wavelet

Structure and Design

Daniel J. Greenhoe

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The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]

[1] Paine (2000) page 63 ⟨Golden Hind⟩

This book is dedicated to Jesus,
who brings us over the waves
and through the storms of life,
if we but put our trust in Him;
and to my wife "Apple", for the
joy she has brought and
continues to bring to my life
and to the lives of our children,
Jonathan and Katie.

— Daniel

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❝Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night? ❞

❝Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine. ❞

Alfred Edward Housman, English poet (1859–1936) [2]

❝The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning. ❞

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]

❝As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.❞

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

❝This is all we can do for you now.❞

Translated note written in Czech found in a World War II Nazi fired shell, one of eleven in the fuel tanks of the B-17 bomber "Tondelayo", that somehow made it safely back to base instead of bursting into flames over Kassel, Germany—all the shells being devoid of explosive charge.[5] In like manner, perhaps this text in some small way may help someone find success in accomplishing a task with significant returns to the well-being of humanity.

[2] quote: Housman (1936) page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961) page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967) page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

[5] Bendiner (1980) page 139, https://www.truthorfiction.com/fall-of-fortresses-bendiner/

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SYMBOLS

❝*rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.*❞

❝*Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.*❞

René Descartes (1596–1650), French philosopher and mathematician [6]

❝*In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.*❞

Gottfried Leibniz (1646–1716), German mathematician, [7]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \ldots$

…continued on next page…

[6] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[7] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$
$\mathbb{Z}^{-}$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_{\mathsf{o}}$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_{\mathsf{e}}$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^{+}$	positive real numbers	$(0, \infty)$
$\mathbb{R}^{-}$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^{*}$	extended real numbers	$\mathbb{R}^{*} \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
$\circledR$	relation	
$\circledwedge$	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \triangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\rightarrow$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathscr{D}(\circledR)$	domain of a relation $\circledR$	
$\mathscr{I}(\circledR)$	image of a relation $\circledR$	
$\mathscr{R}(\circledR)$	range of a relation $\circledR$	
$\mathscr{N}(\circledR)$	null space of a relation $\circledR$	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\nsubseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^{c}	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	

...continued on next page...

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symbol	description	
0	"false" condition	
$\neg$	logical NOT operation	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\leq$	reflexive ordering relation	"less than or equal to"
$\geq$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lVert\lVert\cdot\rVert\rVert$	operator norm	
$\langle\triangle \mid \triangledown\rangle$	inner-product	
$\mathsf{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathscr{T}(X)$	set of topologies on a set X	
$\mathscr{R}(X)$	set of rings of sets on a set X	
$\mathscr{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathscr{S}_{\mathsf{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathscr{I}_{\mathsf{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathscr{B}_{\mathsf{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathscr{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		

...continued on next page...

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symbol	description
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition M.2 page 294)
$\hat{\mathbf{F}}$	*Fourier Series* operator (Definition K.1 page 273)
$\check{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition R.1 page 393)
$\mathbf{Z}$	*Z-Transform* operator (Definition Q.1 page 379)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\check{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\check{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

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Part I

Basic Structure

CHAPTER 1

TRANSLATION AND DILATION

❝Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:❞

❝I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them.❞

René Descartes, philosopher and mathematician (1596–1650) [1]

1.1 Definitions

The space $L^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $L^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition 1.1 page 3). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left| Y^X \right| = | Y |^{| X |}$.

Definition 1.1. *Let X and Y be sets.*

DEF

The space Y^X represents the set of all functions with DOMAIN *X and* RANGE *Y such that*

$$Y^X \triangleq \{\mathsf{f}(x) | \mathsf{f}(x) : X \to Y\}$$

Definition 1.2. [2] *Let X be a set.*

[1] quote: Descartes (1637b)
translation: Descartes (1637c) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

[2] Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) page 22, de la Vallée-Poussin (1915) page 440

DEF *The* ***indicator function*** $\mathbb{1} \in \{0,1\}^{2^X}$ *is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \quad \forall x \in X,\ A \in 2^X \\ 0 & \text{for } x \notin A \quad \forall x \in X,\ A \in 2^X \end{cases}$$

The indicator function $\mathbb{1}$ *is also called the* ***characteristic function****.*

Definition 1.3. [3]

DEF $\mathbf{T}_\tau$ *is a* ***translation operator*** *on* $\mathbb{C}^{\mathbb{C}}$ *if* $\mathbf{T}_\tau \mathsf{f}(x) \triangleq \mathsf{f}(x-\tau) \quad \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}}$.

$\mathbf{D}_\alpha$ *is a* ***dilation operator*** *on* $\mathbb{C}^{\mathbb{C}}$ *if* $\mathbf{D}_\alpha \mathsf{f}(x) \triangleq \mathsf{f}(\alpha x) \quad \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}}$.

Moreover, $\mathbf{T} \triangleq \mathbf{T}_1$ *and* $\mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2$.

Example 1.1. Let **T** and **D** be defined as in Definition 1.3 (page 4).

EX

1.2 Algebraic properties

Proposition 1.1. *Let* $\mathbf{T}_\tau$ *be a* TRANSLATION OPERATOR *(Definition 1.3 page 4).*

PRP $$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) \qquad \forall \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) \text{ is PERIODIC with period } \tau\right)$$

✎PROOF:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n\tau+\tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 1.3 page 4)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m\tau) && \text{where } m \triangleq n-1 && \Longrightarrow \quad n = m+1 \\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m \mathsf{f}(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 1.3 page 4)}
\end{aligned}$$

✏

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exist as operators, as demonstrated next.

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2002) page 473, Packer (2004) page 260, Zayed (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a) page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996) page 81, Dai and Larson (1998) page 2

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1.3 Operator properties

Proposition 1.2 (transversal operator inverses). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 1.3 page 4.*

PRP

$\mathbf{T}$ *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \textit{(translation operator inverse).}$$

$\mathbf{D}$ *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \textit{(dilation operator inverse).}$$

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\Longrightarrow \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\Longrightarrow \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

✏

Proposition 1.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 1.3 page 4.*
Let $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP

$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^j x - n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

1.4 Linear operator properties

Theorem 1.1.

THM

Mapping			*Domain*	*Property*
$\mathbf{T}_\tau\left[\mathsf{f}(x)+\mathsf{g}(x)\right]$	$=$	$\left[\mathbf{T}_\tau\mathsf{f}(x)\right]+\left[\mathbf{T}_\tau\mathsf{g}(x)\right]$	$\forall x\in\mathbb{C}$	(ADDITIVE)
$\mathbf{T}_\tau\left[\alpha\mathsf{g}(x)\right]$	$=$	$c\mathbf{T}_\tau\mathsf{g}(x)$	$\forall x,\alpha\in\mathbb{C}\ c\in\mathbb{R}$	(HOMOGENEOUS)
$\mathbf{D}_\alpha\left[\mathsf{f}(x)+\mathsf{g}(x)\right]$	$=$	$\left[\mathbf{D}_\alpha\mathsf{f}(x)\right]+\left[\mathbf{D}_\alpha\mathsf{g}(x)\right]$	$\forall x\in\mathbb{C}$	(ADDITIVE)
$\mathbf{D}_\alpha\left[\alpha\mathsf{g}(x)\right]$	$=$	$c\mathbf{D}_\alpha\mathsf{g}(x)$	$\forall x,\alpha\in\mathbb{C}\ c\in\mathbb{R}$	(HOMOGENEOUS)

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✎PROOF:

$$\begin{aligned}
\mathbf{T}_\tau\big[\mathsf{f}(x)+\mathsf{g}(x)\big] &\triangleq \mathsf{f}(x-\tau)+\mathsf{g}(x-\tau) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&\triangleq \big[\mathbf{T}_\tau\mathsf{f}(x)\big]+\big[\mathbf{T}_\tau\mathsf{g}(x)\big] && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
\mathbf{T}_\tau\big[c\mathsf{g}(x)\big] &\triangleq c\mathsf{g}(x-\tau) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&\triangleq c\mathbf{T}_\tau\mathsf{g}(x) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
\mathbf{D}_\alpha\big[\mathsf{f}(x)+\mathsf{g}(x)\big] &\triangleq \mathsf{f}(\alpha x)+\mathsf{g}(\alpha x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&\triangleq \big[\mathbf{D}_\alpha\mathsf{f}(x)\big]+\big[\mathbf{D}_\alpha\mathsf{g}(x)\big] && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
\mathbf{D}\big[\alpha\mathsf{g}(x)\big] &\triangleq \alpha\mathsf{g}(\alpha x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&\triangleq \alpha\mathbf{D}\mathsf{g}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}
\end{aligned}$$

✏

Corollary 1.1.

COR

1. $\mathbf{T}0 = 0$ *and*
2. $\mathbf{T}(-\mathsf{f}) = -(\mathbf{T}\mathsf{f})$ $\forall x\in X$ *and*
3. $\mathbf{T}(\mathsf{f}-\mathsf{g}) = \mathbf{T}\mathsf{f}-\mathbf{L}\mathsf{g}$ $\forall x,y\in X$ *and*
4. $\mathbf{T}\left(\sum_{n=1}^{N}\alpha_n\mathsf{f}_n\right) = \sum_{n=1}^{N}\alpha_n\big(\mathbf{T}\mathsf{f}_n\big)$ $x_n\in X,\ \alpha_n\in\mathbb{F}$

✎PROOF:

1. By Theorem 1.1 (page 5), $\mathbf{T}$ is *additive* and *homogeneous*.
2. By (1) and by Definition D.4 (page 171), $\mathbf{T}$ is a *linear* operator.
3. Then the properties follow by Theorem D.1 (page 171).

✏

Corollary 1.2.

COR

1. $\mathbf{D}0 = 0$ *and*
2. $\mathbf{D}(-\mathsf{f}) = -(\mathbf{D}\mathsf{f})$ $\forall x\in X$ *and*
3. $\mathbf{D}(\mathsf{f}-\mathsf{g}) = \mathbf{D}\mathsf{f}-\mathbf{L}\mathsf{g}$ $\forall x,y\in X$ *and*
4. $\mathbf{D}\left(\sum_{n=1}^{N}\alpha_n\mathsf{f}_n\right) = \sum_{n=1}^{N}\alpha_n\big(\mathbf{D}\mathsf{f}_n\big)$ $x_n\in X,\ \alpha_n\in\mathbb{F}$

✎PROOF:

1. By Theorem 1.1 (page 5), $\mathbf{D}$ is *additive* and *homogeneous*.
2. By (1) and by Definition D.4 (page 171), $\mathbf{D}$ is a *linear* operator.
3. Then the properties follow by Theorem D.1 (page 171).

✏

Proposition 1.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 4.*

PRP

$$\mathbf{D}^j\mathbf{T}^n\big[\mathsf{f}\mathsf{g}\big] = 2^{-j/2}\,\big[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\big]\,\big[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\big] \qquad \forall j,n\in\mathbb{Z},\ \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

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✎PROOF:

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}(x)\mathsf{g}(x)\right] &= 2^{j/2}\mathsf{f}\left(2^jx-n\right)\mathsf{g}\left(2^jx-n\right) && \text{by Proposition 1.3 page 5}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^jx-n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^jx-n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition 1.3 page 5}
\end{aligned}$$

✏

In general the operators **T** and **D** are *noncommutative* (**TD** $\neq$ **DT**), as demonstrated by Counterexample 1.1 (next) and Proposition 1.5 (page 7).

Counterexample 1.1.

CNT

As illustrated to the right, it is **not** always true that **TD** = **DT**:

Proposition 1.5 (commutator relation). [4] *Let* **T** *and* **D** *be as in Definition 1.3 page 4.*

PRP

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j && \forall j,n\in\mathbb{Z}\\
\mathbf{T}^n\mathbf{D}^j &= \mathbf{D}^j\mathbf{T}^{2^jn} && \forall n,j\in\mathbb{Z}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^{2^jn}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^jx-2^jn) && \text{by Proposition 1.4 page 6}\\
&= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by \textit{distributivity} of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 144)}\\
&= \mathbf{T}^n2^{j/2}\,\mathsf{f}\left(2^jx\right) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^jx-n) && \text{by Proposition 1.4 page 6}\\
&= 2^{j/2}\,\mathsf{f}\left(2^j\left[x-2^{-j/2}n\right]\right) && \text{by \textit{distributivity} of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 144)}\\
&= \mathbf{T}^{2^{-j/2}n}2^{j/2}\,\mathsf{f}\left(2^jx\right) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}
\end{aligned}$$

✏

1.5 Inner product space properties

In an inner product space, every operator has an *adjoint* (Proposition D.3 page 183) and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator **U** coincide, then **U** is said to be *unitary* (Definition D.14 page 193). And in this case, **U** has several nice properties (see Proposition 1.9 and Theorem 1.2 page 10). Proposition 1.6 (next) gives the adjoints of **D** and **T**, and Proposition 1.7 (page 8) demonstrates that both **D** and **T** are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ (Corollary M.1 page 295) and the *rotation matrix operator* (Example D.5 page 195).

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a) page 641, Goodman et al. (1993b) page 110

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Proposition 1.6. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 1.3 page 4) with* ADJOINT $\mathbf{T}^*$ *and* $\mathbf{D}$ *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$ *(Definition D.8 page 179).*

PRP				
	$\mathbf{T}^*\mathsf{f}(x)$	$=$	$\mathsf{f}(x+1)$	$\forall \mathsf{f}\in L^2_{\mathbb{R}}$ (TRANSLATION OPERATOR ADJOINT)
	$\mathbf{D}^*\mathsf{f}(x)$	$=$	$\frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right)$	$\forall \mathsf{f}\in L^2_{\mathbb{R}}$ (DILATION OPERATOR ADJOINT)

✎PROOF:

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^* \quad \text{(Definition D.8 page 179)} \\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 4)} \\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1 \\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^* \quad \text{(Definition D.8 page 179)} \\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} \quad \text{(Definition 1.3 page 4)} \\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\,\mathsf{du} && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^*\mathsf{dx} && \text{where } x = 2u \\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}$$

✏

Proposition 1.7. [5] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 (page 4).*
Let $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 1.2 (page 5).*

PRP		
	$\mathbf{T}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$	$(\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $L^2_{\mathbb{R}})$.
	$\mathbf{D}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$	$(\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $L^2_{\mathbb{R}})$.

✎PROOF:

$$\begin{aligned}
&\mathbf{T}^{-1} = \mathbf{T}^* && \text{by Proposition 1.2 page 5 and Proposition 1.6 page 8} \\
&\implies \quad \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition D.14 page 193)}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{D}^{-1} = \mathbf{D}^* && \text{by Proposition 1.2 page 5 and Proposition 1.6 page 8} \\
&\implies \quad \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition D.14 page 193)}
\end{aligned}$$

✏

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

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1.6 Normed linear space properties

Proposition 1.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR *(Definition 1.3 page 4)*.

PRP

$$\left\{\begin{array}{ll} (1). & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) \quad \textit{and} \\ (2). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f} \in L^2_{\mathbb{R}}$$

✎PROOF:

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$\begin{aligned} \mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)} \\ &= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis} \end{aligned}$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$\begin{aligned} \|\mathsf{f}(x) - \mathsf{f}(x+h)\| &= \|\mathsf{f}(x) - \mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis} \\ &= \|0\| \\ &= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\| \\ &\leq \varepsilon \\ &\Longrightarrow \forall h > 0, \exists \varepsilon \quad \text{such that} \quad \|\mathsf{f}(x) - \mathsf{f}(x+h)\| < \varepsilon \\ &\overset{\text{def}}{\Longleftrightarrow} \mathsf{f}(x) \text{ is } \textit{continuous} \end{aligned}$$

3. Proof that (1,2) $\Longrightarrow$ *constant* property:
   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \neq \mathsf{f}(y)$.
   (b) Let $(\!(x_n)\!)_{n\in\mathbb{N}}$ be a sequence with limit x and $(\!(y_n)\!)_{n\in\mathbb{N}}$ a sequence with limit y
   (c) Then

$$\begin{aligned} 0 &< \|\mathsf{f}(x) - \mathsf{f}(y)\| && \text{by assumption in item (3a) page 9} \\ &= \lim_{n\to\infty} \left\|\mathsf{f}\left(x_n\right) - \mathsf{f}\left(y_n\right)\right\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 9} \\ &= \lim_{n\to\infty} \left\|\mathsf{f}\left(2^m x_n\right) - \mathsf{f}\left(2^\ell y_n\right)\right\| \quad \forall m, \ell \in \mathbb{Z} && \text{by (1)} \\ &= 0 \end{aligned}$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

✏

Remark 1.1.

REM

In Proposition 1.8 page 9, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample 1.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT

Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{otherwise.} \end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.

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Counterexample 1.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\setminus\mathbb{Q}$ the set of *irrational numbers*.

CNT

Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ -1 & \text{for } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.

Proposition 1.9 (Operator norm)**.** *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 4. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 1.2 page 5. Let* $\mathbf{T}^*$ *and* $\mathbf{D}^*$ *be as in Proposition 1.6 page 8. Let* $\|\cdot\|$ *and* $\langle\triangle\,|\,\triangledown\rangle$ *be as in Definition E.1 page 199. Let* $|||\cdot|||$ *be the operator norm (Definition D.6 page 175) induced by* $\|\cdot\|$.

PRP

$$|||\mathbf{T}||| = |||\mathbf{D}||| = \left|\left|\left|\mathbf{T}^*\right|\right|\right| = \left|\left|\left|\mathbf{D}^*\right|\right|\right| = \left|\left|\left|\mathbf{T}^{-1}\right|\right|\right| = \left|\left|\left|\mathbf{D}^{-1}\right|\right|\right| = 1$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 1.7 page 8) and from Theorem D.25 page 194 and Theorem D.26 page 194. ✏

Theorem 1.2. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 4.*
Let $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 1.2 page 5. Let* $\|\cdot\|$ *and* $\langle\triangle\,|\,\triangledown\rangle$ *be as in Definition E.1 page 199.*

THM

1.	$\|\mathbf{T}\mathsf{f}\|$	$=$	$\|\mathbf{D}\mathsf{f}\|$	$=$	$\|\mathsf{f}\|$	$\forall \mathsf{f}\in L^2_{\mathbb{R}}$	(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}\mathsf{f} - \mathbf{T}\mathsf{g}\|$	$=$	$\|\mathbf{D}\mathsf{f} - \mathbf{D}\mathsf{g}\|$	$=$	$\|\mathsf{f} - \mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in L^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
3.	$\left\|\mathbf{T}^{-1}\mathsf{f} - \mathbf{T}^{-1}\mathsf{g}\right\|$	$=$	$\left\|\mathbf{D}^{-1}\mathsf{f} - \mathbf{D}^{-1}\mathsf{g}\right\|$	$=$	$\|\mathsf{f} - \mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in L^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
4.	$\langle\mathbf{T}\mathsf{f}\,\|\,\mathbf{T}\mathsf{g}\rangle$	$=$	$\langle\mathbf{D}\mathsf{f}\,\|\,\mathbf{D}\mathsf{g}\rangle$	$=$	$\langle\mathsf{f}\,\|\,\mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in L^2_{\mathbb{R}}$	(SURJECTIVE)
5.	$\left\langle\mathbf{T}^{-1}\mathsf{f}\,\middle\|\,\mathbf{T}^{-1}\mathsf{g}\right\rangle$	$=$	$\left\langle\mathbf{D}^{-1}\mathsf{f}\,\middle\|\,\mathbf{D}^{-1}\mathsf{g}\right\rangle$	$=$	$\langle\mathsf{f}\,\|\,\mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in L^2_{\mathbb{R}}$	(SURJECTIVE)

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 1.7 page 8) and from Theorem D.25 page 194 and Theorem D.26 page 194. ✏

Proposition 1.10. *Let* $\mathbf{T}$ *be as in Definition 1.3 page 4. Let* $\mathbf{A}^*$ *be the* ADJOINT *(Definition D.8 page 179) of an operator* $\mathbf{A}$. *Let the property "*SELF ADJOINT*" be defined as in Definition D.11 (page 187).*

PRP

$$\left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)^* \qquad \left(\textit{The operator}\left[\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right]\textit{ is }\text{SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle\left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)\mathsf{f}(x)\,\middle|\,\mathsf{g}(x)\right\rangle &= \left\langle\sum_{n\in\mathbb{Z}}\mathsf{f}(x-n)\,\middle|\,\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&= \left\langle\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n)\,\middle|\,\mathsf{g}(x)\right\rangle && \text{by } \textit{commutative} \text{ property} && \text{(Definition A.5 page 144)}\\
&= \sum_{n\in\mathbb{Z}}\langle\mathsf{f}(x+n)\,|\,\mathsf{g}(x)\rangle && \text{by } \textit{additive} \text{ property of } \langle\triangle\,|\,\triangledown\rangle\\
&= \sum_{n\in\mathbb{Z}}\langle\mathsf{f}(u)\,|\,\mathsf{g}(u-n)\rangle && \text{where } u \triangleq x+n\\
&= \left\langle\mathsf{f}(u)\,\middle|\,\sum_{n\in\mathbb{Z}}\mathsf{g}(u-n)\right\rangle && \text{by } \textit{additive} \text{ property of } \langle\triangle\,|\,\triangledown\rangle\\
&= \left\langle\mathsf{f}(x)\,\middle|\,\sum_{n\in\mathbb{Z}}\mathsf{g}(x-n)\right\rangle && \text{by change of variable: } u\to x
\end{aligned}$$

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$$= \left\langle \mathsf{f}(x) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{g}(x) \right\rangle \qquad \text{by definition of } \mathbf{T} \qquad \text{(Definition 1.3 page 4)}$$

$$\Longleftrightarrow \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* \qquad \text{by definition of } \textit{adjoint} \qquad \text{(Proposition D.3 page 183)}$$

$$\Longleftrightarrow \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) \text{ is } \textit{self-adjoint} \qquad \text{by definition of } \textit{self-adjoint} \qquad \text{(Definition D.11 page 187)}$$

✏

1.7 Fourier transform properties

Proposition 1.11. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 4.*
Let $\mathbf{L}$ *be the* LAPLACE TRANSFORM *(Definition L.1 page 277).*

PRP

1.	$\mathbf{LT}^n$	$=$	$e^{-sn}\mathbf{L}$			$\forall n\in\mathbb{Z}$	
2.	$\mathbf{LD}^j$	$=$	$\mathbf{D}^{-j}\mathbf{L}$			$\forall j\in\mathbb{Z}$	
3.	$\mathbf{DL}$	$=$	$\mathbf{LD}^{-1}$			$\forall n\in\mathbb{Z}$	
4.	$\mathbf{LD}^{-1}\mathbf{L}^{-1}$	$=$	$\mathbf{L}^{-1}\mathbf{D}^{-1}\mathbf{L}$	$=$	$\mathbf{D}$	$\forall n\in\mathbb{Z}$	($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5.	$\mathbf{DLD}$	$=$	$\mathbf{D}^{-1}\mathbf{LD}^{-1}$	$=$	$\mathbf{L}$	$\forall n\in\mathbb{Z}$	

✎PROOF:

$$\begin{aligned}
\mathbf{LT}^n\mathsf{f}(x) &= \mathbf{L}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&= \int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathsf{dx} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathsf{du} && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathsf{du}\right]\\
&= e^{-sn}\,\mathbf{L}\mathsf{f}(x) && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{LD}^j\mathsf{f}(x) &= \mathbf{L}\left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right]e^{-sx}\,\mathsf{dx} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathsf{du} && \text{let } u \triangleq 2^j x \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathsf{du}\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathsf{du}\right] && \text{by Proposition 1.6 page 8 and} && \text{Proposition 1.7 page 8}\\
&= \mathbf{D}^{-j}\,\mathbf{L}\,\mathsf{f}(x) && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{DL}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathsf{dx}\right] && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathsf{dx} && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\frac{1}{2}\,\mathsf{du} && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2} \mathsf{f}\left(\frac{u}{2}\right) \right] e^{-su}\, \mathrm{du} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\, e^{-su}\, \mathrm{du} && \text{by Proposition 1.6 page 8 and} && \text{Proposition 1.7 page 8} \\
&= \mathbf{L}\mathbf{D}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)} \\
\mathbf{L}^{-1}\mathbf{D}^{-1}\mathbf{L} &= \mathbf{L}^{-1}\mathbf{L}\mathbf{D} && \text{by previous result} \\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition D.3 page 170)} \\
\mathbf{L}\mathbf{D}^{-1}\mathbf{L}^{-1} &= \mathbf{D}\mathbf{L}\mathbf{L}^{-1} && \text{by previous result} \\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition D.3 page 170)} \\
\mathbf{D}\mathbf{L}\mathbf{D} &= \mathbf{D}\mathbf{D}^{-1}\mathbf{L} && \text{by previous result} \\
&= \mathbf{L} && \text{by definition of operator inverse} && \text{(Definition D.3 page 170)} \\
\mathbf{D}^{-1}\mathbf{L}\mathbf{D}^{-1} &= \mathbf{D}^{-1}\mathbf{D}\mathbf{L} && \text{by previous result} \\
&= \mathbf{L} && \text{by definition of operator inverse} && \text{(Definition D.3 page 170)}
\end{aligned}
$$

✏

Corollary 1.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 4. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of some function* $\mathsf{f} \in \mathbf{L}^2_{\mathbb{R}}$ *(Definition E.1 page 199).*

COR

1. $\tilde{\mathbf{F}}\mathbf{T}^n = e^{-i\omega n}\tilde{\mathbf{F}}$
2. $\tilde{\mathbf{F}}\mathbf{D}^j = \mathbf{D}^{-j}\tilde{\mathbf{F}}$
3. $\mathbf{D}\tilde{\mathbf{F}} = \tilde{\mathbf{F}}\mathbf{D}^{-1}$
4. $\mathbf{D} = \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} = \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}}$
5. $\tilde{\mathbf{F}} = \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} = \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}$

✎PROOF: These results follow directly from Proposition 1.11 page 11 with $\tilde{\mathbf{F}} = \mathbf{L}|_{s=i\omega}$. ✏

Proposition 1.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 4. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of some function* $\mathsf{f} \in \mathbf{L}^2_{\mathbb{R}}$ *(Definition E.1 page 199).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = \frac{1}{2^{j/2}} e^{-i\frac{\omega}{2^j}n} \tilde{\mathsf{f}}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$
\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n\mathsf{f}(x) && \text{by Corollary 1.3 page 12 (3)} \\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}\mathsf{f}(x) && \text{by Corollary 1.3 page 12 (3)} \\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathsf{f}}(\omega) \\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{\mathsf{f}}\left(2^{-j}\omega\right) && \text{by Proposition 1.2 page 5}
\end{aligned}
$$

✏

Proposition 1.13. *Let* $\mathbf{T}$ *be the translation operator (Definition 1.3 page 4). Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f} \in \mathbf{L}^2_{\mathbb{R}}$. *Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT *(Definition R.1 page 393) of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$ *(Definition F.3 page 205).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) \qquad \forall (\!(a_n)\!)\in\ell^2_{\mathbb{R}}, \phi(x)\in\mathbf{L}^2_{\mathbb{R}}$$

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✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} a_n \mathbf{T}^n \phi(x) &= \sum_{n\in\mathbb{Z}} a_n \tilde{\mathbf{F}}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \tilde{\mathbf{F}}\phi(x) && \text{by Corollary 1.3 page 12} \\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right] \tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega) \\
&= \check{\mathsf{a}}(\omega)\tilde{\phi}(\omega) && \text{by definition of } DTFT \text{ (Definition R.1 page 393)}
\end{aligned}$$

✏

Definition 1.4. *Let* $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition E.1 page 199)*. *Let* $\ell^2_{\mathbb{R}}$ *be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition E.1 page 199)*.

DEF $\mathbf{S}$ *is the* ***sampling operator*** *in* $\ell^2_{\mathbb{R}}{}^{L^2_{\mathbb{R}}}$ *if* $\quad [\mathbf{S}\mathsf{f}(x)](n) \triangleq \mathsf{f}\left(\frac{2\pi}{\tau}n\right) \qquad \forall \mathsf{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}, \tau\in\mathbb{R}^+$

Theorem 1.3 (Poisson Summation Formula—PSF). [6] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f}(x) \in L^2_{\mathbb{R}}$. *Let* $\mathbf{S}$ *be the* SAMPLING OPERATOR *(Definition 1.4 page 13)*.

THM
$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_\tau \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)$ then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:

   Note that $\mathsf{h}(x)$ is *periodic* with period τ (Lemma G.3 page 220). Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of PSF (this theorem—Theorem 1.3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}} \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) && \text{by (1) lemma page 13} \\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \left(\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)\right]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition K.1 page 273)} \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau \mathsf{f}(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,\mathrm{d}u\right] && \text{where } u \triangleq x+n\tau \implies x = u-n\tau \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}} \cancelto{1}{e^{i2\pi kn}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}ku}\,\mathrm{d}u\right]
\end{aligned}$$

[6] Andrews et al. (2001) page 624, Knapp (2005b) page 389, Lasser (1996) page 254, Rudin (1987) pages 194–195, Folland (1992) page 337

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$$= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,\mathrm{d}u\right]}_{[\tilde{\mathbf{F}}\mathsf{f}]\left(\frac{2\pi}{\tau}k\right)} \qquad \text{by evaluation of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem K.1 page 274)}$$

$$= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}\mathsf{f}(x)]\left(\frac{2\pi}{\tau}k\right)\right] \qquad \text{by definition of } \tilde{\mathbf{F}} \qquad \text{(Definition M.2 page 294)}$$

$$= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\mathsf{f} \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition 1.4 page 13)}$$

$$= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} \qquad \text{by evaluation of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem K.1 page 274)}$$

✏

Theorem 1.4 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f}(x) \in L^2_{\mathbb{R}}$.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}}\mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega-\frac{2\pi}{\tau}n\right)}_{\textit{summation in "frequency"}} = \underbrace{\frac{\tau}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\textit{summation in "time"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(\omega) \triangleq \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)$, then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:
   Note that $\mathsf{h}(\omega)$ is periodic with period $2\pi/T$:
   $$\mathsf{h}\left(\omega+\frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}+\frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+(n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right) \triangleq \mathsf{h}(\omega)$$
   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.
2. Proof of IPSF (this theorem—Theorem 1.4):
   $$\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)$$
   $$= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right) \qquad \text{by (1) lemma page 14}$$
   $$= \hat{\mathbf{F}}^{-1}\underbrace{\left[\sqrt{\frac{\tau}{2\pi}}\int_0^{\frac{2\pi}{\tau}}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\,\mathrm{d}\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)\right]} \qquad \text{by definition of } \hat{\mathbf{F}} \qquad \text{(Definition K.1 page 273)}$$
   $$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{\tau}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)e^{-i\omega Tk}\,\mathrm{d}\omega\right]$$
   $$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)}\tilde{\mathsf{f}}(u)e^{-i\left(u-\frac{2\pi}{\tau}n\right)Tk}\,\mathrm{d}u\right] \qquad \text{where } u \triangleq \omega+\frac{2\pi}{\tau}n \implies \omega = u-\frac{2\pi}{\tau}n$$
   $$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi nk}}\int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)}\tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{d}u\right]$$

[7] Gauss (1900) page 88

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$$
\begin{aligned}
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\int_{\mathbb{R}}\tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{du}\right] \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\tilde{\mathsf{f}}(u)e^{iu(-\tau k)}\,\mathrm{du}\right]}_{[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)\right] && \text{by value of } \tilde{\mathbf{F}}^{-1} && \text{(Theorem M.1 page 295)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\,\tilde{\mathsf{f}} && \text{by definition of } \mathbf{S} && \text{(Definition 1.4 page 13)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\mathsf{f}(x) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathsf{f}(-k\tau) && \text{by definition of } \mathbf{S} && \text{(Definition 1.4 page 13)} \\
&= \sqrt{\tau}\,\frac{1}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}}\mathsf{f}(-k\tau)e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem K.1 page 274)} \\
&= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}}\mathsf{f}(-k\tau)e^{ik\tau\omega} && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem K.1 page 274)} \\
&= \frac{\tau}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\mathsf{f}(m\tau)e^{-i\omega m\tau} && \text{let } m \triangleq -k
\end{aligned}
$$

✏

Remark 1.2. The left hand side of the *Poisson Summation Formula* (Theorem 1.3 page 13) is very similar to the *Zak Transform* $\mathbf{Z}$: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark 1.3. A generalization of the *Poisson Summation Formula* (Theorem 1.3 page 13) is the **Selberg Trace Formula**.[9]

1.8 Basis theory properties

Example 1.2 (linear functions). [10] Let $\mathbf{T}$ be the *translation operator* (Definition 1.3 page 4). Let $\mathscr{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX

1. $\{x, \mathbf{T}x\}$ is a *basis* for $\mathscr{L}(\mathbb{C},\mathbb{C})$ and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x \quad \forall \mathsf{f} \in \mathscr{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$
\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition 1.3 page 4} \\
&= (ax+b)|_{x=1}\,x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of } \mathsf{f} \\
&= (a+b)x - b(x-1) \\
&= ax + bx - bx + b \\
&= ax + b \\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of } \mathsf{f}
\end{aligned}
$$

✏

[8] Janssen (1988) page 24, Zayed (1996) page 482
[9] Lax (2002) page 349, Selberg (1956), Terras (1999)
[10] Higgins (1996) page 2 ⟨1.1 General introduction⟩

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Example 1.3 (Cardinal Series). Let $\mathbf{T}$ be the *translation operator* (Definition 1.3 page 4). The *Paley-Wiener* class of functions $\boldsymbol{PW}_\sigma^2$ (Definition P.5 page 375) are those functions which are "*bandlimited*" with respect to their Fourier transform (Definition M.2 page 294). The cardinal series forms an orthogonal basis for such a space (Theorem P.3 page 375). The *Fourier coefficients* (Definition N.11 page 316) for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of $\mathsf{f}(x)$ taken at regular intervals (Theorem P.4 page 375). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \,|\, \mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\,\mathrm{d}x \triangleq \mathsf{f}(n)$$

EX

1. $\left\{ \mathbf{T}^n \frac{\sin(\pi x)}{\pi x} \,\middle|\, n\in\mathbb{N} \right\}$ is a *basis* for $\boldsymbol{PW}_\sigma^2$ and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin(\pi x)}{\pi x}}_{\textit{Cardinal series}}$ $\quad \forall \mathsf{f}\in\boldsymbol{PW}_\sigma^2,\ \sigma\le\frac{1}{2}$

✎PROOF: See Theorem P.3 page 375. ✏

Example 1.4 (Fourier Series).

EX

(1). $\left\{ \mathbf{D}_n e^{ix} \,\middle|\, n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and

(2). $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix}$ $\quad \forall x\in(0:2\pi), \mathsf{f}\in\boldsymbol{L}(0:2\pi)$ where

$$\alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x)\mathbf{D}_n e^{-ix}\,\mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}(0:2\pi)$$

✎PROOF: See Theorem K.1 page 274. ✏

Example 1.5 (Fourier Transform). [11]

EX

(1). $\left\{ \mathbf{D}_\omega e^{ix} \,\middle|\, \omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and

(2). $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)\mathbf{D}_x e^{i\omega}\,\mathrm{d}\omega$ $\quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where

$$\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}_\omega e^{-ix}\,\mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$$

Example 1.6 (Gabor Transform). [12]

EX

(1). $\left\{ \left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{ix}\right) \,\middle|\, \tau,\omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and

(2). $\mathsf{f}(x) = \int_{\mathbb{R}} \mathsf{G}(\tau,\omega)\,\mathbf{D}_x e^{i\omega}\,\mathrm{d}\omega$ $\quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where

$$\mathsf{G}(\tau,\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{-ix}\right)\,\mathrm{d}x \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$$

Example 1.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX

(1). $\left\{ \mathbf{D}^k\mathbf{T}^n\psi(x) \,\middle|\, k,n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and

(2). $\mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha_{k,n}\mathbf{D}^k\mathbf{T}^n\psi(x)$ $\quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where

$$\alpha_n \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}^k\mathbf{T}^n\psi^*(x)\,\mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$$

[11] cross reference: Definition M.2 page 294

[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

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CHAPTER 2

MRA STRUCTURES

> *❝...on fait la science avec des faits comme une maison avec des pierres ; mais une accumulation de faits n'est pas plus une science qu'un tas de pierres n'est une maison.❞*

> *❝Science is built up of facts, as a house is built of stones; but an accumulation of facts is no more a science than a heap of stones is a house.❞*

Jules Henri Poincaré (1854-1912), physicist and mathematician [1]

2.1 Introduction

In 1989, Stéphane G. Mallat introduced the *Multiresolution Analysis* (MRA, Definition 2.1 page 18) method for wavelet construction. The MRA has become the dominate wavelet construction method. This text uses the MRA method extensively, and combines the MRA "scaling subspaces" (Definition 2.1 page 18) with "wavelet subspaces" (Definition 3.1 page 39) to form a subspace structure as represented by the *Hasse diagram* to the right. The *Fast Wavelet Transform* combines both sets of subspaces as well, providing the results of projections onto both wavelet and MRA subspaces. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA.[2]

The MRA is an **analysis** of the linear space $L^2_{\mathbb{R}}$. An analysis of a linear space X is any sequence $(\!(V_j)\!)_{j\in\mathbb{Z}}$ of linear subspaces of X. The partial or complete reconstruction of X from $(\!(V_j)\!)_{j\in\mathbb{Z}}$ is a **synthesis**.[3] An analysis is completely *characterized* by a *transform*. For example, a Fourier analysis is a sequence of subspaces with sinusoidal bases. Examples of subspaces in a Fourier analysis

[1] quote: Poincaré (1902a) ⟨Chapter IX, paragraph 7⟩
translation: Poincaré (1902b) page 141
image: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Poincare.html

[2] Lemarié (1990), Mallat (1999) page 240

[3] The word *analysis* comes from the Greek word ἀνάλυσις, meaning "dissolution" (Perschbacher (1990) page 23

include $V_1 = \text{span}\{e^{ix}\}$, $V_{2.3} = \text{span}\{e^{i2.3x}\}$, $V_{\sqrt{2}} = \text{span}\left\{e^{i\sqrt{2}x}\right\}$, etc. A **transform** is loosely defined as a function that maps a family of functions into an analysis. A very useful transform (a "*Fourier transform*") for Fourier Analysis is (Definition M.2 page 294)

$$[\tilde{\mathbf{F}}\mathsf{f}](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathsf{d}x$$

2.2 Definition

A multiresolution analysis provides "coarse" approximations of a function in a linear space $L^2_{\mathbb{R}}$ at multiple "scales" or "resolutions". Key to this process is a sequence of *scaling functions*. Most traditional transforms feature a single *scaling function* $\phi(x)$ set equal to one ($\phi(x) = 1$). This allows for convenient representation of the most basic functions, such as constants.[4] A multiresolution system, on the other hand, uses a generalized form of the scaling concept:

1. Instead of the scaling function simply being set *equal to unity* ($\phi(x) = 1$), a multiresolution system (Definition 2.3 page 27) is often constructed in such a way that the scaling function $\phi(x)$ forms a *partition of unity* (Definition S.1 page 403) such that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x) = 1$.
2. Instead of there being *just one* scaling function, there is an entire sequence of scaling functions $\left(\!\left(\mathbf{D}^j\phi(x)\right)\!\right)_{j\in\mathbb{Z}}$, each corresponding to a different "*resolution*".

Definition 2.1. [5] *Let $\left(\!\left(V_j\right)\!\right)_{j\in\mathbb{Z}}$ be a sequence of subspaces on $L^2_{\mathbb{R}}$ (Definition E.1 page 199). Let A^- be the* CLOSURE *of a set A.*

DEF

*The sequence $\left(\!\left(V_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **multiresolution analysis** on $L^2_{\mathbb{R}}$ if*

1. $V_j = V_j^-$ $\quad \forall j\in\mathbb{Z}$ $\quad$ (CLOSED) $\quad$ *and*
2. $V_j \subset V_{j+1}$ $\quad \forall j\in\mathbb{Z}$ $\quad$ (LINEARLY ORDERED) $\quad$ *and*
3. $\left(\bigcup_{j\in\mathbb{Z}} V_j\right)^- = L^2_{\mathbb{R}}$ $\quad$ (DENSE *in* $L^2_{\mathbb{R}}$) $\quad$ *and*
4. $\mathsf{f} \in V_j \iff \mathbf{D}\mathsf{f} \in V_{j+1}$ $\quad \forall j\in\mathbb{Z}, \mathsf{f}\in L^2_{\mathbb{R}}$ $\quad$ (SELF-SIMILAR) $\quad$ *and*
5. $\exists\phi$ such that $\left\{\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\right\}$ *is a* RIESZ BASIS *for* V_0.

A MULTIRESOLUTION ANALYSIS *is also called an **MRA**.*
*An element V_j of $\left(\!\left(V_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **scaling subspace** of the space $L^2_{\mathbb{R}}$.*
*The pair $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right)\right)$ is a **multiresolution analysis space**, or **MRA space**.*
*The function ϕ is the **scaling function** of the* MRA SPACE.

The traditional definition of the *MRA* also includes the following:

1. $\mathsf{f} \in V_j \iff \mathbf{T}^n\mathsf{f} \in V_j$ $\quad \forall n,j\in\mathbb{Z}, \mathsf{f}\in L^2_{\mathbb{R}}$ $\quad$ (*translation invariant*)
2. $\bigcap_{j\in\mathbb{Z}} V_j = \{\mathbb{0}\}$ $\quad$ (*greatest lower bound* is $\mathbf{0}$)

⟨entry 359⟩), which in turn means "the resolution or separation into component parts" (Black et al. (2009), http://dictionary.reference.com/browse/dissolution)

[4] Jawerth and Sweldens (1994) page 8

[5] Hernández and Weiss (1996) page 44, Mallat (1999) page 221 ⟨Definition 7.1⟩, Mallat (1989) page 70, Meyer (1992) page 21 ⟨Definition 2.2.1⟩, Christensen (2003) page 284 ⟨Definition 13.1.1⟩, Bachman et al. (2002) pages 451–452 ⟨Definition 7.7.6⟩, Walnut (2002) pages 300–301 ⟨Definition 10.16⟩, Daubechies (1992) pages 129–140 ⟨Riesz basis: page 139⟩

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However, Proposition 2.1 (next) demonstrates that both of these follow from the *MRA* as defined in Definition 2.1.

Proposition 2.1. [6]

PRP

$$\left.\begin{array}{c}(\!(V_j)\!)_{j\in\mathbb{Z}} \text{ is an MRA} \\ \text{(Definition 2.1 page 18)}\end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} 1. & \mathsf{f}\in V_j \iff \mathbf{T}^n\mathsf{f}\in V_j \quad \forall n,j\in\mathbb{Z}, \mathsf{f}\in L^2_{\mathbb{R}} & \text{(TRANSLATION INVARIANT) and} \\ 2. & \bigcap_{j\in\mathbb{Z}} V_j = \{\mathbb{0}\} & \text{(GREATEST LOWER BOUND is } \mathbf{0}\text{)} \end{array}\right.$$

✎PROOF: Proof for (1):

$$\begin{array}{llll}
\mathbf{T}^n\mathsf{f}\in V_j & & & \\
\iff \mathbf{T}^n\mathsf{f}\in \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^m\phi|_{m\in\mathbb{Z}}\right\} & & \text{by definition of } \{\phi\} & \text{(Definition 2.1 page 18)} \\
\iff \exists\,(\!(\alpha_n)\!)_{n\in\mathbb{Z}} \quad \text{such that} \quad \mathbf{T}^n\mathsf{f}(x) & = \sum_{k\in\mathbb{Z}} \alpha_k\mathbf{D}^j\mathbf{T}^k\phi(x) & \text{by definition of } \{\phi\} & \text{(Definition 2.1 page 18)} \\
\iff \exists\,(\!(\alpha_n)\!)_{n\in\mathbb{Z}} \quad \text{such that} \quad \mathsf{f}(x) & = \mathbf{T}^{-n}\sum_{k\in\mathbb{Z}} \alpha_k\mathbf{D}^j\mathbf{T}^k\phi(x) & \text{by definition of } \mathbf{T} & \text{(Definition 1.3 page 4)} \\
& = \sum_{k\in\mathbb{Z}} \alpha_k\mathbf{T}^{-n}\mathbf{D}^j\mathbf{T}^k\phi(x) & & \\
& = \sum_{k\in\mathbb{Z}} \alpha_k\mathbf{D}^j\mathbf{T}^{k-2n}\phi(x) & \text{by } \textit{commutator relation} & \text{(Proposition 1.5 page 7)} \\
& = \sum_{\ell\in\mathbb{Z}} \alpha_{\ell+2n}\mathbf{D}^j\mathbf{T}^{\ell}\phi(x) & \text{where } \ell\triangleq k-2n \implies & k=\ell+2n \\
& = \sum_{\ell\in\mathbb{Z}} \beta_{\ell}\mathbf{D}^j\mathbf{T}^{\ell}\phi(x) & \text{where } \beta_\ell\triangleq\alpha_{\ell+2n} & \\
& \iff \mathsf{f}\in V_j & \text{by def. of } \{\mathbf{T}^n\phi\} & \text{(Definition 2.1 page 18)}
\end{array}$$

Proof for (2):

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace V_j such that
   $\mathbf{V}_j = \left\{\mathbf{P}_j\mathsf{f}|\mathsf{f}\in L^2_{\mathbb{R}}\right\}$

2. lemma: Functions with *compact support* are *dense* in $L^2_{\mathbb{R}}$. Therefore, we only need to prove that the proposition is true for functions with support in $[-R:R]$, for all $R>0$.

3. For some function $\mathsf{f}\in L^2_{\mathbb{R}}$, let $(\!(\mathsf{f}_n)\!)_{n\in\mathbb{Z}}$ be a sequence of functions in $L^2_{\mathbb{R}}$ with *compact support* such that
   $\operatorname{supp}\mathsf{f}_n\subseteq[-R:R]$ for some $R>0$ and $\mathsf{f}(x)=\lim_{n\to\infty}(\!(\mathsf{f}_n(x))\!)$.

4. lemma: $\bigcap V_j=\{\mathbb{0}\} \iff \lim_{j\to-\infty}\left\|\mathbf{P}_j\mathsf{f}\right\|=0 \quad \forall\mathsf{f}\in L^2_{\mathbb{R}}$. Proof:

$$\begin{array}{lll}
\bigcap_{j\in\mathbb{Z}} V_j = \bigcap_{j\in\mathbb{Z}}\left\{\mathbf{P}_j\mathsf{f}|\mathsf{f}\in L^2_{\mathbb{R}}\right\} & \text{by definition of } V_j & \text{(definition 1 page 19)} \\
= \lim_{j\to-\infty}\left\{\mathbf{P}_j\mathsf{f}|\mathsf{f}\in L^2_{\mathbb{R}}\right\} & \text{by definition of } \cap & \\
= \mathbb{0} \iff \lim_{j\to-\infty}\left\|\mathbf{P}_j\mathsf{f}\right\|=0 & \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\| & \text{(Definition D.5 page 174)}
\end{array}$$

5. lemma: $\lim_{j\to-\infty}\left\|\mathbf{P}_j\mathsf{f}\right\|=0 \quad \forall\mathsf{f}\in L^2_{\mathbb{R}}$. Proof:

[6] Hernández and Weiss (1996) page 45 ⟨Theorem 1.6⟩, Wojtaszczyk (1997) pages 19–28 ⟨Proposition 2.14⟩, Pinsky (2002) pages 313–314 ⟨Lemma 6.4.28⟩

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Let $\mathbb{1}_{A(x)}$ be the *set indicator function* (Definition 1.2 page 3)

$$\lim_{j\to-\infty} \|\mathbf{P}_j \mathsf{f}\|^2$$

$$= \lim_{j\to-\infty} \left\|\mathbf{P}_j \lim_{n\to\infty} (\!|\mathsf{f}_n|\!)\right\|^2 \qquad \text{by definition 3 page 19}$$

$$\le \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbf{P}_j \lim_{n\to\infty} (\!|\mathsf{f}_n|\!) \,|\, \mathbf{D}^j\mathbf{T}^n\phi \right\rangle\right|^2 \qquad \text{by } \textit{frame property} \qquad \text{(Proposition N.5 page 326)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \lim_{n\to\infty} (\!|\mathsf{f}_n|\!) \,|\, \mathbf{D}^j\mathbf{T}^n\phi \right\rangle\right|^2 \qquad \text{by definition of } \mathbf{P}_j \qquad \text{(definition 1 page 19)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbb{1}_{[-R:R]}(x) \lim_{n\to\infty} (\!|\mathsf{f}_n|\!) \,|\, \mathbf{D}^j\mathbf{T}^n\phi(x) \right\rangle\right|^2 \qquad \text{by definition of } (\!|\mathsf{f}_n|\!) \qquad \text{(definition 3 page 19)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \lim_{n\to\infty} (\!|\mathsf{f}_n|\!) \,|\, \mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x) \right\rangle\right|^2 \qquad \text{prop. of } \langle\triangle\,|\,\triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(Definition E.1 page 199)}$$

$$\le \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left\|\lim_{n\to\infty} (\!|\mathsf{f}_n|\!)\right\|^2 \left\|\mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x)\right\|^2 \qquad \text{by } \textit{CS Inequality}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x)\right\|^2 \qquad \text{by definition of } (\!|\mathsf{f}_n|\!) \qquad \text{(definition 3 page 19)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\left[\underbrace{\mathbf{D}^j\mathbf{D}^{-j}}_{\mathbf{I}}\mathbb{1}_{[-R:R]}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\phi(x)\right]\right\|^2 \qquad \text{by property of } \mathbf{D} \qquad \text{(Proposition 1.2 page 5)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|2^{j/2}\mathbf{D}^j\left\{\left[\mathbf{D}^{-j}\mathbb{1}_{[-R:R]}(x)\right]\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 \qquad \text{by Proposition 1.4 page 6}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\left\{2^{j/2}2^{-j/2}\mathbb{1}_{[-R:R]}(2^{-j}x)\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 \qquad \text{by property of } \mathbf{D} \qquad \text{(Proposition 1.2 page 5)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\left\{\left[\underbrace{\mathbf{T}^n\mathbf{T}^{-n}}_{\mathbf{I}}\mathbb{1}_{[-R:R]}(2^{-j}x)\right]\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 \qquad \text{by property of } \mathbf{T} \qquad \text{(Proposition 1.2 page 5)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\left\{\left[\mathbf{T}^n\mathbb{1}_{[-R:R]}(2^{-j}x+n)\right]\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 \qquad \text{by property of } \mathbf{T} \qquad \text{(Proposition 1.2 page 5)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\mathbf{T}^n\left\{\mathbb{1}_{[-R:R]}(2^{-j}x+n)\phi(x)\right\}\right\|^2 \qquad \text{by property of } \mathbf{D} \qquad \text{(Proposition 1.2 page 5)}$$

$$= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbb{1}_{[-R:R]}(2^{-j}x+n)\phi(x)\right\|^2 \qquad \text{by } \textit{unitary} \text{ prop.} \qquad \text{(Theorem 1.2 page 10)}$$

$$= B\,\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \lim_{j\to-\infty} \left\|\mathbb{1}_{[-2^jR+n:2^jR+n]}(u)\phi(2^{-j}(u-n))\right\|^2 \qquad u \triangleq 2^jx+n \implies \qquad x = 2^{-j}(u-n)$$

$$= B\,\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \lim_{j\to-\infty} \int_{-2^jR+n}^{2^jR+n} \left|\phi(2^{-j}(u-n))\right|^2 \,\mathrm{du}$$

$$= B\,\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \int_{n}^{n} |\phi(0)|^2 \,\mathrm{du}$$

$$= 0$$

6. Final step in proof that $\bigcap V_j = \{\mathbb{0}\}$: by (4) lemma page 19 and (5) lemma page 20

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Proposition 2.2. [7]

PRP

$$\left\{\begin{array}{lll} (1). & (\mathbf{T}^n\phi) \text{ is a RIESZ SEQUENCE} & \text{and} \\ (2). & \tilde{\phi}(\omega) \text{ is CONTINUOUS at } 0 & \text{and} \\ (3). & \tilde{\phi}(0) \neq 0 & \end{array}\right\} \Longrightarrow \left\{ \left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^{-} = \boldsymbol{L}^2_{\mathbb{R}} \quad (\text{DENSE in } \boldsymbol{L}^2_{\mathbb{R}}) \right\}$$

✎PROOF:

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace $\boldsymbol{V}_j$ such that
$$\mathbf{V}_j = \left\{\mathbf{P}_j \mathsf{f} \,|\, \mathsf{f} \in \boldsymbol{H}\right\}$$

2. definition: Choose $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ such that $\mathsf{f} \perp \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j$. Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier Transform* (Definition M.2 page 294) of $\mathsf{f}(x)$.

3. lemma: The function f (definition 2 page 21) *exists* because the set of functions that can be chosen to be f at least contains 0 (it is not the emptyset). Proof:
$$\begin{aligned} \mathsf{f}(x) = 0 \quad &\Longrightarrow \left\langle \mathsf{f} \,\middle|\, \left\{ \mathsf{h} \in \boldsymbol{L}^2_{\mathbb{R}} \,|\, \mathsf{h} \in \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right\} \right\rangle \\ &= \left\langle 0 \,\middle|\, \left\{ \mathsf{h} \in \boldsymbol{L}^2_{\mathbb{R}} \,|\, \mathsf{h} \in \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right\} \right\rangle \\ &= 0 \\ &\Longrightarrow \quad \mathsf{f} \perp \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \\ &\Longrightarrow \quad \mathsf{f} \text{ exists} \end{aligned}$$

4. lemma: $\left\|\mathbf{P}_j \mathsf{f}\right\| = 0 \quad \forall_{j\in\mathbb{Z}}$. Proof:
$$\begin{aligned} \left\|\mathbf{P}_j \mathsf{f}\right\| &= \|0\| && \text{by definition of } \mathsf{f} && \text{(definition 2 page 21)} \\ &= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\| \end{aligned}$$

5. definition: Choose some function $\mathsf{g} \in \boldsymbol{L}^2_{\mathbb{R}}$ such that $\tilde{\mathsf{g}}(\omega) = \tilde{\mathsf{f}}(\omega)\mathbb{1}_{[-R:R]}$ (Definition 1.2 page 3) for some $R > 0$ and such that $\|\mathsf{f} - \mathsf{g}\| < \varepsilon$. Let $\tilde{\mathsf{g}}(\omega)$ be the *Fourier Transform* (Definition M.2 page 294) of $\mathsf{g}(x)$.

6. lemma: The function g (definition 5 page 21) *exists*. Proof: For some (possibly very large) R,
$$\begin{aligned} \varepsilon &> \left\|\tilde{\mathsf{f}}(\omega) - \tilde{\mathsf{g}}(\omega)\right\| && \text{by definition of } \mathsf{g} && \text{(definition 5 page 21)} \\ &= \left\|\tilde{\mathbf{F}}\mathsf{f}(x) - \tilde{\mathbf{F}}\mathsf{g}(x)\right\| && \text{by definition of } \tilde{\mathsf{f}} \text{ and } \tilde{\mathsf{g}} && \text{(definition 2 page 21), (definition 5 page 21)} \\ &= \left\|\tilde{\mathbf{F}}\left[\mathsf{f}(x) - \mathsf{g}(x)\right]\right\| && \text{by } \textit{linearity} \text{ of } \tilde{\mathbf{F}} && \text{(Definition D.4 page 171)} \\ &= \|\mathsf{f}(x) - \mathsf{g}(x)\| && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem M.2 page 295)} \\ &\Longrightarrow \quad \mathsf{g} \text{ exists} && \text{because it's possible to satisfy definition 5 page 21} \end{aligned}$$

7. lemma: $\left\|\mathbf{P}_j \mathsf{g}\right\| < \varepsilon \quad \forall_{j\in\mathbb{Z}}$ for sufficiently large R. Proof:
$$\begin{aligned} \varepsilon &> \|\mathsf{f} - \mathsf{g}\| && \text{by definition of } \mathsf{g} && \text{(definition 5 page 21)} \\ &\geq \left\|\mathbf{P}_j\left[\mathsf{f} - \mathsf{g}\right]\right\| && \text{by property of } \textit{projection operators} && \text{(Definition D.10 page 185)} \\ &= \left\|\mathbf{P}_j\mathsf{f} - \mathbf{P}_j\mathsf{g}\right\| && \text{by } \textit{additive} \text{ property of } \mathbf{P}_j && \text{(Definition D.4 page 171)} \\ &\geq \left|\left\|\mathbf{P}_j\mathsf{f}\right\| - \left\|\mathbf{P}_j\mathsf{g}\right\|\right| && \text{by } \textit{Reverse Triangle Inequality} \\ &= \left|0 - \left\|\mathbf{P}_j\mathsf{g}\right\|\right| && \text{by ((4) lemma page 21)} \\ &= \left\|\mathbf{P}_j\mathsf{g}\right\| && \text{by } \textit{strictly positive} \text{ property of } \|\cdot\| && \text{(Definition D.5 page 174)} \end{aligned}$$

[7] Wojtaszczyk (1997) pages 28–31 ⟨Proposition 2.15⟩

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8. lemma: $\mathsf{g} = 0$. Proof:

$$
\begin{aligned}
0 &= \lim_{j\to\infty} \left\|\mathbf{P}_j \mathsf{g}\right\|^2 && \text{by (7) lemma page 21} \\
&\geq \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbf{P}_j \mathsf{g} \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by } \textit{frame property} \quad \text{(Proposition N.5 page 326)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \mathsf{g} \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by definition of } \mathbf{P}_j \quad \text{(item (1) page 21)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathbf{F}} \mathsf{g} \,|\, \tilde{\mathbf{F}} \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} \quad \text{(Theorem M.2 page 295)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathsf{g}}(\omega) \,|\, 2^{-j/2} e^{-i2^{-j}\omega n} \tilde{\phi}(2^{-j}\omega) \right\rangle\right|^2 && \text{by Proposition 1.12 page 12} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathsf{g}}(\omega) \tilde{\phi}^*(2^{-j}\omega) \,|\, 2^{-j/2} e^{-i2^{-j}\omega n} \right\rangle\right|^2 && \text{by property of } \langle \triangle \,|\, \triangledown \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \\
&= \lim_{j\to\infty} A \left\|\tilde{\mathsf{g}}(\omega) \tilde{\phi}^*(2^{-j}\omega)\right\|^2 && \text{by } \textit{Parseval's Identity} \quad \text{(Theorem N.9 page 318)} \\
&= A \left\|\tilde{\mathsf{g}}(\omega) \tilde{\phi}^*(0)\right\|^2 && \text{by left hypothesis (2)} \\
&= A \left|\tilde{\phi}^*(0)\right|^2 \left\|\tilde{\mathsf{g}}(\omega)\right\|^2 && \text{by } \textit{homogeneous} \text{ property of } \|\cdot\| \\
&= A \left|\tilde{\phi}(0)\right|^2 \left\|\mathsf{g}\right\|^2 && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} \quad \text{(Theorem M.2 page 295)} \\
&\implies \left\|\mathsf{g}\right\| = 0 && \text{by left hypothesis (3)} \\
&\iff \mathsf{g} = 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|
\end{aligned}
$$

9. Final step in proof that $\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}$:

$$
\begin{aligned}
&\mathsf{g} = 0 && \text{by (8) lemma page 22} \\
&\implies \mathsf{f} = 0 && \text{by definition of } \mathsf{g} \quad \text{(definition 5 page 21)} \\
&\implies \left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}
\end{aligned}
$$

✏

Definition 2.1 defines an MRA on the space $\boldsymbol{L}^2_{\mathbb{R}}$, which is a special case of a *separable Hilbert space*. A Hilbert space is a *linear space* that is equipped with an *inner product*, is *complete* with respect to the *metric* induced by the inner product, and contains a subset that is *dense* in $\boldsymbol{L}^2_{\mathbb{R}}$.

An *inner product* on a linear space endows the linear space with a *topology*. The sum such as $\sum_{n=1}^{N} \alpha_n \mathsf{f}_n$ is finite and thus suitable for a finite linear space only. An infinite space requires an infinite sum $\sum_{n=1}^{\infty} \alpha_n \phi_n$, and an infinite sum is defined in terms of a limit (Definition H.1 page 224). The limit, in turn, is defined in terms of a *topology*. The *inner product* induces a *norm* (Definition D.5 page 174) which induces a *metric* which induces a topology.

Definition 2.1 defines each subspace $\boldsymbol{V}_j$ to be *closed* ($\boldsymbol{V}_j = \boldsymbol{V}_j^-$) in $\boldsymbol{L}^2_{\mathbb{R}}$. As one might imagine, the properties of *completeness* and *closure* are closely related. Moreover, Every *complete* sequence is also *bounded*, and so each subspace $\boldsymbol{V}_j$ is *bounded* as well.

Proposition 2.3. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right)\right)$ *be an* MRA SPACE.

PRP *Each subspace* $\boldsymbol{V}_j$ *is* COMPLETE.

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✎PROOF:

1. By definition Definition 2.1, $L^2_{\mathbb{R}}$ is *complete*.
2. In any metric space, (which includes all inner product spaces such as $L^2_{\mathbb{R}}$), a *closed* subspace of a *complete* metric space is itself also *complete*.
3. In any *complete* metric space X (which includes all Hilbert spaces such as $L^2_{\mathbb{R}}$), the two properties coincide—that is, a subspace is complete *if and only if* it is closed in the space X.
4. So because $L^2_{\mathbb{R}}$ is *complete* and each V_j is *closed*, then each V_j is also *complete*.

✏

2.3 Order structure

A *multiresolution analysis* (Definition 2.1 page 18) together with the set inclusion relation $\subseteq$ forms the *linearly ordered set* $(\{V_j\}, \subseteq)$, illustrated to the right by a *Hasse diagram*. Subspaces V_j increase in "size" with increasing j. That is, they contain more and more vectors (functions) for larger and larger j—with the upper limit of this sequence being $L^2_{\mathbb{R}}$. Alternatively, we can say that approximation within a subspace V_j yields greater "*resolution*" for increasing j.

The *least upper bound* (*l.u.b.*) of the linearly ordered set $(\{V_j\}, \subseteq)$ is $L^2_{\mathbb{R}}$ (Definition 2.1 page 18):

$$\left(\bigcup_{j\in\mathbb{Z}} V_j\right)^{-} = L^2_{\mathbb{R}}.$$

The *greatest lower bound* (*g.l.b.*) of the linearly ordered set $(\{V_j\}, \subseteq)$ is $\mathbf{0}$ (Proposition 2.1 page 19):

$$\bigcap_{j\in\mathbb{Z}} V_j = \mathbf{0}.$$

All linear subspaces contain the zero vector. So the intersection of any two subspaces must at least contain $\mathbb{0}$. If the intersection of any two linear subspaces X and Y is exactly $\{\mathbb{0}\}$, then for any vector in the sum of those subspaces ($u \in X \dotplus Y$) there are **unique** vectors $\mathsf{f} \in X$ and $\mathsf{g} \in Y$ such that $u = \mathsf{f} + \mathsf{g}$. This is *not* necessarily true if the intersection contains more than just $\{\mathbb{0}\}$.

2.4 Dilation equation

Several functions in mathematics exhibit a kind of *self-similar* or *recursive* property:

- If a function $\mathsf{f}(x)$ is *linear*, then (Example 1.2 page 15)
  $$\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x.$$
- If a function $\mathsf{f}(x)$ is sufficiently *bandlimited*, then the *Cardinal series* (Example 1.3 page 16) demonstrates
  $$\mathsf{f}(x) = \sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin[\pi(x)]}{\pi(x)}.$$

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B-splines (Theorem O.3 page 344) are another example:

$$\mathsf{N}_n(x) = \frac{1}{n}x\mathsf{N}_{n-1}(x) - \frac{1}{n}x\mathbf{T}\mathsf{N}_{n-1}(x) + \frac{n+1}{n}\mathbf{T}\mathsf{N}_{n-1}(x) \qquad \forall n\in\mathbb{W}\setminus\{1\}, \forall x\in\mathbb{R}.$$

The scaling function $\phi(x)$ (Definition 2.1 page 18) also exhibits a kind of *self-similar* property. By Definition 2.1 page 18, the dilation $\mathbf{D}\mathsf{f}$ of each vector f in V_0 is in V_1. If $\left\{\mathbf{T}^n\phi\,|\,_{n\in\mathbb{Z}}\right\}$ is a basis for V_0, then $\left\{\mathbf{D}\mathbf{T}^n\phi\,|\,_{n\in\mathbb{Z}}\right\}$ is a basis for V_1, $\left\{\mathbf{D}^2\mathbf{T}^n\phi\,|\,_{n\in\mathbb{Z}}\right\}$ is a basis for V_2, ...; and in general $\left\{\mathbf{D}^j\mathbf{T}^m\phi|j\in\mathbb{Z}\right\}$ is a basis for V_j. Also, if ϕ is in V_0, then it is also in V_1 (because $V_0\subset V_1$). And because ϕ is in V_1 and because $\left\{\mathbf{D}\mathbf{T}^n\phi\,|\,_{n\in\mathbb{Z}}\right\}$ is a basis for V_1, ϕ is a linear combination of the elements in $\left\{\mathbf{D}\mathbf{T}^n\phi\,|\,_{n\in\mathbb{Z}}\right\}$. That is, ϕ can be represented as a linear combination of translated and dilated versions of itself. The resulting equation is called the *dilation equation* (Definition 2.2, next).[8]

Definition 2.2. [9] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!)\right)$ *be a* MULTIRESOLUTION ANALYSIS SPACE *with scaling function* ϕ *(Definition 2.1 page 18). Let* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE *(Definition F.2 page 204) in* $\ell^2_{\mathbb{R}}$ *(Definition F.3 page 205).*

DEF

The EQUATION $\left\{\phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) \qquad \forall x\in\mathbb{R}\right\}$ *is called the* ***dilation equation****.*

It is also called the ***refinement equation****,* ***two-scale difference equation****, and* ***two-scale relation****.*

Remark 2.1.

REM

The *dilation equation* under the defintions of $\mathbf{T}$ and $\mathbf{D}$ evaluates to

$$\phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\phi(2x-n).$$

✎PROOF:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\phi(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n\phi(2x-n) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}
\end{aligned}$$

✏

Theorem 2.1 (dilation equation)**.** *Let an* MRA SPACE *and* SCALING FUNCTION *be as defined in Definition 2.1 page 18.*

THM

$$\left\{\begin{array}{l}\left(L^2_{\mathbb{R}}, (\!(V_j)\!)\right) \text{ is an MRA SPACE} \\ \text{with SCALING FUNCTION } \phi\end{array}\right\} \implies \underbrace{\left\{\begin{array}{l}\exists (\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}} \text{ such that} \\ \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) \qquad \forall x\in\mathbb{R}\end{array}\right\}}_{\text{DILATION EQUATION IN "TIME"}}$$

✎PROOF:

$$\begin{aligned}
\phi &\in V_0 && \text{by definition of MRA} && \text{(Definition 2.1 page 18)} \\
&\subseteq V_1 && \text{by definition of MRA} && \text{(Definition 2.1 page 18)} \\
&\triangleq \operatorname{span}\left\{\mathbf{D}\mathbf{T}^n\phi(x)\,|\,_{n\in\mathbb{Z}}\right\} && \text{by definition of } V_j && \text{(Definition 2.1 page 18)} \\
&\implies \exists (\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}} \text{ such that } \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) && \text{by definition of span} && \text{(Definition N.2 page 305)}
\end{aligned}$$

✏

[8] The property of *translation invariance* is of particular significance in the theory of *normed linear spaces* (a Hilbert space is a complete normed linear space equipped with an inner product).

[9] Jawerth and Sweldens (1994) page 7

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Lemma 2.1. [10] *Let $\phi(x)$ be a function in $L^2_{\mathbb{R}}$ (Definition E.1 page 199). Let $\tilde{\phi}(\omega)$ be the* FOURIER TRANSFORM *(Definition M.2 page 294) of $\phi(x)$. Let $\check{\mathsf{h}}(\omega)$ be the* DISCRETE TIME FOURIER TRANSFORM *(Definition R.1 page 393) of a sequence $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$.*

LEM

$$(A)\quad \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \phi(x) \quad \forall x\in\mathbb{R} \quad \Longleftrightarrow \quad \tilde{\phi}(\omega) = \frac{\sqrt{2}}{2}\,\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \quad \forall\omega\in\mathbb{R} \qquad (1)$$

$$\Longleftrightarrow \quad \tilde{\phi}(\omega) = \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\,\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right) \quad \forall n\in\mathbb{N},\omega\in\mathbb{R} \qquad (2)$$

✎PROOF:

1. Proof that (A) $\Longrightarrow$ (1):

$$\begin{aligned}
\tilde{\phi}(\omega) &\triangleq \tilde{\mathbf{F}}\phi \\
&= \tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n\phi(x) && \text{by (A)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \tilde{\mathbf{F}}\mathbf{DT}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \underbrace{\frac{\sqrt{2}}{2}e^{-i\frac{\omega}{2}n}\phi\left(\frac{\omega}{2}\right)}_{\tilde{\mathbf{F}}\mathbf{DT}^n\phi(x)} && \text{by Proposition 1.12 page 12} \\
&= \frac{\sqrt{2}}{2}\underbrace{\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\frac{\omega}{2}n}\right]}_{\check{\mathsf{h}}(\omega/2)}\tilde{\phi}\left(\frac{\omega}{2}\right) \\
&= \frac{\sqrt{2}}{2}\,\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } DTFT \text{ (Definition R.1 page 393)}
\end{aligned}$$

2. Proof that (A) $\Longleftarrow$ (1):

$$\begin{aligned}
\phi(x) &= \tilde{\mathbf{F}}^{-1}\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega) \\
&= \tilde{\mathbf{F}}^{-1}\frac{\sqrt{2}}{2}\,\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by (1)} \\
&= \tilde{\mathbf{F}}^{-1}\frac{\sqrt{2}}{2}\sum_{n\in\mathbb{Z}} h_n e^{-i\frac{\omega}{2}n}\,\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } DTFT \quad \text{(Definition R.1 page 393)} \\
&= \frac{\sqrt{2}}{2}\sum_{n\in\mathbb{Z}} h_n \tilde{\mathbf{F}}^{-1}e^{-i\frac{\omega}{2}n}\,\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by property of linear operators} \\
&= \frac{\sqrt{2}}{2}\sum_{n\in\mathbb{Z}} h_n \tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathbf{DT}^n\phi && \text{by Proposition 1.12 page 12} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n\phi(x) && \text{by definition of } \textit{operator inverse}
\end{aligned}$$

3. Proof that (1) $\Longrightarrow$ (2):

   (a) Proof for $N = 1$ case:

$$\begin{aligned}
\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\Bigg|_{N=1} &= \frac{\sqrt{2}}{2}\,\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \\
&= \tilde{\phi}(\omega) && \text{by (1)}
\end{aligned}$$

[10] Mallat (1999) page 228

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(b) Proof that [N case] $\Longrightarrow$ [$N+1$ case]:

$$\tilde{\phi}\left(\frac{\omega}{2^{N+1}}\right)\prod_{n=1}^{N+1}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right)=\left[\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\right]\underbrace{\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^{N+1}}\right)\tilde{\phi}\left(\frac{\omega}{2^{N+1}}\right)}_{\tilde{\phi}(\omega/2^N)}$$

$$=\tilde{\phi}(\omega/2^N)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right)$$

$$=\tilde{\phi}(\omega) \qquad \text{by [}N\text{ case] hypothesis}$$

4. Proof that (1) $\Longleftarrow$ (2):

$$\tilde{\phi}(\omega)=\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\Bigg|_{N=1} \qquad \text{by (2)}$$

$$=\tilde{\phi}\left(\frac{\omega}{2}\right)\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)$$

$$=\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$$

✏

Lemma 2.2. *Let $\phi(x)$ be a function in $\mathbf{L}^2_{\mathbb{R}}$ (Definition E.1 page 199). Let $\tilde{\phi}(\omega)$ be the* FOURIER TRANSFORM *(Definition M.2 page 294) of $\phi(x)$. Let $\check{\mathsf{h}}(\omega)$ be the* DISCRETE TIME FOURIER TRANSFORM *(Definition R.1 page 393) of $(\!(\mathsf{h}_n)\!)$. Let $\prod_{n=1}^{\infty}x_n \triangleq \lim_{N\to\infty}\prod_{n=1}^{N}x_n$, with respect to the standard norm in $\mathbf{L}^2_{\mathbb{R}}$.*

LEM

$$\left\{\begin{array}{l}\tilde{\phi}(\omega)=C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\\ \forall C>0,\omega\in\mathbb{R} \qquad (A)\end{array}\right\} \Longrightarrow \phi(x)=\sum_{n\in\mathbb{Z}}\mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) \quad \forall x\in\mathbb{R} \quad (1)$$

$$\Longleftrightarrow \tilde{\phi}(\omega)=\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \quad \forall\omega\in\mathbb{R} \quad (2)$$

$$\Longleftrightarrow \tilde{\phi}(\omega)=\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right) \quad \forall n\in\mathbb{N},\omega\in\mathbb{R} \quad (3)$$

✎PROOF:

1. Proof that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3): by Lemma 2.1 page 25
2. Proof that (A) $\Longrightarrow$ (2):

$$\tilde{\phi}(\omega)=C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right) \qquad \text{by left hypothesis}$$

$$=C\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^{n+1}}\right)$$

$$=C\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega/2}{2^n}\right)$$

$$=\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\left[C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega/2}{2^n}\right)\right]$$

$$=\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \qquad \text{by left hypothesis}$$

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✏

Proposition 2.4. *Let $\phi(x)$ be a function in $L^2_{\mathbb{R}}$ (Definition E.1 page 199). Let $\tilde{\phi}(\omega)$ be the* FOURIER TRANSFORM *(Definition M.2 page 294) of $\phi(x)$. Let $\check{\mathsf{h}}(\omega)$ be the* DISCRETE TIME FOURIER TRANSFORM *(Definition R.1 page 393) of $(\!(\mathsf{h}_n)\!)$. Let $\prod_{n=1}^{\infty} x_n \triangleq \lim_{N\to\infty}\prod_{n=1}^{N} x_n$, with respect to the standard norm in $L^2_{\mathbb{R}}$.*

PRP

$$\left\{\begin{array}{l}\tilde{\phi}(\omega) \text{ is}\\ \text{CONTINUOUS}\\ \text{at } \omega = 0\end{array}\right\} \implies \left\{\begin{array}{lrcll} & \phi(x) & = & \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi(x) & \forall x\in\mathbb{R} \quad (1)\\ \iff & \tilde{\phi}(\omega) & = & \frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) & \forall\omega\in\mathbb{R} \quad (2)\\ \iff & \tilde{\phi}(\omega) & = & \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \forall n\in\mathbb{N},\omega\in\mathbb{R} \quad (3)\\ \iff & \tilde{\phi}(\omega) & = & \tilde{\phi}(0)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \omega\in\mathbb{R} \quad (4)\end{array}\right\}$$

✎PROOF:

1. Proof that (1) $\iff$ (2) $\iff$ (3): by Lemma 2.1 page 25
2. Proof that (3) $\implies$ (4):

$$\begin{aligned}\tilde{\phi}(0)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right) &= \lim_{N\to\infty}\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^n}\right) && \text{by } \textit{continuity} \text{ and definition of } \prod_{n=1}^{\infty} x_n\\ &= \tilde{\phi}(\omega) && \text{by (3) and Lemma 2.1 page 25}\end{aligned}$$

3. Proof that (2) $\impliedby$ (4): by Lemma 2.2 page 26

✏

Definition 2.3 (next) formally defines the coefficients that appear in Theorem 2.1 (page 24).

Definition 2.3. *Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!)\right)$ be a multiresolution analysis space with scaling function ϕ. Let $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ be a sequence of coefficients such that $\phi = \sum_{n\in\mathbb{Z}}\mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi$.*

DEF

*A **multiresolution system** is the tuple $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$. The sequence $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ is the **scaling coefficient sequence**. A multiresolution system is also called an **MRA system**. An* MRA SYSTEM *is an **orthonormal MRA system** if $\left\{\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\}$ is* ORTHONORMAL.

Theorem 2.2. *Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ be an* MRA SYSTEM *(Definition 2.3 page 27). Let $\operatorname{span} A$ be the* LINEAR SPAN *(Definition N.2 page 305) of a set A.*

THM

$$\underbrace{\operatorname{span}\left\{\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} = V_0}_{\left\{\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} \text{ is a BASIS for } V_0} \implies \underbrace{\operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} = V_j \quad \forall j\in\mathbb{W}}_{\left\{\mathbf{D}^j\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} \text{ is a BASIS for } V_j}$$

✎PROOF: Proof is by induction:[11]

[11] Smith (2011) page 4

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1. induction basis (proof for $j = 0$ case):

$$\begin{aligned}\operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi \middle| {\scriptstyle n\in\mathbb{Z}}\right\}\Big|_{j=0} &= \operatorname{span}\left\{\mathbf{T}^n\phi \middle| {\scriptstyle n\in\mathbb{Z}}\right\} \\ &= V_0 && \text{by left hypothesis}\end{aligned}$$

2. induction step (proof that j case $\Longrightarrow$ $j+1$ case):

$$\begin{aligned}
&\operatorname{span}\left\{\mathbf{D}^{j+1}\mathbf{T}^n\phi \middle| {\scriptstyle n\in\mathbb{Z}}\right\} \\
&= \left\{\mathsf{f} \in L^2_{\mathbb{R}} \,|\, \exists\,(\alpha_n) \quad \text{such that} \quad \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j+1}\mathbf{T}^n\phi\right\} && \text{by definition of span} \quad \text{(Definition N.2 page 305)} \\
&= \left\{\mathsf{f} \in L^2_{\mathbb{R}} \,|\, \exists\,(\alpha_n) \quad \text{such that} \quad \mathsf{f}(x) = \mathbf{D}\sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \left\{\mathsf{f} \in L^2_{\mathbb{R}} \,|\, \exists\,(\alpha_n) \quad \text{such that} \quad \mathbf{D}^{-1}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \left\{[\mathbf{D}\mathsf{f}] \in L^2_{\mathbb{R}} \,|\, \exists\,(\alpha_n) \quad \text{such that} \quad \mathbf{D}^{-1}[\mathbf{D}\mathsf{f}(x)] = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \mathbf{D}\left\{\mathsf{f} \in L^2_{\mathbb{R}} \,|\, \exists\,(\alpha_n) \quad \text{such that} \quad \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi\right\} \\
&= \mathbf{D}\operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi \middle| {\scriptstyle n\in\mathbb{Z}}\right\} && \text{by definition of span} \quad \text{(Definition N.2 page 305)} \\
&= \mathbf{D}V_j && \text{by induction hypothesis} \\
&= V_{j+1} && \text{by \textit{self-similar} property} \quad \text{(Definition 2.1 page 18)}
\end{aligned}$$

✏

Example 2.1.

EX

In the *Haar* MRA, the scaling function $\phi(x)$ is the *pulse function*

$$\phi(x) = \begin{cases} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise.}\end{cases}$$

In the subspace V_j ($j \in \mathbb{Z}$) the scaling functions are

$$\mathbf{D}^j\phi(x) = \begin{cases} (2)^{j/2} & \text{for } x \in \left[0 : \left(2^{-j}\right)\right) \\ 0 & \text{otherwise.}\end{cases}$$

The scaling subspace V_0 is the span $V_0 \triangleq \operatorname{span}\left\{\mathbf{T}^n\phi \middle| {\scriptstyle n\in\mathbb{Z}}\right\}$. The scaling subspace V_j is the span $V_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi \middle| n \in \mathbb{Z}\right\}$. Note that $\left\|\mathbf{D}^j\mathbf{T}^n\phi\right\|$ for each resolution j and shift n is unity:

$$\begin{aligned}
\left\|\mathbf{D}^j\mathbf{T}^n\phi\right\|^2 &= \|\phi\|^2 && \text{by \textit{unitary} properties of } \mathbf{T} \text{ and } \mathbf{D} \quad \text{(Theorem 1.2 page 10)} \\
&= \int_0^1 |1|^2 \,\mathrm{d}x && \text{by definition of } \|\cdot\| \text{ on } L^2_{\mathbb{R}} \quad \text{(Definition E.1 page 199)} \\
&= 1
\end{aligned}$$

Let $\mathsf{f}(x) = \sin(\pi x)$. Suppose we want to project $\mathsf{f}(x)$ onto the subspaces V_0, V_1, V_2, ...

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The values of the transform coefficients for the subspace V_j are given by

$$
\begin{aligned}
\big[\mathbf{R}_j \mathsf{f}(x)\big](n) &= \frac{1}{\|\mathbf{D}^j\mathbf{T}^n\phi\|^2}\left\langle \mathsf{f}(x) \,|\, \mathbf{D}^j\mathbf{T}^n\phi\right\rangle \\
&= \frac{1}{\cancel{\|\phi\|^2}^{\,1}}\left\langle \mathsf{f}(x) \,|\, 2^{j/2}\phi\big(2^j x - n\big)\right\rangle && \text{by Proposition 1.3 page 5} \\
&= 2^{j/2}\left\langle \mathsf{f}(x) \,|\, \phi\big(2^j x - n\big)\right\rangle \\
&= 2^{j/2}\int_{2^{-j}n}^{2^{-j}(n+1)} \mathsf{f}(x)\,\mathrm{dx} \\
&= 2^{j/2}\int_{2^{-j}n}^{2^{-j}(n+1)} \sin(\pi x)\,\mathrm{dx} \\
&= 2^{j/2}\left(-\frac{1}{\pi}\right)\cos(\pi x)\Big|_{2^{-j}n}^{2^{-j}(n+1)} \\
&= \frac{2^{j/2}}{\pi}\big[\cos\big(2^{-j}n\pi\big) - \cos\big(2^{-j}(n+1)\pi\big)\big]
\end{aligned}
$$

And the projection $\mathbf{A}_n\mathsf{f}(x)$ of the function $\mathsf{f}(x)$ onto the subspace V_j is

$$
\begin{aligned}
\mathbf{A}_j\mathsf{f}(x) &= \sum_{n\in\mathbb{Z}}\left\langle \mathsf{f}(x) \,|\, \mathbf{D}^j\mathbf{T}^n\phi\right\rangle \mathbf{D}^j\mathbf{T}^n\phi \\
&= \frac{2^{j/2}}{\pi}\sum_{n\in\mathbb{Z}}\big[\cos\big(2^{-j}n\pi\big) - \cos\big(2^{-j}(n+1)\pi\big)\big]2^{j/2}\phi\big(2^j x - n\big) \\
&= \frac{2^j}{\pi}\sum_{n\in\mathbb{Z}}\big[\cos\big(2^{-j}n\pi\big) - \cos\big(2^{-j}(n+1)\pi\big)\big]\phi\big(2^j x - n\big)
\end{aligned}
$$

The transforms of $\sin(\pi x)$ into the subspaces V_0, V_1, and V_2, as well as the approximations in those subspaces are as illustrated in Figure 2.1 (page 30).

2.5 Necessary Conditions

Theorem 2.3 (admissibility condition)**.** *Let* $\check{\mathsf{h}}(z)$ *be the* Z-TRANSFORM *(Definition Q.1 page 379) and* $\check{\mathsf{h}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition R.1 page 393) of a sequence* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$.

THM

$$
\left\{\left(L^2_{\mathbb{R}},\ (\!(V_j)\!),\ \phi,\ (\!(h_n)\!)\right) \text{ is an MRA SYSTEM (Definition 2.3 page 27)}\right\}
$$

$$
\underbrace{\overset{\Longrightarrow}{\not\Longleftarrow}\ \left\{\sum_{n\in\mathbb{Z}}\mathsf{h}_n = \sqrt{2}\right\}}_{\text{(1) ADMISSIBILITY in “time”}}
\iff
\underbrace{\left\{\check{\mathsf{h}}(z)\Big|_{z=1} = \sqrt{2}\right\}}_{\text{(2) ADMISSIBILITY in “z domain”}}
\iff
\underbrace{\left\{\check{\mathsf{h}}(\omega)\Big|_{\omega=0} = \sqrt{2}\right\}}_{\text{(3) ADMISSIBILITY in “frequency”}}
$$

✎PROOF:

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Figure 2.1: Projections of $\sin(\pi x)$ on Haar subspaces (Example 2.1 page 28)

1. Proof that MRA system $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{h}_n &= \frac{\int_{\mathbb{R}} \phi(x)\,\mathrm{dx}}{\int_{\mathbb{R}} \phi(x)\,\mathrm{dx}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \\
&= \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{dx}} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(x)\,\mathrm{dx} \\
&= \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{dx}} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \frac{\sqrt{2}}{\sqrt{2}} \phi(2y-n) 2\,\mathrm{dy} && \text{let } y \triangleq \frac{x+n}{2} \implies x = 2y-n \implies \mathrm{dx} = 2\,\mathrm{dy} \\
&= \frac{2}{\sqrt{2}} \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{dx}} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \phi(y)\,\mathrm{dy} && \text{by definitions of } \mathbf{T} \text{ and } \mathbf{D} \text{ (Definition 1.3 page 4)} \\
&= \sqrt{2} \frac{1}{\int_{\mathbb{R}} \phi(x)\,\mathrm{dx}} \int_{\mathbb{R}} \phi(y)\,\mathrm{dy} && \text{by } \textit{dilation equation} \text{ (Theorem 2.1 page 24)} \\
&= \sqrt{2}
\end{aligned}$$

2. Alternate proof that MRA system $\implies$ (1):
   Let $\mathsf{f}(x) \triangleq 1 \quad \forall x \in \mathbb{R}$.

$$\begin{aligned}
\langle \phi \,|\, \mathsf{f} \rangle &= \left\langle \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \phi \,|\, \mathsf{f} \right\rangle && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 24)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \mathbf{DT}^n \phi \,|\, \mathsf{f} \rangle && \text{by linearity of } \langle \triangle \,|\, \triangledown \rangle \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \phi \,|\, (\mathbf{DT}^n)^* \mathsf{f} \rangle && \text{by definition of operator adjoint} && \text{(Theorem D.13 page 184)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \phi \,|\, (\mathbf{T}^*)^n \mathbf{D}^* \mathsf{f} \rangle && \text{by property of operator adjoint} && \text{(Theorem D.13 page 184)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,|\, (\mathbf{T}^{-1})^n \mathbf{D}^{-1} \mathsf{f} \right\rangle && \text{by unitary property of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Proposition 1.7 page 8)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,|\, (\mathbf{T}^{-1})^n \tfrac{\sqrt{2}}{2} \mathsf{f} \right\rangle && \text{because } \mathsf{f} \text{ is a constant hypothesis} && \text{and by Proposition 1.2 page 5}
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,\middle|\, \tfrac{\sqrt{2}}{2}\mathsf{f} \right\rangle && \text{by } \mathsf{f}(x) = 1 \text{ definition} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \tfrac{\sqrt{2}}{2} \langle \phi \,|\, \mathsf{f} \rangle && \text{by property of } \langle \triangle \,|\, \triangledown \rangle \\
&= \tfrac{\sqrt{2}}{2} \langle \phi \,|\, \mathsf{f} \rangle \sum_{n\in\mathbb{Z}} \mathsf{h}_n \\
&\Longrightarrow \sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2}
\end{aligned}
$$

3. Proof that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3): by Proposition R.2 page 395.

4. Proof for $\not\Longleftarrow$ part: by Counterexample 2.1 page 31.

✏

Counterexample 2.1. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ be an *MRA system* (Definition 2.3 page 27).

CNT

$$
\left\{ \langle\!\langle \mathsf{h}_n \rangle\!\rangle \triangleq \sqrt{2}\bar{\delta}_{n-1} \triangleq \begin{cases} \sqrt{2} & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases} \right\} \Longrightarrow \{\phi(x) = 0\}
$$

which means

$$
\left\{ \sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2} \right\} \not\Longrightarrow \left\{ \left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right) \text{ is an MRA system for } \boldsymbol{L}^2_{\mathbb{R}}. \right\}
$$

✎PROOF:

$$
\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 24)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x - n) && \text{by definitions of } \mathbf{D} \text{ and } \mathbf{T} && \text{(Definition 1.3 page 4)} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{\sqrt{2}\bar{\delta}_{n-1}}_{\langle\!\langle \mathsf{h}_n \rangle\!\rangle} \phi(2x - n) && \text{by definitions of } \langle\!\langle \mathsf{h}_n \rangle\!\rangle \\
&= \sqrt{2}\phi(2x - 1) && \text{by definition of } \phi(x) \\
\Longrightarrow \phi(x) &= 0
\end{aligned}
$$

This implies $\phi(x) = 0$, which implies that $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ is *not* an *MRA system* for $\boldsymbol{L}^2_{\mathbb{R}}$ because

$$
\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right)^{-} = \left(\bigcup_{j\in\mathbb{Z}} \operatorname{span}\left\{ \mathbf{D}^j \mathbf{T}^n \phi \,\middle|\, n\in\mathbb{Z} \right\} \right)^{-} \neq \boldsymbol{L}^2_{\mathbb{R}}
$$

(the *least upper bound* is *not* $\boldsymbol{L}^2_{\mathbb{R}}$). ✏

Theorem 2.4 (Quadrature condition in "time"). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27)*.

THM

$$
\sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}^*_k \left\langle \phi \,\middle|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \langle \phi \,|\, \mathbf{T}^n \phi \rangle \qquad \forall n\in\mathbb{Z}
$$

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✎PROOF:

$$
\begin{aligned}
\langle \phi \mid \mathbf{T}^n\phi\rangle &= \left\langle \sum_{m\in\mathbb{Z}} \mathsf{h}_m \mathbf{D}\mathbf{T}^m\phi \mid \mathbf{T}^n \sum_{k\in\mathbb{Z}} \mathsf{h}_k \mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 24)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \mathbf{D}\mathbf{T}^m\phi \mid \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi\right\rangle && \text{by properties of } \langle\triangle\mid\triangledown\rangle\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \mid (\mathbf{D}\mathbf{T}^m)^* \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi\right\rangle && \text{by definition of operator adjoint} && \text{(Proposition D.3 page 183)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \mid (\mathbf{D}\mathbf{T}^m)^* \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi\right\rangle && \text{by Proposition 1.5 page 7}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \mid \mathbf{T}^{*m}\mathbf{D}^*\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi\right\rangle && \text{by operator star-algebra properties} && \text{(Theorem D.13 page 184)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \mid \mathbf{T}^{-m}\mathbf{D}^{-1}\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi\right\rangle && \text{by Proposition 1.7 page 8}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \mid \mathbf{T}^{2n-m+k}\phi\right\rangle
\end{aligned}
$$

✏

Theorem 2.5 (next) presents the *quadrature necessary conditions* of a *wavelet system*. These relations simplify dramatically in the special case of an *orthonormal wavelet system* (Theorem R.4 page 399).

Theorem 2.5 (Quadrature condition in "frequency"). [12] *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, \langle\!\langle \mathbf{V}_j\rangle\!\rangle, \phi, \langle\!\langle h_n\rangle\!\rangle\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27). Let* $\tilde{\mathsf{x}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition R.1 page 393) for a sequence* $\langle\!\langle x_n\rangle\!\rangle_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$. *Let* $\tilde{\mathsf{S}}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM *(Definition T.3 page 411) of* ϕ.

THM

$$\left|\check{\mathsf{h}}(\omega)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)$$

(*Note:* $\tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1$ *for* ORTHONORMAL *MRA* *(Lemma 6.1 page 85)*)

✎PROOF:

$$
\begin{aligned}
&2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)\\
&= 2(2\pi)\sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(2\omega+2\pi n)\right|^2 && \text{by Theorem T.1 page 411}\\
&= 2(2\pi)\sum_{n\in\mathbb{Z}} \left|\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right)\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 && \text{by Lemma 2.1 page 25}\\
&= 2\pi\sum_{n\in\mathbb{Z}_e} \left|\check{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}_o} \left|\check{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\\
&= 2\pi\sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega+2\pi n)\right|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega+2\pi n+\pi)\right|^2 \left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2\\
&= 2\pi\sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega)\right|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}} \left|\check{\mathsf{h}}(\omega+\pi)\right|^2 \left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 && \text{by Proposition R.1 page 393}\\
&= \left|\check{\mathsf{h}}(\omega)\right|^2\left(2\pi\sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega+2\pi n)\right|^2\right) + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2\left(2\pi\sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right)\\
&= \left|\check{\mathsf{h}}(\omega)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) && \text{by Theorem T.1 page 411}
\end{aligned}
$$

✏

[12] Chui (1992) page 135, Goswami and Chan (1999) page 110

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2.6 Sufficient conditions

Theorem 2.6 (next) gives a set of *sufficient* conditions on the *scaling function* (Definition 2.1 page 18) ϕ to generate an *MRA*. Theorem 6.2 (page 88) provides a set of sufficient conditions on the *scaling coefficients* (Definition 2.3 page 27) $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ to generate an *MRA*; howbeit, this set results in the more restrictive *orthonormal* MRA.

Theorem 2.6. [13] *Let* $\mathsf{V}_j \triangleq \mathsf{span}\,\{\,\mathbf{T}\phi(x)|\ {\scriptstyle n\in\mathbb{Z}}\,\}$ *(Definition N.2 page 305)*.

THM

$$\left\{\begin{array}{ll} (1). & (\!(\mathbf{T}^n\phi)\!) \text{ is a RIESZ SEQUENCE (Definition N.14 page 323)} \quad \text{and} \\ (2). & \exists\,(\!(\mathsf{h}_n)\!) \quad \text{such that} \quad \phi(x) = \sum_{n\in\mathbb{Z}} h_n\mathbf{D}\mathbf{T}^n\phi(x) \quad \text{and} \\ (3). & \tilde{\phi}(\omega) \text{ is CONTINUOUS at } 0 \quad \text{and} \\ (4). & \tilde{\phi}(0) \neq 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} (\!(\mathsf{V}_j)\!)_{j\in\mathbb{Z}} \text{ is an MRA} \\ \text{(Definition 2.1 page 18)} \end{array}\right\}$$

✎PROOF: For this to be true, each of the conditions in the definition of an *MRA* (Definition 2.1 page 18) must be satisfied:

1. Proof that each V_j is *closed*: by definition of span
2. Proof that $(\!(\mathsf{V}_j)\!)$ is *linearly ordered*:
$$\mathsf{V}_j \subseteq \mathsf{V}_{j+1} \iff \mathsf{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi\right\} \subseteq \mathsf{span}\left\{\mathbf{D}^{j+1}\mathbf{T}^n\phi\right\} \qquad \iff (2)$$
3. Proof that $\bigcup_{j\in\mathbb{Z}} \mathsf{V}_j$ is *dense* in $\mathsf{L}^2_{\mathbb{R}}$: by Proposition 2.2 page 21
4. Proof of *self-similar* property:
$$\left\{\mathsf{f} \in \mathsf{V}_j \iff \mathbf{D}\mathsf{f} \in \mathsf{V}_{j+1}\right\} \iff \mathsf{f} \in \mathsf{span}\{\mathbf{T}^n\phi\} \iff \mathbf{D}\mathsf{f} \in \mathsf{span}\{\mathbf{D}\mathbf{T}^n\phi\} \qquad \iff (2)$$
5. Proof for *Riesz basis*: by (1) and Proposition 2.2 page 21.

✏

2.7 Support size

The *support* of a function is what it's non-zero part "sits" on. If the support of the scaling coefficients $(\!(\mathsf{h}_n)\!)$ goes from say $[0,3]$ in $\mathbb{Z}$, what is the support of the scaling function $\phi(x)$? The answer is $[0,3]$ in $\mathbb{R}$—essentially the same as the support of $(\!(\mathsf{h}_n)\!)$ except that the two functions have different domains ($\mathbb{Z}$ versus $\mathbb{R}$). This concept is defined in Definition 2.4 (next definition), and proven in Theorem 2.7 (next theorem).

Definition 2.4. *Let* $\left(\mathsf{L}^2_{\mathbb{R}},\ (\!(\mathsf{V}_j)\!),\ \phi,\ (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27). Let* X^- *represent the* CLOSURE *of a set* X *in* $\mathsf{L}^2_{\mathbb{R}}$, $\vee X$ *the* LEAST UPPER BOUND *of an* ORDERED SET $(X,\leq)$, $\wedge X$ *the* GREATEST LOWER BOUND *of* $(X,\leq)$, *and*

[13] Wojtaszczyk (1997) page 28 ⟨Theorem 2.13⟩, Pinsky (2002) page 313 ⟨Theorem 6.4.27⟩

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$$\lfloor x \rfloor \triangleq \bigvee \{n \in \mathbb{Z} | n \le x\} \quad \forall x \in \mathbb{R} \quad \text{(FLOOR of } x\text{)}$$
$$\lceil x \rceil \triangleq \bigwedge \{n \in \mathbb{Z} | n \ge x\} \quad \forall x \in \mathbb{R} \quad \text{(CEILING of } x\text{)}.$$

DEF *The set* $\mathsf{supp f}$ *of a function* $\mathsf{f} \in Y^X$ *is the* ***support*** *of* f *if*

$$\mathsf{supp f} \triangleq \begin{cases} \{x \in \mathbb{R} | \mathsf{f}(x) \neq 0\}^- & \text{for } X = \mathbb{R} \quad \text{(domain of } \mathsf{f} \text{ is } \mathbb{R}) \quad \text{and} \\ \left\{x \in \mathbb{R} | \mathsf{f}(\lfloor x \rfloor) \neq 0 \text{ and } \mathsf{f}(\lceil x \rceil) \neq 0\right\}^- & \text{for } X = \mathbb{Z} \quad \text{(domain of } \mathsf{f} \text{ is } \mathbb{Z}) \quad . \end{cases}$$

Theorem 2.7 (support size). [14] *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, \langle\!\langle \mathbf{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27)*. *Let* $\mathsf{supp f}$ *be the support of a function* f *(Definition 2.4 page 33)*.

THM $\mathsf{supp}\phi = \mathsf{supp h}$

✎PROOF:

1. Definitions: $\mathsf{supp}\phi \triangleq [a, b]$
   $\mathsf{supp h} \triangleq [k, m]$.

2. lemma: $\mathsf{supp}\phi(x) = [a, b] \iff \mathsf{supp}\phi(2x) = \left[\frac{a}{2}, \frac{b}{2}\right]$

3. lemma: $\mathsf{supp}[\lambda\phi(x)] = \mathsf{supp}[\phi(x)] \quad \forall \lambda \in \mathbb{R}\backslash 0$

4. Proof that $k = a$:

$$\begin{aligned}
a &= \bigwedge \mathsf{supp}\phi(x) && \text{by definition of } a && \text{(item (1) page 34)} \\
&\triangleq \bigwedge \mathsf{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \phi(x)\right] && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 24)} \\
&= \bigwedge \mathsf{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 4)} \\
&= \bigwedge \mathsf{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by (3) lemma} \\
&= \bigwedge \mathsf{supp}\left[\mathsf{h}_k \phi(2x-k)\right] && \text{because } n = k \text{ is the } \textit{least value} \text{ of } n \text{ for which } \mathsf{h}_n \neq 0 \\
&= \bigwedge \mathsf{supp}[\phi(2x-k)] && \text{by (3) lemma} \\
&= \bigwedge \mathsf{supp}\left[\phi\left(2\left[x - \frac{k}{2}\right]\right)\right] \\
&= \bigwedge \left\{t | \phi\left(2\left[x - \frac{k}{2}\right]\right) \neq 0\right\} && \text{by definition of } \mathsf{supp} && \text{(Definition 2.4 page 33)} \\
&= x \quad \text{such that} \quad x - \frac{k}{2} = \frac{a}{2} && \text{by (2) lemma} \\
&= \frac{k}{2} + \frac{a}{2} \\
&\implies && \frac{k}{2} = a - \frac{a}{2} \\
&\iff && k = a
\end{aligned}$$

[14] Mallat (1999) pages 243–244

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5. Proof that $m = b$:

$$
\begin{aligned}
b &= \bigvee \operatorname{supp}\phi(x) && \text{by definition of } b && \text{(item (1) page 34)}\\
&\triangleq \bigvee \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n\phi(x)\right] && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 24)}\\
&= \bigvee \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&= \bigvee \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by (3) lemma}\\
&= \bigvee \operatorname{supp}\left[\mathsf{h}_m \phi(2x-m)\right] && \text{because } n = m \text{ is the } \textit{greatest value} \text{ of } n \text{ for which } \mathsf{h}_n \neq 0\\
&= \bigvee \operatorname{supp}[\phi(2x-m)] && \text{by (3) lemma}\\
&= \bigvee \operatorname{supp}\left[\phi\left(2\left[x-\frac{m}{2}\right]\right)\right]\\
&= \bigvee \left\{t \,|\, \phi\left(2\left[x-\frac{m}{2}\right]\right) \neq 0\right\} && \text{by definition of } \operatorname{supp} && \text{(Definition 2.4 page 33)}\\
&= x \quad \text{such that} \quad x - \frac{m}{2} = \frac{b}{2} && \text{by (2) lemma}\\
&= \frac{m}{2} + \frac{b}{2}\\
&\Longrightarrow \qquad \frac{m}{2} = b - \frac{b}{2}\\
&\Longleftrightarrow \qquad m = b
\end{aligned}
$$

✏

Example 2.2.

EX

B-spline $\mathsf{N}_1(x)$ (Example O.2 page 338) | *B-spline* $\mathsf{N}_2(x)$ (Example O.3 page 338) | *Daubechies-3 scaling function* (Example 7.5 page 126)

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CHAPTER 3

WAVELET STRUCTURES

"The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit. The problem is the overall design."

Freeman Dyson (1923–2020), physicist and mathematician [1]

3.1 Introduction

3.1.1 What are wavelets?

In Fourier analysis, *continuous* dilations (Definition 1.3 page 4) of the *complex exponential* (Definition I.5 page 240) form a *basis* (Definition N.7 page 310) for the *space of square integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 199) such that

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}_\omega e^{ix} | \omega \in \mathbb{R}\right\}.$$

In Fourier series analysis (Theorem K.1 page 274), *discrete* dilations of the complex exponential form a basis for $\boldsymbol{L}^2_{\mathbb{R}}(0 : 2\pi)$ such that

$$\boldsymbol{L}^2_{\mathbb{R}}(0 : 2\pi) = \operatorname{span}\left\{\mathbf{D}_j e^{ix} \middle| j \in \mathbb{Z}\right\}.$$

In Wavelet analysis, for some *mother wavelet* (Definition 3.1 page 39) $\psi(x)$,

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}_\omega \mathbf{T}_\tau \psi(x) | \omega, \tau \in \mathbb{R}\right\}.$$

However, the ranges of parameters ω and τ can be much reduced to the countable set $\mathbb{Z}$ resulting in a *dyadic* wavelet basis such that for some mother wavelet $\psi(x)$,

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}^j \mathbf{T}^n \psi(x) | j, n \in \mathbb{Z}\right\}.$$

This text deals almost exclusively with dyadic wavelets. Wavelets that are both *dyadic* and *com-*

[1] quote: Albers and Dyson (1994) page 20
image: http://www.isepp.org/Media/Speaker%20Images/95-96%20Images/dyson.jpg

pactly supported have the attractive feature that they can be easily implemented in hardware or software by use of the *Fast Wavelet Transform* (Figure V.1 page 425).

3.1.2 Analyses

An analysis can be partially characterized by its order structure with respect to an order relation such as the set inclusion relation $\subseteq$. Most transforms have a very simple M-n order structure, as illustrated to the right. The M-n lattices for $n \geq 3$ are *modular* but not *distributive*. Analyses typically have one subspace that is a *scaling* subspace; and this subspace is often simply a family of constants (as is the case with Fourier Analysis).

A special characteristic of wavelet analysis is that there is not just one scaling subspace, but an entire sequence of scaling subspaces. These scaling subspaces are *linearly ordered* with respect to the ordering relation $\subseteq$. In wavelet theory, this structure is called a *multiresolution analysis*, or *MRA* (Definition 2.1 page 18).
The MRA was introduced by Stéphane G. Mallat in 1989. The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan, and later as the *Gaussian Pyramid* by Burt and Adelson in the 1980s in the West.[2]

A second special characteristic of wavelet analysis is that it's order structure with respect to the $\subseteq$ relation is not a simple M-n lattice (as is with the case of Fourier and other analyses). Rather, it is a lattice of the form illustrated to the right. This lattice is *non-complemented*, *non-distributive*, *non-modular*, and *non-Boolean* (Proposition 3.1 page 41).

The wavelet subspace structure is similar in form to that of the *Primorial numbers*,[3] illustrated to the right by a *Hasse diagram*.

An analysis can be represented using three different structures:

① sequence of subspaces
② sequence of basis coefficients
③ sequence of basis vectors

[2] Mallat (1989) page 70, Iijima (1959), Burt and Adelson (1983), Adelson and Burt (1981), Lindeberg (1993), Alvarez et al. (1993), Guichard et al. (2012) pages 23–24 ⟨§3.2.1 Scale-Space Extrema⟩, Guichard et al. (20xx) pages 77–78 ⟨§5.2.1 Scale-Space Extrema⟩, Weickert (1999) ⟨historical survey⟩

[3] Sloane (2014) ⟨http://oeis.org/A002110⟩

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These structures are isomorphic to each other, and can therefore be used interchangeably.

Figure 3.1 (page 48) illustrate the order structures of some analyses, including two wavelet analyses:

3.2 Definition

The term "wavelet" comes from the French word "*ondelette*", meaning "small wave". And in essence, wavelets are "small waves" (as opposed to the "long waves" of Fourier analysis) that form a basis for the Hilbert space $L^2_{\mathbb{R}}$.[5]

Definition 3.1. [6] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 1.3 page 4.*

DEF

A function $\psi(x)$ *in* $L^2_{\mathbb{R}}$ *is a* ***wavelet function*** *for* $L^2_{\mathbb{R}}$ *if*

$$\left\{\mathbf{D}^j\mathbf{T}^n\psi\,|_{j,n\in\mathbb{Z}}\right\} \text{ is a RIESZ BASIS for } L^2_{\mathbb{R}}.$$

In this case, ψ *is also called the* ***mother wavelet*** *of the basis* $\left\{\mathbf{D}^j\mathbf{T}^n\psi\,|_{j,n\in\mathbb{Z}}\right\}$. *The sequence of subspaces* $(\!(W_j)\!)_{j\in\mathbb{Z}}$ *is the* ***wavelet analysis*** *induced by* ψ, *where each subspace* W_j *is defined as*

$$W_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\psi\,\middle|\,n\in\mathbb{Z}\right\}.$$

A *wavelet analysis* $(\!(W_j)\!)$ is often constructed from a *multiresolution anaysis* (Definition 2.1 page 18) $(\!(V_j)\!)$ under the relationship

$$V_{j+1} = V_j \hat{+} W_j, \qquad \text{where } \hat{+} \text{ is subspace addition (\textit{Minkowski addition})}.$$

By this relationship alone, $(\!(W_j)\!)$ is in no way uniquely defined in terms of a multiresolution analysis $(\!(V_j)\!)$. In general there are many possible complements of a subspace V_j. To uniquely define such a wavelet subspace, one or more additional constraints are required. One of the most common additional constraints is *orthogonality*, such that V_j and W_j are orthogonal to each other (see CHAPTER 6 page 81).

3.3 Dilation equation

Suppose $(\!(\mathbf{T}^n\psi)\!)_{n\in\mathbb{Z}}$ is a basis for W_0. By Definition 3.1 page 39, the wavelet subspace W_0 is contained in the scaling subspace V_1. By Definition 2.1 page 18, the sequence $(\!(\mathbf{DT}^n\phi)\!)_{n\in\mathbb{Z}}$ is a basis for V_1. Because W_0 is contained in V_1, the sequence $(\!(\mathbf{DT}^n\phi)\!)_{n\in\mathbb{Z}}$ is also a basis for W_0.

Theorem 3.1 (wavelet dilation equation)**.** *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 2.3 page 27) and* $(\!(W_j)\!)_{j\in\mathbb{Z}}$ *be a* WAVELET ANALYSIS *(Definition 3.1 page 39) with respect to*

[5] Strang and Nguyen (1996) page ix, Atkinson and Han (2009) page 191

[6] Wojtaszczyk (1997) page 17 ⟨Definition 2.1⟩

$\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \phi, \left(\!\left(h_n \right)\!\right)\right)$ *and with* WAVELET FUNCTION ψ *(Definition 3.1 page 39)*.

THM $\exists \left(\!\left(g_n \right)\!\right)_{n\in\mathbb{Z}}$ such that $\psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{DT}^n \phi$

✎PROOF:

$$\begin{aligned}
\psi &\in W_0 && \text{by Definition 3.1 page 39}\\
&\subseteq V_1 && \text{by Definition 3.1 page 39}\\
&= \operatorname{span}\left(\!\left(\mathbf{DT}^n \phi(x) \right)\!\right)_{n\in\mathbb{Z}} && \text{by Definition 2.1 page 18 (MRA)}\\
&\implies \exists \left(\!\left(g_n \right)\!\right)_{n\in\mathbb{Z}} \quad \text{such that} \quad \psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{DT}^n \phi
\end{aligned}$$

✏

A *wavelet system* (next definition) consists of two subspace sequences:

- A **multiresolution analysis** $\left(\!\left(V_j \right)\!\right)$ (Definition 2.1 page 18) provides "coarse" approximations of a function in $L^2_{\mathbb{R}}$ at different "scales" or resolutions.
- A **wavelet analysis** $\left(\!\left(W_j \right)\!\right)$ provides the "detail" of the function missing from the approximation provided by a given scaling subspace (Definition 3.1 page 39).

Definition 3.2. *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \phi, \left(\!\left(h_n \right)\!\right)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 2.1 page 18) and* $\left(\!\left(W_j \right)\!\right)_{j\in\mathbb{Z}}$ *a wavelet analysis (Definition 3.1 page 39) with respect to* $\left(\!\left(V_j \right)\!\right)_{j\in\mathbb{Z}}$. *Let* $\left(\!\left(g_n \right)\!\right)_{n\in\mathbb{Z}}$ *be a sequence of coefficients.*

DEF *A* ***wavelet system*** *is the tuple* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \left(\!\left(W_j \right)\!\right), \phi, \psi, \left(\!\left(h_n \right)\!\right), \left(\!\left(g_n \right)\!\right)\right)$ *and the sequence* $\left(\!\left(g_n \right)\!\right)_{n\in\mathbb{Z}}$ *that satisfies the equation* $\psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{DT}^n \phi$ *is the* ***wavelet coefficient sequence****.*

Remark 3.1. The pair of coefficient sequences $\left(\left(\!\left(h_n \right)\!\right), \left(\!\left(g_n \right)\!\right)\right)$ generates the scaling function $\phi(x)$ (Definition 2.1 page 18) and the wavelet function $\psi(x)$ (Definition 3.1 page 39). These functions in turn generate the multiresolution analysis $\left(\!\left(V_j \right)\!\right)$ (Definition 2.1 page 18) and the wavelet analysis $\left(\!\left(W_j \right)\!\right)$ (Definition 3.1 page 39). Therefore, the coefficient sequence pair $\left(\left(\!\left(h_n \right)\!\right), \left(\!\left(g_n \right)\!\right)\right)$ totally defines a wavelet system $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \left(\!\left(W_j \right)\!\right), \phi, \psi, \left(\!\left(h_n \right)\!\right), \left(\!\left(g_n \right)\!\right)\right)$ (Definition 3.2 page 40).

Furthermore, especially in the case of orthonormal wavelets, the wavelet coefficient sequence $\left(\!\left(g_n \right)\!\right)_{n\in\mathbb{Z}}$ is often defined in terms of the scaling coefficient sequence $\left(\!\left(h_n \right)\!\right)_{n\in\mathbb{Z}}$ in a very simple and straightforward manner. Therefore, in the case of an orthonormal wavelet system, the coefficient scaling sequence $\left(\!\left(h_n \right)\!\right)_{n\in\mathbb{Z}}$ often totally defines the entire wavelet system. And in this case, designing a wavelet system is only a matter of finding a handful of scaling coefficients $\left(\!\left(h_1, h_2, \ldots, h_n \right)\!\right)$…because once you have these, you can generate everything else.

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3.4 Order structure

The *wavelet system* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ (Definition 3.2 page 40) together with the set inclusion relation $\subseteq$ forms an *ordered set*, illustrated to the right by a *Hasse diagram*.

Proposition 3.1. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system with order relation* $\subseteq$. *The lattice* $\boldsymbol{L} \triangleq \left(\left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \vee, \wedge; \subseteq\right)$ *has the following properties:*

PRP

1. $\boldsymbol{L}$ *is* NONDISTRIBUTIVE.
2. $\boldsymbol{L}$ *is* NONMODULAR.
3. $\boldsymbol{L}$ *is* NONCOMPLEMENTED.
4. $\boldsymbol{L}$ *is* NONBOOLEAN.

✎PROOF:

1. Proof that $\boldsymbol{L}$ is *nondistributive*:
   (a) $\boldsymbol{L}$ contains the $N5$ lattice.
   (b) Because $\boldsymbol{L}$ contains the $N5$ lattice, $\boldsymbol{L}$ is *nondistributive*.
2. Proof that $\boldsymbol{L}$ is *nonmodular* and *nondistributive*:
   (a) $\boldsymbol{L}$ contains the $N5$ lattice.
   (b) Because $\boldsymbol{L}$ contains the $N5$ lattice, $\boldsymbol{L}$ is *nonmodular*.
3. Proof that $\boldsymbol{L}$ is *noncomplemented*:
$$\begin{aligned} x' &= y' = v' = z \\ z' &= \{x, y, v\} \\ x'' &= \left(x'\right)' \\ &= z' \\ &= \{x, y, v\} \\ &\neq x \end{aligned}$$

4. Proof that $\boldsymbol{L}$ is *nonBoolean*:
   (a) $\boldsymbol{L}$ is *nondistributive* (item (1)).
   (b) Because $\boldsymbol{L}$ is *nondistributive*, it is *nonBoolean*.

✏

3.5 Subspace algebraic structure

Theorem 3.2. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40). Let* $\boldsymbol{V}_1 \hat{+} \boldsymbol{V}_2$ *represent* MINKOWSKI ADDITION *of two subspaces* $\boldsymbol{V}_1$ *and* $\boldsymbol{V}_2$ *of a Hilbert space* $\boldsymbol{H}$.

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THM

$$\begin{aligned} L^2_{\mathbb{R}} &= \lim_{j\to\infty} V_j && \text{(}L^2_{\mathbb{R}}\text{ is equivalent to one very large scaling subspace)} \\ &= V_j \dotplus W_j \dotplus W_{j+1} \dotplus W_{j+2} \dotplus \cdots && \left(\begin{array}{l} L^2_{\mathbb{R}}\text{ is equivalent to one scaling space} \\ \text{and a sequence of wavelet subspaces} \end{array}\right) \\ &= \cdots \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus \cdots && \text{(}L^2_{\mathbb{R}}\text{ is equivalent to a sequence of wavelet subspaces)} \end{aligned}$$

✎PROOF:

1. Proof for (1):

$$L^2_{\mathbb{R}} = \lim_{j\to\infty} V_j \qquad \text{by Definition 2.1 page 18}$$

2. Proof for (2):

$$\begin{aligned} \underbrace{V_j \dotplus W_j}_{V_{j+1}} \dotplus W_{j+1} \dotplus W_{j+2} \dotplus \cdots &= \underbrace{V_{j+1} \dotplus W_{j+1}}_{V_{j+2}} \dotplus W_{j+2} \dotplus W_{j+3} \dotplus \cdots \\ &= \underbrace{V_{j+2} \dotplus W_{j+2}}_{V_{j+3}} \dotplus W_{j+3} \dotplus W_{j+4} \dotplus \cdots \\ &= \underbrace{V_{j+3} \dotplus W_{j+3}}_{V_{j+4}} \dotplus W_{j+4} \dotplus W_{j+5} \dotplus \cdots \\ &= \underbrace{V_{j+5} \dotplus W_{j+5}}_{V_{j+5}} \dotplus W_{j+6} \dotplus W_{j+6} \dotplus \cdots \\ &= \lim_{j\to\infty} V_{j+5} \dotplus W_{j+5} \dotplus W_{j+6} \dotplus W_{j+6} \dotplus \cdots \\ &= L^2_{\mathbb{R}} \end{aligned}$$

3. Proof for (3):

$$\begin{aligned} L^2_{\mathbb{R}} &= \underbrace{V_0}_{V_{-1} \dotplus W_{-1}} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots && \text{by (2)} \\ &= \underbrace{V_{-1}}_{V_{-2} \dotplus W_{-2}} W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\ &= \underbrace{V_{-2}}_{V_{-3} \dotplus W_{-3}} W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\ &= \underbrace{V_{-3}}_{V_{-4} \dotplus W_{-4}} W_{-3} \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\ &\vdots \\ &= \cdots \dotplus W_{-3} \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \end{aligned}$$

✏

Remark 3.2. In the special case that two subspaces W_1 and W_2 are *orthogonal* to each other, then the *subspace addition* operation $W_1 \dotplus W_2$ is frequently expressed as $W_1 \oplus W_2$. In the case of an *orthonormal wavelet system* (Definition 6.1 page 81), the expressions in Theorem 3.2 (page 41) could be expressed as

$$\begin{aligned} L^2_{\mathbb{R}} &= \lim_{j\to\infty} V_j \\ &= V_j \oplus W_j \oplus W_{j+1} \oplus W_{j+2} \oplus \cdots \\ &= \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots . \end{aligned}$$

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3.6 Necessary conditions

Theorem 3.3 (quadrature conditions in "time"). *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system* (Definition 3.2 page 40).

THM

1. $$\sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \phi \,|\, \mathbf{T}^n\phi \right\rangle \quad \forall n\in\mathbb{Z}$$
2. $$\sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \psi \,|\, \mathbf{T}^n\psi \right\rangle \quad \forall n\in\mathbb{Z}$$
3. $$\sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \phi \,|\, \mathbf{T}^n\psi \right\rangle \quad \forall n\in\mathbb{Z}$$

✎PROOF:

1. Proof for (1): by Theorem 2.4 page 31.
2. Proof for (2):

$$\begin{aligned}
\langle \psi \,|\, \mathbf{T}^n\psi\rangle &= \left\langle \sum_{m\in\mathbb{Z}} g_m \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} g_k \mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by } \textit{wavelet dilation equation} && \text{(Theorem 3.1 page 39)}\\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by properties of } \langle\triangle\,|\,\triangledown\rangle && \text{(Definition D.9 page 182)}\\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by def. of operator adjoint} && \text{(Proposition D.3 page 183)}\\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by Proposition 1.5 page 7}\\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{*m}\mathbf{D}^*\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by operator star-algebra prop.} && \text{(Theorem D.13 page 184)}\\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{-m}\mathbf{D}^{-1}\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by Proposition 1.7 page 8}\\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
&\langle \phi \,|\, \mathbf{T}^n\psi\rangle\\
&= \left\langle \sum_{m\in\mathbb{Z}} h_m \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} g_k \mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by Theorem 2.1 page 24} && \text{and Theorem 3.1 page 39}\\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by properties of } \langle\triangle\,|\,\triangledown\rangle && \text{(Definition D.9 page 182)}\\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \mathbf{T}^n\mathbf{D}\mathbf{T}^k\phi \right\rangle && \text{by definition of operator adjoint} && \text{(Proposition D.3 page 183)}\\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by Proposition 1.5 page 7}\\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{*m}\mathbf{D}^*\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by operator star-algebra properties} && \text{(Theorem D.13 page 184)}\\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{-m}\mathbf{D}^{-1}\mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k\phi \right\rangle && \text{by Proposition 1.7 page 8}\\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle
\end{aligned}$$

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✏

Proposition 3.2. *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, \left(\!\left(\mathbf{V}_j\right)\!\right), \left(\!\left(\mathbf{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system. Let* $\tilde{\phi}(\omega)$ *and* $\tilde{\psi}(\omega)$ *be the* FOURIER TRANSFORMS *(Definition M.2 page 294) of* $\phi(x)$ *and* $\psi(x)$*, respectively. Let* $\breve{\mathsf{g}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition R.1 page 393) of* $\left(\!\left(g_n\right)\!\right)$.

PRP

$$\tilde{\psi}(\omega) = \frac{\sqrt{2}}{2}\breve{\mathsf{g}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$$

✎PROOF:

$$\begin{aligned}
\tilde{\psi}(\omega) &\triangleq \tilde{\mathbf{F}}\psi \\
&= \tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} g_n\mathbf{D}\mathbf{T}^n\phi && \text{by } \textit{wavelet dilation equation} && \text{(Theorem 3.1 page 39)} \\
&= \sum_{n\in\mathbb{Z}} g_n\tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n\phi \\
&= \sum_{n\in\mathbb{Z}} g_n\mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{T}^n\phi && \text{by Corollary 1.3 page 12} \\
&= \sum_{n\in\mathbb{Z}} g_n\mathbf{D}^{-1}e^{-i\omega n}\tilde{\mathbf{F}}\phi && \text{by Corollary 1.3 page 12} \\
&= \sum_{n\in\mathbb{Z}} g_n\sqrt{2}\left(\mathbf{D}^{-1}e^{-i\omega n}\right)\left(\mathbf{D}^{-1}\tilde{\mathbf{F}}\phi\right) && \text{by Proposition 1.4 page 6} \\
&= \sqrt{2}\left(\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}} g_n e^{-i\omega n}\right)\left(\mathbf{D}^{-1}\tilde{\mathbf{F}}\phi\right) \\
&= \sqrt{2}\left(\mathbf{D}^{-1}\breve{\mathbf{F}}\left(\!\left(g_n\right)\!\right)\right)\left(\mathbf{D}^{-1}\tilde{\mathbf{F}}\phi\right) && \text{by definition of } \breve{\mathbf{F}} && \text{(Definition R.1 page 393)} \\
&= \sqrt{2}\frac{\sqrt{2}}{2}\breve{\mathsf{g}}\left(\frac{\omega}{2}\right)\frac{\sqrt{2}}{2}\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by property of } \mathbf{D} && \text{(Proposition 1.2 page 5)} \\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{g}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)
\end{aligned}$$

✏

Theorem 3.4 (next) presents the *quadrature* necessary conditions of a wavelet system. These relations simplify dramatically in the special case of an *orthonormal wavelet system* (Theorem R.4 page 399).

Theorem 3.4 (Quadrature conditions in "frequency"). [7] *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, \left(\!\left(\mathbf{V}_j\right)\!\right), \left(\!\left(\mathbf{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system. Let* $\breve{\mathsf{x}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition R.1 page 393) for a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$. *Let* $\tilde{\mathsf{S}}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM *(Definition T.3 page 411) of* ϕ, $\tilde{\mathsf{S}}_{\psi\psi}(\omega)$ *be the* AUTO-POWER SPECTRUM *of* ψ, *and* $\tilde{\mathsf{S}}_{\phi\psi}(\omega)$ *be the* CROSS-POWER SPECTRUM *of* ϕ *and* ψ.

THM

1. $\left|\breve{\mathsf{h}}(\omega)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)$
2. $\left|\breve{\mathsf{g}}(\omega)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\breve{\mathsf{g}}(\omega+\pi)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\psi\psi}(2\omega)$
3. $\breve{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega)\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \breve{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\phi\psi}(2\omega)$

✎PROOF:

1. Proof for (1): by Theorem 2.5 page 32.

[7] Chui (1992) page 135, Goswami and Chan (1999) page 110

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2. Proof for (2):

$$
\begin{aligned}
2\tilde{S}_{\psi\psi}(2\omega) &\triangleq 2(2\pi)\sum_{n\in\mathbb{Z}}|\tilde{\psi}(2\omega+2\pi n)|^2\\
&= 2(2\pi)\sum_{n\in\mathbb{Z}}\left|\frac{\sqrt{2}}{2}\breve{g}\left(\frac{2\omega+2\pi n}{2}\right)\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \qquad \text{by Lemma 2.1 page 25}\\
&= 2\pi\sum_{n\in\mathbb{Z}_e}\left|\breve{g}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 +\\
&\qquad 2\pi\sum_{n\in\mathbb{Z}_o}\left|\breve{g}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\\
&= 2\pi\sum_{n\in\mathbb{Z}}|\breve{g}(\omega+2\pi n)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}|\breve{g}(\omega+2\pi n+\pi)|^2\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2\\
&= 2\pi\sum_{n\in\mathbb{Z}}|\breve{g}(\omega)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}|\breve{g}(\omega+\pi)|^2\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2\\
&= |\breve{g}(\omega)|^2\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2+\right)|\breve{g}(\omega+\pi)|^2\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right)\\
&= |\breve{g}(\omega)|^2\tilde{S}_{\phi\phi}(\omega) + |\breve{g}(\omega+\pi)|^2\tilde{S}_{\phi\phi}(\omega+\pi) \qquad \text{by Theorem T.1 page 411}
\end{aligned}
$$

3. Proof for (3):

$$
\begin{aligned}
2\tilde{S}_{\phi\psi}(2\omega) &= 2(2\pi)\sum_{n\in\mathbb{Z}}\tilde{\phi}(2\omega+2\pi n)\tilde{\psi}^*(2\omega+2\pi n)\\
&= 2(2\pi)\sum_{n\in\mathbb{Z}}\frac{\sqrt{2}}{2}\breve{h}(\omega+\pi n)\,\tilde{\phi}(\omega+\pi n)\,\frac{\sqrt{2}}{2}\breve{g}^*(\omega+\pi n)\,\tilde{\phi}^*(\omega+\pi n) \qquad \text{by Lemma 2.1 page 25}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{h}(\omega+\pi n)\,\breve{g}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2\\
&= 2\pi\sum_{n\in\mathbb{Z}_o}\breve{h}(\omega+\pi n)\,\breve{g}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2\\
&\qquad + 2\pi\sum_{n\in\mathbb{Z}_e}\breve{h}(\omega+\pi n)\,\breve{g}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2\\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{h}(\omega+2\pi n+\pi)\,\breve{g}^*(\omega+2\pi n+\pi)\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2\\
&\qquad + 2\pi\sum_{n\in\mathbb{Z}}\breve{h}(\omega+2\pi n)\,\breve{g}^*(\omega+2\pi n)\left|\tilde{\phi}(\omega+2\pi n)\right|^2\\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{h}(\omega+\pi)\,\breve{g}^*(\omega+\pi)\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}\breve{h}(\omega)\,\breve{g}^*(\omega)\left|\tilde{\phi}(\omega+2\pi n)\right|^2\\
&= \breve{h}(\omega)\,\breve{g}^*(\omega)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2\right)\\
&\qquad + \breve{h}(\omega+\pi)\,\breve{g}^*(\omega+\pi)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right)\\
&= \breve{h}(\omega)\breve{g}^*(\omega)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2\right) + \breve{h}(\omega+\pi)\breve{g}^*(\omega+\pi)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right)\\
&= \breve{h}(\omega)\breve{g}^*(\omega)\tilde{S}_{\phi\phi}(\omega) + \breve{h}(\omega+\pi)\breve{g}^*(\omega+\pi)\tilde{S}_{\phi\phi}(\omega+\pi) \qquad \text{by Theorem T.1 page 411}
\end{aligned}
$$

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3.7 Sufficient condition

In this text, an often used sufficient condition for designing the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 3.2 page 40) is the *conjugate quadrature filter condition* (Definition Q.6 page 386). It expresses the sequence $(\!(g_n)\!)$ in terms of the *scaling coefficient sequence* (Definition 2.3 page 27) and a "shift" integer N as $g_n = \pm(-1)^n h^*_{N-n}$. The *CQF condition* has the following "nice" properties:

1. Given a *scaling coefficient sequence* $(\!(h_n)\!)$ (Definition 2.3 page 27), it is extremely simple to compute the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 3.2 page 40).
2. If $\{\mathbf{T}\phi\}$ of a *wavelet system* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ (Definition 3.2 page 40) is *orthonormal* and $\left((\!(g_n)\!), (\!(h_n)\!), N\right)$ satisfies the *CQF condition*, then $\left\{\mathbf{T}^n\psi\right\}$ is also *orthnormal* (Theorem 6.5 page 103).
3. If $\{\mathbf{T}\phi\}$ of a *wavelet system* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ (Definition 3.2 page 40) is *orthonormal* and $\left((\!(g_n)\!), (\!(h_n)\!), N\right)$ satisfies the *CQF condition*, then the *wavelet subspace* W_0 is *orthnormal* to the *scaling subspace* V_0 ($W_0 \perp V_0$) (Theorem 6.5 page 103).

Theorem 3.5. *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40). Let* $\breve{\mathsf{g}}(\omega)$ *be the* DTFT *(Definition R.1 page 393) and* $\breve{\mathsf{g}}(z)$ *the* Z-TRANSFORM *(Definition Q.1 page 379) of* $(\!(g_n)\!)$.

THM

$$\underbrace{g_n = \pm(-1)^n h^*_{N-n},\ N\in\mathbb{Z}}_{\text{CONJUGATE QUADRATURE FILTER}} \iff \breve{\mathsf{g}}(\omega) = \pm(-1)^N e^{-i\omega N}\breve{\mathsf{h}}^*(\omega+\pi)\Big|_{\omega=\pi} \qquad (1)$$

$$\implies \sum_{n\in\mathbb{Z}}(-1)^n g_n = \sqrt{2} \qquad (2)$$

$$\iff \breve{\mathsf{g}}(z)\Big|_{z=-1} = \sqrt{2} \qquad (3)$$

$$\iff \breve{\mathsf{g}}(\omega)\Big|_{\omega=\pi} = \sqrt{2} \qquad (4)$$

✎PROOF:

1. Proof that CQF $\iff$ (1): by Theorem Q.5 page 387

2. Proof that CQF $\implies$ (4):

$$\begin{aligned}
\breve{\mathsf{g}}(\pi) &= \breve{\mathsf{g}}(\omega)\Big|_{\omega=\pi} \\
&= \pm(-1)^N e^{-i\omega N}\breve{\mathsf{h}}^*(\omega+\pi)\Big|_{\omega=\pi} && \text{by } \textit{CQF theorem} && \text{(Theorem Q.5 page 387)} \\
&= \pm(-1)^N e^{-i\pi N}\breve{\mathsf{h}}^*(2\pi) \\
&= \pm(-1)^N(-1)^N\breve{\mathsf{h}}^*(0) && \text{by } \textit{DTFT periodicity} && \text{(Proposition R.1 page 393)} \\
&= \sqrt{2} && \text{by } \textit{admissibility condition} && \text{(Theorem 2.3 page 29)}
\end{aligned}$$

3. Proof that (2) $\iff$ (3) $\iff$ (4): by Proposition R.4 page 396

✏

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3.8 Support size

Theorem 3.6 (support size). [8] *Let* $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j \rangle\!\rangle, \langle\!\langle W_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40) induced by the* CQF CONDITIONS *(Theorem 3.5 page 46). Let* $\mathrm{supp}\,\mathsf{f}$ *be the support of a function* f *(Definition 2.4 page 33).*

THM

$$\begin{aligned} \mathrm{supp}\phi &= \mathrm{supp}\,\mathsf{h} \\ \mathrm{supp}\psi &= \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right] \end{aligned}$$

✎PROOF:

1. Proof that $\mathrm{supp}\phi = \mathrm{supp}\,\mathsf{h}$: by Theorem 2.7 (page 34)
2. Proof that $\mathrm{supp}\psi = \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right]$:

$$\begin{aligned}
\mathrm{supp}\psi(x) &= \mathrm{supp}\left[\sum_{n\in\mathbb{Z}} g_n \mathbf{D}\mathbf{T}^n\phi(x)\right] && \text{by } \textit{wavelet dilation equation} && \text{(Theorem 3.1 page 39)} \\
&= \mathrm{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} g_n\phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 4)} \\
&= \mathrm{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \pm(-1)^N \mathsf{h}(N-n)\phi(2x-n)\right] && \text{by } \textit{CQF conditions} && \text{(Theorem 3.5 page 46)} \\
&= \mathrm{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}(N-n)\phi(2x-n)\right] && \text{by (3) lemma (page 34)} \\
&= \left\{x\in\mathbb{R} \,\middle|\, \sum_{n\in\mathbb{Z}} \mathsf{h}(N-n)\phi(2x-n) \neq 0\right\}^{-} && \text{by definition of supp} && \text{(Definition 2.4 page 33)} \\
&= \left[\frac{n_1}{2}+\frac{N-n_2}{2} : \frac{n_2}{2}+\frac{N-n_1}{2}\right] \\
&= \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right]
\end{aligned}$$

✏

Example 3.1. Figure 3.2 (page 49) illustrates some *Daubechies-p wavelet functions* with *p vanishing moments*. Note the following (Theorem 3.5 page 46):

$$\begin{aligned}
p=1 \quad \sum_{n=0}^{n=1} g_n &= \frac{1}{2}\left(\sqrt{2}-\left(-\sqrt{2}\right)\right) &&= \sqrt{2} \\
p=2 \quad \sum_{n=0}^{n=3} g_n &= \frac{\sqrt{2}}{8}\left((1-\sqrt{3})+(3-\sqrt{3})+(3+\sqrt{3})+(1+\sqrt{3})\right) &&= \sqrt{2}
\end{aligned}$$

[8] Mallat (1999) pages 243–244

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Figure 3.1: examples of the order structures of some analyses

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Figure 3.2: *Daubechies-p wavelet functions* with *p vanishing moments* (Example 3.1 page 47)

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Part II

Design Constraints

CHAPTER 4

PARTITION OF UNITY CONSTRAINT

4.1 Definition and motivation

Figure 4.1: Implications of scaling function properties

A very common property of scaling functions (Definition 2.1 page 18) is the *partition of unity* property (Definition S.1 page 403). The partition of unity is a kind of generalization of *orthonormality*; that is, *all* orthonormal scaling functions form a partition of unity (Theorem 6.1 page 87). But the partition of unity property is not just a consequence of orthonormality, but also a generalization of orthonormality, in that if you remove the orthonormality constraint, the partition of unity is still a reasonable constraint in and of itself.

There are two reasons why the partition of unity property is a reasonable constraint on its own:

- Without a partition of unity, it is difficult to represent a function as simple as a constant.[1]
- For a multiresolution system $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$, the partition of unity property is equivalent to $\sum_{n\in\mathbb{Z}}(-1)^n h_n = 0$ (Theorem 4.1 page 54). As viewed from the perspective of discrete time signal processing (APPENDIX Q page 379), this implies that the scaling coefficients form a "*lowpass filter*"; lowpass filters provide a kind of "coarse approximation" of a function. And that is what the scaling function is "supposed" to do—to provide a coarse approximation at some resolution or "scale" (Definition 2.1 page 18).

[1] Jawerth and Sweldens (1994) page 8

Definition 4.1. [2]

DEF

A function $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ *forms a* ***partition of unity*** *if*

$$\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) = 1 \qquad \forall x \in \mathbb{R}.$$

4.2 Scaling functions with partition of unity

The Z transform (Definition Q.1 page 379) of a sequence $(\!(h_n)\!)$ with sum $\sum_{n\in\mathbb{Z}}(-1)^n \mathsf{h}_n = 0$ has a zero at $z = -1$. Somewhat surprisingly, the *partition of unity* and *zero at* $z = -1$ properties are actually equivalent (next theorem).

Theorem 4.1. [3] *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, (\!(\mathbf{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 2.3 page 27). Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f} \in \mathbf{L}^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION *(Definition N.12 page 316). Let* c *be some contant in* $\mathbb{R}\backslash 0$.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n \phi = c}_{\text{(1) PARTITION OF UNITY}} \iff \underbrace{\sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n = 0}_{\text{(2) ZERO AT } z=-1} \iff \underbrace{\sum_{n\in\mathbb{Z}} \mathsf{h}_{2n} = \sum_{n\in\mathbb{Z}} \mathsf{h}_{2n+1} = \frac{\sqrt{2}}{2}}_{\text{(3) sum of even = sum of odd = } \frac{\sqrt{2}}{2}}$$

✎PROOF: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

1. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \phi &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \left[\sum_{m\in\mathbb{Z}} \mathsf{h}_m \mathbf{D}\mathbf{T}^m \phi\right] && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 24)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathbf{D}\mathbf{T}^m \phi \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^m \phi && \text{by } \textit{commutator relation} && \text{(Proposition 1.5 page 7)}\\
&= \mathbf{D} \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{T}^{2n}\mathbf{T}^m \phi \\
&= \mathbf{D} \sum_{m\in\mathbb{Z}} \mathsf{h}_m \left[\sqrt{\frac{2\pi}{2}} \hat{\mathbf{F}}^{-1}\mathbf{S}_2 \tilde{\mathbf{F}}(\mathbf{T}^m \phi)\right] && \text{by } \textit{PSF} && \text{(Theorem 1.3 page 13)}\\
&= \sqrt{\pi}\mathbf{D} \sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1}\mathbf{S}_2 e^{-i\omega m} \tilde{\mathbf{F}}\phi && \text{by Corollary 1.3 page 12}
\end{aligned}$$

[2] Kelley (1955) page 171, Munkres (2000) page 225, Jänich (1984) page 116, Willard (1970) page 152 ⟨item 20C⟩, Willard (2004) page 152 ⟨item 20C⟩

[3] Jawerth and Sweldens (1994) page 8, Chui (1992) page 123

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$$= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1} e^{-i\frac{2\pi}{2}km}\mathbf{S}_2\tilde{\mathbf{F}}\phi \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition 1.4 page 13)}$$

$$= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1} (-1)^{km}\mathbf{S}_2\tilde{\mathbf{F}}\phi$$

$$= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \left[\frac{\sqrt{2}}{2}\sum_{k\in\mathbb{Z}} (-1)^{km}\left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\frac{2\pi}{2}kx}\right] \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem K.1 page 274)}$$

$$= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}} \left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\sum_{m\in\mathbb{Z}} (-1)^{km}\mathsf{h}_m$$

$$= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e} \left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\sum_{m\in\mathbb{Z}} (-1)^{km}\mathsf{h}_m + \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o} \left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\sum_{m\in\mathbb{Z}} (-1)^{km}\mathsf{h}_m$$

$$= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e} \left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}} \mathsf{h}_m}_{\sqrt{2}} + \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o} \left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}} (-1)^{m}\mathsf{h}_m}_{0}$$

$$= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e} \left(\mathbf{S}_2\tilde{\mathbf{F}}\phi\right)e^{i\pi kx} \qquad \text{by Theorem 2.3 (page 29) and right hypothesis}$$

$$= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e} \tilde{\phi}\left(\frac{2\pi}{2}k\right)e^{i\pi kx} \qquad \text{by definitions of } \tilde{\mathbf{F}} \text{ and } \mathbf{S}_2$$

$$= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}} \tilde{\phi}(2\pi k)e^{i2\pi kx} \qquad \text{by definition of } \mathbb{Z}_e$$

$$= \frac{1}{\sqrt{2}}\mathbf{D}\left\{\sqrt{2\pi}\sum_{k\in\mathbb{Z}} \tilde{\phi}(2\pi k)e^{i2\pi kx}\right\}$$

$$= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n\in\mathbb{Z}} \phi(x+n) \qquad \text{by } PSF \qquad \text{(Theorem 1.3 page 13))}$$

$$= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n} \mathbf{T}^n\phi \qquad \text{by definition of } \mathbf{T} \qquad \text{(Definition 1.3 page 4)}$$

The above equation sequence demonstrates that

$$\mathbf{D}\sum_n \mathbf{T}^n\phi = \sqrt{2}\sum_n \mathbf{T}^n\phi$$

(essentially that $\sum_n \mathbf{T}^n\phi$ is equal to it's own dilation). This implies that $\sum_n \mathbf{T}^n\phi$ is a constant (Proposition 1.8 page 9).

2. Proof that (1) $\Longrightarrow$ (2):

$$c = \sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi \qquad \text{by left hypothesis}$$

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\phi \qquad \text{by } PSF \qquad \text{(Theorem 1.3 page 13)}$$

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\underbrace{\sqrt{2}\left(\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\omega n}\right)\left(\mathbf{D}^{-1}\tilde{\mathbf{F}}\phi\right)}_{\tilde{\mathbf{F}}\phi} \qquad \text{by Lemma 2.1 page 25}$$

$$= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\omega n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) \qquad \text{by Corollary 1.3 page 12}$$

$$= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\frac{\omega}{2}n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) \qquad \text{by evaluation of } \mathbf{D}^{-1} \qquad \text{(Proposition 1.2 page 5)}$$

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$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\frac{2\pi k}{2}n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition 1.4 page 13)}$$

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\right)\left(\mathbf{S}\mathbf{D}^{-1}\mathbf{F}\phi\right)$$

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\right)\left(\mathbf{S}\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{\omega}{2}\right)\right) \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition 1.4 page 13)}$$

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\right)\left(\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{2\pi k}{2}\right)\right)$$

$$= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem K.1 page 274)}$$

$$= \sqrt{\pi}\sum_{k\text{ even}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} + \sqrt{\pi}\sum_{k\text{ odd}}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx}$$

$$= \sqrt{\pi}\sum_{k\text{ even}}\left(\cancelto{\sqrt{2}}{\sum_{n\in\mathbb{Z}} \mathsf{h}_n}\right)\tilde{\phi}(\pi k)e^{i2\pi kx} + \sqrt{\pi}\sum_{k\text{ odd}}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{n}\right)\tilde{\phi}(\pi k)e^{i2\pi kx}$$

$$= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sqrt{2}\,\tilde{\phi}(\pi 2k)e^{i2\pi 2kx} + \sqrt{\pi}\sum_{k\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{n}\right)\tilde{\phi}(\pi[2k+1])e^{i2\pi[2k+1]x} \qquad \text{by Theorem 2.3 page 29}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\tilde{\phi}(0) + \sqrt{\pi}e^{i2\pi x}\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{n}\sum_{k\in\mathbb{Z}}\tilde{\phi}(\pi[2k+1])e^{i4\pi kx} \qquad \text{by left hypothesis and Theorem S.1 page 404}$$

$$\Longrightarrow \qquad \left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^{n}\right) = 0 \qquad \text{because the right side must equal } c$$

3. Proof that (2) $\Longrightarrow$ (3):

$$\sum_{n\in\mathbb{Z}_e} \mathsf{h}_n = \sum_{n\in\mathbb{Z}_o} \mathsf{h}_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \qquad \text{by (2) and Proposition R.4 page 396}$$

$$= \frac{\sqrt{2}}{2} \qquad \text{by } \textit{admissibility condition} \text{ (Theorem 2.3 page 29)}$$

4. Proof that (2) $\Longleftarrow$ (3):

$$\frac{\sqrt{2}}{2} = \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n\mathsf{h}_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n\mathsf{h}_n}_{\text{odd terms}} \qquad \text{by (3)}$$

$$\Longrightarrow \sum_{n\in\mathbb{Z}}(-1)^n\mathsf{h}_n = 0 \qquad \text{by Proposition R.4 page 396}$$

✏

Not every function that forms a *partition of unity* is a *basis* for an *MRA*, as formerly stated next and demonstrated by Counterexample 4.1 (page 57) and Counterexample 4.2 (page 59).

