Trigonometric Systems

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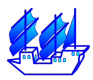
The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]



[1] Paine (2000) page 63 ⟨Golden Hind⟩

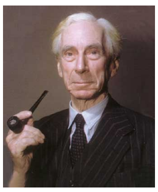
❝Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?❞

❝Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.❞

Alfred Edward Housman, English poet (1859–1936) [2]

❝The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.❞

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



❝As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.❞

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

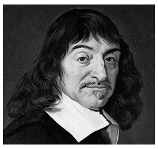
[2] quote: Housman (1936) page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961) page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967) page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝*rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.*❞



❝*Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.*❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝*In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.*❞

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \dots$

...continued on next page...

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$
$\mathbb{Z}^{-}$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_o$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_e$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^{+}$	positive real numbers	$(0, \infty)$
$\mathbb{R}^{-}$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^{*}$	extended real numbers	$\mathbb{R}^{*} \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
$\oslash$	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \nabla)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\nsubseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^c	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	

...continued on next page...

symbol	description	
0	"false" condition	
$\neg$	logical NOT operation	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\le$	reflexive ordering relation	"less than or equal to"
$\ge$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathscr{T}(X)$	set of topologies on a set X	
$\mathscr{R}(X)$	set of rings of sets on a set X	
$\mathscr{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathscr{S}_{\mathsf{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathscr{I}_{\mathsf{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathscr{B}_{\mathsf{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathscr{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		

...continued on next page...

symbol	description
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition 3.2 page 44)
$\hat{\mathbf{F}}$	*Fourier Series* operator (Definition 5.1 page 73)
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition 6.1 page 77)
$\mathbf{Z}$	*Z-Transform* operator (Definition F.4 page 132)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $\left(\!\left(x_n \in \mathbb{C}\right)\!\right)_{n\in\mathbb{Z}}$
$\breve{\mathsf{x}}(z)$	*Z-Transform* of a sequence $\left(\!\left(x_n \in \mathbb{C}\right)\!\right)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

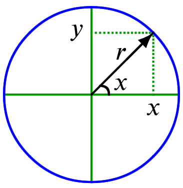
CHAPTER 1

TRIGONOMETRIC FUNCTIONS

1.1 Definition Candidates

There are several ways of defining the sine and cosine functions, including the following:[1]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[2]



$$\begin{aligned}\cos x &\triangleq \frac{x}{r}\\ \sin x &\triangleq \frac{y}{r}\end{aligned}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[3]
$$\cos x \triangleq \mathbf{R}_{\mathrm{e}} e^{ix} \qquad \sin x \triangleq \mathbf{I}_{\mathrm{m}}\left(e^{ix}\right)$$

3. **Polynomial:** Let $\displaystyle\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of *Taylor expansion*s such that[4]

$$\begin{aligned}\cos(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\ \sin(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

[1] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means “bay” or “inlet”—thus resulting in the Latin translation *sinus*, which also means “bay” or “inlet”. Reference: Boyer and Merzbach (1991) page 252

[2] Abramowitz and Stegun (1972) page 78

[3] Euler (1748)

[4] Rosenlicht (1968) page 157, Abramowitz and Stegun (1972) page 74

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \prod_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of a product of factors such that[5]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1-\left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[6]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-(n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-(n\pi)^2}\right)}_{\cot(x)}\sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{d}{dx}$ such that

$$\begin{array}{llllll}
\cos(x) & \triangleq & f(x) & \text{where} & \underbrace{\frac{d^2}{dx^2}f+f=0}_{\text{differential equation}} & \underbrace{f(0)=1}_{\text{1st initial condition}} & \underbrace{\left[\frac{d}{dx}f\right](0)=0}_{\text{2nd initial condition}} \\
\sin(x) & \triangleq & g(x) & \text{where} & \underbrace{\frac{d^2}{dx^2}g+g=0}_{\text{differential equation}} & \underbrace{g(0)=0}_{\text{1st initial condition}} & \underbrace{\left[\frac{d}{dx}g\right](0)=1}_{\text{2nd initial condition}}
\end{array}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[7]

$$\begin{array}{llllll}
\cos(x) & \triangleq & f^{-1}(x) & \text{where} & f(x) & \triangleq \underbrace{\int_x^1 \sqrt{\frac{1}{1-y^2}}\,dy}_{\arccos(x)} \\
\sin(x) & \triangleq & g^{-1}(x) & \text{where} & g(x) & \triangleq \underbrace{\int_0^x \sqrt{\frac{1}{1-y^2}}\,dy}_{\arcsin(x)}
\end{array}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{d}{dx}$ (Definition 1.1 page 3). Support for such an approach includes the following:

- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{d}{dx}$ (Theorem 1.1 page 4).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem 1.3 page 6).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem 1.4 page 7).

[5] Abramowitz and Stegun (1972) page 75
[6] Abramowitz and Stegun (1972) page 75
[7] Abramowitz and Stegun (1972) page 79

- The complex exponential function is a solution of a second order homogeneous differential equation (Definition 1.4 page 8).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section 1.6 page 16).

1.2 Definitions

Definition 1.1. [8] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and* $\frac{d}{dx} \in \boldsymbol{C}^{\boldsymbol{C}}$ *the differentiation operator.*

DEF

The function $f \in \mathbb{C}^{\mathbb{C}}$ *is the **cosine** function* $\cos(x) \triangleq f(x)$ *if*

$$\begin{array}{lrcll}
1. & \frac{d^2}{dx^2}f + f & = & 0 & \text{(second order homogeneous differential equation) and}\\
2. & f(0) & = & 1 & \text{(first initial condition) and}\\
3. & \left[\frac{d}{dx}f\right](0) & = & 0 & \text{(second initial condition).}
\end{array}$$

Definition 1.2. [9] *Let $\boldsymbol{C}$ and $\frac{d}{dx} \in \boldsymbol{C}^{\boldsymbol{C}}$ be defined as in definition of* $\cos(x)$ (Definition 1.1 page 3).

DEF

The function $f \in \mathbb{C}^{\mathbb{C}}$ *is the **sine** function* $\sin(x) \triangleq f(x)$ *if*

$$\begin{array}{lrcll}
1. & \frac{d^2}{dx^2}f + f & = & 0 & \text{(second order homogeneous differential equation) and}\\
2. & f(0) & = & 0 & \text{(first initial condition) and}\\
3. & \left[\frac{d}{dx}f\right](0) & = & 1 & \text{(second initial condition).}
\end{array}$$

Definition 1.3. [10]

DEF

Let π ("pi") be defined as the element in $\mathbb{R}$ such that

$$\begin{array}{lrcll}
(1). & \cos\left(\frac{\pi}{2}\right) & = & 0 & \text{and}\\
(2). & \pi & > & 0 & \text{and}
\end{array}$$

(3). *π is the **smallest** of all elements in $\mathbb{R}$ that satisfies (1) and (2).*

1.3 Basic properties

Lemma 1.1. [11] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and* $\frac{d}{dx} \in \boldsymbol{C}^{\boldsymbol{C}}$ *the differentiation operator.*

LEM

$$\left\{\frac{d^2}{dx^2}f + f = 0\right\} \iff$$

$$\left\{\begin{aligned}
f(x) &= \underbrace{[f](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{d}{dx}f\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}}\\
&= \left(f(0) + \left[\frac{d}{dx}f\right](0)x\right) - \left(\frac{f(0)}{2!}x^2 + \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3\right) + \left(\frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5\right)\cdots
\end{aligned}\right\}$$

[8] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[10] Rosenlicht (1968) page 158
[11] Rosenlicht (1968) page 156, Liouville (1839)

✎PROOF: Let $\mathsf{f}'(x) \triangleq \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x)$.

$$\mathsf{f}'''(x) = -\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](x)$$
$$\mathsf{f}^{(4)}(x) = -\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](x) \qquad = -\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) = \mathsf{f}(x)$$

1. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) + \mathsf{f}(x) = 0 \implies \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\mathsf{f}(x) &= \sum_{n=0}^{\infty}\frac{\mathsf{f}^{(n)}(0)}{n!}x^n && \text{by Taylor expansion (Theorem B.13 page 107)}\\
&= \mathsf{f}(0) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)x - \frac{\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](0)}{2!}x^2 - \frac{\mathsf{f}^3(0)}{3!}x^3 + \frac{\mathsf{f}^4(0)}{4!}x^4 + \frac{\mathsf{f}^5(0)}{5!}x^5 - \cdots\\
&= \mathsf{f}(0) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)x - \frac{\mathsf{f}(0)}{2!}x^2 - \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{3!}x^3 + \frac{\mathsf{f}(0)}{4!}x^4 + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{5!}x^5 - \cdots\\
&= \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]
\end{aligned}$$

2. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) + \mathsf{f}(x) = 0 \impliedby \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}[\mathsf{f}(x)]\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right] && \text{by right hypothesis}\\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)\mathsf{f}(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n-1}\right]\\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n-1)!}x^{2n-1}\right]\\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]\\
&= -\mathsf{f}(x) && \text{by right hypothesis}
\end{aligned}$$

✏

Theorem 1.1 (Taylor series for cosine/sine). [12]

THM

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && \forall x\in\mathbb{R}\\
\sin(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \forall x\in\mathbb{R}
\end{aligned}$$

[12] Rosenlicht (1968) page 157

✎PROOF:

$$\cos(x) = \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \qquad \text{by Lemma 1.1 page 3}$$

$$= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \text{by cos initial conditions (Definition 1.1 page 3)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\sin(x) = \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \qquad \text{by Lemma 1.1 page 3}$$

$$= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \text{by sin initial conditions (Definition 1.2 page 3)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

✎

Theorem 1.2. [13]

THM		
	$\cos(0) = 1$	$\cos(-x) = \cos(x) \quad \forall x\in\mathbb{R}$ (EVEN)
	$\sin(0) = 0$	$\sin(-x) = -\sin(x) \quad \forall x\in\mathbb{R}$ (ODD)

✎PROOF:

$$\cos(0) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=0} \qquad \text{by \textit{Taylor series for cosine} (Theorem 1.1 page 4)}$$

$$= 1$$

$$\sin(0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big|_{x=0} \qquad \text{by \textit{Taylor series for sine} (Theorem 1.1 page 4)}$$

$$= 0$$

$$\cos(-x) = 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \cdots \qquad \text{by \textit{Taylor series for cosine} (Theorem 1.1 page 4)}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \cos(x) \qquad \text{by \textit{Taylor series for cosine} (Theorem 1.1 page 4)}$$

$$\sin(-x) = (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots \qquad \text{by \textit{Taylor series for sine} (Theorem 1.1 page 4)}$$

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \sin(x) \qquad \text{by \textit{Taylor series for sine} (Theorem 1.1 page 4)}$$

✎

Lemma 1.2. [14]

LEM		
	$\cos(1) > 0$	$x \in (0:2) \implies \sin(x) > 0$
	$\cos(2) < 0$	

[13] Rosenlicht (1968) page 157

[14] Rosenlicht (1968) page 158

✎PROOF:

$$\begin{aligned}
\cos(1) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=1} && \text{by \textit{Taylor series for cosine}} && \text{(Theorem 1.1 page 4)}\\
&= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots\\
&> 0\\
\cos(2) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=2} && \text{by \textit{Taylor series for cosine}} && \text{(Theorem 1.1 page 4)}\\
&= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots\\
&< 0
\end{aligned}$$

$$\begin{aligned}
x \in (0:2) &\implies \text{each term in the sequence } \left(\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots\right) \text{ is} > 0\\
&\implies \sin(x) > 0
\end{aligned}$$

✏

Proposition 1.1. *Let π be defined as in Definition 1.3 (page 3).*

PRP
(A). *The value π* ***exists*** *in $\mathbb{R}$.*
(B). $2 < \pi < 4$.

✎PROOF:

$$\begin{aligned}
&\cos(1) > 0 && \text{by Lemma 1.2 page 5}\\
&\cos(2) < 0 && \text{by Lemma 1.2 page 5}\\
&\implies 1 < \frac{\pi}{2} < 2\\
&\implies 2 < \pi < 4
\end{aligned}$$

✏

Theorem 1.3. [15] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $\mathsf{f}'(0) \triangleq \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)$.*

THM
$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \mathsf{f}'(0)\sin(x)\right\} \qquad \forall \mathsf{f} \in \boldsymbol{C}, \forall x \in \mathbb{R}$$

✎PROOF:

1. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) = -\mathsf{f}(x) \implies \mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sin(x)$:

$$\begin{aligned}
\mathsf{f}(x) &= \mathsf{f}(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by left hypothesis and Lemma 1.1 page 3}\\
&= \mathsf{f}(0)\cos x + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sin x && \text{by definitions of cos and sin (Definition 1.1 page 3, Definition 1.2 page 3)}
\end{aligned}$$

[15] Rosenlicht (1968) page 157.

2. Proof that $\frac{d^2}{dx^2}f = -f \Longleftarrow f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$
\begin{aligned}
f(x) &= f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x && \text{by right hypothesis}\\
&= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)}\\
\implies \quad & \frac{d^2}{dx^2}f + f = 0 && \text{by Lemma 1.1 page 3}
\end{aligned}
$$

Remark 1.1. The general solution for the *non-homogeneous* equation $\frac{d^2}{dx^2}f(x) + f(x) = g(x)$ with initial conditions $f(a) = 1$ and $f'(a) = \rho$ is $f(x) = \cos(x) + \rho\sin(x) + \int_a^x g(y)\sin(x-y)\,dy$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992) page 371, Liouville (1839). Volterra equation references: Pedersen (2000) page 99, Lalescu (1908), Lalescu (1911)

Theorem 1.4. [16] *Let $\frac{d}{dx} \in C^C$ be the differentiation operator.*

THM
$$\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x\in\mathbb{R} \qquad \frac{d}{dx}\sin(x) = \cos(x) \quad \forall x\in\mathbb{R} \qquad \cos^2(x) + \sin^2(x) = 1 \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$
\begin{aligned}
\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} && \text{by Taylor series (Theorem 1.1 page 4)}\\
&= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}\\
&= -\sin(x) && \text{by Taylor series (Theorem 1.1 page 4)}\\
\frac{d}{dx}\sin(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by Taylor series (Theorem 1.1 page 4)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= \cos(x) && \text{by Taylor series (Theorem 1.1 page 4)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\left[\cos^2(x) + \sin^2(x)\right] &= -2\cos(x)\sin(x) + 2\sin(x)\cos(x)\\
&= 0\\
&\implies \cos^2(x) + \sin^2(x) \text{ is } \textit{constant}\\
&\implies \cos^2(x) + \sin^2(x)\\
&= \cos^2(0) + \sin^2(0)\\
&= 1 + 0 = 1 && \text{by Theorem 1.2 page 5}
\end{aligned}
$$

Proposition 1.2.

PRP
$$\sin\left(\frac{\pi}{2}\right) = 1$$

[16] Rosenlicht (1968) page 157

✎PROOF:

$$
\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2) + 0} \\
&= \pm\sqrt{\sin^2(\pi/2) + \cos^2(\pi/2)} && \text{by definition of } \pi \quad \text{(Definition 1.3 page 3)} \\
&= \pm\sqrt{1} && \text{by Theorem 1.4 page 7} \\
&= \pm 1 \\
&= 1 && \text{by Lemma 1.2 page 5}
\end{aligned}
$$

✏

1.4 The complex exponential

Definition 1.4.

DEF *The function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq \mathsf{f}(x)$ *if*

1. $\frac{d^2}{dx^2}\mathsf{f} + \mathsf{f} = 0$ *(second order homogeneous differential equation)* *and*
2. $\mathsf{f}(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{d}{dx}\mathsf{f}\right](0) = i$ *(second initial condition).*

Theorem 1.5 (Euler's Identity). [17]

THM $$e^{ix} = \cos(x) + i\sin(x) \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$
\begin{aligned}
\exp(ix) &= \mathsf{f}(0)\cos(x) + \left[\frac{d}{dx}\mathsf{f}\right](0)\sin(x) && \text{by Theorem 1.3 page 6} \\
&= \cos(x) + i\sin(x) && \text{by Definition 1.4 page 8}
\end{aligned}
$$

✏

Proposition 1.3.

PRP $$e^{-i\pi/2} = -i \quad \Big| \quad e^{i\pi/2} = i$$

✎PROOF:

$$
\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by } \textit{Euler's Identity} \text{ (Theorem 1.5 page 8)} \\
&= 0 + i && \text{by Theorem 1.2 (page 5) and Proposition 1.2 (page 7)} \\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by } \textit{Euler's Identity} \text{ (Theorem 1.5 page 8)} \\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem 1.2 page 5} \\
&= 0 - i && \text{by Theorem 1.2 (page 5) and Proposition 1.2 (page 7)}
\end{aligned}
$$

✏

Corollary 1.1.

COR $$e^{ix} = \sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x\in\mathbb{R}$$

[17] Euler (1748), Bottazzini (1986) page 12

✎PROOF:

$$\begin{aligned}
\boxed{e^{ix}} &= \cos(x) + i\sin(x) && \text{by } \textit{Euler's Identity} && \text{(Theorem 1.5 page 8)}\\
&= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by } \textit{Taylor series} && \text{(Theorem 1.1 page 4)}\\
&= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} && = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!}\\
&= \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} && = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}
\end{aligned}$$

✏

Corollary 1.2 (Euler formulas). [18]

COR

$$\cos(x) = \mathbf{R_e}\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \sin(x) = \mathbf{I_m}\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{R_e}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}+\left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} && \text{by definition of } \Re && \text{(Definition A.5 page 91)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} && \text{by } \textit{Euler's Identity} && \text{(Theorem 1.5 page 8)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} && = \frac{\cos(x)}{2}+\frac{\cos(x)}{2} && = \boxed{\cos(x)}\\
\boxed{\mathbf{I_m}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}-\left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} && \text{by definition of } \Im && \text{(Definition A.5 page 91)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} && \text{by } \textit{Euler's Identity} && \text{(Theorem 1.5 page 8)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} && = \frac{i\sin(x)}{2i}+\frac{i\sin(x)}{2i} && = \boxed{\sin(x)}
\end{aligned}$$

✏

Theorem 1.6. [19]

THM

$$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

✎PROOF:

$$\begin{aligned}
e^{\alpha}\, e^{\beta} &= \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) && \text{by Corollary 1.1 page 8}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{\alpha^k}{k!}\, \frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{n!}{n!}\, \frac{\alpha^k}{k!}\, \frac{\beta^{n-k}}{(n-k)!}
\end{aligned}$$

[18] Euler (1748), Bottazzini (1986) page 12
[19] Rudin (1987) page 1

$$
\begin{aligned}
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\,\alpha^k\,\beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}\alpha^k\,\beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the \textit{Binomial Theorem}} && \text{(Theorem B.14 page 107)} \\
&= e^{\alpha+\beta} && \text{by Corollary 1.1 page 8}
\end{aligned}
$$

✏

1.5 Trigonometric Identities

Theorem 1.7 (shift identities).

THM						
	$\cos\left(x+\frac{\pi}{2}\right) = -\sin x \quad \forall x\in\mathbb{R}$			$\sin\left(x+\frac{\pi}{2}\right) = \cos x \quad \forall x\in\mathbb{R}$		
	$\cos\left(x-\frac{\pi}{2}\right) = \sin x \quad \forall x\in\mathbb{R}$			$\sin\left(x-\frac{\pi}{2}\right) = -\cos x \quad \forall x\in\mathbb{R}$		

✎PROOF:

$$
\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)} + e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= \frac{e^{ix}e^{i\frac{\pi}{2}} + e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \frac{e^{ix}(i) + e^{-ix}(-i)}{2} && \text{by Proposition 1.3 page 8} \\
&= \frac{e^{ix} - e^{-ix}}{-2i} \\
&= -\sin x && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)} + e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}} + e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \frac{e^{ix}(-i) + e^{-ix}(i)}{2} && \text{by Proposition 1.3 page 8} \\
&= \frac{e^{ix} - e^{-ix}}{2i} \\
&= \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
\sin\left(x+\frac{\pi}{2}\right) &= \cos\left(\left[x+\frac{\pi}{2}\right] - \frac{\pi}{2}\right) && \text{by previous result} \\
&= \cos(x) \\
\sin\left(x-\frac{\pi}{2}\right) &= -\cos\left(\left[x-\frac{\pi}{2}\right] + \frac{\pi}{2}\right) && \text{by previous result} \\
&= -\cos(x)
\end{aligned}
$$

✏

Theorem 1.8 (product identities).

THM

(A).	$\cos x \cos y$	$=$	$\frac{1}{2}\cos(x-y)$	$+$	$\frac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$
(B).	$\cos x \sin y$	$=$	$-\frac{1}{2}\sin(x-y)$	$+$	$\frac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
(C).	$\sin x \cos y$	$=$	$\frac{1}{2}\sin(x-y)$	$+$	$\frac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
(D).	$\sin x \sin y$	$=$	$\frac{1}{2}\cos(x-y)$	$-$	$\frac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary 1.2 page 9)
(algebraic method requiring *complex number system* $\mathbb{C}$):

$$\begin{aligned}
\cos x\cos y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right) && \text{by \textit{Euler formulas} (Corollary 1.2 page 9)}\\
&= \frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(-x+y)}+e^{i(-x-y)}}{4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{4}+\frac{e^{i(x-y)}+e^{-i(x-y)}}{4}\\
&= \frac{2\cos(x+y)}{4}+\frac{2\cos(x-y)}{4} && \text{by \textit{Euler formulas} (Corollary 1.2 page 9)}\\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem 1.3 page 6)
(differential equation method requiring only *real number system* $\mathbb{R}$):

$$\begin{aligned}
&\mathsf{f}(x) \triangleq \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)\\
&\implies \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) = -\tfrac{1}{2}\sin(x-y)-\tfrac{1}{2}\sin(x+y) && \text{by Theorem 1.4 page 7}\\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x) = -\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y) && \text{by Theorem 1.4 page 7}\\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x)+\mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property}\\
&\implies \underbrace{\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)}_{\mathsf{f}(x)} = \underbrace{[\tfrac{1}{2}\cos(0-y)+\tfrac{1}{2}\cos(0+y)]}_{\mathsf{f}''(0)}\cos(x)+\underbrace{[-\tfrac{1}{2}\sin(0-y)-\tfrac{1}{2}\sin(0+y)]}_{\mathsf{f}'(0)}\sin(x)\\
&\implies \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y) = \cos y\cos x + 0\sin(x)\\
&\implies \cos x\cos y = \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)
\end{aligned}$$

3. Proof for (B) using *Euler formulas* (Corollary 1.2 page 9):

$$\begin{aligned}
\sin x\sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary 1.2 page 9}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary 1.2 page 9}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x \sin y &= \cos(x)\cos\left(y - \frac{\pi}{2}\right) && \text{by } \textit{shift identities} && \text{(Theorem 1.7 page 10)}\\
&= \frac{1}{2}\cos\left(x + y - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(x - y + \frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y) - \frac{1}{2}\sin(x-y) && \text{by } \textit{shift identities} && \text{(Theorem 1.7 page 10)}\\
\sin x \cos y &= \cos y \sin x\\
&= \frac{1}{2}\sin(y+x) - \frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y) && \text{by Theorem 1.2 page 5}
\end{aligned}$$

✏

Proposition 1.4.

PRP				
	(A). $\cos(\pi) = -1$	(C). $\cos(2\pi) = 1$	(E). $e^{i\pi} = -1$
	(B). $\sin(\pi) = 0$	(D). $\sin(2\pi) = 0$	(F). $e^{i2\pi} = 0$

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1 + 1 + \cos(\pi)\\
&= -1 + 2[{}^1\!/\!_2\cos({}^\pi\!/\!_2 - {}^\pi\!/\!_2) + {}^1\!/\!_2\cos({}^\pi\!/\!_2 + {}^\pi\!/\!_2)] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= -1 + 2\cos({}^\pi\!/\!_2)\cos({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= -1 + 2(0)(0) && \text{by definition of } \pi && \text{(Definition 1.3 page 3)}\\
&= -1\\
\sin(\pi) &= 0 + \sin(\pi)\\
&= 2[-{}^1\!/\!_2\sin({}^\pi\!/\!_2 - {}^\pi\!/\!_2) + {}^1\!/\!_2\sin({}^\pi\!/\!_2 + {}^\pi\!/\!_2)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= 2\cos({}^\pi\!/\!_2)\sin({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= 2(0)\sin({}^\pi\!/\!_2) && \text{by definition of } \pi && \text{(Definition 1.3 page 3)}\\
&= 0\\
\cos(2\pi) &= 1 + \cos(2\pi) - 1\\
&= 2[{}^1\!/\!_2\cos(\pi - \pi) + {}^1\!/\!_2\cos(\pi + \pi)] - 1 && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= 2\cos(\pi)\cos(\pi) - 1 && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= 2(-1)(-1) - 1 && \text{by (A)}\\
&= 1\\
\sin(2\pi) &= 0 + \sin(2\pi)\\
&= 2[{}^1\!/\!_2\sin(\pi - \pi) + {}^1\!/\!_2\sin(\pi + \pi)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem 1.2 page 5)}\\
&= 2\sin(\pi)\cos(\pi) && \text{by } \textit{product identities} && \text{(Theorem 1.8 page 10)}\\
&= 2(0)(-1) && \text{by (A) and (B)}\\
&= 0\\
e^{i\pi} &= \cos(\pi) + i\sin(\pi) && \text{by } \textit{Euler's Identity} && \text{(Theorem 1.5 page 8)}\\
&= -1 + 0 && \text{by (A) and (B)}\\
&= -1\\
e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) && \text{by } \textit{Euler's Identity} && \text{(Theorem 1.5 page 8)}\\
&= 1 + 0 && \text{by (C) and (D)}\\
&= 1
\end{aligned}$$

✏

Theorem 1.9 (double angle formulas). [20]

THM

(A).	$\cos(x+y)$	$=$	$\cos x \cos y - \sin x \sin y$	$\forall x,y\in\mathbb{R}$
(B).	$\sin(x+y)$	$=$	$\sin x \cos y + \cos x \sin y$	$\forall x,y\in\mathbb{R}$
(C).	$\tan(x+y)$	$=$	$\dfrac{\tan x + \tan y}{1-\tan x \tan y}$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *product identities* (Theorem 1.8 page 10).

$$\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x+y)}_{\cos(x+y)}+\underbrace{\frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x-y)}_{0} \\
&= \left[\frac{1}{2}\cos(x-y)+\frac{1}{2}\cos(x+y)\right]-\left[\frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)\right] \\
&= \cos x\cos y - \sin x \sin y && \text{by Theorem 1.8 page 10}
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem 1.3 page 6):

$$\begin{aligned}
\mathsf{f}(x) \triangleq \cos(x+y) &\implies \frac{d}{dx}\mathsf{f}(x) = -\sin(x+y) && \text{by Theorem 1.4 page 7} \\
&\implies \frac{d^2}{dx^2}\mathsf{f}(x) = -\cos(x+y) && \text{by Theorem 1.4 page 7} \\
&\implies \frac{d^2}{dx^2}\mathsf{f}(x) + \mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property} \\
&\implies \cos(x+y) = \cos y \cos x - \sin y \sin x && \text{by Theorem 1.3 page 6} \\
&\implies \cos(x+y) = \cos x \cos y - \sin x \sin y && \text{by \textit{commutative} property}
\end{aligned}$$

3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x-\frac{\pi}{2}+y\right) && \text{by \textit{shift identities} (Theorem 1.7 page 10)} \\
&= \cos\left(x-\frac{\pi}{2}\right)\cos(y) - \sin\left(x-\frac{\pi}{2}\right)\sin(y) && \text{by (A)} \\
&= \sin(x)\cos(y)+\cos(x)\sin(y) && \text{by \textit{shift identities} (Theorem 1.7 page 10)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\
&= \frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} && \text{by (A)} \\
&= \left(\frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}\right)\left(\frac{\cos x\cos y}{\cos x \cos y}\right) \\
&= \frac{\frac{\sin x \cos y}{\cos x \cos y}+\frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y}-\frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}}{1-\frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x + \tan y}{1-\tan x \tan y}
\end{aligned}$$

✏

Theorem 1.10 (trigonometric periodicity).

THM

(A).	$\cos(x+M\pi)$	$=$	$(-1)^M\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		(D).	$\cos(x+2M\pi)$	$=$	$\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
(B).	$\sin(x+M\pi)$	$=$	$(-1)^M\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		(E).	$\sin(x+2M\pi)$	$=$	$\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
(C).	$e^{i(x+M\pi)}$	$=$	$(-1)^M e^{ix}$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		(F).	$e^{i(x+2M\pi)}$	$=$	e^{ix}	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$

[20] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

✎PROOF:

1. Proof for (A):

   (a) $M = 0$ case: $\cos(x + 0\pi) \quad = \quad \cos(x) \quad = \quad (-1)^0\cos(x)$

   (b) Proof for $M > 0$ cases (by induction):

      i. Base case $M = 1$:

$$\begin{aligned}
\cos(x+\pi) &= \cos x\cos\pi - \sin x\sin\pi && \text{by *double angle formulas*} && \text{(Theorem 1.9 page 13)}\\
&= \cos x(-1) - \sin x(0) && \text{by } \cos\pi = -1 \text{ result} && \text{(Proposition 1.4 page 12)}\\
&= (-1)^1\cos x
\end{aligned}$$

      ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\cos(x+[M+1]\pi) &= \cos([x+\pi]+M\pi)\\
&= (-1)^M\cos(x+\pi) && \text{by induction hypothesis } (M \text{ case})\\
&= (-1)^M(-1)\cos(x) && \text{by base case (item (1(b)i) page 14)}\\
&= (-1)^{M+1}\cos(x)\\
&\implies M+1 \text{ case}
\end{aligned}$$

   (c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\cos(x+M\pi) &\triangleq \cos(x-N\pi) && \text{by definition of } N\\
&= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by *double angle formulas*} && \text{(Theorem 1.9 page 13)}\\
&= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem 1.2 page 5}\\
&= \cos(x)\cos(0+N\pi) + \sin(x)\sin(0+N\pi)\\
&= \cos(x)(-1)^N\cos(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \ge 0 \text{ results} && \text{(item (1b) page 14)}\\
&= (-1)^N\cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem 1.2 page 5)}\\
&\triangleq (-1)^{-M}\cos(x) && \text{by definition of } N\\
&= (-1)^M\cos(x)
\end{aligned}$$

   (d) Proof using complex exponential:

$$\begin{aligned}
\cos(x+M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by *Euler formulas*} && \text{(Corollary 1.2 page 9)}\\
&= e^{iM\pi}\left[\frac{e^{ix}+e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem 1.6 page 9)}\\
&= \left(e^{i\pi}\right)^M\cos x && \text{by *Euler formulas*} && \text{(Corollary 1.2 page 9)}\\
&= (-1)^M\cos x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition 1.4 page 12)}
\end{aligned}$$

2. Proof for (B):

   (a) $M = 0$ case: $\sin(x + 0\pi) \quad = \quad \sin(x) \quad = \quad (-1)^0\sin(x)$

   (b) Proof for $M > 0$ cases (by induction):

      i. Base case $M = 1$:

$$\begin{aligned}
\sin(x+\pi) &= \sin x\cos\pi + \cos x\sin\pi && \text{by *double angle formulas*} && \text{(Theorem 1.9 page 13)}\\
&= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} && \text{(Proposition 1.4 page 12)}\\
&= (-1)^1\sin x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\sin(x+[M+1]\pi) &= \sin([x+\pi]+M\pi) \\
&= (-1)^M\sin(x+\pi) && \text{by induction hypothesis (M case)} \\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 14)} \\
&= (-1)^{M+1}\sin(x) \\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M<0$ cases: Let $N \triangleq -M \ldots \implies N>0$.

$$\begin{aligned}
\sin(x+M\pi) &\triangleq \sin(x-N\pi) && \text{by definition of } N \\
&= \sin(x)\sin(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem 1.9 page 13)} \\
&= \sin(x)\sin(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem 1.2 page 5} \\
&= \sin(x)\sin(0+N\pi) + \sin(x)\sin(0+N\pi) \\
&= \sin(x)(-1)^N\sin(0) + \sin(x)(-1)^N\sin(0) && \text{by } M\geq 0 \text{ results} && \text{(item (2b) page 14)} \\
&= (-1)^N\sin(x) && \text{by } \sin(0){=}1,\ \sin(0){=}0 \text{ results} && \text{(Theorem 1.2 page 5)} \\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N \\
&= (-1)^M\sin(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)} - e^{-i(x+M\pi)}}{2i} && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= e^{iM\pi}\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \left(e^{i\pi}\right)^M \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary 1.2 page 9)} \\
&= (-1)^M \sin x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition 1.4 page 12)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem 1.6 page 9)} \\
&= \left(e^{i\pi}\right)^M\left(e^{ix}\right) \\
&= (-1)^M e^{ix} && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition 1.4 page 12)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):
$$\begin{array}{rclcll}
\cos(i(x+2M\pi)) &=& (-1)^{2M}\cos(ix) &=& \cos(ix) & \text{by (A)} \\
\sin(i(x+2M\pi)) &=& (-1)^{2M}\sin(ix) &=& \sin(ix) & \text{by (B)} \\
e^{i(x+2M\pi)} &=& (-1)^{2M}e^{ix} &=& e^{ix} & \text{by (C)}
\end{array}$$

✏

Theorem 1.11 (half-angle formulas/squared identities).

THM

(A). $\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$	(C). $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$
(B). $\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$	

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x) + \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem 1.8 page 10)} \\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem 1.2 page 5)} \\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x) - \tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem 1.8 page 10)} \\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem 1.2 page 5)} \\
\cos^2 x + \sin^2 x &= \tfrac{1}{2}[1+\cos(2x)] + \tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)} \\
& && \text{note: see also} && \text{Theorem 1.4 page 7}
\end{aligned}$$

✏

1.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

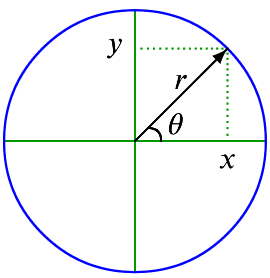
$$\begin{aligned}
\langle \cos(x) \,|\, \sin(x) \rangle &= \int_{-\pi}^{+\pi} \cos(x)\sin(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x-x)\,\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x+x)\,\mathrm{d}x && \text{by Theorem 1.8 page 10} \\
&= \cancelto{0}{\frac{1}{2}\int_{-\pi}^{+\pi} \sin(0)\,\mathrm{d}x} + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(2x)\,\mathrm{d}x \\
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi} \cos(2x) \\
&= -\frac{1}{4}[\cos(2\pi) - \cos(-2\pi)] \\
&= 0
\end{aligned}$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$



$$\begin{aligned}
\cos\theta &\triangleq \frac{x}{r} & \sec\theta &\triangleq \frac{r}{x} \\
\sin\theta &\triangleq \frac{y}{r} & \csc\theta &\triangleq \frac{r}{y} \\
\tan\theta &\triangleq \frac{y}{x} & \cot\theta &\triangleq \frac{x}{y}
\end{aligned}$$

1.7 Trigonometric functions of complex numbers

Definition 1.5. [21]

DEF

$$\begin{aligned}
\cosh(z) &\triangleq \frac{e^z + e^{-z}}{2} && \forall z \in \mathbb{C} \\
\sinh(z) &\triangleq \frac{e^z - e^{-z}}{2} && \forall z \in \mathbb{C}
\end{aligned}$$

[21] Saxelby (1920) page 225

Theorem 1.12. [22]

THM

$$\begin{array}{rcll}
\cosh(ix) &=& \cos(x) & \forall x \in\mathbb{R}\\
\sinh(ix) &=& i\sin(x) & \forall x \in\mathbb{R}\\
\cos(ix) &=& \cosh(x) & \forall x \in\mathbb{R}\\
\sin(ix) &=& i\sinh(x) & \forall x \in\mathbb{R}\\
\cos(x+iy) &=& \cos(x)\cosh(y)-i\sin(x)\sinh(y) & \forall x,y \in\mathbb{R}\\
\sin(x+iy) &=& \sin(x)\cosh(y)+i\cos(x)\sinh(y) & \forall x,y \in\mathbb{R}
\end{array}$$

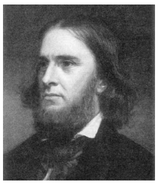
✎PROOF:

$$\begin{aligned}
\cosh(ix) &\triangleq \frac{e^{ix}+e^{-ix}}{2} && \text{by definition of } \cosh(x) && \text{(Definition 1.5 page 16)}\\
&= \cos(x) && \text{by \textit{Euler's Identity}} && \text{(Theorem 1.5 page 8)}\\
\sinh(ix) &\triangleq \frac{e^{ix}-e^{-ix}}{2} && \text{by definition of } \sinh(x) && \text{(Definition 1.5 page 16)}\\
&\triangleq i\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by definition of } \sinh(x) && \text{(Definition 1.5 page 16)}\\
&= i\sin(x) && \text{by \textit{Euler's Identity}} && \text{(Theorem 1.5 page 8)}\\
\cos(ix) &\triangleq \frac{e^{iix}+e^{-iix}}{2} && \text{by \textit{Euler's Identity}} && \text{(Theorem 1.5 page 8)}\\
&= \frac{e^{-x}+e^{x}}{2}\\
&= \frac{e^{x}+e^{-x}}{2}\\
&\triangleq \cosh(x) && \text{by definition of } \cosh(x) && \text{(Definition 1.5 page 16)}\\
\sin(ix) &\triangleq \frac{e^{iix}-e^{-iix}}{2i} && \text{by \textit{Euler's Identity}} && \text{(Theorem 1.5 page 8)}\\
&= \frac{e^{-x}-e^{x}}{2i}\\
&= -(-i^2)\left[\frac{e^{x}-e^{-x}}{2i}\right]\\
&= i\left[\frac{e^{x}-e^{-x}}{2}\right]\\
&\triangleq i\sinh(x) && \text{by definition of } \cosh(x) && \text{(Definition 1.5 page 16)}\\
\cos(x+iy) &= \cos(x)\cos(iy)-\sin(x)\sin(iy) && \text{by \textit{double angle formulas}} && \text{(Theorem 1.9 page 13)}\\
&= \cos(x)\cosh(y)-i\sin(x)\sinh(y) && \text{by previous results}\\
\sin(x+iy) &= \sin(x)\cos(iy)+\cos(x)\sin(iy) && \text{by \textit{double angle formulas}} && \text{(Theorem 1.9 page 13)}\\
&= \sin(x)\cosh(y)+i\cos(x)\sinh(y) && \text{by previous results}
\end{aligned}$$

✏

[22] https://proofwiki.org/wiki/Cosine_of_Complex_Number, https://proofwiki.org/wiki/Sine_of_Complex_Number, Saxelby (1920) pages 416–417

1.8 The power of the exponential



❝Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.❞

Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [23]

❝Young man, in mathematics you don't understand things. You just get used to them.❞

John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [24]

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary 1.3. [25]

COR
$$e^{i\pi} + 1 = 0$$

✎PROOF:

$$\begin{aligned}
e^{ix}\Big|_{x=\pi} &= [\cos x + i \sin x]_{x=\pi} && \text{by Euler's Identity (Theorem 1.5 page 8)}\\
&= -1 + i \cdot 0 && \text{by Proposition 1.4 page 12}\\
&= -1
\end{aligned}$$

⇨

There are many transforms available, several of them integral transforms $[\mathbf{A}\mathsf{f}](s) \triangleq \int_t \mathsf{f}(s)\kappa(t,s)\,\mathsf{d}s$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:

① The *Laplace Transform* with kernel $\kappa(t, s) \triangleq e^{st}$
② The *Fourier Transform* with kernel $\kappa(t, \omega) \triangleq e^{i\omega t}$.

[23] quote: Kasner and Newman (1940) page 104
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

[24] quote: Zukav (1980) page 208
image: http://en.wikipedia.org/wiki/John_von_Neumann

The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."

[25] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem 1.13 page 19).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem 1.13. [26] *Let* $\mathbf{L}$ *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*

$$\check{\mathsf{h}}(s) \triangleq \left\langle \mathsf{h}(t,\omega) \,|\, e^{st} \right\rangle \qquad \text{(Laplace transform)}.$$

THM

$$\left\{\begin{array}{ll} 1. & \mathbf{L} \text{ is linear and} \\ 2. & \mathbf{L} \text{ is time-invariant} \end{array}\right\} \implies \left\{ \mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{\text{eigenvalue}} \; \underbrace{e^{st}}_{\text{eigenvector}} \right\}$$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \,|\, \mathsf{h}((t;u),s)\rangle && \text{by linear hypothesis} \\
&= \langle e^{su} \,|\, \mathsf{h}((t-u),s)\rangle && \text{by time-invariance hypothesis} \\
&= \left\langle e^{s(t-v)} \,|\, \mathsf{h}(v,s)\right\rangle && \text{let } v = t-u \implies u = t-v \\
&= e^{st}\ \langle e^{-sv} \,|\, \mathsf{h}(v,s)\rangle && \text{by additivity of } \langle \triangle \,|\, \nabla\rangle \\
&= \langle \mathsf{h}(v,s) \,|\, e^{-sv}\rangle^*\ e^{st} && \text{by conjugate symmetry of } \langle \triangle \,|\, \nabla\rangle \\
&= \left\langle \mathsf{h}(v,s) \,|\, e^{(-s)v}\right\rangle^*\ e^{st} && \\
&= \check{\mathsf{h}}^*(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}$$

✏

[26] Mallat (1999) page 2, ...page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

CHAPTER 2

TRIGONOMETRIC POLYNOMIALS

❝*I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives.*❞
Charles Hermite (1822 – 1901), French mathematician, in an 1893 letter to Stieltjes, in response to the "pathological" everywhere continuous but nowhere differentiable *Weierstrass functions* $\mathsf{f}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$.[1]

2.1 Trigonometric expansion

Theorem 2.1 (DeMoivre's Theorem).

THM
$$\left(re^{ix}\right)^n = r^n(\cos nx + i\sin nx) \qquad \forall r,x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}\left(re^{ix}\right)^n &= r^n e^{inx} \\ &= r^n\left(\cos nx + i\sin nx\right) && \text{by Euler's identity (Theorem 1.5 page 8)}\end{aligned}$$

✏

The cosine with argument nx can be expanded as a polynomial in $\cos(x)$ (next).

Theorem 2.2 (trigonometric expansion). [2]

[1] quote: Hermite (1893)
translation: Lakatos (1976) page 19
image: http://www-groups.dcs.sx-and.ac.uk/~history/PictDisplay/Hermite.html

[2] Rivlin (1974) page 3 ⟨(1.8)⟩

THM

$$\cos(nx) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

$$\sin(nx) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\sin x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(nx) &= \Re\left(\cos nx + i\sin nx\right)\\
&= \Re\left(e^{inx}\right)\\
&= \Re\left[\left(e^{ix}\right)^n\right]\\
&= \Re\left[(\cos x + i\sin x)^n\right]\\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n}\binom{n}{k}(\cos x)^{n-k}(i\sin x)^k\right]\\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n} i^k\binom{n}{k}\cos^{n-k}x\sin^k x\right]\\
&= \Re\Bigg[\sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + i\sum_{k\in\{1,5,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\\
&\quad - \sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + -i\sum_{k\in\{3,7,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\Bigg]\\
&= \sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x - \sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\\
&= \sum_{k\in\{0,2,\ldots,n\}}\binom{n}{k}(-1)^{\frac{k}{2}}\cos^{n-k}x\sin^k x\\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\sin^{2k}x\\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x(1-\cos^2 x)^k\\
&= \left[\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\right]\left[\sum_{m=0}^{k}\binom{k}{m}(-1)^m\cos^{2m}x\right]\\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}x
\end{aligned}$$

$$\begin{aligned}
\sin(nx) &= \cos\left(nx - \frac{\pi}{2}\right)\\
&= \cos\left(n\left[x - \frac{\pi}{2n}\right]\right)\\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}\left(n\left[x - \frac{\pi}{2n}\right]\right)
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(nx - \frac{\pi}{2} \right)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \sin^{n-2(k-m)} (nx)$$

✏

Example 2.1.

EX

$$\begin{aligned} \cos 5x &= 16\cos^5 x - 20\cos^3 x + 5\cos x \\ \sin 5x &= 16\sin^5 x - 20\sin^3 x + 5\sin x. \end{aligned}$$

✎PROOF:

1. Proof using *DeMoivre's Theorem* (Theorem 2.1 page 21):

$$\begin{aligned}
&\cos 5x + i\sin 5x \\
&= e^{i5x} \\
&= \left(e^{ix}\right)^5 \\
&= (\cos x + i\sin x)^5 \\
&= \sum_{k=0}^{5} \binom{5}{k} [\cos x]^{5-k} [i\sin x]^k \\
&= \binom{5}{0}[\cos x]^{5-0}[i\sin x]^0 + \binom{5}{1}[\cos x]^{5-1}[i\sin x]^1 + \binom{5}{2}[\cos x]^{5-2}[i\sin x]^2 + \\
&\binom{5}{3}[\cos x]^{5-3}[i\sin x]^3 + \binom{5}{4}[\cos x]^{5-4}[i\sin x]^4 + \binom{5}{5}[\cos x]^{5-5}[i\sin x]^5 \\
&= 1\cos^5 x + i5\cos^4 x\sin x - 10\cos^3 x\sin^2 x - i10\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i1\sin^5 x \\
&= \left[\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\right] + i\left[5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)(1-\cos^2 x)\right] + \\
&i\left[5(1-\sin^2 x)(1-\sin^2 x)\sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)\right] + \\
&i\left[5(1 - 2\sin^2 x + \sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5(\cos x - 2\cos^3 x + \cos^5 x)\right] + \\
&i\left[5(\sin x - 2\sin^3 x + \sin^5 x) - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \underbrace{\left[16\cos^5 x - 20\cos^3 x + 5\cos x\right]}_{\cos 5x} + i\underbrace{\left[16\sin^5 x - 20\sin^3 x + 5\sin x\right]}_{\sin 5x}
\end{aligned}$$

2. Proof using trigonometric expansion (Theorem 2.2 page 21):

$$\begin{aligned}
\cos 5x &= \sum_{k=0}^{\lfloor \frac{5}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{2} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{5-2(k-m)} \\
&= (-1)^0 \binom{5}{0}\binom{0}{0}\cos^5 x + (-1)^1 \binom{5}{2}\binom{1}{0}\cos^3 x + (-1)^2 \binom{5}{2}\binom{1}{1}\cos^5 x + \\
&(-1)^2 \binom{5}{4}\binom{2}{0}\cos^1 x + (-1)^3 \binom{5}{4}\binom{2}{1}\cos^3 x + (-1)^4 \binom{5}{4}\binom{2}{2}\cos^5 x
\end{aligned}$$

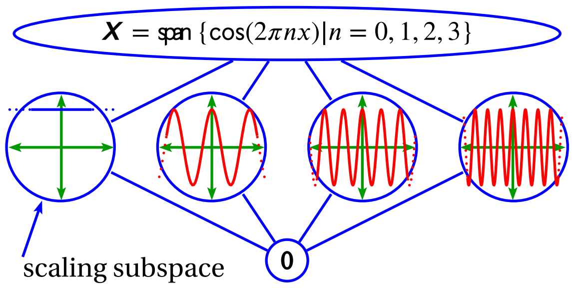


Figure 2.1: Lattice of harmonic cosines $\{\cos(nx) | n = 0, 1, 2, \ldots\}$

$$
\begin{aligned}
&= +(1)(1)\cos^5 x - (10)(1)\cos^3 x + (10)(1)\cos^5 x + (5)(1)\cos x - (5)(2)\cos^3 x + (5)(1)\cos^5 x \\
&= +(1 + 10 + 5)\cos^5 x + (-10 - 10)\cos^3 x + 5\cos x \\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}
$$

✏

Example 2.2. [3]

n	$\cos nx$		polynomial in $\cos x$	n	$\cos nx$		polynomial in $\cos x$
0	$\cos 0x$	$=$	1	4	$\cos 4x$	$=$	$8\cos^4 x - 8\cos^2 x + 1$
1	$\cos 1x$	$=$	$\cos^1 x$	5	$\cos 5x$	$=$	$16\cos^5 x - 20\cos^3 x + 5\cos x$
2	$\cos 2x$	$=$	$2\cos^2 x - 1$	6	$\cos 6x$	$=$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
3	$\cos 3x$	$=$	$4\cos^3 x - 3\cos x$	7	$\cos 7x$	$=$	$64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

✎PROOF:

$$
\begin{aligned}
\cos 2x &= \sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{2-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^2 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^0 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^2 x \\
&= +(1)(1)\cos^2 x - (1)(1) + (1)(1)\cos^2 x \\
&= 2\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 3x &= \sum_{k=0}^{\lfloor \frac{3}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{3-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^3 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^1 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^3 x \\
&= +\binom{3}{0}\binom{0}{0} \cos^3 x - \binom{3}{2}\binom{1}{0} \cos^1 x + \binom{3}{2}\binom{1}{1} \cos^3 x \\
&= +(1)(1)\cos^3 x - (3)(1)\cos^1 x + (3)(1)\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}
$$

$$
\cos 4x = \sum_{k=0}^{\lfloor \frac{4}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{4}{2k} \binom{k}{m} (\cos x)^{4-2(k-m)}
$$

[3] Abramowitz and Stegun (1972) page 795, Guillemin (1957) page 593 ⟨(21)⟩, Sloane (2014) ⟨http://oeis.org/A039991⟩, Sloane (2014) ⟨http://oeis.org/A028297⟩

$$
\begin{aligned}
&= \sum_{k=0}^{2}\sum_{m=0}^{k}(-1)^{k+m}\binom{4}{2k}\binom{k}{m}(\cos x)^{4-2(k-m)} \\
&= (-1)^{0+0}\binom{4}{2\cdot 0}\binom{0}{0}(\cos x)^{4-2(0-0)} + (-1)^{1+0}\binom{4}{2\cdot 1}\binom{1}{0}(\cos x)^{4-2(1-0)} \\
&\quad + (-1)^{1+1}\binom{4}{2\cdot 1}\binom{1}{1}(\cos x)^{4-2(1-1)} + (-1)^{2+0}\binom{4}{2\cdot 2}\binom{2}{0}(\cos x)^{4-2(2-0)} \\
&\quad + (-1)^{2+1}\binom{4}{2\cdot 2}\binom{2}{1}(\cos x)^{4-2(2-1)} + (-1)^{2+2}\binom{4}{2\cdot 2}\binom{2}{2}(\cos x)^{4-2(2-2)} \\
&= (1)(1)\cos^4 x - (6)(1)\cos^2 x + (6)(1)\cos^4 x + (1)(1)\cos^0 x - (1)(2)\cos^2 x + (1)(1)\cos^4 x \\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}
$$

$$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$$ see Example 2.1 page 23

$$
\begin{aligned}
\cos 6x &= \sum_{k=0}^{\lfloor \frac{6}{2} \rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{6}{2k}\binom{k}{m}(\cos x)^{6-2(k-m)} \\
&= (-1)^0\binom{6}{0}\binom{0}{0}\cos^6 x + (-1)^1\binom{6}{2}\binom{1}{0}\cos^4 x + (-1)^2\binom{6}{2}\binom{1}{1}\cos^6 x + (-1)^2\binom{6}{4}\binom{2}{0}\cos^2 x + \\
&(-1)^3\binom{6}{4}\binom{2}{1}\cos^4 x + (-1)^4\binom{6}{4}\binom{2}{2}\cos^6 x + (-1)^3\binom{6}{6}\binom{3}{0}\cos^0 x + (-1)^4\binom{6}{6}\binom{3}{1}\cos^2 x + \\
&(-1)^5\binom{6}{6}\binom{3}{2}\cos^4 x + (-1)^6\binom{6}{6}\binom{3}{3}\cos^6 x \\
&= +(1)(1)\cos^6 x - (15)(1)\cos^4 x + (15)(1)\cos^6 x + (15)(1)\cos^2 x - (15)(2)\cos^4 x + (15)(1)\cos^6 x \\
&\quad - (1)(1)\cos^0 x + (1)(3)\cos^2 x - (1)(3)\cos^4 x + (1)(1)\cos^6 x \\
&= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 7x &= \sum_{k=0}^{\lfloor \frac{7}{2} \rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{3}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{7-2(k-m)} \\
&= (-1)^0\binom{7}{0}\binom{0}{0}\cos^7 x + (-1)^1\binom{7}{2}\binom{1}{0}\cos^5 x + (-1)^2\binom{7}{2}\binom{1}{1}\cos^7 x + (-1)^2\binom{7}{4}\binom{2}{0}\cos^3 x \\
&\quad + (-1)^3\binom{7}{4}\binom{2}{1}\cos^5 x + (-1)^4\binom{7}{4}\binom{2}{2}\cos^7 x + (-1)^3\binom{7}{6}\binom{3}{0}\cos^1 x + (-1)^4\binom{7}{6}\binom{3}{1}\cos^3 x \\
&\quad + (-1)^5\binom{7}{6}\binom{3}{2}\cos^5 x + (-1)^6\binom{7}{6}\binom{3}{3}\cos^7 x \\
&= (1)(1)\cos^7 x - (21)(1)\cos^5 x + (21)(1)\cos^7 x + (35)(1)\cos^3 x \\
&\quad - (35)(2)\cos^5 x + (35)(1)\cos^7 x - (7)(1)\cos^1 x + (7)(3)\cos^3 x \\
&\quad - (7)(3)\cos^5 x + (7)(1)\cos^7 x \\
&= (1 + 21 + 35 + 7)\cos^7 x - (21 + 70 + 21)\cos^5 x + (35 + 21)\cos^3 x - (7)\cos^1 x \\
&= 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x
\end{aligned}
$$

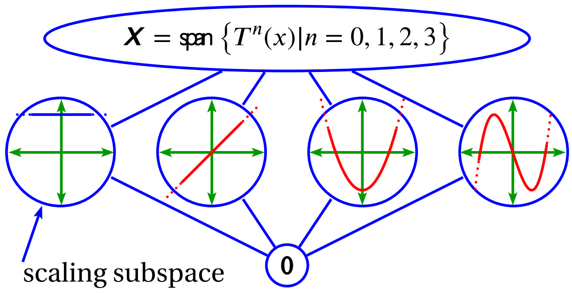


Figure 2.2: Lattice of Chebyshev polynomials $\{T_n(x)|n=0,1,2,3\}$

Note: Trigonometric expansion of $\cos(nx)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[4]

```
trigexpand(cos(2*x));
trigexpand(cos(3*x));
trigexpand(cos(4*x));
trigexpand(cos(5*x));
trigexpand(cos(6*x));
trigexpand(cos(7*x));
```

Definition 2.1.

DEF *The nth* ***Chebyshev polynomial of the first kind*** *is defined as*

$$T_n(x) \triangleq \cos nx \qquad \text{where} \qquad \cos x \triangleq x$$

Theorem 2.3. [5] *Let $T_n(x)$ be a* CHEBYSHEV POLYNOMIAL *with $n \in \mathbb{W}$.*

THM

$$\begin{array}{lcl} n \text{ is EVEN} & \implies & T_n(x) \text{ is EVEN.} \\ n \text{ is ODD} & \implies & T_n(x) \text{ is ODD.} \end{array}$$

Example 2.3. Let $T_n(x)$ be a *Chebyshev polynomial* with $n \in \mathbb{W}$.

EX

$$\begin{array}{lcl|lcl} T_0(x) & = & 1 & T_4(x) & = & 8x^4 - 8x^2 + 1 \\ T_1(x) & = & x & T_5(x) & = & 16x^5 - 20x^3 + 5x \\ T_2(x) & = & 2x^2 - 1 & T_6(x) & = & 32x^6 - 48x^4 + 18x^2 - 1 \\ T_3(x) & = & 4x^3 - 3x & & & \end{array}$$

✎PROOF: Proof of these equations follows directly from Example 2.2 (page 24). ✏

2.2 Trigonometric reduction

Theorem 2.2 (page 21) showed that $\cos nx$ can be expressed as a polynomial in $\cos x$. Conversely, Theorem 2.4 (next) shows that a polynomial in $\cos x$ can be expressed as a linear combination of $(\!(\cos nx)\!)_{n\in\mathbb{Z}}$.

