned}$$

17.2.5 Indicator functions

By the *axiom of extension*, a set is uniquely defined by the elements that are in that set. Thus, we are often interested in the Boolean result of whether an element is in a set A, or is not in A, but exclude the possibility of both being true. That a statement is either true or false but definitely not both is called *the law of the excluded middle* and is a fundamental property of all *Boolean algebras*

$(\{1,0\},\vee,\wedge)$. [27] The *indicator function* (next definition) is a convenient "indicator" of whether or not a particular element is in a set, and has several interesting properties (Theorem 17.11 page 272).

Definition 17.12. [28] *Let X be a set.*

DEF

*The **indicator function** $\mathbb{1} \in \{0,1\}^{2^X}$ is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \qquad \forall x\in X,\ A\in 2^X \\ 0 & \text{for } x \notin A \qquad \forall x\in X,\ A\in 2^X \end{cases}$$

*The indicator function $\mathbb{1}$ is also called the **characteristic function**.*

Theorem 17.11. [29] *Let $\mathbb{1}$ be the* INDICATOR FUNCTION *(Definition 17.12 page 272). Let $x \vee y$ represent the maximum of $\{x,y\}$.*

THM

$$\begin{array}{rclrcl} \mathbb{1}_\varnothing &=& 0 & \mathbb{1}_X &=& 1 \\ \mathbb{1}_{A\cup B} &=& \mathbb{1}_A \vee \mathbb{1}_B & \mathbb{1}_{A\cap B} &=& \mathbb{1}_A \mathbb{1}_B \\ \mathbb{1}_{A\triangle B} &=& \mathbb{1}_A \mathbb{1}_B & \mathbb{1}_{A\setminus B} &=& \mathbb{1}_A\left(1-\mathbb{1}_B\right) \\ \mathbb{1}_{A^c} &=& 1-\mathbb{1}_A & & & \end{array}$$

✎PROOF:

$$\begin{aligned} \mathbb{1}_{A\cup B}(x) &\triangleq \begin{cases} 1 & \text{for } x \in A\cup B \quad \forall x\in X \\ 0 & \text{for } x \notin A\cup B \quad \forall x\in X \end{cases} && \text{by Definition 17.12} \\ &= \begin{cases} 1 & \text{for } x \in A \vee x \in B \quad \forall x\in X \\ 0 & \text{otherwise} \end{cases} && \text{by Definition 16.5 page 216} \\ &= \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{otherwise} \end{cases} \vee \begin{cases} 1 & \text{for } x \in B \\ 0 & \text{otherwise} \end{cases} \\ &= \mathbb{1}_A(x) \vee \mathbb{1}_B(x) && \text{by Definition 17.12} \end{aligned}$$

$$\begin{aligned} \mathbb{1}_{A\cap B}(x) &\triangleq \begin{cases} 1 & \text{for } x \in A\cap B \quad \forall x\in X \\ 0 & \text{for } x \notin A\cap B \quad \forall x\in X \end{cases} && \text{by Definition 17.12} \\ &= \begin{cases} 1 & \text{for } x \in A \wedge x \in B \quad \forall x\in X \\ 0 & \text{otherwise} \end{cases} && \text{by Definition 16.5 page 216} \\ &= \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{otherwise} \end{cases} \wedge \begin{cases} 1 & \text{for } x \in B \\ 0 & \text{otherwise} \end{cases} \\ &= \mathbb{1}_A(x) \wedge \mathbb{1}_B(x) \\ &= \mathbb{1}_A \mathbb{1}_B && \text{by Definition 17.12} \end{aligned}$$

$$\begin{aligned} \mathbb{1}_{A^c}(x) &= \begin{cases} 1 & \text{for } x \in A^c \quad \forall x\in X \\ 0 & \text{for } x \notin A^c \quad \forall x\in X \end{cases} && \text{by Definition 17.12} \\ &= \begin{cases} 1 & \text{for } x \notin A \quad \forall x\in X \\ 0 & \text{for } x \in A \quad \forall x\in X \end{cases} && \text{by Definition 16.5 page 216} \\ &= 1-\mathbb{1}_A \end{aligned}$$

$$\begin{aligned} \mathbb{1}_{A\setminus B} &= \mathbb{1}_{A\cap B^c} \\ &= \mathbb{1}_A \mathbb{1}_{B^c} \\ &= \mathbb{1}_A\left(1-\mathbb{1}_B\right) \end{aligned}$$

[27] excluded middle: Theorem 14.2 page 199

[28] Feller (1971), page 104, Aliprantis and Burkinshaw (1998), page 126, Hausdorff (1937), page 22, de la Vallée-Poussin (1915) page 440

[29] Aliprantis and Burkinshaw (1998), page 126, Hausdorff (1937), pages 22–23

$$\begin{aligned}
\mathbb{1}_{A\triangle B} &= \mathbb{1}_{(A\setminus B^c)\cup(B\setminus A^c)}\\
&= \left(\mathbb{1}_{A\setminus B^c}\right)\vee\left(\mathbb{1}_{B\setminus A^c}\right)\\
&= \left[\mathbb{1}_A\left(1-\mathbb{1}_{B^c}\right)\right]\vee\left[\mathbb{1}_B\left(1-\mathbb{1}_{A^c}\right)\right]\\
&= \left[\mathbb{1}_A\left(1-1+\mathbb{1}_B\right)\right]\vee\left[\mathbb{1}_B\left(1-1+\mathbb{1}_A\right)\right]\\
&= \left[\mathbb{1}_A\mathbb{1}_B\right]\vee\left[\mathbb{1}_B\mathbb{1}_A\right]\\
&= \mathbb{1}_A\mathbb{1}_B
\end{aligned}$$

$$\begin{aligned}
\mathbb{1}_{\varnothing} &= \mathbb{1}_{A\setminus A}\\
&= \mathbb{1}_A\left(1-\mathbb{1}_A\right)\\
&= \mathbb{1}_A-\mathbb{1}_A\mathbb{1}_A\\
&= \mathbb{1}_A-\mathbb{1}_A\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{1}_X &= \mathbb{1}_{A\cup A^c}\\
&= \mathbb{1}_A\vee\mathbb{1}_{A^c}\\
&= \mathbb{1}_A\vee\left(1-\mathbb{1}_A\right)\\
&= 1
\end{aligned}$$

⇨

17.2.6 Calculus of functions

Definition 17.13. [30] *Let Y^X be the set of all functions from a set X to a set Y.*

DEF

$[-\mathsf{f}](x)$	$\triangleq$	$-[\mathsf{f}(x)]$	$\forall x\in X, \mathsf{f}\in Y^X$	(NEGATION)
$\left[\mathsf{f}\ \dot{+}\ \mathsf{g}\right](x)$	$\triangleq$	$\mathsf{f}(x)+\mathsf{g}(x)$	$\forall x\in X\ \mathsf{f},\mathsf{g}\in Y^X$	(FUNCTION ADDITION)
$\left[\mathsf{f}-\mathsf{g}\right](x)$	$\triangleq$	$\mathsf{f}(x)+\left[-\mathsf{g}\right](x)$	$\forall x\in X\ \mathsf{f},\mathsf{g}\in Y^X$	(FUNCTION SUBTRACTION)
$\left[\mathsf{g}\mathsf{f}\right](x)$	$\triangleq$	$\mathsf{g}[\mathsf{f}(x)]$	$\forall x\in X\ \mathsf{f},\mathsf{g}\in Y^X$	(FUNCTION MULTIPLICATION)
$[\alpha\mathsf{f}](x)$	$\triangleq$	$\alpha[\mathsf{f}(x)]$	$\forall x\in X, \alpha\in\mathbf{Y}\ \mathsf{f}\in Y^X$	(SCALAR MULTIPLICATION)

Definition 17.14. *Let f be a function in X^X with inverse relation f^{-1} and let $\mathbf{I}$ be the identity function in X^X.*

DEF

$$\mathsf{f}^n \triangleq \begin{cases}\mathbf{I} & \text{for } n=0\\ \prod_1^n \mathsf{f} & \text{for } n\in\mathbb{N}\\ \left(\mathsf{f}^{-1}\right)^n & \text{for } n\in\mathbb{Z}^-\end{cases}$$

Theorem 17.12. [31] *Let X, Y, and Z be sets.*

THM

1.	$(\mathsf{f}\mathsf{g})^{-1}$	$=$	$\left(\mathsf{g}^{-1}\right)\left(\mathsf{f}^{-1}\right)$	$\forall \mathsf{f}\in Y^X, \mathsf{g}\in Z^Y$	(IDEMPOTENT)
2.	$\mathsf{h}(\mathsf{g}\mathsf{f})$	$=$	$(\mathsf{h}\mathsf{g})\mathsf{f}$	$\forall \mathsf{f}\in X^W, \mathsf{g}\in Y^X, \mathsf{h}\in Z^Y$	(ASSOCIATIVE)
3.	$\left(\mathsf{f}\ \dot{+}\ \mathsf{g}\right)\mathsf{h}$	$=$	$(\mathsf{f}\mathsf{h})\ \dot{+}\ (\mathsf{g}\mathsf{h})$	$\forall \mathsf{f},\mathsf{g}\in Y^X, \mathsf{h}\in Z^Y$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathsf{f}\mathsf{g})$	$=$	$(\alpha\mathsf{f})\mathsf{g}$	$\forall \mathsf{f}\in Y^X, \mathsf{g}\in Z^Y$	(HOMOGENOUS)

✎PROOF:

[30] Michel and Herget (1993) page 409, Cayley (1858), Riesz (1913), Hilbert et al. (1927) page 6
[31] Kelley (1955) page 8, Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

1. Proof of the *idempotent* property:

   (a) Note that $\mathsf{f}\mathsf{g} = \mathsf{f} \circ \mathsf{g}$, where $\circ$ is the *composition function* (Definition 17.3 page 256).

   (b) The result follows from Theorem 17.2 (page 257), where it is demonstrated to be true for the more general case of *relations*.

2. Proof of the *associative* property: This result follows from Theorem 17.2 (page 257), where it is demonstrated to be true for the more general case of *relations*.