Proposition 4.1.

PRP $\phi(x)$ *generates a* PARTITION OF UNITY $\quad\not\Longrightarrow\quad$ $\phi(x)$ *generates an* MRA *system.*

✎PROOF: By Counterexample 4.1 (page 57) and Counterexample 4.2 (page 59). ✏

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4.3 Counter Examples

There are functions that form a *partition of unity*, but yet do *not* satisfy the *dilation equation* (Theorem 2.1 page 24) …and as such are *not* scaling functions (Definition 2.1 page 18). This section gives some of these counter examples.

Counterexample 4.1. Let a function ϕ be defined in terms of the sine function (Definition I.3 page 235) as follows:

CNT

$$\phi(x) \triangleq \begin{cases} \sin^2\left(\frac{\pi}{2}x\right) & \text{for } x \in [0:2] \\ 0 & \text{otherwise} \end{cases}$$

Then $\int_{\mathbb{R}} \phi(x)\,\mathrm{dx} = 1$ and ϕ induces a *partition of unity*

but $\left\{\mathbf{T}^n\phi \,|\, {}_{n\in\mathbb{Z}}\right\}$ does **not** generate an *MRA*.

✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 3) on a set A.

1. Proof that $\int_{\mathbb{R}} \phi(x)\,\mathrm{dx} = 1$: by Example S.3 (page 406)
2. Proof that $\phi(x)$ forms a *partition of unity*: by Example S.3 (page 406)
3. Proof that $\phi(x) \notin \mathrm{span}\left\{\mathbf{DT}^n\phi(x) \,|\, {}_{n\in\mathbb{Z}}\right\}$ (and so does not generate an *MRA*):
   (a) Note that the *support* (Definition 2.4 page 33) of ϕ is $\mathrm{supp}\,\phi = [0:2]$.
   (b) Therefore, the *support* of $(\!(h_n)\!)$ is $\mathrm{supp}\,(\!(h_n)\!) = \{0,1,2\}$ (Theorem 2.7 page 34).
   (c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\left\{h_0, h_1, h_2\right\}$ (the rest would be 0).

Here would be the values of $\left\{h_1, h_2, h_3\right\}$:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n) \\
&= \sum_{n=0}^{2} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 2.7 page 34}
\end{aligned}$$

   (d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x = \frac{1}{2}$, $x = 1$, and

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$x = \frac{3}{2}$ (see the diagram in item (3c) page 57):

$$\begin{aligned}
\frac{\sqrt{2}}{2} \cdot \frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(1-n)\right)\mathbb{1}_{[0:2]}(1-n) \\
&= \mathsf{h}_0\sin^2\left(\frac{\pi}{2}(1-0)\right)\mathbb{1}_{[0:2]}(1-0) + \mathsf{h}_1\sin^2\left(\frac{\pi}{2}(1-1)\right)\mathbb{1}_{[0:2]}(1-1) \\
&\qquad + \mathsf{h}_2\sin^2\left(\frac{\pi}{2}(1-2)\right)\mathbb{1}_{[0:2]}(1-2) \\
&= \mathsf{h}_0 \cdot 1 \cdot 1 + \mathsf{h}_1 \cdot 0 \cdot 1 + \mathsf{h}_2(-1) \cdot 0 \\
&= \mathsf{h}_0
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}}{2} \cdot 1 &= \frac{\sqrt{2}}{2}(1)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{2}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi(1) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2-n)\right)\mathbb{1}_{[0:2]}(2-n) \\
&= \mathsf{h}_0\sin^2\left(\frac{\pi}{2}(2-0)\right)\mathbb{1}_{[0:2]}(2-0) + \mathsf{h}_1\sin^2\left(\frac{\pi}{2}(2-1)\right)\mathbb{1}_{[0:2]}(2-1) \\
&\qquad + \mathsf{h}_2\sin^2\left(\frac{\pi}{2}(2-2)\right)\mathbb{1}_{[0:2]}(2-2) \\
&= \mathsf{h}_0 \cdot 0 \cdot 1 + \mathsf{h}_1 \cdot 1 \cdot 1 + \mathsf{h}_2 \cdot 0 \cdot 1 \\
&= \mathsf{h}_1
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}}{2} \cdot \frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{-\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{3\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{3}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(3-n)\right)\mathbb{1}_{[0:2]}(3-n) \\
&= \mathsf{h}_0\sin^2\left(\frac{\pi}{2}(3-0)\right)\mathbb{1}_{[0:2]}(3-0) + \mathsf{h}_1\sin^2\left(\frac{\pi}{2}(3-1)\right)\mathbb{1}_{[0:2]}(3-1) \\
&\qquad + \mathsf{h}_2\sin^2\left(\frac{\pi}{2}(3-2)\right)\mathbb{1}_{[0:2]}(3-2) \\
&= \mathsf{h}_0 \cdot (-1) \cdot 0 + \mathsf{h}_1 \cdot 0 \cdot 1 + \mathsf{h}_2 1 \cdot 1 \\
&= \mathsf{h}_2
\end{aligned}$$

(e) These values for (h_0, h_1, h_2) are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 2.1 page 24). In particular,

$$\phi(x) \neq \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \phi(x)$$

(see the diagram in item (3c) page 57)

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✏

Counterexample 4.2 (raised sine). [4] Let a function f be defined in terms of a shifted cosine function (Definition I.2 page 235) as follows:

CNT

$$\phi(x) \triangleq \begin{cases} \frac{1}{2}\left\{1+\cos[\pi(|x-1|)]\right\} & \text{for} \quad 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Then ϕ forms a *partition of unity*:

but $\left\{\mathbf{T}^n\phi \mid n\in\mathbb{Z}\right\}$ does **not** generate an *MRA*.

✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 3) on a set A.

1. Proof that $\phi(x)$ forms a *partition of unity*:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x+1) && \text{by Proposition 1.1 page 4} \\
&= \sum_{n\in\mathbb{Z}} \phi(x+1-n) && \text{by Definition 1.3 page 4} \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2}\{1+\cos[\pi(|x-1+1-n|)]\}\mathbb{1}_{[0:2)}(x+1-n) && \text{by definition of } \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2}\{1+\cos[\pi(|x-n|)]\}\mathbb{1}_{[-1:1)}(x-n) && \text{by Definition 1.2 page 3} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{\frac{1}{2}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n|-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_{[-1:1)}(x-n)\Big|_{\beta=1}}_{\textit{raised cosine (Example S.4 page 407) with } \beta=1} \\
&= 1 && \text{by Example S.4 page 407}
\end{aligned}$$

2. Proof that $\phi(x) \notin \operatorname{span}\{\mathbf{DT}^n\phi(x) \mid n\in\mathbb{Z}\}$ (and so does not generate an *MRA*):

   (a) Note that the *support* (Definition 2.4 page 33) of ϕ is $\operatorname{supp}\phi = [0:2]$.

   (b) Therefore, the *support* of $(\!(h_n)\!)$ is $\operatorname{supp}(\!(h_n)\!) = \{0,1,2\}$ (Theorem 2.7 page 34).

   (c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\{h_0, h_1, h_2\}$ (the rest would be 0). Here would be the values of $\{h_1, h_2, h_3\}$:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \frac{1}{2}\left\{1+\cos[\pi(|x-1|)]\right\}\mathbb{1}_{[0:2]}(x) && \text{by definition of } \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{1+\cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n) && \text{by Definition 1.3 page 4} \\
&= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{1+\cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 2.7 page 34}
\end{aligned}$$

[4] Proakis (2001) pages 560–561

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(d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$ (see the diagram in item (3c) page 57):

$$\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{ 1 + \cos[\pi(|2x-1-n|)] \right\} \mathbb{1}_{[0:2]}(2x-n)\Big|_{x=\frac{1}{2}} \\
&= \mathsf{h}_0 \frac{\sqrt{2}}{2}\left\{ 1 + \cos[1-1-0] \right\} \\
&= \mathsf{h}_0\sqrt{2} \\
&\implies \mathsf{h}_0 = \frac{\sqrt{2}}{4}
\end{aligned}$$

$$\begin{aligned}
1 &= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{ 1 + \cos[\pi(|2x-1-n|)] \right\} \mathbb{1}_{[0:2]}(2x-n)\Big|_{x=1} \\
&= \mathsf{h}_1 \frac{\sqrt{2}}{2}\left\{ 1 + \cos[2-1-1] \right\} \\
&= \mathsf{h}_1\sqrt{2} \\
&\implies \mathsf{h}_1 = \frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{ 1 + \cos[\pi(|2x-1-n|)] \right\} \mathbb{1}_{[0:2]}(2x-n)\Big|_{x=\frac{3}{2}} \\
&= \mathsf{h}_2 \frac{\sqrt{2}}{2}\left\{ 1 + \cos[1-1-0] \right\} \\
&= \mathsf{h}_2\sqrt{2} \\
&\implies \mathsf{h}_2 = \frac{\sqrt{2}}{4}
\end{aligned}$$

(e) These values for $(\!(h_0, h_1, h_2)\!)$ are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 2.1 page 24). In particular (see diagram),
$\phi(x) \neq \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \phi(x)$.

✏

4.4 Examples

Under a very weak set of constraints, the B-spline $\mathsf{N}_0(x)$ naturally results (next example).

Example 4.1 (2 coefficient case/Haar wavelet system/order 0 B-spline wavelet system). [5]
Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a *wavelet system*.

[5] Haar (1910), Wojtaszczyk (1997) pages 14–15 ⟨"Sources and comments"⟩

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EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:1] & \text{(Theorem 2.7 page 34) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.1 page 54)} \end{array}\right\} \Longrightarrow \left\{\begin{array}{c|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{2} \\ 1 & \frac{\sqrt{2}}{2} \\ \text{other} & 0 \end{array}\right\}$$

✎PROOF:

1. Proof that (1) $\Longrightarrow$ that only h_0 and h_1 are non-zero: by Theorem 2.7 page 34.

2. Proof for values of h_0 and h_1: Under the constraint of two non-zero scaling coefficients, a scaling function design is fully constrained using the *admissibility equation* (Theorem 2.3 page 29) and the *partition of unity* constraint (Definition S.1 page 403). The partition of unity formed by $\phi(x)$ is illustrated in Example O.13 (page 353).

3. Here are the equations:

$$\begin{array}{rcll} \mathsf{h}_0 + \mathsf{h}_1 & = & \sqrt{2} & \text{(admissibility equation} \quad \text{Theorem 2.3} \quad \text{page 29)} \\ \mathsf{h}_0 - \mathsf{h}_1 & = & 0 & \text{(partition of unity/zero at } -1 \quad \text{Theorem 4.1} \quad \text{page 54)} \end{array}$$

4. Here are the calculations for the coefficients:

$$\begin{array}{lll} (h_0 + h_1) + (h_0 - h_1) = 2h_0 & = \sqrt{2} & \text{(add two equations together)} \\ (h_0 + h_1) - (h_0 - h_1) = 2h_1 & = \sqrt{2} & \text{(subtract second from first)} \end{array}$$

5. Note: h_0 and h_1 can also be produced using other systems of equations including the following:

   (a) *admissibility condition* and *orthonormality* (Example 6.3 page 104)

   (b) *Daubechies-p1* wavelets computed using spectral techniques (Example 7.3 page 123)

✏

Example 4.2. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \langle\!\langle \boldsymbol{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be a *wavelet system*.

EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:2] & \text{(Theorem 2.7 page 34) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.1 page 54) and} \\ 4. & \check{\mathsf{h}}(z) \text{ has a root at } z = -2 & \end{array}\right\} \Longrightarrow \left\{\begin{array}{c|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{6} \cdot 1 \\ 1 & \frac{\sqrt{2}}{6} \cdot 3 \\ 2 & \frac{\sqrt{2}}{6} \cdot 2 \\ \text{other} & 0 \end{array}\right\}$$

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✎PROOF:

1. By (1) and Theorem 2.7 (page 34), there are 3 non-zero coefficients.
2. By (3), $\check{\mathsf{h}}(z)$ has a zero at $z=-1$ and as such has a factor $(z+1)$.
3. By (4), $\check{\mathsf{h}}(z)$ has a zero at $z=-1/2$ and as such has a factor $(z+2)$.
4. By (2), $\sum h_n = \sqrt{2}$.
5. Putting it all together …

$$\check{\mathsf{h}}(z) = \sqrt{2}\left[\frac{(z+1)(z+2)}{(1+1)(1+2)z^2}\right] = \frac{\sqrt{2}}{6}\left[\frac{z^2+3z+2}{z^2}\right] = \frac{\sqrt{2}}{6}\left[1+3z^{-1}+2z^{-2}\right]$$

✏

Example 4.3. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \langle\!\langle \boldsymbol{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be a *wavelet system*.

EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:2] & \text{(Theorem 2.7 page 34) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.1 page 54) and} \\ 4. & \check{\mathsf{h}}(z) \text{ has a root at } z=-0.5 & \end{array}\right\} \Longrightarrow \left\{\begin{array}{c|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{6}\cdot 2 \\ 1 & \frac{\sqrt{2}}{6}\cdot 3 \\ 2 & \frac{\sqrt{2}}{6}\cdot 1 \\ \text{other} & 0 \end{array}\right\}$$

Note that in contrast to Example 4.2, the zero of condition (4) is *inside* the unit circle, making $\check{\mathsf{h}}(z)$ *minimum phase* (Definition Q.2 page 382) and, in agreement with *Robinson's Energy Delay Theorem* (Theorem Q.3 page 382), pushing the energy of the resulting $\phi(x)$ more towards the origin.

✎PROOF:

1. By (1) and Theorem 2.7 (page 34), there are 3 non-zero coefficients.
2. By (3), $\check{\mathsf{h}}(z)$ has a zero at $z=-1$ and as such has a factor $(z+1)$.
3. By (4), $\check{\mathsf{h}}(z)$ has a zero at $z=-1/2$ and as such has a factor $(z+1/2)$.
4. By (2), $\sum h_n = \sqrt{2}$.

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5. Putting it all together …

$$\check{\mathsf{h}}(z) = \sqrt{2}\left[\frac{(z+1)\left(z+\frac{1}{2}\right)}{(1+1)\left(1+\frac{1}{2}\right)z^2}\right] = \frac{\sqrt{2}}{3}\left[\frac{z^2+\frac{3}{2}z+\frac{1}{2}}{z^2}\right] = \frac{\sqrt{2}}{3}\left[1+\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right] = \frac{\sqrt{2}}{6}\left[2+3z^{-1}+z^{-2}\right]$$

✏

Example 4.4. Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a *wavelet system.*

EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:2] & \text{(Theorem 2.7 page 34) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.1 page 54) and} \\ 4. & \check{\mathsf{h}}(z) \text{ has a root at } z=-1/128 & \end{array}\right\} \Longrightarrow \left\{\begin{array}{c|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{258}\cdot 128 \\ 1 & \frac{\sqrt{2}}{258}\cdot 129 \\ 2 & \frac{\sqrt{2}}{258}\cdot 1 \\ \text{other} & 0 \end{array}\right\}$$

Note that the root $z = -1/128$, being so close to the origin, has very little effect on the resulting scaling function, which in turn looks very much like the *Haar scaling function* in Example 4.1.

✎PROOF:

$$\check{\mathsf{h}}(z) = \sqrt{2}\left[\frac{(z+1)\left(z+\frac{1}{128}\right)}{(1+1)\left(1+\frac{1}{128}\right)z^2}\right] = \frac{\sqrt{2}}{\frac{129}{64}}\left[\frac{z^2+\frac{129}{128}z+\frac{1}{128}}{z^2}\right] = \frac{\sqrt{2}}{129}\left[\frac{64z^2+\frac{129}{2}z+\frac{1}{2}}{z^2}\right] = \frac{\sqrt{2}}{258}\left[128+129z^{-1}+z^{-2}\right]$$

✏

Example 4.5. Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a *wavelet system.*

EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:2] & \text{(Theorem 2.7 page 34) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.1 page 54) and} \\ 4. & \check{\mathsf{h}}(z) \text{ has a root at } z=-128 & \end{array}\right\} \Longrightarrow \left\{\begin{array}{c|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{258}\cdot 1 \\ 1 & \frac{\sqrt{2}}{258}\cdot 129 \\ 2 & \frac{\sqrt{2}}{258}\cdot 128 \\ \text{other} & 0 \end{array}\right\}$$

Note two details:

1. The root $z = -128$, being so far from the unit circle $|z| = 1$, has very little effect on the resulting scaling function, which in turn looks very much like the *Haar scaling function* in Example 4.1.

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2. The energy of this scaling function, with zero far outside the unit circle at $z = -128$, is pushed farther away from the origin as compared to the case where there the comparable zero is inside the unit circle (*minimum phase*) at $z = -1/128$ (Example 4.4 page 63).

✎PROOF:

$$\check{\mathsf{h}}(z) = \sqrt{2}\left[\frac{(z+1)(z+128)}{(1+1)(1+128)z^2}\right] = \frac{\sqrt{2}}{2\times 129}\left[\frac{z^2+129z+128}{z^2}\right] = \frac{\sqrt{2}}{258}\left[1+129z^{-1}+128z^{-2}\right]$$

✏

Example 4.6. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \langle\!\langle \boldsymbol{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be a *wavelet system*.

EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:3] & \text{(Theorem 2.7 page 34) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.1 page 54) and} \\ 4. & \check{\mathsf{h}}(z) \text{ has 2 roots at } z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} & \end{array}\right\} \Longrightarrow \left\{\begin{array}{c|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{6}\cdot 1 \\ 1 & \frac{\sqrt{2}}{6}\cdot 2 \\ 2 & \frac{\sqrt{2}}{6}\cdot 2 \\ 3 & \frac{\sqrt{2}}{6}\cdot 1 \\ \text{other} & 0 \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left[\frac{(z+1)\left(z+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\left(z+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}{(1+1)\left(1+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}z^4\right] \\
&= \sqrt{2}\left[\frac{(z+1)\left(z^2+\left(\frac{1}{2}+\frac{1}{2}+0\right)z+\frac{1}{4}\left|1+i\sqrt{3}\right|^2\right)}{(2)\left(1+\left(\frac{1}{2}+\frac{1}{2}\right)+\frac{1}{4}\left|1+i\sqrt{3}\right|^2\right)}z^4\right] \\
&= \sqrt{2}\left[\frac{(z+1)\left(z^2+z+\frac{1+3}{4}\right)}{(2)\left(1+(1)+\frac{1+3}{4}\right)}z^4\right] \\
&= \frac{\sqrt{2}}{6}\left[z^3+2z^2+2z+1\right]
\end{aligned}$$

✏

Example 4.7. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \langle\!\langle \boldsymbol{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be a *wavelet system*.

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EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:3] & \text{(Theorem 2.7 page 34)} \quad \text{and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29)} \quad \text{and} \\ 3. & \textit{partition of unity} & \text{(Theorem 4.1 page 54)} \quad \text{and} \\ 4. & \check{\mathsf{h}}(z) \text{ has 2 roots at } z = \pm i & \end{array}\right\} \Longrightarrow \left\{\begin{array}{|c|c|} \hline n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{4}\cdot 1 \\ 1 & \frac{\sqrt{2}}{4}\cdot 1 \\ 2 & \frac{\sqrt{2}}{4}\cdot 1 \\ 3 & \frac{\sqrt{2}}{4}\cdot 1 \\ \text{other} & 0 \\ \hline \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} \check{\mathsf{h}}(z) &= \sqrt{2}\left[\frac{(z+1)(z-i)(z+i)}{(1+1)(1-i)(1+i)}z^3\right] \\ &= \sqrt{2}\left[\frac{(z+1)\left(z^2+1\right)}{(1+1)\left(1^2+1\right)}z^3\right] \\ &= \frac{\sqrt{2}}{4}\left[\frac{z^3+z^2+z+1}{z^3}\right] \\ &= \frac{\sqrt{2}}{4}\left[1+1z^{-1}+1z^{-2}+1z^{-3}\right] \end{aligned}$$

✏

Example 4.8. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ be a *wavelet system*.

EX

$$\left\{\begin{array}{lll} 1. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29)} \quad \text{and} \\ 2. & \check{\mathsf{h}}(z) \text{ has the roots of } z^3+1=0 & \end{array}\right\} \Longrightarrow \left\{\begin{array}{|c|c|} \hline n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{2}\cdot 1 \\ 1 & \frac{\sqrt{2}}{2}\cdot 0 \\ 2 & \frac{\sqrt{2}}{2}\cdot 0 \\ 3 & \frac{\sqrt{2}}{2}\cdot 1 \\ \text{other} & 0 \\ \hline \end{array}\right\}$$

✎PROOF: See Lemma C.1 (page 162). ✏

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CHAPTER 5

VANISHING MOMENTS CONSTRAINT

One of the most common wavelet design constraints is the number of *vanishing moments* p on the wavelet function such that

$$\langle \psi(x) \mid x^n \rangle = 0 \qquad \text{for} \quad n = 0, 1, 2, \dots, p-1.$$

This chapter investigates wavelet design under the vanishing moment constraint.

5.1 Moments

Definition 5.1. [1]

DEF *The quantity M_n is the n**th moment** of a function $\mathsf{f}(x) \in \mathbf{L}^2_{\mathbb{R}}$ if*

$$M_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\, \mathsf{dx} \qquad \textit{for } n \in \mathbb{W}.$$

Definition 5.2. [2] *Let M_n be the* nTH MOMENT *(Definition 5.1 page 67) of a function $\mathsf{f}(x) \in \mathbf{L}^2_{\mathbb{R}}$ (Definition E.1 page 199).*

DEF *The function $\mathsf{f}(x)$ has an n**th vanishing moment** if $M_n = 0$.*
*The function $\mathsf{f}(x)$ has p **vanishing moments** if $M_0 = M_1 = \cdots = M_{p-1} = 0$ and $M_p \neq 0$.*

Let $\left(\mathbf{L}^2_{\mathbb{R}}, (\!(\mathbf{V}_j)\!), (\!(\mathbf{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a *wavelet system* (Definition 3.2 page 40). Let $\mathbf{P}$ be the *projection operator* that generates the subspace $\mathbf{V}_0$ and $\mathbf{Q}$ the *projection operator* that generates the subspace $\mathbf{W}_0$. The number of *vanishing moments* of a wavelet is important when using wavelets for the analysis and synthesis of a function $\mathsf{f}(x)$ as in

$$\mathsf{f}(x) = \underbrace{\sum_{n\in\mathbb{Z}} \underbrace{\langle \mathsf{f}(x) \mid \mathbf{T}^n \phi(x) \rangle}_{\text{Fourier coefficient}} \mathbf{T}^n \phi(x)}_{\textit{Fourier expansion} \text{ (Definition N.11 page 316) of } \mathbf{P}\mathsf{f}(x)} + \underbrace{\sum_{k=0}^{\infty} \sum_{n\in\mathbb{Z}} \underbrace{\left\langle \mathsf{f}(x) \mid \mathbf{D}^k \mathbf{T}^n \psi(x) \right\rangle}_{\text{Fourier coefficient}} \mathbf{D}^k \mathbf{T}^n \psi(x)}_{\textit{Fourier expansion} \text{ (Definition N.11 page 316) of } \mathbf{Q}\mathsf{f}(x)}$$

[1] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83

[2] Mallat (1999) page 241

All differentiable functions have a *Taylor polynomial expansion* (Theorem C.13 page 167). The *Fourier coefficients* (Definition N.11 page 316) of this polynomial decrease as the order of each term increases. The inner-product of the function and a wavelet with many vanishing moments results in a smaller value, giving a better approximation for a fixed number of Fourier coefficients.

Theorem 5.1.

THM

1. *a function* $\mathsf{f}(x)$ *has* p *vanishing moments* *and*
2. *a polynomial* $\mathsf{q}(x)$ *is of order* $p-1$ *or less*

$$\left.\begin{array}{l}\text{1. a function } \mathsf{f}(x) \text{ has } p \text{ vanishing moments and}\\ \text{2. a polynomial } \mathsf{q}(x) \text{ is of order } p-1 \text{ or less}\end{array}\right\} \implies \underbrace{\langle \mathsf{f} \mid \mathsf{q}\rangle = 0}_{\mathsf{f} \text{ is ORTHOGONAL to } \mathsf{q}}$$

✎PROOF:

$$\left\langle \mathsf{f}(x) \,\middle|\, \sum_{n=0}^{p-1} a_n x^n \right\rangle = \sum_{n=0}^{p-1} a_n^* \langle \mathsf{f}(x) \mid x^n\rangle = \sum_{n=0}^{p-1} a_n^* \cdot 0 = 0$$

✏

5.2 Vanishing moments and the wavelet function

The number of vanishing moments a wavelet has is closely related to how the derivatives of the Fourier transforms of the wavelet and wavelet coefficients behave (next theorem).

Theorem 5.2. [3] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system. Let* $\tilde{\psi}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of* $\psi(x)$. *Let* $\breve{\mathsf{h}}(\omega)$ *and* $\breve{\mathsf{g}}(\omega)$ *be the* DTFT*s (Definition R.1 page 393) of* $(\!(h_n)\!)$ *and* $(\!(g_n)\!)$, *respectively.*

THM

$$\left\{\begin{array}{llll} (A) & \underbrace{\langle \psi(x) \mid x^n\rangle = 0}_{\psi(x) \text{ has a vanishing } n\text{th moment}} & \iff \left[\frac{d}{d\omega}\right]^n \tilde{\psi}(\omega)\Big|_{\omega=0} = 0 & (B)\\ & & \iff \left[\frac{d}{d\omega}\right]^n \breve{\mathsf{g}}(\omega)\Big|_{\omega=0} = 0 & (C)\\ & & \iff \sum_{k\in\mathbb{Z}} k^n g_n = 0 & (D)\end{array}\right\} \quad \forall n \in \mathbb{W}$$

✎PROOF:

1. Proof that $(A) \iff (B)$: by Lemma M.2 page 299
2. Proof that $(B) \implies (C)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \tilde{\psi}(2\omega)\Big|_{\omega=0} && \text{by left hypothesis}\\
&= \left[\frac{d}{d\omega}\right]^n \breve{\mathsf{g}}(\omega)\tilde{\phi}(\omega)\Big|_{\omega=0} && \text{by Proposition 3.2 page 44}\\
&= \sum_{k=0}^{n} \binom{n}{k} \breve{\mathsf{g}}^{(k)}(\omega)\tilde{\phi}^{(n-k)}(\omega)\Bigg|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma E.2 page 201)}\\
&= \left[\breve{\mathsf{g}}^{(n)}(\omega)\tilde{\phi}(\omega) + \sum_{k=0}^{n-1} \binom{n}{k} \breve{\mathsf{g}}^{(k)}(\omega)\tilde{\phi}^{(n-k)}(\omega)\right]_{\omega=0}\\
&= \breve{\mathsf{g}}^{(n)}(0)\tilde{\phi}(0) + \sum_{k=0}^{n-1} \binom{n}{k} \breve{\mathsf{g}}^{(k)}(0)\tilde{\phi}^{(n-k)}(0)
\end{aligned}$$

[3] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

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$$\begin{array}{rcl}
\Longrightarrow & \breve{g}^{(0)}(0) & = & 0 \\
\Longrightarrow & \breve{g}^{(1)}(0) & = & 0 \\
\Longrightarrow & \breve{g}^{(2)}(0) & = & 0 \\
\Longrightarrow & \breve{g}^{(3)}(0) & = & 0 \\
\Longrightarrow & \breve{g}^{(4)}(0) & = & 0 \\
\vdots & \vdots & & \\
\Longrightarrow & \breve{g}^{(n)}(0) & = & 0 \\
\Longrightarrow & \breve{g}^{(n)}(0) & = & 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{array}$$

3. Proof that $(B) \Longleftarrow (C)$:

$$\begin{aligned}
\left[\frac{d}{d\omega}\right]^n \tilde{\psi}(2\omega)\Big|_{\omega=0} &= \left[\frac{d}{d\omega}\right]^n \breve{g}(\omega)\tilde{\phi}(\omega)\Big|_{\omega=0} && \text{by Proposition 3.2 page 44} \\
&= \sum_{k=0}^{n} \binom{n}{k} \breve{g}^{(k)}(\omega)\tilde{\phi}^{(n-k)}(\omega)\Big|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \qquad \text{(Lemma E.2 page 201)} \\
&= \sum_{k=0}^{n} \binom{n}{k} 0\ \tilde{\phi}^{(n-k)}(\omega)\Big|_{\omega=0} && \text{by right hypothesis} \\
&= 0
\end{aligned}$$

4. Proof that $(C) \Longleftrightarrow (D)$: by Theorem R.5 page 401

✏

5.3 Vanishing moments and the scaling function

5.3.1 Results

Lemma 5.1. *Let* $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j \rangle\!\rangle, \langle\!\langle W_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ *be a wavelet system and* M_n *the* nTH MOMENT *(Definition 5.1 page 67) of* $\phi(x)$.

LEM

$$\left[\frac{d}{d\omega}\right]^n \breve{h}(\omega)\Big|_{\omega=\pi} = 0 \qquad \Longrightarrow \qquad \left[\frac{d}{d\omega}\right]^n \tilde{\phi}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)M_n\delta_k$$

✎PROOF:

1. By Lemma M.1 page 298, $\left[\frac{d}{d\omega}\right]^n \tilde{\phi}(\omega)\Big|_{\omega=0} = \frac{1}{\sqrt{2\pi}}(-i)^n M_n$.

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2. Then,

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\phi}(\omega)\Big|_{\omega=2\pi k} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)\Big|_{\omega=2\pi k} && \text{by Proposition 3.2 page 44}\\
&= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}\left(\frac{\omega}{2}\right)\tilde{\phi}^{(n-m)}\left(\frac{\omega}{2}\right)\Bigg|_{\omega=2\pi k} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma E.2 page 201)}\\
&= \sum_{m=0}^{n}\binom{n}{m}\underbrace{\breve{\mathsf{h}}^{(m)}(\pi k)}_{0 \text{ for } k \text{ odd}}\tilde{\phi}^{(n-m)}(\pi k)\\
&= \begin{cases}\displaystyle\sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(\pi k)\tilde{\phi}^{(n-m)}(\pi k) & \text{for } k \text{ even}\\ 0 & \text{for } k \text{ odd}\end{cases} && \text{by left hypothesis}
\end{aligned}$$

3. And so

$$\begin{aligned}
\tilde{\phi}^{(n)}(2\pi 0) &= (-i)^n M_n\\
\tilde{\phi}^{(n)}(2\pi 1) &= 0\\
\tilde{\phi}^{(n)}(2\pi 2) &= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(2\pi 1)\underbrace{\tilde{\phi}^{(n-m)}(2\pi 1)}_{0} = 0\\
\tilde{\phi}^{(n)}(2\pi 3) &= 0\\
\tilde{\phi}^{(n)}(2\pi 4) &= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(2\pi 2)\underbrace{\tilde{\phi}^{(n-m)}(2\pi 2)}_{0} = 0\\
\tilde{\phi}^{(n)}(2\pi 5) &= 0\\
\tilde{\phi}^{(n)}(2\pi 6) &= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(2\pi 3)\underbrace{\tilde{\phi}^{(n-m)}(2\pi 3)}_{0} = 0\\
&\vdots\ \vdots\\
\tilde{\phi}^{(n)}(2\pi k) &= \bar{\delta}_k(-i)^n M_n
\end{aligned}$$

✏

Theorem 5.3. [4] *Let* $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j\rangle\!\rangle, \langle\!\langle W_j\rangle\!\rangle, \phi, \psi, \langle\!\langle h_n\rangle\!\rangle, \langle\!\langle g_n\rangle\!\rangle\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40). Let* $\breve{\mathsf{h}}(\omega)$ *and* $\breve{\mathsf{g}}(\omega)$ *be the* DTFTs *(Definition R.1 page 393) of* $\langle\!\langle h_n\rangle\!\rangle$ *and* $\langle\!\langle g_n\rangle\!\rangle$*, respectively.*

THM

Let $g_n = \pm(-1)^n h^*_{N-n}$ (CQF CONDITION, *Definition Q.6 page 386*)*. Then*

$$\left\{\begin{aligned}
&\underbrace{\langle\psi(x)\,|\,x^n\rangle = 0}_{(1)\ n\text{TH VANISHING MOMENT}} && \iff \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{g}}(\omega)\Big|_{\omega=0} = 0 && (2)\\
& && \iff \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{h}}(\omega)\Big|_{\omega=\pi} = 0 && (3)\\
& && \iff \sum_{k\in\mathbb{Z}}(-1)^k k^n h_k = 0 && (4)\\
& && \iff \sum_{k\in\mathbb{Z}} k^n g_k = 0 && (5)
\end{aligned}\right\} \quad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof that (1) $\iff$ (2): by Theorem 5.2 (page 68)

[4] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

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Figure 5.1: Zero locations for B-cardinal spline $N_p(x)$ and Daubechies-p scaling coefficients

2. Proof that (2) $\iff$ (3): by Theorem Q.6 page 389
3. Proof that (3) $\iff$ (4): by Theorem R.5 page 401
4. Proof that (2) $\iff$ (5): by Theorem R.5 page 401; see also Theorem 5.2 page 68.
5. Proof that CQF $\not\Leftarrow$ (1): by Theorem R.5 page 401; see also Theorem 5.2 page 68.

✏

These relationships imply that a compactly supported scaling function with p vanishing moments has p zeros at $z = -1$ (due to the spline component) as illustrated in Figure 5.1.

5.3.2 Scaling function and polynomials

Linear combinations of $\left\{ \mathbf{T}^n\phi(x) \,|\, n\in\mathbb{Z} \right\}$ of a wavelet system with p vanishing moments can perfectly represent any polynomial of order $p-1$ or less (Theorem 5.4 page 72). Example 5.1 page 72 illustrates this using the Strang-Fix condition (Lemma 5.2 page 71) to represent any polynomial of dgree 4 or less.

Lemma 5.2. [5] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system and* M_n *the* nTH MOMENT *(Definition 5.1 page 67) of* $\phi(x)$.

$$\text{LEM}\quad \left\{\begin{array}{l}\psi \text{ has } p\\ \text{vanishing}\\ \text{moments}\end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} 1. & \displaystyle\sum_{k\in\mathbb{Z}}(x-k)^n\phi(x-k) & = M_n \quad \text{and} \\ 2. & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\phi}(\omega)\Big|_{\omega=2\pi k} & = \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k M_n \end{array}\right\} \quad \text{for } n = 1,2,\ldots,p-1$$

✎PROOF:

1. Proof that (1) $\iff$ (2): by Lemma M.3 page 300
2. Proof for ($\Longrightarrow$) assertion:

$$\begin{aligned}
\langle \psi(x) \,|\, x^n\rangle = 0 \iff & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{\mathrm{h}}(\omega)\Big|_{\omega=\pi} = 0 && \text{by Theorem 5.3 page 70}\\
\Longrightarrow & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\phi}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)M_n\delta_k && \text{by Lemma 5.1 page 69}\\
\iff & \sum_{k\in\mathbb{Z}}(x-k)^n\phi(x-k) = M_n && \text{by } \textit{Strang-Fix condition} \quad \text{(Lemma M.3 page 300)}
\end{aligned}$$

[5] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83, Goswami and Chan (1999) page 102, Mallat (1999) pages 241–243

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Theorem 5.4. *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, \langle\!\langle \mathbf{V}_j \rangle\!\rangle, \langle\!\langle \mathbf{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ *be a wavelet system with* p *vanishing moments. Let* $\mathsf{q}(x)$ *be a* POLYNOMIAL. *Let*

$$Q_p \triangleq \left\{\mathsf{q}(x) \middle| \mathsf{q}(x) \text{ is a POLYNOMIAL of order } p-1 \text{ or less}\right\}.$$

THM

$$\left\{\begin{array}{ll} 1. & \psi \text{ has } p \text{ vanishing moments} \\ 2. & \mathsf{q}(x) \text{ is of order } p-1 \end{array} \quad \text{and} \right\} \implies \left\{\begin{array}{ll} 1. & \text{There exists } \langle\!\langle b_n \rangle\!\rangle_0^{p-1} \text{ such that} \\ & \mathsf{q}(x) = \sum_{n=0}^{p-1} b_n \sum_{k\in\mathbb{Z}} k^n \phi(x-k) \\ 2. & \left\{\mathbf{T}^n\phi(x) \middle| n\in\mathbb{Z}\right\} \text{ is a BASIS for } Q_p. \end{array} \quad \text{and} \right\}$$

✎PROOF: This follows from Lemma 5.2 page 71 (Strang-Fix condition). See Example 5.1 page 72. ✏

Example 5.1. Let $\left(\mathbf{L}^2_{\mathbb{R}}, \langle\!\langle \mathbf{V}_j \rangle\!\rangle, \langle\!\langle \mathbf{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be a *wavelet system* (Definition 3.2 page 40) in which $\psi(x)$ has $p = 5$ vanishing moments.

EX

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \phi(x-k) &= M_0 \\
\sum_{k\in\mathbb{Z}} k\phi(x-k) &= M_0 x - M_1 \\
\sum_{k\in\mathbb{Z}} k^2\phi(x-k) &= M_0 x^2 - 2M_1 x + M_2 \\
\sum_{k\in\mathbb{Z}} k^3\phi(x-k) &= M_0 x^3 - 3M_1 x^2 + 3M_2 x - M_3 \\
\sum_{k\in\mathbb{Z}} k^4\phi(x-k) &= M_0 x^4 - 4M_1 x^3 + 6M_2 x^2 - 4M_3 x + M_4 \\
\sum_{k\in\mathbb{Z}} k^5\phi(x-k) &= M_0 x^5 - 5M_1 x^4 + 10M_2 x^3 - 10M_3 x^2 + 5M_4 x - M_5
\end{aligned}$$

These equations can be represented in matrix algebra form as

EX

$$\begin{bmatrix} \sum_{k\in\mathbb{Z}} k^4\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k^3\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k^2\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k\phi(x-k) \\ \sum_{k\in\mathbb{Z}} \phi(x-k) \end{bmatrix} = \begin{bmatrix} M_0 & -4M_1 & 6M_2 & -4M_3 & M_4 \\ 0 & M_0 & -3M_1 & 3M_2 & -M_3 \\ 0 & 0 & M_0 & -2M_1 & M_2 \\ 0 & 0 & 0 & M_0 & -M_1 \\ 0 & 0 & 0 & 0 & M_0 \end{bmatrix} \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{bmatrix}$$

The operator matrix is triangular and therefore it's inverse always exists. The monomials $1, x, x^2, x^3$ and x^4 can be computed by computing the inverse of the operator matrix.

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EX

$$
\begin{aligned}
1 &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}}\phi(x-k) \\
x &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}}k\phi(x-k) + \frac{M_1}{M_0} \\
x^2 &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}}k^2\phi(x-k) + \frac{2M_1}{M_0^2}\sum_{k\in\mathbb{Z}}k\phi(x-k) + \frac{2M_1^2-M_2M_0}{M_0^2} \\
x^3 &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}}k^3\phi(x-k) + \frac{3M_1}{M_0^2}\sum_{k\in\mathbb{Z}}k^2\phi(x-k) + \frac{6M_1^2-3M_2M_0}{M_0^3}\sum_{k\in\mathbb{Z}}k\phi(x-k) \\
&\quad + \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^3}
\end{aligned}
$$

Note that each monomial x^n is simply a linear combination of $(\!(\phi(x-k))\!)_{k\in\mathbb{Z}}$.

✎PROOF:

$$
\begin{aligned}
M_0 &= \sum_{k\in\mathbb{Z}}(x-k)^0\phi(x-k) \\
&= \sum_{k\in\mathbb{Z}}\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= \sum_{k\in\mathbb{Z}}(x-k)^1\phi(x-k) \\
&= t\sum_{k\in\mathbb{Z}}\phi(x-k) - \sum_{k\in\mathbb{Z}}k\phi(x-k) \\
&= M_0t - \sum_{k\in\mathbb{Z}}k\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_2 &= \sum_{k\in\mathbb{Z}}(x-k)^2\phi(x-k) \\
&= x^2\sum_{k\in\mathbb{Z}}\phi(x-k) - 2t\sum_{k\in\mathbb{Z}}k\phi(x-k) + \sum_{k\in\mathbb{Z}}k^2\phi(x-k) \\
&= x^2M_0 - 2t(M_0t-M_1) + \sum_{k\in\mathbb{Z}}k^2\phi(x-k) \\
&= -M_0x^2 + 2M_1x + \sum_{k\in\mathbb{Z}}k^2\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= \sum_{k\in\mathbb{Z}}(x-k)^3\phi(x-k) \\
&= x^3\sum_{k\in\mathbb{Z}}\phi(x-k) - 3x^2\sum_{k\in\mathbb{Z}}k\phi(x-k) + 3x\sum_{k\in\mathbb{Z}}k^2\phi(x-k) - \sum_{k\in\mathbb{Z}}k^3\phi(x-k) \\
&= x^3M_0 - 3x^2\left[M_0t-M_1\right] + 3x\left[M_0x^2-2M_1x+M_2\right] - \sum_{k\in\mathbb{Z}}k^3\phi(x-k) \\
&= (M_0-3M_0+3M_0)x^3 + (3M_1-6M_1)x^2 + 3M_2x - \sum_{k\in\mathbb{Z}}k^3\phi(x-k) \\
&= M_0x^3 - 3M_1x^2 + 3M_2t - \sum_{k\in\mathbb{Z}}k^3\phi(x-k)
\end{aligned}
$$

$$
M_4 = \sum_{k\in\mathbb{Z}}(x-k)^4\phi(x-k)
$$

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$$
\begin{aligned}
&= x^4 \sum_{k\in\mathbb{Z}} \phi(x-k) - 4x^3 \sum_{k\in\mathbb{Z}} k\phi(x-k) + 6x^2 \sum_{k\in\mathbb{Z}} k^2\phi(x-k) - 4x \sum_{k\in\mathbb{Z}} k^3\phi(x-k) + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= x^4 M_0 - 4x^3\left[M_0 t - M_1\right] + 6x^2\left[M_0x^2 - 2M_1x + M_2\right] - 4x\left[M_0x^3 - 3M_1x^2 + 3M_2t - M_3\right]\\
&\qquad + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= x^4 M_0 + \left[-4M_0x^4 + 4M_1x^3\right] + \left[6M_0x^4 - 12M_1x^3 + 6M_2x^2\right]\\
&\qquad + \left[-4M_0x^4 + 12M_1x^3 - 12M_2x^2 + 4M_3x\right] + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= (M_0 - 4M_0 + 6M_0 - 4M_0)x^4 + (4M_1 - 12M_1 + 12M_1)x^3 + (6M_2 - 12M_2)x^2 + 4M_3x\\
&\qquad + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= -M_0x^4 + 4M_1x^3 - 6M_2x^2 + 4M_3x + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_5 &= \sum_{k\in\mathbb{Z}} (x-k)^5\phi(x-k)\\
&= x^5 \sum_{k\in\mathbb{Z}} \phi(x-k) - 5x^4 \sum_{k\in\mathbb{Z}} k\phi(x-k) + 10x^3 \sum_{k\in\mathbb{Z}} k^2\phi(x-k) - 10x^2 \sum_{k\in\mathbb{Z}} k^3\phi(x-k)\\
&\qquad + 5t \sum_{k\in\mathbb{Z}} k^4\phi(x-k) - \sum_{k\in\mathbb{Z}} k^5\phi(x-k)\\
&= x^5 M_0 - 5x^4\left[M_0x - M_1\right] + 10x^3\left[M_0x^2 - 2M_1x + M_2\right] - 10x^2\left[M_0x^3 - 3M_1x^2 + 3M_2x - M_3\right]\\
&\qquad + 5t\left[M_0x^4 - 4M_1x^3 + 6M_2x^2 - 4M_3x + M_4\right] - \sum_{k\in\mathbb{Z}} k^5\phi(x-k)\\
&= x^5 M_0\left[-5M_0x^5 + 5M_1x^4\right] + \left[10M_0x^5 - 20M_1x^4 + 10M_2x^3\right]\\
&\qquad + \left[-10M_0x^5 + 30M_1x^4 - 30M_2x^3 + 10M_3x^2\right]\\
&\qquad + \left[5M_0x^5 - 20M_1x^4 + 30M_2x^3 - 20M_3x^2 + 5M_4x\right] - \sum_{k\in\mathbb{Z}} k^5\phi(x-k)\\
&= (M_0 - 5M_0 + 10M_0 + 5M_0 - 10M_0)x^5 + (5M_1 - 20M_1 + 30M_1 - 20M_1)x^4\\
&\qquad + (10M_2 - 30M_2 + 30M_2)x^3 + (10M_3 - 20M_3)x^2 + 5M_4x - \sum_{k\in\mathbb{Z}} k^5\phi(x-k)\\
&= M_0x^5 - 5M_1x^4 + 10M_2x^3 - 10M_3x^2 + 5M_4x - \sum_{k\in\mathbb{Z}} k^5\phi(x-k)
\end{aligned}
$$

$$
\begin{bmatrix}
M_0 & -3M_1 & 3M_2 & -M_3 & 1 & 0 & 0 & 0\\
0 & M_0 & -2M_1 & M_2 & 0 & 1 & 0 & 0\\
0 & 0 & M_0 & -M_1 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{bmatrix}
\to
\begin{bmatrix}
M_0 & -3M_1 & 3M_2 & 0 & 1 & 0 & 0 & \frac{M_3}{M_0}\\
0 & M_0 & -2M_1 & 0 & 0 & 1 & 0 & -\frac{M_2}{M_0}\\
0 & 0 & M_0 & 0 & 0 & 0 & 1 & \frac{M_1}{M_0}\\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{bmatrix}
\to
$$

$$
\begin{bmatrix}
M_0 & -3M_1 & 0 & 0 & 1 & 0 & -\frac{3M_2}{M_0} & -\frac{3M_2M_1}{M_0^2} + \frac{M_3}{M_0}\\
0 & M_0 & 0 & 0 & 0 & 1 & \frac{2M_1}{M_0} & \frac{2M_1^2}{M_0^2} - \frac{M_2}{M_0}\\
0 & 0 & M_0 & 0 & 0 & 0 & 1 & \frac{M_1}{M_0}\\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{bmatrix}
\to
$$

$$
\begin{bmatrix}
M_0 & 0 & 0 & 0 & 1 & \frac{3M_1}{M_0} & \frac{6M_1^2}{M_0^2} - \frac{3M_2}{M_0} & \frac{6M_1^3}{M_0^3} - \frac{3M_1M_2}{M_0^2} - \frac{3M_2M_1}{M_0^2} + \frac{M_3}{M_0}\\
0 & M_0 & 0 & 0 & 0 & 1 & \frac{2M_1}{M_0} & \frac{2M_1^2}{M_0^2} - \frac{M_2}{M_0}\\
0 & 0 & M_0 & 0 & 0 & 0 & 1 & \frac{M_1}{M_0}\\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{bmatrix}
\to
$$

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$$\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{3M_1}{M_0^2} & \frac{6M_1^2}{M_0^3}-\frac{3M_2}{M_0^2} & \frac{6M_1^3}{M_0^4}-\frac{3M_1M_2}{M_0^3}-\frac{3M_2M_1}{M_0^3}+\frac{M_3}{M_0^2} \\
0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{2M_1}{M_0^2} & \frac{2M_1^2}{M_0^3}-\frac{M_2}{M_0^2} \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{M_1}{M_0^2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0}
\end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{3M_1}{M_0^2} & \frac{6M_1^2-3M_2M_0}{M_0^3} & \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^4} \\
0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{2M_1}{M_0^2} & \frac{2M_1^2-M_2M_0}{M_0^3} \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{M_1}{M_0^2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0}
\end{array}\right]$$

$$\begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix}
\frac{1}{M_0} & \frac{3M_1}{M_0^2} & \frac{6M_1^2-3M_2M_0}{M_0^3} & \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^4} \\
0 & \frac{1}{M_0} & \frac{2M_1}{M_0^2} & \frac{2M_1^2-M_2M_0}{M_0^3} \\
0 & 0 & \frac{1}{M_0} & \frac{M_1}{M_0^2} \\
0 & 0 & 0 & \frac{1}{M_0}
\end{bmatrix} \begin{bmatrix} \sum_{k\in\mathbb{Z}} k^3\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k^2\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k\phi(x-k) \\ \sum_{k\in\mathbb{Z}} \phi(x-k) \end{bmatrix}$$

$$\begin{aligned}
x^3 &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^3\phi(x-k) + \frac{3M_1}{M_0^2}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{6M_1^2-3M_2M_0}{M_0^3}\sum_{k\in\mathbb{Z}} k\phi(x-k) \\
&\quad + \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^4}\underbrace{\sum_{k\in\mathbb{Z}}\phi(x-k)}_{M_0} \\
&= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^3\phi(x-k) + \frac{3M_1}{M_0^2}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{6M_1^2-3M_2M_0}{M_0^3}\sum_{k\in\mathbb{Z}} k\phi(x-k) \\
&\quad + \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^3} \\
x^2 &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{2M_1}{M_0^2}\sum_{k\in\mathbb{Z}} \phi(x-k) + \frac{2M_1^2-M_2M_0}{M_0^2} \\
x &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k\phi(x-k) + \frac{M_1}{M_0}
\end{aligned}$$

✏

Conjecture 5.1. [6] Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \langle\!\langle \boldsymbol{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be a wavelet system with p vanishing moments and let M_n be the nth moment (Definition 5.1 page 67) of $\phi(x)$.

CNJ

$$\sum_{k\in\mathbb{Z}} k^n\phi(x-k) = \sum_{m=0}^{n}\binom{n}{m}(-1)^m M_m x^{n-m}$$

[6] Zumkeller (2005) http://oeis.org/A110555 $\langle T(n,k) \triangleq \sum_{k=0,1,\ldots,n}\binom{n}{k}(-1)^k\rangle$

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✎PROOF: This conjecture is supported by the results of Example 5.1 (page 72). However, I have no proof at this time.

$$
\begin{aligned}
(-1)^n M_n &= (-1)^n \sum_{k\in\mathbb{Z}} (x-k)^n \phi(x-k) \\
&= (-1)^n \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n} \binom{n}{m} x^{n-m} (-k)^m \phi(x-k) \\
&= (-1)^n \sum_{m=0}^{n} \binom{n}{m} (-1)^m x^{n-m} \sum_{k\in\mathbb{Z}} k^m \phi(x-k) \\
&= \left[(-1)^{2n} \sum_{k\in\mathbb{Z}} k^n \phi(x-k) \right] + (-1)^n \sum_{m=0}^{n-1} \binom{n}{m} (-1)^m x^{n-m} \sum_{k\in\mathbb{Z}} k^m \phi(x-k) \\
&= \left[\sum_{k\in\mathbb{Z}} k^n \phi(x-k) \right] + (-1)^n \sum_{m=0}^{n-1} \binom{n}{m} (-1)^m x^{n-m} \sum_{k\in\mathbb{Z}} k^m \phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} k^n \phi(x-k) &= -(-1)^n \sum_{m=0}^{n-1} \binom{n}{m} (-1)^m x^{n-m} \sum_{k\in\mathbb{Z}} k^m \phi(x-k) + (-1)^n M_n \\
&= \sum_{m=0}^{n-1} \binom{n}{m} (-1)^m \underbrace{\left[(-1)^{n-1} \sum_{k\in\mathbb{Z}} k^m \phi(x-k) \right]}_{M_m?} x^{n-m} + (-1)^n M_n
\end{aligned}
$$

$$
\begin{aligned}
M_m &\stackrel{?}{=} (-1)^{n-1} \sum_{k\in\mathbb{Z}} k^m \phi(x-k) \\
M_m &= \sum_{k\in\mathbb{Z}} (x-k)^m \phi(x-k) \\
&= \sum_{k\in\mathbb{Z}} \sum_{p=0}^{m} \binom{m}{p} x^{m-p} (-k)^p \phi(x-k) \\
&= (-1)^m \sum_{k\in\mathbb{Z}} k^m \phi(x-k) + \underbrace{\sum_{k\in\mathbb{Z}} \sum_{p=0}^{m-1} \binom{m}{p} x^{m-p} (-k)^p \phi(x-k)}_{0?}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} k^n \phi(x-k) &= -(-1)^n \sum_{m=0}^{n-1} \binom{n}{m} (-1)^m x^{n-m} \sum_{k\in\mathbb{Z}} k^m \phi(x-k) + (-1)^n M_n \\
&= \sum_{m=0}^{n-1} \binom{n}{m} (-1)^m (-1)^{n-1} \left[\sum_{k\in\mathbb{Z}} k^m \phi(x-k) \right] x^{n-m} + (-1)^n M_n \\
&= \sum_{m=0}^{n-1} \binom{n}{m} (-1)^m (-1)^{n-1} \left[\sum_{p=0}^{m-1} \binom{m}{p} (-1)^p (-1)^{m-1} \left[\sum_{k\in\mathbb{Z}} k^p \phi(x-k) \right] x^{m-p} + (-1)^m M_m \right] x^{n-m} \\
&\quad + (-1)^n M_n
\end{aligned}
$$

✏

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5.3.3 Scaling function and continuity

Theorem 5.5. *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40). Let* $\tilde{\phi}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of* $\phi(x)$.

THM
$$\left\{\begin{array}{lll} (1). & \psi \text{ has } p \geq 2 \text{ VANISHING MOMENTS} & \text{and} \\ (2). & g_n = (-1)^n h^*_{N-n} & \text{[CQF CONDITION (Definition Q.6 page 386)]} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \tilde{\phi}(\omega) \text{ is CONTINUOUS} \\ \text{at } \omega = 0 \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
(1) \text{ and } (2) &\Longrightarrow \left[\frac{d}{d\omega}\right]^n \tilde{\phi}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k M_n && \text{by Lemma 5.2 page 71} \\
&\Longrightarrow \frac{d}{d\omega}\tilde{\phi}(\omega)\Big|_{\omega=0} = \frac{1}{\sqrt{2\pi}}(-i)^1 M_1 \\
&\Longrightarrow \frac{d}{d\omega}\tilde{\phi}(\omega)\Big|_{\omega=0} = 0 && \text{by (1) and def. of } \textit{vanishing moments} \text{ (Definition 5.2 page 67)} \\
&\Longrightarrow \frac{d}{d\omega}\tilde{\phi}(\omega)\Big|_{\omega=0} < \infty \\
&\Longrightarrow \tilde{\phi}(\omega) \text{ is } \textit{continuous} \text{ at } \omega = 0
\end{aligned}$$

✏

5.4 Sufficient conditions

How can we design a *wavelet system* (Definition 3.2 page 40) to have p *vanishing moments*? One way is to design the *scaling coefficient sequence* (Definition 2.3 page 27) $(\!(h_n)\!)$ in the *z-domain* (Definition Q.1 page 379) such that $\check{\mathsf{h}}(z)$ includes the factor $\left(\frac{1+z^{-1}}{2}\right)^p$ (next theorem). This factor is included in the $\check{\mathsf{h}}(z)$ of the *Daubechies-p wavelet system* (Definition 7.1 page 120). A similar factor (Lemma O.5 page 350) also appears in the *Fourier transform* (Definition M.2 page 294) of *B-splines* (Definition O.2 page 335).

Lemma 5.3. [7] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40). Let* $(\!(q_n)\!)_{n\in\mathbb{Z}}$ *be a sequence with* Z-TRANSFORM *(Definition Q.1 page 379)* $\check{\mathsf{q}}(z)$.

LEM
$$\left\{\begin{array}{llll} (1). & \check{\mathsf{h}}(z) & = & \left(\dfrac{1+z^{-1}}{2}\right)^p \check{\mathsf{q}}(z) \quad \text{and} \\ (2). & g_n & = & (-1)^n h^*_{N-n} \quad \text{[CQF CONDITION (Definition Q.6 page 386)]} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \psi \text{ has } p \\ \text{vanishing moments} \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
& \check{\mathsf{h}}(z) = \left(\frac{1+z^{-1}}{2}\right)^p \check{\mathsf{q}}(z) && \text{by (1)} \\
\Longrightarrow\quad & \check{\mathsf{h}}(\omega) = \left(\frac{1+e^{i\omega}}{2}\right)^p \tilde{\mathsf{q}}(\omega) && \text{by Definition R.1 page 393} \\
\Longrightarrow\quad & \left[\frac{d}{d\omega}\right]^n \check{\mathsf{h}}(\omega)\Big|_{\omega=\pi} = \left[\frac{d}{d\omega}\right]^n \left(\frac{1+e^{i\omega}}{2}\right)^p \tilde{\mathsf{q}}(\omega)\Big|_{\omega=\pi} \\
& = \sum_{k=0}^{n} \binom{n}{k} \left[\left[\frac{d}{d\omega}\right]^k \left(\frac{1+e^{i\omega}}{2}\right)^p\right] \left[\left[\frac{d}{d\omega}\right]^{n-k} \tilde{\mathsf{q}}(\omega)\right]\Bigg|_{\omega=\pi} && \text{by Lemma E.2 page 201}
\end{aligned}$$

[7] Daubechies (1992) page 155, Vidakovic (1999) pages 80–82

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Figure 5.2: Some B-splines

$$= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{p!}{2^p (p-k)!} \prod_{k=1}^{p} \left[i e^{i\omega} \left(1 + e^{i\omega}\right)^k \right] \right] \left[\frac{\mathrm{d}}{\mathrm{d}\omega}^{n-k} \tilde{\mathrm{q}}(\omega) \right] \Bigg|_{\omega=\pi}$$

$$= 0$$

$\Longleftrightarrow$ ψ has p vanishing moments by Theorem 5.3 page 70

✏

5.5 Examples

Example 5.2 (order 3 B-spline wavelet system). [8] The following figures illustrate scaling and wavelet coefficients and functions for a *B-spline*.

EX

n	h_n	g_n
0	$\left(\frac{\sqrt{2}}{16}\right)$	$\left(\frac{\sqrt{2}}{16}\right)$
1	$4\left(\frac{\sqrt{2}}{16}\right)$	$-4\left(\frac{\sqrt{2}}{16}\right)$
2	$6\left(\frac{\sqrt{2}}{16}\right)$	$6\left(\frac{\sqrt{2}}{16}\right)$
3	$4\left(\frac{\sqrt{2}}{16}\right)$	$-4\left(\frac{\sqrt{2}}{16}\right)$
4	$\left(\frac{\sqrt{2}}{16}\right)$	$\left(\frac{\sqrt{2}}{16}\right)$

✎PROOF:

[8] Strang (1989) page 616

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1. These results follow from Theorem O.11 page 362.

$$\left(\begin{array}{l|ccccccccc} n=0: & & & & & 1 & & & & \\ n=1: & & & & 1 & & 1 & & & \\ n=2: & & & 1 & & 2 & & 1 & & \\ n=3: & & 1 & & 3 & & 3 & & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array}\right)$$

2. Alternative …

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left[\frac{(z+1)^4}{(1+1)^4 z^4}\right] \\
&= \frac{\sqrt{2}}{2^4}\left[\frac{\sum_{n=0}^{4}\binom{4}{n}z^n}{z^4}\right] && \text{by } \textit{Binomial Theorem} \text{ (Theorem C.14 page 167)} \\
&= \frac{\sqrt{2}}{16}\left[\frac{z^4+4z^3+6z^2+4z^{+}1}{z^4}\right] && \text{by definition of } \binom{n}{k} \text{ (Definition B.3 page 146)} \\
&= \frac{\sqrt{2}}{16}\left[1+4z^{-1}+6z^{-2}+4z^{-3}+1z^{-4}\right]
\end{aligned}$$

✏

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CHAPTER 6

ORTHONORMALITY CONSTRAINT

6.1 Definition

In any Hilbert space $\boldsymbol{L}^2_{\mathbb{R}}$ with a subspace $\boldsymbol{Y}$, $\boldsymbol{Y} \dotplus \boldsymbol{Y}^{\perp} = \boldsymbol{L}^2_{\mathbb{R}}$ (*Projection Theorem*). The subspace $\boldsymbol{Y}^{\perp}$ is the *orthogonal complement* of $\boldsymbol{Y}$ in $\boldsymbol{L}^2_{\mathbb{R}}$. In wavelet theory, the orthogonal complement of a scaling subspace $\boldsymbol{V}_n$ in the subspace $\boldsymbol{V}_{n+1}$ is the *wavelet subspace* $\boldsymbol{W}_n$ (next definition). The wavelet subspace $\boldsymbol{W}_n$ is *unique* with respect to $\boldsymbol{V}_n$ because in a Hilbert space, $\boldsymbol{W}_n^{\perp} = \boldsymbol{V}_n^{\perp\perp} = \boldsymbol{V}_n$.

Definition 3.2 (page 40) defines a *general* wavelet system. An *orthonormal wavelet system* is simply a wavelet system with additional constraints of orthonormality (Definition 6.1 page 81).

Definition 6.1. *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40) and* $\boldsymbol{\Gamma} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *its component* MRA SYSTEM *(Definition 2.3 page 27). Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION *(Definition N.12 page 316).*

DEF

1. *The* MRA SYSTEM $\boldsymbol{\Gamma}$ *is* ***orthonormal*** *if* $\langle \phi \,|\, \mathbf{T}^n \phi \rangle = \bar{\delta}_n \quad \forall n \in \mathbb{Z}.$
2. *The* WAVELET SYSTEM $\boldsymbol{\Omega}$ *is* ***orthonormal*** *if* $\langle \psi \,|\, \mathbf{T}^n \psi \rangle = \bar{\delta}_n \quad \forall n \in \mathbb{Z}$
   and $\langle \phi \,|\, \mathbf{T}^n \psi \rangle = 0 \quad \forall n \in \mathbb{Z}.$

In the case of (1), $\boldsymbol{\Gamma}$ *is an* ***orthonormal MRA system****.*
In the case of (2), $\boldsymbol{\Omega}$ *is an* ***orthonormal wavelet system****.*

6.2 Orthonormal MRA systems

6.2.1 Scaling coefficients from scaling function

The *dilation equation* (Theorem 2.1 page 24) demonstrates that if we know the *scaling coefficients* $\left(\!\left(\mathsf{h}_n\right)\!\right)$ (Definition 2.3 page 27), we can (recursively) compute the scaling function ϕ (Definition 2.1 page 18). The converse is in general not so easy to compute. However, in an *orthonormal MRA system*, we can compute the scaling coefficients $\left(\!\left(\mathsf{h}_n\right)\!\right)$ from the scaling function ϕ (next).

Proposition 6.1. [1] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27).*

[1] Mallat (1999) page 228 ⟨(7.29)⟩, Wojtaszczyk (1997) page 38

PRP

$$\underbrace{\langle \phi \,|\, \mathbf{T}^n\phi\rangle = \bar{\delta}_n}_{\{\mathbf{T}^n\phi\}\ \textit{is}\ \text{ORTHONORMAL}} \quad\Longrightarrow\quad \mathsf{h}_n = \langle \phi \,|\, \mathbf{DT}^n\phi\rangle = \frac{\sqrt{2}}{2}\int_{\mathbb{R}} \phi(x)\phi^*(2x-n)\,\mathrm{dt} \qquad \forall n\in\mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\langle \phi \,|\, \mathbf{DT}^n\phi\rangle &= \left\langle \sum_{m\in\mathbb{Z}} h_m \mathbf{DT}^m\phi \,|\, \mathbf{DT}^n\phi \right\rangle && \text{by the } \textit{dilation equation} && \text{(Theorem 2.1 page 24)}\\
&= \sum_{m\in\mathbb{Z}} h_m \langle \mathbf{DT}^m\phi \,|\, \mathbf{DT}^n\phi\rangle && \text{by } \textit{additive} \text{ property of } \langle\triangle\,|\,\triangledown\rangle && \text{(Definition D.9 page 182)}\\
&= \sum_{m\in\mathbb{Z}} h_m \left\langle \sqrt{2}\phi(2x-m) \,|\, \sqrt{2}\phi(2x-n)\right\rangle && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{x\in\mathbb{R}} \sqrt{2}\phi(2x-m)\sqrt{2}\phi^*(2x-n)\,\mathrm{dx} && \text{by definition of } \langle\triangle\,|\,\triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)}\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{u\in\mathbb{R}} \sqrt{2}\phi(u-m)\sqrt{2}\phi^*(u-n)\tfrac{1}{2}\,\mathrm{du} && \text{where } u \triangleq 2x \implies && \mathrm{du} = 2\,\mathrm{dx}\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n)\,\mathrm{du}\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n)\,\mathrm{du}\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{v\in\mathbb{R}} \phi(v)\phi^*(v-(n-m))\,\mathrm{dv} && \text{where } v \triangleq u-m \implies && u = v+m\\
&= \sum_{m\in\mathbb{Z}} h_m \langle \phi \,|\, \mathbf{T}^{n-m}\phi\rangle && \text{by definition of } \langle\triangle\,|\,\triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)}\\
&= \sum_{m\in\mathbb{Z}} h_m \bar{\delta}_{m-n} && \text{by left hypothesis}\\
&= \mathsf{h}_n && \text{by definition of } \bar{\delta} && \text{(Definition N.12 page 316)}
\end{aligned}$$

$$\begin{aligned}
\langle \phi \,|\, \mathbf{DT}^n\phi\rangle &= \langle \phi \,|\, \mathbf{D}\phi(x-n)\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 4)}\\
&= \left\langle \phi \,|\, \tfrac{\sqrt{2}}{2}\phi(2x-n)\right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 4)}\\
&= \frac{\sqrt{2}}{2}\int_{\mathbb{R}} \phi(x)\phi^*(2x-n)\,\mathrm{dt} && \text{by definition of } \langle\triangle\,|\,\triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)}
\end{aligned}$$

✏

Example 6.1 (Haar scaling function).

EX

Let $\phi(x) \triangleq \begin{cases} 1 & \text{for } x \in [0:1)\\ 0 & \text{otherwise.}\end{cases}$

Then, $\mathsf{h}_n = \begin{cases} \frac{\sqrt{2}}{2} & \text{for } n \in \{0,1\}\\ 0 & \text{otherwise.}\end{cases}$

✎PROOF:

$\mathsf{h}_n = \langle \phi \,|\, \mathbf{DT}^n\phi\rangle$ by Proposition 6.1 page 81

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$$
\begin{aligned}
&= \left\langle \mathbb{1}_{[0,1)}(x) \,|\, \mathbf{DT}^{-n}\mathbb{1}_{[0,1)}(x)\right\rangle && \text{by definition of } \phi \\
&= \left\langle \mathbb{1}_{[0,1)}(x) \,|\, \sqrt{2}\mathbb{1}_{[0,1)}(2x-n)\right\rangle && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 4)} \\
&= \sqrt{2}\left\langle \mathbb{1}_{[0,1)}(x) \,|\, \mathbb{1}_{[0,1)}(2x-n)\right\rangle && \text{by properties of } \langle\triangle\,|\,\nabla\rangle && \text{(Definition D.9 page 182)} \\
&= \sqrt{2}\int_{\mathbb{R}} \mathbb{1}_{[0,1)}(x)\,\mathbb{1}_{[0,1)}(2x-n)\,\mathrm{dx} && \text{by definition of } \langle\triangle\,|\,\nabla\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)} \\
&= \sqrt{2}\int_0^1 \mathbb{1}_{[0,1)}(2x-n)\,\mathrm{dx} && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 3)} \\
&= \begin{cases} \sqrt{2}\int_0^{\frac{1}{2}} 1\,\mathrm{dx} &= \frac{\sqrt{2}}{2} \quad \text{for } n=0 \\ \sqrt{2}\int_{\frac{1}{2}}^{1} 1\,\mathrm{dx} &= \frac{\sqrt{2}}{2} \quad \text{for } n=1 \\ 0 &= 0 \quad \text{otherwise} \end{cases} && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 3)}
\end{aligned}
$$

✏

6.2.2 Necessary conditions

Proposition 6.2. [2] *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27). Let* $\tilde{\phi}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of* $\phi(x)$.

PRP

$$
\left\{\begin{array}{ll} (A). & \left(\!\left(\mathbf{T}^n\phi\right)\!\right) \text{ is ORTHONORMAL} \quad \text{and} \\ (B). & \tilde{\phi}(\omega) \text{ is CONTINUOUS at } 0 \end{array}\right\} \implies \left\{\begin{array}{lll} (1). & \left|\tilde{\phi}(0)\right| = \dfrac{1}{\sqrt{2\pi}} & \text{and} \\ (2). & \left|\displaystyle\int_{\mathbb{R}} \phi(x)\,\mathrm{dx}\right| = 1 & \end{array}\right\}
$$

✎PROOF:

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace $\boldsymbol{V}_j$ such that
   $\mathbf{V}_j = \left\{\mathbf{P}_j \mathsf{f} \,|\, \mathsf{f} \in \boldsymbol{H}\right\}$

2. Let $\mathsf{g}(x)$ a function in $\boldsymbol{L}^2_{\mathbb{R}}$ such that its *Fourier transform* (Definition M.2 page 294) $\tilde{\mathsf{g}}(\omega)$ has support $\operatorname{supp}\tilde{\mathsf{g}} \subseteq [-1 : 1]$

3. lemma: The set $\left\{\dfrac{1}{2^{j/2}\sqrt{2\pi}} e^{-i\frac{\omega}{2^j}n} \,\middle|\, n\in\mathbb{Z}\right\}$ is an *orthonormal basis* for functions in $\boldsymbol{L}^2_{\mathbb{R}}$ that have support in $\left[-2^j\pi : 2^j\pi\right]$. Proof: This follows from Theorem K.3 page 276.

$$
\begin{aligned}
\int_{-2^j\pi}^{2^j\pi} \left[\frac{1}{2^{j/2}\sqrt{2\pi}} e^{-i\frac{\omega}{2^j}n}\right]\left[\frac{1}{2^{j/2}\sqrt{2\pi}} e^{-i\frac{\omega}{2^j}m}\right]^* \mathrm{d}\omega &= \int_{-2^j\pi}^{2^j\pi} \frac{1}{2^j(2\pi)} e^{-i\frac{\omega}{2^j}(n-m)}\,\mathrm{d}\omega \\
&= \frac{1}{2^j(2\pi)}\left(2^j\pi + 2^j\pi\right)\bar{\delta}_{nm} \\
&= \bar{\delta}_{nm}
\end{aligned}
$$

[2] Wojtaszczyk (1997) page 31 ⟨Proposition 2.16⟩, Pinsky (2002) pages 315–316 ⟨6.4.3.1 Additional remarks⟩

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4. Proof for (1): Let $A \triangleq [-1 : 1]$.

$$\begin{aligned}
\|g\|^2 &= \left\|\lim_{j\to\infty} \mathbf{P}_j g\right\|^2 \\
&= \lim_{j\to\infty} \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbf{P}_j g \mid \mathbf{D}^j\mathbf{T}^n\phi\right\rangle\right|^2 && \text{by (A) and } \textit{Parseval's Identity} && \text{(Theorem N.9 page 318)} \\
&= \lim_{j\to\infty} \sum_{n\in\mathbb{Z}} \left|\left\langle g \mid \mathbf{D}^j\mathbf{T}^n\phi\right\rangle\right|^2 && \text{by definition of } \mathbf{P}_j && \text{(item (1) page 83)} \\
&= \lim_{j\to\infty} \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathbf{F}} g \mid \tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\phi\right\rangle\right|^2 && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem M.2 page 295)} \\
&= \lim_{j\to\infty} \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{g}(\omega) \,\middle|\, \frac{1}{2^{j/2}} e^{-i\frac{\omega}{2^j}n} \tilde{\phi}\left(\frac{\omega}{2^j}\right)\right\rangle\right|^2 && \text{by Proposition 1.12 page 12} \\
&= \lim_{j\to\infty} \sum_{n\in\mathbb{Z}} \left|\int_{\mathbb{R}} \tilde{g}(\omega)\tilde{\phi}^*\left(\frac{\omega}{2^j}\right) \frac{1}{2^{j/2}} e^{i\frac{\omega}{2^j}n}\right|^2 d\omega && \text{by definition of } \langle\triangle \mid \triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)} \\
&= \lim_{j\to\infty} \sum_{n\in\mathbb{Z}} \left|\left\langle \sqrt{2\pi}\tilde{g}(\omega)\tilde{\phi}^*\left(\frac{\omega}{2^j}\right) \,\middle|\, \frac{1}{2^{j/2}\sqrt{2\pi}} e^{-i\frac{\omega}{2^j}n}\right\rangle\right|^2 && \text{by definition of } \langle\triangle \mid \triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)} \\
&= \lim_{j\to\infty} \left\|\sqrt{2\pi}\tilde{g}(\omega)\tilde{\phi}^*\left(\frac{\omega}{2^j}\right)\right\|^2 && \text{by (3) lemma and } \textit{Parseval's Identity} && \text{(Theorem N.9 page 318)} \\
&= \left\|\sqrt{2\pi}\tilde{g}(\omega)\tilde{\phi}^*(0)\right\|^2 \\
&= 2\pi\left|\tilde{\phi}^*(0)\right|^2 \|\tilde{g}(\omega)\|^2 && \text{by } \textit{homogeneous} \text{ property of } \|\cdot\| && \text{(Definition D.5 page 174)} \\
&= 2\pi\left|\tilde{\phi}(0)\right|^2 \|g\|^2 && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem M.2 page 295)} \\
&\implies \left|\tilde{\phi}(0)\right| = \frac{1}{\sqrt{2\pi}}
\end{aligned}$$

5. Proof for (2):

$$\begin{aligned}
\left|\int_{\mathbb{R}} \phi(x)\, dx\right| &= \left|\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi(x)e^{-i\omega x}\, dx\right|_{\omega=0} \\
&= \left|\sqrt{2\pi}\tilde{\phi}(\omega)\right|_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)} \\
&= \left|\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\right| && \text{by (1)} && \text{(item (4) page 84)} \\
&= 1
\end{aligned}$$

✏

Every *MRA system* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$, regardless of whether it is *orthonormal* or not, has the following necessary conditions on $\langle\!\langle h_n \rangle\!\rangle$:

1. $\displaystyle\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}$ — *admissibility condition* (Theorem 2.3 page 29)
2. $\displaystyle\langle\phi \mid \mathbf{T}^n\phi\rangle = \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h_k^* \left\langle\phi \mid \mathbf{T}^{2n-m+k}\phi\right\rangle$ — *quadrature condition* in "time" (Theorem 2.4 page 31)
3. $\displaystyle\tilde{\phi}(\omega) = \frac{\sqrt{2}}{2}\breve{h}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$ — *frequency dilation equation* (Lemma 2.1 page 25)
4. $\displaystyle\left|\breve{h}(\omega)\right|^2\tilde{S}_{\phi\phi}(\omega) + \left|\breve{h}(\omega+\pi)\right|^2\tilde{S}_{\phi\phi}(\omega+\pi) = 2\tilde{S}_{\phi\phi}(2\omega)$ — *quadrature condition* in "frequency" (Theorem 2.5 page 32)

In an *orthonormal MRA system*, the *orthonormal* property is equivalent to $\tilde{S}_{\phi\phi}(\omega) = 1$ (where $\tilde{S}_{\phi\phi}$ is the *auto-power spectrum*), and so the quadrature conditions simplify considerably (next theorem).

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The *orthonormal quadrature conditions* in "time" and "frequency" are also equivalent to each other and are implied by the $\tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1$ condition. However, the *orthonormal quadrature conditions* are *not* equivalent to the $\tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1$ condition (Counterexample 6.1 page 85).

Lemma 6.1 (orthonormal quadrature conditions). [3] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27). Let* $\check{\mathsf{h}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *of* $\langle\!\langle \mathsf{h}_n \rangle\!\rangle$ *(Definition R.1 page 393). Let* $\tilde{\mathsf{S}}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM *of* $\phi(x)$ *(Definition T.3 page 411).*

LEM

$$\underbrace{\langle \phi \,|\, \mathbf{T}^n \phi \rangle = \bar{\delta}_n \quad \forall n \in \mathbb{Z}}_{\text{(1) } \{\mathbf{T}^n\phi\} \text{ is ORTHONORMAL}} \quad \Longleftrightarrow \quad \underbrace{\tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1}_{\text{(2) auto-power spectrum is 1}}$$

$$\underset{\not\Longleftarrow}{\Longrightarrow} \quad \underbrace{\left\{ \sum_{m\in\mathbb{Z}} \mathsf{h}_m \mathsf{h}^*_{m-2n} = \bar{\delta}_n \quad \forall n \in \mathbb{Z} \right\}}_{\text{(3) ORTHOGONAL QUADRATURE CONDITION in "time"}} \quad \Longleftrightarrow \quad \underbrace{\left\{ \left|\check{\mathsf{h}}(\omega)\right|^2 + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2 = 2 \right\}}_{\text{(4) ORTHOGONAL QUADRATURE CONDITION in "frequency"}}$$

✎PROOF:

1. Proof that (1) $\Longleftrightarrow$ (2): by Theorem T.3 page 417.

2. Proof that [(1)/(2)] $\Longrightarrow$ (3):

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} h_m \mathsf{h}^*_{m-2n} &= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} \mathsf{h}^*_k \bar{\delta}_{k-m+2n} && \text{by definition of } \bar{\delta} && \text{(Definition N.12 page 316)} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} \mathsf{h}^*_k \left\langle \phi \,|\, \mathbf{T}^{2n-m+k} \phi \right\rangle && \text{by (1)} \\
&= \langle \phi \,|\, \mathbf{T}^n \phi \rangle && \text{by } \textit{quadrature condition in "time"} && \text{(Theorem 2.4 page 31)} \\
&= \bar{\delta}_n && \text{by (1)}
\end{aligned}$$

3. Proof that (3) $\not\Longrightarrow$ [(1)/(2)]: by Counterexample 6.1 page 85.

4. Proof that (3) $\Longleftrightarrow$ (4): by Lemma 6.2 page 95.

✎

Counterexample 6.1. [4] Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ be an *MRA system* (Definition 2.3 page 27).

CNT

Let $\phi(x) \triangleq \begin{cases} \frac{1}{3} & \text{for } x \in [0:3) \\ 0 & \text{otherwise.} \end{cases}$

$$\text{Then,} \left\{ \begin{aligned} \check{\mathsf{h}}(\omega) + \check{\mathsf{h}}(\omega+\pi) &= 2 && \text{but} \\ \tilde{\mathsf{S}}_{\phi\phi}(\omega) &= \tfrac{1}{9}[3 + 4\cos(\omega) + 2\cos(2\omega)] && \text{and so} \\ \check{\mathsf{h}}(\omega) + \check{\mathsf{h}}(\omega+\pi) = 2 &\not\Longrightarrow \tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1. \end{aligned} \right\}$$

[3] Chui (1992) page 135, Mallat (1999) pages 229–238, Goswami and Chan (1999) page 110, Vaidyanathan (1990) page 65

[4] Vidakovic (1999) page 57 ⟨Remark 3.3.3⟩

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✎PROOF:

$$\begin{aligned}
\tilde{\phi}(\omega) &\triangleq \tilde{\mathbf{F}}\phi(x) \\
&\triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)} \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathbb{1}_{[0:3)}e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \phi(x) \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{3} e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 3)} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-i\omega x}}{-i\omega}\right)\Bigg|_0^3 \\
&= \left(\frac{3i}{\sqrt{2\pi}}\right)\left(\frac{e^{-i3\omega}-1}{3\omega}\right)
\end{aligned}$$

$$\begin{aligned}
\breve{\mathsf{h}}(\omega) &= \sqrt{2}\frac{\tilde{\phi}(2\omega)}{\tilde{\phi}(\omega)} && \text{by Lemma 2.1 page 25} \\
&= \sqrt{2}\frac{\left(\frac{3i}{\sqrt{2\pi}}\right)\left(\frac{e^{-i4\omega}-1}{6\omega}\right)}{\left(\frac{3i}{\sqrt{2\pi}}\right)\left(\frac{e^{-i2\omega}-1}{3\omega}\right)} && \text{by Lemma O.5 page 350} \\
&= \frac{\sqrt{2}}{2}\left(\frac{e^{-i6\omega}-1}{e^{-i3\omega}-1}\right) \\
&= \frac{\sqrt{2}}{2}\left(\frac{e^{-i6\omega}-1}{e^{-i3\omega}-1}\right)\left(\frac{e^{-i3\omega}+1}{e^{-i3\omega}+1}\right) \\
&= \frac{\sqrt{2}}{2}\left(\frac{\left(e^{-i6\omega}-1\right)\left(e^{-i3\omega}+1\right)}{e^{-i6\omega}-1}\right) \\
&= \frac{\sqrt{2}}{2}\left(1+e^{-i3\omega}\right) \\
&= \left(\frac{\sqrt{2}}{2}\right)e^{-i\frac{3}{2}\omega}\left(e^{i\frac{3}{2}\omega}+e^{-i\frac{3}{2}\omega}\right) \\
&= \left(\frac{\sqrt{2}}{2}\right)e^{-i\frac{3}{2}\omega}\left[2\cos\left(\frac{3}{2}\omega\right)\right] \\
&= \sqrt{2}e^{-i\frac{3}{2}\omega}\cos\left(\frac{3}{2}\omega\right)
\end{aligned}$$

$$\begin{aligned}
&\breve{\mathsf{h}}(\omega+\pi) \\
&= \frac{\sqrt{2}}{2}\left(1+e^{-i3(\omega+\pi)}\right) && \text{by previous } \breve{\mathsf{h}}(\omega) \text{ result} \\
&= \frac{\sqrt{2}}{2}\left(1+e^{-i3\omega}e^{-i3\pi}\right) \\
&= \frac{\sqrt{2}}{2}\left(1-e^{-i3\omega}\right) \\
&= \frac{\sqrt{2}}{2}e^{-i\frac{3}{2}\omega}\left(e^{i\frac{3}{2}\omega}-e^{-i\frac{3}{2}\omega}\right) \\
&= \frac{\sqrt{2}}{2}e^{-i\frac{3}{2}\omega}\left[2i\sin\left(\frac{3}{2}\omega\right)\right] \\
&= i\sqrt{2}e^{-i\frac{3}{2}\omega}\sin\left(\frac{3}{2}\omega\right)
\end{aligned}$$

$$\left|\breve{\mathsf{h}}(\omega)\right|^2+\left|\breve{\mathsf{h}}(\omega+\pi)\right|^2$$

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$$
\begin{aligned}
&= \left|\sqrt{2}e^{-i\frac{3}{2}\omega}\cos\left(\frac{3}{2}\omega\right)\right|^2 + \left|i\sqrt{2}e^{-i\frac{3}{2}\omega}\sin\left(\frac{3}{2}\omega\right)\right|^2 \\
&= 2\cos^2\left(\frac{3}{2}\omega\right) + 2\sin^2\left(\frac{3}{2}\omega\right) \\
&= 2\left[\cos^2\left(\frac{3}{2}\omega\right) + \sin^2\left(\frac{3}{2}\omega\right)\right] \\
&= 2 && \text{by } \textit{half-angle formulas} && \text{(Theorem I.11 page 248)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathsf{S}}_{\phi\phi}(\omega) &\triangleq \check{\mathsf{S}}_{\phi\phi}(z)\big|_{z=e^{in\omega}} && \text{by definition of } \tilde{\mathsf{S}}_{\phi\phi} && \text{(Definition T.3 page 411)} \\
&\triangleq \sum_{n\in\mathbb{Z}} \langle \phi \,|\, \mathbf{T}^n\phi\rangle\, z^{-n}\Big|_{z=e^{i\omega}} && \text{by definition of } \tilde{\mathsf{S}}_{\phi\phi} && \text{(Definition T.2 page 411)} \\
&\triangleq \sum_{n\in\mathbb{Z}} \left[\int_{\mathbb{R}} \frac{1}{3}\mathbb{1}_{[0:3)}(x)\frac{1}{3}\mathbb{1}_{[-n:3-n)}(x)\,\mathrm{d}x\right] z^{-n}\Big|_{z=e^{i\omega}} && \text{by definition of } \phi \\
&= \frac{1}{9}\sum_{n\in\mathbb{Z}} \left[\int_0^3 \mathbb{1}_{[-n:3-n)}(x)\,\mathrm{d}x\right] z^{-n}\Big|_{z=e^{i\omega}} && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 3)} \\
&= \frac{1}{9}\left[z^{-2} + 2z^{-1} + 3 + 2z + z^2\right]\Big|_{z=e^{i\omega}} \\
&= \frac{1}{9}\left[e^{-i2\omega} + 2e^{-i\omega} + 3 + 2e^{-i\omega} + e^{-i2\omega}\right] \\
&= \frac{1}{9}[3 + 4\cos(\omega) + 2\cos(2\omega)]
\end{aligned}
$$

✏

Theorem 6.1. *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ *be a wavelet system* (Definition 3.2 page 40).
Let $\check{\mathsf{h}}(z)$ *be the* Z-TRANSFORM (Definition Q.1 page 379) *of* h.

THM

$$
\underbrace{\langle \phi \,|\, \mathbf{T}^n\phi\rangle = \bar{\delta}_n \qquad \forall n\in\mathbb{Z}}_{\{\mathbf{T}^n\phi\}\ \textit{is}\ \text{ORTHONORMAL}} \quad\Longrightarrow\quad
\left\{
\begin{array}{llll}
1. & \displaystyle\sum_{m\in\mathbb{Z}} (-1)^m \mathsf{h}_m & = 0 & \text{(LOW-PASS FILTER)} \quad \textit{and} \\
2. & \displaystyle\sum_{m\in\mathbb{Z}} \mathbf{T}^m\phi & = 1 & \text{(PARTITION OF UNITY)} \quad \textit{and} \\
3. & \check{\mathsf{h}}(z)\big|_{z=-1} & = 0 & \text{(ZERO AT } z=-1\text{)}
\end{array}
\right\}
$$

✎PROOF:

1. Proof for (1):

$$
\begin{aligned}
0 &= \left[-2 + \left|\check{\mathsf{h}}(\omega)\right|^2 + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2\right]_{\omega=0} && \text{by } \textit{orthonormal quadrature condition} && \text{(Lemma 6.2 page 95)} \\
&= -2 + \left|\check{\mathsf{h}}(0)\right|^2 + \left|\check{\mathsf{h}}(\pi)\right|^2 \\
&= -2 + \left|\sqrt{2}\right|^2 + \left|\check{\mathsf{h}}(\pi)\right|^2 && \text{by } \textit{admissibility condition} && \text{(Theorem 2.3 page 29)} \\
&= 0 + \left|\check{\mathsf{h}}(\pi)\right|^2 \\
&= \left|\sum_{n\in\mathbb{Z}} h_n e^{-i\omega n}\right|^2_{\omega=\pi} && \text{by definition of } \check{\mathsf{h}} && \text{(Definition R.1 page 393)} \\
&= \left|\sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n\right|^2 && \text{because } e^{-i\pi n} = \left(e^{-i\pi}\right)^n = (-1)^n \\
\Longrightarrow \sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n &= \pm 0 = 0
\end{aligned}
$$

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2. Proof for (2): by (1) and Theorem 4.1 page 54.

3. Proof for (3): by (1) and Proposition R.4 page 396.

✏

6.2.3 Sufficient conditions

The *admissibility condition* (Theorem 2.3 page 29) and *quadrature condition* (Lemma 6.2 page 95) give **necessary** conditions for $(\!|\mathsf{h}_n|\!)$ to generate an orthogonal scaling function. Theorem 6.2 (next) shows that these two conditions are not only **necessary**, but also **sufficient** conditions.

Theorem 6.2. [5] *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *and* $\mathbf{D}$ *the* DILATION OPERATOR *(Definition 1.3 page 4). Let* $\left\{\mathsf{h}_n \middle| n\in\mathbb{Z}\right\}$ *be a sequence in* $\ell^2_{\mathbb{F}}$ *over some field* $\mathbb{F}$ *and with* DTFT *(Definition R.1 page 393)* $\check{\mathsf{h}}(\omega)$. *Let* $\phi(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *with* FOURIER TRANSFORM *(Definition M.2 page 294)* $\tilde{\phi}(\omega)$ *and defined in terms of* $(\!|\mathsf{h}_n|\!)$ *as*

$$\tilde{\phi}(\omega) \triangleq \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right) \qquad \textit{and} \qquad \boldsymbol{V}_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi(x)\middle| n\in\mathbb{Z}\right\}\ \forall j\in\mathbb{Z}.$$

Let an MRA *be defined as in Definition 2.1 page 18.*

THM

$$\left\{\begin{array}{lll} (A). & \sum_{n\in\mathbb{Z}}\mathsf{h}_n = \sqrt{2} & \textit{and} \\ (B). & \sum_{m\in\mathbb{Z}}\mathsf{h}_m h^*_{m-2n} = \bar{\delta}_n & \textit{and} \\ (C). & \tilde{\phi}(\omega)\ \textit{is}\ \text{CONTINUOUS}\ \textit{at}\ 0 & \textit{and} \\ (D). & \inf_{\omega\in[-\pi/2:\pi/2]}\left|\check{\mathsf{h}}(\omega)\right| > 0 & \end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\left(\boldsymbol{L}^2_{\mathbb{R}},\ (\!|\boldsymbol{V}_j|\!),\ \phi,\ (\!|h_n|\!)\right)\ \textit{is an} \\ \text{ORTHONORMAL MRA SYSTEM} \\ \textit{(Definition 2.3 page 27)}\end{array}\right\}$$

✎PROOF:

1. Proof that (A) $\Longleftarrow$ (right hypothesis): by Theorem 2.3 page 29.

2. Proof that (B) $\Longleftarrow$ (right hypothesis): by Lemma 6.2 page 95.

3. Proof that $(\!|\mathbf{T}^n\phi|\!)$ is *orthonormal*:

   (a) lemma: $(\!|\mathbf{T}^n\phi|\!)$ is *orthonormal* $\Longleftrightarrow$ $\int_{\mathbb{R}}\left|\tilde{\phi}(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega = \bar{\delta}_n$. Proof:

$$\begin{aligned}
\bar{\delta}_n &= \langle\phi\,|\,\mathbf{T}^n\phi\rangle && \text{by left hypothesis} && \text{(for } \Longrightarrow \text{ case)}\\
&= \left\langle\tilde{\mathbf{F}}\phi\,\middle|\,\tilde{\mathbf{F}}\mathbf{T}^n\phi\right\rangle && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem M.2 page 295)}\\
&= \left\langle\tilde{\phi}(\omega)\,\middle|\,e^{-i\omega n}\tilde{\phi}(\omega)\right\rangle && \text{by Corollary 1.3 page 12} && \\
&= \int_{\mathbb{R}}\tilde{\phi}(\omega)\left[\tilde{\phi}(\omega)e^{-i\omega n}\right]^*\mathrm{d}\omega && \text{by definition of } \langle\triangle\,|\,\triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition D.9 page 182)}\\
&= \int_{\mathbb{R}}\left|\tilde{\phi}(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega && && \\
&= \bar{\delta}_n && \text{by right hypothesis} && \text{(for } \Longleftarrow \text{ case)}
\end{aligned}$$

[5] Mallat (1999) pages 229–234, Mallat (1989) page 76, Mallat (2009) pages 271–276 ⟨Theorem 7.2⟩, Meyer (1992)

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(b) definition: $\tilde{\phi}_N(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \prod_{k=1}^{N} \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right) \mathbb{1}_A(\omega)$ and where $A \triangleq \left[-2^N\pi : 2^N\pi\right]$.

(c) definition: Let $\mathtt{I}_N(n) \triangleq \int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N-1} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega$

(d) lemma: $\mathtt{I}_N(n) = 2\pi\bar{\delta}_n \quad \forall N{\in}\mathbb{N}$. Proof:

i. $N = 1$ case:

$$\begin{aligned}
\mathtt{I}_1(n) &\triangleq \int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N-1} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega\Bigg|_{N=1} && \text{by definition of } \mathtt{I}_N \quad \text{(definition 3c page 89)} \\
&= \int_0^{2\pi} e^{in\omega} \prod_{k=1}^{0} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \\
&= \int_0^{2\pi} e^{in\omega} \mathrm{d}\omega \\
&= 2\pi\bar{\delta}_n
\end{aligned}$$

ii. Proof that $\mathtt{I}_{N+1}(n) = \mathtt{I}_N(n)$:

$$\begin{aligned}
&\mathtt{I}_{N+1}(n) \\
&= \int_0^{2^{N+1}\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \qquad \text{by definition of } \mathtt{I}_N \text{ (definition 3c page 89)} \\
&= \int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega + \int_{\omega=2^N\pi}^{\omega=2^{N+1}\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \\
&= \int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega + \int_{v=0}^{v=2^N\pi} e^{i(v-2^N\pi)n} \prod_{k=1}^{N} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{v+2^N\pi}{2^k}\right)\right|^2 \mathrm{d}v \\
&\quad \text{where } v \triangleq \omega - 2^N\pi \implies \omega = v + 2^N\pi \\
&= \int_0^{2^N\pi} e^{in\omega} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^N}\right)\right|^2 \prod_{k=1}^{N-1} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \\
&+ \int_{v=0}^{v=2^N\pi} e^{in(v} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{v}{2^N}+\pi\right)\right| \prod_{k=1}^{N-1} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{v}{2^k}+2^{N-k}\pi\right)\right|^2 \mathrm{d}v \\
&= \int_0^{2^N\pi} e^{in\omega} \frac{1}{2}\left[\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^N}\right)\right|^2 + \left|\breve{\mathsf{h}}\left(\frac{\omega}{2^N}+\pi\right)\right|^2\right] \prod_{k=1}^{N-1} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \qquad \text{by Proposition R.1 page 393} \\
&= \int_0^{2^N\pi} e^{in\omega} \frac{1}{2}[2] \prod_{k=1}^{N-1} \frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \qquad \text{by (B) and } \textit{orthonormal quad. cond.} \text{ (Lemma 6.1 page 85)} \\
&= \mathtt{I}_N(n) \qquad \text{by definition of } \mathtt{I}_N(n) \text{ (definition 3c page 89)}
\end{aligned}$$

iii. Proof that $\mathtt{I}_N(n) = 2\pi\bar{\delta}_n$:

$$\begin{aligned}
\mathtt{I}_N(n) &= \mathtt{I}_1(n) && \text{by induction and (3(d)ii) lemma page 89} \\
&= 2\pi\bar{\delta}_n && \text{by (3(d)i) lemma page 89}
\end{aligned}$$

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(e) lemma: $\int_{\mathbb{R}} \left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega = \bar{\delta}_n \quad \forall N{\in}\mathbb{N}$. Proof:

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega \\
&= \int_{\mathbb{R}} e^{in\omega}\left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\mathbb{1}_{[-2^N\pi:2^N\pi]}\right|^2 \mathrm{d}\omega \qquad \text{by definition of } \tilde{\phi}_N(\omega) \text{ (definition 3b page 89)} \\
&= \frac{1}{2\pi}\int_{-2^N\pi}^{2^N\pi} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-2^N\pi}^{0} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{v=0}^{v=2^N\pi} e^{i(v-2^N\pi)n}\prod_{k=1}^{N}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{v}{2^k}-2^{N-k}\pi\right)\right|^2 \mathrm{d}v \\
&\qquad + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \qquad \text{where } v \triangleq \omega + 2^N\pi \implies \omega = v - 2^N\pi \\
&= \frac{1}{2\pi}\int_{v=0}^{v=2^N\pi} e^{ivn}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{v}{2^k}-\pi\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{v}{2^k}\right)\right|^2 \mathrm{d}v \\
&\qquad + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^N}\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \qquad \text{by Proposition R.1 page 393} \\
&= \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^N}-\pi+2\pi\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \\
&\qquad + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^N}\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \qquad \text{by Proposition R.1 page 393} \\
&= \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left[\left|\check{\mathrm{h}}\left(\frac{\omega}{2^N}\right)\right|^2 + \left|\check{\mathrm{h}}\left(\frac{\omega}{2^N}+\pi\right)\right|^2\right]\prod_{k=1}^{N-1}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}[2]\prod_{k=1}^{N-1}\frac{1}{2}\left|\check{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \mathrm{d}\omega \qquad \text{by (B) and Lemma 6.2 page 95} \\
&= \frac{1}{2\pi}\mathtt{I}_N(n) \qquad \text{by definition of } \mathtt{I}_N(n) \text{ (definition 3c page 89)} \\
&= \bar{\delta}_n \qquad \text{by (3d) lemma page 89}
\end{aligned}
$$

(f) lemma: $\tilde{\phi}(\omega)$ is in $\boldsymbol{L}^2_{\mathbb{R}}$ (alternatively, proof that $\left|\tilde{\phi}(\omega)\right|^2$ is *integrable*). Proof:

$$
\begin{aligned}
\int_{\mathbb{R}} \left|\tilde{\phi}(\omega)\right|^2 \mathrm{d}\omega &= \int_{\mathbb{R}} \lim_{N\to\infty}\left|\tilde{\phi}_N(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \tilde{\phi}_N(\omega) && \text{(definition 3b page 89)} \\
&\le \lim_{N\to\infty}\int_{\mathbb{R}} \left|\tilde{\phi}_N(\omega)\right|^2 \mathrm{d}\omega && \text{by } \textit{Fatou's Lemma} \\
&= 1 && \text{by (3e) lemma page 90} \\
\implies \quad \tilde{\phi}(\omega) \text{ is in } \boldsymbol{L}^2_{\mathbb{R}} && && \text{by definition of } \langle\triangle\,|\,\nabla\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)}
\end{aligned}
$$

(g) lemma (*upper-bound condition*): $\exists C \ge 1$ such that $\left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right| \le C\left|\tilde{\phi}(\omega)\right|^2$. Proof:

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i. Proof for $|\omega| > 2^N\pi$:

$$\begin{aligned}
\left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right| &= \left|\tilde{\phi}_N(\omega)\right|^2 && \text{by definition of } |\cdot| \\
&= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\mathbb{1}_{[-2^N\pi:2^N\pi]}(\omega)\right|^2 && \text{by def. of } \tilde{\phi}_N(\omega) \quad \text{(definition 3b page 89)} \\
&= 0 \qquad \forall|\omega| > 2^N\pi && \text{by definition of } \mathbb{1} \quad \text{(Definition 1.2 page 3)} \\
&\le \left|\tilde{\phi}(\omega)\right|^2 && \text{by definition of } |\cdot|
\end{aligned}$$

ii. lemma: $\left|\tilde{\phi}(\omega)\right|^2 \ge \frac{1}{2\pi C}\ {\scriptstyle \forall|\omega|\le\pi} \quad\Longrightarrow\quad \left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right| \le C\left|\tilde{\phi}(\omega)\right|^2\ {\scriptstyle \forall|\omega|\le 2^N\pi}$

$$\begin{aligned}
\left|\tilde{\phi}(\omega)\right|^2 &= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by definition of } \tilde{\phi}(\omega) \\
&= \left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by Lemma 2.2 page 26} \\
&= 2\pi\left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\right|^2\left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \\
&= 2\pi\left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\right|^2\left|\tilde{\phi}_N(\omega)\right|^2 && \text{by def. of } \tilde{\phi}_N(\omega) \quad \text{(definition 3b page 89)} \\
&\ge (2\pi)\frac{1}{2\pi C}\left|\tilde{\phi}_N(\omega)\right|^2 && \text{by left hypothesis} \\
&= \frac{1}{C}\left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right|
\end{aligned}$$

$\Longrightarrow$ right hypothesis

iii. Proof for $0 \le |\omega| < \varepsilon$:

A. lemma: There exists $\varepsilon > 0$ such that $\left|\check{\mathsf{h}}(\omega)\right|^2 \le 2 = \left|\check{\mathsf{h}}(0)\right|^2$. Proof:

$$\begin{aligned}
\left|\check{\mathsf{h}}(\omega)\right|^2 &\le \left|\check{\mathsf{h}}(\omega)\right|^2 + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2 \\
&= 2 && \text{by (B) and (Lemma 6.1 page 85)} \\
&= \left|\check{\mathsf{h}}(0)\right|^2 && \text{by (A) and Proposition R.2 page 395}
\end{aligned}$$

B. It follows from (3(g)iiiA) lemma that there exists $\varepsilon > 0$ such that $\forall|\omega| \le \varepsilon$

$$\begin{aligned}
0 &\ge \log_e\left(\tfrac{1}{2}\left|\check{\mathsf{h}}(\omega)\right|^2\right) && \text{by (3(g)iiiA) lemma} \\
-|\omega| &\le \log_e\left(\tfrac{1}{2}\left|\check{\mathsf{h}}(\omega)\right|^2\right) \le 0
\end{aligned}$$

C. last step in proof for $0 < \varepsilon < |\omega| \le 2^N\pi$:

$$\begin{aligned}
\left|\tilde{\phi}(\omega)\right|^2 &= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by definition of } \tilde{\phi}(\omega) \\
&= \frac{1}{2\pi}\prod_{k=1}^{\infty}\frac{1}{2}\left|\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \\
&= \frac{1}{2\pi}\exp\log_e\left[\prod_{k=1}^{\infty}\frac{1}{2}\left|\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\right] \\
&= \frac{1}{2\pi}\exp\left\{\log_e\left[\prod_{k=1}^{\infty}\frac{1}{2}\left|\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\right]\right\}
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2\pi}\exp\left\{\sum_{k=1}^{\infty}\log_e\left[\frac{1}{2}\left|\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\right]\right\}\\
&\geq \frac{1}{2\pi}\exp\left\{\log_e\left[\frac{1}{2}\left|\check{\mathsf{h}}\left(\frac{\omega}{2^0}\right)\right|^2\right]\right\}\\
&\geq \frac{1}{2\pi}e^{-|\omega|}\\
&\geq \frac{1}{2\pi}e^{-\varepsilon}\\
&\Longrightarrow \quad \text{upper bound condition} \qquad \text{by (3(g)ii) lemma page 91}
\end{aligned}
$$

iv. Proof for $\varepsilon \leq |\omega| \leq \pi$:

A. Choose N such that $2^{-N}\pi < \varepsilon$

B. Let $K = \inf_{\omega\in[-\omega/2:\omega/2]}\left|\check{\mathsf{h}}(\omega)\right| > 0$

C. Then

$$
\begin{aligned}
\left|\tilde{\phi}(\omega)\right|^2 &= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by definition of } \tilde{\phi}(\omega)\\
&= \left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by Lemma 2.2 page 26}\\
&= 2\pi\left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\right|^2\left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\\
&\geq 2\pi\frac{1}{2\pi}e^{-\varepsilon}\frac{1}{2\pi}\prod_{k=1}^{N}\frac{1}{2}\left|\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by item (3(g)ivA) and (3(g)iiiC) lemma page 91}\\
&\geq \frac{1}{2\pi}e^{-\varepsilon}\prod_{k=1}^{N}\frac{1}{2}|K|^2\\
&= \frac{1}{2\pi}e^{-\varepsilon}\frac{K^{2N}}{2^N}
\end{aligned}
$$

(h) Final step in proof for (2) (proof that $(\!|\mathbf{T}^n\phi|\!)$ is *orthonormal*):

$$
\begin{aligned}
\int_{\mathbb{R}}\left|\tilde{\phi}(\omega)e^{in\omega}\right|^2\,\mathrm{d}\omega &= \int_{\mathbb{R}}\lim_{N\to\infty}\left|\tilde{\phi}_N(\omega)e^{in\omega}\right|^2\,\mathrm{d}\omega\\
&= \lim_{N\to\infty}\int_{\mathbb{R}}\left|\tilde{\phi}_N(\omega)e^{in\omega}\right|^2\,\mathrm{d}\omega && \text{by (3g) lemma and } \textit{Dominated Convergence Theorem}\\
&= \bar{\delta}_n && \text{by (3e) lemma page 90}\\
\Longrightarrow (2) && \text{by (3a) lemma page 88}
\end{aligned}
$$

4. Proof that $(\!|V_j|\!)$ is an *MRA* (Definition 2.1 page 18):

(a) $V_j \triangleq \operatorname{span}(\!|\mathbf{T}^n\mathbf{D}^j\phi|\!)$, so we just need to prove that ϕ is a *scaling function* (Definition 2.1 page 18), and then the claim that $(\!|V_j|\!)$ is an *MRA* follows from Theorem 2.6 page 33.

(b) Proof that $\{\mathbf{T}^n\phi\}$ is an *MRA*:

i. Proof that $(\!|\mathbf{T}^n\phi|\!)$ is a *Riesz basis*: by (1) (*orthonormal* property, item (3) page 88) and by Theorem N.13 page 324.

ii. Proof that $\exists\,(\!|\mathsf{h}_n|\!)$ such that $\phi(x) = \sum_{n\in\mathbb{Z}} h_n\mathbf{D}\mathbf{T}^n\phi(x)$: by definition of $\tilde{\phi}(\omega)$, hypothesis (C), and Lemma 2.2 page 26.

iii. Proof that $\tilde{\phi}(\omega)$ is *continuous* at 0: by hypothesis (C).

iv. Proof that $\tilde{\phi}(0) \neq 0$: by hypothesis (D).

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5. Proof that $(\!(\mathsf{h}_n)\!)$ is a *scaling coefficient sequence* (Definition 2.3 page 27):

$$\begin{aligned}
\tilde{\phi}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right) && \text{by definition of } \tilde{\phi}(\omega)\\
\Longrightarrow\quad \phi(x) &= \sum_{n\in\mathbb{Z}}\mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) && \text{by Lemma 2.2 page 26}\\
\Longrightarrow\quad & (\!(\mathsf{h}_n)\!) \text{ is a } \textit{scaling coefficient sequence} && \text{by (Definition 2.3 page 27)}
\end{aligned}$$

✏

6.2.4 Orthonormal basis from Riesz basis

The definition of a general wavelet system (Definition 3.2 page 40) requires an MRA. The last condition of the definition of an MRA (Definition 2.1 page 18) only requires that a *Riesz basis* exist for the scaling subspace V_0. The MRA definition does *not* require this basis to be *orthonormal*. However, the definition of an *orthonormal wavelet system* (Definition 6.1 page 81) does require an orthonormal basis for V_0. The good news is that if you have a Riesz basis for V_0 (which of course you do since you have an MRA), then from this Riesz basis you can generate an orthonormal basis. One way of doing this is given by Theorem 6.3 (next). That is, Theorem 6.3 gives a sufficient condition for generating an orthonormal basis $\{\mathbf{T}^n\phi\}$ from a Riesz basis $\{\mathbf{T}^n\theta\}$.

Theorem 6.3 (Battle-Lemarié orthogonalization). [6] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27). Let* $\tilde{\phi}(\omega)$ *be the Fourier Transform (Definition M.2 page 294) of an arbitrary function* $\phi \in L^2_{\mathbb{R}}$.

THM

$$\left\{\begin{array}{ll}
1. & \left\{\mathbf{T}^n\theta \,|\, n\in\mathbb{Z}\right\} \textit{ is a } \text{RIESZ BASIS } \textit{for } V_0 \quad \textit{and}\\
2. & \tilde{\phi}(\omega) \triangleq \dfrac{\tilde{\theta}(\omega)}{\sqrt{2\pi\sum_n\left|\tilde{\theta}(\omega+2\pi n)\right|^2}}
\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\left\{\mathbf{D}^m\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\right\}\\ \textit{is an } \text{ORTHONORMAL BASIS } \textit{for } V_m\end{array}\right\}$$

✎PROOF:

1. Proof that $\{\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\}$ is orthonormal: by Theorem N.15 page 327
2. Proof that $\{\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\}$ is a basis for V_0: by Lemma N.1 page 311.
3. Proof that $\{\mathbf{D}^m\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\}$ is a basis for V_m: by Theorem 2.2 page 27.

✏

6.3 Orthonormal wavelet systems

6.3.1 Subspace properties

Definition 6.1 (page 81) requires that the scaling subspace V_0 and the wavelet subspace W_0 be orthogonal. However, Proposition 6.3 (next) shows that this constraint implies that for all scales n, V_n

[6] Vidakovic (1999) page 71, Mallat (1989) page 72, Mallat (1999) page 225

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is orthogonal to W_n:

Proposition 6.3. [7] *Let* $\left(L^2_{\mathbb{R}}, (V_j), (W_j), \phi, \psi, (h_n), (g_n)\right)$ *be a wavelet system* (Definition 3.2 page 40).

PRP

$$\underbrace{V_0 \perp W_0}_{\text{orthogonal subspaces}} \quad \Longrightarrow \quad \underbrace{V_n \perp W_n \quad \forall n \in \mathbb{Z}}_{\text{orthogonal subspaces}}$$

✎PROOF:

$$\begin{aligned} 0 &= \langle \phi(x) \,|\, \psi(x) \rangle && \text{because } V_0 \perp W_0 \\ &= \left\langle \phi(2^j u) \,|\, \psi(2^j u) \right\rangle && \text{where } u = 2^{-j}x \iff x = 2^j u \\ \Longrightarrow \quad & V_j \perp W_j \end{aligned}$$

✏

6.3.2 Wavelet coefficients from wavelet and scaling functions

In an *orthonormal MRA system* (Definition 6.1 page 81), the scaling coefficients (h_n) can be computed from ϕ (Proposition 6.1 page 81). In an *orthonormal wavelet system* (Definition 6.1 page 81), the wavelet coefficients (g_n) can be computed from ϕ and ψ (next proposition).

Proposition 6.4. *Let* $\left(L^2_{\mathbb{R}}, (V_j), (W_j), \phi, \psi, (h_n), (g_n)\right)$ *be a* WAVELET SYSTEM (Definition 3.2 page 40).

PRP

$$\underbrace{\langle \phi \,|\, \mathbf{T}^n \phi \rangle = \bar{\delta}_n}_{\{\mathbf{T}^n\phi\} \text{ is ORTHONORMAL}} \quad \Longrightarrow \quad \mathsf{g}_n = \langle \psi \,|\, \mathbf{D}\mathbf{T}^n \phi \rangle \quad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\langle \psi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle &= \left\langle \sum_{m\in\mathbb{Z}} g_m \mathbf{D}\mathbf{T}^m\phi \,\middle|\, \mathbf{D}\mathbf{T}^n\phi \right\rangle && \text{by Theorem 3.1 page 39} \\
&= \sum_{m\in\mathbb{Z}} g_m \langle \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle \\
&= \sum_{m\in\mathbb{Z}} g_m \left\langle \sqrt{2}\phi(2x-m) \,\middle|\, \sqrt{2}\phi(2x-n) \right\rangle && \text{by definitions of } \mathbf{T} \text{ and } \mathbf{D} \quad \text{(Definition 1.3 page 4)} \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{x\in\mathbb{R}} \sqrt{2}\phi(2x-m)\sqrt{2}\phi^*(2x-n)\,\mathrm{dx} && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ in } L^2_{\mathbb{R}} \quad \text{(Definition E.1 page 199)} \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{u\in\mathbb{R}} \sqrt{2}\phi(u-m)\sqrt{2}\phi^*(u-n)\tfrac{1}{2}\,\mathrm{du} && u = 2x \qquad \mathrm{du} = 2\,\mathrm{dx} \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n)\,\mathrm{du} \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n)\,\mathrm{du} \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{v\in\mathbb{R}} \phi(v)\phi^*(v-(n-m))\,\mathrm{dv} && \text{where } v \triangleq u-m \implies u = v+m \\
&= \sum_{m\in\mathbb{Z}} g_m \langle \phi \,|\, \mathbf{T}^{n-m}\phi \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ in } L^2_{\mathbb{R}} \quad \text{(Definition E.1 page 199)}
\end{aligned}$$

[7] Jawerth and Sweldens (1994) pages 10–11

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$$= \sum_{m\in\mathbb{Z}} g_m \bar{\delta}_{m-n} \qquad \text{by left hypothesis}$$
$$= g_n$$

✏

Example 6.2 (Haar wavelet).

EX

$$\text{Let } \phi(x) \triangleq \begin{cases} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } \psi(x) \triangleq \begin{cases} 1 & \text{for } x \in \left[0:\frac{1}{2}\right) \\ -1 & \text{for } x \in \left[\frac{1}{2}:1\right) \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Then, } g_n = \begin{cases} \frac{\sqrt{2}}{2} & \text{for } n = 0 \\ -\frac{\sqrt{2}}{2} & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

✎PROOF:

$$\begin{aligned}
g_n &= \langle \psi \mid \mathbf{DT}^n \phi \rangle && \text{by Proposition 6.4 page 94} \\
&= \left\langle \mathbb{1}_{[0:1)}(x) - 2\mathbb{1}_{\left[\frac{1}{2}:1\right)}(x) \mid \sqrt{2}\mathbb{1}_{[0,1)}(2x-n) \right\rangle && \text{by definition of } \phi \text{ and } \psi \\
&= \sqrt{2}\left\langle \mathbb{1}_{[0:1)}(x) \mid \mathbb{1}_{[0,1)}(2x-n) \right\rangle - 2\sqrt{2}\left\langle \mathbb{1}_{\left[\frac{1}{2}:1\right)}(x) \mid \mathbb{1}_{[0,1)}(2x-n) \right\rangle && \text{by property of } \langle \triangle \mid \triangledown \rangle \\
&= h_n - 2\sqrt{2}\int_{\mathbb{R}} \mathbb{1}_{\left[\frac{1}{2}:1\right)}(x)\mathbb{1}_{[0,1)}(2x-n)\,dx && \text{by Example 6.1 page 82} \\
&= h_n - 2\sqrt{2}\int_{\left[\frac{1}{2}:1\right)} (x)\mathbb{1}_{[0,1)}(2x-n)\,dx && \text{by definition of } \mathbb{1} \quad \text{(Definition 1.2 page 3)} \\
&= h_n - 2\sqrt{2}\tfrac{1}{2}\bar{\delta}_{n-1} && \text{by definition of } \mathbb{1} \quad \text{(Definition 1.2 page 3)} \\
&= \begin{cases} \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2} & \text{for } n = 0 \\ \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} & \text{for } n = 1 \\ 0 = 0 & \text{otherwise} \end{cases} && \text{by definition of } \bar{\delta} \quad \text{(Definition N.12 page 316)}
\end{aligned}$$

✏

6.3.3 Necessary conditions

Lemma 6.2 (orthonormal quadrature conditions). [8] *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \langle\!\langle \boldsymbol{V}_j \rangle\!\rangle, \langle\!\langle \boldsymbol{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ *be a wavelet system* (Definition 3.2 page 40). *Let* $\check{\mathsf{h}}(\omega)$ *and* $\tilde{\mathsf{g}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORMS

[8] Chui (1992) page 135, Mallat (1999) pages 229–238, Goswami and Chan (1999) page 110, Vaidyanathan (1990) page 65

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(Definition R.1 page 393) of $(\!(\mathsf{h}_n)\!)$ *and* $(\!(\mathsf{g}_n)\!)$*, respectively.*

LEM

$$\underbrace{\left\{\begin{array}{lll} 1. & \langle \phi \mid \mathbf{T}^n\phi\rangle = \bar{\delta}_n & \textit{and} \\ 2. & \langle \psi \mid \mathbf{T}^n\psi\rangle = \bar{\delta}_n & \textit{and} \\ 3. & \langle \phi \mid \mathbf{T}^n\psi\rangle = 0 & \end{array}\right\}}_{\Omega \textit{ is } \text{ORTHONORMAL} \textit{ (Definition 6.1 page 81)}} \implies \underbrace{\left\{\begin{array}{lll} A. & \sum_{m\in\mathbb{Z}} \mathsf{h}_m\mathsf{h}^*_{m-2n} = \bar{\delta}_n & \textit{and} \\ B. & \sum_{m\in\mathbb{Z}} \mathsf{g}_m\mathsf{g}^*_{m-2n} = \bar{\delta}_n & \textit{and} \\ C. & \sum_{m\in\mathbb{Z}} \mathsf{h}_m\mathsf{g}^*_{m-2n} = 0 & \end{array}\right\}}_{\text{ORTHONORMAL QUADRATURE CONDITIONS } \textit{in "time"}} \quad \forall n\in\mathbb{Z}$$

$$\iff \underbrace{\left\{\begin{array}{lllll} a. & |\check{\mathsf{h}}(\omega)|^2 & + & |\check{\mathsf{h}}(\omega+\pi)|^2 & = 2 \quad \textit{and} \\ b. & |\tilde{\mathsf{g}}(\omega)|^2 & + & |\tilde{\mathsf{g}}(\omega+\pi)|^2 & = 2 \quad \textit{and} \\ c. & \check{\mathsf{h}}(\omega)\tilde{\mathsf{g}}^*(\omega) & + & \check{\mathsf{h}}(\omega+\pi)\tilde{\mathsf{g}}^*(\omega+\pi) & = 0 \end{array}\right\}}_{\text{ORTHONORMAL QUADRATURE CONDITIONS } \textit{in "frequency"}}$$

✎PROOF:

1. Proof that (1,2,3) $\implies$ (A): by Lemma 6.1 page 85.
2. Proof that (1,2,3) $\implies$ (B):

$$\begin{aligned} \sum_{m\in\mathbb{Z}} g_m g^*_{m-2n} &= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g^*_k \bar{\delta}_{k-m+2n} && \text{by definition of } \bar{\delta}_n && \text{(Definition N.12 page 316)} \\ &= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g^*_k \left\langle \phi \mid \mathbf{T}^{2n-m+k}\phi\right\rangle && \text{by hypothesis (1)} \\ &= \langle \psi \mid \mathbf{T}^n\psi\rangle && \text{by } \textit{quadrature conditions in "time"} && \text{(Theorem 3.3 page 43)} \\ &= \bar{\delta}_n && \text{by hypothesis (2)} \end{aligned}$$

3. Proof that (1,2,3) $\implies$ (C):

$$\begin{aligned} \sum_{m\in\mathbb{Z}} h_m g^*_{m-2n} &= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g^*_k \bar{\delta}_{k-m+2n} && \text{by definition of } \bar{\delta}_n && \text{(Definition N.12 page 316)} \\ &= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g^*_k \left\langle \phi \mid \mathbf{T}^{2n-m+k}\phi\right\rangle && \text{by hypothesis (1)} \\ &= \langle \phi \mid \mathbf{T}^n\psi\rangle && \text{by } \textit{quadrature conditions in "time"} && \text{(Theorem 3.3 page 43)} \\ &= 0 && \text{by hypothesis (3)} \end{aligned}$$

4. Proof that (A,B,C) $\iff$ (a,b,c): by Lemma 6.2 page 95.

✏

Proposition 6.5. [9] *Let* $\Omega \triangleq \left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40). Let* $\tilde{\mathsf{v}}(\omega)$ *be a* 2π PERIODIC *and* MEASURABLE *function in* $L^2_{\mathbb{R}}$ *(Definition E.1 page 199).*

PRP

Let $V_0 \perp W_0$ *and* $\{\mathbf{T}^n\phi\}$ *be* ORTHONORMAL. *Then*

$$\underbrace{\mathsf{f}\in W_0}_{(A)} \iff \left\{\begin{array}{ll} (1). & \mathsf{f}\in V_1 \quad \textit{and} \\ (2). & \textit{There exists a } \text{MEASURABLE} \textit{ function } \tilde{\mathsf{v}}(\omega)\in L^2_{\mathbb{R}} \textit{ such that} \\ & \tilde{\mathsf{v}}(\omega) = \tilde{\mathsf{v}}(\omega+2\pi) \ (2\pi \text{ PERIODIC}) \textit{ and} \\ & \tilde{\mathsf{f}}(\omega) = \frac{\sqrt{2}}{2} e^{i\frac{\omega}{2}} \tilde{\mathsf{v}}(\omega)\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \end{array}\right\}$$

[9] Wojtaszczyk (1997) pages 32–36 ⟨Proposition 2.18⟩, Hernández and Weiss (1996) pages 55–57 ⟨Lemma 2.11⟩

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✎PROOF:

1. Proof that (A) $\Longrightarrow$ (1):

$$\begin{aligned}
\mathsf{f} \in \boldsymbol{W}_0 &\Longrightarrow \mathsf{f} \in \boldsymbol{W}_0 \hat{+} \boldsymbol{V}_0 && \text{by definition of } \textit{Minkowski addition } \hat{+} && \text{(Definition W.2 page 427)}\\
&\Longleftrightarrow \mathsf{f} \in \boldsymbol{V}_1 && \text{because } \boldsymbol{V}_1 = \boldsymbol{V}_0 \hat{+} \boldsymbol{W}_0 && \text{(Definition 3.1 page 39)}
\end{aligned}$$

2. Proof that (A) $\Longrightarrow$ (2):

(a) There exists a 2π *periodic* function $\tilde{\mathsf{a}}(\omega)$ such that $\tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$. Proof:

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}\mathsf{f}(x) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)}\\
&= \tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n \mathbf{D}\mathbf{T}^n\phi && \text{by (A), } \mathsf{f} \in \boldsymbol{W}_0 \subseteq \boldsymbol{V}_1 && \\
&= \sum_{n\in\mathbb{Z}} a_n \tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n\phi && && \\
&= \sum_{n\in\mathbb{Z}} a_n e^{i\frac{\omega}{2}}\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by Proposition 1.12 page 12} && \\
&= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \textit{DTFT} && \text{(Definition R.1 page 393)}
\end{aligned}$$

(b) $\tilde{\mathsf{a}}(\omega)\breve{\mathsf{h}}^*(\omega) + \tilde{\mathsf{a}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega+\pi) = 0$. Proof:

$$\begin{aligned}
0 &= \mathsf{S}_{\mathsf{f}\phi}(\omega) && \text{by } \boldsymbol{V}_0 \perp \boldsymbol{W}_0 \text{ hypothesis and Theorem T.3 page 417}\\
&= \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\phi}^*(\omega+2\pi n) && \text{by Theorem T.1 page 411}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\tilde{\mathsf{a}}\left(\frac{\omega+2\pi n}{2}\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)}_{\tilde{\mathsf{f}}(\omega+2\pi n)}\underbrace{\breve{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}\right)\tilde{\phi}^*\left(\frac{\omega+2\pi n}{2}\right)}_{\tilde{\phi}^*(\omega+2\pi n)} && \text{by item (2a) page 97}\\
&= \underbrace{\sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}+2\pi n\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\right|^2}_{\text{even terms}}\\
&\quad + \underbrace{\sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}+2\pi n+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\right|^2}_{\text{odd terms}}\\
&= \sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\right|^2 + \sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\right|^2\\
&= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}\right) + \tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)\\
&= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}\right) + \tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right) && \text{by } \textit{orthonormality} \text{ and Theorem T.3 page 417}\\
&\Longrightarrow \tilde{\mathsf{a}}(\omega)\breve{\mathsf{h}}^*(\omega) + \tilde{\mathsf{a}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega+\pi) = 0
\end{aligned}$$

(c) The vectors $[\tilde{\mathsf{a}}(\omega), \tilde{\mathsf{a}}(\omega+\pi)]$ and $\left[\breve{\mathsf{h}}(\omega), \breve{\mathsf{h}}(\omega+\pi)\right]$ are *orthogonal*. Proof:

$$\begin{aligned}
\begin{bmatrix} \breve{\mathsf{h}}(\omega) \\ \breve{\mathsf{h}}(\omega+\pi) \end{bmatrix}^H \begin{bmatrix} \tilde{\mathsf{a}}(\omega) \\ \tilde{\mathsf{a}}(\omega+\pi) \end{bmatrix} &= \begin{bmatrix} \breve{\mathsf{h}}^*(\omega) & \breve{\mathsf{h}}^*(\omega+\pi) \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{a}}(\omega) \\ \tilde{\mathsf{a}}(\omega+\pi) \end{bmatrix}\\
&= \tilde{\mathsf{a}}(\omega)\breve{\mathsf{h}}^*(\omega) + \tilde{\mathsf{a}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega+\pi)\\
&= 0 && \text{by item (2b) page 97}
\end{aligned}$$

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(d) There exists a 2π *periodic* function $\tilde{\mathsf{b}}(\omega)$ such that $[\tilde{\mathsf{a}}(\omega), \tilde{\mathsf{a}}(\omega+\pi)] = \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega+\pi), -\breve{\mathsf{h}}^*(\omega)\right]$. Proof: Because the two vectors are *orthogonal* (item (2c) page 97), if one vector is rotated by 90°, then for fixed ω it is a scalar multiple of the other…

$$\begin{aligned}
[\tilde{\mathsf{a}}(\omega), \tilde{\mathsf{a}}(\omega+\pi)] &= \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega), \breve{\mathsf{h}}^*(\omega+\pi)\right] \underbrace{\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}}_{\textit{rotation matrix}} && \text{by item (2c) page 97} \\
&= \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega), \breve{\mathsf{h}}^*(\omega+\pi)\right] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega+\pi), -\breve{\mathsf{h}}^*(\omega)\right]
\end{aligned}$$

(e) By item (2d), $[\tilde{\mathsf{a}}(\omega+\pi), \tilde{\mathsf{a}}(\omega)] = \tilde{\mathsf{b}}(\omega+\pi)\left[\breve{\mathsf{h}}^*(\omega), -\breve{\mathsf{h}}^*(\omega+\pi)\right]$. Proof:

$$\begin{aligned}
[\tilde{\mathsf{a}}(\omega+\pi), \tilde{\mathsf{a}}(\omega)] &= [\tilde{\mathsf{a}}(\omega+\pi), \tilde{\mathsf{a}}(\omega+\pi+\pi)] \\
&= \tilde{\mathsf{b}}(\omega+\pi)\left[\breve{\mathsf{h}}^*(\omega+\pi+\pi), -\breve{\mathsf{h}}^*(\omega+\pi)\right] && \text{by item (2d) page 98} \\
&= \tilde{\mathsf{b}}(\omega+\pi)\left[\breve{\mathsf{h}}^*(\omega), -\breve{\mathsf{h}}^*(\omega+\pi)\right]
\end{aligned}$$

(f) By item (2d) and item (2e), $\tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi)$. Proof:

$$\underbrace{\begin{aligned}
\tilde{\mathsf{a}}(\omega) &= \tilde{\mathsf{b}}(\omega)\breve{\mathsf{h}}^*(\omega+\pi) && \text{by item (2d)} \\
\tilde{\mathsf{a}}(\omega) &= -\tilde{\mathsf{b}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega+\pi) && \text{by item (2e)}
\end{aligned}}_{\Downarrow \atop \tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi)}
\qquad
\underbrace{\begin{aligned}
\tilde{\mathsf{a}}(\omega+\pi) &= -\tilde{\mathsf{b}}(\omega)\breve{\mathsf{h}}^*(\omega) && \text{by item (2d)} \\
\tilde{\mathsf{a}}(\omega+\pi) &= \tilde{\mathsf{b}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega) && \text{by item (2e)}
\end{aligned}}_{\Downarrow \atop \tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi)}$$

(g) Note that $\tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi) \iff e^{-i(\omega+\pi)}\tilde{\mathsf{b}}(\omega+\pi) = e^{-i(\omega)}\tilde{\mathsf{b}}(\omega)$. Proof:

$$\begin{aligned}
-\tilde{\mathsf{b}}(\omega+\pi) &= -e^{i(\omega+\pi)}e^{-i(\omega+\pi)}\tilde{\mathsf{b}}(\omega+\pi) \\
&= -e^{i(\omega+\pi)}e^{-i\omega}\tilde{\mathsf{b}}(\omega) && \text{by right hypothesis} \\
&= e^{-i\pi}e^{i(\omega+\pi)}e^{-i\omega}\tilde{\mathsf{b}}(\omega) \\
&= e^{i\omega}e^{-i\omega}\tilde{\mathsf{b}}(\omega) \\
&= \tilde{\mathsf{b}}(\omega)
\end{aligned}$$

$$\begin{aligned}
e^{-i(\omega+\pi)}\tilde{\mathsf{b}}(\omega+\pi) &= -e^{-i\omega}\tilde{\mathsf{b}}(\omega+\pi) \\
&= e^{-i\omega}\tilde{\mathsf{b}}(\omega) && \text{by left hypothesis}
\end{aligned}$$

(h) Let $\tilde{\mathsf{v}}(\omega) \triangleq \sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right)$. Then (2) is true …

i. $\tilde{\mathsf{v}}(\omega)$ is 2π *periodic*. Proof:

$$\begin{aligned}
\tilde{\mathsf{v}}(\omega+2\pi) &\triangleq \sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right) && \text{by definition of } \tilde{\mathsf{v}}(\omega) \\
&= \sqrt{2}e^{-i\frac{\omega+2\pi}{2}}\tilde{\mathsf{b}}\left(\frac{\omega+2\pi}{2}\right) \\
&= \sqrt{2}e^{-i\frac{\omega}{2}+\pi}\tilde{\mathsf{b}}\left(\frac{\omega}{2}+\pi\right) \\
&= -\sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}+\pi\right) \\
&= -\sqrt{2}e^{-i\frac{\omega}{2}}\left[-\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right)\right] && \text{by item (2f) page 98} \\
&= \sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right) && \text{by item (2f) page 98} \\
&\triangleq \tilde{\mathsf{v}}(\omega) && \text{by definition of } \tilde{\mathsf{v}}(\omega)
\end{aligned}$$

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ii. $\tilde{\mathsf{f}}(\omega) = \frac{\sqrt{2}}{2}e^{i\frac{\omega}{2}}\tilde{\mathsf{v}}(\omega)\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$. Proof:

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by item (2a) page 97}\\
&= \tilde{\mathsf{b}}\left(\frac{\omega}{2}\right)\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by item (2d) page 98}\\
&= \frac{\sqrt{2}}{2}e^{i\frac{\omega}{2}}\tilde{\mathsf{v}}(\omega)\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \tilde{\mathsf{v}}(\omega)
\end{aligned}$$

3. lemma: proof that (2) $\Longrightarrow$ $\mathsf{f} \perp V_0$:

$$\begin{aligned}
&\mathsf{S}_{\mathsf{f}\phi}(\omega)\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{\phi}(\omega+2\pi n)\tilde{\mathsf{f}}^*(\omega+2\pi n) && \text{by Theorem T.1 page 411}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{\phi}(\omega+2\pi n)\left[\frac{\sqrt{2}}{2}e^{i\left(\frac{\omega+2\pi n}{2}\right)}\tilde{\mathsf{v}}(\omega+2\pi n)\check{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right]^* && \text{by (A)}\\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\sum_{n\in\mathbb{Z}}\tilde{\phi}(\omega+2\pi n)\left[e^{-i\left(\frac{\omega}{2}+\pi n\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right] && \text{by (C)}\\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\underbrace{\sum_{n\in\mathbb{Z}_e}\tilde{\phi}(\omega+2\pi n)\left[e^{-i\left(\frac{\omega}{2}+\pi n\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right]}_{\text{even terms}}\\
&\quad+\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\underbrace{\sum_{n\in\mathbb{Z}_o}\tilde{\phi}(\omega+2\pi n)\left[e^{-i\left(\frac{\omega}{2}+\pi n\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right]}_{\text{odd terms}}\\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\underbrace{\sum_{n\in\mathbb{Z}}\tilde{\phi}(\omega+4\pi n)\left[e^{-i\left(\frac{\omega}{2}+2\pi n\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\right]}_{\text{even terms}}\\
&\quad+\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\underbrace{\sum_{n\in\mathbb{Z}}\tilde{\phi}(\omega+4\pi n+2\pi)\left[e^{-i\left(\frac{\omega}{2}+2\pi n+\pi\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n+2\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\right]}_{\text{odd terms}}\\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)e^{-i\left(\frac{\omega}{2}\right)}\sum_{n\in\mathbb{Z}}\tilde{\phi}(\omega+4\pi n)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\\
&\quad-\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\check{\mathsf{h}}\left(\frac{\omega}{2}\right)e^{-i\left(\frac{\omega}{2}\right)}\sum_{n\in\mathbb{Z}}\tilde{\phi}(\omega+4\pi n+2\pi)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right) && \text{by Proposition R.1 page 393}\\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)e^{-i\left(\frac{\omega}{2}\right)}\sum_{n\in\mathbb{Z}}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\\
&\quad-\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\check{\mathsf{h}}\left(\frac{\omega}{2}\right)e^{-i\left(\frac{\omega}{2}\right)}\sum_{n\in\mathbb{Z}}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\\
&= \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2\right]\\
&\quad-\frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+\pi+2\pi n\right)\right|^2\right] && \text{by Proposition R.1 page 393}\\
&= \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}\right)-\frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)\\
&= \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\check{\mathsf{h}}\left(\frac{\omega}{2}\right)-\frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\check{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right) && \text{by Theorem T.3 page 417}
\end{aligned}$$

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$= 0$ by Lemma 6.2 page 95

$\iff \langle \mathsf{f} \mid \mathbf{T}^n\phi\rangle = \bar{\delta}_n$ by Theorem T.3 page 417

$\iff \mathsf{f} \perp \operatorname{span}\{\mathbf{T}^n\phi\}$

$\iff \mathsf{f} \perp V_0$

4. Proof that (A) $\Longleftarrow$ (1,2):
   (a) by (1), $\mathsf{f} \in V_1 = V_0 \hat{+} W_0$
   (b) by hypothesis, $V_0 \perp W_0$
   (c) by item (3) page 99, $\mathsf{f} \perp V_0$
   (d) so $\mathsf{f} \in W_0$

✏

Theorem 6.4. [10] *Let* $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j\rangle\!\rangle, \langle\!\langle W_j\rangle\!\rangle, \phi, \psi, \langle\!\langle h_n\rangle\!\rangle, \langle\!\langle g_n\rangle\!\rangle\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 40). Let* $\tilde{\mathsf{v}}(\omega)$ *be a measurable function in* $L^2_{\mathbb{R}}$ *(Definition E.1 page 199).*

THM

If $\{\phi|\mathbf{T}^n\phi\} = \bar{\delta}_n$ *then*

$$\left\{\begin{array}{lll} (A). & \tilde{\psi}(\omega) = \frac{\sqrt{2}}{2}e^{i\frac{\omega}{2}}\underbrace{\tilde{\mathsf{v}}(\omega)\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}\right)}_{\tilde{\mathsf{g}}\left(\frac{\omega}{2}\right)\ \text{(Definition 3.2 page 40)}} & \text{and} \\ (B). & \tilde{\mathsf{v}}(\omega) = \tilde{\mathsf{v}}(\omega+2\pi) & \text{and} \\ (C). & |\tilde{\mathsf{v}}(\omega)| = 1 & \end{array}\right\} \iff \underbrace{\left\{\begin{array}{lll} (1). & \langle\psi \mid \mathbf{T}^n\psi\rangle = \bar{\delta}_n & \text{and} \\ (2). & \langle\phi \mid \mathbf{T}^n\psi\rangle = 0 & \text{and} \\ (3). & \operatorname{span}\{\mathbf{T}^n\psi\} = W_0 & \end{array}\right\}}_{\substack{V_0 \perp W_0 \text{ and} \\ \psi \text{ is an ORTHONORMAL WAVELET for } L^2_{\mathbb{R}}}}$$

✎PROOF:

1. lemma: proof that (A) and $\{\mathbf{T}^n\phi\} = \bar{\delta}_n \implies \tilde{\mathsf{S}}_{\psi\psi}(\omega) = |\tilde{\mathsf{v}}(\omega)|^2$:

$$\begin{aligned}
&\tilde{\mathsf{S}}_{\psi\psi}(\omega) \\
&= 2\pi\sum_{n\in\mathbb{Z}}|\tilde{\psi}(\omega+2\pi n)|^2 && \text{by Theorem T.1 page 411} \\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\frac{\sqrt{2}}{2}e^{i\left(\frac{\omega+2\pi n}{2}\right)}\tilde{\mathsf{v}}(\omega+2\pi n)\check{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right|^2 && \text{by (A)} \\
&= \pi\sum_{n\in\mathbb{Z}}\underbrace{\left|e^{i\left(\frac{\omega+2\pi n}{2}\right)}\right|^2}_{1}|\tilde{\mathsf{v}}(\omega+2\pi n)|^2\left|\check{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right|^2 && \text{by property of } |\cdot| \\
&= \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}}\left|\check{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right|^2 && \text{by (B) and (C)} \\
&= \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}_e}\underbrace{\left|\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+\pi n\right)\right|^2}_{\text{even terms}} + \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}_o}\underbrace{\left|\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+\pi n\right)\right|^2}_{\text{odd terms}} \\
&= \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}}\underbrace{\left|\check{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2}_{\text{even terms}} + \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}}\underbrace{\left|\check{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n+\pi\right)\right|^2}_{\text{odd terms}}
\end{aligned}$$

[10] Wojtaszczyk (1997) pages 32–37 ⟨Section 2.4, Theorem 2.20⟩, Hernández and Weiss (1996) page 57 ⟨PROPOSITION 2.13⟩, MALLAT (1999) PAGES 236–238 ⟨THEOREM 7.3⟩, DAUBECHIES (1992) PAGE 133

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$$= \frac{|\tilde{v}(\omega)|^2}{2}\left\{\left|\check{h}^*\left(\frac{\omega}{2}\right)\right|^2 \underbrace{\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2\right]}_{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}\right)} + \left|\check{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2 \underbrace{\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+\pi+2\pi n\right)\right|^2\right]}_{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)}\right\}$$

because *DTFT* is 2π *periodic* (Proposition R.1 page 393)

$$= \frac{|\tilde{v}(\omega)|^2}{2}\left\{\left|\check{h}^*\left(\frac{\omega}{2}\right)\right|^2 \tilde{S}_{\phi\phi}\left(\frac{\omega}{2}\right) + \left|\check{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2 \tilde{S}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)\right\}$$

$$= \frac{|\tilde{v}(\omega)|^2}{2}\left\{\left|\check{h}^*\left(\frac{\omega}{2}\right)\right|^2 \cancelto{1}{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}\right)} + \left|\check{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2 \cancelto{1}{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)}\right\} \qquad \text{by Theorem T.3 page 417}$$

$$= |\tilde{v}(\omega)|^2 \qquad \text{by Lemma 6.2 page 95}$$

2. Proof that (A,B,C) $\Longrightarrow$ (1):

$$\begin{aligned}
\tilde{S}_{\psi\psi}(\omega) &= |\tilde{v}(\omega)|^2 && \text{by item (1) page 100}\\
&= 1 && \text{by (B)}\\
\Longleftrightarrow \langle\tilde{\psi}\,|\,\mathbf{T}^n\tilde{\psi}\rangle &= \bar{\delta}_n && \text{by Theorem T.3 page 417}
\end{aligned}$$

3. Proof that (A,B) $\Longrightarrow$ (2): by Proposition 6.5 page 96 with $\tilde{f}(\omega) \triangleq \tilde{\psi}(\omega)$.