Theorem 2.4 (trigonometric reduction)**.**

[4] maxima pages 157–158 ⟨10.5 Trigonometric Functions⟩

[5] Rivlin (1974) page 5 ⟨(1.13)⟩, Süli and Mayers (2003) page 242 ⟨Lemma 8.2⟩, Davidson and Donsig (2010) page 222 ⟨exercise 10.7.A(a)⟩

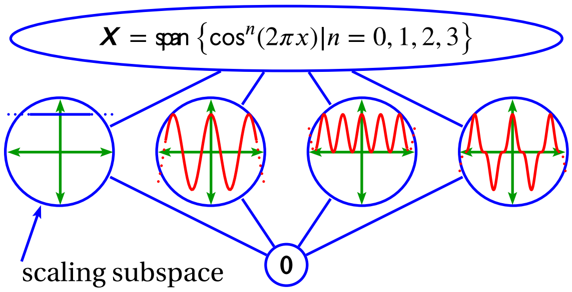


Figure 2.3: Lattice of exponential cosines $\left\{\cos^n x|n = 0, 1, 2, 3\right\}$

THM

$$\begin{aligned}
\cos^n x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases}
\dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & \text{for } n \text{ even} \\
\dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k}\cos[(n-2k)x] & \text{for } n \text{ odd}
\end{cases}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\cos^n x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n \\
&= \mathbf{R}_e\left[\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)x}e^{-ikx}\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\cos[(n-2k)x] + i\sin[(n-2k)x]\right)\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] + i\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sin[(n-2k)x]\right] \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases}
\dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ even} \\
\dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ odd}
\end{cases}
\end{aligned}$$

✏

Example 2.4. [6]

[6] Abramowitz and Stegun (1972) page 795, Sloane (2014) ⟨http://oeis.org/A100257⟩, Sloane (2014) ⟨http://oeis.org/A008314⟩

EX

n	$\cos^n x$	trigonometric reduction	n	$\cos^n x$	trigonometric reduction
0	$\cos^0 x$	$= 1$	4	$\cos^4 x$	$= \dfrac{\cos 4x + 4\cos 2x + 3}{2^3}$
1	$\cos^1 x$	$= \cos x$	5	$\cos^5 x$	$= \dfrac{\cos 5x + 5\cos 3x + 10\cos x}{2^4}$
2	$\cos^2 x$	$= \dfrac{\cos 2x + 1}{2}$	6	$\cos^6 x$	$= \dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{2^5}$
3	$\cos^3 x$	$= \dfrac{\cos 3x + 3\cos x}{2^2}$	7	$\cos^7 x$	$= \dfrac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{2^6}$

✎PROOF:

$$\begin{aligned}
\cos^0 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=0} \\
&= \frac{1}{2^0}\sum_{k=0}^{0}\binom{0}{k}\cos[(0-2k)x] \\
&= \binom{0}{0}\cos[(0-2\cdot 0)x] \\
&= 1 \\
\cos^1 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=1} \\
&= \frac{1}{2^1}\sum_{k=0}^{1}\binom{1}{k}\cos[(1-2k)x] \\
&= \frac{1}{2}\left[\binom{1}{0}\cos[(1-2\cdot 0)x] + \binom{1}{1}\cos[(1-2\cdot 1)x]\right] \\
&= \frac{1}{2}\left[1\cos x + 1\cos(-x)\right] \\
&= \frac{1}{2}(\cos x + \cos x) \\
&= \cos x \\
\cos^2 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=2} \\
&= \frac{1}{2^2}\sum_{k=0}^{2}\binom{2}{k}\cos([2-2k]x) \\
&= \frac{1}{2^2}\left[\binom{2}{0}\cos([2-2\cdot 0]x) + \binom{2}{1}\cos([2-2\cdot 1]x) + \binom{2}{2}\cos([2-2\cdot 2]x) + \right] \\
&= \frac{1}{2^2}\left[1\cos(2x) + 2\cos(0x) + 1\cos(-2x)\right] \\
&= \frac{1}{2^2}\left[\cos(2x) + 2 + \cos(2x)\right] \\
&= \frac{1}{2}\left[\cos(2x) + 1\right] \\
\cos^3 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=3} \\
&= \frac{1}{2^3}\sum_{k=0}^{3}\binom{3}{k}\cos([3-2k]x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^3}\left[1\cos(3x) + 3\cos(1x) + 3\cos(-1x) + 1\cos(-3x)\right]\\
&= \frac{1}{2^3}\left[\cos(3x) + 3\cos(x) + 3\cos(x) + \cos(3x)\right]\\
&= \frac{1}{2^2}\left[\cos(3x) + 3\cos(x)\right]\\
\cos^4 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=4}\\
&= \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}\cos([4-2k]x)\\
&= \frac{1}{2^4}\left[1\cos(4x) + 4\cos(2x) + 6\cos(0x) + 4\cos(-2x) + 1\cos(-4x)\right]\\
&= \frac{1}{2^3}\left[\cos(4x) + 4\cos(2x) + 3\right]\\
\cos^5 x &= \frac{1}{2^{5-1}}\sum_{k=0}^{\lfloor\frac{5}{2}\rfloor}\binom{5}{k}\cos[(5-2k)x]\\
&= \frac{1}{16}\sum_{k=0}^{2}\binom{5}{k}\cos[(5-2k)x]\\
&= \frac{1}{16}\left[\binom{5}{0}\cos 5x + \binom{5}{1}\cos 3x + \binom{5}{2}\cos x\right]\\
&= \frac{1}{16}\left[\cos 5x + 5\cos 3x + 10\cos x\right]\\
\cos^6 x &= \frac{1}{2^6}\binom{6}{\frac{6}{2}} + \frac{1}{2^{6-1}}\sum_{k=0}^{\frac{6}{2}-1}\binom{6}{k}\cos[(6-2k)x]\\
&= \frac{1}{2^6}\binom{6}{3} + \frac{1}{2^5}\sum_{k=0}^{2}\binom{6}{k}\cos[(6-2k)x]\\
&= \frac{1}{64}20 + \frac{1}{32}\left[\binom{6}{0}\cos 6x + \binom{6}{1}\cos 4x\binom{6}{2}\cos 2x\right]\\
&= \frac{1}{32}\left[\cos 6x + 6\cos 4x + 15\cos 2x + 10\right]\\
\cos^7 x &= \frac{1}{2^{7-1}}\sum_{k=0}^{\lfloor\frac{7}{2}\rfloor}\binom{7}{k}\cos[(7-2k)x]\\
&= \frac{1}{64}\sum_{k=0}^{2}\binom{7}{k}\cos[(7-2k)x]\\
&= \frac{1}{64}\left[\binom{7}{0}\cos 7x + \binom{7}{1}\cos 5x + \binom{7}{2}\cos 3x + \binom{7}{3}\cos x\right]\\
&= \frac{1}{64}\left[\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right]
\end{aligned}
$$

Note: Trigonometric reduction of $\cos^n(x)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[7]

```
trigreduce((cos(x))^2);
trigreduce((cos(x))^3);
trigreduce((cos(x))^4);
trigreduce((cos(x))^5);
trigreduce((cos(x))^6);
trigreduce((cos(x))^7);
```

[7] http://maxima.sourceforge.net/docs/manual/en/maxima_15.html
maxima page 158 ⟨10.5 Trigonometric Functions⟩

✏

2.3 Spectral Factorization

Theorem 2.5 (Fejér-Riesz spectral factorization). [8] *Let* $[0,\infty) \subsetneq \mathbb{R}$ *and*

$$\mathsf{p}\left(e^{ix}\right) \triangleq \sum_{n=-N}^{N} a_n e^{inx} \qquad \textit{(Laurent trigonometric polynomial order } 2N\textit{)}$$

$$\mathsf{q}\left(e^{ix}\right) \triangleq \sum_{n=1}^{N} b_n e^{inx} \qquad \textit{(standard trigonometric polynomial order } N\textit{)}$$

THM
$$\mathsf{p}\left(e^{ix}\right) \in [0,\infty) \quad \forall x \in [0,2\pi] \quad \Longrightarrow \quad \left\{\begin{array}{ll} \exists \left(\!\left(b_n\right)\!\right)_{n\in\mathbb{Z}} & \text{such that} \\ \mathsf{p}\left(e^{ix}\right) = \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right) & \forall x \in \mathbb{R} \end{array}\right.$$

✎PROOF:

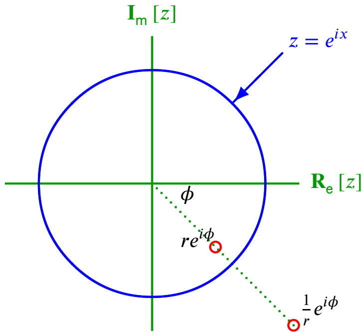
1. Proof that $a_n = a^*_{-n}$ ($\left(\!\left(a_n\right)\!\right)_{n\in\mathbb{Z}}$ is *Hermitian symmetric*):
   Let $a_n \triangleq r_n e^{i\phi_n}$, $r_n, \phi_n \in \mathbb{R}$. Then

$$\begin{aligned}
\mathsf{p}\left(e^{inx}\right) &\triangleq \sum_{n=-N}^{N} a_n e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{i\phi_n} e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{inx+\phi_n} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\sum_{n=-N}^{N} r_n \sin(nx+\phi_n) \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\underbrace{\left[r_0\sin(0x+\phi_0) + \sum_{n=1}^{N} r_n\sin(nx+\phi_n) + \sum_{n=1}^{N} r_{-n}\sin(-nx+\phi_{-n})\right]}_{\text{imaginary part must equal 0 because } p(x)\in\mathbb{R}} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\underbrace{\left[r_0\sin(\phi_0) + \sum_{n=1}^{N} r_n\sin(nx+\phi_n) - \sum_{n=1}^{N} r_{-n}\sin(nx-\phi_{-n})\right]}_{\Longrightarrow\ r_n = r_{-n},\ \phi_n = -\phi_{-n}\ \Longrightarrow\ a_n = a^*_{-n},\ a_0\in\mathbb{R}}
\end{aligned}$$

2. Because the coefficients $\left(\!\left(c_n\right)\!\right)_{n\in\mathbb{Z}}$ are *Hermitian symmetric* and by Theorem B.7 (page 104), the zeros of $P(z)$ occur in *conjugate reciprocol pairs*. This means that if $\sigma \in \mathbb{C}$ is a zero of $P(z)$ ($P(\sigma) = 0$), then $\frac{1}{\sigma^*}$ is also a zero of $P(z)$ ($P\left(\frac{1}{\sigma^*}\right) = 0$). In the complex z plane, this relationship means zeros are reflected across the unit circle such that

$$\frac{1}{\sigma^*} = \frac{1}{(re^{i\phi})^*} = \frac{1}{r}\frac{1}{e^{-i\phi}} = \frac{1}{r}e^{i\phi}$$

[8] Pinsky (2002) pages 330–331



3. Because the zeros of $\mathsf{p}(z)$ occur in conjugate recipricol pairs, $\mathsf{p}\left(e^{ix}\right)$ can be factored:

$$\begin{aligned}
\mathsf{p}\left(e^{ix}\right) &= \mathsf{p}(z)\big|_{z=e^{ix}} \\
&= z^{-N} C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z - \frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N} z^{-1}\left(z - \frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(1 - \frac{1}{\sigma_n^*}z^{-1}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z^{-1} - \sigma_n^*\right)\left(-\frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= \left[C \prod_{n=1}^{N}\left(-\frac{1}{\sigma_n^*}\right)\right]\left[\prod_{n=1}^{N}(z-\sigma_n)\right]\left[\prod_{n=1}^{N}\left(\frac{1}{z^*} - \sigma_n\right)\right]^*\bigg|_{z=e^{ix}} \\
&= \left[C_2 \prod_{n=1}^{N}(z-\sigma_n)\right]\left[C_2 \prod_{n=1}^{N}\left(\frac{1}{z^*} - \sigma_n\right)\right]^*\bigg|_{z=e^{ix}} \\
&= \mathsf{q}(z)\mathsf{q}^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{ix}} \\
&= \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right)
\end{aligned}$$

✎

2.4 Dirichlet Kernel

❝*Dirichlet alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous proof is. Rather we learn it first from him. When Gauss says he has proved something it is clear; when Cauchy says it, one can wager as much pro as con; when Dirichlet says it, it is certain.*❞

Carl Gustav Jacob Jacobi (1804–1851), Jewish-German mathematician [9]

[9] quote: Schubring (2005) page 558
image: http://en.wikipedia.org/wiki/File:Carl_Jacobi.jpg, public domain

The *Dirichlet Kernel* is critical in proving what is not immediately obvious in examining the Fourier Series—that for a broad class of periodic functions, a function can be recovered from (with uniform convergence) its Fourier Series analysis.

Definition 2.2. [10]

DEF *The **Dirichlet Kernel** $\mathsf{D}_n \in \mathbb{R}^{\mathbb{W}}$ with period τ is defined as*

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}kx}$$

Proposition 2.1. [11] *Let D_n be the* Dirichlet Kernel *with period τ (Definition 2.2 page 32).*

PRP
$$\mathsf{D}_n(x) = \frac{1}{\tau}\frac{\sin\left(\frac{\pi}{\tau}[2n+1]x\right)}{\sin\left(\frac{\pi}{\tau}x\right)}$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &\triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \mathsf{D}_n \quad \text{(Definition 2.2 page 32)}\\
&= \frac{1}{\tau}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}(k-n)x} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}kx} = \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\sum_{k=0}^{2n}\left(e^{i\frac{2\pi}{\tau}x}\right)^k\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\frac{1-\left(e^{i\frac{2\pi}{\tau}x}\right)^{2n+1}}{1-e^{i\frac{2\pi}{\tau}x}} && \text{by \textit{geometric series}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\frac{1-e^{i\frac{2\pi}{\tau}(2n+1)x}}{1-e^{i\frac{2\pi}{\tau}x}} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(\frac{e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{i\frac{\pi}{\tau}x}}\right)\frac{e^{-i\frac{\pi}{\tau}(2n+1)x}-e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{-i\frac{\pi}{\tau}x}-e^{i\frac{\pi}{\tau}x}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(e^{i\frac{2\pi n}{\tau}x}\right)\frac{-2i\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{-2i\sin\left[\frac{\pi}{\tau}x\right]} = \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}x\right]}
\end{aligned}$$

✏

Proposition 2.2. [12] *Let D_n be the* Dirichlet Kernel *with period τ (Definition 2.2 page 32).*

PRP
$$\int_0^{\tau}\mathsf{D}_n(x)\,\mathrm{d}x = 1$$

✎PROOF:

$$\begin{aligned}
\int_0^{\tau}\mathsf{D}_n(x)\,\mathrm{d}x &\triangleq \int_0^{\tau}\frac{1}{\tau}\sum_{k=-n}^{n}e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \mathsf{D}_n \text{ (Definition 2.2 page 32)}\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\cos\left(\frac{2\pi}{\tau}nx\right)+i\sin\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x
\end{aligned}$$

[10] Katznelson (2004) page 14, Heil (2011) pages 443–444, Folland (1992) pages 33–34
[11] Katznelson (2004) page 14, Heil (2011) page 444, Folland (1992) page 34
[12] Bruckner et al. (1997) pages 620–621

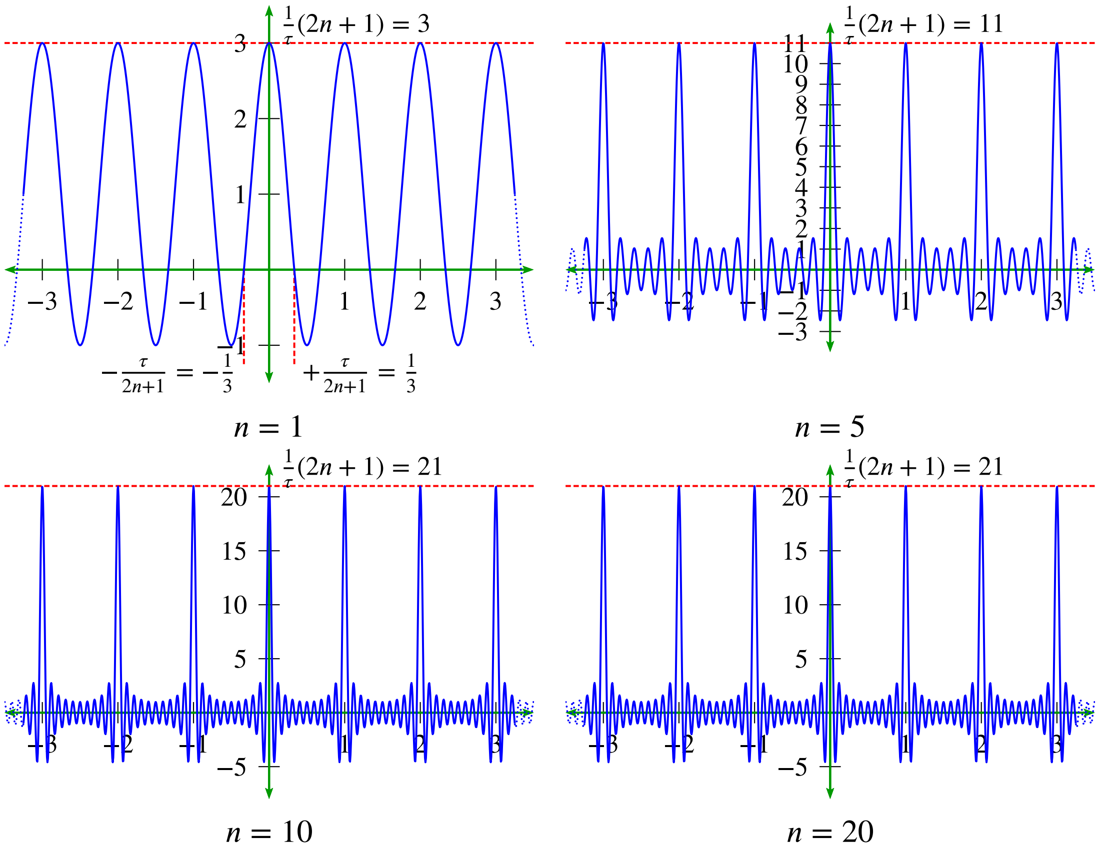


Figure 2.4: D_n function for $N = 1, 5, 10, 20$. $\mathsf{D}_n \to$ comb. (See Proposition 2.1 page 32).

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x \\
&= \frac{1}{\tau}\sum_{k=-n}^{n} \left.\frac{\sin\left(\frac{2\pi}{\tau}nx\right)}{\frac{2\pi}{\tau}n}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left[\frac{\sin\left(\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n} - \frac{\sin\left(-\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n}\right] \\
&= \frac{1}{\tau}\frac{\tau}{2}\sum_{k=-n}^{n}\left[\frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi n)}{\pi n}\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n)}{\pi n}\right]_{k=0} \\
&= 1
\end{aligned}
$$

Proposition 2.3. *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition 2.2 page 32). *Let* w_N *(the "*WIDTH*" of* $\mathsf{D}_n(x)$*) be the distance between the two points where the center pulse of* $\mathsf{D}_n(x)$ *intersects the* x *axis.*

PRP

$$
\begin{aligned}
\mathsf{D}_n(0) &= \frac{1}{\tau}(2n+1) \\
w_n &= \frac{2\tau}{2n+1}
\end{aligned}
$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(0) &= \mathsf{D}_n(x)\big|_{t=0} \\
&= \frac{1}{\tau}\left.\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by Proposition 2.1 page 32} \\
&= \frac{1}{\tau}\left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\frac{\mathrm{d}}{\mathrm{d}x}\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\left.\frac{\cos\left[\frac{\pi}{\tau}(2n+1)x\right]}{\cos\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{1}{1} \\
&= \frac{1}{\tau}(2n+1)
\end{aligned}$$

The center pulse of kernel $\mathsf{D}_n(x)$ intersects the x axis at

$$t = \pm\frac{\tau}{(2n+1)}$$

which implies

$$w_n = \frac{\tau}{2n+1} + \frac{\tau}{2n+1} = \frac{2\tau}{(2n+1)}.$$

✏

Proposition 2.4. [13] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ *(Definition 2.2 page 32).*

PRP: $\mathsf{D}_n(x) = \mathsf{D}_n(-x)$ (D_n *is an* EVEN *function*)

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]} && \text{by Proposition 2.1 page 32} \\
&= \frac{1}{\tau}\frac{-\sin\left[-\frac{\pi}{\tau}(2n+1)x\right]}{-\sin\left[-\frac{\pi}{\tau}t\right]} && \text{because } \sin x \text{ is an } \textit{odd} \text{ function} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)(-x)\right]}{\sin\left[\frac{\pi}{\tau}(-x)\right]} \\
&= \mathsf{D}_n(-x) && \text{by Proposition 2.1 page 32}
\end{aligned}$$

✏

[13] Bruckner et al. (1997) pages 620–621

2.5 Trigonometric summations

Theorem 2.6 (Lagrange trigonometric identities). [14]

THM

$$\sum_{n=0}^{N-1}\cos(nx) = \frac{1}{2} + \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)+\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)+\cos\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=0}^{N-1}\Re e^{inx} = \Re\sum_{n=0}^{N-1}e^{inx} = \Re\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Re\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \Re\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-i\frac{1}{2}x}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Re\left[\left(e^{i\frac{1}{2}(N-1)x}\right)\left(\frac{-i\frac{1}{2}\sin\left(\frac{1}{2}Nx\right)}{-i\frac{1}{2}\sin\left(\frac{1}{2}x\right)}\right)\right] \\
&= \cos\left(\frac{1}{2}(N-1)x\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{-\frac{1}{2}\sin\left(-\frac{1}{2}x\right)+\frac{1}{2}\sin\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by } \textit{product identities} \quad \text{(Theorem 1.8 page 10)} \\
&= \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=0}^{N-1}\Im e^{inx} = \Im\sum_{n=0}^{N-1}e^{inx} = \Im\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Im\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \Im\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-ix/2}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Im\left[\left(e^{i(N-1)x/2}\right)\left(\frac{-\frac{1}{2}i\sin\left(\frac{1}{2}Nx\right)}{-\frac{1}{2}i\sin\left(\frac{1}{2}x\right)}\right)\right]
\end{aligned}$$

[14] Muniz (1953) page 140 ⟨"Lagrange's Trigonometric Identities"⟩, Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (14), (13)⟩

$$= \sin\left(\frac{(N-1)x}{2}\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right)$$

$$= \frac{\frac{1}{2}\cos\left(-\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} \qquad \text{by } \textit{product identities} \qquad \text{(Theorem 1.8 page 10)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}$$

Note that these results (summed with indices from $n=0$ to $n=N-1$) are compatible with Muniz (1953) page 140 (summed with indices from $n=1$ to $n=N$) as demonstrated next:

$$\sum_{n=0}^{N-1}\cos(nx) = \sum_{n=1}^{N}\cos(nx) + [\cos(0x) - \cos(Nx)]$$

$$= \left[-\frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + [\cos(0x) - \cos(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \left(1-\frac{1}{2}\right) + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\cos(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\left[\sin\left(\left[\frac{1}{2}-N\right]x\right) + \sin\left[\left(\frac{1}{2}+N\right)x\right]\right]}{2\sin\left(\frac{1}{2}x\right)} \qquad \text{by Theorem 1.8 page 10}$$

$$= \frac{1}{2} + \frac{\sin\left(\frac{1}{2}[2N-1]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \sum_{n=1}^{N}\sin(nx) + [\sin(0x) - \sin(Nx)]$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} + [0 - \sin(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\sin(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - \left[\cos\left(\left[\frac{1}{2}-N\right]x\right) - \cos\left(\left[\frac{1}{2}+N\right]x\right)\right]}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

✏

Theorem 2.7. [15]

[15] Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (16) and (17)⟩

THM

$$\sum_{n=0}^{N-1}\cos(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] - \sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + \sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)-\sin(nx)\sin(y)] && \text{by } \textit{double angle formulas} \quad \text{(Theorem 1.9 page 13)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx) - \sin(y)\sum_{n=0}^{N-1}\sin(nx)\\
\sum_{n=0}^{N-1}\sin(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)+\sin(nx)\sin(y)] && \text{by } \textit{double angle formulas} \quad \text{(Theorem 1.9 page 13)}\\
&= \cos(y)\sum_{n=0}^{N-1}\cos(nx) + \sin(y)\sum_{n=0}^{N-1}\sin(nx)
\end{aligned}$$

✏

Corollary 2.1 (Summation around unit circle).

THM

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right) = 0 \quad \begin{matrix}\forall\theta\in\mathbb{R}\\ \forall M\in\mathbb{N}\end{matrix}$$

$$\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right) = \frac{N}{2} \quad \begin{matrix}\forall\theta\in\mathbb{R}\\ \forall M\in\mathbb{N}\end{matrix}$$

✎PROOF:

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\\
&= \cos(\theta)\sum_{n=0}^{N-1}\cos\left(\frac{2nM\pi}{N}\right) - \sin(\theta)\sum_{n=0}^{N-1}\sin\left(\frac{2nM\pi}{N}\right) && \text{by Theorem 1.9 page 13}\\
&= \cos(\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] - \sin(\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{2M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] && \text{by Theorem 2.6 page 35}\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right] - \sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{\cos\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{M\pi}{N}\right)}{\sin\left(\frac{M\pi}{N}\right)}\right] - \sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)\right] && \text{by } \textit{trigonometric periodicity} \text{ (Theorem 1.10 page 13)}\\
&= \cos(\theta)[0]-\sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=0}^{N-1} \sin\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by \textit{shift identities}} \quad \text{(Theorem 1.7 page 10)}\\
&= \sum_{n=0}^{N-1} \cos\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2}\\
&= 0 && \text{by previous result}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \cos\left(\theta + \frac{2nM\pi}{N}\right)\sin\left(\theta + \frac{2nM\pi}{N}\right)\\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] - \left[\theta + \frac{2nM\pi}{N}\right]\right) + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] + \left[\theta + \frac{2nM\pi}{N}\right]\right) && \text{by Theorem 1.8 page 10}\\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \cancelto{0}{\sin(0)} + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(2\theta + \frac{4nM\pi}{N}\right)\\
&= \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) && \text{by Theorem 1.9 page 13}\\
&= \cos(2\theta)\left[\frac{1}{2} + \frac{\sin\left(\left[N - \frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{4M\pi}{N}\right) + \frac{\cos\left(\left[N - \frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{4M\pi}{N}\right)}\right] && \text{by Theorem 2.6 page 35}\\
&= \cos(2\theta)\left[\frac{1}{2} - \frac{\sin\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{\cos\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2} - \frac{1}{2}\frac{\sin\left(\frac{2M\pi}{N}\right)}{\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity}} \ \text{(Theorem 1.10 page 13)}\\
&= \cos(\theta)[0] - \sin(\theta)[0]\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2nM\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos\left(2\theta + \frac{4nM\pi}{N}\right)\right] && \text{by Theorem 1.11 page 15}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos(2\theta)\cos\left(\frac{4nM\pi}{N}\right) - \sin(2\theta)\sin\left(\frac{4nM\pi}{N}\right)\right] && \text{by Theorem 1.9 page 13}\\
&= \frac{1}{2}\sum_{n=0}^{N-1} 1 + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\cos\left(\frac{4nM\pi}{N}\right) - \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)\\
&= \left[\frac{1}{2}\sum_{n=0}^{N-1} 1\right] + \frac{1}{2}\cos(2\theta)0 - \frac{1}{2}\sin(2\theta)0 && \text{by previous results}\\
&= \frac{N}{2}
\end{aligned}
$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos^2\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by } \textit{shift identities} \text{ (Theorem 1.7 page 10)} \\
&= \sum_{n=0}^{N-1} \cos^2\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2} \\
&= \frac{N}{2} && \text{by previous result}
\end{aligned}$$

✏

2.6 Summability Kernels

Definition 2.3. [16] *Let $(\kappa_n)_{n\in\mathbb{Z}}$ be a sequence of* CONTINUOUS 2π PERIODIC *functions.*

DEF

The sequence $(\kappa_n)_{n\in\mathbb{Z}}$ is a ***summability kernel*** *if*

1. $\frac{1}{2\pi}\int_0^{2\pi} \kappa_n(x)\,dx = 1 \quad \forall n\in\mathbb{Z}$ *and*
2. $\frac{1}{2\pi}\int_0^{2\pi} |\kappa_n(x)|\,dx \in \mathbb{R} \quad \forall n\in\mathbb{Z}$ *and*
3. $\lim_{n\to\infty}\int_\delta^{2\pi-\delta} |\kappa_n(x)|\,dx = 0 \quad \forall n\in\mathbb{Z}, 0<\delta<\pi$

Theorem 2.8. [17] *Let $(\kappa_n)_{n\in\mathbb{Z}}$ be a sequence. Let $\mathbb{T}$ be the quotient $\mathbb{R}/2\pi\mathbb{Z}$.*

THM

$$\left.\begin{array}{ll} 1. & \mathsf{f} \in L^1(\mathbb{T}) \quad \textit{and} \\ 2. & (\kappa_n) \text{ is a summability kernel} \end{array}\right\} \implies \mathsf{f}(x) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbb{T}} \kappa_n(x)\mathsf{f}(x-x)\,dx$$

The *Dirichlet kernel* (Definition 2.2 page 32) is *not* a *summability kernel*. Examples of kernels that *are* summability kernels include

1. *Fejér's kernel* (Definition 2.4 page 39)
2. *de la Vallée Poussin kernel* (Definition 2.5 page 41)
3. *Jackson kernel* (Definition 2.6 page 41)
4. *Poisson kernel* (Definition 2.7 page 41 .)

Definition 2.4. [18]

DEF

Fejér's kernel *K_n is defined as*

$$\mathsf{K}_n(x) \triangleq \sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx}$$

Proposition 2.5. [19] *Let K_n be Fejér's kernel* (Definition 2.4 page 39).

PRP

$$\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$$

[16] Cerdà (2010) page 56, Katznelson (2004) page 10, de Reyna (2002) page 21, Walnut (2002) pages 40–41, Heil (2011) page 440, Istrăţescu (1987) page 309
[17] Katznelson (2004) page 11
[18] Katznelson (2004) page 12
[19] Katznelson (2004) page 12, Heil (2011) page 448

✎PROOF:

1. Lemma: Proof that $\sin^2\frac{x}{2} \equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right)$:

$$\begin{aligned}
\sin^2\frac{x}{2} &\equiv \left(\frac{e^{-i\frac{x}{2}} - e^{+i\frac{x}{2}}}{2i}\right)^2 && \text{by Euler Formulas (Corollary 1.2 page 9)}\\
&\equiv \frac{-1}{4}\left(e^{-2i\frac{x}{2}} - 2e^{-i\frac{x}{2}}e^{i\frac{x}{2}} + e^{2i\frac{x}{2}}\right)\\
&\equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right) \;:
\end{aligned}$$

2. Lemma:

$$2|k| - |k+1| - |k-1| = \begin{cases} -2 & \text{for } k = 0\\ 0 & \text{for } k \in \mathbb{Z}\backslash 0\end{cases}$$

3. Proof that $\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$:

$$\begin{aligned}
&-4(n+1)\left(\sin\frac{1}{2}x\right)^2 \mathsf{K}_n(x)\\
&= -4(n+1)\left(\frac{-1}{4}\right)\left(e^{-ix} - 2 + e^{ix}\right)\mathsf{K}_n(x) && \text{by item (1)}\\
&= (n+1)\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx} && \text{by Definition 2.4}\\
&= (n+1)\frac{1}{n+1}\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= e^{-ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}e^{ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= \sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k-1)x} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k+1)x}\\
&= \sum_{k=-n-1}^{k=n-1}(n+1-|k+1|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n+1}^{k=n+1}(n+1-|k-1|)e^{ikx}\\
&= \underbrace{e^{-i(n+1)x}}_{k=-n-1} + \underbrace{2e^{-inx}}_{k=-n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad \underbrace{-2e^{-inx}}_{k=-n} + \underbrace{-2e^{inx}}_{k=n} - 2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad \underbrace{e^{i(n+1)x}}_{k=n+1} + \underbrace{2e^{inx}}_{k=n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}\\
&= e^{-i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad -2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}
\end{aligned}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} [(n+1-|k+1|) - 2(n+1-|k|) + (n+1-|k-1|)]\, e^{ikx}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} (2|k| - |k+1| - |k-1|)\, e^{ikx}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} - 2 \qquad \text{by item (2)}$$

$$= -4\left(\sin\frac{n+1}{2}x\right)^2 \qquad \text{by item (1)}$$

✎

Definition 2.5. [20] *Let* K_n *be* FEJÉR'S KERNEL *(Definition 2.4 page 39).*

DEF

*The **de la Vallée Poussin kernel** V_n is defined as*

$$\mathsf{V}_n(x) \triangleq 2K_{2n+1}(x) - K_n(x)$$

Definition 2.6. [21] *Let* K_n *be* FEJÉR'S KERNEL *(Definition 2.4 page 39).*

DEF

*The **Jackson kernel** J_n is defined as*

$$\mathsf{J}_n(x) \triangleq \|\mathsf{K}_n\|^{-2}\, \mathsf{K}_n^2(x)$$

Definition 2.7. [22]

DEF

*The **Poisson kernel** P is defined as*

$$P(r,x) \triangleq \sum_{k\in\mathbb{Z}} r^{|k|} e^{ikx}$$

[20] Katznelson (2004) page 16
[21] Katznelson (2004) page 17
[22] Katznelson (2004) page 16

CHAPTER 3

FOURIER TRANSFORM

❝ *Up to this point we have supposed that the function whose development is required in a series of sines of multiple arcs can be developed in a series arranged according to powers of the variable x, ...We can extend the same results to any functions, even to those which are discontinuous and entirely arbitrary. ...even entirely arbitrary functions may be developed in series of sines of multiple arcs.* ❞

Joseph Fourier (1768–1830) [1]

3.1 Introduction

Historically, before the Fourier Transform was the Taylor Expansion (transform). The Taylor Expansion demonstrates that for **analytic** functions, knowledge of the derivatives of a function at a location $x = a$ allows you to determine (predict) arbitrarily closely all the points $\mathsf{f}(x)$ in the vicinity of $x = a$. But analytic functions are by definition functions for which all their derivatives exist. Thus, if a function is *discontinuous*, it is simply not a candidate for the Taylor Expansion. And some 300 years ago, mathematician giants of the day were fairly content with this.

But then in came an engineer named Joseph Fourier whose day job was working as a governor of lower Egypt under Napolean. He claimed that, rather than expansion based on derivatives, one could expand based on integrals over sinusoids, and that this would work not just for analytic functions, but for **discontinuous** ones as well![2]

Needless to say, this did not go over too well initially in the mathematical community. But over time (on the order of 200 or so years), the Fourier Transform has in many ways won the day.

[3]

[1] quote: Fourier (1878) page 184,186 ⟨§219,220⟩
image: http://en.wikipedia.org/wiki/File:Fourier2.jpg, public domain

[2] Robinson (1982) page 886

[3] Caricature of Legendre (left) and Fourier (right), 1820, by Julien-Léopold Boilly (1796–1874). “*Album de 73*

3.2 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on $\mathscr{B}$, and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \middle| \int_{\mathbb{R}} |\mathsf{f}|^2 \, \mathrm{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \mid \nabla \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \mathrm{d}\mu$ such that

$$\langle \mathsf{f} \mid \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\, \mathrm{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \mid \nabla \rangle\right)$ is a *Hilbert space*.

Definition 3.1. *Let κ be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF — *The function κ is the **Fourier kernel** if* $\quad \kappa(x,\omega) \triangleq e^{i\omega x} \quad \forall x,\omega \in \mathbb{R}$

Definition 3.2. [4] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF — *The **Fourier Transform** operator $\tilde{\mathbf{F}}$ is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \, \mathrm{d}x \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

*This definition of the Fourier Transform is also called the **unitary Fourier Transform**.*

Remark 3.1 (**Fourier transform scaling factor**). [5] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B \int_{\mathbb{R}} \mathsf{F}(\omega) e^{i\omega x}\, \mathrm{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x}\, \mathrm{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x}\, \mathrm{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int \cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem 3.2 page 45)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

Portraits-Charge Aquarelle's des Membres de l'Institute (watercolor portrait #29). Biliotheque de l'Institut de France." Public domain. https://en.wikipedia.org/wiki/File:Legendre_and_Fourier_(1820).jpg

[4] Bachman et al. (2002) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965) page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

[5] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxi–xxxii, Knapp (2005b) pages 374–375

3.3 Operator properties

Theorem 3.1 (Inverse Fourier transform). [6] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition 3.2 page 44). *The inverse $\tilde{\mathbf{F}}^{-1}$ of $\tilde{\mathbf{F}}$ is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\,\mathrm{d}\omega \qquad \forall\tilde{\mathsf{f}}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem 3.2. *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator with inverse $\tilde{\mathbf{F}}^{-1}$ and adjoint $\tilde{\mathbf{F}}^{*}$.*

THM
$$\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\langle \tilde{\mathbf{F}}\mathsf{f} \mid \mathsf{g}\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathrm{d}x \mid \mathsf{g}(\omega)\right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 44}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,\left\langle e^{-i\omega x} \mid \mathsf{g}(\omega)\right\rangle\,\mathrm{d}x && \text{by \textit{additive property} of } \langle\triangle\mid\nabla\rangle\\
&= \int_{\mathbb{R}} \mathsf{f}(x)\,\frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \mid e^{-i\omega x}\right\rangle^{*}\,\mathrm{d}x && \text{by \textit{conjugate symmetric property} of } \langle\triangle\mid\nabla\rangle\\
&= \left\langle \mathsf{f}(x) \mid \frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \mid e^{-i\omega x}\right\rangle\right\rangle && \text{by definition of } \langle\triangle\mid\nabla\rangle\\
&= \left\langle \mathsf{f} \mid \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^{*}}\mathsf{g}\right\rangle && \text{by Theorem 3.1 page 45}
\end{aligned}$$

✏

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* [7] (Corollary 3.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem 3.3 page 45).

Corollary 3.1. *Let $\mathbf{I}$ be the identity operator and let $\tilde{\mathbf{F}}$ be the Fourier Transform operator with adjoint $\tilde{\mathbf{F}}^{*}$ and inverse $\tilde{\mathbf{F}}^{-1}$.*

COR
$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{*}\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \text{ \textit{is unitary}})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^{*} = \tilde{\mathbf{F}}^{-1}$ (Theorem 3.2 page 45). ✏

Theorem 3.3. *Let $\tilde{\mathbf{F}}$ be the Fourier transform operator with adjoint $\tilde{\mathbf{F}}^{*}$ and inverse $\tilde{\mathbf{F}}$. Let $|||\cdot|||$ be the operator norm with respect to the vector norm $\|\cdot\|$ with respect to the Hilbert space $\left(\mathbb{C}^{\mathbb{R}}, \langle\triangle\mid\nabla\rangle\right)$. Let $\mathscr{R}(\mathbf{A})$ be the range of an operator $\mathbf{A}$.*

THM
$$\begin{array}{rclcll}
\mathscr{R}(\mathbf{F}\tau) &=& \mathscr{R}(\tilde{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| &=& |||\tilde{\mathbf{F}}^{-1}||| &=& 1 & (\text{UNITARY})\\
\langle\tilde{\mathbf{F}}\mathsf{f} \mid \tilde{\mathbf{F}}\mathsf{g}\rangle &=& \langle\tilde{\mathbf{F}}^{-1}\mathsf{f} \mid \tilde{\mathbf{F}}^{-1}\mathsf{g}\rangle &=& \langle\mathsf{f}\mid\mathsf{g}\rangle & (\text{PARSEVAL'S EQUATION})\\
\|\tilde{\mathbf{F}}\mathsf{f}\| &=& \|\tilde{\mathbf{F}}^{-1}\mathsf{f}\| &=& \|\mathsf{f}\| & (\text{PLANCHEREL'S FORMULA})\\
\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\| &=& \|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\| &=& \|\mathsf{f}-\mathsf{g}\| & (\text{ISOMETRIC})
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary 3.1 page 45) and from the properties of unitary operators (Theorem G.26 page 170). ✏

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[6] Chorin and Hald (2009) page 13

[7] *unitary operators*: Definition G.14 page 169

3.4 Transversal properties

Theorem 3.4 (Shift relations). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition 3.2 page 44).

THM
$$\begin{aligned}\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega)\\ \left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r)\end{aligned}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition E.1 page 117).

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= \mathbf{L}[\mathsf{f}(x-y)](s)\big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}\\
&= e^{-sy}\left[\mathbf{L}\mathsf{f}(x)\right](s)\big|_{s=i\omega} && \text{by Laplace \textit{translation} property} && \text{(Theorem E.2 page 118)}\\
&= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 44)}\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\mathbf{L}\left(e^{irx}\mathsf{g}(x)\right)\right](s)\big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}\\
&= \left[\left[\mathbf{L}\mathsf{g}(x)\right](s-r)\right]\big|_{s=i\omega} && \text{by Laplace \textit{dilation} property} && \text{(Theorem E.2 page 118)}\\
&= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 44)}
\end{aligned}$$

✏

Theorem 3.5 (Complex conjugate). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

THM
$$\tilde{\mathbf{F}}\mathsf{f}^*(-x) = -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* \qquad \forall \mathsf{f}\in\mathcal{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(-x)e^{-i\omega x}\,\mathsf{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 44)}\\
&= \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(u)e^{i\omega u}(-1)\,\mathsf{d}u && \text{where } u\triangleq -x \implies \mathsf{d}x = -\,\mathsf{d}u\\
&= -\left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(u)e^{-i\omega u}\,\mathsf{d}u\right]^*\\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 44)}
\end{aligned}$$

✏

3.5 Convolution relations

Theorem F.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem 3.6 (convolution theorem). [8] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition 3.2 page 44) *and $\star$ the convolution operator* (Definition D.1 page 115).

[8] Bachman et al. (2002) pages 269–270 ⟨5.2.3 Convolutions to Products⟩, Bachman (1964) page 8, Bracewell (1978) page 110

THM

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g} \in \mathcal{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g} \in \mathcal{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition E.1 page 117).

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega) &= \mathbf{L}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](s)\Big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)} \\
&= \sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s)\Big|_{s=i\omega} && \text{by } \textit{Laplace convolution} \text{ result} && \text{(Theorem E.6 page 129)} \\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) \\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s)|_{s=i\omega} \\
&= \frac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s) \star \left[\mathbf{L}\mathsf{g}\right](s)\Big|_{s=i\omega} \\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)
\end{aligned}$$

⇨

3.6 Calculus relations

Theorem 3.7. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* (*Definition 3.2 page 44*).

THM

$$\left\{\lim_{t\to-\infty} \mathsf{x}(t) = 0\right\} \implies \left\{\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right] = i\omega\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)\right\}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition E.1 page 117).

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right] &\triangleq \mathbf{L}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right](s)\Big|_{s=i\omega} && \text{by definitions of } \mathbf{L} \text{ and } \tilde{\mathbf{F}} && \text{(Definition E.1 page 117)} \\
&= s[\mathbf{L}\mathsf{x}(t)](s)|_{s=i\omega} && \text{by Theorem E.7 page 130} \\
&= i\omega\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)
\end{aligned}$$

⇨

Theorem 3.8. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* (*Definition 3.2 page 44*).

THM

$$\tilde{\mathbf{F}}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u = \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)$$

Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition E.1 page 117). ✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u &\triangleq \mathbf{L}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u\Big|_{s=i\omega} \\
&= \frac{1}{s}[\mathbf{L}\mathsf{x}(t)](s)\Big|_{s=i\omega} && \text{by Theorem E.7 page 130} \\
&= \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}(t)\right](\omega)
\end{aligned}$$

⇨

3.7 Real valued functions

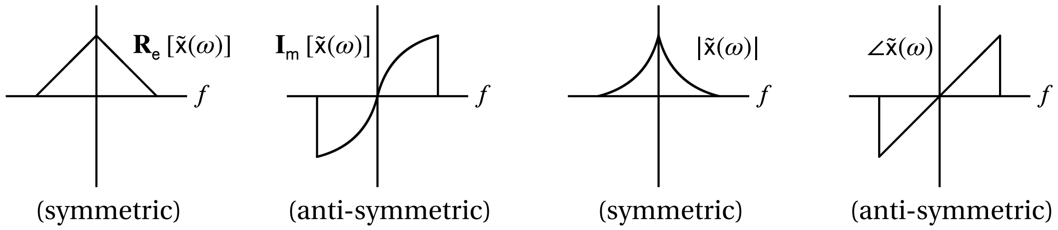


Figure 3.1: Fourier transform components of real-valued signal

Theorem 3.9. *Let* $\mathsf{f}(x)$ *be a function in* $L^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM

$$\left\{\begin{array}{l}\mathsf{f}(x) \text{ is REAL-VALUED} \\ (\mathsf{f} \in \mathbb{R}^{\mathbb{R}})\end{array}\right\} \implies \left\{\begin{array}{rcll} \tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & \text{(HERMITIAN SYMMETRIC)} \\ \mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(SYMMETRIC)} \\ \mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(ANTI-SYMMETRIC)} \\ \left|\tilde{\mathsf{f}}(\omega)\right| &=& \left|\tilde{\mathsf{f}}(-\omega)\right| & \text{(SYMMETRIC)} \\ \angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & \text{(ANTI-SYMMETRIC).} \end{array}\right\}$$

✎PROOF:

$$\begin{array}{lclclclcl}
\tilde{\mathsf{f}}(\omega) & \triangleq & [\tilde{\mathbf{F}}\mathsf{f}(x)](\omega) & \triangleq & \left\langle \mathsf{f}(x) \mid e^{i\omega x}\right\rangle & = & \left\langle \mathsf{f}(x) \mid e^{i(-\omega)x}\right\rangle^* & \triangleq & \tilde{\mathsf{f}}^*(-\omega) \\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] & = & \mathbf{R_e}\left[\tilde{\mathsf{f}}^*(-\omega)\right] & = & \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] & = & \mathbf{I_m}\left[\tilde{\mathsf{f}}^*(-\omega)\right] & = & -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
\left|\tilde{\mathsf{f}}(\omega)\right| & = & \left|\tilde{\mathsf{f}}^*(-\omega)\right| & = & \left|\tilde{\mathsf{f}}(-\omega)\right| \\
\angle\tilde{\mathsf{f}}(\omega) & = & \angle\tilde{\mathsf{f}}^*(-\omega) & = & -\angle\tilde{\mathsf{f}}(-\omega)
\end{array}$$

✏

3.8 Moment properties

Definition 3.3. [9]

DEF

The quantity M_n *is the* n**th moment** *of a function* $\mathsf{f}(x) \in L^2_{\mathbb{R}}$ *if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\,\mathrm{d}x \qquad \text{for } n \in \mathbb{W}.$$

Lemma 3.1. [10] *Let* M_n *be the* nTH MOMENT *(Definition 3.3 page 48) and* $\tilde{\mathsf{f}}(\omega) \triangleq \left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)$ *the* FOURIER TRANSFORM *(Definition 3.2 page 44) of a function* $\mathsf{f}(x)$ *in* $L^2_{\mathbb{R}}$ *(Definition C.1 page 111).*

LEM

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\,\mathrm{d}x \;=\; \sqrt{2\pi}(i)^n \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} \qquad \forall n \in \mathbb{W}, \mathsf{f} \in L^2_{\mathbb{R}}$$

[9] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83
[10] Goswami and Chan (1999) pages 38–39

✎PROOF:

$$\begin{aligned}
\sqrt{2\pi}(i)^n \left[\left[\tfrac{d}{d\omega}\right]^n \tilde{f}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\tfrac{d}{d\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 3.2 page 44)}\\
&= (i)^n \int_{\mathbb{R}} f(x) \left[\left[\tfrac{d}{d\omega}\right]^n e^{-i\omega x}\right] dx \bigg|_{\omega=0} \\
&= (i)^n \int_{\mathbb{R}} f(x) \left[(-i)^n x^n e^{-i\omega x}\right] dx \bigg|_{\omega=0} \\
&= (-i^2)^n \int_{\mathbb{R}} f(x) x^n\, dx \\
&= \int_{\mathbb{R}} x^n f(x)\, dx \\
&\triangleq M_n && \text{by definition of } M_n \quad \text{(Definition 3.3 page 48)}
\end{aligned}$$

✏

Lemma 3.2. [11] *Let M_n be the nTH MOMENT (Definition 3.3 page 48) and $\tilde{f}(\omega) \triangleq [\tilde{\mathbf{F}}f](\omega)$ the FOURIER TRANSFORM (Definition 3.2 page 44) of a function $f(x)$ in $L^2_{\mathbb{R}}$ (Definition C.1 page 111).*

LEM
$$M_n = 0 \quad\Longleftrightarrow\quad \left[\tfrac{d}{d\omega}\right]^n \tilde{f}(\omega)\bigg|_{\omega=0} = 0 \qquad \forall n \in \mathbb{W}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
0 &= \langle f(x) \,|\, x^n \rangle && \text{by left hypothesis}\\
&= \sqrt{2\pi}(-i)^{-n} \left[\tfrac{d}{d\omega}\right]^n \tilde{f}(\omega)\bigg|_{\omega=0} && \text{by Lemma 3.1 page 48}\\
\Longrightarrow & \left[\tfrac{d}{d\omega}\right]^n \tilde{f}(\omega)\bigg|_{\omega=0} = 0
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\tfrac{d}{d\omega}\right]^n \tilde{f}(\omega)\bigg|_{\omega=0} && \text{by right hypothesis}\\
&= \left[\tfrac{d}{d\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx \bigg|_{\omega=0} && \text{by definition of } \tilde{f}(\omega)\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \left[\tfrac{d}{d\omega}\right]^n e^{-i\omega x}\, dx \bigg|_{\omega=0}\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \left[(-i)^n x^n e^{-i\omega x}\right] dx \bigg|_{\omega=0}\\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) x^n\, dx\\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \langle f(x) \,|\, x^n \rangle && \text{by definition of } \langle \cdot \,|\, \cdot \rangle \text{ in } L^2_{\mathbb{R}} \quad \text{(Definition C.1 page 111)}
\end{aligned}$$

✏

[11] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

Lemma 3.3 (Strang-Fix condition). [12] *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* M_n *the* nTH MOMENT *(Definition 3.3 page 48) of* $\mathsf{f}(x)$*. Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 4.3 page 56).*

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "time"}} \iff \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "frequency"}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by definition of } \tilde{\mathsf{f}}(\omega) && \text{(Definition 3.2 page 44)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k)\bar{\delta}_k && \text{by } PSF && \text{(Theorem 4.2 page 64)} \\
&= \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[(-i)^n \bar{\delta}_k \mathsf{M}_n\right] e^{-i2\pi kx} && \text{by definition of } \bar{\delta} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } PSF && \text{(Theorem 4.2 page 64)}
\end{aligned}$$

✏

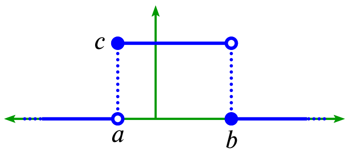
[12] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83, Mallat (1999) pages 241–243, Fix and Strang (1969)

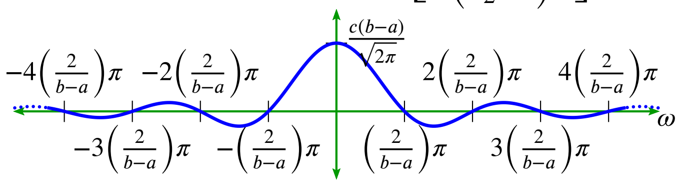
3.9 Examples

Example 3.1 (rectangular pulse). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)} \right]$$





✎PROOF:

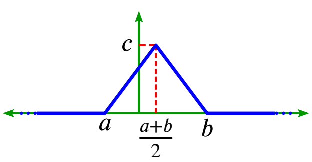
$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by \textit{shift relation}} \quad \text{(Theorem 3.4 page 46)} \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by definition of } \mathsf{f}(x) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} \quad \text{(Definition 4.2 page 56)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 3.2 page 44)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} ce^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \quad \text{(Definition 4.2 page 56)} \\
&= \frac{c}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\frac{1}{-i\omega}e^{-i\omega x}\Big|_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \\
&= \frac{2c}{\sqrt{2\pi}\omega}e^{-i\left(\frac{a+b}{2}\right)\omega}\left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i}\right] \\
&= \frac{c(b-a)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right] && \text{by \textit{Euler formulas}} \quad \text{(Corollary 1.2 page 9)}
\end{aligned}$$

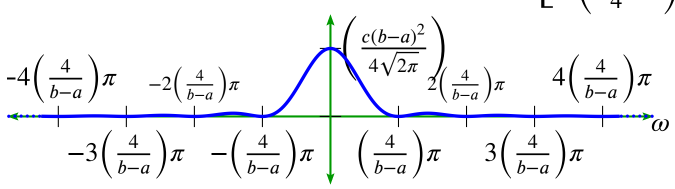
✏

Example 3.2 (triangle). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \left\{ \begin{array}{ll} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{array} \right\} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)} \right]^2$$





✎PROOF:

$$\tilde{\mathsf{f}}(\omega) = \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) \qquad \text{by definition of } \tilde{\mathsf{f}}(\omega)$$

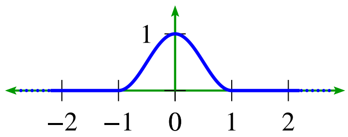
$$
\begin{aligned}
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x-\frac{a+b}{2}\right)\right](\omega) && \text{by \textit{shift relation}} && \text{(Theorem 3.4 page 46)}\\
&= \tilde{\mathbf{F}}\left[c\left(1-\frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) && \text{by definition of } \mathsf{f}(x)\\
&= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\star\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega)\\
&= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] && \text{by \textit{convolution theorem}} && \text{(Theorem F.2 page 134)}\\
&= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2\\
&= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 && \text{by \textit{Rectangular pulse} ex.} && \text{Example 3.1 page 51}\\
&= \frac{c(b-a)^2}{4\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2
\end{aligned}
$$

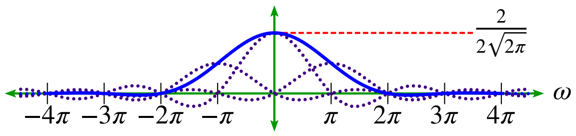
✏

Example 3.3. Let a function f be defined in terms of the cosine function (Definition 1.1 page 3) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases}\cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x|\le 1\\ 0 & \text{otherwise}\end{cases}
\qquad
\tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\text{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\text{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\text{sinc}(\omega-\pi)}\right]$$





✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 4.2 page 56) on a set A.

$$
\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition 3.2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathsf{f}(x)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\cos^2\left(\frac{\pi}{2}x\right)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \text{ (Definition 4.2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left[\frac{e^{i\frac{\pi}{2}x}+e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\,\mathrm{d}x && \text{by Corollary 1.2 page 9}\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1}\left[2+e^{i\pi x}+e^{-i\pi x}\right]e^{-i\omega x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1}2e^{-i\omega x}+e^{-i(\omega+\pi)x}+e^{-i(\omega-\pi)x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\left[2\frac{e^{-i\omega x}}{-i\omega}+\frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}\\
&= \frac{1}{2\sqrt{2\pi}}\left[2\frac{e^{-i\omega}-e^{+i\omega}}{-2i\omega}+\frac{e^{-i(\omega+\pi)}-e^{+i(\omega+\pi)}}{-2i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)}-e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}
\end{aligned}
$$

$$= \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\text{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\text{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\text{sinc}(\omega-\pi)}\right]$$

✎

Example 3.4. [13]

EX
$$\tilde{\mathbf{F}}\left[e^{-\alpha|x|}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right]$$

✎PROOF:

1. Proof using *Laplace Transform*:

$$\begin{aligned}
\sqrt{2\pi}\tilde{\mathbf{F}}\left[e^{-\alpha|x|}\right] &\triangleq \left[\sqrt{2\pi}\right]\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\alpha|x|}e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 44)}\\
&= \left[\int_{\mathbb{R}} e^{-\alpha|x|}e^{-sx}\,\mathrm{d}x\right]_{s=i\omega}\\
&= \left[\frac{2\alpha}{\alpha^2-s^2}\right]_{s=i\omega} && \forall \mathbf{R_e}(s)\in(-\alpha : \alpha) && \text{by Corollary E.9 page 127}\\
&= \frac{2\alpha}{\alpha^2+\omega^2} && \text{because } s=i\omega \text{ is in } (-\alpha : \alpha)
\end{aligned}$$

2. Alternate proof:

$$\begin{aligned}
\sqrt{2\pi}\tilde{\mathbf{F}}\left[e^{-\alpha|x|}\right] &\triangleq \left[\sqrt{2\pi}\right]\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 3.2 page 44)}\\
&= \int_{-\infty}^{0} e^{-\alpha(-x)}e^{-i\omega x}\,\mathrm{d}x + \int_{0}^{\infty} e^{-\alpha(x)}e^{-i\omega x}\,\mathrm{d}x\\
&= \int_{-\infty}^{0} e^{x(\alpha-i\omega)}\,\mathrm{d}x + \int_{0}^{\infty} e^{x(-\alpha-i\omega)}\,\mathrm{d}x\\
&= \left.\frac{e^{x(\alpha-i\omega)}}{\alpha-i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{x(-\alpha-i\omega)}}{-\alpha-i\omega}\right|_{0}^{\infty} && \text{by Fundamental Theorem of Calculus}\\
&= \left[\frac{1}{\alpha-i\omega}-0\right] + \left[0-\frac{1}{-\alpha-i\omega}\right]\\
&= \left[\frac{1}{\alpha-i\omega}\right]\left[\frac{\alpha-i\omega}{\alpha-i\omega}\right] + \left[\frac{1}{\alpha+i\omega}\right]\left[\frac{\alpha+i\omega}{\alpha+i\omega}\right]\\
&= \frac{\alpha-i\omega}{\alpha^2+\omega^2} + \frac{\alpha+i\omega}{\alpha^2+\omega^2}\\
&= \left[\frac{2\alpha}{\alpha^2+\omega^2}\right]
\end{aligned}$$

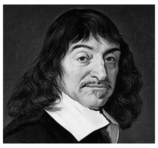
✎

[13] https://math.stackexchange.com/questions/4015842/

CHAPTER 4

TRANSVERSAL OPERATORS

❝*Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:*❞



❝*I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them.*❞

René Descartes, philosopher and mathematician (1596–1650) [1]

4.1 Families of Functions

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition C.1 page 111). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition 4.1 page 55). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition 4.1. *Let X and Y be sets.*

DEF *The space Y^X represents the set of all functions with* DOMAIN *X and* RANGE *Y such that*

$$Y^X \triangleq \{\mathsf{f}(x)|\mathsf{f}(x) : X \to Y\}$$

[1] quote: Descartes (1637b)
translation: Descartes (1637c) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

Definition 4.2. [2] *Let X be a set.*

DEF

*The **indicator function** $\mathbb{1} \in \{0,1\}^{2^X}$ is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A & \forall x \in X,\ A \in 2^X \\ 0 & \text{for } x \notin A & \forall x \in X,\ A \in 2^X \end{cases}$$

*The indicator function $\mathbb{1}$ is also called the **characteristic function**.*

4.2 Definitions and algebraic properties

Much of the wavelet theory developed in this text is constructed using the **translation operator T** and the **dilation operator D** (next).