3. Proof of the *right distributive* property:

$$
\begin{aligned}
\left[(\mathsf{f} \mathring{+} \mathsf{g})\mathsf{h}\right]x &= (\mathsf{f} \mathring{+} \mathsf{g})(\mathsf{h}x) && \text{by Definition 17.13 page 273}\\
&= [\mathsf{f}(\mathsf{h}x)] \mathring{+} \left[\mathsf{g}(\mathsf{h}x)\right] && \text{by Definition 17.13 page 273}\\
&= [(\mathsf{f}\mathsf{h})x] \mathring{+} \left[(\mathsf{g}\mathsf{h})x\right] && \text{by Definition 17.13 page 273}
\end{aligned}
$$

4. Proof of the *homogeneous* property:

$$
\begin{aligned}
\left[\alpha\left[\mathsf{f}\mathsf{g}\right]\right](x) &= \alpha\left[\left[\mathsf{f}\mathsf{g}\right](x)\right] && \text{by Definition 17.13 page 273}\\
&= \alpha\left[\mathsf{f}\left[\mathsf{g}(x)\right]\right] && \text{by Definition 17.13 page 273}\\
&= [\alpha\mathsf{f}]\left[\mathsf{g}(x)\right] && \text{by Definition 17.13 page 273}\\
&= \left[[\alpha\mathsf{f}]\mathsf{g}\right](x) && \text{by Definition 17.13 page 273}
\end{aligned}
$$

✏

Theorem 17.13. *Let $\mathcal{A} \triangleq X^X$ be the set of functions on X^X.*

THM

1. $(\mathcal{A}, \mathring{+})$ *is an additive group.*
2. $(\mathcal{A}, \mathring{+}, \cdot)$ *is a ring.*
3. $(\mathcal{A}, \mathring{+})$ *is a linear space.*
4. $(\mathcal{A}, \mathring{+}, \cdot)$ *is an algebra.*

✎Proof:

1. additive group:

1.	$\mathsf{f} \mathring{+} \mathbb{0}$	$=$	$\mathbb{0} + \mathsf{f} = \mathsf{f}$	$\forall \mathsf{f}\in\mathcal{A}$	($\mathbb{0} \in \mathcal{A}$ is the identity element)
2.	$\mathsf{f} \mathring{+} (-\mathsf{f})$	$=$	$(-\mathsf{f}) + \mathsf{f} = \mathbb{0}$	$\forall \mathsf{f}\in\mathcal{A}$	($(-\mathsf{f})$ is the inverse of f)
3.	$(\mathsf{f} \mathring{+} \mathsf{g}) + \mathsf{h}$	$=$	$\mathsf{f} \mathring{+} (\mathsf{g} + \mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h}\in\mathcal{A}$	($(\mathcal{A}, \cdot)$ is associative)

2. ring:

1.	$(\mathcal{A}, +, *)$ is a group with respect to $(\mathcal{A}, +)$				(additive group)
2.	$\mathsf{f}(\mathsf{g}\mathsf{h})$	$=$	$(\mathsf{f}\mathsf{g})\mathsf{h}$	$\forall \mathsf{f},\mathsf{g},\mathsf{h}\in\mathcal{A}$	(associative with respect to $*$)
3.	$\mathsf{f}(\mathsf{g} + \mathsf{h})$	$=$	$(\mathsf{f}\mathsf{g}) + (\mathsf{f}\mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h}\in\mathcal{A}$	($*$ is left distributive over $+$)
4.	$(\mathsf{f} \mathring{+} \mathsf{g})\mathsf{h}$	$=$	$(\mathsf{f}\mathsf{h}) + (\mathsf{g}\mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h}\in\mathcal{A}$	($*$ is right distributive over $+$).

3. linear space:

1.			$(\mathsf{f} \mathring{+} \mathsf{g}) \mathring{+} \mathsf{h}$	$=$	$\mathsf{f} \mathring{+} (\mathsf{g} \mathring{+} \mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h}\in\mathcal{A}$	($\mathring{+}$ is *associative*)
2.			$\mathsf{f} \mathring{+} \mathsf{g}$	$=$	$\mathsf{g} \mathring{+} \mathsf{f}$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{A}$	($\mathring{+}$ is *commutative*)
3.	$\exists \mathbb{0} \in X$	such that	$\mathsf{f} \mathring{+} \mathbb{0}$	$=$	f	$\forall \mathsf{f}\in X\mathcal{A}$	($\mathring{+}$ identity)
4.	$\exists \mathsf{g} \in X$	such that	$\mathsf{f} \mathring{+} \mathsf{g}$	$=$	$\mathbb{0}$	$\forall \mathsf{f}\in\mathcal{A}$	($\mathring{+}$ inverse)
5.			$\alpha \otimes (\mathsf{f} \mathring{+} \mathsf{g})$	$=$	$(\alpha \otimes \mathsf{f}) \mathring{+} (\alpha \otimes \mathsf{g})$	$\forall \alpha\in S$ and $\mathsf{f},\mathsf{g}\in\mathcal{A}$	($\otimes$ *distributes* over $\mathring{+}$)
6.			$(\alpha + \beta) \otimes \mathsf{f}$	$=$	$(\alpha \otimes \mathsf{f}) \mathring{+} (\beta \otimes \mathsf{f})$	$\forall \alpha,\beta\in S$ and $\mathsf{f}\in\mathcal{A}$	($\otimes$ pseudo-distributes over $+$)
7.			$\alpha(\beta \otimes \mathsf{f})$	$=$	$(\alpha \cdot \beta) \otimes \mathsf{f}$	$\forall \alpha,\beta\in S$ and $\mathsf{f}\in\mathcal{A}$	($\cdot$ associates with $\otimes$)
8.			$1 \otimes \mathsf{f}$	$=$	f	$\forall \mathsf{f}\in\mathcal{A}$	($\otimes$ identity)

4. algebra:

1.	$(\mathsf{fg})\mathsf{h}$	$=$	$\mathsf{f}(\mathsf{gh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	(*associative*)
2.	$\mathsf{f}(\mathsf{g} \dotplus \mathsf{h})$	$=$	$(\mathsf{fg}) + (\mathsf{fh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	(*left distributive*)
3.	$(\mathsf{f} \dotplus \mathsf{g})\mathsf{h}$	$=$	$(\mathsf{fh}) + (\mathsf{gh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	(*right distributive*)
4.	$\alpha(\mathsf{gh})$	$=$	$(\alpha\mathsf{g})\mathsf{h} = \mathsf{g}(\alpha\mathsf{h})$	$\forall \mathsf{g,h} \in \mathcal{A}$ and $\alpha \in \mathbb{F}$	(scalar *commutative*)

⇒

Theorem 17.14. *Let* $\mathcal{A} \triangleq \left\{\mathsf{f} \in X^X \middle| \exists \mathsf{f}^{-1} \text{ such that } \mathsf{f}^{-1}\mathsf{f} \doteq \mathsf{f}\mathsf{f}^{-1} \doteq \mathbf{I}\right\}$ *be the set of invertible functions on* X^X.

THM *$(\mathcal{A}, \cdot)$ is a (multiplicative) group.*

✎PROOF:

1. multiplicative group:

1.	$\mathsf{f}\mathbf{I} = \mathbf{I}\mathsf{f} = \mathsf{f}$	$\forall \mathsf{f} \in \mathcal{A}$	($\mathbf{I} \in \mathcal{A}$ is the identity element)
2.	$\mathsf{f}^{-1}\mathsf{f} = \mathsf{f}\mathsf{f}^{-1} = \mathbf{I}$	$\forall \mathsf{f} \in \mathcal{A}$	(f^{-1} is the inverse of f)
3.	$(\mathsf{fg})\mathsf{h} = \mathsf{f}(\mathsf{gh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	($(\mathcal{A}, \cdot)$ is associative)

2. field:

1.	$(X, +, *)$ is a ring		(ring)
2.	$\boldsymbol{xy} = \boldsymbol{yx}$	$\forall \boldsymbol{x,y} \in X$	(commutative with respect to $*$)
3.	$(X \setminus \{0\}, *)$ is a group		(group with respect to $*$).

⇒

Theorem 17.15. *Let* $\boldsymbol{D}(\mathsf{f})$ *be the domain of an function* f *and* $\boldsymbol{I}(\mathsf{f})$ *the image of* f.

THM

$$\begin{aligned}
\boldsymbol{D}\left(\bigcup_{i\in I} \mathsf{f}_i\right) &= \bigcup_{i\in I} \boldsymbol{D}(\mathsf{f}_i) & \boldsymbol{I}\left(\bigcup_{i\in I} \mathsf{f}_i\right) &= \bigcup_{i\in I} \boldsymbol{I}(\mathsf{f}_i)\\
\boldsymbol{D}\left(\bigcap_{i\in I} \mathsf{f}_i\right) &\subseteq \bigcap_{i\in I} \boldsymbol{D}(\mathsf{f}_i) & \boldsymbol{I}\left(\bigcap_{i\in I} \mathsf{f}_i\right) &\subseteq \bigcap_{i\in I} \boldsymbol{I}(\mathsf{f}_i)\\
\boldsymbol{D}(\mathsf{f} \setminus \mathsf{g}) &\supseteq \boldsymbol{D}(\mathsf{f}) \setminus \boldsymbol{D}(\mathsf{g}) & \boldsymbol{I}(\mathsf{f} \setminus \mathsf{g}) &\supseteq \boldsymbol{I}(\mathsf{f}) \setminus \boldsymbol{I}(\mathsf{g})
\end{aligned}$$

✎PROOF: These results follow from Theorem 17.3 (page 258). ⇒

Definition 17.15. [32] *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be linear spaces over a field* $\mathbb{F}$ *and with dual spaces*

$$\begin{aligned}
\boldsymbol{X}^* &\triangleq \left\{\mathsf{f}(\boldsymbol{x}; \boldsymbol{x}^*) \in \mathbb{F}^{\boldsymbol{X}} \middle| \boldsymbol{x}^* \in \boldsymbol{X}^*\right\} && \textit{(set of functionals with parameter } \boldsymbol{x}^* \textit{ from } \boldsymbol{X} \textit{ to } \mathbb{F})\\
\boldsymbol{Y}^* &\triangleq \left\{\mathsf{g}(\boldsymbol{y}; \boldsymbol{y}^*) \in \mathbb{F}^{\boldsymbol{Y}} \middle| \boldsymbol{y}^* \in \boldsymbol{Y}^*\right\}. && \textit{(set of functionals with parameter } \boldsymbol{y}^* \textit{ from } \boldsymbol{Y} \textit{ to } \mathbb{F})
\end{aligned}$$

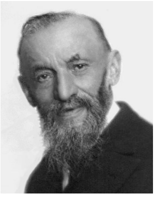
Let $\mathsf{f} \in Y^X$ *be a function.*

DEF *A function* f^* *in* $\boldsymbol{X}^{*\boldsymbol{Y}^*}$ *is the* ***conjugate*** *of the function* f *if*

$$\mathsf{g}(\mathsf{f}\boldsymbol{x}; \boldsymbol{y}^*) = \mathsf{f}(\boldsymbol{x}; \mathsf{f}^*\boldsymbol{y}^*) \qquad \forall \boldsymbol{x} \in \boldsymbol{X}, \mathsf{f} \in \boldsymbol{X}^*, \mathsf{g} \in \boldsymbol{Y}^*$$

[32] Michel and Herget (1993) page 420, Giles (2000), page 171

17.3 Tempered Distributions



❝*I am sure that something must be found. There must exist a notion of generalized functions which are to functions what the real numbers are to the rationals.*❞
Giuseppe Peano (1858–1932), Italian mathematician[33]

Definition 17.16. [34]

DEF

*A **test function** is any function ϕ that satisfies*

1. $\phi \in \mathbb{C}^{\mathbb{R}}$
2. ϕ *is* INFINITELY DIFFERENTIABLE.

*The set of all test functions is denoted $\mathbb{C}^{\infty}(\mathbb{R})$. A test function ϕ belongs to the **Schwartz class** S if, for some set of constants $\{C_{n,k} | n, k \in \mathbb{W}\}$,*

$$(1+|x|)^n \left|\phi^{(k)}\right| \le C_{n,k} \qquad \forall n, k \in \mathbb{W},\ \forall x \in \mathbb{R}$$

Definition 17.17. [35] *Let S be the* SCHWARTZ CLASS *of functions (Definition 17.16).*

DEF

$\mathsf{d}[\cdot]$ *is a **tempered distribution** if*

1. $\mathsf{d}[\alpha_1\phi_1 + \alpha_2\phi_2] = \mathsf{d}[\alpha_1\phi_1] + \mathsf{d}[\alpha_2\phi_2] \qquad \forall \phi_1,\phi_2 \in S, \alpha_1,\alpha_2 \in \mathbb{R}$ (LINEAR) *and*
2. $\lim_{n\to\infty} \phi_n = \phi \implies \lim_{n\to\infty} \mathsf{d}[\phi_n] = \mathsf{d}[\phi] \qquad \forall \phi_1,\phi_2 \in S$ (CONTINUOUS)

Definition 17.18. [36] *Let S be the* SCHWARTZ CLASS *of functions (Definition 17.16).*

DEF

*Two tempered distributions d_1 and d_2 are **equal** if*

$$\mathsf{d}[\phi_1] = \mathsf{d}[\phi_2] \qquad \forall \phi_1, \phi_2 \in S$$

Theorem 17.16 (next) demonstrates that all continuous and what we might call "well behaved" functions generate a tempered distribution.