4. lemma: Proof that $\{\mathbf{T}^n\phi\} = \bar{\delta}_n$ and (A,B,C) $\Longrightarrow$ for all $(\!(a_n)\!)$, there exists $(\!(b_n)\!)$ and $(\!(c_n)\!)$ such that $\tilde{a}\left(\frac{\omega}{2}\right) = \tilde{b}(\omega)\check{h}\left(\frac{\omega}{2}\right) + \tilde{c}(\omega)\tilde{g}\left(\frac{\omega}{2}\right)$

   (a) First note the following relations hold in this case:

$$\begin{aligned}
|\check{h}(\omega)|^2 + |\check{h}(\omega+\pi)|^2 &= 2 && \text{by } \{\mathbf{T}^n\phi\} = \bar{\delta}_n \text{ and by Lemma 6.2 page 95}\\
|\tilde{g}(\omega)|^2 + |\tilde{g}(\omega+\pi)|^2 &= 2 && \text{by (1) and by Lemma 6.2 page 95}\\
\check{h}(\omega)\tilde{g}^*(\omega) + \check{h}(\omega+\pi)\tilde{g}^*(\omega+\pi) &= 0 && \text{by (2) and by Lemma 6.2 page 95}\\
\tilde{g}\left(\frac{\omega}{2}\right) = e^{i\frac{\omega}{2}}\tilde{v}(\omega)\check{h}^*\left(\frac{\omega}{2}+\pi\right) & && \text{by (A)}
\end{aligned}$$

   (b) Then, the following values of $(\!(b_n)\!)$ and $(\!(c_n)\!)$ are solutions for all $(\!(a_n)\!)$

$$\begin{aligned}
b(\omega) &= \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\check{h}^*\left(\tfrac{\omega}{2}\right) + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)\\
c(\omega) &= \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\tilde{g}^*\left(\tfrac{\omega}{2}\right) + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)\tilde{g}^*\left(\tfrac{\omega}{2}+\pi\right)
\end{aligned}$$

Proof:

$$\begin{aligned}
&\tilde{b}(\omega)\check{h}\left(\frac{\omega}{2}\right) + \tilde{c}(\omega)\tilde{g}\left(\frac{\omega}{2}\right)\\
&= \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\left|\check{h}\left(\tfrac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)\check{h}\left(\tfrac{\omega}{2}\right)\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)\\
&\quad + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\left|\tilde{g}\left(\tfrac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)\tilde{g}\left(\tfrac{\omega}{2}\right)\tilde{g}^*\left(\tfrac{\omega}{2}+\pi\right) \qquad \text{by item (4b) page 101}\\
&= \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\left|\check{h}\left(\tfrac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)\check{h}\left(\tfrac{\omega}{2}\right)\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)\\
&\quad + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\left|e^{i\frac{\omega}{2}}\tilde{v}(\omega)\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)e^{i\frac{\omega}{2}}\tilde{v}(\omega)\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)e^{-i\left(\frac{\omega}{2}+\pi\right)}\tilde{v}^*(\omega+2\pi)\check{h}\left(\tfrac{\omega}{2}+2\pi\right)\\
&= \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\left|\check{h}\left(\tfrac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)\check{h}\left(\tfrac{\omega}{2}\right)\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)\\
&\quad + \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\left|\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)\right|^2 - \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}+\pi\right)\tilde{v}(\omega)\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)|\tilde{v}(\omega)|^2\check{h}\left(\tfrac{\omega}{2}\right)\\
&= \tfrac{1}{2}\tilde{a}\left(\tfrac{\omega}{2}\right)\left[\left|\check{h}\left(\tfrac{\omega}{2}\right)\right|^2 + \left|\check{h}^*\left(\tfrac{\omega}{2}+\pi\right)\right|^2\right]
\end{aligned}$$

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$$
\begin{aligned}
&+\tfrac{1}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\left[\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)-\check{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\right]\\
&=\tfrac{1}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)[2]+\tfrac{1}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)[0] \qquad \text{by Lemma 6.2 page 95}\\
&=\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)
\end{aligned}
$$

5. Proof that (A,B,C) $\Longrightarrow$ (3):

$$
\begin{aligned}
&\mathsf{span}\{\mathbf{T}^n\psi\} = \mathcal{W}_0\\
&\Longleftrightarrow\ \mathcal{V}_1 = \mathcal{V}_0 \hat{+} \underbrace{\mathsf{span}\{\mathbf{T}^n\psi\}}_{\mathcal{W}_0} \qquad \text{by Definition 3.1 page 39}\\
&\Longleftrightarrow\ \underbrace{\mathsf{span}\{\mathbf{D}\mathbf{T}^n\phi\}}_{\mathcal{V}_1} = \underbrace{\mathsf{span}\{\mathbf{T}^n\phi\}}_{\mathcal{V}_0} \hat{+} \underbrace{\mathsf{span}\{\mathbf{T}^n\psi\}}_{\mathcal{W}_0} \qquad \text{by Definition 2.1 page 18}\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \sum_{n\in\mathbb{Z}} a_n\mathbf{D}\mathbf{T}^n\phi = \sum_{n\in\mathbb{Z}} b_n\mathbf{T}^n\phi + \sum_{n\in\mathbb{Z}} c_n\mathbf{T}^n\psi \qquad \text{by Definition N.2 page 305}\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \tilde{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}} a_n\mathbf{D}\mathbf{T}^n\phi\right] = \tilde{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}} b_n\mathbf{T}^n\phi\right] + \tilde{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}} c_n\mathbf{T}^n\psi\right] \qquad \text{by Definition M.2 page 294}\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n\phi = \sum_{n\in\mathbb{Z}} b_n\tilde{\mathbf{F}}\mathbf{T}^n\phi + \sum_{n\in\mathbb{Z}} c_n\tilde{\mathbf{F}}\mathbf{T}^n\psi\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \sum_{n\in\mathbb{Z}} a_n\left[\frac{\sqrt{2}}{2}e^{-i\frac{\omega}{2}n}\tilde{\phi}\left(\frac{\omega}{2}\right)\right] = \sum_{n\in\mathbb{Z}} b_n e^{-i\omega n}\tilde{\phi}(\omega) + \sum_{n\in\mathbb{Z}} c_n e^{-i\omega n}\tilde{\psi}(\omega)\\
&\qquad \text{by Corollary 1.3 page 12}\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \frac{\sqrt{2}}{2}\tilde{\phi}\left(\frac{\omega}{2}\right)\sum_{n\in\mathbb{Z}} a_n e^{-i\frac{\omega}{2}n} = \tilde{\phi}(\omega)\sum_{n\in\mathbb{Z}} b_n e^{-i\omega n} + \tilde{\psi}(\omega)\sum_{n\in\mathbb{Z}} c_n e^{-i\omega n}\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \frac{\sqrt{2}}{2}\tilde{\phi}\left(\frac{\omega}{2}\right)\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right) = \tilde{\phi}(\omega)\tilde{\mathsf{b}}(\omega) + \tilde{\psi}(\omega)\tilde{\mathsf{c}}(\omega) \qquad \text{by def. of } DTFT \text{ (Definition R.1 page 393)}\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \text{(by Lemma 2.1 page 25)}\\
&\qquad \frac{\sqrt{2}}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) = \frac{\sqrt{2}}{2}\tilde{\mathsf{b}}(\omega)\check{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) + \frac{\sqrt{2}}{2}\tilde{\mathsf{c}}(\omega)\tilde{\mathsf{g}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)\\
&\Longleftrightarrow\ \begin{matrix}\forall(a_n)\exists(b_n),(c_n)\\ \text{such that}\end{matrix} \quad \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right) = \tilde{\mathsf{b}}(\omega)\check{\mathsf{h}}\left(\frac{\omega}{2}\right) + \tilde{\mathsf{c}}(\omega)\tilde{\mathsf{g}}\left(\frac{\omega}{2}\right)\\
&\Longrightarrow\ (3) \qquad \text{by item (4) page 101}
\end{aligned}
$$

6. Proof that (A,B) $\Longleftarrow$ (3):

$$
\begin{aligned}
\mathsf{span}\{\mathbf{T}^n\psi\} = \mathcal{W}_0 &\Longrightarrow \psi \in \mathcal{W}_0\\
&\Longrightarrow \text{(A) and (C)} \qquad \text{by Proposition 6.5 page 96}
\end{aligned}
$$

7. Proof that (C) $\Longleftarrow$ (3) and $\{\mathbf{T}^n\phi\} = \bar{\delta}_n$:

$$
\begin{aligned}
1 &= \tilde{\mathsf{S}}_{\psi\psi}(\omega) \qquad \text{by right hypothesis and Theorem T.3 page 417}\\
&= |\tilde{\mathsf{v}}(\omega)|^2 \qquad \text{by item (1) page 100}
\end{aligned}
$$

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6.3.4 Sufficient conditions

If we know the *scaling coefficient sequence* $(\!(\mathsf{h}_n)\!)$ (Definition 2.3 page 27), then Theorem 6.4 page 100 gives a necessary and sufficient condition for the *wavelet coefficient sequence* $(\!(\mathsf{g}_n)\!)$ (Definition 3.2 page 40) to generate a *wavelet system* (Definition 3.2 page 40). The actual value of $(\!(\mathsf{g}_n)\!)$ depends on $(\!(\mathsf{h}_n)\!)$ and the choice of $\tilde{\mathsf{v}}(\omega)$ (Theorem 6.4 page 100). Here are some possible choices:[11]

1.	$\tilde{\mathsf{v}}(\omega)$	$\triangleq$	1	$\Longrightarrow$	g_n	$= (-1)^n \mathsf{h}^*_{-1-n}$	
2.	$\tilde{\mathsf{v}}(\omega)$	$\triangleq$	$e^{i\omega}$	$\Longrightarrow$	g_n	$= (-1)^n \mathsf{h}^*_{-3-n}$	
3.	$\tilde{\mathsf{v}}(\omega)$	$\triangleq$	$e^{-i\omega}$	$\Longrightarrow$	g_n	$= (-1)^n \mathsf{h}^*_{1-n}$	*conjugate mirror filter* (CMF)
4.	$\tilde{\mathsf{v}}(\omega)$	$\triangleq$	$\pm(-1)^{N+1} e^{-i\frac{\omega}{2}(N+1)}$	$\Longrightarrow$	g_n	$= \pm(-1)^n \mathsf{h}^*_{N-n}$	*conjugate quadrature filter* (CQF), $N \in \mathbb{Z}$, (Definition Q.6 page 386)

Theorem 6.5. [12] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 6.1 page 81)*.

THM

$$\left\{\begin{array}{lll} (A). & \underbrace{\langle \phi \,|\, \mathbf{T}^n \phi\rangle = \bar{\delta}_n}_{\{\mathbf{T}^n\phi\} \text{ is ORTHONORMAL}} & \text{and} \\ (B). & \mathsf{g}_n = \underbrace{\pm(-1)^n \mathsf{h}^*_{N-n}}_{\text{CQF condition}} & \forall n \in \mathbb{Z} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lllll} (1). & \langle \phi \,|\, \mathbf{T}^n \psi\rangle & = & 0 & \text{and} \\ (2). & \langle \psi \,|\, \mathbf{T}^n \psi\rangle & = & \bar{\delta}_n & \text{and} \\ (3). & \operatorname{span}\left\{\mathbf{T}^n\psi \,|\, n\in\mathbb{Z}\right\} & = & W_0 & \end{array}\right\} \forall n\in\mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\mathsf{g}_n = \pm(-1)^n h^*_{N-n} &\Longleftrightarrow \tilde{\mathsf{g}}(\omega) = \pm(-1)^N e^{-i\omega N} \check{\mathsf{h}}^*(\omega+\pi) && \text{by } \textit{CQF theorem} \text{ (Theorem Q.5 page 387)} \\
&\Longrightarrow \tilde{\psi}(2\omega) = \frac{\sqrt{2}}{2} \underbrace{\pm(-1)^N e^{-i\omega N} \check{\mathsf{h}}^*(\omega+\pi)}_{\tilde{\mathsf{g}}(\omega)} \tilde{\phi}(\omega) && \text{by Proposition 3.2 page 44} \\
&\Longrightarrow \tilde{\psi}(2\omega) = \frac{\sqrt{2}}{2} \underbrace{\left[\pm(-1)^N e^{-i\omega(N+1)}\right]}_{v(2\omega)} e^{i\omega} \check{\mathsf{h}}^*(\omega+\pi)\, \tilde{\phi}(\omega) \\
&\Longrightarrow \left\{\begin{array}{lllll} (1). & \langle \phi \,|\, \mathbf{T}^n \psi\rangle & = & 0 & \text{and} \\ (2). & \langle \psi \,|\, \mathbf{T}^n \psi\rangle & = & \bar{\delta}_n & \text{and} \\ (3). & \operatorname{span}\{\mathbf{T}^n\psi \,|\, n\in\mathbb{Z}\} & = & W_0 & \end{array}\right\} && \text{by Theorem 6.4 page 100}
\end{aligned}$$

✏

Theorem 6.6. *Let* $\Omega \triangleq \left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a tuple.*

THM

$$\left\{\begin{array}{llllll} (A). & \sum_n \mathsf{h}_n & = & \sqrt{2} & & [\text{ADMISSIBILITY } \textit{(Theorem 2.3 page 29)}] & \textit{and} \\ (B). & \sum_{m\in\mathbb{Z}} \mathsf{h}_m h^*_{m-2n} & = & \bar{\delta}_n & & [\text{ORTHO. QUADRATURE CONDITION } \textit{(Lemma 6.1 page 85)}] & \textit{and} \\ (C). & \sum_{k\in\mathbb{Z}} (-1)^k k^n \mathsf{h}_k & = & 0 & \forall n=0,1,\ldots,p-1 & [p \text{ VANISHING MOMENTS } \textit{(Theorem 5.3 page 70)}] & \textit{and} \\ (D). & \inf_{\omega\in[-\pi/2:\pi/2]} \left|\tilde{\mathsf{h}}(\omega)\right| & > & 0 & & [\text{LOW-PASS RESPONSE}] & \textit{and} \\ (E). & \mathsf{g}_n & = & (-1)^n h^*_{N-n} & & [\text{CQF CONDITION } \textit{(Definition Q.6 page 386)}] & \end{array}\right\}$$

$$\Longrightarrow \left\{\begin{array}{ll} (1). & \Omega \textit{ is an } \text{ORTHONORMAL WAVELET SYSTEM} \quad \textit{and} \\ (2). & \psi \textit{ has } p \text{ VANISHING MOMENTS} \end{array}\right\}$$

[11] Strang and Nguyen (1996) page 109 ⟨"A Brief History of h_1"⟩, Vidakovic (1999) page 59
[12] Mallat (1999) pages 236–238

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✎PROOF:

1. lemma: $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ is an *orthonormal MRA system*. Proof:
   (a) If $p = 1$, then this is the *Haar MRA system*.
   (b) If $p \geq 2$, then proof follows from Theorem 6.2 page 88:
      i. Proof that $\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}$: by left hypothesis (A)
      ii. Proof that $\sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n$: by left hypothesis (B)
      iii. Proof that $\tilde{\phi}(\omega)$ is *continuous* at 0:

$$\begin{aligned}
(C) &\implies \sum_{k\in\mathbb{Z}} (-1)^k k^n \mathsf{h}_k = 0 \quad \forall_{n=0,1,\ldots,p-1,\, p\geq 2} \\
&\implies \psi \text{ has } p \textit{ vanishing moments} && \text{by Theorem 5.3 page 70} \\
&\implies \tilde{\phi}(\omega) \text{ is } \textit{continuous} \text{ at } \omega = 0 && \text{by Theorem 5.5 page 77}
\end{aligned}$$

      iv. Proof that $\inf_{\omega\in[-\pi/2:\pi/2]} \left|\check{\mathsf{h}}(\omega)\right| > 0$: by left hypothesis (D)
2. Proof for (1): by (1) lemma, (E), and Theorem 6.5 page 103.
3. Proof for (2): by (C) and Theorem 5.3 page 70.

✏

6.4 Examples

The scaling coefficients can be directly computed using simultaneous equations generated from the following necessary conditions.[13] They can be solved using a symbolic equation software package[14] or by hand.

Example 6.3 (2 coefficient case—Haar wavelet system). Let $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j \rangle\!\rangle, \langle\!\langle W_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be an *orthogonal* wavelet system with two non-zero scaling coefficients.

EX

$$\left\{\begin{array}{lll}
1. & \operatorname{supp}\phi(x) = [0 : 1] & \text{(Theorem 3.6 page 47)} \quad \text{and} \\
2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 29)} \quad \text{and} \\
3. & \left\{\mathbf{T}^n\phi\right\} \text{ is } \textit{orthonormal} & \text{(Lemma 6.2 page 95)} \quad \text{and} \\
4. & \mathsf{g}_n = \pm(-1)^n \mathsf{h}^*_{N-n} \;\forall_{n\in\mathbb{Z}} & \text{(Theorem 6.5 page 103)}
\end{array}\right\} \implies \left\{\begin{array}{c|c|c}
n & \mathsf{h}_n & \mathsf{g}_n \\ \hline
0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\text{other} & 0 & 0
\end{array}\right\}$$

[13] Vidakovic (1999) pages 92–93

[14]

Maxima™	http://maxima.sourceforge.net/	free and open source
Macsyma™	http://www.scientek.com/macsyma/mxmain.htm	a very non-free relative of *Maxima*
MuPAD™	http://www.mupad.de/	30 day free trial
Maple™	http://www.maplesoft.com/	
Mathematica™	http://www.wolfram.com/	

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✎PROOF:

1. Proof that (1) $\Longrightarrow$ that only h_0 and h_1 are non-zero: by Theorem 3.6 page 47.
2. Proof for values of h_0 and h_1:

$$\begin{aligned}
\mathsf{h}_0 + \mathsf{h}_1 &= \sqrt{2} && \text{by admissibility condition—Theorem 2.3 page 29}\\
\mathsf{h}_0^2 + \mathsf{h}_1^2 &= 1 && \text{orthogonal quadrature condition—Lemma 6.2 page 95}
\end{aligned}$$

$$\begin{aligned}
\mathsf{h}_0^2 + \left(\sqrt{2} - h_0\right)^2 &= 1\\
2h_0^2 - 2\sqrt{2}h_0 + 1 &= 0\\
\mathsf{h}_0 &= \frac{2\sqrt{2} \pm \sqrt{\left(2\sqrt{2}\right)^2 - 4\cdot 2\cdot 1}}{2\cdot 2}\\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
\mathsf{h}_1 &= \sqrt{2} - \mathsf{h}_0\\
&= \sqrt{2} - \frac{\sqrt{2}}{2}\\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

3. Note: h_0 and h_1 are the *Haar scaling coefficients*. They can also be produced using other systems of equations including the following:
   (a) Admissibility condition and *partition of unity*—Example 4.1 (page 60)
   (b) *Daubechies-p*1 wavelets computed using spectral techniques—Example 7.3 (page 123)
4. Proof for values of g_0 and g_1: by (4) and Theorem 6.5 page 103.

✏

Example 6.4. [15] In this example, we constrain the system to have $p = 2$ vanishing moments. This same wavelet system is identical to the *Daubechies-p*2 wavelet system which is computed using a spectral factorization technique—Example 7.4 (page 124). Here are the equations:

$$\begin{array}{ccccccccccl}
\mathsf{h}_0 & + & \mathsf{h}_1 & + & \mathsf{h}_2 & + & \mathsf{h}_3 & = & \sqrt{2} & & \text{(admissibility—Theorem 2.3 page 29)}\\
\mathsf{h}_0 & - & \mathsf{h}_1 & + & \mathsf{h}_2 & - & \mathsf{h}_3 & = & 0 & & \text{(vanishing 0th moment/partition of unity—page 54)}\\
 & - & \mathsf{h}_1 & + & 2h_2 & - & 3h_3 & = & 0 & & \text{(vanishing 1st moment—Theorem 5.3 page 70)}\\
\mathsf{h}_0^2 & + & \mathsf{h}_1^2 & + & \mathsf{h}_2^2 & + & \mathsf{h}_3^2 & = & 1 & & (m = 0 \text{ ortho. quad. con.—Lemma 6.2 page 95})
\end{array}$$

Then the scaling coefficients have the following values:

EX	n	$\sqrt{2}\mathsf{h}_n$	$\sqrt{2}\mathsf{g}_n$
	0	$\frac{1+\sqrt{3}}{4}$	$\frac{1-\sqrt{3}}{4}$
	1	$\frac{3+\sqrt{3}}{4}$	$-\frac{3-\sqrt{3}}{4}$
	2	$\frac{3-\sqrt{3}}{4}$	$\frac{3+\sqrt{3}}{4}$
	3	$\frac{1-\sqrt{3}}{4}$	$-\frac{1+\sqrt{3}}{4}$

[15] Vidakovic (1999) page 92, Soman et al. (2010) page 83

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✎PROOF:

1. Solve the 3 linear equations in terms of h_3:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} \\ 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} \\ 0 & -2 & 0 & -2 & -\sqrt{2} \\ 0 & -1 & 2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -2 & \sqrt{2} \\ 0 & 0 & -4 & 4 & -\sqrt{2} \\ 0 & 1 & -2 & 3 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 3 & -2 & \sqrt{2} \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & -1 & \frac{\sqrt{2}}{4} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & \frac{\sqrt{2}}{4} \\ 0 & 1 & 0 & 1 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & -1 & \frac{\sqrt{2}}{4} \end{bmatrix}$$

$$\begin{aligned} \mathsf{h}_0 &= \frac{\sqrt{2}}{4} - \mathsf{h}_3 \\ \mathsf{h}_1 &= \frac{\sqrt{2}}{2} - \mathsf{h}_3 \\ \mathsf{h}_2 &= \frac{\sqrt{2}}{4} + \mathsf{h}_3 \end{aligned}$$

2. Solve the forth quadratic equation in terms of h_3:

$$\begin{aligned} 1 &= \left(\frac{\sqrt{2}}{4} - h_3\right)^2 + \left(\frac{\sqrt{2}}{2} - h_3\right)^2 + \left(\frac{\sqrt{2}}{4} + h_3\right)^2 + \mathsf{h}_3^2 \\ &= \left(\frac{1}{8} - \sqrt{2}2h_3 + h_3^2\right) + \left(\frac{1}{2} - \sqrt{2}h_3 + h_3\right) + \left(\frac{1}{8} + \frac{\sqrt{2}}{2} + h_3^2\right) + \mathsf{h}_3^2 \\ &= 4h_3^2 - \sqrt{2}h_3 + \frac{3}{4} \end{aligned}$$

$$4h_3^2 - \sqrt{2}h_3 - \frac{1}{4} = 0$$

$$\begin{aligned} \mathsf{h}_3 &= \frac{\sqrt{2} \pm \sqrt{2+4}}{8} \\ &= \frac{\sqrt{2} \pm \sqrt{6}}{8} \\ &= \frac{\sqrt{2}}{8}(1 \pm \sqrt{3}) \end{aligned}$$

3. Simplify the results:

$$\begin{aligned} \mathsf{h}_3 &= \frac{\sqrt{2}}{8}(1-\sqrt{3}) \\ \mathsf{h}_2 &= \frac{\sqrt{2}}{4} + \mathsf{h}_3 = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8}(1-\sqrt{3}) = \frac{\sqrt{2}}{8}(2+1-\sqrt{3}) = \frac{\sqrt{2}}{8}(3-\sqrt{3}) \\ \mathsf{h}_1 &= \frac{\sqrt{2}}{2} - \mathsf{h}_3 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}(1-\sqrt{3}) = \frac{\sqrt{2}}{8}(4-1+\sqrt{3}) = \frac{\sqrt{2}}{8}(3+\sqrt{3}) \\ \mathsf{h}_0 &= \frac{\sqrt{2}}{4} - \mathsf{h}_3 = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{8}(1-\sqrt{3}) = \frac{\sqrt{2}}{8}(2-1+\sqrt{3}) = \frac{\sqrt{2}}{8}(1+\sqrt{3}) \end{aligned}$$

4. Solution using *Maxima*™ (2 sets of solutions):

```
solve([h0+h1+h2+h3=sqrt(2),h0-h1+h2-h3=0,-h1+2*h2-3*h3=0,h0^2+h1^2+h2^2+h3^2=1],
        [h0,h1,h2,h3]);
```


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$$\begin{aligned} h_0 &= -\frac{\sqrt{6}-\sqrt{2}}{8} \\ h_1 &= -\frac{\sqrt{6}-3\sqrt{2}}{8} \\ h_2 &= \frac{\sqrt{6}+3\sqrt{2}}{8} \\ h_3 &= \frac{\sqrt{6}+\sqrt{2}}{8} \end{aligned} \qquad \begin{aligned} h_0 &= \frac{\sqrt{6}+\sqrt{2}}{8} \\ h_1 &= \frac{\sqrt{6}+3\sqrt{2}}{8} \\ h_2 &= -\frac{\sqrt{6}-3\sqrt{2}}{8} \\ h_3 &= -\frac{\sqrt{6}-\sqrt{2}}{8} \end{aligned}$$

✏

Example 6.5 (4 scaling coefficient case). This example attempts to use the admissibility condition (Theorem 2.3 page 29) and orthonormal quadrature condition (Lemma 6.2 page 95) to design a 4 coefficient wavelet system. However, it fails due to the failure of the equations to be independent.

Let $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j \rangle\!\rangle, \langle\!\langle W_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ be an *orthogonal* wavelet system with four non-zero scaling coefficients.

1. We can use the following four equations to try to compute the values of the four scaling coefficients, and then compute the four wavelet coefficients from the four scaling coefficients. Note that the second, third, and fourth equations all assume orthonormality.

$$\begin{array}{rcrcrcrcrl} h_0 & + & h_1 & + & h_2 & + & h_3 & = & \sqrt{2} & \text{(admissibility—Theorem 2.3 page 29)} \\ h_0 & - & h_1 & + & h_2 & - & h_3 & = & 0 & \text{(zero at } -1 \text{ from orthonormality—page 87)} \\ h_0^2 & + & h_1^2 & + & h_2^2 & + & h_3^2 & = & 1 & (m=0 \text{ ortho. quad. con.—page 95)} \\ h_0h_2 & + & h_1h_3 & & & & & = & 0 & (m=1 \text{ ortho. quad. con.—page 95)} \end{array}$$

2. These equations are linearly independent, however they are still dependent [16] as shown next:

$$\begin{aligned} &\left[(h_0+h_1+h_2+h_3)+(h_0-h_1+h_2-h_3)\right]^2 \\ &\qquad + \left[(h_0+h_1+h_2+h_3)-(h_0-h_1+h_2-h_3)\right]^2 \\ &= [2h_0+2h_2]^2+[2h_1+2h_3]^2 \\ &= 4[h_0+h_2]^2+4[h_1+h_3]^2 \\ &= 4[h_0^2+2h_0h_2+h_2^2]+4[h_1^2+2h_1h_3+h_3^2] \\ &= 4[h_0^2+h_1^2+h_2^2+h_3^2+2h_0h_2+2h_1h_3] \\ &= 4(h_0^2+h_1^2+h_2^2+h_3^2)+8(h_0h_2+h_1h_3) \end{aligned}$$

$\Longrightarrow$ The last two equations are (non-linearly) dependent on the first two.

3. Because of this dependence, the system of equations has one degree of freedom.

4. Solutions using *Maxima*

```
solve([h0+h1+h2+h3=sqrt(2),h0-h1+h2-h3=0,h0^2+h1^2+h2^2+h3^2=1,h0*h2+h1*h3=0],
      [h0,h1,h2,h3]
     );
```

16 email

Many many thanks to 謝欣霖 (Xiè Xīn Lín) for pointing this out to me in a 2005 October 13 email. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from 謝欣霖's 2005 October 13 email.)

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(a) *Maxima* first solution:

$$\mathsf{h}_1 = \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2 + 4\sqrt{2}h_0 + 1} + 1\right)$$
$$h_2 = \frac{2h_0 - \sqrt{2}}{2}$$
$$h_3 = \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2 + 4\sqrt{2}h_0 + 1} - 1\right)$$

(b) *Maxima* second solution:

$$\mathsf{h}_1 = \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2 + 4\sqrt{2}h_0 + 1} - 1\right)$$
$$h_2 = -\frac{2h_0 - \sqrt{2}}{2}$$
$$h_3 = \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2 + 4\sqrt{2}h_0 + 1} + 1\right)$$

Example 6.6. [17] This example gives the equations for a 6 coefficient wavelet system. See also the six coefficient *Daubechies-p3* wavelet system which is computed using a spectral factorization technique (Example 7.5 page 126).

$$\begin{array}{rcrcrcrcrcrcll}
\mathsf{h}_0 & + & \mathsf{h}_1 & + & \mathsf{h}_2 & + & \mathsf{h}_3 & + & \mathsf{h}_4 & + & \mathsf{h}_5 & = & \sqrt{2} & \text{(admissibility)} \\
\mathsf{h}_0 & - & \mathsf{h}_1 & + & \mathsf{h}_2 & - & \mathsf{h}_3 & + & \mathsf{h}_4 & - & \mathsf{h}_5 & = & 0 & \text{(vanishing 0th moment)} \\
 & - & \mathsf{h}_1 & + & 2h_2 & - & 3h_3 & + & 4h_4 & - & 5h_5 & = & 0 & \text{(vanishing 1st moment)} \\
 & - & \mathsf{h}_1 & + & 4h_2 & - & 9h_3 & + & 16h_4 & - & 25h_5 & = & 0 & \text{(vanishing 2nd moment)} \\
\mathsf{h}_0^2 & + & \mathsf{h}_1^2 & + & \mathsf{h}_2^2 & + & \mathsf{h}_3^2 & + & \mathsf{h}_4^2 & + & \mathsf{h}_5^2 & = & 1 & (m = 0 \text{ orthonormal quadrature condition}) \\
\mathsf{h}_0 h_2 & + & \mathsf{h}_1 h_3 & + & \mathsf{h}_2 h_4 & + & \mathsf{h}_3 h_5 & & & & & = & 0 & (m = 1 \text{ orthonormal quadrature condition})
\end{array}$$

[17] Soman et al. (2010) page 84

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CHAPTER 7

MINIMUM PHASE CONSTRAINT

❝ I regard as quite useless the reading of large treatises of pure analysis: too large a number of methods pass at once before the eyes. It is in the works of applications that one must study them; one judges their ability there and one apprises the manner of making use of them.❞

Joseph Louis Lagrange (1736-1813), mathematician [1]

7.1 General structure

This chapter introduces a wavelet design technique that represents the orthogonal quadrature condition in terms of a polynomial in z. It is the technique that was used by Ingrid Daubechies in her celebrated *Daubechies-p* class of wavelets (see Section 7.2 page 120). The technique can be summarized as follows:

1. **Design requirements.** Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ be a wavelet system. We want to design this system to have two properties:
   (a) The scaling functions $\left(\!\left(\mathbf{T}^n\phi\right)\!\right)$ are **orthonormal** (Definition 6.1 page 81).
   (b) The wavelet function ψ has p *vanishing moments* (Definition 5.2 page 67).
2. **Form of solution.** Lemma 5.3 (page 77) demonstrates that the presence of the factor $\left(\frac{1+z^{-1}}{2}\right)^p$ in the z transform $\check{\mathsf{h}}(z)$ of the scaling coefficient sequence $\left(\!\left(h_n\right)\!\right)$ is a sufficient condition for ψ to have p vanishing moments. Therefore, we constrain $\check{\mathsf{h}}(z)$ to be of the following form:

$$\check{\mathsf{h}}(z) = \sqrt{2}\ \underbrace{\left(\frac{1+z^{-1}}{2}\right)^p}_{\substack{\text{provides } p \\ \text{vanishing moments} \\ \text{(Lemma 5.3 page 77)}}}\ \underbrace{\mathsf{Q}(z)}_{\text{provides orthogonality}}$$

[1] quote: Stopple (2003) page xi
image: http://en.wikipedia.org/wiki/Image:Langrange_portrait.jpg, public domain

So we need to compute the factor $\mathsf{Q}(z)$. Once we have $\mathsf{Q}(z)$, we can multiply the two factors, and the scaling coefficient sequence $(\!(h_n)\!)$ will simply be the coefficients of the polynomial $\check{\mathsf{h}}(z)$:

$$\check{\mathsf{h}}(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + \cdots + h_{2p-1} z^{-2p+1}$$

3. **Computing $\mathsf{Q}(z)$.**

   (a) The frequency representation of the orthogonal quadrature condition $\left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi)\right|^2 = 2$ (Lemma 6.1 page 85) is *periodic* in ω with period 2π and is even about 0. Because it's *periodic*, the quadrature condition can be expressed as a *Fourier Series* (Definition K.1 page 273). Because it's *even*, the quadrature condition can be expressed using only cosine terms.

   (b) Any *harmonic* cosine representation of a function may alternatively be expressed as a *trigonometric expansion* (Theorem J.2 page 253 with examples in Example J.1 page 255 and Example J.2 page 256). Therefore, the frequency orthogonal quadrature condition can be represented as a polynomial in $\cos\omega$.

   (c) As indicated by the trigonometric identity $\sin^2\theta \equiv \frac{1}{2}(1-\cos 2\theta)$ (Theorem I.11 page 248), any polynomial in $\cos\omega$ can alternatively be represented as a polynomial in $\sin^2\frac{\omega}{2}$. Therefore, the frequency orthogonal quadrature condition can be represented as a polynomial in $\sin^2\frac{\omega}{2}$.

   (d) With y defined as $y \triangleq \sin^2\frac{\omega}{2}$, the orthogonal quadrature condition together with the p vanishing moments constraint can be represented as (Lemma 7.2 page 112)

   $$(1-y)^p P(y) + y^p P(1-y) = 1 \quad \text{for some polynomial } \mathsf{P}(y).$$

   (e) The forms $(1-y)^p P(y)$ and $y^p P(1-y)$ are also used in *Hermite interpolation* (Section P.2 page 371) with the result of the endpoints at $y=0$ and $y=1$ being increasingly "flatter" with increasing p. In particular, the first $p-1$ derivatives at the endpoints are 0 (Theorem P.1 page 372).

   (f) There are an infinite number of solutions for $\mathsf{P}(y)$. These solutions are of the form (Lemma 7.3 page 113)

   $$\mathsf{P}(y) \triangleq \mathsf{P}_p(y) + y^p \mathsf{R}\left(\frac{1}{2} - y\right) \quad \text{where} \quad \left\{ \begin{array}{l} \mathsf{P}_p(y) \triangleq \displaystyle\sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k \quad \text{and} \\ \underbrace{\mathsf{R}(y) = -\mathsf{R}(-y)}_{\mathsf{R}\text{ is any } \textit{odd} \text{ polynomial}} \end{array} \right\}$$

   Furthermore, $\mathsf{P}(y)$ has minimum order when $\mathsf{R}(y) = 0$.

   (g) Examples of *odd polynomials* include the *Chebyshev polynomials* (Definition J.1 page 258) $\mathsf{T}_n(x)$ with n *odd* (Theorem J.3 page 258).

   (h) **Spectral factorization.** The quantity y can be expressed in terms of z as (Lemma 7.1 page 111)

   $$y = \left(\frac{1-z}{2}\right)\left(\frac{1-z^{-1}}{2}\right).$$

   This implies that for each root of a polynomial in y, there are two roots in the equivalent polynomial in z, and that these roots occur in recipricol pairs (if z is a root, then z^{-1} is also a root). Therefore, any polynomial $\mathsf{P}(y)$ in y can be factored into two factors in z such that $\mathsf{P}(y) = \mathsf{Q}(z)\mathsf{Q}(z^{-1})$. Once $\mathsf{P}(y)$ has been computed, we can then proceed to compute $\mathsf{Q}(z)$ with help from Theorem 7.1 (page 118).

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The quadrature condition can be expressed as a polynomial in $y = \sin^2\frac{\omega}{2}$ (Lemma 7.2 page 112). The first term in this polynomial quadrature condition is a low-pass response and the second term is a high pass; and they meet in the middle at $\omega = \frac{\pi}{2}$.

$$\underbrace{(1-y)^p\mathsf{P}(y)}_{\text{low-pass}} + \underbrace{y^p\mathsf{P}(1-y)}_{\text{high-pass}} = 1$$

The low-pass and high-pass terms are especially smooth at $\omega = 0$ ($y = 0$) and $\omega = \pi$ ($y = 1$) in that the first $p-1$ derivatives at both points are zero for both terms. This is illustrated in Figure P.1 (page 371).

Lemma 7.1.

LEM

$$\left.\begin{array}{rcl} y & \triangleq & \sin^2\left(\frac{\omega}{2}\right) \quad \textit{and} \\ z & \triangleq & e^{i\omega} \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} y & = & \frac{2-z-z^{-1}}{4} = \left(\frac{1-z}{2}\right)\left(\frac{1-z^{-1}}{2}\right) \\ 1-y & = & \cos^2\left(\frac{\omega}{2}\right) \\ & = & \frac{2+z+z^{-1}}{4} = \left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right) \\ \frac{1}{2}-y & = & \frac{1}{2}\cos\omega \\ z & = & 1-2y \pm \sqrt{y(y-1)} \end{array}\right.$$

✎PROOF:

$$\begin{aligned} y &\triangleq \sin^2\left(\frac{\omega}{2}\right) && \text{by definition of } y \\ &= \frac{1}{2} - \frac{1}{2}\cos(\omega) && \text{by } \textit{half-angle formulas} \quad \text{(Theorem I.11 page 248)} \\ &= \frac{1}{2} - \frac{1}{2}\frac{e^{i\omega}+e^{-i\omega}}{2} && \text{by } \textit{Euler formulas} \quad \text{(Corollary I.2 page 241)} \\ &= \frac{2}{4} - \frac{z+z^{-1}}{4} && \text{by definition of } z \\ &= \frac{2-z-z^{-1}}{4} \\ &= \left(\frac{1-z}{2}\right)\left(\frac{1-z^{-1}}{2}\right) \end{aligned}$$

$$\begin{aligned} 1-y &= 1-\sin^2\left(\frac{\omega}{2}\right) && \text{by definition of } y \\ &= \cos^2\left(\frac{\omega}{2}\right) && \text{by } \textit{half-angle formulas} \quad \text{(Theorem I.11 page 248)} \\ &= \frac{1}{2}(1+\cos\omega) && \text{by } \textit{half-angle formulas} \quad \text{(Theorem I.11 page 248)} \end{aligned}$$

$$\begin{aligned} 1-y &= 1-\frac{2-z-z^{-1}}{4} && \text{by previous result} \\ &= \frac{4-2+z+z^{-1}}{4} \\ &= \frac{2+z+z^{-1}}{4} \\ &= \left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right) \end{aligned}$$

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$$\begin{aligned}
\frac{1}{2} - y &= \frac{1}{2} - \sin^2\left(\frac{\omega}{2}\right) && \text{by definition of } y \\
&= \frac{1}{2} - \frac{1}{2}(1-\cos\omega) && \text{by } \textit{half-angle formulas} \quad \text{(Theorem I.11 page 248)} \\
&= \frac{1}{2}\cos\omega
\end{aligned}$$

✏

Lemma 6.1 (page 85) demonstrates that all orthogonal scaling functions satisfy the *orthonormal quadrature condition* $\left|\tilde{h}(\omega)\right|^2 + \left|\tilde{h}(\omega+\pi)\right|^2 = 2$. When designing orthogonal scaling functions with compact support, it is very useful to be able to express this quadrature condition as a polynomial in $\sin^2\frac{\omega}{2}$. Lemma 7.2 (next) does this by expressing the quadrature condition as a polynomial in y where $y = \sin^2\frac{\omega}{2}$.

Lemma 7.2. *Let* $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j \rangle\!\rangle, \langle\!\langle W_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ *be a wavelet system. Let* $y \triangleq \sin^2\left(\frac{\omega}{2}\right)$ *and* $\mathsf{P}(y)$ *a polynomial in* y.

LEM

$$\left\{\begin{array}{ll} 1. & \underbrace{\left|\tilde{h}(\omega)\right|^2 + \left|\tilde{h}(\omega+\pi)\right|^2 = 2}_{\text{QUADRATURE CONDITION } \textit{in "frequency"}} \quad \textit{and} \\ 2. & \psi \textit{ has } p \text{ VANISHING MOMENTS} \end{array}\right\} \iff \left\{\begin{array}{ll} 1. & \textit{there exists } \mathsf{P}(y) \textit{ such that} \\ & \left|\tilde{h}(\omega)\right|^2 = 2(1-y)^p P(y) \quad \textit{and} \\ 2. & \underbrace{(1-y)^p P(y) + y^p P(1-y) = 1}_{\text{QUADRATURE CONDITION } \textit{as polynomial in } y} \end{array}\right\}$$

✎PROOF:

1. Because $|\mathsf{Q}(\omega)|^2$ is *periodic*, it has a *Fourier series* expansion (Definition K.1 page 273). Because it is real, it has no imaginary part. Therefore, $|\mathsf{Q}(\omega)|^2$ can be expressed as a harmonic cosine polynomial as

$$|\mathsf{Q}(\omega)|^2 = \sum_{n=0}^{\infty} a_n \cos\omega n$$

2. Proof for $\Longrightarrow$ part:

$$\begin{aligned}
\left|\tilde{h}(\omega)\right|^2 &= \left|\sqrt{2}\left(\frac{1+e^{i\omega}}{2}\right)^p \mathsf{Q}(\omega)\right|^2 && \text{by Lemma 5.3 page 77} \\
&= 2\left|\left(\frac{1+e^{i\omega}}{2}\right)^p\right|^2 |\mathsf{Q}(\omega)|^2 \\
&= 2\left(\frac{1+e^{i\omega}}{2}\right)^p \left(\frac{1+e^{-i\omega}}{2}\right)^p |\mathsf{Q}(\omega)|^2 \\
&= 2\left(\frac{1+e^{i\omega}}{2}\right)^p \left(\frac{1+e^{-i\omega}}{2}\right)^p \sum_{n=0}^{\infty} c_n e^{-i\omega n} && \text{by item (1)} \\
&= 2\left(\frac{1+e^{i\omega}+e^{-i\omega}+1}{4}\right)^p \sum_{n=0}^{\infty} a_n \cos\omega n && \text{by item (1)} \\
&= 2\left(\frac{2+2\cos\omega}{4}\right)^p \sum_{n=0}^{\infty} a_n \cos\omega n \\
&= 2\left(\cos^2\frac{\omega}{2}\right)^p P'(\cos\omega) && \text{by } \textit{trigonometric expansion} \quad \text{(Theorem J.2 page 253)} \\
&= 2\left(1-\sin^2\frac{\omega}{2}\right)^p P'\left(1-2\sin^2\frac{\omega}{2}\right) && \text{by } \textit{half-angle formulas} \quad \text{(Theorem I.11 page 248)}
\end{aligned}$$

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$$= 2\left(1-\sin^2\frac{\omega}{2}\right)^p P\left(\sin^2\frac{\omega}{2}\right)$$
$$= 2(1-y)^p P(y)$$

$$\begin{aligned}
\left|\tilde{\mathsf{h}}(\omega+\pi)\right|^2 &= 2\left(1-\sin^2\frac{\omega+\pi}{2}\right)^p P\left(\sin^2\frac{\omega+\pi}{2}\right) && \text{by previous result}\\
&= 2\left(1-\cos^2\frac{\omega}{2}\right)^p P\left(\cos^2\frac{\omega}{2}\right) && \text{by Theorem I.7 page 242}\\
&= 2\left(\sin^2\frac{\omega}{2}\right)^p P\left(1-\sin^2\frac{\omega}{2}\right) && \text{by \textit{half-angle formulas}} \quad \text{(Theorem I.11 page 248)}\\
&= 2y^p P(1-y) && \text{by definition of } y
\end{aligned}$$

$$\begin{aligned}
2 &= \left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi))\right|^2 && \text{by left hypothesis}\\
&= (1-y)^p P(y) + y^p P(1-y) && \text{by previous results}
\end{aligned}$$

3. Proof for $\Longleftarrow$ part:

$$\begin{aligned}
1 &= (1-y)^p P(y) + y^p P(1-y) && \text{by right hypothesis}\\
&= \left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi))\right|^2 && \text{by previous results}
\end{aligned}$$

✏

In the polynomial quadrature condition, what are the possible solutions of $\mathsf{P}(y)$? Lemma 7.3 (next) answers this question.

Lemma 7.3. [2] *Let* P, Q, *and* R *be polynomials over* $\mathbb{R}$.

LEM

$$\underbrace{(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) = 1}_{\textit{quadrature condition as polynomial in } y} \quad \Longleftrightarrow \quad \begin{cases} 1. \quad \mathsf{P}(y) \triangleq \mathsf{P}_p(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right) \quad \textit{where} \\ \qquad \mathsf{P}_p(y) \triangleq \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k \quad \textit{and} \\ 2. \quad \underbrace{\mathsf{R}(y) = -\mathsf{R}(-y)}_{\mathsf{R}\ \textit{is any } \textbf{odd} \textit{ polynomial.}} \end{cases}$$

Furthermore, $\mathsf{P}(y)$ *has minimum order when* $\mathsf{R}(y) = 0$.

✎PROOF:

1. Proof that $(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) = 1 \implies \mathsf{P}(y) = \mathsf{P}_p(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)$:

   (a) lemma: proof that $\frac{1}{(1-y)^p} = \sum_{k=0}^{\infty}\binom{p-1+k}{k}y^k$:

$$\begin{aligned}
\frac{1}{(1-y)^p} &= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}\frac{1}{(1-y)^p}\right]_{y=0} y^k && \text{by \textit{Maclaurin series} (Theorem C.13 page 167)}\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}(1-y)^{-p}\right]_{y=0} y^k\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[(-p)(-1)\frac{\mathrm{d}^{k-1}}{\mathrm{d}y^{k-1}}(1-y)^{-p-1}\right]_{y=0} y^k
\end{aligned}$$

[2] Daubechies (1992) page 171, Pinsky (2002) pages 331–333, Pinsky (2008) pages 331–333 ⟨6.5.2.1 Daubechies recipe⟩, Chui (1992) pages 175–176

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$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left[(p)(-p-1)(-1)\frac{\mathrm{d}^{k-2}}{\mathrm{d}y^{k-2}}(1-y)^{-p-2}\right]_{y=0} y^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left[(p)(p+1)(p+2)\frac{\mathrm{d}^{k-3}}{\mathrm{d}y^{k-3}}(1-y)^{-p-3}\right]_{y=0} y^k \\
&\vdots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left[(p)(p+1)(p+2)\cdots(p+k-2)\frac{\mathrm{d}}{\mathrm{d}y}(1-y)^{-p-(k-1)}\right]_{y=0} y^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left[(p)(p+1)(p+2)\cdots(p+k-2)(p+k-1)(1-y)^{-p-k}\right]_{y=0} y^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(p)(p+1)(p+2)\cdots(p+k-2)(p+k-1)\, y^k \\
&= \sum_{k=0}^{\infty} \frac{(p+k-1)!}{k!(p-1)!}\, y^k \\
&= \sum_{k=0}^{\infty} \frac{(p+k-1)!}{k!(p+k-1-k)!}\, y^k \\
&= \sum_{k=0}^{\infty} \binom{p+k-1}{k}\, y^k \qquad \text{by definition of } \binom{n}{k} \text{ (Definition B.3 page 146)}
\end{aligned}
$$

(b) There are an infinite number of solutions for $\mathsf{P}(y)$. These solutions are here expressed as

$$\mathsf{P}(y) \triangleq \underbrace{\mathsf{P}_p(y)}_{\text{order } p-1} + \underbrace{y^p \mathsf{R}\left(\frac{1}{2}-y\right)}_{\text{order} \geq p}$$

(c) Note that $(1-y)^p$ and y^p are *relatively prime* and are both of order p. Therefore by *Bézout's Theorem* (Theorem C.4 page 161), there exists a unique solution for $\mathsf{P}(y)$ of order $p-1$. This unique solution is here labeled $\mathsf{P}_p(y)$. The more general solution (of order $\geq p$) is here expressed as $\mathsf{P}_p(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)$.

$$\underbrace{(1-y)^p}_{\text{order } p}\ \underbrace{\mathsf{P}_p(y)}_{\text{order } p-1} + \underbrace{y^p}_{\text{order } p}\ \underbrace{\mathsf{P}_p(1-y)}_{\text{order } p-1} = 1$$

(d) We can compute the unique order $p-1$ solution $\mathsf{P}_p(y)$ of $\mathsf{P}(y)$ by using a Maclaurin expansion of $\mathsf{P}(y)$. Let $\mathbf{T}\mathsf{P}(y)$ represent a polynomial $\mathsf{P}(y)$ with all terms of order $\geq p$ truncated.

$$
\begin{aligned}
\underbrace{\mathsf{P}_p(y)}_{\text{order } p-1} &= \frac{1}{(1-y)^p} - \frac{y^p\mathsf{P}_p(1-y)}{(1-y)^p} && \text{by (1c) lemma} \\
&= \mathbf{T}\left\{\frac{1}{(1-y)^p} - \underbrace{\cancelto{0}{\frac{y^p\mathsf{P}_p(1-y)}{(1-y)^p}}}_{\text{all terms order} \geq p}\right\} && \text{by } \textit{Bézout's Theorem} \quad \text{(Theorem C.4 page 161)} \\
&= \mathbf{T}\left\{\frac{1}{(1-y)^p}\right\} && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 4)} \\
&= \mathbf{T}\left\{\sum_{k=0}^{\infty}\binom{p+k-1}{k} y^k\right\} && \text{by item (1a) page 113} \\
&= \sum_{k=0}^{p-1}\binom{p+k-1}{k} y^k && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 4)}
\end{aligned}
$$

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(e) Alternatively, specific cases of $\mathsf{P}_p(y)$ can also be calculated using the *Extended Euclid's Algorithm*:

i. $p = 2$ case: See Example C.1 (page 160).

ii. $p = 3$ case: See Example C.2 (page 160).

iii. $p = 4$ case: See Example C.3 (page 161).

(f) Proof that $\mathsf{R}(y)$ must be an odd polynomial:

$$
\begin{aligned}
&(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y)\\
&= (1-y)^p\left[\mathsf{P}_p(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)\right] + y^p\left[\mathsf{P}_p(1-y) + (1-y)^p\mathsf{R}\left(\frac{1}{2}-[1-y]\right)\right]\\
&= (1-y)^p\mathsf{P}_p(y) + (1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + y^p\mathsf{P}_p(1-y) + y^p(1-y)^p\mathsf{R}\left(-\frac{1}{2}+y\right)\\
&= \underbrace{\left[(1-y)^p\mathsf{P}_p(y) + y^p\mathsf{P}_p(1-y)\right]}_{1} + \left[(1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + y^p(1-y)^p\mathsf{R}\left(y-\frac{1}{2}\right)\right]\\
&= 1 + (1-y)^p y^p\left[\mathsf{R}\left(\frac{1}{2}-y\right) + \mathsf{R}\left(y-\frac{1}{2}\right)\right]\\
&= 1\\
&\iff \left[\mathsf{R}\left(\frac{1}{2}-y\right) + \mathsf{R}\left(y-\frac{1}{2}\right) = 0\right]\\
&\iff \mathsf{R}(y) = -\mathsf{R}(-y) \qquad (\mathsf{R}(y) \text{ is odd})
\end{aligned}
$$

2. Proof that $(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) = 1 \impliedby \mathsf{P}(y) = \mathsf{P}_p(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)$:

(a) Examples of specific cases with $\mathsf{R}(y) = 0$:

$p = 1$ case:

$$
\begin{aligned}
&(1-y)^1\mathsf{P}_p(y) + y^1\mathsf{P}_p(1-y)\\
&\quad= (1-y)^1\sum_{k=0}^{k=1-1}\binom{1+k-1}{k}y^k + y^1\sum_{k=0}^{k=1-1}\binom{1+k-1}{k}(1-y)^k\\
&\quad= (1-y)^1(1) + y^1(1)\\
&\quad= 1
\end{aligned}
$$

$p = 2$ case:

$$
\begin{aligned}
&(1-y)^2\mathsf{P}_p(y) + y^2\mathsf{P}_p(1-y)\\
&\quad= (1-y)^2\sum_{k=0}^{k=2-1}\binom{2+k-1}{k}y^k + y^2\sum_{k=0}^{k=2-1}\binom{2+k-1}{k}(1-y)^k\\
&\quad= (1-y)^2(1+2y) + y^2[1+2(1-y)]\\
&\quad= (1-2y+y^2)(1+2y) + y^2(3-2y)\\
&\quad= \underbrace{1-2y+y^2+2y-4y^2+2y^3}_{(1-y)^p\mathsf{P}_p(y)} + \underbrace{3y^2-2y^3}_{y^p\mathsf{P}_p(1-y)}\\
&\quad= \underbrace{1+0y-3y^2+2y^3}_{(1-y)^p\mathsf{P}_p(y)} + \underbrace{3y^2-2y^3}_{y^p\mathsf{P}_p(1-y)}\\
&\quad= 1
\end{aligned}
$$

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$p = 3$ case:

$$
\begin{aligned}
&(1-y)^3\mathsf{P}_p(y) + y^3\mathsf{P}_p(1-y)\\
&= (1-y)^3 \sum_{k=0}^{k=3-1} \binom{3+k-1}{k} y^k + y^3 \sum_{k=0}^{k=3-1} \binom{3+k-1}{k}(1-y)^k\\
&= (1-y)^3(1+3y+6y^2) + y^3\left[1+3(1-y)+6(1-y)^2\right]\\
&= \underbrace{(1-3y+3y^2-y^3)(1+3y+6y^2)}_{(1-y)^p\mathsf{P}_p(y)} + \underbrace{y^3\left[4-3y+6(1-2y+y^2)\right]}_{y^p\mathsf{P}_p(1-y)}\\
&= \underbrace{1-3y+3y^2-y^3+3y-9y^2+9y^3-3y^4+6y^2-18y^3+18y^4-6y^5}_{(1-y)^p\mathsf{P}_p(y)}\\
&\quad + \underbrace{y^3\left[10-15y+6y^2\right]}_{y^p\mathsf{P}_p(1-y)}\\
&= \underbrace{1+0y+0y^2-10y^3+15y^4-6y^5}_{(1-y)^p\mathsf{P}_p(y)} + \underbrace{10y^3-15y^4+6y^5}_{y^p\mathsf{P}_p(1-y)}\\
&= 1
\end{aligned}
$$

(b) Proof for $\mathsf{R}(y) = 0$ case (using induction): [3]

i. Define $\mathsf{P}_p(y) \triangleq \sum_{k=0}^{p} \binom{p+k}{k} y^k$.

ii. lemma: proof that $(1-y)\mathsf{P}_p(y) = \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p(1-2y)$:

$$
\begin{aligned}
&(1-y)\mathsf{P}_p(y)\\
&= \mathsf{P}_p(y) - y\mathsf{P}_p(y)\\
&= \sum_{k=0}^{p} \binom{p+k}{k} y^k - \sum_{k=0}^{p} \binom{p+k}{k} y^{k+1}\\
&= \underbrace{1 + \sum_{k=1}^{p} \binom{p+k}{k} y^k}_{\sum_{k=0}^{p}\binom{p+k}{k} y^k} \underbrace{- \sum_{k=1}^{p} \binom{p+k-1}{k-1} y^k - \binom{2p}{p} y^p}_{-\sum_{k=0}^{p}\binom{p+k}{k} y^{k+1}}\\
&= 1 + \sum_{k=1}^{p} \left[\binom{p+k}{k} - \binom{p+k-1}{k-1}\right] y^k - \binom{2p}{p} y^{p+1}\\
&= 1 + \sum_{k=1}^{p} \left[\underbrace{\binom{p+k-1}{k} + \binom{p+k-1}{k-1}}_{\binom{p+k}{k}} - \binom{p+k-1}{k-1}\right] y^k - \binom{2p}{p} y^{p+1}
\end{aligned}
$$

by Pascal's Rule page 147

$$
\begin{aligned}
&= 1 + \sum_{k=1}^{p} \binom{p+k-1}{k} y^k - \binom{2p}{p} y^{p+1}\\
&= \sum_{k=0}^{p} \binom{p+k-1}{k} y^k - \binom{2p}{p} y^{p+1}
\end{aligned}
$$

[3] Many thanks to Chip Eastham for his extremely valuable help with this proof.

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$$
\begin{aligned}
&= \sum_{k=0}^{p-1} \binom{p+k-1}{k} y^k + \binom{2p-1}{p} y^p - \binom{2p}{p} y^{p+1} \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p - \frac{(2p)!}{p!p!} y^{p+1} \qquad \text{by definition of } \mathsf{P}_p(y) \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p - \frac{2p}{p}\frac{(2p-1)!}{p!(p-1)!} y^{p+1} \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p - 2\binom{2p-1}{p} y^{p+1} \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p (1-2y)
\end{aligned}
$$

iii. lemma: proof that $y\mathsf{P}_p(1-y) = \mathsf{P}_{p-1}(1-y) - \binom{2p-1}{p}(1-y)^p(1-2y)$:

$$
\begin{aligned}
y\mathsf{P}_p(1-y) &= \left[(1-u)\mathsf{P}_p(u)\right]_{u \triangleq 1-y} \\
&= \left[(1-u)\mathsf{P}_p(u)\right]_{u \triangleq 1-y} \\
&= \left[\mathsf{P}_{p-1}(u) + \binom{2p-1}{p} u^p (1-2u)\right]_{u \triangleq 1-y} \qquad \text{by previous lemma} \\
&= \mathsf{P}_{p-1}(1-y) + \binom{2p-1}{p}(1-y)^p(-1+2y) \\
&= \mathsf{P}_{p-1}(1-y) - \binom{2p-1}{p}(1-y)^p(1-2y)
\end{aligned}
$$

iv. Proof that $p-1$ case $\implies$ p case:

$$
\begin{aligned}
&(1-y)^{p+1}\mathsf{P}_p(y) + y^{p+1}\mathsf{P}_p(1-y) \\
&= (1-y)^p(1-y)\mathsf{P}_p(y) + y^p y\mathsf{P}_p(1-y) \\
&= (1-y)^p \underbrace{\left[\mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p(1-2y)\right]}_{\text{by item (2(b)ii) page 116}} + y^p \underbrace{\left[\mathsf{P}_{p-1}(1-y) - \binom{2p-1}{p}(1-y)^p(1-2y)\right]}_{\text{by item (2(b)iii) page 117}} \\
&= \underbrace{(1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y)}_{1} + \binom{2p-1}{p}(1-2y)\underbrace{\left[(1-y)^p y^p - y^p(1-y)^p\right]}_{0} \\
&= 1 \qquad \text{by induction hypothesis}
\end{aligned}
$$

v. Therefore by induction, $(1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y) = 1$ is true for all $p \in \mathbb{N}$.

(c) Proof for $\mathsf{R}(y) \neq 0$ case:

$$
\begin{aligned}
&(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) \\
&= (1-y)^p\left[\mathsf{P}_{p-1}(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)\right] + y^p\left[\mathsf{P}_{p-1}(1-y) + (1-y)^p\mathsf{R}\left(\frac{1}{2} - [1-y]\right)\right] \\
&= (1-y)^p\mathsf{P}_{p-1}(y) + (1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + y^p\mathsf{P}_{p-1}(1-y) + y^p(1-y)^p\mathsf{R}\left(-\frac{1}{2}+y\right) \\
&= \left[(1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y)\right] + \underbrace{\left[(1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + -y^p(1-y)^p\mathsf{R}\left(y-\frac{1}{2}\right)\right]}_{0 \text{ because } \mathsf{R}(y) \text{ is an odd polynomial}} \\
&= (1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y) \\
&= 1 \qquad \text{by item (2b) page 116}
\end{aligned}
$$

Example 7.1. [4] Here are some examples of the minimal polynomial $\mathsf{P}_p(y)$ for varying number of vanishing moments p:

EX

p	$\mathsf{P}_p(y) = \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k$
1	1
2	$1 + 2y$
3	$1 + 3y + 6y^2$
4	$1 + 4y + 10y^2 + 20y^3$
5	$1 + 5y + 15y^2 + 35y^3 + 70y^4$
6	$1 + 6y + 21y^2 + 56y^3 + 126y^4 + 252y^5$
7	$1 + 7y + 28y^2 + 84y^3 + 210y^4 + 462y^5 + 924y^6$
8	$1 + 8y + 36y^2 + 120y^3 + 330y^4 + 792y^5 + 1716y^6 + 3432y^7$
9	$1 + 9y + 45y^2 + 165y^3 + 495y^4 + 1287y^5 + 3003y^6 + 6435y^7 + 12870y^8$
10	$1 + 10y + 55y^2 + 220y^3 + 715y^4 + 2002y^5 + 5005y^6 + 11440y^7 + 24310y^8 + 48620y^9$

Example 7.2. Here are some examples of the minimal polynomial $\mathsf{P}_p(y)$ evaluated at $y \triangleq \frac{-z+2-z^{-1}}{4}$:

EX

p	$\mathsf{P}_p\left(\frac{z-2+z^{-1}}{-4}\right)$
1	1
2	$\frac{z^2-4z-1}{-2z}$
3	$\frac{3z^4-18z^3+38z^2-18z+3}{2^3z^2}$
4	$\frac{5z^6-40z^5+131z^4-208z^3+131z^2-40z+5}{-2^4z^3}$
5	$\frac{35z^8-350z^7+1520z^6-3650z^5+5018z^4-3650z^3+1520z^2-350z+35}{2^7z^4}$
6	$\frac{63z^{10}-756z^9+4067z^8-12768z^7+25374z^6-32216z^5+25374z^4-12768z^3+4067z^2-756z+63}{-2^8z^5}$

Theorem 7.1 (next) gives a method for computing scaling functions with compact support and p vanishing moments. This method results in an infinite number of possible solutions. For each value p, there are exactly $2^{\lfloor p/2 \rfloor}$ solutions that are of minimum order $2p-1$. Of these $2^{\lfloor p/2 \rfloor}$ solutions, there is one solution that is also *minimum phase* (Definition Q.2 page 382). This is the Daubechies-p scaling function. All the $2^{\lfloor p/2 \rfloor}$ solutions occur when $\mathsf{R}(y) = 0$. An infinite number of other solutions are also available when $\mathsf{R}(y) \neq 0$.

Theorem 7.1. *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, \langle\!\langle \mathbf{V}_j \rangle\!\rangle, \langle\!\langle \mathbf{W}_j \rangle\!\rangle, \phi, \psi, \langle\!\langle h_n \rangle\!\rangle, \langle\!\langle g_n \rangle\!\rangle\right)$ *be an* ***orthonormal*** *wavelet system.*

THM

1. *If* ψ *has* p *vanishing moments, then* $\langle\!\langle h_n \rangle\!\rangle_{n\in\mathbb{Z}}$ *must be of a form that satisfies*

$$i) \quad \sum_n h_n z^{-n} = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z)$$

$$ii) \quad \mathsf{Q}(z)\mathsf{Q}(z^{-1}) = \left[\sum_{k=0}^{p-1}\binom{p-1+k}{k} y^k + y^p R\left(\frac{1}{2}-y\right)\right]\Bigg|_{y=\frac{2-z-z^{-1}}{4}}$$

2. *The support of* $\langle\!\langle h_n \rangle\!\rangle$ *is minimized when* $R(y) = 0$.

3. *If* ψ *has* p *vanishing moments, then the minimum support size of* $\langle\!\langle h_n \rangle\!\rangle$ *is* $2p-1$ *(*$2p$ *non-zero elements).*

4. *If* $\langle\!\langle h_n \rangle\!\rangle$ *has support size* $2p-1$ *(*$2p$ *non-zero coefficients) then* ψ *has maximum of* p *vanishing moments.*

✎PROOF:

[4] Pinsky (2002) page 333, Pinsky (2008) page 333 ⟨Exercise 6.5.11: $\mathsf{P}_1(y), \mathsf{P}_2(y), \ldots, \mathsf{P}_6(y)$⟩, Sloane (2009) http://oeis.org/A000580

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1. Proof for the form of $(\!(h_n)\!)$:

   (a) For $y \triangleq \sin^2\frac{\omega}{2}$

$$\underbrace{\left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi)\right|^2 = 2}_{\text{quadrature condition in frequency domain}} \qquad \Longleftrightarrow \qquad \underbrace{(1-y)^p\ddot{\mathsf{Q}}(y) + y^p\ddot{\mathsf{Q}}(1-y) = 1}_{\text{quadrature condition as a polynomial in } y}$$

   (b) The polynomial $\ddot{\mathsf{Q}}(\omega)$ is non-negative and therefore can be factorized into two factors:

$$\ddot{\mathsf{Q}}(y)\Big|_{y=\frac{2-z-z^{-1}}{4}} = \mathsf{Q}\left(z\right)\mathsf{Q}\left(\frac{1}{z^*}\right) \qquad \text{by } \textit{Fejér-Riesz factorization} \text{ (Theorem J.5 page 262)}$$

   (c)

$$|\check{\mathsf{h}}(z)|^2 = 2\left(\frac{1+z^{-1}}{2}\right)^p \ddot{\mathsf{Q}}(z) \qquad = 2\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}\left(z\right)\mathsf{Q}\left(\frac{1}{z^*}\right)$$

$$\check{\mathsf{h}}(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z)$$

2. Proof that the support of $(\!(h_n)\!)$ is minimized when $R(y) = 0$:
   This follows directly from *Bezout's Theorem* (Lemma C.4 page 161).

$$\begin{aligned}
Q\left(z\right)Q\left(\frac{1}{z^*}\right) &= \ddot{\mathsf{Q}}(y)\Big|_{y=\frac{2-z-z^{-1}}{4}} && \text{by Theorem J.5 page 262}\\
&= \left[\sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k + y^p \cancelto{0}{R\left(\frac{1}{2}-y\right)}\right]_{y=\frac{2-z-z^{-1}}{4}} && \text{by Lemma 7.3 page 113}\\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}\left(\frac{2-z-z^{-1}}{4}\right)^k && \text{by Lemma 7.3 page 113}
\end{aligned}$$

3. Proof that the minimum support size of $(\!(h_n)\!)$ is $2p-1$ ($2p$ non-zero elements).

$$\underbrace{\ddot{\mathsf{Q}}(y)\Big|_{y=\frac{2-z-z^{-1}}{4}}}_{\text{order } p-1 \text{ in } y} = \underbrace{\ddot{\mathsf{Q}}\left(\frac{2-z-z^{-1}}{4}\right)}_{\text{order } 2p-2 \text{ in } z}$$

$$= \underbrace{\mathsf{Q}\left(z\right)}_{\text{order } p-1}\underbrace{\mathsf{Q}\left(\frac{1}{z^*}\right)}_{\text{order } p-1} \qquad \text{by } \textit{Fejér-Riesz factorization} \text{ (Theorem J.5 page 262)}$$

$$\underbrace{\check{\mathsf{h}}(z)}_{\text{order } 2p-1} = \underbrace{\sqrt{2}\left(\frac{z+1}{2}\right)^p}_{\text{order } p}\underbrace{Q(z)}_{\text{order } p-1}$$

7.2 Design details

Definition 7.1. *Let* $\check{\mathsf{h}}(z)$ *be the* Z-TRANSFORM *(Definition Q.1 page 379) of a sequence* $(\!(h_n)\!)_{n\in\mathbb{Z}}$. *Let* $Q(z)$ *be a polynomial with real coefficients and*

$$y \triangleq \sin^2\left(\frac{\omega}{2}\right)$$

$$P(y) \triangleq \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k$$

$$\dot{Q} \triangleq \left\{Q(z)|Q(z)Q(z^{-1}) = P\left(\frac{2-z-z^{-1}}{4}\right\}\right) \qquad \text{(FEJÉR-RIESZ SPECTRAL FACTORIZATIONS of P)}$$

$$\mathbf{R}Q \triangleq \left\{z_n|z_n \text{ is a zero of } Q(z)\right\} \qquad \text{(roots of } Q(z))$$

DEF

$\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *is a* ***Daubechies-p wavelet system*** *if*

1. $\check{\mathsf{h}}(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p Q(z)$ *and*
2. $Q(z)$ *is the polynomial in* $\dot{Q}$ such that $\forall z_n \in \mathbf{R}Q,\ |z_n| < 1$ *and*
3. $g_n = (-1)^n h_{2p-1-n}$

$\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *is also called a* D_p ***wavelet system***.

The Daubechies-p wavelets of Definition 7.1 page 120 can be calculated [5] by the following steps:

1. Compute the polynomial $\mathsf{P}(y)$. This polynomial has $p-1$ roots in y.

$$P(y) \triangleq \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k \qquad \text{where } \binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}$$

2. Compute $\mathsf{P}\left(\frac{2-z-z^{-1}}{4}\right)$. This polynomial has $2p-2$ roots in z.

   (a) Because the coefficients of $\mathsf{P}\left(\frac{2-z-z^{-1}}{4}\right)$ are real, all of its roots occur in *complex conjugate pairs*:

$$\text{root at } z_1 = re^{i\theta} \iff \text{root at } z_1^* = \left(re^{i\theta}\right)^* = re^{-i\theta}$$

[5] For an actual implementation using *Octave*, see Section Y.1 (page 441).

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(b) All of the roots of $\mathsf{P}\left(\frac{2-z-z^{-1}}{4}\right)$ occur in *conjugate reciprocal pairs*:

$$P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = re^{i\theta} \quad \Longleftrightarrow$$
$$P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = \left(re^{i\theta}\right)^{-1} = \frac{1}{r}e^{-i\theta}$$

3. By the *Fejér-Riesz spectral factorization theorem* (Theorem J.5 page 262), [6] $\mathsf{P}\left(\frac{2-z-z^{-1}}{4}\right)$ can be factored in the form
$$P\left(\frac{2-z-z^{-1}}{4}\right) = Q(z)Q(z^{-1}).$$

4. Form $Q(z)$ from the $p-1$ zeros *inside* the unit circle.

5. The scaling coefficients $(\!(h_n)\!)$ are the coefficients of the polynomial
$$\check{\mathsf{h}}(z) = \underbrace{\sqrt{2}\underbrace{\left(\frac{1+z^{-1}}{2}\right)^{p}}_{\text{order } p}\underbrace{Q(z)}_{\text{order } p-1}}_{\text{order } 2p-1\ (2p \text{ coefficients})}$$

As the number of vanishing moments approaches infinity ($p \to \infty$), the zeros of the Daubechies-p scaling coefficients approach two asymptotic arcs as described in the next theorem and illustrated in Figure 7.1 (page 122).

[6] *Fejér-Riesz spectral factorization* (Theorem J.5 page 262)

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Figure 7.1: Zero locations for B-cardinal spline $N_p(x)$ (left) and Daubechies-p (right) scaling coefficients

Lemma 7.4. [7] *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$.

LEM

$\boldsymbol{\Omega}$ *is a* DAUBECHIES-p WAVELET SYSTEM *(Definition 7.1 page 120)* $\Longrightarrow$
with increasing vanishing moments parameter p*, the zeros of* $Q(z)Q(z^{-1})$ *asymptotically approach the area of the complex plain enclosed by the arcs*

$$|z-1| = \sqrt{2} \qquad \text{and} \qquad |z+1| = \sqrt{2}$$

Theorem 7.2. *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$. *Let* $\operatorname{supp}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *be defined as in Definition 2.4 page 33.*

THM

$\{\boldsymbol{\Omega}$ *is a* DAUBECHIES-p WAVELET SYSTEM *(Definition 7.1 page 120)*$\}$ $\Longrightarrow$

$$\left\{\begin{array}{lll} (1). & \boldsymbol{\Omega} \text{ is an ORTHONORMAL WAVELET SYSTEM.} & \text{and} \\ (2). & \check{\mathsf{h}}(\omega) \text{ has MINIMUM PHASE.} & \text{and} \\ (3). & \operatorname{supp}\left(\!\left(h_n\right)\!\right) = 2p-1 \ \left(\left(\!\left(h_n\right)\!\right) \text{ has } 2p \text{ non-zero elements}\right) & \text{and} \\ (4). & \text{the support size of } \left(\!\left(h_n\right)\!\right) \text{ is the smallest possible} & \\ & \text{for wavelets with } p \text{ vanishing moments.} & \end{array}\right\}$$

✎PROOF:

1. proof for (1): For $p = 1$, this is the *Haar wavelet system*. For $p \geq 2$, this follows from Theorem 6.2 page 88.

   (a) Proof that $\left(\!\left(h_n\right)\!\right)$ satisfies $\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}$:

$$\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z=1} &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p Q(z)\Bigg|_{z=1} && \text{by def. of } \textit{Daubechies-p wavelet system} \quad \text{(Definition 7.1 page 120)} \\
&= \sqrt{2}\left(\frac{1+1^{-1}}{2}\right)^p Q(1) \\
&= \sqrt{2}Q(1) \\
&= \sqrt{2}\sqrt{Q(1)Q^*(1^{-1})} \\
&= \sqrt{2}\sqrt{Q(z)Q^*\left(z^{-1}\right)}\Bigg|_{z=1} \\
&= \sqrt{2}\sqrt{P\left(\frac{2-z-z^{-1}}{4}\right)}\Bigg|_{z=1}
\end{aligned}$$

[7] Strang and Nguyen (1996) page 170 ⟨0961408871⟩, Vidakovic (1999) page 78 ⟨Fig. 3.14⟩, Shen and Strang (1996) page 3823, Kateb and Lemarié-Rieusset (1997)

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$$= \sqrt{2}\sqrt{P(0)}$$

$$= \sqrt{2}\sqrt{\sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k}\Bigg|_{y=0} \qquad \text{by } \textit{orthonormal quadrature conditions} \quad \text{(Lemma 7.3 page 113)}$$

$$= \sqrt{2}\sqrt{(1\cdot 0^0+0)} \qquad \text{by definition of } \binom{n}{k} \quad \text{(Definition B.3 page 146)}$$

$$= \sqrt{2}\sqrt{(1+0)}$$

$$= \sqrt{2}$$

$$\Longrightarrow \sum_{n\in\mathbb{Z}} h_n = \sqrt{2} \qquad \text{by } \textit{orthonormal quadrature conditions} \quad \text{(Lemma 6.1 page 85)}$$

(b) Proof that $(\!(h_n)\!)$ satisfies $\sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n$:

$\boldsymbol{\Omega}$ is a D_p *wavelet system*

$$\Longrightarrow (1-y)^p P(y) + y^p P(1-y) = 1 \qquad \text{by Definition 7.1 page 120 and Lemma 7.3 page 113}$$

$$\Longleftrightarrow \left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi)\right|^2 = 2 \qquad \text{by Lemma 7.2 page 112}$$

$$\Longleftrightarrow \sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n \qquad \text{by } \textit{orthonormal quadrature conditions} \quad \text{(Lemma 6.1 page 85)}$$

(c) Proof that $\tilde{\phi}(\omega)$ is *continuous* at 0:

$$\check{\mathsf{h}}(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p Q(z) \qquad \text{by Definition 7.1 page 120}$$

$$\Longrightarrow \psi \text{ has } p \textit{ vanishing moments} \qquad \text{by Lemma 5.3 page 77}$$

$$\Longrightarrow \tilde{\phi}(\omega) \text{ is } \textit{continuous} \text{ at } \omega = 0 \qquad \text{by Theorem 5.5 page 77}$$

(d) Proof that $\inf_{\omega\in[-\pi/2:\pi/2]} \left|\check{\mathsf{h}}(\omega)\right| > 0$:
by Lemma 7.4 page 122—note that the zeros are never on the unit circle in the range $[-\pi/2 : \pi/2]$.

2. Proof for (2): The zeros are chosen to be *inside* the unit circle, thus giving $\tilde{\mathsf{h}}(\omega)$ *minimum phase*.
3. Proof for (3): See Theorem 7.1 page 118.
4. Proof for (4): See Theorem 7.1 page 118.

✍

7.3 Examples

Example 7.3. The **Daubechies**-1 wavelet system yields the following results.

EX

n	h_n	g_n
0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
1	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$

The **Daubechies**-1 wavelet system is equivalent to the *Haar wavelet system*, and it is possible to compute this wavelet system using "time-domain" techniques as well:

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EX

n	h_n	g_n
0	$\frac{\sqrt{2}}{8}(1+\sqrt{3})$	$\frac{\sqrt{2}}{8}(1-\sqrt{3})$
1	$\frac{\sqrt{2}}{8}(3+\sqrt{3})$	$-\frac{\sqrt{2}}{8}(3-\sqrt{3})$
2	$\frac{\sqrt{2}}{8}(3-\sqrt{3})$	$\frac{\sqrt{2}}{8}(3+\sqrt{3})$
3	$\frac{\sqrt{2}}{8}(1-\sqrt{3})$	$-\frac{\sqrt{2}}{8}(1+\sqrt{3})$

Figure 7.2: *Daubechies-2 wavelet system* (Example 7.4 page 124)

1. Admissibility condition and *orthonormality*—Example 6.3 (page 104)

2. Admissibility condition and *partition of unity*—Example 4.1 (page 60)

✎PROOF:

$$
\begin{aligned}
Q(z)Q(z^{-1}) &= P(y)\big|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{0}\binom{1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= 1 \\
&= \underbrace{1}_{Q(z)}\cdot\underbrace{1}_{Q(z^{-1})}
\end{aligned}
$$

$$
\begin{aligned}
\check{h}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \cancelto{1}{Q(z)} \\
&= \underbrace{\frac{1}{\sqrt{2}}}_{h_0} + z^{-1}\underbrace{\frac{1}{\sqrt{2}}}_{h_1}
\end{aligned}
$$

✏

Example 7.4. The **Daubechies-2 wavelet system** yields the results illustrated in Figure 7.2 (page 124). It is also possible to compute this wavelet system using a "time-domain" technique as well—see Example 4.1 (page 60).

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✎PROOF:

$$
\begin{aligned}
\mathsf{Q}(z)\mathsf{Q}(z^{-1}) &= P(y)\big|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{1}\binom{1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= (1+2y)\big|_{y=\frac{2-z-z^{-1}}{4}} \\
&= 1+\frac{2-z-z^{-1}}{2} \\
&= \frac{4-z-z^{-1}}{2} \\
&= \frac{4z-z^2-1}{2z} \\
&= \frac{z^2-4z+1}{-2z} \\
&= \frac{1}{-2z}\left[z-(2-\sqrt{3})\right]\left[z-(2+\sqrt{3})\right] \\
&= \frac{1}{-2}\left[z-(2-\sqrt{3})\right]\left[1-(2+\sqrt{3})z^{-1}\right] \\
&= \frac{1}{2}\left[z-(2-\sqrt{3})\right]\left[(2+\sqrt{3})z^{-1}-1\right] \\
&= \frac{2+\sqrt{3}}{2}\left[z-(2-\sqrt{3})\right]\left[\frac{2+\sqrt{3}}{2+\sqrt{3}}z^{-1}-\frac{1}{2+\sqrt{3}}\right] \\
&= \frac{2+\sqrt{3}}{2}\left[z-(2-\sqrt{3})\right]\left[z^{-1}-(2-\sqrt{3})\right] \\
&= \underbrace{\frac{1+\sqrt{3}}{\sqrt{2}}\left[z-(2-\sqrt{3})\right]}_{\mathsf{Q}(z)}\underbrace{\frac{1+\sqrt{3}}{\sqrt{2}}\left[z^{-1}-(2-\sqrt{3})\right]}_{\mathsf{Q}(z^{-1})}
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p\mathsf{Q}(z) \\
&= \frac{\sqrt{2}}{4}\left(\frac{z+1}{z}\right)^2\mathsf{Q}(z) \\
&= \frac{\sqrt{2}}{4z^2}\frac{1+\sqrt{3}}{2}\left[z-(2-\sqrt{3})\right][z+1]^2 \\
&= \frac{\sqrt{2}}{8z^2}(1+\sqrt{3})\left[z-(2-\sqrt{3})\right]\left[z^2+z+1\right] \\
&= \frac{\sqrt{2}}{8z^2}(1+\sqrt{3})[z^3+z^2(\sqrt{3})+z(-3+2\sqrt{2})+(-2+\sqrt{3})] \\
&= \frac{\sqrt{2}}{8z^2}[z^3(1+\sqrt{3})+z^2(3+\sqrt{3})+z(3-\sqrt{3})+(1-\sqrt{3})] \\
&= \underbrace{z\frac{\sqrt{2}}{8}(1+\sqrt{3})}_{h_0}+\underbrace{\frac{\sqrt{2}}{8}(3+\sqrt{3})}_{h_1}+\underbrace{z^{-1}\frac{\sqrt{2}}{8}(3-\sqrt{3})}_{h_2}+\underbrace{z^{-2}\frac{\sqrt{2}}{8}(1-\sqrt{3})}_{h_3}
\end{aligned}
$$

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EX

n	h_n	g_n
0	0.3326705530	0.0352262919
1	0.8068915093	0.0854412739
2	0.4598775021	−0.1350110200
3	−0.1350110200	−0.4598775021
4	−0.0854412739	0.8068915093
5	0.0352262919	−0.3326705530

Figure 7.3: *Daubechies-3 wavelet system* (Example 7.5 page 126)

Example 7.5. The **Daubechies-3 wavelet system** yields the results illustrated in Figure 7.3 (page 126).

✎PROOF:

$$
\begin{aligned}
Q(z)Q(z^{-1}) &= P(y)\big|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{2}\binom{2+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \frac{2!}{0!(2-0)!}y^0 + \frac{3!}{1!(3-1)!}y^1 + \frac{4!}{2!(4-2)!}y^2\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \left[1+3y+6y^2\right]_{y=\frac{2-z-z^{-1}}{4}} \\
&= 1+3\left(\frac{2-z-z^{-1}}{4}\right)+6\left(\frac{2-z-z^{-1}}{4}\right)^2 \\
&= 1+3\left(\frac{z^2-2z+1}{-4z}\right)+6\left(\frac{z^2-2z+1}{-4z}\right)^2 \\
&= 1+\left(\frac{-3z^2+6z-3}{4z}\right)+6\left(\frac{z^4-4z^3+6z^2-4z+1}{16z^2}\right) \\
&= \frac{16z^2}{16z^2}+\left(\frac{-12z^3+24z^2-12z}{16z^2}\right)+\left(\frac{6z^4-24z^3+36z^2-24z+6}{16z^2}\right) \\
&= \frac{6z^4-36z^3+76z^2-36z+6}{16z^2} \\
&= \frac{3}{8z^2}\left(z^4-6z^3+\frac{38}{3}z^2-6z+1\right) \\
&= \frac{3}{8z^2}\left(z^4-6z^3+\frac{38}{3}z^2-6z+1\right) \\
&= \frac{3}{8z^2}(z-r_1)(z-r_1^*)(z-r_2)(z-r_2^*)
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{3}{8}(z-r_1)(z-r_1^*)(1-z^{-1}r_2)(1-z^{-1}r_2^*)\\
&= \frac{3r_2r_2^*}{8}(z-r_1)(z-r_1^*)(r_2^{-1}-z^{-1})(r_2^{*-1}-z^{-1})\\
&= \frac{3r_2r_2^*}{8}(z-r_1)(z-r_1^*)(\frac{1}{r_2}-z^{-1})(\frac{1}{r_2^*}-z^{-1})\\
&= \frac{3|r_2|^2}{8}(z-r_1)(z-r_1^*)(r_1^*-z^{-1})(r_1-z^{-1})\\
&= \frac{3|r_2|^2}{8}(z-r_1)(z-r_1^*)(z^{-1}-r_1^*)(z^{-1}-r_1)\\
&= \underbrace{\sqrt{\frac{3|r_2|^2}{8}}(z-r_1)(z-r_1^*)}_{\mathsf{Q}(z)}\underbrace{\sqrt{\frac{3|r_2|^2}{8}}(z^{-1}-r_1^*)(z^{-1}-r_1)}_{\mathsf{Q}(z^{-1})}
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= 0.287251 + i0.152892 = \frac{1}{r_2^*}\\
r_2 &= 2.712749 + i1.443887 = \frac{1}{r_1^*}
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z)\\
&= \sqrt{2}\left(\frac{z+1}{2z}\right)^3 \mathsf{Q}(z)\\
&= \sqrt{2}\sqrt{\frac{3|r_2|^2}{8}}\left(\frac{z+1}{2z}\right)^3 (z-r_1)(z-r_1^*)\\
&= \left(\frac{\sqrt{3|r_2|^2}}{16}\right)\frac{(z+1)^3(z-r_1)(z-r_1^*)}{z^3}\\
&= \left(\frac{\sqrt{3|r_2|^2}}{16}\right)\frac{(z^3+3z^2+3z+1)(z^2-2z\mathbf{R}_{\mathsf{e}}\left[r_1\right]+|r_1|^2)}{z^3}\\
&= \frac{h_0z^5+h_1z^4+h_2z^3+h_3z^2+h_4z+h_5}{z^3}
\end{aligned}
$$

$$
\begin{aligned}
h_0 &= +0.3326705529500830\\
h_1 &= +0.8068915093110932\\
h_2 &= +0.4598775021184915\\
h_3 &= -0.1350110200102552\\
h_4 &= -0.0854412738820268\\
h_5 &= +0.0352262918857096
\end{aligned}
$$

✏

Example 7.6. The **Daubechies-4 wavelet system** yields the results illustrated in Figure 7.4 (page 128).

✎PROOF:

$$
P(y)|_{y=\frac{2-z-z^{-1}}{4}} = \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}}
$$

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EX

n	h_n	g_n
0	0.2303778133	−0.0105974018
1	0.7148465706	−0.0328830117
2	0.6308807679	0.0308413818
3	−0.0279837694	0.1870348117
4	−0.1870348117	−0.0279837694
5	0.0308413818	−0.6308807679
6	0.0328830117	0.7148465706
7	−0.0105974018	−0.2303778133

Figure 7.4: *Daubechies-4 wavelet system* (Example 7.6 page 127)

$$= \sum_{k=0}^{3} \binom{3+k}{k} y^k \Bigg|_{y=\frac{2-z-z^{-1}}{4}}$$

$$= \frac{3!}{0!(3-0)!}y^0 + \frac{4!}{1!(4-1)!}y^1 + \frac{5!}{2!(5-2)!}y^2 + \frac{6!}{3!(6-3)!}y^3 \Bigg|_{y=\frac{2-z-z^{-1}}{4}}$$

$$= \left[1 + 4y + 10y^2 + 20y^3\right]_{y=\frac{2-z-z^{-1}}{4}}$$

$$= 1 + 4\left(\frac{2-z-z^{-1}}{4}\right) + 10\left(\frac{2-z-z^{-1}}{4}\right)^2 + 20\left(\frac{2-z-z^{-1}}{4}\right)^3$$

$$\begin{aligned}
r_1 &= +0.3288759177860292 \\
r_2 &= +0.2840962981918215 + 0.2432282259103822i \\
r_3 &= +0.2840962981918215 - 0.2432282259103822i && = r_2^* \\
r_4 &= +3.0406604616474535 && = \frac{1}{r_1} \\
r_5 &= +2.0311355120914394 - 1.7389508076448230i && = \frac{1}{r_2} \\
r_6 &= +2.0311355120914394 + 1.7389508076448230i && = r_5^*
\end{aligned}$$

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z) \\
&= \sqrt{2}\left(\frac{z+1}{2z}\right)^4 \mathsf{Q}(z) \\
&= A_0\left[\frac{z+1}{2}\right]^4 \frac{(z-r_1)(z-r_2)(z-r_2^*)}{z^4} \\
&= \frac{h_0 z^7 + h_1^6 + h_2 z^5 + h_3 z^4 + h_4 z^3 + h_5 z^2 + h_6 z + h_7}{z^4}
\end{aligned}$$

$$\begin{aligned}
h_0 &= +0.2303778133088954 \\
h_1 &= +0.7148465705529126
\end{aligned}$$

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EX

n	h_n	g_n
0	0.0544158422	−0.0001174768
1	0.3128715909	−0.0006754494
2	0.6756307363	−0.0003917404
3	0.5853546837	0.0048703530
4	−0.0158291053	0.0087460940
5	−0.2840155430	−0.0139810279
6	0.0004724846	−0.0440882539
7	0.1287474266	0.0173693010
8	−0.0173693010	0.1287474266
9	−0.0440882539	−0.0004724846
10	0.0139810279	−0.2840155430
11	0.0087460940	0.0158291053
12	−0.0048703530	0.5853546837
13	−0.0003917404	−0.6756307363
14	0.0006754494	0.3128715909
15	−0.0001174768	−0.0544158422

Figure 7.5: *Daubechies*-8 *wavelet system* (Example 7.7 page 129)

$$h_2 = +0.6308807679298577$$
$$h_3 = -0.0279837694168571$$
$$h_4 = -0.1870348117190903$$
$$h_5 = +0.0308413818355609$$
$$h_6 = +0.0328830116668849$$
$$h_7 = -0.0105974017850690$$

✏

Example 7.7. The **Daubechies-8 wavelet system** yields the results illustrated in Figure 7.5 (page 129).

✎PROOF:

$$\begin{aligned}
\mathsf{Q}(z)\mathsf{f}(Q^{-1}) &= P\left(\frac{2-z-z^{-1}}{4}\right)\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{7}\binom{7+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \left[1+8y+36y^2+120y^3+330y^4+792y^5+1716y^6+3432y^7\right]_{y=\frac{2-z-z^{-1}}{4}}
\end{aligned}$$

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r_1	=	$+0.3654035130742002$		r_8	=	$+2.7367005631344137$	
r_2	=	$+0.3577427639711839$	$+0.1159310245530727i$	r_9	=	$+1.0379714430713773$	$+1.7304352168829922i$
r_3	=	$+0.3577427639711839$	$-0.1159310245530727i$	r_{10}	=	$+1.0379714430713773$	$-1.7304352168829922i$
r_4	=	$+0.3298169959381100$	$+0.2476497421117944i$	r_{11}	=	$+1.9388494686323963$	$+1.4558242201373777i$
r_5	=	$+0.3298169959381100$	$-0.2476497421117944i$	r_{12}	=	$+1.9388494686323963$	$-1.4558242201373777i$
r_6	=	$+0.2549176775413241$	$+0.4249813706999120i$	r_{13}	=	$+2.5296496127413031$	$+0.8197646490830808i$
r_7	=	$+0.2549176775413241$	$-0.4249813706999120i$	r_{14}	=	$+2.5296496127413031$	$-0.8197646490830808i$

$$
\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z) \\
&= \sqrt{2}\left(\frac{z+1}{2z}\right)^8 \mathsf{Q}(z) \\
&= A_0\,\frac{(z+1)^8(z-r_1)(z-r_2)(z-r_2^*)(z-r_4)(z-r_4^*)(z-r_6)(z-r_6^*)}{z^8} \\
&= \frac{h_0z^{15}+h_1z^{14}+h_2z^{13}+\cdots+h_{13}z^2+h_{14}z+h_{15}}{z^8}
\end{aligned}
$$

h_0	=	$+0.0544158422430650$	h_6	=	$+0.0004724845739543$	h_{11}	=	$+0.0087460940474007$
h_1	=	$+0.3128715909140905$	h_7	=	$+0.1287474266204180$	h_{12}	=	-0.0048703529934458
h_2	=	$+0.6756307362969000$	h_8	=	-0.0173693010017349	h_{13}	=	-0.0003917403733767
h_3	=	$+0.5853546836540295$	h_9	=	-0.0440882539307343	h_{14}	=	$+0.0006754494064499$
h_4	=	-0.0158291052560285	h_{10}	=	$+0.0139810279173888$	h_{15}	=	-0.0001174767841246
h_5	=	-0.2840155429611567						

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EX

p	n	h_n
5	0	0.1601023980
	1	0.6038292698
	2	0.7243085284
	3	0.1384281459
	4	−0.2422948871
	5	−0.0322448696
	6	0.0775714938
	7	−0.0062414902
	8	−0.0125807520
	9	0.0033357253
6	0	0.1115407434
	1	0.4946238904
	2	0.7511339080
	3	0.3152503517
	4	−0.2262646940
	5	−0.1297668676
	6	0.0975016056
	7	0.0275228655
	8	−0.0315820393
	9	0.0005538422
	10	0.0047772575
	11	−0.0010773011
7	0	0.0778520541
	1	0.3965393195
	2	0.7291320908
	3	0.4697822874
	4	−0.1439060039
	5	−0.2240361850
	6	0.0713092193
	7	0.0806126092
	8	−0.0380299369
	9	−0.0165745416
	10	0.0125509986
	11	0.0004295780
	12	−0.0018016407
	13	0.0003537138

p	n	h_n
8	0	0.0544158422
	1	0.3128715909
	2	0.6756307363
	3	0.5853546837
	4	−0.0158291053
	5	−0.2840155430
	6	0.0004724846
	7	0.1287474266
	8	−0.0173693010
	9	−0.0440882539
	10	0.0139810279
	11	0.0087460940
	12	−0.0048703530
	13	−0.0003917404
	14	0.0006754494
	15	−0.0001174768
9	0	0.0380779474
	1	0.2438346746
	2	0.6048231237
	3	0.6572880781
	4	0.1331973858
	5	−0.2932737833
	6	−0.0968407832
	7	0.1485407493
	8	0.0307256815
	9	−0.0676328291
	10	0.0002509471
	11	0.0223616621
	12	−0.0047232048
	13	−0.0042815037
	14	0.0018476469
	15	0.0002303858
	16	−0.0002519632
	17	0.0000393473

p	n	h_n
10	0	0.0266700579
	1	0.1881768001
	2	0.5272011889
	3	0.6884590395
	4	0.2811723437
	5	−0.2498464243
	6	−0.1959462744
	7	0.1273693403
	8	0.0930573646
	9	−0.0713941472
	10	−0.0294575368
	11	0.0332126741
	12	0.0036065536
	13	−0.0107331755
	14	0.0013953517
	15	0.0019924053
	16	−0.0006858567
	17	−0.0001164669
	18	0.0000935887
	19	−0.0000132642
11	0	0.0186942978
	1	0.1440670212
	2	0.4498997644
	3	0.6856867749
	4	0.4119643689
	5	−0.1622752450
	6	−0.2742308468
	7	0.0660435882
	8	0.1498120125
	9	−0.0464799551
	10	−0.0664387857
	11	0.0313350902
	12	0.0208409044
	13	−0.0153648209
	14	−0.0033408589
	15	0.0049284177
	16	−0.0003085929
	17	−0.0008930233
	18	0.0002491525
	19	0.0000544391
	20	−0.0000346350
	21	0.0000044943

p	n	h_n
12	0	0.0131122580
	1	0.1095662728
	2	0.3773551352
	3	0.6571987226
	4	0.5158864784
	5	−0.0447638857
	6	−0.3161784538
	7	−0.0237792573
	8	0.1824786059
	9	0.0053595697
	10	−0.0964321201
	11	0.0108491303
	12	0.0415462775
	13	−0.0122186491
	14	−0.0128408252
	15	0.0067114990
	16	0.0022486072
	17	−0.0021795036
	18	0.0000065451
	19	0.0003886531
	20	−0.0000885041
	21	−0.0000242415
	22	0.0000127770
	23	−0.0000015291

Table 7.1: Daubechies-p scaling coefficients $(\!(h_n)\!)$

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Figure 7.6: *Daubechies-p wavelet system* pole zero plots

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CHAPTER 8

QUASI-SYMMETRY CONSTRAINT

8.1 Design details

The Daubechies-p wavelets (CHAPTER 7 page 109) are in general *asymmetric* because they are constructed under a *minimum phase* constraint—all zeros for polynomial $Q(z)$ are selected to be inside the unit circle. Suppose we want to design filters such that the number of vanishing moments is the same, but we are willing to lift the minimum phase condition to gain more symmetry. That is, instead of choosing the zeros from $Q(z)Q(z^{-1})$ that are inside the unit circle only, we choose some that are inside and some that are outside to give better balance. However, for zeros that are complex, we still select them in conjugate pairs so that the coefficients of $Q(z)$ are real.

Definition 8.1. *Let $Q(z)$ be a polynomial with real coefficients and*

$$
\begin{aligned}
P(y) &\triangleq \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k \\
\dot{Q} &\triangleq \left\{ Q(z) | Q(z)Q(z^{-1}) = P\left(\frac{2-z-z^{-1}}{4}\right\}\right) && \textit{(Fejér-Riesz spectral factorizations of } P\textit{)} \\
\mathbf{R}Q &\triangleq (\!(r_n)\!)_{r_n \textit{ is a root of } Q(z)} && \textit{(roots of } Q(z)\textit{)} \\
[\mathbf{A}Q](\omega) &\triangleq \operatorname{atan}\left(\frac{\mathbf{I}_{\mathsf{m}}\left[Q(e^{i\omega})\right]}{\mathbf{R}_{\mathsf{e}}\left[Q(e^{i\omega})\right]}\right) && \textit{(phase of } Q(e^{i\omega})\textit{)} \\
\mathbf{E}Q &\triangleq \inf_{m,c\in\mathbb{R}} \|[\mathbf{A}Q](\omega) - [m\omega + c]\|^2 && \textit{(linear phase estimation error of } Q(z)\textit{)} \\
\mathbf{M}Q &\triangleq \sum_{r_n \in \mathbf{R}Q} |r_n|^2 && \textit{(magnitude of roots of } Q(z)\textit{)} \\
\dot{E} &\triangleq \left\{\varepsilon = \mathbf{E}Q | Q \in \dot{Q}\right\} && \textit{(set of all phase errors)} \\
\dot{M} &\triangleq \left\{m = \mathbf{M}Q | Q \in \dot{Q}\right\} && \textit{(set of all root magnitudes)}
\end{aligned}
$$

*Then the **Symlet-p scaling function** is the sequence $(\!(h_n)\!)_{n\in\mathbb{Z}}$ with z-transform $\check{\mathsf{h}}(z) = \sum_n h_n z^{-n}$ that satisfies*

DEF

1. $\check{\mathsf{h}}(z) = \left(\frac{1+z^{-1}}{2}\right)^p Q(z)$
2. $Q(z) = \arg\min_{Q(z)\in\dot{Q}} \left\{\mathbf{M}Q \in \dot{M} \,|\, \mathbf{E}Q = \min \dot{E}\right\}$

The Symlets of Definition 8.1 page 133 can be implemented [1] by the following steps: [2]

1. Compute the polynomial $P(y)$. This polynomial has $p-1$ roots in y.

$$P(y) \triangleq \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k \qquad \text{where } \binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}$$

2. Compute $P\left(\frac{2-z-z^{-1}}{4}\right)$. This polynomial has $2p-2$ roots in z.

(a) Because the coefficients of $P\left(\frac{2-z-z^{-1}}{4}\right)$ are real, all of its roots occur in *complex conjugate pairs*:

$$\text{root at } z_1 = re^{i\theta} \iff \text{root at } z_1^* = \left(re^{i\theta}\right)^* = re^{-i\theta}$$

(b) All of the roots of $P\left(\frac{2-z-z^{-1}}{4}\right)$ occur in *conjugate reciprocal pairs*:

$$P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = re^{i\theta} \qquad \iff$$
$$P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = \left(re^{i\theta}\right)^{-1} = \frac{1}{r}e^{-i\theta}$$

[1] For an actual implementation using *Octave*, see Section Y.1 (page 441).
[2] Daubechies (1992) pages 254–257

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3. By the *Fejér-Riesz spectral factorization* theorem (Theorem J.5 page 262) $P\left(\frac{2-z-z^{-1}}{4}\right)$ can be factored in the form
$$P\left(\frac{2-z-z^{-1}}{4}\right) = Q(z)Q(z^{-1}).$$
Find all such factors of $P\left(\frac{2-z-z^{-1}}{4}\right)$ under the following constraints:
   (a) The selected $p-1$ roots for $Q(z)$ occur in complex conjugate pairs (so $Q(z)$ will have real coefficients).
   (b) $Q(z)$ contains exactly one root from each conjugate reciprocal pair of $P\left(\frac{2-z-z^{-1}}{4}\right)$.

   The above two constraints imply that there are $2^{\lfloor p/2 \rfloor}$ choices of roots for $Q(z)$.

4. Find the two choices that result in a phase that is the closest (in the least square sense) to a straight line. There are two minimum error choices because for each choice, selecting the complementary choice (selecting the zeros that were discarded and discarding the zeros that were selected) will result in the exact same amount of phase error.

5. Of these two choices, select the one where the sum of the magnitude of the $p-1$ roots is least. [3]

6. The scaling coefficients $(\!(h_n)\!)$ are the coefficients of the polynomial
$$\check{\mathsf{h}}(z) = \underbrace{\underbrace{\left(\frac{1+z^{-1}}{2}\right)^p}_{\text{order } p}\underbrace{Q(z)}_{\text{order } p-1}}_{\text{order } 2p-1\ (2p \text{ coefficients})}$$

Theorem 8.1.

THM

Symlet-p wavelets have the following properties:

1. *Symlet-p wavelets have optimum linear phase (in the least squares sense) among all the Bezout polynomial factorization solutions.*
2. *$(\!(h_n)\!)$ has support size $2p-1$ ($(\!(h_n)\!)$ has $2p$ non-zero elements).*
3. *The support size of $(\!(h_n)\!)$ is the smallest possible for wavelets with p vanishing moments.*

[3] This step is arbitrary and not required by Daubechies' definition of Symlets.

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Figure 8.1: Daubechies-4 and Symlet-4 wavelet systems (Example 8.1 page 136)

✎PROOF:

1. Proof that $\tilde{\mathsf{h}}(\omega)$ has optimum **linear phase**: The optimization is performed exhaustively (all the phases are computed and linear error is measured). This is not a rigorous proof.
2. Proof that $\langle h_n \rangle$ has support size $2p-1$: See Theorem 7.1 page 118.
3. Proof that the support size of $\langle h_n \rangle$ is the smallest possible for wavelets with p vanishing moments. See Theorem 7.1 page 118.

✏

8.2 Examples

Example 8.1. Figure 8.1 (page 136) compares the Daubechies-4 and Symlet-4 wavelet structures:

Example 8.2. Figure 8.2 (page 138) compares the Daubechies-8 and Symlet-8 wavelet structures.

Example 8.3. Figure 8.3 (page 138) compares the Daubechies-12 and Symlet-12 wavelet structures.

Example 8.4. Figure 8.4 (page 139) compares the Daubechies-16 and Symlet-16 wavelet structures.

Example 8.5. Figure 8.5 (page 139) and Figure 8.6 (page 140) show the pole-zero plots for Symlet-4, Symlet-5,..., Symlet-17 structures.

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EX

p	n	h_n
1	0	$\frac{1}{\sqrt{2}}$
	1	$\frac{1}{\sqrt{2}}$
2	0	$\frac{\sqrt{2}}{8}(1+\sqrt{3})$
	1	$\frac{\sqrt{2}}{8}(3+\sqrt{3})$
	2	$\frac{\sqrt{2}}{8}(3-\sqrt{3})$
	3	$\frac{\sqrt{2}}{8}(1-\sqrt{3})$
3	0	0.3326705530
	1	0.8068915093
	2	0.4598775021
	3	−0.1350110200
	4	−0.0854412739
	5	0.0352262919
4	0	−0.0757657148
	1	−0.0296355276
	2	0.4976186676
	3	0.8037387518
	4	0.2978577956
	5	−0.0992195436
	6	−0.0126039673
	7	0.0322231006
5	0	0.0273330683
	1	0.0295194909
	2	−0.0391342493
	3	0.1993975340
	4	0.7234076904
	5	0.6339789635
	6	0.0166021058
	7	−0.1753280899
	8	−0.0211018340
	9	0.0195388827
6	0	0.0154041093
	1	0.0034907121
	2	−0.1179901111
	3	−0.0483117426
	4	0.4910559419
	5	0.7876411410
	6	0.3379294217
	7	−0.0726375228
	8	−0.0210602925
	9	0.0447249018
	10	0.0017677119
	11	−0.0078007083
7	0	0.0120154193
	1	0.0172133763
	2	−0.0649080035
	3	−0.0641312898
	4	0.3602184609
	5	0.7819215933
	6	0.4836109157
	7	−0.0568044769
	8	−0.1010109209
	9	0.0447423495
	10	0.0204642076
	11	−0.0181266051
	12	−0.0032832978
	13	0.0022918340
8	0	−0.0033824160
	1	−0.0005421323
	2	0.0316950878
	3	0.0076074873
	4	−0.1432942384
	5	−0.0612733591
	6	0.4813596513
	7	0.7771857517
	8	0.3644418948
	9	−0.0519458381
	10	−0.0272190299
	11	0.0491371797
	12	0.0038087520
	13	−0.0149522583
	14	−0.0003029205
	15	0.0018899503
9	0	0.0014009155
	1	0.0006197809
	2	−0.0132719678
	3	−0.0115282102
	4	0.0302248789
	5	0.0005834627
	6	−0.0545689584
	7	0.2387609146
	8	0.7178970828
	9	0.6173384491
	10	0.0352724880
	11	−0.1915508313
	12	−0.0182337708
	13	0.0620777893
	14	0.0088592675
	15	−0.0102640640
	16	−0.0004731545
	17	0.0010694900
10	0	0.0008625782
	1	0.0007154205
	2	−0.0070567641
	3	0.0005956828
	4	0.0496861266
	5	0.0262403651
	6	−0.1215521056
	7	−0.0150192388
	8	0.5137098734
	9	0.7669548366
	10	0.3402160130
	11	−0.0878787115
	12	−0.0670899078
	13	0.0338423547
	14	−0.0008687521
	15	−0.0230054614
	16	−0.0011404298
	17	0.0050716492
	18	0.0003401493
	19	−0.0004101159
11	0	0.0006871194
	1	0.0013826742
	2	−0.0039185532
	3	−0.0027931771
	4	0.0372023572
	5	0.0509417072
	6	−0.0540827111
	7	−0.0286938383
	8	0.4078687490
	9	0.7685266798
	10	0.4520007834
	11	−0.0815151575
	12	−0.1499464788
	13	0.0182541524
	14	0.0237215478
	15	−0.0273470351
	16	−0.0085852863
	17	0.0098741222
	18	0.0024053043
	19	−0.0016456213
	20	−0.0002460505
	21	0.0001222747

Table 8.1: Symlet-p scaling coefficients $(\!(h_n)\!)$

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Figure 8.2: Daubechies-8 and Symlet-8 wavelet systems (Example 8.2 page 136)

Figure 8.3: Daubechies-12 and Symlet-12 wavelet systems (Example 8.3 page 136)

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Figure 8.4: Daubechies-16 and Symlet-16 wavelet systems (Example 8.4 page 136)

Figure 8.5: Pole-zero plots for Symlet-4, Symlet-5, Symlet-6, and Symlet-7 structures (Example 8.5 page 136)

Figure 8.6: Pole-zero plots for Symlet-8, Symlet-9,..., Symlet-17 structures (Example 8.5 page 136)

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Part III

Appendices

APPENDIX A

ALGEBRAIC STRUCTURES

❝ In this book, learned reader, you have the rules of algebra... It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one.... this most abstruse and unsurpassed treasury of the entire arithmetic being brought to light and, as in a theater, exposed to the sight of all... ❞

Gerolamo Cardano (1501–1576), Italian mathematician, physician, and astrologer [1]

A set together with one or more operations forms several standard mathematical structures:

$$\textit{group} \supseteq \textit{ring} \supseteq \textit{commutative ring} \supseteq \textit{integral domain} \supseteq \textit{field}$$

Definition A.1. [2] *Let X be a set and $\diamond : X \times X \to X$ be an operation on X.*

DEF

*The pair $(X, \diamond)$ is a **group** if*

1.	$\exists e \in X$ such that	$e \diamond x = x \diamond e = x$	$\forall x \in X$	(IDENTITY *element*)	*and*
2.	$\exists (-x) \in X$ such that	$(-x) \diamond x = x \diamond (-x) = e$	$\forall x \in X$	(INVERSE *element*)	*and*
3.		$x \diamond (y \diamond z) = (x \diamond y) \diamond z$	$\forall x,y,z \in X$	(ASSOCIATIVE)	

Definition A.2. [3] *Let $+ : X \times X \to X$ and $\cdot : X \times X \to X$ be operations on a set X. Furthermore, let the operation $\cdot$ also be represented by juxtapostion as in $a \cdot b \equiv ab$.*

DEF

*The triple $(X, +, \cdot)$ is a **ring** if*

1.	$(X, +)$ *is a group.*			*(additive group)*	*and*
2.	$x(yz) = (xy)z$		$\forall x,y,z \in X$	(ASSOCIATIVE *with respect to* $\cdot$)	*and*
3.	$x(y+z) = (xy) + (xz)$		$\forall x,y,z \in X$	($\cdot$ *is* LEFT DISTRIBUTIVE *over* $+$)	*and*
4.	$(x+y)z = (xz) + (yz)$		$\forall x,y,z \in X$	($\cdot$ *is* RIGHT DISTRIBUTIVE *over* $+$).	

Definition A.3. [4]

---

[1] quote: Cardano (1545) page 1
image: http://en.wikipedia.org/wiki/Image:Cardano.jpg

[2] Durbin (2000) page 29

[3] Durbin (2000) pages 114–115

[4] Durbin (2000) page 118

DEF

A triple $(X,+,\cdot)$ *is a* ***commutative ring*** *if*

1. $(X,+,\cdot)$ *is a* RING *and*
2. $xy = yx \quad \forall x,y \in X$ (COMMUTATIVE).

Definition A.4. [5] *Let* R *be a* COMMUTATIVE RING *(Definition A.3 page 143).*

DEF

A function $|\cdot|$ *in* $\mathbb{R}^{\mathbb{R}}$ *is an* ***absolute value*** *(or* ***modulus****) if*

1. $|x| \geq 0 \quad x \in \mathbb{R}$ (NON-NEGATIVE) *and*
2. $|x| = 0 \iff x = 0 \quad x \in \mathbb{R}$ (NONDEGENERATE) *and*
3. $|xy| = |x| \cdot |y| \quad x,y \in \mathbb{R}$ (HOMOGENEOUS / SUBMULTIPLICATIVE) *and*
4. $|x+y| \leq |x| + |y| \quad x,y \in \mathbb{R}$ (SUBADDITIVE / TRIANGLE INEQUALITY)

Definition A.5. [6]

DEF

The structure $F \triangleq (X,+,\cdot,0,1)$ *is a* ***field*** *if*

1. $(X,+,\cdot)$ *is a ring* *(ring)* *and*
2. $xy = yx \quad \forall x,y \in X$ *(commutative with respect to* $\cdot$*)* *and*
3. $(X \setminus \{0\},\cdot)$ *is a group* *(group with respect to* $\cdot$*).*

Definition A.6. [7] *Let* $V = (F,+,\cdot)$ *be a* VECTOR SPACE *and* $\otimes : V \times V \to V$ *be a vector-vector multiplication operator.*
An ***algebra*** *is any pair* $(V,\otimes)$ *that satisfies* ($\otimes$ *is represented by juxtaposition*)

DEF

1. $(\boldsymbol{ux})\boldsymbol{y} = \boldsymbol{u}(\boldsymbol{xy}) \quad \forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in V$ (ASSOCIATIVE) *and*
2. $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}) = (\boldsymbol{ux}) + (\boldsymbol{uy}) \quad \forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in V$ (LEFT DISTRIBUTIVE) *and*
3. $(\boldsymbol{u}+\boldsymbol{x})\boldsymbol{y} = (\boldsymbol{uy}) + (\boldsymbol{xy}) \quad \forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in V$ (RIGHT DISTRIBUTIVE) *and*
4. $\alpha(\boldsymbol{xy}) = (\alpha\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x}(\alpha\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y} \in V$ *and* $\alpha \in F$ (SCALAR COMMUTATIVE) .

[5] Cohn (2002) page 312
[6] Durbin (2000) page 123, Weber (1893)
[7] Abramovich and Aliprantis (2002) page 3, Michel and Herget (1993) page 56

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APPENDIX B

BINOMIAL RELATIONS

B.1 Factorials

Definition B.1 (factorial)**.**

DEF

*The **factorial** $n!$ is defined as*

$$n! \triangleq \begin{cases} n(n-1)(n-2)\cdots 1 & \text{for } n \in \mathbb{Z},\ n \geq 1 \\ 1 & \text{for } n \in \mathbb{Z},\ n = 0 \\ 0 & \text{for } n \in \mathbb{Z},\ n \leq -1 \end{cases}$$

Definition B.2. [1] *The quantities "x **to the** m **falling**", "x **to the** m **rising**", "x **to the** m **central**" are defined as follows:*

DEF

$$x^{\underline{m}} \triangleq \left\{ \begin{array}{ll} \underbrace{x(x-1)(x-2)\cdots(x-m+1)}_{m \text{ factors}} & \forall x \in \mathbb{C}, m \in \mathbb{N} \\ 1 & \forall x \in \mathbb{C}, m = 0 \end{array} \right\} \quad (\text{"}x \textbf{ to the } m \textbf{ falling"})$$

$$x^{\overline{m}} \triangleq \left\{ \begin{array}{ll} \underbrace{x(x+1)(x+2)\cdots(x+m-1)}_{m \text{ factors}} & \forall x \in \mathbb{C}, m \in \mathbb{N} \\ 1 & \forall x \in \mathbb{C}, m = 0 \end{array} \right\} \quad (\text{"}x \textbf{ to the } m \textbf{ rising"})$$

$$x^{\overline{\underline{m}}} \triangleq \left\{ \begin{array}{ll} \underbrace{x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1\right)}_{m \text{ factors}} & \forall x \in \mathbb{C}, m \in \mathbb{N} \\ 1 & \forall x \in \mathbb{C}, m = 0 \end{array} \right\} \quad (\text{"}x \textbf{ to the } m \textbf{ central"})$$

The rising and central expressions may be expressed in terms of the falling expression (next).

Proposition B.1. [2]

PRP

$$x^{\overline{m}} = (-1)^m x^{\underline{m}} \qquad x^{\overline{\underline{m}}} = x\left(x+\frac{m}{2}-1\right)^{\underline{(m-1)}}$$

[1] Graham et al. (1994) pages 47–48 ⟨equations (2.43), (2.44)⟩, Knuth (1992b) page 414 ⟨(2.11), (2.12)⟩, Aigner (2007) page 10, Steffensen (1950) page 8 ⟨descending, ascending, and central factorials⟩, Steffensen (1927) page 8 ⟨descending, ascending, and central factorials⟩

[2] Steffensen (1950) page 8 ⟨(3)⟩

✎PROOF:

$$\begin{aligned}(-1)^m(-x)^{\underline{m}} &= (-1)^m[(-x)(-x-1)(-x-2)\cdots(-x-m+1)] && \text{by Definition B.2 page 145}\\ &= (-1)^m(-1)^m[(x)(x+1)(x+2)\cdots(x+m-1)]\\ &= x^{\overline{m}} && \text{by Definition B.2 page 145}\end{aligned}$$

$$\begin{aligned}x\left(x+\frac{m}{2}-1\right)^{\underline{(m-1)}} &= x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-1-1\right)\cdots\left(x+\frac{m}{2}-1-(m-1)+1\right) && \text{by Definition B.2 page 145}\\ &= x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1)\right)\\ &= x^{\overline{m}}\end{aligned}$$

✏

B.2 Binomial identities

Definition B.3 (Binomial coefficient). [3] *Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{Z}$ the set of integers. Let $x^{\underline{m}}$ represent "x to the m falling" (Definition B.2). Let $n!$ represent "n factorial" (Definition B.1).*

DEF

*The **binomial coefficient** $\binom{x}{k}$ is defined as*

$$\binom{x}{k} \triangleq \begin{cases} \dfrac{x^{\underline{k}}}{k!} & \forall x\in\mathbb{C} \quad k\in\mathbb{W} \quad (k=0,1,2,3,\ldots)\\ 0 & \forall x\in\mathbb{C} \quad k\in\mathbb{Z}^- \quad (k=-1,-2,-3,\ldots)\end{cases}$$

*The value x is called the **upper index** and the value k is called the **lower index**.*

Proposition B.2. *Let $\binom{n}{k}$ be the* BINOMIAL COEFFICIENT *(Definition B.3 page 146).*

PRP

1. $\binom{x}{0} = 1 \quad \forall x\in\mathbb{C}$	2. $\binom{n}{n} = 1 \quad \forall n\in\mathbb{W}$
3. $\binom{x}{1} = x \quad \forall x\in\mathbb{C}$	4. $\binom{x}{k} = 0 \quad \forall x\in\mathbb{C},\ x<k$

✎PROOF:

1. Proof that $\binom{x}{0}=1$:

$$\begin{aligned}\binom{x}{0} &= \frac{x^{\underline{0}}}{0!} && \text{by Definition B.3 page 146}\\ &= \frac{x^{\underline{0}}}{1} && \text{by Definition B.1 page 145}\\ &= 1 && \text{by Definition B.2 page 145}\end{aligned}$$

[3] Graham et al. (1994) page 154 ⟨equation (5.1)⟩, Aigner (2007) page 10 ⟨(1)⟩, Coolidge (1949) pages 149–150, Stifel (1544)

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2. Proof that $\binom{n}{n} = 1$:

$$\begin{aligned}
\binom{n}{n} &= \frac{n^{\underline{n}}}{n!} && \text{by Definition B.3 page 146}\\
&= \frac{n(n-1)\cdots(n-n+1)}{n!} && \text{by Definition B.2 page 145}\\
&= \frac{n(n-1)\cdots(1)}{n(n-1)\cdots(1)} && \text{by Definition B.1 page 145}\\
&= 1
\end{aligned}$$

3. Proof that $\binom{x}{1} = x$:

$$\begin{aligned}
\binom{x}{1} &= \frac{x^{\underline{1}}}{1!} && \text{by Definition B.3 page 146}\\
&= \frac{x^{\underline{1}}}{1} && \text{by Definition B.1 page 145}\\
&= x && \text{by Definition B.2 page 145}
\end{aligned}$$

4. Proof that $\binom{x}{k} = 0$, $\forall x < k$:

$$\begin{aligned}
\binom{x}{k} &= \frac{x^{\underline{k}}}{k!} && \text{by Definition B.3 page 146}\\
&= \frac{x(x-1)\cdots(0)\cdots(x-k+1)}{k!} && \text{by Definition B.2 page 145}\\
&= 0
\end{aligned}$$

✏

Theorem B.1. [4] *Let $\binom{n}{k}$ be the* BINOMIAL COEFFICIENT *(Definition B.3 page 146).*

THM				
1.	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\forall n,k\in\mathbb{Z}, n\geq k\geq 0$	(FACTORIAL EXPANSION)
2.	$\binom{n}{k} = \binom{n}{n-k}$	$\forall n,k\in\mathbb{Z}, n\geq 0$	(SYMMETRY)
3.	$\binom{n+x+1}{n} = \binom{n+x}{n} + \binom{n+x}{n-1}$	$\forall n\in\mathbb{Z}, x\in\mathbb{C}$	(PASCAL'S RULE)
4.	$\binom{x+1}{k+1} = \binom{x}{k+1} + \binom{x}{k}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(PASCAL'S IDENTITY / STIFEL FORMULA)
5.	$\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}$	$\forall k,m\in\mathbb{Z}, x\in\mathbb{C}$	(TRINOMIAL REVISION)
6.	$\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(ABSORPTION IDENTITY)
7.	$\binom{x}{k} = (-1)^k\binom{k-x-1}{k}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(UPPER NEGATION)
8.	$\binom{x}{k} = \binom{x-2}{k-2} + 2\binom{x-2}{k-1} + \binom{x-2}{k}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(SECOND-ORDER PASCAL'S IDENTITY)
9.	$\binom{x-1}{k-1}\binom{x}{k+1}\binom{x+1}{k} = \binom{x-1}{k}\binom{x}{k-1}\binom{x+1}{k+1}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(HEXAGON IDENTITY)

✎PROOF:

[4] Graham et al. (1994) page 174 ⟨Table 174⟩, Gallier (2010) page 221, Gross (2008) page 227 ⟨Table 4.1.2⟩, Coolidge (1949) pages 149–150, Stifel (1544), Balakrishnan (1996) page 43 ⟨Pascal's Rule⟩, Harris et al. (2008) page 143 ⟨*hexagon identity*, (2.15)⟩, Ferland (2009) page 216 ⟨*second-order pascal identity*⟩

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1. Proof for *factorial expansion*:

$$\begin{aligned}
\binom{n}{k} &\triangleq \frac{n^{\underline{k}}}{k!} && \forall n,k \in \mathbb{Z},\ n \geq k \geq 0 && \text{by Definition B.3}\\
&= \frac{n(n-1)(n-2)\cdot(n-k+1)}{k!} && \forall n,k \in \mathbb{Z},\ n \geq k \geq 0 && \text{by Definition B.2}\\
&= \frac{n(n-1)(n-2)\cdot(n-k+1)(n-k)(n-k-1)\cdots 1}{k!(n-k)!} && \forall n,k \in \mathbb{Z},\ n \geq k \geq 0 && \text{by Definition B.2}\\
&= \frac{n!}{k!(n-k)!} && \forall n,k \in \mathbb{Z},\ n \geq k \geq 0 && \text{by Definition B.1}
\end{aligned}$$

2. Proof for *symmetry* property:

   (a) Proof for $n,k \in \mathbb{Z},\ n \geq k \geq 0$: (use item (1) page 148)

$$\begin{aligned}
\binom{n}{n-k} &= \frac{n!}{(n-k)!(n-(n-k)!} && \forall n,k \in \mathbb{Z},\ n \geq k \geq 0 && \text{by item (1) page 148}\\
&= \frac{n!}{k!(n-k)!} && \forall n,k \in \mathbb{Z},\ n \geq k \geq 0 && \\
&= \binom{n}{k} && \forall n,k \in \mathbb{Z},\ n \geq k \geq 0 && \text{by item (1) page 148}
\end{aligned}$$

   (b) Proof for $n,k \in \mathbb{Z},\ n \geq 0 > k$:

$$\begin{aligned}
\binom{n}{n-k} &= \frac{n^{\underline{n-k}}}{(n-k)!} && \forall n,k \in \mathbb{Z},\ n \geq 0 > k && \text{by Definition B.3 page 146}\\
&= \frac{n(n-1)(n-2)\cdots 0\cdots(n-n+k+1)}{(n-k)!} && \forall n,k \in \mathbb{Z},\ n \geq 0 > k && \text{by Definition B.2 page 145}\\
&= 0 && && \\
&= \binom{n}{k} && \forall n,k \in \mathbb{Z},\ n \geq 0 > k && \text{by Definition B.3 page 146}
\end{aligned}$$

   (c) Proof for $n,k \in \mathbb{Z},\ n \geq 0 > k$:

$$\begin{aligned}
\binom{n}{k} &= \frac{n^{\underline{k}}}{k!} && \forall n,k \in \mathbb{Z},\ k > n \geq 0 && \text{by Definition B.3 page 146}\\
&= \frac{n(n-1)(n-2)\cdots 0\cdots(n-k+1)}{(n-k)!} && \forall n,k \in \mathbb{Z},\ k > n \geq 0 && \text{by Definition B.2 page 145}\\
&= 0 && && \\
&= \binom{n}{n-k} && \forall n,k \in \mathbb{Z},\ k > n \geq 0 && \text{by Definition B.3 page 146}
\end{aligned}$$

3. Proof for *Pascal's Rule*:

   (a) Proof for $n < 0,\ x \in \mathbb{C}$:

$$\begin{aligned}
\binom{n+x}{n} + \binom{n+x}{n-1} &= 0 + 0 && \text{by Definition B.3 page 146}\\
&= \binom{n+x+1}{n} && \text{by Definition B.3 page 146}
\end{aligned}$$

   (b) Proof for $n = 0,\ x \in \mathbb{C}$:

$$\begin{aligned}
\binom{n+x}{n} + \binom{n+x}{n-1} &= \binom{n+x}{0} + \binom{n+x}{-1} && \text{by } n = 0 \text{ hypothesis}\\
&= 1 + 0 && \text{by Definition B.3 page 146}\\
&= \binom{n+x+1}{0} && \text{by Definition B.3 page 146}\\
&= \binom{n+x+1}{n} && \text{by } n = 0 \text{ hypothesis}
\end{aligned}$$

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(c) Proof for $n > 0$, $x \in \mathbb{C}$:

$$
\begin{aligned}
&\binom{n+x}{n} + \binom{n+x}{n-1} \\
&\triangleq \frac{n+x^{\underline{n}}}{n!} + \frac{n+x^{\underline{n-1}}}{(n-1)!} && \text{by Definition B.3 page 146} \\
&\triangleq \frac{(n+x)(n+x-1)\cdots(n+x-n+1)}{n!} \\
&\quad + \frac{(n+x)(n+x-1)\cdots(n+x-n+1+1)}{(n-1)!} && \text{by Definition B.2 page 145} \\
&= \frac{[(n+x)(n+x-1)\cdots(x+1)] + [(n+x)(n+x-1)\cdots(x+2)n]}{n!} \\
&= \frac{[(x+1)+n][(n+x)(n+x-1)\cdots(x+2)]}{n!} \\
&= \frac{(n+x+1)(n+x)(n+x-1)\cdots(x+2)}{n!} \\
&\triangleq \frac{(n+x+1)^{\underline{n}}}{n!} && \text{by Definition B.2 page 145} \\
&\triangleq \binom{n+x+1}{n} && \text{by Definition B.3 page 146}
\end{aligned}
$$

4. Proof for *Pascal's Identity*:

$$
\begin{aligned}
\binom{x+1}{k+1} &= \binom{k+y+1}{k+1} && \text{where } y \triangleq x-k \implies x = y+k \\
&= \binom{y+k}{k+1} + \binom{y+k}{k} && \text{by Pascal's Rule (item (3))} \\
&= \binom{x}{k+1} + \binom{x}{k} && \text{by definition of } m
\end{aligned}
$$

5. Proof for *Trinomial revision*:

(a) Proof for $k < 0$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= \binom{x}{m}0 && \text{by } k<0 \text{ hypothesis and Definition B.3 page 146} \\
&= \cancelto{0}{\binom{x}{k}}\binom{x-k}{m-k} && \text{by } k<0 \text{ hypothesis and Definition B.3 page 146}
\end{aligned}
$$

(b) Proof for $k \geq 0$, $m < 0$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= 0\binom{m}{k} && \text{by } m<0 \text{ hypothesis and Definition B.3 page 146} \\
&= \binom{x}{k}\cancelto{0}{\binom{x-k}{m-k}} && \text{by } k \geq 0, m<0 \text{ hypothesis and Definition B.3 page 146}
\end{aligned}
$$

(c) Proof for $m < k$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= \binom{x}{m}0 && \text{by Proposition B.2 page 146} \\
&= \binom{x}{k}\cancelto{0}{\binom{x-k}{m-k}} && \text{by } m<k \text{ hypothesis and Definition B.3 page 146}
\end{aligned}
$$

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(d) Proof for remaining cases:

$$
\begin{aligned}
&\binom{x}{m}\binom{m}{k} \\
&= \frac{x^{\underline{m}}}{m!}\frac{m^{\underline{k}}}{k!} && \text{by Definition B.3 page 146} \\
&= \frac{x(x-1)\cdots(x-m+1)}{m!}\frac{m(m-1)\cdots(m-k+1)}{k!} && \text{by Definition B.2 page 145} \\
&= \frac{x(x-1)\cdots(x-m+1)}{(m-k)!}\frac{1}{k!} \\
&= \frac{x(x-1)\cdots(x-k+1)}{k!}\frac{(x-k)(x-k-1)\cdots(x-m+1)}{(m-k)!} \\
&= \frac{x(x-1)\cdots(x-k+1)}{k!}\frac{(x-k)(x-k-1)\cdots((x-k)-(m-k)+1)}{(m-k)!} \\
&\triangleq \frac{x^{\underline{k}}}{k!}\frac{(x-k)^{\underline{m-k}}}{(m-k)!} && \text{by Definition B.2 page 145} \\
&\triangleq \binom{x}{k}\binom{x-k}{m-k} && \text{by Definition B.3 page 146}
\end{aligned}
$$

6. Proof for *Absorption identity*:

$$
\begin{aligned}
\frac{x}{k}\binom{x-1}{k-1} &= \frac{1}{k}\binom{x}{1}\binom{n-1}{k-1} && \text{by Proposition B.2 page 146} \\
&= \frac{1}{k}\binom{x}{k}\binom{k}{1} && \text{by Trinomial revision (item (5))} \\
&= \frac{1}{k}\binom{x}{k}k && \text{by Proposition B.2 page 146} \\
&= \binom{x}{k}
\end{aligned}
$$

7. Proof for *Upper Negation*:

$$
\begin{aligned}
(-1)^k\binom{k-x-1}{k} &\triangleq (-1)^k\frac{(k-x-1)^{\underline{k}}}{k!} && \text{by Definition B.3} \\
&\triangleq (-1)^k\frac{(k-x-1)(k-x-2)(k-x-3)\cdots(k-x-1-k+1)}{k!} && \text{by Definition B.2} \\
&= (-1)^k\frac{(k-x-1)(k-x-2)(k-x-3)\cdots(-x)}{k!} \\
&= (-1)^k(-1)^k\frac{(x)(x-1)\cdots(x(x-k+3)(x-k+2)(x-k+1)}{k!} \\
&\triangleq \frac{x^{\underline{k}}}{k!} && \text{by Definition B.2} \\
&\triangleq \binom{x}{k} && \text{by Definition B.3}
\end{aligned}
$$

8. Proof for *2nd Order Pascal's Identity*:

$$
\begin{aligned}
&\binom{n-2}{k-2}+2\binom{n-2}{k-1}+\binom{n-2}{k} \\
&\triangleq \frac{(x-2)^{\underline{(k-2)}}}{(k-2)!}+\frac{(x-2)^{\underline{(k-1)}}}{(k-1)!}+\frac{(x-2)^{\underline{k}}}{k!} \\
&\triangleq \frac{(x-2)(x-1)\cdots(x-k+2+1)}{(k-2)!}+2\frac{(x-2)(x-1)\cdots(x-k+1+1)}{(k-1)!}+\frac{(x-2)\cdots(x-k+1)}{k!} \\
&= \frac{(x-2)\cdots(x-2-k+2+1)k(k-1)+2(x-2)\cdots(x-2-k+1+1)k+(x-2)\cdots(x-k-1)}{k!} \\
&= \frac{(x-2)(x-1)\cdots(x-k+1)k(k-1)+2(x-2)(x-1)\cdots(x-k)k+(n-2)(n-1)\cdots(x-k-1)}{k!}
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+2(x-k)k+(x-k)(x-k-1)]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+2(x-k)k-(x-k)k+(x-k)(x-1)]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+(x-k)k+(x-k)(x-1)]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)]\left[k^2-k+kx-k^2+x^2-x-kx+k\right]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)]\left[x^2-x\right]}{k!}\\
&= \frac{x(x-1)(x-2)(x-1)\cdots(x-k+1)}{k!}\\
&\triangleq \frac{n^{\underline{k}}}{k!}\\
&\triangleq \binom{n}{k}
\end{aligned}
$$

9. Proof for *Hexagon Identity*:

$$
\begin{aligned}
&\binom{x-1}{k-1}\binom{x}{k+1}\binom{x+1}{k}\\
&\triangleq \left[\frac{(x-1)^{\underline{(k-1)}}}{(k-1)!}\right]\left[\frac{x^{\underline{(k+1)}}}{(k+1)!}\right]\left[\frac{(x+1)^{\underline{k}}}{k!}\right]\\
&\triangleq \left[\frac{(x-1)\cdots(x-1-k+1+1)}{(k-1)!}\right]\left[\frac{x(x-1)\cdots(x-k-1+1)}{(k+1)!}\right]\left[\frac{(x+1)(x)\cdots(x+1-k+1)}{k!}\right]\\
&= \left[\frac{(x-1)\cdots(x-k+2)(x-k+1)}{(k-1)!}\right]\left[\frac{x(x-1)\cdots(x-k)}{(k+1)!}\right]\left[\frac{(x+1)(x)(x-1)\cdots(x-k+2)}{k!}\right]\\
&= \left[\frac{(x)(x-1)\cdots(x-k+2)}{(k-1)!}\right]\left[\frac{(x+1)x(x-1)\cdots(x-k)(x-k+1)}{(k+1)!}\right]\left[\frac{(x-1)\cdots(x-k)}{k!}\right]\\
&\triangleq \left[\frac{x^{\underline{(k-1)}}}{(k-1)!}\right]\left[\frac{(x+1)^{\underline{(k+1)}}}{(k+1)!}\right]\left[\frac{(x-1)^{\underline{k}}}{k!}\right]\\
&\triangleq \binom{x}{k-1}\binom{x+1}{k+1}\binom{x-1}{k}
\end{aligned}
$$

✏

From Pascal's Recursion we can construct *Pascal's Triangle*:[5]

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

[5] Pascal (1655), Granville (1992), Granville (1997), Edwards (2002), Hall and Knight (1894) pages 320–321 ⟨article 393⟩

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B.3 Binomial summations

Theorem B.2. [6] *Let* $\binom{n}{k}$ *be the* BINOMIAL COEFFICIENT *(Definition B.3 page 146).*

THM

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \qquad \textit{(row sum)}$$

$$\sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m+1} \qquad \textit{(upper sum / column sum)}$$

$$\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n} \qquad \textit{(parallel summation formula / southeast diagonal)}$$

$$\sum_{k=0}^{m}\binom{n-k}{m-k} = \binom{n+1}{m} \qquad \textit{(northwest diagonal)}$$

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k} \qquad \textit{(Vandermonde's convolution)}$$

$$\sum_{i=-j}^{n-j}\binom{m}{j+i}\binom{n}{k-i} = \binom{m+n}{j+k} \qquad \textit{(alternate Vandermonde's convolution)}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

✎PROOF:

1. Proof for *row sum* relation:

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}x^k &= \sum_{k=0}^{n}\binom{n}{k}x^k\Bigg|_{x=1}\\ &= (1+x)^n|_{x=1} && \text{by } \textit{Binomial Theorem} \quad \text{(Theorem C.14 page 167)}\\ &= (1+1)^n\\ &= 2^n\end{aligned}$$

2. Proof for *upper sum* relation (proof by induction):

   (a) Proof for $(n,m)=(0,0)$ case:
   $$\sum_{0}^{0}\binom{k}{m} = \binom{0}{0} = 1 = \binom{0+1}{0+1}$$

   (b) Proof for $(n,m)=(1,0)$ case:
   $$\sum_{0}^{1}\binom{k}{m} = \binom{1}{0}+\binom{1}{1} = 2 = \binom{1+1}{0+1}$$

   (c) Proof for $(n,m)=(1,1)$ case:
   $$\sum_{0}^{1}\binom{k}{m} = \binom{1}{1} = 1 = \binom{1+1}{1+1}$$

[6] Graham et al. (1994) page 169 ⟨Table 169⟩, Gallier (2010) pages 218–223, Gross (2008) page 227 ⟨Table 4.1.2⟩, Harris et al. (2008) pages 137–142, Knuth (1992a), Vandermonde (1772), Zhū (1303)

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(d) Proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
\sum_{k=m}^{n+1}\binom{k}{m} &= \binom{n+1}{m} + \sum_{k=m}^{n} \\
&= \binom{n+1}{m} + \binom{n+1}{m+1} && \text{by left hypothesis} \\
&= \binom{n+2}{m+1} && \text{by Pascal's recursion (Theorem B.1 page 147)}
\end{aligned}$$

3. Proof for *Parallel summation formula* (Proof by induction):

(a) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n}$ is true for $n=0$:

$$\begin{aligned}
\sum_{k=0}^{n}\binom{m+k}{k}\bigg|_{n=0} &= \binom{m+0}{0} \\
&= \frac{(m+0)!}{(m-0)!\;0!} && \text{by Definition B.3 page 146} \\
&= \frac{(m+1)!}{(m+1-0)!\;0!} \\
&= \binom{m+1}{0} && \text{by Definition B.3 page 146} \\
&= \binom{n+m+1}{n}\bigg|_{n=0}
\end{aligned}$$

(b) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n}$ is true for $n=1$:

$$\begin{aligned}
\sum_{k=0}^{n}\binom{m+k}{k}\bigg|_{n=1} &= \binom{m+0}{0} + \binom{m+1}{1} \\
&= \binom{m+1}{0} + \binom{m+1}{1} \\
&= \binom{m+1+1}{1} && \text{by Pascal's Rule page 147} \\
&= \binom{n+m+1}{n}\bigg|_{n=1}
\end{aligned}$$

(c) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n} \implies \sum_{k=0}^{n+1}\binom{m+k}{k} = \binom{(n+1)+m+1}{n+1}$:

$$\begin{aligned}
\sum_{k=0}^{n+1}\binom{m+k}{k} &= \binom{m}{0} + \sum_{k=1}^{n+1}\binom{m+k}{k} \\
&= \binom{m}{0} + \sum_{k=0}^{n}\binom{m+k+1}{k+1} \\
&= \binom{m}{0} + \sum_{k=0}^{n}\binom{m+k}{k} - \binom{m}{0} + \binom{m+n+1}{n+1} \\
&= \binom{n+m+1}{n} + \binom{m+n+1}{n+1} && \text{by left hypothesis} \\
&= \binom{n+m+2}{n+1} && \text{by Pascal's Rule page 147} \\
&= \binom{(n+1)+m+1}{(n+1)}
\end{aligned}$$

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4. Proof for *Vandermonde's convolution*:

$$\begin{aligned}
\sum_{k=0}^{m+n}\binom{m+n}{k}x^k &= \sum_{k=0}^{m+n}\binom{m+n}{k}(1)^{m+n-k}x^k \\
&= (1+x)^{m+n} && \text{by } \textit{Binomial Theorem} \quad \text{(Theorem C.14 page 167)} \\
&= (1+x)^m(1+x)^n \\
&= \underbrace{\left[\sum_{k=0}^{m}\binom{m}{k}x^k\right]}_{(1+x)^m}\underbrace{\left[\sum_{j=0}^{n}\binom{n}{j}x^j\right]}_{(1+x)^n} && \text{by } \textit{Binomial Theorem} \quad \text{(Theorem C.14 page 167)} \\
&= \sum_{k=0}^{m+n}\left[\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}\right]x^k && \text{by Theorem H.3 page 228}
\end{aligned}$$

$$\implies \binom{m+n}{k} = \sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}$$

5. Proof for *alternate Vandermonde's convolution*:

$$\begin{aligned}
\binom{m+n}{j+k} &= \binom{m+n}{u} && \text{where } u \triangleq j+k \implies k = u-j \\
&= \sum_{v=0}^{n}\binom{m}{v}\binom{n}{u-v} \\
&= \sum_{v=0}^{n}\binom{m}{v}\binom{n}{j+k-v} \\
&= \sum_{i+j=0}^{i+j=n}\binom{m}{j+i}\binom{n}{k-i} && \text{where } i \triangleq v-j \implies v = i+j \\
&= \sum_{i=-j}^{i=n-j}\binom{m}{j+i}\binom{n}{k-i}
\end{aligned}$$

6. Proof that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$:

$$\begin{aligned}
\binom{2n}{n} &= \binom{n+n}{n} \\
&= \sum_{k=0} n\binom{n}{k}\binom{n}{n-k} && \text{by Vandermonde's convolution (item (4) page 154)} \\
&= \sum_{k=0} n\binom{n}{k}\binom{n}{k} && \text{by item (2)} \\
&= \sum_{k=0}^{n}\binom{n}{k}^2
\end{aligned}$$

✏

Theorem B.3. [7]

THM $$\sum_{k=1}^{n}\frac{1}{k+1} < \ln(n+1) < \sum_{k=1}^{n}\frac{1}{k}$$

[7] Rivlin (1969) page 60

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Figure B.1: $\ln(n+1)$

✎PROOF: The summations are simply lower and upper bounds of the integral of $\frac{1}{x}$ in the range $[1,\ n+1]$. This is illustrated in Figure B.1.

1. Proof that $\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k}$:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k} &> \int_1^{n+1} \frac{1}{x}\,\mathrm{dx}\\ &= \ln x\Big|_1^{n+1}\\ &= \ln(n+1) - \cancelto{0}{\ln(1)}\\ &= \ln(n+1)\end{aligned}$$

2. Proof that $\sum_{k=1}^{n} \frac{1}{k+1} < \ln(n+1)$:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k+1} &< \int_1^{n+1} \frac{1}{x}\,\mathrm{dx}\\ &= \ln(n+1) - \cancelto{0}{\ln(1)}\\ &= \ln(n+1)\end{aligned}$$

✏

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APPENDIX C

POLYNOMIALS

C.1 Definitions

Definition C.1. [1] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

DEF

A function p *in* $\mathbb{F}^{\mathbb{F}}$ *is a* ***polynomial*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if it is of the form*

$$\mathsf{p}(x) \triangleq \sum_{n=0}^{N} \alpha_n x^n \qquad \alpha_n \in \mathbb{F},\ \alpha_N \neq 0.$$

The ***degree*** *of* p *is* N. *A* ***coefficient*** *of* p *is any element of* $(\!|\alpha_n|\!)_1^N$.
The ***leading coefficient*** *of* p *is* α_N.

Definition C.2. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

DEF

A polynomial p *of degree* N *over the field* $\mathbb{F}$ *and*
a polynomial q *of degree* M *over the field* $\mathbb{F}$ *are* ***equal*** *if*

1. $N = M$ *and*
2. $\alpha_n = \beta_n$ *for* $n = 0, 1, \ldots, N$.

The expression $\mathsf{p}(x) = \mathsf{q}(x)$ *(or* $\mathsf{p} = \mathsf{q}$*) denotes that* p *and* q *are* EQUAL.

[1] Barbeau (1989) page 1, Fuhrmann (2012) page 11, Borwein and Erdélyi (1995) page 2
[2] Fuhrmann (2012) page 11

C.2 Ring properties

C.2.1 Polynomial Arithmetic

Theorem C.1 (polynomial addition). [3] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

THM

$$\underbrace{\left(\sum_{n=0}^{N} \alpha_n x^n\right)}_{\mathsf{p}(x)} + \underbrace{\left(\sum_{n=0}^{M} \beta_n x^n\right)}_{\mathsf{q}(x)} = \underbrace{\sum_{n=0}^{\max(N,M)} \gamma_n x^n}_{\mathsf{p}(x)+\mathsf{q}(x)} \quad \text{where} \quad \gamma_n \triangleq \begin{cases} \alpha_n + \beta_n & \text{for } n \le \min(N,M) \\ \alpha_n & \text{for } n > M \\ \beta_n & \text{for } n > N \end{cases}$$

$$\text{for all} \quad x,\ \alpha_n,\ \beta_n \in \mathbb{F}$$

Polynomial multiplication is equivalent to convolution (Definition F.4 page 205) of the coefficients (Definition C.1 page 157). [4]

Theorem C.2 (polynomial multiplication). [5] *Let* $(\!(\alpha_n \in \mathbb{C})\!)$, $(\!(b_n \in \mathbb{C})\!)$, *and* $x \in \mathbb{C}$.

THM

$$\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{n=0}^{M} \beta_n x^n\right) = \sum_{n=0}^{N+M} \underbrace{\left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right)}_{\textit{Cauchy product}} x^n$$

✎PROOF:

$$\begin{aligned}
\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{m=0}^{M} \beta_m x^m\right) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n \beta_m x^{n+m} \\
&= \sum_{n=0}^{N}\sum_{k=n}^{M+n} \alpha_n \beta_{k-n} x^k \qquad\qquad k \triangleq n+m \iff m = k-n \\
&= \sum_{n=0}^{N+M} \left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right) x^n
\end{aligned}$$

Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	β_0	β_1	β_2	β_3	$\cdots$	β_M
α_0	$\alpha_0\beta_0$	$\alpha_0\beta_1 x$	$\alpha_0\beta_2 x^2$	$\alpha_0\beta_3 x^3$	$\cdots$	$\alpha_0\beta_M x^M$
α_1	$\alpha_1\beta_0 x$	$\alpha_1\beta_1 x^2$	$\alpha_1\beta_2 x^3$	$\alpha_1\beta_3 x^4$	$\cdots$	$\alpha_1\beta_M x^{1+M}$
α_2	$\alpha_2\beta_0 x^2$	$\alpha_2\beta_1 x^3$	$\alpha_2\beta_2 x^4$	$\alpha_2\beta_3 x^5$	$\cdots$	$\alpha_2\beta_M x^{2+M}$
α_3	$\alpha_3\beta_0 x^3$	$\alpha_3\beta_1 x^4$	$\alpha_3\beta_2 x^5$	$\alpha_3\beta_3 x^6$	$\cdots$	$\alpha_3\beta_M x^{3+M}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_N	$\alpha_N\beta_0 x^N$	$\alpha_N\beta_1 x^{N+1}$	$\alpha_N\beta_2 x^{N+2}$	$\alpha_N\beta_3 x^{N+3}$	$\cdots$	$\alpha_N\beta_M x^{N+M}$

1. The expression $\sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n\beta_m x^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

[3] Fuhrmann (2012) page 11

[4] *Convolution*: In fact, using *GNU Octave*™ or *MatLab*™, polynomial multiplication can be performed using convolution. For example, the operation $(x^3 + 5x^2 + 7x + 9)(4x^2 + 11)$ can be calculated in *GNU Octave*™ or *MatLab*™ with conv([1 5 7 9],[4 0 11])

[5] Apostol (1975) page 237

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2. If we switched the order of summation to $\sum_{m=0}^{M}\sum_{n=0}^{N}\alpha_n\beta_m x^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

3. For $N = M = \infty$, the expression $\sum_{n=0}^{N+M}\left(\sum_{k=0}^{n}\alpha_k\beta_{n-k}\right)x^n$ is equivalent to adding *diagonally* starting from the upper left corner and proceeding towards the lower right.

4. For finite N and M...

   (a) The upper limit on the inner summation puts two constraints on k:
   $$\left\{\begin{array}{lll} k & \le & n \quad \text{and} \\ k & \le & N \end{array}\right\} \implies k \le \min(n, N)$$

   (b) The lower limit on the inner summation also puts two constraints on k:
   $$\left\{\begin{array}{lll} k & \ge & 0 \quad \text{and} \\ k & \ge & n - M \end{array}\right\} \implies k \ge \max(0, n - M)$$

✍

Polynomial division can be performed in a manner very similar to integer division (both integers and polynomials are *rings*).