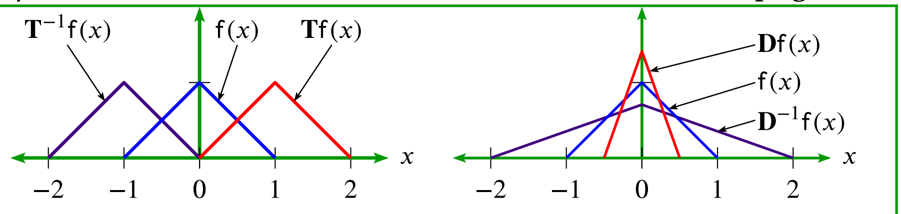
Definition 4.3. [3]

DEF

$$\begin{array}{llll} \mathbf{T}_\tau \text{ is a \textbf{translation operator} on } \mathbb{C}^{\mathbb{C}} \text{ if} & \mathbf{T}_\tau \mathsf{f}(x) \triangleq \mathsf{f}(x-\tau) & \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}} & . \\ \mathbf{D}_\alpha \text{ is a \textbf{dilation operator} on } \mathbb{C}^{\mathbb{C}} \text{ if} & \mathbf{D}_\alpha \mathsf{f}(x) \triangleq \mathsf{f}(\alpha x) & \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}} & . \\ \text{Moreover, } \mathbf{T} \triangleq \mathbf{T}_1 \quad \text{and} \quad \mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2 & & & . \end{array}$$

Example 4.1. Let **T** and **D** be defined as in Definition 4.3 (page 56).

EX



Proposition 4.1. *Let $\mathbf{T}_\tau$ be a* TRANSLATION OPERATOR *(Definition 4.3 page 56).*

PRP

$$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) \qquad \forall \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) \text{ is PERIODIC with period } \tau \right)$$

✎PROOF:

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x - n\tau + \tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 4.3 page 56)} \\ &= \sum_{m\in\mathbb{Z}} \mathsf{f}(x - m\tau) && \text{where } m \triangleq n-1 && \implies n = m+1 \\ &= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m \mathsf{f}(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 4.3 page 56)} \end{aligned}$$

⇒

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exist as operators, as demonstrated next.

[2] Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) page 22, de la Vallée-Poussin (1915) page 440

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2002) page 473, Packer (2004) page 260, Zayed (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a) page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996) page 81, Dai and Larson (1998) page 2

Proposition 4.2 (transversal operator inverses)**.** *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 4.3 page 56.*

PRP

$\mathbf{T}$ *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(translation operator inverse).}$$

$\mathbf{D}$ *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(dilation operator inverse).}$$

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} \quad \text{(Definition 4.3 page 56)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} \quad \text{(Definition 4.3 page 56)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\implies \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} \quad \text{(Definition 4.3 page 56)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} \quad \text{(Definition 4.3 page 56)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\implies \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

✎

Proposition 4.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 4.3 page 56.*
Let $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP

$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^jx-n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

4.3 Linear space properties

Proposition 4.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 56.*

PRP

$$\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}\mathsf{g}\right] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

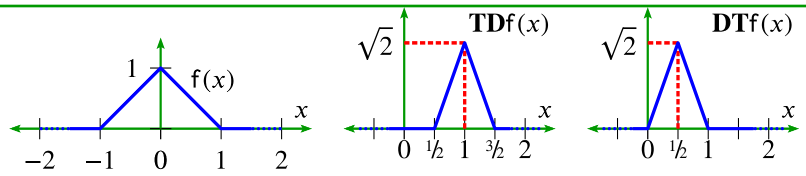
✎PROOF:

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}(x)\mathsf{g}(x)\right] &= 2^{j/2}\mathsf{f}\left(2^jx-n\right)\mathsf{g}\left(2^jx-n\right) && \text{by Proposition 4.3 page 57}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^jx-n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^jx-n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition 4.3 page 57}
\end{aligned}$$

In general the operators **T** and **D** are *noncommutative* (**TD** ≠ **DT**), as demonstrated by Counterexample 4.1 (next) and Proposition 4.5 (page 58).

Counterexample 4.1.

CNT As illustrated to the right, it is **not** always true that **TD** = **DT**:



Proposition 4.5 (commutator relation). [4] *Let* **T** *and* **D** *be as in Definition 4.3 page 56.*

PRP

$$\begin{array}{rcll} \mathbf{D}^j\mathbf{T}^n &=& \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j & \forall j,n\in\mathbb{Z} \\ \mathbf{T}^n\mathbf{D}^j &=& \mathbf{D}^j\mathbf{T}^{2^j n} & \forall n,j\in\mathbb{Z} \end{array}$$

✎PROOF:

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^{2^j n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - 2^j n) && \text{by Proposition 4.4 page 57} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) \quad \text{(Definition ?? page ??)} \\ &= \mathbf{T}^n 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} \quad \text{(Definition 4.3 page 56)} \\ &= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} \quad \text{(Definition 4.3 page 56)} \end{aligned}$$

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - n) && \text{by Proposition 4.4 page 57} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j\left[x-2^{-j/2}n\right]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) \quad \text{(Definition ?? page ??)} \\ &= \mathbf{T}^{2^{-j/2}n} 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} \quad \text{(Definition 4.3 page 56)} \\ &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} \quad \text{(Definition 4.3 page 56)} \end{aligned}$$

4.4 Inner product space properties

In an inner product space, every operator has an *adjoint* (Proposition G.3 page 159) and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator **U** coincide, then **U** is said to be *unitary* (Definition G.14 page 169). And in this case, **U** has several nice properties (see Proposition 4.9 and Theorem 4.1 page 61). Proposition 4.6 (next) gives the adjoints of **D** and **T**, and Proposition 4.7 (page 59) demonstrates that both **D** and **T** are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ (Corollary 3.1 page 45) and the *rotation matrix operator* (Example G.5 page 171).

Proposition 4.6. *Let* **T** *be the* TRANSLATION OPERATOR *(Definition 4.3 page 56) with* ADJOINT $\mathbf{T}^*$ *and* **D** *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$ *(Definition G.8 page 155).*

PRP

$$\begin{array}{rclll} \mathbf{T}^*\mathsf{f}(x) &=& \mathsf{f}(x+1) & \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \text{(TRANSLATION OPERATOR ADJOINT)} \\ \mathbf{D}^*\mathsf{f}(x) &=& \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) & \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \text{(DILATION OPERATOR ADJOINT)} \end{array}$$

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a) page 641, Goodman et al. (1993b) page 110

✎PROOF:

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u\\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^* && \text{(Definition G.8 page 155)}\\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 56)}\\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1\\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u\\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^* && \text{(Definition G.8 page 155)}\\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 56)}\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\,\mathsf{d}u && \text{by definition of } \langle \triangle \,|\, \nabla\rangle\\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^*\,\mathsf{d}x && \text{where } x = 2u\\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle\\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}$$

✏

Proposition 4.7. [5] *Let* **T** *and* **D** *be as in Definition 4.3 (page 56). Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 4.2 (page 57).*

PRP
- $\mathbf{T}$ *is* UNITARY *in* $\boldsymbol{L}^2_{\mathbb{R}}$ $(\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $\boldsymbol{L}^2_{\mathbb{R}})$.
- $\mathbf{D}$ *is* UNITARY *in* $\boldsymbol{L}^2_{\mathbb{R}}$ $(\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $\boldsymbol{L}^2_{\mathbb{R}})$.

✎PROOF:

$$\begin{aligned}
&\mathbf{T}^{-1} = \mathbf{T}^* && \text{by Proposition 4.2 page 57 and Proposition 4.6 page 58}\\
&\implies \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition G.14 page 169)}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{D}^{-1} = \mathbf{D}^* && \text{by Proposition 4.2 page 57 and Proposition 4.6 page 58}\\
&\implies \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition G.14 page 169)}
\end{aligned}$$

✏

4.5 Normed linear space properties

Proposition 4.8. *Let* **D** *be the* DILATION OPERATOR *(Definition 4.3 page 56).*

PRP
$$\left\{\begin{array}{lll}(1). & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) & \textit{and}\\ (2). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} & \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$$

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

✎PROOF:

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$
\begin{aligned}
\mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 56)}\\
&= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis}
\end{aligned}
$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$
\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(x+h)\| &= \|\mathsf{f}(x) - \mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis}\\
&= \|0\|\\
&= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|\\
&\le \varepsilon\\
&\Longrightarrow \forall h > 0,\ \exists \varepsilon \quad \text{such that} \quad \|\mathsf{f}(x) - \mathsf{f}(x+h)\| < \varepsilon\\
&\overset{\text{def}}{\Longleftrightarrow} \mathsf{f}(x) \text{ is } \textit{continuous}
\end{aligned}
$$

3. Proof that (1,2) $\Longrightarrow$ *constant* property:

   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \neq \mathsf{f}(y)$.

   (b) Let $(\!(x_n)\!)_{n\in\mathbb{N}}$ be a sequence with limit x and $(\!(y_n)\!)_{n\in\mathbb{N}}$ a sequence with limit y

   (c) Then

$$
\begin{aligned}
0 &< \|\mathsf{f}(x) - \mathsf{f}(y)\| && \text{by assumption in item (3a) page 60}\\
&= \lim_{n\to\infty} \left\|\mathsf{f}(x_n) - \mathsf{f}(y_n)\right\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 60}\\
&= \lim_{n\to\infty} \left\|\mathsf{f}(2^m x_n) - \mathsf{f}(2^\ell y_n)\right\| \quad \forall m,\ell \in \mathbb{Z} && \text{by (1)}\\
&= 0
\end{aligned}
$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

✎

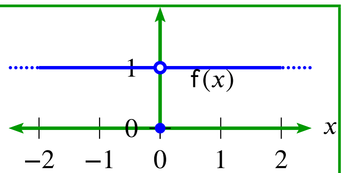
Remark 4.1.

REM In Proposition 4.8 page 59, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample 4.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x = 0\\ 1 & \text{otherwise.}\end{cases}$

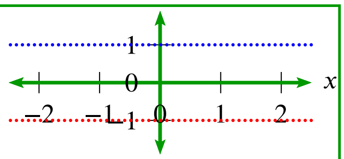
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Counterexample 4.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\backslash\mathbb{Q}$ the set of *irrational numbers*.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x \in \mathbb{Q}\\ -1 & \text{for } x \in \mathbb{R}\backslash\mathbb{Q}.\end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Proposition 4.9 (Operator norm). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 56. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 4.2 page 57. Let* $\mathbf{T}^*$ *and* $\mathbf{D}^*$ *be as in Proposition 4.6 page 58. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition C.1 page 111. Let* $|||\cdot|||$ *be the operator norm* (Definition G.6 page 151) *induced by* $\|\cdot\|$.

PRP: $$|||\mathbf{T}||| = |||\mathbf{D}||| = |||\mathbf{T}^*||| = |||\mathbf{D}^*||| = |||\mathbf{T}^{-1}||| = |||\mathbf{D}^{-1}||| = 1$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 4.7 page 59) and from Theorem G.25 page 170 and Theorem G.26 page 170. ⇒

Theorem 4.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 56.*
Let $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 4.2 page 57. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition C.1 page 111.*

THM:
$$\begin{array}{rlclcll}
1. & \|\mathbf{T}\mathsf{f}\| &=& \|\mathbf{D}\mathsf{f}\| &=& \|\mathsf{f}\| & \forall \mathsf{f}\in\mathcal{L}^2_{\mathbb{R}} \quad \text{(ISOMETRIC IN LENGTH)}\\
2. & \|\mathbf{T}\mathsf{f}-\mathbf{T}\mathsf{g}\| &=& \|\mathbf{D}\mathsf{f}-\mathbf{D}\mathsf{g}\| &=& \|\mathsf{f}-\mathsf{g}\| & \forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}} \quad \text{(ISOMETRIC IN DISTANCE)}\\
3. & \left\|\mathbf{T}^{-1}\mathsf{f}-\mathbf{T}^{-1}\mathsf{g}\right\| &=& \left\|\mathbf{D}^{-1}\mathsf{f}-\mathbf{D}^{-1}\mathsf{g}\right\| &=& \|\mathsf{f}-\mathsf{g}\| & \forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}} \quad \text{(ISOMETRIC IN DISTANCE)}\\
4. & \langle\mathbf{T}\mathsf{f}\mid\mathbf{T}\mathsf{g}\rangle &=& \langle\mathbf{D}\mathsf{f}\mid\mathbf{D}\mathsf{g}\rangle &=& \langle\mathsf{f}\mid\mathsf{g}\rangle & \forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}} \quad \text{(SURJECTIVE)}\\
5. & \left\langle\mathbf{T}^{-1}\mathsf{f}\mid\mathbf{T}^{-1}\mathsf{g}\right\rangle &=& \left\langle\mathbf{D}^{-1}\mathsf{f}\mid\mathbf{D}^{-1}\mathsf{g}\right\rangle &=& \langle\mathsf{f}\mid\mathsf{g}\rangle & \forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}} \quad \text{(SURJECTIVE)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 4.7 page 59) and from Theorem G.25 page 170 and Theorem G.26 page 170. ⇒

Proposition 4.10. *Let* $\mathbf{T}$ *be as in Definition 4.3 page 56. Let* $\mathbf{A}^*$ *be the* ADJOINT (Definition G.8 page 155) *of an operator* $\mathbf{A}$. *Let the property "*SELF ADJOINT*" be defined as in Definition G.11 (page 163).*

PRP: $$\left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)^* \qquad \left(\text{The operator } \left[\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right] \text{ is SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle\left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)\mathsf{f}(x)\mid\mathsf{g}(x)\right\rangle
&= \left\langle\sum_{n\in\mathbb{Z}}\mathsf{f}(x-n)\mid\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 56)}\\
&= \left\langle\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n)\mid\mathsf{g}(x)\right\rangle && \text{by \textit{commutative} property} && \text{(Definition ?? page ??)}\\
&= \sum_{n\in\mathbb{Z}}\langle\mathsf{f}(x+n)\mid\mathsf{g}(x)\rangle && \text{by \textit{additive} property of } \langle\triangle\mid\nabla\rangle\\
&= \sum_{n\in\mathbb{Z}}\langle\mathsf{f}(u)\mid\mathsf{g}(u-n)\rangle && \text{where } u \triangleq x+n\\
&= \left\langle\mathsf{f}(u)\;\middle|\;\sum_{n\in\mathbb{Z}}\mathsf{g}(u-n)\right\rangle && \text{by \textit{additive} property of } \langle\triangle\mid\nabla\rangle\\
&= \left\langle\mathsf{f}(x)\;\middle|\;\sum_{n\in\mathbb{Z}}\mathsf{g}(x-n)\right\rangle && \text{by change of variable: } u\to x\\
&= \left\langle\mathsf{f}(x)\;\middle|\;\sum_{n\in\mathbb{Z}}\mathbf{T}^n\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 56)}\\
\iff \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) &= \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)^* && \text{by definition of \textit{adjoint}} && \text{(Proposition G.3 page 159)}\\
\iff \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) &\text{ is \textit{self-adjoint}} && \text{by definition of \textit{self-adjoint}} && \text{(Definition G.11 page 163)}
\end{aligned}$$

⇒

4.6 Fourier transform properties

Proposition 4.11. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 56.*
Let $\mathbf{B}$ *be the* TWO-SIDED LAPLACE TRANSFORM *defined as* $[\mathbf{B}\mathsf{f}](s) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathsf{d}x$.

PRP

1.	$\mathbf{B}\mathbf{T}^n = e^{-sn}\mathbf{B}$			$\forall n\in\mathbb{Z}$	
2.	$\mathbf{B}\mathbf{D}^j = \mathbf{D}^{-j}\mathbf{B}$			$\forall j\in\mathbb{Z}$	
3.	$\mathbf{D}\mathbf{B} = \mathbf{B}\mathbf{D}^{-1}$			$\forall n\in\mathbb{Z}$	
4.	$\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B}$	$=$	$\mathbf{D}$	$\forall n\in\mathbb{Z}$	($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5.	$\mathbf{D}\mathbf{B}\mathbf{D} = \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1}$	$=$	$\mathbf{B}$	$\forall n\in\mathbb{Z}$	

✎PROOF:

$$\begin{aligned}
\mathbf{B}\mathbf{T}^n\mathsf{f}(x) &= \mathbf{B}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 4.3 page 56)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathsf{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathsf{d}u && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathsf{d}u\right]\\
&= e^{-sn}\,\mathbf{B}\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}\mathbf{D}^j\mathsf{f}(x) &= \mathbf{B}\left[2^{j/2}\mathsf{f}\left(2^jx\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 56)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\mathsf{f}\left(2^jx\right)\right]e^{-sx}\,\mathsf{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathsf{d}u && \text{let } u \triangleq 2^jx \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathsf{d}u\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathsf{d}u\right] && \text{by Proposition 4.6 page 58 and} && \text{Proposition 4.7 page 59}\\
&= \mathbf{D}^{-j}\,\mathbf{B}\,\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{D}\mathbf{B}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathsf{d}x\right] && \text{by definition of } \mathbf{B}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathsf{d}x && \text{by definition of } \mathbf{D} && \text{(Definition 4.3 page 56)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\frac{1}{2}\,\mathsf{d}u && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{u}{2}\right)\right]e^{-su}\,\mathsf{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\,e^{-su}\,\mathsf{d}u && \text{by Proposition 4.6 page 58 and} && \text{Proposition 4.7 page 59}\\
&= \mathbf{B}\mathbf{D}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{B}\mathbf{D} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition G.3 page 146)}\\
\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} &= \mathbf{D}\mathbf{B}\mathbf{B}^{-1} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition G.3 page 146)}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{DBD} &= \mathbf{DD}^{-1}\mathbf{B} && \text{by previous result} \\
&= \mathbf{B} && \text{by definition of operator inverse} && \text{(Definition G.3 page 146)} \\
\mathbf{D}^{-1}\mathbf{BD}^{-1} &= \mathbf{D}^{-1}\mathbf{DB} && \text{by previous result} \\
&= \mathbf{B} && \text{by definition of operator inverse} && \text{(Definition G.3 page 146)}
\end{aligned}
$$

✎

Corollary 4.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 56. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 44) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition C.1 page 111).*

COR

1.	$\tilde{\mathbf{F}}\mathbf{T}^n$	$=$	$e^{-i\omega n}\tilde{\mathbf{F}}$		
2.	$\tilde{\mathbf{F}}\mathbf{D}^j$	$=$	$\mathbf{D}^{-j}\tilde{\mathbf{F}}$		
3.	$\mathbf{D}\tilde{\mathbf{F}}$	$=$	$\tilde{\mathbf{F}}\mathbf{D}^{-1}$		
4.	$\mathbf{D}$	$=$	$\tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1}$	$=$	$\tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}}$
5.	$\tilde{\mathbf{F}}$	$=$	$\mathbf{D}\tilde{\mathbf{F}}\mathbf{D}$	$=$	$\mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}$

✎PROOF: These results follow directly from Proposition 4.11 page 62 with $\tilde{\mathbf{F}} = \mathbf{B}|_{s=i\omega}$. ✎

Proposition 4.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 4.3 page 56. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 44) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition C.1 page 111).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = \frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{\mathsf{f}}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$
\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n\mathsf{f}(x) && \text{by Corollary 4.1 page 63 (3)} \\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}\mathsf{f}(x) && \text{by Corollary 4.1 page 63 (3)} \\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathsf{f}}(\omega) \\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{\mathsf{f}}\left(2^{-j}\omega\right) && \text{by Proposition 4.2 page 57}
\end{aligned}
$$

✎

Proposition 4.13. *Let* $\mathbf{T}$ *be the translation operator (Definition 4.3 page 56). Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 44) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\breve{a}(\omega)$ *be the* DTFT *(Definition 6.1 page 77) of a sequence* $(a_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$ *(Definition F.2 page 131).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{a}(\omega)\tilde{\phi}(\omega) \qquad \forall (a_n)\in\ell^2_{\mathbb{R}}, \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

$$
\begin{aligned}
\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\tilde{\mathbf{F}}\phi(x) && \text{by Corollary 4.1 page 63} \\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega) \\
&= \breve{a}(\omega)\tilde{\phi}(\omega) && \text{by definition of } \textit{DTFT} \text{ (Definition 6.1 page 77)}
\end{aligned}
$$

✎

Definition 4.4. *Let* $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition C.1 page 111). Let* $\ell^2_{\mathbb{R}}$ *be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition C.1 page 111).*

DEF: $\mathbf{S}$ *is the* ***sampling operator*** *in* $\ell^{2}_{\mathbb{R}}{}^{L^2_{\mathbb{R}}}$ *if* $[\mathbf{S}\mathsf{f}(x)](n) \triangleq \mathsf{f}\left(\frac{2\pi}{\tau}n\right) \qquad \forall \mathsf{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)},\ \tau\in\mathbb{R}^+$

Theorem 4.2 (Poisson Summation Formula—PSF). [6] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition 3.2 page 44) of a function* $\mathsf{f}(x) \in L^2_{\mathbb{R}}$. *Let* $\mathbf{S}$ *be the* SAMPLING OPERATOR *(Definition 4.4 page 64).*

THM:
$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_\tau \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)$ then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:

   Note that $\mathsf{h}(x)$ is *periodic* with period τ. Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of PSF (this theorem—Theorem 4.2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) && \text{by (1) lemma page 64}\\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \left(\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,\mathsf{d}x\right]}_{\hat{\mathbf{F}}[\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition 5.1 page 73)}\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau \mathsf{f}(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,\mathsf{d}x\right]\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,\mathsf{d}u\right] && \text{where } u \triangleq x+n\tau \implies x = u-n\tau\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi kn}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}ku}\,\mathsf{d}u\right]\\
&= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,\mathsf{d}u\right]}_{[\tilde{\mathbf{F}}\mathsf{f}]\left(\frac{2\pi}{\tau}k\right)} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 5.1 page 74)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}\mathsf{f}(x)]\left(\frac{2\pi}{\tau}k\right)\right] && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 3.2 page 44)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \mathbf{S} \quad \text{(Definition 4.4 page 64)}\\
&= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 5.1 page 74)}
\end{aligned}$$

✏

[6] Andrews et al. (2001) page 624, Knapp (2005b) page 389, Lasser (1996) page 254, Rudin (1987) pages 194–195, Folland (1992) page 337

Theorem 4.3 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition 3.2 page 44) *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM
$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)}_{\text{summation in “frequency”}} = \underbrace{\frac{\tau}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\text{summation in “time”}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)$, then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:
   Note that $\mathsf{h}(\omega)$ is periodic with period $2\pi/T$:
   $$\mathsf{h}\left(\omega + \frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau} + \frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + (n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \triangleq \mathsf{h}(\omega)$$
   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of IPSF (this theorem—Theorem 4.3):

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \\
&= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) && \text{by (1) lemma page 65} \\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\sqrt{\frac{\tau}{2\pi}}\int_0^{\frac{2\pi}{\tau}} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\,\mathsf{d}\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)\right]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition 5.1 page 73)} \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{\tau}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega Tk}\,\mathsf{d}\omega\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-i\left(u-\frac{2\pi}{\tau}n\right)Tk}\,\mathsf{d}u\right] && \text{where } u \triangleq \omega + \frac{2\pi}{\tau}n \implies \omega = u - \frac{2\pi}{\tau}n \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi nk}}\int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathsf{d}u\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathsf{d}u\right] \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{iu(-\tau k)}\,\mathsf{d}u\right]}_{[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)\right] && \text{by value of } \tilde{\mathbf{F}}^{-1} \quad \text{(Theorem 3.1 page 45)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}} && \text{by definition of } \mathbf{S} \quad \text{(Definition 4.4 page 64)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\mathsf{f}(x) && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 3.2 page 44)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathsf{f}(-k\tau) && \text{by definition of } \mathbf{S} \quad \text{(Definition 4.4 page 64)} \\
&= \sqrt{\tau}\,\frac{1}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau)e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} && \text{by definition of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 5.1 page 74)}
\end{aligned}$$

[7] Gauss (1900) page 88

$$= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}} \sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau)e^{ik\tau\omega} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem 5.1 page 74)}$$

$$= \frac{\tau}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \mathsf{f}(m\tau)e^{-i\omega m\tau} \qquad \text{let } m \triangleq -k$$

⇰

Remark 4.2. The left hand side of the *Poisson Summation Formula* (Theorem 4.2 page 64) is very similar to the *Zak Transform* **Z**: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark 4.3. A generalization of the *Poisson Summation Formula* (Theorem 4.2 page 64) is the **Selberg Trace Formula**.[9]

4.7 Basis theory properties

Example 4.2 (linear functions). [10] Let **T** be the *translation operator* (Definition 4.3 page 56). Let $\mathscr{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX	
1. $\{x, \mathbf{T}x\}$ is a *basis* for $\mathscr{L}(\mathbb{C},\mathbb{C})$	and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x$	$\forall \mathsf{f} \in \mathscr{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition 4.3 page 56}\\
&= (ax+b)|_{x=1}\,x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of f}\\
&= (a+b)x - b(x-1)\\
&= ax+bx-bx+b\\
&= ax+b\\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of f}
\end{aligned}$$

⇰

Example 4.3 (Cardinal Series). Let **T** be the *translation operator* (Definition 4.3 page 56). The *Paley-Wiener* class of functions $\boldsymbol{PW}^2_\sigma$ are those functions which are "*bandlimited*" with respect to their Fourier transform (Definition 3.2 page 44). The cardinal series forms an orthogonal basis for such a space. The *Fourier coefficients* for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of $\mathsf{f}(x)$ taken at regular intervals. In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \,|\, \mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\,\mathsf{d}x \triangleq \mathsf{f}(n)$$

EX	
1. $\left\{ \mathbf{T}^n \frac{\sin(\pi x)}{\pi x} \middle\vert_{n\in\mathbb{N}} \right\}$ is a *basis* for $\boldsymbol{PW}^2_\sigma$	and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin(\pi x)}{\pi x}}_{\text{Cardinal series}}$	$\forall \mathsf{f} \in \boldsymbol{PW}^2_\sigma,\ \sigma \le \frac{1}{2}$

[8] Janssen (1988) page 24, Zayed (1996) page 482
[9] Lax (2002) page 349, Selberg (1956), Terras (1999)
[10] Higgins (1996) page 2 ⟨1.1 General introduction⟩

Example 4.4 (Fourier Series).

EX

(1). $\left\{ \mathbf{D}_n e^{ix} \middle| n \in \mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and

(2). $$\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix} \qquad \forall x \in (0:2\pi), \mathsf{f} \in \boldsymbol{L}(0:2\pi) \quad \text{where}$$

$$\alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x) \mathbf{D}_n e^{-ix} \,\mathsf{d}x \qquad \forall \mathsf{f} \in \boldsymbol{L}(0:2\pi)$$

✎PROOF: See Theorem 5.1 page 74. ✏

Example 4.5 (Fourier Transform). [11]

EX

(1). $\left\{ \mathbf{D}_\omega e^{ix} | \omega \in \mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and

(2). $$\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) \mathbf{D}_x e^{i\omega} \,\mathsf{d}\omega \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \quad \text{where}$$

$$\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \mathbf{D}_\omega e^{-ix} \,\mathsf{d}x \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$$

Example 4.6 (Gabor Transform). [12]

EX

(1). $\left\{ \left(\mathbf{T}_\tau e^{-\pi x^2} \right) \left(\mathbf{D}_\omega e^{ix} \right) \middle| \tau, \omega \in \mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and

(2). $$\mathsf{f}(x) = \int_{\mathbb{R}} \mathsf{G}(\tau, \omega) \mathbf{D}_x e^{i\omega} \,\mathsf{d}\omega \qquad \forall x \in \mathbb{R}, \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \quad \text{where}$$

$$\mathsf{G}(\tau, \omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x) \left(\mathbf{T}_\tau e^{-\pi x^2} \right) \left(\mathbf{D}_\omega e^{-ix} \right) \,\mathsf{d}x \qquad \forall x \in \mathbb{R}, \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$$

Example 4.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX

(1). $\left\{ \mathbf{D}^k \mathbf{T}^n \psi(x) \middle| k, n \in \mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and

(2). $$\mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha_{k,n} \mathbf{D}^k \mathbf{T}^n \psi(x) \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \quad \text{where}$$

$$\alpha_n \triangleq \int_{\mathbb{R}} \mathsf{f}(x) \mathbf{D}^k \mathbf{T}^n \psi^*(x) \,\mathsf{d}x \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$$

[11] cross reference: Definition 3.2 page 44

[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

4.8 Cardinal Series and Sampling

4.8.1 Cardinal series basis

The *Paley-Wiener* class of functions (next definition) are those with a bandlimited Fourier transform. The cardinal series forms an orthogonal basis for such a space (Theorem 4.5 page 68). In a *frame* $(\boldsymbol{x}_n)_{n\in\mathbb{Z}}$ with *frame operator* $\mathbf{S}$ on a *Hilbert Space* $\boldsymbol{H}$ with *inner product* $\langle\triangle\,|\,\nabla\rangle$, a function $\mathsf{f}(x)$ in the space spanned by the frame can be represented by

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \underbrace{\left\langle \mathsf{f}\,|\,\mathbf{S}^{-1}\boldsymbol{x}_n\right\rangle}_{\text{“Fourier coefficient”}} \boldsymbol{x}_n.$$

If the frame is *orthonormal* (giving an *orthonormal basis*), then $\mathbf{S} = \mathbf{S}^{-1} = \mathbf{I}$ and

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f}\,|\,\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n.$$

In the case of the cardinal series, the *Fourier coefficients* are particularly simple—these coefficients are samples of f taken at regular intervals (Theorem 4.6 page 69). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x)\,|\,\delta(x-n\tau)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n\tau)\,\mathrm{d}t \triangleq \mathsf{f}(n\tau)$$

Definition 4.5. [13]

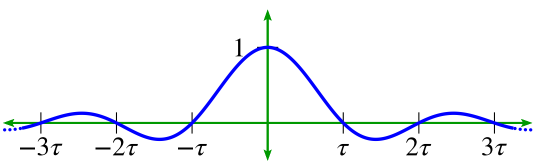
DEF: *A function* $\mathsf{f}\in\mathbb{C}^{\mathbb{C}}$ *is in the* ***Paley-Wiener*** *class of functions* $\boldsymbol{PW}_\sigma^p$ *if there exists* $\mathsf{F}\in\boldsymbol{L}^p(-\sigma:\sigma)$ *such that*

$$\mathsf{f}(x) = \int_{-\sigma}^{\sigma} \mathsf{F}(\omega)e^{ix\omega}\,\mathrm{d}\omega \qquad (\mathsf{f} \textit{ has a BANDLIMITED Fourier transform } \mathsf{F} \textit{ with bandwidth } \sigma)$$

for $p\in[1:\infty)$ *and* $\sigma\in(0:\infty)$.

Theorem 4.4 (Paley-Wiener Theorem for Functions). [14] *Let* f *be an* ENTIRE FUNCTION *(the domain of* f *is the entire complex plane* $\mathbb{C}$*). Let* $\sigma\in\mathbb{R}^+$.

THM:
$$\left\{\mathsf{f}\in\boldsymbol{PW}_\sigma^2\right\} \iff \left\{\begin{array}{lll} 1. & \exists C\in\mathbb{R}^+ \text{ such that } |\mathsf{f}(z)|\le Ce^{\sigma|z|} & (\text{EXPONENTIAL TYPE}) \text{ and} \\ 2. & \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \end{array}\right\}$$



Theorem 4.5 (Cardinal sequence). [15]

THM:
$$\left\{\frac{1}{\tau}\ge 2\sigma\right\} \implies \textit{The sequence } \left(\!\!\left(\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)}\right)\!\!\right)_{n\in\mathbb{Z}} \textit{ is an ORTHONORMAL BASIS for } \boldsymbol{PW}_\sigma^2.$$

[13] Higgins (1996) page 52 ⟨Definition 6.15⟩

[14] Boas (1954) page 103 ⟨6.8.1 Theorem of Paley and Wiener⟩, Katznelson (2004) page 212 ⟨7.4 Theorem⟩, Zygmund (2002) pages 272–273 ⟨(7·2) Theorem of Paley-Wiener⟩, Yosida (1980) page 161, Rudin (1987) page 375 ⟨19.3 Theorem⟩, Young (2001) page 85 ⟨Theorem 18⟩

[15] Higgins (1996) page 52 ⟨Definition 6.15⟩, Hardy (1941) ⟨*orthonormality*⟩, Higgins (1985) page 56 ⟨H1.; historical notes⟩

Theorem 4.6 (Sampling Theorem). [16]

THM

$$\left\{\begin{array}{ll} 1. & \mathsf{f} \in \boldsymbol{PW}_\sigma^2 \quad \textit{and} \\ 2. & \dfrac{1}{\tau} \geq 2\sigma \end{array}\right\} \implies \mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)}}_{\text{CARDINAL SERIES}}.$$

✎PROOF:

$$\text{Let} \qquad \mathsf{s}(x) \triangleq \frac{\sin\left[\frac{\pi}{\tau}x\right]}{\frac{\pi}{\tau}x} \iff \tilde{\mathsf{s}}(\omega) = \begin{cases} \tau & : |f| \leq \frac{1}{2\tau} \\ 0 & : \text{otherwise} \end{cases}$$

1. Proof that the set is *orthonormal*: see Hardy (1941)
2. Proof that the set is a *basis*:

$$\begin{aligned}
\mathsf{f}(x) &= \int_\omega \tilde{\mathsf{f}}(\omega)e^{i\omega t}\, \mathrm{d}\omega && \text{by } \textit{inverse Fourier transform} && \text{(Theorem 3.1 page 45)}\\
&= \int_\omega \mathbf{T}\tilde{\mathsf{f}}_\mathsf{d}(\omega)\tilde{\mathsf{s}}(\omega)e^{i\omega t}\, \mathrm{d}\omega && \text{if } W \leq \frac{1}{2T} \\
&= \mathbf{T}\mathsf{f}_\mathsf{d}(x) \star \mathsf{s}(x) && \text{by } \textit{Convolution theorem} && \text{(Theorem F.2 page 134)}\\
&= \mathbf{T}\int_u [\mathsf{f}_\mathsf{d}(u)]\mathsf{s}(x-u)\, \mathrm{d}u && \text{by } \textit{convolution} \text{ definition} && \text{(Definition D.1 page 115)}\\
&= \mathbf{T}\int_u \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(u)\delta(u-n\tau)\right] \mathsf{s}(x-u)\, \mathrm{d}u && \text{by } \textit{sampling} \text{ definition} && \text{(Theorem 4.7 page 70)}\\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \int_u \mathsf{f}(u)\mathsf{s}(x-u)\delta(u-n\tau)\, \mathrm{d}u \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\mathsf{s}(x-n\tau) && \text{by prop. of } \textit{Dirac delta} \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)} && \text{by definition of } \mathsf{s}(x)
\end{aligned}$$

✏

4.8.2 Sampling

Definition 4.6. [17] *Let $\delta(x)$ be the* DIRAC DELTA *distribution.*

DEF

*The **Shah Function** $Ш(x)$ is defined as* $Ш(x) \triangleq \sum_{n\in\mathbb{Z}} \delta(x-n)$

[16] Whittaker (1915), Kotelnikov (1933), Whittaker (1935), Shannon (1948) ⟨Theorem 13⟩, Shannon (1949) page 11 II (1991) page 1, Nashed and Walter (1991), Higgins (1996) page 5, Young (2001) pages 90–91 ⟨THE PALEY-WIENER SPACE⟩, Papoulis (1980) pages 418–419 ⟨The Sampling Theorem⟩. The *sampling theorem* was "discovered" and published by multiple people: Nyquist in 1928 (DSP?), Whittaker in 1935 (interpolation theory), and Shannon in 1949 (communication theory). references: Mallat (1999) page 43, Oppenheim and Schafer (1999) page 143.

[17] Bracewell (1978) page 77 ⟨The sampling or replicating symbol Ш(x)⟩, Córdoba (1989)191. Note: The symbol Ш is the Cyrillic upper case "sha" character, which has been assigned Unicode location U+0428. Reference: http://unicode.org/cldr/utility/character.jsp?a=0428

If $\mathsf{f}_\mathsf{d}(x)$ is the function $\mathsf{f}(x)$ sampled at rate $1/T$, then $\tilde{\mathsf{f}}_\mathsf{d}(\omega)$ is simply $\tilde{\mathsf{f}}(\omega)$ *replicated* every $1/T$ Hertz and *scaled* by $1/T$. This is proven in Theorem 4.7 (next) and illustrated in Figure 4.1 (page 70).

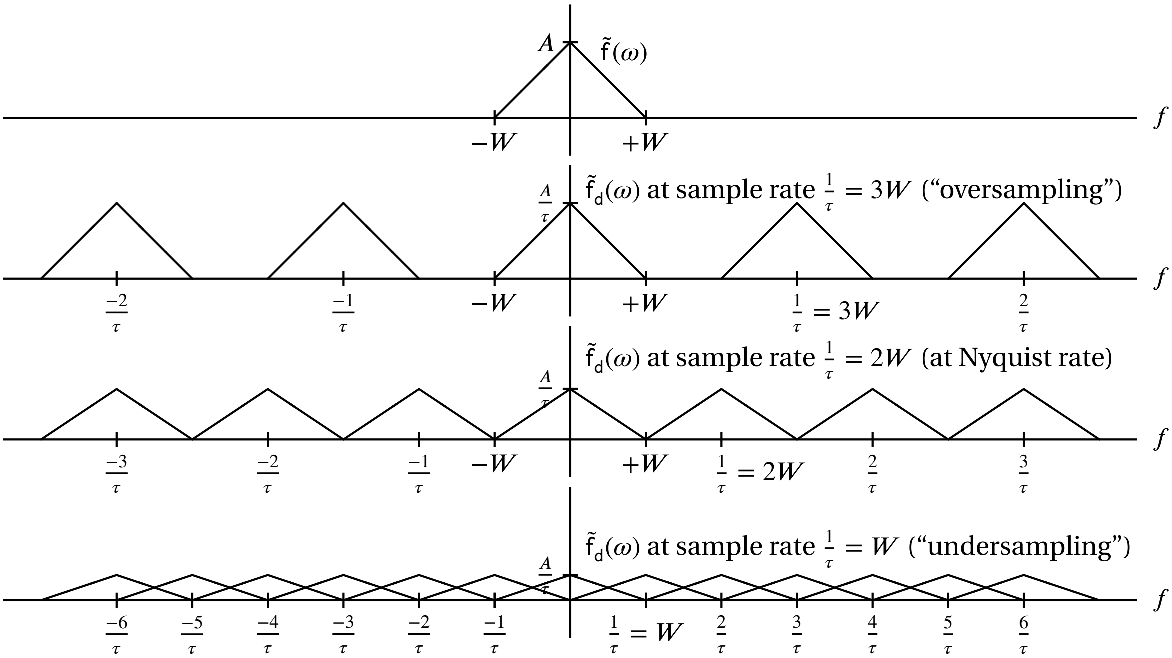


Figure 4.1: Sampling in frequency domain

Theorem 4.7. *Let* $\mathsf{f}, \mathsf{f}_\mathsf{d} \in \boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}, \tilde{\mathsf{f}}_\mathsf{d} \in \boldsymbol{L}^2_{\mathbb{R}}$ *be their respective fourier transforms. Let* $\mathsf{f}_\mathsf{d}(x)$ *be the* ***sampled*** $\mathsf{f}(x)$ *such that*

$$\mathsf{f}_\mathsf{d}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x - n\tau).$$

THM
$$\left\{ \mathsf{f}_\mathsf{d}(x) \triangleq \mathsf{f}(x)Ш(x) \triangleq \mathsf{f}(x) \sum_{n\in\mathbb{Z}} \delta(x - n\tau) \right\} \implies \left\{ \tilde{\mathsf{f}}_\mathsf{d}(\omega) = \frac{2\pi}{\tau} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right) \right\}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{f}}_\mathsf{d}(\omega) &\triangleq \int_t \mathsf{f}_\mathsf{d}(x)e^{-i\omega t}\,\mathsf{d}t \\
&= \int_t \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x-n\tau)\right] e^{-i\omega t}\,\mathsf{d}t \\
&= \sum_{n\in\mathbb{Z}} \int_t \mathsf{f}(x)\delta(x-n\tau)e^{-i\omega t}\,\mathsf{d}t \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau} && \text{by definition of } \delta \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) && \text{by } \textit{IPSF} \quad \text{(Theorem 4.3 page 65)} \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)
\end{aligned}$$

⇨

Suppose a waveform $\mathsf{f}(x)$ is sampled at every time T generating a sequence of sampled values $\mathsf{f}(n\tau)$. Then in general, we can *approximate* $\mathsf{f}(x)$ by using interpolation between the points $\mathsf{f}(n\tau)$. Interpolation can be performed using several interpolation techniques.

In general all techniques lead only to an approximation of $\mathsf{f}(x)$. However, if $\mathsf{f}(x)$ is *bandlimited* with bandwidth $W \le \frac{1}{2T}$, then $\mathsf{f}(x)$ is *perfectly reconstructed* (not just approximated) from the sampled values $\mathsf{f}(n\tau)$ (Theorem 4.6 page 69).

CHAPTER 5

FOURIER SERIES

> ❝ ...et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur.❞


MÉMOIRE

SUR LA

PROPAGATION DE LA CHALEUR

DANS LES CORPS SOLIDES.

Par M. FOURIER (¹).

L'auteur de ce Mémoire s'est proposé de soumettre la Théorie de la chaleur à l'Analyse mathématique et de vérifier, par l'expérience, les résultats du calcul. Pour exposer l'état de la question, supposons une barre de métal, cylindrique et d'une longueur indéterminée, plongée par une de ses extrémités dans un fluide entretenu à une température constante : la chaleur se répandra successivement dans la barre; et, sans la perte qui a lieu à sa surface et à son autre extrémité, elle prendrait dans toute son étendue la température constante du foyer; mais, à cause de cette perte, la chaleur ne s'étendra d'une manière sensible que jusqu'à une distance du foyer dépendante de la grosseur de la barre, de la conductibilité du métal et de son degré de poli, qui influe sur le rayonnement; de sorte que des thermomètres placés dans l'écar-

> ❝ ...and the innovation of the subject, together with its importance, convinced the committee to crown this work. By observing however that the way in which the author arrives at his equations is not free from difficulties, and the analysis of which, to integrate them, still leaves something to be desired, either relative to generality, or even on the side of rigour.❞

A competition awards committee consisting of the mathematical giants Lagrange, Laplace, Legendre, and others, commenting on Fourier's 1807 landmark paper *Dissertation on the propagation of heat in solid bodies* that introduced the *Fourier Series*. [1]

5.1 Definition

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition 5.1. [2]

DEF *The* ***Fourier Series operator*** $\hat{\mathbf{F}} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{R}}$ *is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{f}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{d}x \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{f} \text{ is periodic with period } \tau\right\}$$

[1] quote: Lagrange et al. (1812b) page 374, Lagrange et al. (1812a) page 112, Kahane (2008) page 199
translation: assisted by Google Translate, Castanedo (2005) ⟨chapter 2 footnote 5⟩
paper: Fourier (1807)

[2] Katznelson (2004) page 3

5.2 Inverse Fourier Series operator

Theorem 5.1. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

THM

The ***inverse Fourier Series*** *operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{x}}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)\in\ell^2_{\mathbb{R}}$$

✎PROOF: The proof of the pointwise convergence of the Fourier Series is notoriously difficult. It was conjectured in 1913 by Nokolai Luzin that the Fourier Series for all square summable periodic functions are pointwise convergent: Luzin (1913)

Fifty-three years later (1966) at a conference in Moscow, Lennart Axel Edvard Carleson presented one of the most spectacular results ever in mathematics; he demonstrated that the Luzin conjecture is indeed correct. Carleson formally published his result that same year: Carleson (1966)

Carleson's proof is expounded upon in Reyna's (2002) 175 page book: de Reyna (2002)

Interestingly enough, Carleson started out trying to disprove Luzin's conjecture. Carleson said this in an interview published in 2001:[3] ❝*Well, the problem of course presents itself already when you are a student and I was thinking of the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I suddenly realized why there should be a counter-example and how you should produce it. I thought I really understood what was the background and then to my amazement I could prove that this "correct" counter-example couldn't exist and therefore I suddenly realized that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. The most important aspect in solving a mathematical problem is the conviction of what is the true result! Then it took like 2 or 3 years using the technique that had been developed during the past 20 years or so. It is actually a problem related to analytic functions basically even though it doesn't look that way.*❞

For now, if you just want some intuitive justification for the Fourier Series, and you can somehow imagine that the Dirichlet kernel generates a *comb function* of *Dirac delta* functions, then perhaps what follows may help (or not). It is certainly not mathematically rigorous and is by no means a real proof (but at least it is less than 175 pages).

$$\begin{aligned}
\left[\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{x}\right](x) &= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{d}x\right]}_{\hat{\mathbf{F}}\mathsf{x}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 5.1 page 73)}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}\,\mathsf{d}u\right]e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \hat{\mathbf{F}}^{-1}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}e^{i\frac{2\pi}{\tau}nx}\,\mathsf{d}u\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{i\frac{2\pi}{\tau}n(x-u)}\,\mathsf{d}u
\end{aligned}$$

[3] Carleson and Engquist (2001)

$$
\begin{aligned}
&= \int_0^\tau \mathsf{x}(u) \underbrace{\frac{1}{\tau}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi}{\tau}n(x-u)}}_{\lim_{N\to\infty}\mathsf{D}_n(x)} \, \mathsf{d}u \\
&= \int_0^\tau \mathsf{x}(u) \left[\sum_{n\in\mathbb{Z}} \delta(x-u-n\tau)\right] \mathsf{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{u=0}^{u=\tau} \mathsf{x}(u)\delta(x-u-n\tau)\, \mathsf{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{v-n\tau=0}^{v-n\tau=\tau} \mathsf{x}(v-n\tau)\delta(x-v)\, \mathsf{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v-n\tau)\delta(x-v)\, \mathsf{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v)\delta(x-v)\, \mathsf{d}v && \text{because } \mathsf{x} \text{ is periodic with period } \tau \\
&= \int_{\mathbb{R}} \mathsf{x}(v)\delta(x-v)\, \mathsf{d}v \\
&= \mathsf{x}(x) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} \quad \text{(Definition G.3 page 146)}
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](n) &= \hat{\mathbf{F}}\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) e^{i\frac{2\pi}{\tau}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau \left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) e^{i\frac{2\pi}{\tau}kx}\right] e^{-i\frac{2\pi}{\tau}nx}\, \mathsf{d}x && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition 5.1 page 73)} \\
&= \frac{1}{\tau}\int_0^\tau \left[\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) e^{i\frac{2\pi}{\tau}(k-n)x}\right] \mathsf{d}x \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) \left[\frac{1}{\tau}\int_0^\tau e^{i\frac{2\pi}{\tau}(k-n)x}\, \mathsf{d}x\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) \frac{1}{\tau}\left[\frac{1}{i\frac{2\pi}{\tau}(k-n)} e^{i\frac{2\pi}{\tau}(k-n)x}\right]_0^\tau \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k) \frac{1}{i2\pi(k-n)}\left[e^{i2\pi(k-n)} - 1\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\, \bar{\delta}(k-n) \lim_{x\to 0}\left[\frac{e^{i2\pi x}-1}{i2\pi x}\right] \\
&= \tilde{\mathsf{x}}(n) \left.\frac{\frac{\mathsf{d}}{\mathsf{d}x}\left(e^{i2\pi x}-1\right)}{\frac{\mathsf{d}}{\mathsf{d}x}(i2\pi x)}\right|_{x=0} && \text{by \textit{l'Hôpital's rule}} \\
&= \tilde{\mathsf{x}}(n) \left.\frac{i2\pi e^{i2\pi x}}{i2\pi}\right|_{x=0} \\
&= \tilde{\mathsf{x}}(n) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} \quad \text{(Definition G.3 page 146)}
\end{aligned}
$$

✎

Theorem 5.2.

THM *The **Fourier Series adjoint** operator $\hat{\mathbf{F}}^*$ is given by*

$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathsf{x}(x) \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{x}(x) e^{-i\frac{2\pi}{\tau}nx} \,\mathsf{d}x \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 5.1 page 73)}\\
&= \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{x}(x) \left\langle e^{-i\frac{2\pi}{\tau}nx} \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} \mathsf{d}x && \text{by additivity property of } \langle \triangle \,|\, \triangledown \rangle\\
&= \int_0^\tau \mathsf{x}(x) \frac{1}{\sqrt{\tau}} \left\langle \tilde{\mathsf{y}}(n) \,|\, e^{-i\frac{2\pi}{\tau}nx} \right\rangle_{\mathbb{Z}}^* \mathsf{d}x && \text{by property of } \langle \triangle \,|\, \triangledown \rangle\\
&= \int_0^\tau \mathsf{x}(x) \left[\hat{\mathbf{F}}^{-1}\tilde{\mathsf{y}}(n)\right]^* \mathsf{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem 5.1 page 74)}\\
&= \left\langle \mathsf{x}(x) \,|\, \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^*}\tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{R}}
\end{aligned}$$

✏

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary* [4] (Corollary 5.1 page 76).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary 5.2 page 76).

Corollary 5.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\hat{\mathbf{F}}$ *be the Fourier Series operator with adjoint* $\hat{\mathbf{F}}^*$.

COR: $\left\{\hat{\mathbf{F}}\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^*\hat{\mathbf{F}} = \mathbf{I}\right\}$ $\quad$ ($\hat{\mathbf{F}}$ *is* ***unitary*** *...and thus also* NORMAL *and* ISOMETRIC)

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$ (Theorem 5.2 page 75). ✏

Corollary 5.2. *Let* $\hat{\mathbf{F}}$ *be the Fourier series operator with adjoint* $\hat{\mathbf{F}}^*$ *and inverse* $\hat{\mathbf{F}}^{-1}$.

COR:

$$\begin{array}{rclcll}
\mathscr{R}(\hat{\mathbf{F}}) &=& \mathscr{R}(\hat{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
\left\|\left|\hat{\mathbf{F}}\right|\right\| &=& \left\|\left|\hat{\mathbf{F}}^{-1}\right|\right\| &=& 1 & \text{(UNITARY)}\\
\left\langle \hat{\mathbf{F}}\boldsymbol{x} \,|\, \hat{\mathbf{F}}\boldsymbol{y} \right\rangle &=& \left\langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \,|\, \hat{\mathbf{F}}^{-1}\boldsymbol{y} \right\rangle &=& \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle & \text{(PARSEVAL'S EQUATION)}\\
\left\|\hat{\mathbf{F}}\boldsymbol{x}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\right\| &=& \|\boldsymbol{x}\| & \text{(PLANCHEREL'S FORMULA)}\\
\left\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\right\| &=& \|\boldsymbol{x} - \boldsymbol{y}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary 5.1 page 76) and from the properties of unitary operators (Theorem G.26 page 170). ✏

5.3 Fourier series for compactly supported functions

Theorem 5.3.

THM: *The set*

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i\frac{2\pi}{\tau}nx} \,\middle|\, n\in\mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathsf{f}(x)$ *with support in* $[0 : \tau]$.

[4] *unitary operators*: Definition G.14 page 169

CHAPTER 6

DISCRETE TIME FOURIER TRANSFORM

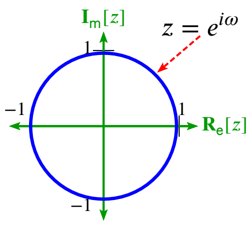
6.1 Definition

Definition 6.1.

DEF *The **discrete-time Fourier transform** $\check{\mathbf{F}}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$\left[\check{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition 6.1 page 77) to the definition of the Z-transform (Definition F.4 page 132), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.



6.2 Properties

Proposition 6.1 (DTFT periodicity). *Let* $\check{x}(\omega) \triangleq \check{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM (*Definition 6.1 page 77*) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP
$$\underbrace{\check{x}(\omega) = \check{x}(\omega + 2\pi n)}_{\text{PERIODIC with period } 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\check{x}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} &&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi nm}} \\
&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} &&= \check{x}(\omega)
\end{aligned}$$

⇨

Theorem 6.1. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 77) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

$$\text{THM}\quad \left\{\ \breve{x}(\omega) \triangleq \breve{\mathbf{F}}\,(\!(x[n])\!)\ \right\} \implies \left\{\begin{array}{llcll} (1). & \breve{\mathbf{F}}\,(\!(x[-n])\!) & = & \tilde{x}(-\omega) & \text{and} \\ (2). & \breve{\mathbf{F}}\,(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}\,(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 77)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}\\
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x^*[n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 77)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\textbf{-algebras} && \text{(Definition A.3 page 90)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{-i(-\omega)n}\right)^*\\
&\triangleq \tilde{x}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x^*[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 77)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[-n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\textbf{-algebras} && \text{(Definition A.3 page 90)}\\
&= \left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n = -m\\
&\triangleq \tilde{x}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

✏

Theorem 6.2. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 77) of a sequence* $(\!(x[n])\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

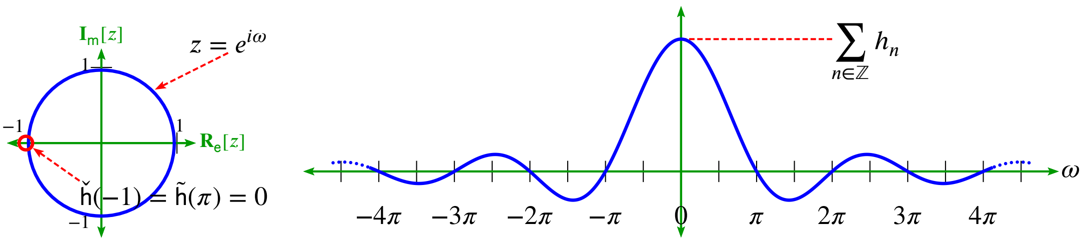
$$\text{THM}\quad \left\{\begin{array}{lll} (1). & \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\,(\!(x[n])\!) & \text{and} \\ (2). & (\!(x[n])\!) \text{ is REAL-VALUED} & \end{array}\right\} \implies \left\{\begin{array}{llcllll} (1). & \breve{\mathbf{F}}\,(\!(x[-n])\!) & = & \tilde{x}(-\omega) & & & \text{and} \\ (2). & \breve{\mathbf{F}}\,(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & = & \tilde{x}(\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}\,(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & = & \tilde{x}(-\omega) & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}\,(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 77)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}
\end{aligned}$$

$$
\begin{aligned}
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}\\[2ex]
\boxed{\tilde{x}^*(-\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(x^*[n])\!)} && \text{by Theorem 6.1 page 78}\\
&= \breve{\mathbf{F}}\,(\!(x[n])\!) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{x}(\omega)} && \text{by definition of } \tilde{x}(\omega) \qquad \text{(Definition 6.1 page 77)}\\[2ex]
\boxed{\tilde{x}^*(\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(x^*[-n])\!)} && \text{by Theorem 6.1 page 78}\\
&= \breve{\mathbf{F}}\,(\!(x[-n])\!) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{x}(-\omega)} && \text{by result (1)}
\end{aligned}
$$

✏



Proposition 6.2. *Let* $\breve{x}(z)$ *be the* Z-TRANSFORM (*Definition F.4 page 132*) *and* $\breve{x}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM (*Definition 6.1 page 77*) *of* $(\!(x_n)\!)$.