Theorem 17.16. [37] *Let f be a function in $\mathbb{C}^{\mathbb{R}}$. Let T_f be defined as*

$$\mathsf{T}_\mathsf{f}[\phi] \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\phi(x)\, \mathrm{d}x.$$

THM

$$\left.\begin{array}{lll} 1. & \mathsf{f} \text{ is CONTINUOUS} & \textit{and} \\ 2. & \exists n, M \text{ such that } |\mathsf{f}(x)| \le M(1+|x|)^n & \forall x \in \mathbb{R} \end{array}\right\} \implies \mathsf{T}_\mathsf{f}[\phi] \text{ is a tempered distribution.}$$

✎PROOF:

1. Proof that T_f is *linear*:

$$\begin{aligned} \mathsf{T}_\mathsf{f}[\phi_1+\phi_2] &= \int_{\mathbb{R}} \mathsf{f}(x)\big(\phi_1(x)+\phi_2(x)\big)\, \mathrm{d}x && \text{by definition of } \mathsf{T}_\mathsf{f} \\ &= \int_{\mathbb{R}} \mathsf{f}(x)\phi_1(x)\, \mathrm{d}x + \int_{\mathbb{R}} \mathsf{f}(x)\phi_2(x)\, \mathrm{d}x && \text{by linearity of } \int \\ &= \mathsf{T}_\mathsf{f}[\phi_1] + \mathsf{T}_\mathsf{f}[\phi_2] && \text{by definition of } \mathsf{T}_\mathsf{f} \end{aligned}$$

[33] quote: Duistermaat and Kolk (2010) page ix
image http://en.wikipedia.org/wiki/File:Giuseppe_Peano.jpg, public domain
[34] Vretblad (2003) page 200
[35] Vretblad (2003) pages 203–204 ⟨Definition 8.3⟩
[36] Vretblad (2003) page 206
[37] Vretblad (2003) page 204

2. Proof that $\mathsf{T_f}$ is *cotinuous*:

$$\begin{aligned}
\lim_{n\to\infty}\left|\mathsf{T_f}[\phi_n] - \mathsf{T_f}[\phi]\right| &= \lim_{n\to\infty}\left|\int_{\mathbb{R}} \mathsf{f}(x)\phi_n(x)\,\mathsf{d}x - \int_{\mathbb{R}} \mathsf{f}(x)\phi(x)\,\mathsf{d}x\right| && \text{by definition of } \mathsf{T_f}\\
&= \lim_{n\to\infty}\left|\int_{\mathbb{R}} \mathsf{f}(x)\big(\phi_n(x) - \phi(x)\,\mathsf{d}x\big)\right| && \text{by linearity of } \int\\
&\le \lim_{n\to\infty}\int_{\mathbb{R}} M(1+|x|)^m\left|\phi_n(x) - \phi(x)\right|\,\mathsf{d}x\\
&= \int_{\mathbb{R}} M(1+|x|)^{m+2}\left|\phi_n(x) - \phi(x)\right|\frac{1}{(1+|x|)^2}\,\mathsf{d}x\\
&\le \lim_{n\to\infty}\max_x\left\{M(1+|x|)^{m+2}\left|\phi_n(x) - \phi(x)\right|\right\}\int_{\mathbb{R}}\frac{1}{(1+|x|)^2}\,\mathsf{d}x\\
&= 0
\end{aligned}$$

✏

Definition 17.19. [38]

DEF *The **Dirac delta distribution** $\delta \in \mathbb{C}^{\mathbb{R}}$ is defined as*
$$\delta[\phi] \triangleq \phi(0)$$

One could argue that a tempered distribution d behaves *as if* it satisfies the following relation:
$$\mathsf{d}[\phi] \risingdotseq \int_{\mathbb{R}} \mathsf{d}(x)\phi(x)\,\mathsf{d}x.$$
This is not technically correct because in general d is not a function that can be evaluated at a given point x (and hence the here undefined relation "$\risingdotseq$"). But despite this failure, the notation is still very useful in that distributions do behave "as if" they are defined by the above integral relation.

Using this notation, the Dirac delta distribution looks likes this:
$$\delta[\phi] \triangleq \phi(0) \risingdotseq \int_{\mathbb{R}} \delta(x)\phi(x)\,\mathsf{d}x$$

We could also define another "scaled" and "translated" distribution δ_{ab} such that
$$\delta_{ab}[\phi] \triangleq b\phi(ab) \risingdotseq \int_{\mathbb{R}} \delta\left(\frac{x}{b} - a\right)\phi(x)\,\mathsf{d}x$$
because

$$\begin{aligned}
\int_{\mathbb{R}} \delta\left(\frac{x}{b} - a\right)\phi(x)\,\mathsf{d}x &= \int_{\mathbb{R}} \delta(u-a)\phi(ub)b\,\mathsf{d}u && \text{where } u = \frac{x}{b}\\
&= b\int_{\mathbb{R}} \delta(u-a)\phi(ub)\,\mathsf{d}u\\
&= b\phi(ab)
\end{aligned}$$

17.4 Literature

Literature survey:

[38] Vretblad (2003) page 205 ⟨Example 8.13⟩, Friedlander and Joshi (1998) page 8

1. Reference books:
   - Maddux (2006)
   - Suppes (1972) ⟨0486616304⟩Chapter 3: *Relations and Functions*
   - Kelley (1955) pages 6–13
2. Pioneering papers on relations:
   - de Morgan (1864a)
     - de Morgan (1864b)
   - Peirce (1883a)
     - Peirce (1883c)
     - Peirce (1883b)
   - Schröder (1895)
3. Axiomization of calculus of relations:
   - Tarski (1941)
4. Historically oriented presentations:
   - Maddux (1991)
   - Pratt (1992) pages 248–254
5. Theory of Distributions
   - Vretblad (2003)
   - Hömander (2003) ⟨Referenced by Vretblad(2003) as a standard work.⟩
   - Knapp (2005)
6. Miscellaneous:
   - Peirce (1870a)
     - Peirce (1870b)
     - Peirce (1870c)

APPENDIX A

INTERVALS AND CONVEXITY

A.1 Intervals

In the real number system, for $a \leq b$, the *interval* $[a:b]$ is the set a and b and all the numbers inbetween, as in $[a:b] \triangleq \{x \in \mathbb{R} | a \leq x \leq b\}$. This concept can be easily generalized:

- In an **ordered set** (Definition 4.2 page 58), if two elements x and y are *comparable* and $x \leq y$, then we say that x and y and all the elements inbetween, as determined by the ordering relation $\leq$, are the interval $[a:b]$ (Definition A.1 page 279).
- In a **lattice** (Definition 5.3 page 73), the concept of the *interval* can be generalized even further. In an arbitrary ordered set, the interval $[x:y]$ of item (A.1) is restricted to the case in which x and y are *comparable* (Definition 4.2 page 58). This restriction can be lifted (Definition A.2 page 279) with the additional structure of upper and lower bounds provided by lattices.
- A **metric space** in general has no *order relation* $\leq$ (Definition 4.2 page 58). But intervals can still be defined (Definition A.4 page 280) in a metric space in terms of the *triangle inequality*.
- A **linear space** over a real or complex field in general has no *order relation* that compares *vectors* in the space, but the standard order relation $\leq$ for real numbers $\mathbb{R}$ can still be used (Definition A.5 page 280) to define an interval in a linear space.

Definition A.1 (intervals on ordered sets). [1] *Let* $(X, \leq)$ *be an* ORDERED SET *(Definition 4.2 page 58).*

DEF

The set	$[x:y]$	$\triangleq$	$\{z \in X | x \leq z \leq y\}$	*is called a **closed interval***	*and*
The set	$(x:y]$	$\triangleq$	$\{z \in X | x < z \leq y\}$	*is called a **half-open interval***	*and*
The set	$[x:y)$	$\triangleq$	$\{z \in X | x \leq z < y\}$	*is called a **half-open interval***	*and*
The set	$(x:y)$	$\triangleq$	$\{z \in X | x < z < y\}$	*is called an **open interval**.*	

Definition A.2 (intervals on lattices). [2] *Let* $(X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition 5.3 page 73).*

DEF

The set $[x:y]$	$\triangleq$	$\{z \in X | x \wedge y \leq z \leq x \vee y\}$	*is called a **closed interval**.*
The set $(x:y]$	$\triangleq$	$\{z \in X | x \wedge y < z \leq x \vee y\}$	*is called a **half-open interval**.*
The set $[x:y)$	$\triangleq$	$\{z \in X | x \wedge y \leq z < x \vee y\}$	*is called a **half-open interval**.*
The set $(x:y)$	$\triangleq$	$\{z \in X | x \wedge y < z < x \vee y\}$	*is called an **open interval**.*

When x and y *are* comparable and $x \leq y$, then Definition A.2 (previous) simplifies to item (A.1)

[1] Apostol (1975) page 4, Ore (1935) page 409
[2] Duthie (1942) page 2, Ore (1935) page 425 ⟨*quotient structures*⟩

(page 279).

Definition A.3. [3] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *with dual* $\mathbf{L}^*$. *Let* $[x : y]$ *be a* CLOSED INTERVAL (Definition A.2 page 279) *on set* X. *The sublattices* $\mathbf{L}[x : y]$ *and* $\mathbf{L}^*[x : y]$ *are defined as follows:*

DEF
$$\begin{array}{lcll} \mathbf{L}[x : y] & \triangleq & \{z \in \mathbf{L} \mid z \in [x : y]\} & \forall x,y \in X \\ \mathbf{L}^*[x : y] & \triangleq & \left\{z \in \mathbf{L}^* \mid z \in [x : y]\right\} & \forall x,y \in X \end{array}$$

Definition A.4. [4]

DEF
In a METRIC SPACE (X, d),
the set $[a : b]$ *is the* ***closed interval*** *from* x *to* y *and is defined as*
$$[x : y] \triangleq \{z \in X \mid \mathsf{d}(x,z) + \mathsf{d}(z,y) = \mathsf{d}(x,y)\}.$$
An element $z \in X$ *is* ***geodesically between*** x *and* y *if* $z \in [x : y]$.

Definition A.5. [5]

DEF
In a LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$,
$$\begin{array}{lcll} \left[\boldsymbol{x} : \boldsymbol{y}\right] & \triangleq & \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} \mid 0 \leq \lambda \leq 1\} & \text{is called a \textbf{closed interval} and} \\ (\boldsymbol{x} : \boldsymbol{y}] & \triangleq & \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} \mid 0 < \lambda \leq 1\} & \text{is called a \textbf{half-open interval} and} \\ \left[\boldsymbol{x} : \boldsymbol{y}\right) & \triangleq & \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} \mid 0 \leq \lambda < 1\} & \text{is called a \textbf{half-open interval} and} \\ (\boldsymbol{x} : \boldsymbol{y}) & \triangleq & \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} = \boldsymbol{z} \mid 0 < \lambda < 1\} & \text{is called an \textbf{open interval}.} \end{array}$$

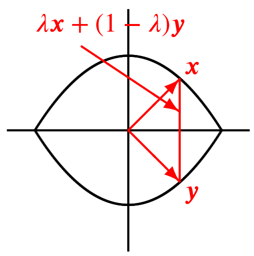
A.2 Convex sets

Using the concept of the *interval* (previous section), we can define the *convex set* (next definition).

Definition A.6. [6] *Let* X *be a* SET *in an* ORDERED SET $(X, \leq)$, *a* LATTICE $(X, \vee, \wedge; \leq)$, *a* METRIC SPACE (X, d), *or a* LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

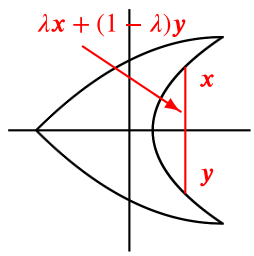
DEF
A subset $D \subseteq X$ *is a* ***convex set*** *in* X *if*
$$x, y \in D \quad \Longrightarrow \quad [x : y] \subseteq D.$$
A set that is ***not*** *convex is* ***concave***.