Definition C.3 (Polynomial division)**.** *The quantities of polynomial division are defined as follows:*

DEF

$$\frac{\mathsf{d}(x)}{\mathsf{p}(x)} = \mathsf{q}(x) + \frac{\mathsf{r}(x)}{\mathsf{p}(x)} \qquad \textit{where} \qquad \left\{\begin{array}{lll} \mathsf{d}(x) & \textit{is the } \textbf{\textit{dividend}} & \textit{and} \\ \mathsf{p}(x) & \textit{is the } \textbf{\textit{divisor}} & \textit{and} \\ \mathsf{q}(x) & \textit{is the } \textbf{\textit{quotient}} & \textit{and} \\ \mathsf{r}(x) & \textit{is the } \textbf{\textit{remainder}}. & \end{array}\right\}$$

C.2.2 Greatest common divisor

Theorem C.3 (Extended Euclidean Algorithm)**.** [6]

Let $\mathsf{r}_1(x)$ *and* $\mathsf{r}_2(x)$ *be polynomials. The following algorithm computes their greatest common divisor* $\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$*, and factors* $\mathsf{a}(x)$ *and* $\mathsf{b}(x)$ *such that*

$$\mathsf{r}_1(x)\mathsf{a}(x) + \mathsf{r}_2(x)\mathsf{b}(x) = \gcd(\mathsf{r}_1, \mathsf{r}_2)$$

THM

n	*remainder* $\mathsf{r}_n = \mathsf{r}_{n-2} - \mathsf{q}_n\mathsf{r}_{n-1}$	*quotient* q_n	*factor* $\alpha_n = a_{n-2} - \mathsf{q}_n\alpha_{n-1}$	*factor* $\beta_n = b_{n-2} - \mathsf{q}_n\beta_{n-1}$
1	$\mathsf{r}_1(x)$	–	1	0
2	$\mathsf{r}_2(x)$	–	0	1
3	$\mathsf{r}_1 - q_3\mathsf{r}_2$	q_3	1	$-q_3$
4	$\mathsf{r}_2 - q_4 r_3$	q_4	$-\mathsf{q}_4$	$1 + \mathsf{q}_4\mathsf{q}_1$
5	$\mathsf{r}_1 - \mathsf{q}_5\mathsf{r}_2$	q_5	$1 + \mathsf{q}_5\mathsf{q}_4$	$-q_3 - \mathsf{q}_5(1 + \mathsf{q}_4\mathsf{q}_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$	q_n	$\mathsf{a}(x) = a_{n-2} - q_n\alpha_{n-1}$	$\mathsf{b}(x) = b_{n-2} - q_n\beta_{n-1}$
$n+1$	0	q_{n+1}		

[6] Wicker (1995) page 53, Fuhrmann (2012) page 11

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✎PROOF:

$$\begin{aligned} \mathsf{r}_1 &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3 \\ &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3 \end{aligned}$$

✏

Example C.1. Let

$$\mathsf{u}(x) \triangleq (1-x)^2 \qquad \mathsf{v}(x) \triangleq x^2.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^2}_{\mathsf{u}(x)}\underbrace{(1+2x)}_{\mathsf{a}(x)} + \underbrace{(x^2)}_{\mathsf{v}(x)}\underbrace{(3-2x)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^2 = 1-2x+x^2 = \mathsf{u}(x)$	–	1	0
0	$x^2 = \mathsf{v}(x)$	–	0	1
1	$1-2x$	1	1	-1
2	$\frac{1}{2}x$	$-\frac{1}{2}x$	$\frac{1}{2}x$	$1-\frac{1}{2}x$
3	$1 = \gcd(\,(1-x)^2,\ x^2\,)$	-4	$1+2x = \mathsf{a}(x)$	$3-2x = \mathsf{b}(x)$
4	0	$\frac{1}{2}x$	–	–

✏

Example C.2. Let

$$\mathsf{u}(x) \triangleq (1-x)^3 \qquad \mathsf{v}(x) \triangleq x^3.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^3}_{\mathsf{u}(x)}\underbrace{(1+3x+6x^2)}_{\mathsf{a}(x)} + \underbrace{(x^3)}_{\mathsf{v}(x)}\underbrace{(10-15x+6x^2)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^3 = 1-3x+3x^2-x^3$	–	1	0
0	x^3	–	0	1
1	$1-3x+3x^2$	-1	1	1
2	$-\frac{1}{3}x+x^2$	$\frac{1}{3}x$	$-\frac{1}{3}x$	$1-\frac{1}{3}x$
3	$1-2x$	3	$1+x$	$-2+x$
4	$\frac{1}{6}x$	$-\frac{1}{2}x$	$\frac{1}{6}x+\frac{1}{2}x^2$	$1-\frac{4}{3}y+\frac{1}{2}x^2$
5	$1 = \gcd((1-x)^3, x^3)$	-12	$1+3x+6x^2 = \mathsf{a}(x)$	$10-15x+6x^2 = \mathsf{b}(x)$
6	0	$\frac{1}{6}x$		

✏

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Example C.3. Let

$$\mathsf{u}(x) \triangleq (1-x)^4 \qquad \mathsf{v}(x) \triangleq x^4.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^4}_{\mathsf{u}(x)}\underbrace{(1+4x+10x^2+20x^3)}_{\mathsf{a}(x)} + \underbrace{(x^4)}_{\mathsf{v}(x)}\underbrace{(35-84x+70x^2-20x^3)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^4 = 1-4x+6^2-4x^3+x^4$	–	1	0
0	x^4	–	0	1
1	$1-4x+6x^2-4x^3$	1	1	-1
2	$\frac{1}{4}x - x^2 + \frac{3}{2}x^3$	$-\frac{1}{4}x$	$\frac{1}{4}x$	$1-\frac{1}{4}x$
3	$1-\frac{10}{3}x+\frac{10}{3}x^2$	$-\frac{8}{3}$	$1+\frac{2}{3}x$	$\frac{5}{3}-\frac{2}{3}x$
4	$-\frac{1}{5}x+\frac{1}{2}x^2$	$\frac{3}{2}\cdot\frac{3}{10}x$	$-\frac{1}{5}x-\frac{3}{10}x^2$	$1-x+\frac{3}{10}x^2$
5	$1-2x$	$\frac{20}{3}$	$1+2x+2x^2$	$-5+6x-2x^2$
6	$\frac{1}{20}x$	$-\frac{1}{4}x$	$\frac{1}{20}x+\frac{1}{5}x^2+\frac{1}{2}x^3$	$1-\frac{9}{4}x+\frac{18}{10}x^2-\frac{1}{2}x^3$
7	$1 = \gcd((1-x)^4, x^4)$	-40	$1+4x+10x^2+20x^3$	$35-84x+70x^2-20x^3$
8	0	$\frac{1}{20}x$	–	–

✏

❝*Infinitesimal analysis was considered so attractive and important because of its numerous and useful applications; as such, it attracted upon itself all research attention and efforts. Concurrently, algebraic analysis appeared to be a field where nothing remained to be done, or where whatever remained to be done would have only been worthless speculation. ...Nevertheless, the major contributors to infinitesimal analysis are well aware of the need to improve algebraic analysis: Their own progress depends upon it.*❞

Étienne Bézout, 1779 [7]

Theorem C.4 (Bézout's Identity). [8] [9] *Let* $\mathsf{p}_1(x)$ *be a polynomial of degree* n_1 *and* $\mathsf{p}_2(x)$ *be a polynomial of degree* n_2.

[7] quote: Bézout (1779a)
translation: Bézout (1779b) page xv
image: http://en.wikipedia.org/wiki/File:Etienne_Bezout2.jpg, public domain

[8] Bourbaki (2003b) page 2 ⟨Theorem 1 Chapter VII⟩, Fuhrmann (2012) pages 15–17 ⟨Corollary 1.31, Corollary 1.38⟩, Adhikari and Adhikari (2003) page 182, Warner (1990) page 381, Daubechies (1992) page 169, Mallat (1999) page 250

[9] Historical information: Bézout (1779a) ⟨???⟩, Bézout (1779b) ⟨???⟩, Bachet (1621) ⟨???⟩, Childs (2009) pages 37–46 ⟨some history on page 46⟩, http://serge.mehl.free.fr/chrono/Bachet.html, http://serge.mehl.free.fr/chrono/Bezout.html

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THM

$$\underbrace{\gcd(\mathsf{p}_1(x),\ \mathsf{p}_2(x)) = 1}_{\mathsf{p}_1(x)\ \textit{and}\ \mathsf{p}_2(x)\ \textit{are relatively prime}} \quad \Longrightarrow \quad \begin{cases} 1. & \exists \mathsf{q}_1(x), \mathsf{q}_2(x) \quad \text{such that} \\ & \underset{\substack{\uparrow \\ \textit{degree}\ n_1}}{\mathsf{p}_1(x)}\underset{\substack{\downarrow \\ \textit{degree}\ n_2 - 1}}{\mathsf{q}_1(x)} + \underset{\substack{\uparrow \\ \textit{degree}\ n_2}}{\mathsf{p}_2(x)}\underset{\substack{\downarrow \\ \textit{degree}\ n_1 - 1}}{\mathsf{q}_2(x)} = 1 \\ 2. & \textit{order of}\ \mathsf{q}_1(x) = n_2 - 1 \\ 3. & \textit{order of}\ \mathsf{q}_2(x) = n_1 - 1 \end{cases}$$

✎PROOF: No proof at this time. ✏

C.3 Roots

❝Neither the true nor the false roots are always real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation $x^3 - 6x^2 + 13x - 10 = 0$ as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remain always imaginary.❞

René Descartes (1596–1650), French philosopher and mathematician[10]

Theorem C.5 (Fundamental Theorem of Algebra). [11] *Let* $\mathsf{p}(x)$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

THM

$$\{\textit{degree of}\ \mathsf{p}(x)\ \textit{is}\ N\} \quad \Longrightarrow \quad \left\{ \begin{array}{l} \underbrace{\exists\ (\!|x_n|\!)_1^N\ \textit{such that}\ \mathsf{p}(x_n) = 0\ \textit{for}\ n = 1, 2, \ldots, N}_{\mathsf{p}(x)\ \textit{has}\ N\ \textit{zeros}} \\ \textit{where}\ x_n\ \textit{and}\ x_m\ \textit{are not necessarily distinct for}\ n \neq m. \end{array} \right\}$$

Corollary C.1. *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

COR

$$\left\{ \underbrace{\begin{array}{l} \textit{There exists}\ (\!|x_n|\!)_1^N \\ \textit{such that}\ \mathsf{p}(x_n) = 0\ \textit{for}\ n = 0, 1, \ldots, N \\ \textit{and where}\ x_n\ \textit{and}\ x_m\ \textit{are} \\ \textit{not necessarily distinct for}\ n \neq m. \end{array}}_{N\ \textit{zeros of}\ \mathsf{p}(x)} \right\} \Longrightarrow \left\{ \mathsf{p}(x) = \left(\frac{\alpha_0}{\prod_{n=1}^{N}(-x_n)} \right) \underbrace{\prod_{n=1}^{N}(x - x_n)}_{N\ \textit{factors}} \right\}$$

Lemma C.1.

LEM

$$\{x^N + 1 = 0\} \quad \Longrightarrow \quad \left\{ \textit{roots of}\ x = \left\{ \exp\left[i\frac{\pi}{N}(2n+1) \right] | n = 0, 1, \ldots, N-1 \right\} \right\}$$

[10] quote: Descartes (1637a)
English: Descartes (1954) page 175
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

[11] Prasolov (2004) pages 1–2 ⟨Section 1.1.1⟩, Borwein and Erdélyi (1995) page 11 ⟨Theorem 1.2.1⟩

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Figure C.1: Roots of $x^n + 1 = 0$

✎PROOF:

$$
\begin{aligned}
e^{iN\theta_n - i2\pi n} &= -1 && n \in \mathbb{Z} \\
N\theta_n - 2\pi n &= \pi && n = 0, 1, \ldots, N-1 \\
N\theta_n &= 2\pi n + \pi \\
\theta_n &= \frac{\pi}{N}(2n+1)
\end{aligned}
$$

✏

Theorem C.6. *Let* $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \le n \le N\}$ *and* $\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$.

THM

$$\underbrace{\alpha_n = \alpha_{-n}^* \quad \forall n \in I}_{(\alpha_n) \text{ is Hermitian symmetric}} \qquad \Longleftrightarrow \qquad \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C}$$

✎PROOF:

1. Proof that $\alpha_n = \alpha_{-n}^* \implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$
\begin{aligned}
\mathsf{p}(x) &\triangleq \sum_{n=-N}^{N} \alpha_n x^n && \text{by definition of } \mathsf{p}(x) \\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha_{-n} x^{-n} \\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha_n^* x^{-n} && \text{by left hypothesis} \\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n^* x^{-n} + \sum_{n=1}^{N} \alpha_n x^n \\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n^* \left(\frac{1}{x}\right)^n + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x}\right)^{-n} \\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha_n^* \left(\frac{1}{x^*}\right)^{-n}\right]^* \\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha_{-n} \left(\frac{1}{x^*}\right)^{-n}\right]^* && \text{by left hypothesis} \\
&= \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^*
\end{aligned}
$$

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Figure C.2: Recipricol conjugate pairs

$$= \mathsf{p}^*\left(\frac{1}{x^*}\right) \qquad \text{by definition of } \mathsf{p}(x)$$

2. Proof that $\alpha_n = \alpha_{-n}^* \Longleftarrow \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\sum_{n=-N}^{N} \alpha_n x^n &\triangleq \mathsf{p}(x) && \text{by definition of } \mathsf{p}(x)\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by right hypothesis}\\
&\triangleq \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^* && \text{by definition of } \mathsf{p}(x)\\
&= \sum_{n=-N}^{N} \alpha_n^* \left(\frac{1}{x}\right)^n\\
&= \sum_{n=-N}^{N} \alpha_{-n}^* x^n && \text{by symmetry of summation indices}\\
\Longrightarrow \quad \alpha_n &= \alpha_{-n}^* && \text{by matching of polynomial coefficients}
\end{aligned}$$

✏

Theorem C.7. *Let $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \le n \le N\}$ and*

$$\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$$

THM

$$\underbrace{\alpha_n = \alpha_{-n}^* \quad \forall n \in I}_{(\alpha_n) \text{ is Hermitian symmetric}} \quad \Longrightarrow \quad \underbrace{\left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]}_{\text{roots occur in conjugate recipricol pairs}}$$

✎PROOF:

$$\begin{aligned}
\alpha_n = \alpha_{-n}^* \quad \forall n \in I & && \text{by left hypothesis}\\
\Longrightarrow \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C} & && \text{by Theorem C.6 page 163}\\
\Longrightarrow \left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right] & &&
\end{aligned}$$

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If σ is a zero of $\mathsf{p}(x)$, then so is $\frac{1}{\sigma^*}$ because

$$\mathsf{p}\left(\frac{1}{\sigma^*}\right) = \mathsf{p}^*(\sigma) = 0^* = 0.$$

Figure C.3: Conjugate pairs of roots

Theorem C.8 (next) states that the roots of real polynomials occur in complex conjugate pairs; this is illustrated in Figure C.3 (page 165).

Theorem C.8. [12] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial.*

THM

$$\left[\underbrace{(\!(\alpha_n \in \mathbb{R})\!)_{n=0,1,\ldots,N}}_{\textit{coefficients are real}}\right] \Longrightarrow \left[\underbrace{\mathsf{p}(x_0) = 0 \iff \mathsf{p}(x_0^*) = 0}_{\textit{zeros occur in conjugate pairs}}\right]$$

Theorem C.9 (Routh-Hurwitz Criterion)**.** [13] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$ *and*

$$d_0 \triangleq \alpha_0 \qquad d_1 \triangleq \alpha_1 \qquad d_2 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} \qquad d_3 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} \qquad d_4 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 \end{vmatrix}$$

$$d_n \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ \alpha_{2n-3} & \alpha_{2n-4} & \cdots & \alpha_{n-2} \\ \alpha_{2n-1} & \alpha_{2n-2} & \cdots & \alpha_n \end{vmatrix}$$

Let $\mathbf{S}(\!(x_n)\!)$ *be the number of sign changes of some sequence* $(\!(x_n)\!)$ *after eliminating all zero elements* $(x_n = 0)$*.*

THM

$$\underbrace{\left|\left\{x_n \middle| \mathsf{p}(x_n) = 0,\ \Re[x_n] > 0\right\}\right|}_{\textit{number of roots in right half plane}} = \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ d_1 d_2,\ d_2 d_3,\ \ldots,\ d_{p-2} d_{p-1},\ \alpha_p\right)\!\right)}_{\textit{number of sign changes}}$$

$$= \underbrace{\mathbf{S}\left(\!\left(d_0,\ d_1,\ \frac{d_2}{d_1},\ \frac{d_3}{d_2},\ \ldots,\ \frac{d_p}{d_{p-1}}\right)\!\right)}_{\textit{number of sign changes}}$$

[12] Korn and Korn (1968) page 17
[13] Korn and Korn (1968) page 17

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Theorem C.10 (Descartes rule of signs). [14] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$.

THM

$$\underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Re[x_n] > 0\right\},\ \Im[x_n] = 0\right|}_{\text{number of roots on right real axis}} = \underbrace{\mathbf{S}(\alpha_n)}_{\text{number of sign changes – even integer}} - 2m \qquad \text{where } m \in \mathbb{W}$$

Theorem C.11. [15] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$.

THM

$$\underbrace{\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1}\ \geq 0}_{\text{first } k \text{ coefficients are nonnegative}} \implies \begin{cases} \underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Im[x_n] = 0\right\}\right|}_{\text{number of real roots}} < \underbrace{1 + \left(\frac{q}{\alpha_0}\right)^{\frac{1}{k}}}_{\text{upper bound}} \\ \text{where } q \triangleq \underbrace{\max\left\{\ |\alpha_n|\ |\alpha_n < 0\right\}}_{\text{largest negative coefficient}} \end{cases}$$

Theorem C.12 (Rolle's Theorem). [16] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$. *The number of real zeros of* $\mathsf{p}'(x)$ *between any two real consecutive real zeros of* $\mathsf{p}(x)$ *is* ***odd***.

Definition C.4. [17] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

DEF

$\frac{\mathsf{p}(x)}{\mathsf{q}(x)}$ *is a* ***rational function***
if $\mathsf{p}(x)$ *and* $\mathsf{q}(x)$ *are* POLYNOMIALS *over* $(\mathbb{F}, +, \cdot, 0, 1)$.

Example C.4.

EX

An example of a rational function using polynomials in x^{-1} is

$$A(x) = \frac{b_0 + \beta_1 x^{-1} + \beta_2 x^{-2} + \beta_3 x^{-3}}{1 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \alpha_3 x^{-3}}$$

This can be expressed as a rational function using polynomials in x by multiplying numerator and denominator by x^3:

$$A(x) = \frac{x^3}{x^3} A(x) = \frac{b_0 x^3 + \beta_1 x^2 + \beta_2 x + \beta_3}{x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3}$$

Definition C.5.

DEF

The ***zeros*** *of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $B(x)$.
The ***poles*** *of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $A(x)$.

C.4 Polynomial expansions

"Thus, if a straight-line is cut at random, then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces."
Euclid (~300BC), Greek mathematician, demonstrating the *Binomial theorem* for exponent $n = 2$ as in $(x + y)^2 = x^2 + 2xy + y^2$. [18]

[14] Korn and Korn (1968) page 17
[15] Korn and Korn (1968) page 18
[16] Korn and Korn (1968) page 18
[17] Fuhrmann (2012) page 22

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Theorem C.13 (Taylor Series). [19] *Let $\mathbf{C}$ be the space of all* ANALYTIC *functions and $\frac{\mathrm{d}}{\mathrm{d}x}$ in $\mathbf{C}^{\mathbf{C}}$ the* DIFFERENTIATION OPERATOR.

THM

*A **Taylor series** about the point a of a function $\mathsf{f} \in \mathbf{C}^{\mathbf{C}}$ is* $\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}^n \mathsf{f}\right](a)}{n!}(x-a)^n \quad \forall a \in \mathbb{R}, \mathsf{f} \in \mathbf{C}$

*A **Maclaurin series** is a* TAYLOR SERIES *about the point $a = 0$.*

Theorem C.14 (Binomial Theorem). [20]

THM

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad \text{where} \qquad \binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$$

✎PROOF: This theorem is proven using two different techniques. Either is sufficient. The first requires the Maclaurin series resulting in a more compact proof, but requires the additional (here unproven) Maclaurin series. The second proof uses induction resulting in a longer proof, but does not require any external theorem.

1. Proof using Maclaurin series:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}y^k} \left[(x+y)^n\right]_{y=0} y^k && \text{by Maclaurin series (Theorem C.13 page 167)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left[n(n-1)(n-2)\cdots(n-k+1)(x+y)^{n-k}\right]_{y=0} y^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{n!}{(n-k)!} x^{n-k} y^k \\
&= \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k && \text{by definition of } \binom{n}{k} \text{ (Definition B.3 page 146)} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k + \cancelto{0}{\sum_{k=n+1}^{\infty} \binom{n}{k} x^{n-k} y^k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k && \text{because } (x+y)^n \text{ has order } n
\end{aligned}$$

2. Proof using induction:

   (a) Proof that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ is true for $n = 0$:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \Bigg|_{n=0} &= \binom{0}{0} x^0 y^{0-0} \\
&= 1 \\
&= (x+y)^n \big|_{n=0}
\end{aligned}$$

[18] quote: Euclid (circa 300BC) ⟨Book II, Proposition 4⟩, Coolidge (1949) page 147
image: http://commons.wikimedia.org/wiki/File:Euklid-von-Alexandria_1.jpg, public domain

[19] Flanigan (1983) page 221 ⟨Theorem 15⟩, Strichartz (1995) page 281, Sohrab (2003) page 317 ⟨Theorem 8.4.9⟩, Taylor (1715), Taylor (1717), Maclaurin (1742)

[20] Graham et al. (1994) page 162 ⟨(5.12)⟩, Rotman (2010) page 84 ⟨Proposition 2.5⟩, Bourbaki (2003a) page 99 ⟨Corollary 1⟩, Warner (1990) pages 189–190 ⟨Theorem 21.1⟩, Metzler et al. (1908) page 169 ⟨any real exponent⟩, Coolidge (1949)

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(b) Proof that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ is true for $n = 1$:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \Bigg|_{n=1} &= \binom{1}{0} x^0 y^{1-0} + \binom{1}{1} x^1 y^{1-1} \\
&= y + x \\
&= (x+y)^n|_{n=1}
\end{aligned}$$

(c) Proof that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \implies (x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n}{k} x^k y^{n+1-k}$:

$$\begin{aligned}
&\sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] x^k y^{n+1-k} \qquad \text{by } \textit{Pascal's Rule} \text{ (Theorem B.1 page 147)} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&= x^{n+1} + y^{n+1} + \left[\sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n+1-(k+1)} - x^{n+1} \right] + \left[\sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} - y^{n+1} \right] \\
&= x \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= x(x+y)^n + y(x+y)^n \qquad \text{by left hypothesis} \\
&= (x+y)(x+y)^n \\
&= (x+y)^{n+1}
\end{aligned}$$

✏

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APPENDIX D

OPERATORS ON LINEAR SPACES

❝And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.❞

Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

D.1 Operators on linear spaces

D.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition D.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD *(Definition A.5 page 144). Let* X *be a set, let* $+$ *be an* OPERATOR *(Definition D.2 page 170) in* X^{X^2}*, and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991) page 41, Halmos (1948) pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

Figure D.1: Some operator types

DEF

The structure $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\big)$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1.	$\exists \mathbb{0} \in X$	such that	$\boldsymbol{x} + \mathbb{0}$	$=$	$\boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	(+ IDENTITY)
2.	$\exists \boldsymbol{y} \in X$	such that	$\boldsymbol{x} + \boldsymbol{y}$	$=$	$\mathbb{0}$	$\forall \boldsymbol{x} \in X$	(+ INVERSE)
3.			$(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$	$=$	$\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$	$\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$	(+ *is* ASSOCIATIVE)
4.			$\boldsymbol{x} + \boldsymbol{y}$	$=$	$\boldsymbol{y} + \boldsymbol{x}$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(+ *is* COMMUTATIVE)
5.			$1 \cdot \boldsymbol{x}$	$=$	$\boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	($\cdot$ IDENTITY)
6.			$\alpha \cdot (\beta \cdot \boldsymbol{x})$	$=$	$(\alpha \cdot \beta) \cdot \boldsymbol{x}$	$\forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$	($\cdot$ ASSOCIATES *with* $\cdot$)
7.			$\alpha \cdot (\boldsymbol{x} + \boldsymbol{y})$	$=$	$(\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y})$	$\forall \alpha \in S$ *and* $\boldsymbol{x}, \boldsymbol{y} \in X$	($\cdot$ DISTRIBUTES *over* $+$)
8.			$(\alpha + \beta) \cdot \boldsymbol{x}$	$=$	$(\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x})$	$\forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$	($\cdot$ PSEUDO-DISTRIBUTES *over* $+$)

The set X *is called the* ***underlying set****. The elements of* X *are called* ***vectors****. The elements of* $\mathbb{F}$ *are called* ***scalars****. A linear space is also called a* ***vector space****. If* $\mathbb{F} \triangleq \mathbb{R}$*, then* $\boldsymbol{\Omega}$ *is a* ***real linear space****. If* $\mathbb{F} \triangleq \mathbb{C}$*, then* $\boldsymbol{\Omega}$ *is a* ***complex linear space****.*

Definition D.2. [3]

DEF

A function $\mathbf{A}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *is an* ***operator*** *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *if* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both* LINEAR SPACES *(Definition D.1 page 169).*

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}\boldsymbol{x} = \mathbf{B}\boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{X}\boldsymbol{Y}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}\boldsymbol{x} = \mathbf{I}$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

Definition D.3. [4] *Let* $\boldsymbol{X}^{\boldsymbol{X}}$ *be the set of all operators with from a* LINEAR SPACE $\boldsymbol{X}$ *to* $\boldsymbol{X}$*. Let* $\mathbf{I}$ *be an operator in* $\boldsymbol{X}^{\boldsymbol{X}}$*. Let* $\mathbb{1}(\boldsymbol{X})$ *be the* IDENTITY ELEMENT *in* $\boldsymbol{X}^{\boldsymbol{X}}$*.*

DEF

$\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\quad \mathbf{I} = \mathbb{1}(\boldsymbol{X})$.

[3] Heil (2011) page 42
[4] Michel and Herget (1993) page 411

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D.1.2 Linear operators

Definition D.4. [5] *Let* $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\big)$ *and* $\boldsymbol{Y} \triangleq \big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\big)$ *be linear spaces.*

DEF

An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *is* ***linear*** *if*

1. $\mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$ (ADDITIVE) *and*
2. $\mathbf{L}(\alpha\boldsymbol{x}) = \alpha\mathbf{L}\boldsymbol{x} \quad \forall \boldsymbol{x} \in \boldsymbol{X}, \ \forall \alpha \in \mathbb{F}$ (HOMOGENEOUS).

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ *such that*

$$\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \big\{\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}} \,|\, \mathbf{L} \text{ is linear}\big\} \quad .$$

Theorem D.1. [6] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$*.*

THM

$$\{\mathbf{L} \text{ is LINEAR}\} \implies \left\{\begin{array}{llll} 1. & \mathbf{L}\mathbb{0} = \mathbb{0} & & \text{and} \\ 2. & \mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x}) & \forall \boldsymbol{x} \in \boldsymbol{X} & \text{and} \\ 3. & \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X} & \text{and} \\ 4. & \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n \big(\mathbf{L}\boldsymbol{x}_n\big) & \boldsymbol{x}_n \in \boldsymbol{X}, \alpha_n \in \mathbb{F} & \end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned} \mathbf{L}\mathbb{0} &= \mathbf{L}(0 \cdot \mathbb{0}) && \text{by additive identity property} \\ &= 0 \cdot (\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \qquad \text{(Definition D.4 page 171)} \\ &= \mathbb{0} && \text{by } \textit{additive identity} \text{ property} \end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned} \mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1 \cdot \boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \\ &= -1 \cdot (\mathbf{L}\boldsymbol{x}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \qquad \text{(Definition D.4 page 171)} \\ &= -(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y}$:

$$\begin{aligned} \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x} + (-\boldsymbol{y})) && \text{by } \textit{additive inverse} \text{ property} \\ &= \mathbf{L}(\boldsymbol{x}) + \mathbf{L}(-\boldsymbol{y}) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \qquad \text{(Definition D.4 page 171)} \\ &= \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} && \text{by item (2)} \end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n \big(\mathbf{L}\boldsymbol{x}_n\big)$:

   (a) Proof for $N = 1$:

$$\begin{aligned} \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\big(\alpha_1 \boldsymbol{x}_1\big) && \text{by } N = 1 \text{ hypothesis} \\ &= \alpha_1 \big(\mathbf{L}\boldsymbol{x}_1\big) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \qquad \text{(Definition D.4 page 171)} \end{aligned}$$

[5] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[6] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

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(b) Proof that N case $\implies N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1}\alpha_n\boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1}+\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right)\\
&= \mathbf{L}(\alpha_{N+1}\boldsymbol{x}_{N+1})+\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \quad \text{(Definition D.4 page 171)}\\
&= \alpha_{N+1}\mathbf{L}(\boldsymbol{x}_{N+1})+\sum_{n=1}^{N}\mathbf{L}(\alpha_n\boldsymbol{x}_n) && \text{by left } N+1 \text{ hypothesis}\\
&= \sum_{n=1}^{N+1}\mathbf{L}(\alpha_n\boldsymbol{x}_n)
\end{aligned}$$

✏

Theorem D.2. [7] *Let $\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathscr{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
	$\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	*is a linear space*	*(space of linear transforms)*
	$\mathscr{N}(\mathbf{L})$	*is a linear subspace of* $\boldsymbol{X}$	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$
	$\mathscr{I}(\mathbf{L})$	*is a linear subspace of* $\boldsymbol{Y}$	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$

✎PROOF:

1. Proof that $\mathscr{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0}\in\mathscr{N}(\mathbf{L}) \implies \mathscr{N}(\mathbf{L})\neq\varnothing$
   (b) $\mathscr{N}(\mathbf{L})\triangleq\{\boldsymbol{x}\in\boldsymbol{X}\,|\,\mathbf{L}\boldsymbol{x}=\mathbb{0}\}\subseteq\boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathscr{N}(\mathbf{L}) \implies \mathbb{0}=\mathbf{L}(\boldsymbol{x}+\boldsymbol{y})=\mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x}\in\mathscr{N}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\boldsymbol{X} \implies \mathbb{0}=\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathscr{N}(\mathbf{L})$
2. Proof that $\mathscr{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0}\in\mathscr{I}(\mathbf{L}) \implies \mathscr{I}(\mathbf{L})\neq\varnothing$
   (b) $\mathscr{I}(\mathbf{L})\triangleq\{\boldsymbol{y}\in\boldsymbol{Y}\,|\,\exists\boldsymbol{x}\in\boldsymbol{X} \text{ such that } \boldsymbol{y}=\mathbf{L}\boldsymbol{x}\}\subseteq\boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathscr{I}(\mathbf{L}) \implies \exists\boldsymbol{v}\in\boldsymbol{X} \text{ such that } \mathbf{L}\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x}\in\mathscr{I}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\mathscr{I}(\mathbf{L}) \implies \exists\boldsymbol{x}\in\boldsymbol{X} \text{ such that } \boldsymbol{y}=\mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y}=\alpha\mathbf{L}\boldsymbol{x}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathscr{I}(\mathbf{L})$

✏

Example D.1. [8] Let $\mathscr{C}([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX	
	$\mathscr{C}([a:b],\mathbb{R})$ is a linear space.

Theorem D.3. [9] *Let $\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathscr{N}(\mathbf{L})$ be the* NULL SPACE *of a linear operator $\mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM			
	$\mathbf{L}\boldsymbol{x}=\mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y}\in\mathscr{N}(\mathbf{L})$
	$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathscr{N}(\mathbf{L})=\{\mathbb{0}\}$

[7] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩
[8] Eidelman et al. (2004) page 3
[9] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

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✎PROOF:

1. Proof that $\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \implies \boldsymbol{x} - \boldsymbol{y} \in \mathscr{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} && \text{by Theorem D.1 page 171}\\
&= \mathbb{0} && \text{by left hypothesis}\\
&\implies \quad \boldsymbol{x}-\boldsymbol{y} \in \mathscr{N}(\mathbf{L}) && \text{by definition of } \textit{Null Space}
\end{aligned}$$

2. Proof that $\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \impliedby \boldsymbol{x} - \boldsymbol{y} \in \mathscr{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}\boldsymbol{y} &= \mathbf{L}\boldsymbol{y} + \mathbb{0} && \text{by definition of linear space (Definition D.1 page 169)}\\
&= \mathbf{L}\boldsymbol{y} + \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) && \text{by right hypothesis}\\
&= \mathbf{L}\boldsymbol{y} + (\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y}) && \text{by Theorem D.1 page 171}\\
&= (\mathbf{L}\boldsymbol{y} - \mathbf{L}\boldsymbol{y}) + \mathbf{L}\boldsymbol{x} && \text{by } \textit{associative} \text{ and } \textit{commutative} \text{ properties (Definition D.1 page 169)}\\
&= \mathbf{L}\boldsymbol{x}
\end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathscr{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned}
\mathbf{L} \text{ is } \textit{injective} &\iff \{(\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \quad \iff \quad \boldsymbol{x} = \boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y} \in X\}\\
&\iff \left\{\left[\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} = \mathbb{0} \quad \iff \quad (\boldsymbol{x}-\boldsymbol{y}) = \mathbb{0}\right] \quad \forall \boldsymbol{x},\boldsymbol{y} \in X\right\}\\
&\iff \left\{\left[\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbb{0} \quad \iff \quad (\boldsymbol{x}-\boldsymbol{y}) = \mathbb{0}\right] \quad \forall \boldsymbol{x},\boldsymbol{y} \in X\right\}\\
&\iff \mathscr{N}(\mathbf{L}) = \{\mathbb{0}\}
\end{aligned}$$

✏

Theorem D.4. [10] *Let $\boldsymbol{W}$, $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ be linear spaces over a field $\mathbb{F}$.*

THM					
	1.	$\mathbf{L}(\mathbf{MN})$	$= (\mathbf{LM})\mathbf{N}$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Z},\boldsymbol{W}), \mathbf{M}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(ASSOCIATIVE)
	2.	$\mathbf{L}\big(\mathbf{M} \mathbin{\dot{+}} \mathbf{N}\big)$	$= (\mathbf{LM}) \mathbin{\dot{+}} (\mathbf{LN})$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(LEFT DISTRIBUTIVE)
	3.	$\big(\mathbf{L} \mathbin{\dot{+}} \mathbf{M}\big)\mathbf{N}$	$= (\mathbf{LN}) \mathbin{\dot{+}} (\mathbf{MN})$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(RIGHT DISTRIBUTIVE)
	4.	$\alpha(\mathbf{LM})$	$= (\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L}\in\mathscr{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{MN}) = (\mathbf{LM})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}\big(\mathbf{M} \mathbin{\dot{+}} \mathbf{N}\big) = (\mathbf{LM}) \mathbin{\dot{+}} (\mathbf{LN})$:

$$\begin{aligned}
\big[\mathbf{L}\big(\mathbf{M} \mathbin{\dot{+}} \mathbf{N}\big)\big]\boldsymbol{x} &= \mathbf{L}\big[\big(\mathbf{M} \mathbin{\dot{+}} \mathbf{N}\big)\boldsymbol{x}\big]\\
&= \mathbf{L}\big[(\mathbf{M}\boldsymbol{x}) \mathbin{\dot{+}} (\mathbf{N}\boldsymbol{x})\big]\\
&= [\mathbf{L}(\mathbf{M}\boldsymbol{x})] \mathbin{\dot{+}} [\mathbf{L}(\mathbf{N}\boldsymbol{x})] && \text{by } \textit{additive} \text{ property Definition D.4 page 171}\\
&= [(\mathbf{LM})\boldsymbol{x}] \mathbin{\dot{+}} [(\mathbf{LN})\boldsymbol{x}]
\end{aligned}$$

3. Proof that $\big(\mathbf{L} \mathbin{\dot{+}} \mathbf{M}\big)\mathbf{N} = (\mathbf{LN}) \mathbin{\dot{+}} (\mathbf{MN})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{LM}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{LM}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned}
[\alpha(\mathbf{LM})]\boldsymbol{x} &= \alpha[(\mathbf{LM})\boldsymbol{x}]\\
&= \mathbf{L}[\alpha(\mathbf{M}\boldsymbol{x})] && \text{by } \textit{homogeneous} \text{ property Definition D.4 page 171}\\
&= \mathbf{L}[(\alpha\mathbf{M})\boldsymbol{x}]\\
&= [\mathbf{L}(\alpha\mathbf{M})]\boldsymbol{x}
\end{aligned}$$

[10] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

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Theorem D.5 (Fundamental theorem of linear equations). [11] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\dim \mathcal{I}(\mathbf{L}) + \dim \mathcal{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1, 2, \dots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \dots, \psi_p\}$ is a basis for $\mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
&\text{Let } p \triangleq \dim \boldsymbol{X}.\\
&\text{Let } n \triangleq \dim \mathcal{N}(\mathbf{L}).
\end{aligned}$$

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\}\\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \dots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\}\\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \dots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L}\psi_k \right\}\\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \dots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L}\psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L}\psi_k \right\}\\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \dots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L}\psi_k + \mathbb{0} \right\}\\
&= p - n\\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ✏

D.2 Operators on Normed linear spaces

D.2.1 Operator norm

Definition D.5. [12] *Let $\boldsymbol{V} = (X, \mathbb{F}, \hat{+}, \cdot)$ be a linear space and $\mathbb{F}$ be a field with absolute value function $|\cdot| \in \mathbb{R}^{\mathbb{F}}$ (Definition A.4 page 144).*

DEF

*A **norm** is any functional $\|\cdot\|$ in $\mathbb{R}^{\boldsymbol{X}}$ that satisfies*

1.	$\|\boldsymbol{x}\| \geq 0$	$\forall \boldsymbol{x} \in X$	(STRICTLY POSITIVE)	*and*
2.	$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	(NONDEGENERATE)	*and*
3.	$\|a\boldsymbol{x}\| = \lvert a\rvert \, \|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X, a \in \mathbb{C}$	(HOMOGENEOUS)	*and*
4.	$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(SUBADDITIVE/*triangle inquality*).	

*A **normed linear space** is the pair $(\boldsymbol{V}, \|\cdot\|)$.*

[11] Michel and Herget (1993) page 99

[12] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

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Definition D.6. [13] *Let $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [14]

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} \qquad \forall \mathbf{A} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition D.1 (next) shows that the functional defined in Definition D.6 (previous) is a *norm* (Definition D.5 page 174).

Proposition D.1. [15] *Let $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ and $\boldsymbol{Y} \triangleq \big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1.	$\|\|\|\mathbf{A}\|\|\| \ge 0$		$\forall \mathbf{A}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(NON-NEGATIVE)	*and*
2.	$\|\|\|\mathbf{A}\|\|\| = 0$	$\iff \mathbf{A} \mathrel{\mathring{=}} \mathbb{0}$	$\forall \mathbf{A}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(NONDEGENERATE)	*and*
3.	$\|\|\|\alpha\mathbf{A}\|\|\| = \lvert\alpha\rvert \, \|\|\|\mathbf{A}\|\|\|$		$\forall \mathbf{A}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)	*and*
4.	$\|\|\|\mathbf{A} \mathring{+} \mathbf{B}\|\|\| \le \|\|\|\mathbf{A}\|\|\| + \|\|\|\mathbf{B}\|\|\|$		$\forall \mathbf{A}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \ne \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)} \\ &> 0 \end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \mathrel{\mathring{=}} \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)} \\ &= \sup_{x\in\boldsymbol{X}} \{\|\mathbb{0}x\| \mid \|x\| \le 1\} \\ &= 0 \end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha| \, |||\mathbf{A}|||$:

$$\begin{aligned} |||\alpha\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\alpha\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)} \\ &= \sup_{x\in\boldsymbol{X}} \{|\alpha| \, \|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)} \\ &= |\alpha| \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of sup} \\ &= |\alpha| \; |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)} \end{aligned}$$

[13] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225

[14] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990) page 290

[15] Rudin (1991) page 93

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4. Proof that $|||\mathbf{A} \mathbin{\mathring{+}} \mathbf{B}||| \le |||\mathbf{A}||| + |||\mathbf{B}|||$:

$$\begin{aligned}
|||\mathbf{A} \mathbin{\mathring{+}} \mathbf{B}||| &\triangleq \sup_{x\in X}\left\{\|(\mathbf{A} \mathbin{\mathring{+}} \mathbf{B})x\| \mid \|x\| \le 1\right\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)}\\
&= \sup_{x\in X}\{\|\mathbf{A}x + \mathbf{B}x\| \mid \|x\| \le 1\}\\
&\le \sup_{x\in X}\{\|\mathbf{A}x\| + \|\mathbf{B}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)}\\
&\le \sup_{x\in X}\{\|\mathbf{A}x\| \mid \|x\| \le 1\} + \sup_{x\in X}\{\|\mathbf{B}x\| \mid \|x\| \le 1\}\\
&\triangleq |||\mathbf{A}||| + |||\mathbf{B}||| && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)}
\end{aligned}$$

✏

Lemma D.1. *Let* $(\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *and* $\boldsymbol{Y} \triangleq \big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$.

LEM

$$|||\mathbf{L}||| = \sup_{x}\{\|\mathbf{L}x\| \mid \|x\| = 1\} \qquad \forall x \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF: [16]

1. Proof that $\sup_x\{\|\mathbf{L}x\| \mid \|x\| \le 1\} \ge \sup_x\{\|\mathbf{L}x\| \mid \|x\| = 1\}$:

$$\sup_{x}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} \ge \sup_{x}\{\|\mathbf{L}x\| \mid \|x\| = 1\} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ y \in X \,\middle|\, \begin{array}{ll} 1. & \|\mathbf{L}y\| = \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} \text{ and} \\ 2. & 0 < \|y\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_x\{\|\mathbf{L}x\| \mid \|x\| \le 1\} \le \sup_x\{\|\mathbf{L}x\| \mid \|x\| = 1\}$:

$$\begin{aligned}
\sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} &= \|\mathbf{L}y\| && \text{by definition of set } Y\\
&= \frac{\|y\|}{\|y\|}\|\mathbf{L}y\|\\
&= \|y\|\left\|\frac{1}{\|y\|}\mathbf{L}y\right\| && \text{by homogeneous property (page 174)}\\
&= \|y\|\left\|\mathbf{L}\frac{y}{\|y\|}\right\| && \text{by homogeneous property (page 171)}\\
&\le \|y\|\sup_{y\in Y}\left\{\left\|\mathbf{L}\frac{y}{\|y\|}\right\|\right\} && \text{by definition of supremum}\\
&= \|y\|\sup_{y\in Y}\left\{\left\|\mathbf{L}\frac{y}{\|y\|}\right\| \,\middle|\, \left\|\frac{y}{\|y\|}\right\| = 1\right\} && \text{because } \left\|\frac{y}{\|y\|}\right\| = 1 \text{ for all } y \in Y\\
&\le \sup_{y\in Y}\left\{\left\|\mathbf{L}\frac{y}{\|y\|}\right\| \,\middle|\, \left\|\frac{y}{\|y\|}\right\| = 1\right\} && \text{because } 0 < \|y\| \le 1\\
&\le \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| = 1\} && \text{because } \frac{y}{\|y\|} \in X \quad \forall y \in Y
\end{aligned}$$

16 **email**

Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

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4. By (1) and (3),

$$\sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| = 1\}$$

✏

Proposition D.2. [17] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathscr{L}(\boldsymbol{X},\boldsymbol{X}), |||\cdot|||)$.

PRP	$\|\|\|\mathbf{I}\|\|\| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 175)}\\
&= \sup\{\|x\| \mid \|x\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition D.3 page 170)}\\
&= 1
\end{aligned}$$

✏

Theorem D.6. [18] *Let* $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM				
	$\|\mathbf{L}x\|$	$\le$	$\|\|\|\mathbf{L}\|\|\|\ \|x\|$	$\forall \mathbf{L} \in \mathscr{L}(\boldsymbol{X},\boldsymbol{Y}),\ x \in \boldsymbol{X}$
	$\|\|\|\mathbf{KL}\|\|\|$	$\le$	$\|\|\|\mathbf{K}\|\|\|\ \|\|\|\mathbf{L}\|\|\|$	$\forall \mathbf{K},\mathbf{L} \in \mathscr{L}(\boldsymbol{X},\boldsymbol{Y})$

✎PROOF:

1. Proof that $\|\mathbf{L}x\| \le |||\mathbf{L}|||\ \|x\|$:

$$\begin{aligned}
\|\mathbf{L}x\| &= \frac{\|x\|}{\|x\|}\|\mathbf{L}x\|\\
&= \|x\|\left\|\frac{1}{\|x\|}\mathbf{L}x\right\| && \text{by property of norms}\\
&= \|x\|\left\|\mathbf{L}\frac{x}{\|x\|}\right\| && \text{by property of linear operators}\\
&\triangleq \|x\|\,\|\mathbf{L}y\| && \text{where } y \triangleq \frac{x}{\|x\|}\\
&\le \|x\|\sup_y \|\mathbf{L}y\| && \text{by definition of supremum}\\
&= \|x\|\sup_y\{\|\mathbf{L}y\| \mid \|y\| = 1\} && \text{because } \|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1\\
&\triangleq \|x\|\,\|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

[17] Michel and Herget (1993) page 410
[18] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

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2. Proof that $|||\mathbf{KL}||| \le |||\mathbf{K}|||\ |||\mathbf{L}|||$:

$$
\begin{aligned}
|||\mathbf{KL}||| &\triangleq \sup_{x\in X}\{\|(\mathbf{KL})x\| \mid \|x\| \le 1\} && \text{by Definition D.6 page 175 } (|||\cdot|||)\\
&= \sup_{x\in X}\{\|\mathbf{K}(\mathbf{L}x)\| \mid \|x\| \le 1\} \\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ \|\mathbf{L}x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ \|x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&= \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ 1 \mid \|x\| \le 1\} && \text{by definition of sup}\\
&= |||\mathbf{K}|||\ |||\mathbf{L}||| && \text{by definition of sup}
\end{aligned}
$$

✏

D.2.2 Bounded linear operators

Definition D.7. [19] *Let $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ be a normed space of linear operators.*

DEF

An operator $\mathbf{B}$ *is* ***bounded*** *if* $|||\mathbf{B}||| < \infty$.

The quantity $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ *is the set of all* ***bounded linear operators*** *on* $(\boldsymbol{X},\boldsymbol{Y})$ *such that*

$$\mathscr{B}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \{\mathbf{L} \in \mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \mid |||\mathbf{L}||| < \infty\}.$$

Theorem D.7. [20] *Let $(\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ be the set of linear operators over normed linear spaces $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ and $\boldsymbol{Y} \triangleq \big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$.*

THM

The following conditions are all EQUIVALENT*:*

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $x_0 \in X$ $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point* $x \in X$*)* $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
3. $|||\mathbf{L}||| < \infty$ *(*$\mathbf{L}$ *is* BOUNDED*)* $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}x\| \le M\,\|x\|$ $\quad \forall \mathbf{L}\in\mathscr{L}(\boldsymbol{X},\boldsymbol{Y}), x\in X$

✎PROOF:

1. Proof that 1 $\Longrightarrow$ 2:

$$
\begin{aligned}
\epsilon &> \big\|\mathbf{L}x - \mathbf{L}x_0\big\| && \text{by hypothesis 1}\\
&= \big\|\mathbf{L}(x - x_0)\big\| && \text{by linearity (Definition D.4 page 171)}\\
&= \big\|\mathbf{L}(x + y - x_0 - y)\big\| \\
&= \big\|\mathbf{L}(x + y) - \mathbf{L}(x_0 + y)\big\| && \text{by linearity (Definition D.4 page 171)}\\
&\Longrightarrow \mathbf{L} \text{ is continuous at point } x + y \\
&\Longrightarrow \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}
$$

2. Proof that 2 $\Longrightarrow$ 1: obvious:

[19] Rudin (1991) pages 92–93
[20] Aliprantis and Burkinshaw (1998) page 227

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3. Proof that 4 $\implies$ 2:[21]

$$\begin{aligned}
|||\mathbf{L}\boldsymbol{x}||| \le M\,||\boldsymbol{x}|| &\implies |||\mathbf{L}(\boldsymbol{x}-\boldsymbol{y})||| \le M\,||\boldsymbol{x}-\boldsymbol{y}|| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le M\,||\boldsymbol{x}-\boldsymbol{y}|| && \text{by linearity of } \mathbf{L} \text{ (Definition D.4 page 171)}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } M\,||\boldsymbol{x}-\boldsymbol{y}|| < \epsilon\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } ||\boldsymbol{x}-\boldsymbol{y}|| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that 3 $\implies$ 4:

$$\begin{aligned}
||\mathbf{L}\boldsymbol{x}|| &\le \underbrace{|||\mathbf{L}|||}_{M}\ ||\boldsymbol{x}|| && \text{by Theorem D.6 page 177}\\
&= M\ ||\boldsymbol{x}|| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that 1 $\implies$ 3:[22]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{||\mathbf{L}\boldsymbol{x}|| \mid ||\boldsymbol{x}|| \le 1\} = \infty\\
&\implies \exists (\boldsymbol{x}_n) \quad \text{such that} \quad \|\boldsymbol{x}_n\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\|\mathbf{L}\boldsymbol{x}_n\| \mid \|\boldsymbol{x}_n\| \le 1\right\} = \infty\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \|\mathbf{L}\boldsymbol{x}_n\|\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \|\mathbf{L}\boldsymbol{x}_n\| \ge n\\
&\implies \frac{1}{n}\|\boldsymbol{x}_n\| = \frac{1}{n} \text{ and } \frac{1}{n}\|\mathbf{L}\boldsymbol{x}_n\| \ge 1\\
&\implies \left\|\frac{\boldsymbol{x}_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{\boldsymbol{x}_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

✍

D.2.3 Adjoints on normed linear spaces

Definition D.8. *Let $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *of $\boldsymbol{X}$.*

DEF

$\mathbf{B}^*$ *is the* ***adjoint*** *of an operator* $\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ *if*

$$\mathsf{f}(\mathbf{B}\boldsymbol{x}) = \left[\mathbf{B}^*\mathsf{f}\right](\boldsymbol{x}) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ \boldsymbol{x} \in \boldsymbol{X}$$

Theorem D.8. [23] *Let $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *$\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM

$$\begin{aligned}
(\mathbf{A} \mathbin{\mathring{+}} \mathbf{B})^* &= \mathbf{A}^* \mathbin{\mathring{+}} \mathbf{B}^* && \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\lambda\mathbf{A})^* &= \lambda\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\mathbf{A}\mathbf{B})^* &= \mathbf{B}^*\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{Y})
\end{aligned}$$

[21] Bollobás (1999) page 29
[22] Aliprantis and Burkinshaw (1998) page 227
[23] Bollobás (1999) page 156

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✎PROOF:

$$
\begin{aligned}
\left[\mathbf{A} \mathbin{\mathring{+}} \mathbf{B}\right]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \mathbin{\mathring{+}} \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition D.8 page 179)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition D.4 page 171)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} && \\
&= \mathbf{A}^{*}\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^{*}\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 179)}\\
&= \left[\mathbf{A}^{*} + \mathbf{B}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{linear functional} &&
\end{aligned}
$$

$$
\begin{aligned}
[\lambda\mathbf{A}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 179)}\\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} && \\
&= \left[\lambda\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 179)}
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{AB}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{AB}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 179)}\\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of } \textit{linear operators} && \text{(Definition D.4 page 171)}\\
&= \left[\mathbf{A}^{*}\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 179)}\\
&= \mathbf{B}^{*}\left[\mathbf{A}^{*}\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 179)}\\
&= \left[\mathbf{B}^{*}\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 179)}
\end{aligned}
$$

✏

Theorem D.9. [24] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^{*}$ be the adjoint of an operator $\mathbf{B}$.*

THM

$$\left|\left|\left|\mathbf{B}\right|\right|\right| = \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| \qquad \forall \mathbf{B} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF:

$$
\begin{aligned}
\left|\left|\left|\mathbf{B}\right|\right|\right| &\triangleq \sup\left\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\right\} && \text{by Definition D.6 page 175}\\
&\overset{?}{=} \sup\left\{|g(\mathbf{B}\boldsymbol{x}; \boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x}; \mathbf{B}^{*}\boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&\triangleq \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&= \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \left\|\boldsymbol{y}^{*}\right\| \le 1\right\}\\
&\triangleq \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| && \text{by Definition D.6 page 175}
\end{aligned}
$$

✏

D.2.4 More properties

❝Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.❞

Stanislaus M. Ulam (1909–1984), Polish mathematician [25]

[24] Rudin (1991) page 98

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Theorem D.10 (Mazur-Ulam theorem). [26] *Let $\phi \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ be a function on normed linear spaces $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. Let $\mathbf{I} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$ be the identity operator on $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.*

THM

$$\left.\begin{array}{ll} 1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\textit{bijective}} \quad \textit{and} \\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}}_{\textit{isometric}} \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\textit{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$
   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$
2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$
3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned} \left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis} \\ &= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1} \end{aligned}$$

4. Proof that $\mathrm{g}\boldsymbol{z} = \boldsymbol{z}$:

$$\begin{aligned} 2\lambda &= 2\sup\{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\} && \text{by definition of } \lambda \text{ item (2)} \\ &\le 2\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\| && \text{by definition of sup} \\ &= \|2\boldsymbol{z} - 2\mathrm{g}\boldsymbol{z}\| \\ &= \|\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}\boldsymbol{z}\| && \text{by definition of } \psi \text{ item (1)} \\ &= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}^{-1}\mathrm{g}\boldsymbol{z}\right\| && \text{by item (3)} \\ &= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| && \text{by definition of } \mathrm{g}^{-1} \\ &= \left\|\psi\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| \\ &= \left\|\mathrm{g}^{*}\boldsymbol{z} - \boldsymbol{z}\right\| \\ &\le \lambda && \text{by definition of } \lambda \text{ item (2)} \\ &\implies 2\lambda \le \lambda \\ &\implies \lambda = 0 \\ &\implies \mathrm{g}\boldsymbol{z} = \boldsymbol{z} \end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) = \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}$:

$$\begin{aligned} \phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) &= \\ &= \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y} \end{aligned}$$

[25] quote: Ulam (1991) page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[26] Oikhberg and Rosenthal (2007) page 598, Väisälä (2003) page 634, Giles (2000) page 11, Dunford and Schwartz (1957) page 91, Mazur and Ulam (1932)

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6. Proof that $\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}$:

$$\begin{aligned}\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) &= \\ &= [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}\end{aligned}$$

✏

Theorem D.11 (Neumann Expansion Theorem). [27] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{ll} 1. & \mathbf{A} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X}) \quad \textit{(A is bounded)} \\ 2. & |||\mathbf{A}||| < 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} 1. & (\mathbf{I}-\mathbf{A})^{-1} & \textit{exists} \\ 2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \leq \dfrac{1}{1-|||\mathbf{A}|||} \\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = \displaystyle\sum_{n=0}^{\infty}\mathbf{A}^n \\ & \textit{with uniform convergence} & \end{array}\right.$$

D.3 Operators on Inner product spaces

D.3.1 General Results

Definition D.9. [28] *Let* $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\big)$ *be a linear space.*

DEF

A function $\langle\triangle\,|\,\triangledown\rangle \in \mathbb{F}^{X\times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

1.	$\langle \boldsymbol{x}\,	\,\boldsymbol{x}\rangle \geq 0$	$\forall \boldsymbol{x}\in X$	*(non-negative)*	*and*
2.	$\langle \boldsymbol{x}\,	\,\boldsymbol{x}\rangle = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x}\in X$	*(nondegenerate)*	*and*
3.	$\langle \alpha\boldsymbol{x}\,|\,\boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}\,|\,\boldsymbol{y}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y}\in X, \forall\alpha\in\mathbb{C}$	*(homogeneous)*	*and*
4.	$\langle \boldsymbol{x}+\boldsymbol{y}\,|\,\boldsymbol{u}\rangle = \langle \boldsymbol{x}\,|\,\boldsymbol{u}\rangle + \langle \boldsymbol{y}\,|\,\boldsymbol{u}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X$	*(additive)*	*and*
5.	$\langle \boldsymbol{x}\,|\,\boldsymbol{y}\rangle = \langle \boldsymbol{y}\,|\,\boldsymbol{x}\rangle^*$	$\forall \boldsymbol{x},\boldsymbol{y}\in X$	*(conjugate symmetric).*	

An inner product is also called a ***scalar product****.*
An ***inner product space*** *is the pair* $(\boldsymbol{\Omega}, \langle\triangle\,|\,\triangledown\rangle)$.

Theorem D.12. [29] *Let* $\mathbf{A},\mathbf{B} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\triangledown\rangle\big)$.

THM

$$\begin{array}{lllllll} \langle \mathbf{B}\boldsymbol{x}\,|\,\boldsymbol{x}\rangle & = & 0 & \forall\boldsymbol{x}\in X & \iff & \mathbf{B}\boldsymbol{x} & = & \mathbb{0} \quad \forall\boldsymbol{x}\in X \\ \langle \mathbf{A}\boldsymbol{x}\,|\,\boldsymbol{x}\rangle & = & \langle \mathbf{B}\boldsymbol{x}\,|\,\boldsymbol{x}\rangle & \forall\boldsymbol{x}\in X & \iff & \mathbf{A} & = & \mathbf{B} \end{array}$$

✎PROOF:

[27] Michel and Herget (1993) page 415
[28] Haaser and Sullivan (1991) page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72
[29] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

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1. Proof that $\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 \implies \mathbf{B}\boldsymbol{x} = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(\boldsymbol{x}+\mathbf{B}\boldsymbol{x}) \mid (\boldsymbol{x}+\mathbf{B}\boldsymbol{x})\rangle + i\,\langle \mathbf{B}(\boldsymbol{x}+i\mathbf{B}\boldsymbol{x}) \mid (\boldsymbol{x}+i\mathbf{B}\boldsymbol{x})\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}\boldsymbol{x}+\mathbf{B}^2\boldsymbol{x}) \mid \boldsymbol{x}+\mathbf{B}\boldsymbol{x}\right\rangle\right\} + i\left\{\left\langle \mathbf{B}\boldsymbol{x}+i\mathbf{B}^2\boldsymbol{x}) \mid \boldsymbol{x}+i\mathbf{B}\boldsymbol{x}\right\rangle\right\} && \text{by Definition D.4 page 171}\\
&= \left\{\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle + \langle \mathbf{B}\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\rangle + \left\langle \mathbf{B}^2\boldsymbol{x} \mid \boldsymbol{x}\right\rangle + \left\langle \mathbf{B}^2\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\right\rangle\right\} && \text{by Definition D.9 page 182}\\
&\quad + i\left\{\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle - i\,\langle \mathbf{B}\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\rangle + i\left\langle \mathbf{B}^2\boldsymbol{x} \mid \boldsymbol{x}\right\rangle - i^2\left\langle \mathbf{B}^2\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\rangle + \left\langle \mathbf{B}^2\boldsymbol{x} \mid \boldsymbol{x}\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\rangle + i\left\langle \mathbf{B}^2\boldsymbol{x} \mid \boldsymbol{x}\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\rangle + \left\langle \mathbf{B}^2\boldsymbol{x} \mid \boldsymbol{x}\right\rangle\right\} + \left\{\langle \mathbf{B}\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\rangle - \left\langle \mathbf{B}^2\boldsymbol{x} \mid \boldsymbol{x}\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}\boldsymbol{x} \mid \mathbf{B}\boldsymbol{x}\rangle\\
&= 2\,\|\mathbf{B}\boldsymbol{x}\|^2\\
&\implies \mathbf{B}\boldsymbol{x} = \mathbb{0} && \text{by Definition D.5 page 174}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 \impliedby \mathbf{B}\boldsymbol{x} = \mathbb{0}$: by property of inner products.

3. Proof that $\langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle \implies \mathbf{A} \mathrel{\hat{=}} \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle - \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}\boldsymbol{x} - \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by \textit{additivity} property of } \langle \triangle \mid \triangledown\rangle \text{ (Definition D.9 page 182)}\\
&= \langle (\mathbf{A}-\mathbf{B})\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A}-\mathbf{B})\boldsymbol{x} &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle \impliedby \mathbf{A} \mathrel{\hat{=}} \mathbf{B}$:

$$\langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle \qquad \text{by } \mathbf{A} \mathrel{\hat{=}} \mathbf{B} \text{ hypothesis}$$

✏

D.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition D.3 page 183). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem D.13 page 184).
- Both support decomposition into "real" and "imaginary" parts (Theorem **??** page **??**).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *Null Space* of an operator (Theorem D.14 page 185).

Proposition D.3. [30] *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *(Definition D.7 page 178) on a* HILBERT SPACE $\boldsymbol{H}$.

PRP *An operator $\mathbf{B}^*$ is the **adjoint** of $\mathbf{B} \in \mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ if*

$$\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y}\rangle = \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y}\rangle \qquad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{H}.$$

✎PROOF:

[30] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000) page 182, von Neumann (1929) page 49, Stone (1932) page 41

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1. For fixed y, $\mathsf{f}(\boldsymbol{x}) \triangleq \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is a *functional* in $\mathbb{F}^{\boldsymbol{X}}$.
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle &\triangleq \mathsf{f}(\mathbf{B}\boldsymbol{x}) \\ &\triangleq \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{operator adjoint} && \text{(Definition D.8 page 179)} \\ &= \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle\end{aligned}$$

✏

Example D.2.

EX

In matrix algebra ("linear algebra")

- The inner product operation $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is represented by $\boldsymbol{y}^H\boldsymbol{x}$.
- The linear operator is represented as a matrix $\boldsymbol{A}$.
- The operation of $\boldsymbol{A}$ on a vector $\boldsymbol{x}$ is represented as $\boldsymbol{A}\boldsymbol{x}$.
- The adjoint of matrix $\boldsymbol{A}$ is the Hermitian matrix $\boldsymbol{A}^H$.

✎PROOF:

$$\langle \boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H\boldsymbol{A}\boldsymbol{x} = [(\boldsymbol{A}\boldsymbol{x})^H\boldsymbol{y}]^H = [\boldsymbol{x}^H\boldsymbol{A}^H\boldsymbol{y}]^H = (\boldsymbol{A}^H\boldsymbol{y})^H\boldsymbol{x} \triangleq \left\langle \boldsymbol{x} \mid \boldsymbol{A}^H\boldsymbol{y} \right\rangle$$

✏

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition ?? page ??). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[31]

Theorem D.13 (operator star-algebra). [32] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM

The pair $(\boldsymbol{H}, *)$ *is a* $*$-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \mathbin{\dot{+}} \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(INVOLUTARY)	

✎PROOF:

$$\begin{aligned}\left\langle \boldsymbol{x} \mid (\mathbf{A} \mathbin{\dot{+}} \mathbf{B})^*\boldsymbol{y} \right\rangle &= \left\langle (\mathbf{A} \mathbin{\dot{+}} \mathbf{B})\boldsymbol{x} \mid \boldsymbol{y} \right\rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)} \\ &= \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 182)} \\ &= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of operator addition} \\ &= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} + \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 182)} \\ &= \langle \boldsymbol{x} \mid (\mathbf{A}^* + \mathbf{B}^*)\boldsymbol{y} \rangle && \text{by definition of operator addition}\end{aligned}$$

$$\begin{aligned}\langle \boldsymbol{x} \mid (\alpha\mathbf{A})^*\boldsymbol{y} \rangle &= \langle (\alpha\mathbf{A})\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)} \\ &= \langle \alpha(\mathbf{A}\boldsymbol{x}) \mid \boldsymbol{y} \rangle && \text{by definition of scalar multiplication} \\ &= \alpha \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 182)} \\ &= \alpha \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)} \\ &= \langle \boldsymbol{x} \mid \alpha^*\mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 182)}\end{aligned}$$

[31] Sakai (1998) page 1

[32] Halmos (1998) pages 39–40, Rudin (1991) page 311

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$$\begin{aligned}
\langle \boldsymbol{x} \mid (\mathbf{AB})^* \boldsymbol{y}\rangle &= \langle (\mathbf{AB})\boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)}\\
&= \langle \mathbf{A}(\mathbf{B}\boldsymbol{x}) \mid \boldsymbol{y}\rangle && \text{by definition of operator multiplication} \\
&= \langle (\mathbf{B}\boldsymbol{x}) \mid \mathbf{A}^*\boldsymbol{y}\rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)}\\
&= \langle \boldsymbol{x} \mid \mathbf{B}^*\mathbf{A}^*\boldsymbol{y}\rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \mid \mathbf{A}^{**}\boldsymbol{y}\rangle &= \langle \mathbf{A}^*\boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)}\\
&= \langle \boldsymbol{y} \mid \mathbf{A}^*\boldsymbol{x}\rangle^* && \text{by definition of inner product} && \text{(Definition D.9 page 182)}\\
&= \langle \mathbf{A}\boldsymbol{y} \mid \boldsymbol{x}\rangle^* && \text{by definition of adjoint} && \text{(Proposition D.3 page 183)}\\
&= \langle \boldsymbol{x} \mid \mathbf{A}\boldsymbol{y}\rangle && \text{by definition of inner product} && \text{(Definition D.9 page 182)}
\end{aligned}$$

✏

Theorem D.14. [33] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM
$$\mathcal{N}(\mathbf{A}) = \mathcal{I}(\mathbf{A}^*)^\perp$$
$$\mathcal{N}(\mathbf{A}^*) = \mathcal{I}(\mathbf{A})^\perp$$

✎PROOF:

$$\begin{aligned}
\mathcal{I}(\mathbf{A}^*)^\perp &= \left\{y \in \boldsymbol{H} \mid \langle \boldsymbol{y} \mid \boldsymbol{u}\rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}^*)\right\}\\
&= \left\{y \in \boldsymbol{H} \mid \langle \boldsymbol{y} \mid \mathbf{A}^*\boldsymbol{x}\rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H}\right\}\\
&= \{y \in \boldsymbol{H} \mid \langle \mathbf{A}\boldsymbol{y} \mid \boldsymbol{x}\rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H}\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition D.3 page 183)}\\
&= \{y \in \boldsymbol{H} \mid \mathbf{A}\boldsymbol{y} = 0\}\\
&= \mathcal{N}(\mathbf{A}) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\mathbf{A})^\perp &= \{y \in \boldsymbol{H} \mid \langle \boldsymbol{y} \mid \boldsymbol{u}\rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A})\}\\
&= \{y \in \boldsymbol{H} \mid \langle \boldsymbol{y} \mid \mathbf{A}\boldsymbol{x}\rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H}\} && \text{by definition of } \mathcal{I}\\
&= \left\{y \in \boldsymbol{H} \mid \langle \mathbf{A}^*\boldsymbol{y} \mid \boldsymbol{x}\rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H}\right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition D.3 page 183)}\\
&= \left\{y \in \boldsymbol{H} \mid \mathbf{A}^*\boldsymbol{y} = 0\right\}\\
&= \mathcal{N}(\mathbf{A}^*) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

✏

D.4 Special Classes of Operators

D.4.1 Projection operators

Definition D.10. [34] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF
$\mathbf{P}$ *is a* ***projection*** *operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[33] Rudin (1991) page 312

[34] Rudin (1991) page 126 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 26, Halmos (1958) page 73 ⟨§41. Projections⟩

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Theorem D.15. [35] *Let $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ with* NULL SPACE *$\mathscr{N}(\mathbf{P})$ and* IMAGE SET *$\mathscr{I}(\mathbf{P})$.*

THM

$$\left.\begin{array}{llll} 1. & \mathbf{P}^2 = \mathbf{P} & (\mathbf{P}\text{ is a projection operator}) & \text{and} \\ 2. & \boldsymbol{\Omega} = \boldsymbol{X} \hat{+} \boldsymbol{Y} & (\boldsymbol{Y}\text{ compliments }\boldsymbol{X}\text{ in }\boldsymbol{\Omega}) & \text{and} \\ 3. & \mathbf{P}\boldsymbol{\Omega} = \boldsymbol{X} & (\mathbf{P}\text{ projects onto }\boldsymbol{X}) & \end{array}\right\} \Longrightarrow \left\{\begin{array}{llll} 1. & \mathscr{I}(\mathbf{P}) = \boldsymbol{X} & & \text{and} \\ 2. & \mathscr{N}(\mathbf{P}) = \boldsymbol{Y} & & \text{and} \\ 3. & \boldsymbol{\Omega} = \mathscr{I}(\mathbf{P}) \hat{+} \mathscr{N}(\mathbf{P}) & & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} \mathscr{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\ &= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\ &= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\ &= \boldsymbol{\Omega}_1 \end{aligned}$$

$$\begin{aligned} \mathscr{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\} \\ &= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_2 \end{aligned}$$

✏

Theorem D.16. [36] *Let $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X},\boldsymbol{Y})$.*

THM

$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P}\text{ is a projection operator}} \quad \Longleftrightarrow \quad \underbrace{(\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})}_{(\mathbf{I}-\mathbf{P})\text{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})$:

$$\begin{aligned} (\mathbf{I}-\mathbf{P})^2 &= (\mathbf{I}-\mathbf{P})(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{I}(\mathbf{I}-\mathbf{P}) + (-\mathbf{P})(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{I}-\mathbf{P}-\mathbf{P}\mathbf{I}+\mathbf{P}^2 \\ &= \mathbf{I}-\mathbf{P}-\mathbf{P}+\mathbf{P} && \text{by left hypothesis} \\ &= \mathbf{I}-\mathbf{P} \end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})$:

$$\begin{aligned} \mathbf{P}^2 &= \underbrace{\mathbf{I}-\mathbf{P}-\mathbf{P}+\mathbf{P}^2}_{(\mathbf{I}-\mathbf{P})^2} - (\mathbf{I}-\mathbf{P}-\mathbf{P}) \\ &= (\mathbf{I}-\mathbf{P})^2 - (\mathbf{I}-\mathbf{P}-\mathbf{P}) \\ &= (\mathbf{I}-\mathbf{P}) - (\mathbf{I}-\mathbf{P}-\mathbf{P}) && \text{by right hypothesis} \\ &= \mathbf{P} \end{aligned}$$

✏

[35] Michel and Herget (1993) pages 120–121
[36] Michel and Herget (1993) page 121

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Theorem D.17. [37] *Let $\boldsymbol{H}$ be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathscr{N}(\mathbf{P})$, *and* IMAGE SET $\mathscr{I}(\mathbf{P})$.

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

1.	$\mathbf{P}^* = \mathbf{P}$		(**P** *is* SELF-ADJOINT)	$\Longleftrightarrow$
2.	$\mathbf{P}^*\mathbf{P} = \mathbf{P}\mathbf{P}^*$		(**P** *is* NORMAL)	$\Longleftrightarrow$
3.	$\mathscr{I}(\mathbf{P}) = \mathscr{N}(\mathbf{P})^\perp$			$\Longleftrightarrow$
4.	$\langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \|\mathbf{P}\boldsymbol{x}\|^2$	$\forall \boldsymbol{x}\in X$		

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}\mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)}\\ &= \mathbf{P}\mathbf{P}^* && \text{by Theorem D.13 page 184}\end{aligned}$$

Proof that (1) $\Longrightarrow$ (3):

$$\begin{aligned}\mathscr{I}(\mathbf{P}) &= \mathscr{N}(\mathbf{P}^*)^\perp && \text{by Theorem D.14 page 185}\\ &= \mathscr{N}(\mathbf{P})^\perp && \text{by (1)}\end{aligned}$$

Proof that (3) $\Longrightarrow$ (4):

Proof that (4) $\Longrightarrow$ (1):

✏

D.4.2 Self Adjoint Operators

Definition D.11. [38] *Let* $\mathbf{B} \in \mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be* ***self-adjoint*** *or* ***hermitian*** *if* $\mathbf{B} \mathrel{\overset{\circ}{=}} \mathbf{B}^*$.

Example D.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathbf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}.$$

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}}\mathsf{f}(u)\,\mathsf{du}$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

Theorem D.18. [39] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.

THM

$$\left\{\underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S}\text{ is }\textbf{self adjoint}}\right\} \Longrightarrow \left\{\begin{array}{lll} 1. & \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x}\rangle \in \mathbb{R} & \text{(the hermitian quadratic form of } \mathbf{S} \text{ is REAL-VALUED)}\\ 2. & \lambda_n \in \mathbb{R} & \text{(eigenvalues of } \mathbf{S} \text{ are REAL-VALUED)}\\ 3. & \lambda_n \neq \lambda_m \Longrightarrow \langle \psi_n \mid \psi_m\rangle = 0 & \text{(eigenvectors are ORTHOGONAL)}\end{array}\right\}$$

[37] Rudin (1991) page 314

[38] Historical works regarding self-adjoint operators: von Neumann (1929) page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932) page 50 ⟨"self-adjoint transformations"⟩

[39] Lax (2002) pages 315–316, Keener (1988) pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩, Bertero and Boccacci (1998) page 225 ⟨§"9.2 SVD of a matrix …If all eigenvectors are normalized…"⟩

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✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle \in \mathbb{R}$:

$$\begin{aligned}
\langle \boldsymbol{x} \mid \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&= \langle \boldsymbol{x} \mid \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \mid \triangledown \rangle \text{ Definition D.9 page 182}
\end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned}
\lambda_n \left\|\psi_n\right\|^2 &= \lambda_n \langle \psi_n \mid \psi_n \rangle && \text{by definition} \\
&= \langle \lambda_n \psi_n \mid \psi_n \rangle && \text{by definition of } \langle \triangle \mid \triangledown \rangle \text{ Definition D.9 page 182} \\
&= \langle \mathbf{S}\psi_n \mid \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \mid \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \mid \lambda_n \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \lambda_n^* \langle \psi_n \mid \psi_n \rangle && \text{by definition of } \langle \triangle \mid \triangledown \rangle \text{ Definition D.9 page 182} \\
&= \lambda_n^* \left\|\psi_n\right\|^2 && \text{by definition}
\end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \mid \psi_m \rangle &= \langle \lambda_n \psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \triangledown \rangle \text{ Definition D.9 page 182} \\
&= \langle \mathbf{S}\psi_n \mid \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \mid \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \mid \lambda_m \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \lambda_m^* \langle \psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \triangledown \rangle \text{ Definition D.9 page 182} \\
&= \lambda_m \langle \psi_n \mid \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \mid \psi_m \rangle = 0$.

✏

D.4.3 Normal Operators

Definition D.12. [40] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{N}^*$ be the adjoint of an operator $\mathbf{N} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF $\quad \mathbf{N}$ *is* ***normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

Theorem D.19. [41] *Let $\mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE *$\boldsymbol{H}$. Let $\mathscr{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$ and $\mathscr{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathscr{B}(\boldsymbol{H}, \boldsymbol{H})$.*

THM $\quad \underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \iff \left\|\mathbf{N}^*\boldsymbol{x}\right\| = \|\mathbf{N}\boldsymbol{x}\| \quad \forall \boldsymbol{x} \in \boldsymbol{H}$

[40] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969) page 167, Frobenius (1878), Frobenius (1968) page 391

[41] Rudin (1991) pages 312–313

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✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \mid \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition D.3 page 183 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \mid \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis } (\mathbf{N} \text{ is normal})\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition D.3 page 183 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition D.3 page 183 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem D.13 page 184 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis } (\|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Proposition D.3 page 183 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem D.20. [42] *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathscr{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathscr{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathscr{N}(\mathbf{N}^*) = \mathscr{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same Null Space}}$$

✎PROOF:

$$\begin{aligned}
\mathscr{N}(\mathbf{N}^*) &= \{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} && \text{by definition of } \textit{Null Space}\\
&= \{\boldsymbol{x} | \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} && \text{by definition of } \|\cdot\| \text{ (Definition D.5 page 174)}\\
&= \{\boldsymbol{x} | \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\}\\
&= \{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} && \text{by definition of } \|\cdot\| \text{ (Definition D.5 page 174)}\\
&= \mathscr{N}(\mathbf{N}) && \text{by definition of } \textit{Null Space } \mathscr{N}
\end{aligned}$$

✏

Theorem D.21. [43] *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathscr{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathscr{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m\rangle = 0}_{\textit{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. Rudin (1991) page 313 claims both to be true.

[42] Rudin (1991) pages 312–313
[43] Rudin (1991) pages 312–313

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1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\qquad = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\qquad = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem D.13 page 184} \\
&\qquad = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem D.13 page 184} \\
&\qquad = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\Longrightarrow \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\Longleftrightarrow \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle\psi_n \,|\, \psi_m\rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle\psi_n \,|\, \psi_m\rangle &= \langle\lambda_n\psi_n \,|\, \psi_m\rangle && \text{by definition of } \langle\triangle \,|\, \triangledown\rangle \text{ Definition D.9 page 182} \\
&= \langle\mathbf{N}\psi_n \,|\, \psi_m\rangle && \text{by definition of eigenpairs} \\
&= \langle\psi_n \,|\, \mathbf{N}^*\psi_m\rangle && \text{by Proposition D.3 page 183 (definition of adjoint)} \\
&= \langle\psi_n \,|\, \lambda_m^*\psi_m\rangle && \text{by (4.)} \\
&= \lambda_m \langle\psi_n \,|\, \psi_m\rangle && \text{by definition of } \langle\triangle \,|\, \triangledown\rangle \text{ Definition D.9 page 182}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle\psi_n \,|\, \psi_m\rangle = 0$.

✏

D.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition D.13. *Let* $\big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *and* $\big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be* NORMED LINEAR SPACES *(Definition D.5 page 174)*.

DEF *An operator* $\mathbf{M} \in \mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is* ***isometric*** *if*

$$\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \qquad \forall_{\boldsymbol{x}\in X}.$$

Theorem D.22. [44] *Let* $\big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *and* $\big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathscr{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM

$$\underbrace{\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall_{\boldsymbol{x}\in X}}_{\textit{isometric in length}} \quad \Longleftrightarrow \quad \underbrace{\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall_{\boldsymbol{x},\boldsymbol{y}\in X}}_{\textit{isometric in distance}}$$

✎PROOF:

1. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \implies \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| &= \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\| && \text{by definition of linear operators (Definition D.4 page 171)} \\
&= \|\mathbf{M}\boldsymbol{u}\| && \text{let } \boldsymbol{u} \triangleq \boldsymbol{x} - \boldsymbol{y} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by left hypothesis}
\end{aligned}$$

[44] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

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2. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \Longleftarrow \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \|\mathbf{M}(\boldsymbol{x} - \mathbb{0})\| \\
&= \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition D.4 page 171)} \\
&= \|\boldsymbol{x} - \mathbb{0}\| && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition D.13 page 190) in the more general normed linear spaces, while Theorem D.22 (page 190) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \iff \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem D.23 (next) demonstrates two additional equivalent properties.

Theorem D.23. [45] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.*

THM

*The following conditions are all **equivalent**:*

1.	$\mathbf{M}^*\mathbf{M}$	$=$	$\mathbf{I}$			$\iff$
2.	$\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle$	$=$	$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	*($\mathbf{M}$ is surjective)*	$\iff$
3.	$\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|$	$=$	$\|x - y\|$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	*(isometric in distance)*	$\iff$
4.	$\|\mathbf{M}\boldsymbol{x}\|$	$=$	$\|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X$	*(isometric in length)*	

✎PROOF:

1. Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}
\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle &= \langle \boldsymbol{x} \mid \mathbf{M}^*\mathbf{M}\boldsymbol{y} \rangle && \text{by Proposition D.3 page 183 (definition of adjoint)} \\
&= \langle \boldsymbol{x} \mid \mathbf{I}\boldsymbol{y} \rangle && \text{by (1)} \\
&= \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by Definition D.3 page 170 (definition of } \mathbf{I}\text{)}
\end{aligned}$$

2. Proof that (2) $\Longrightarrow$ (4):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \sqrt{\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x} \rangle} && \text{by definition of } \|\cdot\| \\
&= \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle} && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
4\,\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle &= \|\mathbf{M}\boldsymbol{x} + \mathbf{M}\boldsymbol{y}\|^2 - \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|^2 + i\,\|\mathbf{M}\boldsymbol{x} + i\mathbf{M}\boldsymbol{y}\|^2 - i\,\|\mathbf{M}\boldsymbol{x} - i\mathbf{M}\boldsymbol{y}\|^2 && \text{by polarization id.} \\
&= \|\mathbf{M}(\boldsymbol{x} + \boldsymbol{y})\|^2 - \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\|^2 + i\,\|\mathbf{M}(\boldsymbol{x} + i\boldsymbol{y})\|^2 - i\,\|\mathbf{M}(\boldsymbol{x} - i\boldsymbol{y})\|^2 && \text{by Definition D.4} \\
&= \|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 + i\,\|\boldsymbol{x} + i\boldsymbol{y}\|^2 - i\,\|\boldsymbol{x} - i\boldsymbol{y}\|^2 && \text{by left hypothesis}
\end{aligned}$$

4. Proof that (3) $\iff$ (4): by Theorem D.22 page 190

[45] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

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5. Proof that (4) $\Longrightarrow$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition D.3 page 183 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem D.13 page 184 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y = 0\\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Definition D.3 page 170 (definition of } \mathbf{I})\\
\Longrightarrow \qquad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}$$

✏

Theorem D.24. [46] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathscr{L}(\boldsymbol{X}, \boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M}\text{ is isometric}} \quad \Longrightarrow \quad \begin{cases} |||\mathbf{M}||| = 1 & \text{(UNIT LENGTH) and}\\ |\lambda| = 1 & \forall \lambda \in \Lambda \end{cases}$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \Longrightarrow |||\mathbf{M}||| = 1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{\boldsymbol{x}\in X}\{\|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Definition D.6 page 175}\\
&= \sup_{\boldsymbol{x}\in X}\{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Theorem D.23 page 191}\\
&= \sup_{\boldsymbol{x}\in X}\{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x}, \lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|}\ \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem D.23 page 191}\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|}\ |\lambda|\ \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

✏

Example D.4 (One sided shift operator). [47] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ $(0, 1, 2, \ldots)$ and shift operators defined as

$$\begin{aligned}
&1. \quad \mathbf{S}_r\,(\!(x_0, x_1, x_2, \ldots)\!) \triangleq (\!(0, x_0, x_1, x_2, \ldots)\!) && \text{(right shift operator)}\\
&2. \quad \mathbf{S}_l\,(\!(x_0, x_1, x_2, \ldots)\!) \triangleq (\!(x_1, x_2, x_3, \ldots)\!) && \text{(left shift operator)}
\end{aligned}$$

EX

1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

[46] Michel and Herget (1993) page 432
[47] Michel and Herget (1993) page 441

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✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r (\!(x_0, x_1, x_2, \ldots)\!) \mid (\!(y_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots)\!) \rangle &= \langle (\!(0, x_0, x_1, x_2, \ldots)\!) \mid (\!(y_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots)\!) \rangle \\
&= \sum_{n=1}^{\infty} \boldsymbol{x}_{n-1}\ \boldsymbol{y}_n^* \\
&= \sum_{n=0}^{\infty} \boldsymbol{x}_n\ \boldsymbol{y}_{n+1}^* \\
&= \sum_{n=0}^{\infty} \boldsymbol{x}_n\ \boldsymbol{y}_{n+1}^* \\
&= \langle (\!(x_0, x_1, x_2, \ldots)\!) \mid (\!(y_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \ldots)\!) \rangle \\
&= \left\langle (\!(x_0, x_1, x_2, \ldots)\!) \mid \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} (\!(y_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots)\!) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✏

D.4.5 Unitary operators

Definition D.14. [48] *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF *The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition D.4. *Let $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP $\left.\begin{array}{l}\mathbf{U} \text{ is UNITARY and} \\ \mathbf{V} \text{ is UNITARY}\end{array}\right\} \implies (\mathbf{UV})$ *is* UNITARY.

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\left(\mathbf{V}^*\mathbf{U}^*\right) && \text{by Theorem D.8 page 179} \\
&= \mathbf{U}\left(\mathbf{V}\mathbf{V}^*\right)\mathbf{U}^* && \text{by associative property} \\
&= \mathbf{U}\mathbf{I}\mathbf{U}^* && \text{by definition of } \textit{unitary} \text{ operators} \quad \text{(Definition D.14 page 193)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} \quad \text{(Definition D.14 page 193)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \left(\mathbf{V}^*\mathbf{U}^*\right)(\mathbf{UV}) && \text{by Theorem D.8 page 179} \\
&= \mathbf{V}^*\left(\mathbf{U}^*\mathbf{U}\right)\mathbf{V} && \text{by associative property} \\
&= \mathbf{V}^*\mathbf{I}\mathbf{V} && \text{by definition of } \textit{unitary} \text{ operators} \quad \text{(Definition D.14 page 193)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} \quad \text{(Definition D.14 page 193)}
\end{aligned}$$

[48] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

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✏

Theorem D.25. [49] *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathscr{I}(\mathbf{U})$ be the* IMAGE SET *of* $\mathbf{U}$.

THM

If $\mathbf{U}$ *is a* ***bounded linear operator*** *($\mathbf{U} \in \mathscr{B}(\boldsymbol{H},\boldsymbol{H})$), then the following conditions are* ***equivalent****:*

1.	$\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$			(UNITARY)	$\Longleftrightarrow$
2.	$\langle \mathbf{U}\boldsymbol{x} \mid \mathbf{U}\boldsymbol{y}\rangle = \langle \mathbf{U}^*\boldsymbol{x} \mid \mathbf{U}^*\boldsymbol{y}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$	*and*	$\mathscr{I}(\mathbf{U}) = X$	(SURJECTIVE)	$\Longleftrightarrow$
3.	$\lVert \mathbf{U}\boldsymbol{x} - \mathbf{U}\boldsymbol{y}\rVert = \lVert \mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\rVert = \lVert \boldsymbol{x} - \boldsymbol{y}\rVert$	*and*	$\mathscr{I}(\mathbf{U}) = X$	(ISOMETRIC IN DISTANCE)	$\Longleftrightarrow$
4.	$\lVert \mathbf{U}\boldsymbol{x}\rVert = \lVert \boldsymbol{x}\rVert$	*and*	$\mathscr{I}(\mathbf{U}) = X$	(ISOMETRIC IN LENGTH)	

✎PROOF:

1. Proof that (1) $\Longrightarrow$ (2):
   (a) $\langle \mathbf{U}\boldsymbol{x} \mid \mathbf{U}\boldsymbol{y}\rangle = \langle \mathbf{U}^*\boldsymbol{x} \mid \mathbf{U}^*\boldsymbol{y}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ by Theorem D.23 (page 191).
   (b) Proof that $\mathscr{I}(\mathbf{U}) = X$:

$$\begin{aligned} X &\supseteq \mathscr{I}(\mathbf{U}) && \text{because } \mathbf{U} \in X^X \\ &\supseteq \mathscr{I}(\mathbf{U}\mathbf{U}^*) \\ &= \mathscr{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}) \\ &= X && \text{by Definition D.3 page 170 (definition of } \mathbf{I}) \end{aligned}$$

2. Proof that (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4): by Theorem D.23 page 191.
3. Proof that (3) $\Longrightarrow$ (1):
   (a) Proof that $\lVert \mathbf{U}\boldsymbol{x} - \mathbf{U}\boldsymbol{y}\rVert = \lVert \boldsymbol{x} - \boldsymbol{y}\rVert \Longrightarrow \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem D.23 page 191
   (b) Proof that $\lVert \mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\rVert = \lVert \boldsymbol{x} - \boldsymbol{y}\rVert \Longrightarrow \mathbf{U}\mathbf{U}^* = \mathbf{I}$:

$$\begin{aligned} \lVert \mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\rVert = \lVert \boldsymbol{x} - \boldsymbol{y}\rVert \Longrightarrow \quad \mathbf{U}^{**}\mathbf{U}^* &= \mathbf{I} && \text{by Theorem D.23 page 191} \\ \mathbf{U}\mathbf{U}^* &= \mathbf{I} && \text{by Theorem D.13 page 184} \end{aligned}$$

✏

Theorem D.26. *Let $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathbf{U}$ *be a bounded linear operator in $\mathscr{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathscr{N}(\mathbf{U})$ the* NULL SPACE *of* $\mathbf{U}$, *and $\mathscr{I}(\mathbf{U})$ the* IMAGE SET *of* $\mathbf{U}$.

THM

$$\underbrace{\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U}\text{ is }\textbf{unitary}} \Longrightarrow \left\{ \begin{array}{rclcll} \mathbf{U}^{-1} &=& \mathbf{U}^* & & & \textit{and} \\ \mathscr{I}(\mathbf{U}) &=& \mathscr{I}(\mathbf{U}^*) &=& X & \textit{and} \\ \mathscr{N}(\mathbf{U}) &=& \mathscr{N}(\mathbf{U}^*) &=& \{\mathbb{0}\} & \textit{and} \\ \lVert\lVert \mathbf{U} \rVert\rVert &=& \lVert\lVert \mathbf{U}^* \rVert\rVert &=& 1 & \text{(UNIT LENGTH)} \end{array} \right\}$$

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both *isometric* and *normal*:

$$\begin{array}{rclcrcl} \mathbf{U}^*\mathbf{U} & & & = & \mathbf{I} & \Longrightarrow & \mathbf{U} \text{ is } \textit{isometric} \\ \mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & \mathbf{I} & \Longrightarrow & \mathbf{U}^* \text{ is } \textit{isometric} \\ \mathbf{U}^{-1} & = & \mathbf{U}^* & & & \Longrightarrow & \mathbf{U}^{-1} \text{ is } \textit{isometric} \\ & & & & & & \\ \mathbf{U}^*\mathbf{U} & = & \mathbf{U}\mathbf{U}^* & = & & \Longrightarrow & \mathbf{U} \text{ is } \textit{normal} \\ \mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & & \Longrightarrow & \mathbf{U}^* \text{ is } \textit{normal} \\ \mathbf{U}^{-1} & = & \mathbf{U}^* & & & \Longrightarrow & \mathbf{U}^{-1} \text{ is } \textit{normal} \end{array}$$

[49] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

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2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathscr{I}(\mathbf{U}) = \mathscr{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem D.25 page 194.
3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathscr{N}(\mathbf{U}) = \mathscr{N}(\mathbf{U}^*) = \mathscr{N}(\mathbf{U}^{-1})$:

$$\begin{aligned}
\mathscr{N}(\mathbf{U}^*) &= \mathscr{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both } \textit{normal} \text{ and by Theorem D.20 page 189}\\
&= \mathscr{I}(\mathbf{U})^\perp && \text{by Theorem D.14 page 185}\\
&= X^\perp && \text{by above result}\\
&= \{\mathbb{0}\}
\end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem D.24 page 192.

✏

Example D.5 (Rotation matrix). [50]

EX

$$\underbrace{\left\{\mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right\}}_{\textbf{rotation matrix } \mathbf{R}_\theta\,:\,\mathbb{R}^2 \to \mathbb{R}^2} \implies \left\{\begin{array}{llll} (1). & \mathbf{R}^{-1}{}_\theta & = \mathbf{R}_{-\theta} & \text{and} \\ (2). & \mathbf{R}^*{}_\theta & = \mathbf{R}^{-1}{}_\theta & (\mathbf{R} \text{ is unitary}) \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R}\\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of } \textit{Hermetian transpose} \text{ operator } H\\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{(Theorem I.2 page 237)}\\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R}\\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✏

Example D.6. [51] Let $\mathbf{A}$ and $\mathbf{B}$ be matrix operators.

EX

$$\underbrace{\begin{array}{c|c} \mathbf{A} \triangleq \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} & \mathbf{B} \triangleq \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \mathbf{A} \text{ is a } \textit{rotation operator}. & \mathbf{B} \text{ is a } \textit{reflection operator}. \end{array}}_{\text{Both } \mathbf{A} \text{ and } \mathbf{B} \text{ are } \textbf{unitary}.}$$

Example D.7. Examples of *Fredholm integral operators* include

EX

1.	**Fourier Transform**	$[\tilde{\mathbf{F}}\mathsf{x}](f)$	$=$	$\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-i2\pi ft}\,\mathsf{dt}$	$\kappa(t,f)$	$=$	$e^{-i2\pi ft}$
2.	**Inverse Fourier Transform**	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t)$	$=$	$\int_{f\in\mathbb{R}} \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathsf{df}$	$\kappa(f,t)$	$=$	$e^{i2\pi ft}$
3.	**Laplace operator**	$[\mathbf{L}\mathsf{x}](s)$	$=$	$\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathsf{dt}$	$\kappa(t,s)$	$=$	e^{-st}

Example D.8 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathsf{f}(x) \triangleq \mathsf{f}(x-1) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

[50] Noble and Daniel (1988) page 311
[51] Gel'fand (1963) page 4, Gelfand et al. (2018) page 4

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EX					
1.	$\mathbf{T}^{-1}\mathsf{f}(x)$	$=$	$\mathsf{f}(x+1)$	$\forall \mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}}$	(inverse translation operator)
2.	$\mathbf{T}^*$	$=$	$\mathbf{T}^{-1}$		($\mathbf{T}$ is invertible)
3.	$\mathbf{T}^*\mathbf{T}$	$=$	$\mathbf{T}\mathbf{T}^* = \mathbf{I}$		($\mathbf{T}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$
$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned}
\langle \mathbf{T}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \langle \mathsf{f}(x-1) \,|\, \mathsf{g}(x)\rangle && \text{by definition of } \mathbf{T}\\
&= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathsf{dx}\\
&= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathsf{dx}\\
&= \langle \mathsf{f}(x) \,|\, \mathsf{g}(x+1)\rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example D.9 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall \mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX					
1.	$\mathbf{D}^{-1}\mathsf{f}(x)$	$=$	$\frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$	$\forall \mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}}$	(inverse dilation operator)
2.	$\mathbf{D}^*$	$=$	$\mathbf{D}^{-1}$		($\mathbf{D}$ is invertible)
3.	$\mathbf{D}^*\mathbf{D}$	$=$	$\mathbf{D}\mathbf{D}^* = \mathbf{I}$		($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x) \,\middle|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathsf{dx}\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathsf{du} && \text{let } u \triangleq 2x \quad\Longrightarrow\quad \mathsf{dx} = \frac{1}{2}\,\mathsf{du}\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\mathsf{du}\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

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✏

Example D.10 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq (\!(x_{n-1})\!)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq (\!(x_{n+1})\!)_{n\in\mathbb{Z}}\,.$$

$$\begin{aligned}
\langle \mathbf{D}\,(\!(x_n)\!) \,|\, (\!(y_n)\!)\rangle &= \langle (\!(x_{n-1})\!) \,|\, (\!(y_n)\!)\rangle && \text{by definition of } \mathbf{D}\\
&= \sum_n \boldsymbol{x}_{n-1}\, \boldsymbol{y}_n^* \\
&= \sum_n \boldsymbol{x}_n\, \boldsymbol{y}_{n+1}^* \\
&= \left\langle (\!(x_n)\!) \,|\, (\!(y_{n+1})\!) \right\rangle \\
&= \left\langle (\!(x_n)\!) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\, (\!(y_n)\!) \right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary. ✏

Example D.11 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator (Theorem K.1 page 274)

$$[\tilde{\mathbf{F}}\mathsf{x}](f) \triangleq \int_t \mathsf{x}(t)\underbrace{e^{-i2\pi f t}}_{\kappa(t,f)}\,\mathsf{d}t \qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](t) \triangleq \int_f \tilde{\mathsf{x}}(f)\underbrace{e^{i2\pi f t}}_{\kappa^*(t,f)}\,\mathsf{d}f\,.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{x} \,|\, \tilde{\mathsf{y}}\right\rangle &= \left\langle \int_t \mathsf{x}(t)e^{-i2\pi f t}\,\mathsf{d}t \,|\, \tilde{\mathsf{y}}(f)\right\rangle \\
&= \int_t \mathsf{x}(t)\left\langle e^{-i2\pi f t} \,|\, \tilde{\mathsf{y}}(f)\right\rangle\,\mathsf{d}t \\
&= \int_t \mathsf{x}(t)\int_f e^{-i2\pi f t}\tilde{\mathsf{y}}^*(f)\,\mathsf{d}f\,\mathsf{d}t \\
&= \int_t \mathsf{x}(t)\left[\int_f e^{i2\pi f t}\tilde{\mathsf{y}}(f)\,\mathsf{d}f\right]^*\,\mathsf{d}t \\
&= \left\langle \mathsf{x}(t) \,|\, \int_f \tilde{\mathsf{y}}(f)e^{i2\pi f t}\,\mathsf{d}f\right\rangle \\
&= \left\langle \mathsf{x} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathsf{y}}\right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✏

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D.5 Operator order

Definition D.15. [52] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF

$\mathbf{P}$ *is* ***positive*** *if* $\langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x}\rangle \geq 0 \;\forall \boldsymbol{x} \in \boldsymbol{X}$.
This condition is denoted $\mathbf{P} \geq 0$.

Theorem D.27. [53]

THM

$$\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \quad \Longrightarrow \quad \begin{cases} (\mathbf{P}+\mathbf{Q}) \geq 0 & & \text{((}\mathbf{P}+\mathbf{Q}\text{) is positive)} \\ \mathbf{A}^*\mathbf{P}\mathbf{A} \geq 0 & \forall \mathbf{A} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X}) & \text{(}\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive)} \\ \mathbf{A}^*\mathbf{A} \geq 0 & \forall \mathbf{A} \in \mathscr{B}(\boldsymbol{X},\boldsymbol{X}) & \text{(}\mathbf{A}^*\mathbf{A} \text{ is positive)} \end{cases}$$

✎PROOF:

$\langle (\mathbf{P}+\mathbf{Q})\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x}\rangle + \langle \mathbf{Q}\boldsymbol{x} \mid \boldsymbol{x}\rangle$	by additive property of $\langle \triangle \mid \triangledown\rangle$ (Definition D.9 page 182)
$\geq \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x}\rangle$	by left hypothesis
≥ 0	by left hypothesis
$\langle \mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{P}\mathbf{A}\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x}\rangle$	by definition of adjoint (Proposition D.3 page 183)
$= \langle \mathbf{P}\boldsymbol{y} \mid \boldsymbol{y}\rangle$	where $\boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x}$
≥ 0	by left hypothesis
$\langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle$	by definition of $\mathbf{I}$ (Definition D.3 page 170)
≥ 0	by non-negative property of $\langle \triangle \mid \triangledown\rangle$ (Definition D.9 page 182)
$\Longrightarrow$ $\mathbf{I}$ is positive	
$\langle \mathbf{A}^*\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle$	by definition of $\mathbf{I}$ (Definition D.3 page 170)
≥ 0	by two previous results

✏

Definition D.16. [54] *Let* $\mathbf{A}, \mathbf{B} \in \mathscr{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be* BOUNDED *operators.*

DEF

$\mathbf{A} \geq \mathbf{B}$ *("*$\mathbf{A}$ *is greater than or equal to* $\mathbf{B}$*") if*
$\mathbf{A} - \mathbf{B} \geq 0$ *("(*$\mathbf{A} - \mathbf{B}$*) is positive")*

[52] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩
[53] Michel and Herget (1993) page 429
[54] Michel and Herget (1993) page 429

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APPENDIX E

CALCULUS

Definition E.1. *Let $\mathbb{R}$ be the set of real numbers, $\mathscr{B}$ the set of* BOREL SETS *on $\mathbb{R}$, and μ the standard* BOREL MEASURE *on $\mathscr{B}$. Let $\mathbb{R}^{\mathbb{R}}$ be as in Definition 1.1 page 3.*

DEF

*The **space of Lebesgue square-integrable functions** $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ (or $\boldsymbol{L}^2_{\mathbb{R}}$) is defined as*

$$\boldsymbol{L}^2_{\mathbb{R}} \triangleq \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \middle| \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \mathsf{d}\mu < \infty \right\}.$$

*The **standard inner product** $\langle \triangle \mid \triangledown \rangle$ on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\, \mathsf{dx}.$$

*The **standard norm** $\|\cdot\|$ on $\boldsymbol{L}^2_{\mathbb{R}}$ is defined as $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$*

Definition E.2. *Let $\mathsf{f}(x)$ be a* FUNCTION *in $\mathbb{R}^{\mathbb{R}}$.*

DEF

$$\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon \to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition E.1.

PRP

$$\left\{ \begin{array}{ll} (1). & \mathsf{f}(x) \text{ is CONTINUOUS and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC about a point } a} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lrcll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) & \text{(ANTI-SYMMETRIC about } a) \\ (2). & \mathsf{f}'(a) & = & 0 & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned} \mathsf{f}'(a+x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\ &= -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\ &= -\mathsf{f}(a-x) \end{aligned}$$

$$\begin{aligned} \mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\ &= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result} \end{aligned}$$

$$= 0$$

✏

Lemma E.1.

LEM $\mathsf{f}(x)$ *is* INVERTIBLE $\Longrightarrow \left\{ \dfrac{\mathsf{d}}{\mathsf{dy}}\mathsf{f}^{-1}(y) = \dfrac{1}{\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{dy}}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to 0}\frac{\mathsf{f}^{-1}(y+\varepsilon)-\mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{dy}} && \text{(Definition E.2 page 199)}\\
&= \lim_{\delta\to 0}\left.\frac{1}{\left[\dfrac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1}\\
&\triangleq \left.\frac{1}{\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}(x)}\right|_{x\triangleq\mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{dx}} && \text{(Definition E.2 page 199)}\\
&= \frac{1}{\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✏

Theorem E.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

THM $\displaystyle\int_{[0:1)^n}\mathsf{f}^{(n)}\left(\sum_{k=1}^{n}x_k\right)\mathsf{dx}_1\,\mathsf{dx}_2\cdots\mathsf{dx}_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n\in\mathbb{N}$

✎PROOF: Proof by induction:

1. Base case …proof for $n = 1$ case:

$$\begin{aligned}
\int_{[0:1)}\mathsf{f}^{(1)}(x)\,\mathsf{dx} &= \mathsf{f}(1)-\mathsf{f}(0) && \text{by } \textit{Fundamental theorem of calculus}\\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1)+(-1)^{1+0}\binom{1}{0}\mathsf{f}(0)\\
&= \sum_{k=0}^{1}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

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2. Induction step …proof that n case $\implies$ $n+1$ case:

$$\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \mathsf{d}x_1\ \mathsf{d}x_2 \cdots\ \mathsf{d}x_{n+1}$$

$$= \int_{[0:1)^n} \left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right) \mathsf{d}x_{n+1}\right] \mathsf{d}x_1\ \mathsf{d}x_2 \cdots\ \mathsf{d}x_n$$

$$= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right] \mathsf{d}x_1\ \mathsf{d}x_2 \cdots\ \mathsf{d}x_n \qquad \text{by } \textit{Fundamental theorem of calculus}$$

$$= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(1 + \sum_{k=1}^{n} x_k\right) - \mathsf{f}^{(n)}\left(0 + \sum_{k=1}^{n} x_k\right)\right] \mathsf{d}x_1\ \mathsf{d}x_2 \cdots\ \mathsf{d}x_n$$

$$= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1) - \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \text{by induction hypothesis}$$

$$= \sum_{m=1}^{m=n+1}(-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m) + \sum_{k=0}^{n}(-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \text{where } m \triangleq k+1 \implies k = m-1$$

$$= \left[\mathsf{f}(n+1) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right] + \left[(-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right]$$

$$= \mathsf{f}(n+1) + (-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\text{use } \textit{Stifel formula}}\mathsf{f}(k)$$

$$= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1) + (-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) \qquad \text{by } \textit{Stifel formula}$$

(Theorem B.1 page 147)

$$= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)$$

✏

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma E.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots$, *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example E.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\mathsf{f}(x)\mathsf{g}(x)\right] = \mathsf{f}'''(x)\mathsf{g}(x) + 3\mathsf{f}''(x)\mathsf{g}'(x) + 3\mathsf{f}'(x)\mathsf{g}''(x) + \mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem E.2 (Leibniz integration rule). [3]

[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)

[3] Flanders (1973) page 615 ⟨(1.1)⟩ Talvila (2001), Knapp (2005b) page 389 ⟨Chapter VII⟩, Protter and Morrey (2012) page 422 ⟨Leibniz Rule. Theorem 1.⟩, http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

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THM

$$\frac{\mathrm{d}}{\mathrm{dx}}\int_{\mathrm{a}(x)}^{\mathrm{b}(x)} \mathrm{g}(t)\ \mathrm{dt} = \mathrm{g}[\mathrm{b}(x)]\mathrm{b}'(x) - \mathrm{g}[\mathrm{a}(x)]\mathrm{a}'(x)$$

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APPENDIX F

CONVOLUTION

F.1 Convolution over continuous domains

F.1.1 Definition

Definition F.1. [1] *Let $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF

*The **convolution** operation $\star$ is defined as*

$$[\mathsf{f} \star \mathsf{g}](x) \triangleq \mathsf{f}(x) \star \mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{du} \qquad \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

F.1.2 Properties

Theorem F.1. [2] *Let $\star$ be the* CONVOLUTION *operatation* (Definition F.1 page 203). *Let $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

THM

$$\begin{aligned}
\mathsf{f} \star \mathsf{g} &= \mathsf{g} \star \mathsf{f} && \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)} && \text{(COMMUTATIVE)}\\
\mathsf{f} \star (\mathsf{g} \star \mathsf{h}) &= (\mathsf{f} \star \mathsf{g}) \star \mathsf{h} && \forall \mathsf{f},\mathsf{gh}\in L^2_{(\mathbb{R},\mathscr{B},\mu)} && \text{(ASSOCIATIVE)}\\
(\alpha\mathsf{f}) \star \mathsf{g} &= \alpha(\mathsf{f} \star \mathsf{g}) = \mathsf{f} \star (\alpha\mathsf{g}) && \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)},\ \alpha\in\mathbb{C} && \text{(HOMOGENEOUS)}\\
\mathsf{f} \star (\mathsf{g} + \mathsf{h}) &= (\mathsf{f} \star \mathsf{g}) + (\mathsf{f} \star \mathsf{h}) && \forall \mathsf{f},\mathsf{g},\mathsf{h}\in L^2_{(\mathbb{R},\mathscr{B},\mu)} && \text{(DISTRIBUTIVE)}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{f} \star \mathsf{g} &\triangleq \int_{u=-\infty}^{u=\infty} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{du} && \text{by definition of } \star && \text{(Definition F.1 page 203)}\\
&\triangleq \int_{x-v=-\infty}^{x-v=\infty} \mathsf{f}(x-v)\mathsf{g}(v)(-1)\,\mathsf{dv} && \text{where } v \triangleq x-u && \implies u = x-v,\ \mathsf{du} = -\,\mathsf{dv}\\
&= -\int_{-v=-\infty}^{-v=\infty} \mathsf{f}(x-v)\mathsf{g}(v)\,\mathsf{dv}
\end{aligned}$$

[1] Bachman et al. (2002) page 268 ⟨Definition 5.2.1, but with $1/2\pi$ scaling factor⟩, Bachman (1964) page 6, Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩

[2] Bachman et al. (2002) pages 268–270, Schatzman (2002) page 147 ⟨7.2.1 Convolution⟩

$$= -\int_{v=\infty}^{v=-\infty} \mathsf{f}(x-v)\mathsf{g}(v)\,\mathsf{dv}$$
$$= \int_{v=-\infty}^{v=\infty} \mathsf{g}(v)\mathsf{f}(x-v)\,\mathsf{dv}$$
$$\triangleq \mathsf{g}\star\mathsf{f} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$

$$\mathsf{f}\star(\mathsf{g}\star\mathsf{h}) \triangleq \mathsf{f}\star\int_{u\in\mathbb{R}} \mathsf{g}(u)\mathsf{h}(x-u)\,\mathsf{du} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$
$$\triangleq \int_{v\in\mathbb{R}} \mathsf{f}(v)\int_{u\in\mathbb{R}} \mathsf{g}(u)\mathsf{h}(x-v-u)\,\mathsf{du}\,\mathsf{dv} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$
$$= \int_{v\in\mathbb{R}}\int_{(w-v)\in\mathbb{R}} \mathsf{f}(v)\mathsf{g}(w-v)\mathsf{h}(x-w)\,\mathsf{d}\omega\,\mathsf{dv} \qquad \text{where } w\triangleq u+v \quad \Longrightarrow u=w-v$$
$$= \int_{w\in\mathbb{R}} \underbrace{\left[\int_{v\in\mathbb{R}} \mathsf{f}(v)\mathsf{g}(w-v)\,\mathsf{dv}\right]}_{\text{function of } w}\mathsf{h}(x-w)\,\mathsf{d}\omega$$
$$\triangleq \left[\int_{v\in\mathbb{R}} \mathsf{f}(v)\mathsf{g}(w-v)\,\mathsf{dv}\right]\star\mathsf{h} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$
$$\triangleq \left[\mathsf{f}\star\mathsf{g}\right]\star\mathsf{h} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$

$$[\alpha\mathsf{f}]\star\mathsf{g} \triangleq \int_{\mathbb{R}} [\alpha\mathsf{f}(u)]\mathsf{g}(x-u)\,\mathsf{du} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$
$$= \alpha\int_{\mathbb{R}} [\alpha\mathsf{f}(u)]\mathsf{g}(x-u)\,\mathsf{du} \qquad \text{by } \textit{homogeneous} \text{ property of } \int \mathsf{dt} \text{ operator}$$
$$= \boxed{\alpha\left[\mathsf{f}\star\mathsf{g}\right]} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$
$$\triangleq \alpha\left[\int_{\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{du}\right] \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$
$$\triangleq \int_{\mathbb{R}} \mathsf{f}(u)\left[\alpha\mathsf{g}(x-u)\right]\mathsf{du} \qquad \text{by } \textit{homogeneous} \text{ property of } \int \mathsf{dt} \text{ operator}$$
$$\triangleq \boxed{\mathsf{f}\star\left[\alpha\mathsf{g}\right]} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$

$$\mathsf{f}\star(\mathsf{g}+\mathsf{h}) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\mathsf{g}(x-u)+\mathsf{h}(x-u)\right]\mathsf{du} \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$
$$= \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{du} + \int_{u\in\mathbb{R}} \mathsf{h}(u)\mathsf{g}(x-u)\,\mathsf{du} \qquad \text{by } \textit{additive} \text{ property of } \int \mathsf{dt} \text{ operator}$$
$$= (\mathsf{f}\star\mathsf{g})+(\mathsf{f}\star\mathsf{h}) \qquad \text{by definition of } \star \quad \text{(Definition F.1 page 203)}$$

✏

F.2 Convolution over discrete domains

Definition F.2. [3] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF *A function f in X^Y is a* ***sequence*** *over X if $Y=\mathbb{Z}$.*
A sequence may be denoted in the form $(\!(x_n)\!)_{n\in\mathbb{Z}}$ or simply as $(\!(x_n)\!)$.

[3] Bromwich (1908) page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31

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Definition F.3. [4] *Let* $(\mathbb{F},+,\cdot,0,1)$ *be a* FIELD *(Definition A.5 page 144).*

DEF

The ***space of all absolutely square summable sequences*** $\ell^2_{\mathbb{F}}$ *over* $\mathbb{F}$ *is defined as*

$$\ell^2_{\mathbb{F}} \triangleq \left\{ (x_n)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}$$

The space $\ell^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\ell^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\ell^2_{\mathbb{R}}$ is isomorphic to $L^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition F.4. *Let* $(x_n)_{n\in\mathbb{Z}}$ *and* $(y_n)_{n\in\mathbb{Z}}$ *be sequences (Definition F.2 page 204) in the space* $\ell^2_{\mathbb{R}}$ *(Definition F.3 page 205).*

DEF

The ***convolution*** $\star$ *of sequences* (x_n) *and* (y_n) *is defined as*

$$((x_n)\star(y_n))_{m\in\mathbb{Z}} \triangleq \left(\sum_{n\in\mathbb{Z}} x_n y_{m-n}\right)_{m\in\mathbb{Z}} \qquad \forall (x_n)_{n\in\mathbb{Z}},(y_n)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}$$

Theorem F.2. *Let* $\star$ *be the discrete* CONVOLUTION *operatation (Definition F.4 page 205).*

THM

$$\begin{aligned}
(x_n)\star(y_n) &= (y_n)\star(x_n) && \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}} && \text{(COMMUTATIVE)}\\
(x_n)\star\left[(y_n)\star(z_n)\right] &= \left[(x_n)\star(y_n)\right]\star(z_n) && \forall (x_n),(y_n)(z_n)\in\ell^2_{\mathbb{R}} && \text{(ASSOCIATIVE)}\\
\left[\alpha(x_n)\right]\star(y_n) &= \alpha\left[(x_n)\star(y_n)\right] = (x_n)\star\left[\alpha(y_n)\right] && \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}},\alpha\in\mathbb{C} && \text{(HOMOGENEOUS)}\\
(x_n)\star\left[(y_n)+(z_n)\right] &= \left[(x_n)\star(y_n)\right]+\left[(x_n)\star(z_n)\right] && \forall (x_n),(y_n),(z_n)\in\ell^2_{\mathbb{R}} && \text{(DISTRIBUTIVE)}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
((x_n)\star(y_n))_{m\in\mathbb{Z}} &\triangleq \left(\sum_{n=-\infty}^{n=\infty}(x_n)(y_{m-n})\right)_{m\in\mathbb{Z}} && \text{by definition of } \star && \text{(Definition F.4 page 205)}\\
&\triangleq \left(\sum_{m-k=-\infty}^{m-k=\infty}(x_{m-k})(y_k)\right)_{m\in\mathbb{Z}} && \text{where } k\triangleq m-n && \implies n=m-k\\
&= \left(\sum_{-k=-\infty}^{-k=\infty}(y_k)(x_{m-k})\right)_{m\in\mathbb{Z}}\\
&= \left(\sum_{k=-\infty}^{k=\infty}(y_k)(x_{m-k})\right)_{m\in\mathbb{Z}}\\
&\triangleq ((y_n)\star(x_n))_{m\in\mathbb{Z}} && \text{by definition of } \star && \text{(Definition F.4 page 205)}
\end{aligned}$$

$$\begin{aligned}
&(x_n)\star\left[(y_n)\star(z_n)\right]\\
&\triangleq (x_n)\star\left(\sum_{n\in\mathbb{Z}}(y_n)(z_{m-n})\right)_{m\in\mathbb{Z}} && \text{by definition of } \star && \text{(Definition F.4 page 205)}\\
&\triangleq \left(\sum_{k\in\mathbb{Z}}(x_k)\sum_{n\in\mathbb{Z}}(y_n)(z_{m-k-n})\right)_{m\in\mathbb{Z}} && \text{by definition of } \star && \text{(Definition F.4 page 205)}\\
&= \left(\sum_{k\in\mathbb{Z}}(x_k)\sum_{p-k\in\mathbb{Z}}(y_{p-k})(z_{m-p})\right)_{m\in\mathbb{Z}} && \text{where } p\triangleq k+n && \implies n=p-k
\end{aligned}$$

[4] Kubrusly (2011) page 347 ⟨Example 5.K⟩

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$$= \left(\sum_{k\in\mathbb{Z}} \sum_{p-k\in\mathbb{Z}} (\!(x_k)\!)\,(\!(y_{p-k})\!)\,(\!(z_{m-p})\!) \right)_{m\in\mathbb{Z}}$$

$$= \left(\sum_{p\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} (\!(x_k)\!)\,(\!(y_{p-k})\!)\,(\!(z_{m-p})\!) \right)_{m\in\mathbb{Z}}$$

$$= \sum_{p\in\mathbb{Z}} (\!((x_k) \star (y_k))\!)_p\,(\!(z_{m-p})\!) \qquad \text{by definition of } \star \qquad \text{(Definition F.4 page 205)}$$

$$= \sum_{p\in\mathbb{Z}} (\!((x_n) \star (y_n))\!)_p\,(\!(z_{m-p})\!) \qquad \text{by change of variable} \qquad k \to n$$

$$= \left[(\!(x_n)\!) \star (\!(y_n)\!)\right] \star (\!(z_n)\!) \qquad \text{by definition of } \star \qquad \text{(Definition F.4 page 205)}$$

$$\begin{aligned}
\left[\alpha\,(\!(x_n)\!)\right] \star (\!(y_n)\!) &\triangleq \left(\sum_{n\in\mathbb{Z}} \left[\alpha\,(\!(x_n)\!)\right] (\!(y_{m-n})\!) \right)_{m\in\mathbb{Z}} && \text{by definition of } \star \quad \text{(Definition F.4 page 205)}\\
&= \alpha \left(\sum_{n\in\mathbb{Z}} \left[\alpha\,(\!(x_n)\!)\right] (\!(y_{m-n})\!) \right)_{m\in\mathbb{Z}} && \text{by } \textit{homogeneous} \text{ property of } \sum \text{ operator}\\
&= \boxed{\alpha\left[(\!(x_n)\!) \star (\!(y_n)\!)\right]} && \text{by definition of } \star \quad \text{(Definition F.4 page 205)}\\
&\triangleq \alpha \left(\sum_{n\in\mathbb{Z}} (\!(x_n)\!)\,(\!(y_{m-n})\!) \right)_{m\in\mathbb{Z}} && \text{by definition of } \star \quad \text{(Definition F.4 page 205)}\\
&\triangleq \left(\sum_{n\in\mathbb{Z}} (\!(x_n)\!)\,\left[\alpha\,(\!(y_{m-n})\!)\right] \right)_{m\in\mathbb{Z}} && \text{by } \textit{homogeneous} \text{ property of } \sum \text{ operator}\\
&\triangleq \boxed{(\!(x_n)\!) \star \left[\alpha\,(\!(y_n)\!)\right]} && \text{by definition of } \star \quad \text{(Definition F.4 page 205)}
\end{aligned}$$

$$\begin{aligned}
(\!(x_n)\!) \star \left[(\!(y_n)\!) + (\!(z_n)\!)\right] &\triangleq \sum_{n\in\mathbb{Z}} (\!(x_n)\!)\left[(\!(y_{m-n})\!) + (\!(z_{m-n})\!)\right] && \text{by definition of } \star \quad \text{(Definition F.4 page 205)}\\
&= \sum_{n\in\mathbb{Z}} (\!(x_n)\!)\,(\!(y_{m-n})\!) + \sum_{n\in\mathbb{Z}} (\!(z_n)\!)\,(\!(y_{m-n})\!) && \text{by } \textit{additive} \text{ property of } \sum \text{ operator}\\
&= \left[(\!(x_n)\!) \star (\!(y_n)\!)\right] + \left[(\!(x_n)\!) \star (\!(z_n)\!)\right] && \text{by definition of } \star \quad \text{(Definition F.4 page 205)}
\end{aligned}$$

✏

Proposition F.1. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition F.4 page 205). Let $\ell^2_{\mathbb{R}}$ be the set of* ABSOLUTELY SUMMABLE *sequences (Definition F.3 page 205).*

PRP

$$\left\{ \begin{array}{lll} \text{(A).} & \mathsf{x}(n) \in \ell^2_{\mathbb{R}} & \textit{and} \\ \text{(B).} & \mathsf{y}(n) \in \ell^2_{\mathbb{R}} & \end{array} \right\} \Longrightarrow \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] = \mathsf{x}[-n] \star \mathsf{y}(n) \right\}$$

✎PROOF:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \Longrightarrow k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by } \textit{absolutely summable} \text{ hypothesis} && \text{(Definition F.3 page 205)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \Longrightarrow \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of } \textit{convolution} \star && \text{(Definition F.4 page 205)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n]
\end{aligned}$$

✏

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F.3 References

1. History of convolution: Dominguez-Torres (2015)
2. Doetsch (1958)
3. Dominguez-Torres (2010)
4. Dimovski (2012)

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APPENDIX G

FINITE SUMS

❝ I think that it was Harald Bohr who remarked to me that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove." ❞

G.H. Hardy (1877–1947) in his "Presidential Address" to the London Mathematical Society on November 8, 1928, about a remark that he suggested was from Harald Bohr (1887–1951), Danish mathematician pictured to the left. [1]

G.1 Summation

Definition G.1. [2] *Let + be an addition operator on a tuple* $(\!| x_n |\!)_m^N$.

DEF

The ***summation*** *of* $(\!| x_n |\!)$ *from index* m *to index* N *with respect to* $+$ *is*

$$\sum_{n=m}^{N} x_n \triangleq \begin{cases} 0 & \text{for } N < m \\ \left(\sum_{n=m}^{N-1} x_n\right) + x_N & \text{for } N \geq m \end{cases}$$

Theorem G.1 (Generalized associative property)**.** [3] *Let + be an addition operator on a tuple* $(\!| x_n |\!)_m^N$.

THM

$$\underbrace{+ \text{ is ASSOCIATIVE} \implies \sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right) = \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n \quad \text{for } m < L < M \leq N}_{\sum_{n=m}^{N} \text{ is ASSOCIATIVE}}$$

---

[1] quote: Hardy (1929) page 64
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Bohr_Harald.html

[2] reference: Berberian (1961) page 8 ⟨Definition I.3.1⟩
"∑" notation: Fourier (1820) page 280

[3] Berberian (1961) pages 9–10 ⟨Theorem I.3.1⟩

✎PROOF:

1. Proof for $N < m$ case: $\sum_{n=m}^{N} x_n = 0$.

2. Proof for $N = m$ case: $\sum_{n=m}^{m} x_n = \left(\sum_{n=m}^{m-1} x_n\right) + x_m = 0 + x_m = x_m$.

3. Proof for $N = m+1$ case: $\sum_{n=m}^{m+1} x_n = \left(\sum_{n=m}^{m} x_n\right) + x_{m+1} = x_m + x_{m+1}$

4. Proof for $N = m+2$ case:

$$\begin{aligned}
\sum_{n=m}^{m+2} x_n &= \left(\sum_{n=m}^{m+1} x_n\right) + x_{m+2} && \text{by Definition G.1 page 209}\\
&= \left(x_m + x_{m+1}\right) + x_{m+2} && \text{by item (3)}\\
&= x_m + \left(x_{m+1} + x_{m+2}\right) && \text{by left hypothesis}
\end{aligned}$$

5. Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\sum_{n=m}^{N+1} x_n &= \underbrace{\left(\sum_{n=m}^{N} x_n\right)}_{\textit{associative}} + x_{N+1} && \text{by Definition G.1 page 209}\\
&= \left(\sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right)\right) + x_{N+1} && = \left(\left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n\right) + x_{N+1}\\
&= \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N} x_n + x_{N+1}\right) && = \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N+1} x_n\right)
\end{aligned}$$

✏

G.2 Means

G.2.1 Weighted ϕ-means

Definition G.2. [4]

DEF

*The $(\!|\lambda_n|\!)_1^N$ weighted ϕ-**mean** of a tuple $(\!|x_n|\!)_1^N$ is defined as*

$$\mathsf{M}_\phi\left((\!|x_n|\!)\right) \triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right)$$

where ϕ is a CONTINUOUS *and* STRICTLY MONOTONIC *function in $\mathbb{R}^{\mathbb{R}^{\vdash}}$ and $(\!|\lambda_n|\!)_{n=1}^N$ is a sequence of weights for which $\sum_{n=1}^{N} \lambda_n = 1$.*

Lemma G.1. [5] *Let $\mathsf{M}_\phi\left((\!|x_n|\!)\right)$ be the $(\!|\lambda_n|\!)_1^N$ weighted ϕ-mean of a tuple $(\!|x_n|\!)_1^N$.*

[4] Bollobás (1999) page 5

[5] Pečarić et al. (1992) page 107, Bollobás (1999) page 5, Hardy et al. (1952) page 75

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LEM

$$\begin{array}{llll}
\phi\psi^{-1} \textit{ is } \textsc{convex} & \textit{and} & \phi \textit{ is } \textsc{increasing} & \Longrightarrow \mathsf{M}_\phi\big(\llparenthesis x_n \rrparenthesis\big) \geq \mathsf{M}_\psi\big(\llparenthesis x_n \rrparenthesis\big) \\
\phi\psi^{-1} \textit{ is } \textsc{convex} & \textit{and} & \phi \textit{ is } \textsc{decreasing} & \Longrightarrow \mathsf{M}_\phi\big(\llparenthesis x_n \rrparenthesis\big) \leq \mathsf{M}_\psi\big(\llparenthesis x_n \rrparenthesis\big) \\
\phi\psi^{-1} \textit{ is } \textsc{concave} & \textit{and} & \phi \textit{ is } \textsc{increasing} & \Longrightarrow \mathsf{M}_\phi\big(\llparenthesis x_n \rrparenthesis\big) \leq \mathsf{M}_\psi\big(\llparenthesis x_n \rrparenthesis\big) \\
\phi\psi^{-1} \textit{ is } \textsc{concave} & \textit{and} & \phi \textit{ is } \textsc{decreasing} & \Longrightarrow \mathsf{M}_\phi\big(\llparenthesis x_n \rrparenthesis\big) \geq \mathsf{M}_\psi\big(\llparenthesis x_n \rrparenthesis\big)
\end{array}$$

✎PROOF:

1. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *increasing*:

$$\begin{aligned}
\mathsf{M}_\phi\big(\llparenthesis x_n \rrparenthesis\big) &\triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi && \text{(Definition G.2 page 210)} \\
&= \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1} \\
&\geq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^{N} \lambda_n \psi(x_n)\right) && \text{by } \textit{Jensen's Inequality} \\
&= \left(\psi^{-1}\sum_{n=1}^{N} \lambda_n \psi(x_n)\right) && \text{by definition of } \psi^{-1} \\
&\triangleq \mathsf{M}_\psi\big(\llparenthesis x_n \rrparenthesis\big) && \text{by definition of } \mathsf{M}_\psi && \text{(Definition G.2 page 210)}
\end{aligned}$$

2. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *decreasing*:

$$\begin{aligned}
\mathsf{M}_\phi\big(\llparenthesis x_n \rrparenthesis\big) &\triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi && \text{(Definition G.2 page 210)} \\
&= \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1} \\
&\leq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^{N} \lambda_n \psi(x_n)\right) && \text{by } \textit{Jensen's Inequality} \text{ and because } \phi^{-1} \text{ is } \textit{decreasing} && \text{(by hypothesis)} \\
&= \left(\psi^{-1}\sum_{n=1}^{N} \lambda_n \psi(x_n)\right) && \text{by definition of } \psi^{-1} \\
&\triangleq \mathsf{M}_\psi\big(\llparenthesis x_n \rrparenthesis\big) && \text{by definition of } \mathsf{M}_\psi && \text{(Definition G.2 page 210)}
\end{aligned}$$

✏

One of the most well known inequalities in mathematics is *Minkowski's Inequality* (Theorem G.5 page 217). In 1946, H.P. Mulholland submitted a result[6] that generalizes Minkowski's Inequality to an equal weighted ϕ-mean. And Milovanović and Milovanovć (1979) generalized this even further to a *weighted* ϕ-mean (Theorem G.2, next).

Theorem G.2. [7]

THM

$$\left\{\begin{array}{lll|lll}
(1). & \phi \textit{ is } \textsc{convex} & \textit{and} & (3). & \phi(0) = 0 & \textit{and} \\
(2). & \phi \textit{ is } \textsc{strictly monotonic} & \textit{and} & (4). & \log\circ\phi\circ\exp \textit{ is } \textsc{convex} &
\end{array}\right\} \Longrightarrow$$

$$\left\{\phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n + y_n)\right) \leq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right) + \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(y_n)\right)\right\}$$

[6] Mulholland (1950)

[7] Milovanović and Milovanović (1979), Bullen (2003) page 306 ⟨Theorem 9⟩

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G.2.2 Power means

Definition G.3. [8] *Let* $\mathsf{M}_{\phi(x;r)}\big(\langle x_n\rangle\big)$ *be the* $\langle\lambda_n\rangle_1^N$ *weighted* ϕ*-mean of a* NON-NEGATIVE *tuple* $\langle x_n\rangle_1^N$ *(Definition G.2 page 210).*

DEF

A mean $\mathsf{M}_{\phi(x;r)}\big(\langle x_n\rangle\big)$ *is a* ***power mean*** *with parameter* r *if* $\phi(x) \triangleq x^r$*. That is,*

$$\mathsf{M}_{\phi(x;r)}\big(\langle x_n\rangle\big) = \left(\sum_{n=1}^{N} \lambda_n (x_n)^r\right)^{\frac{1}{r}}$$

Theorem G.3. [9] *Let* $\mathsf{M}_{\phi(x;r)}\big(\langle x_n\rangle\big)$ *be* POWER MEAN *with parameter* r *of an* N*-tuple* $\langle x_n\rangle_1^N$*. Let* $\mathbb{R}^*$ *be the set of extended real numbers* $\big(\mathbb{R}^* \triangleq \mathbb{R}\cup\{-\infty,\ \infty\}\big)$.[10]

THM

$$\mathsf{M}_{\phi(x;r)}\big(\langle x_n\rangle\big) \triangleq \left(\sum_{n=1}^{N} \lambda_n (x_n)^r\right)^{\frac{1}{r}} \quad \textit{is} \text{ CONTINUOUS } \textit{and} \text{ STRICTLY INCREASING } \textit{in } \mathbb{R}^*.$$

$$\mathsf{M}_{\phi(x;r)}\big(\langle x_n\rangle\big) = \begin{cases} \min\limits_{n=1,2,\ldots,N} \langle x_n\rangle & \textit{for } r=-\infty \\ \prod\limits_{n=1}^{N} x_n^{\lambda_n} & \textit{for } r=0 \\ \max\limits_{n=1,2,\ldots,N} \langle x_n\rangle & \textit{for } r=+\infty \end{cases}$$

✎PROOF:

1. Proof that $M_{\phi(x;r)}$ is *strictly increasing* in r:
   (a) Let r and s be such that $-\infty < r < s < \infty$.
   (b) Let $\phi_r \triangleq x^r$ and $\phi_s \triangleq x^s$. Then $\phi_r\phi_s^{-1} = x^{\frac{r}{s}}$.
   (c) The composite function $\phi_r\phi_s^{-1}$ is *convex* or *concave* depending on the values of r and s:

	$r<0$ (ϕ_r decreasing)	$r>0$ (ϕ_r increasing)
$s<0$	*convex*	(not possible)
$s>0$	*convex*	*concave*

   (d) Therefore by Lemma G.1 (page 210),
   $$-\infty < r < s < \infty \quad\Longrightarrow\quad \mathsf{M}_{\phi(x;r)}\big(\langle x_n\rangle\big) < \mathsf{M}_{\phi(x;s)}\big(\langle x_n\rangle\big).$$
2. Proof that $\mathsf{M}_{\phi(x;r)}$ is continuous in r for $r\in\mathbb{R}\backslash 0$: The sum of continuous functions is continuous. For the cases of $r\in\{-\infty,\ 0,\ \infty\}$, see the items that follow.
3. Lemma: $\mathsf{M}_{\phi(x;-r)}\big(\langle x_n\rangle\big) = \big\{\mathsf{M}_{\phi(x;r)}\big(\langle x_n^{-1}\rangle\big)\big\}^{-1}$. Proof:

$$\begin{aligned}
\big\{\mathsf{M}_{\phi(x;r)}\big(\langle x_n^{-1}\rangle\big)\big\}^{-1} &= \left\{\left(\sum_{n=1}^{N} \lambda_n (x_n^{-1})^r\right)^{\frac{1}{r}}\right\}^{-1} && \text{by definition of } \mathsf{M}_\phi \\
&= \left(\sum_{n=1}^{N} \lambda_n (x_n)^{-r}\right)^{\frac{1}{-r}} \\
&= \mathsf{M}_{\phi(x;-r)}\big(\langle x_n\rangle\big) && \text{by definition of } \mathsf{M}_\phi
\end{aligned}$$

[8] Bullen (2003) page 175, Bollobás (1999) page 6
[9] Bullen (2003) pages 175–177 ⟨see also page 203⟩, Bollobás (1999) pages 6–8, Besso (1879), Bienaymé (1840) page 68
[10] Rana (2002) pages 385–388 ⟨Appendix A⟩

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4. Proof that $\lim_{r\to\infty} \mathsf{M}_\phi\big(\langle\!\langle x_n \rangle\!\rangle\big) = \max_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle$:

   (a) Let $x_m \triangleq \max_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle$

   (b) Note that $\lim_{r\to\infty} \mathsf{M}_\phi \le \max_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle$ because

$$\begin{aligned}
\lim_{r\to\infty} \mathsf{M}_\phi\big(\langle\!\langle x_n \rangle\!\rangle\big) &= \lim_{r\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi \\
&\le \lim_{r\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x) \triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing} \\
&= \lim_{r\to\infty} \left(x_m^r \underbrace{\sum_{n=1}^{N} \lambda_n}_{1}\right)^{\frac{1}{r}} && \text{because } x_m \text{ is a constant} \\
&= \lim_{r\to\infty} \left(x_m^r \cdot 1\right)^{\frac{1}{r}} \\
&= x_m \\
&= \max_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}$$

   (c) But also note that $\lim_{r\to\infty} \mathsf{M}_\phi \ge \max_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle$ because

$$\begin{aligned}
\lim_{r\to\infty} \mathsf{M}_\phi\big(\langle\!\langle x_n \rangle\!\rangle\big) &= \lim_{r\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi \\
&\ge \lim_{r\to\infty} \left(w_m x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x) \triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing} \\
&= \lim_{r\to\infty} w_m^{\frac{1}{r}} x_m^{\frac{r}{r}} \\
&= x_m \\
&= \max_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}$$

   (d) Combining items (b) and (c) we have $\lim_{r\to\infty} \mathsf{M}_\phi = \max_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle$.

5. Proof that $\lim_{r\to-\infty} \mathsf{M}_\phi\big(\langle\!\langle x_n \rangle\!\rangle\big) = \min_{n\in\mathbb{Z}} \langle\!\langle x_n \rangle\!\rangle$:

$$\begin{aligned}
\lim_{r\to-\infty} \mathsf{M}_{\phi(x;r)}\big(\langle\!\langle x_n \rangle\!\rangle\big) &= \lim_{r\to\infty} \mathsf{M}_{\phi(x;-r)}\big(\langle\!\langle x_n \rangle\!\rangle\big) && \text{by change of variable } r \\
&= \lim_{r\to\infty} \left\{\mathsf{M}_{\phi(x;r)}\big(\langle\!\langle x_n^{-1} \rangle\!\rangle\big)\right\}^{-1} && \text{by Lemma in item (3) page 212} \\
&= \lim_{r\to\infty} \frac{1}{\mathsf{M}_{\phi(x;r)}\big(\langle\!\langle x_n^{-1} \rangle\!\rangle\big)} \\
&= \frac{\lim_{r\to\infty} 1}{\lim_{r\to\infty} \mathsf{M}_{\phi(x;r)}\big(\langle\!\langle x_n^{-1} \rangle\!\rangle\big)} && \text{by property of lim [11]} \\
&= \frac{1}{\max_{n\in\mathbb{Z}} \langle\!\langle x_n^{-1} \rangle\!\rangle} && \text{by item (4)}
\end{aligned}$$

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$$= \frac{1}{\left(\min\limits_{n\in\mathbb{Z}} (\!| x_n |\!)\right)^{-1}}$$
$$= \min_{n\in\mathbb{Z}} (\!| x_n |\!)$$

6. Proof that $\lim\limits_{r\to 0} \mathsf{M}_\phi\left((\!| x_n |\!)\right) = \prod\limits_{n=1}^{N} x_n^{\lambda_n}$:

$$\lim_{r\to 0} \mathsf{M}_\phi\left((\!| x_n |\!)\right) = \lim_{r\to 0} \exp\left\{\ln\left\{\mathsf{M}_\phi\left((\!| x_n |\!)\right)\right\}\right\}$$

$$= \lim_{r\to 0} \exp\left\{\ln\left\{\left(\sum_{n=1}^{N} \lambda_n \left(x_n^r\right)\right)^{\frac{1}{r}}\right\}\right\} \qquad \text{by definition of } \mathsf{M}_\phi$$

$$= \exp\left\{\frac{\frac{\partial}{\partial r}\ln\left(\sum\limits_{n=1}^{N} \lambda_n \left(x_n^r\right)\right)}{\frac{\partial}{\partial r} r}\right\}_{r=0} \qquad \text{by l'Hôpital's rule}^{12}$$

$$= \exp\left\{\frac{\sum\limits_{n=1}^{N} \lambda_n \frac{\partial}{\partial r}\left(x_n^r\right)}{\sum\limits_{n=1}^{N} \lambda_n \left(x_n^r\right)}\right\}_{r=0} \qquad = \exp\left\{\frac{\sum\limits_{n=1}^{N} \lambda_n \frac{\partial}{\partial r}\exp\left(\ln\left(x_n^r\right)\right)}{\sum\limits_{n=1}^{N} \lambda_n}\right\}_{r=0}$$

$$= \exp\left\{\frac{\sum\limits_{n=1}^{N} \lambda_n \frac{\partial}{\partial r}\exp\left(r\ln\left(x_n\right)\right)}{1}\right\}_{r=0} \qquad = \exp\left\{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r}\exp\left(r\ln\left(x_n\right)\right)\right\}_{r=0}$$

$$= \exp\left\{\sum_{n=1}^{N} \lambda_n \exp\left\{r\ln x_n\right\}\ln\left(x_n\right)\right\}_{r=0} \qquad = \exp\left\{\sum_{n=1}^{N} \lambda_n \ln\left(x_n\right)\right\}$$

$$= \exp\left\{\sum_{n=1}^{N} \ln\left(x_n^{\lambda_n}\right)\right\} \qquad = \exp\left\{\ln\prod_{n=1}^{N} x_n^{\lambda_n}\right\} = \prod_{n=1}^{N} x_n^{\lambda_n}$$

✏

Definition G.4. *Let $(\!| x_n |\!)_1^N$ be a tuple. Let $(\!| \lambda_n |\!)_1^N$ be a tuple of weighting values.*

[11] Rudin (1976) page 85 ⟨4.4 Theorem⟩
[12] Rudin (1976) page 109 ⟨5.13 Theorem⟩

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DEF

*The **harmonic mean** of* $(\!| x_n |\!)$ *is defined as* $\mu_h \triangleq \left(\sum_{n=1}^{N} \lambda_n \frac{1}{x_n}\right)^{-1}$ *where* $\sum_{n=1}^{N} \lambda_n = 1$

*The **geometric mean** of* $(\!| x_n |\!)$ *is defined as* $\mu_g \triangleq \sum_{n=1}^{N} x_n^{\lambda_n}$ *where* $\sum_{n=1}^{N} \lambda_n = 1$

*The **arithmetic mean** of* $(\!| x_n |\!)$ *is defined as* $\mu_a \triangleq \sum_{n=1}^{N} \lambda_n x_n$ *where* $\sum_{n=1}^{N} \lambda_n = 1$

*The **average** of* $(\!| x_n |\!)$ *is defined as* $\mu_a \triangleq \frac{1}{N} \sum_{n=1}^{N} x_n$

G.3 Inequalities on power means

Corollary G.1. [13] *Let* $(\!| x_n |\!)_1^N$ *be a tuple. Let* $(\!| \lambda_n |\!)_1^N$ *be a tuple of weighting values.*

COR

$$\min (\!| x_n |\!) \leq \underbrace{\left(\sum_{n=1}^{N} \lambda_n \frac{1}{x_n}\right)^{-1}}_{\text{harmonic mean}} \leq \underbrace{\prod_{n=1}^{N} x_n^{\lambda_n}}_{\text{geometric mean}} \leq \underbrace{\sum_{n=1}^{N} \lambda_n x_n}_{\text{arithmetic mean}} \leq \max (\!| x_n |\!) \quad \text{where} \quad \sum_{n=1}^{N} \lambda_n = 1$$

✎PROOF:

1. These five means are all special cases of the *power mean* $\mathsf{M}_{\phi(x:r)}$ (Definition G.3 page 212):
   $r = \infty$: $\max (\!| x_n |\!)$
   $r = 1$: arithmetic mean
   $r = 0$: geometric mean
   $r = -1$: harmonic mean
   $r = -\infty$: $\min (\!| x_n |\!)$
2. The inequalities follow directly from Theorem G.3 (page 212).
3. Generalized AM-GM inequality: If one is only concerned with the arithmetic mean and geometric mean, their relationship can be established directly using *Jensen's Inequality*:

$$\sum_{n=1}^{N} \lambda_n x_n = b^{\log_b\left(\sum_{n=1}^{N} \lambda_n x_n\right)} \geq b^{\left(\sum_{n=1}^{N} \lambda_n \log_b x_n\right)} \quad \text{by } \textit{Jensen's Inequality}$$

$$= \prod_{n=1}^{N} b^{(\lambda_n \log_b x_n)} = \prod_{n=1}^{N} b^{(\log_b x_n)\lambda_n} = \prod_{n=1}^{N} x_n^{\lambda_n}$$

✏

Lemma G.2 (Young's Inequality). [14]

[13] Bullen (2003) page 71, Bollobás (1999) page 5, Cauchy (1821) pages 457–459 ⟨Note II, theorem 17⟩, Jensen (1906) page 183

[14] Carothers (2000) page 43, Tolsted (1964) page 5, Maligranda (1995) page 257, Hardy et al. (1952) ⟨Theorem 24⟩, Young (1912) page 226

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LEM

$$xy < \frac{x^p}{p} + \frac{y^q}{q} \quad \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \text{but } y \neq x^{p-1}$$

$$xy = \frac{x^p}{p} + \frac{y^q}{q} \quad \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \text{and } y = x^{p-1}$$

✎PROOF:

1. Proof that $\frac{1}{p-1} = q - 1$:

$$\begin{aligned}
\frac{1}{p} + \frac{1}{q} = 1 &\iff \frac{q}{q} + \frac{q}{p} = q \\
&\iff q\left(1 - \frac{1}{p}\right) = 1 \\
&\iff q = \frac{1}{1 - \frac{1}{p}} \\
&\iff q = \frac{p}{p-1} \\
&\iff q - 1 = \frac{p}{p-1} - \frac{p-1}{p-1} \\
&\iff q - 1 = \frac{p - (p-1)}{p-1} \\
&\iff q - 1 = \frac{1}{p-1}
\end{aligned}$$

2. Proof that $v = u^{p-1} \iff u = v^{q-1}$:

$$\begin{aligned}
u &= v^{\frac{1}{p-1}} && \text{by left hypothesis} \\
&= v^{q-1} && \text{by item (1)}
\end{aligned}$$

3. Proof that $v = u^{p-1}$ is propemonotonically increasing in u and $u = v^{q-1}$ is propemonotonically increasing in v:

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u} u^{p-1} &&= (p-1)u^{p-2} &&> 0 \\
\frac{\mathrm{d}u}{\mathrm{d}v} &= \frac{\mathrm{d}}{\mathrm{d}v} v^{q-1} &&= (q-1)v^{q-2} &&> 0
\end{aligned}$$

4. Proof that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$:

$$\begin{aligned}
xy &\leq \int_0^x u^{p-1}\ \mathrm{du} + \int_0^y v^{q-1}\ \mathrm{dv} \\
&= \left.\frac{u^p}{p}\right|_0^x + \left.\frac{v^q}{q}\right|_0^y \\
&= \frac{x^p}{p} + \frac{y^q}{q}
\end{aligned}$$

✏

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Theorem G.4 (Hölder's Inequality). [15] *Let* $(\!| x_n \in \mathbb{C} |\!)_1^N$ *and* $(\!| y_n \in \mathbb{C} |\!)_1^N$ *be complex* N*-tuples.*

THM

$$\underbrace{\sum_{n=1}^{N} |x_n y_n|}_{\|\boldsymbol{x}\cdot\boldsymbol{y}\|_1} \le \underbrace{\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}}_{\|\boldsymbol{x}\|_p} \underbrace{\left(\sum_{n=1}^{N} |y_n|^q\right)^{\frac{1}{q}}}_{\|\boldsymbol{y}\|_q} \qquad \textit{with} \qquad \frac{1}{p}+\frac{1}{q}=1 \qquad \forall\, 1<p<\infty$$

✎PROOF: Let $\|x_n\|_p \triangleq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$.

$$\begin{aligned}
\sum_{n=1}^{N} |x_n y_n| &= \|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q \sum_{n=1}^{N} \frac{|x_n y_n|}{\|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q} \\
&= \|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q \sum_{n=1}^{N} \frac{|x_n|}{\|(\!| x_n |\!)\|_p} \frac{|y_n|}{\|(\!| y_n |\!)\|_q} \\
&\le \|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q \sum_{n=1}^{N} \left(\frac{1}{p}\frac{|x_n|^p}{\|(\!| x_n |\!)\|_p^p} + \frac{1}{q}\frac{|y_n|^q}{\|(\!| y_n |\!)\|_q^q}\right) && \text{by } \textit{Young's Inequality} \quad \text{(Lemma G.2 page 215)} \\
&= \|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q \left(\frac{1}{p}\cdot\frac{\sum |x_n|^p}{\|(\!| x_n |\!)\|_p^p} + \frac{1}{q}\cdot\frac{\sum |y_n|^q}{\|(\!| y_n |\!)\|_q^q}\right) \\
&= \|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q \left(\frac{1}{p}\frac{\|(\!| x_n |\!)\|_p^p}{\|(\!| x_n |\!)\|_p^p} + \frac{1}{q}\frac{\|(\!| y_n |\!)\|_q^q}{\|(\!| y_n |\!)\|_q^q}\right) && \text{by definition of } \|\cdot\| \\
&= \|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q \underbrace{\left(\frac{1}{p}+\frac{1}{q}\right)}_{1} \\
&= \|(\!| x_n |\!)\|_p \, \|(\!| y_n |\!)\|_q && \text{by } \frac{1}{p}+\frac{1}{q}=1 \text{ constraint}
\end{aligned}$$

✏

Theorem G.5 (Minkowski's Inequality for sequences). [16] *Let* $(\!| x_n \in \mathbb{C} |\!)_1^N$ *and* $(\!| y_n \in \mathbb{C} |\!)_1^N$ *be complex* N*-tuples.*

THM

$$\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{\frac{1}{p}} \qquad \forall\, 1<p<\infty$$

✎PROOF:

1. Define $q \triangleq \dfrac{p}{p-1}$
2. Define $\|\boldsymbol{x}\|_p \triangleq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$.