PRP

$$
\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\breve{x}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{x}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}, c\in\mathbb{R}
$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$
\begin{aligned}
\breve{x}(z)\Big|_{z=1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \text{by definition of } \breve{x}(z) \text{ (Definition F.4 page 132)}\\
&= \sum_{n\in\mathbb{Z}} x_n && \text{because } z^n = 1 \text{ for all } n\in\mathbb{Z}\\
&= c && \text{by hypothesis (1)}
\end{aligned}
$$

2. Proof that (2) $\implies$ (3):

$$
\begin{aligned}
\breve{x}(\omega)\Big|_{\omega=0} &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} && \text{by definition of } \breve{x}(\omega) \qquad \text{(Definition 6.1 page 77)}\\
&= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1}\\
&= \breve{x}(z)\Big|_{z=1} && \text{by definition of } \breve{x}(z) \qquad \text{(Definition F.4 page 132)}\\
&= c && \text{by hypothesis (2)}
\end{aligned}
$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=0} \\
&= \check{x}(\omega) && \text{by definition of } \check{x}(\omega) && \text{(Definition 6.1 page 77)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

✏

Proposition 6.3. *If the coefficients are **real**, then the magnitude response (MR) is **symmetric**.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{h}(-\omega)\right| &\triangleq \left|\check{h}(z)\right|_{z=e^{-i\omega}} &&\triangleq \left|\sum_{m\in\mathbb{Z}} x[m]z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} x[m]e^{i\omega m}\right| &&= \left|\left(\sum_{m\in\mathbb{Z}} x^*[m]e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } real} &&= \left|\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right| \\
&\triangleq \left|\check{h}(z)\right|_{z=e^{-i\omega}} &&\triangleq \left|\tilde{h}(\omega)\right|
\end{aligned}$$

✏

Proposition 6.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}}(-1)^n x_n = c}_{\text{(1) in “time”}} \iff \underbrace{\check{x}(z)|_{z=-1} = c}_{\text{(2) in “z domain”}} \iff \underbrace{\check{x}(\omega)|_{\omega=\pi} = c}_{\text{(3) in “frequency”}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{x}(z)|_{z=-1} &= \left.\sum_{n\in\mathbb{Z}} x_n z^{-n}\right|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \sum_{n\in\mathbb{Z}} z^{-n} x_n \Big|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n \Big|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\implies$ (4):

(a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

(b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

(e) Solutions to these equations give

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_{2n} \triangleq A &= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)\\
\sum_{n\in\mathbb{Z}} x_{2n+1} \triangleq B &= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)
\end{aligned}$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}}\\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1}\\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1}\\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)}\\
&= c
\end{aligned}$$

✏

Lemma 6.1. *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* DTFT (Definition 6.1 page 77) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

LEM

$$\underbrace{(x_n \in \mathbb{R})_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{\mathsf{x}}(\omega)|^2 = |\breve{\mathsf{x}}(-\omega)|^2}_{\text{EVEN}} \qquad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
|\breve{\mathsf{x}}(\omega)|^2 &= |\breve{\mathsf{x}}(z)|^2\Big|_{z=e^{i\omega}}\\
&= \breve{\mathsf{x}}(z)\breve{\mathsf{x}}^*(z)\Big|_{z=e^{i\omega}}\\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}}\\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^*(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}}\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}}\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right]\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right]\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right] \\
&= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2\sum_{n\in\mathbb{Z}}\sum_{m>n} x_n x_m \cos[\omega(m-n)]
\end{aligned}$$

Since cos is real and even, then $|\breve{x}(\omega)|^2$ must also be real and even. ✏

Theorem 6.3 (inverse DTFT). [2] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 77) of a sequence $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\breve{x}^{-1}$ be the inverse of $\breve{x}$.*

T H M

$$\underbrace{\left\{ \breve{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\breve{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x_n)\!)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega) e^{i\omega n}\, d\omega \quad \forall \alpha\in\mathbb{R} \right\}}_{(\!(x_n)\!) = \breve{\mathbf{F}}^{-1}\breve{\mathbf{F}}(\!(x_n)\!)} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega)e^{i\omega n}\, d\omega &= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m}\right]}_{\breve{x}(\omega)} e^{i\omega n}\, d\omega && \text{by definition of } \breve{x}(\omega) \\
&= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\, d\omega \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\, d\omega \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \left[2\pi\bar{\delta}_{m-n}\right] \\
&= x_n
\end{aligned}$$

✏

Theorem 6.4 (orthonormal quadrature conditions). [3] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 77) of a sequence $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\bar{\delta}_n$ be the* KRONECKER DELTA FUNCTION *at n.*

T H M

$$\begin{array}{lclcrcll}
\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} & = & 0 & \iff & \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) & = & 0 & \forall n\in\mathbb{Z}, \forall (\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}} \\
\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} & = & \bar{\delta}_n & \iff & |\breve{x}(\omega)|^2 + |\breve{x}(\omega+\pi)|^2 & = & 2 & \forall n\in\mathbb{Z}, \forall (\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}}
\end{array}$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

1. Proof that $2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)$:

$$\begin{aligned}
&2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} \\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n}
\end{aligned}$$

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩
[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

$$
\begin{aligned}
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n}\left(1+e^{i\pi n}\right) \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n \\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_m z^{m} + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_m e^{+i(\omega+\pi)m} \\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^* \\
&\triangleq \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi)
\end{aligned}
$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) = 0$:

$$
\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}
$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) = 0$:

$$
\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 0 && \text{by right hypothesis}
\end{aligned}
$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\check{x}(\omega)|^2 + |\check{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$
\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n} \\
&= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}
$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\check{x}(\omega)|^2 + |\check{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$
\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 2 && \text{by right hypothesis}
\end{aligned}
$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

6.3 Derivatives

Theorem 6.5. [4] *Let* $\breve{x}(\omega)$ *be the* DTFT (Definition 6.1 page 77) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

THM

$$\begin{array}{lrcll}
\text{(A)} & \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=0} = 0 & \iff & \sum_{k\in\mathbb{Z}} k^n x_k = 0 & \text{(B)} \quad \forall n\in\mathbb{W} \\
\text{(C)} & \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} = 0 & \iff & \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 & \text{(D)} \quad \forall n\in\mathbb{W}
\end{array}$$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=0} && \text{by hypothesis (A)} \\
&= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=0} && \text{by definition of } \breve{x}(\omega) \text{ (Definition 6.1 page 77)} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\Big|_{\omega=0} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=0} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=0} &= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=0} && \text{by definition of } \tilde{g} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\right]\Big|_{\omega=0} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=0} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k \\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

3. Proof that $(C) \implies (D)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} && \text{by hypothesis (C)} \\
&= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=\pi} && \text{by definition of } \breve{x} \text{ (Definition 6.1 page 77)} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\Big|_{\omega=\pi} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=\pi}
\end{aligned}$$

[4] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

$$= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]$$
$$= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k$$

4. Proof that $(C) \Longleftarrow (D)$:

$$\begin{aligned}
\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \check{\mathsf{x}}(\omega)\Big|_{\omega=\pi} &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Bigg|_{\omega=\pi} && \text{by definition of } \check{\mathsf{x}} \text{ (Definition 6.1 page 77)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega k}\Bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k\\
&= 0 && \text{by hypothesis (D)}
\end{aligned}$$

✎

APPENDIX A

NORMED ALGEBRAS

A.1 Algebras

All *linear spaces* are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition A.1. [3] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF: *An algebra $\boldsymbol{A}$ is **unital** if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition A.2. [4] *Let $\boldsymbol{A}$ be an* UNITAL ALGEBRA *(Definition A.1 page 89) with unit e.*

DEF:

$$\begin{array}{llcll}
\text{The \textbf{spectrum} of } x \in \boldsymbol{A} \text{ is} & \sigma(x) & \triangleq & \left\{\lambda \in \mathbb{C} \mid \lambda e - x \text{ is not invertible}\right\}. & \\
\text{The \textbf{resolvent} of } x \in \boldsymbol{A} \text{ is} & \rho_x(\lambda) & \triangleq & (\lambda e - x)^{-1} & \forall \lambda \notin \sigma(x). \\
\text{The \textbf{spectral radius} of } x \in \boldsymbol{A} \text{ is} & r(x) & \triangleq & \sup\{|\lambda| \mid \lambda \in \sigma(x)\}. &
\end{array}$$

[1] Fuchs (1995) page 2
[2] Hazewinkel (2000) page v
[3] Folland (1995) page 1
[4] Folland (1995) pages 3–4

A.2 Star-Algebras

Definition A.3. [5] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF

*The pair $(\boldsymbol{A}, *)$ is a **$*$-algebra**, or **star-algebra**, if*

1.	$(x+y)^* = x^* + y^*$	$\forall x,y \in \boldsymbol{A}$	(DISTRIBUTIVE)	and
2.	$(\alpha x)^* = \bar{\alpha} x^*$	$\forall x \in \boldsymbol{A}, \alpha \in \mathbb{C}$	(CONJUGATE LINEAR)	and
3.	$(xy)^* = y^* x^*$	$\forall x,y \in \boldsymbol{A}$	(ANTIAUTOMORPHIC)	and
4.	$x^{**} = x$	$\forall x \in \boldsymbol{A}$	(INVOLUTORY)	

*The operator $*$ is called an **involution** on the algebra $\boldsymbol{A}$.*

Proposition A.1. [6] *Let $(\boldsymbol{A}, *)$ be an* UNITAL $*$-ALGEBRA.

PRP

$$x \text{ is invertible} \implies \begin{cases} 1. & x^* \text{ is INVERTIBLE} \quad \forall x \in \boldsymbol{A} \quad \text{and} \\ 2. & (x^*)^{-1} = (x^{-1})^* \quad \forall x \in \boldsymbol{A} \end{cases}$$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= (x\,e^*)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= (x^*\,e^{**})^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= (x^*\,e)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)}\\
e^*\,x &= (e^*\,x)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= (e^{**}\,x^*)^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= (e\,x^*)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= (x^*)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)}
\end{aligned}$$

2. Proof that $(x^*)^{-1} = (x^{-1})^*$:

$$\begin{aligned}
(x^{-1})^*\,(x^*) &= \left[x\,(x^{-1})\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition A.3 page 90)}\\
&= e^*\\
&= e && \text{by item (1) page 90}\\
(x^*)\,(x^{-1})^* &= \left[x^{-1}\,x\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition A.3 page 90)}\\
&= e^*\\
&= e && \text{by item (1) page 90}
\end{aligned}$$

⇰

Definition A.4. [7] *Let $(\boldsymbol{A}, \|\cdot\|)$ be a* $*$-ALGEBRA *(Definition A.3 page 90).*

DEF

- *An element $x \in \boldsymbol{A}$ is **hermitian** or **self-adjoint** if $x^* = x$.*
- *An element $x \in \boldsymbol{A}$ is **normal** if $xx^* = x^*x$.*
- *An element $x \in \boldsymbol{A}$ is a **projection** if $xx = x$ (*INVOLUTORY*) and $x^* = x$ (*HERMITIAN*).*

[5] Rickart (1960) page 178, Gelfand and Naimark (1964), page 241

[6] Folland (1995) page 5

[7] Rickart (1960) page 178, Gelfand and Naimark (1964), page 242

Theorem A.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (Definition A.3 page 90).

THM

$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are HERMITIAN}} \implies \begin{cases} x + y = (x+y)^* & (x+y \text{ is self adjoint}) \\ x^* = (x^*)^* & (x^* \text{ is self adjoint}) \\ \underbrace{xy = (xy)^*}_{(xy) \text{ is HERMITIAN}} \iff \underbrace{xy = yx}_{\text{commutative}} & \end{cases}$$

✎PROOF:

$$\begin{aligned} (x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 90)} \\ &= x + y && \text{by left hypothesis} \end{aligned}$$

$$(x^*)^* = x \qquad \text{by } \textit{involutory} \text{ property of } * \qquad \text{(Definition A.3 page 90)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned} xy &= (xy)^* && \text{by left hypothesis} \\ &= y^* x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition A.3 page 90)} \\ &= yx && \text{by left hypothesis} \end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned} (xy)^* &= (yx)^* && \text{by left hypothesis} \\ &= x^* y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition A.3 page 90)} \\ &= xy && \text{by left hypothesis} \end{aligned}$$

✏

Definition A.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (Definition A.3 page 90).

DEF

The ***real part*** *of* x *is defined as* $\mathbf{R}_{\mathrm{e}} x \triangleq \frac{1}{2}\left(x + x^*\right)$

The ***imaginary part*** *of* x *is defined as* $\mathbf{I}_{\mathrm{m}} x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem A.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA (Definition A.3 page 90).

THM

$$\begin{aligned} \mathbf{R}_{\mathrm{e}} x &= \left(\mathbf{R}_{\mathrm{e}} x\right)^* && \forall x \in \boldsymbol{A} && (\mathbf{R}_{\mathrm{e}} x \text{ is HERMITIAN}) \\ \mathbf{I}_{\mathrm{m}} x &= \left(\mathbf{I}_{\mathrm{m}} x\right)^* && \forall x \in \boldsymbol{A} && (\mathbf{I}_{\mathrm{m}} x \text{ is HERMITIAN}) \end{aligned}$$

✎PROOF:

$$\begin{aligned} \left(\mathbf{R}_{\mathrm{e}} x\right)^* &= \left(\frac{1}{2}(x + x^*)\right)^* && \text{by definition of } \mathfrak{R} && \text{(Definition A.5 page 91)} \\ &= \frac{1}{2}(x^* + x^{**}) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 90)} \\ &= \frac{1}{2}(x^* + x) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)} \\ &= \mathbf{R}_{\mathrm{e}} x && \text{by definition of } \mathfrak{R} && \text{(Definition A.5 page 91)} \\ \left(\mathbf{I}_{\mathrm{m}} x\right)^* &= \left(\frac{1}{2i}(x - x^*)\right)^* && \text{by definition of } \mathfrak{I} && \text{(Definition A.5 page 91)} \end{aligned}$$

[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998) page 42

$$
\begin{aligned}
&= \frac{1}{2i}\left(x^* - x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= \frac{1}{2i}\left(x^* - x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition A.3 page 90)}\\
&= \mathbf{I}_{\mathsf{m}} x && \text{by definition of } \Im && \text{(Definition A.5 page 91)}
\end{aligned}
$$

✏

Theorem A.3 (Hermitian representation).[11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition A.3 page 90).*

THM	$a = x + iy \quad \Longleftrightarrow \quad x = \mathbf{R}_{\mathsf{e}} a \quad \textit{and} \quad y = \mathbf{I}_{\mathsf{m}} a$

✎PROOF:

Proof that $a = x + iy \implies x = \mathbf{R}_{\mathsf{e}} a$ and $y = \mathbf{I}_{\mathsf{m}} a$:

$$
\begin{aligned}
& & a &= x + iy && \text{by left hypothesis}\\
&\implies & a^* &= (x + iy)^* && \text{by definition of } \textit{adjoint} && \text{(Definition A.4 page 90)}\\
& & &= x^* - iy^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition A.3 page 90)}\\
& & &= x - iy && \text{by Theorem A.2 page 91}\\
&\implies & x &= a - iy && \text{by solving for } x \text{ in } a = x + iy \text{ equation}\\
& & x &= a^* + iy && \text{by solving for } x \text{ in } a^* = x - iy \text{ equation}\\
&\implies & x + x &= a + a^* && \text{by adding previous 2 equations}\\
&\implies & 2x &= a + a^* && \text{by solving for } x \text{ in previous equation}\\
&\implies & x &= \frac{1}{2}\left(a + a^*\right)\\
& & &= \mathbf{R}_{\mathsf{e}} a && \text{by definition of } \Re && \text{(Definition A.5 page 91)}
\end{aligned}
$$

$$
\begin{aligned}
& & iy &= a - x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation}\\
& & iy &= -a^* + x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation}\\
&\implies & iy + iy &= a - a^* && \text{by adding previous 2 equations}\\
&\implies & y &= \frac{1}{2i}\left(a - a^*\right) && \text{by solving for } iy \text{ in previous equations}\\
& & &= \mathbf{I}_{\mathsf{m}} a && \text{by definition of } \Im && \text{(Definition A.5 page 91)}
\end{aligned}
$$

Proof that $a = x + iy \impliedby x = \mathbf{R}_{\mathsf{e}} a$ and $y = \mathbf{I}_{\mathsf{m}} a$:

$$
\begin{aligned}
x + iy &= \mathbf{R}_{\mathsf{e}} a + i\,\mathbf{I}_{\mathsf{m}} a && \text{by right hypothesis}\\
&= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\mathbf{R}_{\mathsf{e}} a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\mathbf{I}_{\mathsf{m}} a} && \text{by definition of } \Re \text{ and } \Im && \text{(Definition A.5 page 91)}\\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \cancelto{0}{\left(\frac{1}{2}a^* - \frac{1}{2}a^*\right)}\\
&= a
\end{aligned}
$$

✏

[11] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Neumark (1943b) page 7

A.3 Normed Algebras

Definition A.6. [12] *Let A be an algebra.*

DEF

*The pair $(A, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in A \qquad \text{(multiplicative condition)}$$

*A normed algebra $(A, \|\cdot\|)$ is a **Banach algebra** if $(A, \|\cdot\|)$ is also a Banach space.*

Proposition A.2.

PRP

$(A, \|\cdot\|)$ *is a normed algebra* $\implies$ *multiplication is **continuous** in* $(A, \|\cdot\|)$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x-y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(A, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 93)}\\
&= \|z(x-y)\| && && \\
&\le \|z\|\;\|x-y\| && \text{by definition of } \textit{normed algebra} && \text{(Definition A.6 page 93)}\\
&\triangleq \|z\|\;\delta && \text{by definition of } \delta && \text{(item (2) page 93)}\\
&\le \epsilon && \text{for some value of } \delta > 0 &&
\end{aligned}$$

✏

Theorem A.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l} (A, \|\cdot\|) \text{ is a Banach algebra} \\ \text{every nonzero } x \in A \text{ is invertible} \end{array}\right\} \implies A \equiv \mathbb{C} \quad (A \text{ is isomorphic to } \mathbb{C})$$

A.4 C* Algebras

Definition A.7. [14]

DEF

*The triple $(A, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(A, \|\cdot\|)$ *is a Banach algebra* and
2. $(A, *)$ *is a $*$-algebra* and
3. $\|x^*x\| = \|x\|^2 \quad \forall x \in A$.

*A C^* **algebra** $(A, \|\cdot\|, *)$ is also called a **C star algebra**.*

---

[12] Rickart (1960) page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩

[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩

[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem A.5. [15] *Let $\boldsymbol{A}$ be an algebra.*

THM	
$(\boldsymbol{A}, \|\cdot\|, *)$ *is a* C^* ***algebra*** $\implies$	$\|x^*\| = \|x\|$

✎PROOF:

$$
\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\ \|x\|^2 \\
&= \frac{1}{\|x\|}\ \|x^*x\| && \text{by definition of } C^*\text{-algebra} && \text{(Definition A.7 page 93)} \\
&\le \frac{1}{\|x\|}\ \|x^*\|\,\|x\| && \text{by definition of normed algebra} && \text{(Definition A.6 page 93)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by involution property of } * && \text{(Definition A.3 page 90)}
\end{aligned}
$$

✏

[15] Folland (1995) page 1, Gelfand and Neumark (1943b) page 4, Gelfand and Neumark (1943a)

APPENDIX B

POLYNOMIALS

B.1 Definitions

Definition B.1. [1] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

D E F

A function p *in* $\mathbb{F}^{\mathbb{F}}$ *is a* ***polynomial*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if it is of the form*

$$\mathsf{p}(x) \triangleq \sum_{n=0}^{N} \alpha_n x^n \qquad \alpha_n \in \mathbb{F},\ \alpha_N \neq 0.$$

The ***degree*** *of* p *is* N. *A* ***coefficient*** *of* p *is any element of* $⦅\alpha_n⦆_1^N$.
The ***leading coefficient*** *of* p *is* α_N.

Definition B.2. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

D E F

A polynomial p *of degree* N *over the field* $\mathbb{F}$ *and*
a polynomial q *of degree* M *over the field* $\mathbb{F}$ *are* ***equal*** *if*

1. $N = M$ *and*
2. $\alpha_n = \beta_n \qquad \text{for} \qquad n = 0, 1, \ldots, N$.

The expression $\mathsf{p}(x) = \mathsf{q}(x)$ *(or* $\mathsf{p} = \mathsf{q}$*) denotes that* p *and* q *are* EQUAL.

---

[1] Barbeau (1989) page 1, Fuhrmann (2012) page 11, Borwein and Erdélyi (1995) page 2
[2] Fuhrmann (2012) page 11

B.2 Ring properties

B.2.1 Polynomial Arithmetic

Theorem B.1 (polynomial addition). [3] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

$$\underbrace{\left(\sum_{n=0}^{N} \alpha_n x^n\right)}_{\mathsf{p}(x)} + \underbrace{\left(\sum_{n=0}^{M} \beta_n x^n\right)}_{\mathsf{q}(x)} = \underbrace{\sum_{n=0}^{\max(N,M)} \gamma_n x^n}_{\mathsf{p}(x)+\mathsf{q}(x)} \quad \text{where} \quad \gamma_n \triangleq \begin{cases} \alpha_n + \beta_n & \text{for } n \le \min(N,M) \\ \alpha_n & \text{for } n > M \\ \beta_n & \text{for } n > N \end{cases}$$
$$\text{for all} \quad x,\ \alpha_n,\ \beta_n \in \mathbb{F}$$

Polynomial multiplication is equivalent to convolution (Definition F.3 page 131) of the coefficients (Definition B.1 page 95). [4]

Theorem B.2 (polynomial multiplication). [5] *Let* $(\!(\alpha_n \in \mathbb{C})\!)$, $(\!(b_n \in \mathbb{C})\!)$, *and* $x \in \mathbb{C}$.

$$\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{n=0}^{M} \beta_n x^n\right) = \sum_{n=0}^{N+M} \underbrace{\left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right)}_{\text{Cauchy product}} x^n$$

✎PROOF:

$$\begin{aligned}
\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{m=0}^{M} \beta_m x^m\right) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n \beta_m x^{n+m} \\
&= \sum_{n=0}^{N}\sum_{k=n}^{M+n} \alpha_n \beta_{k-n} x^k && k \triangleq n+m \iff m = k-n \\
&= \sum_{n=0}^{N+M}\left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right)x^n
\end{aligned}$$

Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	β_0	β_1	β_2	β_3	$\cdots$	β_M
α_0	$\alpha_0\beta_0$	$\alpha_0\beta_1 x$	$\alpha_0\beta_2 x^2$	$\alpha_0\beta_3 x^3$	$\cdots$	$\alpha_0\beta_M x^M$
α_1	$\alpha_1\beta_0 x$	$\alpha_1\beta_1 x^2$	$\alpha_1\beta_2 x^3$	$\alpha_1\beta_3 x^4$	$\cdots$	$\alpha_1\beta_M x^{1+M}$
α_2	$\alpha_2\beta_0 x^2$	$\alpha_2\beta_1 x^3$	$\alpha_2\beta_2 x^4$	$\alpha_2\beta_3 x^5$	$\cdots$	$\alpha_2\beta_M x^{2+M}$
α_3	$\alpha_3\beta_0 x^3$	$\alpha_3\beta_1 x^4$	$\alpha_3\beta_2 x^5$	$\alpha_3\beta_3 x^6$	$\cdots$	$\alpha_3\beta_M x^{3+M}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_N	$\alpha_N\beta_0 x^N$	$\alpha_N\beta_1 x^{N+1}$	$\alpha_N\beta_2 x^{N+2}$	$\alpha_N\beta_3 x^{N+3}$	$\cdots$	$\alpha_N\beta_M x^{N+M}$

1. The expression $\sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n \beta_m x^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

[3] Fuhrmann (2012) page 11

[4] *Convolution*: In fact, using *GNU Octave*™ or *MatLab*™, polynomial multiplication can be performed using convolution. For example, the operation $(x^3 + 5x^2 + 7x + 9)(4x^2 + 11)$ can be calculated in *GNU Octave*™ or *MatLab*™ with conv([1 5 7 9],[4 0 11])

[5] Apostol (1975) page 237

2. If we switched the order of summation to $\sum_{m=0}^{M}\sum_{n=0}^{N}\alpha_n\beta_m x^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

3. For $N = M = \infty$, the expression $\sum_{n=0}^{N+M}\left(\sum_{k=0}^{n}\alpha_k\beta_{n-k}\right)x^n$ is equivalent to adding *diagonally* starting from the upper left corner and proceeding towards the lower right.

4. For finite N and M…

   (a) The upper limit on the inner summation puts two constraints on k:
   $$\left\{\begin{array}{lllc} k & \le & n & \text{and} \\ k & \le & N & \end{array}\right\} \implies k \le \min(n, N)$$

   (b) The lower limit on the inner summation also puts two constraints on k:
   $$\left\{\begin{array}{lllc} k & \ge & 0 & \text{and} \\ k & \ge & n-M & \end{array}\right\} \implies k \ge \max(0, n-M)$$

✏

Polynomial division can be performed in a manner very similar to integer division (both integers and polynomials are *rings*).

Definition B.3 (Polynomial division)**.** *The quantities of polynomial division are defined as follows:*

DEF
$$\frac{\mathsf{d}(x)}{\mathsf{p}(x)} = \mathsf{q}(x) + \frac{\mathsf{r}(x)}{\mathsf{p}(x)} \qquad \textit{where} \qquad \left\{\begin{array}{lll} \mathsf{d}(x) & \textit{is the } \textbf{\textit{dividend}} & \textit{and} \\ \mathsf{p}(x) & \textit{is the } \textbf{\textit{divisor}} & \textit{and} \\ \mathsf{q}(x) & \textit{is the } \textbf{\textit{quotient}} & \textit{and} \\ \mathsf{r}(x) & \textit{is the } \textbf{\textit{remainder}}. & \end{array}\right\}$$

The ring of integers $\mathbb{Z}$ contains some special elements called *primes* which can only be divided[6] by themselves or 1.

Rings of polynomials have a similar elements called *primitive polynomials*.

Definition B.4.

DEF
A ***primitive polynomial*** *is any polynomial $p(x)$ that satisfies*
1. *$p(x)$ cannot be factored*
2. *the smallest order polynomial that $p(x)$ can divide is $x^{2^n-1}+1 = 0$.*

Example B.1. [7] Some examples of primitive polynomials over $GF(2)$ are

EX

order	primitive polynomial
2	$\mathsf{p}(x) = x^2 + x + 1$
3	$\mathsf{p}(x) = x^3 + x + 1$
4	$\mathsf{p}(x) = x^4 + x + 1$
5	$\mathsf{p}(x) = x^5 + x^2 + 1$
5	$\mathsf{p}(x) = x^5 + x^4 + x^2 + x + 1$
16	$\mathsf{p}(x) = x^{16} + x^{15} + x^{13} + x^4 + 1$
31	$\mathsf{p}(x) = x^{31} + x^{28} + 1$

An m-sequence is the remainder when dividing any non-zero polynomial by a primitive polynomial. We can define an *equivalence relation* on polynomials which defines two polynomials as *equivalent with respect to* $p(x)$ when their remainders are equal.

[6] The expression "a divides b" means that b/a has remainder 0.
[7] Wicker (1995) pages 465–475

Definition B.5 (Equivalence relation). *Let* $\dfrac{\alpha_1(x)}{p(x)} = \mathsf{q}_1(x) + \dfrac{r_1(x)}{p(x)}$ *and* $\dfrac{\alpha_2(x)}{p(x)} = \mathsf{q}_2(x) + \dfrac{r_2(x)}{p(x)}$.

Then $\alpha_1(x) \equiv \alpha_2(x)$ *with respect to* $p(x)$ *if* $r_1(x) = r_2(x)$.

Using the equivalence relation of Definition B.5, we can develop two very useful equivalent representations of polynomials over GF(2). We will call these two representations the *exponential* representation and the *polynomial* representation.

Example B.2. By Definition B.5 and under $p(x) = x^3 + x + 1$, we have the following equivalent representations:

EX

$$\begin{array}{rcrcccrcl}
\frac{x^0}{x^3+x+1} & = & 0 & + & \frac{1}{x^3+x+1} & \Longrightarrow & x^0 & \equiv & 1 \\
\frac{x^1}{x^3+x+1} & = & 0 & + & \frac{x}{x^3+x+1} & \Longrightarrow & x^1 & \equiv & x \\
\frac{x^2}{x^3+x+1} & = & 0 & + & \frac{x^2}{x^3+x+1} & \Longrightarrow & x^2 & \equiv & x^2 \\
\frac{x^3}{x^3+x+1} & = & 1 & + & \frac{x+1}{x^3+x+1} & \Longrightarrow & x^3 & \equiv & x+1 \\
\frac{x^4}{x^3+x+1} & = & x & + & \frac{x^2+x}{x^3+x+1} & \Longrightarrow & x^4 & \equiv & x^2+x \\
\frac{x^5}{x^3+x+1} & = & x^2+1 & + & \frac{x^2+x+1}{x^3+x+1} & \Longrightarrow & x^5 & \equiv & x^2+x+1 \\
\frac{x^6}{x^3+x+1} & = & x^3+x+1 & + & \frac{x^2+1}{x^3+x+1} & \Longrightarrow & x^6 & \equiv & x^2+1 \\
\frac{x^7}{x^3+x+1} & = & x^4+x^2+x+1 & + & \frac{1}{x^3+x+1} & \Longrightarrow & x^7 & \equiv & 1
\end{array}$$

Notice that $x^7 \equiv x^0$, and so a cycle is formed with $2^3 - 1 = 7$ elements in the cycle. The monomials to the left of the $\equiv$ are the *exponential* representation and the polynomials to the right are the *polynomial* representation. Additionally, the polynomial representation may be put in a vector form giving a *vector* representation. The vectors may be interpreted as a binary number and represented as a decimal numeral.

EX

exponential	polynomial	vector	decimal
x^0	1	[001]	1
x^1	x	[010]	2
x^2	x^2	[100]	4
x^3	$x+1$	[011]	3
x^4	x^2+x	[110]	6
x^5	x^2+x+1	[111]	7
x^6	x^2+1	[101]	5

Example B.3. We can generate an m-sequence of length $2^3 - 1 = 7$ by dividing 1 by the primitive polynomial $x^3 + x + 1$.

$$
\begin{array}{r|l}
 & x^{-3}+x^{-5}+x^{-6}+\quad x^{-7}+x^{-10}+x^{-12}+x^{-13}+x^{-14}+x^{-17}+\cdots \\ \cline{2-2}
x^3+x+1 & 1 \\
 & 1+x^{-2}+x^{-3} \\ \cline{2-2}
 & x^{-2}+x^{-3} \\
 & x^{-2}+x^{-4}+x^{-5} \\ \cline{2-2}
 & x^{-3}+x^{-4}+x^{-5} \\
 & x^{-3}+x^{-5}+x^{-6} \\ \cline{2-2}
 & x^{-4}+x^{-6} \\
 & x^{-4}+x^{-6}+x^{-7} \\ \cline{2-2}
 & x^{-7} \\
 & x^{-7}+x^{-9}+x^{-10} \\ \cline{2-2}
 & x^{-9}+x^{-10} \\
 & x^{-9}+x^{-11}+x^{-12} \\ \cline{2-2}
 & x^{-10}+x^{-11}+x^{-12} \\
 & x^{-10}+x^{-12}+x^{-13} \\ \cline{2-2}
 & x^{-11}+x^{-13} \\
 & x^{-11}+x^{-13}+x^{-14} \\ \cline{2-2}
 & x^{-14} \\
 & \vdots
\end{array}
$$

The coefficients, starting with the x^{-1} term, of the resulting polynomial form the m-sequence

$$0010111\ 0010111\ \cdots$$

which repeats every $2^3 - 1 = 7$ elements.

Note that the division operation in Example B.3 can be performed using vector notation rather than polynomial notation.

Example B.4. Generate an m-sequence of length $2^3-1 = 7$ by dividing 1 by the primitive polynomial $x^3 + x + 1$ using vector notation.

$$
\begin{array}{rccccccccccccccccccc}
 & & . & 0&0&1&0&1&1&1&0&0&1&0&1&1&1&0&\cdots \\ \cline{2-19}
1011\;\big| & 1 & . & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0& \\
 & 1 & & 0&1&1 \\ \cline{2-6}
 & 0 & & 0&1&1&0 \\
 & & & 0&0&0&0 \\ \cline{4-7}
 & & & 0&1&1&0&0 \\
 & & & &1&0&1&1 \\ \cline{5-8}
 & & & &0&1&1&1&0 \\
 & & & & &1&0&1&1 \\ \cline{6-9}
 & & & & &0&1&0&1&0 \\
 & & & & & &1&0&1&1 \\ \cline{7-10}
 & & & & & &0&0&0&1&0 \\
 & & & & & & &0&0&0&0 \\ \cline{8-11}
 & & & & & & &0&0&1&0&0 \\
 & & & & & & & &0&0&0&0 \\ \cline{9-12}
 & & & & & & & &0&1&0&0&0 \\
 & & & & & & & & &1&0&1&1 \\ \cline{10-13}
 & & & & & & & & &0&0&1&1&0 \\
 & & & & & & & & & &0&0&0&0 \\ \cline{11-14}
 & & & & & & & & & &0&1&1&0&0 \\
 & & & & & & & & & & &1&0&1&1 \\ \cline{12-15}
 & & & & & & & & & & &0&1&1&1&0 \\
 & & & & & & & & & & & &0&0&0&0 \\
 & & & & & & & & & & & & &\vdots
\end{array}
$$

The coefficients, starting to the right of the binary point, is again the sequence

$$0010111\ 0010111\ \cdots.$$

B.2.2 Greatest common divisor

Theorem B.3 (Extended Euclidean Algorithm). [8]

Let $\mathsf{r}_1(x)$ *and* $\mathsf{r}_2(x)$ *be polynomials. The following algorithm computes their greatest common divisor* $\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$, *and factors* $\mathsf{a}(x)$ *and* $\mathsf{b}(x)$ *such that*

$$\mathsf{r}_1(x)\mathsf{a}(x) + \mathsf{r}_2(x)\mathsf{b}(x) = \gcd(\mathsf{r}_1, \mathsf{r}_2)$$

THM

n	*remainder* $\mathsf{r}_n = \mathsf{r}_{n-2} - \mathsf{q}_n\mathsf{r}_{n-1}$	*quotient* q_n	*factor* $\alpha_n = a_{n-2} - \mathsf{q}_n\alpha_{n-1}$	*factor* $\beta_n = b_{n-2} - \mathsf{q}_n\beta_{n-1}$
1	$\mathsf{r}_1(x)$	$-$	1	0
2	$\mathsf{r}_2(x)$	$-$	0	1
3	$\mathsf{r}_1 - q_3\mathsf{r}_2$	q_3	1	$-q_3$
4	$\mathsf{r}_2 - q_4 r_3$	q_4	$-\mathsf{q}_4$	$1 + \mathsf{q}_4\mathsf{q}_1$
5	$\mathsf{r}_1 - \mathsf{q}_5\mathsf{r}_2$	q_5	$1 + \mathsf{q}_5\mathsf{q}_4$	$-q_3 - \mathsf{q}_5(1 + \mathsf{q}_4\mathsf{q}_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$	q_n	$\mathsf{a}(x) = a_{n-2} - q_n\alpha_{n-1}$	$\mathsf{b}(x) = b_{n-2} - q_n\beta_{n-1}$
$n+1$	0	q_{n+1}		

✎PROOF:

$$\begin{aligned}\mathsf{r}_1 &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3 \\ &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3\end{aligned}$$

✏

Example B.5. Let

$$\mathsf{u}(x) \triangleq (1-x)^2 \qquad \mathsf{v}(x) \triangleq x^2.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^2}_{\mathsf{u}(x)}\underbrace{(1+2x)}_{\mathsf{a}(x)} + \underbrace{(x^2)}_{\mathsf{v}(x)}\underbrace{(3-2x)}_{\mathsf{b}(x)} = \underbrace{1}_{\gcd}$$

Because $\gcd(\mathsf{u}, \mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$r_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^2 = 1 - 2x + x^2 = \mathsf{u}(x)$	$-$	1	0
0	$x^2 = \mathsf{v}(x)$	$-$	0	1
1	$1 - 2x$	1	1	-1
2	$\frac{1}{2}x$	$-\frac{1}{2}x$	$\frac{1}{2}x$	$1 - \frac{1}{2}x$
3	$1 = \gcd(\ (1-x)^2,\ x^2\ )$	-4	$1 + 2x = \mathsf{a}(x)$	$3 - 2x = \mathsf{b}(x)$
4	0	$\frac{1}{2}x$	$-$	$-$

✏

[8] Wicker (1995) page 53, Fuhrmann (2012) page 11

Example B.6. Let

$$\mathsf{u}(x) \triangleq (1-x)^3 \qquad \mathsf{v}(x) \triangleq x^3.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^3}_{\mathsf{u}(x)}\underbrace{(1+3x+6x^2)}_{\mathsf{a}(x)} + \underbrace{(x^3)}_{\mathsf{v}(x)}\underbrace{(10-15x+6x^2)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^3 = 1-3x+3x^2-x^3$	$-$	1	0
0	x^3	$-$	0	1
1	$1-3x+3x^2$	-1	1	1
2	$-\frac{1}{3}x+x^2$	$\frac{1}{3}x$	$-\frac{1}{3}x$	$1-\frac{1}{3}x$
3	$1-2x$	3	$1+x$	$-2+x$
4	$\frac{1}{6}x$	$-\frac{1}{2}x$	$\frac{1}{6}x+\frac{1}{2}x^2$	$1-\frac{4}{3}y+\frac{1}{2}x^2$
5	$1 = \gcd((1-x)^3, x^3)$	-12	$1+3x+6x^2 = \mathsf{a}(x)$	$10-15x+6x^2 = \mathsf{b}(x)$
6	0	$\frac{1}{6}x$		

✏

Example B.7. Let

$$\mathsf{u}(x) \triangleq (1-x)^4 \qquad \mathsf{v}(x) \triangleq x^4.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^4}_{\mathsf{u}(x)}\underbrace{(1+4x+10x^2+20x^3)}_{\mathsf{a}(x)} + \underbrace{(x^4)}_{\mathsf{v}(x)}\underbrace{(35-84x+70x^2-20x^3)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^4 = 1-4x+6^2-4x^3+x^4$	$-$	1	0
0	x^4	$-$	0	1
1	$1-4x+6x^2-4x^3$	1	1	-1
2	$\frac{1}{4}x-x^2+\frac{3}{2}x^3$	$-\frac{1}{4}x$	$\frac{1}{4}x$	$1-\frac{1}{4}x$
3	$1-\frac{10}{3}x+\frac{10}{3}x^2$	$-\frac{8}{3}$	$1+\frac{2}{3}x$	$\frac{5}{3}-\frac{2}{3}x$
4	$-\frac{1}{5}x+\frac{1}{2}x^2$	$\frac{3}{2}\cdot\frac{3}{10}x$	$-\frac{1}{5}x-\frac{3}{10}x^2$	$1-x+\frac{3}{10}x^2$
5	$1-2x$	$\frac{20}{3}$	$1+2x+2x^2$	$-5+6x-2x^2$
6	$\frac{1}{20}x$	$-\frac{1}{4}x$	$\frac{1}{20}x+\frac{1}{5}x^2+\frac{1}{2}x^3$	$1-\frac{9}{4}x+\frac{18}{10}x^2-\frac{1}{2}x^3$
7	$1 = \gcd((1-x)^4, x^4)$	-40	$1+4x+10x^2+20x^3$	$35-84x+70x^2-20x^3$
8	0	$\frac{1}{20}x$	$-$	$-$

✏

❝*Infinitesimal analysis was considered so attractive and important because of its numerous and useful applications; as such, it attracted upon itself all research attention and efforts. Concurrently, algebraic analysis appeared to be a field where nothing remained to be done, or where whatever remained to be done would have only been worthless speculation. ...Nevertheless, the major contributors to infinitesimal analysis are well aware of the need to improve algebraic analysis: Their own progress depends upon it.*❞

Étienne Bézout, 1779 [9]

Theorem B.4 (Bézout's Identity). [10] [11] *Let* $\mathsf{p}_1(x)$ *be a polynomial of degree* n_1 *and* $\mathsf{p}_2(x)$ *be a polynomial of degree* n_2.

THM

$$\underbrace{\gcd(\mathsf{p}_1(x),\ \mathsf{p}_2(x)) = 1}_{\mathsf{p}_1(x) \textit{ and } \mathsf{p}_2(x) \textit{ are relatively prime}} \implies \left\{\begin{array}{ll} 1. & \exists \mathsf{q}_1(x), \mathsf{q}_2(x) \quad \text{such that} \\ & \overset{\substack{\textit{degree } n_2 - 1 \\ \downarrow}}{\underset{\substack{\uparrow \\ \textit{degree } n_1}}{\mathsf{p}_1(x)}\mathsf{q}_1(x)} + \overset{\substack{\textit{degree } n_1 - 1 \\ \downarrow}}{\underset{\substack{\uparrow \\ \textit{degree } n_2}}{\mathsf{p}_2(x)}\mathsf{q}_2(x)} = 1 \\ 2. & \textit{order of } \mathsf{q}_1(x) = n_2 - 1 \\ 3. & \textit{order of } \mathsf{q}_2(x) = n_1 - 1 \end{array}\right.$$

✎PROOF: No proof at this time. ✏

B.3 Roots

❝*Neither the true nor the false roots are always real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation* $x^3 - 6x^2 + 13x - 10 = 0$ *as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remain always imaginary.*❞

René Descartes (1596–1650), French philosopher and mathematician[12]

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[9] quote: Bézout (1779a)
translation: Bézout (1779b) page xv
image: http://en.wikipedia.org/wiki/File:Etienne_Bezout2.jpg, public domain

[10] Bourbaki (2003b) page 2 ⟨Theorem 1 Chapter VII⟩, Fuhrmann (2012) pages 15–17 ⟨Corollary 1.31, Corollary 1.38⟩, Adhikari and Adhikari (2003) page 182, Warner (1990) page 381, Daubechies (1992) page 169, Mallat (1999) page 250

[11] Historical information: Bézout (1779a) ⟨???⟩, Bézout (1779b) ⟨???⟩, Bachet (1621) ⟨???⟩, Childs (2009) pages 37–46 ⟨some history on page 46⟩, http://serge.mehl.free.fr/chrono/Bachet.html, http://serge.mehl.free.fr/chrono/Bezout.html

[12] quote: Descartes (1637a)
English: Descartes (1954) page 175
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

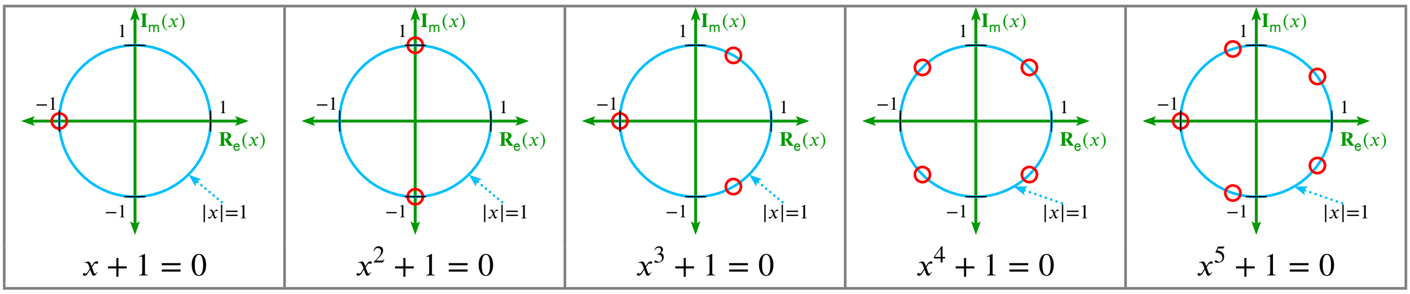


Figure B.1: Roots of $x^n + 1 = 0$

Theorem B.5 (Fundamental Theorem of Algebra). [13] *Let* $\mathrm{p}(x)$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

$$\text{THM}\quad \left\{\textit{degree of } \mathrm{p}(x) \textit{ is } N\right\} \implies \left\{\begin{array}{l}\underbrace{\exists \left(x_n\right)_1^N \textit{ such that } \mathrm{p}(x_n) = 0 \textit{ for } n = 1, 2, \ldots, N}_{\mathrm{p}(x) \textit{ has } N \textit{ zeros}} \\ \textit{where } x_n \textit{ and } x_m \textit{ are not necessarily distinct for } n \neq m.\end{array}\right\}$$

Corollary B.1. *Let* $\mathrm{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

$$\text{COR}\quad \left\{\underbrace{\begin{array}{l}\textit{There exists } \left(x_n\right)_1^N \\ \textit{such that } \mathrm{p}(x_n) = 0 \textit{ for } n = 0, 1, \ldots, N \\ \textit{and where } x_n \textit{ and } x_m \textit{ are} \\ \textit{not necessarily distinct for } n \neq m.\end{array}}_{N \textit{ zeros of } \mathrm{p}(x)}\right\} \implies \left\{\mathrm{p}(x) = \left(\frac{\alpha_0}{\prod_{n=1}^{N}(-x_n)}\right)\underbrace{\prod_{n=1}^{N}(x - x_n)}_{N \textit{ factors}}\right\}$$

Lemma B.1.

$$\text{LEM}\quad \left\{x^N + 1 = 0\right\} \implies \left\{\textit{roots of } x = \left\{\exp\left[i\frac{\pi}{N}(2n+1)\right] \middle| n = 0, 1, \ldots, N-1\right\}\right\}$$

✎PROOF:

$$\begin{aligned} e^{iN\theta_n - i2\pi n} &= -1 && n \in \mathbb{Z} \\ N\theta_n - 2\pi n &= \pi && n = 0, 1, \ldots, N-1 \\ N\theta_n &= 2\pi n + \pi \\ \theta_n &= \frac{\pi}{N}(2n+1) \end{aligned}$$

✏

Theorem B.6. *Let* $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \leq n \leq N\}$ *and* $\mathrm{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$.

$$\text{THM}\quad \underbrace{\alpha_n = \alpha_{-n}^* \quad \forall n \in I}_{\left(\alpha_n\right) \textit{ is Hermitian symmetric}} \iff \mathrm{p}(x) = \mathrm{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C}$$

✎PROOF:

[13] Prasolov (2004) pages 1–2 ⟨Section 1.1.1⟩, Borwein and Erdélyi (1995) page 11 ⟨Theorem 1.2.1⟩

1. Proof that $\alpha_n = \alpha^*_{-n} \implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\mathsf{p}(x) &\triangleq \sum_{n=-N}^{N} \alpha_n x^n && \text{by definition of } \mathsf{p}(x)\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha_{-n} x^{-n}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha^*_n x^{-n} && \text{by left hypothesis}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n x^{-n} + \sum_{n=1}^{N} \alpha_n x^n\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x}\right)^{-n}\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x^*}\right)^{-n}\right]^*\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha_{-n} \left(\frac{1}{x^*}\right)^{-n}\right]^* && \text{by left hypothesis}\\
&= \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^*\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by definition of } \mathsf{p}(x)
\end{aligned}$$

2. Proof that $\alpha_n = \alpha^*_{-n} \impliedby \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\sum_{n=-N}^{N} \alpha_n x^n &\triangleq \mathsf{p}(x) && \text{by definition of } \mathsf{p}(x)\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by right hypothesis}\\
&\triangleq \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^* && \text{by definition of } \mathsf{p}(x)\\
&= \sum_{n=-N}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n\\
&= \sum_{n=-N}^{N} \alpha^*_{-n} x^n && \text{by symmetry of summation indices}\\
\implies \quad \alpha_n &= \alpha^*_{-n} && \text{by matching of polynomial coefficients}
\end{aligned}$$

✎

Theorem B.7. *Let $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \le n \le N\}$ and*

$$\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$$

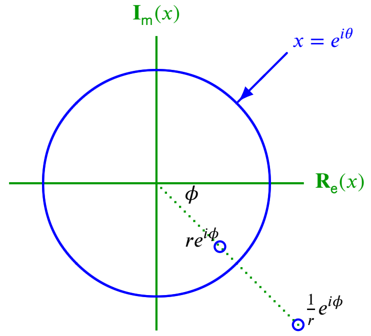


Figure B.2: Recipricol conjugate zero pairs

THM

$$\underbrace{\alpha_n = \alpha^*_{-n} \quad \forall n \in I}_{(\alpha_n) \text{ is Hermitian symmetric}} \implies \underbrace{\left[\sigma \textit{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \textit{ is a root of } \mathsf{p}(x)\right]}_{\text{roots occur in conjugate reciprocal pairs}}$$

✎PROOF:

$$\begin{aligned}
&\alpha_n = \alpha^*_{-n} \quad \forall n \in I && \text{by left hypothesis}\\
&\implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C} && \text{by Theorem B.6 page 103}\\
&\implies \left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]
\end{aligned}$$

If σ is a zero of $\mathsf{p}(x)$, then so is $\frac{1}{\sigma^*}$ because

$$\mathsf{p}\left(\frac{1}{\sigma^*}\right) = \mathsf{p}^*(\sigma) = 0^* = 0.$$

✏

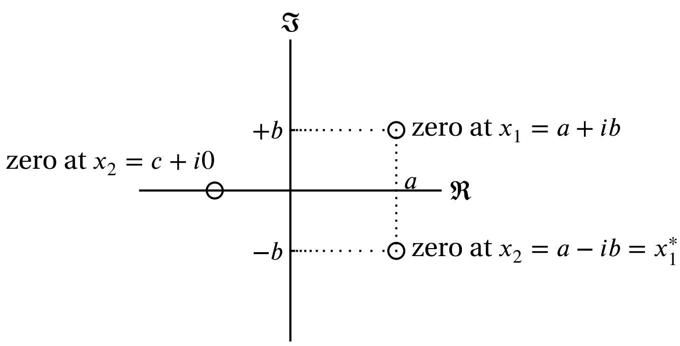


Figure B.3: Conjugate pairs of roots

Theorem B.8 page 105 (next) states that the roots of real polynomials occur in complex conjugate pairs. This is illustrated in Figure B.3.

Theorem B.8. [14] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial.*

[14] Korn and Korn (1968) page 17

THM
$$\left[\underbrace{(\alpha_n \in \mathbb{R})_{n=0,1,\ldots,N}}_{\text{coefficients are real}}\right] \implies \left[\underbrace{\mathsf{p}(x_0)=0 \iff \mathsf{p}(x_0^*)=0}_{\text{zeros occur in conjugate pairs}}\right]$$

Theorem B.9 (Routh-Hurwitz Criterion). [15] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$ *and*

$$d_0 \triangleq \alpha_0 \qquad d_1 \triangleq \alpha_1 \qquad d_2 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} \qquad d_3 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} \qquad d_4 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 \end{vmatrix}$$

$$d_n \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ \alpha_{2n-3} & \alpha_{2n-4} & \cdots & \alpha_{n-2} \\ \alpha_{2n-1} & \alpha_{2n-2} & \cdots & \alpha_n \end{vmatrix}$$

Let $\mathbf{S}(\!(x_n)\!)$ *be the number of sign changes of some sequence* $(\!(x_n)\!)$ *after eliminating all zero elements* $(x_n = 0)$*.*

THM
$$\underbrace{\left|\left\{x_n | \mathsf{p}(x_n)=0,\ \Re[x_n]>0\right\}\right|}_{\text{number of roots in right half plane}} = \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ d_1d_2,\ d_2d_3,\ \ldots,\ d_{p-2}d_{p-1},\ \alpha_p\right)\!\right)}_{\text{number of sign changes}}$$
$$= \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ \frac{d_2}{d_1},\ \frac{d_3}{d_2},\ \ldots,\ \frac{d_p}{d_{p-1}}\right)\!\right)}_{\text{number of sign changes}}$$

Theorem B.10 (Descartes rule of signs). [16] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$*.*

THM
$$\underbrace{\left|\left\{x_n | \mathsf{p}(x_n)=0,\ \Re[x_n]>0\right\},\ \Im[x_n]=0\right|}_{\text{number of roots on right real axis}} = \underbrace{\mathbf{S}(\!(\alpha_n)\!) - 2m}_{\text{number of sign changes − even integer}} \qquad \text{where } m \in \mathbb{W}$$

Theorem B.11. [17] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$*.*

THM
$$\underbrace{\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1}\ \geq 0}_{\text{first } k \text{ coefficients are nonnegative}} \implies \begin{cases} \underbrace{\left|\left\{x_n | \mathsf{p}(x_n)=0,\ \Im[x_n]=0\right\}\right|}_{\text{number of real roots}} < \underbrace{1+\left(\frac{q}{\alpha_0}\right)^{\frac{1}{k}}}_{\text{upper bound}} \\ \text{where } q \triangleq \underbrace{\max\left\{\ |\alpha_n|\ |\alpha_n<0\right\}}_{\text{largest negative coefficient}} \end{cases}$$

Theorem B.12 (Rolle's Theorem). [18] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$*. The number of real zeros of* $\mathsf{p}'(x)$ *between any two real consecutive real zeros of* $\mathsf{p}(x)$ *is* ***odd****.*

Definition B.6. [19] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD*.*

DEF
$\dfrac{\mathsf{p}(x)}{\mathsf{q}(x)}$ *is a* ***rational function***
if $\mathsf{p}(x)$ *and* $\mathsf{q}(x)$ *are* POLYNOMIALS *over* $(\mathbb{F}, +, \cdot, 0, 1)$*.*

[15] Korn and Korn (1968) page 17
[16] Korn and Korn (1968) page 17
[17] Korn and Korn (1968) page 18
[18] Korn and Korn (1968) page 18
[19] Fuhrmann (2012) page 22

Example B.8.

EX

An example of a rational function using polynomials in x^{-1} is

$$A(x) = \frac{b_0 + \beta_1 x^{-1} + \beta_2 x^{-2} + \beta_3 x^{-3}}{1 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \alpha_3 x^{-3}}$$

This can be expressed as a rational function using polynomials in x by multiplying numerator and denominator by x^3:

$$A(x) = \frac{x^3}{x^3} A(x) = \frac{b_0 x^3 + \beta_1 x^2 + \beta_2 x + \beta_3}{x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3}$$

Definition B.7.

DEF

The ***zeros*** *of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $B(x)$.

The ***poles*** *of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $A(x)$.