Example A.1. Consider the *Euclidean space* $\mathbb{R}^2$ (a special case of a *linear space*).



$\Leftarrow$ {The figure to the left is a *convex* set in $\mathbb{R}^2$.

The figure to the right is a *concave* set in $\mathbb{R}^2$.} $\Rightarrow$



Example A.2. In a *metric space*, examples of *convex sets* are convex *balls*. Examples include those balls generated by the following metrics:

- Taxi-cab metric
- Euclidean metric
- Sup metric
- Tangential metric

[3] Maeda and Maeda (1970), page 1
[4] van de Vel (1993) page 8
[5] Barvinok (2002) page 2
[6] Barvinok (2002) page 5

Examples of metrics generating balls which are *not* convex include the following:

- Parabolic metric
- Exponential metric

A.3 Convex functions

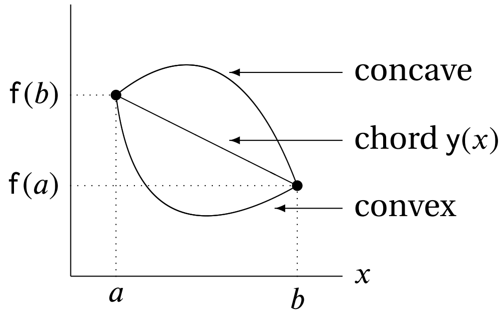


Figure A.1: Convex and concave functions

Definition A.7. [7] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *and* D *a* CONVEX SET (Definition A.6 page 280) *in* X.

DEF

A function $f \in F^D$ *is* ***convex*** *if*

$$f(\lambda x + [1-\lambda] y) \le \lambda f(x) + (1-\lambda) f(y) \qquad \forall x, y \in D \text{ and } \forall \lambda \in (0,1)$$

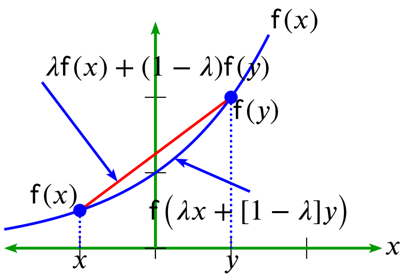
A function $g \in F^D$ *is* ***strictly convex*** *if*

$$g(\lambda x + [1-\lambda] y) = \lambda g(x) + (1-\lambda) g(y) \qquad \forall x, y \in D,\ x \ne y,\ \text{and } \forall \lambda \in (0,1)$$

A function $f \in F^D$ *is* ***concave*** *if* $-f$ *is* CONVEX.

A function $f \in F^D$ *is* ***affine*** *if* f *is* CONVEX *and* CONCAVE.

Example A.3. The function $f(x) = 2^x$ is a *convex function* (Definition A.7 page 281), as illustrated to the right.



Definition A.8. [8] *Let* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

DEF

The ***epigraph*** $\text{epi}(f)$ *and* ***hypograph*** $\text{hyp}(f)$ *of a functional* $f \in \mathbb{R}^X$ *are defined as*

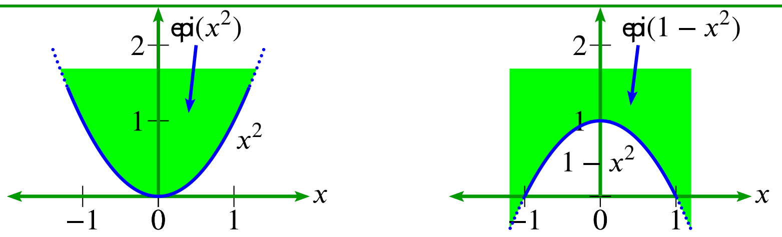
$$\text{epi}(f) \triangleq \{(x,y) \in X \times \mathbb{R} \mid y \ge f(x)\}$$
$$\text{hyp}(f) \triangleq \{(x,y) \in X \times \mathbb{R} \mid y \le f(x)\}$$

Example A.4.

[7] Simon (2011) page 2, Barvinok (2002) page 2, Bollobás (1999), page 3, Jensen (1906), page 176, Clarkson (1936) ⟨strictly convex⟩

[8] Beer (1993) page 13 ⟨§1.3⟩, Aubin and Frankowska (2009) page 222, Aubin (2011) page 223

EX



Proposition A.1. [9] *Let* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* f *be a* FUNCTIONAL *in* $\mathbb{R}^X$.

PRP

$$\left\{\begin{array}{l}\mathsf{f} \textit{ is a} \\ \text{CONVEX FUNCTION}\end{array}\right\} \iff \left\{\begin{array}{l}\mathsf{epi}(\mathsf{f}) \textit{ is a} \\ \text{CONVEX SET}\end{array}\right\}$$

Often a function can be proven to be *convex* or *concave*. *Convex* and *concave* functions are defined in Definition A.9 (page 282) (next) and illustrated in Figure A.1 (page 281).

Definition A.9. *Let*

$$\mathsf{y}(x) = \frac{\mathsf{f}(b) - \mathsf{f}(a)}{b-a}(x-a) + \mathsf{f}(a).$$

A function $\mathsf{f} : \mathbb{R} \to \mathbb{R}$ *is*

DEF

(1).	***convex***	*in* $(a:b)$ *if*	$\mathsf{f}(x) \le \mathsf{y}(x)$	*for* $x \in (a:b)$
(2).	***concave***	*in* $(a:b)$ *if*	$\mathsf{f}(x) \ge \mathsf{y}(x)$	*for* $x \in (a:b)$
(3).	***strictly convex***	*in* $(a:b)$ *if*	$\mathsf{f}(x) < \mathsf{y}(x)$	*for* $x \in (a:b)$
(4).	***strictly concave***	*in* $(a:b)$ *if*	$\mathsf{f}(x) > \mathsf{y}(x)$	*for* $x \in (a:b)$

Theorem A.1 (Jensen's Inequality). [10] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE, D *a subset of* X, *and* f *a functional in* $\mathbb{F}^D$. *Let* $\sum$ *be the* SUMMATION OPERATOR.

THM

$$\left\{\begin{array}{lll} 1. & D \text{ is CONVEX} & \textit{and} \\ 2. & \mathsf{f} \text{ is CONVEX} & \textit{and} \\ 3. & \displaystyle\sum_{n=1}^{N} \lambda_n = 1 & \text{(WEIGHTS)} \end{array}\right\} \implies \mathsf{f}\left(\sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n\right) \le \sum_{n=1}^{N} \lambda_n \mathsf{f}(\boldsymbol{x}_n) \qquad \forall \boldsymbol{x}_n \in D,\ N \in \mathbb{N}$$

✎PROOF: Proof is by induction:

1. Proof that statement is true for $N = 1$:

$$\begin{aligned}\mathsf{f}\left(\sum_{n=1}^{N=1} \lambda_n \boldsymbol{x}_n\right) &= \mathsf{f}(\lambda_1 \boldsymbol{x}_1) \\ &\le \mathsf{f}(\lambda_1 \boldsymbol{x}_1) \\ &= \sum_{n=1}^{N=1} \lambda_n \mathsf{f}(\boldsymbol{x}_n)\end{aligned}$$

[9] Udriste (1994) page 63, Kurdila and Zabarankin (2005) page 178 ⟨Proposition 6.1.1⟩, Rockafellar (1970) page 23 ⟨Section 4 Convex Functions⟩, Çinlar and Vanderbei (2013) page 86 ⟨5.4 Theorem⟩

[10] Mitrinović et al. (2010) page 6, Bollobás (1999) page 3, Lay (1982) page 7, Jensen (1906), pages 179–180

2. Proof that statement is true for $N = 2$:

$$\begin{aligned}
\mathsf{f}\left(\sum_{n=1}^{N=2} \lambda_n \boldsymbol{x}_n\right) &= \mathsf{f}(\lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2) \\
&\le \lambda_1 \mathsf{f}(\boldsymbol{x}_1) + \lambda_2 \mathsf{f}(\boldsymbol{x}_2) && \text{by convexity hypothesis} \\
&= \sum_{n=1}^{N=2} \lambda_n \mathsf{f}(\boldsymbol{x}_n)
\end{aligned}$$

3. Proof that if the statement is true for N, then it is also true for $N+1$:

$$\begin{aligned}
\mathsf{f}\left(\sum_{n=1}^{N+1} \lambda_n \boldsymbol{x}_n\right) &= \mathsf{f}\left(\sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n + \lambda_{N+1}\boldsymbol{x}_{N+1}\right) \\
&= \mathsf{f}\left([1-\lambda_{N+1}]\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \boldsymbol{x}_n + \lambda_{N+1}\boldsymbol{x}_{N+1}\right) \\
&\le [1-\lambda_{N+1}]\mathsf{f}\left(\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \boldsymbol{x}_n\right) + \lambda_{N+1}\mathsf{f}(\boldsymbol{x}_{N+1}) && \text{by convexity hypothesis} \\
&\le [1-\lambda_{N+1}]\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \mathsf{f}(\boldsymbol{x}_n) + \lambda_{N+1}\mathsf{f}(\boldsymbol{x}_{N+1}) && \text{by "true for } N\text{" hypothesis} \\
&= \sum_{n=1}^{N} \lambda_n \mathsf{f}(\boldsymbol{x}_n) + \lambda_{N+1}\mathsf{f}(\boldsymbol{x}_{N+1}) \\
&= \sum_{n=1}^{N+1} \lambda_n \mathsf{f}(\boldsymbol{x}_n)
\end{aligned}$$

4. Since the statement is true for $N = 1$, $N = 2$, and true for $N \implies$ true for $N+1$, then it is true for $N = 1, 2, 3, 4, \ldots$

✑

The next theorem gives another form of convex functions that is a little less intuitive but provides powerful analytic results.

Theorem A.2. *Let* $\mathsf{f} : \mathbb{R} \to \mathbb{R}$. *For every* $x_1, x_2 \in (a, b)$ *and* $\lambda \in [0, 1]$

THM	
T H M	f *is convex in* $(a,b) \iff \mathsf{f}(\lambda x_1 + (1-\lambda)x_2) \le \lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)$

✎PROOF:

1. prove f is convex $\implies \mathsf{f}(\lambda x_1 + (1-\lambda)x_2) \le \lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)$:

$$\begin{aligned}
\mathsf{f}(\lambda x_1 + (1-\lambda)x_2) &\le \frac{\mathsf{f}(b)-\mathsf{f}(a)}{b-a}[\lambda x_1 + (1-\lambda)x_2 - a] + \mathsf{f}(a) \\
&= \frac{\mathsf{f}(x_2)-\mathsf{f}(x_1)}{x_2-x_1}[\lambda x_1 + (1-\lambda)x_2 - x_1] + \mathsf{f}(x_1) \\
&= \frac{\mathsf{f}(x_2)-\mathsf{f}(x_1)}{x_2-x_1}[(x_2-x_1)(1-\lambda)] + \mathsf{f}(x_1) \\
&= (1-\lambda)\mathsf{f}(x_2) - (1-\lambda)\mathsf{f}(x_1) + \mathsf{f}(x_1) \\
&= \lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)
\end{aligned}$$

2. prove f is convex $\Longleftarrow$ $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda \mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)$:

   Let $x = \lambda(b-a) + a$ Notice that as λ varies from 0 to 1, x varies from b to a. So free variable λ works as a change of variable for free variable x.

$$\begin{aligned}
\lambda &= \frac{x-a}{b-a} \\
\mathsf{f}(x) &= \mathsf{f}(\lambda(b-a)+a) \\
&\le \lambda\mathsf{f}(b) + (1-\lambda)\mathsf{f}(a) \\
&= \lambda[\mathsf{f}(b) - \mathsf{f}(a)] + \mathsf{f}(a) \\
&= \frac{\mathsf{f}(b)-\mathsf{f}(a)}{b-a}(x-a) + \mathsf{f}(a)
\end{aligned}$$

✏

Taking the second derivative of a function provides a convenient test for whether that function is convex.