[15] Bullen (2003) page 178 ⟨2.1⟩, Carothers (2000) page 44, Tolsted (1964) page 6, Maligranda (1995) page 257, Hardy et al. (1952) ⟨Theorem 11⟩, Hölder (1889)

[16] Bullen (2003) page 179, Carothers (2000) page 44, Tolsted (1964) page 7, Maligranda (1995) page 258, Hardy et al. (1952) ⟨Theorem 24⟩, Minkowski (1910) page 115

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3. Proof that $\|x_n + y_n\|_p \le \|x_n\|_p + \|y_n\|_p$:

$$\boxed{\|x_n + y_n\|_p^p}$$

$$= \sum_{n=1}^{N} |x_n + y_n|^p \qquad \text{by definition of } \|\cdot\|_p \qquad \text{(definition 2 page 217)}$$

$$= \sum_{n=1}^{N} |x_n + y_n||x_n + y_n|^{p-1} \qquad \text{by } \textit{homogeneous} \text{ property of } |\cdot|$$

$$\boxed{\le} \sum_{n=1}^{N} |x_n||x_n + y_n|^{p-1} + \sum_{n=1}^{N} |y_n||x_n + y_n|^{p-1} \qquad \text{by } \textit{subadditive} \text{ property of } |\cdot|$$

$$= \sum_{n=1}^{N} |x_n(x_n + y_n)^{p-1}| + \sum_{n=1}^{N} |y_n(x_n + y_n)^{p-1}| \qquad \text{by } \textit{homogeneous} \text{ property of } |\cdot|$$

$$\boxed{\le} \|x_n\|_p \|(x_n + y_n)^{p-1}\|_q + \|y_n\|_p \|(x_n + y_n)^{p-1}\|_q \qquad \text{by } \textit{Hölder's Inequality} \qquad \text{(Theorem G.4 page 217)}$$

$$= \left(\|x_n\|_p + \|y_n\|_p\right) \|(x_n + y_n)^{p-1}\|_q$$

$$= \left(\|x_n\|_p + \|y_n\|_p\right)\left(\sum_{n=1}^{N} |(x_n + y_n)^{p-1}|^q\right)^{\frac{1}{q}} \qquad \text{by definition of } \|\cdot\|_p \qquad \text{(definition 2 page 217)}$$

$$= \left(\|x_n\|_p + \|y_n\|_p\right)\left(\sum_{n=1}^{N} |(x_n + y_n)^{p-1}|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \qquad \text{by definition 1}$$

$$= \left(\|x_n\|_p + \|y_n\|_p\right)\left(\sum_{n=1}^{N} |(x_n + y_n)|^p\right)^{\frac{p-1}{p}}$$

$$= \left(\|x_n\|_p + \|y_n\|_p\right) \|x_n + y_n\|^{p-1} \qquad \text{by definition of } \|\cdot\|_p \qquad \text{(definition 2 page 217)}$$

$$\Longrightarrow \quad \boxed{\|x_n + y_n\|_p \le \|x_n\|_p + \|y_n\|_p}$$

✏

"Cauchy is the only one occupied with pure mathematics: Poisson, Fourier, Ampere, etc., busy themselves exclusively with magnetism and other physical subjects. Mr. Laplace writes nothing now, I believe."

Niels Henrik Abel in an 1826 letter [17]

Theorem G.6 (Cauchy-Schwarz Inequality for sequences). [18] *Let* $(\!|x_n \in \mathbb{C}|\!)_1^N$ *and* $(\!|y_n \in \mathbb{C}|\!)_1^N$ *be complex* N*-tuples.*

[17] quote: Bell (1986) page 318 ⟨Chapter 17. "Genius and Poverty" "Abel (1802–1829)"⟩, Boyer and Merzbach (2011) page 462 ⟨without "Mr. Laplace…" portion⟩. image: http://en.wikipedia.org/wiki/File:Augustin-Louis_Cauchy_1901.jpg, public domain

[18] Aliprantis and Burkinshaw (1998) page 278, Scharz (1885), Bouniakowsky (1859), Hardy et al. (1952) page 25 ⟨Theorem 11⟩, Cauchy (1821) page 455 ⟨???⟩

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THM

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 \leq \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \qquad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$$

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 = \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \iff \exists a \in \mathbb{C} \text{ such that } \boldsymbol{y} = a\boldsymbol{x} \qquad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$$

✎PROOF:

1. The *Cauchy-Schwarz Inequality for sequences* is a special case of the *Hölder inequality* (Theorem G.4 page 217) for $p = q = 2$.

2. Alternatively, the *Cauchy-Schwarz inequality for sequences* is a special case of the *Cauchy-Schwarz inequality for inner-product spaces*:

   (a) $\langle x_n \mid y_n\rangle \triangleq \sum_{n=1}^{N} x_n y_n$ is an inner-product and $\left(\llparenthesis x_n \rrparenthesis, \langle \triangle \mid \triangledown\rangle\right)$ is an inner-product space.

   (b) By the more general *Cauchy-Schwarz Inequality for inner-product spaces*,

$$\begin{aligned}
\left(\sum_{n=1}^{N} a_n\lambda_n\right)^2 &\triangleq \langle a_n \mid \lambda_n\rangle^2 && \text{by definition of } \langle x_n \mid y_n\rangle \\
&\boxed{\leq} \|x_n\|^2 \|y_n\|^2 && \text{by } \textit{Cauchy-Schwarz Inequality for inner-product spaces} \text{ (Theorem ?? page ??)} \\
&\triangleq \left(\sum_{n=1}^{N} x_n^2\right)\left(\sum_{n=1}^{N} y_n^2\right) && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Not only does the *Hölder inequality* imply the *Cauchy-Schwarz inequality*, but somewhat surprisingly, the converse is also true: The Cauchy-Schwarz inequality implies the Hölder inequality.[19]

✏

Proposition G.1. [20]

PRP

$$(x+y)^p \leq 2^p(x^p + y^p) \qquad \forall x, y \geq 0,\ 1 < p < \infty$$

✎PROOF:

$$\begin{aligned}
(x+y)^p &\leq (2\max\{x,y\})^p \\
&= 2^p(\max\{x,y\})^p \\
&= 2^p(\max\{x^p, y^p\}) \\
&\leq 2^p(x^p + y^p)
\end{aligned}$$

✏

[19] Bullen (2003) pages 183–185 ⟨Theorem 5⟩
[20] Carothers (2000) page 43

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G.4 Power Sums

Theorem G.7 (Geometric Series). [21]

THM
$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \qquad \forall r \in \mathbb{C}\backslash 0$$

✎PROOF:

$$\boxed{\sum_{k=0}^{n-1} r^k} = \left(\frac{1}{1-r}\right)\left[(1-r)\sum_{k=0}^{n-1} r^k\right] = \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - r\sum_{k=0}^{n-1} r^k\right] = \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - \left(\sum_{k=0}^{n-1} r^k - 1 + r^n\right)\right]$$
$$= \left(\frac{1}{1-r}\right)\left[1-r^n\right] = \boxed{\frac{1-r^n}{1-r}}$$

✏

Lemma G.3. *Let* $\mathsf{f}(x)$ *be a function.*

LEM
$$\mathsf{S}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) = \mathsf{S}(x+\tau) \qquad (\mathsf{S}(x) \text{ is PERIODIC with period } \tau)$$

✎PROOF:

$$\mathsf{S}(x+\tau) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+\tau+n\tau) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+(n+1)\tau) = \sum_{m\in\mathbb{Z}} \mathsf{f}(x+m\tau) \quad (\text{where } m \triangleq n+1)$$
$$\triangleq \mathsf{S}(x)$$

✏

Proposition G.2 (Power Sums). [22]

PRP
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \forall n\in\mathbb{N} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \quad \forall n\in\mathbb{N}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n\in\mathbb{N} \qquad \sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \quad \forall n\in\mathbb{N}$$

✎PROOF:

1. Proof that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$: (proof by induction)

$$\sum_{k=1}^{n=1} k = 1 = \frac{1(1+1)}{2} = \frac{n(n+1)}{2}\bigg|_{n=1}$$
$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + \left(n+1\right) = \underbrace{\left(\frac{n(n+1)}{2}\right) + \left(n+1\right)}_{\text{by left hypothesis}} = \left(n+1\right)\left(\frac{n}{2}+1\right)$$
$$= \left(n+1\right)\left(\frac{n+2}{2}\right) = \frac{(n+1)(n+2)}{2}$$

[21] Hall and Knight (1894) page 39 ⟨article 55⟩
[22] Amann and Escher (2008) pages 51–57, Menini and Oystaeyen (2004) page 91 ⟨Exercises 5.36–5.39⟩

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2. Proof that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$: (proof by induction)

$$\sum_{k=1}^{n=1} k^2 = 1 = \frac{1(1+1)(2+1)}{6} = \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1}$$

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + \left(n+1\right)^2 = \underbrace{\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(n+1\right)^2}_{\text{by left hypothesis}} = \left(n+1\right)\left(\frac{n(2n+1)+6(n+1)}{6}\right)$$

$$= \left(n+1\right)\left(\frac{2n^2+7n+6}{6}\right) = \left(n+1\right)\left(\frac{(n+2)(2n+3)}{6}\right) = \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}$$

✍

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APPENDIX H

INFINITE SUMS

"If, for always increasing values of n, the sum s_n approaches a certain limit s, the series will be called convergent and the limit in question will be called the sum of the series."

Cauchy [1]

"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes..."

Niels Henrik Abel, in a January 16, 1826 letter to Holmboé [2]

"The series is divergent; therefore we may be able to do something with it."

Oliver Heaviside (1850–1925) [3]

---

[1] quote: Cauchy (1821) page 114, González-Velasco (1992) pages 429–430
image: http://en.wikipedia.org, public domain

[2] quote: Kline (1972) page 973 ⟨Chapter 47⟩
image: http://en.wikipedia.org/wiki/File:Niels_Henrik_Abel.jpg, public domain

[3] quote: Kline (1972) page 1096 ⟨Chapter 47⟩
image: http://en.wikipedia.org/wiki/File:Oliver_Heaviside2.jpg, public domain

> *"Some modern appraisals of the cavalier style of 18th-century mathematicians in handling infinite series conveys the impression that these poor men set their brains aside when confronted by them."*
>
> Ivor Grattan-Guinness (1990)[4]

H.1 Convergence

An infinite summation $\sum_{n=1}^{\infty} x_n$ is meaningless outside some topological space (e.g. metric space, normed space, etc.). The sum $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ is an abbreviation for $\lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{x}_n$ (next definition); and the concept of *limit* is also itself meaningless outside of a *topological space*.

Definition H.1. [5] *Let $(X, \boldsymbol{T})$ be a topological space and* lim *be the limit generated by the topology $\boldsymbol{T}$.*

DEF

$$\sum_{n=1}^{\infty} \boldsymbol{x}_n \triangleq \sum_{n\in\mathbb{N}} \boldsymbol{x}_n \triangleq \lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{x}_n$$

$$\sum_{n=-\infty}^{\infty} \boldsymbol{x}_n \triangleq \sum_{n\in\mathbb{Z}} \boldsymbol{x}_n \triangleq \lim_{N\to\infty} \left(\sum_{n=0}^{N} \boldsymbol{x}_n\right) + \left(\lim_{N\to-\infty} \sum_{n=-1}^{N} \boldsymbol{x}_n\right)$$

In general, the order of summation of an infinite series *does* matter.

Definition H.2. [6] *Let P be the set of all* PERMUTATIONS *in $\mathbb{N}^{\mathbb{N}}$.*

DEF

*A series $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ is **absolutely convergent** if* $\quad \sum_{n=1}^{\infty} \boldsymbol{x}_n = \sum_{n=1}^{\infty} \boldsymbol{x}_{\mathsf{p}(n)} \quad \forall \mathsf{p} \in P$

*A series is **conditionally convergent** if it is* CONVERGENT *but not* ABSOLUTELY CONVERGENT.

Theorem H.1 (Riemann Series Theorem). [7] *Let $\mathsf{p}(n)$ be a permutation on $\mathbb{N}$. Let $(\!(a_n)\!)_{n\in\mathbb{Z}}$ be a sequence of real numbers.*

THM

$$\left\{\begin{array}{l}\sum_{n=1}^{\infty} a_n \text{ is} \\ \text{CONDITIONALLY CONVERGENT}\end{array}\right\} \implies \left\{\begin{array}{l}\text{For every } x \in \mathbb{R} \text{ there exists } \mathsf{p} \\ \text{such that } \sum_{n=1}^{\infty} a_{\mathsf{p}(n)} = x \\ \text{or such that } \sum_{n=1}^{\infty} a_{\mathsf{p}(n)} \text{ is DIVERGENT}\end{array}\right\}$$

Theorem H.2. [8]

THM

$$\sum_{n=1}^{\infty} |\boldsymbol{x}_n| < \infty \implies \left\{\sum_{n=1}^{\infty} \boldsymbol{x}_n \text{ is ABSOLUTELY CONVERGENT.}\right\}$$

[4] Grattan-Guinness (1990) page 163

[5] Klauder (2010) page 4, Kubrusly (2001) page 43, Bachman and Narici (1966) pages 3–4

[6] Kadets and Kadets (1997) page 5 ⟨THEOREM 1.1.1 (RIEMANN'S THEOREM)⟩, BROMWICH (1908) PAGE 64 ⟨IV. ABSOLUTE CONVERGENCE.⟩, SZÁSZ AND BARLAZ (1952) PAGE 2

[7] Kadets and Kadets (1997) page 5 ⟨THEOREM 1.1.1 (RIEMANN'S THEOREM)⟩, BROMWICH (1908) PAGE 68 ⟨ARTICLE 28. RIEMANN'S THEOREM⟩

[8] Kadets and Kadets (1997) page 5 ⟨THEOREM 1.1.1 (RIEMANN'S THEOREM)⟩

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Example H.1 (Logarithmic Series). [9] Consider the sum $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n}$. To which value this sum converges, or whether it even converges at all, depends on the order in which the terms are summed. This is demonstrated by the following series:

EX

If the series is added in the given order, the result is $\ln 2$:

$$\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} \triangleq \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{(n-1)}\frac{1}{n} = \ln 2$$

But if the order is changed, the sum can be any real value:

Let x be any real value (even an irrational one such as π or $\sqrt{2}$).

$$\text{Let}\quad \mathsf{p}(N) \triangleq \begin{cases} \text{next unused odd value} & \text{if } \sum_{n=1}^{N-1}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} \le x \\ \text{next unused even value} & \text{if } \sum_{n=1}^{N-1}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} > x \end{cases}$$

$$\sum_{n=1}^{\infty}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} \triangleq \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} = x$$

The series can even be summed in such a way that it does not converge at all:

Let $\mathsf{q}(n)$ be a permutation that partitions the natural numbers into odd and even values such that $(\!(x_{\mathsf{q}(n)})\!) = (\!(1,3,5,\ldots,2,4,6,\ldots)\!)$.

$$\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} = \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n-1}}_{\infty} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n}}_{\infty} \quad\Longrightarrow\quad \text{diverges}$$

✎PROOF:

1. Proof that $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} = \ln 2$ using polynomial expansion:[10]

   (a) Lemma: Proof that $\frac{1}{1+x} = \sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}$:

$$\begin{aligned}(1+x)\left(\sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}\right) &= \sum_{k=0}^{2n-1}(-1)^k x^k + \sum_{k=0}^{2n-1}(-1)^k x^{k+1} + x^{2n}\\ &= 1 + \sum_{k=1}^{2n-1}(-1)^k x^k - \sum_{k=1}^{2n-1}(-1)^k x^k + x^{2n}\\ &= 1 + x^{2n}\end{aligned}$$

   (b) Lemma: Proof that $\lim_{n\to\infty}\int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x = 0$:

$$\begin{aligned}0 &< \lim_{n\to\infty}\int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x\\ &< \lim_{n\to\infty}\int_0^1 x^{2n}\,\mathrm{d}x\end{aligned}$$

[9] Bromwich (1908) pages 51–52 ⟨Article 21 Example 1⟩, Hall and Knight (1894) page 191 ⟨article 223⟩, Jolley (1961) pages 14–15 ⟨item (71)⟩, Sloane (2014) ⟨http://oeis.org/A002939⟩⟨$2n(2n-1)$⟩, Graham et al. (1994) page 99 ⟨n.w. diagonal of *spiral function*⟩, Many many thanks to Po-Ning Chen (Chinese: ???, pinyin: Chén Bó Nǐng) for his consultation regarding this series.

[10] Bromwich (1908) pages 51–52 ⟨Article 21 Example 1⟩

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$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{x^{2n+1}}{2n+1}\bigg|_0^1 \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \\
&= 0
\end{aligned}
$$

(c) Proof that sum = ln 2:

$$
\begin{aligned}
\ln 2 &= \ln 2 - \ln 1 \\
&= \int_1^2 \frac{1}{x}\,\mathrm{dx} \\
&= \int_0^1 \frac{1}{x+1}\,\mathrm{dx} \\
&= \lim_{n\to\infty} \int_0^1 \left\{ \sum_{k=0}^{2n-1} (-1)^k x^k + \frac{x^{2n}}{1+x} \right\} \mathrm{dx} && \text{by item (1a)} \\
&= \sum_{k=0}^{2n-1} (-1)^k \int_0^1 x^k\,\mathrm{dx} + \lim_{n\to\infty} \int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{dx} \\
&= \lim_{n\to\infty} \sum_{k=0}^{2n-1} (-1)^k \frac{1}{k} + 0 && \text{by item (1b)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^{k-1} \frac{1}{k} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}
\end{aligned}
$$

2. Proof that $\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{1}{n} = \ln 2$ using Taylor expansion:[11]

(a) Lemma: Proof that $\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n} x^n$:

$$
\begin{aligned}
\ln(x+1) &= \sum_{n=0}^{\infty} \frac{\left[\mathbf{D}^n \ln(x+1)\right](0)}{n!} x^n && \text{by } \textit{Taylor series expansion} \text{ (Theorem C.13 page 167)} \\
&= \frac{\ln(1+0)}{0!} x^0 + \frac{\frac{1}{1+0}}{1!} x^1 - \frac{\frac{1}{(1+0)^2}}{2!} x^2 + \frac{\frac{2}{(1+0)^3}}{3!} x^3 - \frac{\frac{6}{(1+0)^4}}{4!} x^4 + \frac{\frac{24}{(1+0)^5}}{5!} x^5 - \frac{\frac{120}{(1+0)^6}}{6!} x^6 + \cdots \\
&= 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n} x^n
\end{aligned}
$$

[11] Hall and Knight (1894) page 191 ⟨article 223⟩

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(b) Proof that $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} = \ln 2$:

$$\begin{aligned}
&= \underbrace{\left(1-\frac{1}{2}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{\frac{1}{12}} + \underbrace{\left(\frac{1}{5}-\frac{1}{6}\right)}_{\frac{1}{30}} + \underbrace{\left(\frac{1}{7}-\frac{1}{8}\right)}_{\frac{1}{56}} + \ldots \\
&= \sum_{n=1}^{\infty}(-1)^{(n+1)}\frac{1}{n}x^n\Big|_{x=1} \\
&= \ln 2 && \text{by Lemma in item 1} \\
&\approx 0.693147
\end{aligned}$$

3. Proof that $\sum_{n=1}^{\infty}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} = x$: If the partial sum is less than x, positive values are added. If the partial sum is greater than x, negative values are added. The limit is x.

4. Proof that $\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} = \infty - \infty$:

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} &= \underbrace{\sum_{n=1}^{\infty}(-1)^{(2n-1-1)}\frac{1}{2n-1}}_{\text{odd indices}} + \underbrace{\sum_{n=1}^{\infty}(-1)^{(2n-1)}\frac{1}{2n}}_{\text{even indices}} \\
&= \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n-1}}_{\infty} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n}}_{\infty} \\
&\to \text{(diverges)}
\end{aligned}$$

✏

Divergent series could even result in decisions that may be considered extremely irrational, as demonstrated by *St. Petersberg Paradox* (next).

Example H.2 (St. Petersburg Paradox). [12] There is a lottery with a prize pot of \$1. A coin is tossed. If the coin is a tail, the money in the lottery is doubled (\$2, \$4, \$8, \$16, …). If the coin is a head, you win the money and the game is finished.

How much money would you be willing to play this game? The answer to this question for some people may depend on the expected value of how much money would be won. But the expected value of the amount of money you would win is

$$\tfrac{1}{2}\times\$1 + \tfrac{1}{4}\times\$2 + \tfrac{1}{8}\times\$4 + \tfrac{1}{16}\times\$8 + \ldots = \tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\ldots = \infty$$

Since the expected value of the win is infinity, you should be willing to pay any finite amount of money to play this game (even trillions of dollars). But yet common sense would tell most people that this would be an unwise investment.

[12] Székely (1986) pages 27–28, Bernoulli (1783) pages 31–32 ⟨§17⟩, de Montmort (1713) page 402 ⟨1713 letter from Nicolas Bernoulli⟩

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H.2 Multiplication

Theorem H.3. [13] *Let* $(\!|x_n|\!)_1^N$ *and* $(\!|y_n|\!)_1^N$ *be sequences over a ring* $(\mathbb{X},+,\times)$.

THM

$$\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) = \sum_{n=0}^{p+q} \underbrace{\left(\sum_{k=\max(0,n-q)}^{\min(n,p)} x_k y_{n-k}\right)}_{\textit{Cauchy product}}$$

✎PROOF:

1.
$$\begin{aligned}
\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) &= \sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m} \\
&= \sum_{n=0}^{p}\sum_{k=n}^{q+n} x_n y_{k-n} \qquad k = n+m \qquad m = k-n \\
&\vdots \\
&= \sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)
\end{aligned}$$

2. Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	y_0	y_1	y_2	y_3	$\cdots$	y_q
x_0	x_0y_0	x_0y_1	x_0y_2	x_0y_3	$\cdots$	x_0y_q
x_1	x_1y_0	x_1y_1	x_1y_2	x_1y_3	$\cdots$	x_1y_q
x_2	x_2y_0	x_2y_1	x_2y_2	x_2y_3	$\cdots$	x_2y_q
x_3	x_3y_0	x_3y_1	x_3y_2	x_3y_3	$\cdots$	x_3y_q
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_p	x_py_0	x_py_1	x_py_2	x_py_3	$\cdots$	x_py_q

(a) The expression $\sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

(b) If we switched the order of summation to $\sum_{m=0}^{q}\sum_{n=0}^{p} x_n y_m z^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

(c) However the final result expression $\sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)$ is equivalent to adding *diagonally* starting from the upper left corner and proceding to the lower right.

(d) Upper limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:
$$\left.\begin{array}{rcl} k & \le & n \\ k & \le & p \end{array}\right\} \implies k \le \min(n,p)$$

(e) Lower limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:
$$\left.\begin{array}{rcl} k & \ge & 0 \\ k & \ge & n-q \end{array}\right\} \implies k \ge \max(0,n-q)$$

✏

[13] Apostol (1975) page 204 ⟨Definition 8.45⟩

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Corollary H.1. *Let* $(\!| x_n \in \mathbb{C} |\!)$ *and* $(\!| y_n \in \mathbb{C} |\!)$.

COR
$$\left(\sum_{n=0}^{\infty} x_n\right)\left(\sum_{m=0}^{\infty} y_m\right) = \sum_{n=0}^{\infty}\underbrace{\left(\sum_{k=0}^{n} x_k y_{n-k}\right)}_{\textit{Cauchy product}}$$

✎PROOF:

$$\left(\sum_{n=0}^{p=\infty} x_n\right)\left(\sum_{m=0}^{q=\infty} y_m\right) = \sum_{n=0}^{\infty}\left(\sum_{k=\max(0,n-\infty)}^{\min(n,\infty)} x_k y_{n-k}\right) \qquad \text{by Theorem H.3 page 228}$$
$$= \sum_{n=0}^{\infty}\left((\sum_{k=0}^{n} x_k y_{n-k}\right)$$

✏

Theorem H.4. [14] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM
$$\left\{\begin{array}{ll} X \textit{ is } \text{ABSOLUTELY CONVERGENT} & \textit{and} \\ Y \textit{ is } \text{ABSOLUTELY CONVERGENT} & \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} Z \textit{ is } \text{ABSOLUTELY CONVERGENT} & \textit{and} \\ Z = XY. & \end{array}\right\}$$

Theorem H.5. [15] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM
$$\left\{\begin{array}{lll} 1. & X \textit{ is } \text{ABSOLUTELY CONVERGENT} & \textit{and} \\ 2. & Y \textit{ is } \text{CONVERGENT} & \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} 1. & Z \textit{ is } \text{CONVERGENT} & \textit{and} \\ 2. & Z = XY & \end{array}\right\}$$

Theorem H.6. [16] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM
$$\left\{\begin{array}{lll} 1. & X \textit{ is } \text{CONVERGENT} & \textit{and} \\ 2. & Y \textit{ is } \text{CONVERGENT} & \textit{and} \\ 3. & Z \textit{ is } \text{CONVERGENT} & \end{array}\right\} \Longrightarrow \{Z = XY\}$$

H.3 Summability

Cauchy and Abel, the 19th century champions of rigour in analysis, firmly rejected any and all divergent sums. However in more recent times, certain classes of divergent sums have been found to be extremely useful. Often such sums are ones that are said to be *summable*.

Definition H.3. [17]

[14] Hardy (1949) pages 227–228 ⟨THEOREM 160⟩, Bromwich (1908) PAGE 66 ⟨ARTICLE 27.⟩, Cauchy (1821) PAGES 147–148 ⟨6.e THÉORÈME⟩

[15] Hardy (1949) page 228 ⟨THEOREM 161⟩, Bromwich (1908) PAGES 85–86 ⟨ARTICLE 35.⟩, Mertens (1875)

[16] Hardy (1949) page 228 ⟨THEOREM 162⟩, Abel (1826)

[17] Zygmund (2002a) pages 75–76, Hardy (1949) page 96 ⟨5.4 Cesàro means⟩, Whittaker and Watson (1920) pages 155–156 ⟨8.43, 8.431⟩, Cesàro (1890)

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DEF

The series $\sum_{n=0}^{\infty} x_n$ *is* ***summable by the*** k***-th arithmetic mean of Cesàro to limit*** x*, or* ***summable*** (C,k) ***to the limit*** x*, if*

$$\lim_{n\to\infty} \frac{S_n^k}{A_n^k} = x \quad \text{for } n \in \mathbb{W} \text{ and where}$$

$$S_n^k \triangleq \begin{cases} \sum_{m=0}^{n} x_m & \text{for } k = 0 \\ \sum_{m=0}^{n} S_m^{k-1} & \text{for } k = 1,2,3,\ldots \end{cases} \quad \text{and} \quad A_n^k \triangleq \begin{cases} 1 & \text{for } k = 0 \\ \sum_{m=0}^{n} A_m^{k-1} & \text{for } k = 1,2,3,\ldots \end{cases}$$

Proposition H.1. [18]

PRP

$$\sum_{n=0}^{\infty} x_n \quad \text{is summable } (C,0) \text{ to the limit } x \text{ if } \lim_{N\to\infty} \sum_{n=0}^{N} x_n = x \qquad \text{(normal convergence)}$$

$$\sum_{n=0}^{\infty} x_n \quad \text{is summable } (C,1) \text{ to the limit } x \text{ if } \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} s_n = x \qquad \text{(arthimetic mean)}$$

$$\text{where } s_n \triangleq \sum_{m=0}^{n} x_m$$

Definition H.4. [19]

DEF

The series $\sum_{n=0}^{\infty} a_n$ *is* ***summable by Euler's method to limit*** a *if*

$$\lim_{x\to 1-0} \sum_{n=0}^{\infty} a_n x^n = a$$

Example H.3. [20]

EX

The series $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ is *divergent*
(it is *not* summable $(C,0)$),
but yet it *is* summable $(C,1)$ to the limit $\frac{1}{2}$.
It is also summable by Euler's method to the limit $\frac{1}{2}$.

✎PROOF:

1. Proof for Cesàro summability:

   (a) Note that the sequence of partial sums s_n is $s_0 = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0$, $s_4 = 1$, ... That is
   $$s_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

[18] Zygmund (2002a) pages 75–76, Thomson et al. (2008) page 129 ⟨Definition 3.54⟩, Szász and Barlaz (1952) page 13
[19] Whittaker and Watson (1920) page 155 ⟨8.42⟩
[20] Thomson et al. (2008) page 130 ⟨Example 3.56⟩, Whittaker and Watson (1920) page 155 ⟨8.42⟩

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(b) Then

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} s_k &= \lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=0}^{2n} s_k \\
&= \lim_{n\to\infty}\frac{1}{2n+1}\left(\underbrace{\sum_{k=0}^{n} s_{2k}}_{\text{even terms}} + \underbrace{\sum_{k=0}^{n-1} s_{2k+1}}_{\text{odd terms}}\right) \\
&= \lim_{n\to\infty}\frac{1}{2n+1}\left(\sum_{k=0}^{n} 1 + \sum_{k=0}^{n-1} 0\right) \\
&= \lim_{n\to\infty}\frac{n+1}{2n+1} \\
&= \frac{1}{2}
\end{aligned}$$

2. Proof for Euler summability:

$$\begin{aligned}
\lim_{x\to 1-0}\sum_{n=0}^{\infty}(-1)^n &= \lim_{x\to 1-0}\lim_{n\to\infty}\left(\frac{1}{1+x} = \sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}\right) \\
&= \lim_{x\to 1-0}\frac{1}{1+x} && \text{by item (1a) (page 225) of Example H.1} \\
&= \frac{1}{2}
\end{aligned}$$

✍

H.4 Convergence in Banach spaces

The properties of *strong convergence* and *weak convergence* are defined on *sequences*. An infinite sum $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ in a Banach space is the limit of a sequence of partial sums $\left(\left(\sum_{n=1}^{N} \boldsymbol{x}_n\right)\right)$, so the properties of strong and weak convergence apply to infinite sums as well. Definition H.5 (next) assigns special equality symbols for these sums.

Definition H.5. *Let* $\boldsymbol{B} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be a Banach space.*

DEF

The expression $\boldsymbol{x} \overset{\star}{=} \sum_{n=1}^{\infty} \boldsymbol{x}_n$ *denotes that the sum* ***converges strongly*** *to* $\boldsymbol{x}$.

The expression $\boldsymbol{x} \dot{=} \sum_{n=1}^{\infty} \boldsymbol{x}_n$ *denotes that the sum* ***converges weakly*** *to* $\boldsymbol{x}$.

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H.5 Convergence tests for real sequences

Theorem H.7 (*comparison test*). [21]

THM

$$\left\{\begin{array}{lll} 1. & \sum_{n=1}^{\infty} (\!(y_n)\!) \text{ CONVERGES} & \textit{and} \\ 2. & x_n \le y_n \quad \forall n \in \mathbb{N} & \end{array}\right\} \Longrightarrow \sum_{n=1}^{\infty} (\!(x_n)\!) \text{ CONVERGES}$$

[21] Bonar et al. (2006) page 26 ⟨Theorem 1.53 (Limit Comparison Test Strengthened)⟩, Heinbockel (2010) page 152 ⟨Comparison Tests⟩

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APPENDIX I

TRIGONOMETRIC FUNCTIONS

I.1 Definition Candidates

Definition I.1 (Hermitian components). [1] *Let* $(\mathbb{F}, *)$ *be a* $*$*-algebra a* (STAR ALGEBRA).

DEF	*The* ***real part*** *of* x *is defined as*	$\Re x$	$\triangleq \frac{1}{2}\left(x + x^*\right)$	$\forall x \in \mathbb{F}$
	The ***imaginary part*** *of* x *is defined as*	$\Im x$	$\triangleq \frac{1}{2i}\left(x - x^*\right)$	$\forall x \in \mathbb{F}$

There are several ways of defining the sine and cosine functions, including the following:[2]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[3]

$$\cos x \triangleq \frac{x}{r} \qquad \sin x \triangleq \frac{y}{r}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[4]

$$\cos x \triangleq \mathbf{R}_{\mathrm{e}} e^{ix} \qquad \sin x \triangleq \mathbf{I}_{\mathrm{m}}\left(e^{ix}\right)$$

3. **Polynomial:** Let $\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space (Definition H.1 page 224). The sine

[1] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964) page 242

[2] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means "bay" or "inlet"—thus resulting in the Latin translation *sinus*, which also means "bay" or "inlet". Reference: Boyer and Merzbach (1991) page 252

[3] Abramowitz and Stegun (1972) page 78

[4] Euler (1748)

and cosine functions can be defined in terms of *Taylor expansions* such that[5]

$$\begin{aligned}\cos(x) &\triangleq \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} &&= 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\\ \sin(x) &\triangleq \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} &&= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\end{aligned}$$

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty}\prod_{n=0}^{N} x_n$ in some topological space (Definition H.1 page 224). The sine and cosine functions can be defined in terms of a product of factors such that[6]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1-\left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[7]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-(n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-(n\pi)^2}\right)}_{\cot(x)}\sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{d}{dx}$ such that

$$\begin{aligned}\cos(x) &\triangleq f(x) \quad \text{where} \quad \underbrace{\frac{d^2}{dx^2}f+f=0}_{\text{differential equation}} \quad \underbrace{f(0)=1}_{\text{1st initial condition}} \quad \underbrace{\left[\frac{d}{dx}f\right](0)=0}_{\text{2nd initial condition}}\\ \sin(x) &\triangleq g(x) \quad \text{where} \quad \underbrace{\frac{d^2}{dx^2}g+g=0}_{\text{differential equation}} \quad \underbrace{g(0)=0}_{\text{1st initial condition}} \quad \underbrace{\left[\frac{d}{dx}g\right](0)=1}_{\text{2nd initial condition}}\end{aligned}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[8]

$$\begin{aligned}\cos(x) &\triangleq f^{-1}(x) \quad \text{where} \quad f(x) \triangleq \underbrace{\int_x^1\sqrt{\frac{1}{1-y^2}}\,dy}_{\arccos(x)}\\ \sin(x) &\triangleq g^{-1}(x) \quad \text{where} \quad g(x) \triangleq \underbrace{\int_0^x\sqrt{\frac{1}{1-y^2}}\,dy}_{\arcsin(x)}\end{aligned}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{d}{dx}$ (Definition I.2 page 235). Support for such an approach includes the following:

[5] Rosenlicht (1968) page 157, Abramowitz and Stegun (1972) page 74
[6] Abramowitz and Stegun (1972) page 75
[7] Abramowitz and Stegun (1972) page 75
[8] Abramowitz and Stegun (1972) page 79

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- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{d}{dx}$ (Theorem I.1 page 237).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem I.3 page 238).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem I.4 page 239).
- The complex exponential function is a solution of a second order homogeneous differential equation (Definition I.5 page 240).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section I.6 page 248).

I.2 Definitions

Definition I.2. [9] *Let C be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{d}{dx} \in C^C$ the differentiation operator.*

DEF

*The function $f \in \mathbb{C}^{\mathbb{C}}$ is the **cosine** function $\cos(x) \triangleq f(x)$ if*

1. $\frac{d^2}{dx^2}f + f = 0$ *(second order homogeneous differential equation)* *and*
2. $f(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{d}{dx}f\right](0) = 0$ *(second initial condition).*

Definition I.3. [10] *Let C and $\frac{d}{dx} \in C^C$ be defined as in definition of $\cos(x)$ (Definition I.2 page 235).*

DEF

*The function $f \in \mathbb{C}^{\mathbb{C}}$ is the **sine** function $\sin(x) \triangleq f(x)$ if*

1. $\frac{d^2}{dx^2}f + f = 0$ *(second order homogeneous differential equation)* *and*
2. $f(0) = 0$ *(first initial condition)* *and*
3. $\left[\frac{d}{dx}f\right](0) = 1$ *(second initial condition).*

Definition I.4. [11]

DEF

Let π ("pi") be defined as the element in $\mathbb{R}$ such that

(1). $\cos\left(\frac{\pi}{2}\right) = 0$ *and*

(2). $\pi > 0$ *and*

(3). *π is the **smallest** of all elements in $\mathbb{R}$ that satisfies (1) and (2).*

I.3 Basic properties

Lemma I.1. [12] *Let C be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{d}{dx} \in C^C$ the differentiation operator.*

[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[10] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[11] Rosenlicht (1968) page 158
[12] Rosenlicht (1968) page 156, Liouville (1839)

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LEM

$$\left\{\frac{d^2}{dx^2}f + f = 0\right\} \quad \Longleftrightarrow$$

$$\left\{\begin{aligned} f(x) &= \underbrace{[f](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{d}{dx}f\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \\ &= \left(f(0) + \left[\frac{d}{dx}f\right](0)x\right) - \left(\frac{f(0)}{2!}x^2 + \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3\right) + \left(\frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5\right)\cdots \end{aligned}\right\}$$

✎PROOF: Let $f'(x) \triangleq \frac{d}{dx}f(x)$.

$$\begin{aligned} f'''(x) &= -\left[\frac{d}{dx}f\right](x) \\ f^{(4)}(x) &= -\left[\frac{d}{dx}f\right](x) \qquad = -\left[\frac{d^2}{dx^2}f\right](x) = f(x) \end{aligned}$$

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \implies f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \qquad \text{by Taylor expansion (Theorem C.13 page 167)} \\ &= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{\left[\frac{d^2}{dx^2}f\right](0)}{2!}x^2 - \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 - \cdots \\ &= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{f(0)}{2!}x^2 - \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3 + \frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5 - \cdots \\ &= f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \end{aligned}$$

2. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \impliedby f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned} \left[\frac{d^2}{dx^2}f\right](x) &= \frac{d}{dx}\frac{d}{dx}[f(x)] \\ &= \frac{d}{dx}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \qquad \text{by right hypothesis} \\ &= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)f(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n-1}\right] \\ &= \sum_{n=1}^{\infty}(-1)^n\left[\frac{f(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n-1)!}x^{2n-1}\right] \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \\ &= -f(x) \qquad \text{by right hypothesis} \end{aligned}$$

✏

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Theorem I.1 (Taylor series for cosine/sine). [13]

THM

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \forall x\in\mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\cos(x) = \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \qquad \text{by Lemma I.1 page 235}$$

$$= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \text{by cos initial conditions (Definition I.2 page 235)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\sin(x) = \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \qquad \text{by Lemma I.1 page 235}$$

$$= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \text{by sin initial conditions (Definition I.3 page 235)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

✏

Theorem I.2. [14]

THM

$\cos(0) = 1$	$\cos(-x) = \cos(x)$	$\forall x\in\mathbb{R}$	(EVEN)
$\sin(0) = 0$	$\sin(-x) = -\sin(x)$	$\forall x\in\mathbb{R}$	(ODD)

✎PROOF:

$$\cos(0) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=0} \qquad \text{by } \textit{Taylor series for cosine} \qquad \text{(Theorem I.1 page 237)}$$

$$= 1$$

$$\sin(0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big|_{x=0} \qquad \text{by } \textit{Taylor series for sine} \qquad \text{(Theorem I.1 page 237)}$$

$$= 0$$

$$\cos(-x) = 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \cdots \qquad \text{by } \textit{Taylor series for cosine} \qquad \text{(Theorem I.1 page 237)}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \cos(x) \qquad \text{by } \textit{Taylor series for cosine} \qquad \text{(Theorem I.1 page 237)}$$

$$\sin(-x) = (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots \qquad \text{by } \textit{Taylor series for sine} \qquad \text{(Theorem I.1 page 237)}$$

[13] Rosenlicht (1968) page 157
[14] Rosenlicht (1968) page 157

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$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \sin(x) \qquad \text{by } \textit{Taylor series for sine} \qquad \text{(Theorem I.1 page 237)}$$

✏

Lemma I.2. [15]

LEM		
$\cos(1) > 0$ $\cos(2) < 0$	$x \in (0:2) \implies \sin(x) > 0$	

✎PROOF:

$$\cos(1) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=1} \qquad \text{by } \textit{Taylor series for cosine} \qquad \text{(Theorem I.1 page 237)}$$

$$= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots$$

$$> 0$$

$$\cos(2) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=2} \qquad \text{by } \textit{Taylor series for cosine} \qquad \text{(Theorem I.1 page 237)}$$

$$= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots$$

$$< 0$$

$$x \in (0:2) \implies \text{each term in the sequence } \left(\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots\right) \text{ is } > 0$$

$$\implies \sin(x) > 0$$

✏

Proposition I.1. *Let π be defined as in Definition I.4 (page 235).*

PRP	
(A).	*The value π **exists** in $\mathbb{R}$.*
(B).	$2 < \pi < 4$.

✎PROOF:

$$\cos(1) > 0 \qquad \text{by Lemma I.2 page 238}$$

$$\cos(2) < 0 \qquad \text{by Lemma I.2 page 238}$$

$$\implies 1 < \frac{\pi}{2} < 2$$

$$\implies 2 < \pi < 4$$

✏

Theorem I.3. [16] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathsf{d}}{\mathsf{dx}} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $\mathsf{f}'(0) \triangleq \left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)$.*

THM	
$\left\{\frac{\mathsf{d}^2}{\mathsf{dx}^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \mathsf{f}'(0)\sin(x)\right\}$	$\forall \mathsf{f} \in \boldsymbol{C}, \forall x \in \mathbb{R}$

[15] Rosenlicht (1968) page 158
[16] Rosenlicht (1968) page 157.

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✎PROOF:

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) = -f(x) \implies f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$f(x) = f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} \qquad \text{by left hypothesis and Lemma I.1 page 235}$$

$$= f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x \qquad \text{by definitions of cos and sin (Definition I.2 page 235, Definition I.3 page 235)}$$

2. Proof that $\frac{d^2}{dx^2}f = -f \impliedby f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$f(x) = f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x \qquad \text{by right hypothesis}$$

$$= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)}$$

$$\implies \frac{d^2}{dx^2}f + f = 0 \qquad \text{by Lemma I.1 page 235}$$

✏

Remark I.1. The general solution for the *non-homogeneous* equation $\frac{d^2}{dx^2}f(x)+f(x) = g(x)$ with initial conditions $f(a) = 1$ and $f'(a) = \rho$ is $f(x) = \cos(x) + \rho\sin(x) + \int_a^x g(y)\sin(x-y)\,dy$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992) page 371, Liouville (1839). Volterra equation references: Pedersen (2000) page 99, Lalescu (1908), Lalescu (1911)

Theorem I.4. [17] *Let $\frac{d}{dx} \in C^C$ be the differentiation operator.*

THM

$$\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x\in\mathbb{R} \;\Big|\; \frac{d}{dx}\sin(x) = \cos(x) \quad \forall x\in\mathbb{R} \;\Big|\; \cos^2(x)+\sin^2(x) = 1 \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \qquad \text{by } \textit{Taylor series} \quad \text{(Theorem I.1 page 237)}$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}$$

$$= -\sin(x) \qquad \text{by } \textit{Taylor series} \quad \text{(Theorem I.1 page 237)}$$

$$\frac{d}{dx}\sin(x) = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \text{by } \textit{Taylor series} \quad \text{(Theorem I.1 page 237)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$= \cos(x) \qquad \text{by } \textit{Taylor series} \quad \text{(Theorem I.1 page 237)}$$

[17] Rosenlicht (1968) page 157

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$$\begin{aligned}
\frac{d}{dx}\left[\cos^2(x)+\sin^2(x)\right] &= -2\cos(x)\sin(x)+2\sin(x)\cos(x)\\
&= 0\\
&\implies \cos^2(x)+\sin^2(x) \text{ is } \textit{constant}\\
&\implies \cos^2(x)+\sin^2(x)\\
&= \cos^2(0)+\sin^2(0)\\
&= 1+0=1 && \text{by Theorem I.2 page 237}
\end{aligned}$$

✏

Proposition I.2.

PRP $\sin\left(\frac{\pi}{2}\right) = 1$

✎PROOF:

$$\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2)+0}\\
&= \pm\sqrt{\sin^2(\pi/2)+\cos^2(\pi/2)} && \text{by definition of } \pi \quad \text{(Definition I.4 page 235)}\\
&= \pm\sqrt{1} && \text{by Theorem I.4 page 239}\\
&= \pm 1\\
&= 1 && \text{by Lemma I.2 page 238}
\end{aligned}$$

✏

I.4 The complex exponential

Definition I.5.

DEF *The function* $f \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq f(x)$ *if*

1. $\frac{d^2}{dx^2}f + f = 0$ *(second order homogeneous differential equation)* *and*
2. $f(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{d}{dx}f\right](0) = i$ *(second initial condition).*

Theorem I.5 (Euler's Identity). [18]

THM $e^{ix} = \cos(x) + i\sin(x) \qquad \forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\exp(ix) &= f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x) && \text{by Theorem I.3 page 238}\\
&= \cos(x) + i\sin(x) && \text{by Definition I.5 page 240}
\end{aligned}$$

✏

Proposition I.3.

PRP $e^{-i\pi/2} = -i \;\Big|\; e^{i\pi/2} = i$

[18] Euler (1748), Bottazzini (1986) page 12

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✎PROOF:

$$\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by } \textit{Euler's Identity} \text{ (Theorem I.5 page 240)}\\
&= 0 + i && \text{by Theorem I.2 (page 237) and Proposition I.2 (page 240)}\\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by } \textit{Euler's Identity} \text{ (Theorem I.5 page 240)}\\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem I.2 page 237}\\
&= 0 - i && \text{by Theorem I.2 (page 237) and Proposition I.2 (page 240)}
\end{aligned}$$

✏

Corollary I.1.

COR $$e^{ix} = \sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{e^{ix}} &= \cos(x) + i\sin(x) && \text{by } \textit{Euler's Identity} && \text{(Theorem I.5 page 240)}\\
&= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i\underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by } \textit{Taylor series} && \text{(Theorem I.1 page 237)}\\
&= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} && = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!}\\
&= \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} && = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}
\end{aligned}$$

✏

Corollary I.2 (Euler formulas). [19]

COR $$\cos(x) = \mathbf{R}_e\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \Big| \qquad \sin(x) = \mathbf{I}_m\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{R}_e\left(e^{ix}\right)} &\triangleq \frac{e^{ix}+\left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} && \text{by definition of } \Re\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} && \text{by } \textit{Euler's Identity} && \text{(Theorem I.5 page 240)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} && = \frac{\cos(x)}{2} + \frac{\cos(x)}{2} && = \boxed{\cos(x)}\\
\boxed{\mathbf{I}_m\left(e^{ix}\right)} &\triangleq \frac{e^{ix}-\left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} && \text{by definition of } \Im\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} && \text{by } \textit{Euler's Identity} && \text{(Theorem I.5 page 240)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} && = \frac{i\sin(x)}{2i} + \frac{i\sin(x)}{2i} && = \boxed{\sin(x)}
\end{aligned}$$

✏

[19] Euler (1748), Bottazzini (1986) page 12

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Theorem I.6. [20]

THM $e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta \in \mathbb{C}$

✎PROOF:

$$
\begin{aligned}
e^{\alpha}\, e^{\beta} &= \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) && \text{by Corollary I.1 page 241}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{\alpha^k}{k!}\,\frac{\beta^{n-k}}{(n-k)!} && \text{by Corollary H.1 page 229}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{n!}{n!}\frac{\alpha^k}{k!}\,\frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\,\alpha^k\,\beta^{n-k}\\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}\alpha^k\,\beta^{n-k}\\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the } \textit{Binomial Theorem} \quad \text{(Theorem C.14 page 167)}\\
&= e^{\alpha+\beta} && \text{by Corollary I.1 page 241}
\end{aligned}
$$

✏

I.5 Trigonometric Identities

Theorem I.7 (shift identities)**.**

THM

$$
\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= -\sin x \qquad \forall x\in\mathbb{R} & \qquad \sin\left(x+\frac{\pi}{2}\right) &= \cos x \qquad \forall x\in\mathbb{R}\\
\cos\left(x-\frac{\pi}{2}\right) &= \sin x \qquad \forall x\in\mathbb{R} & \qquad \sin\left(x-\frac{\pi}{2}\right) &= -\cos x \qquad \forall x\in\mathbb{R}
\end{aligned}
$$

✎PROOF:

$$
\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)} + e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by } \textit{Euler formulas} \quad \text{(Corollary I.2 page 241)}\\
&= \frac{e^{ix}e^{i\frac{\pi}{2}} + e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} \quad \text{(Theorem I.6 page 242)}\\
&= \frac{e^{ix}(i) + e^{-ix}(-i)}{2} && \text{by Proposition I.3 page 240}\\
&= \frac{e^{ix} - e^{-ix}}{-2i}\\
&= -\sin x && \text{by } \textit{Euler formulas} \quad \text{(Corollary I.2 page 241)}\\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)} + e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by } \textit{Euler formulas} \quad \text{(Corollary I.2 page 241)}\\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}} + e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} \quad \text{(Theorem I.6 page 242)}
\end{aligned}
$$

[20] Rudin (1987) page 1

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$$= \frac{e^{ix}(-i) + e^{-ix}(i)}{2} \qquad \text{by Proposition I.3 page 240}$$
$$= \frac{e^{ix} - e^{-ix}}{2i}$$
$$= \sin x \qquad \text{by } \textit{Euler formulas} \qquad \text{(Corollary I.2 page 241)}$$
$$\sin\left(x + \frac{\pi}{2}\right) = \cos\left(\left[x + \frac{\pi}{2}\right] - \frac{\pi}{2}\right) \qquad \text{by previous result}$$
$$= \cos(x)$$
$$\sin\left(x - \frac{\pi}{2}\right) = -\cos\left(\left[x - \frac{\pi}{2}\right] + \frac{\pi}{2}\right) \qquad \text{by previous result}$$
$$= -\cos(x)$$

✏

Theorem I.8 (product identities).

THM							
	(A).	$\cos x \cos y$	$=$	$\frac{1}{2}\cos(x-y)$	$+$	$\frac{1}{2}\cos(x+y)$	$\forall x,y \in \mathbb{R}$
	(B).	$\cos x \sin y$	$=$	$-\frac{1}{2}\sin(x-y)$	$+$	$\frac{1}{2}\sin(x+y)$	$\forall x,y \in \mathbb{R}$
	(C).	$\sin x \cos y$	$=$	$\frac{1}{2}\sin(x-y)$	$+$	$\frac{1}{2}\sin(x+y)$	$\forall x,y \in \mathbb{R}$
	(D).	$\sin x \sin y$	$=$	$\frac{1}{2}\cos(x-y)$	$-$	$\frac{1}{2}\cos(x+y)$	$\forall x,y \in \mathbb{R}$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary I.2 page 241)
(algebraic method requiring *complex number system* $\mathbb{C}$):

$$\cos x \cos y = \left(\frac{e^{ix} + e^{-ix}}{2}\right)\left(\frac{e^{iy} + e^{-iy}}{2}\right) \qquad \text{by } \textit{Euler formulas} \qquad \text{(Corollary I.2 page 241)}$$
$$= \frac{e^{i(x+y)} + e^{i(x-y)} + e^{i(-x+y)} + e^{i(-x-y)}}{4}$$
$$= \frac{e^{i(x+y)} + e^{-i(x+y)}}{4} + \frac{e^{i(x-y)} + e^{-i(x-y)}}{4}$$
$$= \frac{2\cos(x+y)}{4} + \frac{2\cos(x-y)}{4} \qquad \text{by } \textit{Euler formulas} \qquad \text{(Corollary I.2 page 241)}$$
$$= \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$$

2. Proof for (A) using *Volterra integral equation* (Theorem I.3 page 238)
(differential equation method requiring only *real number system* $\mathbb{R}$):

$$f(x) \triangleq \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y)$$
$$\Longrightarrow \frac{d}{dx}f(x) = -\tfrac{1}{2}\sin(x-y) - \tfrac{1}{2}\sin(x+y) \qquad \text{by Theorem I.4 page 239}$$
$$\Longrightarrow \frac{d^2}{dx^2}f(x) = -\tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y) \qquad \text{by Theorem I.4 page 239}$$
$$\Longrightarrow \frac{d^2}{dx^2}f(x) + f(x) = 0 \qquad \text{by } \textit{additive inverse} \text{ property}$$
$$\Longrightarrow \underbrace{\tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y)}_{f(x)} = \underbrace{[\tfrac{1}{2}\cos(0-y) + \tfrac{1}{2}\cos(0+y)]}_{f''(0)}\cos(x) + \underbrace{[-\tfrac{1}{2}\sin(0-y) - \tfrac{1}{2}\sin(0+y)]}_{f'(0)}\sin(x)$$
$$\Longrightarrow \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y) = \cos y \cos x + 0\sin(x)$$
$$\Longrightarrow \cos x \cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y)$$

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3. Proof for (B) using *Euler formulas* (Corollary I.2 page 241):

$$\begin{aligned}
\sin x \sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary I.2 page 241}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary I.2 page 241}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x \sin y &= \cos(x)\cos\left(y-\frac{\pi}{2}\right) && \text{by \textit{shift identities}} && \text{(Theorem I.7 page 242)}\\
&= \frac{1}{2}\cos\left(x+y-\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(x-y+\frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y)-\frac{1}{2}\sin(x-y) && \text{by \textit{shift identities}} && \text{(Theorem I.7 page 242)}\\
\sin x \cos y &= \cos y \sin x\\
&= \frac{1}{2}\sin(y+x)-\frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y)+\frac{1}{2}\sin(x-y) && \text{by Theorem I.2 page 237}
\end{aligned}$$

✏

Proposition I.4.

PRP												
	(A).	$\cos(\pi)$	$=$	-1	(C).	$\cos(2\pi)$	$=$	1	(E).	$e^{i\pi}$	$=$	-1
	(B).	$\sin(\pi)$	$=$	0	(D).	$\sin(2\pi)$	$=$	0	(F).	$e^{i2\pi}$	$=$	0

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1+1+\cos(\pi)\\
&= -1+2[\tfrac{1}{2}\cos(\tfrac{\pi}{2}-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(\tfrac{\pi}{2}+\tfrac{\pi}{2})] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem I.2 page 237)}\\
&= -1+2\cos(\tfrac{\pi}{2})\cos(\tfrac{\pi}{2}) && \text{by \textit{product identities}} && \text{(Theorem I.8 page 243)}\\
&= -1+2(0)(0) && \text{by definition of } \pi && \text{(Definition I.4 page 235)}\\
&= -1\\
\sin(\pi) &= 0+\sin(\pi)\\
&= 2[-\tfrac{1}{2}\sin(\tfrac{\pi}{2}-\tfrac{\pi}{2})+\tfrac{1}{2}\sin(\tfrac{\pi}{2}+\tfrac{\pi}{2})] && \text{by } \sin(0)=0 \text{ result} && \text{(Theorem I.2 page 237)}\\
&= 2\cos(\tfrac{\pi}{2})\sin(\tfrac{\pi}{2}) && \text{by \textit{product identities}} && \text{(Theorem I.8 page 243)}\\
&= 2(0)\sin(\tfrac{\pi}{2}) && \text{by definition of } \pi && \text{(Definition I.4 page 235)}\\
&= 0\\
\cos(2\pi) &= 1+\cos(2\pi)-1\\
&= 2[\tfrac{1}{2}\cos(\pi-\pi)+\tfrac{1}{2}\cos(\pi+\pi)]-1 && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem I.2 page 237)}\\
&= 2\cos(\pi)\cos(\pi)-1 && \text{by \textit{product identities}} && \text{(Theorem I.8 page 243)}\\
&= 2(-1)(-1)-1 && \text{by (A)}\\
&= 1
\end{aligned}$$

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$$\begin{aligned}
\sin(2\pi) &= 0 + \sin(2\pi) \\
&= 2[\tfrac{1}{2}\sin(\pi-\pi) + \tfrac{1}{2}\sin(\pi+\pi)] && \text{by } \sin(0)=0 \text{ result} && \text{(Theorem I.2 page 237)} \\
&= 2\sin(\pi)\cos(\pi) && \text{by } \textit{product identities} && \text{(Theorem I.8 page 243)} \\
&= 2(0)(-1) && \text{by (A) and (B)} \\
&= 0 \\
e^{i\pi} &= \cos(\pi) + i\sin(\pi) && \text{by } \textit{Euler's Identity} && \text{(Theorem I.5 page 240)} \\
&= -1 + 0 && \text{by (A) and (B)} \\
&= -1 \\
e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) && \text{by } \textit{Euler's Identity} && \text{(Theorem I.5 page 240)} \\
&= 1 + 0 && \text{by (C) and (D)} \\
&= 1
\end{aligned}$$

✏

Theorem I.9 (double angle formulas). [21]

THM					
	(A).	$\cos(x+y)$	$=$	$\cos x\cos y - \sin x\sin y$	$\forall x,y\in\mathbb{R}$
	(B).	$\sin(x+y)$	$=$	$\sin x\cos y + \cos x\sin y$	$\forall x,y\in\mathbb{R}$
	(C).	$\tan(x+y)$	$=$	$\dfrac{\tan x + \tan y}{1 - \tan x\tan y}$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *product identities* (Theorem I.8 page 243).

$$\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x+y)}_{\cos(x+y)} + \underbrace{\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x-y)}_{0} \\
&= \left[\frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\right] - \left[\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\right] \\
&= \cos x\cos y - \sin x\sin y && \text{by Theorem I.8 page 243}
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem I.3 page 238):

$$\begin{aligned}
f(x) \triangleq \cos(x+y) &\implies \tfrac{d}{dx}f(x) = -\sin(x+y) && \text{by Theorem I.4 page 239} \\
&\implies \tfrac{d^2}{dx^2}f(x) = -\cos(x+y) && \text{by Theorem I.4 page 239} \\
&\implies \tfrac{d^2}{dx^2}f(x) + f(x) = 0 && \text{by } \textit{additive inverse} \text{ property} \\
&\implies \cos(x+y) = \cos y\cos x - \sin y\sin x && \text{by Theorem I.3 page 238} \\
&\implies \cos(x+y) = \cos x\cos y - \sin x\sin y && \text{by } \textit{commutative} \text{ property}
\end{aligned}$$

[21] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

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3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x-\frac{\pi}{2}+y\right) && \text{by } \textit{shift identities} \text{ (Theorem I.7 page 242)}\\
&= \cos\left(x-\frac{\pi}{2}\right)\cos(y)-\sin\left(x-\frac{\pi}{2}\right)\sin(y) && \text{by (A)}\\
&= \sin(x)\cos(y)+\cos(x)\sin(y) && \text{by } \textit{shift identities} \text{ (Theorem I.7 page 242)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)}\\
&= \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y} && \text{by (A)}\\
&= \left(\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\right)\left(\frac{\cos x\cos y}{\cos x\cos y}\right)\\
&= \frac{\frac{\sin x\cos y}{\cos x\cos y}+\frac{\cos x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y}-\frac{\sin x\sin y}{\cos x\cos y}} = \frac{\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}}{1-\frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x+\tan y}{1-\tan x\tan y}
\end{aligned}$$

✏

Theorem I.10 (trigonometric periodicity)**.**

THM												
	(A).	$\cos(x+M\pi)$	$=$	$(-1)^M\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$	(D).	$\cos(x+2M\pi)$	$=$	$\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
	(B).	$\sin(x+M\pi)$	$=$	$(-1)^M\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$	(E).	$\sin(x+2M\pi)$	$=$	$\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
	(C).	$e^{i(x+M\pi)}$	$=$	$(-1)^M e^{ix}$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$	(F).	$e^{i(x+2M\pi)}$	$=$	e^{ix}	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$

✎PROOF:

1. Proof for (A):

   (a) $M=0$ case: $\cos(x+0\pi) = \cos(x) = (-1)^0\cos(x)$

   (b) Proof for $M>0$ cases (by induction):

      i. Base case $M=1$:

$$\begin{aligned}
\cos(x+\pi) &= \cos x\cos\pi-\sin x\sin\pi && \text{by } \textit{double angle formulas} && \text{(Theorem I.9 page 245)}\\
&= \cos x(-1)-\sin x(0) && \text{by } \cos\pi=-1 \text{ result} && \text{(Proposition I.4 page 244)}\\
&= (-1)^1\cos x
\end{aligned}$$

      ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\cos(x+[M+1]\pi) &= \cos([x+\pi]+M\pi)\\
&= (-1)^M\cos(x+\pi) && \text{by induction hypothesis } (M \text{ case})\\
&= (-1)^M(-1)\cos(x) && \text{by base case (item (1(b)i) page 246)}\\
&= (-1)^{M+1}\cos(x)\\
&\implies M+1 \text{ case}
\end{aligned}$$

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(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\cos(x+M\pi) &\triangleq \cos(x-N\pi) && \text{by definition of } N \\
&= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by } \textit{double angle formulas} \quad \text{(Theorem I.9 page 245)} \\
&= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem I.2 page 237} \\
&= \cos(x)\cos(0+N\pi) + \sin(x)\sin(0+N\pi) \\
&= \cos(x)(-1)^N\cos(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} \quad \text{(item (1b) page 246)} \\
&= (-1)^N\cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} \quad \text{(Theorem I.2 page 237)} \\
&\triangleq (-1)^{-M}\cos(x) && \text{by definition of } N \\
&= (-1)^M\cos(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\cos(x+M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by } \textit{Euler formulas} \quad \text{(Corollary I.2 page 241)} \\
&= e^{iM\pi}\left[\frac{e^{ix}+e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} \quad \text{(Theorem I.6 page 242)} \\
&= \left(e^{i\pi}\right)^M\cos x && \text{by } \textit{Euler formulas} \quad \text{(Corollary I.2 page 241)} \\
&= (-1)^M\cos x && \text{by } e^{i\pi} = -1 \text{ result} \quad \text{(Proposition I.4 page 244)}
\end{aligned}$$

2. Proof for (B):

(a) $M = 0$ case: $\sin(x+0\pi) = \sin(x) = (-1)^0\sin(x)$

(b) Proof for $M > 0$ cases (by induction):

i. Base case $M = 1$:

$$\begin{aligned}
\sin(x+\pi) &= \sin x\cos\pi + \cos x\sin\pi && \text{by } \textit{double angle formulas} \quad \text{(Theorem I.9 page 245)} \\
&= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} \quad \text{(Proposition I.4 page 244)} \\
&= (-1)^1\sin x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies M+1$ case:

$$\begin{aligned}
\sin(x+[M+1]\pi) &= \sin([x+\pi]+M\pi) \\
&= (-1)^M\sin(x+\pi) && \text{by induction hypothesis } (M \text{ case}) \\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 247)} \\
&= (-1)^{M+1}\sin(x) \\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\sin(x+M\pi) &\triangleq \sin(x-N\pi) && \text{by definition of } N \\
&= \sin(x)\sin(-N\pi) - \sin(x)\sin(-N\pi) && \text{by } \textit{double angle formulas} \quad \text{(Theorem I.9 page 245)} \\
&= \sin(x)\sin(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem I.2 page 237} \\
&= \sin(x)\sin(0+N\pi) + \sin(x)\sin(0+N\pi) \\
&= \sin(x)(-1)^N\sin(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} \quad \text{(item (2b) page 247)} \\
&= (-1)^N\sin(x) && \text{by } \sin(0)=1,\ \sin(0)=0 \text{ results} \quad \text{(Theorem I.2 page 237)} \\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N \\
&= (-1)^M\sin(x)
\end{aligned}$$

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(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)} - e^{-i(x+M\pi)}}{2i} && \text{by } \textit{Euler formulas} && \text{(Corollary I.2 page 241)}\\
&= e^{iM\pi}\left[\frac{e^{ix} - e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem I.6 page 242)}\\
&= \left(e^{i\pi}\right)^{M}\sin x && \text{by } \textit{Euler formulas} && \text{(Corollary I.2 page 241)}\\
&= (-1)^{M}\sin x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition I.4 page 244)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem I.6 page 242)}\\
&= \left(e^{i\pi}\right)^{M}\left(e^{ix}\right)\\
&= (-1)^{M}e^{ix} && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition I.4 page 244)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):

$$\begin{aligned}
\cos(i(x+2M\pi)) &= (-1)^{2M}\cos(ix) &&= \cos(ix) && \text{by (A)}\\
\sin(i(x+2M\pi)) &= (-1)^{2M}\sin(ix) &&= \sin(ix) && \text{by (B)}\\
e^{i(x+2M\pi)} &= (-1)^{2M}e^{ix} &&= e^{ix} && \text{by (C)}
\end{aligned}$$

✏

Theorem I.11 (half-angle formulas/squared identities)**.**

THM			
	(A). $\cos^2 x = \frac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$	*(C).* $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$	
	(B). $\sin^2 x = \frac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$		

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x) + \tfrac{1}{2}\cos(x+x) && \text{by } \textit{product identities} && \text{(Theorem I.8 page 243)}\\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem I.2 page 237)}\\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x) - \tfrac{1}{2}\cos(x+x) && \text{by } \textit{product identities} && \text{(Theorem I.8 page 243)}\\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem I.2 page 237)}\\
\cos^2 x + \sin^2 x &= \tfrac{1}{2}[1+\cos(2x)] + \tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)}\\
& && \text{note: see also} && \text{Theorem I.4 page 239}
\end{aligned}$$

✏

I.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

$$\begin{aligned}
\langle \cos(x) \,|\, \sin(x)\rangle &= \int_{-\pi}^{+\pi} \cos(x)\sin(x)\,dx\\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x-x)\,dx + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x+x)\,dx && \text{by Theorem I.8 page 243}\\
&= \frac{1}{2}\underbrace{\int_{-\pi}^{+\pi} \sin(0)\,dx}_{0} + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(2x)\,dx
\end{aligned}$$

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$$
\begin{aligned}
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi}\cos(2x)\\
&= -\frac{1}{4}[\cos(2\pi)-\cos(-2\pi)]\\
&= 0
\end{aligned}
$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$
\begin{aligned}
\cos\theta &\triangleq \frac{x}{r} & \sec\theta &\triangleq \frac{r}{x}\\
\sin\theta &\triangleq \frac{y}{r} & \csc\theta &\triangleq \frac{r}{y}\\
\tan\theta &\triangleq \frac{y}{x} & \cot\theta &\triangleq \frac{x}{y}
\end{aligned}
$$

I.7 Trigonometric functions of complex numbers

Definition I.6. [22]

DEF

$$
\begin{aligned}
\cosh(z) &\triangleq \frac{e^z+e^{-z}}{2} && \forall z\in\mathbb{C}\\
\sinh(z) &\triangleq \frac{e^z-e^{-z}}{2} && \forall z\in\mathbb{C}
\end{aligned}
$$

Theorem I.12. [23]

THM

$$
\begin{aligned}
\cosh(ix) &= \cos(x) && \forall x\in\mathbb{R}\\
\sinh(ix) &= i\sin(x) && \forall x\in\mathbb{R}\\
\cos(ix) &= \cosh(x) && \forall x\in\mathbb{R}\\
\sin(ix) &= i\sinh(x) && \forall x\in\mathbb{R}\\
\cos(x+iy) &= \cos(x)\cosh(y)-i\sin(x)\sinh(y) && \forall x,y\in\mathbb{R}\\
\sin(x+iy) &= \sin(x)\cosh(y)+i\cos(x)\sinh(y) && \forall x,y\in\mathbb{R}
\end{aligned}
$$

[22] Saxelby (1920) page 225

[23] https://proofwiki.org/wiki/Cosine_of_Complex_Number, https://proofwiki.org/wiki/Sine_of_Complex_Number, Saxelby (1920) pages 416–417

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✎PROOF:

$\cosh(ix)$	$\triangleq \dfrac{e^{ix}+e^{-ix}}{2}$	by definition of $\cosh(x)$	(Definition I.6 page 249)
	$= \cos(x)$	by *Euler's Identity*	(Theorem I.5 page 240)
$\sinh(ix)$	$\triangleq \dfrac{e^{ix}-e^{-ix}}{2}$	by definition of $\sinh(x)$	(Definition I.6 page 249)
	$\triangleq i\left[\dfrac{e^{ix}-e^{-ix}}{2i}\right]$	by definition of $\sinh(x)$	(Definition I.6 page 249)
	$= i\sin(x)$	by *Euler's Identity*	(Theorem I.5 page 240)
$\cos(ix)$	$\triangleq \dfrac{e^{iix}+e^{-iix}}{2}$	by *Euler's Identity*	(Theorem I.5 page 240)
	$= \dfrac{e^{-x}+e^{x}}{2}$		
	$= \dfrac{e^{x}+e^{-x}}{2}$		
	$\triangleq \cosh(x)$	by definition of $\cosh(x)$	(Definition I.6 page 249)
$\sin(ix)$	$\triangleq \dfrac{e^{iix}-e^{-iix}}{2i}$	by *Euler's Identity*	(Theorem I.5 page 240)
	$= \dfrac{e^{-x}-e^{x}}{2i}$		
	$= -(-i^2)\left[\dfrac{e^{x}-e^{-x}}{2i}\right]$		
	$= i\left[\dfrac{e^{x}-e^{-x}}{2}\right]$		
	$\triangleq i\sinh(x)$	by definition of $\cosh(x)$	(Definition I.6 page 249)
$\cos(x+iy)$	$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$	by *double angle formulas*	(Theorem I.9 page 245)
	$= \cos(x)\cosh(y) - i\sin(x)\sinh(y)$	by previous results	
$\sin(x+iy)$	$= \sin(x)\cos(iy) + \cos(x)\sin(iy)$	by *double angle formulas*	(Theorem I.9 page 245)
	$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$	by previous results	

✏

I.8 The power of the exponential

❝Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.❞

Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [24]

[24] quote: Kasner and Newman (1940) page 104
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

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"Young man, in mathematics you don't understand things. You just get used to them."

John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [25]

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary I.3. [26]

COR $e^{i\pi} + 1 = 0$

✎PROOF:

$$\begin{aligned} e^{ix}\big|_{x=\pi} &= [\cos x + i\sin x]_{x=\pi} && \text{by Euler's Identity (Theorem I.5 page 240)} \\ &= -1 + i \cdot 0 && \text{by Proposition I.4 page 244} \\ &= -1 \end{aligned}$$

✏

There are many transforms available, several of them integral transforms $[\mathbf{A}\mathsf{f}](s) \triangleq \int_t \mathsf{f}(s)\kappa(t,s)\,\mathrm{d}s$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:

① The *Laplace Transform* with kernel $\kappa(t,\ s) \triangleq e^{st}$

② The *Fourier Transform* with kernel $\kappa(t,\ \omega) \triangleq e^{i\omega t}$.

Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem I.13 page 251).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem I.13. [27] *Let* $\mathbf{L}$ *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*

$\check{\mathsf{h}}(s) \triangleq \left\langle \mathsf{h}(t,\omega) \,\middle|\, e^{st} \right\rangle$ (LAPLACE TRANSFORM).

---

[25] quote: Zukav (1980) page 208
image: http://en.wikipedia.org/wiki/John_von_Neumann

The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."

[26] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

[27] Mallat (1999) page 2, …page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

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THM

$$\left\{\begin{array}{ll} 1. & \mathbf{L} \textit{ is } \text{LINEAR } \textit{and} \\ 2. & \mathbf{L} \textit{ is } \text{TIME-INVARIANT} \end{array}\right\} \implies \left\{\mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{eigenvalue} \underbrace{e^{st}}_{eigenvector}\right\}$$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \mid \mathsf{h}((t;u),s)\rangle && \text{by linear hypothesis} \\
&= \langle e^{su} \mid \mathsf{h}((t-u),s)\rangle && \text{by time-invariance hypothesis} \\
&= \left\langle e^{s(t-v)} \mid \mathsf{h}(v,s)\right\rangle && \text{let } v = t-u \implies u = t-v \\
&= e^{st}\ \langle e^{-sv} \mid \mathsf{h}(v,s)\rangle && \text{by additivity of } \langle\triangle \mid \triangledown\rangle \\
&= \langle \mathsf{h}(v,s) \mid e^{-sv}\rangle^*\ e^{st} && \text{by conjugate symmetry of } \langle\triangle \mid \triangledown\rangle \\
&= \left\langle \mathsf{h}(v,s) \mid e^{(-s)v}\right\rangle^*\ e^{st} \\
&= \check{\mathsf{h}}^*(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}$$

✏

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APPENDIX J

TRIGONOMETRIC POLYNOMIALS

❝ I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives.❞

Charles Hermite (1822 – 1901), French mathematician, in an 1893 letter to Stieltjes, in response to the "pathological" everywhere continuous but nowhere differentiable *Weierstrass functions* $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$.[1]

J.1 Trigonometric expansion

Theorem J.1 (DeMoivre's Theorem)**.**

THM

$$\left(re^{ix}\right)^n = r^n(\cos nx + i\sin nx) \qquad \forall r,x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}\left(re^{ix}\right)^n &= r^n e^{inx} \\ &= r^n\left(\cos nx + i\sin nx\right) \qquad \text{by Euler's identity (Theorem I.5 page 240)}\end{aligned}$$

✏

The cosine with argument nx can be expanded as a polynomial in $\cos(x)$ (next).

Theorem J.2 (trigonometric expansion)**.** [2]

---

[1] quote: Hermite (1893)
translation: Lakatos (1976) page 19
image: http://www-groups.dcs.sx-and.ac.uk/~history/PictDisplay/Hermite.html

[2] Rivlin (1974) page 3 ⟨(1.8)⟩

THM

$$\cos(nx) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \qquad \forall n \in \mathbb{W} \text{ and } x \in \mathbb{R}$$

$$\sin(nx) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\sin x)^{n-2(k-m)} \qquad \forall n \in \mathbb{W} \text{ and } x \in \mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(nx) &= \Re\left(\cos nx + i \sin nx\right) \\
&= \Re\left(e^{inx}\right) \\
&= \Re\left[\left(e^{ix}\right)^n\right] \\
&= \Re\left[(\cos x + i \sin x)^n\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n} \binom{n}{k} (\cos x)^{n-k} (i \sin x)^k\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n} i^k \binom{n}{k} \cos^{n-k} x \sin^k x\right] \\
&= \Re\Bigg[\sum_{k\in\{0,4,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x + i \sum_{k\in\{1,5,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x \\
&\quad - \sum_{k\in\{2,6,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x + -i \sum_{k\in\{3,7,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x\Bigg] \\
&= \sum_{k\in\{0,4,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x - \sum_{k\in\{2,6,\ldots,n\}} \binom{n}{k} \cos^{n-k} x \sin^k x \\
&= \sum_{k\in\{0,2,\ldots,n\}} \binom{n}{k} (-1)^{\frac{k}{2}} \cos^{n-k} x \sin^k x \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \cos^{n-2k} x \sin^{2k} x \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \cos^{n-2k} x (1 - \cos^2 x)^k \\
&= \left[\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \cos^{n-2k} x\right] \left[\sum_{m=0}^{k} \binom{k}{m} (-1)^m \cos^{2m} x\right] \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} x
\end{aligned}$$

$$\begin{aligned}
\sin(nx) &= \cos\left(nx - \frac{\pi}{2}\right) \\
&= \cos\left(n\left[x - \frac{\pi}{2n}\right]\right) \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(n\left[x - \frac{\pi}{2n}\right]\right)
\end{aligned}$$

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$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(nx - \frac{\pi}{2}\right)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \sin^{n-2(k-m)} (nx)$$

✏

Example J.1.

EX

$$\begin{aligned} \cos 5x &= 16\cos^5 x - 20\cos^3 x + 5\cos x \\ \sin 5x &= 16\sin^5 x - 20\sin^3 x + 5\sin x. \end{aligned}$$

✎PROOF:

1. Proof using *DeMoivre's Theorem* (Theorem J.1 page 253):

$$\begin{aligned}
&\cos 5x + i\sin 5x \\
&= e^{i5x} \\
&= \left(e^{ix}\right)^5 \\
&= (\cos x + i\sin x)^5 \\
&= \sum_{k=0}^{5} \binom{5}{k} [\cos x]^{5-k} [i\sin x]^k \\
&= \binom{5}{0}[\cos x]^{5-0}[i\sin x]^0 + \binom{5}{1}[\cos x]^{5-1}[i\sin x]^1 + \binom{5}{2}[\cos x]^{5-2}[i\sin x]^2 + \\
&\binom{5}{3}[\cos x]^{5-3}[i\sin x]^3 + \binom{5}{4}[\cos x]^{5-4}[i\sin x]^4 + \binom{5}{5}[\cos x]^{5-5}[i\sin x]^5 \\
&= 1\cos^5 x + i5\cos^4 x\sin x - 10\cos^3 x\sin^2 x - i10\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i1\sin^5 x \\
&= \left[\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\right] + i\left[5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)(1-\cos^2 x)\right] + \\
&i\left[5(1-\sin^2 x)(1-\sin^2 x)\sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)\right] + \\
&i\left[5(1 - 2\sin^2 x + \sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5(\cos x - 2\cos^3 x + \cos^5 x)\right] + \\
&i\left[5(\sin x - 2\sin^3 x + \sin^5 x) - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \underbrace{\left[16\cos^5 x - 20\cos^3 x + 5\cos x\right]}_{\cos 5x} + i\underbrace{\left[16\sin^5 x - 20\sin^3 x + 5\sin x\right]}_{\sin 5x}
\end{aligned}$$

2. Proof using trigonometric expansion (Theorem J.2 page 253):

$$\begin{aligned}
\cos 5x &= \sum_{k=0}^{\lfloor \frac{5}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{2} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{5-2(k-m)} \\
&= (-1)^0 \binom{5}{0}\binom{0}{0}\cos^5 x + (-1)^1 \binom{5}{2}\binom{1}{0}\cos^3 x + (-1)^2 \binom{5}{2}\binom{1}{1}\cos^5 x + \\
&(-1)^2 \binom{5}{4}\binom{2}{0}\cos^1 x + (-1)^3 \binom{5}{4}\binom{2}{1}\cos^3 x + (-1)^4 \binom{5}{4}\binom{2}{2}\cos^5 x
\end{aligned}$$

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Figure J.1: Lattice of harmonic cosines $\{\cos(nx)|n = 0, 1, 2, \ldots\}$

$$= +(1)(1)\cos^5 x - (10)(1)\cos^3 x + (10)(1)\cos^5 x + (5)(1)\cos x - (5)(2)\cos^3 x + (5)(1)\cos^5 x$$
$$= +(1 + 10 + 5)\cos^5 x + (-10 - 10)\cos^3 x + 5\cos x$$
$$= 16\cos^5 x - 20\cos^3 x + 5\cos x$$

✏

Example J.2. [3]

EX	n	$\cos nx$		polynomial in $\cos x$	n	$\cos nx$		polynomial in $\cos x$
	0	$\cos 0x$	$=$	1	4	$\cos 4x$	$=$	$8\cos^4 x - 8\cos^2 x + 1$
	1	$\cos 1x$	$=$	$\cos^1 x$	5	$\cos 5x$	$=$	$16\cos^5 x - 20\cos^3 x + 5\cos x$
	2	$\cos 2x$	$=$	$2\cos^2 x - 1$	6	$\cos 6x$	$=$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
	3	$\cos 3x$	$=$	$4\cos^3 x - 3\cos x$	7	$\cos 7x$	$=$	$64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

✎PROOF:

$$\cos 2x = \sum_{k=0}^{\left\lfloor \frac{2}{2} \right\rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{2-2(k-m)}$$
$$= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^2 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^0 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^2 x$$
$$= +(1)(1)\cos^2 x - (1)(1) + (1)(1)\cos^2 x$$
$$= 2\cos^2 x - 1$$

$$\cos 3x = \sum_{k=0}^{\left\lfloor \frac{3}{2} \right\rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{3-2(k-m)}$$
$$= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^3 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^1 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^3 x$$
$$= +\binom{3}{0}\binom{0}{0} \cos^3 x - \binom{3}{2}\binom{1}{0} \cos^1 x + \binom{3}{2}\binom{1}{1} \cos^3 x$$
$$= +(1)(1)\cos^3 x - (3)(1)\cos^1 x + (3)(1)\cos^3 x$$
$$= 4\cos^3 x - 3\cos x$$

$$\cos 4x = \sum_{k=0}^{\left\lfloor \frac{4}{2} \right\rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{4}{2k} \binom{k}{m} (\cos x)^{4-2(k-m)}$$

[3] Abramowitz and Stegun (1972) page 795, Guillemin (1957) page 593 ⟨(21)⟩, Sloane (2014) ⟨http://oeis.org/A039991⟩, Sloane (2014) ⟨http://oeis.org/A028297⟩

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$$
\begin{aligned}
&= \sum_{k=0}^{2}\sum_{m=0}^{k}(-1)^{k+m}\binom{4}{2k}\binom{k}{m}(\cos x)^{4-2(k-m)}\\
&= (-1)^{0+0}\binom{4}{2\cdot 0}\binom{0}{0}(\cos x)^{4-2(0-0)} + (-1)^{1+0}\binom{4}{2\cdot 1}\binom{1}{0}(\cos x)^{4-2(1-0)}\\
&\quad + (-1)^{1+1}\binom{4}{2\cdot 1}\binom{1}{1}(\cos x)^{4-2(1-1)} + (-1)^{2+0}\binom{4}{2\cdot 2}\binom{2}{0}(\cos x)^{4-2(2-0)}\\
&\quad + (-1)^{2+1}\binom{4}{2\cdot 2}\binom{2}{1}(\cos x)^{4-2(2-1)} + (-1)^{2+2}\binom{4}{2\cdot 2}\binom{2}{2}(\cos x)^{4-2(2-2)}\\
&= (1)(1)\cos^4 x - (6)(1)\cos^2 x + (6)(1)\cos^4 x + (1)(1)\cos^0 x - (1)(2)\cos^2 x + (1)(1)\cos^4 x\\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}
$$

$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ see Example J.1 page 255

$$
\begin{aligned}
\cos 6x &= \sum_{k=0}^{\left\lfloor \frac{6}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{6}{2k}\binom{k}{m}(\cos x)^{6-2(k-m)}\\
&= (-1)^0\binom{6}{0}\binom{0}{0}\cos^6 x + (-1)^1\binom{6}{2}\binom{1}{0}\cos^4 x + (-1)^2\binom{6}{2}\binom{1}{1}\cos^6 x + (-1)^2\binom{6}{4}\binom{2}{0}\cos^2 x +\\
&\quad (-1)^3\binom{6}{4}\binom{2}{1}\cos^4 x + (-1)^4\binom{6}{4}\binom{2}{2}\cos^6 x + (-1)^3\binom{6}{6}\binom{3}{0}\cos^0 x + (-1)^4\binom{6}{6}\binom{3}{1}\cos^2 x +\\
&\quad (-1)^5\binom{6}{6}\binom{3}{2}\cos^4 x + (-1)^6\binom{6}{6}\binom{3}{3}\cos^6 x\\
&= +(1)(1)\cos^6 x - (15)(1)\cos^4 x + (15)(1)\cos^6 x + (15)(1)\cos^2 x - (15)(2)\cos^4 x + (15)(1)\cos^6 x\\
&\quad - (1)(1)\cos^0 x + (1)(3)\cos^2 x - (1)(3)\cos^4 x + (1)(1)\cos^6 x\\
&= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 7x &= \sum_{k=0}^{\left\lfloor \frac{7}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)}\\
&= \sum_{k=0}^{3}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{7-2(k-m)}\\
&= (-1)^0\binom{7}{0}\binom{0}{0}\cos^7 x + (-1)^1\binom{7}{2}\binom{1}{0}\cos^5 x + (-1)^2\binom{7}{2}\binom{1}{1}\cos^7 x + (-1)^2\binom{7}{4}\binom{2}{0}\cos^3 x\\
&\quad + (-1)^3\binom{7}{4}\binom{2}{1}\cos^5 x + (-1)^4\binom{7}{4}\binom{2}{2}\cos^7 x + (-1)^3\binom{7}{6}\binom{3}{0}\cos^1 x + (-1)^4\binom{7}{6}\binom{3}{1}\cos^3 x\\
&\quad + (-1)^5\binom{7}{6}\binom{3}{2}\cos^5 x + (-1)^6\binom{7}{6}\binom{3}{3}\cos^7 x\\
&= (1)(1)\cos^7 x - (21)(1)\cos^5 x + (21)(1)\cos^7 x + (35)(1)\cos^3 x\\
&\quad - (35)(2)\cos^5 x + (35)(1)\cos^7 x - (7)(1)\cos^1 x + (7)(3)\cos^3 x\\
&\quad - (7)(3)\cos^5 x + (7)(1)\cos^7 x\\
&= (1+21+35+7)\cos^7 x - (21+70+21)\cos^5 x + (35+21)\cos^3 x - (7)\cos^1 x\\
&= 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x
\end{aligned}
$$

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Figure J.2: Lattice of Chebyshev polynomials $\{T_n(x) | n = 0, 1, 2, 3\}$

Note: Trigonometric expansion of $\cos(nx)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[4]

```
trigexpand(cos(2*x));
trigexpand(cos(3*x));
trigexpand(cos(4*x));
trigexpand(cos(5*x));
trigexpand(cos(6*x));
trigexpand(cos(7*x));
```

✏

Definition J.1.

DEF *The nth* ***Chebyshev polynomial of the first kind*** *is defined as*

$$T_n(x) \triangleq \cos nx \qquad \textit{where} \qquad \cos x \triangleq x$$

Theorem J.3. [5] *Let $T_n(x)$ be a* CHEBYSHEV POLYNOMIAL *with $n \in \mathbb{W}$.*

THM

$$\begin{array}{lll} n \textit{ is } \text{EVEN} & \Longrightarrow & T_n(x) \textit{ is } \text{EVEN}. \\ n \textit{ is } \text{ODD} & \Longrightarrow & T_n(x) \textit{ is } \text{ODD}. \end{array}$$

Example J.3. Let $T_n(x)$ be a *Chebyshev polynomial* with $n \in \mathbb{W}$.

EX

$$\begin{array}{lll|lll} T_0(x) & = & 1 & T_4(x) & = & 8x^4 - 8x^2 + 1 \\ T_1(x) & = & x & T_5(x) & = & 16x^5 - 20x^3 + 5x \\ T_2(x) & = & 2x^2 - 1 & T_6(x) & = & 32x^6 - 48x^4 + 18x^2 - 1 \\ T_3(x) & = & 4x^3 - 3x & & & \end{array}$$

✎PROOF: Proof of these equations follows directly from Example J.2 (page 256). ✏

J.2 Trigonometric reduction

Theorem J.2 (page 253) showed that $\cos nx$ can be expressed as a polynomial in $\cos x$. Conversely, Theorem J.4 (next) shows that a polynomial in $\cos x$ can be expressed as a linear combination of $(\!(\cos nx)\!)_{n\in\mathbb{Z}}$.

Theorem J.4 (trigonometric reduction)**.**

[4] maxima pages 157–158 ⟨10.5 Trigonometric Functions⟩

[5] Rivlin (1974) page 5 ⟨(1.13)⟩, Süli and Mayers (2003) page 242 ⟨Lemma 8.2⟩, Davidson and Donsig (2010) page 222 ⟨exercise 10.7.A(a)⟩

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Figure J.3: Lattice of exponential cosines $\left\{\cos^n x | n = 0,1,2,3\right\}$

THM

$$\cos^n x = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x]$$

$$= \begin{cases} \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ even} \\ \frac{1}{2^{n-1}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k}\cos[(n-2k)x] & \textit{for } n \textit{ odd} \end{cases}$$

✎PROOF:

$$\begin{aligned}
\cos^n x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n \\
&= \mathbf{R}_e\left[\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)x}e^{-ikx}\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\cos[(n-2k)x]+i\sin[(n-2k)x]\right)\right] \\
&= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x]+i\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sin[(n-2k)x]\right] \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases} \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ even} \\ \frac{1}{2^{n-1}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k}\cos[(n-2k)x] & : \quad n \text{ odd} \end{cases}
\end{aligned}$$

✏

Example J.4. [6]

[6] Abramowitz and Stegun (1972) page 795, Sloane (2014) ⟨http://oeis.org/A100257⟩, Sloane (2014) ⟨http://oeis.org/A008314⟩

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EX

n	$\cos^n x$	trigonometric reduction	n	$\cos^n x$	trigonometric reduction
0	$\cos^0 x$	$= 1$	4	$\cos^4 x$	$= \dfrac{\cos 4x + 4\cos 2x + 3}{2^3}$
1	$\cos^1 x$	$= \cos x$	5	$\cos^5 x$	$= \dfrac{\cos 5x + 5\cos 3x + 10\cos x}{2^4}$
2	$\cos^2 x$	$= \dfrac{\cos 2x + 1}{2}$	6	$\cos^6 x$	$= \dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{2^5}$
3	$\cos^3 x$	$= \dfrac{\cos 3x + 3\cos x}{2^2}$	7	$\cos^7 x$	$= \dfrac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{2^6}$

✎PROOF:

$$\begin{aligned}
\cos^0 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\Bigg|_{n=0}\\
&= \frac{1}{2^0}\sum_{k=0}^{0}\binom{0}{k}\cos[(0-2k)x]\\
&= \binom{0}{0}\cos[(0-2\cdot 0)x]\\
&= 1\\
\cos^1 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\Bigg|_{n=1}\\
&= \frac{1}{2^1}\sum_{k=0}^{1}\binom{1}{k}\cos[(1-2k)x]\\
&= \frac{1}{2}\left[\binom{1}{0}\cos[(1-2\cdot 0)x] + \binom{1}{1}\cos[(1-2\cdot 1)x]\right]\\
&= \frac{1}{2}[1\cos x + 1\cos(-x)]\\
&= \frac{1}{2}(\cos x + \cos x)\\
&= \cos x\\
\cos^2 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\Bigg|_{n=2}\\
&= \frac{1}{2^2}\sum_{k=0}^{2}\binom{2}{k}\cos([2-2k]x)\\
&= \frac{1}{2^2}\left[\binom{2}{0}\cos([2-2\cdot 0]x) + \binom{2}{1}\cos([2-2\cdot 1]x) + \binom{2}{2}\cos([2-2\cdot 2]x) +\right]\\
&= \frac{1}{2^2}[1\cos(2x) + 2\cos(0x) + 1\cos(-2x)]\\
&= \frac{1}{2^2}[\cos(2x) + 2 + \cos(2x)]\\
&= \frac{1}{2}[\cos(2x) + 1]\\
\cos^3 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\Bigg|_{n=3}\\
&= \frac{1}{2^3}\sum_{k=0}^{3}\binom{3}{k}\cos([3-2k]x)
\end{aligned}$$

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$$= \frac{1}{2^3}\left[1\cos(3x) + 3\cos(1x) + 3\cos(-1x) + 1\cos(-3x)\right]$$
$$= \frac{1}{2^3}\left[\cos(3x) + 3\cos(x) + 3\cos(x) + \cos(3x)\right]$$
$$= \frac{1}{2^2}\left[\cos(3x) + 3\cos(x)\right]$$

$$\cos^4 x = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\Bigg|_{n=4}$$
$$= \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}\cos([4-2k]x)$$
$$= \frac{1}{2^4}\left[1\cos(4x) + 4\cos(2x) + 6\cos(0x) + 4\cos(-2x) + 1\cos(-4x)\right]$$
$$= \frac{1}{2^3}\left[\cos(4x) + 4\cos(2x) + 3\right]$$

$$\cos^5 x = \frac{1}{2^{5-1}}\sum_{k=0}^{\lfloor\frac{5}{2}\rfloor}\binom{5}{k}\cos[(5-2k)x]$$
$$= \frac{1}{16}\sum_{k=0}^{2}\binom{5}{k}\cos[(5-2k)x]$$
$$= \frac{1}{16}\left[\binom{5}{0}\cos 5x + \binom{5}{1}\cos 3x + \binom{5}{2}\cos x\right]$$
$$= \frac{1}{16}\left[\cos 5x + 5\cos 3x + 10\cos x\right]$$

$$\cos^6 x = \frac{1}{2^6}\binom{6}{\frac{6}{2}} + \frac{1}{2^{6-1}}\sum_{k=0}^{\frac{6}{2}-1}\binom{6}{k}\cos[(6-2k)x]$$
$$= \frac{1}{2^6}\binom{6}{3} + \frac{1}{2^5}\sum_{k=0}^{2}\binom{6}{k}\cos[(6-2k)x]$$
$$= \frac{1}{64}20 + \frac{1}{32}\left[\binom{6}{0}\cos 6x + \binom{6}{1}\cos 4x\binom{6}{2}\cos 2x\right]$$
$$= \frac{1}{32}\left[\cos 6x + 6\cos 4x + 15\cos 2x + 10\right]$$

$$\cos^7 x = \frac{1}{2^{7-1}}\sum_{k=0}^{\lfloor\frac{7}{2}\rfloor}\binom{7}{k}\cos[(7-2k)x]$$
$$= \frac{1}{64}\sum_{k=0}^{2}\binom{7}{k}\cos[(7-2k)x]$$
$$= \frac{1}{64}\left[\binom{7}{0}\cos 7x + \binom{7}{1}\cos 5x + \binom{7}{2}\cos 3x + \binom{7}{3}\cos x\right]$$
$$= \frac{1}{64}\left[\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right]$$

Note: Trigonometric reduction of $\cos^n(x)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[7]

```
trigreduce((cos(x))^2);
trigreduce((cos(x))^3);
trigreduce((cos(x))^4);
trigreduce((cos(x))^5);
trigreduce((cos(x))^6);
trigreduce((cos(x))^7);
```

[7] http://maxima.sourceforge.net/docs/manual/en/maxima_15.html
maxima page 158 ⟨10.5 Trigonometric Functions⟩

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✏

J.3 Spectral Factorization

Theorem J.5 (Fejér-Riesz spectral factorization). [8] *Let* $[0,\infty) \subsetneq \mathbb{R}$ *and*

$$\mathsf{p}\left(e^{ix}\right) \triangleq \sum_{n=-N}^{N} a_n e^{inx} \qquad \textit{(Laurent trigonometric polynomial order } 2N\textit{)}$$

$$\mathsf{q}\left(e^{ix}\right) \triangleq \sum_{n=1}^{N} b_n e^{inx} \qquad \textit{(standard trigonometric polynomial order } N\textit{)}$$

THM

$$\mathsf{p}\left(e^{ix}\right) \in [0,\infty) \quad \forall x \in [0,2\pi] \quad \Longrightarrow \quad \begin{cases} \exists \left(\!\left(b_n\right)\!\right)_{n\in\mathbb{Z}} \text{ such that} \\ \mathsf{p}\left(e^{ix}\right) = \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right) \qquad \forall x \in \mathbb{R} \end{cases}$$

✎PROOF:

1. Proof that $a_n = a_{-n}^*$ ($\left(\!\left(a_n\right)\!\right)_{n\in\mathbb{Z}}$ is *Hermitian symmetric*):
   Let $a_n \triangleq r_n e^{i\phi_n}$, $r_n, \phi_n \in \mathbb{R}$. Then

$$\begin{aligned}
\mathsf{p}\left(e^{inx}\right) &\triangleq \sum_{n=-N}^{N} a_n e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{i\phi_n} e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{inx+\phi_n} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \sum_{n=-N}^{N} r_n \sin(nx+\phi_n) \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(0x+\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) + \sum_{n=1}^{N} r_{-n} \sin(-nx+\phi_{-n}) \right]}_{\text{imaginary part must equal 0 because } p(x)\in\mathbb{R}} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) - \sum_{n=1}^{N} r_{-n} \sin(nx-\phi_{-n}) \right]}_{\Longrightarrow\ r_n = r_{-n},\ \phi_n = -\phi_{-n} \ \Longrightarrow\ a_n = a_{-n}^*,\ a_0 \in \mathbb{R}}
\end{aligned}$$

2. Because the coefficients $\left(\!\left(c_n\right)\!\right)_{n\in\mathbb{Z}}$ are *Hermitian symmetric* and by Theorem C.7 (page 164), the zeros of $P(z)$ occur in *conjugate recipricol pairs*. This means that if $\sigma \in \mathbb{C}$ is a zero of $P(z)$ ($P(\sigma) = 0$), then $\frac{1}{\sigma^*}$ is also a zero of $P(z)$ ($P\left(\frac{1}{\sigma^*}\right) = 0$). In the complex z plane, this relationship means zeros are reflected across the unit circle such that

$$\frac{1}{\sigma^*} = \frac{1}{(re^{i\phi})^*} = \frac{1}{r}\frac{1}{e^{-i\phi}} = \frac{1}{r}e^{i\phi}$$

[8] Pinsky (2002) pages 330–331

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3. Because the zeros of $\mathsf{p}(z)$ occur in conjugate recipricol pairs, $\mathsf{p}\left(e^{ix}\right)$ can be factored:

$$
\begin{aligned}
\mathsf{p}\left(e^{ix}\right) &= \mathsf{p}(z)\big|_{z=e^{ix}} \\
&= z^{-N} C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z - \frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N} z^{-1}\left(z - \frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(1 - \frac{1}{\sigma_n^*} z^{-1}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z^{-1} - \sigma_n^*\right)\left(-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= \left[C \prod_{n=1}^{N}\left(-\frac{1}{\sigma_n^*}\right)\right]\left[\prod_{n=1}^{N}(z-\sigma_n)\right]\left[\prod_{n=1}^{N}\left(\frac{1}{z^*} - \sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \left[C_2 \prod_{n=1}^{N}(z-\sigma_n)\right]\left[C_2 \prod_{n=1}^{N}\left(\frac{1}{z^*} - \sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}(z)\mathsf{q}^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right)
\end{aligned}
$$

✏

J.4 Dirichlet Kernel

"Dirichlet alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous proof is. Rather we learn it first from him. When Gauss says he has proved something it is clear; when Cauchy says it, one can wager as much pro as con; when Dirichlet says it, it is certain."

Carl Gustav Jacob Jacobi (1804–1851), Jewish-German mathematician [9]

[9] quote: Schubring (2005) page 558
image: http://en.wikipedia.org/wiki/File:Carl_Jacobi.jpg, public domain

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The *Dirichlet Kernel* is critical in proving what is not immediately obvious in examining the Fourier Series—that for a broad class of periodic functions, a function can be recovered from (with uniform convergence) its Fourier Series analysis.

Definition J.2. [10]

DEF *The **Dirichlet Kernel** $\mathsf{D}_n \in \mathbb{R}^{\mathbb{W}}$ with period τ is defined as*

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}kx}$$

Proposition J.1. [11] *Let D_n be the* DIRICHLET KERNEL *with period τ (Definition J.2 page 264).*

PRP
$$\mathsf{D}_n(x) = \frac{1}{\tau}\frac{\sin\left(\frac{\pi}{\tau}[2n+1]x\right)}{\sin\left(\frac{\pi}{\tau}x\right)}$$

✎PROOF:

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx} \qquad \text{by definition of } \mathsf{D}_n \qquad \text{(Definition J.2 page 264)}$$

$$= \frac{1}{\tau}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}(k-n)x} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}kx} = \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\sum_{k=0}^{2n}\left(e^{i\frac{2\pi}{\tau}x}\right)^k$$

$$= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\frac{1-\left(e^{i\frac{2\pi}{\tau}x}\right)^{2n+1}}{1-e^{i\frac{2\pi}{\tau}x}} \qquad \text{by } \textit{geometric series} \qquad \text{(Theorem G.7 page 220)}$$

$$= \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\frac{1-e^{i\frac{2\pi}{\tau}(2n+1)x}}{1-e^{i\frac{2\pi}{\tau}x}} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(\frac{e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{i\frac{\pi}{\tau}x}}\right)\frac{e^{-i\frac{\pi}{\tau}(2n+1)x}-e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{-i\frac{\pi}{\tau}x}-e^{i\frac{\pi}{\tau}x}}$$

$$= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(e^{i\frac{2\pi n}{\tau}x}\right)\frac{-2i\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{-2i\sin\left[\frac{\pi}{\tau}x\right]} = \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}x\right]}$$

✏

Proposition J.2. [12] *Let D_n be the* DIRICHLET KERNEL *with period τ (Definition J.2 page 264).*

PRP
$$\int_0^\tau \mathsf{D}_n(x)\,\mathrm{dx} = 1$$

✎PROOF:

$$\int_0^\tau \mathsf{D}_n(x)\,\mathrm{dx} \triangleq \int_0^\tau \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{dx} \qquad \text{by definition of } \mathsf{D}_n \text{ (Definition J.2 page 264)}$$

$$= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{dx}$$

$$= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right) + i\sin\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{dx}$$

[10] Katznelson (2004) page 14, Heil (2011) pages 443–444, Folland (1992) pages 33–34
[11] Katznelson (2004) page 14, Heil (2011) page 444, Folland (1992) page 34
[12] Bruckner et al. (1997) pages 620–621

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Figure J.4: D_n function for $N = 1, 5, 10, 20$. $\mathsf{D}_n \to$ comb. (See Proposition J.1 page 264).

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{dx} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left.\frac{\sin\left(\frac{2\pi}{\tau}nx\right)}{\frac{2\pi}{\tau}n}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left[\frac{\sin\left(\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n} - \frac{\sin\left(-\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n}\right] \\
&= \frac{1}{\tau}\frac{\tau}{2}\sum_{k=-n}^{n}\left[\frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi n)}{\pi n}\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n)}{\pi n}\right]_{k=0} \\
&= 1
\end{aligned}
$$

✍

Proposition J.3. *Let* D_n *be the* DIRICHLET KERNEL *with period* τ *(Definition J.2 page 264). Let* w_N *(the "*WIDTH*" of* $\mathsf{D}_n(x)$*) be the distance between the two points where the center pulse of* $\mathsf{D}_n(x)$ *intersects the* x *axis.*

PRP

$$
\begin{aligned}
\mathsf{D}_n(0) &= \frac{1}{\tau}(2n+1) \\
w_n &= \frac{2\tau}{2n+1}
\end{aligned}
$$

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✎PROOF:

$$
\begin{aligned}
\mathsf{D}_n(0) &= \mathsf{D}_n(x)\big|_{t=0} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} && \text{by Proposition J.1 page 264} \\
&= \frac{1}{\tau}\frac{\frac{\mathsf{d}}{\mathsf{dx}}\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\frac{\mathsf{d}}{\mathsf{dx}}\sin\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{\cos\left[\frac{\pi}{\tau}(2n+1)x\right]}{\cos\left[\frac{\pi}{\tau}t\right]}\Bigg|_{t=0} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{1}{1} \\
&= \frac{1}{\tau}(2n+1)
\end{aligned}
$$

The center pulse of kernel $\mathsf{D}_n(x)$ intersects the x axis at

$$t = \pm\frac{\tau}{(2n+1)}$$

which implies

$$w_n = \frac{\tau}{2n+1} + \frac{\tau}{2n+1} = \frac{2\tau}{(2n+1)}.$$

✏

Proposition J.4. [13] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ *(Definition J.2 page 264).*

PRP	$\mathsf{D}_n(x) = \mathsf{D}_n(-x)$ (*D_n is an* EVEN *function*)

✎PROOF:

$$
\begin{aligned}
\mathsf{D}_n(x) &= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]} && \text{by Proposition J.1 page 264} \\
&= \frac{1}{\tau}\frac{-\sin\left[-\frac{\pi}{\tau}(2n+1)x\right]}{-\sin\left[-\frac{\pi}{\tau}t\right]} && \text{because } \sin x \text{ is an } \textit{odd} \text{ function} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)(-x)\right]}{\sin\left[\frac{\pi}{\tau}(-x)\right]} \\
&= \mathsf{D}_n(-x) && \text{by Proposition J.1 page 264}
\end{aligned}
$$

✏

[13] Bruckner et al. (1997) pages 620–621

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J.5 Trigonometric summations

Theorem J.6 (Lagrange trigonometric identities). [14]

THM

$$\sum_{n=0}^{N-1}\cos(nx) = \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)+\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)+\cos\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=0}^{N-1}\Re e^{inx} = \Re\sum_{n=0}^{N-1}e^{inx} = \Re\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Re\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \quad \text{(Theorem G.7 page 220)} \\
&= \Re\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-i\frac{1}{2}x}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Re\left[\left(e^{i\frac{1}{2}(N-1)x}\right)\left(\frac{-i\frac{1}{2}\sin\left(\frac{1}{2}Nx\right)}{-i\frac{1}{2}\sin\left(\frac{1}{2}x\right)}\right)\right] \\
&= \cos\left(\frac{1}{2}(N-1)x\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{-\frac{1}{2}\sin\left(-\frac{1}{2}x\right)+\frac{1}{2}\sin\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by } \textit{product identities} \quad \text{(Theorem I.8 page 243)} \\
&= \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=0}^{N-1}\Im e^{inx} = \Im\sum_{n=0}^{N-1}e^{inx} = \Im\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Im\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \quad \text{(Theorem G.7 page 220)} \\
&= \Im\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-ix/2}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \Im\left[\left(e^{i(N-1)x/2}\right)\left(\frac{-\frac{1}{2}i\sin\left(\frac{1}{2}Nx\right)}{-\frac{1}{2}i\sin\left(\frac{1}{2}x\right)}\right)\right]
\end{aligned}$$

[14] Muniz (1953) page 140 ⟨"Lagrange's Trigonometric Identities"⟩, Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (14), (13)⟩

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$$= \sin\left(\frac{(N-1)x}{2}\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right)$$

$$= \frac{\frac{1}{2}\cos\left(-\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} \qquad \text{by } \textit{product identities} \qquad \text{(Theorem I.8 page 243)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}$$

Note that these results (summed with indices from $n = 0$ to $n = N-1$) are compatible with Muniz (1953) page 140 (summed with indices from $n = 1$ to $n = N$) as demonstrated next:

$$\sum_{n=0}^{N-1}\cos(nx) = \sum_{n=1}^{N}\cos(nx) + [\cos(0x) - \cos(Nx)]$$

$$= \left[-\frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + [\cos(0x) - \cos(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \left(1-\frac{1}{2}\right) + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\cos(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\left[\sin\left(\left[\frac{1}{2}-N\right]x\right) + \sin\left[\left(\frac{1}{2}+N\right)x\right]\right]}{2\sin\left(\frac{1}{2}x\right)} \qquad \text{by Theorem I.8 page 243}$$

$$= \frac{1}{2} + \frac{\sin\left(\frac{1}{2}[2N-1]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \sum_{n=1}^{N}\sin(nx) + [\sin(0x) - \sin(Nx)]$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} + [0 - \sin(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\sin(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - \left[\cos\left(\left[\frac{1}{2}-N\right]x\right) - \cos\left(\left[\frac{1}{2}+N\right]x\right)\right]}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

✏

Theorem J.7. [15]

[15] Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (16) and (17)⟩

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THM

$$\sum_{n=0}^{N-1}\cos(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]-\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]+\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\sum_{n=0}^{N-1}\cos(nx+y) = \sum_{n=0}^{N-1}[\cos(nx)\cos(y)-\sin(nx)\sin(y)] \qquad \text{by } \textit{double angle formulas} \qquad \text{(Theorem I.9 page 245)}$$

$$= \cos(y)\sum_{n=0}^{N-1}\cos(nx)-\sin(y)\sum_{n=0}^{N-1}\sin(nx)$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \sum_{n=0}^{N-1}[\cos(nx)\cos(y)+\sin(nx)\sin(y)] \qquad \text{by } \textit{double angle formulas} \qquad \text{(Theorem I.9 page 245)}$$

$$= \cos(y)\sum_{n=0}^{N-1}\cos(nx)+\sin(y)\sum_{n=0}^{N-1}\sin(nx)$$

✏

Corollary J.1 (Summation around unit circle).

THM

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right) = \sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)=0 \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

$$\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right)=\frac{N}{2} \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

✎PROOF:

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)$$

$$= \cos(\theta)\sum_{n=0}^{N-1}\cos\left(\frac{2nM\pi}{N}\right)-\sin(\theta)\sum_{n=0}^{N-1}\sin\left(\frac{2nM\pi}{N}\right) \qquad \text{by Theorem I.9 page 245}$$

$$= \cos(\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{2M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] \qquad \text{by Theorem J.6 page 267}$$

$$= \cos(\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{\cos\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]$$

$$= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{M\pi}{N}\right)}{\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)\right] \qquad \text{by } \textit{trigonometric periodicity} \quad \text{(Theorem I.10 page 246)}$$

$$= \cos(\theta)[0]-\sin(\theta)[0]$$

$$= 0$$

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$$\begin{aligned}
\sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1}\cos\left(\theta-\frac{\pi}{2}+\frac{2nM\pi}{N}\right) && \text{by \textit{shift identities}} \quad \text{(Theorem I.7 page 242)}\\
&= \sum_{n=0}^{N-1}\cos\left(\phi+\frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta-\frac{\pi}{2}\\
&= 0 && \text{by previous result}
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)\\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(\left[\theta+\frac{2nM\pi}{N}\right]-\left[\theta+\frac{2nM\pi}{N}\right]\right)+\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(\left[\theta+\frac{2nM\pi}{N}\right]+\left[\theta+\frac{2nM\pi}{N}\right]\right) && \text{by Theorem I.8 page 243}\\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\cancelto{0}{\sin(0)}+\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(2\theta+\frac{4nM\pi}{N}\right)\\
&= \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)+\frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right) && \text{by Theorem I.9 page 245}\\
&= \cos(2\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{4M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{4M\pi}{N}\right)}\right] && \text{by Theorem J.6 page 267}\\
&= \cos(2\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{2M\pi}{N}-4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]-\sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)-\frac{\cos\left(\frac{2M\pi}{N}-4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{2M\pi}{N}\right)}{\sin\left(\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity}}\\
&\qquad\qquad\qquad

$$\begin{aligned}
\sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos^2\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by } \textit{shift identities} \text{ (Theorem I.7 page 242)} \\
&= \sum_{n=0}^{N-1} \cos^2\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2} \\
&= \frac{N}{2} && \text{by previous result}
\end{aligned}$$

✏

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APPENDIX K

FOURIER SERIES

❝ *...et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur.*❞

MEMOIRE
PROPAGATION DE LA CHALEUR
DANS LES CORPS SOLIDES.
Par M. FOURIER (*).

❝ *...and the innovation of the subject, together with its importance, convinced the committee to crown this work. By observing however that the way in which the author arrives at his equations is not free from difficulties, and the analysis of which, to integrate them, still leaves something to be desired, either relative to generality, or even on the side of rigour.*❞

A competition awards committee consisting of the mathematical giants Lagrange, Laplace, Legendre, and others, commenting on Fourier's 1807 landmark paper *Dissertation on the propagation of heat in solid bodies* that introduced the *Fourier Series.* [1]

K.1 Definition

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition K.1. [2]

DEF

The ***Fourier Series operator*** $\hat{\mathbf{F}} : L^2_{\mathbb{R}} \to \ell^2_{\mathbb{R}}$ *is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}} \int_0^{\tau} \mathsf{f}(x) e^{-i\frac{2\pi}{\tau}nx}\, \mathsf{dx} \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in L^2_{\mathbb{R}} | \mathsf{f} \text{ is periodic with period } \tau\right\}$$

[1] quote: Lagrange et al. (1812b) page 374, Lagrange et al. (1812a) page 112, Kahane (2008) page 199
translation: assisted by Google Translate, Castanedo (2005) ⟨chapter 2 footnote 5⟩
paper: Fourier (1807)

[2] Katznelson (2004) page 3

K.2 Inverse Fourier Series operator

Theorem K.1. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

THM

*The **inverse Fourier Series** operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{x}}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)\in\ell^2_{\mathbb{R}}$$

✎PROOF: The proof of the pointwise convergence of the Fourier Series is notoriously difficult. It was conjectured in 1913 by Nokolai Luzin that the Fourier Series for all square summable periodic functions are pointwise convergent: Luzin (1913)

Fifty-three years later (1966) at a conference in Moscow, Lennart Axel Edvard Carleson presented one of the most spectacular results ever in mathematics; he demonstrated that the Luzin conjecture is indeed correct. Carleson formally published his result that same year: Carleson (1966)

Carleson's proof is expounded upon in Reyna's (2002) 175 page book: de Reyna (2002)

Interestingly enough, Carleson started out trying to disprove Luzin's conjecture. Carleson said this in an interview published in 2001:[3] ❝*Well, the problem of course presents itself already when you are a student and I was thinking of the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I suddenly realized why there should be a counter-example and how you should produce it. I thought I really understood what was the back ground and then to my amazement I could prove that this "correct" counter-example couldn't exist and therefore I suddenly realized that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. The most important aspect in solving a mathematical problem is the conviction of what is the true result! Then it took like 2 or 3 years using the technique that had been developed during the past 20 years or so. It is actually a problem related to analytic functions basically even though it doesn't look that way.*❞

For now, if you just want some intuitive justification for the Fourier Series, and you can somehow imagine that the Dirichlet kernel generates a *comb function* of *Dirac delta* functions, then perhaps what follows may help (or not). It is certainly not mathematically rigorous and is by no means a real proof (but at least it is less than 175 pages).

$$\begin{aligned}
\left[\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{x}\right](x) &= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}\mathsf{x}} && \text{by definition of } \hat{\mathbf{F}} \qquad \text{(Definition K.1 page 273)}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}\,\mathrm{d}u\right]e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \hat{\mathbf{F}}^{-1}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{i\frac{2\pi}{\tau}n(x-u)}\,\mathrm{d}u
\end{aligned}$$

[3] Carleson and Engquist (2001)

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$$= \int_0^\tau \mathsf{x}(u) \underbrace{\frac{1}{\tau}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi}{\tau}n(x-u)}}_{\lim_{N\to\infty}\mathsf{D}_n(x)} \,\mathsf{du}$$

$$= \int_0^\tau \mathsf{x}(u)\left[\sum_{n\in\mathbb{Z}} \delta(x-u-n\tau)\right] \mathsf{du}$$

$$= \sum_{n\in\mathbb{Z}} \int_{u=0}^{u=\tau} \mathsf{x}(u)\delta(x-u-n\tau)\,\mathsf{du}$$

$$= \sum_{n\in\mathbb{Z}} \int_{v-n\tau=0}^{v-n\tau=\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathsf{dv} \qquad \text{where } v \triangleq u+n\tau$$

$$= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathsf{dv} \qquad \text{where } v \triangleq u+n\tau$$

$$= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v)\delta(x-v)\,\mathsf{dv} \qquad \text{because } \mathsf{x} \text{ is periodic with period } \tau$$

$$= \int_{\mathbb{R}} \mathsf{x}(v)\delta(x-v)\,\mathsf{dv}$$

$$= \mathsf{x}(x)$$

$$= \mathbf{I}\tilde{\mathsf{x}}(n) \qquad \text{by definition of } \mathbf{I}$$

(Definition D.3 page 170)

$$\left[\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](n) = \hat{\mathbf{F}}\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] \qquad \text{by definition of } \hat{\mathbf{F}}^{-1}$$

$$= \frac{1}{\sqrt{\tau}}\int_0^\tau \left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{dx} \qquad \text{by definition of } \hat{\mathbf{F}}$$

(Definition K.1 page 273)

$$= \frac{1}{\tau}\int_0^\tau \left[\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}(k-n)x}\right] \mathsf{dx}$$

$$= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\left[\frac{1}{\tau}\int_0^\tau e^{i\frac{2\pi}{\tau}(k-n)x}\,\mathsf{dx}\right]$$

$$= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\frac{1}{\tau}\left[\frac{1}{i\frac{2\pi}{\tau}(k-n)}e^{i\frac{2\pi}{\tau}(k-n)x}\right]_0^\tau$$

$$= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\frac{1}{i2\pi(k-n)}\left[e^{i2\pi(k-n)}-1\right]$$

$$= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\,\bar{\delta}(k-n)\lim_{x\to 0}\left[\frac{e^{i2\pi x}-1}{i2\pi x}\right]$$

$$= \tilde{\mathsf{x}}(n)\left.\frac{\frac{\mathsf{d}}{\mathsf{dx}}\left(e^{i2\pi x}-1\right)}{\frac{\mathsf{d}}{\mathsf{dx}}(i2\pi x)}\right|_{x=0} \qquad \text{by } \textit{l'Hôpital's rule}$$

$$= \tilde{\mathsf{x}}(n)\left.\frac{i2\pi e^{i2\pi x}}{i2\pi}\right|_{x=0}$$

$$= \tilde{\mathsf{x}}(n)$$

$$= \mathbf{I}\tilde{\mathsf{x}}(n) \qquad \text{by definition of } \mathbf{I}$$

(Definition D.3 page 170)

✍

Theorem K.2.

THM *The **Fourier Series adjoint** operator $\hat{\mathbf{F}}^*$ is given by*

$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

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✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathsf{x}(x) \mid \tilde{\mathsf{y}}(n)\right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{d}x \mid \tilde{\mathsf{y}}(n)\right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition K.1 page 273)}\\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)\left\langle e^{-i\frac{2\pi}{\tau}nx} \mid \tilde{\mathsf{y}}(n)\right\rangle_{\mathbb{Z}}\,\mathsf{d}x && \text{by additivity property of } \langle\triangle\mid\triangledown\rangle\\
&= \int_0^\tau \mathsf{x}(x)\frac{1}{\sqrt{\tau}}\left\langle \tilde{\mathsf{y}}(n) \mid e^{-i\frac{2\pi}{\tau}nx}\right\rangle_{\mathbb{Z}}^*\,\mathsf{d}x && \text{by property of } \langle\triangle\mid\triangledown\rangle\\
&= \int_0^\tau \mathsf{x}(x)\left[\hat{\mathbf{F}}^{-1}\tilde{\mathsf{y}}(n)\right]^*\,\mathsf{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem K.1 page 274)}\\
&= \left\langle \mathsf{x}(x) \mid \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^*}\tilde{\mathsf{y}}(n)\right\rangle_{\mathbb{R}}
\end{aligned}$$

✎

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary* [4] (Corollary K.1 page 276).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary K.2 page 276).

Corollary K.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\hat{\mathbf{F}}$ *be the Fourier Series operator with adjoint* $\hat{\mathbf{F}}^*$.

COR $\left\{\hat{\mathbf{F}}\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^*\hat{\mathbf{F}} = \mathbf{I}\right\}$ ($\hat{\mathbf{F}}$ *is* ***unitary*** *...and thus also* NORMAL *and* ISOMETRIC)

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$ (Theorem K.2 page 275). ✎

Corollary K.2. *Let* $\hat{\mathbf{F}}$ *be the Fourier series operator with adjoint* $\hat{\mathbf{F}}^*$ *and inverse* $\hat{\mathbf{F}}^{-1}$.

COR

$$\begin{aligned}
\mathscr{R}(\hat{\mathbf{F}}) &= \mathscr{R}(\hat{\mathbf{F}}^{-1}) &&= \boldsymbol{L}^2_{\mathbb{R}} && \\
\left|\!\left|\!\left|\hat{\mathbf{F}}\right|\!\right|\!\right| &= \left|\!\left|\!\left|\hat{\mathbf{F}}^{-1}\right|\!\right|\!\right| &&= 1 && \text{(UNITARY)}\\
\left\langle \hat{\mathbf{F}}\boldsymbol{x} \mid \hat{\mathbf{F}}\boldsymbol{y}\right\rangle &= \left\langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \mid \hat{\mathbf{F}}^{-1}\boldsymbol{y}\right\rangle &&= \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{(PARSEVAL'S EQUATION)}\\
\left\|\hat{\mathbf{F}}\boldsymbol{x}\right\| &= \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\right\| &&= \|\boldsymbol{x}\| && \text{(PLANCHEREL'S FORMULA)}\\
\left\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\right\| &= \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\right\| &&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{(ISOMETRIC)}
\end{aligned}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary K.1 page 276) and from the properties of unitary operators (Theorem D.26 page 194). ✎

K.3 Fourier series for compactly supported functions

Theorem K.3.

THM *The set*

$$\left\{ \frac{1}{\sqrt{\tau}}e^{i\frac{2\pi}{\tau}nx} \,\middle|\, n\in\mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathsf{f}(x)$ *with support in* $[0 : \tau]$.

[4] *unitary operators*: Definition D.14 page 193

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APPENDIX L

LAPLACE TRANSFORM

> *❝La langue de l analyse, la plus parfaite de toutes les langues, tant par elle-même un puissant instrument de découvertes; ses notations, lorsqu'elles sont nécessaires et heureusement imaginées, sont des germes de nouveaux calculs.❞*
>
> *❝The language of analysis, most perfect of all, being in itself a powerful instrument of discoveries, its notations, especially when they are necessary and happily imagined, are the seeds of new calculi.❞*
>
> Pierre-Simon Laplace[1]

L.1 Operator Definition

Definition L.1. [2] *Let* $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF *The* ***Laplace Transform*** *operator* $\mathbf{L}$ *is here defined as*

$$[\mathbf{L}\mathsf{f}](s) \triangleq \int_{x\in\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathsf{d}x \qquad \forall \mathsf{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Such integrals may *converge* for certain values of s and *diverge* for others.

Definition L.2. *Let* $\mathbf{L}\big[\mathsf{g}(x)\big]$ *be the* LAPLACE TRANSFORM *(Definition L.1 page 277) of a function* $\mathsf{g}(x)$.

DEF *The set* $\mathbf{Roc}\mathbf{L}\big[\mathsf{g}(x)\big]$ *of all* s *for which* $\mathbf{L}\big[\mathsf{g}(x)\big]$ CONVERGES *is the* ***Region of Convergence*** *of* $\mathbf{L}\big[\mathsf{g}(x)\big]$.

In this text, the region of convergence may in places be specified using the *open interval* $(A : B)$ and *closed interval* $[A : B]$.

[1] Laplace (1814) page xxxi ⟨Introduction⟩, Laplace (1812), Laplace (1902) pages 48–49, Moritz (1914) page 200 ⟨Quote 1222., but "conceived" not "imagined", and "are so many germs" not "are the seeds"⟩, https://todayinsci.com/L/Laplace_Pierre/LaplacePierre-Analysis-Quotations.htm, https://translate.google.com/,

[2] Bracewell (1978) page 219 ⟨Chapter 11 The Laplace transform⟩, van der Pol and Bremmer (1959) page 13 ⟨5. Strip of convergence of the Laplace integral⟩, Levy (1958) page 2 ⟨"two-sided transformation"⟩, Betten (2008b) page 295 ⟨(B.1)⟩

Remark L.1. A scaling factor $\frac{1}{\sqrt{2\pi}}$ in front of $\int_{\mathbb{R}}$ in Definition L.1 is not typically found in references offering definitions of the Laplace Transform, and is not included here either. That is not to say, however, that it's not a good idea. Including it would make the operator $\mathbf{L}$ more directly compatible with the *Unitary Fourier Transform* operator $\tilde{\mathbf{F}}$ (Definition M.2 page 294). Note also that a $\frac{1}{2\pi}$ scaling factor is included in [Bachman et al. (2002) page 268] in their definition of *convolution* (Definition F.1 page 203, Section L.8 page 289).

L.2 Operator Inverse

Theorem L.1. [3]

THM

$$\mathsf{g}(x) = \mathbf{L}^{-1}[\mathsf{G}(s)] \triangleq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathsf{G}(s)e^{sx}ds \quad \textit{for some } c \in \mathbb{R}^+$$

L.3 Transversal properties

Theorem L.2. [4] *Let* $\mathbf{L}\big[\mathsf{g}(x)\big]$ *be the* LAPLACE TRANSFORM *(Definition L.1 page 277) of a function* $\mathsf{g}(x)$. *Let the* REGION OF CONVERGENCE *of* $\mathbf{L}[\mathsf{g}(x)](s)$ *be* $A \le \mathbf{R}_{\mathsf{e}}(s) \le B$ *with* $(A, B) \in \mathbb{R}^2$.

THM

Mapping	*Region of Convergence*	*Domain*	*Property*
$\mathbf{L}\big[\mathsf{g}(x-\alpha)\big] = e^{-\alpha s}\mathbf{L}\big[\mathsf{g}(x)\big](s)$	*for* $\mathbf{R}_{\mathsf{e}}(s) \in [A : B]$	$\forall x,\alpha\in\mathbb{C}$	(TRANSLATION)
$\mathbf{L}\big[\mathsf{g}(\alpha x)\big] = \frac{1}{\lvert\alpha\rvert}\mathbf{L}\big[\mathsf{g}(x)\big]\left(\frac{s}{\alpha}\right)$	*for* $\mathbf{R}_{\mathsf{e}}\left(\frac{s}{\alpha}\right) \in [A : B]$	$\forall x,\alpha\in\mathbb{C}$	(DILATION)

✎PROOF:

$$\begin{aligned}
\mathbf{L}\big[\mathsf{g}(x-\alpha)\big] &\triangleq \int_{x=-\infty}^{x=\infty} \mathsf{g}(x-\alpha)e^{-sx}\,\mathrm{dx} && && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \int_{u+\alpha=-\infty}^{u+\alpha=\infty} \mathsf{g}(u)e^{-s(\alpha+u)}\,\mathrm{du} && && \text{where } u \triangleq x-\alpha && \implies x = \alpha + u\\
&= e^{-\alpha s}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-su}\,\mathrm{du} && \forall A \le \mathbf{R}_{\mathsf{e}}(s) \le B && \text{by property of exponents} && b^{x+\alpha} = b^x b^\alpha\\
&\triangleq e^{-\alpha s}\int_{x=-\infty}^{x=\infty} \mathsf{g}(x)e^{-sx}\,\mathrm{dx} && \forall A \le \mathbf{R}_{\mathsf{e}}(s) \le B && \text{by change of variable} && u \to x\\
&\triangleq e^{-\alpha s}\big[\mathbf{L}\mathsf{g}(x)\big] && \forall A \le \mathbf{R}_{\mathsf{e}}(s) \le B && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\mathsf{g}(\alpha x)\big] &\triangleq \int_{x=-\infty}^{x=\infty} \mathsf{g}(\alpha x)e^{-sx}\,\mathrm{dx} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \int_{u/\alpha=-\infty}^{u/\alpha=\infty} \mathsf{g}(u)e^{-s(u/\alpha)}\frac{1}{\alpha}\,\mathrm{du} && \text{where } u \triangleq \alpha x && \implies x = \frac{u}{\alpha}\\
&= \frac{1}{\alpha}\int_{u/\alpha=-\infty}^{u/\alpha=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathrm{du}
\end{aligned}$$

[3] Bracewell (1978) page 220 ⟨Chapter 11 The Laplace transform⟩

[4] Bracewell (1978) page 224 ⟨Table 11.1: "Shift" and "Similarity" entries⟩, Levy (1958) page 15 ⟨Equation 0.8⟩

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$$= \begin{cases} \frac{1}{\alpha}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{du} & \text{if } \alpha \ge 0 \qquad \forall A \le \mathbf{R}_{\mathsf{e}}\left(\frac{s}{\alpha}\right) \le B \\ \frac{1}{\alpha}\int_{u=\infty}^{u=-\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{du} & \text{otherwise} \quad \forall A \le \mathbf{R}_{\mathsf{e}}\left(\frac{s}{\alpha}\right) \le B \end{cases}$$

$$= \begin{cases} \frac{1}{\alpha}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{du} & \text{if } \alpha \ge 0 \qquad \forall A \le \mathbf{R}_{\mathsf{e}}\left(\frac{s}{\alpha}\right) \le B \\ -\frac{1}{\alpha}\int_{u=-\infty}^{u=\infty} \mathsf{g}(u)e^{-(s/\alpha)u}\,\mathsf{du} & \text{otherwise} \quad \forall A \le \mathbf{R}_{\mathsf{e}}\left(\frac{s}{\alpha}\right) \le B \end{cases}$$

$$= \frac{1}{|\alpha|}\int_{x\in\mathbb{R}} \mathsf{g}(x)e^{-(s/\alpha)x}\,\mathsf{dx} \qquad \forall A \le \mathbf{R}_{\mathsf{e}}\left(\frac{s}{\alpha}\right) \le B \quad \text{by change of variable} \qquad u \to x$$

$$\triangleq \frac{1}{|\alpha|}\left[\mathbf{L}\mathsf{g}(x)\right]\left(\frac{s}{\alpha}\right) \qquad \forall A \le \mathbf{R}_{\mathsf{e}}\left(\frac{s}{\alpha}\right) \le B \quad \text{by definition of } \mathbf{L} \qquad \text{(Definition L.1 page 277)}$$

✏

Corollary L.1. [5] *Let* $\mathbf{L}$, $\mathsf{G}(s)$, A, *and* B *be defined as in Theorem L.2 (page 278).*

COR	*Mapping*	*Region of Convergence*	*Domain*	*Property*
	$\mathbf{L}\left[\mathsf{g}(-x)\right] = \mathsf{G}(-s)$	*for* $\mathbf{R}_{\mathsf{e}}(s) \in [-B : -A]$	$\forall x,\alpha\in\mathbb{C}$	(REVERSAL)

✎PROOF:

$$\begin{aligned} \mathbf{L}\left[\mathsf{g}(-x)\right] &= \mathbf{L}\left[\mathsf{g}([-1]x)\right] & \mathbf{R}_{\mathsf{e}}(s) \in [A : B] & \quad \text{by definition of unary operator } - \\ &= \mathbf{L}\left[\frac{1}{|-1|}\mathsf{g}\left(\frac{x}{-1}\right)\right] & \mathbf{R}_{\mathsf{e}}\left(\frac{s}{-1}\right) \in [A : B] & \quad \text{by } \textit{dilation} \text{ property (Theorem L.2 page 278)} \\ &= \mathsf{G}(-s) & \mathbf{R}_{\mathsf{e}}(s) \in [-B : -A] & \end{aligned}$$

✏

L.4 Linear properties

Theorem L.3. [6] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator (Definition L.1 page 277). Let* $\mathsf{G}(s) \triangleq \left[\mathbf{L}\mathsf{g}(x)\right]$ *and* $\mathsf{F}(s) \triangleq [\mathbf{L}\mathsf{f}(x)]$. *Let the* REGION OF CONVERGENCE *of* $\mathsf{G}(s)$ *be* $A \le \mathbf{R}_{\mathsf{e}}(s) \le B$ *and the* REGION OF CONVERGENCE *of* $\mathsf{F}(s)$ *be* $C \le \mathbf{R}_{\mathsf{e}}(s) \le D$.

THM	*Mapping*	*Region of Convergence*	*Domain*	*Property*
	$\mathbf{L}\left[\mathsf{f}(x)+\mathsf{g}(x)\right] = \mathsf{F}(s)+\mathsf{G}(s)$	*for* $\mathbf{R}_{\mathsf{e}}(s) \in [A : B] \cap [C : D]$	$\forall x\in\mathbb{C}$	(ADDITIVE)
	$\mathbf{L}\left[\alpha\mathsf{g}(x)\right] = \alpha\mathsf{G}(s)$	*for* $\mathbf{R}_{\mathsf{e}}(s) \in [A : B]$	$\forall x,\alpha\in\mathbb{C}$	(HOMOGENEOUS)

Corollary L.2 (Linear Properties)**.** *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator (Definition L.1 page 277). Let* A *and* B *be real numbers such that* $[A : B]$ *is the* REGION OF CONVERGENCE *of* $\mathbf{L}\left[\mathsf{g}(x)\right]$. *Let* C *and* D *be real numbers such that* $[C : D]$ *is the* REGION OF CONVERGENCE *of* $\mathbf{L}[\mathsf{f}(x)]$. *Let* A_n *and* B_n *be real numbers such that* $\left[A_n : B_n\right]$ *is the* REGION OF CONVERGENCE *of* $\mathbf{L}\left[\mathsf{g}_n(x)\right]$.

[5] Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩

[6] Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩, Betten (2008a) page 296 ⟨(B.6)⟩, Levy (1958) page 13 ⟨Equation 0.2⟩, van der Pol and Bremmer (1959) page 22 ⟨Introduction⟩, Shafii-Mousavi (2015) page 7 ⟨Theorem 1.4⟩

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COR	Mapping			Region of Convergence	Domain
	$\mathbf{L}[0]$	$=$	0	for $\mathbf{R}_e(s) \in [A : B]$	$\forall x \in \mathbb{C}$
	$\mathbf{L}\big[-g(x)\big]$	$=$	$-\mathbf{L}\big[g(x)\big]$	for $\mathbf{R}_e(s) \in [A : B]$	$\forall x \in \mathbb{C}$
	$\mathbf{L}\big[f(x) - g(x)\big]$	$=$	$\mathbf{L}\big[g(x)\big] - \mathbf{L}[f(x)]$	for $\mathbf{R}_e(s) \in [A : B] \cap [C : D]$	$\forall x \in \mathbb{C}$
	$\mathbf{L}\left[\sum_{n=1}^{N} \alpha_n g_n(x)\right]$	$=$	$\sum_{n=1}^{N} \alpha_n \mathbf{L}\big[g_n(x)\big]$	for $\mathbf{R}_e(s) \in \bigcap_{n=1}^{N} \big[A_n : B_n\big]$	$\forall x, \alpha_n \in \mathbb{C}$

✎PROOF:

1. By Theorem L.3 (page 279), the operator *Laplace Transform* operator $\mathbf{L}$ is *additive* and *homogeneous*.
2. By item (1) and Definition D.4 (page 171), $\mathbf{L}$ is *linear*.
3. By item (2) and Theorem D.1 (page 171), the four properties listed follow.

✏

L.5 Modulation properties

Theorem L.4. [7] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277). *Let* $G(s) \triangleq \big[\mathbf{L}g(x)\big]$. *Let the* REGION OF CONVERGENCE *of* $G(s)$ *be* $A \le \mathbf{R}_e(s) \le B$.

THM	Mapping			Region of Convergence	Domain	Property
	$\mathbf{L}\big[e^{-\alpha x} g(x)\big]$	$=$	$G(s+\alpha)$	for $A - \mathbf{R}_e(\alpha) \le \mathbf{R}_e(s) \le B - \mathbf{R}_e(\alpha)$	$\forall x, \alpha \in \mathbb{C}$	(MODULATION)

✎PROOF:

$$\begin{aligned}
\mathbf{L}\big[e^{-\alpha x} g(x)\big] &\triangleq \int_{x\in\mathbb{R}} e^{-\alpha x} g(x) e^{-sx}\,dx && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \int_{x\in\mathbb{R}} g(x) e^{-(s+\alpha)x}\,dx && A \le \mathbf{R}_e(s+\alpha) \le B && b^{x+y} = b^x b^y\\
&\triangleq \big[\mathbf{L}g(x)\big](s+\alpha) && A - \mathbf{R}_e(\alpha) \le \mathbf{R}_e(s) \le B - \mathbf{R}_e(\alpha) && \text{(Definition L.1 page 277)}\\
&\triangleq G(s+\alpha) && A - \mathbf{R}_e(\alpha) \le \mathbf{R}_e(s) \le B - \mathbf{R}_e(\alpha) && \text{by definition of } G(s)
\end{aligned}$$

✏

Corollary L.3. [8] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277). *Let* $G(s) \triangleq \big[\mathbf{L}g(x)\big]$ *Let the* REGION OF CONVERGENCE *of* $G(s)$ *be* $A \le \mathbf{R}_e(s) \le B$.

COR	Mapping			Region of Convergence	Domain
	$\mathbf{L}\big[\cos(\omega_o x) g(x)\big]$	$=$	$\frac{1}{2}G(s - i\omega_o) + \frac{1}{2}G(s + i\omega_o)$	for $\mathbf{R}_e(s) \in [A : B]$	$\forall x \omega_o \in \mathbb{C}$
	$\mathbf{L}\big[\sin(\omega_o x) g(x)\big]$	$=$	$-\frac{i}{2}G(s - i\omega_o) + \frac{i}{2}G(s + i\omega_o)$	for $\mathbf{R}_e(s) \in [A : B]$	$\forall x \omega_o \in \mathbb{C}$
	$\mathbf{L}\big[\cosh(\omega_o x) g(x)\big]$	$=$	$\frac{1}{2}G(s - \omega_o) + \frac{1}{2}G(s + \omega_o)$	for $\mathbf{R}_e(s) \in \big[A + \big\lvert\mathbf{R}_e(\omega_o)\big\rvert : B - \big\lvert\mathbf{R}_e(\omega_o)\big\rvert\big]$	$\forall x \omega_o \in \mathbb{C}$
	$\mathbf{L}\big[\sinh(\omega_o x) g(x)\big]$	$=$	$\frac{1}{2}G(s - \omega_o) - \frac{1}{2}G(s + \omega_o)$	for $\mathbf{R}_e(s) \in \big[A + \big\lvert\mathbf{R}_e(\omega_o)\big\rvert : B - \big\lvert\mathbf{R}_e(\omega_o)\big\rvert\big]$	$\forall x \omega_o \in \mathbb{C}$

[7] Bracewell (1978) page 224 ⟨Table 11.1: "Modulation" entry⟩, Levy (1958) page 19 ⟨Equation 1.2⟩
[8] Bracewell (1978) page 224 ⟨Table 11.1 Theorems for the Laplace Transform⟩

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✎PROOF:

1. Mappings:

$$\begin{aligned}
\mathbf{L}\big[\cosh(\omega_o x)g(x)\big] &= \mathbf{L}\left[\left(\frac{e^{\omega_o x}+e^{-\omega_o x}}{2}\right)g(x)\right] && \text{by definition of } \cosh(x) && \text{(Definition I.6 page 249)}\\
&= \frac{1}{2}\mathbf{L}\big[e^{\omega_o x}g(x)\big](s)+\frac{1}{2}\mathbf{L}\big[e^{-\omega_o x}g(x)\big](s) && \text{by } \textit{additive} \text{ property} && \text{(Theorem L.3 page 279)}\\
&= \frac{1}{2}\mathbf{L}\big[g(x)\big](s-\omega)+\frac{1}{2}\mathbf{L}\big[g(x)\big](s+\omega) && \text{by } \textit{modulation} \text{ prop.} && \text{(Theorem L.4 page 280)}\\
&= \frac{1}{2}G(s-\omega_o)+\frac{1}{2}G(s+\omega_o) && \text{by definition of } G(s)
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\sinh(\omega_o x)g(x)\big] &= \mathbf{L}\left[\left(\frac{e^{\omega_o x}-e^{-\omega_o x}}{2}\right)g(x)\right] && \text{by definition of } \sinh(x) && \text{(Definition I.6 page 249)}\\
&= \frac{1}{2}\mathbf{L}\big[e^{\omega_o x}g(x)\big](s)-\frac{1}{2}\mathbf{L}\big[e^{-\omega_o x}g(x)\big](s) && \text{by } \textit{additive} \text{ property} && \text{(Theorem L.3 page 279)}\\
&= \frac{1}{2}\mathbf{L}\big[g(x)\big](s-\omega)-\frac{1}{2}\mathbf{L}\big[g(x)\big](s+\omega) && \text{by } \textit{modulation} \text{ prop.} && \text{(Theorem L.4 page 280)}\\
&= \frac{1}{2}G(s-\omega_o)-\frac{1}{2}G(s+\omega_o) && \text{by definition of } G(s)
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\cos(\omega_o x)g(x)\big] &= \mathbf{L}\big[\cosh(i\omega_o x)g(x)\big] && \text{by Theorem I.12 page 249}\\
&= \frac{1}{2}G(s-i\omega_o)+\frac{1}{2}G(s+i\omega_o) && \text{by } \mathbf{L}\big[\cos(\omega_o x)g(x)\big] \text{ result}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\sin(\omega_o x)g(x)\big] &= \mathbf{L}\big[-i^2\sin(\omega_o x)g(x)\big]\\
&= -i\mathbf{L}\big[i\sin(\omega_o x)g(x)\big] && \text{by } \textit{homogeneous} \text{ property} && \text{(Theorem L.3 page 279)}\\
&= -i\mathbf{L}\big[\sinh(i\omega_o x)g(x)\big] && \text{by Theorem I.12 page 249}\\
&= -\frac{i}{2}G(s-i\omega_o)+\frac{i}{2}G(s+i\omega_o) && \text{by } \mathbf{L}\big[\sin(\omega_o x)g(x)\big] \text{ result}
\end{aligned}$$

2. Region of Convergence of $\mathbf{L}\big[\cos(\omega_o x)g(x)\big]$ and $\mathbf{L}\big[\sin(\omega_o x)g(x)\big]$:

$$\begin{aligned}
&\mathbf{RocL}\big[{}^{\cos}\!/_{\sin}(\omega_o x)g(x)\big]\\
&= \mathbf{RocL}\left[\left(\frac{e^{i\omega_o x}\pm e^{-i\omega_o x}}{2}\right)g(x)\right] && \text{by } \textit{Euler's Identity} && \text{(Theorem I.5 page 240)}\\
&= \mathbf{Roc}\left(\mathbf{L}\left[\frac{e^{i\omega_o x}}{2}g(x)\right]\pm\mathbf{L}\left[\frac{e^{-i\omega_o x}}{2}g(x)\right]\right) && \text{by } \textit{additive} \text{ property} && \text{(Theorem L.3 page 279)}\\
&= \mathbf{RocL}\left[\left(\frac{e^{-i\omega_o x}}{2}\right)g(x)\right]\cap\mathbf{RocL}\left[\left(\frac{e^{i\omega_o x}}{2}\right)g(x)\right]\\
&= \big[A-\mathbf{R}_e(i\omega)\,:\,B-\mathbf{R}_e(i\omega)\big]\cap\big[A-\mathbf{R}_e(-i\omega)\,:\,B-\mathbf{R}_e(-i\omega)\big]\\
&= [A-0\,:\,B-0]\cap[A-0\,:\,B-0]\\
&= [A\,:\,B]
\end{aligned}$$

3. Region of Convergence of $\mathbf{L}\big[\cosh(\omega_o x)g(x)\big]$ and $\mathbf{L}\big[\sinh(\omega_o x)g(x)\big]$:

$$\begin{aligned}
&\mathbf{RocL}\big[{}^{\cosh}\!/_{\sinh}(\omega_o x)g(x)\big]\\
&= \mathbf{RocL}\left[\left(\frac{e^{\omega_o x}\pm e^{-\omega_o x}}{2}\right)g(x)\right] && \text{by def. } \cosh(x),\ \sinh(x) && \text{(Definition I.6 page 249)}
\end{aligned}$$

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$$
\begin{aligned}
&= \mathbf{Roc}\left(\mathbf{L}\left[\frac{e^{\omega_o x}}{2}g(x)\right] \pm \mathbf{L}\left[\frac{e^{-\omega_o x}}{2}g(x)\right]\right) && \text{by } \textit{additive} \text{ property} && \text{(Theorem L.3 page 279)}\\
&= \mathbf{Roc}\mathbf{L}\left[\left(\frac{e^{-\omega_o x}}{2}\right)g(x)\right] \cap \mathbf{Roc}\mathbf{L}\left[\left(\frac{e^{\omega_o x}}{2}\right)g(x)\right]\\
&= \left[A - \mathbf{R}_e(\omega_o) \,:\, B - \mathbf{R}_e(\omega_o)\right] \cap \left[A - \mathbf{R}_e(-\omega_o) \,:\, B - \mathbf{R}_e(-\omega_o)\right]\\
&= \begin{cases} \left[A + \mathbf{R}_e(\omega_o) \,:\, B - \mathbf{R}_e(\omega_o)\right] & \text{for } \omega \geq 0\\ \left[A - \mathbf{R}_e(\omega_o) \,:\, B + \mathbf{R}_e(\omega_o)\right] & \text{otherwise}\end{cases}\\
&= \left[A + \left|\mathbf{R}_e(\omega_o)\right| \,:\, B - \left|\mathbf{R}_e(\omega_o)\right|\right] && \text{by definition of } |x|
\end{aligned}
$$

✏

L.6 Causality properties

Definition L.3. [9] *The **Heaviside step function** $\mu(x)$ or **unit step function** is defined as*

DEF

$$\mu(x) \triangleq \begin{cases} 1 & \forall x \geq 0\\ 0 & \textit{otherwise}\end{cases}$$

Theorem L.5. [10] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277) *and* $\mu(x)$ *the* UNIT STEP *function* (Definition L.3 page 282).