B.4 Polynomial expansions

❝*Thus, if a straight-line is cut at random, then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces.*❞

Euclid (~300BC), Greek mathematician, demonstrating the *Binomial theorem* for exponent $n = 2$ as in $(x+y)^2 = x^2 + 2xy + y^2$. [20]

Theorem B.13 (Taylor Series). [21] *Let* $\boldsymbol{C}$ *be the space of all* ANALYTIC *functions and* $\frac{d}{dx}$ *in* $\boldsymbol{C}^{\boldsymbol{C}}$ *the* DIFFERENTIATION OPERATOR.

THM

A ***Taylor series*** *about the point* a *of a function* $\mathsf{f} \in \boldsymbol{C}^{\boldsymbol{C}}$ *is*

$$\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\left[\frac{d}{dx}^n \mathsf{f}\right](a)}{n!}(x-a)^n \qquad \forall a \in \mathbb{R}, \mathsf{f} \in \boldsymbol{C}$$

A ***Maclaurin series*** *is a* TAYLOR SERIES *about the point* $a = 0$.

Theorem B.14 (Binomial Theorem). [22]

THM

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad \text{where} \qquad \binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$$

✎PROOF: This theorem is proven using two different techniques. Either is sufficient. The first requires the Maclaurin series resulting in a more compact proof, but requires the additional (here unproven) Maclaurin series. The second proof uses induction resulting in a longer proof, but does not require any external theorem.

[20] quote: Euclid (circa 300BC) ⟨Book II, Proposition 4⟩, Coolidge (1949) page 147
image: http://commons.wikimedia.org/wiki/File:Euklid-von-Alexandria_1.jpg, public domain

[21] Flanigan (1983) page 221 ⟨Theorem 15⟩, Strichartz (1995) page 281, Sohrab (2003) page 317 ⟨Theorem 8.4.9⟩, Taylor (1715), Taylor (1717), Maclaurin (1742)

[22] Graham et al. (1994) page 162 ⟨(5.12)⟩, Rotman (2010) page 84 ⟨Proposition 2.5⟩, Bourbaki (2003a) page 99 ⟨Corollary 1⟩, Warner (1990) pages 189–190 ⟨Theorem 21.1⟩, Metzler et al. (1908) page 169 ⟨any real exponent⟩, Coolidge (1949)

1. Proof using Maclaurin series:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}y^k}\Big[(x+y)^n\Big]_{y=0} y^k && \text{by Maclaurin series (Theorem B.13 page 107)}\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\Big[n(n-1)(n-2)\cdots(n-k+1)(x+y)^{n-k}\Big]_{y=0} y^k\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{n!}{(n-k)!} x^{n-k}y^k\\
&= \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k}y^k && \text{by definition of } \binom{n}{k}\\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k + \cancelto{0}{\sum_{k=n+1}^{\infty} \binom{n}{k} x^{n-k}y^k}\\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k && \text{because } (x+y)^n \text{ has order } n
\end{aligned}$$

2. Proof using induction:

(a) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ is true for $n=0$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right|_{n=0} &= \binom{0}{0}x^0y^{0-0}\\
&= 1\\
&= (x+y)^n\big|_{n=0}
\end{aligned}$$

(b) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ is true for $n=1$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right|_{n=1} &= \binom{1}{0}x^0y^{1-0} + \binom{1}{1}x^1y^{1-1}\\
&= y+x\\
&= (x+y)^n\big|_{n=1}
\end{aligned}$$

(c) Proof that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \implies (x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n}{k}x^k y^{n+1-k}$:

$$\begin{aligned}
&\sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n+1-k}\\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n+1-k}\\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]x^k y^{n+1-k} && \text{by \textit{Pascal's Rule}}\\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}x^k y^{n+1-k} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n+1-k}\\
&= x^{n+1} + y^{n+1} + \left[\sum_{k=0}^{n}\binom{n}{k}x^{k+1} y^{n+1-(k+1)} - x^{n+1}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k} - y^{n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= x\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} + y\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \\
&= x(x+y)^n + y(x+y)^n && \text{by left hypothesis} \\
&= (x+y)(x+y)^n \\
&= (x+y)^{n+1}
\end{aligned}
$$

✏

APPENDIX C

CALCULUS

Definition C.1. *Let* $\mathbb{R}$ *be the set of real numbers,* $\mathscr{B}$ *the set of* BOREL SETS *on* $\mathbb{R}$*, and* μ *the standard* BOREL MEASURE *on* $\mathscr{B}$*. Let* $\mathbb{R}^{\mathbb{R}}$ *be as in Definition* 4.1 *page* 55*.*

DEF

The ***space of Lebesgue square-integrable functions*** $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *(or* $\boldsymbol{L}^2_{\mathbb{R}}$*) is defined as*

$$\boldsymbol{L}^2_{\mathbb{R}} \triangleq \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \mathsf{d}\mu < \infty \right\}.$$

The ***standard inner product*** $\langle \triangle \mid \nabla \rangle$ *on* $\boldsymbol{L}^2_{\mathbb{R}}$ *is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\,\mathsf{d}x.$$

The ***standard norm*** $\|\cdot\|$ *on* $\boldsymbol{L}^2_{\mathbb{R}}$ *is defined as* $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$

Definition C.2. *Let* $\mathsf{f}(x)$ *be a* FUNCTION *in* $\mathbb{R}^{\mathbb{R}}$*.*

DEF

$$\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon \to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition C.1.

PRP

$$\left\{ \begin{array}{ll} (1). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} \quad \textit{and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC } \textit{about a point a}} \end{array} \right\} \implies \left\{ \begin{array}{llcll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) & (\text{ANTI-SYMMETRIC } \textit{about a}) \\ (2). & \mathsf{f}'(a) & = & 0 & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned} \mathsf{f}'(a+x) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\ &= -\mathsf{f}(a-x) \end{aligned}$$

$$\begin{aligned} \mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\ &= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result} \end{aligned}$$

$$= 0$$

✏

Lemma C.1.

LEM
$$\mathsf{f}(x) \text{ is INVERTIBLE} \implies \left\{ \frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) = \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to 0}\frac{\mathsf{f}^{-1}(y+\varepsilon)-\mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}y} && \text{(Definition C.2 page 111)}\\
&= \lim_{\delta\to 0}\left.\frac{1}{\left[\frac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1}\\
&\triangleq \left.\frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x)}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition C.2 page 111)}\\
&= \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✏

Theorem C.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

THM
$$\int_{[0:1)^n}\mathsf{f}^{(n)}\left(\sum_{k=1}^{n}x_k\right)\mathsf{d}x_1\,\mathsf{d}x_2\cdots\,\mathsf{d}x_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n\in\mathbb{N}$$

✎PROOF: Proof by induction:

1. Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\int_{[0:1)}\mathsf{f}^{(1)}(x)\,\mathsf{d}x &= \mathsf{f}(1)-\mathsf{f}(0) && \text{by *Fundamental theorem of calculus*}\\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1) + (-1)^{1+0}\binom{1}{0}\mathsf{f}(0)\\
&= \sum_{k=0}^{1}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

2. Induction step …proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \mathsf{d}x_1\, \mathsf{d}x_2 \cdots \mathsf{d}x_{n+1} \\
&= \int_{[0:1)^n} \left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right) \mathsf{d}x_{n+1}\right] \mathsf{d}x_1\, \mathsf{d}x_2 \cdots \mathsf{d}x_n \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right] \mathsf{d}x_1\, \mathsf{d}x_2 \cdots \mathsf{d}x_n && \text{by \textit{Fundamental theorem of calculus}} \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(1 + \sum_{k=1}^{n} x_k\right) - \mathsf{f}^{(n)}\left(0 + \sum_{k=1}^{n} x_k\right)\right] \mathsf{d}x_1\, \mathsf{d}x_2 \cdots \mathsf{d}x_n \\
&= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1) - \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by induction hypothesis} \\
&= \sum_{m=1}^{m=n+1} (-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m) + \sum_{k=0}^{n} (-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{where } m \triangleq k+1 \implies k = m-1 \\
&= \left[\mathsf{f}(n+1) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right] + \left[(-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right] \\
&= \mathsf{f}(n+1) + (-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n} (-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\text{use \textit{Stifel formula}}}\mathsf{f}(k) \\
&= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1) + (-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by \textit{Stifel formula}} \\
&= \sum_{k=0}^{n+1} (-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)
\end{aligned}$$

✏

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma C.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots$, *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example C.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\big[\mathsf{f}(x)\mathsf{g}(x)\big] = \mathsf{f}'''(x)\mathsf{g}(x) + 3\mathsf{f}''(x)\mathsf{g}'(x) + 3\mathsf{f}'(x)\mathsf{g}''(x) + \mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem C.2 (Leibniz integration rule). [3]

[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)

[3] Flanders (1973) page 615 ⟨(1.1)⟩ Talvila (2001), Knapp (2005b) page 389 ⟨Chapter VII⟩, Protter and Morrey (2012) page 422 ⟨Leibniz Rule. Theorem 1.⟩, `http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html`

THM

$$\frac{\mathrm{d}}{\mathrm{dx}} \int_{a(x)}^{b(x)} g(t)\, dt = g[b(x)]b'(x) - g[a(x)]a'(x)$$

APPENDIX D

CONVOLUTION

D.1 Definition

Definition D.1. [1]

DEF

The ***convolution operation*** *is defined as*

$$\left[\mathsf{f}\star\mathsf{g}\right](x)\triangleq\mathsf{f}(x)\star\mathsf{g}(x)\triangleq\int_{u\in\mathbb{R}}\mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{d}u\qquad\forall\mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

D.2 Properties

Theorem D.1. [2]

THM

$$\begin{aligned}
\mathsf{f}\star\mathsf{g} &= \mathsf{g}\star\mathsf{f} && \text{(COMMUTATIVE)}\\
\mathsf{f}\star(\mathsf{g}\star\mathsf{h}) &= (\mathsf{g}\star\mathsf{g})\star\mathsf{h} && \text{(ASSOCIATEVE)}\\
(\alpha\mathsf{f})\star\mathsf{g} &= \alpha(\mathsf{f}\star\mathsf{g})=\mathsf{f}\star(\alpha\mathsf{g}) && \forall\alpha\in\mathbb{C}\\
\mathsf{f}\star(\mathsf{g}+\mathsf{h}) &= (\mathsf{f}\star\mathsf{g})+(\mathsf{f}\star\mathsf{h}) && \text{(DISTRIBUTIVE)}
\end{aligned}$$

[1] Bachman et al. (2002) page 268 ⟨Definition 5.2.1, but with $1/2\pi$ scaling factor⟩, Bachman (1964) page 6, Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩

[2] Bachman et al. (2002) pages 268–270

APPENDIX E

LAPLACE TRANSFORM

> *“La langue de l analyse, la plus parfaite de toutes les langues, tant par elle-même un puissant instrument de découvertes; ses notations, lorsqu'elles sont nécessaires et heureusement imaginées, sont des germes de nouveaux calculs.”*
>
> *“The language of analysis, most perfect of all, being in itself a powerful instrument of discoveries, its notations, especially when they are necessary and happily imagined, are the seeds of new calculi.”*
>
> Pierre-Simon Laplace[1]

E.1 Operator Definition

Definition E.1. [2] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF *The **Laplace Transform** operator* $\mathbf{L}$ *is here defined as*

$$[\mathbf{L}\mathsf{f}](s) \triangleq \int_{x\in\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathsf{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Such integrals may *converge* for certain values of s and *diverge* for others.

Definition E.2. *Let* $\mathbf{L}\left[\mathsf{g}(x)\right]$ *be the* LAPLACE TRANSFORM (Definition E.1 page 117) *of a function* $\mathsf{g}(x)$.

DEF *The set* $\mathbf{RoC}\mathbf{L}\left[\mathsf{g}(x)\right]$ *of all s for which* $\mathbf{L}\left[\mathsf{g}(x)\right]$ CONVERGES *is the **Region of Convergence** of* $\mathbf{L}\left[\mathsf{g}(x)\right]$.

In this text, the region of convergence may in places be specified using the *open interval* $(A : B)$ and *closed interval* $[A : B]$.

[1] Laplace (1814) page xxxi ⟨Introduction⟩, Laplace (1812), Laplace (1902) pages 48–49, Moritz (1914) page 200 ⟨Quote 1222., but “conceived” not “imagined”, and “are so many germs” not “are the seeds”⟩, `https://todayinsci.com/L/Laplace_Pierre/LaplacePierre-Analysis-Quotations.htm`, `https://translate.google.com/`,

[2] Bracewell (1978) page 219 ⟨Chapter 11 The Laplace transform⟩, van der Pol and Bremmer (1959) page 13 ⟨5. Strip of convergence of the Laplace integral⟩, Levy (1958) page 2 ⟨“two-sided transformation”⟩, Betten (2008b) page 295 ⟨(B.1)⟩

Remark E.1. A scaling factor $\frac{1}{\sqrt{2\pi}}$ in front of $\int_{\mathbb{R}}$ in Definition E.1 is not typically found in references offering definitions of the Laplace Transform, and is not included here either. That is not to say, however, that it's not a good idea. Including it would make the operator $\mathbf{L}$ more directly compatible with the *Unitary Fourier Transform* operator $\tilde{\mathbf{F}}$ (Definition 3.2 page 44). Note also that a $\frac{1}{2\pi}$ scaling factor is included in [Bachman et al. (2002) page 268] in their definition of *convolution* (Definition D.1 page 115, Section E.8 page 129).

E.2 Operator Inverse

Theorem E.1. [3]

THM
$$\mathsf{g}(x) = \mathbf{L}^{-1}[\mathsf{G}(s)] \triangleq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathsf{G}(s)e^{sx}\,ds \quad \textit{for some } c \in \mathbb{R}^+$$

E.3 Transversal properties

Theorem E.2. [4] *Let* $\mathbf{L}\left[\mathsf{g}(x)\right]$ *be the* LAPLACE TRANSFORM *(Definition E.1 page 117) of a function* $\mathsf{g}(x)$*. Let the* REGION OF CONVERGENCE *of* $\mathbf{L}[\mathsf{g}(x)](s)$ *be* $A \le \mathbf{R_e}(s) \le B$ *with* $(A,B) \in \mathbb{R}^2$*.*

THM

Mapping			*Region of Convergence*	*Domain*	*Property*
$\mathbf{L}\left[\mathsf{g}(x-\alpha)\right]$	$=$	$e^{-\alpha s}\mathbf{L}\left[\mathsf{g}(x)\right](s)$	*for* $\mathbf{R_e}(s) \in [A:B]$	$\forall x,\alpha \in \mathbb{C}$	(TRANSLATION)
$\mathbf{L}\left[\mathsf{g}(\alpha x)\right]$	$=$	$\frac{1}{\lvert\alpha\rvert}\mathbf{L}\left[\mathsf{g}(x)\right]\left(\frac{s}{\alpha}\right)$	*for* $\mathbf{R_e}\left(\frac{s}{\alpha}\right) \in [A:B]$	$\forall x,\alpha \in \mathbb{C}$	(DILATION)

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[\mathsf{g}(x-\alpha)\right] &\triangleq \int_{x=-\infty}^{x=\infty} \mathsf{g}(x-\alpha)e^{-sx}\,\mathsf{d}x && && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}\\
&= \int_{u+\alpha=-\infty}^{u+\alpha=\infty} \mathsf{g}(u)e^{-s(\alpha+u)}\,\mathsf{d}u && && \text{where } u \triangleq x-\alpha && \implies x = \alpha + u\\
&= e^{-\alpha s}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-su}\,\mathsf{d}u && \forall A \le \mathbf{R_e}(s) \le B && \text{by property of exponents} && b^{x+\alpha} = b^x b^\alpha\\
&\triangleq e^{-\alpha s}\int_{x=-\infty}^{x=\infty} \mathsf{g}(x)e^{-sx}\,\mathsf{d}x && \forall A \le \mathbf{R_e}(s) \le B && \text{by change of variable} && u \to x\\
&\triangleq e^{-\alpha s}\left[\mathbf{L}\mathsf{g}(x)\right] && \forall A \le \mathbf{R_e}(s) \le B && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\left[\mathsf{g}(\alpha x)\right] &\triangleq \int_{x=-\infty}^{x=\infty} \mathsf{g}(\alpha x)e^{-sx}\,\mathsf{d}x && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}\\
&= \int_{u/\alpha=-\infty}^{u/\alpha=\infty} \mathsf{g}(u)e^{-s(u/\alpha)}\frac{1}{\alpha}\,\mathsf{d}u && \text{where } u \triangleq \alpha x && \implies x = \frac{u}{\alpha}\\
&= \frac{1}{\alpha}\int_{u/\alpha=-\infty}^{u/\alpha=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{d}u
\end{aligned}$$

[3] Bracewell (1978) page 220 ⟨Chapter 11 The Laplace transform⟩
[4] Bracewell (1978) page 224 ⟨Table 11.1: "Shift" and "Similarity" entries⟩, Levy (1958) page 15 ⟨Equation 0.8⟩

$$
\begin{aligned}
&= \begin{cases} \frac{1}{\alpha}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{d}u & \text{if } \alpha \ge 0 \quad \forall A \le \mathbf{R_e}\left(\frac{s}{\alpha}\right) \le B \\ \frac{1}{\alpha}\int_{u=\infty}^{u=-\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{d}u & \text{otherwise} \quad \forall A \le \mathbf{R_e}\left(\frac{s}{\alpha}\right) \le B \end{cases} \\
&= \begin{cases} \frac{1}{\alpha}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{d}u & \text{if } \alpha \ge 0 \quad \forall A \le \mathbf{R_e}\left(\frac{s}{\alpha}\right) \le B \\ -\frac{1}{\alpha}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{d}u & \text{otherwise} \quad \forall A \le \mathbf{R_e}\left(\frac{s}{\alpha}\right) \le B \end{cases} \\
&= \frac{1}{|\alpha|}\int_{x\in\mathbb{R}} \mathsf{g}(x)e^{-(s/\alpha)x}\,\mathsf{d}x \qquad \forall A \le \mathbf{R_e}\left(\frac{s}{\alpha}\right) \le B \qquad \text{by change of variable} \qquad u \to x \\
&\triangleq \frac{1}{|\alpha|}\left[\mathbf{L}\mathsf{g}(x)\right]\left(\frac{s}{\alpha}\right) \qquad \forall A \le \mathbf{R_e}\left(\frac{s}{\alpha}\right) \le B \qquad \text{by definition of } \mathbf{L} \qquad \text{(Definition E.1 page 117)}
\end{aligned}
$$

✎

Corollary E.1. [5] *Let* $\mathbf{L}$, $\mathsf{G}(s)$, A, *and* B *be defined as in Theorem E.2 (page 118).*

COR	*Mapping*			*Region of Convergence*	*Domain*	*Property*
	$\mathbf{L}\left[\mathsf{g}(-x)\right]$	$=$	$\mathsf{G}(-s)$	*for* $\mathbf{R_e}(s) \in [-B : -A]$	$\forall x,\alpha\in\mathbb{C}$	(REVERSAL)

✎PROOF:

$$
\begin{aligned}
\mathbf{L}\left[\mathsf{g}(-x)\right] &= \mathbf{L}\left[\mathsf{g}([-1]x)\right] && \mathbf{R_e}(s) \in [A : B] && \text{by definition of unary operator } - \\
&= \mathbf{L}\left[\frac{1}{|-1|}\mathsf{g}\left(\frac{x}{-1}\right)\right] && \mathbf{R_e}\left(\frac{s}{-1}\right) \in [A : B] && \text{by } \textit{dilation} \text{ property (Theorem E.2 page 118)} \\
&= \mathsf{G}(-s) && \mathbf{R_e}(s) \in [-B : -A]
\end{aligned}
$$

✎

E.4 Linear properties

Theorem E.3. [6] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117). *Let* $\mathsf{G}(s) \triangleq \left[\mathbf{L}\mathsf{g}(x)\right]$ *and* $\mathsf{F}(s) \triangleq [\mathbf{L}\mathsf{f}(x)]$. *Let the* REGION OF CONVERGENCE *of* $\mathsf{G}(s)$ *be* $A \le \mathbf{R_e}(s) \le B$ *and the* REGION OF CONVERGENCE *of* $\mathsf{F}(s)$ *be* $C \le \mathbf{R_e}(s) \le D$.

THM	*Mapping*			*Region of Convergence*	*Domain*	*Property*
	$\mathbf{L}\left[\mathsf{f}(x)+\mathsf{g}(x)\right]$	$=$	$\mathsf{F}(s)+\mathsf{G}(s)$	*for* $\mathbf{R_e}(s) \in [A : B] \cap [C : D]$	$\forall x,\alpha\in\mathbb{C}$	(ADDITIVE)
	$\mathbf{L}\left[\alpha\mathsf{g}(x)\right]$	$=$	$\alpha\mathsf{G}(s)$	*for* $\mathbf{R_e}(s) \in [A : B]$	$\forall x,\alpha\in\mathbb{C}$	(HOMOGENEOUS)

Corollary E.2 (Linear Properties). *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117). *Let* A *and* B *be real numbers such that* $[A : B]$ *is the* REGION OF CONVERGENCE *of* $\mathbf{L}\left[\mathsf{g}(x)\right]$. *Let* C *and* D *be real numbers such that* $[C : D]$ *is the* REGION OF CONVERGENCE *of* $\mathbf{L}[\mathsf{f}(x)]$. *Let* A_n *and* B_n *be real numbers such that* $\left[A_n : B_n\right]$ *is the* REGION OF CONVERGENCE *of* $\mathbf{L}\left[\mathsf{g}_n(x)\right]$.

[5] Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩
[6] Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩, Betten (2008a) page 296 ⟨(B.6)⟩, Levy (1958) page 13 ⟨Equation 0.2⟩, van der Pol and Bremmer (1959) page 22 ⟨Introduction⟩, Shafii-Mousavi (2015) page 7 ⟨Theorem 1.4⟩

COR

Mapping			Region of Convergence	Domain
$\mathbf{L}[0]$	$=$	0	*for* $\mathbf{R_e}(s) \in [A : B]$	$\forall x \in \mathbb{C}$
$\mathbf{L}\left[-\mathsf{g}(x)\right]$	$=$	$-\mathbf{L}\left[\mathsf{g}(x)\right]$	*for* $\mathbf{R_e}(s) \in [A : B]$	$\forall x \in \mathbb{C}$
$\mathbf{L}\left[\mathsf{f}(x) - \mathsf{g}(x)\right]$	$=$	$\mathbf{L}\left[\mathsf{g}(x)\right] - \mathbf{L}[\mathsf{f}(x)]$	*for* $\mathbf{R_e}(s) \in [A : B] \cap [C : D]$	$\forall x \in \mathbb{C}$
$\mathbf{L}\left[\sum_{n=1}^{N} \alpha_n \mathsf{g}_n(x)\right]$	$=$	$\sum_{n=1}^{N} \alpha_n \mathbf{L}\left[\mathsf{g}_n(x)\right]$	*for* $\mathbf{R_e}(s) \in \bigcap_{n=1}^{N} \left[A_n : B_n\right]$	$\forall x, \alpha_n \in \mathbb{C}$

✎PROOF:

1. By Theorem E.3 (page 119), the operator *Laplace Transform* operator $\mathbf{L}$ is *additive* and *homogeneous*.
2. By item (1) and Definition G.4 (page 147), $\mathbf{L}$ is *linear*.
3. By item (2) and Theorem G.1 (page 147), the four properties listed follow.

✏

E.5 Modulation properties

Theorem E.4. [7] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117). *Let* $\mathsf{G}(s) \triangleq \left[\mathbf{L}\mathsf{g}(x)\right]$. *Let the* REGION OF CONVERGENCE *of* $\mathsf{G}(s)$ *be* $A \leq \mathbf{R_e}(s) \leq B$.

THM

Mapping			Region of Convergence	Domain	Property
$\mathbf{L}\left[e^{-\alpha x}\mathsf{g}(x)\right]$	$=$	$\mathsf{G}(s+\alpha)$	*for* $A - \mathbf{R_e}(\alpha) \leq \mathbf{R_e}(s) \leq B - \mathbf{R_e}(\alpha)$	$\forall x, \alpha \in \mathbb{C}$	(MODULATION)

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[e^{-\alpha x}\mathsf{g}(x)\right] &\triangleq \int_{x\in\mathbb{R}} e^{-\alpha x}\mathsf{g}(x)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}\\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)e^{-(s+\alpha)x}\,\mathrm{d}x && A \leq \mathbf{R_e}(s+\alpha) \leq B && b^{x+y} = b^x b^y\\
&\triangleq \left[\mathbf{L}\mathsf{g}(x)\right](s+\alpha) && A - \mathbf{R_e}(\alpha) \leq \mathbf{R_e}(s) \leq B - \mathbf{R_e}(\alpha) && \text{(Definition E.1 page 117)}\\
&\triangleq \mathsf{G}(s+\alpha) && A - \mathbf{R_e}(\alpha) \leq \mathbf{R_e}(s) \leq B - \mathbf{R_e}(\alpha) && \text{by definition of } \mathsf{G}(s)
\end{aligned}$$

✏

Corollary E.3. [8] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117). *Let* $\mathsf{G}(s) \triangleq \left[\mathbf{L}\mathsf{g}(x)\right]$ *Let the* REGION OF CONVERGENCE *of* $\mathsf{G}(s)$ *be* $A \leq \mathbf{R_e}(s) \leq B$.

COR

Mapping			Region of Convergence	Domain
$\mathbf{L}\left[\cos(\omega_o x)\mathsf{g}(x)\right]$	$=$	$\frac{1}{2}\mathsf{G}(s - i\omega_o) + \frac{1}{2}\mathsf{G}(s + i\omega_o)$	*for* $\mathbf{R_e}(s) \in [A : B]$	$\forall x\omega_o \in \mathbb{C}$
$\mathbf{L}\left[\sin(\omega_o x)\mathsf{g}(x)\right]$	$=$	$-\frac{i}{2}\mathsf{G}(s - i\omega_o) + \frac{i}{2}\mathsf{G}(s + i\omega_o)$	*for* $\mathbf{R_e}(s) \in [A : B]$	$\forall x\omega_o \in \mathbb{C}$
$\mathbf{L}\left[\cosh(\omega_o x)\mathsf{g}(x)\right]$	$=$	$\frac{1}{2}\mathsf{G}(s - \omega_o) + \frac{1}{2}\mathsf{G}(s + \omega_o)$	*for* $\mathbf{R_e}(s) \in \left[A + \left\lvert\mathbf{R_e}(\omega_o)\right\rvert : B - \left\lvert\mathbf{R_e}(\omega_o)\right\rvert\right]$	$\forall x\omega_o \in \mathbb{C}$
$\mathbf{L}\left[\sinh(\omega_o x)\mathsf{g}(x)\right]$	$=$	$\frac{1}{2}\mathsf{G}(s - \omega_o) - \frac{1}{2}\mathsf{G}(s + \omega_o)$	*for* $\mathbf{R_e}(s) \in \left[A + \left\lvert\mathbf{R_e}(\omega_o)\right\rvert : B - \left\lvert\mathbf{R_e}(\omega_o)\right\rvert\right]$	$\forall x\omega_o \in \mathbb{C}$

[7] Bracewell (1978) page 224 ⟨Table 11.1: “Modulation” entry⟩, Levy (1958) page 19 ⟨Equation 1.2⟩
[8] Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩

✎PROOF:

1. Mappings:

$$\begin{aligned}
\mathbf{L}\left[\cosh(\omega_o x)\mathsf{g}(x)\right] &= \mathbf{L}\left[\left(\frac{e^{\omega_o x}+e^{-\omega_o x}}{2}\right)\mathsf{g}(x)\right] && \text{by definition of } \cosh(x) && \text{(Definition 1.5 page 16)}\\
&= \frac{1}{2}\mathbf{L}\left[e^{\omega_o x}\mathsf{g}(x)\right](s)+\frac{1}{2}\mathbf{L}\left[e^{-\omega_o x}\mathsf{g}(x)\right](s) && \text{by \textit{additive} property} && \text{(Theorem E.3 page 119)}\\
&= \frac{1}{2}\mathbf{L}\left[\mathsf{g}(x)\right](s-\omega)+\frac{1}{2}\mathbf{L}\left[\mathsf{g}(x)\right](s+\omega) && \text{by \textit{modulation} prop.} && \text{(Theorem E.4 page 120)}\\
&= \frac{1}{2}\mathsf{G}(s-\omega_o)+\frac{1}{2}\mathsf{G}(s+\omega_o) && \text{by definition of } \mathsf{G}(s)
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\left[\sinh(\omega_o x)\mathsf{g}(x)\right] &= \mathbf{L}\left[\left(\frac{e^{\omega_o x}-e^{-\omega_o x}}{2}\right)\mathsf{g}(x)\right] && \text{by definition of } \sinh(x) && \text{(Definition 1.5 page 16)}\\
&= \frac{1}{2}\mathbf{L}\left[e^{\omega_o x}\mathsf{g}(x)\right](s)-\frac{1}{2}\mathbf{L}\left[e^{-\omega_o x}\mathsf{g}(x)\right](s) && \text{by \textit{additive} property} && \text{(Theorem E.3 page 119)}\\
&= \frac{1}{2}\mathbf{L}\left[\mathsf{g}(x)\right](s-\omega)-\frac{1}{2}\mathbf{L}\left[\mathsf{g}(x)\right](s+\omega) && \text{by \textit{modulation} prop.} && \text{(Theorem E.4 page 120)}\\
&= \frac{1}{2}\mathsf{G}(s-\omega_o)-\frac{1}{2}\mathsf{G}(s+\omega_o) && \text{by definition of } \mathsf{G}(s)
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\left[\cos(\omega_o x)\mathsf{g}(x)\right] &= \mathbf{L}\left[\cosh(i\omega_o x)\mathsf{g}(x)\right] && \text{by Theorem 1.12 page 17}\\
&= \frac{1}{2}\mathsf{G}(s-i\omega_o)+\frac{1}{2}\mathsf{G}(s+i\omega_o) && \text{by } \mathbf{L}\left[\cos(\omega_o x)\mathsf{g}(x)\right] \text{ result}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\left[\sin(\omega_o x)\mathsf{g}(x)\right] &= \mathbf{L}\left[-i^2\sin(\omega_o x)\mathsf{g}(x)\right]\\
&= -i\mathbf{L}\left[i\sin(\omega_o x)\mathsf{g}(x)\right] && \text{by \textit{homogeneous} property} && \text{(Theorem E.3 page 119)}\\
&= -i\mathbf{L}\left[\sinh(i\omega_o x)\mathsf{g}(x)\right] && \text{by Theorem 1.12 page 17}\\
&= -\frac{i}{2}\mathsf{G}(s-i\omega_o)+\frac{i}{2}\mathsf{G}(s+i\omega_o) && \text{by } \mathbf{L}\left[\sin(\omega_o x)\mathsf{g}(x)\right] \text{ result}
\end{aligned}$$

2. Region of Convergence of $\mathbf{L}\left[\cos(\omega_o x)\mathsf{g}(x)\right]$ and $\mathbf{L}\left[\sin(\omega_o x)\mathsf{g}(x)\right]$:

$$\begin{aligned}
&\mathbf{RocL}\left[{}^{\cos}\!/\!{}_{\sin}(\omega_o x)\mathsf{g}(x)\right]\\
&= \mathbf{RocL}\left[\left(\frac{e^{i\omega_o x}\pm e^{-i\omega_o x}}{2}\right)\mathsf{g}(x)\right] && \text{by \textit{Euler's Identity}} && \text{(Theorem 1.5 page 8)}\\
&= \mathbf{Roc}\left(\mathbf{L}\left[\frac{e^{i\omega_o x}}{2}\mathsf{g}(x)\right]\pm\mathbf{L}\left[\frac{e^{-i\omega_o x}}{2}\mathsf{g}(x)\right]\right) && \text{by \textit{additive} property} && \text{(Theorem E.3 page 119)}\\
&= \mathbf{RocL}\left[\left(\frac{e^{-i\omega_o x}}{2}\right)\mathsf{g}(x)\right]\cap\mathbf{RocL}\left[\left(\frac{e^{i\omega_o x}}{2}\right)\mathsf{g}(x)\right]\\
&= \left[A-\mathbf{R_e}(i\omega) : B-\mathbf{R_e}(i\omega)\right]\cap\left[A-\mathbf{R_e}(-i\omega) : B-\mathbf{R_e}(-i\omega)\right]\\
&= [A-0 : B-0]\cap[A-0 : B-0]\\
&= [A : B]
\end{aligned}$$

3. Region of Convergence of $\mathbf{L}\left[\cosh(\omega_o x)\mathsf{g}(x)\right]$ and $\mathbf{L}\left[\sinh(\omega_o x)\mathsf{g}(x)\right]$:

$$\begin{aligned}
&\mathbf{RocL}\left[{}^{\cosh}\!/\!{}_{\sinh}(\omega_o x)\mathsf{g}(x)\right]\\
&= \mathbf{RocL}\left[\left(\frac{e^{\omega_o x}\pm e^{-\omega_o x}}{2}\right)\mathsf{g}(x)\right] && \text{by def. } \cosh(x),\ \sinh(x) && \text{(Definition 1.5 page 16)}
\end{aligned}$$

$$= \mathbf{Roc}\left(\mathbf{L}\left[\frac{e^{\omega_o x}}{2}\mathrm{g}(x)\right] \pm \mathbf{L}\left[\frac{e^{-\omega_o x}}{2}\mathrm{g}(x)\right]\right) \quad \text{by } \textit{additive} \text{ property} \quad \text{(Theorem E.3 page 119)}$$

$$= \mathbf{RocL}\left[\left(\frac{e^{-\omega_o x}}{2}\right)\mathrm{g}(x)\right] \cap \mathbf{RocL}\left[\left(\frac{e^{\omega_o x}}{2}\right)\mathrm{g}(x)\right]$$

$$= \left[A - \mathbf{R}_e(\omega_o) : B - \mathbf{R}_e(\omega_o)\right] \cap \left[A - \mathbf{R}_e(-\omega_o) : B - \mathbf{R}_e(-\omega_o)\right]$$

$$= \begin{cases} \left[A + \mathbf{R}_e(\omega_o) : B - \mathbf{R}_e(\omega_o)\right] & \text{for } \omega \geq 0 \\ \left[A - \mathbf{R}_e(\omega_o) : B + \mathbf{R}_e(\omega_o)\right] & \text{otherwise} \end{cases}$$

$$= \left[A + \left|\mathbf{R}_e(\omega_o)\right| : B - \left|\mathbf{R}_e(\omega_o)\right|\right] \quad \text{by definition of } |x|$$

✏

E.6 Causality properties

Definition E.3. [9] *The* ***Heaviside step function*** $\mu(x)$ *or* ***unit step function*** *is defined as*

DEF

$$\mu(x) \triangleq \begin{cases} 1 & \forall x \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Theorem E.5. [10] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117) *and* $\mu(x)$ *the* UNIT STEP *function* (Definition E.3 page 122).

THM

	Mapping			Region of Convergence	Domain
(1).	$\mathbf{L}[\mu(x)]$	$=$	$\frac{1}{s}$	for $\mathbf{R}_e(s) > 0$	$\forall x \in \mathbb{R}$
(2).	$\mathbf{L}[\mu(-x)]$	$=$	$-\frac{1}{s}$	for $\mathbf{R}_e(s) < 0$	$\forall x \in \mathbb{R}$

✎PROOF:

$$\mathbf{L}[\mu(x)] \triangleq \int_{\mathbb{R}} \mu(x) e^{-sx}\, \mathrm{d}x \quad \text{by definition of } \mathbf{L} \quad \text{(Definition E.1 page 117)}$$

$$= \int_0^\infty e^{-sx}\, \mathrm{d}x \quad \text{by definition of } \mu(x) \quad \text{(Definition E.3 page 122)}$$

$$= \left.\frac{e^{-sx}}{-s}\right|_0^\infty \quad \text{by } \textit{Fundamental Theorem of Calculus}$$

$$= \lim_{x\to\infty}\left[\frac{e^{-sx}}{-s}\right] - \left(\frac{e^0}{-s}\right)$$

$$= 0 + \frac{1}{s} \qquad \forall \mathbf{R}_e(s) > 0$$

$$= \frac{1}{s} \qquad \forall \mathbf{R}_e(s) > 0$$

$$\mathbf{L}[\mu(-x)] = \mathbf{L}[\mu(x)](-s) \qquad \mathbf{R}_e(s) < 0 \quad \text{by } \textit{reversal} \text{ property} \quad \text{(Corollary E.1 page 119)}$$

$$= \frac{-1}{s} \qquad \mathbf{R}_e(s) < 0 \quad \text{by (1)}$$

✏

[9] Betten (2008a) page 285

[10] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩

Corollary E.4. [11] *Let* $\mathbf{L}$ *be the* Laplace Transform *operator* (Definition E.1 page 117) *and* $\mu(x)$ *the* unit step *function.*

COR	Mapping			Region of Convergence	Domain
	$\mathbf{L}\left[e^{-\alpha x}\mu(x)\right]$	$=$	$\dfrac{1}{s+\alpha}$	for $\mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)$	$\forall x\in\mathbb{R};\alpha\in\mathbb{C}$
	$\mathbf{L}\left[e^{-\alpha x}\mu(-x)\right]$	$=$	$\dfrac{1}{s+\alpha}$	for $\mathbf{R_e}(s) < \mathbf{R_e}(\alpha)$	$\forall x\in\mathbb{R};\alpha\in\mathbb{C}$

✎Proof:

$$\begin{aligned}
\mathbf{L}\left[e^{-\alpha x}\mu(x)\right](s) &= \mathbf{L}[\mu(x)](s+\alpha) && \text{by } \textit{modulation} \quad \text{(Theorem E.4 page 120)}\\
&= \frac{1}{s+\alpha} && \forall\mathbf{R_e}(s)\in\big(0-\mathbf{R_e}(\alpha) : \infty-\mathbf{R_e}(\alpha)\big) \text{ by Theorem E.5 page 122}\\
&= \frac{1}{s+\alpha} && \forall\mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\left[e^{-\alpha x}\mu(-x)\right](s) &= \mathbf{L}[\mu(-x)](s+\alpha) && \text{by } \textit{modulation} \quad \text{(Theorem E.4 page 120)}\\
&= \frac{-1}{s+\alpha} && \forall\mathbf{R_e}(s)\in\big(-\infty-\mathbf{R_e}(\alpha) : 0-(-\mathbf{R_e}(\alpha))\big) \text{ by Theorem E.5 page 122}\\
&= \frac{-1}{s+\alpha} && \forall\mathbf{R_e}(s) < \mathbf{R_e}(\alpha) \quad \text{by } \textit{anti-causality} \quad \text{(Theorem E.5 page 122)}
\end{aligned}$$

✎

Corollary E.5. [12] *Let* $\mathbf{L}$ *be the* Laplace Transform *operator* (Definition E.1 page 117) *and* $\mu(x)$ *the* unit step *function.*

COR		Mapping			Region of Convergence	Domain
	(1).	$\mathbf{L}\left[\cos(\omega_o x)\mu(x)\right]$	$=$	$\dfrac{s}{s^2+\omega_o^2}$	for $\mathbf{R_e}(s) > 0$	$x,\omega_o\in\mathbb{R}$
	(2).	$\mathbf{L}\left[\sin(\omega_o x)\mu(x)\right]$	$=$	$\dfrac{\omega_o}{s^2+\omega_o^2}$	for $\mathbf{R_e}(s) > 0$	$x,\omega_o\in\mathbb{R}$
	(3).	$\mathbf{L}\left[\cos(\omega_o x)\mu(-x)\right]$	$=$	$\dfrac{-s}{s^2+\omega_o^2}$	for $\mathbf{R_e}(s) < 0$	$x,\omega_o\in\mathbb{R}$
	(4).	$\mathbf{L}\left[\sin(\omega_o x)\mu(-x)\right]$	$=$	$\dfrac{-\omega_o}{s^2+\omega_o^2}$	for $\mathbf{R_e}(s) < 0$	$x,\omega_o\in\mathbb{R}$

✎Proof:

$$\begin{aligned}
\mathbf{L}\left[\cos(\omega_o x)\mu(x)\right](s) &= \frac{1}{2}\mathbf{L}[\mu(x)](s-i\omega_o) + \frac{1}{2}\mathbf{L}[\mu(x)](s+i\omega_o) && \text{by } \textit{modulation} \quad \text{(Corollary E.3 page 120)}\\
&= \frac{1}{2}\left[\frac{1}{s-i\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+i\omega_o}\right] && \mathbf{R_e}(s) > 0 \quad \text{by } \textit{causal} \text{ prop.} \quad \text{(Theorem E.5 page 122)}\\
&= \frac{1}{2}\left[\frac{1}{s-i\omega_o}\right]\left[\frac{s+i\omega_o}{s+i\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+i\omega_o}\right]\left[\frac{s-i\omega_o}{s-i\omega_o}\right] && (\textit{Rationalizing the Denominator})\\
&= \frac{1}{2}\left[\frac{(s+i\omega_o)+(s-i\omega_o)}{s^2+\omega_o^2}\right] && \mathbf{R_e}(s) > 0\\
&= \frac{s}{s^2+\omega_o^2} && \mathbf{R_e}(s) > 0
\end{aligned}$$

[11] van der Pol and Bremmer (1959) page 22 ⟨Introduction⟩, Shafii-Mousavi (2015) page 3 ⟨Table 1, using One-Sided Laplace Transform⟩, van der Pol and Bremmer (1959) page 26 ⟨(8) seems to have an error: $\frac{s}{s+\alpha}$⟩

[12] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩, Shafii-Mousavi (2015) page 3 ⟨Table 1, using One-Sided Laplace Transform⟩

$$
\begin{aligned}
\mathbf{L}\left[\sin(\omega_o x)\mu(x)\right](s) &= -\frac{i}{2}\mathbf{L}[\mu(x)](s-i\omega_o) + \frac{i}{2}\mathbf{L}[\mu(x)](s+i\omega_o) && \text{by } \textit{modulation} \quad \text{(Corollary E.3 page 120)}\\
&= -\frac{i}{2}\left[\frac{1}{s-i\omega_o}\right] + \frac{i}{2}\left[\frac{1}{s+i\omega_o}\right] \qquad \mathbf{R}_e(s) > 0 && \text{by } \textit{causal} \text{ prop.} \quad \text{(Theorem E.5 page 122)}\\
&= -\frac{i}{2}\left[\frac{1}{s-i\omega_o}\right]\left[\frac{s+i\omega_o}{s+i\omega_o}\right] + \frac{i}{2}\left[\frac{1}{s+i\omega_o}\right]\left[\frac{s-i\omega_o}{s-i\omega_o}\right] && (\textit{Rationalizing the Denominator})\\
&= \frac{i}{2}\left[\frac{-(s+i\omega_o)+(s-i\omega_o)}{s^2+\omega_o^2}\right] && \mathbf{R}_e(s) > 0\\
&= \frac{\omega_o}{s^2+\omega_o^2} && \mathbf{R}_e(s) > 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}\left[\mu(-x)\cos(\omega_o x)\right](s) &= \mathbf{L}\left[\mu(-x)\cos(\omega_o(-x))\right](s) && \text{by } \textit{even} \text{ property of } \cos(x) \quad \text{(Theorem 1.2 page 5)}\\
&= \mathbf{L}\left[\mu(x)\cos(\omega_o x)\right](-s) && \text{by } \textit{reversal} \text{ property} \quad \text{(Corollary E.1 page 119)}\\
&= \frac{(-s)}{(-s)^2+\omega_o^2} \qquad \mathbf{R}_e(s) < 0 && \text{by (1)}\\
&= \frac{-s}{s^2+\omega_o^2} \qquad \mathbf{R}_e(s) < 0\\
\mathbf{L}\left[\sin(\omega_o x)\mu(-x)\right](s) &= \mathbf{L}\left[-\sin(\omega_o(-x))\mu(-x)\right](s) && \text{by } \textit{odd} \text{ property of } \sin(x) \quad \text{(Theorem 1.2 page 5)}\\
&= -\mathbf{L}\left[\sin(\omega_o(-x))\mu(-x)\right](s) && \text{by } \textit{homogeneous} \text{ property} \quad \text{(Theorem E.3 page 119)}\\
&= -\mathbf{L}\left[\sin(\omega_o x)\mu(x)\right](-s) && \text{by } \textit{reversal} \text{ property} \quad \text{(Corollary E.1 page 119)}\\
&= -\left[\frac{\omega_o}{(-s)^2+\omega_o^2}\right] \qquad \mathbf{R}_e(s) < 0 && \text{by (2)}\\
&= \frac{-\omega_o}{s^2+\omega_o^2} \qquad \mathbf{R}_e(s) < 0
\end{aligned}
$$

✏

Corollary E.6. [13] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117) *and* $\mu(x)$ *the* UNIT STEP *function.*

COR

	Mapping			Region of Convergence	Domain
(1).	$\mathbf{L}\left[\cosh(\omega_o x)\mu(x)\right]$	=	$\frac{s}{s^2-\omega_o^2}$	*for* $\mathbf{R}_e(s) > \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$
(2).	$\mathbf{L}\left[\sinh(\omega_o x)\mu(x)\right]$	=	$\frac{\omega_o}{s^2-\omega_o^2}$	*for* $\mathbf{R}_e(s) > \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$
(3).	$\mathbf{L}\left[\sinh(\omega_o x)\mu(-x)\right]$	=	$\frac{-s}{s^2-\omega_o^2}$	*for* $\mathbf{R}_e(s) < \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$
(4).	$\mathbf{L}\left[\sinh(\omega_o x)\mu(-x)\right]$	=	$\frac{-\omega_o}{s^2-\omega_o^2}$	*for* $\mathbf{R}_e(s) < \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$

✎PROOF:

1. Mappings for $\mathbf{L}\left[\cosh(\omega_o x)\mu(x)\right]$ and $\mathbf{L}\left[\sinh(\omega_o x)\mu(x)\right]$:

$$
\begin{aligned}
\mathbf{L}\left[\cosh(\omega_o x)\mu(x)\right](s) &= \frac{1}{2}\mathbf{L}[\mu(x)](s-\omega_o) + \frac{1}{2}\mathbf{L}[\mu(x)](s+\omega_o) && \text{by } \textit{modulation} \quad \text{(Corollary E.3 page 120)}\\
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+\omega_o}\right] && \text{by } \textit{causal} \text{ property} \quad \text{(Theorem E.5 page 122)}
\end{aligned}
$$

[13] Shafii-Mousavi (2015) page 3 ⟨Table 1, using One-Sided Laplace Transform⟩

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right]\left[\frac{s+\omega_o}{s+\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+\omega_o}\right]\left[\frac{s-\omega_o}{s-\omega_o}\right] \\
&= \frac{1}{2}\left[\frac{(s+\omega_o)+(s-\omega_o)}{s^2-\omega_o^2}\right] \\
&= \frac{s}{s^2-\omega_o^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}\left[\sinh(\omega_o x)\mu(x)\right](s) &= \frac{1}{2}\mathbf{L}[\mu(x)](s-\omega_o) - \frac{1}{2}\mathbf{L}[\mu(x)](s+\omega_o) && \text{by } \textit{modulation} && \text{(Corollary E.3 page 120)} \\
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right] - \frac{1}{2}\left[\frac{1}{s+\omega_o}\right] && \text{by } \textit{causal} \text{ property} && \text{(Theorem E.5 page 122)} \\
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right]\left[\frac{s+\omega_o}{s+\omega_o}\right] - \frac{1}{2}\left[\frac{1}{s+\omega_o}\right]\left[\frac{s-\omega_o}{s-\omega_o}\right] \\
&= \frac{1}{2}\left[\frac{(s+\omega_o)-(s-\omega_o)}{s^2-\omega_o^2}\right] \\
&= \frac{\omega_o}{s^2-\omega_o^2}
\end{aligned}
$$

2. Region of Convergence of $\mathbf{L}\left[\cosh(\omega_o x)\mu(x)\right]$ and $\mathbf{L}\left[\sinh(\omega_o x)\mu(x)\right]$:

$$
\begin{aligned}
\mathbf{RoC}\mathbf{L}\left[\cosh(\omega_o x)\mu(x)\right] &= \left[A + \left|\mathbf{R}_e(\omega_o)\right| : B - \left|\mathbf{R}_e(\omega_o)\right|\right] && \text{by Corollary E.3 page 120} \\
&= \left(0 + \left|\mathbf{R}_e(\omega_o)\right| : \infty - \left|\mathbf{R}_e(\omega_o)\right|\right) && \text{by Theorem E.5 page 122} \\
&= \left(\left|\mathbf{R}_e(\omega_o)\right| : \infty\right) \\
&\implies \mathbf{RoC}\mathbf{L}\left[\cosh(\omega_o x)\mu(x)\right] > \left|\mathbf{R}_e(\omega_o)\right|
\end{aligned}
$$

3. Mappings for $\mathbf{L}\left[\cosh(\omega_o x)\mu(-x)\right]$ and $\mathbf{L}\left[\sinh(\omega_o x)\mu(-x)\right]$:

$$
\begin{aligned}
\mathbf{L}\left[\cosh(\omega_o x)\mu(-x)\right](s) &= \mathbf{L}\left[\cosh(\omega_o(-x))\mu(-x)\right](s) \\
&= \mathbf{L}\left[\cosh(\omega_o x)\mu(x)\right](-s) && \text{by } \textit{reversal} \text{ property} && \text{(Corollary E.1 page 119)} \\
&= \frac{(-s)}{(-s)^2-\omega_o^2} && \text{by previous result} \\
&= \frac{-s}{s^2-\omega_o^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}\left[\sinh(\omega_o x)\mu(-x)\right](s) &= \mathbf{L}\left[-\sinh(\omega_o(-x))\mu(-x)\right](s) \\
&= -\mathbf{L}\left[\sinh(\omega_o(-x))\mu(-x)\right](s) && \text{by } \textit{homogeneous} \text{ property} && \text{(Theorem E.3 page 119)} \\
&= -\mathbf{L}\left[\sinh(\omega_o x)\mu(x)\right](-s) && \text{by } \textit{reversal} \text{ property} && \text{(Corollary E.1 page 119)} \\
&= \frac{-\omega_o}{(-s)^2-\omega_o^2} && \text{by previous result} \\
&= \frac{-\omega_o}{s^2-\omega_o^2}
\end{aligned}
$$

4. Region of Convergence of $\mathbf{L}\left[\cosh(\omega_o x)\mu(-x)\right]$ and $\mathbf{L}\left[\sinh(\omega_o x)\mu(-x)\right]$:

$$
\begin{aligned}
\mathbf{RoC}\mathbf{L}\left[\cosh(\omega_o x)\mu(-x)\right] &= \left[A + \left|\mathbf{R}_e(\omega_o)\right| : B - \left|\mathbf{R}_e(\omega_o)\right|\right] && \text{by Corollary E.3 page 120} \\
&= \left(-\infty + \left|\mathbf{R}_e(\omega_o)\right| : 0 - \left|\mathbf{R}_e(\omega_o)\right|\right) && \text{by Theorem E.5 page 122} \\
&= \left(-\infty : \left|\mathbf{R}_e(\omega_o)\right|\right) \\
&\implies \mathbf{RoC}\mathbf{L}\left[\cosh(\omega_o x)\mu(-x)\right] < \left|\mathbf{R}_e(\omega_o)\right|
\end{aligned}
$$

Corollary E.7. [14] *Let* **L** *be the* LAPLACE TRANSFORM *operator (Definition E.1 page 117) and* $\mu(x)$ *the* UNIT STEP *function.*

	Mapping			*Region of Convergence*		*Domain*
(1).	$\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha x}\mu(x)\right]$	$=$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$	*for*	$\mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)$	$x,\omega_o \in \mathbb{R};\ \alpha \in \mathbb{C}$
(2).	$\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha x}\mu(x)\right]$	$=$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$	*for*	$\mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)$	$x,\omega_o \in \mathbb{R};\ \alpha \in \mathbb{C}$
(3).	$\mathbf{L}\left[\cos(\omega_o x)e^{\alpha x}\mu(-x)\right]$	$=$	$\dfrac{-s+\alpha}{(s-\alpha)^2+\omega_o^2}$	*for*	$\mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)$	$x,\omega_o \in \mathbb{R};\ \alpha \in \mathbb{C}$
(4).	$\mathbf{L}\left[\sin(\omega_o x)e^{\alpha x}\mu(-x)\right]$	$=$	$\dfrac{-\omega_o}{(s-\alpha)^2+\omega_o^2}$	*for*	$\mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)$	$x,\omega_o \in \mathbb{R};\ \alpha \in \mathbb{C}$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha x}\mu(x)\right](s) &= \mathbf{L}\left[\mu(x)\cos(\omega_o x)\right](s+\alpha) && \text{by } \textit{modulation} \text{ property (Theorem E.4 page 120)}\\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) \in \left(0-\mathbf{R_e}(\alpha) : \infty - \mathbf{R_e}(\alpha)\right) \text{ by Corollary E.5}\\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)\\
\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha x}\mu(x)\right](s) &= \mathbf{L}\left[\mu(x)\sin(\omega_o x)\right](s+\alpha) && \text{by } \textit{modulation} \text{ property (Theorem E.4 page 120)}\\
&= \frac{\omega_o}{(s+\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) \in \left(0-\mathbf{R_e}(\alpha) : \infty - \mathbf{R_e}(\alpha)\right) \text{ by Corollary E.5}\\
&= \frac{\omega_o}{(s+\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)\\
\mathbf{L}\left[\cos(\omega_o x)e^{\alpha x}\mu(-x)\right](s) &= \mathbf{L}\left[\mu(-x)\cos(\omega_o x)\right](s-\alpha) && \text{by } \textit{modulation} \text{ property (Theorem E.4 page 120)}\\
&= \frac{-(s-\alpha)}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) \in \left(0-\mathbf{R_e}(\alpha) : \infty - \mathbf{R_e}(\alpha)\right) \text{ by Corollary E.5}\\
&= \frac{-s+\alpha}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)\\
\mathbf{L}\left[\sin(\omega_o x)e^{\alpha x}\mu(-x)\right](s) &= \mathbf{L}\left[\mu(-x)\sin(\omega_o x)\right](s-\alpha) && \text{by } \textit{modulation} \text{ property (Theorem E.4 page 120)}\\
&= \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) \in \left(0-\mathbf{R_e}(\alpha) : \infty - \mathbf{R_e}(\alpha)\right) \text{ by Corollary E.5}\\
&= \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)
\end{aligned}$$

✏

Corollary E.8. *Let* **L** *be the* LAPLACE TRANSFORM *operator (Definition E.1 page 117).*

$\mathbf{L}\left[\cos(\omega_o x)\right]$	***is divergent***	$\forall s \in \mathbb{C}$	$\forall x,\omega_o \in \mathbb{R}$
$\mathbf{L}\left[\sin(\omega_o x)\right]$	***is divergent***	$\forall s \in \mathbb{C}$	$\forall x,\omega_o \in \mathbb{R}\backslash\{0\}$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[\cos(\omega_o x)\right] &= \underbrace{\mathbf{L}\left[\mu(x)\cos(\omega_o x)\right]}_{\forall \mathbf{R_e}(s) > 0} + \underbrace{\mathbf{L}\left[\mu(-x)\cos(\omega_o x)\right]}_{\forall \mathbf{R_e}(s) < 0} && \text{by Corollary E.5 page 123}\\
&= \underbrace{\frac{s}{s^2+\omega_o^2}}_{\forall \mathbf{R_e}(s) > 0} + \underbrace{\frac{-s}{s^2+\omega_o^2}}_{\forall \mathbf{R_e}(s) < 0} && \text{by Corollary E.5 page 123}
\end{aligned}$$

[14] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩

$$
\begin{aligned}
&= \begin{cases} 0 & \forall \mathbf{R_e}(s) \in (-\infty : 0) \cap (0 : \infty) = \varnothing \\ \infty & \forall s \in \mathbb{C} \end{cases} \\
&\implies \mathbf{L}\left[\cos(\omega_o x)\right] \text{ is } \textbf{divergent } \forall s \in \mathbb{C} \\
\mathbf{L}\left[\sin(\omega_o x)\right] &= \underbrace{\mathbf{L}\left[\mu(x)\sin(\omega_o x)\right]}_{\forall \mathbf{R_e}(s) > 0} + \underbrace{\mathbf{L}\left[\mu(-x)\sin(\omega_o x)\right]}_{\forall \mathbf{R_e}(s) < 0} && \text{by Corollary E.5 page 123} \\
&= \underbrace{\frac{\omega_o}{s^2+\omega_o^2}}_{\forall \mathbf{R_e}(s) > 0} + \underbrace{\frac{-\omega_o}{s^2+\omega_o^2}}_{\forall \mathbf{R_e}(s) < 0} && \text{by Corollary E.5 page 123} \\
&= \begin{cases} 0 & \forall \mathbf{R_e}(s) \in (-\infty : 0) \cap (0 : \infty) = \varnothing \\ \infty & \forall s \in \mathbb{C} \end{cases} \\
&\implies \mathbf{L}\left[\sin(\omega_o x)\right] \text{ is } \textbf{divergent } \forall s \in \mathbb{C}
\end{aligned}
$$

✏

E.7 Exponential decay properties

Corollary E.9. [15] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117) *and* $\mu(x)$ *the* UNIT STEP *function. Let* $A \triangleq \mathbf{R_e}(\alpha)$.