Theorem A.3. [11]

THM $\mathsf{f}''(x) > 0 \implies \mathsf{f}$ *is convex*

✎PROOF:

$$\begin{aligned}
\mathsf{f}(x) &= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x-x_0) + \mathsf{f}''(c)(x-x_0)^2 \\
&\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x-x_0) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x - \lambda x_1 - (1-\lambda)x_2)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}(x_1) &\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x_1 - \lambda x_1 - (1-\lambda)x_2) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(1-\lambda)(x_1 - x_2) \\
&= \mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2 - x_1)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}(x_2) &\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x_2 - \lambda x_1 - (1-\lambda)x_2) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2 - x_1)
\end{aligned}$$

$$\begin{aligned}
\lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2) &\ge \lambda\left[\mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2-x_1)\right] + (1-\lambda)\left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] \\
&= \lambda\left[\mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2-x_1)\right] + \left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] - \lambda\left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] \\
&= \mathsf{f}(x_0) \\
&= \mathsf{f}(\lambda x_1 + (1-\lambda)x_2)
\end{aligned}$$

By Theorem A.2 (page 283), $\mathsf{f}(x)$ is convex. ✏

A.4 Literature

Literature survey:

[11] Cover and Thomas (1991), pages 24–25

1. Abstract convexity:
   Edelman and Jamison (1985)
   van de Vel (1993)
   Hörmander (1994)
2. Order convexity (lattice theory):
   Edelman (1986)
3. Metric convexity:
   Menger (1928)
   Blumenthal (1970) page 41 ⟨?⟩
   Khamsi and Kirk (2001) pages 35–38

APPENDIX B

NORMED ALGEBRAS

B.1 Algebras

All *linear spaces* are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition B.1. [3] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF *An algebra $\boldsymbol{A}$ is **unital** if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition B.2. [4] *Let $\boldsymbol{A}$ be an* UNITAL ALGEBRA *(Definition B.1 page 287) with unit e.*

DEF
$$\begin{array}{llcll}
\text{The } \textbf{spectrum} \text{ of } x \in \boldsymbol{A} \text{ is} & \sigma(x) & \triangleq & \{\lambda \in \mathbb{C} | \lambda e - x \text{ is not invertible}\}. & \\
\text{The } \textbf{resolvent} \text{ of } x \in \boldsymbol{A} \text{ is} & \rho_x(\lambda) & \triangleq & (\lambda e - x)^{-1} & \forall \lambda \notin \sigma(x). \\
\text{The } \textbf{spectral radius} \text{ of } x \in \boldsymbol{A} \text{ is} & r(x) & \triangleq & \sup\{|\lambda| | \lambda \in \sigma(x)\}. &
\end{array}$$

[1] Fuchs (1995) page 2
[2] Hazewinkel (2000) page v
[3] Folland (1995) page 1
[4] Folland (1995) pages 3–4

B.2 Star-Algebras

Definition B.3. [5] *Let* $\boldsymbol{A}$ *be an* ALGEBRA.

DEF

The pair $(\boldsymbol{A}, *)$ *is a* ***-algebra**, or **star-algebra**, if*

1. $(x+y)^* = x^* + y^* \quad \forall x,y\in\boldsymbol{A}$ (DISTRIBUTIVE) and
2. $(\alpha x)^* = \bar{\alpha} x^* \quad \forall x\in\boldsymbol{A}, \alpha\in\mathbb{C}$ (CONJUGATE LINEAR) and
3. $(xy)^* = y^* x^* \quad \forall x,y\in\boldsymbol{A}$ (ANTIAUTOMORPHIC) and
4. $x^{**} = x \quad \forall x\in\boldsymbol{A}$ (INVOLUTORY)

The operator $*$ *is called an* ***involution*** *on the algebra* $\boldsymbol{A}$.

Proposition B.1. [6] *Let* $(\boldsymbol{A}, *)$ *be an* UNITAL $*$-ALGEBRA.

PRP

$$x \text{ is invertible} \implies \left\{\begin{array}{lll} 1. & x^* \text{ is INVERTIBLE} & \forall x\in\boldsymbol{A} \quad \text{and} \\ 2. & \left(x^*\right)^{-1} = \left(x^{-1}\right)^* & \forall x\in\boldsymbol{A} \end{array}\right.$$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= \left(x\,e^*\right)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \left(x^*\,e^{**}\right)^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \left(x^*\,e\right)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \left(x^*\right)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)}\\
e^*\,x &= \left(e^*\,x\right)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \left(e^{**}\,x^*\right)^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \left(e\,x^*\right)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \left(x^*\right)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)}
\end{aligned}$$

2. Proof that $\left(x^*\right)^{-1} = \left(x^{-1}\right)^*$:

$$\begin{aligned}
\left(x^{-1}\right)^*\left(x^*\right) &= \left[x\left(x^{-1}\right)\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition B.3 page 288)}\\
&= e^*\\
&= e && \text{by item (1) page 288}\\
\left(x^*\right)\left(x^{-1}\right)^* &= \left[x^{-1}\,x\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition B.3 page 288)}\\
&= e^*\\
&= e && \text{by item (1) page 288}
\end{aligned}$$

✏

Definition B.4. [7] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition B.3 page 288)*.

DEF

- *An element* $x \in \boldsymbol{A}$ *is* ***hermitian*** *or* ***self-adjoint*** *if* $x^* = x$.
- *An element* $x \in \boldsymbol{A}$ *is* ***normal*** *if* $xx^* = x^*x$.
- *An element* $x \in \boldsymbol{A}$ *is a* ***projection*** *if* $xx = x$ (INVOLUTORY) *and* $x^* = x$ (HERMITIAN).

[5] Rickart (1960), page 178, Gelfand and Naimark (1964), page 241
[6] Folland (1995) page 5
[7] Rickart (1960), page 178, Gelfand and Naimark (1964), page 242

Theorem B.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition B.3 page 288).*

THM

$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are hermitian}} \implies \left\{\begin{array}{lll} x+y &=& (x+y)^* \quad (x+y \text{ is self adjoint}) \\ x^* &=& (x^*)^* \quad (x^* \text{ is self adjoint}) \\ \underbrace{xy = (xy)^*}_{(xy) \text{ is hermitian}} & \iff & \underbrace{xy = yx}_{\text{commutative}} \end{array}\right.$$

✎PROOF:

$$\begin{aligned} (x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition B.3 page 288)} \\ &= x + y && \text{by left hypothesis} \end{aligned}$$

$$(x^*)^* = x \qquad \text{by } \textit{involutory} \text{ property of } * \qquad \text{(Definition B.3 page 288)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned} xy &= (xy)^* && \text{by left hypothesis} \\ &= y^*x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition B.3 page 288)} \\ &= yx && \text{by left hypothesis} \end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned} (xy)^* &= (yx)^* && \text{by left hypothesis} \\ &= x^*y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition B.3 page 288)} \\ &= xy && \text{by left hypothesis} \end{aligned}$$

✏

Definition B.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition B.3 page 288).*

DEF

*The **real part** of x is defined as* $\mathbf{R}_e x \triangleq \frac{1}{2}\left(x + x^*\right)$

*The **imaginary part** of x is defined as* $\mathbf{I}_m x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem B.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition B.3 page 288).*

THM

$$\begin{array}{llll} \Re x &=& (\Re x)^* \quad \forall x \in \boldsymbol{A} & (\Re x \text{ is hermitian}) \\ \Im x &=& (\Im x)^* \quad \forall x \in \boldsymbol{A} & (\Im x \text{ is hermitian}) \end{array}$$

✎PROOF:

$$\begin{aligned} (\Re x)^* &= \left(\frac{1}{2}(x + x^*)\right)^* && \text{by definition of } \Re && \text{(Definition B.5 page 289)} \\ &= \frac{1}{2}(x^* + x^{**}) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition B.3 page 288)} \\ &= \frac{1}{2}(x^* + x) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)} \\ &= \Re x && \text{by definition of } \Re && \text{(Definition B.5 page 289)} \end{aligned}$$

$$(\Im x)^* = \left(\frac{1}{2i}(x - x^*)\right)^* \qquad \text{by definition of } \Im \qquad \text{(Definition B.5 page 289)}$$

[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960), page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998) page 42

$$\begin{aligned}
&= \frac{1}{2i}\left(x^* - x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \frac{1}{2i}\left(x^* - x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition B.3 page 288)}\\
&= \Im x && \text{by definition of } \Im && \text{(Definition B.5 page 289)}
\end{aligned}$$

✏

Theorem B.3 (Hermitian representation). [11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition B.3 page 288).*

THM
$$a = x + iy \iff x = \Re a \quad \textit{and} \quad y = \Im a$$

✎PROOF:

Proof that $a = x + iy \implies x = \Re a$ and $y = \Im a$:

$$\begin{aligned}
& & a &= x + iy && \text{by left hypothesis}\\
&\implies & a^* &= (x+iy)^* && \text{by definition of } \textit{adjoint} && \text{(Definition B.4 page 288)}\\
& & &= x^* - iy^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition B.3 page 288)}\\
& & &= x - iy && \text{by Theorem B.2 page 289}\\
&\implies & x &= a - iy && \text{by solving for } x \text{ in } a = x+iy \text{ equation}\\
& & x &= a^* + iy && \text{by solving for } x \text{ in } a^* = x-iy \text{ equation}\\
&\implies & x + x &= a + a^* && \text{by adding previous 2 equations}\\
&\implies & 2x &= a + a^* && \text{by solving for } x \text{ in previous equation}\\
&\implies & x &= \frac{1}{2}\left(a + a^*\right)\\
& & &= \Re a && \text{by definition of } \Re && \text{(Definition B.5 page 289)}\\
\\
& & iy &= a - x && \text{by solving for } iy \text{ in } a = x+iy \text{ equation}\\
& & iy &= -a^* + x && \text{by solving for } iy \text{ in } a = x+iy \text{ equation}\\
&\implies & iy + iy &= a - a^* && \text{by adding previous 2 equations}\\
&\implies & y &= \frac{1}{2i}\left(a - a^*\right) && \text{by solving for } iy \text{ in previous equations}\\
& & &= \Im a && \text{by definition of } \Im && \text{(Definition B.5 page 289)}
\end{aligned}$$

Proof that $a = x + iy \impliedby x = \Re a$ and $y = \Im a$:

$$\begin{aligned}
x + iy &= \Re a + i\,\Im a && \text{by right hypothesis}\\
&= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\Re a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\Im a} && \text{by definition of } \Re \text{ and } \Im && \text{(Definition B.5 page 289)}\\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \left(\cancelto{0}{\frac{1}{2}a^* - \frac{1}{2}a^*}\right)\\
&= a
\end{aligned}$$

✏

[11] Michel and Herget (1993) page 430, Rickart (1960), page 179, Gelfand and Neumark (1943b), page 7

B.3 Normed Algebras

Definition B.6. [12] *Let $\boldsymbol{A}$ be an algebra.*

DEF

*The pair $(\boldsymbol{A}, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in \boldsymbol{A} \qquad \text{(multiplicative condition)}$$

*A normed algebra $(\boldsymbol{A}, \|\cdot\|)$ is a **Banach algebra** if $(\boldsymbol{A}, \|\cdot\|)$ is also a Banach space.*

Proposition B.2.