THM

Mapping	*Region of Convergence*	*Domain*
$\mathbf{L}[\mu(x)] = \frac{1}{s}$	*for* $\mathbf{R}_e(s) > 0$	$\forall x \in \mathbb{R}$

✎PROOF:

$$
\begin{aligned}
\mathbf{L}[\mu(x)] &\triangleq \int_{\mathbb{R}} \mu(x)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \int_0^\infty e^{-sx}\,\mathrm{d}x && \text{by definition of } \mu(x) && \text{(Definition L.3 page 282)}\\
&= \left.\frac{e^{-sx}}{-s}\right|_0^\infty && \text{by } \textit{Fundamental Theorem of Calculus}\\
&= \lim_{x\to\infty}\left[\frac{e^{-sx}}{-s}\right] - \left(\frac{e^0}{-s}\right)\\
&= 0 + \frac{1}{s} && \forall \mathbf{R}_e(s) > 0\\
&= \frac{1}{s} && \forall \mathbf{R}_e(s) > 0
\end{aligned}
$$

✏

Corollary L.4. *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277) *and* $\mu(x)$ *the* UNIT STEP *function* (Definition L.3 page 282).

THM

	Mapping	*Region of Convergence*	*Domain*
(1).	$\mathbf{L}[\mu(-x)] = -\frac{1}{s}$	*for* $\mathbf{R}_e(s) < 0$	$\forall x \in \mathbb{R}$
(2).	$\mathbf{L}[\mu(x-a)] = \frac{e^{-as}}{s}$	*for* $\mathbf{R}_e(s) < 0$	$\forall a,x \in \mathbb{R}$
(2).	$\mathbf{L}[\mu(a-x)] = -\frac{e^{as}}{s}$	*for* $\mathbf{R}_e(s) > 0$	$\forall a,x \in \mathbb{R}$

[9] Betten (2008a) page 285
[10] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩

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✎PROOF:

$$\begin{aligned}
\mathbf{L}[\mu(-x)] &= \mathbf{L}[\mu(x)](-s) && \mathbf{R}_e(s) < 0 && \text{by } \textit{reversal} \text{ property} && \text{(Corollary L.1 page 279)}\\
&= \frac{-1}{s} && \mathbf{R}_e(s) < 0 && \text{by } \textit{step} \text{ property} && \text{by (Theorem L.5 page 282)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}[\mu(x-a)] &= e^{-as}\mathbf{L}[\mu(x)] && \mathbf{R}_e(s) < 0 && \text{by } \textit{translation} \text{ property} && \text{(Theorem L.2 page 278)}\\
&= \frac{e^{-as}}{s} && \mathbf{R}_e(s) < 0 && \text{by } \textit{step} \text{ property} && \text{by (Theorem L.5 page 282)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}[\mu(a-x)] &= e^{as}\mathbf{L}[\mu(-x)] && \mathbf{R}_e(s) > 0 && \text{by } \textit{translation} \text{ property} && \text{(Theorem L.2 page 278)}\\
&= -\frac{e^{as}}{s} && \mathbf{R}_e(s) > 0 && \text{by previous result}
\end{aligned}$$

✏

Corollary L.5. [11] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* *(Definition L.1 page 277)* *and* $\mu(x)$ *the* UNIT STEP *function.*

COR	*Mapping*		*Region of Convergence*	*Domain*
	$\mathbf{L}\left[e^{-\alpha x}\mu(x)\right]$	$= \dfrac{1}{s+\alpha}$	*for* $\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)$	$\forall x\in\mathbb{R}; \alpha\in\mathbb{C}$
	$\mathbf{L}\left[e^{-\alpha x}\mu(-x)\right]$	$= \dfrac{1}{s+\alpha}$	*for* $\mathbf{R}_e(s) < \mathbf{R}_e(\alpha)$	$\forall x\in\mathbb{R}; \alpha\in\mathbb{C}$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[e^{-\alpha x}\mu(x)\right](s) &= \mathbf{L}[\mu(x)](s+\alpha) && \text{by } \textit{modulation} && \text{(Theorem L.4 page 280)}\\
&= \frac{1}{s+\alpha} && \forall \mathbf{R}_e(s) \in \big(0-\mathbf{R}_e(\alpha) : \infty - \mathbf{R}_e(\alpha)\big) \text{ by Theorem L.5 page 282}\\
&= \frac{1}{s+\alpha} && \forall \mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\left[e^{-\alpha x}\mu(-x)\right](s) &= \mathbf{L}[\mu(-x)](s+\alpha) && \text{by } \textit{modulation} && \text{(Theorem L.4 page 280)}\\
&= \frac{-1}{s+\alpha} && \forall \mathbf{R}_e(s) \in \big(-\infty - \mathbf{R}_e(\alpha) : 0-(-\mathbf{R}_e(\alpha))\big) \text{ by Corollary L.4 page 282}\\
&= \frac{-1}{s+\alpha} && \forall \mathbf{R}_e(s) < \mathbf{R}_e(\alpha) \quad \text{by } \textit{anti-causality} && \text{(Corollary L.4 page 282)}
\end{aligned}$$

✏

Corollary L.6. [12] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* *(Definition L.1 page 277)* *and* $\mu(x)$ *the* UNIT STEP *function.*

COR	*Mapping*		*Region of Convergence*	*Domain*
	(1). $\mathbf{L}\left[\cos(\omega_o x)\mu(x)\right]$	$= \dfrac{s}{s^2+\omega_o^2}$	*for* $\mathbf{R}_e(s) > 0$	$x,\omega_o\in\mathbb{R}$
	(2). $\mathbf{L}\left[\sin(\omega_o x)\mu(x)\right]$	$= \dfrac{\omega_o}{s^2+\omega_o^2}$	*for* $\mathbf{R}_e(s) > 0$	$x,\omega_o\in\mathbb{R}$
	(3). $\mathbf{L}\left[\cos(\omega_o x)\mu(-x)\right]$	$= \dfrac{-s}{s^2+\omega_o^2}$	*for* $\mathbf{R}_e(s) < 0$	$x,\omega_o\in\mathbb{R}$
	(4). $\mathbf{L}\left[\sin(\omega_o x)\mu(-x)\right]$	$= \dfrac{-\omega_o}{s^2+\omega_o^2}$	*for* $\mathbf{R}_e(s) < 0$	$x,\omega_o\in\mathbb{R}$

[11] van der Pol and Bremmer (1959) page 22 ⟨Introduction⟩, Shafii-Mousavi (2015) page 3 ⟨Table 1, using One-Sided Laplace Transform⟩, van der Pol and Bremmer (1959) page 26 ⟨(8) seems to have an error: $\frac{s}{s+\alpha}$⟩

[12] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩, Shafii-Mousavi (2015) page 3 ⟨Table 1, using One-Sided Laplace Transform⟩

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✎PROOF:

$$\begin{aligned}
\mathbf{L}\big[\cos(\omega_o x)\mu(x)\big](s) &= \frac{1}{2}\mathbf{L}[\mu(x)](s-i\omega_o) + \frac{1}{2}\mathbf{L}[\mu(x)](s+i\omega_o) && \text{by } \textit{modulation} \quad \text{(Corollary L.3 page 280)}\\
&= \frac{1}{2}\left[\frac{1}{s-i\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+i\omega_o}\right] && \mathbf{R}_e(s)>0 \quad \text{by } \textit{causal} \text{ prop.} \quad \text{(Theorem L.5 page 282)}\\
&= \frac{1}{2}\left[\frac{1}{s-i\omega_o}\right]\left[\frac{s+i\omega_o}{s+i\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+i\omega_o}\right]\left[\frac{s-i\omega_o}{s-i\omega_o}\right] && (\textit{Rationalizing the Denominator})\\
&= \frac{1}{2}\left[\frac{(s+i\omega_o)+(s-i\omega_o)}{s^2+\omega_o^2}\right] && \mathbf{R}_e(s)>0\\
&= \frac{s}{s^2+\omega_o^2} && \mathbf{R}_e(s)>0
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\sin(\omega_o x)\mu(x)\big](s) &= -\frac{i}{2}\mathbf{L}[\mu(x)](s-i\omega_o) + \frac{i}{2}\mathbf{L}[\mu(x)](s+i\omega_o) && \text{by } \textit{modulation} \quad \text{(Corollary L.3 page 280)}\\
&= -\frac{i}{2}\left[\frac{1}{s-i\omega_o}\right] + \frac{i}{2}\left[\frac{1}{s+i\omega_o}\right] && \mathbf{R}_e(s)>0 \quad \text{by } \textit{causal} \text{ prop.} \quad \text{(Theorem L.5 page 282)}\\
&= -\frac{i}{2}\left[\frac{1}{s-i\omega_o}\right]\left[\frac{s+i\omega_o}{s+i\omega_o}\right] + \frac{i}{2}\left[\frac{1}{s+i\omega_o}\right]\left[\frac{s-i\omega_o}{s-i\omega_o}\right] && (\textit{Rationalizing the Denominator})\\
&= \frac{i}{2}\left[\frac{-(s+i\omega_o)+(s-i\omega_o)}{s^2+\omega_o^2}\right] && \mathbf{R}_e(s)>0\\
&= \frac{\omega_o}{s^2+\omega_o^2} && \mathbf{R}_e(s)>0
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\mu(-x)\cos(\omega_o x)\big](s) &= \mathbf{L}\big[\mu(-x)\cos(\omega_o(-x))\big](s) && \text{by } \textit{even} \text{ property of } \cos(x) \quad \text{(Theorem I.2 page 237)}\\
&= \mathbf{L}\big[\mu(x)\cos(\omega_o x)\big](-s) && \text{by } \textit{reversal} \text{ property} \quad \text{(Corollary L.1 page 279)}\\
&= \frac{(-s)}{(-s)^2+\omega_o^2} && \mathbf{R}_e(s)<0 \quad \text{by (1)}\\
&= \frac{-s}{s^2+\omega_o^2} && \mathbf{R}_e(s)<0
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\sin(\omega_o x)\mu(-x)\big](s) &= \mathbf{L}\big[-\sin(\omega_o(-x))\mu(-x)\big](s) && \text{by } \textit{odd} \text{ property of } \sin(x) \quad \text{(Theorem I.2 page 237)}\\
&= -\mathbf{L}\big[\sin(\omega_o(-x))\mu(-x)\big](s) && \text{by } \textit{homogeneous} \text{ property} \quad \text{(Theorem L.3 page 279)}\\
&= -\mathbf{L}\big[\sin(\omega_o x)\mu(x)\big](-s) && \text{by } \textit{reversal} \text{ property} \quad \text{(Corollary L.1 page 279)}\\
&= -\left[\frac{\omega_o}{(-s)^2+\omega_o^2}\right] && \mathbf{R}_e(s)<0 \quad \text{by (2)}\\
&= \frac{-\omega_o}{s^2+\omega_o^2} && \mathbf{R}_e(s)<0
\end{aligned}$$

✏

Corollary L.7. [13] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277) *and* $\mu(x)$ *the* UNIT STEP *function.*

COR	Mapping	Region of Convergence	Domain
(1).	$\mathbf{L}\big[\cosh(\omega_o x)\mu(x)\big] = \frac{s}{s^2-\omega_o^2}$	for $\mathbf{R}_e(s) > \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$
(2).	$\mathbf{L}\big[\sinh(\omega_o x)\mu(x)\big] = \frac{\omega_o}{s^2-\omega_o^2}$	for $\mathbf{R}_e(s) > \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$
(3).	$\mathbf{L}\big[\sinh(\omega_o x)\mu(-x)\big] = \frac{-s}{s^2-\omega_o^2}$	for $\mathbf{R}_e(s) < \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$
(4).	$\mathbf{L}\big[\sinh(\omega_o x)\mu(-x)\big] = \frac{-\omega_o}{s^2-\omega_o^2}$	for $\mathbf{R}_e(s) < \lvert\omega_o\rvert$	$x,\omega_o\in\mathbb{R}$

[13] Shafii-Mousavi (2015) page 3 ⟨Table 1, using One-Sided Laplace Transform⟩

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✎PROOF:

1. Mappings for $\mathbf{L}\big[\cosh(\omega_o x)\mu(x)\big]$ and $\mathbf{L}\big[\sinh(\omega_o x)\mu(x)\big]$:

$$\begin{aligned}
\mathbf{L}\big[\cosh(\omega_o x)\mu(x)\big](s) &= \frac{1}{2}\mathbf{L}[\mu(x)](s-\omega_o) + \frac{1}{2}\mathbf{L}[\mu(x)](s+\omega_o) && \text{by } \textit{modulation} && \text{(Corollary L.3 page 280)}\\
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+\omega_o}\right] && \text{by } \textit{causal} \text{ property} && \text{(Theorem L.5 page 282)}\\
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right]\left[\frac{s+\omega_o}{s+\omega_o}\right] + \frac{1}{2}\left[\frac{1}{s+\omega_o}\right]\left[\frac{s-\omega_o}{s-\omega_o}\right]\\
&= \frac{1}{2}\left[\frac{(s+\omega_o)+(s-\omega_o)}{s^2-\omega_o^2}\right]\\
&= \frac{s}{s^2-\omega_o^2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\sinh(\omega_o x)\mu(x)\big](s) &= \frac{1}{2}\mathbf{L}[\mu(x)](s-\omega_o) - \frac{1}{2}\mathbf{L}[\mu(x)](s+\omega_o) && \text{by } \textit{modulation} && \text{(Corollary L.3 page 280)}\\
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right] - \frac{1}{2}\left[\frac{1}{s+\omega_o}\right] && \text{by } \textit{causal} \text{ property} && \text{(Theorem L.5 page 282)}\\
&= \frac{1}{2}\left[\frac{1}{s-\omega_o}\right]\left[\frac{s+\omega_o}{s+\omega_o}\right] - \frac{1}{2}\left[\frac{1}{s+\omega_o}\right]\left[\frac{s-\omega_o}{s-\omega_o}\right]\\
&= \frac{1}{2}\left[\frac{(s+\omega_o)-(s-\omega_o)}{s^2-\omega_o^2}\right]\\
&= \frac{\omega_o}{s^2-\omega_o^2}
\end{aligned}$$

2. Region of Convergence of $\mathbf{L}\big[\cosh(\omega_o x)\mu(x)\big]$ and $\mathbf{L}\big[\sinh(\omega_o x)\mu(x)\big]$:

$$\begin{aligned}
\mathbf{RoC}\mathbf{L}\big[\cosh(\omega_o x)\mu(x)\big] &= \big[A + \big|\mathbf{R}_e(\omega_o)\big| \,:\, B - \big|\mathbf{R}_e(\omega_o)\big|\big] && \text{by Corollary L.3 page 280}\\
&= \big(0 + \big|\mathbf{R}_e(\omega_o)\big| \,:\, \infty - \big|\mathbf{R}_e(\omega_o)\big|\big) && \text{by Theorem L.5 page 282}\\
&= \big(\big|\mathbf{R}_e(\omega_o)\big| \,:\, \infty\big)\\
&\implies \mathbf{RoC}\mathbf{L}\big[\cosh(\omega_o x)\mu(x)\big] > \big|\mathbf{R}_e(\omega_o)\big|
\end{aligned}$$

3. Mappings for $\mathbf{L}\big[\cosh(\omega_o x)\mu(-x)\big]$ and $\mathbf{L}\big[\sinh(\omega_o x)\mu(-x)\big]$:

$$\begin{aligned}
\mathbf{L}\big[\cosh(\omega_o x)\mu(-x)\big](s) &= \mathbf{L}\big[\cosh(\omega_o(-x))\mu(-x)\big](s)\\
&= \mathbf{L}\big[\cosh(\omega_o x)\mu(x)\big](-s) && \text{by } \textit{reversal} \text{ property} && \text{(Corollary L.1 page 279)}\\
&= \frac{(-s)}{(-s)^2-\omega_o^2} && \text{by previous result}\\
&= \frac{-s}{s^2-\omega_o^2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\big[\sinh(\omega_o x)\mu(-x)\big](s) &= \mathbf{L}\big[-\sinh(\omega_o(-x))\mu(-x)\big](s)\\
&= -\mathbf{L}\big[\sinh(\omega_o(-x))\mu(-x)\big](s) && \text{by } \textit{homogeneous} \text{ property} && \text{(Theorem L.3 page 279)}\\
&= -\mathbf{L}\big[\sinh(\omega_o x)\mu(x)\big](-s) && \text{by } \textit{reversal} \text{ property} && \text{(Corollary L.1 page 279)}\\
&= \frac{-\omega_o}{(-s)^2-\omega_o^2} && \text{by previous result}\\
&= \frac{-\omega_o}{s^2-\omega_o^2}
\end{aligned}$$

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4. Region of Convergence of $\mathbf{L}\big[\cosh(\omega_o x)\mu(-x)\big]$ and $\mathbf{L}\big[\sinh(\omega_o x)\mu(-x)\big]$:

$$\begin{aligned}
\mathbf{RoCL}\big[\cosh(\omega_o x)\mu(-x)\big] &= \big[A + \big|\mathbf{R}_e(\omega_o)\big| \,:\, B - \big|\mathbf{R}_e(\omega_o)\big|\big] && \text{by Corollary L.3 page 280}\\
&= \big(-\infty + \big|\mathbf{R}_e(\omega_o)\big| \,:\, 0 - \big|\mathbf{R}_e(\omega_o)\big|\big) && \text{by Corollary L.4 page 282}\\
&= \big(-\infty \,:\, \big|\mathbf{R}_e(\omega_o)\big|\big)\\
&\implies \mathbf{RoCL}\big[\cosh(\omega_o x)\mu(-x)\big] < \big|\mathbf{R}_e(\omega_o)\big|
\end{aligned}$$

✏

Corollary L.8. [14] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277) *and* $\mu(x)$ *the* UNIT STEP *function.*

COR	Mapping		Region of Convergence	Domain
(1).	$\mathbf{L}\big[\cos(\omega_o x)e^{-\alpha x}\mu(x)\big]$	$= \dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)$	$x,\omega_o\in\mathbb{R};\alpha\in\mathbb{C}$
(2).	$\mathbf{L}\big[\sin(\omega_o x)e^{-\alpha x}\mu(x)\big]$	$= \dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)$	$x,\omega_o\in\mathbb{R};\alpha\in\mathbb{C}$
(3).	$\mathbf{L}\big[\cos(\omega_o x)e^{\alpha x}\mu(-x)\big]$	$= \dfrac{-s+\alpha}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)$	$x,\omega_o\in\mathbb{R};\alpha\in\mathbb{C}$
(4).	$\mathbf{L}\big[\sin(\omega_o x)e^{\alpha x}\mu(-x)\big]$	$= \dfrac{-\omega_o}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)$	$x,\omega_o\in\mathbb{R};\alpha\in\mathbb{C}$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\big[\cos(\omega_o x)e^{-\alpha x}\mu(x)\big](s) &= \mathbf{L}\big[\mu(x)\cos(\omega_o x)\big](s+\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem L.4 page 280)}\\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s)\in\big(0-\mathbf{R}_e(\alpha) \,:\, \infty-\mathbf{R}_e(\alpha)\big) && \text{by Corollary L.6}\\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)\\
\mathbf{L}\big[\sin(\omega_o x)e^{-\alpha x}\mu(x)\big](s) &= \mathbf{L}\big[\mu(x)\sin(\omega_o x)\big](s+\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem L.4 page 280)}\\
&= \frac{\omega_o}{(s+\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s)\in\big(0-\mathbf{R}_e(\alpha) \,:\, \infty-\mathbf{R}_e(\alpha)\big) && \text{by Corollary L.6}\\
&= \frac{\omega_o}{(s+\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)\\
\mathbf{L}\big[\cos(\omega_o x)e^{\alpha x}\mu(-x)\big](s) &= \mathbf{L}\big[\mu(-x)\cos(\omega_o x)\big](s-\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem L.4 page 280)}\\
&= \frac{-(s-\alpha)}{(s-\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s)\in\big(0-\mathbf{R}_e(\alpha) \,:\, \infty-\mathbf{R}_e(\alpha)\big) && \text{by Corollary L.6}\\
&= \frac{-s+\alpha}{(s-\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)\\
\mathbf{L}\big[\sin(\omega_o x)e^{\alpha x}\mu(-x)\big](s) &= \mathbf{L}\big[\mu(-x)\sin(\omega_o x)\big](s-\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem L.4 page 280)}\\
&= \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s)\in\big(0-\mathbf{R}_e(\alpha) \,:\, \infty-\mathbf{R}_e(\alpha)\big) && \text{by Corollary L.6}\\
&= \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall\mathbf{R}_e(s) > -\mathbf{R}_e(\alpha)
\end{aligned}$$

✏

Corollary L.9. *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277).

COR				
	$\mathbf{L}\big[\cos(\omega_o x)\big]$	*is* ***divergent***	$\forall s\in\mathbb{C}$	$\forall x,\omega_o\in\mathbb{R}$
	$\mathbf{L}\big[\sin(\omega_o x)\big]$	*is* ***divergent***	$\forall s\in\mathbb{C}$	$\forall x,\omega_o\in\mathbb{R}\backslash\{0\}$

[14] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩

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✎PROOF:

$$\begin{aligned}
\mathbf{L}\big[\cos(\omega_o x)\big] &= \underbrace{\mathbf{L}\big[\mu(x)\cos(\omega_o x)\big]}_{\forall \mathbf{R}_e(s)>0} + \underbrace{\mathbf{L}\big[\mu(-x)\cos(\omega_o x)\big]}_{\forall \mathbf{R}_e(s)<0} && \text{by Corollary L.6 page 283}\\
&= \underbrace{\frac{s}{s^2+\omega_o^2}}_{\forall \mathbf{R}_e(s)>0} + \underbrace{\frac{-s}{s^2+\omega_o^2}}_{\forall \mathbf{R}_e(s)<0} && \text{by Corollary L.6 page 283}\\
&= \begin{cases} 0 & \forall \mathbf{R}_e(s) \in (-\infty : 0) \cap (0 : \infty) = \varnothing \\ \infty & \forall s \in \mathbb{C} \end{cases}\\
&\Longrightarrow \mathbf{L}\big[\cos(\omega_o x)\big] \text{ is } \textbf{divergent } \forall s \in \mathbb{C}\\
\mathbf{L}\big[\sin(\omega_o x)\big] &= \underbrace{\mathbf{L}\big[\mu(x)\sin(\omega_o x)\big]}_{\forall \mathbf{R}_e(s)>0} + \underbrace{\mathbf{L}\big[\mu(-x)\sin(\omega_o x)\big]}_{\forall \mathbf{R}_e(s)<0} && \text{by Corollary L.6 page 283}\\
&= \underbrace{\frac{\omega_o}{s^2+\omega_o^2}}_{\forall \mathbf{R}_e(s)>0} + \underbrace{\frac{-\omega_o}{s^2+\omega_o^2}}_{\forall \mathbf{R}_e(s)<0} && \text{by Corollary L.6 page 283}\\
&= \begin{cases} 0 & \forall \mathbf{R}_e(s) \in (-\infty : 0) \cap (0 : \infty) = \varnothing \\ \infty & \forall s \in \mathbb{C} \end{cases}\\
&\Longrightarrow \mathbf{L}\big[\sin(\omega_o x)\big] \text{ is } \textbf{divergent } \forall s \in \mathbb{C}
\end{aligned}$$

✏

L.7 Exponential decay properties

Corollary L.10. [15] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277) *and* $\mu(x)$ *the* UNIT STEP *function. Let* $A \triangleq \mathbf{R}_e(\alpha)$.

COR	*Mapping*	*Region of Convergence*	*Domain*
	$\mathbf{L}\big[e^{-\alpha\lvert x\rvert}\big] = \dfrac{2\alpha}{\alpha^2 - s^2}$	*for* $\mathbf{R}_e(s) \in (-A : A)$	$x,\alpha \in \mathbb{C}$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\big[e^{-\alpha|x|}\big] &= \mathbf{L}\big[e^{-\alpha|x|}\mu(x) + e^{-\alpha|x|}\mu(-x)\big] && \text{by definition of } \mu(x) && \text{(Definition L.3 page 282)}\\
&= \mathbf{L}\big[e^{-\alpha|x|}\mu(x)\big] + \mathbf{L}\big[e^{-\alpha|x|}\mu(-x)\big] && \text{by } \textit{homogeneous} \text{ property} && \text{(Theorem L.3 page 279)}\\
&= \underbrace{\mathbf{L}\big[e^{-\alpha x}\mu(x)\big]}_{\mathbf{R}_e(s)>-\mathbf{R}_e(\alpha)} + \underbrace{\mathbf{L}\big[e^{\alpha x}\mu(-x)\big]}_{\mathbf{R}_e(s)<\mathbf{R}_e(\alpha)} && \text{by Definition L.3 page 282} && \text{and Corollary L.5 page 283}\\
&= \left[\frac{1}{s+\alpha}\right] + \left[\frac{-1}{s-\alpha}\right] && \forall \mathbf{R}_e(s) \in \big(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\big) && \text{by Corollary L.5 page 283}\\
&= \frac{(s-\alpha)-(s+\alpha)}{(s+\alpha)(s-\alpha)} && \forall \mathbf{R}_e(s) \in \big(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\big)\\
&= \frac{2\alpha}{\alpha^2 - s^2} && \forall \mathbf{R}_e(s) \in \big(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\big)
\end{aligned}$$

✏

[15] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩,

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Corollary L.11. [16] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277) *and* $\mu(x)$ *the* UNIT STEP *function. Let* $A \triangleq \mathbf{R}_e(\alpha)$.

COR	Mapping		Region of Convergence	Domain
(1).	$\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(x)\right]$	$= \dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$
(2).	$\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(-x)\right]$	$= \dfrac{-s+\alpha}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$
(3).	$\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha\lvert x\rvert}\right]$	$= \dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2} + \dfrac{-s+\alpha}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$

✎PROOF:

1. Proof for (1):

$$\begin{aligned}
&\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(x)\right](s)\\
&= \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha x}\mu(x)\right](s) && \text{by definition of } \mu(x) && \text{(Definition L.3 page 282)}\\
&= \mathbf{L}\left[\cos(\omega_o x)\mu(x)\right](s+\alpha) && \forall \mathbf{R}_e(s) \in \left(0-\mathbf{R}_e(\alpha) : \infty-\mathbf{R}_e(\alpha)\right) && \text{by } \textit{modulation} \text{ prop.} && \text{(Theorem L.4 page 280)}\\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} && \forall \mathbf{R}_e(s) \in \left(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\right) && \text{by Corollary L.6 page 283}
\end{aligned}$$

2. Proof for (2):

$$\begin{aligned}
&\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(-x)\right]\\
&= \mathbf{L}\left[\cos(\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by definition of } \mu(x) && \text{(Definition L.3 page 282)}\\
&= \mathbf{L}\left[\cos(-\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by } \textit{even} \text{ property of } \cos(x) && \text{(Theorem I.2 page 237)}\\
&= \mathbf{L}\left[e^{\alpha x}\cos(\omega_o(-x))\mu(-x)\right]\\
&= \mathbf{L}\underbrace{\cos(\omega_o(-x))\mu(-x)}_{g(x)}(s-\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem L.4 page 280)}\\
&= \mathbf{L}\underbrace{\cos(\omega_o(-x))\mu(-x)}_{g(x)}(s-\alpha) && \text{by } \textit{modulation} \text{ property} && \text{(Theorem L.4 page 280)}\\
&= \frac{-s+\alpha}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R}_e(s) \in \left(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\right) && \text{by Corollary L.6 and Theorem L.4 page 280}
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
\mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\right] &= \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(x)\right] + \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha|x|}\mu(-x)\right]\\
&= \mathbf{L}\left[\cos(\omega_o x)e^{-\alpha x}\mu(x)\right] + \mathbf{L}\left[\cos(-\omega_o x)e^{\alpha x}\mu(-x)\right]\\
&= \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} + \frac{-s+\alpha}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R}_e(s) \in \left(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\right)
\end{aligned}$$

✏

Corollary L.12. [17] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition L.1 page 277) *and* $\mu(x)$ *the* UNIT STEP *function. Let* $A \triangleq \mathbf{R}_e(\alpha)$.

[16] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩, Levy (1958) page 19 ⟨with $\psi = 0$, $\alpha_0 = \alpha$, and $\alpha_1 = 1$⟩, http://ece-research.unm.edu/bsanthan/ece541/table_ME.pdf

[17] Bracewell (1978) page 227 ⟨Table 11.2 Some Laplace transforms⟩, Levy (1958) page 19 ⟨with $\psi = 0$, $\alpha_0 = \alpha$, and $\alpha_1 = 1$⟩, http://ece-research.unm.edu/bsanthan/ece541/table_ME.pdf

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COR	Mapping	Region of Convergence	Domain
(1).	$\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(x)\right] = \dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$
(2).	$\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha\lvert x\rvert}\mu(-x)\right] = \dfrac{-\omega_o}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$
(3).	$\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha\lvert x\rvert}\right] = \dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2} + \dfrac{-\omega_o}{(s-\alpha)^2+\omega_o^2}$	for $\mathbf{R}_e(s) \in (-A : A)$	$x,\alpha\in\mathbb{R}$

✎PROOF:

1. Proof for (1):

$$\begin{aligned}
&\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(x)\right] \\
&= \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha x}\mu(x)\right] && \text{by definition of } \mu(x) && \text{(Definition L.3 page 282)} \\
&= \frac{s+\alpha}{(\omega_o)^2+\omega_o^2} && \forall \mathbf{R}_e(s) \in \left(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\right) && \text{by Corollary L.6 page 283 and Theorem L.4 page 280}
\end{aligned}$$

2. Proof for (2):

$$\begin{aligned}
&\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(-x)\right] \\
&= \mathbf{L}\left[\sin(-\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by definition of } \mu(x) && \text{(Definition L.3 page 282)} \\
&= \mathbf{L}\left[-\sin(\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by } \textit{odd} \text{ property of } \sin(x) && \text{(Theorem I.2 page 237)} \\
&= -\mathbf{L}\left[\sin(\omega_o x)e^{\alpha x}\mu(-x)\right] && \text{by } \textit{homogeneous} \text{ property} && \text{(Theorem L.3 page 279)} \\
&= \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R}_e(s) \in \left(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\right) && \text{by Theorem L.4 page 280 and Corollary L.6}
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
\mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\right] &= \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(x)\right] + \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha|x|}\mu(-x)\right] \\
&= \mathbf{L}\left[\sin(\omega_o x)e^{-\alpha x}\mu(x)\right] + \mathbf{L}\left[\sin(-\omega_o x)e^{\alpha x}\mu(-x)\right] \\
&= \frac{\omega_o}{(s+\alpha)^2+\omega_o^2} + \frac{-\omega_o}{(s-\alpha)^2+\omega_o^2} && \forall \mathbf{R}_e(s) \in \left(-\mathbf{R}_e(\alpha) : \mathbf{R}_e(\alpha)\right)
\end{aligned}$$

✏

L.8 Product properties

Theorem L.6 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "s domain" and vice-versa.

Theorem L.6 (convolution theorem). [18] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator (Definition L.1 page 277) and* $\star$ *the convolution operator (Definition F.1 page 203). Let* A, B, C, *and* D *be defined as in Corollary L.2 (page 279).*

THM				
	$\mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](s) = [\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s)$	$\forall \mathbf{R}_e(s) \in [A : B] \cap [C : D]$	$\forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$
	$\mathbf{L}\left[\mathsf{f}(x)\mathsf{g}(x)\right](s) = [\mathbf{L}\mathsf{f}](s)\star\left[\mathbf{L}\mathsf{g}\right](s)$	$\forall \mathbf{R}_e(s) \in [A+C : B+D],\ c \in (A : B)$	$\forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$.

[18] Bracewell (1978) page 224, Bachman et al. (2002) pages 268–270, Bachman (1964) page 8

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✎PROOF:

$$\begin{aligned}
\mathbf{L}\big[\mathsf{f}(x) \star \mathsf{g}(x)\big](s) &= \mathbf{L}\left[\int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{du}\right](s) && \text{by definition of } \star && \text{(Definition F.1 page 203)}\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\big[\mathbf{L}\mathsf{g}(x-u)\big](s)\,\mathsf{du} \\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-su}\,\big[\mathbf{L}\mathsf{g}(x)\big](s)\,\mathsf{du} && \text{by } \textit{translation} \text{ property} && \text{(Theorem L.2 page 278)}\\
&= \underbrace{\left(\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathsf{du}\right)}_{[\mathbf{L}\mathsf{f}](s)}\,\big[\mathbf{L}\mathsf{g}\big](s) \\
&= [\mathbf{L}\mathsf{f}](s)\,\big[\mathbf{L}\mathsf{g}\big](s) && \mathbf{R}_{\mathsf{e}}(s) \in [A\,:\,B]\cap[C\,:\,D] && \text{by definition of } \mathbf{L}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s) &= \mathbf{L}\big[\big(\mathbf{L}^{-1}\mathbf{L}\mathsf{f}(x)\big)\,\mathsf{g}(x)\big](s) && \text{by def. of operator inverse} && \text{(Definition D.3 page 170)}\\
&= \mathbf{L}\left[\left(\int_{v\in\mathbb{R}} [\mathbf{L}\mathsf{f}(x)](v)e^{sxv}\,\mathsf{dv}\right)\mathsf{g}(x)\right](s) && \text{by Theorem L.1 page 278} \\
&= \int_{v\in\mathbb{R}} [\mathbf{L}\mathsf{f}(x)](v)\big[\mathbf{L}\big(e^{sxv}\,\mathsf{g}(x)\big)\big](s,v)\,\mathsf{dv} \\
&= \int_{v\in\mathbb{R}} [\mathbf{L}\mathsf{f}(x)](v)\big[\mathbf{L}\mathsf{g}(x)\big](s-v)\,\mathsf{dv} && \text{by Theorem L.2 page 278} \\
&= [\mathbf{L}\mathsf{f}](s) \star \big[\mathbf{L}\mathsf{g}\big](s) && \text{by definition of } \star && \text{(Definition F.1 page 203)}
\end{aligned}$$

✏

L.9 Calculus properties

Theorem L.7. [19] *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* *(Definition L.1 page 277)*.

THM

$$\begin{aligned}
\mathbf{L}\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{g}(x)\right](s) &= s\big[\mathbf{L}\mathsf{g}\big](s) + \lim_{x\to\infty}\left[e^{-sx}\mathsf{g}(x) - e^{sx}\mathsf{g}(-x)\right] \\
\mathbf{L}\left[\int_{u=-\infty}^{u=x} \mathsf{g}(u)\,\mathsf{du}\right](s) &= \frac{1}{s}\big[\mathbf{L}\mathsf{g}\big](s)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{g}(x)\right] &\triangleq \int_{x\in\mathbb{R}} \underbrace{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{g}(x)\right]}_{\mathsf{dv}}\underbrace{e^{-sx}}_{u}\,\mathsf{dx} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \left[\underbrace{e^{-sx}}_{u}\underbrace{\mathsf{g}(x)}_{v}\right]_{x=-\infty}^{x=+\infty} - \left[\int_{x\in\mathbb{R}} \underbrace{\mathsf{g}(x)}_{v}\underbrace{(-s)e^{-sx}\,\mathsf{dx}}_{\mathsf{du}}\right] && \text{by } \textit{Integration by Parts} \\
&= \lim_{x\to\infty}\left[e^{-sx}\mathsf{g}(x)\right] - \lim_{x\to-\infty}\left[e^{-sx}\mathsf{g}(-\infty)\right] + s\int_{x\in\mathbb{R}} \mathsf{g}(x)e^{-sx}\,\mathsf{dx} \\
&= s\big[\mathbf{L}\mathsf{g}\big](s) + \lim_{x\to\infty}\left[e^{-sx}\mathsf{g}(x) - e^{sx}\mathsf{g}(-x)\right] && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}
\end{aligned}$$

[19] Betten (2008b) page 301 ⟨(B.27)⟩, Levy (1958) page 15 ⟨Equation 0.7⟩

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$$
\begin{aligned}
\mathbf{L}\int_{v=-\infty}^{v=x} \mathsf{g}(v)\,\mathsf{dv} &\triangleq \int_{x=-\infty}^{x=+\infty}\left[\int_{v=-\infty}^{v=x} \mathsf{g}(v)\,\mathsf{dv}\right]e^{-sx}\,\mathsf{dx} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \int_{x=-\infty}^{x=+\infty}\left[\int_{v=-\infty}^{v=+\infty} \mathsf{g}(v)\mu(x-v)\,\mathsf{dv}\right]e^{-sx}\,\mathsf{dx} && \text{by definition of } \mu(x) && \text{(Definition L.3 page 282)}\\
&= \int_{v\in\mathbb{R}} \mathsf{g}(v)\left[\int_{x\in\mathbb{R}} \mu(x-v)e^{-sx}\,\mathsf{dx}\right]\mathsf{dv}\\
&= \int_{v\in\mathbb{R}} \mathsf{g}(v)\left[e^{-vs}\frac{1}{s}\right]\mathsf{dv} && \forall\mathbf{R}_{\mathsf{e}}(s)<0 && \text{by Corollary L.4 page 282}\\
&= \frac{1}{s}\int_{v\in\mathbb{R}} \mathsf{g}(v)e^{-vs}\,\mathsf{dv} && \forall\mathbf{R}_{\mathsf{e}}(s)<0 && \text{by Corollary L.4 page 282}\\
&\triangleq \frac{1}{s}\left[\mathbf{L}\mathsf{g}\right](s) && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}
\end{aligned}
$$

✏

L.10 Moment properties

Definition L.4. [20]

DEF *The quantity* M_n *is the* n***th moment*** *of a function* $\mathsf{f}(x)\in\mathbf{L}^2_{\mathbb{R}}$ *if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n\mathsf{f}(x)\,\mathsf{dx} \qquad \text{for } n\in\mathbb{W}.$$

Theorem L.8. [21] *Let* M_n *be the* nTH MOMENT *(Definition L.4 page 291) of a function* $\mathsf{f}(x)\in\mathbf{L}^2_{\mathbb{R}}$.

THM

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n\mathsf{f}(x)\,\mathsf{dx} \;=\; \left[(-1)^n\left[\left[\frac{\mathsf{d}}{\mathsf{d}s}\right]^n\mathbf{L}\mathsf{f}\right](s)\right]_{s=0} \qquad \forall n\in\mathbb{W},\mathsf{f}\in\mathbf{L}^2_{\mathbb{R}}$$

✎PROOF:

$$
\begin{aligned}
(-1)^n\left[\left[\frac{\mathsf{d}}{\mathsf{d}s}\right]^n\mathbf{L}\mathsf{f}\right](s)_{s=0} &= (-1)^n\left[\left[\frac{\mathsf{d}}{\mathsf{d}s}\right]^n\int_{\mathbb{R}}\mathsf{f}(x)e^{-sx}\,\mathsf{dx}\right]_{s=0} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= (-1)^n\left[\int_{\mathbb{R}}\mathsf{f}(x)\left[\left[\frac{\mathsf{d}}{\mathsf{d}s}\right]^n e^{-sx}\right]\mathsf{dx}\right]_{s=0}\\
&= (-1)^n\left[\int_{\mathbb{R}}\mathsf{f}(x)(-x)^n e^{-sx}\,\mathsf{dx}\right]_{s=0}\\
&= (-1)^n\int_{\mathbb{R}}\mathsf{f}(x)(-x)^n\,\mathsf{dx}\\
&= (-1)^{2n}\int_{\mathbb{R}}\mathsf{f}(x)x^n\,\mathsf{dx}\\
&\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n && \text{(Definition M.3 page 298)}
\end{aligned}
$$

✏

Corollary L.13. [22] *Let* M_n *be the* nTH MOMENT *(Definition L.4 page 291) of a function* $\mathsf{f}(x)$ *in* $\mathbf{L}^2_{\mathbb{R}}$.

COR

$$\mathsf{M}_n = 0 \qquad\Longleftrightarrow\qquad \left[\left[\frac{\mathsf{d}}{\mathsf{d}s}\right]^n\mathbf{L}\mathsf{f}\right]_{s=0} = 0 \qquad \forall n\in\mathbb{W}$$

[20] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83
[21] Goswami and Chan (1999) pages 38–39
[22] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

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✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
0 &= (-1)^n \cdot 0 \\
&= (-1)^n \mathsf{M}_n && \text{by left hypothesis} \\
&= (-1)^n \int_{\mathbb{R}} x^n \mathsf{f}(x)\,\mathsf{dx} \\
&= (-1)^n \left[(-1)^n \left[\frac{\mathsf{d}}{\mathsf{ds}}\right]^n \mathbf{L}\mathsf{f}\right]_{s=0} && \text{by Theorem L.8 page 291} \\
&= \left[\left[\frac{\mathsf{d}}{\mathsf{ds}}\right]^n \mathbf{L}\mathsf{f}\right]_{s=0}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\left[\frac{\mathsf{d}}{\mathsf{ds}}\right]^n \mathbf{L}\mathsf{f}\right]_{s=0} && \text{by right hypothesis} \\
&= \left[\left[\frac{\mathsf{d}}{\mathsf{ds}}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-sx}\,\mathsf{dx}\right]_{s=0} && \text{by definition of } \mathbf{L} \quad \text{(Definition L.1 page 277)} \\
&= \left[\int_{\mathbb{R}} \mathsf{f}(x) \left[\frac{\mathsf{d}}{\mathsf{ds}}\right]^n e^{-sx}\,\mathsf{dx}\right]_{s=0} \\
&= \left[\int_{\mathbb{R}} \mathsf{f}(x) \left[(-1)^n x^n e^{-sx}\right]\,\mathsf{dx}\right]_{s=0} \\
&= (-1)^n \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathsf{dx}
\end{aligned}$$

✏

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APPENDIX M

FOURIER TRANSFORM

❝ Up to this point we have supposed that the function whose development is required in a series of sines of multiple arcs can be developed in a series arranged according to powers of the variable x, ...We can extend the same results to any functions, even to those which are discontinuous and entirely arbitrary. ...even entirely arbitrary functions may be developed in series of sines of multiple arcs. ❞

Joseph Fourier (1768–1830) [1]

M.1 Introduction

Historically, before the Fourier Transform was the Taylor Expansion (transform). The Taylor Expansion demonstrates that for **analytic** functions, knowledge of the derivatives of a function at a location $x = a$ allows you to determine (predict) arbitrarily closely all the points $\mathtt{f}(x)$ in the vicinity of $x = a$. But analytic functions are by definition functions for which all their derivatives exist. Thus, if a function is *discontinuous*, it is simply not a candidate for the Taylor Expansion. And some 300 years ago, mathematician giants of the day were fairly content with this.

But then in came an engineer named Joseph Fourier whose day job was working as a governor of lower Egypt under Napolean. He claimed that, rather than expansion based on derivatives, one could expand based on integrals over sinusoids, and that this would work not just for analytic functions, but for **discontinuous** ones as well![2]

Needless to say, this did not go over too well initially in the mathematical community. But over time (on the order of 200 or so years), the Fourier Transform has in many ways won the day.

[3]

[1] quote: Fourier (1878) page 184,186 ⟨§219,220⟩
image: http://en.wikipedia.org/wiki/File:Fourier2.jpg, public domain

[2] Robinson (1982) page 886

[3] Caricature of Legendre (left) and Fourier (right), 1820, by Julien-Léopold Boilly (1796–1874). "*Album de 73*

M.2 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $L^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on $\mathscr{B}$, and

$$L^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \,\middle|\, \int_{\mathbb{R}} |\mathsf{f}|^2 \, d\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \,|\, \triangledown \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} d\mu$ such that

$$\langle \mathsf{f} \,|\, \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, dx,$$

and $\left(L^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \,|\, \triangledown \rangle \right)$ is a *Hilbert space*.

Definition M.1. *Let κ be a* FUNCTION *in $\mathbb{C}^{\mathbb{R}^2}$.*

DEF *The function κ is the* ***Fourier kernel*** *if* $\quad \kappa(x,\omega) \triangleq e^{i\omega x} \quad \forall x,\omega \in \mathbb{R}$

Definition M.2. [4] *Let $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF *The* ***Fourier Transform*** *operator $\tilde{\mathbf{F}}$ is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \, dx \qquad \forall \mathsf{f} \in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

This definition of the Fourier Transform is also called the ***unitary Fourier Transform****.*

Remark M.1 (**Fourier transform scaling factor**). [5] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x} \, dx \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B \int_{\mathbb{R}} \mathsf{F}(\omega) e^{i\omega x} \, d\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \, dx$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x} \, dx$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x} \, dx$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int \cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \, dx$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem M.2 page 295)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

Portraits-Charge Aquarelle's des Membres de l'Institute (watercolor portrait #29). Biliotheque de l'Institut de France." Public domain. https://en.wikipedia.org/wiki/File:Legendre_and_Fourier_(1820).jpg

[4] Bachman et al. (2002) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965) page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

[5] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005b) pages 374–375

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M.3 Operator properties

Theorem M.1 (Inverse Fourier transform). [6] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition M.2 page 294). *The inverse $\tilde{\mathbf{F}}^{-1}$ of $\tilde{\mathbf{F}}$ is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{f}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{f}(\omega)e^{i\omega x}\,d\omega \qquad \forall \tilde{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem M.2. *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator with inverse $\tilde{\mathbf{F}}^{-1}$ and adjoint $\tilde{\mathbf{F}}^*$.*

THM
$$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}f \mid g\right\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\,e^{-i\omega x}\,dx \mid g(\omega)\right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 294}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\,\left\langle e^{-i\omega x} \mid g(\omega)\right\rangle\,dx && \text{by } \textit{additive property} \text{ of } \langle\triangle\mid\triangledown\rangle\\
&= \int_{\mathbb{R}} f(x)\,\frac{1}{\sqrt{2\pi}}\left\langle g(\omega) \mid e^{-i\omega x}\right\rangle^*\,dx && \text{by } \textit{conjugate symmetric property} \text{ of } \langle\triangle\mid\triangledown\rangle\\
&= \left\langle f(x) \mid \frac{1}{\sqrt{2\pi}}\left\langle g(\omega) \mid e^{-i\omega x}\right\rangle\right\rangle && \text{by definition of } \langle\triangle\mid\triangledown\rangle\\
&= \left\langle f \mid \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}g\right\rangle && \text{by Theorem M.1 page 295}
\end{aligned}$$

✏

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* [7] (Corollary M.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem M.3 page 295).

Corollary M.1. *Let $\mathbf{I}$ be the identity operator and let $\tilde{\mathbf{F}}$ be the Fourier Transform operator with adjoint $\tilde{\mathbf{F}}^*$ and inverse $\tilde{\mathbf{F}}^{-1}$.*

COR
$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem M.2 page 295). ✏

Theorem M.3. *Let $\tilde{\mathbf{F}}$ be the Fourier transform operator with adjoint $\tilde{\mathbf{F}}^*$ and inverse $\tilde{\mathbf{F}}$. Let $|||\cdot|||$ be the operator norm with respect to the vector norm $\|\cdot\|$ with respect to the Hilbert space $\left(\mathbb{C}^{\mathbb{R}}, \langle\triangle\mid\triangledown\rangle\right)$. Let $\mathscr{R}(\mathbf{A})$ be the range of an operator $\mathbf{A}$.*

THM
$$\begin{array}{rcccll}
\mathscr{R}(\mathbf{F}\tau) &=& \mathscr{R}(\tilde{\mathbf{F}}^{-1}) &=& L^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| &=& |||\tilde{\mathbf{F}}^{-1}||| &=& 1 & \text{(UNITARY)}\\
\left\langle \tilde{\mathbf{F}}f \mid \tilde{\mathbf{F}}g\right\rangle &=& \left\langle \tilde{\mathbf{F}}^{-1}f \mid \tilde{\mathbf{F}}^{-1}g\right\rangle &=& \langle f \mid g\rangle & \text{(PARSEVAL'S EQUATION)}\\
\left\|\tilde{\mathbf{F}}f\right\| &=& \left\|\tilde{\mathbf{F}}^{-1}f\right\| &=& \|f\| & \text{(PLANCHEREL'S FORMULA)}\\
\left\|\tilde{\mathbf{F}}f - \tilde{\mathbf{F}}g\right\| &=& \left\|\tilde{\mathbf{F}}^{-1}f - \tilde{\mathbf{F}}^{-1}g\right\| &=& \|f-g\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary M.1 page 295) and from the properties of unitary operators (Theorem D.26 page 194). ✏

[6] Chorin and Hald (2009) page 13
[7] *unitary operators*: Definition D.14 page 193

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M.4 Transversal properties

Theorem M.4 (Shift relations). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition M.2 page 294).

THM
$$\begin{aligned}\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega)\\ \left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r)\end{aligned}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition L.1 page 277).

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= \mathbf{L}[\mathsf{f}(x-y)](s)|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= e^{-sy}\,[\mathbf{L}\mathsf{f}(x)](s)|_{s=i\omega} && \text{by Laplace } \textit{translation} \text{ property} && \text{(Theorem L.2 page 278)}\\
&= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)}\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\mathbf{L}\left(e^{irx}\mathsf{g}(x)\right)\right](s)\Big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \left[\left[\mathbf{L}\mathsf{g}(x)\right](s-r)\right]\Big|_{s=i\omega} && \text{by Laplace } \textit{dilation} \text{ property} && \text{(Theorem L.2 page 278)}\\
&= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)}
\end{aligned}$$

✏

Theorem M.5 (Complex conjugate). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

THM
$$\tilde{\mathbf{F}}\mathsf{f}^*(-x) = -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* \quad \forall \mathsf{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(-x)e^{-i\omega x}\,\mathsf{dx} && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)}\\
&= \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(u)e^{i\omega u}(-1)\,\mathsf{du} && \text{where } u \triangleq -x \implies \mathsf{dx} = -\,\mathsf{du}\\
&= -\left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(u)e^{-i\omega u}\,\mathsf{du}\right]^* \\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)}
\end{aligned}$$

✏

M.5 Convolution relations

Theorem Q.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem M.6 (convolution theorem). [8] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition M.2 page 294) *and $\star$ the convolution operator* (Definition F.1 page 203).

[8] Bachman et al. (2002) pages 269–270 ⟨5.2.3 Convolutions to Products⟩, Bachman (1964) page 8, Bracewell (1978) page 110

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THM

$$\underbrace{\tilde{\mathbf{F}}\big[\mathsf{f}(x) \star \mathsf{g}(x)\big](\omega)}_{\textit{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\big[\tilde{\mathbf{F}}\mathsf{f}\big](\omega)\,\big[\tilde{\mathbf{F}}\mathsf{g}\big](\omega)}_{\textit{multiplication in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\tilde{\mathbf{F}}\big[\mathsf{f}(x)\mathsf{g}(x)\big](\omega)}_{\textit{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}\big[\tilde{\mathbf{F}}\mathsf{f}\big](\omega) \star \big[\tilde{\mathbf{F}}\mathsf{g}\big](\omega)}_{\textit{convolution in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition L.1 page 277).

$$\begin{aligned}
\tilde{\mathbf{F}}\big[\mathsf{f}(x) \star \mathsf{g}(x)\big](\omega) &= \mathbf{L}\big[\mathsf{f}(x) \star \mathsf{g}(x)\big](s)\big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition L.1 page 277)}\\
&= \sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\,\big[\mathbf{L}\mathsf{g}\big](s)\Big|_{s=i\omega} && \text{by } \textit{Laplace convolution} \text{ result} && \text{(Theorem L.6 page 289)}\\
&= \sqrt{2\pi}\big[\tilde{\mathbf{F}}\mathsf{f}\big](\omega)\,\big[\tilde{\mathbf{F}}\mathsf{g}\big](\omega)\\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s)|_{s=i\omega}\\
&= \tfrac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s) \star \big[\mathbf{L}\mathsf{g}\big](s)\Big|_{s=i\omega}\\
&= \tfrac{1}{\sqrt{2\pi}}\big[\tilde{\mathbf{F}}\mathsf{f}\big](\omega) \star \big[\tilde{\mathbf{F}}\mathsf{g}\big](\omega)
\end{aligned}$$

✏

M.6 Calculus relations

Theorem M.7. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* *(Definition M.2 page 294)*.

THM

$$\left\{\lim_{t\to-\infty} \mathsf{x}(t) = 0\right\} \quad\Longrightarrow\quad \left\{\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{dt}}\mathsf{x}(t)\right] = i\omega\big[\tilde{\mathbf{F}}\mathsf{x}\big](\omega)\right\}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition L.1 page 277).

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\frac{\mathsf{d}}{\mathsf{dt}}\mathsf{x}(t)\right] &\triangleq \mathbf{L}\left[\frac{\mathsf{d}}{\mathsf{dt}}\mathsf{x}(t)\right](s)\Big|_{s=i\omega} && \text{by definitions of } \mathbf{L} \text{ and } \tilde{\mathbf{F}} && \text{(Definition L.1 page 277)}\\
&= s[\mathbf{L}\mathsf{x}(t)](s)|_{s=i\omega} && \text{by Theorem L.7 page 290}\\
&= i\omega\big[\tilde{\mathbf{F}}\mathsf{x}\big](\omega)
\end{aligned}$$

✏

Theorem M.8. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* *(Definition M.2 page 294)*.

THM

$$\tilde{\mathbf{F}}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{du} = \frac{1}{i\omega}\big[\tilde{\mathbf{F}}\mathsf{x}\big](\omega)$$

Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition L.1 page 277). ✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{du} &\triangleq \mathbf{L}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{du}\Big|_{s=i\omega}\\
&= \frac{1}{s}[\mathbf{L}\mathsf{x}(t)](s)\Big|_{s=i\omega} && \text{by Theorem L.7 page 290}\\
&= \frac{1}{i\omega}\big[\tilde{\mathbf{F}}\mathsf{x}(t)\big](\omega)
\end{aligned}$$

✏

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M.7 Real valued functions

Figure M.1: Fourier transform components of real-valued signal

Theorem M.9. *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM

$$\left\{\begin{array}{l}\mathsf{f}(x) \text{ is REAL-VALUED}\\ (\mathsf{f}\in\mathbb{R}^{\mathbb{R}})\end{array}\right\} \Longrightarrow \left\{\begin{array}{rcll}\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & \text{(HERMITIAN SYMMETRIC)}\\ \mathbf{R}_e\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R}_e\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(SYMMETRIC)}\\ \mathbf{I}_m\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I}_m\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(ANTI-SYMMETRIC)}\\ |\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| & \text{(SYMMETRIC)}\\ \angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & \text{(ANTI-SYMMETRIC).}\end{array}\right\}$$

✎PROOF:

$$\begin{array}{lclclclcl}\tilde{\mathsf{f}}(\omega) &\triangleq& [\tilde{\mathbf{F}}\mathsf{f}(x)](\omega) &\triangleq& \left\langle \mathsf{f}(x)\,|\,e^{i\omega x}\right\rangle &=& \left\langle \mathsf{f}(x)\,|\,e^{i(-\omega)x}\right\rangle^* &\triangleq& \tilde{\mathsf{f}}^*(-\omega)\\ \mathbf{R}_e\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R}_e\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& \mathbf{R}_e\left[\tilde{\mathsf{f}}(-\omega)\right] & & & & \\ \mathbf{I}_m\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{I}_m\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& -\mathbf{I}_m\left[\tilde{\mathsf{f}}(-\omega)\right] & & & & \\ |\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}^*(-\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| & & & & \\ \angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}^*(-\omega) &=& -\angle\tilde{\mathsf{f}}(-\omega) & & & & \end{array}$$

✏

M.8 Moment properties

Definition M.3. [9]

DEF

The quantity M_n *is the* n***th moment*** *of a function* $\mathsf{f}(x)\in\boldsymbol{L}^2_{\mathbb{R}}$ *if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n\mathsf{f}(x)\,\mathrm{d}x \qquad \text{for } n\in\mathbb{W}.$$

Lemma M.1. [10] *Let* M_n *be the* nTH MOMENT *(Definition M.3 page 298) and* $\tilde{\mathsf{f}}(\omega)\triangleq\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)$ *the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ *(Definition E.1 page 199).*

LEM

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n\mathsf{f}(x)\,\mathrm{d}x \;=\; \sqrt{2\pi}(i)^n\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} \qquad \forall n\in\mathbb{W},\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$$

[9] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83
[10] Goswami and Chan (1999) pages 38–39

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✎PROOF:

$$\begin{aligned}
\sqrt{2\pi}(i)^n\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\mathsf{f}}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition M.2 page 294)}\\
&= (i)^n\int_{\mathbb{R}}\mathsf{f}(x)\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\right]\mathrm{d}x\Big|_{\omega=0}\\
&= (i)^n\int_{\mathbb{R}}\mathsf{f}(x)\left[(-i)^n x^n e^{-i\omega x}\right]\mathrm{d}x\Big|_{\omega=0}\\
&= (-i^2)^n\int_{\mathbb{R}}\mathsf{f}(x)x^n\,\mathrm{d}x\\
&= \int_{\mathbb{R}}x^n\mathsf{f}(x)\,\mathrm{d}x\\
&\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \quad \text{(Definition M.3 page 298)}
\end{aligned}$$

✏

Lemma M.2. [11] *Let* M_n *be the* nTH MOMENT *(Definition M.3 page 298) and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ *(Definition E.1 page 199).*

LEM

$$\mathsf{M}_n = 0 \quad\Longleftrightarrow\quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
0 &= \langle \mathsf{f}(x)\,|\,x^n\rangle && \text{by left hypothesis}\\
&= \sqrt{2\pi}(-i)^{-n}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \text{by Lemma M.1 page 298}\\
&\Longrightarrow \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} = 0
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \text{by right hypothesis}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\Big|_{\omega=0} && \text{by definition of } \tilde{\mathsf{f}}(\omega)\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{f}(x)\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\,\mathrm{d}x\Big|_{\omega=0}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{f}(x)\left[(-i)^n x^n e^{-i\omega x}\right]\mathrm{d}x\Big|_{\omega=0}\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{f}(x)x^n\,\mathrm{d}x\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\langle \mathsf{f}(x)\,|\,x^n\rangle && \text{by definition of } \langle\cdot\,|\,\cdot\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{(Definition E.1 page 199)}
\end{aligned}$$

✏

[11] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

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Lemma M.3 (Strang-Fix condition). [12] *Let* $\mathsf{f}(x)$ *be a function in* $L^2_{\mathbb{R}}$ *and* M_n *the* nTH MOMENT *(Definition M.3 page 298) of* $\mathsf{f}(x)$. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 1.3 page 4)*.

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "time"}} \quad\Longleftrightarrow\quad \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "frequency"}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by definition of } \tilde{\mathsf{f}}(\omega) \quad \text{(Definition M.2 page 294)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k)\bar{\delta}_k && \text{by } PSF \quad \text{(Theorem 1.3 page 13)} \\
&= \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[(-i)^n \bar{\delta}_k \mathsf{M}_n\right] e^{-i2\pi kx} && \text{by definition of } \bar{\delta} \quad \text{(Definition N.12 page 316)} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } PSF \quad \text{(Theorem 1.3 page 13)}
\end{aligned}$$

✏

[12] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83, Mallat (1999) pages 241–243, Fix and Strang (1969)

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M.9 Examples

Example M.1 (rectangular pulse). Let $\tilde{f}(\omega)$ be the *Fourier transform* of a function $f(x) \in L^2_{\mathbb{R}}$.

EX

$$f(x) = \begin{cases} c & \text{for } x \in [a : b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)} \right]$$

✎PROOF:

$$\begin{aligned}
\tilde{f}(\omega) &= \tilde{\mathbf{F}}[f(x)](\omega) && \text{by definition of } \tilde{f}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[f\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} && \text{(Theorem M.4 page 296)} \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by definition of } f(x) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 3)} \\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x) e^{-i\omega x}\, dx && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)} \\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} c e^{-i\omega x}\, dx && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 3)} \\
&= \frac{c}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \frac{1}{-i\omega} e^{-i\omega x} \Bigg|_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \\
&= \frac{2c}{\sqrt{2\pi}\omega} e^{-i\left(\frac{a+b}{2}\right)\omega} \left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i} \right] \\
&= \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)} \right] && \text{by } \textit{Euler formulas} && \text{(Corollary I.2 page 241)}
\end{aligned}$$

✏

Example M.2 (triangle). Let $\tilde{f}(\omega)$ be the *Fourier transform* of a function $f(x) \in L^2_{\mathbb{R}}$.

EX

$$f(x) = \left\{ \begin{array}{ll} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a : b) \\ 0 & \text{otherwise} \end{array} \right\} \quad \tilde{f}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)} \right]^2$$

✎PROOF:

$$\tilde{f}(\omega) = \tilde{\mathbf{F}}[f(x)](\omega) \qquad \text{by definition of } \tilde{f}(\omega)$$

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$$= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x-\frac{a+b}{2}\right)\right](\omega) \qquad \text{by } \textit{shift relation} \qquad \text{(Theorem M.4 page 296)}$$

$$= \tilde{\mathbf{F}}\left[c\left(1-\frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) \qquad \text{by definition of } \mathsf{f}(x)$$

$$= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x) \star \mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega)$$

$$= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] \qquad \text{by } \textit{convolution theorem} \qquad \text{(Theorem Q.2 page 380)}$$

$$= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2$$

$$= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 \qquad \text{by } \textit{Rectangular pulse} \text{ ex.} \qquad \text{Example M.1 page 301}$$

$$= \frac{c(b-a)^2}{4\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$

✏

Example M.3. Let a function f be defined in terms of the cosine function (Definition I.2 page 235) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$

✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 3) on a set A.

$$\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x \qquad \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition M.2)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,\mathrm{d}x \qquad \text{by definition of } \mathsf{f}(x)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right)e^{-i\omega x}\,\mathrm{d}x \qquad \text{by definition of } \mathbb{1} \text{ (Definition 1.2)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \left[\frac{e^{i\frac{\pi}{2}x}+e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\,\mathrm{d}x \qquad \text{by Corollary I.2 page 241}$$

$$= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1} \left[2+e^{i\pi x}+e^{-i\pi x}\right]e^{-i\omega x}\,\mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1} 2e^{-i\omega x}+e^{-i(\omega+\pi)x}+e^{-i(\omega-\pi)x}\,\mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}}\left[2\frac{e^{-i\omega x}}{-i\omega}+\frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2\sqrt{2\pi}}\left[2\frac{e^{-i\omega}-e^{+i\omega}}{-2i\omega}+\frac{e^{-i(\omega+\pi)}-e^{+i(\omega+\pi)}}{-2i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)}-e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}$$

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$$= \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$

✏

Example M.4. [13]

EX $$\tilde{\mathbf{F}}\left[e^{-\alpha|x|}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right]$$

✎PROOF:

1. Proof using *Laplace Transform*:

$$\begin{aligned}
\sqrt{2\pi}\tilde{\mathbf{F}}\left[e^{-\alpha|x|}\right] &\triangleq \left[\sqrt{2\pi}\right]\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\alpha|x|}e^{-i\omega x}\,\mathrm{dx} && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)}\\
&= \left[\int_{\mathbb{R}} e^{-\alpha|x|}e^{-sx}\,\mathrm{dx}\right]_{s=i\omega} \\
&= \left[\frac{2\alpha}{\alpha^2-s^2}\right]_{s=i\omega} && \forall \mathbf{R}_{\mathrm{e}}(s)\in(-\alpha\,:\,\alpha) && \text{by Corollary L.10 page 287}\\
&= \frac{2\alpha}{\alpha^2+\omega^2} && \text{because } s=i\omega \text{ is in } (-\alpha\,:\,\alpha)
\end{aligned}$$

2. Alternate proof:

$$\begin{aligned}
\sqrt{2\pi}\tilde{\mathbf{F}}\left[e^{-\alpha|x|}\right] &\triangleq \left[\sqrt{2\pi}\right]\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-i\omega x}\,\mathrm{dx} && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition M.2 page 294)}\\
&= \int_{-\infty}^{0} e^{-\alpha(-x)}e^{-i\omega x}\,\mathrm{dx} + \int_{0}^{\infty} e^{-\alpha(x)}e^{-i\omega x}\,\mathrm{dx} \\
&= \int_{-\infty}^{0} e^{x(\alpha-i\omega)}\,\mathrm{dx} + \int_{0}^{\infty} e^{x(-\alpha-i\omega)}\,\mathrm{dx} \\
&= \left.\frac{e^{x(\alpha-i\omega)}}{\alpha-i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{x(-\alpha-i\omega)}}{-\alpha-i\omega}\right|_{0}^{\infty} && \text{by Fundamental Theorem of Calculus}\\
&= \left[\frac{1}{\alpha-i\omega}-0\right] + \left[0-\frac{1}{-\alpha-i\omega}\right] \\
&= \left[\frac{1}{\alpha-i\omega}\right]\left[\frac{\alpha-i\omega}{\alpha-i\omega}\right] + \left[\frac{1}{\alpha+i\omega}\right]\left[\frac{\alpha+i\omega}{\alpha+i\omega}\right] \\
&= \frac{\alpha-i\omega}{\alpha^2+\omega^2} + \frac{\alpha+i\omega}{\alpha^2+\omega^2} \\
&= \left[\frac{2\alpha}{\alpha^2+\omega^2}\right]
\end{aligned}$$

✏

[13] https://math.stackexchange.com/questions/4015842/

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APPENDIX N

LINEAR COMBINATIONS

N.1 Linear combinations in linear spaces

A *linear space* (Definition D.1 page 169) in general is not equipped with a *topology*. Without a topology, it is not possible to determine whether an *infinite sum* (Definition H.1 page 224) of vectors converges. Therefore in this section (dealing with linear spaces), all definitions related to sums of vectors will be valid for *finite* sums(Definition G.1 page 209) only (finite "N").

Definition N.1. [1] *Let* $\left\{ \boldsymbol{x}_n \in X \middle|_{n=1,2,\dots,N} \right\}$ *be a set of vectors in a* LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

DEF

A vector $\boldsymbol{x} \in X$ *is a* ***linear combination*** *of the vectors in* $\left\{ \boldsymbol{x}_n \right\}$ *if*

there exists $\left\{ \alpha_n \in \mathbb{F} \middle|_{n=1,2,\dots,N} \right\}$ *such that* $$\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n .$$

Definition N.2. [2] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space and* Y *be a subset of* X.

DEF

The ***linear span*** *of* Y *is defined as* $$\operatorname{span} Y \triangleq \left\{ \sum_{\gamma \in \Gamma} \alpha_\gamma \boldsymbol{y}_\gamma \middle| \alpha_\gamma \in \mathbb{F},\ \boldsymbol{y}_\gamma \in Y \right\}.$$

The set Y ***spans*** *a set* A *if* $$A \subseteq \operatorname{span} Y.$$

Proposition N.1. [3] *Let* $\left\{ \boldsymbol{x}_n \in X \middle|_{n=1,2,\dots,N} \right\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

PRP

1. $\operatorname{span}\left\{ \boldsymbol{x}_n \right\}$ *is a* LINEAR SPACE *(Definition D.1 page 169) and*
2. $\operatorname{span}\left\{ \boldsymbol{x}_n \right\}$ *is a* LINEAR SUBSPACE *of* $\boldsymbol{L}$.

Definition N.3. [4] *Let* $\boldsymbol{L} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a* LINEAR SPACE.

DEF

The set $Y \triangleq \left\{ \boldsymbol{x}_n \in X \middle|_{n=1,2,\dots,N} \right\}$ *is* ***linearly independent*** *in* $\boldsymbol{L}$ *if*

$$\left\{ \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = 0 \right\} \implies \left\{ \alpha_1 = \alpha_2 = \cdots = \alpha_N = 0 \right\}.$$

The set Y *is* ***linearly dependent*** *in* $\boldsymbol{L}$ *if* Y *is not linearly independent in* $\boldsymbol{L}$.

[1] Berberian (1961) page 11 ⟨Definition I.4.1⟩, Kubrusly (2001) page 46

[2] Michel and Herget (1993) page 86 ⟨3.3.7 Definition⟩, Kurdila and Zabarankin (2005) page 44, Searcóid (2002) page 71 ⟨Definition 3.2.5—more general definition⟩

[3] Kubrusly (2001) page 46

[4] Bachman and Narici (1966) pages 3–4, Christensen (2003) page 2, Heil (2011) page 156 ⟨Definition 5.7⟩

Figure N.1: Lattice of *linear combinations*

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Definition N.4. [5] *Let* $\{\boldsymbol{x}_n \in X \mid_{n=1,2,\ldots,N}\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF

The set $\{\boldsymbol{x}_n\}$ *is a* ***Hamel basis*** *for* $\boldsymbol{L}$ *if*

1. $\{\boldsymbol{x}_n\}$ SPANS $\boldsymbol{L}$ *(Definition N.2 page 305)* *and*
2. $\{\boldsymbol{x}_n\}$ *is* LINEARLY INDEPENDENT *in* $\boldsymbol{L}$ *(Definition N.1 page 305)* .

A HAMEL BASIS *is also called a* ***linear basis***.

Definition N.5. [6] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* $\boldsymbol{x}$ *be a* VECTOR *in* $\boldsymbol{L}$ *and* $Y \triangleq \{\boldsymbol{x}_n \in X \mid_{n=1,2,\ldots,N}\}$ *be a set of vectors in* $\boldsymbol{L}$.

DEF

The expression $\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$ *is the* ***expansion*** *of* $\boldsymbol{x}$ *on* Y *in* $\boldsymbol{L}$ *if* $\quad \boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$.

In this case, the sequence $(\!(\alpha_n)\!)_{n=1}^{N}$ *is the* ***coordinates*** *of* $\boldsymbol{x}$ *with respect to* Y *in* $\boldsymbol{L}$.

If $\alpha_N \neq 0$, *then* N *is the* ***dimension*** $\dim \boldsymbol{L}$ *of* $\boldsymbol{L}$.

Theorem N.1. [7] *Let* $\{\boldsymbol{x}_n \mid_{n=1,2,\ldots,N}\}$ *be a* HAMEL BASIS *(Definition N.4 page 307) for a* LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

THM

$$\left\{\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = \sum_{n=1}^{N} \beta_n \boldsymbol{x}_n\right\} \quad \Longrightarrow \quad \underbrace{\alpha_n = \beta_n \quad \forall n=1,2,\ldots,N}_{\text{coordinates of } \boldsymbol{x} \text{ are UNIQUE}} \qquad \forall \boldsymbol{x} \in X$$

✎PROOF:

$$\begin{aligned}
\mathbb{0} &= \boldsymbol{x} - \boldsymbol{x} \\
&= \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n - \sum_{n=1}^{N} \beta_n \boldsymbol{x}_n \\
&= \sum_{n=1}^{N} (\alpha_n - \beta_n) \boldsymbol{x}_n \\
&\Longrightarrow \{\boldsymbol{x}_n\} \text{ is } \textit{linearly dependent} \text{ if } (\alpha_n - \beta_n) \neq 0 \qquad \forall n = 1, 2, \ldots, N \\
&\Longrightarrow (\alpha_n - \beta_n) = 0 \qquad \forall n = 1, 2, \ldots, N \qquad \text{(because } \{\boldsymbol{x}_n\} \text{ is a } \textit{basis} \text{ and therefore must be } \textit{linearly independent}) \\
&\Longrightarrow \alpha_n = \beta_n \text{ for } n = 1, 2, \ldots, N
\end{aligned}$$

✏

Theorem N.2. [8] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

THM

$$\left\{\begin{array}{ll} 1. & \{\boldsymbol{x}_n \in X|_{n=1,2,\ldots,N}\} \textit{ is a } \text{HAMEL BASIS} \textit{ for } \boldsymbol{L} \qquad \textit{and} \\ 2. & \{\boldsymbol{y}_n \in X|_{n=1,2,\ldots,M}\} \textit{ is a set of } \text{LINEARLY INDEPENDENT} \textit{ vectors in } \boldsymbol{L} \end{array}\right\}$$

$$\Longrightarrow \quad \left\{\begin{array}{ll} 1. & M \leq N \qquad \textit{and} \\ 2. & M = N \Longrightarrow \{\boldsymbol{y}_n|_{n=1,2,\ldots,M}\} \textit{ is a } \text{BASIS} \textit{ for } \boldsymbol{L} \qquad \textit{and} \\ 3. & M \neq N \Longrightarrow \{\boldsymbol{y}_n|_{n=1,2,\ldots,M}\} \textit{ is } \text{NOT} \textit{ a basis for } \boldsymbol{L} \end{array}\right\}$$

✎PROOF:

[5] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[6] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[7] Michel and Herget (1993) pages 89–90 ⟨Theorem 3.3.25⟩

[8] Michel and Herget (1993) pages 90–91 ⟨Theorem 3.3.26⟩

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1. Proof that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is a *basis* for $\boldsymbol{L}$:
   (a) Proof that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ *spans* $\boldsymbol{L}$:
      i. Because $\{\boldsymbol{x}_n|_{n=1,2,\ldots,N}\}$ is a *basis* for $\boldsymbol{L}$, there exists $\beta \in \mathbb{F}$ and $\{\alpha_n \in \mathbb{F}|_{n=1,2,\ldots,N}\}$ such that
      $$\beta \boldsymbol{y}_1 + \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = 0.$$
      ii. Select an n such that $\alpha_n \neq 0$ and renumber (if necessary) the above indices such that
      $$\boldsymbol{x}_n = -\frac{\beta}{\alpha_n}\boldsymbol{y}_1 + \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n}\boldsymbol{x}_n.$$
      iii. Then, for any $\boldsymbol{y} \in X$, we can write
      $$\begin{aligned}\boldsymbol{y} &= \sum_{n=1}^{N} \gamma_{n\in\mathbb{Z}}\boldsymbol{x}_n \\ &= \left(\sum_{n=1}^{N-1} \gamma_{n\in\mathbb{Z}}\boldsymbol{x}_n\right) + \gamma_{n\in\mathbb{Z}}\left(-\frac{\beta}{\alpha_n}\boldsymbol{y}_1 - \sum_{n=1}^{N-1}\frac{\alpha_n}{\alpha_n}\boldsymbol{x}_n\right) \\ &= -\frac{\beta\gamma_n}{\alpha_n}\boldsymbol{y}_1 + \sum_{n=1}^{N-1}\left(\gamma_n - \frac{\alpha_n\gamma_n}{\alpha_n}\right)\boldsymbol{x}_n \\ &= \delta\boldsymbol{y}_1 + \sum_{n=1}^{N-1}\delta_{n\in\mathbb{Z}}\boldsymbol{x}_n\end{aligned}$$
      iv. This implies that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ *spans* $\boldsymbol{L}$:
   (b) Proof that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is *linearly independent*:
      i. If $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is *linearly dependent*, then there exists $\{\epsilon, \epsilon_1, \ldots, \epsilon_{N-1}\}$ such that
      $$\epsilon\boldsymbol{y}_1 + \left(\sum_{n=1}^{N-1}\epsilon_{n\in\mathbb{Z}}\boldsymbol{x}_n\right) + 0\boldsymbol{x}_n = 0.$$
      ii. item (1(b)i) implies that the coordinate of $\boldsymbol{y}_1$ associated with $\boldsymbol{x}_n$ *is* 0.
      $$\boldsymbol{y}_1 = -\left(\sum_{n=1}^{N-1}\frac{\epsilon_n}{\epsilon}\boldsymbol{x}_n\right) + 0\boldsymbol{x}_n = 0.$$
      iii. item (1(a)i) implies that the coordinate of $\boldsymbol{y}_1$ associated with $\boldsymbol{x}_n$ is *not* 0.
      $$\boldsymbol{y}_1 = -\sum_{n=1}^{N}\frac{\alpha_n}{\beta}\boldsymbol{x}_n.$$
      iv. This implies that item (1(b)i) (that the set is linearly dependent) is *false* because item (1(b)ii) and item (1(b)iii) *contradict* each other.
      v. This implies $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is *linearly independent*.
2. Proof that $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-2}\}$ is a *basis*: Repeat item (1).
3. Suppose $m = n$. Proof that $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_M\}$ is a *basis*: Repeat item (1) $M-1$ times.
4. Proof that $M \not> N$:
   (a) Suppose that $M = N+1$.
   (b) Then because $\{\boldsymbol{y}_n|_{n=1,2,\ldots,N}\}$ is a *basis*, there exists $\{\zeta_n|_{n=1,2,\ldots,N+1}\}$ such that
   $$\sum_{n=1}^{N+1}\zeta_{n\in\mathbb{Z}}\boldsymbol{y}_{n\in\mathbb{Z}} = 0.$$
   (c) This implies that $\{\boldsymbol{y}_n|_{n=1,2,\ldots,N+1}\}$ is *linearly dependent*.

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(d) This implies that $\{\boldsymbol{y}_n|_{n=1,2,\ldots,N+1}\}$ is *not* a basis.

(e) This implies that $M \not> N$.

5. Proof that $M \neq N \implies \{\boldsymbol{y}_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$:

   (a) Proof that $M > N \implies \{\boldsymbol{y}_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$: same as in item (4).

   (b) Proof that $M < N \implies \{\boldsymbol{y}_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$:

      i. Suppose $m = N - 1$.

      ii. Then $\{\boldsymbol{y}_n|_{n=1,2,\ldots,N-1}\}$ is a *basis* and there exists λ such that
      $$\sum_{n=1}^{N} \lambda_{n\in\mathbb{Z}} \boldsymbol{y}_{n\in\mathbb{Z}} = 0.$$

      iii. This implies that $\{\boldsymbol{y}_n|_{n=1,2,\ldots,N}\}$ is *linearly dependent* and is *not* a basis.

      iv. But this contradicts item (3), therefore $M \neq N - 1$.

      v. Because $M = N$ yields a basis but $M = N - 1$ does not, $M < N - 1$ also does not yield a basis.

✏

Corollary N.1. [9] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space.*

COR
$$\underbrace{\left\{\begin{array}{ll} 1. & \{\boldsymbol{x}_n \in X|_{n=1,2,\ldots,N}\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \quad \text{and} \\ 2. & \{\boldsymbol{y}_n \in X|_{n=1,2,\ldots,M}\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \end{array}\right\}}_{\text{(all Hamel bases for } \boldsymbol{L} \text{ have the same number of vectors)}} \implies \{N = M\}$$

✎PROOF: This follows from Theorem N.2 (page 307). ✏

N.2 Bases in topological linear spaces

A linear space supports the concept of the *span* of a set of vectors (Definition N.2 page 305). In a topological linear space $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$, a set A is said to be *total* in $\boldsymbol{\Omega}$ if the span of A is *dense* in $\boldsymbol{\Omega}$. In this case, A is said to be a *total set* or a *complete set*. However, this use of "complete" in a "*complete set*" is not equivalent to the use of "complete" in a "*complete metric space*" .[10] In this text, except for these comments and Definition N.6, "complete" refers to the metric space definition only.

If a set is both *total* and *linearly independent* (Definition N.3 page 305) in $\boldsymbol{\Omega}$, then that set is a *Hamel basis* (Definition N.4 page 307) for $\boldsymbol{\Omega}$.

Definition N.6. [11] *Let* A^- *be the* CLOSURE *of a* A *in a* TOPOLOGICAL LINEAR SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$. *Let* $\operatorname{span} A$ *be the* SPAN *(Definition N.2 page 305) of a set* A.

DEF
A set of vectors A *is* ***total*** *(or* ***complete*** *or* ***fundamental****) in* $\boldsymbol{\Omega}$ *if*
$$(\operatorname{span} A)^- = \boldsymbol{\Omega} \qquad \text{(SPAN of } A \text{ is DENSE in } \boldsymbol{\Omega}\text{)}.$$

[9] Kubrusly (2001) page 52 ⟨Theorem 2.7⟩, Michel and Herget (1993) page 91 ⟨Theorem 3.3.31⟩

[10] Haaser and Sullivan (1991) pages 296–297 ⟨6·Orthogonal Bases⟩, Rynne and Youngson (2008) page 78 ⟨Remark 3.50⟩, Heil (2011) page 21 ⟨Remark 1.26⟩

[11] Young (2001) page 19 ⟨Definition 1.5.1⟩, Sohrab (2003) page 362 ⟨Definition 9.2.3⟩, Gupta (1998) page 134 ⟨Definition 2.4⟩, Bachman and Narici (1966) pages 149–153 ⟨Definition 9.3, Theorems 9.9 and 9.10⟩

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N.3 Schauder bases in Banach spaces

Definition N.7. [12] *Let* $\boldsymbol{B} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be a* BANACH SPACE. *Let* $\overset{\star}{=}$ *represent* STRONG CONVERGENCE *(Definition H.5 page 231) in* $\boldsymbol{B}$.

DEF

The countable set $\big\{\boldsymbol{x}_n \in X \big|_{n\in\mathbb{N}}\big\}$ *is a* ***Schauder basis*** *for* $\boldsymbol{B}$ *if for each* $\boldsymbol{x} \in X$

1. $\exists\, (\!(\alpha_n \in \mathbb{F})\!)_{n\in\mathbb{N}}$ such that $\boldsymbol{x} \overset{\star}{=} \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ (STRONG CONVERGENCE *in* $\boldsymbol{B}$) *and*
2. $\left\{\sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \overset{\star}{=} \sum_{n=1}^{\infty} \beta_n \boldsymbol{x}_n\right\} \implies \big\{(\!(\alpha_n)\!) = (\!(\beta_n)\!)\big\}$ (COEFFICIENT FUNCTIONALS *are* UNIQUE)

In this case, $\sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ *is the* ***expansion*** *of* $\boldsymbol{x}$ *on* $\big\{\boldsymbol{x}_n \big|_{n\in\mathbb{N}}\big\}$ *and*

the elements of $(\!(\alpha_n)\!)$ *are the* ***coefficient functionals*** *associated with the basis* $\{\boldsymbol{x}_n\}$.
Coefficient functionals are also called ***coordinate functionals***.

In a Banach space, the existence of a Schauder basis implies that the space is *separable* (Theorem N.3 page 310). The question of whether the converse is also true was posed by Banach himself in 1932,[13] and became know as "*The basis problem*". This remained an open question for many years. The question was finally answered some 41 years later in 1973 by Per Enflo (University of California at Berkley), with the answer being "no". Enflo constructed a counterexample in which a separable Banach space does *not* have a Schauder basis.[14] Life is simpler in Hilbert spaces where the converse *is* true: a Hilbert space has a Schauder basis *if and only if* it is separable (Theorem N.11 page 323).

Theorem N.3. [15] *Let* $\boldsymbol{B} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\big)$ *be a* BANACH SPACE. *Let* $\mathbb{Q}$ *be the field of rational numbers.*

THM

$$\left\{\begin{array}{ll} 1. & \boldsymbol{B} \text{ has a SCHAUDER BASIS and} \\ 2. & \mathbb{Q} \text{ is DENSE in } \mathbb{F}. \end{array}\right\} \implies \big\{\, \boldsymbol{B} \text{ is SEPARABLE} \,\big\}$$

✎PROOF:

1. lemma:

$$\begin{aligned} \left|\left\{\boldsymbol{x} \middle| \exists\, (\!(\alpha_n \mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty} \left\|\boldsymbol{x} - \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right\| = 0\right\}\right| &= |\,\mathbb{Q}\times\mathbb{N}\,| \\ &= |\,\mathbb{Z}\times\mathbb{Z}\,| \\ &= |\,\mathbb{Z}\,| \\ &= \textit{countably infinite} \end{aligned}$$

[12] Carothers (2005) pages 24–25, Christensen (2003) pages 46–49 ⟨Definition 3.1.1 and page 49⟩, Young (2001) page 19 ⟨Section 6⟩, Singer (1970) page 17, Schauder (1927), Schauder (1928)
[13] Banach (1932a) page 111
[14] Enflo (1973), Lindenstrauss and Tzafriri (1977) pages 84–95 ⟨Section 2.d⟩
[15] Bachman et al. (2002) page 112 ⟨3.4.8⟩, Giles (2000) page 17, Heil (2011) page 21 ⟨Theorem 1.27⟩

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2. remainder of proof:

$\boldsymbol{B}$ has a *Schauder basis* $(\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{N}}$

$$\Longrightarrow \text{ for every } \boldsymbol{x}\in\boldsymbol{B}\text{, there exists } (\!(\alpha_n\in\mathbb{F})\!)_{n\in\mathbb{N}} \text{ such that } \boldsymbol{x} \doteq \sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n \qquad \text{by Definition N.7 page 310}$$

$$\Longrightarrow \text{ for every } \boldsymbol{x}\in\boldsymbol{B}\text{, there exists } (\!(\alpha_n\in\mathbb{F})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0 \qquad \text{by Definition H.5}$$

$$\Longrightarrow \text{ for every } \boldsymbol{x}\in\boldsymbol{B}\text{, there exists } (\!(\alpha_n\in\mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0 \qquad \text{because } \mathbb{Q}^- = \mathbb{F}$$

$$\Longrightarrow \boldsymbol{B} = \left\{\boldsymbol{x}\,|\,\exists\,(\!(\alpha_n\mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0\right\}$$

$$\Longrightarrow \boldsymbol{B} = \left\{\boldsymbol{x}\,|\,\exists\,(\!(\alpha_n\mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \boldsymbol{x} = \lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\}^-$$

$$\Longrightarrow \boldsymbol{B} \text{ is } \textit{separable} \qquad \text{by (1) lemma page 310}$$

✏

Definition N.8. [16] *Let* $\{\boldsymbol{x}_n|_{n\in\mathbb{N}}\}$ *and* $\{\boldsymbol{y}_n|_{n\in\mathbb{N}}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $\big(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\|\cdot\|\big)$.

DEF $\{\boldsymbol{x}_n\}$ *is* ***equivalent*** *to* $\{\boldsymbol{y}_n\}$
if there exists a BOUNDED INVERTIBLE *operator* $\mathbf{R}$ *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *such that* $\quad \mathbf{R}\boldsymbol{x}_n = \boldsymbol{y}_n \quad \forall n\in\mathbb{Z}$

Theorem N.4. [17] *Let* $\{\boldsymbol{x}_n|_{n\in\mathbb{N}}\}$ *and* $\{\boldsymbol{y}_n|_{n\in\mathbb{N}}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $\big(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\|\cdot\|\big)$.

THM $\{\{\boldsymbol{x}_n\}$ *is* EQUIVALENT *to* $\{\boldsymbol{y}_n\}\}$

$$\Longleftrightarrow \quad \left\{\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n \text{ is CONVERGENT} \Longleftrightarrow \sum_{n=1}^{\infty}\alpha_n\boldsymbol{y}_n \text{ is CONVERGENT}\right\}$$

Lemma N.1. [18] *Let* $\boldsymbol{\Omega} \triangleq \big(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\boldsymbol{T}\big)$ *be a topological linear space. Let* $\operatorname{span}A$ *be the* SPAN *of a set* A *(Definition N.2 page 305). Let* $\tilde{\mathsf{f}}(\omega)$ *and* $\tilde{\mathsf{g}}(\omega)$ *be the* FOURIER TRANSFORMS *(Definition M.2 page 294) of the functions* $\mathsf{f}(x)$ *and* $\mathsf{g}(x)$*, respectively, in* $\boldsymbol{L}^2_{\mathbb{R}}$ *(Definition E.1 page 199). Let* $\check{\mathsf{a}}(\omega)$ *be the* DTFT *(Definition R.1 page 393) of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition F.3 page 205).*

LEM
$$\left\{\begin{array}{ll}(1). & \{\mathbf{T}^n\mathsf{f}|_{n\in\mathbb{Z}}\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega} \quad \text{and}\\ (2). & \{\mathbf{T}^n\mathsf{g}|_{n\in\mathbb{Z}}\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\exists\,(\!(a_n)\!)_{n\in\mathbb{Z}} \quad \text{such that}\\ \tilde{\mathsf{f}}(\omega) = \check{\mathsf{a}}(\omega)\tilde{\mathsf{g}}(\omega)\end{array}\right\}$$

✎PROOF: Let $\boldsymbol{V}_0'$ be the space spanned by $\{\mathbf{T}^n\phi|_{n\in\mathbb{Z}}\}$.

$$\begin{aligned}\tilde{\mathsf{f}}(\omega) &\triangleq \tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \tilde{\mathbf{F}} \qquad \text{(Definition M.2 page 294)}\\ &= \tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}}a_n\mathbf{T}\mathsf{g} && \text{by (2)}\\ &= \sum_{n\in\mathbb{Z}}a_n\tilde{\mathbf{F}}\mathbf{T}\mathsf{g}\end{aligned}$$

[16] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[17] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[18] Daubechies (1992) page 140

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$$= \underbrace{\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}}_{\check{a}(\omega)} \underbrace{\tilde{\mathbf{F}}\mathsf{g}}_{\tilde{g}(\omega)} \qquad \text{(Corollary 1.3 page 12)}$$

$$= \check{a}(\omega)\tilde{g}(\omega) \qquad \text{by definition of } \check{\mathbf{F}} \text{ and } \tilde{\mathbf{F}} \qquad \text{(Definition R.1 page 393) (Definition M.2 page 294)}$$

$$\begin{aligned}
V_0 &\triangleq \left\{ \mathsf{f}(x) \middle| \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) \middle| \tilde{\mathbf{F}}\mathsf{f}(x) = \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) \middle| \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{b}}(\omega)\tilde{\mathsf{g}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) \middle| \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{b}}(\omega)\check{\mathsf{a}}(\omega)\tilde{\mathsf{f}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) \middle| \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{c}}(\omega)\tilde{\mathsf{f}}(\omega) \right\} \qquad \text{where } \tilde{\mathsf{c}}(\omega) \triangleq \tilde{\mathsf{b}}(\omega)\check{\mathsf{a}}(\omega) \\
&= \left\{ \mathsf{f}(x) \middle| \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{c}_n \mathsf{f}(x-n) \right\} \\
&\triangleq V_0'
\end{aligned}$$

✏

N.4 Linear combinations in inner product spaces

Definition N.9. *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition D.9 page 182)*.

DEF *Two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* X *are* ***orthogonal*** *if*

$$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \begin{cases} 0 & \text{for } \boldsymbol{x} \neq \boldsymbol{y} \\ c \in \mathbb{F}\backslash 0 & \text{for } \boldsymbol{x} = \boldsymbol{y} \end{cases}$$

In an *inner product space, orthogonality* is a special case of *linear independence*; or alternatively, linear independence is a generalization of orthogonality (next theorem).

Theorem N.5. [19] *Let* $\left\{ \boldsymbol{x}_n \in X \middle|_{n=1,2,\ldots,N} \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 182)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM $\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is ORTHOGONAL} \\ \text{(Definition N.9 page 312)} \end{array} \right\} \implies \left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is LINEARLY INDEPENDENT} \\ \text{(Definition N.1 page 305)} \end{array} \right\}$

✎PROOF:

1. Proof using *Pythagorean theorem*:
   Let $(\!(\alpha_n)\!)_{n\in\mathbb{N}}$ be a sequence with at least one nonzero element.

[19] Aliprantis and Burkinshaw (1998) page 283 ⟨Corollary 32.8⟩, Kubrusly (2001) page 352 ⟨Proposition 5.34⟩

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$$\begin{aligned}
\left\|\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\|^2 &= \sum_{n=1}^{N}\|\alpha_n\boldsymbol{x}_n\|^2 && \text{by left hypoth. and } \textit{Pythagorean Theorem}\\
&= \sum_{n=1}^{N}|\alpha_n|^2\,\|\boldsymbol{x}_n\|^2 && \text{by definition of } \|\cdot\| && \text{(Definition D.5 page 174)}\\
&> 0
\end{aligned}$$

$$\Longrightarrow \sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n \neq 0$$

$$\Longrightarrow (\boldsymbol{x}_n)_{n\in\mathbb{N}} \text{ is } \textit{linearly independent} \quad \text{by definition of } \textit{linear independence} \quad \text{(Definition N.3 page 305)}$$

2. Alternative proof:

$$\begin{aligned}
\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n = \mathbb{0} &\Longrightarrow \left\langle \sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n \,\middle|\, \boldsymbol{x}_m\right\rangle = \langle \mathbb{0} \mid \boldsymbol{x}_m\rangle\\
&\Longrightarrow \sum_{n=1}^{N}\alpha_n\langle \boldsymbol{x}_n \mid \boldsymbol{x}_m\rangle = 0\\
&\Longrightarrow \sum_{n=1}^{N}\alpha_n\bar{\delta}(k-m) = 0\\
&\Longrightarrow \alpha_m = 0 \qquad \text{for } m = 1,2,\ldots,N
\end{aligned}$$

✏

Theorem N.6 (Bessel's Equality). [20] *Let* $\left\{\boldsymbol{x}_n \in X \,\middle|\, {}_{n=1,2,\ldots,N}\right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 182)* $\big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle \mid \triangledown\rangle\big)$ *and with* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.

THM

$$\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \textit{ is } \text{ORTHONORMAL} \\ \textit{(Definition N.9 page 312)} \end{array} \right\} \Longrightarrow \left\{ \underbrace{\left\|\boldsymbol{x} - \sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2}_{\textit{approximation error}} = \|\boldsymbol{x}\|^2 - \sum_{n=1}^{N}|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle|^2 \quad \forall \boldsymbol{x}\in X \right\}$$

✎PROOF:

$$\begin{aligned}
&\left\|\boldsymbol{x} - \sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2\\
&= \|\boldsymbol{x}\|^2 + \left\|\sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2 - 2\Re\left\langle \boldsymbol{x} \,\middle|\, \sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\rangle && \text{by } \textit{polar identity}\\
&= \|\boldsymbol{x}\|^2 + \left\|\sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2 - 2\Re\left[\left(\sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right)^*\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right] && \text{by property of } \langle\triangle \mid \triangledown\rangle && \text{(Definition D.9 page 182)}\\
&= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N}\|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\|^2 - 2\Re\left[\left(\sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right)^*\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right] && \text{by } \textit{Pythagorean Theorem}
\end{aligned}$$

[20] Bachman et al. (2002) page 103, Pedersen (2000) pages 38–39

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$$= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N} \left\|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n\right\|^2 - 2\Re\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle^* \ \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right)$$

$$= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \underbrace{\left\|\boldsymbol{x}_n\right\|^2}_{1} - 2\Re\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle^* \ \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right) \quad \text{by property of } \|\cdot\| \quad \text{(Definition D.5 page 174)}$$

$$= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \cdot 1 - 2\Re\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle^* \ \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right) \quad \text{by def. of } \textit{orthonormality} \quad \text{(Definition N.9 page 312)}$$

$$= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 - 2\Re \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2$$

$$= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 - 2 \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \quad \text{because } |\cdot| \text{ is real}$$

$$= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2$$

✏

Theorem N.7 (Bessel's inequality). [21] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, {\scriptstyle n=1,2,\ldots,N} \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 182)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *and with* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.

THM

$$\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \textit{ is } \textsc{orthonormal} \\ \textit{(Definition N.9 page 312)} \end{array} \right\} \implies \left\{ \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \ \le \ \|\boldsymbol{x}\|^2 \quad \forall \boldsymbol{x} \in X \right\}$$

✎PROOF:

$$0 \le \left\| \boldsymbol{x} - \sum_{n=1}^{N} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n \right\|^2 \quad \text{by definition of } \|\cdot\| \quad \text{(Definition D.5 page 174)}$$

$$= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \quad \text{by } \textit{Bessel's Equality} \quad \text{(Theorem N.6 page 313)}$$

✏

The *Best Approximation Theorem* (next) shows that

- the best sequence for representing a vector is the sequence of projections of the vector onto the sequence of basis functions
- the error of the projection is orthogonal to the projection.

Theorem N.8 (Best Approximation Theorem). [22] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, {\scriptstyle n=1,2,\ldots,N} \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 182)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *and with* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.

[21] Giles (2000) pages 54–55 ⟨3.13 Bessel's inequality⟩, Bollobás (1999) page 147, Aliprantis and Burkinshaw (1998) page 284

[22] Walter and Shen (2001) pages 3–4, Pedersen (2000) page 39, Edwards (1995) pages 94–100, Weyl (1940)

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THM

$$\left\{\begin{array}{c}\{\boldsymbol{x}_n\}\ \textit{is}\\ \text{ORTHONORMAL}\\ \textit{(Definition N.9 page 312)}\end{array}\right\} \implies \left\{\begin{array}{lll} 1. & \arg\min\limits_{(\alpha_n)_{n=1}^N} \left\|\boldsymbol{x}-\sum\limits_{n=1}^N \alpha_n\boldsymbol{x}_n\right\| = \underbrace{(\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle)_{n=1}^N}_{\textit{best } \alpha_n = \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle} & \forall \boldsymbol{x}\in X \quad \textit{and} \\ 2. & \underbrace{\left(\sum\limits_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right)}_{\textit{approximation}} \perp \underbrace{\left(\boldsymbol{x}-\sum\limits_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right)}_{\textit{approximation error}} & \forall \boldsymbol{x}\in X \end{array}\right\}$$

✎PROOF:

1. Proof that $(\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle)$ is the best sequence:

$$\begin{aligned}
&\left\|\boldsymbol{x}-\sum_{n=1}^N \alpha_n\boldsymbol{x}_n\right\|^2\\
&= \|\boldsymbol{x}\|^2 - 2\Re\left\langle \boldsymbol{x}\mid\sum_{n=1}^N \alpha_n\boldsymbol{x}_n\right\rangle + \left\|\sum_{n=1}^N \alpha_n\boldsymbol{x}_n\right\|^2\\
&= \|\boldsymbol{x}\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^*\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\right) + \sum_{n=1}^N \|\alpha_n\boldsymbol{x}_n\|^2 \qquad \text{by } \textit{Pythagorean Theorem}\\
&= \|\boldsymbol{x}\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^*\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\right) + \sum_{n=1}^N |\alpha_n|^2 + \underbrace{\left[\sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2 - \sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2\right]}_{0}\\
&= \left[\|\boldsymbol{x}\|^2 - \sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2 - 2\mathbf{R}_{\mathrm{e}}\left[\alpha_n^*\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\right] + |\alpha_n|^2\right]\\
&= \left[\|\boldsymbol{x}\|^2 - \sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2 - \alpha_n^*\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle - \alpha_n\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle^* + |\alpha_n|^2\right]\\
&= \left\|\boldsymbol{x}-\sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2 + \sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle - \alpha_n|^2 \qquad \text{by } \textit{Bessel's Equality} \qquad \text{(Theorem N.6 page 313)}\\
&\geq \left\|\boldsymbol{x}-\sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2
\end{aligned}$$

2. Proof that the approximation and approximation error are orthogonal:

$$\begin{aligned}
\left\langle \sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n \,\middle|\, \boldsymbol{x}-\sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\rangle &= \left\langle \sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n \,\middle|\, \boldsymbol{x}\right\rangle - \left\langle \sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n \,\middle|\, \sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\rangle\\
&= \sum_{n=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle^*\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle - \sum_{n=1}^N\sum_{m=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\langle \boldsymbol{x}\mid\boldsymbol{x}_m\rangle^*\langle \boldsymbol{x}_n\mid\boldsymbol{x}_m\rangle\\
&= \sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2 - \sum_{n=1}^N\sum_{m=1}^N \langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle\langle \boldsymbol{x}\mid\boldsymbol{x}_m\rangle^*\,\bar{\delta}_{nm}\\
&= \sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2 - \sum_{n=1}^N |\langle \boldsymbol{x}\mid\boldsymbol{x}_n\rangle|^2\\
&= 0
\end{aligned}$$

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✏

N.5 Orthonormal bases in Hilbert spaces

Definition N.10. *Let* $\left\{ \boldsymbol{x}_n \in X \middle| {\scriptstyle n=1,2,\ldots,N} \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 182)* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

DEF

The set $\{\boldsymbol{x}_n\}$ *is an* ***orthogonal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{\boldsymbol{x}_n\}$ *is* ORTHOGONAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.

The set $\{\boldsymbol{x}_n\}$ *is an* ***orthonormal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{\boldsymbol{x}_n\}$ *is* ORTHONORMAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.

Definition N.11. [23] *Let* $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$ *be a Hilbert space.*

DEF

Suppose there exists a set $\left\{ \boldsymbol{x}_n \in X \middle| {\scriptstyle n\in\mathbb{N}} \right\}$ *such that* $\boldsymbol{x} \stackrel{*}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n$.

Then the quantities $\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle$ *are called the* ***Fourier coefficients*** *of* $\boldsymbol{x}$ *and the sum* $\sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n$ *is called the* ***Fourier expansion*** *of* $\boldsymbol{x}$ *or the* ***Fourier series*** *for* $\boldsymbol{x}$.

Definition N.12.

DEF

The ***Kronecker delta function*** $\bar{\delta}_n$ *is defined as*

$$\bar{\delta}_n \triangleq \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \forall n \in \mathbb{Z}$$

Lemma N.2 (Perfect reconstruction)**.** *Let* $\left\{ \boldsymbol{x}_n \in X \middle| {\scriptstyle n\in\mathbb{N}} \right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

LEM

$$\left\{ \begin{array}{ll} (1). & (\boldsymbol{x}_n) \text{ is a BASIS for } \boldsymbol{H} \quad \text{and} \\ (2). & (\boldsymbol{x}_n) \text{ is ORTHONORMAL} \end{array} \right\} \implies \boldsymbol{x} \stackrel{*}{=} \overbrace{\sum_{n=1}^{\infty} \underbrace{\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle}_{\text{Fourier coefficient}} \boldsymbol{x}_n}^{\text{Fourier expansion}} \quad \forall \boldsymbol{x} \in X$$

✎PROOF:

$$\begin{aligned}
\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle &= \left\langle \sum_{m\in\mathbb{Z}} \alpha_m \boldsymbol{x}_m \mid \boldsymbol{x}_n \right\rangle && \text{by left hypothesis (1)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \langle \boldsymbol{x}_m \mid \boldsymbol{x}_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition D.9 page 182)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \bar{\delta}_{n-m} && \text{by left hypothesis (2)} && \text{(Definition N.9 page 312)} \\
&= \alpha_n
\end{aligned}$$

✏

Proposition N.2. [24] *Let* $\left\{ \boldsymbol{x}_n \in X \middle| {\scriptstyle n\in\mathbb{N}} \right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

[23] Fabian et al. (2010) page 27 ⟨Theorem 1.55⟩, Young (2001) page 6, Young (1980) page 6
[24] Han et al. (2007) pages 93–94 ⟨Proposition 3.11⟩

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PRP

$$\underbrace{\|\boldsymbol{x}\|^2 \overset{\star}{=} \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2}_{\text{PARSEVAL FRAME}} \iff \underbrace{\boldsymbol{x} \overset{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n}_{\text{FOURIER EXPANSION (Definition N.11 page 316)}} \qquad \forall x \in X$$

✎PROOF:

1. Proof that *Parseval frame* $\Longleftarrow$ *Fourier expansion*

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &\triangleq \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition of } \|\cdot\| \\
&= \left\langle \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{x} \mid \boldsymbol{x}_n \right\rangle && \text{by right hypothesis} \\
&\overset{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle && \text{by property of } \langle \triangle \mid \triangledown\rangle \\
&\overset{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle^* && \text{by property of } \langle \triangle \mid \triangledown\rangle \\
&\overset{\star}{=} \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 && \text{by property of } \mathbb{C}
\end{aligned}$$

2. Proof that *Parseval frame* $\Longrightarrow$ *Fourier expansion*

   (a) Let $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ be the *standard othornormal basis* such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.

   (b) Let $\mathbf{M}$ be an operator in $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{x} \triangleq \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{e}_n$.

   (c) lemma: $\mathbf{M}$ is *isometric*. Proof:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{e}_n \right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2b) page 317)} \\
&= \sum_{n=1}^{\infty} \left\| \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{e}_n \right\|^2 && \text{by } \textit{Pythagorean Theorem} \\
&= \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \|\boldsymbol{e}_n\|^2 && \text{by } \textit{homogeneous} \text{ property of } \|\cdot\| && \text{(Definition D.5 page 174)} \\
&= \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 && \text{by definition of } \textit{orthonormal} && \text{(Definition N.9 page 312)} \\
&= \|\boldsymbol{x}\|^2 && \text{by Parseval frame hypothesis} \\
&\Longrightarrow \quad \mathbf{M} \text{ is } \textit{isometric} && \text{by definition of } \textit{isometric} && \text{(Definition D.13 page 190)}
\end{aligned}$$

   (d) Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthornormal basis* for $\boldsymbol{H}$.

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(e) Proof for *Fourier expansion*:

$$
\begin{aligned}
\boldsymbol{x} &= \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by } \textit{Fourier expansion} \text{ (Proposition N.3 page 320)} \\
&= \sum_{n=1}^{\infty} \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by (2c) lemma page 317 and Theorem D.23 page 191} \\
&= \sum_{n=1}^{\infty} \left\langle \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \boldsymbol{e}_m \mid \sum_{k=1}^{\infty} \langle \boldsymbol{u}_n \mid \boldsymbol{x}_k \rangle \boldsymbol{e}_k \right\rangle \boldsymbol{u}_n && \text{by item (2b) page 317} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \sum_{k=1}^{\infty} \langle \boldsymbol{u}_n \mid \boldsymbol{x}_k \rangle^* \langle \boldsymbol{e}_m \mid \boldsymbol{e}_k \rangle \boldsymbol{u}_n && \text{by Definition D.9 page 182} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \langle \boldsymbol{u}_n \mid \boldsymbol{x}_m \rangle^* \boldsymbol{u}_n && \text{by item (2a) page 317 and Definition N.9 page 312} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \langle \boldsymbol{x}_m \mid \boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by Definition D.9 page 182} \\
&= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \sum_{n=1}^{\infty} \langle \boldsymbol{x}_m \mid \boldsymbol{u}_n \rangle \boldsymbol{u}_n \\
&= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \boldsymbol{x}_m && \text{by item (2d) page 317}
\end{aligned}
$$

✏

When is a set of orthonormal vectors in a Hilbert space $\boldsymbol{H}$ *total*? Theorem N.9 (next) offers some help.

Theorem N.9 (The Fourier Series Theorem). [25] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n \in \mathbb{N} \right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle \right)$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.

THM

(A) $\left\{ \boldsymbol{x}_n \right\}$ *is* ORTHONORMAL *in* $\boldsymbol{H}$ $\Longrightarrow$

$$
\left\{
\begin{array}{llllll}
 & (1). & \left(\operatorname{span}\left\{ \boldsymbol{x}_n \right\}\right)^{-} & = & \boldsymbol{H} & & (\left\{ \boldsymbol{x}_n \right\} \text{ is TOTAL in } \boldsymbol{H}) \\
\Longleftrightarrow & (2). & \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle & \stackrel{\star}{=} & \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \langle \boldsymbol{y} \mid \boldsymbol{x}_n \rangle^* & \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X} & \text{(GENERALIZED PARSEVAL'S IDENTITY)} \\
\Longleftrightarrow & (3). & \|\boldsymbol{x}\|^2 & \stackrel{\star}{=} & \sum_{n=1}^{\infty} \left| \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \right|^2 & \forall \boldsymbol{x} \in X & \text{(PARSEVAL'S IDENTITY)} \\
\Longleftrightarrow & (4). & \boldsymbol{x} & \stackrel{\star}{=} & \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n & \forall \boldsymbol{x} \in X & \text{(FOURIER SERIES EXPANSION)}
\end{array}
\right\}
$$

✎PROOF:

[25] Bachman and Narici (1966) pages 149–155 ⟨Theorem 9.12⟩, Kubrusly (2001) pages 360–363 ⟨Theorem 5.48⟩, Aliprantis and Burkinshaw (1998) pages 298–299 ⟨Theorem 34.2⟩, Christensen (2003) page 57 ⟨Theorem 3.4.2⟩, Berberian (1961) pages 52–53 ⟨Theorem II§8.3⟩, Heil (2011) pages 34–35 ⟨Theorem 1.50⟩, Bracewell (1978) page 112 ⟨Rayleigh's theorem⟩

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1. Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}
\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle &\stackrel{*}{=} \left\langle \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \,\middle|\, \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m \rangle \boldsymbol{x}_m \right\rangle && \text{by (A) and (1)} \\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \left\langle \boldsymbol{x}_n \,\middle|\, \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m \rangle \boldsymbol{x}_m \right\rangle && \text{by property of } \langle \cdot \mid \cdot \rangle \quad \text{(Definition D.9 page 182)} \\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m \rangle^* \langle \boldsymbol{x}_n \mid \boldsymbol{x}_m \rangle && \text{by property of } \langle \cdot \mid \cdot \rangle \quad \text{(Definition D.9 page 182)} \\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m \rangle^* \bar{\delta}_{mn} && \text{by (A)} \\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \langle \boldsymbol{y} \mid \boldsymbol{x}_n \rangle^* && \text{by definition of } \bar{\delta}_n \quad \text{(Definition N.12 page 316)}
\end{aligned}$$

2. Proof that (2) $\Longrightarrow$ (3):

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &\triangleq \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by definition of } \textit{induced norm} \\
&= \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle^* && \text{by (2)} \\
&= \sum_{n=1}^{\infty} \left| \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \right|^2
\end{aligned}$$

3. Proof that (3) $\Longleftrightarrow$ (4) *not* using (A): by Proposition N.2 page 316

4. Proof that (3) $\Longrightarrow$ (1) (proof by contradiction):

   (a) Suppose $\{\boldsymbol{x}_n\}$ is *not total*.

   (b) Then there must exist a vector $\boldsymbol{y}$ in $\boldsymbol{H}$ such that the set $B \triangleq \{\boldsymbol{x}_n\} \cup \boldsymbol{y}$ is *orthonormal*.

   (c) Then $1 = \|y\|^2 \neq \sum_{n=1}^{\infty} \left| \langle \boldsymbol{y} \mid \boldsymbol{x}_n \rangle \right|^2 = 0$.

   (d) But this contradicts (3), and so $\{\boldsymbol{x}_n\}$ must be *total* and (3) $\Longrightarrow$ (1).

5. Extraneous proof that (3) $\Longrightarrow$ (4) (this proof is not really necessary here):

$$\begin{aligned}
\left\| \boldsymbol{x} - \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 &= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 && \text{by } \textit{Bessel's Equality} \quad \text{(Theorem N.6 page 313)} \\
&= 0 && \text{by (3)} \\
\Longrightarrow \quad \boldsymbol{x} &\stackrel{*}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by definition of } \stackrel{*}{=} \quad \text{(Definition H.5 page 231)}
\end{aligned}$$

6. Extraneous proof that (A) $\Longrightarrow$ (4) (this proof is not really necessary here)

   (a) The sequence $\sum_{n=1}^{N} \left| \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \right|^2$ is *monotonically increasing* in n.

   (b) By Bessel's inequality (page 314), the sequence is upper bounded by $\|\boldsymbol{x}\|^2$:

$$\sum_{n=1}^{N} \left| \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \right|^2 \leq \|\boldsymbol{x}\|^2$$

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(c) Because this sequence is both monotonically increasing and bounded in n, it must equal its bound in the limit as n approaches infinity:

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left|\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle\right|^2 = \|\boldsymbol{x}\|^2 \tag{N.1}$$

(d) If we combine this result with *Bessel's Equality* (Theorem N.6 page 313) we have

$$\begin{aligned}
\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n\right\|^2 &= \|\boldsymbol{x}\|^2-\lim_{N\to\infty}\sum_{n=1}^{N}|\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle|^2 && \text{by } \textit{Bessel's Equality} \text{ (Theorem N.6 page 313)}\\
&= \|\boldsymbol{x}\|^2-\|\boldsymbol{x}\|^2 && \text{by equation (N.1) page 320}\\
&= 0
\end{aligned}$$

✏

Proposition N.3 (Fourier expansion)**.** *Let* $\left\{\boldsymbol{x}_n\in X\,\middle|\,n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\left(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\langle\triangle\mid\triangledown\rangle\right)$.

PRP

$$\underbrace{\{\boldsymbol{x}_n\}\ \textit{is an}\ \text{ORTHONORMAL BASIS}\ \textit{for}\ \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{\boldsymbol{x}\overset{*}{=}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n}_{(1)} \iff \underbrace{\alpha_n=\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle}_{(2)}\right\}$$

✎PROOF:

1. Proof that (1) $\implies$ (2): by Lemma N.2 page 316
2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\left\|\boldsymbol{x}-\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_{n\in\mathbb{Z}}\right\|^2 &= \left\|\boldsymbol{x}-\sum_{n=1}^{\infty}\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_{n\in\mathbb{Z}}\right\|^2 && \text{by right hypothesis}\\
&= \|\boldsymbol{x}\|^2-\sum_{n=1}^{\infty}|\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle|^2 && \text{by Bessel's equality} && \text{(Theorem N.6 page 313)}\\
&= 0 && \text{by } \textit{Parseval's Identity} && \text{(Theorem N.9 page 318)}\\
&\overset{\text{def}}{\iff} \quad \boldsymbol{x}\overset{*}{=}\sum_{n=1}^{\infty}\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n && \text{by definition of } \textit{strong convergence} && \text{(Definition H.5 page 231)}
\end{aligned}$$

✏

Proposition N.4 (Riesz-Fischer Theorem)**.** [26] *Let* $\left\{\boldsymbol{x}_n\in X\,\middle|\,n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\left(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\langle\triangle\mid\triangledown\rangle\right)$.

PRP

$$\underbrace{\{\boldsymbol{x}_n\}\ \textit{is an}\ \text{ORTHONORMAL BASIS}\ \textit{for}\ \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{\sum_{n=1}^{\infty}|\alpha_n|^2<\infty}_{(1)} \iff \underbrace{\begin{array}{l}\exists \boldsymbol{x}\in\boldsymbol{H}\ \text{ such that}\\ \alpha_n=\langle \boldsymbol{x}\mid \boldsymbol{x}_n\rangle\end{array}}_{(2)}\right\}$$

✎PROOF:

[26] Young (2001) page 6

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1. Proof that (1) $\Longrightarrow$ (2):

   (a) If (1) is true, then let $\boldsymbol{x} \triangleq \sum_{n\in\mathbb{N}} \alpha_n \boldsymbol{x}_n$.

   (b) Then

$$\begin{aligned}
\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle &= \left\langle \sum_{m\in\mathbb{N}} \alpha_m \boldsymbol{x}_m \mid \boldsymbol{x}_n \right\rangle && \text{by definition of } \boldsymbol{x} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \langle \boldsymbol{x}_m \mid \boldsymbol{x}_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition D.9 page 182)} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \bar{\delta}_{mn} && \text{by (A)} \\
&= \sum_{m\in\mathbb{N}} \alpha_n && \text{by definition of } \bar{\delta} && \text{(Definition N.12 page 316)}
\end{aligned}$$

2. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |\alpha_n|^2 &= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 && \text{by (2)} \\
&\le \|\boldsymbol{x}\|^2 && \text{by } \textit{Bessel's Inequality} && \text{(Theorem N.7 page 314)} \\
&\le \infty
\end{aligned}$$

✏

Theorem N.10. [27]

THM *All* SEPARABLE HILBERT SPACES *are* ISOMORPHIC. *That is,*

$$\left\{\begin{array}{l} \boldsymbol{X} \text{ is a separable} \\ \text{Hilbert space} \quad \textit{and} \\ \boldsymbol{Y} \text{ is a separable} \\ \text{Hilbert space} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lllll} \textit{there is a BIJECTIVE operator } \mathbf{M} \in \boldsymbol{Y}^{\boldsymbol{X}} \textit{ such that} & & & & \\ (1). \quad \boldsymbol{y} & = & \mathbf{M}\boldsymbol{x} & \forall \boldsymbol{x}\in\boldsymbol{X}, \boldsymbol{y}\in\boldsymbol{Y} & \textit{and} \\ (2). \quad \|\mathbf{M}\boldsymbol{x}\| & = & \|\boldsymbol{x}\| & \forall \boldsymbol{x}\in\boldsymbol{X} & \textit{and} \\ (3). \quad \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle & = & \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle & \forall \boldsymbol{x}\in\boldsymbol{X}, \boldsymbol{y}\in\boldsymbol{Y} & \end{array}\right\}$$

✎PROOF:

1. Let $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\big)$ be a *separable Hilbert space* with *orthonormal basis* $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$.
   Let $\boldsymbol{Y} \triangleq \big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\big)$ be a *separable Hilbert space* with *orthonormal basis* $\{\boldsymbol{y}_n \mid n\in\mathbb{N}\}$.

2. Proof that there exists *bijective* operator $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$:

   (a) Let $\mathbf{M}$ be defined such that $\boldsymbol{y}_n \triangleq \mathbf{M}\boldsymbol{x}_n$.

   (b) Thus $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$.

   (c) Because $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$, $\mathbf{M}$ has an inverse operator $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$ such that $\boldsymbol{x}_n = \mathbf{M}^{-1}\boldsymbol{y}_n$.

3. Proof that $\mathbf{M}$ and $\mathbf{M}^{-1}$ are *bijective* operators between $\boldsymbol{X}$ and $\boldsymbol{Y}$:

[27] Young (2001) page 6

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(a) Proof that $\mathbf{M}$ maps $\boldsymbol{X}$ *into* $\boldsymbol{Y}$:

$$\begin{aligned}
\boldsymbol{x} \in \boldsymbol{X} \iff \boldsymbol{x} &\stackrel{*}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by } \textit{Fourier expansion} && \text{(Theorem N.9 page 318)}\\
&\implies \exists \boldsymbol{y} \in \boldsymbol{Y} \quad \text{such that} \quad \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle = \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle && \text{by } \textit{Riesz-Fischer Thm.} && \text{(Proposition N.4 page 320)}\\
&\implies\\
\boldsymbol{y} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by } \textit{Fourier expansion} && \text{(Theorem N.9 page 318)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{y}_n && \text{by } \textit{Riesz-Fischer Thm.} && \text{(Proposition N.4 page 320)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n && \text{by definition of } \mathbf{M} && \text{(item (2a) page 321)}\\
&= \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by prop. of linear ops.} && \text{(Theorem D.1 page 171)}\\
&= \mathbf{M}\boldsymbol{x} && \text{by definition of } \boldsymbol{x}
\end{aligned}$$

(b) Proof that $\mathbf{M}^{-1}$ maps $\boldsymbol{Y}$ *into* $\boldsymbol{X}$:

$$\begin{aligned}
\boldsymbol{y} \in \boldsymbol{Y} \iff \boldsymbol{y} &\stackrel{*}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by } \textit{Fourier expansion} && \text{(Theorem N.9 page 318)}\\
&\implies \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle = \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle && \text{by } \textit{Riesz-Fischer Thm.} && \text{(Proposition N.4 page 320)}\\
&\implies\\
\boldsymbol{x} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by } \textit{Fourier expansion} && \text{(Theorem N.9 page 318)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{x}_n && \text{by } \textit{Riesz-Fischer Thm.} && \text{(Proposition N.4 page 320)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \mathbf{M}^{-1}\boldsymbol{y}_n && \text{by definition of } \mathbf{M}^{-1} && \text{(item (2c) page 321)}\\
&= \mathbf{M}^{-1} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by prop. of } \textit{linear ops.} && \text{(Theorem D.1 page 171)}\\
&= \mathbf{M}^{-1}\boldsymbol{y} && \text{by definition of } \boldsymbol{y}
\end{aligned}$$

4. Proof for (2):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by } \textit{Fourier expansion} && \text{(Theorem N.9 page 318)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n \right\|^2 && \text{by property of } \textit{linear operators} && \text{(Theorem D.1 page 171)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{y}_n \right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2a) page 321)}\\
&= \sum_{n\in\mathbb{N}} \left| \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \right|^2 && \text{by } \textit{Parseval's Identity} && \text{(Proposition N.4 page 320)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by } \textit{Parseval's Identity} && \text{(Proposition N.4 page 320)}\\
&= \|\boldsymbol{x}\|^2 && \text{by } \textit{Fourier expansion} && \text{(Theorem N.9 page 318)}
\end{aligned}$$

5. Proof for (3): by (2) and Theorem D.23 page 191

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Theorem N.11. [28] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has a* SCHAUDER BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

Theorem N.12. [29] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has an* ORTHONORMAL BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

N.6 Riesz bases in Hilbert spaces

Definition N.13. [30] *Let $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+},$

DEF: $\{\boldsymbol{x}_n\}$ *is a* ***Riesz basis*** *for* $\boldsymbol{H}$ *if* $\{\boldsymbol{x}_n\}$ *is* EQUIVALENT *(Definition N.8 page 311) to some* ORTHONORMAL BASIS *(Definition N.10 page 316) in* $\boldsymbol{H}$.

Definition N.14. [31] *Let $(\!(\boldsymbol{x}_n \in X)\!)_{n\in\mathbb{N}}$ be a sequence of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle\big)$.

DEF: *The sequence $(\!(\boldsymbol{x}_n)\!)$ is a* ***Riesz sequence*** *for* $\boldsymbol{H}$ *if*

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 \qquad \forall (\!(\alpha_n)\!)\in\ell^2_{\mathbb{F}}.$$

Definition N.15. *Let* $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle\big)$ *be an* INNER PRODUCT SPACE *(Definition D.9 page 182)*.

DEF: *The sequences $(\!(\boldsymbol{x}_n \in X)\!)_{n\in\mathbb{Z}}$ and $(\!(\boldsymbol{y}_n \in X)\!)_{n\in\mathbb{Z}}$ are* ***biorthogonal*** *with respect to each other in* $\boldsymbol{X}$ *if* $\langle \boldsymbol{x}_n \,|\, \boldsymbol{y}_m\rangle = \bar{\delta}_{nm}$

Lemma N.3. [32] *Let $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ be a sequence in a* HILBERT SPACE $\boldsymbol{X} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle\big)$. *Let $\{\boldsymbol{y}_n \mid n\in\mathbb{N}\}$ be a sequence in a* HILBERT SPACE $\boldsymbol{Y} \triangleq \big(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle\big)$. *Let*

LEM:

$$\left\{\begin{array}{lll}
\text{(i).} & \{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{X} & \text{and} \\
\text{(ii).} & \text{There exists } A>0 \text{ such that } A\sum_{n\in C}|a_n|^2 \le \left\|\sum_{n\in C}a_n\boldsymbol{x}_n\right\|^2 \quad \text{for finite } C & \text{and} \\
\text{(iii).} & \text{There exists } B>0 \text{ such that } \left\|\sum_{n=1}^{\infty}b_n\boldsymbol{y}_n\right\|^2 \le B\sum_{n=1}^{\infty}|b_n|^2 \quad \forall (\!(b_n)\!)_{n\in\mathbb{N}}\in\ell^2_{\mathbb{F}} & \text{and}
\end{array}\right\} \implies$$

$$\left\{\begin{array}{lll}
\text{(1).} & \mathbf{R}^\circ \text{ is a linear bounded operator that maps from } \operatorname{span}\{\boldsymbol{x}_n\} \text{ to } \operatorname{span}\{\boldsymbol{y}_n\} & \\
 & \quad \text{where } \mathbf{R}^\circ\sum_{n\in C}c_n\boldsymbol{x}_n \triangleq \sum_{n\in C}c_n\boldsymbol{y}_n, \text{ for some sequence } (\!(c_n)\!) \text{ and finite set } C & \text{and} \\
\text{(2).} & \mathbf{R} \text{ has a unique extension to a bounded operator } \mathbf{R} \text{ that maps from } \boldsymbol{X} \text{ to } \boldsymbol{Y} & \text{and} \\
\text{(3).} & |||\mathbf{R}^\circ||| \le \frac{B}{A} & \text{and} \\
\text{(4).} & |||\mathbf{R}||| \le \frac{B}{A} &
\end{array}\right\}$$

[28] Bachman et al. (2002) page 112 ⟨3.4.8⟩, Berberian (1961) page 53 ⟨Theorem II§8.3⟩
[29] Kubrusly (2001) page 357 ⟨Proposition 5.43⟩
[30] Young (2001) page 27 ⟨Definition 1.8.2⟩, Christensen (2003) page 63 ⟨Definition 3.6.1⟩, Heil (2011) page 196 ⟨Definition 7.9⟩
[31] Christensen (2003) pages 66–68 ⟨page 68 and (3.24) on page 66⟩, Wojtaszczyk (1997) page 20 ⟨Definition 2.6⟩
[32] Christensen (2003) pages 65–66 ⟨Lemma 3.6.5⟩

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Theorem N.13. [33] *Let* $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\triangledown\rangle\big)$.

THM

$$\left\{ \begin{array}{l} \{\boldsymbol{x}_n\} \text{ is a RIESZ BASIS} \\ \text{for } \boldsymbol{H} \end{array} \right\} \iff \left\{ \begin{array}{ll} (1). & \{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{H} \quad \text{and} \\ (2). & \exists A,B\in\mathbb{R}^+ \quad \text{such that} \quad \forall (\alpha_n)\in\ell^2_{\mathbb{F}}, \\ & A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 \end{array} \right\}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:
   (a) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (1): all bases for $\boldsymbol{H}$ are *total* in $\boldsymbol{H}$.
   (b) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (2):
      i. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthonormal basis* for $\boldsymbol{H}$.
      ii. Let $\mathbf{R}$ be a *bounded bijective* operator such that $\boldsymbol{x}_n = \mathbf{R}\boldsymbol{u}_n$.
      iii. Proof for upper bound B:

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \left\|\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii))} \\
&= \left\|\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem D.1 page 171} \\
&\le |||\mathbf{R}|||^2\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem D.6 page 177} \\
&= |||\mathbf{R}|||^2\sum_{n=1}^{\infty}\left\|\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by } \textit{Pythagorean Theorem} \\
&= |||\mathbf{R}|||^2\sum_{n=1}^{\infty}|\alpha|^2\left\|\boldsymbol{u}_n\right\|^2 && \text{by } \textit{homogeneous} \text{ property of norms} && \text{(Definition D.5 page 174)} \\
&= \underbrace{|||\mathbf{R}|||^2}_{B}\sum_{n=1}^{\infty}|\alpha|^2 && \text{by definition of } \textit{orthonormality} && \text{(Definition N.9 page 312)}
\end{aligned}$$

      iv. Proof for lower bound A:

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \frac{|||\mathbf{R}^{-1}|||^2}{|||\mathbf{R}^{-1}|||^2}\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{because } |||\mathbf{R}^{-1}||| > 0 && \text{(Proposition D.1 page 175)} \\
&\ge \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{by Theorem D.6 page 177} \\
&= \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii) page 324)} \\
&= \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by property of } \textit{linear operators} && \text{(Theorem D.1 page 171)}
\end{aligned}$$

[33] Young (2001) page 27 ⟨Theorem 1.8.9⟩, Christensen (2003) page 66 ⟨Theorem 3.6.6⟩, Heil (2011) pages 197–198 ⟨Theorem 7.13⟩, Christensen (2008) pages 61–62 ⟨Theorem 3.3.7⟩

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$$= \frac{1}{\left|\left|\left|\mathbf{R}^{-1}\right|\right|\right|^2} \left\|\sum_{n=1}^{\infty} \alpha_n \boldsymbol{u}_n\right\|^2 \qquad \text{by definition of inverse op.} \qquad \text{(Definition D.3 page 170)}$$

$$= \frac{1}{\left|\left|\left|\mathbf{R}^{-1}\right|\right|\right|^2} \sum_{n=1}^{\infty} \left\|\alpha_n \boldsymbol{u}_n\right\|^2 \qquad \text{by } \textit{Pythagorean Theorem}$$

$$= \frac{1}{\left|\left|\left|\mathbf{R}^{-1}\right|\right|\right|^2} \sum_{n=1}^{\infty} \left|\alpha_n\right|^2 \left\|\boldsymbol{u}_n\right\|^2 \qquad \text{by } \|\cdot\| \textit{ homogeneous} \text{ prop.} \qquad \text{(Definition D.5 page 174)}$$

$$= \underbrace{\frac{1}{\left|\left|\left|\mathbf{R}^{-1}\right|\right|\right|^2}}_{A} \sum_{n=1}^{\infty} \left|\alpha_n\right|^2 \qquad \text{by def. of } \textit{orthonormality} \qquad \text{(Definition N.9 page 312)}$$

2. Proof for ($\Longrightarrow$) case:
   (a) Let $\left\{\boldsymbol{u}_n \middle| {}_{n\in\mathbb{N}}\right\}$ be an *orthonormal basis* for $\boldsymbol{H}$.
   (b) Using (2) and Lemma N.3 (page 323), construct an bounded extension operator $\mathbf{R}$ such that $\mathbf{R}\boldsymbol{u}_n = \boldsymbol{x}_n$ for all $n \in \mathbb{N}$.
   (c) Using (2) and Lemma N.3 (page 323), construct an bounded extension operator $\mathbf{S}$ such that $\mathbf{S}\boldsymbol{x}_n = \boldsymbol{u}_n$ for all $n \in \mathbb{N}$.
   (d) Then, $\mathbf{RV}\boldsymbol{x} = \mathbf{VR}\boldsymbol{x} \implies \mathbf{V} = \mathbf{R}^{-1}$, and so $\mathbf{R}$ is a bounded invertible operator
   (e) and $\left\{\boldsymbol{x}_n\right\}$ is a *Riesz sequence*.

✏

Theorem N.14. [34] *Let* $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle \mid \triangledown\rangle\right)$ *be a* SEPARABLE HILBERT SPACE.

THM

$$\left\{\begin{array}{l} (\boldsymbol{x}_n \in \boldsymbol{H})_{n\in\mathbb{Z}} \textit{ is a} \\ \text{RIESZ BASIS } \textit{for } \boldsymbol{H} \end{array}\right\} \implies \left\{\begin{array}{ll} \multicolumn{2}{l}{\textit{There exists } (\boldsymbol{y}_n \in \boldsymbol{H})_{n\in\mathbb{Z}} \textit{ such that}} \\ (1). & (\boldsymbol{x}_n) \textit{ and } (\boldsymbol{y}_n) \textit{ are } \text{BIORTHOGONAL} \quad \textit{and} \\ (2). & (\boldsymbol{y}_n) \textit{ is also a } \text{RIESZ BASIS } \textit{for } \boldsymbol{H} \quad \textit{and} \\ (3). & \exists B>A>0 \quad \text{such that} \\ & A\sum_{n=1}^{\infty}\left|a_n\right|^2 \le \left\|\sum_{n=1}^{\infty} a_n\boldsymbol{x}_n\right\|^2 = \left\|\sum_{n=1}^{\infty} a_n\boldsymbol{y}_n\right\|^2 \le B\sum_{n=1}^{\infty}\left|a_n\right|^2 \\ & \forall (a_n)_{n\in\mathbb{N}} \in \ell^2_{\mathbb{F}} \end{array}\right\}$$

✎PROOF:

1. Proof for (1):
   (a) Let $\boldsymbol{e}_n$ be the *unit vector* in $\boldsymbol{H}$ such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.
   (b) Let $\mathbf{M}$ be an operator on $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{e}_n = \boldsymbol{x}_n$.
   (c) Note that $\mathbf{M}$ is *isometric*, and as such $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall \boldsymbol{x}\in\boldsymbol{H}$.
   (d) Let $\boldsymbol{y}_n \triangleq \left(\mathbf{M}^{-1}\right)^*$.
   (e) Then,

$$\begin{aligned} \langle \boldsymbol{y}_n \mid \boldsymbol{x}_m\rangle &= \left\langle \left(\mathbf{M}^{-1}\right)^* \boldsymbol{e}_n \mid \mathbf{M}\boldsymbol{e}_m \right\rangle \\ &= \left\langle \boldsymbol{e}_n \mid \mathbf{M}^{-1}\mathbf{M}\boldsymbol{e}_m \right\rangle \\ &= \langle \boldsymbol{e}_n \mid \boldsymbol{e}_m\rangle \\ &= \bar{\delta}_{nm} \\ &\implies \left\{\boldsymbol{x}_n\right\} \text{ and } \left\{\boldsymbol{y}_n\right\} \text{ are } \textit{biorthogonal} \qquad \text{by Definition N.9 page 312} \end{aligned}$$

[34] Wojtaszczyk (1997) page 20 ⟨Lemma 2.7(a)⟩

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2. Proof for (3):

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}}\alpha_n\boldsymbol{y}_n\right\| &= \left\|\sum_{n\in\mathbb{Z}}\alpha_n\left(\mathbf{M}^{-1}\right)^*\boldsymbol{e}_n\right\| && \text{by definition of } \boldsymbol{y}_n && \text{(Proposition 1d page 325)}\\
&= \left\|\left(\mathbf{M}^{-1}\right)^*\sum_{n\in\mathbb{Z}}\alpha_n\boldsymbol{e}_n\right\| && \text{by property of } \textit{linear ops.} \\
&= \left\|\sum_{n\in\mathbb{Z}}\alpha_n\boldsymbol{e}_n\right\| && \text{because } \left(\mathbf{M}^{-1}\right)^* \text{ is } \textit{isometric} && \text{(Definition D.13 page 190)}\\
&= \left\|\mathbf{M}\sum_{n\in\mathbb{Z}}\alpha_n\boldsymbol{e}_n\right\| && \text{because } \mathbf{M} \text{ is } \textit{isometric} && \text{(Definition D.13 page 190)}\\
&= \left\|\sum_{n\in\mathbb{Z}}\alpha_n\mathbf{M}\boldsymbol{e}_n\right\| && \text{by property of } \textit{linear operators} \\
&= \left\|\sum_{n\in\mathbb{Z}}\alpha_n\boldsymbol{x}_n\right\| && \text{by definition of } \mathbf{M} \\
&\Longrightarrow \left\{\boldsymbol{y}_n\right\} \text{ is a } \textit{Riesz basis} && \text{by left hypothesis}
\end{aligned}$$

3. Proof for (2): by (3) and definition of *Riesz basis* (Definition N.13 page 323)

✏

Proposition N.5. [35] *Let* $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\triangledown\rangle\right)$.

PRP

$$\left\{\begin{array}{l}\left\{\boldsymbol{x}_n\right\} \textit{ is a } \text{RIESZ BASIS} \textit{ for } \boldsymbol{H} \textit{ with}\\ A\sum_{n=1}^{\infty}\left|a_n\right|^2 \le \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}\left|a_n\right|^2 \\ \forall\{a_n\}\in\ell^2_{\mathbb{F}}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\left\{\boldsymbol{x}_n\right\} \textit{ is a } \text{FRAME} \textit{ for } \boldsymbol{H} \textit{ with}\\ \underbrace{\frac{1}{B}\|\boldsymbol{x}\|^2 \le \sum_{n=1}^{\infty}\left|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle\right|^2 \le \frac{1}{A}\|\boldsymbol{x}\|^2}_{\text{STABILITY CONDITION}} \\ \forall\boldsymbol{x}\in\boldsymbol{H}\end{array}\right\}$$

✎PROOF:

1. Let $\left\{\boldsymbol{y}_n \middle| n\in\mathbb{N}\right\}$ be a *Riesz basis* that is *biorthogonal* to $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ (Theorem N.14 page 325).

2. Let $\boldsymbol{x} \triangleq \sum_{n=1}^{\infty}a_n\boldsymbol{y}_n$.

3. lemma:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle\right|^2 &= \sum_{n=1}^{\infty}\left|\left\langle\sum_{m=1}^{\infty}a_m\boldsymbol{y}_m\,\middle|\,\boldsymbol{x}_n\right\rangle\right|^2 && \text{by definition of } \boldsymbol{x} && \text{(item (2) page 326)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty}a_n\langle\boldsymbol{y}_m\,|\,\boldsymbol{x}_n\rangle\right|^2 && \text{by } \textit{homogeneous} \text{ property of } \langle\triangle\,|\,\triangledown\rangle && \text{(Definition D.9 page 182)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty}a_n\bar{\delta}_{mn}\right|^2 && \text{by definition of } \textit{biorthogonal} && \text{(Definition N.15 page 323)}\\
&= \sum_{n=1}^{\infty}\left|a_n\right|^2 && \text{by definition of } \bar{\delta} && \text{(Definition N.12 page 316)}
\end{aligned}$$

[35] Igari (1996) page 220 ⟨Lemma 9.8⟩, Wojtaszczyk (1997) pages 20–21 ⟨Lemma 2.7(a)⟩

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4. Then

$$
\begin{aligned}
& A\sum_{n=1}^{\infty}|a_n|^2 && \le && \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{x}_n\right\|^2 && \le && B\sum_{n=1}^{\infty}|a_n|^2 && \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 326)} \\
\Longrightarrow\ & A\sum_{n=1}^{\infty}|a_n|^2 && \le && \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{y}_n\right\|^2 && \le && B\sum_{n=1}^{\infty}|a_n|^2 && \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 326)} \\
\Longrightarrow\ & A\sum_{n=1}^{\infty}|a_n|^2 && \le && \|\boldsymbol{x}\|^2 && \le && B\sum_{n=1}^{\infty}|a_n|^2 && \text{by definition of } \boldsymbol{x} \text{ (item (2) page 326)} \\
\Longrightarrow\ & A\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 && \le && \|\boldsymbol{x}\|^2 && \le && B\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 && \text{by (3) lemma} \\
\Longrightarrow\ & \frac{1}{B}\|\boldsymbol{x}\|^2 && \le && \sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 && \le && \frac{1}{A}\|\boldsymbol{x}\|^2
\end{aligned}
$$

✏

Theorem N.15 (Battle-Lemarié orthogonalization). [36] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f}\in L^2_{\mathbb{R}}$.

THM

$$
\left\{\begin{array}{ll}
1. & \{\mathbf{T}^n\mathsf{g}\,|\,n\in\mathbb{Z}\} \text{ is a RIESZ BASIS for } L^2_{\mathbb{R}} \text{ and} \\[2ex]
2. & \tilde{\mathsf{f}}(\omega)\triangleq\dfrac{\tilde{\mathsf{g}}(\omega)}{\sqrt{2\pi\sum\limits_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2}}
\end{array}\right\}
\Longrightarrow
\left\{\begin{array}{l}
\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\} \\
\text{is an ORTHONORMAL BASIS for } L^2_{\mathbb{R}}
\end{array}\right\}
$$

✎PROOF:

1. Proof that $\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}$ is orthonormal:

$$
\begin{aligned}
\tilde{\mathsf{S}}_{\phi\phi}(\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem T.1 page 411} \\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{2\pi\sum\limits_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi(n-m))|^2}}\right|^2 && \text{by left hypothesis} \\
&= \sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{\sum\limits_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2 \\
&= \sum_{n\in\mathbb{Z}}\left|\frac{1}{\sqrt{\sum\limits_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2|\tilde{\mathsf{g}}(\omega+2\pi n)|^2 \\
&= \frac{1}{\sum\limits_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2 \\
&= 1 \\
\Longrightarrow\ & \{\mathsf{f}_n\,|\,n\in\mathbb{Z}\} \text{ is orthonormal} && \text{by Theorem T.3 page 417}
\end{aligned}
$$

[36] Wojtaszczyk (1997) page 25 ⟨Remark 2.4⟩, Vidakovic (1999) page 71, Mallat (1989) page 72, Mallat (1999) page 225, Daubechies (1992) page 140 ⟨(5.3.3)⟩

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2. Proof that $\{\mathbf{T}^n \mathsf{f} \mid n\in\mathbb{Z}\}$ is a basis for V_0: by Lemma N.1 page 311.

✏

N.7 Frames in Hilbert spaces

Definition N.16. [37] *Let* $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\mid\triangledown\rangle)$.

DEF

The set $\{\boldsymbol{x}_n\}$ *is a* ***frame*** *for* $\boldsymbol{H}$ *if* (STABILITY CONDITION)

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\,\|\boldsymbol{x}\|^2 \le \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \le B\,\|\boldsymbol{x}\|^2 \qquad \forall \boldsymbol{x}\in X.$$

The quantities A *and* B *are* ***frame bounds***.
The quantity A' *is the* ***optimal lower frame bound*** *if*
$A' = \sup\{A \in \mathbb{R}^+ \mid A \text{ is a lower frame bound}\}$.
The quantity B' *is the* ***optimal upper frame bound*** *if*
$B' = \inf\{B \in \mathbb{R}^+ \mid B \text{ is an upper frame bound}\}$.
A frame is a ***tight frame*** *if* $A = B$.
A frame is a ***normalized tight frame*** *(or a* ***Parseval frame****) if* $A = B = 1$.
A frame $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ *is an* ***exact frame*** *if for some* $m \in \mathbb{Z}$, $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}\setminus\{\boldsymbol{x}_m\}$ *is* NOT *a frame.*

A frame is a *Parseval frame* (Definition N.16) if it satisfies *Parseval's Identity* (Theorem N.9 page 318). All orthonormal bases are Parseval frames (Theorem N.9 page 318); but not all Parseval frames are orthonormal bases.

Definition N.17. *Let* $\{\boldsymbol{x}_n\}$ *be a* ***frame*** *(Definition N.16 page 328) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\mid\triangledown\rangle)$. *Let* $\mathbf{S}$ *be an* OPERATOR *on* $\boldsymbol{H}$.