COR	Mapping	Region of Convergence	Domain
	$\mathbf{L}\left[e^{-\alpha\lvert x\rvert}\right] = \dfrac{2\alpha}{\alpha^2 - s^2}$	for $\mathbf{R_e}(s) \in (-A : A)$	$x,\alpha \in \mathbb{C}$

✎PROOF:

$$
\begin{aligned}
\mathbf{L}\left[e^{-\alpha|x|}\right] &= \mathbf{L}\left[e^{-\alpha|x|}\mu(x) + e^{-\alpha|x|}\mu(-x)\right] && \text{by definition of } \mu(x) && \text{(Definition E.3 page 122)} \\
&= \mathbf{L}\left[e^{-\alpha|x|}\mu(x)\right] + \mathbf{L}\left[e^{-\alpha|x|}\mu(-x)\right] && \text{by } \textit{homogeneous} \text{ property} && \text{(Theorem E.3 page 119)} \\
&= \underbrace{\mathbf{L}\left[e^{-\alpha x}\mu(x)\right]}_{\mathbf{R_e}(s) > -\mathbf{R_e}(\alpha)} + \underbrace{\mathbf{L}\left[e^{\alpha x}\mu(-x)\right]}_{\mathbf{R_e}(s) < \mathbf{R_e}(\alpha)} && \text{by Definition E.3 page 122} && \text{and Corollary E.4 page 123} \\
&= \left[\frac{1}{s+\alpha}\right] + \left[\frac{-1}{s-\alpha}\right] && \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right) && \text{by Corollary E.4 page 123} \\
&= \frac{(s-\alpha)-(s+\alpha)}{(s+\alpha)(s-\alpha)} && \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right) \\
&= \frac{2\alpha}{\alpha^2 - s^2} && \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right)
\end{aligned}
$$

✏

Corollary E.10. [16] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117) *and* $\mu(x)$ *the* UNIT STEP *function. Let* $A \triangleq \mathbf{R_e}(\alpha)$.

COR		Mapping	Region of Convergence	Domain
	(1).	$\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(x)\right] = \dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$	for $\mathbf{R_e}(s) \in (-A : A)$	$x,\alpha \in \mathbb{R}$
	(2).	$\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(-x)\right] = \dfrac{-s+\alpha}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R_e}(s) \in (-A : A)$	$x,\alpha \in \mathbb{R}$
	(3).	$\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha\lvert x\rvert}\right] = \dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2} + \dfrac{-s+\alpha}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R_e}(s) \in (-A : A)$	$x,\alpha \in \mathbb{R}$

[15] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩,

[16] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩, Levy (1958) page 19 ⟨with $\psi = 0$, $\alpha_0 = \alpha$, and $\alpha_1 = 1$⟩, http://ece-research.unm.edu/bsanthan/ece541/table_ME.pdf

✎PROOF:

1. Proof for (1):

$$\begin{aligned}
&\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(x)\right](s) \\
&= \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha x}\mu(x)\right](s) && \text{by definition of } \mu(x) && \text{(Definition E.3 page 122)} \\
&= \mathbf{L}\left[\cos(\omega_o x)\mu(x)\right](s+\alpha) \quad \forall \mathbf{R_e}(s) \in \left(0 - \mathbf{R_e}(\alpha) : \infty - \mathbf{R_e}(\alpha)\right) && \text{by } \textit{modulation} \text{ prop.} && \text{(Theorem E.4 page 120)} \\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} \quad \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right) && \text{by Corollary E.5 page 123}
\end{aligned}$$

2. Proof for (2):

$$\begin{aligned}
&\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(-x)\right] \\
&= \mathbf{L}\left[\cos(\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by definition of } \mu(x) && \text{(Definition E.3 page 122)} \\
&= \mathbf{L}\left[\cos(-\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by } \textit{even} \text{ property of } \cos(x) && \text{(Theorem 1.2 page 5)} \\
&= \mathbf{L}\left[e^{\alpha x}\cos(\omega_o(-x))\mu(-x)\right] \\
&= \mathbf{L}\underbrace{\cos(\omega_o(-x))\mu(-x)}_{g(x)}(s-\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem E.4 page 120)} \\
&= \mathbf{L}\underbrace{\cos(\omega_o(-x))\mu(-x)}_{g(x)}(s-\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem E.4 page 120)} \\
&= \frac{-s+\alpha}{(s-\alpha)^2+\omega_o^2} \quad \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right) && \text{by Corollary E.5 and Theorem E.4 page 120}
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\right] &= \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(x)\right] + \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(-x)\right] \\
&= \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha x}\mu(x)\right] + \mathbf{L}\left[\cos(-\omega_o x)e^{\alpha x}\mu(-x)\right] \\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} + \frac{-s+\alpha}{(s-\alpha)^2+\omega_o^2} \qquad \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right)
\end{aligned}$$

✏

Corollary E.11. [17] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117) *and* $\mu(x)$ *the* UNIT STEP *function. Let* $A \triangleq \mathbf{R_e}(\alpha)$.

COR

	Mapping			Region of Convergence	Domain
(1).	$\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(x)\right]$	$=$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$	for $\mathbf{R_e}(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$
(2).	$\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(-x)\right]$	$=$	$\dfrac{-\omega_o}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R_e}(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$
(3).	$\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha\lvert x\rvert}\right]$	$=$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2} + \dfrac{-\omega_o}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R_e}(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$

✎PROOF:

1. Proof for (1):

$$\begin{aligned}
&\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(x)\right] \\
&= \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha x}\mu(x)\right] && \text{by definition of } \mu(x) && \text{(Definition E.3 page 122)} \\
&= \frac{s+\alpha}{(\omega_o)^2+\omega_o^2} \quad \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right) && \text{by Corollary E.5 page 123 and Theorem E.4 page 120}
\end{aligned}$$

[17] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩, Levy (1958) page 19 ⟨with $\psi = 0$, $\alpha_0 = \alpha$, and $\alpha_1 = 1$⟩, http://ece-research.unm.edu/bsanthan/ece541/table_ME.pdf

2. Proof for (2):

$$\begin{aligned}
&\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(-x)\right] \\
&= \mathbf{L}\left[\sin(-\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by definition of } \mu(x) && \text{(Definition E.3 page 122)} \\
&= \mathbf{L}\left[-\sin(\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by } \textit{odd} \text{ property of } \sin(x) && \text{(Theorem 1.2 page 5)} \\
&= -\mathbf{L}\left[\sin(\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by } \textit{homogeneous} \text{ property} && \text{(Theorem E.3 page 119)} \\
&= \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right) && \text{by Theorem E.4 page 120 and Corollary E.5}
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\right] &= \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(x)\right] + \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(-x)\right] \\
&= \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha x}\mu(x)\right] + \mathbf{L}\left[\sin(-\omega_o x)e^{\alpha x}\mu(-x)\right] \\
&= \frac{\omega_o}{(s+\alpha)^2+\omega_o^2} + \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R_e}(s) \in \left(-\mathbf{R_e}(\alpha) : \mathbf{R_e}(\alpha)\right)
\end{aligned}$$

✏

E.8 Product properties

Theorem E.6 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "s domain" and vice-versa.

Theorem E.6 (convolution theorem). [18] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition E.1 page 117) *and* $\star$ *the convolution operator* (Definition D.1 page 115). *Let* A, B, C, *and* D *be defined as in Corollary E.2 (page 119).*

THM

$$\begin{aligned}
\mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](s) &= [\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s) && \forall \mathbf{R_e}(s) \in [A:B]\cap[C:D] && \forall \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \\
\mathbf{L}\left[\mathsf{f}(x)\mathsf{g}(x)\right](s) &= [\mathbf{L}\mathsf{f}](s)\star\left[\mathbf{L}\mathsf{g}\right](s) && \forall \mathbf{R_e}(s) \in [A+C:B+D],\ c\in(A:B) && \forall \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](s) &= \mathbf{L}\left[\int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{d}u\right](s) && \text{by definition of } \star && \text{(Definition D.1 page 115)} \\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\mathbf{L}\mathsf{g}(x-u)\right](s)\,\mathsf{d}u \\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-su}\left[\mathbf{L}\mathsf{g}(x)\right](s)\,\mathsf{d}u && \text{by } \textit{translation} \text{ property} && \text{(Theorem E.2 page 118)} \\
&= \underbrace{\left(\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathsf{d}u\right)}_{[\mathbf{L}\mathsf{f}](s)}\left[\mathbf{L}\mathsf{g}\right](s) \\
&= [\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s) && \mathbf{R_e}(s)\in[A:B]\cap[C:D] && \text{by definition of } \mathbf{L}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s) &= \mathbf{L}\left[\left(\mathbf{L}^{-1}\mathbf{L}\mathsf{f}(x)\right)\mathsf{g}(x)\right](s) && \text{by def. of operator inverse} && \text{(Definition G.3 page 146)} \\
&= \mathbf{L}\left[\left(\int_{v\in\mathbb{R}}[\mathbf{L}\mathsf{f}(x)](v)e^{sxv}\,\mathsf{d}v\right)\mathsf{g}(x)\right](s) && \text{by Theorem E.1 page 118}
\end{aligned}$$

[18] Bracewell (1978) page 224, Bachman et al. (2002) pages 268–270, Bachman (1964) page 8

$$\begin{aligned}
&= \int_{v\in\mathbb{R}} [\mathbf{L}f(x)](v)\left[\mathbf{L}\left(e^{sxv}\, g(x)\right)\right](s,v)\,dv \\
&= \int_{v\in\mathbb{R}} [\mathbf{L}f(x)](v)\left[\mathbf{L}g(x)\right](s-v)\,dv && \text{by Theorem E.2 page 118} \\
&= [\mathbf{L}f](s) \star \left[\mathbf{L}g\right](s) && \text{by definition of } \star && \text{(Definition D.1 page 115)}
\end{aligned}$$

✏

E.9 Calculus properties

Theorem E.7. [19] *Let* $\mathbf{L}$ *be the* Laplace Transform *operator* (Definition E.1 page 117).

THM

$$\left\{\lim_{x\to-\infty} g(x) = 0\right\} \implies \begin{cases} \left\{\mathbf{L}\left[\frac{d}{dt}g(x)\right]\right\} &= s\left[\mathbf{L}g\right](s) \\ \mathbf{L}\displaystyle\int_{u=-\infty}^{u=x} g(u)\,du &= \frac{1}{s}\left[\mathbf{L}g\right](s) \end{cases}$$

✎Proof:

$$\begin{aligned}
\mathbf{L}\left[\frac{d}{dx}g(x)\right] &\triangleq \int_{x\in\mathbb{R}} \underbrace{\left[\frac{d}{dx}g(x)\right]}_{dv}\underbrace{e^{-sx}}_{u}\,dx && \text{by definition of } \mathbf{L} \\
&= \underbrace{e^{-sx}}_{u}\underbrace{g(x)}_{v}\Big|_{x=-\infty}^{x=+\infty} - \int_{x\in\mathbb{R}} \underbrace{g(x)}_{v}\underbrace{(-s)e^{-sx}}_{du}\,dx && \text{by \textit{Integration by Parts}} \\
&= \cancelto{0}{e^{-s\infty}}g(\infty) - e^{s\infty}\cancelto{0}{g(-\infty)} - (-s)\int_{x\in\mathbb{R}} g(x)e^{-sx}\,dx && \text{by left hypothesis} \\
&\triangleq s\left[\mathbf{L}g\right](s) && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\int_{u=-\infty}^{u=x} g(u)\,du &\triangleq \int_{x=-\infty}^{x=+\infty}\left[\int_{u=-\infty}^{u=x} g(u)\,du\right]e^{-sx}\,dx && \text{by definition of } \mathbf{L} \\
&= \int_{x=-\infty}^{x=+\infty}\left[\int_{u=-\infty}^{u=+\infty} g(u)\mu(x-u)\,du\right]e^{-sx}\,dx \\
&= \int_{v=-\infty}^{v=+\infty}\int_{u=-\infty}^{u=+\infty} g(u)h(v)e^{-s(u+v)}\,du\,dv && \left(\begin{array}{ll}\text{where} & v \triangleq x-u \\ \implies & x = u+v\end{array}\right) \\
&= \left[\int_{v=-\infty}^{v=+\infty} \mu(v)e^{-sv}\,dv\right]\underbrace{\left[\int_{u=-\infty}^{u=+\infty} g(u)e^{-su}\,du\right]}_{\text{\textit{Laplace Transform} of } g(x)} \\
&= \left[\int_{v=0}^{v=\infty} e^{-sv}\,dv\right]\left[\mathbf{L}g\right](s) \\
&= \frac{1}{-s}e^{-sv}\Big|_{v=0}^{v=\infty}\left[\mathbf{L}g\right](s) && \text{by \textit{Fundamental Theorem of Calculus}} \\
&= \frac{1}{s}\left[\mathbf{L}g\right](s) && \text{by definition of } \mathbf{L} && \text{(Definition E.1 page 117)}
\end{aligned}$$

✏

[19] Betten (2008b) page 301 ⟨(B.27)⟩, Levy (1958) page 15 ⟨Equation 0.7⟩

APPENDIX F

Z TRANSFORM

F.1 Convolution operator

Definition F.1. [1] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF
*A function f in X^Y is a **sequence** over X if $Y = \mathbb{Z}$.*
A sequence may be denoted in the form $\left(x_n\right)_{n\in\mathbb{Z}}$ or simply as $\left(x_n\right)$.

Definition F.2. [2] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a* FIELD *(Definition **??** page **??**).*

DEF
*The **space of all absolutely square summable sequences** $\ell^2_{\mathbb{F}}$ over $\mathbb{F}$ is defined as*

$$\ell^2_{\mathbb{F}} \triangleq \left\{ \left(x_n\right)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} \left|x_n\right|^2 < \infty \right\}$$

The space $\ell^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\ell^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\ell^2_{\mathbb{R}}$ is isomorphic to $\boldsymbol{L}^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition F.3.

DEF
*The **convolution** operation $\star$ is defined as*

$$\left(x_n\right) \star \left(y_n\right) \triangleq \left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right)_{n\in\mathbb{Z}} \qquad \forall \left(x_n\right)_{n\in\mathbb{Z}}, \left(y_n\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

Proposition F.1. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition F.3 page 131).*

PRP
$$\left(x_n\right) \star \left(y_n\right) = \left(y_n\right) \star \left(x_n\right) \qquad \forall \left(x_n\right)_{n\in\mathbb{Z}}, \left(y_n\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

[1] Bromwich (1908) page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[2] Kubrusly (2011) page 347 ⟨Example 5.K⟩

✎PROOF:

$$\begin{aligned}
[x \star y](n) &\triangleq \sum_{m\in\mathbb{Z}} x_m y_{n-m} && \text{by Definition F.3 page 131}\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{where } k \triangleq n-m \implies m = n-k\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{by } \textit{commutativity} \text{ of addition}\\
&= \sum_{m\in\mathbb{Z}} x_{n-m} y_m && \text{by change of variables}\\
&= \sum_{m\in\mathbb{Z}} y_m x_{n-m} && \text{by commutative property of the field over } \mathbb{C}\\
&\triangleq (y \star x)_n && \text{by Definition F.3 page 131}
\end{aligned}$$

✏

Proposition F.2. *Let* $\star$ *be the* CONVOLUTION OPERATOR (Definition F.3 page 131). *Let* $\ell^2_{\mathbb{R}}$ *be the set of* ABSOLUTELY SUMMABLE *sequences* (Definition F.2 page 131).

PRP
$$\left\{\begin{array}{llll} \text{(A).} & \mathsf{x}(n) & \in & \ell^2_{\mathbb{R}} \quad \text{and}\\ \text{(B).} & \mathsf{y}(n) & \in & \ell^2_{\mathbb{R}} \end{array}\right\} \implies \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] = \mathsf{x}[-n] \star \mathsf{y}(n) \right\}$$

✎PROOF:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \implies k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by } \textit{absolutely summable} \text{ hypothesis} && \text{(Definition F.2 page 131)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \implies \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of } \textit{convolution} \star && \text{(Definition F.3 page 131)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n]
\end{aligned}$$

✏

F.2 Z-transform

Definition F.4. [3]

DEF
The ***z-transform*** $\mathbf{Z}$ *of* $(x_n)_{n\in\mathbb{Z}}$ *is defined as*
$$[\mathbf{Z}(x_n)](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\textit{Laurent series}} \qquad \forall (x_n) \in \ell^2_{\mathbb{R}}$$

Theorem F.1. *Let* $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ *be the* Z-TRANSFORM *of* $\mathsf{x}[n]$.

THM
$$\left\{ \check{\mathsf{x}}(z) \triangleq \mathbf{Z}(\mathsf{x}[n]) \right\} \implies \left\{\begin{array}{lrcll}
\text{(1).} & \mathbf{Z}(\alpha\mathsf{x}[n]) & = & \alpha\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(2).} & \mathbf{Z}(\mathsf{x}[n-k]) & = & z^{-k}\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(3).} & \mathbf{Z}(\mathsf{x}[-n]) & = & \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(4).} & \mathbf{Z}(\mathsf{x}^*[n]) & = & \check{\mathsf{x}}^*(z^*) & \forall (x_n)\in\ell^2_{\mathbb{R}} \quad \text{and}\\
\text{(5).} & \mathbf{Z}(\mathsf{x}^*[-n]) & = & \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}}
\end{array}\right\}$$

[3] *Laurent series*: Abramovich and Aliprantis (2002) page 49

✎PROOF:

$$
\begin{aligned}
\alpha \mathbb{Z}\breve{x}(z) &\triangleq \alpha \mathbf{Z}\,(\!(x[n])\!) && \text{by definition of } \breve{x}(z) \\
&\triangleq \alpha \sum_{n\in\mathbb{Z}} x[n]z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator} \\
&\triangleq \sum_{n\in\mathbb{Z}} (\alpha x[n])z^{-n} && \text{by \textit{distributive} property} \\
&\triangleq \mathbf{Z}\,(\!(\alpha x[n])\!) && \text{by definition of } \mathbf{Z} \text{ operator} \\
z^{-k}\breve{x}(z) &= z^{-k}\mathbf{Z}\,(\!(x[n])\!) && \text{by definition of } \breve{x}(z) && \text{(left hypothesis)} \\
&\triangleq z^{-k}\sum_{n=-\infty}^{n=+\infty} x[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
&= \sum_{n=-\infty}^{n=+\infty} x[n]z^{-n-k} \\
&= \sum_{m-k=-\infty}^{m-k=+\infty} x[m-k]z^{-m} && \text{where } m \triangleq n+k && \implies n = m-k \\
&= \sum_{m=-\infty}^{m=+\infty} x[m-k]z^{-m} \\
&= \sum_{n=-\infty}^{n=+\infty} x[n-k]z^{-n} && \text{where } n \triangleq m \\
&\triangleq \mathbf{Z}\,(\!(x[n-k])\!) && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
\mathbf{Z}\,(\!(x^*[n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
&\triangleq \left(\sum_{n\in\mathbb{Z}} x[n](z^*)^{-n}\right)^* && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
&\triangleq \breve{x}^*(z^*) && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
\mathbf{Z}\,(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
&= \sum_{-m\in\mathbb{Z}} x[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition F.2 page 131)} \\
&= \sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition F.2 page 131)} \\
&\triangleq \breve{x}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
\mathbf{Z}\,(\!(x^*[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)} \\
&= \sum_{-m\in\mathbb{Z}} x^*[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x^*[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition F.2 page 131)} \\
&= \sum_{m\in\mathbb{Z}} x^*[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition F.2 page 131)} \\
&= \left(\sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z^*}\right)^{-m}\right)^* && \text{by \textit{absolutely summable} property} && \text{(Definition F.2 page 131)}
\end{aligned}
$$

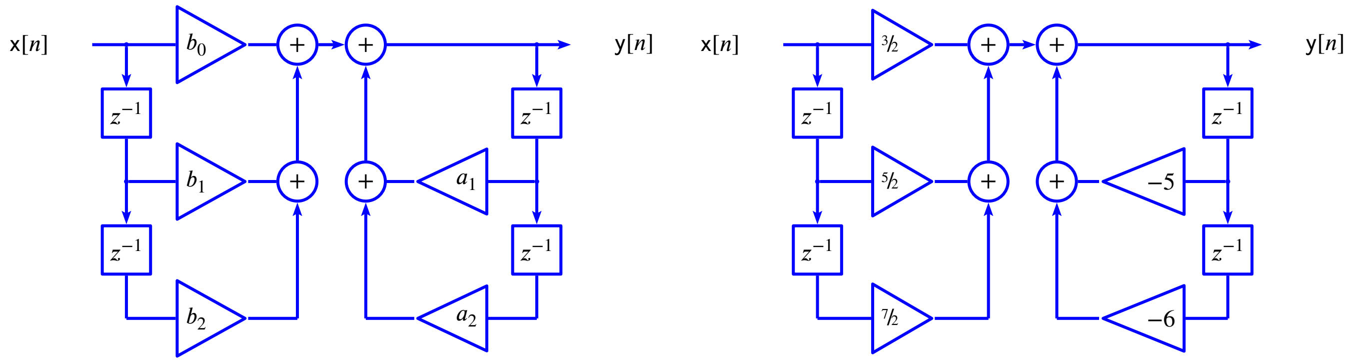


Figure F.1: Direct form 1 order 2 IIR filters

$$\triangleq \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition F.4 page 132)}$$

✏

Theorem F.2 (convolution theorem). *Let* $\star$ *be the convolution operator* (Definition F.3 page 131).

THM

$$\underbrace{\mathbf{Z}\left(\langle\!\langle x_n \rangle\!\rangle \star \langle\!\langle y_n \rangle\!\rangle\right)}_{\textit{sequence convolution}} = \underbrace{\left(\mathbf{Z}\langle\!\langle x_n \rangle\!\rangle\right)\left(\mathbf{Z}\langle\!\langle y_n \rangle\!\rangle\right)}_{\textit{series multiplication}} \qquad \forall \langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}}, \langle\!\langle y_n \rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x \star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by definition of } \star && \text{(Definition F.3 page 131)}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n} &&\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k \triangleq n-m && \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right] &&&&\\
&\triangleq \left[\mathbf{Z}\langle\!\langle x_n \rangle\!\rangle\right]\left[\mathbf{Z}\langle\!\langle y_n \rangle\!\rangle\right] && \text{by definition of } \mathbf{Z} && \text{(Definition F.4 page 132)}
\end{aligned}$$

✏

F.3 From z-domain back to time-domain

$$\check{\mathsf{y}}(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) - a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$$

$$\mathsf{y}[n] = b_0 \mathsf{x}[n] + b_1 \mathsf{x}[n-1] + b_2 \mathsf{x}[n-2] - a_1 \mathsf{y}[n-1] - a_2 \mathsf{y}[n-2]$$

Example F.1. See Figure F.1 (page 134)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2\left(z^2+5z+6\right)} = \frac{\left(\tfrac{3}{2}z^2+\tfrac{5}{2}z+\tfrac{7}{2}\right)}{z^2+5z+6} = \frac{\left(\tfrac{3}{2}+\tfrac{5}{2}z^{-1}+\tfrac{7}{2}z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

F.4 Zero locations

The system property of *minimum phase* is defined in Definition F.5 (next) and illustrated in Figure F.2 (page 135).

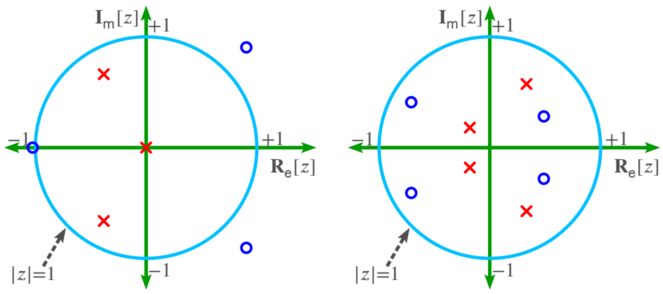


Figure F.2: Minimum Phase filter

Definition F.5. [4] *Let* $\breve{x}(z) \triangleq \mathbf{Z}\,(\!(x_n)\!)$ *be the* Z TRANSFORM (Definition F.4 page 132) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $(\!(z_n)\!)_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\breve{x}(z)$.

DEF: *The sequence* $(\!(x_n)\!)$ *is **minimum phase** if*

$$\underbrace{|z_n| < 1 \qquad \forall n \in \mathbb{Z}}_{\breve{x}(z) \text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem F.3 (Robinson's Energy Delay Theorem). [5] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM:

$$\left\{\begin{array}{ll} \mathsf{p} & \text{is MINIMUM PHASE} \quad \text{and} \\ \mathsf{q} & \text{is NOT } \textit{minimum phase} \end{array}\right\} \implies \underbrace{\sum_{n=0}^{m-1} |a_n|^2}_{\text{"energy" of the first } m \text{ coefficients of } \mathsf{p}(z)} \geq \underbrace{\sum_{n=0}^{m-1} |b_n|^2}_{\text{"energy" of the first } m \text{ coefficients of } \mathsf{q}(z)} \qquad \forall 0 \le m \le N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle (Figure F.3 page 136).

Example F.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure F.3 (page 136).

[4] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36
[5] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976) pages 52–53

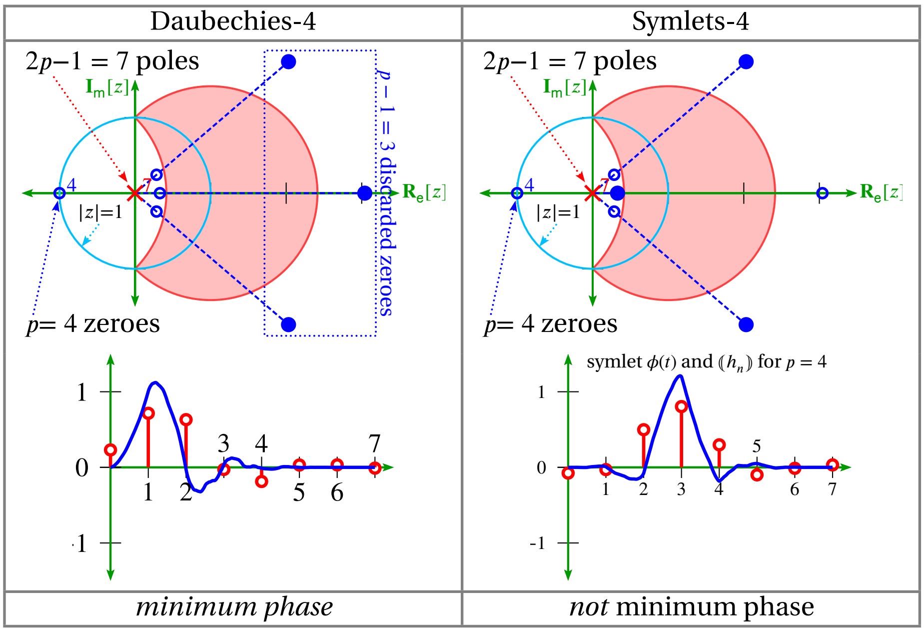


Figure F.3: Daubechies-4 and Symlet-4 scaling functions pole-zero plots

F.5 Pole locations

Definition F.6.

DEF *A filter (or system or operator)* **H** *is* ***causal***
if its current output does not depend on future inputs.

Definition F.7.

DEF *A filter (or system or operator)* **H** *is* ***time-invariant***
if the mapping it performs does not change with time.

Definition F.8.

DEF *An operation* **H** *is* ***linear*** *if any output y_n can be described as a linear combination of inputs x_n as in*

$$\mathsf{y}_n = \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\mathsf{x}(n-m)\,.$$

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem F.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Example F.3. Stable/unstable filters are illustrated in Figure F.4 (page 137).

True or False? This filter has no poles:

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$

$$\mathsf{H}(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$

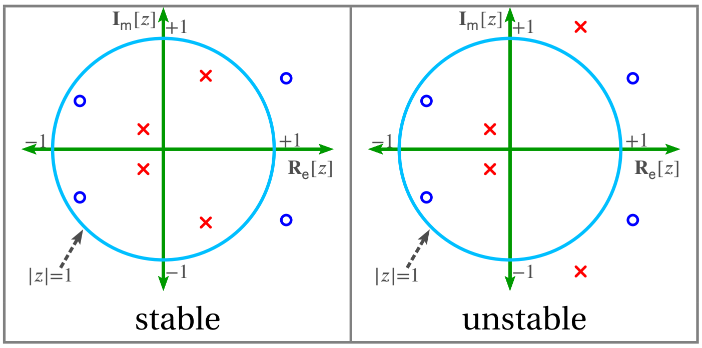


Figure F.4: Pole-zero plot stable/unstable causal LTI filters (Example F.3 page 136)

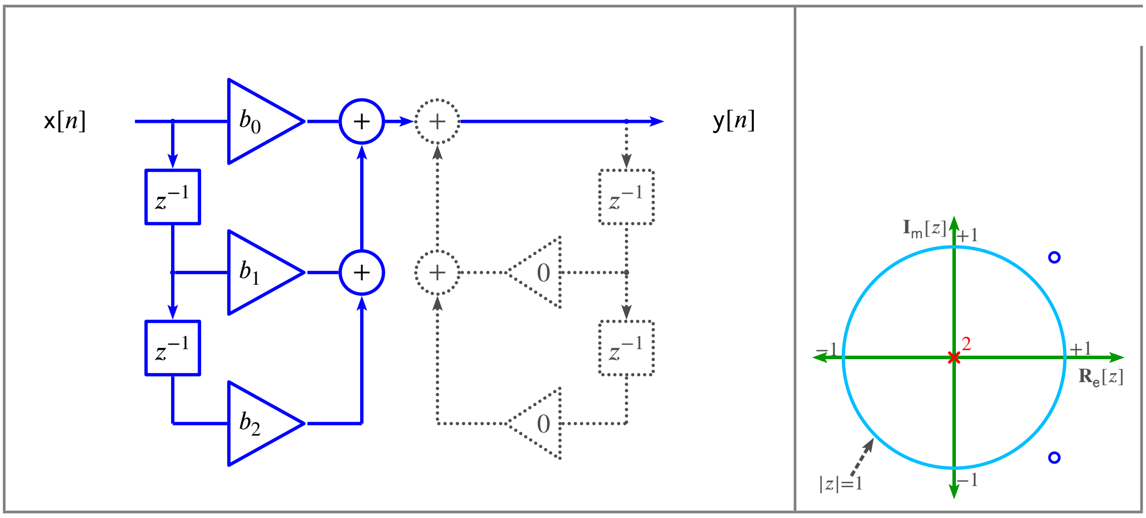


Figure F.5: FIR filters

F.6 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

Proposition F.3.

PRP $\left\{\begin{array}{l}\text{ZEROS } \textit{and} \text{ POLES}\\ \textit{occur in} \text{ CONJUGATE PAIRS}\end{array}\right\} \implies \left\{\begin{array}{l}\text{COEFFICIENTS}\\ \textit{are} \text{ REAL.}\end{array}\right\}$

✎PROOF:

$$\begin{aligned}\left(z-p_1\right)\left(z-p_1^*\right) &= [z-(a+ib)][z-(a-ib)]\\ &= z^2+[-a+ib-ib-a]z-[ib]^2\end{aligned}$$

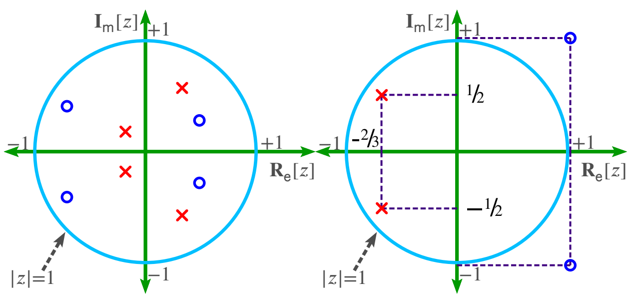


Figure F.6: Conjugate pair structure yielding real coefficients

$$= z^2 - 2az + b^2$$

✏

Example F.4. See Figure F.6 (page 137).

$$\begin{aligned}
H(z) &= G\frac{[z - z_1][z - z_2]}{[z - p_1][z - p_2]} = G\frac{[z - (1 + i)][z - (1 - i)]}{[z - (-2/3 + i1/2)][z - (-2/3 - i1/2)]} \\
&= G\frac{z^2 - z[(1 - i) + (1 + i)] + (1 - i)(1 + i)}{z^2 - z[(-2/3 + i1/2) + (-2/3 + i1/2)] + (-2/3 + i1/2)(-2/3 + i1/2)} \\
&= G\frac{z^2 - 2z + 2}{z^2 - 4/3z + (4/3 + 1/4)} = G\frac{z^2 - 2z + 2}{z^2 - 4/3z + 19/12}
\end{aligned}$$

F.7 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma F.1. *A causal LTI operator* **H** *can be expressed as a rational expression* $\check{h}(z)$.

$$\begin{aligned}
\check{h}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \frac{\sum_{n=0}^{N} b_n z^{-n}}{1 + \sum_{n=1}^{N} a_n z^{-n}}
\end{aligned}$$

A filter operation $\check{h}(z)$ can be expressed as a product of its roots (poles and zeros).

$$\begin{aligned}
\check{h}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \alpha\frac{(z - z_1)(z - z_2)\cdots(z - z_N)}{(z - p_1)(z - p_2)\cdots(z - p_N)}
\end{aligned}$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with $\times$ and zeros with $\bigcirc$. An example is shown in Figure F.7 page 139. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

F.8 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{x}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{y}(z)$ is a *high-pass filter*.

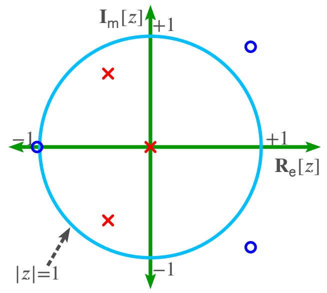


Figure F.7: Pole-zero plot for rational expression with real coefficients

Definition F.9. [6] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES (Definition F.1 page 131) *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ (Definition F.2 page 131).

DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter***.

Any triple $\left((\!(x_n)\!), (\!(y_n)\!), N\right)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition***.

Theorem F.5 (CQF theorem). [7] *Let* $\breve{y}(\omega)$ *and* $\breve{x}(\omega)$ *be the* DTFT*s* (Definition 6.1 page 77) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$, *respectively, in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ (Definition F.2 page 131).

THM

$$\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{\text{(1) CQF in "time"}} \iff \breve{y}(z) = \pm(-1)^N z^{-N}\breve{x}^*\left(\frac{-1}{z^*}\right) \quad \text{(2) CQF in "z-domain"}$$

$$\iff \breve{y}(\omega) = \pm(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi) \quad \text{(3) CQF in "frequency"}$$

$$\iff x_n = \pm(-1)^N(-1)^n y^*_{N-n} \quad \text{(4) "reversed" CQF in "time"}$$

$$\iff \breve{x}(z) = \pm z^{-N}\breve{y}^*\left(\frac{-1}{z^*}\right) \quad \text{(5) "reversed" CQF in "z-domain"}$$

$$\iff \breve{x}(\omega) = \pm e^{-i\omega N}\breve{y}^*(\omega+\pi) \quad \text{(6) "reversed" CQF in "frequency"}$$

$\forall N\in\mathbb{Z}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{y}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-}transform \quad \text{(Definition F.4 page 132)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm\sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies n = N-m\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*
\end{aligned}$$

[6] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

[7] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

$$= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) \qquad \text{by definition of } z\text{-transform} \quad \text{(Definition F.4 page 132)}$$

2. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\check{y}(z) &= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by (2)} \\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-transform} \quad \text{(Definition F.4 page 132)} \\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m^* \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-transform} \quad \text{(Definition F.4 page 132)} \\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x_m^* z^{-(N-m)} \\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{n} x_{N-n}^* z^{-n} && \text{where } n = N - m \implies \quad m \triangleq N - n \\
&\implies x_n = \pm(-1)^n x_{N-n}^*
\end{aligned}$$

3. Proof that (1) $\implies$ (3):

$$\begin{aligned}
\check{y}(\omega) &\triangleq \left.\check{x}(z)\right|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition 6.1 page 77)} \\
&= \left[\pm(-1)^N z^{-N} \check{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)} \\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i\pi} e^{i\omega}\right) \\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i(\omega+\pi)}\right) \\
&= \pm(-1)^N e^{-i\omega N} \check{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition 6.1 page 77)}
\end{aligned}$$

4. Proof that (1) $\implies$ (6):

$$\begin{aligned}
\check{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT \quad \text{(Definition 6.1 page 77)} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x_{N-n}^*}_{CQF} e^{-i\omega n} && \text{by (1)} \\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x_m^* e^{-i\omega(N-m)} && \text{where } m \triangleq N - n \implies \quad n = N - m \\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} (-1)^m x_m^* e^{i\omega m} \\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} e^{i\pi m} x_m^* e^{i\omega m} \\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} x_m^* e^{i(\omega+\pi)m} \\
&= \pm(-1)^N e^{-i\omega N} \left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^* \\
&= \pm(-1)^N e^{-i\omega N} \check{x}^*(\omega+\pi) && \text{by definition of } DTFT \quad \text{(Definition 6.1 page 77)}
\end{aligned}$$

5. Proof that (1) $\Longleftarrow$ (3):

$$\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{\mathsf{y}}(\omega)e^{i\omega n}\,\mathsf{d}\omega && \text{by } \textit{inverse DTFT} && \text{(Theorem 6.3 page 83)}\\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\breve{\mathsf{x}}^*(\omega+\pi)}_{\text{right hypothesis}}e^{i\omega n}\,\mathsf{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{\mathsf{x}}^*(\omega+\pi)e^{i\omega(n-N)}\,\mathsf{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{0}^{2\pi} \breve{\mathsf{x}}^*(v)e^{i(v-\pi)(n-N)}\,\mathsf{d}v && \text{where } v \triangleq \omega+\pi \implies && \omega = v-\pi\\
&= \pm(-1)^N e^{-i\pi(n-N)}\frac{1}{2\pi}\int_{0}^{2\pi} \breve{\mathsf{x}}^*(v)e^{iv(n-N)}\,\mathsf{d}v\\
&= \pm(-1)^N\underbrace{(-1)^N}_{e^{i\pi N}}\underbrace{(-1)^n}_{e^{-i\pi n}}\left[\frac{1}{2\pi}\int_{0}^{2\pi} \breve{\mathsf{x}}(v)e^{iv(N-n)}\,\mathsf{d}v\right]^*\\
&= \pm(-1)^n x^*_{N-n} && \text{by } \textit{inverse DTFT} && \text{(Theorem 6.3 page 83)}
\end{aligned}$$

6. Proof that (1) $\iff$ (4):

$$\begin{aligned}
y_n = \pm(-1)^n x^*_{N-n} &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x^*_{N-n}\\
&\iff \pm(-1)^n y_n = x^*_{N-n}\\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x^*_{N-n}\right)^*\\
&\iff \pm(-1)^n y^*_n = x_{N-n}\\
&\iff x_{N-n} = \pm(-1)^n y^*_n\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m} && \text{where } m \triangleq N-n \implies && n = N-m\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m}\\
&\iff x_m = \pm(-1)^N(-1)^m y^*_{N-m}\\
&\iff x_n = \pm(-1)^N(-1)^n y^*_{N-n} && \text{by change of free variables}
\end{aligned}$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

✏

Theorem F.6. [8] *Let* $\breve{\mathsf{y}}(\omega)$ *and* $\breve{\mathsf{x}}(\omega)$ *be the* DTFT*s* (Definition 6.1 page 77) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$, *respectively, in* $\ell^2_{\mathbb{R}}$ (Definition F.2 page 131).

THM *Let* $y_n = \pm(-1)^n x^*_{N-n}$ (CQF CONDITION, *Definition F.9 page 139*). *Then*

$$\left\{\begin{array}{llcll}
\text{(A)} \quad \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{y}}(\omega)\Big|_{\omega=0} = 0 & \iff & \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\Big|_{\omega=\pi} & = 0 & \text{(B)}\\
& \iff & \sum_{k\in\mathbb{Z}}(-1)^k k^n x_k & = 0 & \text{(C)}\\
& \iff & \sum_{k\in\mathbb{Z}} k^n y_k & = 0 & \text{(D)}
\end{array}\right\} \quad \forall n\in\mathbb{W}$$

✎PROOF:

[8] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

1. Proof that (A) $\implies$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} && \text{by (A)}\\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi)\Big|_{\omega=0} && \text{by \textit{CQF theorem}} \quad \text{(Theorem F.5 page 139)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n}\binom{n}{\ell}\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^\ell \left[e^{-i\omega N}\right]\cdot\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma C.2 page 113)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n}\binom{n}{\ell} -iN^\ell e^{-i\omega N}\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0}\\
&= (\pm)(-1)^N e^{-i0N}\sum_{\ell=0}^{n}\binom{n}{\ell} -iN^\ell \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell}\left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0}
\end{aligned}$$

$$\begin{aligned}
\implies \breve{x}^{(0)}(\pi) &= 0\\
\implies \breve{x}^{(1)}(\pi) &= 0\\
\implies \breve{x}^{(2)}(\pi) &= 0\\
\implies \breve{x}^{(3)}(\pi) &= 0\\
\implies \breve{x}^{(4)}(\pi) &= 0\\
\vdots \quad \vdots &\\
\implies \breve{x}^{(n)}(\pi) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

2. Proof that (A) $\impliedby$ (B):

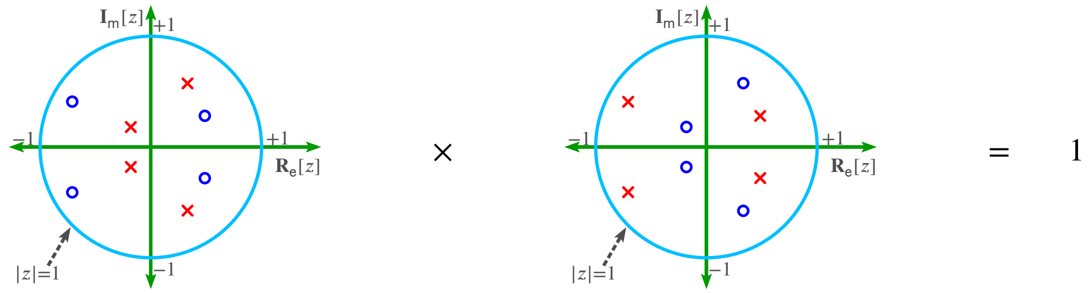
$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} && \text{by (B)}\\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n (\pm)e^{-i\omega N}\breve{y}^*(\omega+\pi)\Big|_{\omega=\pi} && \text{by \textit{CQF theorem}} \quad \text{(Theorem F.5 page 139)}\\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^\ell \left[e^{-i\omega N}\right]\cdot\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma C.2 page 113)}\\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}(-iN)^\ell e^{-i\omega N}\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}\\
&= (\pm)e^{-i\pi N}\sum_{\ell=0}^{n}\binom{n}{\ell} -iN^\ell \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}\\
&= (\pm)(-1)^N\sum_{\ell=0}^{n}\binom{n}{\ell} -iN^\ell \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell}\left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}
\end{aligned}$$

$$\begin{aligned}
\implies \breve{y}^{(0)}(0) &= 0\\
\implies \breve{y}^{(1)}(0) &= 0\\
\implies \breve{y}^{(2)}(0) &= 0\\
\implies \breve{y}^{(3)}(0) &= 0\\
\implies \breve{y}^{(4)}(0) &= 0\\
\vdots \quad \vdots &\\
\implies \breve{y}^{(n)}(0) &= 0\\
\implies \breve{y}^{(n)}(0) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

3. Proof that (B) $\iff$ (C): by Theorem 6.5 page 85

4. Proof that (A) $\iff$ (D): by Theorem 6.5 page 85

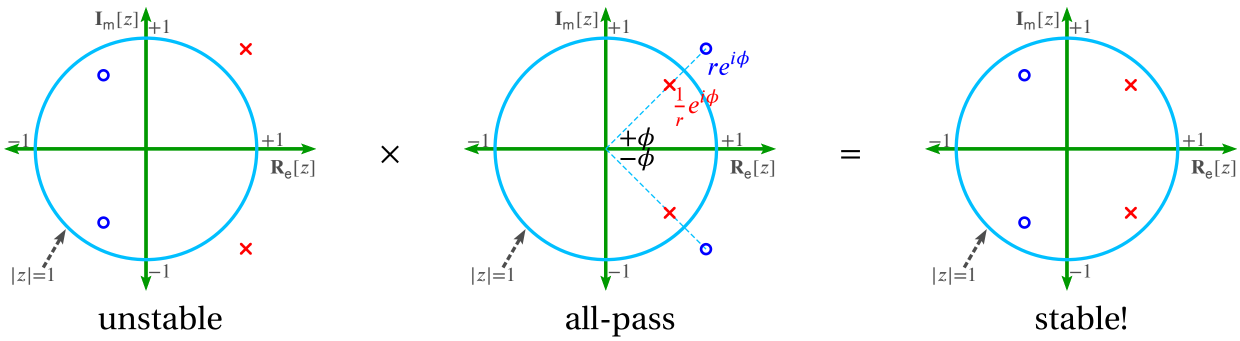
5. Proof that (CQF) $\not\Leftarrow$ (A): Here is a counterexample: $\breve{y}(\omega) = 0$.



$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

F.9 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.



$$\begin{aligned}
|A(z)|_{z=e^{i\omega}} &= \frac{1}{r}\left|\frac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}}
&&= \left|\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|e^{i\phi}\left(\frac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}}
&&= \left|z\left(\frac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|-z\left(\frac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}}
&&= \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\frac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
&= \left|\frac{1}{e^{-iv}}\left(\frac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right|
&&= \left|\frac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right| \\
&= 1
\end{aligned}$$

APPENDIX G

OPERATORS ON LINEAR SPACES

❝*And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.*❞
Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

G.1 Operators on linear spaces

G.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition G.1. [2] *Let* $(\mathbb{F},+,\cdot,0,1)$ *be a* FIELD. *Let* X *be a set, let* $+$ *be an* OPERATOR *(Definition G.2 page 146) in* X^{X^2}, *and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991) page 41, Halmos (1948) pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

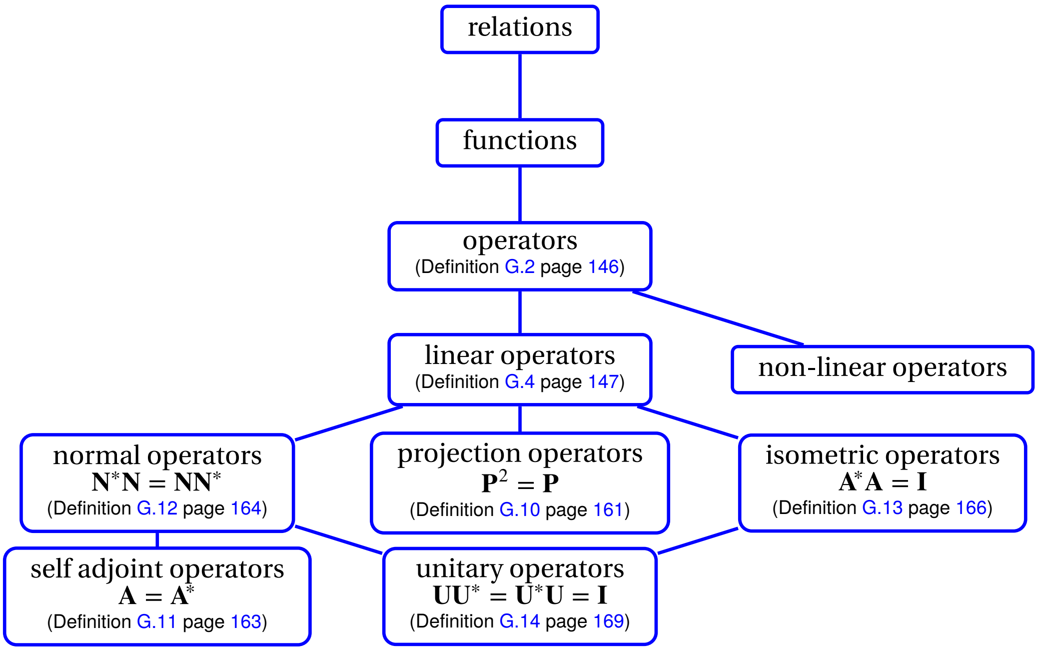


Figure G.1: Some operator types

DEF *The structure* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

$$\begin{array}{rlrcll}
1. & \exists \mathbb{0} \in X \text{ such that} & \boldsymbol{x} + \mathbb{0} &=& \boldsymbol{x} & \forall \boldsymbol{x} \in X \quad (+ \text{ IDENTITY}) \\
2. & \exists \boldsymbol{y} \in X \text{ such that} & \boldsymbol{x} + \boldsymbol{y} &=& \mathbb{0} & \forall \boldsymbol{x} \in X \quad (+ \text{ INVERSE}) \\
3. & & (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} &=& \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) & \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X \quad (+ \text{ is ASSOCIATIVE}) \\
4. & & \boldsymbol{x} + \boldsymbol{y} &=& \boldsymbol{y} + \boldsymbol{x} & \forall \boldsymbol{x},\boldsymbol{y} \in X \quad (+ \text{ is COMMUTATIVE}) \\
5. & & 1 \cdot \boldsymbol{x} &=& \boldsymbol{x} & \forall \boldsymbol{x} \in X \quad (\cdot \text{ IDENTITY}) \\
6. & & \alpha \cdot (\beta \cdot \boldsymbol{x}) &=& (\alpha \cdot \beta) \cdot \boldsymbol{x} & \forall \alpha,\beta \in S \text{ and } \boldsymbol{x} \in X \quad (\cdot \text{ ASSOCIATES with } \cdot) \\
7. & & \alpha \cdot (\boldsymbol{x} + \boldsymbol{y}) &=& (\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y}) & \forall \alpha \in S \text{ and } \boldsymbol{x},\boldsymbol{y} \in X \quad (\cdot \text{ DISTRIBUTES over } +) \\
8. & & (\alpha + \beta) \cdot \boldsymbol{x} &=& (\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x}) & \forall \alpha,\beta \in S \text{ and } \boldsymbol{x} \in X \quad (\cdot \text{ PSEUDO-DISTRIBUTES over } +)
\end{array}$$

The set X *is called the* ***underlying set****. The elements of* X *are called* ***vectors****. The elements of* $\mathbb{F}$ *are called* ***scalars****. A linear space is also called a* ***vector space****. If* $\mathbb{F} \triangleq \mathbb{R}$*, then* $\boldsymbol{\Omega}$ *is a* ***real linear space****. If* $\mathbb{F} \triangleq \mathbb{C}$*, then* $\boldsymbol{\Omega}$ *is a* ***complex linear space****.*

Definition G.2. [3]

DEF *A function* $\mathbf{A}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *is an* ***operator*** *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *if* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both* LINEAR SPACES *(Definition G.1 page 145).*

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}\boldsymbol{x} = \mathbf{B}\boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{X}\boldsymbol{Y}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}\boldsymbol{x} = \mathbf{I}$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

Definition G.3. [4] *Let* $\boldsymbol{X}^{\boldsymbol{X}}$ *be the set of all operators with from a* LINEAR SPACE $\boldsymbol{X}$ *to* $\boldsymbol{X}$*. Let* $\mathbf{I}$ *be an operator in* $\boldsymbol{X}^{\boldsymbol{X}}$*. Let* $①(\boldsymbol{X})$ *be the* IDENTITY ELEMENT *in* $\boldsymbol{X}^{\boldsymbol{X}}$*.*

DEF $\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\qquad \mathbf{I} = ①(\boldsymbol{X})$.

[3] Heil (2011) page 42
[4] Michel and Herget (1993) page 411

G.1.2 Linear operators

Definition G.4. [5] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be linear spaces.*

DEF

An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *is **linear** if*

1. $\mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$ (ADDITIVE) *and*
2. $\mathbf{L}(\alpha\boldsymbol{x}) = \alpha\mathbf{L}\boldsymbol{x} \quad \forall \boldsymbol{x} \in \boldsymbol{X}, \quad \forall \alpha \in \mathbb{F}$ (HOMOGENEOUS).

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ *such that*
$\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \{\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}} | \mathbf{L} \text{ is linear}\}$.

Theorem G.1. [6] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$.

THM

$$\{\mathbf{L} \text{ is LINEAR}\} \implies \left\{ \begin{array}{llll} 1. & \mathbf{L}\mathbb{0} = \mathbb{0} & & \text{and} \\ 2. & \mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x}) & \forall \boldsymbol{x} \in \boldsymbol{X} & \text{and} \\ 3. & \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X} & \text{and} \\ 4. & \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n (\mathbf{L}\boldsymbol{x}_n) & \boldsymbol{x}_n \in \boldsymbol{X}, \alpha_n \in \mathbb{F} & \end{array} \right\}$$

✎PROOF:

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned} \mathbf{L}\mathbb{0} &= \mathbf{L}(0 \cdot \mathbb{0}) && \text{by additive identity property} \\ &= 0 \cdot (\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 147)} \\ &= \mathbb{0} && \text{by } \textit{additive identity} \text{ property} \end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned} \mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1 \cdot \boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \\ &= -1 \cdot (\mathbf{L}\boldsymbol{x}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 147)} \\ &= -(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y}$:

$$\begin{aligned} \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x}+(-\boldsymbol{y})) && \text{by } \textit{additive inverse} \text{ property} \\ &= \mathbf{L}(\boldsymbol{x}) + \mathbf{L}(-\boldsymbol{y}) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 147)} \\ &= \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} && \text{by item (2)} \end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n (\mathbf{L}\boldsymbol{x}_n)$:

   (a) Proof for $N = 1$:

$$\begin{aligned} \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}(\alpha_1 \boldsymbol{x}_1) && \text{by } N = 1 \text{ hypothesis} \\ &= \alpha_1 (\mathbf{L}\boldsymbol{x}_1) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 147)} \end{aligned}$$

[5] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[6] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

(b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1}\alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1} + \sum_{n=1}^{N}\alpha_n \boldsymbol{x}_n\right) \\
&= \mathbf{L}(\alpha_{N+1}\boldsymbol{x}_{N+1}) + \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n \boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \quad \text{(Definition G.4 page 147)} \\
&= \alpha_{N+1}\mathbf{L}(\boldsymbol{x}_{N+1}) + \sum_{n=1}^{N}\mathbf{L}(\alpha_n \boldsymbol{x}_n) && \text{by left } N+1 \text{ hypothesis} \\
&= \sum_{n=1}^{N+1}\mathbf{L}(\alpha_n \boldsymbol{x}_n)
\end{aligned}$$

✏

Theorem G.2. [7] *Let $\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* Null Space *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathscr{I}(\mathbf{L})$ the* image set *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
$\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathscr{N}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{X}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathscr{I}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{Y}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	

✎Proof:

1. Proof that $\mathscr{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0}\in\mathscr{N}(\mathbf{L}) \implies \mathscr{N}(\mathbf{L})\neq\varnothing$
   (b) $\mathscr{N}(\mathbf{L}) \triangleq \{\boldsymbol{x}\in\boldsymbol{X} | \mathbf{L}\boldsymbol{x}=\mathbb{0}\} \subseteq \boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathscr{N}(\mathbf{L}) \implies \mathbb{0}=\mathbf{L}(\boldsymbol{x}+\boldsymbol{y})=\mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x}\in\mathscr{N}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\boldsymbol{X} \implies \mathbb{0}=\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathscr{N}(\mathbf{L})$
2. Proof that $\mathscr{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0}\in\mathscr{I}(\mathbf{L}) \implies \mathscr{I}(\mathbf{L})\neq\varnothing$
   (b) $\mathscr{I}(\mathbf{L}) \triangleq \{\boldsymbol{y}\in\boldsymbol{Y} | \exists\boldsymbol{x}\in\boldsymbol{X} \text{ such that } \boldsymbol{y}=\mathbf{L}\boldsymbol{x}\} \subseteq \boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathscr{I}(\mathbf{L}) \implies \exists\boldsymbol{v}\in\boldsymbol{X} \text{ such that } \mathbf{L}\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x}\in\mathscr{I}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\mathscr{I}(\mathbf{L}) \implies \exists\boldsymbol{x}\in\boldsymbol{X} \text{ such that } \boldsymbol{y}=\mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y}=\alpha\mathbf{L}\boldsymbol{x}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathscr{I}(\mathbf{L})$

✏

Example G.1. [8] Let $\mathscr{C}([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX
$\mathscr{C}([a:b],\mathbb{R})$ is a linear space.