PRP

$(\boldsymbol{A}, \|\cdot\|)$ *is a normed algebra* $\implies$ *multiplication is **continuous** in* $(\boldsymbol{A}, \|\cdot\|)$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x-y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(\boldsymbol{A}, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 291)}\\
&= \|z(x-y)\| \\
&\le \|z\|\;\|x-y\| && \text{by definition of } \textit{normed algebra} && \text{(Definition B.6 page 291)}\\
&\triangleq \|z\|\;\delta && \text{by definition of } \delta && \text{(item (2) page 291)}\\
&\le \epsilon && \text{for some value of } \delta > 0
\end{aligned}$$

✏

Theorem B.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l}(\boldsymbol{A}, \|\cdot\|) \text{ is a Banach algebra}\\ \text{every nonzero } x \in \boldsymbol{A} \text{ is invertible}\end{array}\right\} \implies \boldsymbol{A} \equiv \mathbb{C} \quad (\boldsymbol{A} \text{ is isomorphic to } \mathbb{C})$$

B.4 C* Algebras

Definition B.7. [14]

DEF

*The triple $(\boldsymbol{A}, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(\boldsymbol{A}, \|\cdot\|)$ *is a Banach algebra* and
2. $(\boldsymbol{A}, *)$ *is a $*$-algebra* and
3. $\left\|x^*x\right\| = \|x\|^2 \quad \forall x \in \boldsymbol{A}$.

*A C^* **algebra** $(\boldsymbol{A}, \|\cdot\|, *)$ is also called a **C star algebra**.*

[12] Rickart (1960), page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩
[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩
[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem B.5. [15] *Let $\boldsymbol{A}$ be an algebra.*

THM			
	$(\boldsymbol{A}, \|\cdot\|, *)$ *is a* C^* ***algebra***	$\Longrightarrow$	$\|x^*\| = \|x\|$

✎PROOF:

$$\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\ \|x\|^2 \\
&= \frac{1}{\|x\|}\ \|x^* x\| && \text{by definition of } C^*\text{-}\textit{algebra} && \text{(Definition B.7 page 291)} \\
&\le \frac{1}{\|x\|}\ \|x^*\|\ \|x\| && \text{by definition of } \textit{normed algebra} && \text{(Definition B.6 page 291)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by } \textit{involution} \text{ property of } * && \text{(Definition B.3 page 288)}
\end{aligned}$$

✏

[15] Folland (1995) page 1, Gelfand and Neumark (1943b), page 4, Gelfand and Neumark (1943a)

APPENDIX C

TRANSLATION SPACES

C.1 Translation

C.1.1 Definitions

Definition C.1. *Let X be a set and $\mathbf{I}$ be the identity operator on X.*

DEF

$\mathbf{T}_x$ *is a **translation operator** on X if*

1. $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$ $\quad \forall A \in 2^X$ (IDENTITY) *and*
2. $\mathbf{T}_x \mathbf{T}_y = \mathbf{T}_y \mathbf{T}_x$ $\quad \forall x, y \in X$ (COMMUTATIVE) *and*
3. $\mathbf{T}_x \bigcup_{i \in I} A_i = \bigcup_{i \in I} \mathbf{T}_x A_i$ $\quad \forall A, Y \in 2^X, x \in X$ (DISTRIBUTIVE *over* $\cup$) *and*
4. $\bigcup_{b \in B} \mathbf{T}_b A = \bigcup_{a \in A} \mathbf{T}_a B$ $\quad \forall A, B \in 2^X$ *and*
5. $\mathbf{T}_x (A \cap B) = (\mathbf{T}_x A) \cap (\mathbf{T}_x B)$ $\quad \forall A, B \in 2^X, x \in X$ *and*
6. $\mathbf{T}_x (A^c) = \mathsf{c}(\mathbf{T}_x A)$ $\quad \forall A, B \in 2^X, x \in X.$

*The pair $(X, \mathbf{T})$ is a **translation space** on X.*

Definition C.2. [1] *Let X be a set on which is defined the translation opertor $\mathbf{T}_x$. **Minkowski addition** $\oplus$ and **Minkowski subtraction** $\ominus$ is defined as follows:*

DEF

$$A \oplus B = \bigcup_{b \in B} \mathbf{T}_b A \qquad \forall A, B \in 2^X \qquad \text{(MINKOWSKI ADDITION)}$$

$$A \ominus B = \bigcap_{b \in B} \mathbf{T}_b A \qquad \forall A, B \in 2^X \qquad \text{(MINKOWSKI SUBTRACTION)}$$

Theorem C.1 (next) shows a relationship between Minkowski addition and Minkowski subtraction.

Theorem C.1 (de Morgan relations). [2] *Let $(X, +)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\mathsf{c}(A \oplus B) = A^c \ominus B \qquad \forall A, B \in 2^X$$

$$\mathsf{c}(A \ominus B) = A^c \oplus B \qquad \forall A, B \in 2^X$$

[1] Matheron (1975) page 17
Lay (1982) page 7

[2] Pitas and Venetsanopoulos (1991), page 159

✎PROOF:

$$\begin{aligned}
\mathsf{c}(A \oplus B) &= \mathsf{c}\left(\bigcup_{b\in B} \mathbf{T}_b A\right) && \text{by Definition C.2 page 293}\\
&= \bigcap_{b\in B} \mathsf{c}(\mathbf{T}_b A) && \text{by Demorgan relation page 293}\\
&= \bigcap_{b\in B} \mathbf{T}_b(A^{\mathsf{c}}) && \text{by Definition C.1 page 293}\\
&= A^{\mathsf{c}} \ominus B && \text{by Theorem C.2 page 296}
\end{aligned}$$

$$\begin{aligned}
\mathsf{c}(A \ominus B) &= \mathsf{c}\left(\bigcap_{b\in B} \mathbf{T}_b A\right) && \text{by Definition C.2 page 293}\\
&= \bigcup_{b\in B} \mathsf{c}(\mathbf{T}_b A) && \text{by Demorgan relation page 293}\\
&= \bigcup_{b\in B} \mathbf{T}_b(A^{\mathsf{c}}) && \text{by Definition C.1 page 293}\\
&= A^{\mathsf{c}} \oplus B && \text{by Theorem C.2 page 296}
\end{aligned}$$

✏

C.1.2 Examples

Example C.1 (Translation on groups). [3] Let $\oplus$ be the Minkowski addition operator defined in terms of the *translation operator* $\mathbf{T}$. Let $(X, +)$ be a *group*.

EX
$$\left\{ \mathbf{T}_x A \triangleq \{a + x | a \in A\} \quad \forall A \in 2^X \right\} \implies \left\{ \begin{array}{ll} \mathbf{T}_x \text{ is a translation operator} & \text{and} \\ A \oplus B = \{a + b | a \in A \text{ and } b \in B\} & \forall\, A, B \in 2^X \end{array} \right\}$$

✎PROOF:

1. Proof that $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$:

$$\begin{aligned}
\mathbf{T}_0 A &= \{a + 0 | a \in A\} && \text{by definition of } \mathbf{T}_x\\
&= \{a | a \in A\} && \text{by additive identity property of groups}\\
&= A
\end{aligned}$$

2. Proof that $\mathbf{T}_x\mathbf{T}_y = \mathbf{T}_y\mathbf{T}_x$:

$$\begin{aligned}
\mathbf{T}_x\mathbf{T}_y A &= \mathbf{T}_x \{a + y | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a + y + x | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a + x + y | a \in A\} && \text{by commutative property of groups}\\
&= \mathbf{T}_y \{a + x | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \mathbf{T}_y\mathbf{T}_x \{a | a \in A\} && \text{by definition of } \mathbf{T}_x
\end{aligned}$$

[3] Matheron (1975) pages 16–17
Pitas and Venetsanopoulos (1991) page 159
Lay (1982) page 7

3. Proof that $\mathbf{T}_x \bigcup_{i \in I} A_i = \bigcup_{i \in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcup_i A_i &= \left\{ y + x | y \in \bigcup_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y + x | \bigvee_i y \in A_i \right\} \\
&= \bigcup_i \{ y + x | y \in A_i \} \\
&= \bigcup_i \mathbf{T}_x \{ y | y \in A_i \} \\
&= \bigcup_i \mathbf{T}_x A
\end{aligned}$$

4. Proof that $\bigcup_{b \in B} \mathbf{T}_b A = \bigcup_{a \in A} \mathbf{T}_a B$:

$$\begin{aligned}
\bigcup_{b \in B} \mathbf{T}_b A &= \bigcup_{b \in B} \{ a + b | a \in A \} && \text{by definition of } \mathbf{T}_x \\
&= \{ a + b | a \in A \text{ and } b \in B \} \\
&= \{ b + a | b \in B \text{ and } a \in A \} \\
&= \bigcup_{a \in A} \{ b + a | b \in B \} \\
&= \bigcup_{a \in A} \mathbf{T}_a B
\end{aligned}$$

5. Proof that $\mathbf{T}_x \bigcap_{i \in I} A_i = \bigcap_{i \in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcap_i A_i &= \left\{ y + x | y \in \bigcap_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y + x | \bigwedge_i y \in A_i \right\} \\
&= \bigcap_i \{ y + x | y \in A_i \} \\
&= \bigcap_i \mathbf{T}_x \{ y | y \in A_i \} \\
&= \bigcap_i \mathbf{T}_x A
\end{aligned}$$

6. Proof that $\mathbf{T}_x \left(A^{\mathsf{c}} \right) = \mathsf{c} \left(\mathbf{T}_x A \right)$:

$$\begin{aligned}
\mathbf{T}_x \mathsf{c} A &= \mathbf{T}_x \{ a | a \in A^{\mathsf{c}} \} \\
&= \{ a + x | a \in A^{\mathsf{c}} \} \\
&= \{ a + x | a \notin A \} \\
&= \{ a + x | \neg (a \in A) \} \\
&= \mathsf{c} \{ a + x | a \in A \} \\
&= \mathsf{c} \mathbf{T}_x A
\end{aligned}$$

$$\begin{aligned}
A \oplus B &= \bigcup_{b \in B} \mathbf{T}_b A && \text{by Definition C.2 page 293} \\
&= \{ a + b | a \in A \text{ and } b \in B \} && \text{by Definition C.1 page 293}
\end{aligned}$$

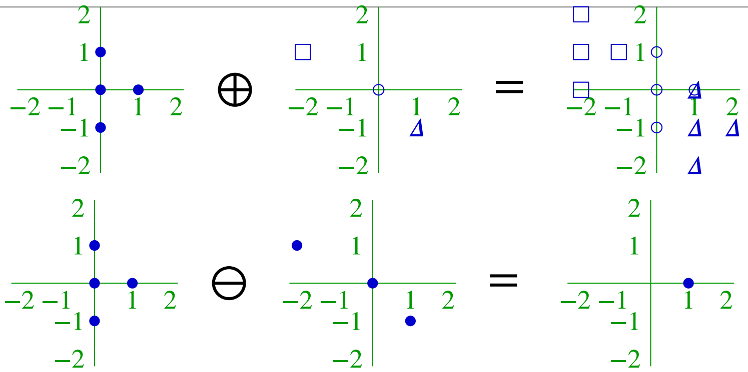


Figure C.1: Illustration for Example C.2 (page 296)

Example C.2. Let

$$A \triangleq \{(0,0),\ (0,1),\ (0,-1),\ (1,1)\}$$
$$B \triangleq \{(0,0),\ (-2,1),\ \ (1,-1)\}$$

Then

$$A \oplus B = \{(0,0),\ (0,1),\ (0,-1),\ (1,1),\ (-2,1),\ (-2,2),\ (-2,0),\ (-1,2),\ (1,-1),\ (1,-2),\ (2,0)\}$$
$$A \ominus B = \{(1,0)\}$$

These relationships are illustrated in Figure C.1 (page 296).