DEF

$\mathbf{S}$ *is a* ***frame operator*** *for* $\{\boldsymbol{x}_n\}$ *if* $\quad \mathbf{S}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n \qquad \forall \mathsf{f}\in\boldsymbol{H}.$

Theorem N.16. [38] *Let* $\mathbf{S}$ *be a* FRAME OPERATOR *(Definition N.17 page 328) of a* FRAME $\{\boldsymbol{x}_n\}$ *(Definition N.16 page 328) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\mid\triangledown\rangle)$.

THM

(1). $\mathbf{S}$ *is* INVERTIBLE. *and*

(2). $\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \left\langle \mathsf{f} \mid \mathbf{S}^{-1}\boldsymbol{x}_n\right\rangle \boldsymbol{x}_n = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n\rangle\, \mathbf{S}^{-1}\boldsymbol{x}_n \quad \forall \mathsf{f} \in \boldsymbol{H}$

Theorem N.17. [39] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\mid\triangledown\rangle)$.

THM

$\{\boldsymbol{x}_n\}$ *is a* FRAME *for* $\operatorname{span}\{\boldsymbol{x}_n\}$.

✎PROOF:

[37] Young (2001) pages 154–155, Christensen (2003) page 88 ⟨Definitions 5.1.1, 5.1.2⟩, Heil (2011) pages 204–205 ⟨Definition 8.2⟩, Jørgensen et al. (2008) page 267 ⟨Definition 12.22⟩, Duffin and Schaeffer (1952) page 343, Daubechies et al. (1986) page 1272

[38] Christensen (2008) pages 100–102 ⟨Theorem 5.1.7⟩

[39] Christensen (2003) page 3

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1. Upper bound: Proof that there exists B such that $\sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \le B\,\|\boldsymbol{x}\|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

$$\begin{aligned}\sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 &\le \sum_{n=1}^{N} \langle \boldsymbol{x}_n \mid \boldsymbol{x}_n\rangle\,\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by } \textit{Cauchy-Schwarz inequality}\\ &= \underbrace{\left\{\sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2\right\}}_{B} \|\boldsymbol{x}\|^2\end{aligned}$$

2. Lower bound: Proof that there exists A such that $A\,\|\boldsymbol{x}\|^2 \le \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

$$\begin{aligned}\sum_{n=1}^{N} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 &= \sum_{n=1}^{N} \left|\left\langle \boldsymbol{x}_n \mid \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\rangle\right|^2 \|\boldsymbol{x}\|^2\\ &\ge \underbrace{\left(\inf_{\boldsymbol{y}}\left\{\sum_{n=1}^{N} \left|\langle \boldsymbol{x}_n \mid \boldsymbol{y}\rangle\right|^2 \mid \|\boldsymbol{y}\| = 1\right\}\right)}_{A} \|\boldsymbol{x}\|^2\end{aligned}$$

✏

Example N.1. Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown\rangle\right)$ be an inner product space with $\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \middle| \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1x_2 + y_1y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition N.17 page 328) with *inverse* $\mathbf{S}^{-1}$.

EX

Let $N \in \{3, 4, 5, \dots\}$, $\theta \in \mathbb{R}$, and $r \in \mathbb{R}^+$ $(r > 0)$.

Let $\boldsymbol{x}_n \triangleq r\begin{bmatrix} \cos(\theta + 2n\pi/N) \\ \sin(\theta + 2n\pi/N) \end{bmatrix} \quad \forall_{n\in\{0,1,\dots,N-1\}}$.

Then, $(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1})$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{Nr^2}{2}$.

Moreover, $\mathbf{S} = \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{S}^{-1} = \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\boldsymbol{v} = \frac{2}{Nr^2}\sum_{n=0}^{N-1} \langle \boldsymbol{v} \mid \boldsymbol{x}_n\rangle\,\boldsymbol{x}_n \quad \forall_{\boldsymbol{v}\in\mathbb{R}^2}$.

✎PROOF:

1. Proof that $(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1})$ is a *tight frame* with *frame bound* $A = \frac{Nr^2}{2}$: Let $\boldsymbol{v} \triangleq (x, y) \in \mathbb{R}^2$.

$$\begin{aligned}\sum_{n=0}^{N-1} \left|\langle \boldsymbol{v} \mid \boldsymbol{x}_n\rangle\right|^2 &\triangleq \sum_{n=0}^{N-1} \left|\boldsymbol{v}^{\mathbf{H}} r\begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix}\right|^2 && \text{by definitions of } \boldsymbol{v} \text{ of } \langle \boldsymbol{y} \mid \boldsymbol{x}\rangle\\ &\triangleq \sum_{n=0}^{N-1} r^2\left|x\cos\left(\theta + \frac{2n\pi}{N}\right) + y\sin\left(\theta + \frac{2n\pi}{N}\right)\right|^2 && \text{by definition of } \boldsymbol{y}^{\mathbf{H}}\boldsymbol{x} \text{ operation}\\ &= r^2x^2\sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2n\pi}{N}\right) + r^2y^2\sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2n\pi}{N}\right) + r^2xy\sum_{n=0}^{N-1} \cos\left(\theta + \frac{2n\pi}{N}\right)\sin\left(\theta + \frac{2n\pi}{N}\right)\\ &= r^2x^2\frac{N}{2} + r^2y^2\frac{N}{2} + r^2xy0 && \text{by Corollary J.1 page 269}\\ &= \left(x^2 + y^2\right)\frac{Nr^2}{2} = \left(\frac{Nr^2}{2}\right)\boldsymbol{v}^{\mathbf{H}}\boldsymbol{v} \triangleq \underbrace{\left(\frac{Nr^2}{2}\right)}_{A}\|\boldsymbol{v}\|^2 && \text{by definition of } \|\boldsymbol{v}\|\end{aligned}$$

2. Proof that $\mathbf{S} = \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

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(a) Let $\boldsymbol{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) lemma: $\mathbf{S}\boldsymbol{e}_1 = \frac{Nr^2}{2}\boldsymbol{e}_1$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_1 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\cos\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \cos^2\left(\theta + \frac{2n\pi}{N}\right) \\ \cos\left(\theta + \frac{2n\pi}{N}\right)\sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} N/2 \\ 0 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_1 \qquad \text{by } \textit{Summation around unit circle} \text{ (Corollary J.1 page 269)}
\end{aligned}$$

(c) lemma: $\mathbf{S}\boldsymbol{e}_2 = \frac{Nr^2}{2}\boldsymbol{e}_2$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_2 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_2 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\sin\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \sin\left(\theta + \frac{2n\pi}{N}\right)\cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin^2\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} 0 \\ N/2 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_2 \qquad \text{by } \textit{Summation around unit circle} \text{ (Corollary J.1 page 269)}
\end{aligned}$$

(d) Complete the proof of item (2) using *Eigendecomposition* $\mathbf{S} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \frac{Nr^2}{2}\boldsymbol{e}_1 \qquad \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_1 \text{ is an } \textit{eigenvector} \text{ of } \mathbf{S} \text{ with } \textit{eigenvalue } \frac{Nr^2}{2} \\
\mathbf{S}\boldsymbol{e}_2 &= \frac{Nr^2}{2}\boldsymbol{e}_2 \qquad \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_2 \text{ is an } \textit{eigenvector} \text{ of } \mathbf{S} \text{ with } \textit{eigenvalue } \frac{Nr^2}{2}
\end{aligned}$$

$$\mathbf{S} = \overbrace{\underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}^{-1}}_{\mathbf{Q}^{-1}}}^{\textit{Eigendecomposition} \text{ of } \mathbf{S}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Proof that $\mathbf{S}^{-1} = \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$\begin{aligned}
\mathbf{S}\mathbf{S}^{-1} &= \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 \qquad \text{by item (2)} \\
\mathbf{S}^{-1}\mathbf{S} &= \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 \qquad \text{by item (2)}
\end{aligned}$$

4. Proof that $\boldsymbol{v} = \frac{2}{Nr^2}\sum_{n=0}^{N-1}(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \rangle \boldsymbol{x}_n = \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,\middle|\, \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n \qquad \text{by item (3)} \\
&= \frac{2}{Nr^2}\sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \frac{2}{Nr^2}\sum_{n=0}^{N-1} (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n \qquad \text{by definition of } \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle
\end{aligned}$$

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✏

Example N.2 (Peace Frame/Mercedes Frame). [40] Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle\triangle \mid \triangledown\rangle\right)$ be an inner product space with $\left\langle (x_1, y_1) \mid (x_2, y_2) \right\rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition N.17 page 328) with *inverse* $\mathbf{S}^{-1}$.

EX

Let $\boldsymbol{x}_1 \triangleq \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\boldsymbol{x}_2 \triangleq \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}$, and $\boldsymbol{x}_3 \triangleq \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$.

Then, $(\!(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\!)$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{3}{2}$.

Moreover, $\mathbf{S} = \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \mathbf{S}^{-1} = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

and $\quad \boldsymbol{v} = \frac{2}{3}\sum_{n=1}^{3} \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \triangleq \frac{2}{3}\sum_{n=1}^{3} \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_n\right) \boldsymbol{x}_n \qquad \forall \boldsymbol{v} \in \mathbb{R}^2.$

✎PROOF:

1. This frame is simply a special case of the frame presented in Example N.1 (page 329) with $r = 1$, $N = 3$, and $\theta = \pi/2$.

2. Let's give it a try! Let $\boldsymbol{v} \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned}
\sum_{n=1}^{3} \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n &= \frac{2}{3}\sum_{n=1}^{3} \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n\right)\boldsymbol{x}_n && \text{by Example N.1 page 329} \\
&= \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_1\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_2\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_3\right)\boldsymbol{x}_3 \\
&= \frac{2}{3}\left(\left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{2}{3}\cdot\frac{1}{2}\left(\left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{1}{3}\left((2)\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \left(-\sqrt{3}-1\right)\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix} + \left(\sqrt{3}-1\right)\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right) \\
&= \frac{1}{6}\begin{bmatrix} 2(0) & + & (-\sqrt{3}-1)(-\sqrt{3}) & + & (\sqrt{3}-1)(\sqrt{3}) \\ 2(2) & + & (-\sqrt{3}-1)(-1) & + & (\sqrt{3}-1)(-1) \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 0 & + & (3+\sqrt{3}) & + & (3-\sqrt{3}) \\ 4 & + & (1+\sqrt{3}) & + & (1-\sqrt{3}) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \triangleq \boldsymbol{v}
\end{aligned}$$

✏

In Example N.1 (page 329) and Example N.2 (page 331), the frame operator $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ were computed. In general however, it is not always necessary or even possible to compute these, as illustrated in Example N.3 (next).

Example N.3. [41] Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle\triangle \mid \triangledown\rangle\right)$ be an inner product space with $\left\langle (x_1, y_1) \mid (x_2, y_2) \right\rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition N.17 page 328) with *inverse* $\mathbf{S}^{-1}$.

[40] Heil (2011) pages 204–205 ⟨$r = 1$ case⟩, Byrne (2005) page 80 ⟨$r = 1$ case⟩, Han et al. (2007) page 91 ⟨Example 3.9, $r = \sqrt{2/3}$ case⟩

[41] Christensen (2003) pages 7–8 ⟨?⟩

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EX

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be *orthonormal* vectors in $\boldsymbol{X} \triangleq \operatorname{span}\{\boldsymbol{p}, \boldsymbol{q}\}$.
Let $\boldsymbol{x}_1 \triangleq \boldsymbol{p}$, $\boldsymbol{x}_2 \triangleq \boldsymbol{p}+\boldsymbol{q}$, and $\boldsymbol{x}_3 \triangleq \boldsymbol{p}-\boldsymbol{q}$.
Then, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, , \boldsymbol{x}_3\}$ is a **frame** for $\boldsymbol{X}$ with *frame bounds* $A = 0$ and $B = 5$.

Moreover,
$\mathbf{S}^{-1}\boldsymbol{x}_1 = \frac{1}{3}\boldsymbol{p}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_2 = \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_3 = \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}$.

✎PROOF:

1. Proof that $(\!(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\!)$ is a *frame* with *frame bounds* $A = 0$ and $B = 5$:

$$\begin{aligned}
\sum_{n=1}^{3} \left|\langle \boldsymbol{v} \,|\, \boldsymbol{x}_n\rangle\right|^2 &\triangleq |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle|^2 && \text{by definitions of } \boldsymbol{x}_1, \boldsymbol{x}_2, \text{ and } \boldsymbol{x}_3 \\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 && \text{by } \textit{additivity} \text{ of } \langle \triangle \,|\, \triangledown \rangle \text{ (Definition D.9 page 182)} \\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right) \\
&\quad + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 - \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right) \\
&= 3|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + 2|\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 \\
&\boxed{\le} 3\,\|\boldsymbol{v}\|\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{v}\|\,\|\boldsymbol{q}\| && \text{by } \textit{CS Inequality} \\
&= \|\boldsymbol{v}\|\,(3\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{q}\|) \\
&= \boxed{5}\,\|\boldsymbol{v}\| && \text{by } \textit{orthonormality} \text{ of } \boldsymbol{p} \text{ and } \boldsymbol{q}
\end{aligned}$$

2. lemma: $\mathbf{S}\boldsymbol{p} = 3\boldsymbol{p}$, $\mathbf{S}\boldsymbol{q} = 2\boldsymbol{q}$, $\mathbf{S}^{-1}\boldsymbol{p} = \frac{1}{3}\boldsymbol{p}$, and $\mathbf{S}^{-1}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{p} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{p} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n \\
&= \langle \boldsymbol{p} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{p} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\,(\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{p} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\,(\boldsymbol{p}-\boldsymbol{q}) \\
&= (1)\boldsymbol{p} + (1+0)(\boldsymbol{p}+\boldsymbol{q}) + (1-0)(\boldsymbol{p}-\boldsymbol{q}) \\
&= 3\boldsymbol{p} \\
\implies \mathbf{S}^{-1}\boldsymbol{p} &= \frac{1}{3}\boldsymbol{p} \\
\mathbf{S}\boldsymbol{q} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{q} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n \\
&= \langle \boldsymbol{q} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{q} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\,(\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{q} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\,(\boldsymbol{p}-\boldsymbol{q}) \\
&= (0)\boldsymbol{q} + (0+1)(\boldsymbol{p}+\boldsymbol{q}) + (0-1)(\boldsymbol{p}-\boldsymbol{q}) \\
&= 2\boldsymbol{q} \\
\implies \mathbf{S}^{-1}\boldsymbol{q} &= \frac{1}{2}\boldsymbol{q}
\end{aligned}$$

3. Remark: Without knowing $\boldsymbol{p}$ and $\boldsymbol{q}$, from (2) lemma it follows that it is not possible to compute $\mathbf{S}$ or $\mathbf{S}^{-1}$ explicitly.

4. Proof that $\mathbf{S}^{-1}\boldsymbol{x}_1 = \frac{1}{3}\boldsymbol{p}$, $\mathbf{S}^{-1}\boldsymbol{x}_2 = \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$ and $\mathbf{S}^{-1}\boldsymbol{x}_3 = \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}$:

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_1 &\triangleq \mathbf{S}^{-1}\boldsymbol{p} && \text{by definition of } \boldsymbol{x}_1 \\
&= \frac{1}{3}\boldsymbol{p} && \text{by (2) lemma} \\
\mathbf{S}^{-1}\boldsymbol{x}_2 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2 \\
&= \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

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$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_3 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2\\
&= \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

5. Check that $\boldsymbol{v} = \sum_n \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \langle \boldsymbol{v} \mid \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \mid \boldsymbol{q} \rangle \boldsymbol{q}$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=1}^{3} \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n\\
&= \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q})\\
&= \left\langle \boldsymbol{v} \mid \frac{1}{3}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \mid \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \mid \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q})\\
&= \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) \langle \boldsymbol{v} \mid \boldsymbol{p} \rangle \boldsymbol{p} + \left(\frac{1}{3}-\frac{1}{3}\right) \langle \boldsymbol{v} \mid \boldsymbol{p} \rangle \boldsymbol{q} + \left(\frac{1}{2}-\frac{1}{2}\right) \langle \boldsymbol{v} \mid \boldsymbol{q} \rangle \boldsymbol{p} + \left(\frac{1}{2}+\frac{1}{2}\right) \langle \boldsymbol{v} \mid \boldsymbol{q} \rangle \boldsymbol{q}\\
&= \langle \boldsymbol{v} \mid \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \mid \boldsymbol{q} \rangle \boldsymbol{q}
\end{aligned}$$

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APPENDIX O

B-SPLINES

O.1 Definitions

Definition O.1. *Let X be a set.*

DEF *The **step function** $\sigma \in \mathbb{R}^{\mathbb{R}}$ is defined as*

$$\sigma(x) \triangleq \mathbb{1}_{[0:\infty)}(x) \qquad \forall x \in \mathbb{R}.$$

Lemma O.1. *Let $\sigma(x)$ be the* STEP FUNCTION *(Definition O.1 page 335).*

LEM $$\{g(x) > 0\} \quad \Longrightarrow \quad \left\{\sigma\left[g(x)f(x)\right] = \sigma[f(x)]\right\} \qquad \forall f,g \in \mathbb{R}^{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\sigma\left[g(x)f(x)\right] &\triangleq \mathbb{1}_{[0:\infty)}\left[g(x)f(x)\right] && \text{by definition of } \sigma(x) && \text{(Definition O.1 page 335)}\\
&\triangleq \begin{cases} 1 & \text{for } g(x)f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 3)}\\
&= \begin{cases} 1 & \text{for } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} && \text{by } g(x) > 0 \text{ hypothesis} \\
&\triangleq \mathbb{1}_{[0:\infty)}[f(x)] && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 3)}\\
&\triangleq \sigma[f(x)] && \text{by definition of } \sigma(x) && \text{(Definition O.1 page 335)}
\end{aligned}$$

✏

Definition O.2. [1] *Let $\mathbb{1}$ be the* SET INDICATOR *function (Definition 1.2 page 3). Let $f(x) \star g(x)$ represent the* CONVOLUTION *operation (Definition F.1 page 203).*

DEF *The n**th order cardinal B-spline** $N_n(x)$ for $n \in \mathbb{W}$ is defined as*

$$N_n(x) \triangleq \left\{ \begin{array}{ll} \mathbb{1}_{[0:1)}(x) & \text{for } n = 0 \\ N_{n-1}(x) \star N_0(x) & \text{for } n \in \mathbb{W}\backslash 0 \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

Lemma O.2. [2]

[1] Chui (1992) page 85 ⟨(4.2.1)⟩, Christensen (2008) page 140, Chui (1988) page 1

[2] Christensen (2008) page 140, Chui (1992) page 85 ⟨(4.2.1)⟩, Chui (1988) page 1, Prasad and Iyengar (1997) page 145

LEM

$$\mathsf{N}_n(x) = \int_{\tau=0}^{\tau=1} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau \qquad \forall n \in \{1,2,3,\ldots\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{N}_n(x) &\triangleq \mathsf{N}_{n-1}(x) \star \mathsf{N}_0(x) && \text{by definition of } \mathsf{N}_n(x) && \text{(Definition O.2 page 335)}\\
&\triangleq \int_{\mathbb{R}} \mathsf{N}_{n-1}(x-\tau)\mathsf{N}_0(\tau)\,\mathrm{d}\tau && \text{by definition of convolution operation } \star && \text{(Definition F.1 page 203)}\\
&\triangleq \int_{\mathbb{R}} \mathsf{N}_{n-1}(x-\tau)\mathbb{1}_{[0:1)}(\tau)\,\mathrm{d}\tau && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition O.2 page 335)}\\
&= \int_{[0:1)} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 3)}\\
&= \int_{[0:1]} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau\\
&\triangleq \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau
\end{aligned}$$

✏

Lemma O.3. *Let* $\mathsf{f}(x)$ *be a* FUNCTION *in* $\mathbb{R}^{\mathbb{R}}$. *Let* $\mathsf{F}(x)$ *be the* ANTI-DERIVATIVE *of* $\mathsf{f}(x)$. *Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition O.1 page 335)*.

LEM

$$\begin{aligned}
&\int_{y=a}^{y=b} \mathsf{f}(x-y)\sigma(x-y)\,\mathrm{d}y\\
&= \left\{\begin{array}{ll} -\displaystyle\int_{y=x-a}^{y=x-b} \mathsf{f}(y)\,\mathrm{d}y & \textit{for } x \ge b\\ -\displaystyle\int_{y=x-a}^{y=0} \mathsf{f}(y)\,\mathrm{d}y & \textit{for } a \le x \le b\\ 0 & \textit{for } x \le a \end{array}\right\} = \left\{\begin{array}{ll} \mathsf{F}(x-a)-\mathsf{F}(x-b) & \textit{for } x \ge b\\ \mathsf{F}(x-a)-\mathsf{F}(0) & \textit{for } a \le x \le b\\ 0 & \textit{for } x \le a \end{array}\right\}\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(0)-\mathsf{F}(x-b)]\sigma(x-b)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\int_{y=a}^{y=b} \mathsf{f}(x-y)\sigma(x-y)\,\mathrm{d}y &= \left\{\begin{array}{ll} \int_{y=a}^{y=b} \mathsf{f}(x-y)\,\mathrm{d}y & \text{for } x \ge b\\ \int_{y=a}^{y=x} \mathsf{f}(x-y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by definition of } \sigma \text{ (Definition O.1 page 335)}\\
&= \left\{\begin{array}{ll} -\int_{u=x-a}^{u=x-b} \mathsf{f}(u)\,\mathrm{d}u & \text{for } x \ge b\\ -\int_{u=x-a}^{u=0} \mathsf{f}(u)\,\mathrm{d}u & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{where } u \triangleq x-y \implies y = x-u\\
&= \left\{\begin{array}{ll} -\int_{y=x-a}^{y=x-b} \mathsf{f}(y)\,\mathrm{d}y & \text{for } x \ge b\\ -\int_{y=x-a}^{y=0} \mathsf{f}(y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by change of dummy variable } (u \to y)\\
&= \left\{\begin{array}{ll} \mathsf{F}(x-a)-\mathsf{F}(x-b) & \text{for } x \ge b\\ \mathsf{F}(x-a)-\mathsf{F}(0) & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by } \textit{Fundamental Theorem of Calculus}\\
&= [\mathsf{F}(x-a)-\mathsf{F}(x-b)]\sigma(x-b) + [\mathsf{F}(x-a)-\mathsf{F}(0)][\sigma(x-a)-\sigma(x-b)]\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(x-a)-\mathsf{F}(x-b)-\mathsf{F}(x-a)+\mathsf{F}(0)]\sigma(x-b)\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(0)-\mathsf{F}(x-b)]\sigma(x-b)
\end{aligned}$$

✏

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Figure O.1: some low order B-splines (Example O.1 page 337)

Lemma O.4. *Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition O.1 page 335).*

LEM
$$\int_{\tau=0}^{\tau=1} (x-\tau-k)^n \sigma(x-\tau-k)\,d\tau = \frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k) - (x-k-1)^{n+1}\sigma(x-k-1)\right]$$

✎PROOF:

$$\begin{aligned}
&\int_{\tau=0}^{\tau=1} (x-\tau-k)^n \sigma(x-\tau-k)\,d\tau \\
&= \int_{y=k}^{y=k+1} (x-y)^n \sigma(x-y)\,dy && \text{where } y \triangleq \tau+k \implies \tau = y-k \\
&= [F(x-k)-F(0)]\sigma(x-k) + [F(0)-F(x-k-1)]\sigma(x-k-1) && \text{by Lemma O.3 (page 336), where } f(x) \triangleq x^n \\
&= \frac{\left[(x-k)^{n+1}-0\right]\sigma(x-k) + \left[0-(x-k-1)^{n+1}\right]\sigma(x-k-1)}{n+1} && \text{because } F(x) \triangleq \int f(x)\,dx = \frac{x^{n+1}}{n+1}+c \\
&= \frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k) - (x-k-1)^{n+1}\sigma(x-k-1)\right]
\end{aligned}$$

✏

Example O.1. [3] Let $\sigma(x)$ be the *step function* (Definition O.1 page 335). Let $\binom{n}{k}$ be the *binomial coefficient* (Definition B.3 page 146). The 0*th order B-spline* (Definition O.2 page 335) $N_0(x)$ can be expressed as follows:

EX
$$N_0(x) = \left\{\begin{array}{ll} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise} \end{array}\right\} = \left\{\sum_{k=0}^{1} (-1)^k \binom{1}{k} (x-k)^0 \sigma(x-k) \quad \forall x \in \mathbb{R}\right\}$$

The B-spline $N_0(x)$ is illustrated in Figure O.1 (page 337).

[3] Schumaker (2007) page 136 ⟨Table 1⟩

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✎PROOF:

$$\begin{aligned}
\mathsf{N}_0(x) &= \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition O.2 page 335)}\\
&= \sigma(x) - \sigma(x-1) && \text{by definition of } \sigma(x) && \text{(Definition O.1 page 335)}\\
&= \left[\binom{1}{0}\sigma(x) - \binom{1}{1}\sigma(x-1)\right] && \text{by definition of } \textit{binomial coefficient } \binom{n}{k} && \text{(Definition B.3 page 146)}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-k)^0\sigma(x-k) && \text{by definition of } \sum \text{ operator}
\end{aligned}$$

✏

Example O.2. [4] Let $\sigma(x)$ be the *step function*. Let $\binom{n}{k}$ be the *binomial coefficient*.
The 1*st order B-spline* $\mathsf{N}_1(x)$ can be expressed as follows:

EX

$$\mathsf{N}_1(x) = \left\{\begin{matrix} x & \text{for } x \in [0:1] \\ -x+2 & \text{for } x \in [1:2] \\ 0 & \text{otherwise}\end{matrix}\right\} = \left\{\sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_1(x)$ is illustrated in Figure O.1 (page 337).

✎PROOF:

$$\begin{aligned}
\mathsf{N}_1(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_0(x-\tau)\,\mathrm{d}\tau && \text{by Lemma O.2 page 335}\\
&= \int_{\tau=0}^{\tau=1} \sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-\tau-k)^0\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by Example O.1 page 337}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}\int_{\tau=0}^{\tau=1}(x-\tau-k)^0\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by } \textit{linearity} \text{ of } \int \mathrm{d}\tau \text{ operator}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}\frac{1}{0+1}\left[(x-k)^{0+1}\sigma(x-k) - (x-k-1)^{0+1}\sigma(x-k-1)\right] && \text{by Lemma O.4 page 337}\\
&= \begin{pmatrix} 1\{(x-0)\sigma(x-0) - (x-1)\sigma(x-1)\} \\ -1\{(x-1)\sigma(x-1) - (x-2)\sigma(x-2)\}\end{pmatrix}\\
&= x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k) && \text{by def. of } \binom{n}{k} \text{ (Definition B.3 page 146)}\\
&= \left\{\begin{matrix} x & \text{for } x \in [0:1) \\ -x+2 & \text{for } x \in [1:2] \\ 0 & \text{otherwise}\end{matrix}\right\} && \text{by def. of } \sigma(x) \text{ (Definition O.1 page 335)}
\end{aligned}$$

✏

Example O.3. [5] Let $\sigma(x)$ be the *step function*. Let $\binom{n}{k}$ be the *binomial coefficient*.
The 2*nd order B-spline* $\mathsf{N}_2(x)$ can be expressed as follows:

EX

$$\mathsf{N}_2(x) = \frac{1}{2}\left\{\begin{matrix} x^2 & \text{for } x \in [0:1) \\ -2x^2+6x-3 & \text{for } x \in [1:2] \\ x^2-6x+9 & \text{for } x \in [2:3] \\ 0 & \text{otherwise}\end{matrix}\right\} = \left\{\frac{1}{2}\sum_{k=0}^{3}(-1)^k\binom{3}{k}(x-k)^2\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_2(x)$ is illustrated in Figure O.1 (page 337).

[4] Christensen (2008) page 148 ⟨Exercise 6.2⟩, Christensen (2010) page 212 ⟨Exercise 10.2⟩, Heil (2011) pages 142–143 ⟨Definition 4.22 (The Schauder System)⟩, Schumaker (2007) page 136 ⟨Table 1⟩, Stoer and Bulirsch (2002) page 124

[5] Christensen (2008) page 148 ⟨Exercise 6.2⟩, Christensen (2010) page 212 ⟨Exercise 10.2⟩, Schumaker (2007) page 136 ⟨Table 1⟩, Stoer and Bulirsch (2002) page 124

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✎PROOF:

$$
\begin{aligned}
\mathsf{N}_2(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_1(x-\tau)\,\mathrm{d}\tau && \text{by Lemma O.2 page 335}\\
&= \int_{\tau=0}^{\tau=1} \sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-\tau-k)\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by Example O.2 page 338}\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\int_{\tau=0}^{\tau=1}(x-\tau-k)\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by \textit{linearity} of } \int \mathrm{d}\tau \text{ operator}\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\frac{1}{1+1}\left[(x-k)^{1+1}\sigma(x-k)-(x-k-1)^{1+1}\sigma(x-k-1)\right] && \text{by Lemma O.4 page 337}\\
&= \frac{1}{2}\left(\begin{array}{rl} 1 & \left\{(x-0)^2\sigma(x-0)-(x-1)^2\sigma(x-1)\right\}\\ -2 & \left\{(x-1)^2\sigma(x-1)-(x-2)^2\sigma(x-2)\right\}\\ +1 & \left\{(x-2)^2\sigma(x-2)-(x-3)^2\sigma(x-3)\right\}\end{array}\right)\\
&= \frac{1}{2}\left[x^2\sigma(x)-3(x-1)^2\sigma(x-1)+3(x-2)^2\sigma(x-2)-(x-3)^2\sigma(x-3)\right]\\
&= \frac{1}{2}\sum_{k=0}^{3}(-1)^k\binom{3}{k}(x-k)^2\sigma(x-k) && \text{by def. of } \binom{n}{k} \text{ (Definition B.3 page 146)}\\
&= \frac{1}{2}\left\{\begin{array}{rl} x^2 & \text{for } x\in[0:1)\\ -2x^2+6x-3 & \text{for } x\in[1:2]\\ x^2-6x+9 & \text{for } x\in[2:3]\\ 0 & \text{otherwise}\end{array}\right\} && \text{by def. of } \sigma(x) \text{ (Definition O.1 page 335)}
\end{aligned}
$$

The final steps of this proof can be calculated "by hand" or by using the free and open source software package *Maxima* along with the script file listed in Section Y.2 (page 451). ✏

O.2 Algebraic properties

Theorem O.1 (next) presents a closed form expression for an *nth order B-spline* $\mathsf{N}_n(x)$ based on the definition of $\mathsf{N}_n(x)$ given in Definition O.2 (page 335). Alternatively, Theorem O.1 could serve as the definition and Definition O.2 as a property.

Theorem O.1. [6] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition O.2 page 335)*. *Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition O.1 page 335)*.

THM

$$\mathsf{N}_n(x) = \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n\sigma(x-k) \qquad \forall n\in\{0,1,2,\ldots\}=\mathbb{W}$$

✎PROOF: Proof follows by induction:

1. base case (choose one):
   - Proof for $n=0$ case: by Example O.1 (page 337).
   - Proof for $n=1$ case: by Example O.2 (page 338).
   - Proof for $n=2$ case: by Example O.3 (page 338).

[6] Christensen (2008) page 142 ⟨Theorem 6.1.3⟩, Chui (1992) page 84 ⟨(4.1.12)⟩

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2. inductive step—proof that n case $\implies$ $n+1$ case:

$$
\begin{aligned}
&\mathsf{N}_{n+1}(x)\\
&= \int_0^1 \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by Lemma O.2 page 335}\\
&= \int_0^1 \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-\tau-k)^n\sigma(x-\tau-k)\,\mathsf{d}\tau && \text{by induction hypothesis}\\
&= \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\int_0^1(x-\tau-k)^n\sigma(x-\tau-k)\,\mathsf{d}\tau && \text{by \textit{linearity} of } \int \mathsf{d}\tau \text{ operator}\\
&= \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k)-(x-k-1)^{n+1}\sigma(x-k-1)\right] && \text{by Lemma O.4 page 337}\\
&= \frac{1}{(n+1)!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\left[(x-k)^{n+1}\sigma(x-k)-(x-k-1)^{n+1}\sigma(x-k-1)\right]\\
&= \frac{1}{(n+1)!}\left[\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)-\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k-1)^{n+1}\sigma(x-k-1)\right]\\
&= \frac{1}{(n+1)!}\left[\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)-\sum_{m=1}^{m=n+2}(-1)^{m-1}\binom{n+1}{m-1}(x-m)^{n+1}\sigma(x-m)\right]\\
&\quad \text{where } m \triangleq k+1 \implies k=m-1\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l}\displaystyle\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)\\ \displaystyle-\sum_{m=1}^{m=n+2}(-1)^{m-1}\left[\binom{n+2}{m}-\binom{n+1}{m}\right](x-m)^{n+1}\sigma(x-m)\end{array}\right) && \begin{array}{l}\text{by \textit{Pascal's identity} /}\\ \text{\textit{Stifel formula}}\\ \text{(Theorem B.1 page 147)}\end{array}\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l}\displaystyle\sum_{m=1}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m)\\ \displaystyle-\sum_{m=1}^{m=n+2}(-1)^m\binom{n+1}{m}(x-m)^{n+1}\sigma(x-m)\\ \displaystyle+\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)\end{array}\right) && \text{note } (-1)^{m-1}=-(-1)^m
\end{aligned}
$$

$$
= \frac{1}{(n+1)!}\left(\begin{array}{l|ll}
\displaystyle\sum_{m=0}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m) & \text{(A)} & \text{desired } n+1 \text{ case}\\
\displaystyle-\quad(-1)^0\binom{n+2}{0}(x-0)^{n+1}\sigma(x-0) & \text{(B)} & \text{cancelled by (F)}\\
\displaystyle-\sum_{m=1}^{m=n+1}(-1)^m\binom{n+1}{m}(x-m)^{n+1}\sigma(x-m) & \text{(C)} & \text{cancelled by (E)}\\
\displaystyle-\quad(-1)^{n+2}\binom{n+1}{n+2}(x-n-2)^{n+1}\sigma(x-n-2) & \text{(D)} & \binom{n+1}{n+2}=0 \text{ by Proposition B.2 page 146}\\
\displaystyle+\sum_{k=1}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k) & \text{(E)} & \text{cancelled by (C)}\\
\displaystyle+\quad(-1)^0\binom{n+1}{0}(x-0)^{n+1}\sigma(x-0) & \text{(F)} & \binom{n+2}{0}=\binom{n+1}{0}=1, \text{ so (F) is cancelled by (B)}
\end{array}\right)
$$

$$
= \frac{1}{(n+1)!}\sum_{m=0}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m) \qquad (n+1 \text{ case})
$$

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✏

Example O.4. [7] Let $\mathsf{N}_3(x)$ be the 3*rd order B-spline* (Definition O.2 page 335).[8]

EX

$$\mathsf{N}_3(x) = \frac{1}{6}\left\{\begin{array}{rl} x^3 & \text{for } 0 \le x \le 1 \\ -3x^3+12x^2-12x+4 & \text{for } 1 \le x \le 2 \\ 3x^3-24x^2+60x-44 & \text{for } 2 \le x \le 3 \\ -x^3+12x^2-48x+64 & \text{for } 3 \le x \le 4 \\ 0 & \text{otherwise} \end{array}\right\}$$

✎PROOF: This expression can be calculated "by hand" using Theorem O.1 (page 339) or by using the free and open source software package *Maxima* along with the script file listed in Section Y.2 (page 451). ✏

Example O.5. Let $\mathsf{N}_4(x)$ be the 4*th order B-spline* (Definition O.2 page 335).

EX

$$\mathsf{N}_4(x) = \frac{1}{24}\left\{\begin{array}{rl} x^4 & \text{for } 0 \le x \le= 1 \\ -4x^4+20x^3-30x^2+20x-5 & \text{for } 1 \le x \le= 2 \\ 6x^4-60x^3+210x^2-300x+155 & \text{for } 2 \le x \le= 3 \\ -4x^4+60x^3-330x^2+780x-655 & \text{for } 3 \le x \le= 4 \\ x^4-20x^3+150x^2-500x+625 & \text{for } 4 \le x \le= 5 \\ 0 & \text{otherwise} \end{array}\right\}$$

✎PROOF: This expression can be calculated "by hand" using Theorem O.1 (page 339) or by using the free and open source software package *Maxima* along with the script file listed in Section Y.2 (page 451). ✏

Example O.6. Let $\mathsf{N}_5(x)$ be the 5*th order B-spline* (Definition O.2 page 335).

EX

$$\mathsf{N}_5(x) = \frac{1}{120}\left\{\begin{array}{rl} x^5 & \text{for } 0 \le x \le= 1 \\ -5x^5+30x^4-60x^3+60x^2-30x+6 & \text{for } 1 \le x \le= 2 \\ 10x^5-120x^4+540x^3-1140x^2+1170x-474 & \text{for } 2 \le x \le= 3 \\ -10x^5+180x^4-1260x^3+4260x^2-6930x+4386 & \text{for } 3 \le x \le= 4 \\ 5x^5-120x^4+1140x^3-5340x^2+12270x-10974 & \text{for } 4 \le x \le= 5 \\ -x^5+30x^4-360x^3+2160x^2-6480x+7776 & \text{for } 5 \le x \le= 6 \\ 0 & \text{otherwise} \end{array}\right\}$$

The 5*th order B-spline* $\mathsf{N}_5(x)$ is illustrated in Figure O.1 (page 337).

✎PROOF: This expression can be calculated "by hand" using Theorem O.1 (page 339) or by using the free and open source software package *Maxima* along with the script file listed in Section Y.2 (page 451). ✏

Example O.7. Let $\mathsf{N}_6(x)$ be the 6*th order B-spline* (Definition O.2 page 335).

EX

$$\mathsf{N}_6(x) = \frac{1}{720}\left\{\begin{array}{rl} x^6 & \text{for } 0 \le x \le 1 \\ -6x^6+42x^5-105x^4+140x^3-105x^2+42x-7 & \text{for } 1 \le x \le 2 \\ 15x^6-210x^5+1155x^4-3220x^3+4935x^2-3990x+1337 & \text{for } 2 \le x \le 3 \\ -20x^6+420x^5-3570x^4+15680x^3-37590x^2+47040x-24178 & \text{for } 3 \le x \le 4 \\ 15x^6-420x^5+4830x^4-29120x^3+96810x^2-168000x+119182 & \text{for } 4 \le x \le 5 \\ -6x^6+210x^5-3045x^4+23380x^3-100065x^2+225750x-208943 & \text{for } 5 \le x \le 6 \\ x^6-42x^5+735x^4-6860x^3+36015x^2-100842x+117649 & \text{for } 6 \le x \le 7 \\ 0 & \text{otherwise} \end{array}\right\}$$

The 6*th order B-spline* $\mathsf{N}_6(x)$ is illustrated in Figure O.1 (page 337).

✎PROOF: This expression can be calculated "by hand" using Theorem O.1 (page 339) or by using the free and open source software package *Maxima* along with the script file listed in Section Y.2 (page 451). ✏

[7] Schumaker (2007) page 136 ⟨Table 1⟩, Shizgal (2015) page 92 ⟨(2.199)⟩, Szabó and Horváth (2004) page 146 ⟨(4)⟩, Wei and Billings (2006) page 578 ⟨Table 1⟩, Maleknejad et al. (2013) ⟨(9)⟩

[8] For help with plotting *B-splines*, see APPENDIX Y (page 441).

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Example O.8. Let $\mathsf{N}_7(x)$ be the 7*th order B-spline* (Definition O.2 page 335).

EX

$$7!\mathsf{N}_7(x) = 5040\mathsf{N}_7(x) =$$
$$\left\{\begin{array}{rrrrrrrrrl}
x^7 & & & & & & & & & \text{for } 0 \le x \le 1\\
-7x^7 & +56x^6 & -168x^5 & +280x^4 & -280x^3 & +168x^2 & -56x & +8 & & \text{for } 1 \le x \le 2\\
21x^7 & -336x^6 & +2184x^5 & -7560x^4 & +15400x^3 & -18648x^2 & +12488x & -3576 & & \text{for } 2 \le x \le 3\\
-35x^7 & +840x^6 & -8400x^5 & +45360x^4 & -143360x^3 & +267120x^2 & -273280x & +118896 & & \text{for } 3 \le x \le 4\\
35x^7 & -1120x^6 & +15120x^5 & -111440x^4 & +483840x^3 & -1238160x^2 & +1733760x & -1027984 & & \text{for } 4 \le x \le 5\\
-21x^7 & +840x^6 & -14280x^5 & +133560x^4 & -741160x^3 & +2436840x^2 & -4391240x & +3347016 & & \text{for } 5 \le x \le 6\\
7x^7 & -336x^6 & +6888x^5 & -78120x^4 & +528920x^3 & -2135448x^2 & +4753336x & -4491192 & & \text{for } 6 \le x \le 7\\
-x^7 & +56x^6 & -1344x^5 & +17920x^4 & -143360x^3 & +688128x^2 & -1835008x & +2097152 & & \text{for } 7 \le x \le 8\\
 & & & & & & & & 0 & \text{otherwise}
\end{array}\right\}$$

The 7*th order B-spline* $\mathsf{N}_7(x)$ is illustrated in Figure O.1 (page 337).

✎PROOF: This expression can be calculated "by hand" using Theorem O.1 (page 339) or by using the free and open source software package *Maxima* along with the script file listed in Section Y.2 (page 451). ✏

Example O.9. [9] The $(n+1)^2$ coefficients of the order $n, n-1, \dots, 0$ monomials of each *B-spline* $\mathsf{N}_n(x)$ multiplied by $n!$ induce an *integer sequence*
$\mathbb{x} \triangleq (\!(1,1,0,-1,2,1,0,0,-2,6,-3,1,-6,9,1,0,0,0,-3,12,-12,4,3,-24,60,-44,-1,12,-48,64,\dots)\!)$
as more fully listed in Table O.1 (page 366). In this sequence $\mathbb{x} \triangleq (\!(x_0, x_1, x_2, \dots)\!)$, the coefficients for the *order n B-spline* $\mathsf{N}_n(x)$ begin at the sequence index value

$$p \triangleq \sum_{k=0}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \text{and end at index value } p + (n+1)^2 - 1.$$

For example, the coefficients for $\mathsf{N}_3(x)$ begin at index value $p \triangleq 0+1+4+9 = 14$ and end at index value $p + 4^2 - 1 = 29$. Using these coefficients gives the following expression for $\mathsf{N}_3(x)$:

$$\mathsf{N}_3(x) = \left[\begin{array}{rrrr|l}
1 & 0 & 0 & 0 & \text{for } 0 \le x < 1\\
-3 & 12 & -12 & 4 & \text{for } 1 \le x < 2\\
3 & -24 & 60 & -44 & \text{for } 2 \le x < 3\\
-1 & 12 & -48 & 64 & \text{for } 3 \le x < 4
\end{array}\right]\left[\begin{array}{c} x^3\\ x^2\\ x\\ 1\end{array}\right] = \left\{\begin{array}{rrrrl}
x^3 & & & & \text{for } 0 \le x < 1\\
-3x^3 & +12x^2 & -12x & +4 & \text{for } 1 \le x < 2\\
3x^3 & -24x^2 & +60x & -44 & \text{for } 2 \le x < 3\\
-x^3 & +12x^2 & -48x & +64 & \text{for } 3 \le x < 4\\
 & & & 0 & \text{otherwise}
\end{array}\right.$$

…which agrees with the result presented in Example O.4 (page 341).

✎PROOF:

1. The coefficients for the sequence $\mathbb{x}$ may be computed with assistance from *Maxima* together with the script file listed in Section Y.2 (page 451).

2. Proof that $\sum_{k=0}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$: The summation is a *power sum*. The relation may be proved using *induction*.[10]

   (a) Base case: n=0 case …

$$\begin{aligned}
\sum_{k=0}^{k=0} k^2 &= 0\\
&= \frac{0(0+1)(2\cdot 0+1)}{6}\\
&= \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=0}
\end{aligned}$$

[9] Greenhoe (2017b)
[10] Greenhoe (2017a) pages 186–187 ⟨Proposition 11.2 (Power Sums)⟩

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(b) Base case: n=1 case …

$$\begin{aligned}\sum_{k=0}^{k=1} k^2 &= 0+1\\ &= \frac{1(1+1)(2\cdot 1+1)}{6}\\ &= \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1}\end{aligned}$$

(c) inductive step—proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}\sum_{k=0}^{n+1} k^2 &= \left(\sum_{k=0}^{n} k^2\right) + \left(n+1\right)^2\\ &= \left(\frac{n(n+1)(2n+1)}{6}\right) + \left(n+1\right)^2 && \text{by } n \text{ case hypothesis}\\ &= \left(n+1\right)\left(\frac{n(2n+1)+6(n+1)}{6}\right)\\ &= \left(n+1\right)\left(\frac{2n^2+7n+6}{6}\right)\\ &= \left(n+1\right)\left(\frac{(n+2)(2n+3)}{6}\right)\\ &= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}\end{aligned}$$

✏

Theorem O.2. [11]

THM $\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_n(x) = \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1) \qquad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}$

✎PROOF:

1. Proof using Lemma O.2 (page 335) and the *Fundamental Theorem of Calculus*:

$$\begin{aligned}\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_n(x) &= \frac{\mathsf{d}}{\mathsf{d}x}\int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathsf{d}\tau && \text{by Lemma O.2 page 335}\\ &= \frac{\mathsf{d}}{\mathsf{d}x}\int_{x-u=0}^{x-u=1} \mathsf{N}_{n-1}(u)(-1)\,\mathsf{d}u && \text{where } u \triangleq x-\tau \implies \tau = x-u\\ &= \frac{\mathsf{d}}{\mathsf{d}x}\int_{u=x-1}^{u=x} \mathsf{N}_{n-1}(u)\,\mathsf{d}u\\ &= \left\{\frac{\mathsf{d}}{\mathsf{d}x}\int \mathsf{N}_{n-1}(u)\,\mathsf{d}u\Big|_{u=x}\right\} - \left\{\frac{\mathsf{d}}{\mathsf{d}x}\int \mathsf{N}_{n-1}(u)\,\mathsf{d}u\Big|_{u=x-1}\right\} && \text{by } \textit{Fundamental Theorem of Calculus}^{12}\\ &= \left\{\mathsf{N}_{n-1}(x)\frac{\mathsf{d}}{\mathsf{d}x}(x)\right\} - \left\{\mathsf{N}_{n-1}(x-1)\frac{\mathsf{d}}{\mathsf{d}x}(x-1)\right\} && \text{by } \textit{Chain Rule}^{13}\\ &= \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)\end{aligned}$$

[11] Höllig (2003) page 25 ⟨3.2⟩, Schumaker (2007) page 121 ⟨Theorem 4.16⟩
[12] Hijab (2011) page 163 ⟨Theorem 4.4.3⟩
[13] Hijab (2011) pages 73–74 ⟨Theorem 3.1.2⟩

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2. Proof using Lemma O.2 (page 335) and *induction*:

   (a) Base case …proof for $n = 1$ case:

$$\begin{aligned}
\mathsf{N}_0(x) - \mathsf{N}_0(x-1) &= \underbrace{\sigma(x) - \sigma(x-1)}_{\mathsf{N}_0(x)} - \underbrace{[\sigma(x-1) - \sigma(x-2)]}_{\mathsf{N}_0(x-1)} && \text{by Example O.1 page 337}\\
&= \sigma(x) - 2\sigma(x-1) + \sigma(x-2)\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\sigma(x-k)\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k)\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_1(x) && \text{by Example O.2 page 338}
\end{aligned}$$

   (b) Base case …proof for $n = 2$ case:

$$\begin{aligned}
&\mathsf{N}_1(x) - \mathsf{N}_1(x-1)\\
&= \underbrace{x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)}_{\mathsf{N}_1(x)}\\
&\quad - \underbrace{[(x-1)\sigma(x-1) - 2(x-2)\sigma(x-2) + (x-3)\sigma(x-3)]}_{\mathsf{N}_1(x-1)} && \text{by Example O.2 page 338}\\
&= x\sigma(x) + [-2x+2-x+1]\sigma(x-1) + [x-2+2x-4]\sigma(x-2) + [-x+3]\sigma(x-3)\\
&= x\sigma(x) + [-3x+3]\sigma(x-1) + [3x-6]\sigma(x-2) + [-x+3]\sigma(x-3)\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\left\{\begin{array}{l}\frac{1}{2}x^2\sigma(x) + \left[-\frac{3}{2}x^2 + 3x - \frac{1}{2}\right]\sigma(x-1) + \left[\frac{3}{2}x^2 - 6x + 3\right]\sigma(x-2)\\ \qquad + \left[-\frac{1}{2}x^2 + 3x - \frac{5}{2}\right]\sigma(x-3)\end{array}\right\}\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_2(x) && \text{by Example O.3 page 338}
\end{aligned}$$

   (c) Proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_{n+1}(x) &= \frac{\mathsf{d}}{\mathsf{d}x}\int_0^1 \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by Lemma O.2 page 335}\\
&= \int_0^1 \frac{\partial}{\partial\tau}\mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by \textit{Leibniz Integration Rule} (Theorem E.2 page 201)}\\
&= \int_0^1 \left[\mathsf{N}_{n-1}(x-\tau) - \mathsf{N}_{n-1}(x-1-\tau)\right]\mathsf{d}\tau && \text{by left hypothesis}\\
&= \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathsf{d}\tau - \int_0^1 \mathsf{N}_{n-1}(x-1-\tau)\,\mathsf{d}\tau\\
&= \mathsf{N}_n(x) - \mathsf{N}_n(x-1) && \text{by Lemma O.2 page 335}
\end{aligned}$$

✏

Theorem O.3 (*B-spline recursion*). [14] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition O.2 page 335)*.

THM

$$\mathsf{N}_n(x) = \frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1) \quad \forall n \in \{1,2,3,\ldots\},\ \forall x \in \mathbb{R}$$

[14] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972)

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✎PROOF:

1. Base case …proof for $n = 1$ case:

$$\frac{x}{1}\mathsf{N}_0(x) + \frac{1+1-x}{1}\mathsf{N}_0(x-1) = \frac{x}{1}\underbrace{[\sigma(x)-\sigma(x-1)]}_{\mathsf{N}_0(x)} + \frac{1+1-x}{1}\underbrace{[\sigma(x-1)-\sigma(x-2)]}_{\mathsf{N}_0(x-1)}$$
$$= x\sigma(x) + [-x-x+2]\sigma(x-1) + [x-2]\sigma(x-2)$$
$$= \mathsf{N}_1(x) \qquad \text{by Example O.2 page 338}$$

2. Induction step …proof that n case $\implies$ $n+1$ case:

$$\frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1) + c_1$$
$$= \int \frac{\mathsf{d}}{\mathsf{dx}}\left\{\frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1)\right\}\mathsf{dx}$$
$$= \int \underbrace{\frac{1}{n+1}\mathsf{N}_n(x) + \frac{x}{n+1}\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_n(x)}_{\frac{\mathsf{d}}{\mathsf{dx}}\frac{x}{n+1}\mathsf{N}_n(x)} + \underbrace{\frac{-1}{n+1}\mathsf{N}_n(x-1) + \frac{n+2-x}{n}\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_n(x-1)}_{\frac{\mathsf{d}}{\mathsf{dx}}\frac{n+2-x}{n+1}\mathsf{N}_n(x-1)}\,\mathsf{dx}$$

by *product rule*

$$= \int \frac{1}{n+1}\underbrace{\left[\frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1)\right]}_{\text{by } n \text{ hypothesis}} + \frac{x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)\right]}_{\text{by Theorem O.2 page 343}}$$
$$- \underbrace{\left[\frac{x-1}{n^2+n}\mathsf{N}_{n-1}(x-1) + \frac{n-x+2}{n(n+1)}\mathsf{N}_{n-1}(x-2)\right]}_{\text{by induction hypothesis}}$$
$$+ \frac{n+2-x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x-1) - \mathsf{N}_{n-1}(x-2)\right]}_{\text{by Theorem O.2 page 343}}\,\mathsf{dx}$$
$$= \int \left[\frac{x}{n(n+1)} + \frac{x}{n+1}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n-x+1}{n(n+1)} - \frac{x-1}{n(n+1)} + \frac{n+2-2x}{n+1}\right]\mathsf{N}_{n-1}(x-1)$$
$$+ \left[\frac{-n-2+x}{n(n+1)} + \frac{-n-2+x}{n+1}\right]\mathsf{N}_{n-1}(x-2)\,\mathsf{dx}$$
$$= \int \left[\frac{x+nx}{n(n+1)}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x+n(n+2-2x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-1)$$
$$+ \left[\frac{-n-2+x+n(-n-2+x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-2)\,\mathsf{dx}$$
$$= \int \left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x}{n}\right]\mathsf{N}_{n-1}(x-1) + \left[\frac{-n-2+x}{n}\right]\mathsf{N}_{n-1}(x-2)\,\mathsf{dx}$$
$$= \int \underbrace{\left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+1-x}{n}\right]\mathsf{N}_{n-1}(x-1)}_{\mathsf{N}_n(x)}$$
$$- \underbrace{\left[\frac{x-1}{n}\right]\mathsf{N}_{n-1}(x-1) - \left[\frac{n+2-x}{n}\right]\mathsf{N}_{n-1}(x-2)}_{\mathsf{N}_{n-1}(x-1)}\,\mathsf{dx}$$
$$= \int \mathsf{N}_n(x) - \mathsf{N}_n(x-1)\,\mathsf{dx} \qquad \text{by } n \text{ hypothesis}$$
$$= \int \frac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_{n+1}(x)\,\mathsf{dx} \qquad \text{by Theorem O.2 page 343}$$
$$= \mathsf{N}_{n+1}(x) + c_2$$

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Proof that $c_1 = c_2$: By item (2) (page 346), $\mathsf{N}_n(x) = 0$ for $x < 0$. Therefore, $c_1 = c_2$.

✏

Theorem O.4 (*B-spline general form*). [15] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition O.2 page 335). Let* $\mathsf{supp}\,\mathsf{f}$ *be the* SUPPORT *of a function* $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$.

THM

1.	$\mathsf{N}_n(x)$	≥ 0	$\forall n \in \mathbb{W}$,	$\forall x \in \mathbb{R}$	(NON-NEGATIVE)
2.	$\mathsf{supp}\mathsf{N}_n(x)$	$= [0 : n+1]$	$\forall n \in \mathbb{W}$		(CLOSED SUPPORT)
3.	$\int_{\mathbb{R}} \mathsf{N}_n(x)\,\mathsf{d}x$	$= 1$	$\forall n \in \mathbb{W}$		(UNIT AREA)
4.	$\mathsf{N}_n\left(\frac{n+1}{2} - x\right)$	$= \mathsf{N}_n\left(\frac{n+1}{2} + x\right)$	$\forall n \in \mathbb{N}$	$\forall x \in \mathbb{R}$	(SYMMETRIC *about* $x = \frac{n+1}{2}$)

✎PROOF:

1. Proof that $\mathsf{N}_n(x) \geq 0$ (proof by induction):

   (a) base case…proof that $\mathsf{N}_0(x) \geq 0$:

   $$\begin{aligned} \mathsf{N}_0(x) &\triangleq \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition O.2 page 335)} \\ &\geq 0 && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 3)} \end{aligned}$$

   (b) inductive step—proof that $\{\mathsf{N}_n(x) \geq 0\} \implies \{\mathsf{N}_{n+1}(x) \geq 0\}$:

   $$\begin{aligned} \mathsf{N}_{n+1}(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by Lemma O.2 page 335} \\ &\geq 0 && \text{by induction hypothesis } (\mathsf{N}_n(x) \geq 0) \end{aligned}$$

2. Proof that $\mathsf{supp}\mathsf{N}_n(x) = [0 : n+1]$ (proof by induction):

   (a) Base case …proof that $\mathsf{supp}\mathsf{N}_0 = [0 : 1]$:

   $$\begin{aligned} \mathsf{supp}\mathsf{N}_0 &\triangleq \mathsf{supp}\,\mathbb{1}_{[0:1)} && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition O.2 page 335)} \\ &= \{[0 : 1)\}^- && \text{by definition of } \textit{support} \text{ operator} \\ &= [0 : 1] && \text{by definition of } \textit{closure} \text{ operator} \end{aligned}$$

   (b) Induction step …proof that $\{\mathsf{supp}\mathsf{N}_n = [0 : n+1]\} \implies \{\mathsf{supp}\mathsf{N}_{n+1} = [0 : n+2]\}$:

   $$\begin{aligned} \mathsf{supp}\mathsf{N}_{n+1}(x) &= \mathsf{supp} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by Lemma O.2 page 335} \\ &= \mathsf{supp} \int_{[0:1]} \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by def. of } \textit{Lebesgue integration} \\ &= \left\{x \in \mathbb{R} \middle| (x-\tau) \in [0 : n+1] \text{ for some } \tau \in [0 : 1]\right\}^- && \text{by induction hypothesis} \\ &= [0 : n+1] \cup [0+1 : n+1+1]^- \\ &= [0 : n+2]^- \\ &= [0 : n+2] && \text{by property of } \textit{closure} \text{ operator} \end{aligned}$$

3. Proof that $\int_{\mathbb{R}} \mathsf{N}_n(x)\,\mathsf{d}x = 1$ (proof by induction):

[15] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972), Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2⟩

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(a) Base case …proof that $\int_{\mathbb{R}} \mathsf{N}_0(x) = 1$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{N}_0(x)\ \mathsf{dx} &= \int_{\mathbb{R}} \mathbb{1}_{[0:1)}\ \mathsf{dx} && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition O.2 page 335)}\\
&= \int_{[0:1)} 1\ \mathsf{dx} && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 3)}\\
&= \int_{[0:1]} 1\ \mathsf{dx} && \text{by property of } \textit{Lebesgue integration}\\
&= 1
\end{aligned}$$

(b) Induction step …proof that $\left\{\int_{\mathbb{R}} \mathsf{N}_n(x) = 1\right\} \implies \left\{\int_{\mathbb{R}} \mathsf{N}_{n+1} = 1\right\}$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{N}_{n+1}(x)\ \mathsf{dx} &= \int_{\mathbb{R}} \int_0^1 \mathsf{N}_n(x-\tau)\ \mathsf{d}\tau\ \mathsf{dx} && \text{by Lemma O.2 page 335}\\
&= \int_0^1 \int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\ \mathsf{dx}\ \mathsf{d}\tau\\
&= \int_0^1 \int_{\mathbb{R}} \mathsf{N}_n(u)\ \mathsf{du}\ \mathsf{d}\tau && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \int_0^1 1\ \mathsf{d}\tau && \text{by induction hypothesis}\\
&= 1
\end{aligned}$$

4. Proof that $\mathsf{N}_n(x)$ is *symmetric* for $n \in \{1, 2, 3, \ldots\}$:

(a) Note that $\mathsf{N}_0(x)$ $(n = 0)$ is *not symmetric* (in particular it fails at $x = 1/2$) because
$\mathsf{N}_0\left(\frac{0+1}{2} - \frac{1}{2}\right) = \mathsf{N}_0(0) = 1 \neq 0 = \mathsf{N}_1(1) = \mathsf{N}_0\left(\frac{0+1}{2} + \frac{1}{2}\right)$

(b) Base case …proof for $n = 1$ case:

$$\begin{aligned}
\mathsf{N}_1\left(\frac{1+1}{2} - x\right) &= \mathsf{N}_1(1-x)\\
&= \left\{\begin{matrix} (1-x) & \text{for } 1-x \in [0:1] \\ -(1-x)+2 & \text{for } 1-x \in [1:2] \\ 0 & \text{otherwise} \end{matrix}\right\} && \text{by Example O.2 page 338}\\
&= \left\{\begin{matrix} -x+1 & \text{for } -x \in [-1:0] \\ x+1 & \text{for } -x \in [0:1] \\ 0 & \text{otherwise} \end{matrix}\right\}\\
&= \left\{\begin{matrix} x+1 & \text{for } x \in [-1:0] \\ -x+1 & \text{for } x \in [0:1] \\ 0 & \text{otherwise} \end{matrix}\right\}\\
&= \left\{\begin{matrix} (1+x) & \text{for } 1+x \in [0:1] \\ -(1+x)+2 & \text{for } 1+x \in [1:2] \\ 0 & \text{otherwise} \end{matrix}\right\}\\
&= \mathsf{N}_1(1+x) && \text{by Example O.2 page 338}\\
&= \mathsf{N}_1\left(\frac{1+1}{2} + x\right)
\end{aligned}$$

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(c) Induction step …proof that $n-1$ case $\implies$ n case:

$$
\begin{aligned}
&\mathsf{N}_n\left(\frac{n+1}{2}+x\right)\\
&= \frac{\frac{n+1}{2}+x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}+x\right) + \frac{n+1-\left(\frac{n+1}{2}+x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}+x-1\right) && \text{by Theorem O.3 page 344}\\
&= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x+\frac{1}{2}\right]\right) + \frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x-\frac{1}{2}\right]\right)\\
&= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x+\frac{1}{2}\right]\right) + \frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x-\frac{1}{2}\right]\right) && \text{by induction hypothesis}\\
&= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\left[\frac{n+1}{2}-x\right]-1\right) + \frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}-x\right)\\
&= \mathsf{N}_n\left(\frac{n+1}{2}-x\right) && \text{by Theorem O.3 page 344}
\end{aligned}
$$

✏

O.3 Projection properties

In the case where $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is to be used as a basis in some subspace of $\boldsymbol{L}^2_{\mathbb{R}}$, one may want to *project* a function $\mathsf{f}(x)$ onto a basis function $\mathsf{N}_n(x-k)$. This is especially true when $(\!(\mathsf{N}_n(x-k))\!)$ is *orthogonal*; but in the case of *B-splines* this is only true when $n=0$ (Theorem O.8 page 358). Nevertheless, projection of a function onto $\mathsf{N}_n(x-k)$, or the projection of $\mathsf{N}_n(x)$ onto another basis fucntion (such as the complex exponential in the case of *Fourier analysis* as in Lemma O.5 page 350), is still useful. Projection in an *inner product space* is typically performed using the *inner product* $\langle \mathsf{f}(x) \mid \mathsf{N}_n(x-k)\rangle$; and in the space $\boldsymbol{L}^2_{\mathbb{R}}$, this inner product is typically defined as an *integral* such that

$$\langle \mathsf{f}(x) \mid \mathsf{N}_n(x-k)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_n(x-k)\,\mathsf{dx}.$$

As it turns out, there is a way to compute this inner product that only involves the function $\mathsf{f}(x)$ and the order parameter n (next theorem).

Theorem O.5. [16] *Let* f *be a* CONTINUOUS *function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* nTH DERIVATIVE *of* $\mathsf{f}(x)$.

THM

(1). $$\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_n(x)\,\mathsf{dx} = \int_{[0:1)^{n+1}} \mathsf{f}(x_1+x_2+\cdots+x_{n+1})\,\mathsf{dx}_1\,\mathsf{dx}_2\cdots\mathsf{dx}_{n+1}$$

(2). $$\int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathsf{dx} = \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)$$

✎PROOF:

1. Proof for (1) (proof by induction):

   (a) Base case …proof for $n=0$ case:

$$\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_0(x)\,\mathsf{dx} = \int_{[0:1)} \mathsf{f}(x)\,\mathsf{dx} \qquad \text{by definition of } \mathsf{N}_0(x) \qquad \text{(Definition O.2 page 335)}$$

[16] Chui (1992) page 85 ⟨(4.2.2), (4.2.3)⟩, Christensen (2008) page 140 ⟨Theorem 6.1.1⟩

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(b) Inductive step—proof that n case $\implies n+1$ case:

$$\begin{aligned}
&\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_{n+1}(x)\,\mathsf{dx} \\
&= \int_{\mathbb{R}} \left[\int_0^1 \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau\right]\mathsf{f}(x)\,\mathsf{dx} && \text{by Lemma O.2 page 335} \\
&= \int_{[0:1)}\int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\mathsf{f}(x)\,\mathsf{dx}\,\mathsf{d}\tau \\
&= \int_{[0:1)}\int_{\mathbb{R}} \mathsf{N}_n(u)\mathsf{f}(u+\tau)\,\mathsf{du}\,\mathsf{d}\tau && \text{where } u \triangleq x-\tau \implies x = u+\tau \\
&= \int_{[0:1)}\int_{[0:1)^{n+1}} \mathsf{f}(u_1+u_2+\cdots+u_{n+1}+\tau)\,\mathsf{du}_1\ \mathsf{du}_2\ \cdots\ \mathsf{du}_{n+1}\ \mathsf{d}\tau && \text{by induction hypothesis} \\
&= \int_{[0:1)^{n+2}} \mathsf{f}(u_1+u_2+\cdots+u_{n+1}+u_{n+2})\,\mathsf{du}_1\ \mathsf{du}_2\ \cdots\ \mathsf{du}_{n+2}\ \mathsf{d}\tau \\
&= \int_{[0:1)^{n+2}} \mathsf{f}(x_1+x_2+\cdots+x_{n+1}+x_{n+2})\,\mathsf{dx}_1\ \mathsf{dx}_2\ \cdots\ \mathsf{dx}_{n+2} && \text{by change of variables } u_k \to x_k
\end{aligned}$$

2. Proof for (2):

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathsf{dx} &= \int_{[0:1)^{n+1}} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n+1} x_k\right)\mathsf{dx}_1\ \mathsf{dx}_2\cdots\ \mathsf{dx}_{n+1} && \text{by (1)} \\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by Theorem E.1 page 200}
\end{aligned}$$

Figure O.2: Zero locations for B-cardinal spline $N_n(x)$ scaling coefficients

O.4 Fourier analysis

Simply put, no matter what new and fancy basis sequences are discovered, the *Fourier transform* never goes out of style. This is largely because the *kernel* of the Fourier transform—the *complex exponential* function—has two properties that makes it extremely special:

- The complex exponential is an eigenvalue of any *linear time invariant* (*LTI*) operator (Theorem I.13 page 251).
- The complex exponential generates a *continuous point spectrum* for the *differential operator*.

Thus, we might expect the projection of the *B-spline* function $\mathsf{N}_n(x)$ onto the complex exponential (essentially the *Fourier transform* of $\mathsf{N}_n(x)$,...next lemma) to be useful. Such a hunch would be confirmed because it is useful for proving that

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- the sequence $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is a *Riesz basis* (Lemma O.6 page 353, Theorem O.8 page 358) and
- the sequence $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is a *multiresolution analysis* (Theorem O.10 page 361).

Lemma O.5. [17] *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* *(Definition M.2 page 294)*.

LEM

$$\tilde{\mathbf{F}}\mathsf{N}_n(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1} = \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1} \triangleq \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1}$$

✎PROOF:

1. Proof using Theorem O.5 page 348:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{N}_n(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition M.2 page 294)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{[0:1)^{n+1}} e^{-i\omega(x_1+x_2+\cdots+x_{n+1})}\,\mathrm{d}x_1\ \mathrm{d}x_2\ \cdots,\ \mathrm{d}x_{n+1} && \text{by Theorem O.5}\\
&= \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{n+1}\left(\int_{[0:1)} e^{-i\omega x_k}\,\mathrm{d}x_k\right) && \text{because } e^{x+y}=e^xe^y\\
&= \frac{1}{\sqrt{2\pi}}\left(\int_0^1 e^{-i\omega x}\,\mathrm{d}x\right)^{n+1} = \frac{1}{\sqrt{2\pi}}\left(\left.\frac{e^{-i\omega x}}{-i\omega}\right|_0^1\right)^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1} = \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}}{i\omega}\right)\right]^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{2i\sin\left(\frac{\omega}{2}\right)}{\frac{2i\omega}{2}}\right)\right]^{n+1} && \text{by } \textit{Euler formulas} \quad \text{(Corollary I.2 page 241)}\\
&= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1}
\end{aligned}$$

2. Proof using *rectangular pulse* example (Example M.1 page 301) and *Convolution Theorem* (Theorem Q.2 page 380):

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \left[\sqrt{2\pi}\right]^n\left[\tilde{\mathbf{F}}\mathsf{N}_0\right]^{n+1} && \text{by } \textit{Convolution Theorem} \quad \text{(Theorem Q.2 page 380)}\\
&= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\frac{c(b-a)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left(\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right)\right]^{n+1} && \text{by } \textit{rectangular pulse} \text{ example with } a=0,\ b=c=1 \quad \text{(Example M.1 page 301)}\\
&= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{\omega}{2}\right)}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{(\omega/2)}\right)\right]^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{(n+1)\omega}{2}\right)}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{(\omega/2)}\right)^{n+1}
\end{aligned}$$

✏

Example O.10. The *Fourier transform magnitude* $\left|\left[\tilde{\mathbf{F}}\mathsf{N}_0\right](\omega)\right|$ of the *0 order B-spline* $\mathsf{N}_0(x)$ is illustrated to the right.

[17] Christensen (2008) page 142 ⟨Corollary 6.1.2⟩

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Example O.11. The *Fourier transform magnitude* $\left|\left[\tilde{\mathbf{F}}\mathsf{N}_1\right](\omega)\right|$ of the *1st order B-spline* $\mathsf{N}_1(x)$ is illustrated to the right.

Example O.12. The *Fourier transform magnitude* $\left|\left[\tilde{\mathbf{F}}\mathsf{N}_2\right](\omega)\right|$ of the *2nd order B-spline* $\mathsf{N}_2(x)$ is illustrated to the right.

O.5 Basis properties

O.5.1 Uniqueness properties

Coefficients of a *basis sequence* are not always *unique*. Take for example a very trivial sequence (α_1, α_2) in which the coefficients are summed. If $\mathsf{f}(x) \triangleq \alpha_1 + \alpha_2$ and $\mathsf{g}(x) \triangleq \beta_1 + \beta_2$,

$$\begin{aligned} &\text{then} \quad \{(\alpha_1,\alpha_2) = (\beta_1,\beta_2)\} \quad \Longrightarrow \quad \mathsf{f}(x) = \mathsf{g}(x) \\ &\text{but} \quad \mathsf{f}(x) = \mathsf{g}(x) \quad \not\Longrightarrow \quad \{(\alpha_1,\alpha_2) = (\beta_1,\beta_2)\}, \end{aligned}$$

because for example if $(\alpha_1, \alpha_2) = (1, 2)$ and $(\beta_1, \beta_2) = (-6, 9)$, then $\mathsf{f}(x) = \mathsf{g}(x)$, but $(\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)$. This example demonstrates that the "if and only if" condition $\Longleftrightarrow$ does not hold and coefficients are not unique in all *basis sequences*. But arguably a minimal requirement for any practical basis sequence is that the coefficients are *unique* (the "if and only if" condition $\Longleftrightarrow$ holds). And indeed, in a *B-spline* basis sequence $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$, the coefficients $(\!(\alpha_k)\!)_{k\in\mathbb{Z}}$ are *unique*, as demonstrated by Theorem O.6 (next).

Theorem O.6. [18] *Let* $\mathsf{N}_n(x)$ *be the* nTH-ORDER B-SPLINE *(Definition O.2 page 335). Let*

$$\mathsf{f}(x) \triangleq \sum_{k\in\mathbb{Z}} \alpha_k \mathsf{N}_n(x-k) \quad \textit{and} \quad \mathsf{g}(x) \triangleq \sum_{k\in\mathbb{Z}} \beta_k \mathsf{N}_n(x-k).$$

THM

$$\underbrace{\left\{\ \mathsf{f}(x) = \mathsf{g}(x) \quad \forall x \in \mathbb{R}\ \right\} \quad \Longleftrightarrow \quad \left\{(\!(\alpha_k)\!)_{k\in\mathbb{Z}} = (\!(\beta_k)\!)_{k\in\mathbb{Z}}\right\}}_{\textit{coefficients are}\ \text{UNIQUE}}$$

✎PROOF:

1. Proof that $\Longleftarrow$ condition holds:

$$\begin{aligned} \mathsf{f}(x) &\triangleq \sum_{k\in\mathbb{Z}} \alpha_k \mathsf{N}_n(x-k) && \text{by definition of } \mathsf{f}(x) \\ &= \sum_{k\in\mathbb{Z}} \beta_k \mathsf{N}_n(x-k) && \text{by right hypothesis} \\ &\triangleq \mathsf{g}(x) && \text{by definition of } \mathsf{g}(x) \end{aligned}$$

2. Proof that $\Longrightarrow$ condition holds (proof by contradiction):

   (a) Suppose it does *not* hold.

[18] Wojtaszczyk (1997) page 55 ⟨Theorem 3.11⟩

(b) Then there exists sequences $(\!(\alpha_k)\!)_{k\in\mathbb{Z}}$ and $(\!(\beta_k)\!)_{k\in\mathbb{Z}}$ such that
$$(\!(\alpha_k)\!) - (\!(\beta_k)\!) \triangleq (\!(alpha_k - \beta_k)\!) \neq (\!(0,0,0,\cdots)\!)$$
but also such that $\mathsf{f}(x) - \mathsf{g}(x) = 0\ \forall x \in \mathbb{R}$.

(c) If this were possible, then

$$\begin{aligned}
0 &= \mathsf{f}(x) - \mathsf{g}(x) \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \mathsf{N}_n(x-m) - \sum_{m\in\mathbb{Z}} \beta_m \mathsf{N}_n(x-m) \\
&= \sum_{m\in\mathbb{Z}} \big(\alpha_m - \beta_m\big)\mathsf{N}_n(x-m) \\
&= \sum_{m=0}^{m=n} \big(\alpha_m - \beta_m\big)\frac{1}{n!}\left[\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n\sigma(x-k)\right] && \text{by Theorem O.1 page 339}
\end{aligned}$$

(d) But this is *impossible* because $\mathsf{N}(x)$ is *non-negative* (Theorem O.4 page 346).

(e) Therefore, there is a contradiction, and the $\Longrightarrow$ condition *does* hold.

✏

O.5.2 Partition of unity properties

In the case in which a sequence of *B-splines* $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is to be used as a *basis* for some subspace of $\boldsymbol{L}^2_{\mathbb{R}}$, arguably one of the most important properties for the sequence to have is the *partition of unity* property such that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1$. This allows for convenient representation of the most basic functions, such as constants.[19] As it turns out, B-splines *do* have this property (next theorem).

Theorem O.7 (*B-spline partition of unity*). [20] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition O.2 page 335)*.

THM
$$\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1 \quad \forall n \in \mathbb{W} \qquad \text{(PARTITION OF UNITY)}$$

✎PROOF:

1. lemma: $\sum_{k\in\mathbb{Z}} \mathsf{N}_0(x-k) = 1$. Proof:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{N}_0(x-k) &= \sum_{k\in\mathbb{Z}} \mathbb{1}_{[0:1)}(x-k) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition O.2 page 335)} \\
&= 1 && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 3)}
\end{aligned}$$

2. Proof for this theorem follows from the $n = 0$ case ((1) lemma page 352), the definition of $\mathsf{N}_n(x)$ (Definition O.2 page 335), and Corollary S.1 (page 404).

3. Alternatively, this theorem can be proved by *induction*:

   (a) Base case ($n = 0$ case): by (1) lemma.

[19] Jawerth and Sweldens (1994) page 8

[20] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972)

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Figure O.3: *auto-power spectrum* $\tilde{S}_n(\omega)$ plots of *B-splines* $\mathsf{N}_n(x)$ (Lemma O.6 page 353) For C and LaTeX source code to generate such a plot, see Section Y.3 (page 454).

(b) Inductive step—proof that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1 \implies \sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) = 1$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) &= \sum_{k\in\mathbb{Z}} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-k-\tau)\,\mathrm{d}\tau && \text{by Lemma O.2 page 335} \\
&= \sum_{k\in\mathbb{Z}} \int_{x-u=0}^{x-u=1} \mathsf{N}_n(u-k)(-1)\,\mathrm{du} && \text{where } u \triangleq x-\tau \implies \tau = x-u \\
&= \sum_{k\in\mathbb{Z}} \int_{u=x-1}^{u=x} \mathsf{N}_n(u-k)\,\mathrm{du} \\
&= \int_{u=x-1}^{u=x} \left(\sum_{k\in\mathbb{Z}} \mathsf{N}_n(u-k)\right) \mathrm{du} \\
&= \int_{u=x-1}^{u=x} 1\,\mathrm{du} && \text{by induction hypothesis} \\
&= 1
\end{aligned}$$

✏

Example O.13. The *partition of unity* property for the 0 *order B-spline* $\mathsf{N}_0(x)$ (Example O.1 page 337) is illustrated to the right.

Example O.14. The *partition of unity* property for the 1*st order B-spline* $\mathsf{N}_1(x)$ (Example O.2 page 338) is illustrated to the right.

Example O.15. The *partition of unity* property for the 2*nd order B-spline* $\mathsf{N}_2(x)$ (Example O.3 page 338) is illustrated to the right.

O.5.3 Riesz basis properties

Lemma O.6. *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition O.2 page 335).*
Let $\tilde{S}_n(\omega) \triangleq 2\pi \sum_{k\in\mathbb{Z}} \left|\tilde{\mathsf{F}}\mathsf{N}_n(\omega - 2\pi k)\right|^2$ *be the* AUTO-POWER SPECTRUM *(Definition T.3 page 411) of* $\mathsf{N}_n(x)$.

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LEM

(1).	$0 < \tilde{S}_n(\omega) \le 1 \quad \forall \omega \in \mathbb{R} \quad , \quad \forall n \in \mathbb{W}$	(3).	$\tilde{S}_n(0) = 1 \quad \forall n \in \mathbb{W}$
(2).	$\tilde{S}_n(\omega) = 1 \quad \forall \omega \in \mathbb{R} \quad , \quad \text{for } n = 0$	(4).	$\tilde{S}_n(\pi) \le \frac{1}{3} \quad \forall n \in \mathbb{W}\setminus\{0\}$

(*Note: see illustration in Figure O.3 page 353.*)

✎PROOF:

1. lemma: $\tilde{S}_n(\omega) = \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}$. Proof:

$$\begin{aligned}
\tilde{S}_n(\omega) &\triangleq 2\pi \sum_{k\in\mathbb{Z}} \left|\tilde{\mathbf{F}}\mathsf{N}_n(\omega - 2\pi k)\right|^2 && \text{by Definition T.3 page 411}\\
&= 2\pi \sum_{k\in\mathbb{Z}} \left|\frac{1}{\sqrt{2\pi}} e^{-i\frac{(n+1)(\omega-2\pi k)}{2}} \left(\frac{\sin\left(\frac{\omega-2\pi k}{2}\right)}{\frac{\omega-2\pi k}{2}}\right)^{n+1}\right|^2 && \text{by Lemma O.5 page 350}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega-2\pi k}{2}\right)}{\frac{\omega-2\pi k}{2}}\right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2} - k\pi\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{(-1)^k \sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}
\end{aligned}$$

2. lemma (one sided series form):

$$\begin{aligned}
\tilde{S}_n(\omega) &= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)} && \text{by (1) lemma}\\
&= \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)} \left(\sum_{k=1}^{\infty} \left[\frac{1}{2k - \frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty} \left[\frac{1}{2k + \frac{\omega}{\pi}}\right]^{2(n+1)}\right)
\end{aligned}$$

3. lemma: $\tilde{S}_n(\omega)$ is *continuous* for all $\omega \in \mathbb{R}$.
   Proof: $\sin(\omega/2)$ and $\omega/2$ are *continuous*, so $\tilde{S}_n(\omega)$ is *continuous* as well.

4. lemma: $\tilde{S}_n(\omega)$ is *periodic* with period 2π (and so we only need to examine $\tilde{S}_n(\omega)$ for $\omega \in [0 : 2\pi)$). Proof of *periodicity*: This follows directly from Proposition T.2 (page 412).

5. lemma: $\tilde{S}_n(-\omega) = \tilde{S}_n(\omega)$ (*symmetric* about 0) and $\tilde{S}_n(\pi - \omega) = \tilde{S}_n(\pi + \omega)$ (*symmetric* about π). Proof: This follows directly from Proposition T.3 (page 413).

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6. Proof that $\tilde{S}_n(0) = 1$:

$$\begin{aligned}
\tilde{S}_n(0) &= \lim_{\omega\to 0} \tilde{S}_n(\omega) && \text{by (3) lemma}\\
&= \lim_{\omega\to 0}\left[\left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right)\right] && \text{by (2) lemma}\\
&= \lim_{\omega\to 0}\left[\frac{\cos\left(\frac{\omega}{2}\right)}{-\frac{1}{2}}\right]^{2(n+1)} + 0 && \text{by } \textit{l'Hôpital's rule}\\
&= (-1)^{2(n+1)} = 1
\end{aligned}$$

7. Proof that $\tilde{S}_n(\pi)$ converges to some value > 0:

(a) Proof that $\tilde{S}_n(\pi) > 0$:

$$\begin{aligned}
\tilde{S}_n(\pi) &= \left[\frac{\sin(\pi/2)}{(\pi/2)}\right]^{2(n+1)} + \left[\frac{\sin(\pi/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\pi}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\pi}{\pi}}\right]^{2(n+1)}\right) && \text{by (2) lemma}\\
&= \left(\frac{2}{\pi}\right)^{2(n+1)}\left[1 + \left(\frac{1}{1}\right)^{2(n+1)} + \left(\frac{1}{3}\right)^{2(n+1)} + \left(\frac{1}{3}\right)^{2(n+1)} + \left(\frac{1}{5}\right)^{2(n+1)} + \left(\frac{1}{5}\right)^{2(n+1)} + \cdots\right]\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \underbrace{\sum_{k=1}^{\infty}\left[\frac{1}{2k-1}\right]^{2(n+1)}}_{\textit{Dirichlet Lambda function } \lambda(2n+2)}\\
&> 0 \qquad \text{because } x^2 > 0 \text{ for all } x \in \mathbb{R}\setminus\{0\}
\end{aligned}$$

(b) Proof that $\tilde{S}_n(\pi)$ *converges*:

$$\begin{aligned}
\tilde{S}_n(\pi) &= 2\left(\frac{2}{\pi}\right)^{2(n+1)}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^{2(n+1)} && \text{by item (7a)}\\
&\le 2\left(\frac{2}{\pi}\right)^{2(n+1)}\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{2(n+1)}\\
&\le 2\left(\frac{2}{\pi}\right)^{2(n+1)}\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{2}\\
&\implies \textit{convergence} && \text{by } \textit{comparison test}
\end{aligned}$$

(c) Tighter bounds for $\tilde{S}_n(\pi)$ for certain values of $n \in \{0,1,2,3,4\}$:

$$\begin{aligned}
\tilde{S}_n(\pi) &= 2\left(\frac{2}{\pi}\right)^{2(n+1)}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^{2(n+1)} && \text{by item (7a)}\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)} U_{2(n+1)} && \text{by Jolley (1961) pages 56–57 ⟨(307)⟩}\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)}\left[\frac{\pi^{2(n+1)}\alpha_{n+1}}{(4)[(2n+2)!]}\right] && \text{by Jolley (1961) pages 56–57 ⟨(307)⟩}\\
&= \frac{2^{2n+1}\alpha_{n+1}}{(2n+2)!}
\end{aligned}$$

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$$= \left\{ \begin{array}{lll} \frac{2^1(1)}{2!} & \text{for } n=0 & (\alpha_1 = 1) \\ \frac{2^3(1)}{4!} & \text{for } n=1 & (\alpha_2 = 1) \\ \frac{2^5(3)}{6!} & \text{for } n=2 & (\alpha_3 = 3) \\ \frac{2^7(17)}{8!} & \text{for } n=3 & (\alpha_4 = 17) \\ \frac{2^9(155)}{10!} & \text{for } n=4 & (\alpha_5 = 155) \end{array} \right\} \qquad \text{by Jolley (1961) page 234 } \langle(1130)\rangle$$

$$= \left\{ \begin{array}{ll} 1 & \text{for } n=0 \\ \frac{1}{3} & \text{for } n=1 \\ \frac{2}{15} & \text{for } n=2 \\ \frac{17}{315} & \text{for } n=3 \\ \frac{62}{2835} & \text{for } n=4 \end{array} \right\} = \left\{ \begin{array}{ll} 1 & \text{for } n=0 \\ 0.3333333333333\cdots & \text{for } n=1 \\ 0.1333333333333\cdots & \text{for } n=2 \\ 0.0539682539682\cdots & \text{for } n=3 \\ 0.0218694885361\cdots & \text{for } n=4 \end{array} \right\}$$

(d) Being important for the $n=0$ case, note that[21]

$$\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^2}_{\textit{Dirichlet Lambda function } \lambda(2)} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

(e) Proof that $\tilde{S}_n(\pi) \le \frac{1}{3}$: because $\tilde{S}_n(\pi) = \frac{1}{3}$ for $n=1$ (item (7c) page 355) and because $\tilde{S}_n(\pi)$ is decreasing for increasing n.

8. lemma: $\tilde{S}_n(\omega)$ *converges* to some value $> 0\ \forall\omega \in \mathbb{R}$. Proof:

(a) For $\omega = 0$, $\tilde{S}_n(\omega) = 1$ by item (6).

(b) Proof that $\tilde{S}_n(\omega) > 0$ for $\omega \in (0 : 2\pi)$:

$$\tilde{S}_n(\omega) = \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right) \quad \text{by (2) lemma}$$

$$> 0$$

(c) Proof that $\tilde{S}_n(\omega)$ *converges*:

i. lemma: $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{2k\pm\frac{\omega}{\pi}}\right]^{2(n+1)}$ *converges*. Proof:

$$\begin{aligned}
\lim_{b\to\infty}\int_1^b \left[\frac{1}{2y\pm\frac{\omega}{\pi}}\right]^{2(n+1)} \mathrm{dy} &= \lim_{b\to\infty}\int_1^b \left[2y\pm\frac{\omega}{\pi}\right]^{-2n-2} \mathrm{dy} \\
&= \lim_{b\to\infty} \left.\frac{\left[2y\pm\frac{\omega}{\pi}\right]^{-2n-1}}{2(-2n-1)}\right|_1^b \\
&= \lim_{b\to\infty}\left(\frac{-1}{2(2n+1)}\right)\left[\frac{1}{\left[2b\pm\frac{\omega}{\pi}\right]^{2n+1}} - \frac{1}{\left[2\pm\frac{\omega}{\pi}\right]^{2n+1}}\right] \\
&= 0 + \frac{1}{2(2n+1)\left[2\pm\frac{\omega}{\pi}\right]^{2n+1}} \\
&< \infty \qquad \forall\omega\in[0:2\pi)
\end{aligned}$$

[21] Nahin (2011) page 153, Bailey et al. (2013a) page 334 ⟨Catalan's Constant⟩, Bailey et al. (2013b) page 849 ⟨Catalan's Constant⟩, Bailey et al. (2011) page 15 ⟨4.1 Catalan's constan⟩, Wells (1987) page 36 ⟨Dictionary entry for π: pages 31–37⟩, Heinbockel (2010) page 94 ⟨(2.27) *Dirichlet Lambda function*⟩

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$$\Longrightarrow \sum_{k=1}^{\infty}\left[\frac{1}{2k \pm \frac{\omega}{\pi}}\right]^{2(n+1)} \quad \textit{converges} \qquad \text{by } \textit{integral test}$$

ii. completion of proof using (8(c)i) lemma …

$$\begin{aligned}
&\tilde{S}_n(\omega)\\
&= \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right) && \text{by (2) lemma}\\
&\Longrightarrow \tilde{S}_n(\omega) \textit{ converges } \forall\omega \in (0:2\pi) && \text{by (8(c)i) lemma}
\end{aligned}$$

9. lemma (an expression for $\tilde{S}_n'(\omega)$):

$$\begin{aligned}
\tilde{S}_n'(\omega) &\triangleq \frac{\mathsf{d}}{\mathsf{d}\omega}\tilde{S}_n(\omega)\\
&= \frac{\mathsf{d}}{\mathsf{d}\omega}\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2(n+1)} && \text{by (1) lemma page 354}\\
&= \sum_{k\in\mathbb{Z}}\frac{\mathsf{d}}{\mathsf{d}\omega}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2(n+1)} && \text{by } \textit{linearity} \text{ of } \frac{\mathsf{d}}{\mathsf{d}\omega} \text{ operator}\\
&= \sum_{k\in\mathbb{Z}}2(n+1)\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\frac{\mathsf{d}}{\mathsf{d}\omega}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right] && \text{by } \textit{power rule}\\
&= 2(n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\frac{1}{2}\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)-\sin\left(\frac{\omega}{2}\right)\left(-\frac{1}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by } \textit{quotient rule}\\
&= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right]
\end{aligned}$$

10. lemma: $\tilde{S}_n'(0) = \tilde{S}_n'(\pi) = 0$. Proof: This follows from Proposition T.3 (page 413). Here is alternate proof:

$$\begin{aligned}
\tilde{S}_n'(0) &= \lim_{\omega\to 0}\tilde{S}_n'(\omega)\\
&= \lim_{\omega\to 0}(n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by (9) lemma}\\
&= \lim_{\omega\to 0}(n+1)\left[\frac{\sin\left(\frac{\omega}{2}\right)}{-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(-\frac{\omega}{2}\right)^2}\right]\\
&= (n+1)\lim_{\omega\to 0}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{-\frac{\omega}{2}}\right]^{2n+1}\lim_{\omega\to 0}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(-\frac{\omega}{2}\right)^2}\right]\\
&= (n+1)[-1]^{2n+1}\lim_{\omega\to 0}\left[\frac{-\frac{1}{2}\sin\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\cos\left(\frac{\omega}{2}\right)\left(-\frac{1}{2}\right)+\cos\left(\frac{\omega}{2}\right)\left(\frac{1}{2}\right)}{-\frac{2}{2}\left(-\frac{\omega}{2}\right)}\right] && \text{by } \textit{l'Hôpital's rule}\\
&= (1)(0)
\end{aligned}$$

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$$= 0$$

$$\tilde{S}_n'(\pi) = (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\pi}{2}\right)}{k\pi-\frac{\pi}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\pi}{2}\right)\left(k\pi-\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)}{\left(k\pi-\frac{\pi}{2}\right)^2}\right] \qquad \text{by (9) lemma}$$

$$= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{1}{k\pi-\frac{\pi}{2}}\right]^{2n+1}\left[\frac{0\left(k\pi-\frac{\pi}{2}\right)+1}{\left(k\pi-\frac{\pi}{2}\right)^2}\right]$$

$$= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\sum_{k\in\mathbb{Z}}\left[\frac{1}{2k-1}\right]^{2n+3}$$

$$= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\left[\left(\frac{1}{1}\right)^{2n+3}+\left(\frac{1}{-1}\right)^{2n+3}+\left(\frac{1}{3}\right)^{2n+3}+\left(\frac{1}{-3}\right)^{2n+3}+\ldots\right]$$

$$= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\sum_{k=1}^{\infty}(-1)^{k+1}\alpha_k \qquad \text{where } \alpha_k \triangleq \left\{\begin{array}{ll}\left(\frac{1}{k}\right)^{2n+3} & \text{for } k \text{ odd}\\ \left(\frac{1}{k-1}\right)^{2n+3} & \text{for } k \text{ even}\end{array}\right\}$$

$$= 0 \qquad \text{because } \lim_{k\to\infty}\alpha_k = 0 \text{ and by } \textit{Alternating Series Test}$$

11. lemma: $\tilde{S}_n(\omega)$ is *decreasing* with respect to $\omega \in [0 : \pi]$. Proof:

$$\tilde{S}_n'(\omega) = (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] \qquad \text{by (9) lemma page 357}$$

$$= \underbrace{(n+1)\left(\sin\frac{\omega}{2}\right)^{2n+1}}_{\geq 0 \text{ for } \omega\in[0:2\pi)}\underbrace{\sum_{k\in\mathbb{Z}}\left[\frac{1}{k\pi-\frac{\omega}{2}}\right]^{2n+2}}_{>0}\left[\underbrace{\left(\cos\frac{\omega}{2}\right)}_{\text{sign change at } \omega=\pi} + \frac{\overbrace{\sin\frac{\omega}{2}}^{>0 \text{ for } \omega\in(0:2\pi)}}{\underbrace{k\pi-\frac{\omega}{2}}_{\textit{decreasing} \text{ w.r.t. } \omega\in\mathbb{R}}}\right]$$

12. lemma: $\tilde{S}_n(\omega)$ is *increasing* with respect to $\omega \in [\pi : 2\pi)$. Proof: This is true because $\tilde{S}_n(\omega)$ is *decreasing* in $[0 : \pi]$ ((11) lemma) and because $\tilde{S}_n(\omega)$ is *symmetric* about $\omega = \pi$ ((5) lemma).

13. Proof that $0 < \tilde{S}_n(\omega) \leq 1$:

    (a) $\tilde{S}_n(\omega) > 0$ by (8) lemma and

    (b) $\tilde{S}_n(0) = 1$ by item (6) and

    (c) $\tilde{S}_n(\omega)$ is *decreasing* from $\omega = 0$ to $\omega = \pi$ by (11) lemma and

    (d) $\tilde{S}_n(\omega)$ is *increasing* from $\omega = \pi$ to $\omega = 2\pi$ by (12) lemma and

    (e) $\tilde{S}_n(2\pi) = 1$ because $\tilde{S}_n(2\pi) = \tilde{S}_n(0)$ by (4) lemma.

✏

Theorem O.8. [22]

THM					
	1. $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ *is a*	RIESZ BASIS	*for* $\operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$		$\forall n \in \mathbb{W}$
	2. $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ *is an*	ORTHONORMAL BASIS	*for* $\operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$	$\Longleftrightarrow$	$n = 0$

[22] Wojtaszczyk (1997) page 56 ⟨Proposition 3.12⟩, Prasad and Iyengar (1997) page 148 ⟨Theorem 6.3⟩, Forster and Massopust (2009) page 66 ⟨Theorem 2.25⟩

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✎PROOF:

1. Proof that $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is a *Riesz basis* for $\operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$:

$$\begin{aligned}
&0 < \tilde{S}_n(\omega) \le 1 && \text{by Lemma O.6 page 353 (1)}\\
&\implies (\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}} \text{ is a } \textit{Riesz basis} \text{ for } \operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}} && \text{by Theorem T.2 page 414}
\end{aligned}$$

2. Proof that $\{n=0\} \iff (\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is an *orthonormal basis* for $\operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$:

$$\begin{aligned}
n=0 &\iff \tilde{S}_n(\omega) = 1 && \text{by Lemma O.6 page 353 (2), (4)}\\
&\iff (\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}} \text{ is an } \textit{orthonormal basis} \text{ for } \operatorname{span}(\!(\mathsf{N}_n(x-k))\!) && \text{by Theorem T.3 page 417}
\end{aligned}$$

✏

O.6 Mutiresolution properties

O.6.1 Introduction

In 1989, Stéphane G. Mallat introduced the *Multiresolution Analysis* (MRA) structure (Definition 2.1 page 18) An MRA is very powerful because it can be used to approximate functions at incrementally increasing "scales" of resolution, and furthermore induces a *wavelet*. In fact, the MRA has become the dominate wavelet construction method. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA. [23]

O.6.2 B-spline dyadic decomposition

One key feature of an MRA is *dyadic decomposition* such that $\mathsf{N}_n(x) = \sum_k \alpha_n \mathsf{N}_n(2x-n)$ for some sequence $(\!(\alpha_n)\!)$. As it turns out, *B-splines* also have this property (next theorem).

Theorem O.9 (*B-spline dyadic decomposition*). [24] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE.

THM

$$\mathsf{N}_n(x) = \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-k) \qquad \forall n\in\mathbb{W},\ \forall x\in\mathbb{R}$$

✎PROOF:

1. Base case ...proof for $n=0$ case:

$$\begin{aligned}
\mathsf{N}_0(x) &= \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathbb{1}_A(x) \quad \text{(Definition 1.2 page 3)}\\
&= \mathbb{1}_{[0:½)}(x) + \mathbb{1}_{[½:1)}(x)\\
&= \mathbb{1}_{[2\times 0:2\times ½)}(2x) + \mathbb{1}_{[2\times ½-1:2\times 1-1)}(2x-1)\\
&= \mathbb{1}_{[0:1)}(2x) + \mathbb{1}_{[0:1)}(2x-1)\\
&= \frac{1}{2^0}\sum_{k=0}^{0+1}\binom{0+1}{k}\mathsf{N}_n(2x-k)
\end{aligned}$$

[23] Mallat (1999) page 240, Definition 2.1 (page 18)
[24] Prasad and Iyengar (1997) pages 151–152 ⟨proof using Fourier transform⟩

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2. Induction step…proof that n case $\implies n+1$ case:

$$
\begin{aligned}
\mathsf{N}_{n+1}(x) &= \int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma O.2 page 335}\\
&= \int_0^1 \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-2\tau-k)\,\mathrm{d}\tau && \text{by induction hypothesis}\\
&= \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\int_{\tau=0}^{\tau=1}\mathsf{N}_n(2x-2\tau-k)\,\mathrm{d}\tau && \text{by \textit{linearity} of } \textstyle\sum \text{ operator}\\
&= \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\int_{u=0}^{u=2}\mathsf{N}_n(2x-u-k)\frac{1}{2}\,\mathrm{d}u && \text{where } u \triangleq 2\tau \implies \tau = \tfrac{1}{2}u\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\int_{u=0}^{u=1}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u + \int_{u=1}^{u=2}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u\right]\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\int_{u=0}^{u=1}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u + \int_{v=0}^{v=1}\mathsf{N}_n(2x-k-v-1)\,\mathrm{d}v\right] && \text{where } v \triangleq u-1 \implies u = v+1\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\mathsf{N}_n(2x-k)+\mathsf{N}_n(2x-k-1)\right] && \text{by Lemma O.2 page 335}\\
&= \frac{1}{2^{n+1}}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-k)+\sum_{m=1}^{n+2}\binom{n+1}{m-1}\mathsf{N}_n(2x-m)\right] && \text{where } m \triangleq k+1 \implies k = m-1\\
&= \frac{1}{2^{n+1}}\left[\underbrace{\sum_{k=1}^{n+1}\left[\binom{n+1}{k}+\binom{n+1}{k-1}\right]\mathsf{N}_n(2x-k)}_{\text{common indices of above two summations}}+\underbrace{\binom{n+1}{0}\mathsf{N}_n(2x-0)}_{k=0\text{ term}}+\underbrace{\binom{n+2}{n+2}\mathsf{N}_n(2x-n-2)}_{m=n+2\text{ term}}\right]\\
&= \frac{1}{2^{n+1}}\left[\underbrace{\sum_{k=1}^{n+1}\binom{n+2}{k}\mathsf{N}_n(2x-k)}_{\text{by \textit{Stifel formual} (Theorem B.1 page 147)}}+\underbrace{\binom{n+2}{0}\mathsf{N}_n(2x-0)}_{\text{because }\binom{n+1}{0}=1=\binom{n+2}{0}}+\binom{n+2}{n+2}\mathsf{N}_n(2x-n-2)\right]\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+2}\binom{n+2}{k}\mathsf{N}_n(2x-k)
\end{aligned}
$$

✏

Example O.16. [25]The 0 *order B-spline dyadic decomposition*

$$\mathsf{N}_0(x) = \frac{1}{1}\sum_{k=0}^{k=1}\binom{1}{k}\mathsf{N}_0(2x-k)$$

is illustrated to the right.

[25] Strang (1989) page 615 ⟨Box function⟩, Strang and Nguyen (1996) page 441 ⟨Box function⟩

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Example O.17. [26]The 1*st order B-spline dyadic decomposition*

$$\mathsf{N}_1(x) = \frac{1}{2}\sum_{k=0}^{k=2}\binom{2}{k}\mathsf{N}_1(2x-k)$$

is illustrated to the right.

Example O.18. The 2*nd order B-spline dyadic decomposition*

$$\mathsf{N}_2(x) = \frac{1}{4}\sum_{k=0}^{k=3}\binom{3}{k}\mathsf{N}_2(2x-k)$$

is illustrated to the right.

O.6.3 B-spline MRA scaling functions

Theorem O.10. *Let f* $N_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition O.2 page 335)*.
Let $V_j \triangleq \operatorname{span}\left(\left(\mathsf{N}_n\left(2^j x - k\right)\right)\right)_{k\in\mathbb{Z}}$.

THM $\left(\left(V_j\right)\right)_{j\in\mathbb{Z}}$ *is a* MULTIRESOLUTION ANALYSIS *on* $L^2_{\mathbb{R}}$ *with* SCALING FUNCTION $\phi(x) \triangleq \mathsf{N}_n(x)$

✎PROOF:

1. lemma: $(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$ is a *Riesz sequence*. Proof: by Theorem O.8 (page 358).
2. lemma: $\exists\,(h_k)$ such that $\mathsf{N}_n(x) = \sum_{k\in\mathbb{Z}} h_k \mathsf{N}_n(2x-k)$. Proof: by Theorem O.9 (page 359). In fact, note that $h_k = \frac{1}{2^n\sqrt{2}}\binom{n+1}{k}$
3. lemma: $\tilde{\mathbf{F}}\mathsf{N}_n(\omega)$ is *continuous* at 0. Proof:

$$\begin{aligned}\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1} && \text{by Lemma O.5 page 350}\\ &\Longrightarrow \textit{continuous} \text{ at } 0 && \text{by known property of sinc function}\end{aligned}$$

4. lemma: $\tilde{\phi}(0) \neq 0$. Proof:

$$\begin{aligned}\tilde{\mathbf{F}}\mathsf{N}_n(0) &= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1}\Bigg|_{\omega=0} && \text{by Lemma O.5 page 350}\\ &= 1\cdot\frac{1}{1/2} = 2 && \text{by } \textit{l'Hôpital's rule}\\ &\neq 0\end{aligned}$$

5. The completion of this proof follows directly from (1) lemma, (2) lemma, (3) lemma, (4) lemma, and Theorem 2.6 (page 33).

✏

[26] Strang (1989) page 615 ⟨Hat function⟩, Strang and Nguyen (1996) page 442 ⟨Hat function⟩, Heil (2011) page 380 ⟨Fig. 12.10⟩

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Figure O.4: *dilation equation* demonstrations for selected B-splines (Example O.19 page 362)

O.6.4 B-spline MRA coefficient sequences

Because each *B-spline* $\mathsf{N}_n(x)$ is the *scaling function* for an *MRA* (Theorem O.10 page 361), each *B-spline* also satisfies the *dilation equation* (Theorem 2.1 page 24) such that

$$\mathsf{N}_n(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \mathsf{N}(2x-k) \quad \text{where} \quad h_k = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2^{n+1}}\binom{n+1}{k} & \text{for } n = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{array}\right\}$$

The resulting *sequence* $(\!(h_k)\!)_{k\in\mathbb{Z}}$ is the *order n B-spline MRA coefficient sequence* induced by the *order n B-spline MRA scaling sequence* $\phi(x) \triangleq \mathsf{N}_n(x)$.[27]

Example O.19. See Figure O.4 (page 362) for some *dilation equation* demonstrations of selected B-splines.

Theorem O.11 (B-spline scaling coefficients)**.** *Let* $\left(\mathbf{L}^2_{\mathbb{R}}, (\!(\mathbf{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 27). Let* $\mathsf{N}_n(x)$ *be a* nTH ORDER B-SPLINE *(Definition O.2 page 335).*

THM

$$\underbrace{\phi(x) \triangleq \mathsf{N}_n(x)}_{\text{(1) B-spline scaling function}} \implies (\!(h_k)\!) = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2^{n+1}}\binom{n}{k} & \text{for } k = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{array}\right. \qquad \text{(2) scaling sequence in "time"}$$

$$\iff \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} = \frac{\sqrt{2}}{2^n}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} \qquad \text{(3) scaling sequence in "z domain"}$$

$$\iff \tilde{\mathsf{h}}(\omega) = 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} \qquad \text{(4) scaling sequence in "frequency"}$$

✎PROOF:

1. Proof that (1) $\implies$ (3): By Theorem O.10 page 361 we know that $\mathsf{N}_n(x)$ is a *scaling function* (Definition 2.1 page 18). So then we know that we can use Lemma 2.1 page 25.

$$\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \sqrt{2}\frac{\tilde{\phi}(2\omega)}{\tilde{\phi}(\omega)} && \text{by Lemma 2.1 page 25} \\
&= \sqrt{2}\frac{\tilde{\mathsf{N}}_n(2\omega)}{\tilde{\mathsf{N}}_n(\omega)} && \text{by (1)} \\
&= \sqrt{2}\frac{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i2\omega}}{2i\omega}\right)^{n+1}}{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1}} && \text{by Lemma O.5 page 350}
\end{aligned}$$

[27] For *Octave/MatLab* code useful for plotting a function given a sequence of coefficients $(\!(h_k)\!)$, see Section Y.1 (page 441).

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$$
\begin{aligned}
&= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{1-z^{-2}}{1-z^{-1}}\right)^{n+1}\Bigg|_{z=e^{i\omega}}\\
&= \frac{\sqrt{2}}{2^{n+1}}\left[\left(\frac{1-z^{-2}}{1-z^{-1}}\right)\left(\frac{1+z^{-1}}{1+z^{-1}}\right)\right]^{n+1}\Bigg|_{z=e^{i\omega}}\\
&= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{\left(1-z^{-2}\right)\left(1+z^{-1}\right)}{1-z^{-2}}\right)^{n+1}\Bigg|_{z=e^{i\omega}}\\
&= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}}
\end{aligned}
$$

2. Proof that (3) $\Longleftrightarrow$ (2):

$$
\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} && \text{by (3)}\\
&= \frac{\sqrt{2}}{2^{n}}\left(\sum_{k=0}^{n+1}\binom{n}{k}z^{-k}\right)\Bigg|_{z\triangleq e^{i\omega}} && \text{by } \textit{binomial theorem}\\
\Longleftrightarrow \qquad h_k &= \frac{\sqrt{2}}{2^{n+1}}\binom{n}{k} && \text{by definition of } Z \textit{ transform} \text{ (Definition Q.1 page 379)}
\end{aligned}
$$

3. Proof that (3) $\Longrightarrow$ (4):

$$
\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition R.1 page 393)}\\
&= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} && \text{by (3)}\\
&= \frac{\sqrt{2}}{2^{n}}\left(1+e^{-i\omega}\right)^{n+1} && \text{by definition of } z\\
&= \frac{\sqrt{2}}{2^{n}}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1}\\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1}
\end{aligned}
$$

4. Proof that (3) $\Longleftarrow$ (4):

$$
\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \check{\mathsf{h}}\left(e^{i\omega}\right)\\
&= \tilde{\mathsf{h}}(\omega)\\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} && \text{by (4)}\\
&= \frac{\sqrt{2}}{2^{n}}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}\left(1+e^{-i\omega}\right)^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}}
\end{aligned}
$$

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✏

Example O.20 (2 coefficient case). [28] Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ be an *MRA system* (Definition 2.3 page 27).

EX

$$\underbrace{\left\{\begin{array}{ll} 1. & \operatorname{supp}\phi(x) = [0:1] \quad \text{and} \\ 2. & \left(\!\left(\phi(x-k)\right)\!\right) \text{ forms a} \\ & \textit{partition of unity} \end{array}\right\}}_{\text{(A)}} \iff \underbrace{h_n = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2} & \text{for } n=0 \\ \frac{\sqrt{2}}{2} & \text{for } n=1 \\ 0 & \text{otherwise} \end{array}\right\}}_{\text{(B)}} \iff \underbrace{\left\{\phi(x) = \mathsf{N}_0(x)\right\}}_{\text{(C)}}$$

✎PROOF:

1. Proof that (A) $\Longrightarrow$ (B):

   (a) lemma: Only h_0 and h_1 are *non-zero*; All other coefficients h_k are 0. Proof: This follows from $\operatorname{supp}\phi(x) = [0:1]$ (Definition 2.4 page 33) and by Theorem 2.7 page 34.

   (b) lemma (equations for $\left(\!\left(h_k\right)\!\right)$): Because $\left(\!\left(h_k\right)\!\right)$ is a *scaling coefficient sequence* (Definition 2.1 page 18), it must satisfy the *admissibility equation* (Theorem 2.3 page 29). And because $\left(\!\left(\phi(x-k)\right)\!\right)$ forms a *partition of unity*, it must satisfy the equations given by Theorem 4.1 (page 54). (1a) lemma and these two constraints yield two simultaneous equations and two unknowns:

   $$\begin{array}{rcll} h_0 + h_1 & = & \sqrt{2} & (\textit{admissibility condition}) \\ h_0 - h_1 & = & 0 & (\textit{partition of unity/zero at} -1/\textit{vanishing 0th moment}) \end{array}$$

   (c) lemma: The equations provided by (1b) lemma can be expressed in matrix algebra form as follows…

   $$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

   (d) lemma: The *inverse* $\mathbf{A}^{-1}$ of $\mathbf{A}$ can be expressed as demonstrated below…

   $$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & -1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$
   $$\Longrightarrow \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (e) Proof for the values of $\left(\!\left(h_k\right)\!\right)$ (B):

   $$\begin{aligned} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} &= \mathbf{A}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} && \text{by (1c) lemma} \\ &= \mathbf{A}^{-1}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} && \text{by (1c) lemma} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} && \text{by (1d) lemma} \\ &= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

[28] Haar (1910), Wojtaszczyk (1997) pages 14–15 ⟨"Sources and comments"⟩

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2. Proof that (B) $\Longrightarrow$ (C):

$$
\begin{aligned}
\text{(B)} \Longrightarrow \phi(x) &= \sum_{k\in\mathbb{Z}} h_k\sqrt{2}\phi(2x-k) && \textit{dilation equation} && \text{(Theorem 2.1 page 24)}\\
&= \sum_{k=0}^{k=1}\left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\phi(2x-k) && \text{by item (1e) page 364}\\
&= \sum_{k=0}^{k=1}\phi(2x-k)\\
&= \sum_{k=0}^{k=1}\binom{1}{k}\phi(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition B.3 page 146)}\\
&\Longrightarrow \text{(D)} && \text{by } \textit{B-spline dyadic decomposition} && \text{(Theorem O.9 page 359)}
\end{aligned}
$$

3. Proof that (B) $\Longleftarrow$ (C):

$$
\begin{aligned}
\text{(C)} \Longrightarrow \mathsf{N}_0(x) &= \sum_{k=0}^{k=1}\binom{1}{k}\mathsf{N}_0(2x-k) && \text{by } \textit{B-spline dyadic decomposition} && \text{(Theorem O.9 page 359)}\\
&= \sum_{k=0}^{k=1}\left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{N}_0(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition B.3 page 146)}\\
&= \sum_{k\in\mathbb{Z}} h_k\sqrt{2}\mathsf{N}_0(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition B.3 page 146)}\\
&\Longrightarrow \text{(B)}
\end{aligned}
$$

4. Proof that (A) $\Longleftarrow$ (C):
   1. Proof that (C) $\Longrightarrow$ $\operatorname{supp}\phi(x) = [0 : 1]$: by Theorem O.4 (page 346)
   2. Proof that (C) $\Longrightarrow$ $(\!(\phi(x-k))\!)$ forms a *partition of unity*: by Theorem O.7 (page 352)

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EX	n=0,	(÷0!)	1;							
	n=1,	(÷1!)	1,	0;						
			−1,	2;						
	n=2,	(÷2!)	1,	0,	0;					
			−2,	6,	−3;					
			1,	−6,	9;					
	n=3,	(÷3!)	1,	0,	0,	0;				
			−3,	12,	−12,	4;				
			3,	−24,	60,	−44;				
			−1,	12,	−48,	64;				
	n=4,	(÷4!)	1,	0,	0,	0,	0;			
			−4,	20,	−30,	20,	−5;			
			6,	−60,	210,	−300,	155;			
			−4,	60,	−330,	780,	−655;			
			1,	−20,	150,	−500,	625;			
	n=5,	(÷5!)	1,	0,	0,	0,	0,	0;		
			−5,	30,	−60,	60,	−30,	6;		
			10,	−120,	540,	−1140,	1170,	−474;		
			−10,	180,	−1260,	4260,	−6930,	4386;		
			5,	−120,	1140,	−5340,	12270,	−10974;		
			−1,	30,	−360,	2160,	−6480,	7776;		
	n=6,	(÷6!)	1,	0,	0,	0,	0,	0,	0;	
			−6,	42,	−105,	140,	−105,	42,	−7;	
			15,	−210,	1155,	−3220,	4935,	−3990,	1337;	
			−20,	420,	−3570,	15680,	−37590,	47040,	−24178;	
			15,	−420,	4830,	−29120,	96810,	−168000,	119182;	
			−6,	210,	−3045,	23380,	−100065,	225750,	−208943;	
			1,	−42,	735,	−6860,	36015,	−100842,	117649;	
	n=7,	(÷7!)	1,	0,	0,	0,	0,	0,	0,	0;
			−7,	56,	−168,	280,	−280,	168,	−56,	8;
			21,	−336,	2184,	−7560,	15400,	−18648,	12488,	−3576;
			−35,	840,	−8400,	45360,	−143360,	267120,	−273280,	118896;
			35,	−1120,	15120,	−111440,	483840,	−1238160,	1733760,	−1027984;
			−21,	840,	−14280,	133560,	−741160,	2436840,	−4391240,	3347016;
			7,	−336,	6888,	−78120,	528920,	−2135448,	4753336,	−4491192;
			−1,	56,	−1344,	17920,	−143360,	688128,	−1835008,	2097152;
	n=8,	(÷8!)	1,	0,	0,	0,	0,	0,	0,	0,	0;
			−8,	72,	−252,	504,	−630,	504,	−252,	72,	−9
			28,	−504,	3780,	−15624,	39690,	−64008,	64260,	−36792,	9207
			−56,	1512,	−17388,	111384,	−436590,	1079064,	−1650348,	1432872,	−541917
			70,	−2520,	39060,	−340200,	1821330,	−6146280,	12800340,	−15082200,	7715619
			−56,	2520,	−49140,	541800,	−3691170,	15903720,	−42324660,	63667800,	−41503131
			28,	−1512,	35532,	−474264,	3929310,	−20674584,	67410252,	−124449192,	99584613
			−8,	504,	−13860,	217224,	−2121210,	13208328,	−51179940,	112731192,	−107948223
			1,	−72,	2268,	−40824,	459270,	−3306744,	14880348,	−38263752,	43046721

Table O.1: Coefficients of the *B-splines* $\mathsf{N}_n(x)$ multiplied by $n!$ (Example O.9 page 342)

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APPENDIX P

INTERPOLATION

P.1 Polynomial interpolation

P.1.1 Lagrange interpolation

Definition P.1. [1] *The* ***Lagrange polynomial*** $L_{P,n}(x)$ *with respect to the* $n+1$ *points* $P = \{(x_k, y_k) | k = 0, 1, 2, \dots, n\}$ *is defined as*

DEF
$$L_{P,n}(x) \triangleq \sum_{k=0}^{n} y_k \prod_{m \neq n} \frac{x - x_m}{x_k - x_m}$$

Proposition P.1. *Let* $L_{P,n}(x)$ *be the Lagrange polynomial with respect to the points* $P = \{(x_k, y_k) | k = 0, 1, 2, \dots, n\}$.

PRP
1. $L_{P,n}(x)$ *is an* n*th order polynomial.*
2. $L_{P,n}(x)$ *intersects all* $n+1$ *points in* P.

Example P.1 (Lagrange interpolation). The Lagrange polynomial $L_{P,3}(x)$ with respect to the 4 points $P = \{(-2, 1),\ (-1, 3),\ (3, 2),\ (5, 4)\}$ is

EX
$$L_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$\begin{aligned}
L_{P,3}(x) &= \sum_{k=0}^{n} y_k \prod_{m \neq n} \frac{x - x_m}{x_k - x_m} \qquad \text{by Definition P.1 page 367} \\
&= y_0 \frac{(x+1)(x-3)(x-5)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{(x+2)(x-3)(x-5)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
&\quad + y_2 \frac{(x+2)(x+1)(x-5)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x+2)(x+1)(x-3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \\
&= 1 \frac{(x+1)(x-3)(x-5)}{(-2+1)(-2-3)(-2-5)} + 3 \frac{(x+2)(x-3)(x-5)}{(-1+2)(-1-3)(-1-5)}
\end{aligned}$$

[1] Waring (1779) page 60, Euler (1783) page 165 ⟨§. 10. Problema 2. Corollarium 3.⟩, Gauss (1866), Lagrange (1877), Matthews and Fink (1992) page 206, Meijering (2002) ⟨historical background⟩

$$
\begin{aligned}
&+2\frac{(x+2)(x+1)(x-5)}{(3+2)(3+1)(3-5)}+4\frac{(x+2)(x+1)(x-3)}{(5+2)(5+1)(5-3)}\\
&=1\underbrace{\frac{x^3-7x^2+7x+15}{-35}}_{\text{roots}=-1,3,5}+3\underbrace{\frac{x^3-6x^2-x+30}{24}}_{\text{roots}=-2,3,5}+2\underbrace{\frac{x^3-2x^2-13x-10}{-40}}_{\text{roots}=-2,-1,5}+4\underbrace{\frac{x^3-7x-6}{84}}_{\text{roots}=-2,-1,3}\\
&=-\frac{x^3-7x^2+7x+15}{35}+\frac{x^3-6x^2-x+30}{8}-\frac{x^3-2x^2-13x-10}{20}+\frac{x^3-7x-6}{21}\\
&=x^3\left(\frac{-8\cdot 20\cdot 21+35\cdot 20\cdot 21-35\cdot 8\cdot 21+35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right)\\
&\quad+x^2\left(\frac{7\cdot 8\cdot 20\cdot 21-6\cdot 35\cdot 20\cdot 21+2\cdot 35\cdot 8\cdot 21+0\cdot 35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right)\\
&\quad+x\left(\frac{-7\cdot 8\cdot 20\cdot 21-35\cdot 20\cdot 21+13\cdot 35\cdot 8\cdot 21-7\cdot 35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right)\\
&\quad+\left(\frac{-15\cdot 8\cdot 20\cdot 21+30\cdot 35\cdot 20\cdot 21+10\cdot 35\cdot 8\cdot 21-6\cdot 35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right)\\
&=\frac{11060}{117600}x^3+\frac{-52920}{117600}x^2+\frac{-980}{117600}x+\frac{415800}{117600}\\
&=\frac{79}{840}x^3+\frac{-378}{840}x^2+\frac{-7}{840}x+\frac{2970}{840}
\end{aligned}
$$

✏

Definition P.2. [2] *The **Newton polynomial** $N_{P,n}(x)$ with respect to the $n+1$ points $P=\{(x_k,y_k)|k=0,1,2,\dots,n\}$ is defined as*

DEF
$$N_{P,n}(x)\triangleq\sum_{k=0}^{n}\alpha_k\prod_{m=0}^{k}(x-x_m)$$

Proposition P.2. *Let $N_{P,n}(x)$ be the Newton polynomial with respect to the points $P=\{(x_k,y_k)|k=0,1,2,\dots,n\}$.*

PRP
1. *$N_{P,n}(x)$ is an nth order polynomial.*
2. *$N_{P,n}(x)$ intersects all $n+1$ points in P.*

Example P.2 (Newton polynomial interpolation).

EX
The Newton polynomial $N_{P,3}(x)$ with respect to the 4 points $P=\{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is $N_{P,3}(x)=\frac{79}{840}x^3+\frac{-378}{840}x^2+\frac{-7}{840}x+\frac{2970}{840}$

✎PROOF:

$$
\begin{aligned}
N_{P,3}(x)&=\sum_{k=0}^{n}\alpha_k\prod_{m=1}^{k}(x-x_m)\\
&=\alpha_0+\alpha_1(x-x_0)+\alpha_2(x-x_0)(x-x_1)+\alpha_3(x-x_0)(x-x_1)(x-x_2)\\
&=\alpha_0+\alpha_1(x+2)+\alpha_2(x+2)(x+1)+\alpha_3(x+2)(x+1)(x-3)\\
&=\alpha_0+\alpha_1(x+2)+\alpha_2(x^2+3x+2)+\alpha_3(x^3-7x-6)\\
&=x^3(\alpha_3)+x^2(\alpha_2)+x(-7\alpha_3+3\alpha_2+\alpha_1)+(-6\alpha_3+2\alpha_2+2\alpha_1+\alpha_0)\\
&=\begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 2 & 3 & 1 & 0\\ -6 & -7 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ x\\ x^2\\ x^3\end{bmatrix}
\end{aligned}
$$

[2] Newton (1711), Fraser (1919) pages 9–17 ⟨Methodus differentialis: "A photographic reproduction of the original Latin text"⟩, Fraser (1919) pages 18–25 ⟨Methodus differentialis: English translation⟩, Fraser (1919) pages 1–8 ⟨historical background and notes⟩, Meijering (2002) ⟨historical background⟩, Matthews and Fink (1992) page 220

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$$
\begin{aligned}
\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} &= \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & 0 \\ 1 & (x_3 - x_0) & (x_3 - x_0)(x_3 - x_1) & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (-1+2) & 0 & 0 \\ 1 & (3+2) & (3+2)(3+1) & 0 \\ 1 & (5+2) & (5+2)(5+1) & (5+2)(5+1)(5-3) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 \\ 1 & 7 & 42 & 84 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 & 0 & 0 & 1 & 0 \\ 1 & 7 & 42 & 84 & 0 & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 5 & 20 & 0 & -1 & 0 & 1 & 0 \\ 0 & 7 & 42 & 84 & -1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 20 & 0 & 4 & -5 & 1 & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & 6-\frac{42}{5} & -7+\frac{42}{4} & -\frac{42}{20} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{12}{5} & \frac{14}{4} & -\frac{42}{20} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{24}{10} & \frac{35}{10} & -\frac{21}{10} & \frac{10}{10} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 1 & -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840} \end{bmatrix}
\end{aligned}
$$

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$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0\\ -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840}\end{bmatrix}\begin{bmatrix}1\\3\\2\\4\end{bmatrix}$$
$$= \begin{bmatrix}1\\2\\-\frac{9}{20}\\ \frac{79}{840}\end{bmatrix}$$

$$N_{P,3}(x) = \begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 2 & 3 & 1 & 0\\ -6 & -7 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\x\\x^2\\x^3\end{bmatrix}$$
$$= \left[\begin{array}{c|c|c|c} 1 & 2 & -\frac{9}{20} & \frac{79}{840}\end{array}\right]\begin{bmatrix} 1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 2 & 3 & 1 & 0\\ -6 & -7 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\x\\x^2\\x^3\end{bmatrix}$$
$$= \left[\begin{array}{c|c|c|c} 1+4-\frac{9}{10}-\frac{79}{140} & 2-\frac{27}{20}-\frac{79}{120} & -\frac{9}{20} & \frac{79}{840}\end{array}\right]\begin{bmatrix}1\\x\\x^2\\x^3\end{bmatrix}$$
$$= \frac{79}{840}x^3 - \frac{378}{840}x^2 - \frac{7}{840}x + \frac{2970}{840}$$

✏

Example P.3 (Least squares polynomial interpolation). [3] The best 3rd order polynomial in the **least squares** $S_{P,3}(x)$ sense with respect to the 4 points $P = \{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is

EX $$S_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$\boldsymbol{x} = \begin{bmatrix}x_0\\x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\-1\\3\\5\end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix}1\\3\\2\\4\end{bmatrix} = \begin{bmatrix}1\\3\\2\\4\end{bmatrix}$$

We want to find a third order polynomial

$$dx^3 + cx^2 + bx + a$$

that best approximates the 4 points in the least squares sense. We define the matrix U (known) and vector $\hat{\theta}$ (to be computed) as follows:

$$U^H \triangleq \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3\\ 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 1 & x_3 & x_3^2 & x_3^3\end{bmatrix} \qquad \hat{\theta} \triangleq \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

[3] van Overschee and de Moor (2012) page 14 ⟨Table 1.1; historical perspective as relates to "subspace identification"⟩

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Then, using *Least squares*, the best coefficients for the polynomial are

$$
\begin{aligned}
\hat{\theta} &= \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
&= R^{-1}W \\
&= (UU^H)^{-1}\,(U\boldsymbol{y}) \\
&= \left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}^H \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}^H \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix}^H \begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix}^H \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix}^H \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix}^H \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2970 \\ -7 \\ -378 \\ 79 \end{bmatrix}
\end{aligned}
$$

✏

P.2 Hermite interpolation

Figure P.1: Polynomial quadrature condition low-pass and high-pass terms

Definition P.3.

DEF

*The **minimal polynomial** $\mathsf{P}_m(x)$ is defined as*

$$\mathsf{P}_m(x) = \sum_{k=0}^{p-1} \binom{p-1+k}{k} x^k$$

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Definition P.4.

DEF

The low pass Hermite interpolation function is defined as $\mathsf{h}_l \triangleq (1-x)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} x^k$

The high pass Hermite interpolation function is defined as $\mathsf{h}_h \triangleq x^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} (1-x)^k$

Example P.4. [4] Here are some examples of the minimal polynomial $\mathsf{P}_m(x)$ for varying number of vanishing moments p:

EX

p	$\mathsf{P}_m(x) = \sum_{k=0}^{p-1} \binom{p-1+k}{k} x^k$
1	1
2	$1 + 2x$
3	$1 + 3x + 6x^2$
4	$1 + 4x + 10x^2 + 20x^3$
5	$1 + 5x + 15x^2 + 35x^3 + 70x^4$
6	$1 + 6x + 21x^2 + 56x^3 + 126x^4 + 252x^5$
7	$1 + 7x + 28x^2 + 84x^3 + 210x^4 + 462x^5 + 924x^6$
8	$1 + 8x + 36x^2 + 120x^3 + 330x^4 + 792x^5 + 1716x^6 + 3432x^7$
9	$1 + 9x + 45x^2 + 165x^3 + 495x^4 + 1287x^5 + 3003x^6 + 6435x^7 + 12870x^8$
10	$1 + 10x + 55x^2 + 220x^3 + 715x^4 + 2002x^5 + 5005x^6 + 11440x^7 + 24310x^8 + 48620x^9$

Example P.5. [5] Low-pass term $(1-y)^p \mathsf{P}_m(y)$:

EX

p	$(1-x)^p \mathsf{P}_m(x) = (1-x)^p \sum_{k=0}^{p-1} \binom{p-1+k}{k} x^k$
1	$1 - x$
2	$1 - 3x^2 + 2x^3$
3	$1 - 10x^3 + 15x^4 - 6x^5$
4	$1 - 35x^4 + 84x^5 - 70x^6 + 20x^7$
5	$1 - 126x^5 + 420x^6 - 540x^7 + 315x^8 - 70x^9$
6	$1 - 462x^6 + 1980x^7 - 3465x^8 + 3080x^9 - 1386x^{10} + 252x^{11}$

Theorem P.1 (*Hermite Interpolation*).

THM

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} x^k\right]\Bigg|_{x=0} = \bar{\delta}_n \quad \text{for } n = 0,1,2,\ldots,p-1$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} x^k\right]\Bigg|_{x=1} = 0 \quad \text{for } n = 0,1,2,\ldots,p-1$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} (1-x)^k\right]\Bigg|_{x=0} = 0 \quad \text{for } n = 0,1,2,\ldots,p-1$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} (1-x)^k\right]\Bigg|_{x=1} = \bar{\delta}_n \quad \text{for } n = 0,1,2,\ldots,p-1$$

✎PROOF: Let

$$\mathsf{f}(x) \triangleq (1-x)^p \sum_{n=0}^{p-1} \binom{p-1+n}{n} x^n$$

[4] Pinsky (2002) page 333, Pinsky (2008) page 333 ⟨Exercise 6.5.11: $P_1(y), P_2(y), \ldots, P_6(y)$⟩, Sloane (2009) http://oeis.org/A000580

[5] Ebert et al. (2003) page 347 ⟨$3t^2 - 2t^3$ and $6t^5 - 15t^4 + 10t^3$⟩, Perlin (2002) ⟨$3t^2 - 2t^3$ and $6t^5 - 15t^4 + 10t^3$⟩, ? http://oeis.org/A091811 ? http://oeis.org/A327809

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$$\mathsf{g}(x) \triangleq x^p \sum_{n=0}^{p-1} \binom{p-1+n}{n}(1-x)^n$$
$$q \triangleq p-1$$

1. Proof that $\mathsf{f}(0) = 1$:

$$\begin{aligned}
\mathsf{f}(0) &= (1-x)^p \sum_{m=0}^{p-1} \binom{p-1+m}{m} x^m \Bigg|_{x=0} \\
&= (1-x)^p \left[\binom{p-1}{0} + \sum_{m=1}^{p-1} \binom{p-1+m}{m} x^m \right] \Bigg|_{x=0} \\
&= 1
\end{aligned}$$

2. Proof that $\mathsf{f}(x) = p \sum_{n=0}^{2p-1} \left[\sum_{k=\max(0,n-q)}^{\min(n,p)} (-1)^k \frac{(p+n-k-1)!}{(p-k)!(n-k)!\,k!} \right] x^n$:

$$\begin{aligned}
(1-x)^p \mathsf{P}_m(x) &= \sum_{n=0}^{p} \binom{p}{n} (-1)^n x^n \sum_{m=0}^{p-1} \binom{p-1+m}{m} x^m \\
&= \sum_{n=0}^{2p-1} \sum_{k=\max(0,n-q)}^{\min(n,p)} \binom{p}{k} (-1)^k \binom{p-1+n-k}{n-k} x^n && \text{by Theorem C.2 page 158} \\
&= \sum_{n=0}^{2p-1} \sum_{k=\max(0,n-q)}^{\min(n,p)} (-1)^k \frac{p!}{(p-k)!k!} \frac{(p-1+n-k)!}{(p-1)!(n-k)!} x^n \\
&= p \sum_{n=0}^{2p-1} \left[\sum_{k=\max(0,n-q)}^{\min(n,p)} (-1)^k \frac{(p+n-k-1)!}{(p-k)!(n-k)!\,k!} \right] x^n
\end{aligned}$$

3. Proof that $\mathsf{f}^{(n)}(0) = \bar{\delta}_n$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(1-x)^p \mathsf{P}_m(x) \right] \Bigg|_{x=0} &= \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[p \sum_{m=0}^{2p-1} \left[\sum_{k=\max(0,m-q)}^{\min(m,p)} (-1)^k \frac{(p+m-k-1)!}{(p-k)!(m-k)!\,k!} \right] x^m \right] \Bigg|_{x=0} && \text{by 1.} \\
&= p \sum_{m=n}^{2p-1} \sum_{k=\max(0,m-q)}^{\min(m,p)} (-1)^k \frac{(p-1+m-k)!}{(p-k)!(m-k)!k!} \frac{m!}{(m-n)!} x^{m-n} \Bigg|_{x=0} \\
&= p \sum_{k=\max(0,n-q)}^{\min(n,p)} (-1)^k \frac{(p-1+n-k)!}{(p-k)!} \frac{n!}{(n-k)!k!} \\
&= p \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(p+n-k-1)!}{(p-k)!} \\
&\stackrel{?}{=} \bar{\delta}_n \qquad \text{for } n = 0, 1, 2, \ldots, p-1
\end{aligned}$$

4. Proof that $\mathsf{f}^{(n)}(0) = \bar{\delta}_n$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
&\frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(1-x)^p \mathsf{P}_m(x) \right] \Bigg|_{x=0} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}} (1-x)^p \right] \left[\frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathsf{P}_m(x) \right] \Bigg|_{x=0} && \text{by Lemma E.2 (Leibnitz rule)}
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}(1-x)^p\right]\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sum_{m=0}^{p-1}\binom{p-1+m}{m}x^m\right]\Bigg|_{x=0} && \text{by definition of } \mathsf{P}_m(x)\\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}(1-x)^{(p-n+k)}\right]\left[\sum_{m=k}^{p-1}\binom{p-1+m}{m}\frac{m!}{(m-k)!}x^{m-k}\right]\Bigg|_{x=0}\\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}\right]\left[\binom{p-1+k}{k}k!\right]\\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}\right]\left[\frac{(p-1+k)!}{(p-1)!k!}\,k!\right]\\
&= (-1)^n p\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(p+k-1)!}{(p+k-n)!}\\
&\stackrel{?}{=} \bar{\delta}_n \qquad \text{for } k=0,1,2,\ldots,p-1
\end{aligned}
$$

5. Proof that $\mathsf{f}^{(n)}(1)=0$ for $n=0,1,2,\ldots,p-1$:

$$
\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^p\mathsf{P}_m(x)\right]\Bigg|_{x=1} &= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1-x)^p\right]\mathsf{P}_m^{(n-k)}(x)\Bigg|_{x=1} && \text{by Lemma E.2 (Leibnitz rule)}\\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^k\frac{p!}{(p-k)!}(1-x)^{p-k}\right]\mathsf{P}_m^{(n-k)}(x)\Bigg|_{x=1}\\
&= \sum_{k=0}^{n}\binom{n}{k}\,0\cdot\mathsf{P}_m^{(n-k)}(x)\Bigg|_{x=1} && \text{by Lemma E.2}\\
&= 0 \qquad \text{for } k=0,1,2,\ldots,p-1
\end{aligned}
$$

✍

P.3 Cardinal Series and Sampling

P.3.1 Cardinal series basis

The *Paley-Wiener* class of functions (next definition) are those with a bandlimited Fourier transform. The cardinal series forms an orthogonal basis for such a space (Theorem P.3 page 375). In a *frame* $(\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{Z}}$ with *frame operator* $\mathbf{S}$ on a *Hilbert Space* $\boldsymbol{H}$ with *inner product* $\langle\triangle\,|\,\triangledown\rangle$, a function $\mathsf{f}(x)$ in the space spanned by the frame can be represented by

$$\mathsf{f}(x)=\sum_{n\in\mathbb{Z}}\underbrace{\left\langle \mathsf{f}\,|\,\mathbf{S}^{-1}\boldsymbol{x}_n\right\rangle}_{\text{“Fourier coefficient”}}\boldsymbol{x}_n.$$

If the frame is *orthonormal* (giving an *orthonormal basis*), then $\mathbf{S}=\mathbf{S}^{-1}=\mathbf{I}$ and

$$\mathsf{f}(x)=\sum_{n\in\mathbb{Z}}\langle \mathsf{f}\,|\,\boldsymbol{x}_n\rangle\,\boldsymbol{x}_n.$$

In the case of the cardinal series, the *Fourier coefficients* (Definition N.11 page 316) are particularly simple—these coefficients are samples of f taken at regular intervals (Theorem P.4 page 375). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x)\,|\,\delta(x-n\tau)\rangle \triangleq \int_{\mathbb{R}}\mathsf{f}(x)\delta(x-n\tau)\,\mathrm{d}t \triangleq \mathsf{f}(n\tau)$$

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Definition P.5. [6]

DEF

A function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is in the* ***Paley-Wiener*** *class of functions* $\boldsymbol{PW}_\sigma^p$ *if there exists* $\mathsf{F} \in \boldsymbol{L}^p(-\sigma : \sigma)$ *such that*

$$\mathsf{f}(x) = \int_{-\sigma}^{\sigma} \mathsf{F}(\omega) e^{ix\omega}\, d\omega \qquad (\mathsf{f} \text{ has a BANDLIMITED Fourier transform } \mathsf{F} \text{ with bandwidth } \sigma)$$

for $p \in [1 : \infty)$ *and* $\sigma \in (0 : \infty)$.

Theorem P.2 (Paley-Wiener Theorem for Functions). [7] *Let* f *be an* ENTIRE FUNCTION *(the domain of* f *is the entire complex plane* $\mathbb{C}$*). Let* $\sigma \in \mathbb{R}^+$.

THM

$$\{\mathsf{f} \in \boldsymbol{PW}_\sigma^2\} \iff \left\{ \begin{array}{ll} 1. & \exists C \in \mathbb{R}^+ \quad \text{such that} \quad |\mathsf{f}(z)| \le C e^{\sigma|z|} \quad \text{(EXPONENTIAL TYPE)} \quad \textit{and} \\ 2. & \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \end{array} \right\}$$

Theorem P.3 (Cardinal sequence). [8]

THM

$$\left\{\frac{1}{\tau} \ge 2\sigma\right\} \implies \textit{The sequence} \quad \left(\frac{\sin\left[\frac{\pi}{\tau}(x - n\tau)\right]}{\frac{\pi}{\tau}(x - n\tau)}\right)_{n\in\mathbb{Z}} \quad \textit{is an ORTHONORMAL BASIS for } \boldsymbol{PW}_\sigma^2.$$

Theorem P.4 (Sampling Theorem). [9]

THM

$$\left\{ \begin{array}{ll} 1. & \mathsf{f} \in \boldsymbol{PW}_\sigma^2 \quad \textit{and} \\ 2. & \frac{1}{\tau} \ge 2\sigma \end{array} \right\} \implies \mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n\tau) \frac{\sin\left[\frac{\pi}{\tau}(x - n\tau)\right]}{\frac{\pi}{\tau}(x - n\tau)}}_{\text{CARDINAL SERIES}}.$$

✎PROOF:

$$\text{Let} \quad \mathsf{s}(x) \triangleq \frac{\sin\left[\frac{\pi}{\tau}x\right]}{\frac{\pi}{\tau}x} \iff \tilde{\mathsf{s}}(\omega) = \begin{cases} \tau & : |f| \le \frac{1}{2\tau} \\ 0 & : \text{otherwise} \end{cases}$$

1. Proof that the set is *orthonormal*: see Hardy (1941)

[6] Higgins (1996) page 52 ⟨Definition 6.15⟩

[7] Boas (1954) page 103 ⟨6.8.1 Theorem of Paley and Wiener⟩, Katznelson (2004) page 212 ⟨7.4 Theorem⟩, Zygmund (2002b) pages 272–273 ⟨(7·2) THEOREM OF PALEY-WIENER⟩, YOSIDA (1980) PAGE 161, RUDIN (1987) PAGE 375 ⟨19.3 THEOREM⟩, YOUNG (2001) PAGE 85 ⟨THEOREM 18⟩

[8] Higgins (1996) page 52 ⟨Definition 6.15⟩, Hardy (1941) ⟨*orthonormality*⟩, Higgins (1985) page 56 ⟨H1.; historical notes⟩

[9] Whittaker (1915), Kotelnikov (1933), Whittaker (1935), Shannon (1948) ⟨Theorem 13⟩, Shannon (1949) page 11 II (1991) page 1, Nashed and Walter (1991), Higgins (1996) page 5, Young (2001) pages 90–91 ⟨THE PALEY-WIENER SPACE⟩, Papoulis (1980) pages 418–419 ⟨The Sampling Theorem⟩. The *sampling theorem* was "discovered" and published by multiple people: Nyquist in 1928 (DSP?), Whittaker in 1935 (interpolation theory), and Shannon in 1949 (communication theory). references: Mallat (1999) page 43, Oppenheim and Schafer (1999) page 143.

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2. Proof that the set is a *basis*:

$$
\begin{aligned}
\mathsf{f}(x) &= \int_\omega \tilde{\mathsf{f}}(\omega)e^{i\omega t}\,\mathrm{d}\omega && \text{by } \textit{inverse Fourier transform} && \text{(Theorem M.1 page 295)}\\
&= \int_\omega \mathbf{T}\tilde{\mathsf{f}}_{\mathsf{d}}(\omega)\tilde{\mathsf{s}}(\omega)e^{i\omega t}\,\mathrm{d}\omega && \text{if } W \le \frac{1}{2T} && \\
&= \mathbf{T}\mathsf{f}_{\mathsf{d}}(x) \star \mathsf{s}(x) && \text{by } \textit{Convolution theorem} && \text{(Theorem Q.2 page 380)}\\
&= \mathbf{T}\int_u [\mathsf{f}_{\mathsf{d}}(u)]\mathsf{s}(x-u)\,\mathrm{d}u && \text{by } \textit{convolution} \text{ definition} && \text{(Definition F.1 page 203)}\\
&= \mathbf{T}\int_u \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(u)\delta(u-n\tau)\right]\mathsf{s}(x-u)\,\mathrm{d}u && \text{by } \textit{sampling} \text{ definition} && \text{(Theorem P.5 page 376)}\\
&= \mathbf{T}\sum_{n\in\mathbb{Z}}\int_u \mathsf{f}(u)\mathsf{s}(x-u)\delta(u-n\tau)\,\mathrm{d}u && && \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\mathsf{s}(x-n\tau) && \text{by prop. of } \textit{Dirac delta} && \\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)} && \text{by definition of } \mathsf{s}(x) &&
\end{aligned}
$$

✏

P.3.2 Sampling

Definition P.6. [10] *Let $\delta(x)$ be the* DIRAC DELTA *distribution.*

DEF *The **Shah Function** $\text{Ш}(x)$ is defined as* $\text{Ш}(x) \triangleq \sum_{n\in\mathbb{Z}} \delta(x-n)$

If $\mathsf{f}_{\mathsf{d}}(x)$ is the function $\mathsf{f}(x)$ sampled at rate $1/T$, then $\tilde{\mathsf{f}}_{\mathsf{d}}(\omega)$ is simply $\tilde{\mathsf{f}}(\omega)$ *replicated* every $1/T$ Hertz and *scaled* by $1/T$. This is proven in Theorem P.5 (next) and illustrated in Figure P.2 (page 377).

Theorem P.5. *Let $\mathsf{f}, \mathsf{f}_{\mathsf{d}} \in \boldsymbol{L}^2_{\mathbb{R}}$ and $\tilde{\mathsf{f}}, \tilde{\mathsf{f}}_{\mathsf{d}} \in \boldsymbol{L}^2_{\mathbb{R}}$ be their respective fourier transforms. Let $\mathsf{f}_{\mathsf{d}}(x)$ be the **sampled** $\mathsf{f}(x)$ such that*

$$\mathsf{f}_{\mathsf{d}}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x-n\tau).$$

THM
$$\left\{\mathsf{f}_{\mathsf{d}}(x) \triangleq \mathsf{f}(x)\text{Ш}(x) \triangleq \mathsf{f}(x)\sum_{n\in\mathbb{Z}}\delta(x-n\tau)\right\} \implies \left\{\tilde{\mathsf{f}}_{\mathsf{d}}(\omega) = \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)\right\}$$

[10] Bracewell (1978) page 77 ⟨The sampling or replicating symbol Ш(x)⟩, Córdoba (1989)191. Note: The symbol Ш is the Cyrillic upper case "sha" character, which has been assigned Unicode location U+0428. Reference: http://unicode.org/cldr/utility/character.jsp?a=0428

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Figure P.2: Sampling in frequency domain

✎PROOF:

$$
\begin{aligned}
\tilde{f}_d(\omega) &\triangleq \int_t f_d(x) e^{-i\omega t}\,dt \\
&= \int_t \left[\sum_{n\in\mathbb{Z}} f(x)\delta(x-n\tau)\right] e^{-i\omega t}\,dt \\
&= \sum_{n\in\mathbb{Z}} \int_t f(x)\delta(x-n\tau) e^{-i\omega t}\,dt \\
&= \sum_{n\in\mathbb{Z}} f(n\tau) e^{-i\omega n\tau} && \text{by definition of } \delta \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right) && \text{by } IPSF \qquad \text{(Theorem 1.4 page 14)} \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega - \frac{2\pi}{\tau}n\right)
\end{aligned}
$$

✏

Suppose a waveform $f(x)$ is sampled at every time T generating a sequence of sampled values $f(n\tau)$. Then in general, we can *approximate* $f(x)$ by using interpolation between the points $f(n\tau)$. Interpolation can be performed using several interpolation techniques.