Theorem G.3. [9] *Let $\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* Null Space *of a linear operator $\mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM		
$\mathbf{L}\boldsymbol{x}=\mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y}\in\mathscr{N}(\mathbf{L})$
$\mathbf{L}$ *is* injective	$\iff$	$\mathscr{N}(\mathbf{L})=\{\mathbb{0}\}$

[7] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩
[8] Eidelman et al. (2004) page 3
[9] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

✎PROOF:

1. Proof that $\mathbf{L}x = \mathbf{L}y \implies x - y \in \mathscr{N}(\mathbf{L})$:

$$\begin{aligned} \mathbf{L}(x-y) &= \mathbf{L}x - \mathbf{L}y && \text{by Theorem G.1 page 147} \\ &= \mathbb{0} && \text{by left hypothesis} \\ \implies \quad & x - y \in \mathscr{N}(\mathbf{L}) && \text{by definition of } \textit{Null Space} \end{aligned}$$

2. Proof that $\mathbf{L}x = \mathbf{L}y \impliedby x - y \in \mathscr{N}(\mathbf{L})$:

$$\begin{aligned} \mathbf{L}y &= \mathbf{L}y + \mathbb{0} && \text{by definition of linear space (Definition G.1 page 145)} \\ &= \mathbf{L}y + \mathbf{L}(x-y) && \text{by right hypothesis} \\ &= \mathbf{L}y + (\mathbf{L}x - \mathbf{L}y) && \text{by Theorem G.1 page 147} \\ &= (\mathbf{L}y - \mathbf{L}y) + \mathbf{L}x && \text{by } \textit{associative} \text{ and } \textit{commutative} \text{ properties (Definition G.1 page 145)} \\ &= \mathbf{L}x \end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathscr{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned} \mathbf{L} \text{ is } \textit{injective} &\iff \{(\mathbf{L}x = \mathbf{L}y \iff x = y) \quad \forall x, y \in X\} \\ &\iff \left\{\left[\mathbf{L}x - \mathbf{L}y = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\} \\ &\iff \left\{\left[\mathbf{L}(x-y) = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\} \\ &\iff \mathscr{N}(\mathbf{L}) = \{\mathbb{0}\} \end{aligned}$$

✏

Theorem G.4. [10] *Let $\boldsymbol{W}$, $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ be linear spaces over a field $\mathbb{F}$.*

THM					
1.	$\mathbf{L}(\mathbf{MN})$	$=$	$(\mathbf{LM})\mathbf{N}$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Z},\boldsymbol{W}), \mathbf{M}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(ASSOCIATIVE)
2.	$\mathbf{L}\left(\mathbf{M} \dotplus \mathbf{N}\right)$	$=$	$(\mathbf{LM}) \dotplus (\mathbf{LN})$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(LEFT DISTRIBUTIVE)
3.	$\left(\mathbf{L} \dotplus \mathbf{M}\right)\mathbf{N}$	$=$	$(\mathbf{LN}) \dotplus (\mathbf{MN})$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{LM})$	$=$	$(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{MN}) = (\mathbf{LM})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}\left(\mathbf{M} \dotplus \mathbf{N}\right) = (\mathbf{LM}) \dotplus (\mathbf{LN})$:

$$\begin{aligned} \left[\mathbf{L}\left(\mathbf{M} \dotplus \mathbf{N}\right)\right]x &= \mathbf{L}\left[\left(\mathbf{M} \dotplus \mathbf{N}\right)x\right] \\ &= \mathbf{L}\left[(\mathbf{M}x) \dotplus (\mathbf{N}x)\right] \\ &= [\mathbf{L}(\mathbf{M}x)] \dotplus [\mathbf{L}(\mathbf{N}x)] && \text{by } \textit{additive} \text{ property Definition G.4 page 147} \\ &= [(\mathbf{LM})x] \dotplus [(\mathbf{LN})x] \end{aligned}$$

3. Proof that $\left(\mathbf{L} \dotplus \mathbf{M}\right)\mathbf{N} = (\mathbf{LN}) \dotplus (\mathbf{MN})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{LM}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{LM}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned} [\alpha(\mathbf{LM})]x &= \alpha[(\mathbf{LM})x] \\ &= \mathbf{L}[\alpha(\mathbf{M}x)] && \text{by } \textit{homogeneous} \text{ property Definition G.4 page 147} \\ &= \mathbf{L}[(\alpha\mathbf{M})x] \\ &= [\mathbf{L}(\alpha\mathbf{M})]x \end{aligned}$$

[10] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

Theorem G.5 (Fundamental theorem of linear equations). [11] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathscr{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\dim \mathscr{I}(\mathbf{L}) + \dim \mathscr{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1, 2, \ldots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for $\mathscr{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathscr{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathscr{I}(\mathbf{L}) &= \dim \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \mathbb{0} \right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathscr{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ✏

G.2 Operators on Normed linear spaces

G.2.1 Operator norm

Definition G.5. [12] *Let $\boldsymbol{V} = (X, \mathbb{F}, \dotplus, \cdot)$ be a linear space and $\mathbb{F}$ be a field with absolute value function $|\cdot| \in \mathbb{R}^{\mathbb{F}}$.*

DEF

*A **norm** is any functional $\|\cdot\|$ in $\mathbb{R}^X$ that satisfies*

1.	$\|\boldsymbol{x}\| \geq 0$	$\forall \boldsymbol{x} \in X$	(STRICTLY POSITIVE)	*and*
2.	$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	(NONDEGENERATE)	*and*
3.	$\|a\boldsymbol{x}\| = \|a\| \, \|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X,\ a \in \mathbb{C}$	(HOMOGENEOUS)	*and*
4.	$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(SUBADDITIVE/*triangle inquality*).	

*A **normed linear space** is the pair $(\boldsymbol{V}, \|\cdot\|)$.*

[11] Michel and Herget (1993) page 99
[12] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

Definition G.6. [13] *Let $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [14]

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \leq 1 \} \qquad \forall \mathbf{A} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition G.1 (next) shows that the functional defined in Definition G.6 (previous) is a *norm* (Definition G.5 page 150).

Proposition G.1. [15] *Let $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ and $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1.	$|||\mathbf{A}||| \geq 0$	$\forall \mathbf{A} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NON-NEGATIVE)	and
2.	$|||\mathbf{A}||| = 0 \iff \mathbf{A} \doteq \mathbb{0}$	$\forall \mathbf{A} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NONDEGENERATE)	and
3.	$|||\alpha\mathbf{A}||| = |\alpha| \, |||\mathbf{A}|||$	$\forall \mathbf{A} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), \alpha \in \mathbb{F}$	(HOMOGENEOUS)	and
4.	$\left|\left|\left|\mathbf{A} \dot{+} \mathbf{B}\right|\right|\right| \leq |||\mathbf{A}||| + |||\mathbf{B}|||$	$\forall \mathbf{A} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \neq \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \leq 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)} \\ &> 0 \end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \doteq \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \leq 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)} \\ &= \sup_{x \in \boldsymbol{X}} \{ \|\mathbb{0}x\| \mid \|x\| \leq 1 \} \\ &= 0 \end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha| \, |||\mathbf{A}|||$:

$$\begin{aligned} |||\alpha\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{ \|\alpha\mathbf{A}x\| \mid \|x\| \leq 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)} \\ &= \sup_{x \in \boldsymbol{X}} \{ |\alpha| \, \|\mathbf{A}x\| \mid \|x\| \leq 1 \} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)} \\ &= |\alpha| \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}x\| \mid \|x\| \leq 1 \} && \text{by definition of sup} \\ &= |\alpha| \; |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)} \end{aligned}$$

[13] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225

[14] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990) page 290

[15] Rudin (1991) page 93

4. Proof that $\left\|\left|\mathbf{A} \mathring{+} \mathbf{B}\right|\right\| \le |||\mathbf{A}||| + |||\mathbf{B}|||$:

$$
\begin{aligned}
\left\|\left|\mathbf{A} \mathring{+} \mathbf{B}\right|\right\| &\triangleq \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \left\|(\mathbf{A} \mathring{+} \mathbf{B})\boldsymbol{x}\right\| \mid \|\boldsymbol{x}\| \le 1 \right\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)} \\
&= \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\} \\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x}\| + \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)} \\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\} + \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\} \\
&\triangleq |||\mathbf{A}||| + |||\mathbf{B}||| && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)}
\end{aligned}
$$

⇰

Lemma G.1. *Let* $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *and* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.

LEM

$$|||\mathbf{L}||| = \sup_{\boldsymbol{x}} \left\{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \right\} \qquad \forall \boldsymbol{x} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF: [16]

1. Proof that $\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \ge \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$:

$$\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \ge \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

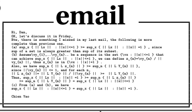
2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ \boldsymbol{y} \in X \,\middle|\, \begin{array}{ll} 1. & \|\mathbf{L}\boldsymbol{y}\| = \sup_{\boldsymbol{x}\in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \text{ and} \\ 2. & 0 < \|\boldsymbol{y}\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \le \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$:

$$
\begin{aligned}
\sup_{\boldsymbol{x}\in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} &= \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of set } Y \\
&= \frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \|\mathbf{L}\boldsymbol{y}\| \\
&= \|\boldsymbol{y}\| \left\| \frac{1}{\|\boldsymbol{y}\|} \mathbf{L}\boldsymbol{y} \right\| && \text{by homogeneous property (page 150)} \\
&= \|\boldsymbol{y}\| \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| && \text{by homogeneous property (page 147)} \\
&\le \|\boldsymbol{y}\| \sup_{\boldsymbol{y}\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \right\} && \text{by definition of supremum} \\
&= \|\boldsymbol{y}\| \sup_{\boldsymbol{y}\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \text{ for all } \boldsymbol{y} \in Y \\
&\le \sup_{\boldsymbol{y}\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } 0 < \|\boldsymbol{y}\| \le 1 \\
&\le \sup_{\boldsymbol{x}\in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{because } \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in X \quad \forall \boldsymbol{y} \in Y
\end{aligned}
$$

[16]



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4. By (1) and (3),

$$\sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| = 1\}$$

✏

Proposition G.2. [17] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathscr{L}(\boldsymbol{X},\boldsymbol{X}), |||\cdot|||)$.

PRP: $|||\mathbf{I}||| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 151)}\\
&= \sup\{\|x\| \mid \|x\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition G.3 page 146)}\\
&= 1
\end{aligned}$$

✏

Theorem G.6. [18] *Let* $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM:

$$\begin{array}{rcll}
\|\mathbf{L}x\| & \le & |||\mathbf{L}|||\ \|x\| & \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}),\ x\in\boldsymbol{X}\\
|||\mathbf{KL}||| & \le & |||\mathbf{K}|||\ |||\mathbf{L}||| & \forall \mathbf{K},\mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})
\end{array}$$

✎PROOF:

1. Proof that $\|\mathbf{L}x\| \le |||\mathbf{L}|||\ \|x\|$:

$$\begin{aligned}
\|\mathbf{L}x\| &= \frac{\|x\|}{\|x\|}\|\mathbf{L}x\|\\
&= \|x\|\left\|\frac{1}{\|x\|}\mathbf{L}x\right\| && \text{by property of norms}\\
&= \|x\|\left\|\mathbf{L}\frac{x}{\|x\|}\right\| && \text{by property of linear operators}\\
&\triangleq \|x\|\ \|\mathbf{L}y\| && \text{where } y \triangleq \frac{x}{\|x\|}\\
&\le \|x\|\sup_y\|\mathbf{L}y\| && \text{by definition of supremum}\\
&= \|x\|\sup_y\{\|\mathbf{L}y\| \mid \|y\| = 1\} && \text{because } \|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1\\
&\triangleq \|x\|\ \|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

[17] Michel and Herget (1993) page 410

[18] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

2. Proof that $\|\|\|\mathbf{KL}\|\|\| \le \|\|\|\mathbf{K}\|\|\| \;\|\|\|\mathbf{L}\|\|\|$:

$$\begin{aligned}
\|\|\|\mathbf{KL}\|\|\| &\triangleq \sup_{x\in X}\{\|(\mathbf{KL})x\| \mid \|x\| \le 1\} && \text{by Definition G.6 page 151 } (\|\|\|\cdot\|\|\|)\\
&= \sup_{x\in X}\{\|\mathbf{K}(\mathbf{L}x)\| \mid \|x\| \le 1\}\\
&\le \sup_{x\in X}\{\|\|\|\mathbf{K}\|\|\| \;\|\mathbf{L}x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&\le \sup_{x\in X}\{\|\|\|\mathbf{K}\|\|\| \;\|\|\|\mathbf{L}\|\|\| \;\|x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&= \sup_{x\in X}\{\|\|\|\mathbf{K}\|\|\| \;\|\|\|\mathbf{L}\|\|\| \;1 \mid \|x\| \le 1\} && \text{by definition of sup}\\
&= \|\|\|\mathbf{K}\|\|\| \;\|\|\|\mathbf{L}\|\|\| && \text{by definition of sup}
\end{aligned}$$

✏

G.2.2 Bounded linear operators

Definition G.7. [19] *Let $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \|\|\|\cdot\|\|\|)$ be a normed space of linear operators.*

DEF

An operator $\mathbf{B}$ *is* ***bounded*** *if* $\|\|\|\mathbf{B}\|\|\| < \infty$.
The quantity $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ *is the set of all* ***bounded linear operators*** *on* $(\boldsymbol{X},\boldsymbol{Y})$ *such that*

$$\mathscr{B}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \{\mathbf{L} \in \mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \mid \|\|\|\mathbf{L}\|\|\| < \infty\}.$$

Theorem G.7. [20] *Let $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \|\|\|\cdot\|\|\|)$ be the set of linear operators over normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

THM

The following conditions are all EQUIVALENT:

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $x_0 \in X$ $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \iff$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point $x \in X$)* $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \iff$
3. $\|\|\|\mathbf{L}\|\|\| < \infty$ *($\mathbf{L}$ is* BOUNDED*)* $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \iff$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}x\| \le M\,\|x\|$ $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), x\in X$

✎PROOF:

1. Proof that 1 $\implies$ 2:

$$\begin{aligned}
\epsilon &> \left\|\mathbf{L}x - \mathbf{L}x_0\right\| && \text{by hypothesis 1}\\
&= \left\|\mathbf{L}(x - x_0)\right\| && \text{by linearity (Definition G.4 page 147)}\\
&= \left\|\mathbf{L}(x + y - x_0 - y)\right\|\\
&= \left\|\mathbf{L}(x+y) - \mathbf{L}(x_0+y)\right\| && \text{by linearity (Definition G.4 page 147)}\\
&\implies \mathbf{L} \text{ is continuous at point } x+y\\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}$$

2. Proof that 2 $\implies$ 1: obvious:

[19] Rudin (1991) pages 92–93
[20] Aliprantis and Burkinshaw (1998) page 227

3. Proof that $4 \implies 2$:[21]

$$\begin{aligned}
|||\mathbf{L}x||| \le M\,\|x\| &\implies |||\mathbf{L}(x-y)||| \le M\,\|x-y\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le M\,\|x-y\| && \text{by linearity of } \mathbf{L} \text{ (Definition G.4 page 147)}\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le \epsilon \text{ whenever } M\,\|x-y\| < \epsilon\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le \epsilon \text{ whenever } \|x-y\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that $3 \implies 4$:

$$\begin{aligned}
\|\mathbf{L}x\| &\le \underbrace{|||\mathbf{L}|||}_{M}\ \|x\| && \text{by Theorem G.6 page 153}\\
&= M\ \|x\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that $1 \implies 3$:[22]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \infty\\
&\implies \exists\,(x_n) \ \text{ such that } \ \|x_n\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\|\mathbf{L}x_n\| \mid \|x_n\| \le 1\right\} = \infty\\
&\implies \|x_n\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \|\mathbf{L}x_n\|\\
&\implies \|x_n\| = 1 \text{ and } \|\mathbf{L}x_n\| \ge n\\
&\implies \frac{1}{n}\|x_n\| = \frac{1}{n} \text{ and } \frac{1}{n}\|\mathbf{L}x_n\| \ge 1\\
&\implies \left\|\frac{x_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

✎

G.2.3 Adjoints on normed linear spaces

Definition G.8. *Let $\mathscr{B}(X, Y)$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces X and Y. Let X^* be the* TOPOLOGICAL DUAL SPACE *of X.*

DEF: $\mathbf{B}^*$ *is the **adjoint** of an operator* $\mathbf{B} \in \mathscr{B}(X, Y)$ *if*

$$\mathsf{f}(\mathbf{B}x) = \left[\mathbf{B}^*\mathsf{f}\right](x) \qquad \forall \mathsf{f} \in X^*,\ x \in X$$

Theorem G.8. [23] *Let $\mathscr{B}(X, Y)$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *X and Y.*

THM:
$$\begin{aligned}
(\mathbf{A} \stackrel{+}{\dotplus} \mathbf{B})^* &= \mathbf{A}^* \stackrel{+}{\dotplus} \mathbf{B}^* && \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(X,Y)\\
(\lambda\mathbf{A})^* &= \lambda\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(X,Y)\\
(\mathbf{A}\mathbf{B})^* &= \mathbf{B}^*\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(X,Y)
\end{aligned}$$

[21] Bollobás (1999) page 29
[22] Aliprantis and Burkinshaw (1998) page 227
[23] Bollobás (1999) page 156

✎PROOF:

$$\begin{aligned}
\left[\mathbf{A} \overset{\circ}{+} \mathbf{B}\right]^* \mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \overset{\circ}{+} \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition G.8 page 155)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition G.4 page 147)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} && \\
&= \mathbf{A}^*\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 155)}\\
&= \left[\mathbf{A}^* + \mathbf{B}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{linear functional} &&
\end{aligned}$$

$$\begin{aligned}
[\lambda\mathbf{A}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 155)}\\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} && \\
&= \left[\lambda\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 155)}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{A}\mathbf{B}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 155)}\\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of } \textit{linear operators} && \text{(Definition G.4 page 147)}\\
&= \left[\mathbf{A}^*\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 155)}\\
&= \mathbf{B}^*\left[\mathbf{A}^*\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 155)}\\
&= \left[\mathbf{B}^*\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 155)}
\end{aligned}$$

✏

Theorem G.9. [24] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^*$ be the adjoint of an operator $\mathbf{B}$.*

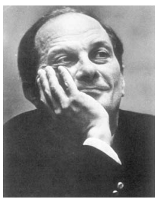
THM
$$\|\|\mathbf{B}\|\| = \|\|\mathbf{B}^*\|\| \qquad \forall \mathbf{B} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF:

$$\begin{aligned}
\|\|\mathbf{B}\|\| &\triangleq \sup\left\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\right\} && \text{by Definition G.6 page 151}\\
&\overset{?}{=} \sup\left\{|\mathsf{g}(\mathbf{B}\boldsymbol{x}; \boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \|\boldsymbol{y}^*\| \le 1\right\} && \\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x}; \mathbf{B}^*\boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \|\boldsymbol{y}^*\| \le 1\right\} && \\
&\triangleq \sup\left\{\|\mathbf{B}^*\boldsymbol{y}^*\| \mid \|\boldsymbol{x}\| \le 1,\ \|\boldsymbol{y}^*\| \le 1\right\} && \\
&= \sup\left\{\|\mathbf{B}^*\boldsymbol{y}^*\| \mid \|\boldsymbol{y}^*\| \le 1\right\} && \\
&\triangleq \|\|\mathbf{B}^*\|\| && \text{by Definition G.6 page 151}
\end{aligned}$$

✏

G.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞

Stanislaus M. Ulam (1909–1984), Polish mathematician [25]

[24] Rudin (1991) page 98

Theorem G.10 (Mazur-Ulam theorem). [26] *Let $\phi \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ be a function on normed linear spaces $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. Let $\mathbf{I} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$ be the identity operator on $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.*

THM

$$\left.\begin{array}{ll} 1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} \quad \text{and} \\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}}_{\text{isometric}} \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$
   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$
2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$
3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned} \left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis} \\ &= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1} \end{aligned}$$

4. Proof that $\mathrm{g}\boldsymbol{z} = \boldsymbol{z}$:

$$\begin{aligned} 2\lambda &= 2\sup\{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\} && \text{by definition of } \lambda \text{ item (2)} \\ &\le 2\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\| && \text{by definition of sup} \\ &= \|2\boldsymbol{z} - 2\mathrm{g}\boldsymbol{z}\| \\ &= \|\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}\boldsymbol{z}\| && \text{by definition of } \psi \text{ item (1)} \\ &= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}^{-1}\mathrm{g}\boldsymbol{z}\right\| && \text{by item (3)} \\ &= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| && \text{by definition of } \mathrm{g}^{-1} \\ &= \left\|\psi\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| \\ &= \left\|\mathrm{g}^{*}\boldsymbol{z} - \boldsymbol{z}\right\| \\ &\le \lambda && \text{by definition of } \lambda \text{ item (2)} \\ &\implies 2\lambda \le \lambda \\ &\implies \lambda = 0 \\ &\implies \mathrm{g}\boldsymbol{z} = \boldsymbol{z} \end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) = \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}$:

$$\begin{aligned} \phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) &= \\ &= \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y} \end{aligned}$$

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[25] quote: Ulam (1991) page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[26] Oikhberg and Rosenthal (2007) page 598, Väisälä (2003) page 634, Giles (2000) page 11, Dunford and Schwartz (1957) page 91, Mazur and Ulam (1932)

6. Proof that $\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}$:

$$\begin{aligned}\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) &= \\ &= [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}\end{aligned}$$

✏

Theorem G.11 (Neumann Expansion Theorem). [27] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{ll} 1. & \mathbf{A} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X}) \quad (\mathbf{A} \text{ is bounded}) \\ 2. & |||\mathbf{A}||| < 1 \end{array}\right\} \implies \left\{\begin{array}{llcl} 1. & (\mathbf{I}-\mathbf{A})^{-1} & & \textit{exists} \\ 2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \le & \dfrac{1}{1-|||\mathbf{A}|||} \\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \displaystyle\sum_{n=0}^{\infty} \mathbf{A}^n \\ & \multicolumn{3}{l}{\textit{with uniform convergence}} \end{array}\right.$$

G.3 Operators on Inner product spaces

G.3.1 General Results

Definition G.9. [28] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space.*

DEF

A function $\langle \triangle \mid \triangledown \rangle \in \mathbb{F}^{X\times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

1.	$\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle \ge 0$	$\forall \boldsymbol{x}\in X$	*(non-negative)*	*and*
2.	$\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x}\in X$	*(nondegenerate)*	*and*
3.	$\langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y}\in X, \forall\alpha\in\mathbb{C}$	*(homogeneous)*	*and*
4.	$\langle \boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{u}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{u}\rangle + \langle \boldsymbol{y} \mid \boldsymbol{u}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X$	*(additive)*	*and*
5.	$\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle = \langle \boldsymbol{y} \mid \boldsymbol{x}\rangle^*$	$\forall \boldsymbol{x},\boldsymbol{y}\in X$	*(conjugate symmetric).*	

An inner product is also called a ***scalar product****.*
An ***inner product space*** *is the pair* $(\boldsymbol{\Omega}, \langle \triangle \mid \triangledown \rangle)$.

Theorem G.12. [29] *Let* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

THM

$$\begin{array}{lclccl} \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 & \forall \boldsymbol{x}\in X & \iff & \mathbf{B}\boldsymbol{x} = \mathbb{0} & \forall \boldsymbol{x}\in X \\ \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle & \forall \boldsymbol{x}\in X & \iff & \mathbf{A} = \mathbf{B} & \end{array}$$

✎PROOF:

[27] Michel and Herget (1993) page 415
[28] Haaser and Sullivan (1991) page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72
[29] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

1. Proof that $\langle \mathbf{B}x \mid x \rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x+\mathbf{B}x) \mid (x+\mathbf{B}x)\rangle + i\,\langle \mathbf{B}(x+i\mathbf{B}x) \mid (x+i\mathbf{B}x)\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}x+\mathbf{B}^2x) \mid x+\mathbf{B}x \right\rangle\right\} + i\left\{\left\langle \mathbf{B}x+i\mathbf{B}^2x) \mid x+i\mathbf{B}x \right\rangle\right\} && \text{by Definition G.4 page 147}\\
&= \left\{\langle \mathbf{B}x \mid x\rangle + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + \left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\} && \text{by Definition G.9 page 158}\\
&\quad + i\left\{\langle \mathbf{B}x \mid x\rangle - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2\left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle\right\} + \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle - \left\langle \mathbf{B}^2x \mid x\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle\\
&= 2\,\|\mathbf{B}x\|^2\\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition G.5 page 150}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \mid x \rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products.

3. Proof that $\langle \mathbf{A}x \mid x \rangle = \langle \mathbf{B}x \mid x \rangle \implies \mathbf{A} \doteq \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \mid x\rangle - \langle \mathbf{B}x \mid x\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}x - \mathbf{B}x \mid x\rangle && \text{by \textit{additivity} property of } \langle \triangle \mid \triangledown\rangle \text{ (Definition G.9 page 158)}\\
&= \langle (\mathbf{A}-\mathbf{B})x \mid x\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A}-\mathbf{B})x &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \mid x \rangle = \langle \mathbf{B}x \mid x \rangle \impliedby \mathbf{A} \doteq \mathbf{B}$:

$$\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \qquad \text{by } \mathbf{A} \doteq \mathbf{B} \text{ hypothesis}$$

⇒

G.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition G.3 page 159). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem G.13 page 160).
- Both support decomposition into "real" and "imaginary" parts (Theorem A.3 page 92).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *Null Space* of an operator (Theorem G.14 page 161).

Proposition G.3. [30] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *(Definition G.7 page 154) on a* HILBERT SPACE $\boldsymbol{H}$.

PRP *An operator $\mathbf{B}^*$ is the **adjoint** of $\mathbf{B} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ if*
$$\langle \mathbf{B}x \mid y\rangle = \langle x \mid \mathbf{B}^* y\rangle \qquad \forall x, y \in \boldsymbol{H}.$$

✎PROOF:

[30] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000) page 182, von Neumann (1929) page 49, Stone (1932) page 41

1. For fixed y, $f(x) \triangleq \langle x \mid y\rangle$ is a *functional* in $\mathbb{F}^X$.
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}
\langle \mathbf{B}x \mid y\rangle &\triangleq f(\mathbf{B}x) \\
&\triangleq \mathbf{B}^* f(x) && \text{by definition of } \textit{operator adjoint} && \text{(Definition G.8 page 155)} \\
&= \langle x \mid \mathbf{B}^* y\rangle
\end{aligned}$$

⇰

Example G.2.

EX

In matrix algebra ("linear algebra")

- The inner product operation $\langle x \mid y\rangle$ is represented by $y^H x$.
- The linear operator is represented as a matrix A.
- The operation of A on a vector x is represented as Ax.
- The adjoint of matrix A is the Hermitian matrix A^H.

✎PROOF:

$$\langle Ax \mid y\rangle \triangleq y^H A x = [(Ax)^H y]^H = [x^H A^H y]^H = (A^H y)^H x \triangleq \left\langle x \mid A^H y\right\rangle$$

⇰

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition A.3 page 90). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[31]

Theorem G.13 (operator star-algebra). [32] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM

The pair $(\boldsymbol{H}, *)$ *is a* $*$-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \dotplus \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(INVOLUTORY)	

✎PROOF:

$$\begin{aligned}
\left\langle x \mid (\mathbf{A} \dotplus \mathbf{B})^* y\right\rangle &= \left\langle (\mathbf{A} \dotplus \mathbf{B})x \mid y\right\rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)} \\
&= \langle \mathbf{A}x \mid y\rangle + \langle \mathbf{B}x \mid y\rangle && \text{by definition of inner product} && \text{(Definition G.9 page 158)} \\
&= \langle x \mid \mathbf{A}^* y\rangle + \langle x \mid \mathbf{B}^* y\rangle && \text{by definition of operator addition} \\
&= \langle x \mid \mathbf{A}^* y + \mathbf{B}^* y\rangle && \text{by definition of inner product} && \text{(Definition G.9 page 158)} \\
&= \langle x \mid (\mathbf{A}^* + \mathbf{B}^*) y\rangle && \text{by definition of operator addition}
\end{aligned}$$

$$\begin{aligned}
\langle x \mid (\alpha\mathbf{A})^* y\rangle &= \langle (\alpha\mathbf{A})x \mid y\rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)} \\
&= \langle \alpha(\mathbf{A}x) \mid y\rangle && \text{by definition of scalar multiplication} \\
&= \alpha\,\langle \mathbf{A}x \mid y\rangle && \text{by definition of inner product} && \text{(Definition G.9 page 158)} \\
&= \alpha\,\langle x \mid \mathbf{A}^* y\rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)} \\
&= \langle x \mid \alpha^* \mathbf{A}^* y\rangle && \text{by definition of inner product} && \text{(Definition G.9 page 158)}
\end{aligned}$$

[31] Sakai (1998) page 1
[32] Halmos (1998) pages 39–40, Rudin (1991) page 311

$$\begin{aligned}
\langle x \,|\, (\mathbf{AB})^* y \rangle &= \langle (\mathbf{AB})x \,|\, y \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)}\\
&= \langle \mathbf{A}(\mathbf{B}x) \,|\, y \rangle && \text{by definition of operator multiplication}\\
&= \langle (\mathbf{B}x) \,|\, \mathbf{A}^* y \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)}\\
&= \langle x \,|\, \mathbf{B}^*\mathbf{A}^* y \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)}
\end{aligned}$$

$$\begin{aligned}
\langle x \,|\, \mathbf{A}^{**} y \rangle &= \langle \mathbf{A}^* x \,|\, y \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)}\\
&= \langle y \,|\, \mathbf{A}^* x \rangle^* && \text{by definition of inner product} && \text{(Definition G.9 page 158)}\\
&= \langle \mathbf{A}y \,|\, x \rangle^* && \text{by definition of adjoint} && \text{(Proposition G.3 page 159)}\\
&= \langle x \,|\, \mathbf{A}y \rangle && \text{by definition of inner product} && \text{(Definition G.9 page 158)}
\end{aligned}$$

✏

Theorem G.14. [33] *Let Y^X be the set of all operators from a linear space X to a linear space Y. Let $\mathscr{N}(\mathbf{L})$ be the* NULL SPACE *of an operator* $\mathbf{L}$ *in* Y^X *and* $\mathscr{I}(\mathbf{L})$ *the* IMAGE SET *of* $\mathbf{L}$ *in* Y^X.

THM
$$\begin{aligned}
\mathscr{N}(\mathbf{A}) &= \mathscr{I}(\mathbf{A}^*)^\perp\\
\mathscr{N}(\mathbf{A}^*) &= \mathscr{I}(\mathbf{A})^\perp
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathscr{I}(\mathbf{A}^*)^\perp &= \left\{ y \in H \,|\, \langle y \,|\, u \rangle = 0 \quad \forall u \in \mathscr{I}(\mathbf{A}^*) \right\}\\
&= \left\{ y \in H \,|\, \langle y \,|\, \mathbf{A}^* x \rangle = 0 \quad \forall x \in H \right\}\\
&= \{ y \in H \,|\, \langle \mathbf{A}y \,|\, x \rangle = 0 \quad \forall x \in H \} && \text{by definition of } \mathbf{A}^* && \text{(Proposition G.3 page 159)}\\
&= \{ y \in H \,|\, \mathbf{A}y = 0 \}\\
&= \mathscr{N}(\mathbf{A}) && \text{by definition of } \mathscr{N}(\mathbf{A})
\end{aligned}$$

$$\begin{aligned}
\mathscr{I}(\mathbf{A})^\perp &= \{ y \in H \,|\, \langle y \,|\, u \rangle = 0 \quad \forall u \in \mathscr{I}(\mathbf{A}) \}\\
&= \{ y \in H \,|\, \langle y \,|\, \mathbf{A}x \rangle = 0 \quad \forall x \in H \} && \text{by definition of } \mathscr{I}\\
&= \left\{ y \in H \,|\, \langle \mathbf{A}^* y \,|\, x \rangle = 0 \quad \forall x \in H \right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition G.3 page 159)}\\
&= \left\{ y \in H \,|\, \mathbf{A}^* y = 0 \right\}\\
&= \mathscr{N}(\mathbf{A}^*) && \text{by definition of } \mathscr{N}(\mathbf{A})
\end{aligned}$$

✏

G.4 Special Classes of Operators

G.4.1 Projection operators

Definition G.10. [34] *Let $\mathscr{B}(X, Y)$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces X and Y. Let $\mathbf{P}$ be a bounded linear operator in $\mathscr{B}(X, Y)$.*

DEF
$\mathbf{P}$ *is a **projection** operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[33] Rudin (1991) page 312

[34] Rudin (1991) page 126 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 26, Halmos (1958) page 73 ⟨§41. Projections⟩

Theorem G.15. [35] *Let* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{P}$ *be a bounded linear operator in* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *with* NULL SPACE $\mathscr{N}(\mathbf{P})$ *and* IMAGE SET $\mathscr{I}(\mathbf{P})$.

THM

$$\left.\begin{array}{lrcll} 1. & \mathbf{P}^2 & = & \mathbf{P} & (\mathbf{P} \textit{ is a projection operator}) \quad \textit{and} \\ 2. & \boldsymbol{\Omega} & = & \boldsymbol{X} \dotplus \boldsymbol{Y} & (\boldsymbol{Y} \textit{ compliments } \boldsymbol{X} \textit{ in } \boldsymbol{\Omega}) \quad \textit{and} \\ 3. & \mathbf{P}\boldsymbol{\Omega} & = & \boldsymbol{X} & (\mathbf{P} \textit{ projects onto } \boldsymbol{X}) \end{array}\right\} \implies \left\{\begin{array}{lrcll} 1. & \mathscr{I}(\mathbf{P}) & = & \boldsymbol{X} & \textit{and} \\ 2. & \mathscr{N}(\mathbf{P}) & = & \boldsymbol{Y} & \textit{and} \\ 3. & \boldsymbol{\Omega} & = & \mathscr{I}(\mathbf{P}) \dotplus \mathscr{N}(\mathbf{P}) & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} \mathscr{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\ &= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\ &= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\ &= \boldsymbol{\Omega}_1 \end{aligned}$$

$$\begin{aligned} \mathscr{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\} \\ &= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_2 \end{aligned}$$

✏

Theorem G.16. [36] *Let* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{P}$ *be a bounded linear operator in* $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.

THM

$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P} \textit{ is a projection operator}} \iff \underbrace{(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})}_{(\mathbf{I} - \mathbf{P}) \textit{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} (\mathbf{I} - \mathbf{P})^2 &= (\mathbf{I} - \mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I}(\mathbf{I} - \mathbf{P}) + (-\mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2 \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} && \text{by left hypothesis} \\ &= \mathbf{I} - \mathbf{P} \end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} \mathbf{P}^2 &= \underbrace{\mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P}^2}_{(\mathbf{I} - \mathbf{P})^2} - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P})^2 - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P}) - (\mathbf{I} - \mathbf{P} - \mathbf{P}) && \text{by right hypothesis} \\ &= \mathbf{P} \end{aligned}$$

✏

[35] Michel and Herget (1993) pages 120–121
[36] Michel and Herget (1993) page 121

Theorem G.17. [37] *Let $\boldsymbol{H}$ be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathcal{N}(\mathbf{P})$, *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

1. $\mathbf{P}^* = \mathbf{P}$ ($\mathbf{P}$ *is* SELF-ADJOINT) $\iff$
2. $\mathbf{P}^*\mathbf{P} = \mathbf{P}\mathbf{P}^*$ ($\mathbf{P}$ *is* NORMAL) $\iff$
3. $\mathcal{I}(\mathbf{P}) = \mathcal{N}(\mathbf{P})^\perp$ $\iff$
4. $\langle \mathbf{P}x \,|\, x\rangle = \|\mathbf{P}x\|^2 \quad \forall x \in X$

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\implies$ (2):

$$\begin{aligned}
\mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)}\\
&= \mathbf{P}\mathbf{P}^* && \text{by Theorem G.13 page 160}
\end{aligned}$$

Proof that (1) $\implies$ (3):

$$\begin{aligned}
\mathcal{I}(\mathbf{P}) &= \mathcal{N}(\mathbf{P}^*)^\perp && \text{by Theorem G.14 page 161}\\
&= \mathcal{N}(\mathbf{P})^\perp && \text{by (1)}
\end{aligned}$$

Proof that (3) $\implies$ (4):

Proof that (4) $\implies$ (1):

✏

G.4.2 Self Adjoint Operators

Definition G.11. [38] *Let* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be **self-adjoint** or **hermitian** if* $\mathbf{B} \overset{\circ}{=} \mathbf{B}^*$.

Example G.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation $\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}$.

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}}\mathsf{f}(u)\,\mathsf{d}u$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

Theorem G.18. [39] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|x\| \triangleq \sqrt{\langle x \,|\, x\rangle}$.

THM

$$\left\{\underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S}\text{ is self adjoint}}\right\} \implies \left\{\begin{array}{lll}
1. & \langle \mathbf{S}x \,|\, x\rangle \in \mathbb{R} & \text{(the hermitian quadratic form of } \mathbf{S} \text{ is REAL-VALUED)}\\
2. & \lambda_n \in \mathbb{R} & \text{(eigenvalues of } \mathbf{S} \text{ are REAL-VALUED)}\\
3. & \lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m\rangle = 0 & \text{(eigenvectors are ORTHOGONAL)}
\end{array}\right\}$$

[37] Rudin (1991) page 314

[38] Historical works regarding self-adjoint operators: von Neumann (1929) page 49, “linearer Operator R selbstadjungiert oder Hermitesch”, Stone (1932) page 50 ⟨“self-adjoint transformations”⟩

[39] Lax (2002) pages 315–316, Keener (1988) pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩, Bertero and Boccacci (1998) page 225 ⟨§“9.2 SVD of a matrix …If all eigenvectors are normalized…”⟩

✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}x \,|\, x \rangle \in \mathbb{R}$:

$$\begin{aligned}
\langle x \,|\, \mathbf{S}x \rangle &= \langle \mathbf{S}x \,|\, x \rangle && \text{by left hypothesis} \\
&= \langle x \,|\, \mathbf{S}x \rangle^* && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 158}
\end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned}
\lambda_n \|\psi_n\|^2 &= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition} \\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 158} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_n \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \lambda_n^* \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 158} \\
&= \lambda_n^* \|\psi_n\|^2 && \text{by definition}
\end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 158} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_m \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \lambda_m^* \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 158} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✑

G.4.3 Normal Operators

Definition G.12. [40] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{N}^*$ be the adjoint of an operator $\mathbf{N} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF: $\mathbf{N}$ *is* ***normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

Theorem G.19. [41] *Let $\mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE *$\boldsymbol{H}$. Let $\mathscr{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ and $\mathscr{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$.*

THM: $$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \text{ is normal}} \iff \|\mathbf{N}^* x\| = \|\mathbf{N}x\| \qquad \forall x \in \boldsymbol{H}$$

[40] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969) page 167, Frobenius (1878), Frobenius (1968) page 391

[41] Rudin (1991) pages 312–313

✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition G.3 page 159 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (}\mathbf{N}\text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition G.3 page 159 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition G.3 page 159 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem G.13 page 160 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis } (\|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by Proposition G.3 page 159 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem G.20. [42] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same Null Space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \left\{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \textit{Null Space}\\
&= \left\{\boldsymbol{x} | \, \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition G.5 page 150)}\\
&= \left\{\boldsymbol{x} | \, \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \\
&= \left\{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition G.5 page 150)}\\
&= \mathcal{N}(\mathbf{N}) && \text{by definition of } \textit{Null Space } \mathcal{N}
\end{aligned}$$

✏

Theorem G.21. [43] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m\rangle = 0}_{\text{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. Rudin (1991) page 313 claims both to be true.

[42] Rudin (1991) pages 312–313

[43] Rudin (1991) pages 312–313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$
\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\quad\; = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem G.13 page 160} \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem G.13 page 160} \\
&\quad\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\Longrightarrow \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\iff \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}
$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$
\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n\psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 158} \\
&= \langle \mathbf{N}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{N}^*\psi_m \rangle && \text{by Proposition G.3 page 159 (definition of adjoint)} \\
&= \langle \psi_n \,|\, \lambda_m^*\psi_m \rangle && \text{by (4.)} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 158}
\end{aligned}
$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✎

G.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition G.13. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES *(Definition G.5 page 150).*

DEF: *An operator* $\mathbf{M} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is* ***isometric*** *if*
$$\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \qquad \forall x \in X.$$

Theorem G.22. [44] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM:
$$\underbrace{\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall x \in X}_{\text{isometric in length}} \iff \underbrace{\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall x, y \in X}_{\text{isometric in distance}}$$

✎PROOF:

1. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \implies \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$
\begin{aligned}
\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| &= \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\| && \text{by definition of linear operators (Definition G.4 page 147)} \\
&= \|\mathbf{M}\boldsymbol{u}\| && \text{let } \boldsymbol{u} \triangleq \boldsymbol{x} - \boldsymbol{y} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by left hypothesis}
\end{aligned}
$$

[44] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

2. Proof that $\|\mathbf{M}x\| = \|x\| \impliedby \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$:

$$\begin{aligned}
\|\mathbf{M}x\| &= \|\mathbf{M}(x - \mathbb{0})\| \\
&= \|\mathbf{M}x - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition G.4 page 147)} \\
&= \|x - \mathbb{0}\| && \text{by right hypothesis} \\
&= \|x\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition G.13 page 166) in the more general normed linear spaces, while Theorem G.22 (page 166) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}x\| = \|x\| \iff \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$ for all $x, y \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem G.23 (next) demonstrates two additional equivalent properties.

Theorem G.23. [45] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$.*

THM

The following conditions are all ***equivalent***:

1.	$\mathbf{M}^*\mathbf{M} = \mathbf{I}$					$\iff$
2.	$\langle \mathbf{M}x \mid \mathbf{M}y \rangle$	$=$	$\langle x \mid y \rangle$	$\forall x,y \in X$	*($\mathbf{M}$ is surjective)*	$\iff$
3.	$\|\mathbf{M}x - \mathbf{M}y\|$	$=$	$\|x - y\|$	$\forall x,y \in X$	*(isometric in distance)*	$\iff$
4.	$\|\mathbf{M}x\|$	$=$	$\|x\|$	$\forall x \in X$	*(isometric in length)*	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}x \mid \mathbf{M}y \rangle &= \langle x \mid \mathbf{M}^*\mathbf{M}y \rangle && \text{by Proposition G.3 page 159 (definition of adjoint)} \\
&= \langle x \mid \mathbf{I}y \rangle && \text{by (1)} \\
&= \langle x \mid y \rangle && \text{by Definition G.3 page 146 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}x\| &= \sqrt{\langle \mathbf{M}x \mid \mathbf{M}x \rangle} && \text{by definition of } \|\cdot\| \\
&= \sqrt{\langle x \mid x \rangle} && \text{by right hypothesis} \\
&= \|x\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}x \mid \mathbf{M}y \rangle &= \|\mathbf{M}x + \mathbf{M}y\|^2 - \|\mathbf{M}x - \mathbf{M}y\|^2 + i\,\|\mathbf{M}x + i\mathbf{M}y\|^2 - i\,\|\mathbf{M}x - i\mathbf{M}y\|^2 && \text{by polarization id.} \\
&= \|\mathbf{M}(x + y)\|^2 - \|\mathbf{M}(x - y)\|^2 + i\,\|\mathbf{M}(x + iy)\|^2 - i\,\|\mathbf{M}(x - iy)\|^2 && \text{by Definition G.4} \\
&= \|x + y\|^2 - \|x - y\|^2 + i\,\|x + iy\|^2 - i\,\|x - iy\|^2 && \text{by left hypothesis}
\end{aligned}$$

4. Proof that (3) $\iff$ (4): by Theorem G.22 page 166

[45] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

5. Proof that (4) $\implies$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition G.3 page 159 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem G.13 page 160 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y=0\\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Definition G.3 page 146 (definition of } \mathbf{I})\\
\implies \qquad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x}\in X
\end{aligned}$$

✏

Theorem G.24. [46] *Let $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathscr{L}(\boldsymbol{X},\boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M}=\mathbf{I}}_{\mathbf{M}\text{ is isometric}} \implies \left\{\begin{array}{rcll} |||\mathbf{M}||| &=& 1 & (\text{UNIT LENGTH}) \quad \textit{and}\\ |\lambda| &=& 1 & \forall\lambda\in\Lambda \end{array}\right.$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M}=\mathbf{I} \implies |||\mathbf{M}|||=1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{x\in X}\{\|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\|=1\} && \text{by Definition G.6 page 151}\\
&= \sup_{x\in X}\{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\|=1\} && \text{by Theorem G.23 page 167}\\
&= \sup_{x\in X}\{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda|=1$: Let $(\boldsymbol{x},\lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|}\;\|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|}\;\|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem G.23 page 167}\\
&= \frac{1}{\|\boldsymbol{x}\|}\;\|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|}\;|\lambda|\;\|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

✏

Example G.4 (One sided shift operator). [47] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ $(0,1,2,\ldots)$ and shift operators defined as

$$\begin{array}{llcll}
1. & \mathbf{S}_r\,(\!(x_0,x_1,x_2,\ldots)\!) & \triangleq & (\!(0,x_0,x_1,x_2,\ldots)\!) & \text{(right shift operator)}\\
2. & \mathbf{S}_l\,(\!(x_0,x_1,x_2,\ldots)\!) & \triangleq & (\!(x_1,x_2,x_3,\ldots)\!) & \text{(left shift operator)}
\end{array}$$

EX

1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

[46] Michel and Herget (1993) page 432
[47] Michel and Herget (1993) page 441

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \,|\, \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \rangle &= \langle \left(\!\left(0, x_0, x_1, x_2, \ldots \right)\!\right) \,|\, \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\; y_n^* \\
&= \sum_{n=0}^{\infty} x_n\; y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\; y_{n+1}^* \\
&= \langle \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \,|\, \left(\!\left(y_1, y_2, y_3, \ldots \right)\!\right) \rangle \\
&= \left\langle \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \,\middle|\, \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✏

G.4.5 Unitary operators

Definition G.14. [48] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF: *The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} \;=\; \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition G.4. *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP: $\left.\begin{array}{l} \mathbf{U} \text{ is UNITARY} \quad \text{and} \\ \mathbf{V} \text{ is UNITARY} \end{array}\right\} \implies (\mathbf{UV}) \text{ is UNITARY.}$

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\big(\mathbf{V}^*\mathbf{U}^*\big) && \text{by Theorem G.8 page 155} \\
&= \mathbf{U}\big(\mathbf{V}\mathbf{V}^*\big)\mathbf{U}^* && \text{by associative property} \\
&= \mathbf{U}\mathbf{I}\mathbf{U}^* && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition G.14 page 169)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition G.14 page 169)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \big(\mathbf{V}^*\mathbf{U}^*\big)(\mathbf{UV}) && \text{by Theorem G.8 page 155} \\
&= \mathbf{V}^*\big(\mathbf{U}^*\mathbf{U}\big)\mathbf{V} && \text{by associative property} \\
&= \mathbf{V}^*\mathbf{I}\mathbf{V} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition G.14 page 169)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition G.14 page 169)}
\end{aligned}$$

[48] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

Theorem G.25. [49] *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathscr{I}(\mathbf{U})$ be the* IMAGE SET *of* $\mathbf{U}$.

THM *If $\mathbf{U}$ is a **bounded linear operator** ($\mathbf{U}\in\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$), then the following conditions are **equivalent**:*

1. $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ (UNITARY) $\iff$
2. $\langle \mathbf{U}x \mid \mathbf{U}y\rangle = \langle \mathbf{U}^*x \mid \mathbf{U}^*y\rangle = \langle x \mid y\rangle$ *and* $\mathscr{I}(\mathbf{U}) = X$ (SURJECTIVE) $\iff$
3. $\|\mathbf{U}x - \mathbf{U}y\| = \|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x-y\|$ *and* $\mathscr{I}(\mathbf{U}) = X$ (ISOMETRIC IN DISTANCE) $\iff$
4. $\|\mathbf{U}x\| = \|x\|$ *and* $\mathscr{I}(\mathbf{U}) = X$ (ISOMETRIC IN LENGTH)

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) $\langle \mathbf{U}x \mid \mathbf{U}y\rangle = \langle \mathbf{U}^*x \mid \mathbf{U}^*y\rangle = \langle x \mid y\rangle$ by Theorem G.23 (page 167).
   (b) Proof that $\mathscr{I}(\mathbf{U}) = X$:

$$\begin{aligned}
X &\supseteq \mathscr{I}(\mathbf{U}) && \text{because } \mathbf{U}\in X^X\\
&\supseteq \mathscr{I}(\mathbf{U}\mathbf{U}^*)\\
&= \mathscr{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I})\\
&= X && \text{by Definition G.3 page 146 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\iff$ (3) $\iff$ (4): by Theorem G.23 page 167.
3. Proof that (3) $\implies$ (1):
   (a) Proof that $\|\mathbf{U}x - \mathbf{U}y\| = \|x-y\| \implies \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem G.23 page 167
   (b) Proof that $\|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x-y\| \implies \mathbf{U}\mathbf{U}^* = \mathbf{I}$:

$$\begin{aligned}
\|\mathbf{U}^*x - \mathbf{U}^*y\| = \|x-y\| \implies\quad \mathbf{U}^{**}\mathbf{U}^* &= \mathbf{I} && \text{by Theorem G.23 page 167}\\
\mathbf{U}\mathbf{U}^* &= \mathbf{I} && \text{by Theorem G.13 page 160}
\end{aligned}$$

Theorem G.26. *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathbf{U}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathscr{N}(\mathbf{U})$ the* NULL SPACE *of $\mathbf{U}$, and $\mathscr{I}(\mathbf{U})$ the* IMAGE SET *of* $\mathbf{U}$.

THM

$$\underbrace{\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U}\text{ is unitary}} \implies \left\{\begin{array}{ccccl}
\mathbf{U}^{-1} &=& \mathbf{U}^* & & \textit{and}\\
\mathscr{I}(\mathbf{U}) &=& \mathscr{I}(\mathbf{U}^*) &=& X \quad \textit{and}\\
\mathscr{N}(\mathbf{U}) &=& \mathscr{N}(\mathbf{U}^*) &=& \{\mathbb{0}\} \quad \textit{and}\\
|||\mathbf{U}||| &=& |||\mathbf{U}^*||| &=& 1 \quad (\text{UNIT LENGTH})
\end{array}\right\}$$

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both *isometric* and *normal*:

$$\begin{array}{lllll}
\mathbf{U}^*\mathbf{U} & & = \mathbf{I} & \implies & \mathbf{U}\text{ is } \textit{isometric}\\
\mathbf{U}\mathbf{U}^* &= \mathbf{U}^*\mathbf{U} & = \mathbf{I} & \implies & \mathbf{U}^*\text{ is } \textit{isometric}\\
\mathbf{U}^{-1} &= \mathbf{U}^* & & \implies & \mathbf{U}^{-1}\text{ is } \textit{isometric}\\
\\
\mathbf{U}^*\mathbf{U} &= \mathbf{U}\mathbf{U}^* & = & \implies & \mathbf{U}\text{ is } \textit{normal}\\
\mathbf{U}\mathbf{U}^* &= \mathbf{U}^*\mathbf{U} & = & \implies & \mathbf{U}^*\text{ is } \textit{normal}\\
\mathbf{U}^{-1} &= \mathbf{U}^* & & \implies & \mathbf{U}^{-1}\text{ is } \textit{normal}
\end{array}$$

[49] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathscr{I}(\mathbf{U}) = \mathscr{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem G.25 page 170.
3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathscr{N}(\mathbf{U}) = \mathscr{N}(\mathbf{U}^*) = \mathscr{N}(\mathbf{U}^{-1})$:

$$\begin{aligned}
\mathscr{N}(\mathbf{U}^*) &= \mathscr{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both } \textit{normal} \text{ and by Theorem G.20 page 165}\\
&= \mathscr{I}(\mathbf{U})^\perp && \text{by Theorem G.14 page 161}\\
&= X^\perp && \text{by above result}\\
&= \{\mathbb{0}\}
\end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem G.24 page 168.

Example G.5 (Rotation matrix). [50]

EX

$$\underbrace{\left\{ \mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}}_{\text{rotation matrix } \mathbf{R}_\theta \,:\, \mathbb{R}^2 \to \mathbb{R}^2} \implies \left\{ \begin{array}{llcll} (1). & \mathbf{R}^{-1}{}_\theta & = & \mathbf{R}_{-\theta} & \text{and} \\ (2). & \mathbf{R}^*{}_\theta & = & \mathbf{R}^{-1}{}_\theta & (\mathbf{R} \text{ is unitary}) \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R}\\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of } \textit{Hermetian transpose} \text{ operator } H\\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{(Theorem 1.2 page 5)}\\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R}\\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✏

Example G.6. [51] Let $\mathbf{A}$ and $\mathbf{B}$ be matrix operators.

EX

$$\underbrace{\underbrace{\mathbf{A} \triangleq \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A} \text{ is a } \textit{rotation operator}.} \quad \underbrace{\mathbf{B} \triangleq \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\mathbf{B} \text{ is a } \textit{reflection operator}.}}_{\text{Both } \mathbf{A} \text{ and } \mathbf{B} \text{ are } \textbf{unitary}.}$$

Example G.7. Examples of *Fredholm integral operators* include

EX

1.	**Fourier Transform**	$[\tilde{\mathbf{F}}\mathsf{x}](f)$	$=$	$\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-i2\pi ft}\,\mathrm{d}t$	$\kappa(t,f) =$	$e^{-i2\pi ft}$
2.	**Inverse Fourier Transform**	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t)$	$=$	$\int_{f\in\mathbb{R}} \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathrm{d}f$	$\kappa(f,t) =$	$e^{i2\pi ft}$
3.	**Laplace operator**	$[\mathbf{L}\mathsf{x}](s)$	$=$	$\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t,s) =$	e^{-st}

Example G.8 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathsf{f}(x) \triangleq \mathsf{f}(x-1) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

[50] Noble and Daniel (1988) page 311
[51] Gel'fand (1963) page 4, Gelfand et al. (2018) page 4

EX

1.	$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$		(inverse translation operator)
2.	$\mathbf{T}^* = \mathbf{T}^{-1}$			($\mathbf{T}$ is invertible)
3.	$\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^* = \mathbf{I}$			($\mathbf{T}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$
$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned}
\langle \mathbf{T}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \langle \mathsf{f}(x-1) \,|\, \mathsf{g}(x)\rangle && \text{by definition of } \mathbf{T}\\
&= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathsf{d}x\\
&= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathsf{d}x\\
&= \langle \mathsf{f}(x) \,|\, \mathsf{g}(x+1)\rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example G.9 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX

1.	$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(inverse dilation operator)
2.	$\mathbf{D}^* = \mathbf{D}^{-1}$		($\mathbf{D}$ is invertible)
3.	$\mathbf{D}^*\mathbf{D} = \mathbf{D}\mathbf{D}^* = \mathbf{I}$		($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x) \,\middle|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathsf{d}x\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathsf{d}u && \text{let } u \triangleq 2x \implies \mathsf{d}x = \frac{1}{2}\,\mathsf{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\,\mathsf{d}u\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

Example G.10 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq (\!(x_{n-1})\!)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n+1}\right)\!\right)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\,(\!(x_n)\!) \,|\, (\!(y_n)\!)\rangle &= \langle(\!(x_{n-1})\!) \,|\, (\!(y_n)\!)\rangle && \text{by definition of } \mathbf{D}\\
&= \sum_n \boldsymbol{x}_{n-1}\ \boldsymbol{y}_n^* \\
&= \sum_n \boldsymbol{x}_n\ \boldsymbol{y}_{n+1}^* \\
&= \left\langle (\!(x_n)\!) \,|\, \left(\!\left(y_{n+1}\right)\!\right)\right\rangle \\
&= \left\langle (\!(x_n)\!) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\,(\!(y_n)\!)\right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary. ✑

Example G.11 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator (Theorem 5.1 page 74)

$$[\tilde{\mathbf{F}}\mathsf{x}](f) \triangleq \int_t \mathsf{x}(t)\underbrace{e^{-i2\pi ft}}_{\kappa(t,f)}\,\mathsf{d}t \qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](t) \triangleq \int_f \tilde{\mathsf{x}}(f)\underbrace{e^{i2\pi ft}}_{\kappa^*(t,f)}\,\mathsf{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{x} \,|\, \tilde{\mathsf{y}}\right\rangle &= \left\langle \int_t \mathsf{x}(t)e^{-i2\pi ft}\,\mathsf{d}t \,|\, \tilde{\mathsf{y}}(f)\right\rangle \\
&= \int_t \mathsf{x}(t)\left\langle e^{-i2\pi ft} \,|\, \tilde{\mathsf{y}}(f)\right\rangle\,\mathsf{d}t \\
&= \int_t \mathsf{x}(t)\int_f e^{-i2\pi ft}\tilde{\mathsf{y}}^*(f)\,\mathsf{d}f\,\mathsf{d}t \\
&= \int_t \mathsf{x}(t)\left[\int_f e^{i2\pi ft}\tilde{\mathsf{y}}(f)\,\mathsf{d}f\right]^*\,\mathsf{d}t \\
&= \left\langle \mathsf{x}(t) \,|\, \int_f \tilde{\mathsf{y}}(f)e^{i2\pi ft}\,\mathsf{d}f\right\rangle \\
&= \left\langle \mathsf{x} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathsf{y}}\right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✑

G.5 Operator order

Definition G.15. [52] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF: $\mathbf{P}$ *is* ***positive*** *if* $\langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \;\forall \boldsymbol{x} \in \boldsymbol{X}$.
This condition is denoted $\mathbf{P} \geq 0$.