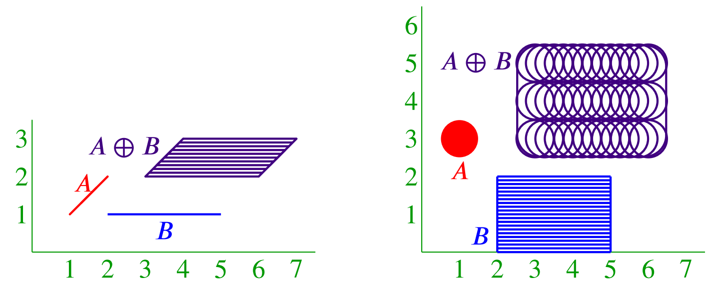


Figure C.2: Illustration for Example C.3 page 296

Example C.3. [4] Two more examples are illustrated in Figure C.2 (page 296).

C.1.3 Additive properties

Theorem C.2. [5] *Let $(X,+)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

THM	$A \oplus \{0\}$	$=$	A	$\forall A \subseteq X$	
	$A \oplus B$	$=$	$B \oplus A$	$\forall A,B \subseteq X$	(COMMUTATIVE)
	$A \oplus (B \oplus C)$	$=$	$(A \oplus B) \oplus C$	$\forall A,B,C \subseteq X$	(ASSOCIATIVE)
	$\mathbf{T}_x(A \oplus B)$	$=$	$\left(\mathbf{T}_x A\right) \oplus B$	$\forall A,B \subseteq X,\ x \in X$	(TRANSLATION INVARIANT)

[4] Lay (1982) page 7
[5] Pitas and Venetsanopoulos (1991), pages 163–164

✎PROOF:

$$\begin{aligned}
A \oplus \{0\} &= A \oplus B|_{B=\{0\}} \\
&= \left.\bigcup_{b\in B} \mathbf{T}_b A\right|_{B=\{0\}} && \text{by Definition C.2 page 293} \\
&= \mathbf{T}_0 A \\
&= A && \text{by Definition C.1 page 293}
\end{aligned}$$

$$\begin{aligned}
A \oplus B &= \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition C.2 page 293} \\
&= \bigcup_{a\in A} \mathbf{T}_a B && \text{by Definition C.1 page 293} \\
&= B \oplus A && \text{by Definition C.2 page 293}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \oplus C) &= \bigcup_{y\in B\oplus C} \mathbf{T}_y A && \text{by Definition C.2 page 293} \\
&= \bigcup_{a\in A} \mathbf{T}_a (B \oplus C) && \text{by Definition C.1 page 293} \\
&= \bigcup_{a\in A} \mathbf{T}_a \left(\bigcup_{c\in C} \mathbf{T}_c B\right) && \text{by Definition C.2 page 293} \\
&= \bigcup_{a\in A} \left(\bigcup_{c\in C} \mathbf{T}_a \mathbf{T}_c B\right) && \text{by Definition C.1 page 293} \\
&= \bigcup_{a\in A} \left(\bigcup_{c\in C} \mathbf{T}_c \mathbf{T}_a B\right) && \text{by Definition C.1 page 293} \\
&= \bigcup_{c\in C} \mathbf{T}_c \left(\bigcup_{a\in A} \mathbf{T}_a B\right) && \text{by Definition C.1 page 293} \\
&= \bigcup_{c\in C} \mathbf{T}_c \left(\bigcup_{b\in B} \mathbf{T}_b A\right) && \text{by Definition C.1 page 293} \\
&= \bigcup_{c\in C} \mathbf{T}_c (A \oplus B) && \text{by Definition C.2 page 293} \\
&= (A \oplus B) \oplus C && \text{by Definition C.2 page 293}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_x (A \oplus B) &= \mathbf{T}_x \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition C.2 page 293} \\
&= \bigcup_{b\in B} \mathbf{T}_x \mathbf{T}_b A && \text{by Definition C.1 page 293} \\
&= \bigcup_{b\in B} \mathbf{T}_b \mathbf{T}_x A && \text{by Definition C.1 page 293} \\
&= \big(\mathbf{T}_x A\big) \oplus B && \text{by Definition C.2 page 293}
\end{aligned}$$

⇒

Theorem C.3. [6] *Let $(X, +)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

[6] Pitas and Venetsanopoulos (1991), page 163

THM

$$\begin{array}{rcll} A \oplus (B \cup C) &=& (A \oplus B) \cup (A \oplus C) & \forall A,B,C \subseteq X \quad (\oplus \text{ is LEFT DISTRIBUTIVE over } \cup) \\ (A \cup B) \oplus C &=& (A \oplus C) \cup (B \oplus C) & \forall A,B,C \subseteq X \quad (\oplus \text{ RIGHT DISTRIBUTIVE over } \cup) \\ A \oplus (B \cap C) &\subseteq& (A \oplus B) \cap (A \oplus C) & \forall A,B,C \subseteq X \\ (A \cap B) \oplus C &\subseteq& (A \oplus C) \cap (B \oplus C) & \forall A,B,C \subseteq X \end{array}$$

✎PROOF:

$$\begin{aligned} (A \cup B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c(A \cup B) && \text{by Definition C.2 page 293} \\ &= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cup (\mathbf{T}_c B) \right] && \text{by Definition C.1 page 293} \\ &= \left(\bigcup_{c \in C} \mathbf{T}_c A \right) \cup \left(\bigcup_{c \in C} \mathbf{T}_c B \right) \\ &= (A \oplus C) \cup (B \oplus C) && \text{by Definition C.2 page 293} \end{aligned}$$

$$\begin{aligned} A \oplus (B \cup C) &= (B \cup C) \oplus A && \text{by Theorem C.2 page 296} \\ &= (B \oplus A) \cup (C \oplus A) && \text{by previous result} \\ &= (A \oplus B) \cup (A \oplus C) && \text{by Theorem C.2 page 296} \end{aligned}$$

$$\begin{aligned} (A \cap B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c(A \cap B) && \text{by Theorem C.2 page 296} \\ &= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cap (\mathbf{T}_c B) \right] && \text{by Definition C.1 page 293} \\ &\subseteq \left(\bigcup_{c \in C} \mathbf{T}_c A \right) \cap \left(\bigcup_{c \in C} \mathbf{T}_c B \right) && \text{by minimax inequality page 76} \\ &= (A \oplus C) \cap (B \oplus C) && \text{by Theorem C.2 page 296} \end{aligned}$$

$$\begin{aligned} A \oplus (B \cap C) &= (B \cap C) \oplus A && \text{by Theorem C.2 page 296} \\ &\subseteq (B \oplus A) \cap (C \oplus A) && \text{by previous result} \\ &= (A \oplus B) \cap (A \oplus C) && \text{by Theorem C.2 page 296} \end{aligned}$$

✏

C.1.4 Subtractive properties

Theorem C.4. [7] *Let $(X, +)$ be a group with with Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\begin{array}{rrcll} & A \ominus \{0\} &=& A & \forall A \subseteq X \\ & A \ominus B &=& B^c \ominus A^c & \forall A,B \subseteq X \\ & \mathbf{T}_x(A \ominus B) &=& (\mathbf{T}_x A) \ominus B & \forall A,B \subseteq X,\ x \in X \quad (\text{TRANSLATION INVARIANT}) \\ A \subseteq B \implies & A \ominus C &\subseteq& B \ominus C & \forall A,B,C \subseteq X \quad (\text{INCREASING}) \end{array}$$

[7] Pitas and Venetsanopoulos (1991), pages 164–165

✎PROOF:

$$\begin{aligned}
A \ominus \{0\} &= \mathsf{c}(A^\mathsf{c} \oplus \{0\}) && \text{by Theorem C.1 page 293}\\
&= \mathsf{c}(A^\mathsf{c}) && \text{by Theorem C.2 page 296}\\
&= A
\end{aligned}$$

$$\begin{aligned}
A \ominus B &= \mathsf{cc}(A \ominus B)\\
&= \mathsf{c}(A^\mathsf{c} \oplus B) && \text{by Theorem C.1 page 293}\\
&= \mathsf{c}(B \oplus A^\mathsf{c}) && \text{by Theorem C.2 page 296}\\
&= B^\mathsf{c} \ominus A^\mathsf{c} && \text{by Theorem C.1 page 293}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_x(A \ominus B) &= \mathbf{T}_x\mathsf{c}\left(A^\mathsf{c} \oplus B\right) && \text{by Theorem C.1 page 293}\\
&= \mathsf{c}\mathbf{T}_x\left(A^\mathsf{c} \oplus B\right) && \text{by Definition C.1 page 293}\\
&= \mathsf{c}\left(\mathbf{T}_x A^\mathsf{c} \oplus B\right) && \text{by Theorem C.2 page 296}\\
&= \mathsf{c}\left(\mathsf{c}\mathbf{T}_x A \oplus B\right) && \text{by Definition C.1 page 293}\\
&= \mathbf{T}_x A \ominus B && \text{by Theorem C.1 page 293}
\end{aligned}$$

$$\begin{aligned}
A \ominus C &= \bigcap_{c\in C} A_c && \text{by Theorem C.2 page 296}\\
&\subseteq \bigcap_{c\in C} B_c && \text{by } A \subseteq B \text{ hypothesis}\\
&= B \ominus C && \text{by Definition C.2 page 293}
\end{aligned}$$

✏

Theorem C.5. [8] *Let $(X,+)$ be a group with with Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$A \ominus (B \cup C)$	$=$	$(A \ominus B) \cap (A \ominus C)$	$\forall A,B,C \subseteq X$	($\ominus$ LEFT DISTRIBUTIVE *over* $\cup$)
$(A \cap B) \ominus C$	$=$	$(A \ominus C) \cap (B \ominus C)$	$\forall A,B,C \subseteq X$	($\ominus$ RIGHT DISTRIBUTIVE *over* $\cap$)
$(A \cup B) \ominus C$	$\supseteq$	$(A \ominus C) \cup (B \ominus C)$	$\forall A,B,C \subseteq X$	
$A \ominus (B \cap C)$	$\supseteq$	$(A \ominus B) \cup (A \ominus C)$	$\forall A,B,C \subseteq X$	

✎PROOF:

$$\begin{aligned}
A \ominus (B \cup C) &= \mathsf{cc}\Big[\ A \ominus (B \cup C)\ \Big]\\
&= \mathsf{c}\Big[\ A^\mathsf{c} \oplus (B \cup C)\ \Big] && \text{by Theorem C.1 page 293}\\
&= \mathsf{c}\Big[\ (A^\mathsf{c} \oplus B) \cup (A^\mathsf{c} \oplus C)\ \Big] && \text{by Theorem C.3 page 297}\\
&= \left[\mathsf{c}\left(A^\mathsf{c} \oplus B\right)\right] \cap \left[\mathsf{c}\left(A^\mathsf{c} \oplus C\right)\right] && \text{by Demorgan relation page 293}\\
&= (A \ominus B) \cap (A \ominus C) && \text{by Theorem C.1 page 293}
\end{aligned}$$

$$\begin{aligned}
(A \cap B) \ominus C &= \mathsf{c}[(A \cap B) \ominus C]\\
&= \mathsf{c}[\mathsf{c}(A \cap B) \oplus C] && \text{by Theorem C.1 page 293}\\
&= \mathsf{c}\left[(A^\mathsf{c} \cup B^\mathsf{c}) \oplus C\right]\\
&= \mathsf{c}\left[(A^\mathsf{c} \oplus C) \cup (B^\mathsf{c} \oplus C)\right] && \text{by Theorem C.3 page 297}\\
&= \mathsf{c}(A^\mathsf{c} \oplus C) \cap \mathsf{c}(B^\mathsf{c} \oplus C)\\
&= (A \ominus C) \cap (B \ominus C) && \text{by Theorem C.1 page 293}
\end{aligned}$$

[8] Pitas and Venetsanopoulos (1991), page 165

$$\begin{aligned}
A \ominus (B \cap C) &= \mathsf{cc}\left[\ A \ominus (B \cap C)\ \right] \\
&= \mathsf{c}\left[\ A^c \oplus (B \cap C)\ \right] && \text{by Theorem C.1 page 293} \\
&\supseteq \mathsf{c}\left[(A^c \oplus B) \cap (A^c \oplus C)\right] && \text{by Theorem C.3 page 297} \\
&= \left[\mathsf{c}(A^c \oplus B)\right] \cup \left[\mathsf{c}(A^c \oplus C)\right] && \text{by Demorgan relation page 293} \\
&= (A \ominus B) \cup (A \ominus C) && \text{by Theorem C.1 page 293}
\end{aligned}$$

$$\begin{aligned}
(A \cup B) \ominus C &= \mathsf{cc}[(A \cup B) \ominus C] \\
&= \mathsf{c}[\mathsf{c}(A \cup B) \oplus C] && \text{by Theorem C.1 page 293} \\
&= \mathsf{c}\left[\left(A^c \cap B^c\right) \oplus C\right] && \text{by Demorgan relation page 293} \\
&\supseteq \mathsf{c}\left[\left(A^c \oplus C\right) \cap \left(B^c \oplus C\right)\right] \\
&= \mathsf{c}\left(A^c \oplus C\right) \cup \mathsf{c}\left(B^c \oplus C\right) \\
&= (A \ominus C) \cup (B \ominus C) && \text{by Theorem C.1 page 293}
\end{aligned}$$

⇨

Theorem C.6. [9] *Let $(X, +)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM				
	$A \ominus (B \oplus C)$	$=$	$(A \ominus B) \ominus C$	$\forall A,B,C \subseteq X$
	$A \oplus (B \ominus C)$	$\subseteq$	$(A \oplus B) \ominus C$	$\forall A,B,C \subseteq X$

✎PROOF:

$$\begin{aligned}
A \ominus (B \oplus C) &= \mathsf{cc}\left[\ A \ominus (B \oplus C)\ \right] \\
&= \mathsf{c}\left[\ A^c \oplus (B \oplus C)\ \right] && \text{by Theorem C.1 page 293} \\
&= \mathsf{c}\left[\ (A^c \oplus B) \oplus C\ \right] && \text{by Theorem C.2 page 296} \\
&= \mathsf{c}(A^c \oplus B) \ominus C && \text{by Theorem C.1 page 293} \\
&= (A \ominus B) \ominus C && \text{by Theorem C.1 page 293}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \ominus C) &= A \oplus \left(\bigcap_{c \in C} \mathbf{T}_c B\right) && \text{by Definition C.2 page 293} \\
&= \left(\bigcap_{c \in C} \mathbf{T}_c B\right) \oplus A && \text{by Theorem C.2 page 296} \\
&= \bigcup_{a \in A} \mathbf{T}_a \left(\bigcap_{c \in C} \mathbf{T}_c B\right) && \text{by Definition C.2 page 293} \\
&= \bigcup_{a \in A} \bigcap_{c \in C} \mathbf{T}_a \mathbf{T}_c B && \text{by Definition C.1 page 293} \\
&\subseteq \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_a \mathbf{T}_c B && \text{by minimax inequality page 76} \\
&= \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_c \mathbf{T}_a B && \text{by Definition C.1 page 293}
\end{aligned}$$

[9] Pitas and Venetsanopoulos (1991), page 166

$$
\begin{aligned}
&= \bigcap_{c\in C} \mathbf{T}_c\left(\bigcup_{a\in A} \mathbf{T}_a B\right) && \text{by Definition C.1 page 293}\\
&= \bigcap_{c\in C} \mathbf{T}_c(B \oplus A) && \text{by Definition C.2 page 293}\\
&= (B \oplus A) \ominus C && \text{by Definition C.2 page 293}\\
&= (A \oplus B) \ominus C && \text{by Theorem C.2 page 296}
\end{aligned}
$$

C.2 Operations

Definition C.3. [10] *Let $(X,+)$ be a group.*

DEF: *The **symmetric set** of A is the set* $\check{A} \triangleq -A \qquad \forall A \subseteq X$

Definition C.4. [11] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and D^s be the symmetric set of set D.*

DEF:
*The **dilation** of A by D is the operation* $A \oplus \check{D} \qquad \forall A, D \subseteq X.$
*The **erosion** of A by E is the operation* $A \ominus \check{E} \qquad \forall A, E \subseteq X.$

Definition C.5. [12] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and B^s be the symmetric set of a set B.*

DEF:
*The **opening** of A with respect to B is the set* $A_B \triangleq \underbrace{\underbrace{(A \ominus \check{B})}_{\text{erosion}} \oplus B}_{\text{dilation}} \qquad \forall A, B \subseteq X.$

*The **closing** of A with respect to B is the set* $A^B \triangleq \underbrace{\underbrace{(A \oplus \check{B})}_{\text{dilation}} \ominus B}_{\text{erosion}} \qquad \forall A, B \subseteq X.$

Theorem C.7. [13] *Let $(X,+)$ be a group with A_B representing the opening of a set A with respect to a set B and A^B representing the closing of a set A with respect to a set B.*

THM:
(complement of the opening)→ $\mathsf{c}(A_B) = (A^{\mathsf{c}})^B$ ←*(closing of the complement)* $\forall A, B \subseteq X$
(complement of the closing)→ $\mathsf{c}(A^B) = (A^{\mathsf{c}})_B$ ←*(opening of the complement)* $\forall A, B \subseteq X$

✎PROOF:

$$
\begin{aligned}
\mathsf{c}(A_B) &= \mathsf{c}\Big[(A \ominus \check{B}) \oplus B \Big] && \text{by Definition C.5 page 301}\\
&= \mathsf{c}(A \ominus \check{B}) \ominus B && \text{by Theorem C.1 page 293}\\
&= \mathsf{c}(A \ominus \check{B}) \ominus B && \text{by Theorem C.1 page 293}\\
&= (A^{\mathsf{c}} \oplus \check{B}) \ominus B && \text{by Theorem C.1 page 293}\\
&= (A^{\mathsf{c}})^B && \text{by Definition C.5 page 301}
\end{aligned}
$$

10 Matheron (1975), page 17
11 Pitas and Venetsanopoulos (1991), page 161
12 Serra (1982), page 50
13 Serra (1982), page 51

$$
\begin{aligned}
\mathsf{c}(A^B) &= \mathsf{c}\Big[\,(A \oplus \breve{B}) \ominus B\,\Big] && \text{by Definition C.5 page 301}\\
&= \mathsf{c}(A \oplus \breve{B}) \oplus B && \text{by Theorem C.1 page 293}\\
&= \mathsf{c}(A \oplus \breve{B}) \oplus B && \text{by Theorem C.1 page 293}\\
&= (A^{\mathsf{c}} \ominus \breve{B}) \oplus B && \text{by Theorem C.1 page 293}\\
&= (A^{\mathsf{c}})_B && \text{by Definition C.5 page 301}
\end{aligned}
$$

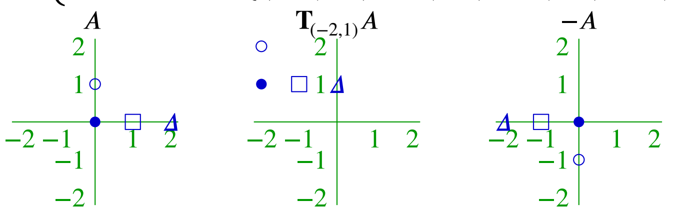
✏

Example C.4.

EX

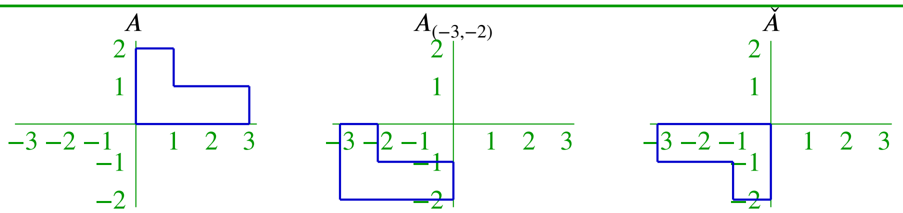
$A \triangleq \{(0,0),\ (1,0),\ (2,0),\ (0,1)\} \implies$

$$\left\{\begin{array}{lcl} A_{(-2,1)} & = & \{(-2,1),\ (-1,1),\ (0,1),\ (-2,2)\} \quad \text{and} \\ \breve{A} & = & \{(0,0),\ (-1,0),\ (-2,0),\ (0,-1)\} \end{array}\right\}$$



Example C.5. An example similar to Example C.4 (page 302) but using solid shapes is illustrated next:

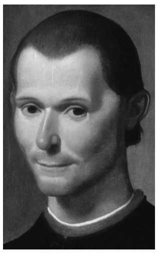
EX



Back Matter

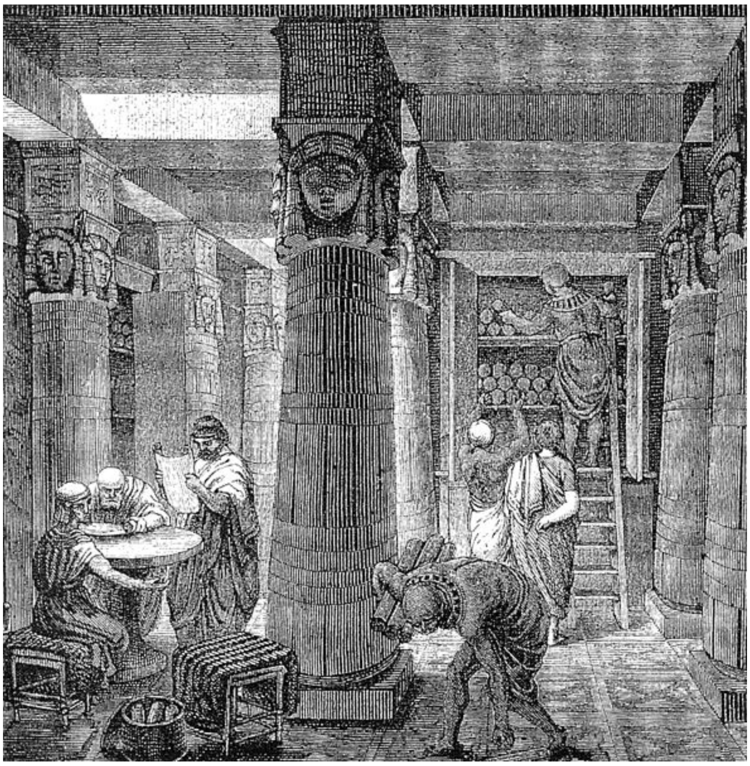
It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.

Niels Henrik Abel (1802–1829), Norwegian mathematician [14]

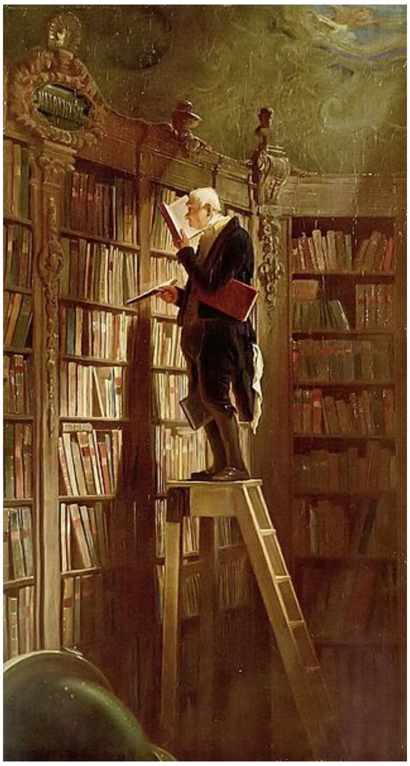


When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.

Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [15]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850

[16]

To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.

Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [17]

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14 quote: Simmons (2007), page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

15 quote: Machiavelli (1961), page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

16 http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

[17] quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

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