Theorem G.27. [53]

THM:
$$\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \implies \begin{cases} (\mathbf{P}+\mathbf{Q}) \geq 0 & & ((\mathbf{P}+\mathbf{Q}) \text{ is positive}) \\ \mathbf{A}^*\mathbf{P}\mathbf{A} \geq 0 & \forall \mathbf{A} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive}) \\ \mathbf{A}^*\mathbf{A} \geq 0 & \forall \mathbf{A} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{A} \text{ is positive}) \end{cases}$$

✎PROOF:

$$\begin{aligned}
\langle (\mathbf{P}+\mathbf{Q})\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle + \langle \mathbf{Q}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by additive property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition G.9 page 158)} \\
&\geq \langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{P}\mathbf{A}\boldsymbol{x} \,|\, \mathbf{A}\boldsymbol{x} \rangle && \text{by definition of adjoint (Proposition G.3 page 159)} \\
&= \langle \mathbf{P}\boldsymbol{y} \,|\, \boldsymbol{y} \rangle && \text{where } \boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{I}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition G.3 page 146)} \\
&\geq 0 && \text{by non-negative property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition G.9 page 158)} \\
&\implies \mathbf{I} \text{ is positive} \\
\langle \mathbf{A}^*\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle &= \langle \mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition G.3 page 146)} \\
&\geq 0 && \text{by two previous results}
\end{aligned}$$

✏

Definition G.16. [54] *Let* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be* BOUNDED *operators.*

DEF: $\mathbf{A} \geq \mathbf{B}$ *("*$\mathbf{A}$ *is greater than or equal to* $\mathbf{B}$*") if*
$\mathbf{A} - \mathbf{B} \geq 0$ *("*$(\mathbf{A} - \mathbf{B})$ *is positive")*

[52] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩
[53] Michel and Herget (1993) page 429
[54] Michel and Herget (1993) page 429

BIBLIOGRAPHY

Yuri A. Abramovich and Charalambos D. Aliprantis. *An Invitation to Operator Theory*. American Mathematical Society, Providence, Rhode Island, 2002. ISBN 0-8218-2146-6. URL http://books.google.com/books?vid=ISBN0821821466.

Milton Abramowitz and Irene A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards, 1972. URL http://www.cs.bham.ac.uk/~aps/research/projects/as/book.php.

M.R. Adhikari and A. Adhikari. *Groups, Rings And Modules With Applications*. Universities Press, Hyderabad, 2 edition, 2003. ISBN 978-8173714290. URL http://books.google.com/books?vid=ISBN8173714290.

Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis*. Acedemic Press, London, 3 edition, 1998. ISBN 9780120502578. URL http://www.amazon.com/dp/0120502577.

George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*, volume 71 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, Cambridge, U.K., new edition, February 15 2001. ISBN 0521789885. URL http://books.google.com/books?vid=ISBN0521789885.

Tom M. Apostol. *Mathematical Analysis*. Addison-Wesley series in mathematics. Addison-Wesley, Reading, 2 edition, 1975. ISBN 986-154-103-9. URL http://books.google.com/books?vid=ISBN0201002884.

Léon Autonne. Sur l'hermitien (on the hermitian). In *Comptes Rendus Des SéAnces De L'AcadéMie Des Sciences*, volume 133, pages 209–268. De L'Académie des sciences (Academy of Sciences), Paris, 1901. URL http://visualiseur.bnf.fr/Visualiseur?O=NUMM-3089. Comptes Rendus Des SéAnces De L'AcadéMie Des Sciences (Reports Of the Meetings Of the Academy of Science).

Léon Autonne. Sur l'hermitien (on the hermitian). *Rendiconti del Circolo Matematico di Palermo*, 16:104–128, 1902. Rendiconti del Circolo Matematico di Palermo (Statements of the Mathematical Circle of Palermo).

Claude Gaspard Bachet. *Arithmétiques de Diophante*. 1621. URL http://www.bsb-muenchen-digital.de/~web/web1008/bsb10081407/images/index.html?digID=bsb10081407.

George Bachman. *Elements of Abstract Harmonic Analysis*. Academic paperbacks. Academic Press, New York, 1964. ISBN 9781483267562. URL http://books.google.com/books?vid=ISBN9781483267562.

George Bachman and Lawrence Narici. *Functional Analysis*. Academic Press textbooks in mathematics; Pure and Applied Mathematics Series. Academic Press, 1 edition, 1966. ISBN 9780486402512. URL http://books.google.com/books?vid=ISBN0486402517. "unabridged republication" available from Dover (isbn 0486402517).

George Bachman, Lawrence Narici, and Edward Beckenstein. *Fourier and Wavelet Analysis*. Universitext Series. Springer, 2002. ISBN 9780387988993. URL http://books.google.com/books?vid=ISBN9780387988993. "Corrected second printing, 2002".

Stefan Banach. Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales (on abstract operations and their applications to the integral equations). *Fundamenta Mathematicae*, 3:133–181, 1922. URL http://matwbn.icm.edu.pl/ksiazki/fm/fm3/fm3120.pdf.

Stefan Banach. *Théorie des opérations linéaires*. Monografje Matematyczne, Warsaw, Poland, 1932a. URL http://matwbn.icm.edu.pl/kstresc.php?tom=1&wyd=10. (Theory of linear operations).

Stefan Banach. *Theory of Linear Operations*, volume 38 of *North-Holland mathematical library*. North-Holland, Amsterdam, 1932b. ISBN 0444701842. URL http://www.amazon.com/dp/0444701842/. English translation of 1932 French edition, published in 1987.

Edward Barbeau. *Polynomials*. Problem Books in Mathematics. Springer, New York, 1989. ISBN 0-387-96919-5. URL http://books.google.com/books?vid=ISBN0387406271.

Adi Ben-Israel and Robert P. Gilbert. *Computer-supported calculus*. Texts and monographs in symbolic computation. Springer, 2002. ISBN 3-211-82924-5. URL http://books.google.com/books?vid=ISBN3211829245.

Sterling Khazag Berberian. *Introduction to Hilbert Space*. Oxford University Press, New York, 1961. URL http://books.google.com/books?vid=ISBN0821819127.

M. Bertero and P. Boccacci. *Introduction to Inverse Problems in Imaging*. CRC Press, 1998. ISBN 9781439822067. URL http://books.google.com/books?vid=ISBN9781439822067.

Josef Betten. *Creep Mechanics*, chapter A. The Heaviside and Dirac Functions, pages 285–294. Springer Science & Business Media, 3 edition, 2008a. ISBN 9783540850519. URL https://link.springer.com/content/pdf/bbm:978-3-540-85051-9/1.pdf. https://www.springer.com/gp/book/9783540850502.

Josef Betten. *Creep Mechanics*, chapter B. The Laplace Transformation, pages 295–343. Springer Science & Business Media, 3 edition, 2008b. ISBN 9783540850519. URL https://link.springer.com/content/pdf/bbm:978-3-540-85051-9/1.pdf. https://www.springer.com/gp/book/9783540850502.

Etienne Bézout. *Théorie Générale des équations Algébriques*. 1779a. URL http://books.google.com.tw/books?id=RDEVAAAAQAAJ. (General Theory of Algebraic Equations).

Etienne Bézout. *General Theory of Algebraic Equations*. Princeton University Press, 1779b. ISBN 978-0691114323. URL http://books.google.com/books?vid=ISBN0691114323. 2006 translation of Théorie générale des équations algébriques (1779).

Ralph Philip Boas. *Entire Functions*, volume 5 of *Pure and Applied Mathematics. A Series of Monographs and Text Books*. Academic Press, 1954. ISBN 9780123745828. URL http://books.google.com/books?vid=ISBN0123745829.

Béla Bollobás. *Linear Analysis; an introductory course*. Cambridge mathematical textbooks. Cambridge University Press, Cambridge, 2 edition, March 1 1999. ISBN 978-0521655774. URL http://books.google.com/books?vid=ISBN0521655773.

Peter B. Borwein and Tamás Erdélyi. *Polynomials and Polynomial Inequalities*. Graduate Texts in Mathematics Series. Springer, 1995. ISBN 9780387945095. URL http://books.google.com/books?vid=ISBN0387945091.

Umberto Bottazzini. *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*. Springer-Verlag, New York, 1986. ISBN 0-387-96302-2. URL http://books.google.com/books?vid=ISBN0387963022.

Nicolas Bourbaki. *Algebra I*. Elements of Mathematics. Springer, 2003a. ISBN 3-540-64243-9. URL http://books.google.com/books?vid=ISBN3540642439.

Nicolas Bourbaki. *Algebra II*. Elements of Mathematics. Springer, 2003b. ISBN 978-3540007067. URL http://books.google.com/books?vid=ISBN3540007067.

Carl Benjamin Boyer and Uta C. Merzbach. *A History of Mathematics*. Wiley, New York, 2 edition, 1991. ISBN 0471543977. URL http://books.google.com/books?vid=ISBN0471543977.

Ronald Newbold Bracewell. *The Fourier transform and its applications*. McGraw-Hill electrical and electronic engineering series. McGraw-Hill, 2, illustrated, international student edition edition, 1978. ISBN 9780070070134. URL http://books.google.com/books?vid=ISBN007007013X.

Thomas John I'Anson Bromwich. *An Introduction to the Theory of Infinite Series*. Macmillan and Company, 1 edition, 1908. ISBN 9780821839768. URL http://www.archive.org/details/anintroductiont00bromgoog.

Andrew M. Bruckner, Judith B. Bruckner, and Brian S. Thomson. *Real Analysis*. Prentice-Hall, Upper Saddle River, N.J., 1997. ISBN 9780134588865. URL http://books.google.com/books?vid=ISBN013458886X.

Florian Cajori. A history of mathematical notations; notations mainly in higher mathematics. In *A History of Mathematical Notations; Two Volumes Bound as One*, volume 2. Dover, Mineola, New York, USA, 1993. ISBN 0-486-67766-4. URL http://books.google.com/books?vid=ISBN0486677664. reprint of 1929 edition by *The Open Court Publishing Company*.

Lennart Axel Edvard Carleson. Convergence and growth of partial sums of fourier series. *Acta Mathematica*, 116(1):135–157, 1966. doi: 10.1007/BF02392815. URL http://dx.doi.org/10.1007/BF02392815.

Lennart Axel Edvard Carleson and Björn Engquist. After the 'golden age': what next? lennart carleson interviewed by björn engquist. In Björn Engquist and Wilfried Schmid, editors, *Mathematics Unlimited—2001 and beyond*, pages 455–461. Springer, Berlin, 2001. ISBN 978-3-540-66913-5. URL https://link.springer.com/chapter/10.1007/978-3-642-56478-9_22.

Peter G. Casazza and Mark C. Lammers. Bracket products for weyl-heisenberg frames. In Hans G. Feichtinger and Thomas Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 71–98. Birkhäuser, 1998. ISBN 9780817639594.

Clémente Ibarra Castanedo. *Quantitative subsurface defect evaluation by pulsed phase thermography: depth retrieval with the phase*. PhD thesis, Université Laval, October 2005. URL http://archimede.bibl.ulaval.ca/archimede/fichiers/23016/23016.html. Faculte' Des Sciences Et De Génie.

Joan Cerdà. *Linear functional analysis*, volume 116 of *Graduate studies in mathematics*. American Mathematical Society, July 16 2010. ISBN 0821851152. URL http://books.google.com/books?vid=ISBN0821851152.

Lindsay Childs. *A Concrete Introduction To Higher Algebra*. Undergraduate texts in mathematics. Springer, New York, 3 edition, 2009. ISBN 978-0-387-74527-5. URL http://books.google.com/books?vid=ISBN0387745270.

Alexandre J. Chorin and Ole H. Hald. *Stochastic Tools in Mathematics and Science*, volume 1 of *Surveys and Tutorials in the Applied Mathematical Sciences*. Springer, New York, 2 edition, 2009. ISBN 978-1-4419-1001-1. URL http://books.google.com/books?vid=ISBN9781441910011.

Ole Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston/Basel/Berlin, 2003. ISBN 0-8176-4295-1. URL http://books.google.com/books?vid=ISBN0817642951.

Charles K. Chui. *An Introduction to Wavelets*. Academic Press, San Diego, California, USA, January 3 1992. ISBN 9780121745844. URL http://books.google.com/books?vid=ISBN0121745848.

Jon F. Claerbout. *Fundamentals of Geophysical Data Processing with Applications to Petroleum Prospecting*. International series in the earth and planetary sciences, Tab Mastering Electronics Series. McGraw-Hill, New York, 1976. ISBN 9780070111172. URL http://sep.stanford.edu/sep/prof/.

J. L. Coolidge. The story of the binomial theorem. *The American Mathematical Monthly*, 56(3): 147–157, March 1949. URL http://www.jstor.org/stable/2305028.

A. Córdoba. Dirac combs. *Letters in Mathematical Physics*, 17(3):191–196, 1989. URL https://doi.org/10.1007/BF00401584. print ISSN 0377-9017 online ISSN 1573-0530.

Xingde Dai and David R. Larson. *Wandering vectors for unitary systems and orthogonal wavelets*. Number 640 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence R.I., July 1998. ISBN 0821808001. URL http://books.google.com/books?vid=ISBN0821808001.

Xingde Dai and Shijie Lu. Wavelets in subspaces. *Michigan Math. J.*, 43(1):81–98, 1996. doi: 10.1307/mmj/1029005391. URL http://projecteuclid.org/euclid.mmj/1029005391.

Ingrid Daubechies. *Ten Lectures on Wavelets*. Society for Industrial and Applied Mathematics, Philadelphia, 1992. ISBN 0-89871-274-2. URL http://www.amazon.com/dp/0898712742.

Kenneth R. Davidson and Allan P. Donsig. *Real Analysis and Applications*. Springer, 2010. ISBN 9781441900050. URL http://books.google.com/books?vid=ISBN1441900055.

Charles Jean de la Vallée-Poussin. Sur l'intégrale de lebesgue. *Transactions of the American Mathematical Society*, 16(4):435–501, October 1915. URL http://www.jstor.org/stable/1988879.

Juan-Arias de Reyna. *Pointwise Convergence of Fourier Series*, volume 1785 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin/Heidelberg/New York, 2002. ISBN 3540432701. URL http://books.google.com/books?vid=ISBN3540432701.

René Descartes. *La géométrie*. 1637a. URL http://historical.library.cornell.edu/math/math_D.html.

René Descartes. *Discours de la méthode pour bien conduire sa raison, et chercher la verite' dans les sciences*. Jan Maire, Leiden, 1637b. URL http://www.gutenberg.org/etext/13846.

René Descartes. *Discourse on the Method of Rightly Conducting the Reason in the Search for Truth in the Sciences*. 1637c. URL http://www.gutenberg.org/etext/59.

René Descartes. *Regulae ad directionem ingenii*. 1684a. URL http://www.fh-augsburg.de/~harsch/Chronologia/Lspost17/Descartes/des_re00.html.

René Descartes. *Rules for Direction of the Mind*. 1684b. URL http://en.wikisource.org/wiki/Rules_for_the_Direction_of_the_Mind.

René Descartes. *The Geometry of Rene Descartes*. Courier Dover Publications, June 1 1954. ISBN 0486600688. URL http://books.google.com/books?vid=isbn0486600688. orginally published by Open Court Publishing, Chicago, 1925; translation of La géométrie.

Jean Alexandre Dieudonné. *Foundations of Modern Analysis*. Academic Press, New York, 1969. ISBN 1406727911. URL http://books.google.com/books?vid=ISBN1406727911.

Bogdan Dumitrescu. *Positive Trigonometric Polynomials and Signal Processing Applications*. Signals and Communication Technology. Springer, 2007. ISBN 978-1-4020-5124-1. URL gen.lib.rus.ec/get?md5=5346e169091b2d928d8333cd053300f9.

Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part 1, General Theory*, volume 7 of *Pure and applied mathematics*. Interscience Publishers, New York, 1957. ISBN 0471226394. URL http://www.amazon.com/dp/0471608483. with the assistance of William G. Bade and Robert G. Bartle.

Yuli Eidelman, Vitali D. Milman, and Antonis Tsolomitis. *Functional Analysis: An Introduction*, volume 66 of *Graduate Studies in Mathematics*. American Mathematical Society, 2004. ISBN 0821836463. URL http://books.google.com/books?vid=ISBN0821836463.

Euclid. *Elements*. circa 300BC. URL http://farside.ph.utexas.edu/euclid.html.

Leonhard Euler. *Introductio in analysin infinitorum*, volume 1. Marcum-Michaelem Bousquet & Socios, Lausannæ, 1748. URL http://www.math.dartmouth.edu/~euler/pages/E101.html. Introduction to the Analysis of the Infinite.

Leonhard Euler. *Introduction to the Analysis of the Infinite*. Springer, 1988. ISBN 0387968245. URL http://books.google.com/books?vid=ISBN0387968245. translation of 1748 Introductio in analysin infinitorum.

David Ewen. *The Book of Modern Composers*. Alfred A. Knopf, New York, 1950. URL http://books.google.com/books?id=yHw4AAAAIAAJ.

David Ewen. *The New Book of Modern Composers*. Alfred A. Knopf, New York, 3 edition, 1961. URL http://books.google.com/books?id=bZIaAAAAMAAJ.

Lorenzo Farina and Sergio Rinaldi. *Positive Linear Systems: Theory and Applications*. Pure and applied mathematics. John Wiley & Sons, 1 edition, July 3 2000. ISBN 9780471384564. URL http://books.google.com/books?vid=ISBN0471384569.

G.L. Fix and G. Strang. Fourier analysis of the finite element method in ritz-galerkin theory. *Studies in Applied Mathematics*, 48:265–273, 1969.

Harley Flanders. Differentiation under the integral sign. *The American Mathematical Monthly*, 80(6):615–627, June–July 1973. URL http://sgpwe.izt.uam.mx/files/users/uami/jdf/proyectos/Derivar_inetegral.pdf. http://www.jstor.org/pss/2319163.

Francis J. Flanigan. *Complex Variables; Harmonic and Analytic Functions*. Dover, New York, 1983. ISBN 9780486613888. URL http://books.google.com/books?vid=ISBN0486613887.

Gerald B. Folland. *Fourier Analysis and its Applications*. Wadsworth & Brooks / Cole Advanced Books & Software, Pacific Grove, California, USA, 1992. ISBN 0-534-17094-3. URL http://www.worldcat.org/isbn/0534170943.

Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1995. ISBN 0-8493-8490-7. URL http://books.google.com/books?vid=ISBN0849384907.

Brigitte Forster and Peter Massopust, editors. *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*. Applied and Numerical Harmonic Analysis. Springer, November 19 2009. ISBN 9780817648909. URL http://books.google.com/books?vid=ISBN0817648909.

Jean-Baptiste-Joseph Fourier. Mémoire sur la propagation de la chaleur dans les corps solides (dissertation on the propagation of heat in solid bodies). In M. Gaston Darboux, editor, *Œuvres De Fourier*, pages 215–221. Ministère de L'instruction Publique, Paris, France, 2 edition, December 21 1807. URL http://gallica.bnf.fr/ark:/12148/bpt6k33707/f220n7.

Jean-Baptiste-Joseph Fourier. *Théorie Analytique de la Chaleur (The Analytical Theory of Heat)*. Chez Firmin Didot, pere et fils, Paris, 1822. URL http://books.google.com/books?vid=04X2vlqZx7hydlQUWEq&id=TDQJAAAAIAAJ.

Jean-Baptiste-Joseph Fourier. *The Analytical Theory of Heat (Théorie Analytique de la Chaleur)*. Cambridge University Press, Cambridge, February 20 1878. URL http://www.archive.org/details/analyticaltheory00fourrich. 1878 English translation of the original 1822 French edition. A 2003 Dover edition is also available: isbn 0486495310.

Ferdinand Georg Frobenius. Uber lineare substitutionen und bilineare formen. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 84:1–63, 1878. ISSN 0075-4102. URL http://www.digizeitschriften.de/home/services/pdfterms/?ID=509796.

Ferdinand Georg Frobenius. Uber lineare substitutionen und bilineare formen. In Jean Pierre Serre, editor, *Gesammelte Abhandlungen (Collected Papers)*, volume I, pages 343–405. Springer, Berlin, 1968. URL http://www.worldcat.org/oclc/253015. reprint of Frobenius' 1878 paper.

Jürgen Fuchs. *Affine Lie Algebras and Quantum Groups: An Introduction, With Applications in Conformal Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1995. ISBN 052148412X. URL http://books.google.com/books?vid=ISBN052148412X.

Paul Abraham Fuhrmann. *A Polynomial Approach to Linear Algebra*. Springer Science+Business Media, LLC, 2 edition, 2012. ISBN 978-1461403371. URL http://books.google.com/books?vid=ISBN1461403375.

Dennis Gabor. Theory of communication. *Journal of the Institution of Electrical Engineers*, 93(26): 429–457, November 1946. URL http://bigwww.epfl.ch/chaudhury/gabor.pdf.

Carl Friedrich Gauss. *Carl Friedrich Gauss Werke*, volume 8. Königlichen Gesellschaft der Wissenschaften, B.G. Teubneur In Leipzig, Göttingen, 1900. URL http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN236010751.

Israel M. Gelfand. Normierte ringe. *Mat. Sbornik*, 9(51)(1):3–24, 1941. URL http://www.mathnet.ru/links/d19894c5545d2f75b3e1037162f86d58/sm6046.pdf.

Israel M. Gelfand and Mark A. Naimark. Normed rings with an involution and their representations. In *Commutative Normed Rings*, number 170 in AMS Chelsea Publishing Series, pages 240–274. Chelsea Publishing Company, Bronx, 1964. ISBN 9780821820223. URL http://books.google.com/books?vid=ISBN0821820222.

Israel M. Gelfand and Mark A. Neumark. On the imbedding of normed rings into the ring of operators in hilbert space. *Mat. Sbornik*, 12(54:2):197–217, 1943a.

Israel M. Gelfand and Mark A. Neumark. On the imbedding of normed rings into the ring of operators in hilbert space. In Robert S. Doran, editor, *C*-algebras: 1943–1993: a Fifty Year Celebration: Ams Special Session Commenorating the First Fifty Years of C*-Algebra Theory January 13–14, 1993*, pages 3–19. American Mathematical Society, 1943b. ISBN 978-0821851753. URL http://books.google.com/books?vid=ISBN0821851756.

Israel M. Gelfand, R. A. Minlos, and Z. .Ya. Shapiro. *Representations of the rotation and Lorentz groups and their applications*. Courier Dover Publications, reprint edition, 2018. ISBN 9780486823850. Izrail' Moiseevich Gel'fand.

Izrail' Moiseevich Gel'fand. *Representations of the rotation and Lorentz groups and their applications*. Pergamon Press book, 1963. 2018 Dover edition available.

John Robilliard Giles. *Introduction to the Analysis of Normed Linear Spaces*. Number 13 in Australian Mathematical Society lecture series. Cambridge University Press, Cambridge, 2000. ISBN 0-521-65375-4. URL http://books.google.com/books?vid=ISBN0521653754.

T. N. T. Goodman, S. L. Lee, and W. S. Tang. Wavelets in wandering subspaces. *Transactions of the A.M.S.*, 338(2):639–654, August 1993a. URL http://www.jstor.org/stable/2154421. Transactions of the American Mathematical Society.

T. N. T. Goodman, S. L. Lee, and W. S. Tang. Wavelets in wandering subspaces. *Advances in Computational Mathematics 1*, pages 109–126, February 1993b.

Jaideva C. Goswami and Andrew K. Chan. *Fundamentals of Wavelets; Theory, Algorithms, and Applications*. John Wiley & Sons, Inc., 1999. ISBN 0-471-19748-3. URL http://vadkudr.boom.ru/Collection/fundwave_contents.html.

Ronald L. Graham, Donald Ervin Knuth, and Oren Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, 2 edition, 1994. ISBN 0201558025. URL http://books.google.com/books?vid=ISBN0201558025.

Ernst Adolph Guillemin. *Synthesis of Passive Networks: Theory and Methods Appropriate to the Realization and Approximation Problems*. John Wiley & Sons, 1957. ISBN 9780882754819. URL http://books.google.com.tw/books?id=JQ4nAAAAMAAJ.

Norman B. Haaser and Joseph A. Sullivan. *Real Analysis*. Dover Publications, New York, 1991. ISBN 0-486-66509-7. URL http://books.google.com/books?vid=ISBN0486665097.

Paul R. Haddad and Ali N. Akansu. *Multiresolution Signal Decomposition: Transforms, Subbands, and Wavelets*. Acedemic Press, October 1 1992. ISBN 0323138365. URL http://books.google.com/books?vid=ISBN0323138365.

Paul R. Halmos. *Finite Dimensional Vector Spaces*. Princeton University Press, Princeton, 1 edition, 1948. ISBN 0691090955. URL http://books.google.com/books?vid=isbn0691090955.

Paul R. Halmos. *Finite Dimensional Vector Spaces*. Springer Science & Business Media, New York, 1 edition, 1958. ISBN 9781461263876. URL http://books.google.com/books?vid=isbn9781461263876. 2012 “Softcover reprint of the hardcover 1st edition 1987”; *Undergraduate Texts in Mathematics*.

Paul R. Halmos. *Intoduction to Hilbert Space and the Theory of Spectral Multiplicity*. Chelsea Publishing Company, New York, 2 edition, 1998. ISBN 0821813781. URL http://books.google.com/books?vid=ISBN0821813781.

Godfrey H. Hardy. *A Mathematician's Apology*. Cambridge University Press, Cambridge, 1940. URL http://www.math.ualberta.ca/~mss/misc/A%20Mathematician's%20Apology.pdf.

Godfrey Harold Hardy. Notes on special systems of orthogonal functions (iv): the orthogonal functions of whittaker's cardinal series. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37(4):331–348, October 1941. URL http://dx.doi.org/10.1017/S0305004100017977.

Felix Hausdorff. *Set Theory*. Chelsea Publishing Company, New York, 3 edition, 1937. ISBN 0828401195. URL http://books.google.com/books?vid=ISBN0828401195. 1957 translation of the 1937 German *Grundzüge der Mengenlehre*.

Michiel Hazewinkel, editor. *Handbook of Algebras*, volume 2. North-Holland, Amsterdam, 1 edition, 2000. ISBN 044450396X. URL http://books.google.com/books?vid=ISBN044450396X.

Jean Van Heijenoort. *From Frege to Gödel : A Source Book in Mathematical Logic, 1879-1931*. Harvard University Press, Cambridge, Massachusetts, 1967. URL http://www.hup.harvard.edu/catalog/VANFGX.html.

Christopher Heil. *A Basis Theory Primer*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, expanded edition edition, 2011. ISBN 9780817646868. URL http://books.google.com/books?vid=ISBN9780817646868.

Christopher E. Heil and David F. Walnut. Continuous and discrete wavelet transforms. *Society for Industrial and Applied Mathematics*, 31(4), December 1989. URL http://citeseer.ist.psu.edu/viewdoc/download?doi=10.1.1.132.1241&rep=rep1&type=pdf.

Charles Hermite. Lettre à stieltjes. In B. Baillaud and H. Bourget, editors, *Correspondance d'Hermite et de Stieltjes*, volume 2, pages 317–319. Gauthier-Villars, Paris, May 20 1893. published in 1905.

J. R. Higgins. Five short stories about the cardinal series. *Bulletin of the American Mathematical Society*, 12(1):45–89, 1985. URL http://www.ams.org/journals/bull/1985-12-01/S0273-0979-1985-15293-0/.

John Rowland Higgins. *Sampling Theory in Fourier and Signal Analysis: Foundations*. Oxford Science Publications. Oxford University Press, August 1 1996. ISBN 9780198596998. URL http://books.google.com/books?vid=ISBN0198596995.

David Hilbert, Lothar Nordheim, and John von Neumann. über die grundlagen der quantenmechanik (on the bases of quantum mechanics). *Mathematische Annalen*, 98:1–30, 1927. ISSN 0025-5831 (print) 1432-1807 (online). URL http://dz-srv1.sub.uni-goettingen.de/cache/toc/D27776.html.

Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990. ISBN 0-521-30586-1. URL http://books.google.com/books?vid=isbn0521305861. Library: QA188H66 1985.

Alfred Edward Housman. *More Poems*. Alfred A. Knopf, 1936. URL http://books.google.com/books?id=rTMiAAAAMAAJ.

Robert J. Marks II. *Introduction to Shannon Sampling and Interpolation Theory*. Springer-Verlag, New York, 1991. ISBN 0-387-97391-5,3-540-97391-5. URL http://marksmannet.com/RobertMarks/REPRINTS/1999_IntroductionToShannonSamplingAndInterpolationTheory.pdf.

Vasile I. Istrăţescu. *Inner Product Structures: Theory and Applications*. Mathematics and Its Applications. D. Reidel Publishing Company, 1987. ISBN 9789027721822. URL http://books.google.com/books?vid=ISBN9027721823.

A. J. E. M. Janssen. The zak transform: A signal transform for sampled time-continuous signals. *Philips Journal of Research*, 43(1):23–69, 1988.

Bjorn Jawerth and Wim Sweldens. An overview of wavelet based multiresolution analysis. *SIAM Review*, 36:377–412, September 1994. URL http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.45.5864&rep=rep1&type=pdf.

Alan Jeffrey and Hui Hui Dai. *Handbook of Mathematical Formulas and Integrals*. Handbook of Mathematical Formulas and Integrals Series. Academic Press, 4 edition, January 18 2008. ISBN 9780080556840. URL http://books.google.com/books?vid=ISBN0080556841.

K. D. Joshi. *Applied Discrete Structures*. New Age International, New Delhi, 1997. ISBN 8122408265. URL http://books.google.com/books?vid=ISBN8122408265.

J.S.Chitode. *Signals And Systems*. Technical Publications, 2009. ISBN 9788184316780. URL http://books.google.com/books?vid=ISBN818431678X.

Jean-Pierre Kahane. Partial differential equations, trigonometric series, and the concept of function around 1800: a brief story about lagrange and fourier. In Dorina Mitrea and V. G. Mazía, editors, *Perspectives In Partial Differential Equations, Harmonic Analysis And Applications: a Volume in Honor of Vladimir G. Maz'ya's 70th birthday*, volume 79 of *Proceedings of Symposia in Pure Mathematics*, pages 187–206. American Mathematical Society, 2008. ISBN 0821844245. URL http://books.google.com/books?vid=ISBN0821844245.

David W. Kammler. *A First Course in Fourier Analysis*. Cambridge University Press, 2 edition, 2008. ISBN 9780521883405. URL http://books.google.com/books?vid=ISBN0521883407.

Edward Kasner and James Roy Newman. *Mathematics and the Imagination*. Simon and Schuster, 1940. ISBN 0486417034. URL http://books.google.com/books?vid=ISBN0486417034. “unabridged and unaltered republication” available from Dover.

Yitzhak Katznelson. *An Introduction to Harmonic Analysis*. Cambridge mathematical library. Cambridge University Press, 3 edition, 2004. ISBN 0521543592. URL http://books.google.com/books?vid=ISBN0521543592.

James P. Keener. *Principles of Applied Mathematics; Transformation and Approximation*. Addison-Wesley Publishing Company, Reading, Massachusets, 1988. ISBN 0201156741. URL http://www.worldcat.org/isbn/0201156741.

Anthony W Knapp. *Advanced Real Analysis*. Cornerstones. Birkhäuser, Boston, Massachusetts, USA, 1 edition, July 29 2005a. ISBN 0817643826. URL http://books.google.com/books?vid=ISBN0817643826.

Anthony W Knapp. *Basic Real Analysis*. Cornerstones. Birkhäuser, Boston, Massachusetts, USA, 1 edition, July 29 2005b. ISBN 0817632506. URL http://books.google.com/books?vid=ISBN0817632506.

Granino A. Korn and Theresea M. Korn. *Mathematical Handbook for Scientists and Engineers; Definitions, Theorems, and Formulas for Reference and Review*. McGraw-Hill, Inc., New York, 2 edition, 1968. ISBN 0486411478. URL http://books.google.com/books?vid=ISBN0486411478.

Vladimir Aleksandrovich Kotelnikov. On the transmission capacity of the 'ether' and of cables in electrical communications. In *Proceedings of the first All-Union Conference on the technological reconstruction of the communications sector and the development of low-current engineering*, Moscow, 1933. URL http://ict.open.ac.uk/classics/1.pdf.

Carlos S. Kubrusly. *The Elements of Operator Theory*. Springer, 1 edition, 2001. ISBN 9780817641740. URL http://books.google.com/books?vid=ISBN0817641742.

Carlos S. Kubrusly. *The Elements of Operator Theory*. Springer, 2 edition, 2011. ISBN 9780817649975. URL http://books.google.com/books?vid=ISBN0817649972.

Joseph-Louis Lagrange, Pierre-Simon Laplace, Étienne Louis Malus, René Just Haüy, and Adrien-Marie Legendre. Proclamation des prix décernés dans la séance publique de 6 janvier 1812. *Esprit des Journaux, Francais et étrangers par Une Societe de Gens Delettres*, 2:111–112, January 6 1812a. URL http://books.google.com/books?id=QpUUAAAAQAAJ.

Joseph-Louis Lagrange, Pierre-Simon Laplace, Étienne Louis Malus, René Just Haüy, and Adrien-Marie Legendre. Proclamation des prix décernés dans la séance publique de 6 janvier 1812. *Mercure De France, Journal Littéraire et Politique*, 50:374–375, January 6 1812b. URL http://books.google.com/books?id=8HxBAAAAcAAJ.

Imre Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge University Press, Cambridge, 1976. ISBN 0521290384. URL http://books.google.com/books?vid=ISBN0521290384.

Traian Lalescu. *Sur les équations de Volterra*. PhD thesis, University of Paris, 1908. advisor was Émile Picard.

Traian Lalescu. *Introduction à la théorie des équations intégrales (Introduction to the Theory of Integral Equations)*. Librairie Scientifique A. Hermann, Paris, 1911. URL http://www.worldcat.org/oclc/1278521. first book about integral equations ever published.

Pierre-Simon Laplace. *Theorie Analytique des Probabilités*. 1812.

Pierre-Simon Laplace. *Theorie Analytique des Probabilités*. Paris, 2 edition, 1814. URL https://archive.org/details/bub_gb_6MRLAAAAMAAJ/.

Pierre-Simon Laplace. *A Philosophical Essay on Probabilities*. Wiley, 1902. URL https://books.google.com/books?id=DfI7AQAAMAAJ.

Rupert Lasser. *Introduction to Fourier Series*, volume 199 of *Monographs and textbooks in pure and applied mathematics*. Marcel Dekker, New York, New York, USA, February 8 1996. ISBN 978-0824796105. URL http://books.google.com/books?vid=ISBN0824796101. QA404.L33 1996.

Peter D. Lax. *Functional Analysis*. John Wiley & Sons Inc., USA, 2002. ISBN 0-471-55604-1. URL http://www.worldcat.org/isbn/0471556041. QA320.L345 2002.

Gottfried W. Leibniz. Symbolismus memorabilis calculi algebraici et infinitesimalis, in comparatione potentiarum et differentiarum; et de lege homogeneorum transcendentali. *Miscellanea Berolinensia ad incrementum scientiarum, ex scriptis Societati Regiae scientarum*, pages 160–165, 1710. URL http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/anzeige/index_html?band=01-misc/1&seite:int=184.

Gottfried Wilhelm Leibniz. Letter to christian huygens, 1679. In Leroy E. Loemker, editor, *Philosophical Papers and Letters*, volume 2 of *The New Synthese Historical Library*, chapter 27, pages 248–249. Kluwer Academic Press, Dordrecht, 2 edition, September 8 1679. ISBN 902770693X. URL http://books.google.com/books?vid=ISBN902770693X.

Ezra C. Levy. *A New Table of Laplace Transformation Pairs*. Space Technology Laboratories, Inc., P.O. Box 95001 Los Angeles 45, California, November 1958. "This Laplace transform table contains 1024 possible transformation pairs in nine sections.…In this table we shall deal with the one-sided Laplace transformation, as contrasted to the two-sided transformation…The reason for this choice is that for the systems that are most commonly considered the solution of interest is that corresponding to real values of time, and which assumes knowledge of the initial conditions, i. e. , the dynamic characteristics of the system at t = 0.".

J. Liouville. Sur l'integration d'une classe d'équations différentielles du second ordre en quantités finies explicites. *Journal De Mathematiques Pures Et Appliquees*, 4:423–456, 1839. URL http://gallica.bnf.fr/ark:/12148/bpt6k16383z.

Lynn H. Loomis and Ethan D. Bolker. *Harmonic analysis*. Mathematical Association of America, 1965. URL http://books.google.com/books?id=MEfvAAAAMAAJ.

Nikolai Luzin. Sur la convergence des séries trigonom etriers de fourier. *C. R. Acad. Sci.*, 156:1655–1658, 1913.

Colin Maclaurin. *Treatise of Fluxions*. W. Baynes, 1742. URL http://www.amazon.com/dp/B000863E7M.

Stéphane G. Mallat. *A Wavelet Tour of Signal Processing*. Elsevier, 2 edition, September 15 1999. ISBN 9780124666061. URL http://books.google.com/books?vid=ISBN012466606X.

maxima. *Maxima Manual version 5.28.0*. 5.28.0 edition. URL http://maxima.sourceforge.net/documentation.html.

Stefan Mazur. Sur les anneaux linéaires. *Comptes rendus de l'Académie des sciences*, 207:1025–1027, 1938.

Stefan Mazur and Stanislaus M. Ulam. Sur les transformations isométriques d'espaces vectoriels normées. *Comptes rendus de l'Académie des sciences*, 194:946–948, 1932.

William Henry Metzler, Edward Drake Roe, and Warren Gardner Bullard. *College Algebra*. Longsmans, Green, & Co., New York, 1908. URL http://archive.org/details/collegealgebra00metzrich.

Anthony N. Michel and Charles J. Herget. *Applied Algebra and Functional Analysis*. Dover Publications, Inc., 1993. ISBN 048667598X. URL http://books.google.com/books?vid=ISBN048667598X. original version published by Prentice-Hall in 1981.

Fred Mintzer. Filters for distortion-free two-band multi-rate filter banks. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 32, 1985.

Robert Édouard Moritz. *Memorabilia Mathematica; Or, The Philomath's Quotation-book*. Macmillan, 1914. URL https://archive.org/details/memorabiliamath02morigoog.

Eddie Ortiz Muniz. A method for deriving various formulas in electrostatics and electromagnetism using lagrange's trigonometric identities. *American Journal of Physics*, 21(140), 1953. doi: 10.1119/1.1933371. URL http://dx.doi.org/10.1119/1.1933371.

M. Zuhair Nashed and Gilbert G. Walter. General sampling theorems for functions in reproducing kernel hilbert spaces. *Mathematics of Control, Signals and Systems*, 4(4):363–390, 1991. URL http://link.springer.com/article/10.1007/BF02570568.

Ben Noble and James W. Daniel. *Applied Linear Algebra*. Prentice-Hall, Englewood Cliffs, NJ, USA, 3 edition, 1988. ISBN 0130412600. URL http://www.worldcat.org/isbn/0130412600.

Timur Oikhberg and Haskell Rosenthal. A metric characterization of normed linear spaces. *Rocky Mountain Journal Of Mathematics*, 37(2):597–608, 2007. URL http://www.ma.utexas.edu/users/rosenthl/pdf-papers/95-oikh.pdf.

Alan V. Oppenheim and Ronald W. Schafer. *Discrete-Time Signal Processing*. Prentice Hall, 2 edition, 1999. ISBN 9780137549207. URL http://www.amazon.com/dp/0137549202.

Judith Packer. Applications of the work of stone and von neumann to wavelets. In Robert S. Doran and Richard V. Kadison, editors, *Operator Algebras, Quantization, and Noncommutative Geometry: A Centennial Celebration Honoring John Von Neumann and Marshall H. Stone : AMS Special Session on Operator Algebras, Quantization, and Noncommutative Geometry, a Centennial Celebration Honoring John Von Neumann and Marshall H. Stone, January 15-16, 2003, Baltimore, Maryland*, volume 365 of *Contemporary mathematics—American Mathematical Society*, pages 253–280, Baltimore, Maryland, 2004. American Mathematical Society. ISBN 9780821834022. URL http://books.google.com/books?vid=isbn0821834029.

Lincoln P. Paine. *Warships of the World to 1900*. Ships of the World Series. Houghton Mifflin Harcourt, 2000. ISBN 9780395984147. URL http://books.google.com/books?vid=ISBN9780395984149.

Athanasios Papoulis. *Circuits and Systems: A Modern Approach*. HRW series in electrical and computer engineering. Holt, Rinehart, and Winston, 1980. ISBN 9780030560972. URL http://books.google.com/books?vid=ISBN0030560977.

Giuseppe Peano. *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva*. Fratelli Bocca Editori, Torino, 1888a. Geometric Calculus: According to the *Ausdehnungslehre* of H. Grassmann.

Giuseppe Peano. *Geometric Calculus: According to the Ausdehnungslehre of H. Grassmann*. Springer (2000), 1888b. ISBN 0817641262. URL http://books.google.com/books?vid=isbn0817641262. originally published in 1888 in Italian.

Michael Pedersen. *Functional Analysis in Applied Mathematics and Engineering*. Chapman & Hall/CRC, New York, 2000. ISBN 9780849371691. URL http://books.google.com/books?vid=ISBN0849371694. Library QA320.P394 1999.

Mark A. Pinsky. *Introduction to Fourier Analysis and Wavelets*. Brooks/Cole, Pacific Grove, 2002. ISBN 9780534376604. URL http://www.amazon.com/dp/0534376606.

Lakshman Prasad and Sundararaja S. Iyengar. *Wavelet Analysis with Applications to Image Processing*. CRC Press LLC, Boca Raton, 1997. ISBN 978-0849331695. URL http://books.google.com/books?vid=ISBN0849331692. Library TA1637.P7 1997.

Victor V. Prasolov. *Polynomials*, volume 11 of *Algorithms and Computation in Mathematics*. Springer, 2004. ISBN 978-3-540-40714-0. URL http://books.google.com/books?vid=ISBN3540407146. translated from Russian (2001, 2nd edition).

Murray H. Protter and Charles B. Jr. Morrey. *Intermediate Calculus*. Undergraduate Texts in Mathematics. Springer Science & Business Media, 2, illustrated edition, 2012. ISBN 9781461210863. URL http://books.google.com/books?vid=ISBN1461210860. "Softcover reprint of the hardcover 2nd edition 1985" (page iv).

Ptolemy. *Ptolemy's Almagest*. Springer-Verlag (1984), New York, circa 100AD. ISBN 0387912207. URL http://gallica.bnf.fr/ark:/12148/bpt6k3974x.

Shie Qian and Dapang Chen. *Joint time-frequency analysis: methods and applications*. PTR Prentice Hall, 1996. ISBN 9780132543842. URL http://books.google.com/books?vid=ISBN0132543842.

Charles Earl Rickart. *General Theory of Banach Algebras*. University series in higher mathematics. D. Van Nostrand Company, Yale University, 1960. URL http://books.google.com/books?id=PVrvAAAAMAAJ.

Theodore J. Rivlin. *The Chebyshev Polynomials*. Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts. John Wiley & Sons, New York, 1974. ISBN 0-471-72470-X. URL http://books.google.com/books?vid=ISBN047172470X.

Enders A. Robinson. *Random Wavelets and Cybernetic Systems*, volume 9 of *Griffins Statistical Monographs & Courses*. Lubrecht & Cramer Limited, London, June 1962. ISBN 0852640757. URL http://books.google.com/books?vid=ISBN0852640757.

Enders A. Robinson. Multichannel z-transforms and minimum delay. *Geophyics*, 31(3):482–500, June 1966. doi: 10.1190/1.1439788. URL http://dx.doi.org/10.1190/1.1439788.

Enders A. Robinson. A historical perspective of spectrum estimation. *Proceedings of the IEEE*, 70 (9):885–907, September 1982. doi: 10.1109/PROC.1982.12423. URL http://www.archive.ece.cmu.edu/~ece792/handouts/Robinson82.pdf.

Maxwell Rosenlicht. *Introduction to Analysis*. Dover Publications, New York, 1968. ISBN 0-486-65038-3. URL http://books.google.com/books?vid=ISBN0486650383.

Joseph J. Rotman. *Advanced Modern Algebra*, volume 114 of *Graduate studies in mathematics*. American Mathematical Society, 2 edition, 2010. ISBN 978-0-8218-4741-1. URL http://books.google.com/books?vid=ISBN0821847414.

Walter Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, New York, New York, USA, 3 edition, 1987. ISBN 9780070542341. URL http://www.amazon.com/dp/0070542341. Library QA300.R8 1976.

Walter Rudin. *Functional Analysis*. McGraw-Hill, New York, 2 edition, 1991. ISBN 0-07-118845-2. URL http://www.worldcat.org/isbn/0070542252. Library QA320.R83 1991.

Shôichirô Sakai. *C*-Algebras and W*-Algebras*. Springer-Verlag, Berlin, 1 edition, 1998. ISBN 9783540636335. URL http://books.google.com/books?vid=ISBN3540636331. reprint of 1971 edition.

Frank Moulton Saxelby. *A course in practical mathematics*. Longmans, Green, and Co., 39 Paternoster Row, London, 6 edition, 1920. URL https://archive.org/details/courseinpractica00saxerich/.

Gert Schubring. *Conflicts Between Generalization, Rigor, and Intuition: Number Concepts Underlying the Development of Analysis in 17th–19th Century France and Germany*. Sources and studies in the history of mathematics and physical sciences. Springer, New York, 1 edition, June 2005. ISBN 0387228365. URL http://books.google.com/books?vid=ISBN0387228365.

Isaac Schur. Uber die charakterischen wurzeln einer linearen substitution mit enier anwendung auf die theorie der integralgleichungen (over the characteristic roots of one linear substitution with an application to the theory of the integral). *Mathematische Annalen*, 66:488–510, 1909. URL http://dz-srv1.sub.uni-goettingen.de/cache/toc/D38231.html.

Atle Selberg. Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to dirichlet series. *Journal of the Indian Mathematical Society*, 20:47–87, 1956.

Morteza Shafii-Mousavi. *Laplace Transforms Essentials*. Essentials. Research & Education Association, Piscataway, New Jersey, USA, 2015. ISBN 9780738672458. URL http://books.google.com/books?vid=ISBN9780738672458.

Claude E. Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27:379–343,623–656, July,October 1948. URL http://cm.bell-labs.com/cm/ms/what/shannonday/shannon1948.pdf.

Claude E. Shannon. Communication in the presence of noise. *Proceedings of the IRE*, 37:10–21, January 1949. ISSN 0096-8390. doi: 10.1109/JRPROC.1949.232969. URL www.stanford.edu/class/ee104/shannonpaper.pdf.

Neil J. A. Sloane. On-line encyclopedia of integer sequences. World Wide Web, 2014. URL http://oeis.org/.

M.J.T. Smith and T.P. Barnwell. A procedure for designing exact reconstruction filter banks for tree-structured subband coders. *IEEE International Conference on Acoustics, Speech and Signal Processing*, 9:421–424, 1984a. T.P. Barnwell is T.P. Barnwell III.

M.J.T. Smith and T.P. Barnwell. The design of digital filters for exact reconstruction in subband coding. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 34(3):434–441, June 1984b. ISSN 0096-3518. doi: 10.1109/TASSP.1986.1164832. T.P. Barnwell is T.P. Barnwell III.

Houshang H. Sohrab. *Basic Real Analysis*. Birkhäuser, Boston, 1 edition, 2003. ISBN 978-0817642112. URL http://books.google.com/books?vid=ISBN0817642110.

Lynn Arthur Steen. Highlights in the history of spectral theory. *The American Mathematical Monthly*, 80(4):359–381, April 1973. ISSN 00029890. URL http://www.jstor.org/stable/2319079.

Marshall Harvey Stone. *Linear transformations in Hilbert space and their applications to analysis*, volume 15 of *American Mathematical Society. Colloquium publications*. American Mathematical Society, New York, 1932. URL http://books.google.com/books?vid=ISBN0821810154. 1990 reprint of the original 1932 edition.

Gilbert Strang and Truong Nguyen. *Wavelets and Filter Banks.* Wellesley-Cambridge Press, Wellesley, MA, 1996. ISBN 9780961408879. URL http://books.google.com/books?vid=ISBN0961408871.

Robert S. Strichartz. *The Way of Analysis.* Jones and Bartlett Publishers, Boston-London, 1995. ISBN 978-0867204711. URL http://books.google.com/books?vid=ISBN0867204710.

Endre Süli and David F. Mayers. *An Introduction to Numerical Analysis.* Cambridge University Press, August 28 2003. ISBN 9780521007948. URL http://books.google.com/books?vid=ISBN0521007941.

Wim Sweldens and Robert Piessens. Wavelet sampling techniques. In *1993 Proceedings of the Statistical Computing Section*, pages 20–29. American Statistical Association, August 1993. URL http://citeseer.ist.psu.edu/18531.html.

Erik Talvila. Necessary and sufficient conditions for differentiating under the integral sign. *The American Mathematical Monthly*, 108(6):544–548, June–July 2001. URL http://arxiv.org/abs/math/0101012.

Brook Taylor. *Methodus Incrementorum Directa et Inversa.* London, 1715. URL https://books.google.com/books?id=oA5tpwAACAAJ.

Brook Taylor. *Methodus Incrementorum Directa et Inversa.* London, 1717. URL https://books.google.com/books?id=zbgYn4lq4-EC.

Audrey Terras. *Fourier Analysis on Finite Groups and Applications.* Number 43 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1999. ISBN 0-521-45718-1. URL http://books.google.com/books?vid=ISBN0521457181.

Brian S. Thomson, Andrew M. Bruckner, and Judith B. Bruckner. *Elementary Real Analysis.* www.classicalrealanalysis.com, 2 edition, 2008. ISBN 9781434843678. URL http://classicalrealanalysis.info/com/Elementary-Real-Analysis.php.

Stanislaw Marcin Ulam. *Adventures of a Mathematician.* University of California Press, Berkeley, 1991. ISBN 0520071549. URL http://books.google.com/books?vid=ISBN0520071549.

P.P. Vaidyanathan. *Multirate Systems and Filter Banks.* Prentice Hall Signal Processing Series. Prentice Hall, 1993. ISBN 0136057187. URL http://books.google.com/books?vid=ISBN0136057187.

Jussi Väisälä. A proof of the mazur-ulam theorem. *The American Mathematical Monthly*, 110(7): 633–635, August–September 2003. URL http://www.helsinki.fi/~jvaisala/mazurulam.pdf.

Balthasar van der Pol and Hendricus Bremmer. *Operational Calculus based on the Two-Sided Laplace Integral.* Cambridge at the University Press, 2nd, reprinted edition, 1959. URL https://archive.org/details/in.ernet.dli.2015.141269/.

Brani Vidakovic. *Statistical Modeling by Wavelets.* John Wiley & Sons, Inc, New York, 1999. ISBN 9780471293651. URL http://www.amazon.com/dp/0471293652.

John von Neumann. Allgemeine eigenwerttheorie hermitescher funktionaloperatoren. *Mathematische Annalen*, 102(1):49–131, 1929. ISSN 0025-5831 (print) 1432-1807 (online). URL http://resolver.sub.uni-goettingen.de/purl?GDZPPN002273535. General eigenvalue theory of Hermitian functional operators.

David F. Walnut. *An Introduction to Wavelet Analysis.* Applied and numerical harmonic analysis. Springer, 2002. ISBN 0817639624. URL http://books.google.com/books?vid=ISBN0817639624.

Seth Warner. *Modern Algebra*. Dover, Mineola, 1990. ISBN 0-486-66341-8. URL http://books.google.com/books?vid=ISBN0486663418. "An unabridged, corrected republication of the work originally published in two volumes by Prentice Hall, Inc., Englewood Cliffs, New Jersey, in 1965".

Edmund Taylor Whittaker. On the functions which are represented by the expansions of the interpolation theory. *Proceedings of the Royal Society Edinburgh*, 35:181–194, 1915.

John Macnaghten Whittaker. *Interpolatory Function Theory*, volume 33 of *Cambridge tracts in mathematics and mathematical physics*. Cambridge University Press, 1935. URL http://books.google.com.tw/books?id=yyPvAAAAMAAJ.

Stephen B. Wicker. *Error Control Systems for Digital Communication and Storage*. Prentice Hall, Upper Saddle River, 1995. ISBN 0-13-200809-2. URL http://www.worldcat.org/isbn/0132008092.

P. Wojtaszczyk. *A Mathematical Introduction to Wavelets*, volume 37 of *London Mathematical Society student texts*. Cambridge University Press, February 13 1997. ISBN 9780521578943. URL http://books.google.com/books?vid=ISBN0521578949.

Kôsaku Yosida. *Functional Analysis*, volume 123 of *Classics in Mathematics*. Springer, 6 reprint revised edition, 1980. ISBN 9783540586548. URL http://books.google.com/books?vid=ISBN3540586547.

Robert M. Young. *An introduction to nonharmonic Fourier series*, volume 93 of *Pure and applied mathematics*. Academic Press, revised first edition, May 16 2001. ISBN 0127729550. URL http://books.google.com/books?vid=ISBN0127729550.

Ahmed I. Zayed. *Handbook of Function and Generalized Function Transformations*. Mathematical Sciences Reference Series. CRC Press, Boca Raton, 1996. ISBN 0849378516. URL http://books.google.com/books?vid=ISBN0849378516.

Ahmed I. Zayed. A prelude to sampling, wavelets, and tomography. In John Benedetto and Ahmed I. Zayed, editors, *Sampling, Wavelets, and Tomography*, Applied and Numerical Harmonic Analysis, pages 1–32. Springer, 2004. ISBN 9780817643041. URL http://books.google.com/books?vid=ISBN0817643044.

Gary Zukav. *The Dancing Wu Li Masters : An Overview of the New Physics*. Bantam Books, New York, 1980. ISBN 055326382X. URL http://books.google.com/books?vid=ISBN055326382X.

Antoni Zygmund. Trigonometric series volume ii. In *Trigonometric Series*, page 364. Cambridge University Press, London/New York/Melbourne, 3 edition, 2002. ISBN 0-521-89053-5. URL http://books.google.com/books?vid=ISBN0521890535.

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