In general all techniques lead only to an approximation of $f(x)$. However, if $f(x)$ is *bandlimited* with bandwidth $W \le \frac{1}{2T}$, then $f(x)$ is *perfectly reconstructed* (not just approximated) from the sampled values $f(n\tau)$ (Theorem P.4 page 375).

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APPENDIX Q

Z TRANSFORM

A digital filter is an operator on a sequence For a wide class of digital filters, this operator is linear. This operation can often be more clearly understood by the use of a special transform called the *Z-transform* (Definition Q.1 page 379). The Z-transform represents linear filters by ratios of polynomials (a polynomial divided by a polynomial) in a free variable z. The roots of the numerator polynomial are called *zeros*; the roots of the denominator polynomial are called *poles* (Definition C.5 page 166). The location in the z-plane of these poles and zeros determine the behavior of the filter operation.

Q.1 Z-transform

Definition Q.1. [1]

DEF

*The **z-transform** $\mathbf{Z}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$\left[\mathbf{Z}\left(\!(x_n)\!\right)\right](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\textit{Laurent series}} \qquad \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}}$$

Theorem Q.1. *Let $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ be the* Z-TRANSFORM *of* $\mathsf{x}[n]$.

THM

$$\left\{ \check{\mathsf{x}}(z) \triangleq \mathbf{Z}\left(\!(\mathsf{x}[n])\!\right) \right\} \implies \left\{ \begin{array}{llcll} (1). & \mathbf{Z}\left(\!(\alpha\mathsf{x}[n])\!\right) & = & \alpha\check{\mathsf{x}}(z) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \textit{and} \\ (2). & \mathbf{Z}\left(\!(\mathsf{x}[n-k])\!\right) & = & z^{-k}\check{\mathsf{x}}(z) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \textit{and} \\ (3). & \mathbf{Z}\left(\!(\mathsf{x}[-n])\!\right) & = & \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \textit{and} \\ (4). & \mathbf{Z}\left(\!(\mathsf{x}^*[n])\!\right) & = & \check{\mathsf{x}}^*\left(z^*\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \quad \textit{and} \\ (5). & \mathbf{Z}\left(\!(\mathsf{x}^*[-n])\!\right) & = & \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} \end{array} \right\}$$

✎PROOF:

$$\begin{aligned} \alpha\mathbb{Z}\check{\mathsf{x}}(z) &\triangleq \alpha\mathbf{Z}\left(\!(\mathsf{x}[n])\!\right) && \text{by definition of } \check{\mathsf{x}}(z) \\ &\triangleq \alpha\sum_{n\in\mathbb{Z}} \mathsf{x}[n]z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator} \end{aligned}$$

[1] *Laurent series*: Abramovich and Aliprantis (2002) page 49

$$\begin{aligned}
&\triangleq \sum_{n\in\mathbb{Z}} (\alpha \mathsf{x}[n]) z^{-n} && \text{by } \textit{distributive} \text{ property} \\
&\triangleq \mathbf{Z}\left(\!\!\left(\alpha \mathsf{x}[n]\right)\!\!\right) && \text{by definition of } \mathbf{Z} \text{ operator} \\
z^{-k}\check{\mathsf{x}}(z) &= z^{-k}\mathbf{Z}\left(\!\!\left(\mathsf{x}[n]\right)\!\!\right) && \text{by definition of } \check{\mathsf{x}}(z) \qquad \text{(left hypothesis)} \\
&\triangleq z^{-k}\sum_{n=-\infty}^{n=+\infty} \mathsf{x}[n] z^{-n} && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
&= \sum_{n=-\infty}^{n=+\infty} \mathsf{x}[n] z^{-n-k} \\
&= \sum_{m-k=-\infty}^{m-k=+\infty} \mathsf{x}[m-k] z^{-m} && \text{where } m \triangleq n+k \qquad \Longrightarrow n = m-k \\
&= \sum_{m=-\infty}^{m=+\infty} \mathsf{x}[m-k] z^{-m} \\
&= \sum_{n=-\infty}^{n=+\infty} \mathsf{x}[n-k] z^{-n} && \text{where } n \triangleq m \\
&\triangleq \mathbf{Z}\left(\!\!\left(\mathsf{x}[n-k]\right)\!\!\right) && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
\mathbf{Z}\left(\!\!\left(\mathsf{x}^*[n]\right)\!\!\right) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}^*[n] z^{-n} && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
&\triangleq \left(\sum_{n\in\mathbb{Z}} \mathsf{x}[n](z^*)^{-n}\right)^* && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
&\triangleq \check{\mathsf{x}}^*(z^*) && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
\mathbf{Z}\left(\!\!\left(\mathsf{x}[-n]\right)\!\!\right) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}[-n] z^{-n} && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
&= \sum_{-m\in\mathbb{Z}} \mathsf{x}[m] z^{m} && \text{where } m \triangleq -n \qquad \Longrightarrow n = -m \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m] z^{m} && \text{by } \textit{absolutely summable} \text{ property} \qquad \text{(Definition F.3 page 205)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m] \left(\frac{1}{z}\right)^{-m} && \text{by } \textit{absolutely summable} \text{ property} \qquad \text{(Definition F.3 page 205)} \\
&\triangleq \check{\mathsf{x}}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
\mathbf{Z}\left(\!\!\left(\mathsf{x}^*[-n]\right)\!\!\right) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}^*[-n] z^{-n} && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)} \\
&= \sum_{-m\in\mathbb{Z}} \mathsf{x}^*[m] z^{m} && \text{where } m \triangleq -n \qquad \Longrightarrow n = -m \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}^*[m] z^{m} && \text{by } \textit{absolutely summable} \text{ property} \qquad \text{(Definition F.3 page 205)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}^*[m] \left(\frac{1}{z}\right)^{-m} && \text{by } \textit{absolutely summable} \text{ property} \qquad \text{(Definition F.3 page 205)} \\
&= \left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m] \left(\frac{1}{z^*}\right)^{-m}\right)^* && \text{by } \textit{absolutely summable} \text{ property} \qquad \text{(Definition F.3 page 205)} \\
&\triangleq \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) && \text{by definition of } \mathbf{Z} \qquad \text{(Definition Q.1 page 379)}
\end{aligned}$$

✍

Theorem Q.2 (convolution theorem). *Let $\star$ be the convolution operator* (Definition F.4 page 205).

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Figure Q.1: Direct form 1 order 2 IIR filters

THM

$$\mathbf{Z}\underbrace{\left(\left(\!\left(x_n\right)\!\right) \star \left(\!\left(y_n\right)\!\right)\right)}_{\textit{sequence convolution}} = \underbrace{\left(\mathbf{Z}\left(\!\left(x_n\right)\!\right)\right)\left(\mathbf{Z}\left(\!\left(y_n\right)\!\right)\right)}_{\textit{series multiplication}} \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}, \left(\!\left(y_n\right)\!\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x \star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by definition of } \star && \text{(Definition F.4 page 205)}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition Q.1 page 379)}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k \triangleq n-m && \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right] && && \\
&\triangleq \left[\mathbf{Z}\left(\!\left(x_n\right)\!\right)\right]\left[\mathbf{Z}\left(\!\left(y_n\right)\!\right)\right] && \text{by definition of } \mathbf{Z} && \text{(Definition Q.1 page 379)}
\end{aligned}$$

✏

Q.2 From z-domain back to time-domain

$$\check{y}(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) - a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Example Q.1. See Figure Q.1 (page 381)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2\left(z^2+5z+6\right)} = \frac{\left(\frac{3}{2}z^2+\frac{5}{2}z+\frac{7}{2}\right)}{z^2+5z+6} = \frac{\left(\frac{3}{2}+\frac{5}{2}z^{-1}+\frac{7}{2}z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

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Q.3 Zero locations

The system property of *minimum phase* is defined in Definition Q.2 (next) and illustrated in Figure Q.2 (page 382).

not minimum phase *minimum phase*

Figure Q.2: Minimum Phase filter

Definition Q.2. [2] *Let* $\check{\mathsf{x}}(z) \triangleq \mathbf{Z}(\langle x_n \rangle)$ *be the* Z TRANSFORM *(Definition Q.1 page 379) of a sequence* $\langle x_n \rangle_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\langle z_n \rangle_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\check{\mathsf{x}}(z)$.

DEF

The sequence $\langle x_n \rangle$ *is* ***minimum phase*** *if*

$$\underbrace{|z_n| < 1 \qquad \forall n \in \mathbb{Z}}_{\check{\mathsf{x}}(z) \text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem Q.3 (Robinson's Energy Delay Theorem)**.** [3] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM

$$\left\{ \begin{array}{ll} \mathsf{p} & \text{is MINIMUM PHASE} \\ \mathsf{q} & \text{is NOT minimum phase} \end{array} \text{ and } \right\} \Longrightarrow \underbrace{\sum_{n=0}^{m-1} |a_n|^2}_{\substack{\text{"energy" of the first } m \text{ coefficients of } \mathsf{p}(z)}} \geq \underbrace{\sum_{n=0}^{m-1} |b_n|^2}_{\substack{\text{"energy" of the first } m \text{ coefficients of } \mathsf{q}(z)}} \qquad \forall 0 \leq m \leq N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle (Figure Q.4 page 384).

Example Q.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure Q.4 (page 384).

[2] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36

[3] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976) pages 52–53

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Figure Q.3: Minimum/non-minimum phase comparison

Q.4 Pole locations

Definition Q.3.

DEF *A filter (or system or operator)* **H** *is* ***causal*** *if its current output does not depend on future inputs.*

Definition Q.4.

DEF *A filter (or system or operator)* **H** *is* ***time-invariant*** *if the mapping it performs does not change with time.*

Definition Q.5.

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Figure Q.4: Another minimum/non-minimum phase comparison

DEF *An operation* **H** *is* ***linear*** *if any output* y_n *can be described as a linear combination of inputs* x_n *as in*

$$y_n = \sum_{m\in\mathbb{Z}} h(m)x(n-m)\ .$$

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem Q.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Figure Q.5: Pole-zero plot stable/unstable causal LTI filters (Example Q.3 page 384)

Example Q.3. Stable/unstable filters are illustrated in Figure Q.5 (page 384).

True or False? This filter has no poles:

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$

Q.5 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

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Figure Q.6: FIR filters

Figure Q.7: Conjugate pair structure yielding real coefficients

Proposition Q.1.

PRP $\left\{\begin{array}{l}\text{ZEROS } \textit{and} \text{ POLES} \\ \textit{occur in} \text{ CONJUGATE PAIRS}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\text{COEFFICIENTS} \\ \textit{are} \text{ REAL.}\end{array}\right\}$

✎PROOF:

$$\begin{aligned}(z-p_1)(z-p_1^*) &= [z-(a+ib)][z-(a-ib)] \\ &= z^2+[-a+ib-ib-a]z-[ib]^2 \\ &= z^2-2az+b^2\end{aligned}$$

✏

Example Q.4. See Figure Q.7 (page 385).

$$\begin{aligned}H(z) &= G\frac{[z-z_1][z-z_2]}{[z-p_1][z-p_2]} = G\frac{[z-(1+i)][z-(1-i)]}{[z-(-\tfrac{2}{3}+i\tfrac{1}{2})][z-(-\tfrac{2}{3}-i\tfrac{1}{2})]} \\ &= G\frac{z^2-z[(1-i)+(1+i)]+(1-i)(1+i)}{z^2-z[(-\tfrac{2}{3}+i\tfrac{1}{2})+(-\tfrac{2}{3}+i\tfrac{1}{2})]+(-\tfrac{2}{3}+i\tfrac{1}{2})(-\tfrac{2}{3}+i\tfrac{1}{2})} \\ &= G\frac{z^2-2z+2}{z^2-\tfrac{4}{3}z+(\tfrac{4}{3}+\tfrac{1}{4})} = G\frac{z^2-2z+2}{z^2-\tfrac{4}{3}z+\tfrac{19}{12}}\end{aligned}$$

Q.6 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma Q.1. *A causal LTI operator* $\mathbf{H}$ *can be expressed as a rational expression* $\check{\mathsf{h}}(z)$.

$$\begin{aligned}\check{\mathsf{h}}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}\\ &= \frac{\sum_{n=0}^{N} b_n z^{-n}}{1 + \sum_{n=1}^{N} a_n z^{-n}}\end{aligned}$$

A filter operation $\check{\mathsf{h}}(z)$ can be expressed as a product of its roots (poles and zeros).

$$\begin{aligned}\check{\mathsf{h}}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}\\ &= \alpha \frac{(z - z_1)(z - z_2)\cdots(z - z_N)}{(z - p_1)(z - p_2)\cdots(z - p_N)}\end{aligned}$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with $\times$ and zeros with $\bigcirc$. An example is shown in Figure Q.8 page 386. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

Figure Q.8: Pole-zero plot for rational expression with real coefficients

Q.7 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{\mathsf{x}}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{\mathsf{y}}(z)$ is a *high-pass filter*.

Definition Q.6. [4] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES *(Definition F.2 page 204) in* $\ell^2_{\mathbb{R}}$ *(Definition F.3 page 205)*.

[4] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

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DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter****.*
Any triple $\big((\!(x_n)\!), (\!(y_n)\!), N\big)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition****.*

Theorem Q.5 (CQF theorem). [5] *Let* $\check{\mathrm{y}}(\omega)$ *and* $\check{\mathrm{x}}(\omega)$ *be the* DTFTs *(Definition R.1 page 393) of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$*, respectively, in* $\ell^2_{\mathbb{R}}$ *(Definition F.3 page 205).*

THM

$$\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{(1)\ \text{CQF in "time"}} \iff \check{\mathrm{y}}(z) = \pm(-1)^N z^{-N}\check{\mathrm{x}}^*\left(\frac{-1}{z^*}\right) \qquad (2)\ \text{CQF in "z-domain"}$$

$$\iff \check{\mathrm{y}}(\omega) = \pm(-1)^N e^{-i\omega N}\check{\mathrm{x}}^*(\omega+\pi) \qquad (3)\ \text{CQF in "frequency"}$$

$$\iff x_n = \pm(-1)^N(-1)^n y^*_{N-n} \qquad (4)\ \text{"reversed" CQF in "time"}$$

$$\iff \check{\mathrm{x}}(z) = \pm z^{-N}\check{\mathrm{y}}^*\left(\frac{-1}{z^*}\right) \qquad (5)\ \text{"reversed" CQF in "z-domain"}$$

$$\iff \check{\mathrm{x}}(\omega) = \pm e^{-i\omega N}\check{\mathrm{y}}^*(\omega+\pi) \qquad (6)\ \text{"reversed" CQF in "frequency"}$$

$\forall N\in\mathbb{Z}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{\mathrm{y}}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-transform} \quad \text{(Definition Q.1 page 379)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm\sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies n = N-m\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*\\
&= \pm(-1)^N z^{-N}\check{\mathrm{x}}^*\left(\frac{-1}{z^*}\right) && \text{by definition of } z\text{-transform} \quad \text{(Definition Q.1 page 379)}
\end{aligned}$$

2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\check{\mathrm{y}}(z) &= \pm(-1)^N z^{-N}\check{\mathrm{x}}^*\left(\frac{-1}{z^*}\right) && \text{by (2)}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-transform} \quad \text{(Definition Q.1 page 379)}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x^*_m \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-transform} \quad \text{(Definition Q.1 page 379)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x^*_m z^{-(N-m)}
\end{aligned}$$

[5] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

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$$= \sum_{m\in\mathbb{Z}} (\pm)(-1)^n x^*_{N-n} z^{-n} \qquad \text{where } n = N-m \implies \qquad m \triangleq N-n$$
$$\implies x_n = \pm(-1)^n x^*_{N-n}$$

3. Proof that (1) $\implies$ (3):

$$\begin{aligned}
\check{y}(\omega) &\triangleq \check{x}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition R.1 page 393)}\\
&= \left[\pm(-1)^N z^{-N}\check{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)}\\
&= \pm(-1)^N e^{-i\omega N}\check{x}\left(e^{i\pi}e^{i\omega}\right)\\
&= \pm(-1)^N e^{-i\omega N}\check{x}\left(e^{i(\omega+\pi)}\right)\\
&= \pm(-1)^N e^{-i\omega N}\check{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition R.1 page 393)}
\end{aligned}$$

4. Proof that (1) $\implies$ (6):

$$\begin{aligned}
\check{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT \quad \text{(Definition R.1 page 393)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x^*_{N-n}}_{CQF} e^{-i\omega n} && \text{by (1)}\\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x^*_m e^{-i\omega(N-m)} && \text{where } m \triangleq N-n \implies \quad n = N-m\\
&= \pm(-1)^N e^{-i\omega N}\sum_{m\in\mathbb{Z}} (-1)^m x^*_m e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N}\sum_{m\in\mathbb{Z}} e^{i\pi m} x^*_m e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N}\sum_{m\in\mathbb{Z}} x^*_m e^{i(\omega+\pi)m}\\
&= \pm(-1)^N e^{-i\omega N}\left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^*\\
&= \pm(-1)^N e^{-i\omega N}\check{x}^*(\omega+\pi) && \text{by definition of } DTFT \quad \text{(Definition R.1 page 393)}
\end{aligned}$$

5. Proof that (1) $\impliedby$ (3):

$$\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \check{y}(\omega)e^{i\omega n}\,d\omega && \text{by } \textit{inverse DTFT} \quad \text{(Theorem R.3 page 399)}\\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\check{x}^*(\omega+\pi)}_{\text{right hypothesis}} e^{i\omega n}\,d\omega && \text{by right hypothesis}\\
&= \pm(-1)^N \frac{1}{2\pi}\int_{-\pi}^{+\pi} \check{x}^*(\omega+\pi)e^{i\omega(n-N)}\,d\omega && \text{by right hypothesis}\\
&= \pm(-1)^N \frac{1}{2\pi}\int_{0}^{2\pi} \check{x}^*(v)e^{i(v-\pi)(n-N)}\,dv && \text{where } v \triangleq \omega+\pi \implies \quad \omega = v-\pi\\
&= \pm(-1)^N e^{-i\pi(n-N)}\frac{1}{2\pi}\int_{0}^{2\pi} \check{x}^*(v)e^{iv(n-N)}\,dv\\
&= \pm(-1)^N \underbrace{(-1)^N}_{e^{i\pi N}}\underbrace{(-1)^n}_{e^{-i\pi n}}\left[\frac{1}{2\pi}\int_{0}^{2\pi} \check{x}(v)e^{iv(N-n)}\,dv\right]^*\\
&= \pm(-1)^n x^*_{N-n} && \text{by } \textit{inverse DTFT} \quad \text{(Theorem R.3 page 399)}
\end{aligned}$$

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6. Proof that (1) $\iff$ (4):

$$\begin{aligned}
y_n = \pm(-1)^n x^*_{N-n} &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x^*_{N-n} \\
&\iff \pm(-1)^n y_n = x^*_{N-n} \\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x^*_{N-n}\right)^* \\
&\iff \pm(-1)^n y^*_n = x_{N-n} \\
&\iff x_{N-n} = \pm(-1)^n y^*_n \\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m} && \text{where } m \triangleq N-n \implies n = N-m \\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m} \\
&\iff x_m = \pm(-1)^N(-1)^m y^*_{N-m} \\
&\iff x_n = \pm(-1)^N(-1)^n y^*_{N-n} && \text{by change of free variables}
\end{aligned}$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

✏

Theorem Q.6. [6] *Let* $\check{y}(\omega)$ *and* $\check{x}(\omega)$ *be the* DTFT*s* (Definition R.1 page 393) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$*, respectively, in* $\ell^2_{\mathbb{R}}$ (Definition F.3 page 205).

THM

Let $y_n = \pm(-1)^n x^*_{N-n}$ (CQF CONDITION, Definition Q.6 page 386). *Then*

$$\left\{\begin{aligned}
&(A) \quad \left[\frac{d}{d\omega}\right]^n \check{y}(\omega)\Big|_{\omega=0} = 0 \iff \left[\frac{d}{d\omega}\right]^n \check{x}(\omega)\Big|_{\omega=\pi} = 0 \quad (B) \\
&\iff \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 \quad (C) \\
&\iff \sum_{k\in\mathbb{Z}} k^n y_k = 0 \quad (D)
\end{aligned}\right\} \quad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof that (A) $\implies$ (B):

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \check{y}(\omega)\Big|_{\omega=0} && \text{by (A)} \\
&= \left[\frac{d}{d\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\check{x}^*(\omega+\pi)\Big|_{\omega=0} && \text{by } \textit{CQF theorem} \quad \text{(Theorem Q.5 page 387)} \\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} \left[\frac{d}{d\omega}\right]^{\ell} \left[e^{-i\omega N}\right] \cdot \left[\frac{d}{d\omega}\right]^{n-\ell} \left[\check{x}^*(\omega+\pi)\right]\Bigg|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma E.2 page 201)} \\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} e^{-i\omega N} \left[\frac{d}{d\omega}\right]^{n-\ell} \left[\check{x}^*(\omega+\pi)\right]\Bigg|_{\omega=0} \\
&= (\pm)(-1)^N \cancel{e^{-i0N}} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} \left[\frac{d}{d\omega}\right]^{n-\ell} \left[\check{x}^*(\omega+\pi)\right]\Bigg|_{\omega=0}
\end{aligned}$$

$$\begin{aligned}
\implies \check{x}^{(0)}(\pi) &= 0 \\
\implies \check{x}^{(1)}(\pi) &= 0 \\
\implies \check{x}^{(2)}(\pi) &= 0 \\
\implies \check{x}^{(3)}(\pi) &= 0 \\
\implies \check{x}^{(4)}(\pi) &= 0 \\
\vdots \quad & \quad \vdots \\
\implies \check{x}^{(n)}(\pi) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

[6] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

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$\times$

$= \quad 1$

$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

2. Proof that (A) $\Longleftarrow$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{\mathrm{x}}(\omega)\Big|_{\omega=\pi} && \text{by (B)} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)e^{-i\omega N}\check{\mathrm{y}}^*(\omega+\pi)\Big|_{\omega=\pi} && \text{by } \textit{CQF theorem} \quad \text{(Theorem Q.5 page 387)} \\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{\ell}\left[e^{-i\omega N}\right]\cdot\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\check{\mathrm{y}}^*(\omega+\pi)\right]\Bigg|_{\omega=\pi} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma E.2 page 201)} \\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}(-iN)^{\ell}e^{-i\omega N}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\check{\mathrm{y}}^*(\omega+\pi)\right]\Bigg|_{\omega=\pi} \\
&= (\pm)\cancel{e^{-i\pi N}}\sum_{\ell=0}^{n}\binom{n}{\ell}-iN^{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\check{\mathrm{y}}^*(\omega+\pi)\right]\Bigg|_{\omega=\pi} \\
&= (\pm)(-1)^N\sum_{\ell=0}^{n}\binom{n}{\ell}-iN^{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\check{\mathrm{y}}^*(\omega+\pi)\right]\Bigg|_{\omega=\pi}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad \check{\mathrm{y}}^{(0)}(0) &= 0 \\
\Longrightarrow \quad \check{\mathrm{y}}^{(1)}(0) &= 0 \\
\Longrightarrow \quad \check{\mathrm{y}}^{(2)}(0) &= 0 \\
\Longrightarrow \quad \check{\mathrm{y}}^{(3)}(0) &= 0 \\
\Longrightarrow \quad \check{\mathrm{y}}^{(4)}(0) &= 0 \\
\vdots \quad & \quad \vdots \\
\Longrightarrow \quad \check{\mathrm{y}}^{(n)}(0) &= 0 \\
\Longrightarrow \quad \check{\mathrm{y}}^{(n)}(0) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

3. Proof that (B) $\Longleftrightarrow$ (C): by Theorem R.5 page 401

4. Proof that (A) $\Longleftrightarrow$ (D): by Theorem R.5 page 401

5. Proof that (CQF) $\not\Longleftarrow$ (A): Here is a counterexample: $\check{\mathrm{y}}(\omega) = 0$.

✏

Q.8 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.

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unstable | all-pass | stable!

$$
\begin{aligned}
|A(z)|_{z=e^{i\omega}} &= \frac{1}{r}\left|\frac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}} &&= \left|\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|e^{i\phi}\left(\frac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} &&= \left|z\left(\frac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|-z\left(\frac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} &&= \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\frac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
&= \left|\frac{1}{e^{-iv}}\left(\frac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right| &&= \left|\frac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right| \\
&= 1
\end{aligned}
$$

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APPENDIX R

DISCRETE TIME FOURIER TRANSFORM

R.1 Definition

Definition R.1.

DEF

The ***discrete-time Fourier transform*** $\breve{\mathbf{F}}$ *of* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *is defined as*

$$\left[\breve{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition R.1 page 393) to the definition of the Z-transform (Definition Q.1 page 379), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.

R.2 Properties

Proposition R.1 (DTFT periodicity)**.** *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition R.1 page 393) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP

$$\underbrace{\breve{x}(\omega) = \breve{x}(\omega + 2\pi n)}_{\text{PERIODIC } \textit{with period } 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}\breve{x}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} &&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi n m}} \\ &= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} &&= \breve{x}(\omega)\end{aligned}$$

✏

Theorem R.1. *Let* $\check{\mathsf{x}}(\omega) \triangleq \check{\mathbf{F}}\big[(\!(\mathsf{x}[n])\!)\big](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition R.1 page 393) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM

$$\left\{\ \tilde{\mathsf{x}}(\omega) \triangleq \check{\mathbf{F}}\,(\!(\mathsf{x}[n])\!)\ \right\} \implies \left\{\begin{array}{llcll} (1). & \check{\mathbf{F}}\,(\!(\mathsf{x}[-n])\!) & = & \tilde{\mathsf{x}}(-\omega) & \textit{and} \\ (2). & \check{\mathbf{F}}\,(\!(\mathsf{x}^*[n])\!) & = & \tilde{\mathsf{x}}^*(-\omega) & \textit{and} \\ (3). & \check{\mathbf{F}}\,(\!(\mathsf{x}^*[-n])\!) & = & \tilde{\mathsf{x}}^*(\omega) & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\check{\mathbf{F}}\,(\!(\mathsf{x}[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}[-n]e^{-i\omega n} && \text{by definition of } \textit{DTFT} && \text{(Definition R.1 page 393)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i(-\omega)m}\\
&\triangleq \tilde{\mathsf{x}}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\check{\mathbf{F}}\,(\!(\mathsf{x}^*[n])\!) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}^*[n]e^{-i\omega n} && \text{by definition of } \textit{DTFT} && \text{(Definition R.1 page 393)}\\
&= \left(\sum_{n\in\mathbb{Z}} \mathsf{x}[n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\textbf{algebras} && \text{(Definition ?? page ??)}\\
&= \left(\sum_{n\in\mathbb{Z}} \mathsf{x}[n]e^{-i(-\omega)n}\right)^*\\
&\triangleq \tilde{\mathsf{x}}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\check{\mathbf{F}}\,(\!(\mathsf{x}^*[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}^*[-n]e^{-i\omega n} && \text{by definition of } \textit{DTFT} && \text{(Definition R.1 page 393)}\\
&= \left(\sum_{n\in\mathbb{Z}} \mathsf{x}[-n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\textbf{algebras} && \text{(Definition ?? page ??)}\\
&= \left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n = -m\\
&\triangleq \tilde{\mathsf{x}}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

✏

Theorem R.2. *Let* $\check{\mathsf{x}}(\omega) \triangleq \check{\mathbf{F}}\big[(\!(\mathsf{x}[n])\!)\big](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition R.1 page 393) of a sequence* $(\!(\mathsf{x}[n])\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM

$$\left\{\begin{array}{lll} (1). & \tilde{\mathsf{x}}(\omega) \triangleq \check{\mathbf{F}}\,(\!(\mathsf{x}[n])\!) & \textit{and} \\ (2). & (\!(\mathsf{x}[n])\!) \textit{ is } \text{REAL-VALUED} & \end{array}\right\} \implies \left\{\begin{array}{llcllcll} (1). & \check{\mathbf{F}}\,(\!(\mathsf{x}[-n])\!) & = & \tilde{\mathsf{x}}(-\omega) & & & & \textit{and} \\ (2). & \check{\mathbf{F}}\,(\!(\mathsf{x}^*[n])\!) & = & \tilde{\mathsf{x}}^*(-\omega) & = & \tilde{\mathsf{x}}(\omega) & & \textit{and} \\ (3). & \check{\mathbf{F}}\,(\!(\mathsf{x}^*[-n])\!) & = & \tilde{\mathsf{x}}^*(\omega) & = & \tilde{\mathsf{x}}(-\omega) & & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\check{\mathbf{F}}\,(\!(\mathsf{x}[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}[-n]e^{-i\omega n} && \text{by definition of } \textit{DTFT} && \text{(Definition R.1 page 393)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i(-\omega)m}
\end{aligned}$$

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$$\triangleq \tilde{\mathsf{x}}(-\omega) \qquad \text{by left hypothesis}$$

$$\begin{aligned}
\boxed{\tilde{\mathsf{x}}^*(-\omega)} &= \boxed{\breve{\mathbf{F}}\left(\!\!\left(\mathsf{x}^*[n]\right)\!\!\right)} && \text{by Theorem R.1 page 394}\\
&= \breve{\mathbf{F}}\left(\!\!\left(\mathsf{x}[n]\right)\!\!\right) && \text{by \textit{real-valued} hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(\omega)} && \text{by definition of } \tilde{\mathsf{x}}(\omega) \quad \text{(Definition R.1 page 393)}
\end{aligned}$$

$$\begin{aligned}
\boxed{\tilde{\mathsf{x}}^*(\omega)} &= \boxed{\breve{\mathbf{F}}\left(\!\!\left(\mathsf{x}^*[-n]\right)\!\!\right)} && \text{by Theorem R.1 page 394}\\
&= \breve{\mathbf{F}}\left(\!\!\left(\mathsf{x}[-n]\right)\!\!\right) && \text{by \textit{real-valued} hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(-\omega)} && \text{by result (1)}
\end{aligned}$$

✍

Proposition R.2. *Let* $\breve{\mathsf{x}}(z)$ *be the* Z-TRANSFORM *(Definition Q.1 page 379) and* $\breve{\mathsf{x}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition R.1 page 393) of* $\left(\!\!\left(x_n\right)\!\!\right)$.

PRP

$$\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall \left(\!\!\left(x_n\right)\!\!\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}, c\in\mathbb{R}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{\mathsf{x}}(z)\Big|_{z=1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \text{by definition of } \breve{\mathsf{x}}(z) \text{ (Definition Q.1 page 379)}\\
&= \sum_{n\in\mathbb{Z}} x_n && \text{because } z^n = 1 \text{ for all } n\in\mathbb{Z}\\
&= c && \text{by hypothesis (1)}
\end{aligned}$$

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} && \text{by definition of } \breve{\mathsf{x}}(\omega) \quad \text{(Definition R.1 page 393)}\\
&= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1}\\
&= \breve{\mathsf{x}}(z)\Big|_{z=1} && \text{by definition of } \breve{\mathsf{x}}(z) \quad \text{(Definition Q.1 page 379)}\\
&= c && \text{by hypothesis (2)}
\end{aligned}$$

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3. Proof that (3) $\Longrightarrow$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} \\
&= \check{\mathsf{x}}(\omega) && \text{by definition of } \check{\mathsf{x}}(\omega) && \text{(Definition R.1 page 393)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

✏

Proposition R.3. *If the coefficients are **real**, then the magnitude response (MR) is **symmetric**.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathsf{h}}(-\omega)\right| &\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{i\omega m}\right| && = \left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}^*[m]e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } \textit{real}} && = \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right| \\
&\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\tilde{\mathsf{h}}(\omega)\right|
\end{aligned}$$

✏

Proposition R.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}}(-1)^n x_n = c}_{\textit{(1) in "time"}} \iff \underbrace{\check{\mathsf{x}}(z)|_{z=-1} = c}_{\textit{(2) in "z domain"}} \iff \underbrace{\check{\mathsf{x}}(\omega)|_{\omega=\pi} = c}_{\textit{(3) in "frequency"}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\textit{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned}
\check{\mathsf{x}}(z)|_{z=-1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

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2. Proof that (2) $\Longrightarrow$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\bigg|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \sum_{n\in\mathbb{Z}} z^{-n} x_n\bigg|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\Longrightarrow$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n\bigg|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\Longrightarrow$ (4):

   (a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

   (b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

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(e) Solutions to these equations give

$$\sum_{n\in\mathbb{Z}} x_{2n} \triangleq A \qquad = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)$$

$$\sum_{n\in\mathbb{Z}} x_{2n+1} \triangleq B \qquad = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1} \\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1} \\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)} \\
&= c
\end{aligned}$$

✏

Lemma R.1. *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* DTFT *(Definition R.1 page 393) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$.

LEM $\underbrace{(\!(x_n \in \mathbb{R})\!)_{n\in\mathbb{Z}}}_{\text{REAL-VALUED } sequence} \quad \Longrightarrow \quad \underbrace{|\check{\mathsf{x}}(\omega)|^2 = |\check{\mathsf{x}}(-\omega)|^2}_{\text{EVEN}} \quad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

$$\begin{aligned}
|\check{\mathsf{x}}(\omega)|^2 &= |\check{\mathsf{x}}(z)|^2\big|_{z=e^{i\omega}} \\
&= \check{\mathsf{x}}(z)\check{\mathsf{x}}^*(z)\big|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^*(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

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$$= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right]$$
$$= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2\sum_{n\in\mathbb{Z}}\sum_{m>n} x_n x_m \cos[\omega(m-n)]$$

Since cos is real and even, then $|\check{\mathrm{x}}(\omega)|^2$ must also be real and even. ✏

Theorem R.3 (inverse DTFT). [2] *Let $\check{\mathrm{x}}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition R.1 page 393) of a sequence $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\tilde{\mathrm{x}}^{-1}$ be the inverse of $\check{\mathrm{x}}$.*

THM

$$\underbrace{\left\{ \check{\mathrm{x}}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\check{\mathrm{x}}(\omega) \triangleq \check{\mathbf{F}}(\!(x_n)\!)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \check{\mathrm{x}}(\omega) e^{i\omega n}\,\mathrm{d}\omega \quad \forall \alpha\in\mathbb{R} \right\}}_{(\!(x_n)\!) = \check{\mathbf{F}}^{-1}\check{\mathbf{F}}(\!(x_n)\!)} \quad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \check{\mathrm{x}}(\omega) e^{i\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m}\right]}_{\check{\mathrm{x}}(\omega)} e^{i\omega n}\,\mathrm{d}\omega \qquad \text{by definition of } \check{\mathrm{x}}(\omega)$$
$$= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\,\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\,\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \left[2\pi\bar{\delta}_{m-n}\right]$$
$$= x_n$$

✏

Theorem R.4 (orthonormal quadrature conditions). [3] *Let $\check{\mathrm{x}}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition R.1 page 393) of a sequence $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\bar{\delta}_n$ be the* KRONECKER DELTA FUNCTION *at n (Definition N.12 page 316).*

THM

$$\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \iff \check{\mathrm{x}}(\omega)\check{\mathrm{y}}^*(\omega) + \check{\mathrm{x}}(\omega+\pi)\check{\mathrm{y}}^*(\omega+\pi) = 0 \qquad \forall n\in\mathbb{Z}, \forall (\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}}$$
$$\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \iff |\check{\mathrm{x}}(\omega)|^2 + |\check{\mathrm{x}}(\omega+\pi)|^2 = 2 \qquad \forall n\in\mathbb{Z}, \forall (\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}}$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩
[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

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1. Proof that $2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi)$:

$$\begin{aligned}
&2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n}\left(1+e^{i\pi n}\right)\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_{m} z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_{m} e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n\\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_{m} z^{m} + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_{m} e^{+i(\omega+\pi)m}\\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^*\\
&\triangleq \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi)
\end{aligned}$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 0 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\check{x}(\omega)|^2 + |\check{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n}\\
&= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\check{x}(\omega)|^2 + |\check{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 2 && \text{by right hypothesis}
\end{aligned}$$

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Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

✏

R.3 Derivatives

Theorem R.5. [4] *Let* $\check{x}(\omega)$ *be the* DTFT *(Definition R.1 page 393) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$.

THM

(A)	$\left[\frac{d}{d\omega}\right]^n \check{x}(\omega)\Big\vert_{\omega=0}$	$=$	0	$\Longleftrightarrow$	$\sum_{k\in\mathbb{Z}} k^n x_k$	$=$	0	*(B)*	$\forall n\in\mathbb{W}$
(C)	$\left[\frac{d}{d\omega}\right]^n \check{x}(\omega)\Big\vert_{\omega=\pi}$	$=$	0	$\Longleftrightarrow$	$\sum_{k\in\mathbb{Z}} (-1)^k k^n x_k$	$=$	0	*(D)*	$\forall n\in\mathbb{W}$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \check{x}(\omega)\Big\vert_{\omega=0} && \text{by hypothesis (A)}\\
&= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big\vert_{\omega=0} && \text{by definition of } \check{x}(\omega) \text{ (Definition R.1 page 393)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\Big\vert_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big\vert_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{d}{d\omega}\right]^n \check{x}(\omega)\Big\vert_{\omega=0} &= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big\vert_{\omega=0} && \text{by definition of } \tilde{g}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\right]\Big\vert_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big\vert_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k\\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

[4] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

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3. Proof that $(C) \implies (D)$:

$$
\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{x}(\omega)\Big|_{\omega=\pi} && \text{by hypothesis (C)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Bigg|_{\omega=\pi} && \text{by definition of } \check{x} \text{ (Definition R.1 page 393)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k
\end{aligned}
$$

4. Proof that $(C) \impliedby (D)$:

$$
\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{x}(\omega)\Big|_{\omega=\pi} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Bigg|_{\omega=\pi} && \text{by definition of } \check{x} \text{ (Definition R.1 page 393)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k\\
&= 0 && \text{by hypothesis (D)}
\end{aligned}
$$

✍

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APPENDIX S

PARTITION OF UNITY

S.1 Definition and motivation

A very common property of scaling functions (Definition 2.1 page 18) is the *partition of unity* property (Definition S.1 page 403). The partition of unity is a kind of generalization of *orthonormality*; that is, *all* orthonormal scaling functions form a partition of unity (Theorem 6.1 page 87). But the partition of unity property is not just a consequence of orthonormality, but also a generalization of orthonormality, in that if you remove the orthonormality constraint, the partition of unity is still a reasonable constraint in and of itself.

There are two reasons why the partition of unity property is a reasonable constraint on its own:

- Without a partition of unity, it is difficult to represent a function as simple as a constant.[1]
- For a multiresolution system $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j \right)\!\right), \phi, \left(\!\left(h_n \right)\!\right)\right)$, the partition of unity property is equivalent to $\sum_{n\in\mathbb{Z}}(-1)^n h_n = 0$ (Theorem 4.1 page 54). As viewed from the perspective of discrete time signal processing (APPENDIX Q page 379), this implies that the scaling coefficients form a "*lowpass filter*"; lowpass filters provide a kind of "coarse approximation" of a function. And that is what the scaling function is "supposed" to do—to provide a coarse approximation at some resolution or "scale" (Definition 2.1 page 18).

Definition S.1. [2]

DEF

A function $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ *forms a* ***partition of unity*** *if*

$$\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) = 1 \qquad \forall x \in \mathbb{R}.$$

[1] Jawerth and Sweldens (1994) page 8

[2] Kelley (1955) page 171, Munkres (2000) page 225, Jänich (1984) page 116, Willard (1970) page 152 ⟨item 20C⟩, Willard (2004) page 152 ⟨item 20C⟩

S.2 Results

Theorem S.1. [3] *Let* $\left(L^2_{\mathbb{R}}, \langle\!\langle V_j \rangle\!\rangle, \phi, \langle\!\langle h_n \rangle\!\rangle\right)$ *be a multiresolution system* (Definition 2.3 page 27). *Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* FOURIER TRANSFORM (Definition M.2 page 294) *of a function* $\mathsf{f} \in L^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f} = c}_{\text{PARTITION OF UNITY in "time"}} \quad \Longleftrightarrow \quad \underbrace{\left[\tilde{\mathbf{F}}\mathsf{f}\right](2\pi n) = \bar{\delta}_n}_{\text{PARTITION OF UNITY in "frequency"}}$$

✎PROOF: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
c &= \sum_{m\in\mathbb{Z}} \mathbf{T}^m\mathsf{f}(x) && \text{by left hypothesis} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m) && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 4)} \\
&= \sqrt{2\pi} \sum_{m\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi m)e^{i2\pi mx} && \text{by } PSF \quad \text{(Theorem 1.3 page 13)} \\
&= \underbrace{\sqrt{2\pi}\,\tilde{\mathsf{f}}(2\pi n)e^{i2\pi nx}}_{\text{real and constant for } n=0} + \underbrace{\sqrt{2\pi} \sum_{m\in\mathbb{Z}\setminus n} \tilde{\mathsf{f}}(2\pi m)e^{i2\pi mx}}_{\text{complex and non-constant}} \\
&\Longrightarrow \sqrt{2\pi}\tilde{\mathsf{f}}(2\pi n) = c\bar{\delta}_n && \text{because } c \text{ is real and constant for all } x
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 4)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi n)e^{-i2\pi nx} && \text{by } PSF \quad \text{(Theorem 1.3 page 13)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \frac{c}{\sqrt{2\pi}}\bar{\delta}_n e^{-i2\pi nx} && \text{by right hypothesis} \\
&= \sqrt{2\pi}\,\frac{c}{\sqrt{2\pi}} e^{-i2\pi 0x} && \text{by definition of } \bar{\delta}_n \quad \text{(Definition N.12 page 316)} \\
&= c
\end{aligned}$$

✏

Corollary S.1.

COR

$$\left\{\begin{array}{l} \exists \mathsf{g} \in L^2_{\mathbb{R}} \text{ such that} \\ \mathsf{f}(x) = \mathbb{1}_{[-1:1)}(x) \star \mathsf{g}(x) \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} \mathsf{f}(x) \text{ generates} \\ a \text{ PARTITION OF UNITY} \end{array}\right\}$$

[3] Jawerth and Sweldens (1994) page 8

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✎PROOF:

$$\begin{aligned}
\mathsf{f}(x) = \mathbb{1}_{[0:1)}(x) \star \mathsf{g}(x) &\implies \tilde{\mathsf{f}}(\omega) = \tilde{\mathbf{F}}\left[\mathbb{1}_{[-1:1)}\right](\omega)\tilde{\mathsf{g}}(\omega) && \text{by } \textit{convolution theorem} && \text{(Theorem Q.2 page 380)}\\
&\iff \tilde{\mathsf{f}}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{\sin(\omega)}{\omega}\tilde{\mathsf{g}}(\omega) && \text{by } \textit{rectangular pulse} \text{ ex.} && \text{(Example M.1 page 301)}\\
&\implies \tilde{\mathsf{f}}(2\pi n) = 0\\
&\iff \mathsf{f}(x) \text{ generates a } \textit{partition of unity} && \text{by Theorem S.1 page 404}
\end{aligned}$$

✏

S.3 Examples

Example S.1. All *B-splines* (Definition O.2 page 335) form a partition of unity (Theorem O.4 page 346). All B-splines of order $n = 1$ or greater can be generated by convolution with a *pulse* function, similar to that specified in Corollary S.1 (page 404) and as illustrated below:

(Example O.13 page 353)

(Example O.14 page 353)

(Example O.15 page 353)

Example S.2. Let a function f be defined in terms of the cosine function (Definition I.2 page 235) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1\\ 0 & \text{otherwise}\end{cases}$$

Then f induces a *partition of unity*:

Note that $\tilde{\mathsf{f}}(\omega) = \dfrac{1}{2\sqrt{2\pi}}\left[\underbrace{\dfrac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\dfrac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)} + \underbrace{\dfrac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)}\right]$

and so $\tilde{\mathsf{f}}(2\pi n) = \frac{1}{\sqrt{2\pi}}\bar{\delta}_n$:

✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 3) on a set A.

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1. Proof that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f} = 1$ (time domain proof):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \cos^2(x)\mathbb{1}_{[-1:1]}(x) && \text{by definition of } \mathsf{f}(x)\\
&= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \cos^2(x)\mathbb{1}_{[-1:1)}(x) && \text{because } \cos^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 1\\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n) && \text{by definition of } \mathbf{T} \text{ (Definition 1.3 page 4)}\\
&= \underbrace{\sum_{n\in\mathbb{Z}_o} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n)}_{\text{odd part}} + \underbrace{\sum_{n\in\mathbb{Z}_e} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n)}_{\text{even part}}\\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-2n)\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-2n-1)\right)\mathbb{1}_{[-1:1)}(x-2n-1)\\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x-n\pi\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x-n\pi-\frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-2n-1)\\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n}\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} (-1)^{2n}\cos^2\left(\frac{\pi}{2}x-\frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-2n-1)\\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n-1) && \text{by Theorem I.11 page 248}\\
&= \cos^2\left(\frac{\pi}{2}x\right)\sum_{n\in\mathbb{Z}} \mathbb{1}_{[-1:1)}(x-2n) + \sin^2\left(\frac{\pi}{2}x\right)\sum_{n\in\mathbb{Z}} \mathbb{1}_{[-1:1)}(x-2n-1)\\
&= \cos^2\left(\frac{\pi}{2}x\right)\cdot 1 + \sin^2\left(\frac{\pi}{2}x\right)\cdot 1\\
&= 1 && \text{by } \textit{square identity} \text{ (Theorem I.11 page 248)}
\end{aligned}$$

2. Proof that $\tilde{\mathsf{f}}(\omega) = \cdots$: by Example M.3 page 302

✏

Example S.3. Let a function f be defined in terms of the sine function (Definition I.3 page 235) as follows:

CNT

$$\mathsf{f}(x) \triangleq \begin{cases} \sin^2\left(\frac{\pi}{2}x\right) & \text{for } x \in [0:2]\\ 0 & \text{otherwise}\end{cases}$$

Then $\int_{\mathbb{R}} \mathsf{f}(x)\,\mathrm{d}x = 1$ and f induces a *partition of unity*

✎PROOF:

1. Proof that $\int_{\mathbb{R}} \mathsf{f}(x)\,\mathrm{d}x = 1$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{f}(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x)\,\mathrm{d}x && \text{by definition of } \mathsf{f}(x)\\
&= \int_0^2 \sin^2\left(\frac{\pi}{2}x\right)\mathrm{d}x && \text{by definition of } \mathbb{1}_{A(x)} \text{ (Definition 1.2 page 3)}\\
&= \int_0^2 \frac{1}{2}[1-\cos(\pi x)]\,\mathrm{d}x && \text{by Theorem I.11 page 248}\\
&= \frac{1}{2}\left[x - \frac{1}{\pi}\sin(\pi x)\right]_0^2
\end{aligned}$$

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$$= \frac{1}{2}[2-0-0-0]$$
$$= 1$$

2. Proof that $\mathsf{f}(x)$ forms a *partition of unity*:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x) && \text{by definition of } \mathsf{f}(x)\\
&= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{because } \sin^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 2\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^{m-1} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{where } m \triangleq n+1 \implies n = m-1\\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m+1)\right)\mathbb{1}_{[0:2)}(x-m+1) && \text{by definition of } \mathbf{T} \text{ (Definition 1.3 page 4)}\\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m)+\frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-m)\\
&= \sum_{m\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-m)\right)\mathbb{1}_{[-1:1)}(x-m) && \text{by Theorem I.11 page 248}\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x) && \text{by definition of } \mathbf{T} \text{ (Definition 1.3 page 4)}\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x) && \text{because } \cos^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 1\\
&= 1 && \text{by Example S.2 page 405}
\end{aligned}$$

✏

Example S.4 (raised cosine). [4] Let a function f be defined in terms of the cosine function (Definition I.2 page 235) as follows:

EX

$$\text{Let } \mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } 0 \le |x| < \frac{1-\beta}{2}\\ \frac{1}{2}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x|-\frac{1-\beta}{2}\right)\right]\right\} & \text{for } \frac{1-\beta}{2} \le |x| < \frac{1+\beta}{2}\\ 0 & \text{otherwise}\end{cases}$$

Then f induces a *partition of unity*:

✎PROOF:

[4] Proakis (2001) pages 560–561

1. definition: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 3) on a set A.
   Let $A \triangleq \left[\frac{1+\beta}{-2} : \frac{1-\beta}{-2}\right)$, $\quad B \triangleq \left[\frac{1-\beta}{-2} : \frac{1-\beta}{2}\right)$, and $\quad C \triangleq \left[\frac{1-\beta}{2} : \frac{1+\beta}{2}\right)$

2. lemma: $\mathbb{1}_A(x-1) = \mathbb{1}_C(x)$. Proof:

$$\begin{aligned}
\mathbb{1}_A(x-1) &\triangleq \begin{cases} 1 & \text{if } -\frac{1+\beta}{2} \le x-1 < -\frac{1-\beta}{2} \\ 0 & \text{otherwise} \end{cases} && \text{by definition of } \mathbb{1} \text{ (Definition 1.2 page 3) and } A \text{ ((2) lemma page 408)} \\
&= \begin{cases} 1 & \text{if } 1-\frac{1+\beta}{2} \le x < 1-\frac{1-\beta}{2} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \frac{1-\beta}{2} \le x < \frac{1+\beta}{2} \\ 0 & \text{otherwise} \end{cases} \\
&\triangleq \mathbb{1}_C(x) && \text{by definition of } \mathbb{1} \text{ (Definition 1.2 page 3) and } C \text{ ((2) lemma page 408)}
\end{aligned}$$

3. lemma: $-1 + \dfrac{1-\beta}{2} = -\beta - \dfrac{1-\beta}{2}$. Proof:

$$-1 + \frac{1-\beta}{2} = \frac{-2+1-\beta}{2} \quad = \frac{-1-\beta}{2} = (-\beta+\beta) - \left(\frac{1+\beta}{2}\right) \quad = -\beta + \frac{2\beta-1-\beta}{2} = -\beta - \frac{1-\beta}{2}$$

4. Proof that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f} = 1$:

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) && \text{by Definition 1.3} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_A(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by definition 1 page 408} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) \\
&\quad + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n-1)\mathbb{1}_A(x-n-1) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by Proposition 1.1} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n-1)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by (2) lemma page 408} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n| - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n-1| - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathbb{1}_B(x-n) && \text{by definition of } \mathsf{f}(x) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left((x-n) - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(-(x-n-1) - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathbb{1}_B(x-n) && \text{by def. of } \mathbb{1}_C(x) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-1 + \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathbb{1}_B(x-n) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\beta - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathbb{1}_B(x-n) && \text{by (3) lemma page 408} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)
\end{aligned}$$

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$$
\begin{aligned}
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}-\frac{\pi\beta}{\beta}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n)\\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)\\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1-\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n)\\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\mathbb{1}_C(x-n)+\frac{1}{2}\sum_{n\in\mathbb{Z}}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n)\\
&= \sum_{n\in\mathbb{Z}}\mathbb{1}_{B\cup C}(x-n)\\
&= 1
\end{aligned}
$$

✍

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APPENDIX T

POWER SPECTRUM FUNCTIONS

T.1 Correlation

Definition T.1 and Definition T.2 define four quantities. In this document, the quantities' notation and terminology are similar to those used in the study of *random processes*.

Definition T.1. [1] *Let* $\langle \triangle \mid \triangledown \rangle$ *be the* STANDARD INNER PRODUCT *in* $\boldsymbol{L}^2_{\mathbb{R}}$ *(Definition E.1 page 199).*

DEF	$\mathsf{R}_{\mathsf{fg}}(n)$	$\triangleq$	$\langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{g}(x) \rangle$,	$n \in \mathbb{Z}$; $\mathsf{f}, \mathsf{g} \in \boldsymbol{L}^2_{\mathbb{F}}$,	*is the* ***cross-correlation function*** *of* f *and* g.
	$\mathsf{R}_{\mathsf{ff}}(n)$	$\triangleq$	$\langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{f}(x) \rangle$,	$n \in \mathbb{Z}$; $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{F}}$,	*is the* ***autocorrelation function*** *of* f.

Definition T.2. [2] *Let* $\mathsf{R}_{\mathsf{fg}}(n)$ *and* $\mathsf{R}_{\mathsf{ff}}(n)$ *be the sequences defined in Definition T.1 page 411. Let* $\mathbf{Z}\,(\!(x_n)\!)$ *be the* Z-TRANSFORM *(Definition Q.1 page 379) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$.

DEF	$\check{\mathsf{S}}_{\mathsf{fg}}(z)$	$\triangleq$	$\mathbf{Z}\left[\mathsf{R}_{\mathsf{fg}}(n)\right]$,	$\mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}$,	*is the* ***complex cross-power spectrum*** *of* f *and* g.
	$\check{\mathsf{S}}_{\mathsf{ff}}(z)$	$\triangleq$	$\mathbf{Z}\left[\mathsf{R}_{\mathsf{ff}}(n)\right]$,	$\mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}$,	*is the* ***complex auto-power spectrum*** *of* f.

T.2 Power Spectrum

Definition T.3. [3] *Let* $\check{\mathsf{S}}_{\mathsf{fg}}(z)$ *and* $\check{\mathsf{S}}_{\mathsf{ff}}(z)$ *be the functions defined in Definition T.2 page 411.*

DEF	$\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$	$\triangleq$	$\check{\mathsf{S}}_{\mathsf{fg}}\left(e^{i\omega}\right)$,	$\forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}$,	*is the* ***cross-power spectrum*** *of* f *and* g.
	$\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$	$\triangleq$	$\check{\mathsf{S}}_{\mathsf{ff}}\left(e^{i\omega}\right)$,	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{F}}$,	*is the* ***auto-power spectrum*** *of* f.

Theorem T.1. [4] *Let* $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ *and* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be defined as in Definition T.3 (page 411).*
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition M.2 page 294) of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{F}}$.

THM	$\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$	$=$	$2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n)$	$\forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}$
	$\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$	$=$	$2\pi \sum_{n\in\mathbb{Z}} \left\lvert\tilde{\mathsf{f}}(\omega+2\pi n)\right\rvert^2$	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{F}}$

[1] Chui (1992) page 134, Papoulis (1991) pages 294–332 ⟨(10-29), (10-169)⟩
[2] Chui (1992) page 134, Papoulis (1991) page 334 ⟨(10-178)⟩
[3] Chui (1992) page 134, Papoulis (1991) page 333 ⟨(10-179)⟩
[4] Chui (1992) page 135

✎PROOF: Let $z \triangleq e^{i\omega}$.

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) &\triangleq \check{\mathsf{S}}_{\mathsf{fg}}(z) && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition T.3 page 411)}\\
&= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n) z^{-n} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition T.2 page 411)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\, z^{-n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition T.3 page 411)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathbf{F}}[\mathsf{f}(x)] \,|\, \tilde{\mathbf{F}}\big[\mathsf{g}(x-n)\big] \right\rangle z^{-n} && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem M.3 page 295)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathsf{f}}(v) \,|\, e^{-ivn}\tilde{\mathsf{g}}(v) \right\rangle z^{-n} && \text{by } \textit{shift relation} && \text{(Theorem M.4 page 296)}\\
&= \sum_{n\in\mathbb{Z}} \sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v)e^{ivu}\,\mathrm{d}v\right]_{u=n} z^{-n} && \text{by definition of } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 199)}\\
&= \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \left[\tilde{\mathbf{F}}^{-1}\left(\sqrt{2\pi}\tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v)\right)\right]_{u=n} e^{-i\omega n} && \text{by Theorem M.1 page 295}\\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n) && \text{by } \textit{IPSF} \text{ with } \tau = 1 && \text{(Theorem 1.4 page 14)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)\big|_{\mathsf{g}=\mathsf{f}} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) && \text{(Definition T.3 page 411)}\\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n)\Bigg|_{\mathsf{g}\triangleq\mathsf{f}} && \text{by previous result}\\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{f}}^*(\omega+2\pi n)\\
&= 2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{because } |z|^2 \triangleq zz^* \quad \forall z\in\mathbb{C}
\end{aligned}$$

✏

Proposition T.1. *Let $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition T.3 (page 411).*

PRP

$$\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) \geq 0 \quad \text{(NON-NEGATIVE)}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= 2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem T.1 page 411}\\
&\geq 0 && \text{because } |z| \geq 0 \quad \forall z\in\mathbb{C}
\end{aligned}$$

✏

Proposition T.2. *Let $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ and $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition T.3 (page 411).*

PRP

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) \quad \text{(PERIODIC } \textit{with period } 2\pi\text{)}\\
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) \quad \text{(PERIODIC } \textit{with period } 2\pi\text{)}
\end{aligned}$$

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✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(\omega+2\pi) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\omega+2\pi+2\pi n)\tilde{g}^*(\omega+2\pi+2\pi n) && \text{by Theorem T.1 page 411}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}[\omega+2\pi(n+1)]\tilde{g}^*[\omega+2\pi(n+1)]\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{f}[\omega+2\pi m]\tilde{g}^*[\omega+2\pi m] && \text{where } m\triangleq n+1\\
&= \tilde{S}_{fg}(\omega) && \text{by Theorem T.1 page 411}\\
\tilde{S}_{ff}(\omega+2\pi) &= \tilde{S}_{fg}(\omega+2\pi)\big|_{g=f}\\
&= \tilde{S}_{fg}(\omega)\big|_{g=f} && \text{by previous result}\\
&= \tilde{S}_{ff}(\omega)
\end{aligned}$$

✏

Proposition T.3. *Let $\tilde{S}_{fg}(\omega)$ and $\tilde{S}_{ff}(\omega)$ be defined as in Definition T.3 (page 411).*

PRP						
	f,g *are real*	$\Longrightarrow$	$\tilde{S}_{fg}(-\omega)$	$=$	$\tilde{S}_{gf}(\omega)$	
	f *is real*	$\Longrightarrow$	$\tilde{S}_{ff}(-\omega)$	$=$	$\tilde{S}_{ff}(\omega)$	(SYMMETRIC *about* 0)
	f,g *are real*	$\Longrightarrow$	$\tilde{S}_{fg}(\pi-\omega)$	$=$	$\tilde{S}_{gf}(\pi+\omega)$	
	f *is real*	$\Longrightarrow$	$\tilde{S}_{ff}(\pi-\omega)$	$=$	$\tilde{S}_{ff}(\pi+\omega)$	(SYMMETRIC *about* π)

✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(-\omega+2\pi n)\tilde{g}^*(-\omega+2\pi n) && \text{by Theorem T.1 page 411}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\omega-2\pi n)\tilde{g}(\omega-2\pi n) && \text{by hypothesis and Theorem M.5 page 296}\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\omega+2\pi m)\tilde{f}^*(\omega+2\pi m) && \text{where } m\triangleq -n\\
&= \tilde{S}_{gf}(\omega) && \text{by Theorem T.1 page 411}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{fg}(\pi-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\pi-\omega+2\pi n)\tilde{g}^*(\pi-\omega+2\pi n) && \text{by Theorem T.1 page 411}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(-\pi+\omega-2\pi n)\tilde{g}(-\pi+\omega-2\pi n) && \text{by hypothesis and Theorem M.5 page 296}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega-2\pi-2\pi n)\tilde{g}(\pi+\omega-2\pi-2\pi n)\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega+2\pi(-n-1))\tilde{g}(\pi+\omega+2\pi(-n-1))\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\pi+\omega+2\pi m)\tilde{f}^*(\pi+\omega+2\pi m) && \text{where } m\triangleq -n-1\\
&= \tilde{S}_{gf}(\pi+\omega) && \text{by Theorem T.1 page 411}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{ff}(-\omega) &= \tilde{S}_{fg}(-\omega)\big|_{g\triangleq f}\\
&= \tilde{S}_{gf}(+\omega)\big|_{g\triangleq f} && \text{by previous result}\\
&= \tilde{S}_{ff}(+\omega) && \text{by definition of } g\ (g\triangleq f)
\end{aligned}$$

$$\tilde{S}_{ff}(\pi-\omega) = \tilde{S}_{fg}(\pi-\omega)\big|_{g\triangleq f}$$

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$$= \tilde{\mathsf{S}}_{\mathsf{gf}}(\pi+\omega)\big|_{\mathsf{g}\triangleq\mathsf{f}} \qquad \text{by previous result}$$
$$= \tilde{\mathsf{S}}_{\mathsf{ff}}(\pi+\omega) \qquad \text{by definition of } \mathsf{g}\ (\mathsf{g}\triangleq\mathsf{f})$$

✏

Proposition T.4. *Let* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be the* AUTO-POWER SPECTRUM *(Definition T.3 page 411) of a function* $\mathsf{f}(x)\in L^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{S}}'_{\mathsf{ff}}(\omega)\triangleq\frac{\mathsf{d}}{\mathsf{d}\omega}\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *(Definition E.2 page 199).*

PRP

$$\left\{\begin{array}{ll}(a). & \mathsf{f}\ \textit{is}\ \text{REAL}\quad \textit{and}\\ (b). & \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)\ \textit{is}\ \text{CONTINUOUS}\ \textit{at}\ \omega=0\end{array}\right\}\implies\left\{\begin{array}{ll}(1). & \tilde{\mathsf{S}}'_{\mathsf{ff}}(0)=0\quad \textit{and}\\ (2). & \underbrace{\tilde{\mathsf{S}}'_{\mathsf{ff}}(\omega)=-\tilde{\mathsf{S}}'_{\mathsf{ff}}(-\omega)\quad\forall\omega\in\mathbb{R}}_{\text{ANTI-SYMMETRIC}\ \textit{about}\ 0}\end{array}\right\}$$

$$\left\{\begin{array}{ll}(c). & \mathsf{f}\ \textit{is}\ \text{REAL}\quad \textit{and}\\ (d). & \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)\ \textit{is}\ \text{CONTINUOUS}\ \textit{at}\ \omega=\pi\end{array}\right\}\implies\left\{\begin{array}{ll}(3). & \tilde{\mathsf{S}}'_{\mathsf{ff}}(\pi)=0\quad \textit{and}\\ (4). & \underbrace{\tilde{\mathsf{S}}'_{\mathsf{ff}}(\pi+\omega)=-\tilde{\mathsf{S}}'_{\mathsf{ff}}(\pi-\omega)\quad\forall\omega\in\mathbb{R}}_{\text{ANTI-SYMMETRIC}\ \textit{about}\ \pi}\end{array}\right\}$$

✎PROOF: This follows from Proposition T.3 (page 413) and Proposition E.1 (page 199). ✏

Theorem T.2 (next) is a major result and provides strong motivation for bothering with *power spectrum* functions in the first place. In particular, the *auto-power spectrum* being *bounded* provides a necessary and sufficient condition for a sequence of functions $(\!(\phi(x-n))\!)_{n\in\mathbb{Z}}$ to be a *Riesz basis* (Definition N.13 page 323) for the *span* $\operatorname{span}(\!(\phi(x-n))\!)$ of the sequence.

Theorem T.2. [5] *Let* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be defined as in Definition T.3 (page 411). Let* $\|\cdot\|$ *be defined as in Definition E.1 (page 199). Let* $0<A<B$.

THM

$$\underbrace{\left\{A\sum_{n\in\mathbb{N}}|a_n|^2\le\left\|\sum_{n\in\mathbb{Z}}a_n\phi(x-n)\right\|^2\le B\sum_{n\in\mathbb{N}}|\alpha_n|^2\quad\forall(\!(a_n)\!)\in\ell^2_{\mathbb{F}}\right\}}_{(\!(\phi(x-n))\!)\ \textit{is a}\ \text{RIESZ BASIS}\ \textit{for}\ \operatorname{span}(\!(\phi(x-n))\!)\ \textit{(Theorem N.13 page 324)}}\quad\Longleftrightarrow\quad\left\{A\le\tilde{\mathsf{S}}_{\phi\phi}(\omega)\le B\right\}$$

✎PROOF:

1. lemma:

$$\left\|\sum_{n\in\mathbb{Z}}a_n\phi(x-n)\right\|^2=\left\|\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}}a_n\phi(x-n)\right\|^2 \qquad \text{because } \tilde{\mathbf{F}} \text{ is } \textit{unitary} \text{ (Theorem M.2 page 295)}$$
$$=\left\|\breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega)\right\|^2 \qquad \text{by Proposition 1.13 page 12}$$
$$=\int_{\mathbb{R}}\left|\breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega)\right|^2\,\mathsf{d}\omega \qquad \text{by definition of } \|\cdot\|$$
$$=\sum_{n\in\mathbb{Z}}\int_0^{2\pi}\left|\breve{\mathsf{a}}(\omega+2\pi n)\tilde{\phi}(\omega+2\pi n)\right|^2\,\mathsf{d}\omega$$
$$=\int_0^{2\pi}\sum_{n\in\mathbb{Z}}|\breve{\mathsf{a}}(\omega+2\pi n)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2\,\mathsf{d}\omega$$
$$=\int_0^{2\pi}\sum_{n\in\mathbb{Z}}|\breve{\mathsf{a}}(\omega)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2\,\mathsf{d}\omega \qquad \text{by Proposition R.1 page 393}$$

[5] Wojtaszczyk (1997) pages 22–23 ⟨Proposition 2.8⟩, Igari (1996) page 219 ⟨Lemma 9.6⟩, Pinsky (2002) page 306 ⟨Theorem 6.4.8⟩

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$$= \int_0^{2\pi} |\breve{a}(\omega)|^2 \frac{1}{2\pi} 2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega + 2\pi n)\right|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \qquad \text{by definition of } \tilde{S}_{\phi\phi}(\omega) \text{ (Theorem T.1 page 411)}$$

2. lemma:

$$\frac{1}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \right|^2 d\omega \qquad \text{by def. of } DTFT \text{ (Definition R.1 page 393)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \right] \left[\sum_{m\in\mathbb{Z}} a_m e^{-i\omega m} \right]^* d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \right] \left[\sum_{m\in\mathbb{Z}} a_m^* e^{i\omega m} \right] d\omega$$

$$= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_n a_m^* \int_0^{2\pi} e^{-i\omega(n-m)} \, d\omega$$

$$= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_n a_m^* 2\pi \bar{\delta}_{nm}$$

$$= \sum_{n\in\mathbb{Z}} |a_n|^2 \qquad \text{by definition of } \bar{\delta} \text{ (Definition N.12 page 316)}$$

3. Proof for ($\Longleftarrow$) case:

$$\boxed{A \sum_{n\in\mathbb{Z}} |a_n|^2} = \frac{A}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 \, d\omega \qquad \text{by (2) lemma page 415}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 A \, d\omega$$

$$\boxed{\le} \frac{1}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \qquad \text{by right hypothesis}$$

$$= \boxed{\left\| \sum_{n\in\mathbb{Z}} a_n \phi(x-n) \right\|^2} \qquad \text{by (1) lemma page 414}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \qquad \text{by (1) lemma page 414}$$

$$\boxed{\le} \frac{1}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 B \, d\omega \qquad \text{by right hypothesis}$$

$$= \frac{B}{2\pi} \int_0^{2\pi} |\breve{a}(\omega)|^2 \, d\omega$$

$$= \boxed{B \sum_{n\in\mathbb{Z}} |a_n|^2} \qquad \text{by (2) lemma page 415}$$

4. Proof for ($\Longrightarrow$) case:

   (a) Let $Y \triangleq \left\{\omega \in [0 : 2\pi] | \tilde{S}_{\phi\phi}(\omega) > \alpha\right\}$
   and $X \triangleq \left\{\omega \in [0 : 2\pi] | \tilde{S}_{\phi\phi}(\omega) < \alpha\right\}$

   (b) Let $\mathbb{1}_{A(x)}$ be the *set indicator* (Definition 1.2 page 3) of a set A.
   Let $(\!(b_n)\!)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem R.3 page 399) of $\mathbb{1}_Y(\omega)$ such that
   $\mathbb{1}_Y(\omega) \triangleq \sum_{n\in\mathbb{N}} b_n e^{-i\omega n} \triangleq \tilde{b}(\omega)$.

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Let $(\!(a_n)\!)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem R.3 page 399) of $\mathbb{1}_X(\omega)$ such that
$\mathbb{1}_X(\omega) \triangleq \sum_{n\in\mathbb{N}} a_n e^{-i\omega n} \triangleq \breve{\mathsf{a}}(\omega)$.

(c) Proof that $\alpha \le B$:
Let $\mu(A)$ be the *measure* of a set A.

$$\begin{aligned}
\boxed{B}\sum_{n\in\mathbb{Z}} |b_n|^2 &\ge \left\|\sum_{n\in\mathbb{Z}} b_n\phi(x-n)\right\|^2 && \text{by left hypothesis} \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\tilde{\mathsf{b}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by (1) lemma page 414} \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\mathbb{1}_Y(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) \quad \text{(item (4b) page 415)} \\
&= \frac{1}{2\pi}\int_Y |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) \quad \text{(item (4b) page 415)} \\
&\boxed{\ge} \frac{\alpha}{2\pi}\mu(Y) && \text{by definition of } Y \quad \text{(item (4a) page 415)} \\
&= \int_0^{2\pi} |\mathbb{1}_Y(\omega)|^2\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) \quad \text{(item (4b) page 415)} \\
&= \int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} b_n e^{-i\omega n}\right|^2 \mathrm{d}\omega && \text{by definition of } (\!(b_n)\!) \quad \text{(item (4b) page 415)} \\
&= \int_0^{2\pi} |\tilde{\mathsf{b}}(\omega)|^2\,\mathrm{d}\omega && \text{by definition of } \tilde{\mathsf{b}}(\omega) \quad \text{(item (4b) page 415)} \\
&= \boxed{\alpha}\sum_{n\in\mathbb{Z}} |b_n|^2 && \text{by (2) lemma page 415}
\end{aligned}$$

(d) Proof that $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$:
- (i). $\tilde{\mathsf{S}}_{\phi\phi}(\omega) > \alpha$ whenever $\omega \in Y$ (item (4a) page 415).
- (ii). But even then, $\alpha \le B$ (item (4c) page 416).
- (iii). So, $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$.

(e) Proof that $A \le \alpha$:
Let $\mu(A)$ be the *measure* of a set A.

$$\begin{aligned}
\boxed{A}\sum_{n\in\mathbb{Z}} |a_n|^2 &\le \left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 && \text{by left hypothesis} \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by (1) lemma page 414} \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\mathbb{1}_X(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) \quad \text{(Definition 1.2 page 3)} \\
&= \frac{1}{2\pi}\int_X |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) \quad \text{(Definition 1.2 page 3)} \\
&\boxed{\le} \frac{\alpha}{2\pi}\mu(X) && \text{by definition of } X \quad \text{(item (4a) page 415)} \\
&= \int_0^{2\pi} |\mathbb{1}_X(\omega)|^2\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) \quad \text{(Definition 1.2 page 3)} \\
&= \int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right|^2 \mathrm{d}\omega && \text{by definition of } (\!(a_n)\!) \quad \text{((2) lemma page 415)} \\
&= \int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2\,\mathrm{d}\omega && \text{by definition of } \breve{\mathsf{a}}(\omega) \quad \text{((2) lemma page 415)}
\end{aligned}$$

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$$= \boxed{\alpha} \sum_{n\in\mathbb{Z}} |a_n|^2 \qquad \text{by (2) lemma page 415}$$

(f) Proof that $A \le \tilde{\mathsf{S}}_{\phi\phi}(\omega)$:
  (i). $\tilde{\mathsf{S}}_{\phi\phi}(\omega) < \alpha$ whenever $\omega \in X$ (item (4a) page 415).
  (ii). But even then, $A \le \alpha$ (item (4e) page 416).
  (iii). So, $A \le \tilde{\mathsf{S}}_{\phi\phi}(\omega)$.

✏

In the case that f and g are *orthonormal*, the spectral density relations simplify considerably (next).

Theorem T.3. [6] *Let* $\tilde{\mathsf{S}}_{\mathsf{ff}}$ *and* $\tilde{\mathsf{S}}_{\mathsf{fg}}$ *be the* SPECTRAL DENSITY FUNCTIONS *(Definition T.3 page 411).*

THM								
	$\langle \mathsf{f}(x) \mid \mathsf{f}(x-n)\rangle$	$=$	$\bar{\delta}_n$	$(\!(\mathsf{f}(x-n))\!)$ *is* ORTHONORMAL)	$\iff$	$\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$	$=$	1 $\quad \forall \mathsf{f}\in L^2_{\mathbb{F}}$
	$\langle \mathsf{f}(x) \mid \mathsf{g}(x-n)\rangle$	$=$	0	$(\!(\mathsf{f}(x)$ *is* ORTHOGONAL *to* $(\!(\mathsf{g}(x-n))\!))$	$\iff$	$\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$	$=$	0 $\quad \forall \mathsf{f},\mathsf{g}\in L^2_{\mathbb{F}}$

✎PROOF:

1. Proof that $\langle \mathsf{f}(x) \mid \mathsf{f}(x-n)\rangle = \bar{\delta}_n \iff \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) = 1$: This follows directly from Theorem T.2 (page 414) with $A = B = 1$ (by *Parseval's Identity* Theorem N.9 page 318 since $\{\mathbf{T}^n\mathsf{f}\}$ is *orthonormal*)

2. Alternate proof that $\langle \mathsf{f}(x) \mid \mathsf{f}(x-n)\rangle = \bar{\delta}_n \implies \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) = 1$:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{ff}}(n)e^{-i\omega n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition T.3 page 411)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \mid \mathsf{f}(x-n)\rangle\, e^{-i\omega n} && \text{by definition of } \mathsf{R}_{\mathsf{ff}} && \text{(Definition T.1 page 411)}\\
&= \sum_{n\in\mathbb{Z}} \bar{\delta}_n e^{-i\omega n} && \text{by left hypothesis}\\
&= 1 && \text{by definition of } \bar{\delta} && \text{(Definition N.12 page 316)}
\end{aligned}$$

3. Alternate proof that $\langle \mathsf{f}(x) \mid \mathsf{f}(x-n)\rangle = \bar{\delta}_n \impliedby \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) = 1$:

$$\begin{aligned}
&\langle \mathsf{f}(x) \mid \mathsf{f}(x-n)\rangle\\
&= \left\langle \tilde{\mathbf{F}}\mathsf{f}(x) \mid \tilde{\mathbf{F}}\mathsf{f}(x-n)\right\rangle && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem M.3 page 295)}\\
&= \left\langle \tilde{\mathsf{f}}(\omega) \mid e^{-i\omega n}\tilde{\mathsf{f}}(\omega)\right\rangle && \text{by } \textit{shift property} \text{ of } \tilde{\mathbf{F}} && \text{(Theorem M.4 page 296)}\\
&= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega n}\tilde{\mathsf{f}}^*(\omega)\,\mathrm{d}\omega && \text{by definition of } \langle \triangle \mid \triangledown\rangle && \text{(Definition E.1 page 199)}\\
&= \int_{\mathbb{R}} \left|\tilde{\mathsf{f}}(\omega)\right|^2 e^{i\omega n}\,\mathrm{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \left|\tilde{\mathsf{f}}(\omega)\right|^2 e^{i\omega n}\,\mathrm{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{0}^{2\pi} \left|\tilde{\mathsf{f}}(u+2\pi n)\right|^2 e^{i(u+2\pi n)n}\,\mathrm{d}u && \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n\\
&= \frac{1}{2\pi}\int_{0}^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(u+2\pi n)\right|^2\right] e^{iun}\cancelto{1}{e^{i2\pi nn}}\,\mathrm{d}u\\
&= \frac{1}{2\pi}\int_{0}^{2\pi} \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)e^{iun}\,\mathrm{d}u && \text{by Theorem T.1 page 411}
\end{aligned}$$

[6] Hernández and Weiss (1996) page 50 ⟨PROPOSITION 2.1.11⟩, Wojtaszczyk (1997) PAGE 23 ⟨COROLLARY 2.9⟩, Igari (1996) PAGES 214–215 ⟨LEMMA 9.2⟩, Pinsky (2002) PAGE 306 ⟨COROLLARY 6.4.9⟩

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$$= \frac{1}{2\pi}\int_0^{2\pi} e^{iun}\,\mathrm{d}u \qquad \text{by right hypothesis}$$
$$= \bar{\delta}_n \qquad \text{by definition of } \bar{\delta} \qquad \text{(Definition N.12 page 316)}$$

4. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle = 0 \implies \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

$$\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n)e^{-i\omega n} \qquad \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} \qquad \text{(Definition T.3 page 411)}$$
$$= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\, e^{-i\omega n} \qquad \text{by definition of } \mathsf{R}_{\mathsf{fg}} \qquad \text{(Definition T.1 page 411)}$$
$$= \sum_{n\in\mathbb{Z}} 0e^{-i\omega n} \qquad \text{by left hypothesis}$$
$$= 0$$

5. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle = 0 \impliedby \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

$$\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle$$
$$= \left\langle \tilde{\mathbf{F}}\mathsf{f}(x) \,|\, \tilde{\mathbf{F}}\mathsf{g}(x-n)\right\rangle \qquad \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} \qquad \text{(Theorem M.3 page 295)}$$
$$= \left\langle \tilde{\mathsf{f}}(\omega) \,|\, e^{-i\omega n}\tilde{\mathsf{g}}(\omega)\right\rangle \qquad \text{by } \textit{shift property} \text{ of } \tilde{\mathbf{F}} \qquad \text{(Theorem M.4 page 296)}$$
$$= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega n}\tilde{\mathsf{g}}^*(\omega)\,\mathrm{d}\omega \qquad \text{by definition of } \langle \triangle \,|\, \triangledown\rangle \qquad \text{(Definition E.1 page 199)}$$
$$= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)\tilde{\mathsf{g}}^*(\omega)e^{i\omega n}\,\mathrm{d}\omega$$
$$= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \tilde{\mathsf{f}}(\omega)\tilde{\mathsf{g}}^*(\omega)e^{i\omega n}\,\mathrm{d}\omega$$
$$= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \tilde{\mathsf{f}}(u+2\pi n)\tilde{\mathsf{g}}^*(u+2\pi n)e^{i(u+2\pi n)n}\,\mathrm{d}u \qquad \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(u+2\pi n)\tilde{\mathsf{g}}^*(u+2\pi n)\right] e^{iun}\cancelto{1}{e^{i2\pi nn}}\,\mathrm{d}u$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \tilde{\mathsf{S}}_{\mathsf{fg}}(u)e^{iun}\,\mathrm{d}u \qquad \text{by Theorem T.1 page 411}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} 0\cdot e^{iun}\,\mathrm{d}u \qquad \text{by right hypothesis}$$
$$= 0$$

✍

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APPENDIX U

POLLEN PARAMETERIZED WAVELETS

David Pollen showed that there are an uncountably infinite number of compactly supported scaling functions (and wavelets) available for coefficient length 4 and greater. This is stated in Theorem U.1, demonstrated in Example U.1 (page 419) and illustrated in Figure U.1 (page 420).

Theorem U.1 (Pollen parameterization theorem). [1]

THM

$$\check{\mathsf{h}}(z) = \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix} E(z^2)\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}$$

where

$$E(z) = \begin{cases} U_1 U_2 \cdots U_{2k} & \textit{for } h_n \textit{ of length } 4k+2,\ k \in \mathbb{W} \\ U_1 U_2 \cdots U_{2k+1} & \textit{for } h_n \textit{ of length } 4k+4,\ k \in \mathbb{W} \end{cases}$$

$$U(z) = \begin{bmatrix} \mathsf{u}(z) & \mathsf{v}(z) \\ -\mathsf{v}(z^{-1}) & \mathsf{u}(z^{-1}) \end{bmatrix}$$

$$\mathsf{u}(z) = \frac{1}{2}\left[(1-\cos\theta)z + (1+\cos\theta)\right]$$

$$\mathsf{v}(z) = \frac{1}{2}\left[(-\sin\theta) + (\sin\theta)z^{-1}\right]$$

Example U.1 (Pollen length-4 scaling coefficients). [2]

EX

n	h_n
0	$\frac{\sqrt{2}}{4}(1-\cos\theta+\sin\theta)$
1	$\frac{\sqrt{2}}{4}(1+\cos\theta+\sin\theta)$
2	$\frac{\sqrt{2}}{4}(1+\cos\theta-\sin\theta)$
3	$\frac{\sqrt{2}}{4}(1-\cos\theta-\sin\theta)$

[1] Klappenecker (1996) page 3

[2] Klappenecker (1996) page 3, Burrus et al. (1998) page 66

✎PROOF:

$$
\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix} E(z^2)\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix} U(z^2)\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}\mathsf{u}(z^2) & \mathsf{v}(z^2)\\ -\mathsf{v}(z^{-2}) & \mathsf{u}(z^{-2})\end{bmatrix}\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\left[(1-\cos\theta)z^2+(1+\cos\theta)\right] & \frac{1}{2}\left[(-\sin\theta)+(\sin\theta)z^{-2}\right]\\ -\frac{1}{2}\left[(-\sin\theta)+(\sin\theta)z^{2}\right] & \frac{1}{2}\left[(1-\cos\theta)z^{-2}+(1+\cos\theta)\right]\end{bmatrix}\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{4}\begin{bmatrix}-\left[(1-\cos\theta)z^2+(1+\cos\theta)\right]+\left[(-\sin\theta)+(\sin\theta)z^{2}\right]\\ -\left[(-\sin\theta)+(\sin\theta)z^{-2}\right]+\left[(1-\cos\theta)z^{-2}+(1+\cos\theta)\right]\end{bmatrix}^T\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{4}\Big[(1-\cos\theta)z+(1+\cos\theta)z^{-1}-(\sin\theta)z^{-1}+(\sin\theta)z\\
&\qquad +\sin\theta-(\sin\theta)z^{-2}+(1-\cos\theta)z^{-2}+(1+\cos\theta)\Big]\\
&= \frac{\sqrt{2}}{4}\Big[(1-\cos\theta+\sin\theta)z+(1+\cos\theta+\sin\theta)\\
&\qquad +(1+\cos\theta-\sin\theta)z^{-1}+(1-\cos\theta-\sin\theta)z^{-2}\Big]
\end{aligned}
$$

✏

Figure U.1: Pollen length-4 wavelet function with varying parameter θ

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Example U.2 (Pollen length-6 scaling coefficients). [3] Length-6 scaling coefficients have two parameters, here designated α and β. The scaling coefficients are as follows.

EX

n	h_n
0	$\frac{1}{4\sqrt{2}}[(1+\cos(\alpha)+\sin(\alpha))(1-\cos(\beta)-\sin(\beta))+2\sin(\beta)\cos(\alpha)]$
1	$\frac{1}{4\sqrt{2}}[(1-\cos(\alpha)+\sin(\alpha))(1+\cos(\beta)-\sin(\beta))-2\sin(\beta)\cos(\alpha)]$
2	$\frac{1}{2\sqrt{2}}[(1+\cos(\alpha-\beta)+\sin(\alpha-\beta)]$
3	$\frac{1}{2\sqrt{2}}[(1+\cos(\alpha-\beta)-\sin(\alpha-\beta)]$
4	$\frac{1}{\sqrt{2}}-h(0)-h(2)$
5	$\frac{1}{\sqrt{2}}-h(1)-h(3)$

[3] Burrus et al. (1998) page 66, Vidakovic (1999) page 95

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APPENDIX V

FAST WAVELET TRANSFORM (FWT)

The Fast Wavelet Transform can be computed using simple discrete filter operations (as a conjugate mirror filter).

Definition V.1 (Wavelet Transform)**.** *Let the wavelet transform* $\mathbf{W} : \{f : \mathbb{R} \to \mathbb{C}\} \to \{w : \mathbb{Z}^2 \to \mathbb{C}\}$ *be defined as* [1]

DEF $$[\mathbf{W}\mathsf{f}](j,n) \triangleq \left\langle \mathsf{f}(x) \,|\, \psi_{k,n}(x) \right\rangle$$

Definition V.2. *The following relations are defined as described below:*

DEF

scaling coefficients	$v_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$v_j(n)$	$\triangleq$	$\left\langle f(x) \,|\, \phi_{j,n}(x) \right\rangle$
wavelet coefficients	$w_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$w_j(n)$	$\triangleq$	$\left\langle f(x) \,|\, \psi_{j,n}(x) \right\rangle$
scaling filter coefficients	$\bar{\mathsf{h}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{h}}(n)$	$\triangleq$	$\mathsf{h}(-n)$
wavelet filter coefficients	$\bar{\mathsf{g}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{g}}(n)$	$\triangleq$	$\mathsf{g}(-n)$

The scaling and wavelet filter coefficients at scale j are equal to the filtered and downsampled (Theorem **??** page **??**) scaling filter coefficients at scale $j+1$:[2]

- The convolution (Definition F.4 page 205) of $v_{j+1}(n)$ with $\bar{\mathsf{h}}(n)$ and then downsampling by 2 produces $v_j(n)$.
- The convolution of $v_{j+1}(n)$ with $\bar{\mathsf{g}}(n)$ and then downsampling by 2 produces $w_j(n)$.

This is formally stated and proved in the next theorem.

[1] Notice that this definition is similar to the definition of transforms of other analysis systems:

Laplace Transform	$\mathscr{L} f(s)$	$\triangleq$	$\langle \mathsf{f}(x) \,|\, e^{sx} \rangle$	$\triangleq$	$\int_x \mathsf{f}(x) e^{-sx}\, \mathsf{dx}$
Continuous Fourier Transform	$\mathscr{F} f(\omega)$	$\triangleq$	$\left\langle \mathsf{f}(x) \,|\, e^{i\omega x} \right\rangle$	$\triangleq$	$\int_x \mathsf{f}(x) e^{-i\omega x}\, \mathsf{dx}$
Fourier Series Transform	$\mathscr{F}_s f(k)$	$\triangleq$	$\left\langle \mathsf{f}(x) \,|\, e^{i\frac{2\pi}{T}kx} \right\rangle$	$\triangleq$	$\int_x \mathsf{f}(x) e^{-i\frac{2\pi}{T}kx}\, \mathsf{dx}$
Z-Transform	$\mathscr{Z} f(z)$	$\triangleq$	$\langle \mathsf{f}(x) \,|\, z^n \rangle$	$\triangleq$	$\sum_n \mathsf{f}(x) z^{-n}$
Discrete Fourier Transform	$\mathscr{F}_d f(k)$	$\triangleq$	$\left\langle f(n) \,|\, e^{i\frac{2\pi}{N}kn} \right\rangle$	$\triangleq$	$\sum_n \mathsf{f}(x) e^{-i\frac{2\pi}{N}kn}$

[2] Mallat (1999) page 257, Burrus et al. (1998) page 35

Theorem V.1.

THM
$$\begin{aligned} v_j(n) &= [\bar{\mathsf{h}} \star v_{j+1}](2n) \\ w_j(n) &= [\bar{\mathsf{g}} \star v_{j+1}](2n) \end{aligned}$$

✎PROOF:

$$\begin{aligned}
v_j(n) &= \left\langle \mathsf{f}(x) \,|\, \phi_{j,n}(x) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\phi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\sqrt{2}\sum_m h(m)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sum_m h(j)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \,|\, \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \,|\, \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m h(m) v_{j+1}(2n+m) \\
&= \sum_p h(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p-2n \\
&= \sum_p \bar{\mathsf{h}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{h}} \star v_{j+1}](2n)
\end{aligned}$$

$$\begin{aligned}
w_j(n) &= \left\langle \mathsf{f}(x) \,|\, \psi_{j,n}(x) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\psi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\sqrt{2}\sum_m \mathsf{g}(j)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sum_m \mathsf{g}(m)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \,|\, \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \,|\, \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m \mathsf{g}(m) v_{j+1}(2n+m) \\
&= \sum_p \mathsf{g}(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p-2n \\
&= \sum_p \bar{\mathsf{g}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{g}} \star v_{j+1}](2n)
\end{aligned}$$

✏

These filtering and downsampling operations are equivalent to the operations performed by a filter bank. Therefore, a filter bank can be used to implement a *Fast Wavelet Transform* (*FWT*), as illustrated in Figure V.1 (page 425).

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Figure V.1: k-Stage Fast Wavelet Transform

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APPENDIX W

TRANSLATION SPACES

W.1 Translation

W.1.1 Definitions

Definition W.1. *Let X be a set and $\mathbf{I}$ be the identity operator on X.*

DEF

$\mathbf{T}_x$ *is a* ***translation operator*** *on X if*

1. $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$ $\quad \forall A \in 2^X$ (IDENTITY) *and*
2. $\mathbf{T}_x \mathbf{T}_y = \mathbf{T}_y \mathbf{T}_x$ $\quad \forall x, y \in X$ (COMMUTATIVE) *and*
3. $\mathbf{T}_x \bigcup_{i \in I} A_i = \bigcup_{i \in I} \mathbf{T}_x A_i$ $\quad \forall A, Y \in 2^X, x \in X$ (DISTRIBUTIVE *over* $\cup$) *and*
4. $\bigcup_{b \in B} \mathbf{T}_b A = \bigcup_{a \in A} \mathbf{T}_a B$ $\quad \forall A, B \in 2^X$ *and*
5. $\mathbf{T}_x (A \cap B) = \left(\mathbf{T}_x A\right) \cap \left(\mathbf{T}_x B\right)$ $\quad \forall A, B \in 2^X, x \in X$ *and*
6. $\mathbf{T}_x \left(A^{\mathsf{c}}\right) = \mathsf{c}\left(\mathbf{T}_x A\right)$ $\quad \forall A, B \in 2^X, x \in X.$

The pair $(X, \mathbf{T})$ is a ***translation space*** *on X.*

Definition W.2. [1] *Let X be a set on which is defined the translation opertor $\mathbf{T}_x$.* ***Minkowski addition*** $\oplus$ *and* ***Minkowski subtraction*** $\ominus$ *is defined as follows:*

DEF

$$A \oplus B = \bigcup_{b \in B} \mathbf{T}_b A \qquad \forall\, A, B \in 2^X \qquad \text{(MINKOWSKI ADDITION)}$$

$$A \ominus B = \bigcap_{b \in B} \mathbf{T}_b A \qquad \forall\, A, B \in 2^X \qquad \text{(MINKOWSKI SUBTRACTION)}$$

Theorem W.1 (next) shows a relationship between Minkowski addition and Minkowski subtraction.

Theorem W.1 (de Morgan relations)**.** [2] *Let $(X, +)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\mathsf{c}(A \oplus B) = A^{\mathsf{c}} \ominus B \qquad \forall\, A, B \in 2^X$$

$$\mathsf{c}(A \ominus B) = A^{\mathsf{c}} \oplus B \qquad \forall\, A, B \in 2^X$$

[1] Matheron (1975) page 17, Lay (1982) page 7

[2] Pitas and Venetsanopoulos (1991) page 159

✎PROOF:

$$\begin{aligned}
\mathsf{c}(A \oplus B) &= \mathsf{c}\left(\bigcup_{b\in B} \mathbf{T}_b A\right) && \text{by Definition W.2 page 427}\\
&= \bigcap_{b\in B} \mathsf{c}\left(\mathbf{T}_b A\right) && \text{by Demorgan relation page 427}\\
&= \bigcap_{b\in B} \mathbf{T}_b\left(A^{\mathsf{c}}\right) && \text{by Definition W.1 page 427}\\
&= A^{\mathsf{c}} \ominus B && \text{by Theorem W.2 page 430}
\end{aligned}$$

$$\begin{aligned}
\mathsf{c}(A \ominus B) &= \mathsf{c}\left(\bigcap_{b\in B} \mathbf{T}_b A\right) && \text{by Definition W.2 page 427}\\
&= \bigcup_{b\in B} \mathsf{c}\left(\mathbf{T}_b A\right) && \text{by Demorgan relation page 427}\\
&= \bigcup_{b\in B} \mathbf{T}_b\left(A^{\mathsf{c}}\right) && \text{by Definition W.1 page 427}\\
&= A^{\mathsf{c}} \oplus B && \text{by Theorem W.2 page 430}
\end{aligned}$$

✏

W.1.2 Examples

Example W.1 (Translation on groups). [3] Let $\oplus$ be the Minkowski addition operator defined in terms of the *translation operator* $\mathbf{T}$. Let $(X,+)$ be a *group*.

EX

$$\left\{\ \mathbf{T}_x A \triangleq \{a+x|a\in A\} \quad \forall A\in 2^X\ \right\} \Longrightarrow \left\{\begin{array}{ll} \mathbf{T}_x \text{ is a translation operator} & \text{and} \\ A\oplus B = \left\{a+b|a\in A \text{ and } b\in B\right\} & \forall\, A,B\in 2^X \end{array}\right\}$$

✎PROOF:

1. Proof that $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$:

$$\begin{aligned}
\mathbf{T}_0 A &= \{a+0|a\in A\} && \text{by definition of } \mathbf{T}_x\\
&= \{a|a\in A\} && \text{by additive identity property of groups}\\
&= A
\end{aligned}$$

2. Proof that $\mathbf{T}_x\mathbf{T}_y = \mathbf{T}_y\mathbf{T}_x$:

$$\begin{aligned}
\mathbf{T}_x\mathbf{T}_y A &= \mathbf{T}_x\,\{a+y|a\in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a+y+x|a\in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a+x+y|a\in A\} && \text{by commutative property of groups}\\
&= \mathbf{T}_y\,\{a+x|a\in A\} && \text{by definition of } \mathbf{T}_y\\
&= \mathbf{T}_y\mathbf{T}_x\,\{a|a\in A\} && \text{by definition of } \mathbf{T}_x
\end{aligned}$$

[3] Matheron (1975) pages 16–17, Pitas and Venetsanopoulos (1991) page 159, Lay (1982) page 7

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3. Proof that $\mathbf{T}_x \bigcup_{i\in I} A_i = \bigcup_{i\in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcup_i A_i &= \left\{ y+x \middle| y \in \bigcup_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y+x \middle| \bigvee_i y \in A_i \right\} \\
&= \bigcup_i \left\{ y+x \middle| y \in A_i \right\} \\
&= \bigcup_i \mathbf{T}_x \left\{ y \middle| y \in A_i \right\} \\
&= \bigcup_i \mathbf{T}_x A
\end{aligned}$$

4. Proof that $\bigcup_{b\in B} \mathbf{T}_b A = \bigcup_{a\in A} \mathbf{T}_a B$:

$$\begin{aligned}
\bigcup_{b\in B} \mathbf{T}_b A &= \bigcup_{b\in B} \{a+b|a\in A\} && \text{by definition of } \mathbf{T}_x \\
&= \left\{a+b \middle| a\in A \text{ and } b\in B\right\} \\
&= \left\{b+a \middle| b\in B \text{ and } a\in A\right\} \\
&= \bigcup_{a\in A} \{b+a|b\in B\} \\
&= \bigcup_{a\in A} \mathbf{T}_a B
\end{aligned}$$

5. Proof that $\mathbf{T}_x \bigcap_{i\in I} A_i = \bigcap_{i\in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcap_i A_i &= \left\{ y+x \middle| y \in \bigcap_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y+x \middle| \bigwedge_i y \in A_i \right\} \\
&= \bigcap_i \left\{ y+x \middle| y \in A_i \right\} \\
&= \bigcap_i \mathbf{T}_x \left\{ y \middle| y \in A_i \right\} \\
&= \bigcap_i \mathbf{T}_x A
\end{aligned}$$

6. Proof that $\mathbf{T}_x\left(A^{\mathsf{c}}\right) = \mathsf{c}\left(\mathbf{T}_x A\right)$:

$$\begin{aligned}
\mathbf{T}_x \mathsf{c} A &= \mathbf{T}_x \left\{a \middle| a\in A^{\mathsf{c}}\right\} \\
&= \left\{a+x \middle| a\in A^{\mathsf{c}}\right\} \\
&= \{a+x|a\notin A\} \\
&= \{a+x|\neg(a\in A)\} \\
&= \mathsf{c}\,\{a+x|a\in A\} \\
&= \mathsf{c}\mathbf{T}_x A
\end{aligned}$$

$$\begin{aligned}
A \oplus B &= \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition W.2 page 427} \\
&= \left\{a+b \middle| a\in A \text{ and } b\in B\right\} && \text{by Definition W.1 page 427}
\end{aligned}$$

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Figure W.1: Illustration for Example W.2 (page 430)

Example W.2. Let

$$A \triangleq \{(0,0),\ (0,1),\ (0,-1),\ (1,1)\}$$
$$B \triangleq \{(0,0),\ (-2,1),\ \ (1,-1)\}$$

Then

$$A \oplus B = \{(0,0),\ (0,1),\ (0,-1),\ (1,1),\ (-2,1),\ (-2,2),\ (-2,0),\ (-1,2),\ (1,-1),\ (1,-2),\ (2,0)\}$$
$$A \ominus B = \{(1,0)\}$$

These relationships are illustrated in Figure W.1 (page 430).

Figure W.2: Illustration for Example W.3 page 430

Example W.3. [4] Two more examples are illustrated in Figure W.2 (page 430).

W.1.3 Additive properties

Theorem W.2. [5] *Let $(X,+)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

THM					
	$A \oplus \{0\}$	$=$	A	$\forall A \subseteq X$	
	$A \oplus B$	$=$	$B \oplus A$	$\forall A,B \subseteq X$	(COMMUTATIVE)
	$A \oplus (B \oplus C)$	$=$	$(A \oplus B) \oplus C$	$\forall A,B,C \subseteq X$	(ASSOCIATIVE)
	$\mathbf{T}_x(A \oplus B)$	$=$	$\left(\mathbf{T}_x A\right) \oplus B$	$\forall A,B \subseteq X,\ x \in X$	(TRANSLATION INVARIANT)

[4] Lay (1982) page 7
[5] Pitas and Venetsanopoulos (1991) pages 163–164

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✎PROOF:

$$
\begin{aligned}
A \oplus \{0\} &= A \oplus B|_{B=\{0\}} \\
&= \bigcup_{b\in B} \mathbf{T}_b A \Big|_{B=\{0\}} && \text{by Definition W.2 page 427} \\
&= \mathbf{T}_0 A \\
&= A && \text{by Definition W.1 page 427}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus B &= \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition W.2 page 427} \\
&= \bigcup_{a\in A} \mathbf{T}_a B && \text{by Definition W.1 page 427} \\
&= B \oplus A && \text{by Definition W.2 page 427}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus (B \oplus C) &= \bigcup_{y\in B\oplus C} \mathbf{T}_y A && \text{by Definition W.2 page 427} \\
&= \bigcup_{a\in A} \mathbf{T}_a (B \oplus C) && \text{by Definition W.1 page 427} \\
&= \bigcup_{a\in A} \mathbf{T}_a \left(\bigcup_{c\in C} \mathbf{T}_c B \right) && \text{by Definition W.2 page 427} \\
&= \bigcup_{a\in A} \left(\bigcup_{c\in C} \mathbf{T}_a \mathbf{T}_c B \right) && \text{by Definition W.1 page 427} \\
&= \bigcup_{a\in A} \left(\bigcup_{c\in C} \mathbf{T}_c \mathbf{T}_a B \right) && \text{by Definition W.1 page 427} \\
&= \bigcup_{c\in C} \mathbf{T}_c \left(\bigcup_{a\in A} \mathbf{T}_a B \right) && \text{by Definition W.1 page 427} \\
&= \bigcup_{c\in C} \mathbf{T}_c \left(\bigcup_{b\in B} \mathbf{T}_b A \right) && \text{by Definition W.1 page 427} \\
&= \bigcup_{c\in C} \mathbf{T}_c (A \oplus B) && \text{by Definition W.2 page 427} \\
&= (A \oplus B) \oplus C && \text{by Definition W.2 page 427}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T}_x (A \oplus B) &= \mathbf{T}_x \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition W.2 page 427} \\
&= \bigcup_{b\in B} \mathbf{T}_x \mathbf{T}_b A && \text{by Definition W.1 page 427} \\
&= \bigcup_{b\in B} \mathbf{T}_b \mathbf{T}_x A && \text{by Definition W.1 page 427} \\
&= \left(\mathbf{T}_x A\right) \oplus B && \text{by Definition W.2 page 427}
\end{aligned}
$$

✏

Theorem W.3. [6] *Let $(X,+)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

[6] Pitas and Venetsanopoulos (1991) page 163

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THM

$A \oplus (B \cup C)$	$=$	$(A \oplus B) \cup (A \oplus C)$	$\forall A,B,C \subseteq X$	($\oplus$ *is* LEFT DISTRIBUTIVE *over* $\cup$)
$(A \cup B) \oplus C$	$=$	$(A \oplus C) \cup (B \oplus C)$	$\forall A,B,C \subseteq X$	($\oplus$ RIGHT DISTRIBUTIVE *over* $\cup$)
$A \oplus (B \cap C)$	$\subseteq$	$(A \oplus B) \cap (A \oplus C)$	$\forall A,B,C \subseteq X$	
$(A \cap B) \oplus C$	$\subseteq$	$(A \oplus C) \cap (B \oplus C)$	$\forall A,B,C \subseteq X$	

✎PROOF:

$$
\begin{aligned}
(A \cup B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c(A \cup B) && \text{by Definition W.2 page 427}\\
&= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cup (\mathbf{T}_c B)\right] && \text{by Definition W.1 page 427}\\
&= \left(\bigcup_{c \in C} \mathbf{T}_c A\right) \cup \left(\bigcup_{c \in C} \mathbf{T}_c B\right) \\
&= (A \oplus C) \cup (B \oplus C) && \text{by Definition W.2 page 427}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus (B \cup C) &= (B \cup C) \oplus A && \text{by Theorem W.2 page 430}\\
&= (B \oplus A) \cup (C \oplus A) && \text{by previous result}\\
&= (A \oplus B) \cup (A \oplus C) && \text{by Theorem W.2 page 430}
\end{aligned}
$$

$$
\begin{aligned}
(A \cap B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c(A \cap B) && \text{by Theorem W.2 page 430}\\
&= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cap (\mathbf{T}_c B)\right] && \text{by Definition W.1 page 427}\\
&\subseteq \left(\bigcup_{c \in C} \mathbf{T}_c A\right) \cap \left(\bigcup_{c \in C} \mathbf{T}_c B\right) && \text{by minimax inequality}\\
&= (A \oplus C) \cap (B \oplus C) && \text{by Theorem W.2 page 430}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus (B \cap C) &= (B \cap C) \oplus A && \text{by Theorem W.2 page 430}\\
&\subseteq (B \oplus A) \cap (C \oplus A) && \text{by previous result}\\
&= (A \oplus B) \cap (A \oplus C) && \text{by Theorem W.2 page 430}
\end{aligned}
$$

✏

W.1.4 Subtractive properties

Theorem W.4. [7] *Let $(X,+)$ be a group with with Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

	$A \ominus \{0\}$	$=$	A	$\forall A \subseteq X$	
	$A \ominus B$	$=$	$B^c \ominus A^c$	$\forall A,B \subseteq X$	
	$\mathbf{T}_x(A \ominus B)$	$=$	$(\mathbf{T}_x A) \ominus B$	$\forall A,B \subseteq X,\ x \in X$	(TRANSLATION INVARIANT)
$A \subseteq B \implies$	$A \ominus C$	$\subseteq$	$B \ominus C$	$\forall A,B,C \subseteq X$	(INCREASING)

[7] Pitas and Venetsanopoulos (1991) pages 164–165

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✎PROOF:

$$\begin{aligned}
A \ominus \{0\} &= \mathsf{c}(A^{\mathsf{c}} \oplus \{0\}) && \text{by Theorem W.1 page 427}\\
&= \mathsf{c}(A^{\mathsf{c}}) && \text{by Theorem W.2 page 430}\\
&= A
\end{aligned}$$

$$\begin{aligned}
A \ominus B &= \mathsf{cc}(A \ominus B)\\
&= \mathsf{c}(A^{\mathsf{c}} \oplus B) && \text{by Theorem W.1 page 427}\\
&= \mathsf{c}(B \oplus A^{\mathsf{c}}) && \text{by Theorem W.2 page 430}\\
&= B^{\mathsf{c}} \ominus A^{\mathsf{c}} && \text{by Theorem W.1 page 427}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_x(A \ominus B) &= \mathbf{T}_x\mathsf{c}\big(A^{\mathsf{c}} \oplus B\big) && \text{by Theorem W.1 page 427}\\
&= \mathsf{c}\mathbf{T}_x\big(A^{\mathsf{c}} \oplus B\big) && \text{by Definition W.1 page 427}\\
&= \mathsf{c}\big(\mathbf{T}_x A^{\mathsf{c}} \oplus B\big) && \text{by Theorem W.2 page 430}\\
&= \mathsf{c}\big(\mathsf{c}\mathbf{T}_x A \oplus B\big) && \text{by Definition W.1 page 427}\\
&= \mathbf{T}_x A \ominus B && \text{by Theorem W.1 page 427}
\end{aligned}$$

$$\begin{aligned}
A \ominus C &= \bigcap_{c \in C} A_c && \text{by Theorem W.2 page 430}\\
&\subseteq \bigcap_{c \in C} B_c && \text{by } A \subseteq B \text{ hypothesis}\\
&= B \ominus C && \text{by Definition W.2 page 427}
\end{aligned}$$

✏

Theorem W.5. [8] *Let $(X,+)$ be a group with with Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM	$A \ominus (B \cup C)$	$=$	$(A \ominus B) \cap (A \ominus C)$	$\forall A,B,C \subseteq X$	($\ominus$ LEFT DISTRIBUTIVE *over* $\cup$)
	$(A \cap B) \ominus C$	$=$	$(A \ominus C) \cap (B \ominus C)$	$\forall A,B,C \subseteq X$	($\ominus$ RIGHT DISTRIBUTIVE *over* $\cap$)
	$(A \cup B) \ominus C$	$\supseteq$	$(A \ominus C) \cup (B \ominus C)$	$\forall A,B,C \subseteq X$	
	$A \ominus (B \cap C)$	$\supseteq$	$(A \ominus B) \cup (A \ominus C)$	$\forall A,B,C \subseteq X$	

✎PROOF:

$$\begin{aligned}
A \ominus (B \cup C) &= \mathsf{cc}\Big[\, A \ominus (B \cup C) \,\Big]\\
&= \mathsf{c}\Big[\, A^{\mathsf{c}} \oplus (B \cup C) \,\Big] && \text{by Theorem W.1 page 427}\\
&= \mathsf{c}\Big[\, (A^{\mathsf{c}} \oplus B) \cup (A^{\mathsf{c}} \oplus C) \,\Big] && \text{by Theorem W.3 page 431}\\
&= \big[\mathsf{c}\big(A^{\mathsf{c}} \oplus B\big)\big] \cap \big[\mathsf{c}\big(A^{\mathsf{c}} \oplus C\big)\big] && \text{by Demorgan relation page 427}\\
&= (A \ominus B) \cap (A \ominus C) && \text{by Theorem W.1 page 427}
\end{aligned}$$

$$\begin{aligned}
(A \cap B) \ominus C &= \mathsf{c}[(A \cap B) \ominus C]\\
&= \mathsf{c}[\mathsf{c}(A \cap B) \oplus C] && \text{by Theorem W.1 page 427}\\
&= \mathsf{c}\big[(A^{\mathsf{c}} \cup B^{\mathsf{c}}) \oplus C\big]\\
&= \mathsf{c}\big[(A^{\mathsf{c}} \oplus C) \cup (B^{\mathsf{c}} \oplus C)\big] && \text{by Theorem W.3 page 431}\\
&= \mathsf{c}(A^{\mathsf{c}} \oplus C) \cap \mathsf{c}(B^{\mathsf{c}} \oplus C)\\
&= (A \ominus C) \cap (B \ominus C) && \text{by Theorem W.1 page 427}
\end{aligned}$$

[8] Pitas and Venetsanopoulos (1991) page 165

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$$
\begin{aligned}
A \ominus (B \cap C) &= \mathsf{cc}\Big[\ A \ominus (B \cap C)\ \Big] \\
&= \mathsf{c}\Big[\ A^{\mathsf{c}} \oplus (B \cap C)\ \Big] && \text{by Theorem W.1 page 427} \\
&\supseteq \mathsf{c}\big[(A^{\mathsf{c}} \oplus B) \cap (A^{\mathsf{c}} \oplus C)\big] && \text{by Theorem W.3 page 431} \\
&= \big[\mathsf{c}(A^{\mathsf{c}} \oplus B)\big] \cup \big[\mathsf{c}(A^{\mathsf{c}} \oplus C)\big] && \text{by Demorgan relation page 427} \\
&= (A \ominus B) \cup (A \ominus C) && \text{by Theorem W.1 page 427}
\end{aligned}
$$

$$
\begin{aligned}
(A \cup B) \ominus C &= \mathsf{cc}[(A \cup B) \ominus C] \\
&= \mathsf{c}[\mathsf{c}(A \cup B) \oplus C] && \text{by Theorem W.1 page 427} \\
&= \mathsf{c}\big[\big(A^{\mathsf{c}} \cap B^{\mathsf{c}}\big) \oplus C\big] && \text{by Demorgan relation page 427} \\
&\supseteq \mathsf{c}\big[\big(A^{\mathsf{c}} \oplus C\big) \cap \big(B^{\mathsf{c}} \oplus C\big)\big] \\
&= \mathsf{c}\big(A^{\mathsf{c}} \oplus C\big) \cup \mathsf{c}\big(B^{\mathsf{c}} \oplus C\big) \\
&= (A \ominus C) \cup (B \ominus C) && \text{by Theorem W.1 page 427}
\end{aligned}
$$

✏

Theorem W.6. [9] *Let $(X,+)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM				
	$A \ominus (B \oplus C)$	$=$	$(A \ominus B) \ominus C$	$\forall A,B,C \subseteq X$
	$A \oplus (B \ominus C)$	$\subseteq$	$(A \oplus B) \ominus C$	$\forall A,B,C \subseteq X$

✎PROOF:

$$
\begin{aligned}
A \ominus (B \oplus C) &= \mathsf{cc}\Big[\ A \ominus (B \oplus C)\ \Big] \\
&= \mathsf{c}\Big[\ A^{\mathsf{c}} \oplus (B \oplus C)\ \Big] && \text{by Theorem W.1 page 427} \\
&= \mathsf{c}\Big[\ (A^{\mathsf{c}} \oplus B) \oplus C\ \Big] && \text{by Theorem W.2 page 430} \\
&= \mathsf{c}(A^{\mathsf{c}} \oplus B) \ominus C && \text{by Theorem W.1 page 427} \\
&= (A \ominus B) \ominus C && \text{by Theorem W.1 page 427}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus (B \ominus C) &= A \oplus \left(\bigcap_{c \in C} \mathbf{T}_c B\right) && \text{by Definition W.2 page 427} \\
&= \left(\bigcap_{c \in C} \mathbf{T}_c B\right) \oplus A && \text{by Theorem W.2 page 430} \\
&= \bigcup_{a \in A} \mathbf{T}_a \left(\bigcap_{c \in C} \mathbf{T}_c B\right) && \text{by Definition W.2 page 427} \\
&= \bigcup_{a \in A} \bigcap_{c \in C} \mathbf{T}_a \mathbf{T}_c B && \text{by Definition W.1 page 427} \\
&\subseteq \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_a \mathbf{T}_c B && \text{by minimax inequality} \\
&= \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_c \mathbf{T}_a B && \text{by Definition W.1 page 427}
\end{aligned}
$$

[9] Pitas and Venetsanopoulos (1991) page 166

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$$= \bigcap_{c\in C} \mathbf{T}_c\left(\bigcup_{a\in A} \mathbf{T}_a B\right) \qquad \text{by Definition W.1 page 427}$$
$$= \bigcap_{c\in C} \mathbf{T}_c(B \oplus A) \qquad \text{by Definition W.2 page 427}$$
$$= (B \oplus A) \ominus C \qquad \text{by Definition W.2 page 427}$$
$$= (A \oplus B) \ominus C \qquad \text{by Theorem W.2 page 430}$$

✏

W.2 Operations

Definition W.3. [10] *Let $(X,+)$ be a group.*

DEF *The **symmetric set** of A is the set* $\check{A} \triangleq -A \qquad \forall A\subseteq X$

Definition W.4. [11] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and D^s be the symmetric set of set D.*

DEF *The **dilation** of A by D is the operation* $A \oplus \check{D} \qquad \forall A,D\subseteq X.$
*The **erosion** of A by E is the operation* $A \ominus \check{E} \qquad \forall A,E\subseteq X.$

Definition W.5. [12] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and B^s be the symmetric set of a set B.*

DEF *The **opening** of A with respect to B is the set* $A_B \triangleq \underbrace{\underbrace{(A \ominus \check{B})}_{\text{erosion}} \oplus B}_{\text{dilation}} \qquad \forall A,B\subseteq X.$

*The **closing** of A with respect to B is the set* $A^B \triangleq \underbrace{\underbrace{(A \oplus \check{B})}_{\text{dilation}} \ominus B}_{\text{erosion}} \qquad \forall A,B\subseteq X.$

Theorem W.7. [13] *Let $(X,+)$ be a group with A_B representing the opening of a set A with respect to a set B and A^B representing the closing of a set A with respect to a set B.*

THM
(complement of the opening)→ $\mathsf{c}(A_B) = (A^{\mathsf{c}})^B$ *←(closing of the complement)* $\forall A,B\subseteq X$
(complement of the closing)→ $\mathsf{c}(A^B) = (A^{\mathsf{c}})_B$ *←(opening of the complement)* $\forall A,B\subseteq X$

✎PROOF:

$$\mathsf{c}(A_B) = \mathsf{c}\Big[\,(A \ominus \check{B}) \oplus B\,\Big] \qquad \text{by Definition W.5 page 435}$$
$$= \mathsf{c}(A \ominus \check{B}) \ominus B \qquad \text{by Theorem W.1 page 427}$$
$$= \mathsf{c}(A \ominus \check{B}) \ominus B \qquad \text{by Theorem W.1 page 427}$$
$$= (A^{\mathsf{c}} \oplus \check{B}) \ominus B \qquad \text{by Theorem W.1 page 427}$$
$$= (A^{\mathsf{c}})^B \qquad \text{by Definition W.5 page 435}$$

[10] Matheron (1975) page 17
[11] Pitas and Venetsanopoulos (1991) page 161
[12] Serra (1982) page 50
[13] Serra (1982) page 51

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$$
\begin{aligned}
\mathsf{c}(A^B) &= \mathsf{c}\Big[\,(A \oplus \check{B}) \ominus B\,\Big] && \text{by Definition W.5 page 435}\\
&= \mathsf{c}(A \oplus \check{B}) \oplus B && \text{by Theorem W.1 page 427}\\
&= \mathsf{c}(A \oplus \check{B}) \oplus B && \text{by Theorem W.1 page 427}\\
&= (A^{\mathsf{c}} \ominus \check{B}) \oplus B && \text{by Theorem W.1 page 427}\\
&= (A^{\mathsf{c}})_B && \text{by Definition W.5 page 435}
\end{aligned}
$$

✍

Example W.4.

EX

$$
A \triangleq \{(0,0),\ (1,0),\ (2,0),\ (0,1)\} \implies
\left\{\begin{array}{lcl} A_{(-2,1)} &=& \{(-2,1),\ (-1,1),\ (0,1),\ (-2,2)\} \quad \text{and} \\ \check{A} &=& \{(0,0),\ (-1,0),\ (-2,0),\ (0,-1)\} \end{array}\right\}
$$

Example W.5. An example similar to Example W.4 (page 436) but using solid shapes is illustrated next:

EX

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APPENDIX X

STROMBERG WAVELETS

Definition X.1.

DEF

$$A_0 \triangleq \left\{\cdots, -\tfrac{3}{2}, -1, -\tfrac{1}{2}, 0, 1, 2, 3, \cdots\right\} \quad \textit{and}$$
$$A_1 \triangleq A_0 \cup \left\{\tfrac{1}{2}\right\}$$

Definition X.2. *The **simple tent function** Λ_σ for $\sigma \in A_0$ is defined as*

DEF

$$\Lambda_\sigma(x) \triangleq \begin{cases} 1-\sigma+x & \textit{for } x \in [\sigma-1 : \sigma] \\ 1+\sigma-x & \textit{for } x \in [\sigma : \sigma+1] \\ 0 & \textit{otherwise} \end{cases}$$

Definition X.3. [1]

DEF

$$S(x) \triangleq \begin{cases} [S(\sigma)-S(\sigma-1)](x-\sigma)+S(\sigma) & \textit{for } x \in [\sigma-1 : \sigma] \\ [S(\sigma+1)-S(\sigma)](x-\sigma)+S(\sigma) & \textit{for } x \in [\sigma : \sigma+1] \end{cases}$$

Theorem X.1. [2]

THM

$$0 = \langle S(x) \mid \Lambda_\sigma(x)\rangle = \begin{cases} S(\sigma-1)+4S(\sigma)+S(\sigma+1) & \textit{for } \sigma \in \{2,3,4,\ldots\} \\ S(0)+6S\left(\frac{1}{2}\right)+13S(1)+4S(2) & \textit{for } \sigma = 1 \\ 2S\left(-\frac{1}{2}\right)+9S(0)+6S\left(\frac{1}{2}\right)+S(1) & \textit{for } \sigma = 0 \\ S\left(\sigma-\frac{1}{2}\right)+4S(\sigma)+S\left(\sigma+\frac{1}{2}\right) & \textit{for } \sigma \in \left\{\ldots, -\frac{5}{2}, -2, -\frac{3}{2}, -1, -\frac{1}{2}\right\} \end{cases}$$

✎PROOF:

1. lemma:

$$\int_{x=\sigma-1}^{x=\sigma} [(A-B)(x-\sigma)+A](1-\sigma+x)\,\mathrm{dx}$$
$$= \int_{x=\sigma-1}^{x=\sigma} [A(1-\sigma+x)+B(\sigma-x)](1-\sigma+x)\,\mathrm{dx}$$
$$= \int_{x=\sigma-1}^{x=\sigma} A\left[x^2+2x(1-\sigma)+(1-\sigma)^2\right]+B\left[-x^2+x(\sigma+\sigma-1)+\sigma(1-\sigma)\right]\,\mathrm{dx}$$

[1] Strömberg (1983) page 479 ⟨4. Explicit Description of τ and ρ, $m=0$⟩

[2] Strömberg (1983), Wojtaszczyk (1997), Jaffard et al. (2001) pages 23–24 ⟨§ 2.5 New Directions of the 1930s: The Franklin System⟩ Greenhoe

$$
\begin{aligned}
&= \left[A\left[\frac{1}{3}x^3 + x^2(1-\sigma) + x(1-\sigma)^2\right] + B\left[-\frac{1}{3}x^3 + \frac{1}{2}x^2(2\sigma-1) + x\sigma(1-\sigma)\right]\right]_{x=\sigma-1}^{x=\sigma} \\
&= A\left[\frac{1}{3}\left(\sigma^3 - \left(\sigma^3 - 3\sigma^2 + 3\sigma - 1\right)\right) + \left(\sigma^2 - \left(\sigma^2 - 2\sigma + 1\right)\right)(1-\sigma) + (1-\sigma)^2(\sigma - (\sigma-1))\right] \\
&+ B\left[-\frac{1}{3}\left(\sigma^3 - \left(\sigma^3 - 3\sigma^2 + 3\sigma - 1\right)\right) + \frac{1}{2}\left(\sigma^2 - \left(\sigma^2 - 2\sigma + 1\right)\right)(2\sigma-1) + \sigma(1-\sigma)\right] \\
&= A\left[\left(\sigma^2 - \sigma + \frac{1}{3}\right) + (2\sigma-1)(1-\sigma) + (-\sigma+1)^2\right] + B\left[-\left(\sigma^2 - \sigma + \frac{1}{3}\right) + \frac{1}{2}(2\sigma-1)(2\sigma-1) + \sigma(-\sigma+1)\right] \\
&= A\left[\sigma^2(1-2+1) + \sigma(-1+2+1-2) + \left(\frac{1}{3} - 1 + 1\right)\right] + B\left[\sigma^2(-1+2-1) + \sigma(1-1-1+1) + \left(-\frac{1}{3} + \frac{1}{2}\right)\right] \\
&= \boxed{\frac{1}{3}A + \frac{1}{6}B}
\end{aligned}
$$

2. lemma:

$$
\begin{aligned}
&\int_{x=\sigma}^{x=\sigma+1} [(C-A)(x-\sigma) + C](1+\sigma-x)\ \mathrm{dx} \\
&= \int_{x=\sigma}^{x=\sigma+1} [C(1-\sigma+x) + A(\sigma-x)](1+\sigma-x)\ \mathrm{dx} \\
&= \int_{x=\sigma}^{x=\sigma+1} C\left[-x^2 + x(\sigma+\sigma+1) + \sigma(-\sigma-1)\right] + A\left[x^2 + x(-\sigma-1-1-\sigma) + (1+\sigma)^2\right]\ \mathrm{dx} \\
&= \left[C\left[-\frac{1}{3}x^3 + \frac{1}{2}x^2(2\sigma+1) - x\sigma(\sigma+1)\right] + A\left[\frac{1}{3}x^3 - x^2(\sigma+1) + x(\sigma+1)^2\right]\right]_{x=\sigma}^{x=\sigma+1} \\
&= C\left[-\frac{1}{3}\left(\left(\sigma^3 + 3\sigma^2 + 3\sigma + 1\right) - \sigma^3\right) + \frac{1}{2}\left(\left(\sigma^2 + 2\sigma + 1\right) - \sigma^2\right)(2\sigma+1) - \sigma(\sigma+1)\right] \\
&+ A\left[\frac{1}{3}\left(\left(\sigma^3 + 3\sigma^2 + 3\sigma + 1\right) - \sigma^3\right) - \left(\left(\sigma^2 + 2\sigma + 1\right) - \sigma^2\right)(\sigma+1) + (\sigma+1)^2\right] \\
&= C\left[-\sigma^2 + \sigma - \frac{1}{3} + \frac{1}{2}(2\sigma+1)(2\sigma+1) - \sigma^2 - \sigma\right] + A\left[\sigma^2 + \sigma + \frac{1}{3} - (2\sigma+1)(\sigma+1) + \sigma^2 + 2\sigma + 1\right] \\
&= C\left[\sigma^2(-1+2-1) + \sigma(-1+1+1-1) + \left(-\frac{1}{3} + \frac{1}{2}\right)\right] + A\left[\sigma^2(1-2+1) + \sigma(1-2-1+2) + \left(\frac{1}{3} - 1 + 1\right)\right] \\
&= \boxed{\frac{1}{6}C + \frac{1}{3}A}
\end{aligned}
$$

3. definitions: Let $A \triangleq S(\sigma)$, $B \triangleq S(\sigma-1)$, and $C \triangleq S(\sigma+1)$.

4. Finish proof…

$$
\begin{aligned}
\langle S(x) \mid \Lambda_\sigma(x)\rangle &= \int_{x\in\mathbb{R}} S(x)\Lambda_\sigma(x)\ \mathrm{dx} && \text{by definition of } \langle x \mid y\rangle \\
&= \int_{x=\sigma-1}^{x=\sigma+1} S(x)\Lambda_\sigma(x)\ \mathrm{dx} && \text{because support of } \Lambda_\sigma \text{ is } [\sigma-1 : \sigma+1] \\
&= \int_{x=\sigma-1}^{x=\sigma} S(x)\Lambda_\sigma(x)\ \mathrm{dx} + \int_{x=\sigma}^{x=\sigma+1} S(x)\Lambda_\sigma(x)\ \mathrm{dx} \\
&\triangleq \int_{x=\sigma-1}^{x=\sigma} [(A-B)(x-\sigma) + A](1-\sigma+x)\ \mathrm{dx} \\
&+ \int_{x=\sigma}^{x=\sigma+1} [(C-A)(x-\sigma) + C](1+\sigma-x)\ \mathrm{dx} && \text{by definition 3} \\
&= \left[\frac{1}{3}A + \frac{1}{6}B\right] + \left[\frac{1}{6}C + \frac{1}{3}A\right] && \text{by (1) lemma and (2) lemma} \\
&= \frac{1}{6}[B + 4A + C] \\
&\triangleq \boxed{\frac{1}{6}[S(\sigma-1) + 4S(\sigma) + S(\sigma+1)]} && \text{by definition 3}
\end{aligned}
$$

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APPENDIX Y

SOURCE CODE

The source code in this appendix for *GNU Octave*. [1] Octave is similar to *MatLab* with some differences:

1. GNU Octave is free.
2. GNU Octave is open-source.
3. GNU Octave uses a separate graphics engine called *GNU-Plot* for all graphing.

Octave code can easily be adapted to MatLab code and vice-versa.

Y.1 GNU Octave script file

```
printf(" ========================================================= \n");
printf("| Daniel J. Greenhoe                                        |\n");
printf("| http://banyan.cm.nctu.edu.tw/~dgreenhoe/wsd/index.html   |\n");
printf("| Experiments with odd polynomials R(1/2-y)                |\n");
printf(" ========================================================= \n");

%-----------------------------------------------------------
% References:
%
%     http://banyan.cm.nctu.edu.tw/~dgreenhoe/wsd/index.html
%
%     C. Sidney Burrus, Ramesh A. Gopinath, and Haitao Guo
%     Introduction to Wavelets and Wavelet Transforms; A Primer
%     Prentice Hall, 1998
%     ISBN 0-13-489600-9
%-----------------------------------------------------------

%=====================================
% Initialization
%=====================================
clear;

%=====================================
% Function Declarations
%=====================================

%-----------------------------------------------------------------------------
%function: downsample x(n) by L
%-----------------------------------------------------------------------------
```

[1] *GNU Octave*: http://www.octave.org/

```
function y = downSample(x,L)
   n = length(x);
   m = ceil(n/L);
   y = 0*x(1:m);
   for i=0:(m-1)
      y(i+1) = x(i*L+1);
   endfor
endfunction

%-----------------------------------------------------------------------
%function: upsample x(n) by L
%-----------------------------------------------------------------------
function y = upSample(x,L)
   n = length(x);
   m = n*L;
   y = zeros(1,m);
   for i=0:(n -1)
      y(i*L+1) = x(i+1);
   endfor
endfunction

%-----------------------------------------------------------------------
%function: Generate scaling function from scaling filter coefficients h(n)
%   p = scaling function
%   h = scaling filter coefficients h(n)
%   k = number of iterations
%   d = density = number phi(t) samples per h(n) sample
%
% reference: Burrus page 67
%    MatLab code:    Appendix C page 258
%    Key equation:   page 67
%
% reference: Rao
%
%-----------------------------------------------------------------------
function p = gen_phi(h,k,d)
   h = 2*h/sum(h);                  %scale to sqrt(2)*sqrt(2)
   m =  length(h)-1;                %
   p = ones(1,d*m)/m;               %<p,1>=1    ref: Rao p.53
   hu = upSample(h,d);              %upsample h(n) to match phi(t) density
   printf('\n');
   for i = 0:(k-1)                  %iterate
      printf('%d ',i);
      fflush(stdout);
      ph  = conv(hu,p);             %convolve
      p   = downSample(ph,2);       %downsample
   endfor
   p = p(1:m*d);
   printf('\n');
endfunction

%-----------------------------------------------------------------------
%function: Generate wavelet function psi(t) from scaling function phi(t)
%          and wavelet filter coefficients g(n)
%   phi = scaling function phi(t)
%   g = wavelet filter coefficients g(n)
%   d = density = number phi(t) samples per g(n) sample
% reference: Burrus page 15
%
%-----------------------------------------------------------------------
function psi = gen_psi(phi,g,d)
   g   = sqrt(2)*g;                 %scale to sqrt(2)*sqrt(2)
   gu  = upSample(g,d);             %upsample h(n) to match phi(t) density
   pg  = conv(gu,phi);              %convolve
   psi = downSample(pg,2);          %downsample
   m =  length(g)-1;                %
   psi = psi(1:m*d);                %truncate
endfunction

%-----------------------------------------------------------------------
%function: Generate wavelet coefficients g(n) from scaling coefficients h(n)
%  references:
%     Mallat page 238
%     Burrus page  15
%     Burrus page  79
%-----------------------------------------------------------------------
function g = h2g_coefs(h)
```


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```
    g = h;
    N = length(h);
    for n=0:(N-1)
        g(n +1) = (-1)**(n)*h(N-1-n +1);
    endfor
endfunction

%-----------------------------------------------------------------------------
%function: Generate filter coefficients from wavelet/scalar coefficients
%    _
%    h(n) = h(-n) with shift by N where N = length of h(n)
%    _
%    g(n) = g(-n) with shift by N
%-----------------------------------------------------------------------------
function hbar = h2hbar(h)
    hbar = zeros(size(h));                  %allocate memory
    N    = length(h)      ;                  %N = length of h(n)
    for n = 0:(N-1)                         %
        hbar(n +1) = h(N-1-n +1);           %hbar(n) = h(N-1-n)
    endfor
endfunction

%-----------------------------------------------------------------------------
%                                                   -z + 2 -1/z
% Convert polynomial in y to polynomial in y = -------------
%                                                        4
%                                  /                 \
%         |                        |  -z + 2 -1/z    |
%     P(y)|                     = P| ------------- |   = Q(z)Q(1/z)
%         |y=[(-z+2-1/z)/4]        \       4         /
%
% Input
% ------
%  P = [p_{n-1} ... p_2 p_1 p_0]
%     ==> P(y)=p_{n-1}y^{m-1}+...+p_2 y^2+p_1y+p_0
%
% Output
% ------
%  QQ = [q_{2n-2} ... q_2 q_1 q_0]
%
%                        q_{2n-2}z^{2n-2} +...+ q_2 z^2 + r_1 z + r_0
%      ==>  Q(z)Q(1/z)=-------------------------------------------
%                                          z^n
%
% Theory
% ------
% This conversion is useful when using polynomials in y = sin^2(w/2).
% Polynomials in sin^2(w/2) can be used to represent any even
% periodic funtion with period 2pi.
%   * sin^2(w/2) = (1/2)(1-cosw)
%
%                     1-cosw      1      e^{w}+e^{-w}      2 - z - 1/z    |
% y = sin^2(w/2) = -------- = --- - --------------- = -------------- |
%                       2       2          4                  4        |z=e^w
%
%-----------------------------------------------------------------------------
function QQ = y2sin2(P)
    n=length(P);                            % number of terms in P(y). num 0s=n-1
    N=2*n-1;                                % number of terms in Q(z)Q(1/z)
    QQ= zeros(1,N);                         % init Q(z)Q(1/z) = 0+0z+...+0z^{N-1}
    q=1;                                    % init q(z)       = 1
    for k=0:n-1                             %
      QQ = [QQ,0](2:N+1);                   % z[-z + 2 -1/z] = -z^2 + 2z - 1
      QQ = QQ + \                           % Q(z)Q(1/z)
        P(n-k)*[zeros(1,N-2*k-1),q]/4^k;%   = SUM p1k_k*[(-z+2-z^-1)/4]^k
      q = conv(q,[-1 2 -1]);                % q(z) = q(z)[-z^2 + 2z - 1]
    endfor                                  %
endfunction

%-----------------------------------------------------------------------------
% Generate Daubechies class scaling coefficients
%  - Daubechies-p: minimum support (2p-1, R(y)=0) and minimum phase
%  - Symmlets-p:   minimum support (2p-1, R(y)=0) and quasi-linear phase
%  - Experimental: non-minimum support (>2p-1, R(y)!=0)
%
%  Input
%  ------
```

```
%  p = number of vanishing moments
%  R = [r_{m-1} ... r_2 r_1 r_0] ==> R(y)=r_{m-1}y^{m-1}+...+r_2 y^2+r_1y+r_0
%
%  Output
%  ------
%  n  = p + m
%
%  QQ = [q_{2n-1} ... q_1 q_0]
%
%                    q_{2n-1}z^{2n-1}+...+r_2 z^2+r_1z+r_0
%    ==>  Q(z)Q(1/z)=--------------------------------------
%                                  z^n
%
%  rQQ= [ r_1 r_2 ... r_{n-1}  r_n r_{n+1}...r_{n-2} ]
%        |<--roots inside -->|<--roots outside  --->|
%        |   unit circle     |   unit circle        |
%
%  A  = [a_p ... a_2 a_1 a_0]
%
%                        (z+1)^p
%    ==>  A(z) = sqrt(2)(---)   = a_pz^p + ... + a_2z^2 + a_1z + a_0
%                        ( 2 )
%
%  rA = [ r_1 r_2 ... r_p ] = roots of A(z)
%
%
%  Theory
%  ------
%  h(n) = scaling coefficients
%
%  H(z) = SUM h(n) z^{-n}
%          n
%       = A(z)Q(z)
%
%                 (z+1)^p
%       = sqrt(2)(---)   Q(z)
%                 ( 2 )
%
%  Finding Q(z) involves factoring a polynomial P(...):
%
%  P(y) = Pm(y) + y^p R(1/2-y)
%
%  In general, R is any polynomial that satisfies R(1/2-y)=-R(y-1/2)
%    (R is an odd polynomial about y=1/2)
%  For Daubechies-p wavelets and symmlets, R(y)=0
%    (giving Daubechies-p and symmlets minimum support)
%
%          p-1 ( p-1+k )
%  Pm(y) = SUM (        ) y^k
%          k=0 (    k   )
%
%  Q(z)Q(1/z) = P([2-z-1/z]/4)
%-------------------------------------------------------------------------
function [n,A,QQ,rA,rQQ] = gen_Dclass(p,R)
                                        % Compute A(z) = sqrt(2)[ (z+1)/2 ]^p
                                        % ------------------------------------
   A=1;                                 % A(z) = (z+1)^0 = 1
   for k=1:p                            %
     A=conv(A,[1 1]);                   % A(z) = (z+1)^k  k=1,2,3,...,p
   endfor                               %
   A = (sqrt(2)/2^p)*A;                 % A(z) = sqrt(2)[(z+1)/2]^p
   rA = sort(roots(A))';                % rA   = roots of A(z)

                                        % Pm(y)
                                        % ------------------------------------
   Pm = zeros(1,p);                     % init Pm(y) = 0y^{p-1}+...+0y^2+0y+0
   for k=0:p-1                          %
     Pm(p-k) = bincoeff(p-1+k,k);       % Pm(y) = SUM {p-1+k choose k} y^k
   endfor                               %

                                        % P( [2-z-1/z]/4 )
                                        % ------------------------------------
   m=length(R);                         %
   P=[zeros(1,m),Pm] + [R,zeros(1,p)];  % P(y) = Pm(y) + y^p R(y)
   while P(1)==0                        % remove leading zero coefficient terms
     n=length(P);                       %   n = number of terms in P(y)
     P=P(2:n);                          %   remove leading zero coefficient
```


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```
    endwhile                                %
    n=length(P);                            % number of terms in P(y). num 0s=n-1
    QQ = y2sin2(P);                         % Q(z)Q(1/z) = P(y)|_y={(-z+2-1/z)/4}

                                            % Compute H(z)=A(z)Q(z)
                                            % ------------------------------------
    rQQ = sort(roots(QQ))';                 % roots of Q(z)Q(1/z) (p-1 roots of Q(z), p-1 roots of Q(1/z))
    rQ  = rQQ(1:n-1);                       % roots of Q(z) (p-1 roots inside unit circle)
    Q   = poly(rQ);                         % convert roots into poly Q(z)
    Q   = sign(real(Q)).*abs(Q);            % eliminate any extraneous imag. components
    H   = conv(A,Q);                        % H(z) = A(z)Q(z)
    rH  = sort(roots(H))';                  % roots of H(z)
    h   = (sqrt(2)/sum(H))*H;               % admissibility cond: SUM h_n = sqrt(2)
endfunction

%-----------------------------------------------------------------------------
%function: Generate Daubechies-p scaling coefficients
%
%  Input
%  ------
%  p:   number of vanishing moments
%
%  Output
%  ------
%  n  = p + m
%  h:   scaling coefficients
%  rQQ: roots of Q(z)Q(z^-1)
%                  |   |_______ p-1 roots outside unit circle
%                  |___________ p-1 roots inside  unit circle
%  rH:  roots of H(z)          p roots at z=-1 and p-1 roots of Q(z)
%
%
%  Theory
%  ------
%  h(n) = scaling coefficients
%
%  H(z) = SUM h(n) z^{-n}
%           n
%       = A(z)Q(z)
%
%                  (z+1)^p
%       = sqrt(2)(---)   Q(z)
%                  ( 2 )
%
%-----------------------------------------------------------------------------
function [h,rQQ,rH] = gen_Dp(p)
  R = [0];                              % for Daubechies-p, R(y)=0
  [n,A,QQ,rA,rQQ] = gen_Dclass(p,R);    %
  rQ  = rQQ(1:p-1);                     % roots of Q(z) (p-1 roots inside unit circle)
  Q   = poly(rQ);                       % convert roots into poly Q(z)
  Q   = sign(real(Q)).*abs(Q);          % eliminate any extraneous imag. components
  H   = conv(A,Q);                      % H(z) = A(z)Q(z)
  rH  = sort(roots(H))';                % roots of H(z)
  h   = (sqrt(2)/sum(H))*H;             % admissibility cond: SUM h_n = sqrt(2)
endfunction

%-----------------------------------------------------------------------------
%function: Generate Symmlets-p scaling coefficients
%
%  Input
%  ------
%  p:   number of vanishing moments
%
%  Output
%  ------
%  h:   scaling coefficients
%  rQQ: roots of Q(z)Q(z^-1)
%                  |   |_______ p-1 roots outside unit circle
%                  |___________ p-1 roots inside  unit circle
%  rH:  roots of H(z)          p roots at z=-1 and p-1 roots of Q(z)
%
%-----------------------------------------------------------------------------
function [h,rQQ,rH] = gen_Sp(p)
                                            % Initialization
                                            % ----------------------
  N = 1024;                                 % number of data points for linear measure
  R = [0];                                  % for Symmlets-p, R(y)=0
```

```
  [n,A,QQ,rA,rQQ]=gen_Dclass(p,R);      %
  Nquads     = floor(p/2);              % number of zero quads
  Nsol       = 2^Nquads;                % number of possible solutions
  rQQ_mat     = zeros(p-1,Nsol);        % allocate matrix of roots for all solutions
  mse        = zeros(1,   Nsol);        % allocate mean square error of each solution
  sqMag      = zeros(1,   Nsol);        % allocate square magnitude of each solution

                                        % Main loop: generate all root selections
                                        % ----------------------------------------
  for n=0:Nsol-1                        % loop once for each possible solution
    t = conj(rQQ(1:2:p-1));             % extract one representative from each quad
    for m=0:Nquads-1                    % take conjugate recipricols of roots
      if( mod(floor(n/2^m),2) )         % if bit k of n is 1,
        t(m+1) = conj(1/t(m+1));        %   then take conjugate recipricol of root
      endif                             %
    endfor                              %

    rNew            = zeros(p-1,1);     % initialize new root column vector
    rNew(1:2:p-1)  = t;                 % rNew=[r1 (real), r2, r2*, r3, r3*,...]'
    rNew(2:2:p-1)  = conj(t(mod(p+1,2)+1:Nquads));
    rMat(:,n+1)    = rNew;              % store new root vector in root matrix

                                        % Linear phase measurements
                                        % -------------------------------
    h = real(poly(rNew));               % convert new roots into a poly in z
    [H,w] = freqz(h,[1],N);             % compute the frequency response
    phase = unwrap(angle(H));           % unwrapped phase of h
    [pCoefs,pVals]=polyfit(w,phase,1); % find the 1st order poly p(w) that best fits phase(w)
    mse(n+1) = (pVals.yf-phase)'*(pVals.yf-phase)/N;  % measure the error
    sqMag(n+1) = (rNew'*rNew);          % measure the magnitude and store
  endfor
                                        % Find best set of roots
                                        % --------------------------
  [err1,i1] = min(mse);                 % find the 1st location of minimum error
  t=mse;                                % copy mse to temporary vector t
  t(i1) = max(t);                       % get 1st minimum out of the way to find 2nd
  [err2,i2] = min(t);                   % find the 2nd location of minimum error
  t= max(mse)*ones(1,Nsol);             % set t to all large values
  if( sqMag(i1)<sqMag(i2) )             % decide which min to use based on smaller magnitude
    iMinErr = i1;                       %   if the magnitude of first is smaller
  else                                  %
    iMinErr = i2;                       %   if the magnitude of the second is smaller
  endif                                 %
  mmse = mse(iMinErr);                  % minimum mean square error
  rNew = rMat(:,iMinErr);               % extract min phase min magnitude vector

                                        % Compute H(z)=A(z)Q(z)
                                        % -----------------------------------
  rQQ = [rNew; conj(1./rNew)]';         % vector w/ both retained and discarded roots
  rQ  = rQQ(1:p-1);                     % roots of Q(z) (p-1 roots inside unit circle)
  Q   = poly(rQ);                       % convert roots into poly Q(z)
  Q   = sign(real(Q)).*abs(Q);          % eliminate any extraneous imag. components
  H   = conv(A,Q);                      % H(z) = A(z)Q(z)
  rH  = sort(roots(H))';                % roots of H(z)
  h   = (sqrt(2)/sum(H))*H;             % admissibility cond: SUM h_n = sqrt(2)
endfunction

%-----------------------------------------------------------------------------
% Generate experimental coefficients with R(y)!=0
%
%  Input
%  ------
%  p:   number of vanishing moments
%  R = [r_{m-1} ... r_2 r_1 r_0] ==> R(y)=r_{m-1}y^{m-1}+...+r_2 y^2+r_1y+r_0
%
%  Output
%  ------
%  h:   scaling coefficients
%  rQQ: roots of Q(z)Q(z^-1)
%                 |   |_______ p-1 roots outside unit circle
%                 |___________ p-1 roots inside  unit circle
%  rH:  roots of H(z)          p roots at z=-1 and p-1 roots of Q(z)
%
%
%  Theory
%  ------
%  h(n) = scaling coefficients
```


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```
%
%  H(z) = SUM h(n) z^{-n}
%           n
%       = A(z)Q(z)
%
%                 (z+1)^p
%       = sqrt(2)(---)   Q(z)
%                 ( 2 )
%
%-----------------------------------------------------------------------
function [n,h,rQQ,rH] = gen_Rp(p,R)
  [n,A,QQ,rA,rQQ] = gen_Dclass(p,R);  %
  rQ   = rQQ(1:n-1);                  % roots of Q(z) (n-1 roots inside unit circle)
  Q    = poly(rQ);                    % convert roots into poly Q(z)
  Q    = sign(real(Q)).*abs(Q);       % eliminate any extraneous imag. components
  H    = conv(A,Q);                   % H(z) = A(z)Q(z)
  rH   = sort(roots(H))';             % roots of H(z)
  h    = (sqrt(2)/sum(H))*H;          % admissibility cond: SUM h_n = sqrt(2)
endfunction

#-----------------------------------------------------------------------
#function: Generate scaling coefficients h(n) from length-4 parameter alpha
#          Pollen wavelets
#  references:
#    http://www2.isye.gatech.edu/~brani/datasoft/DL.pdf  page 2
#    Burrus page  66
#-----------------------------------------------------------------------
function h = gen_pollen4(alpha)
  h =  (sqrt(2)/4) * [\
       1 + cos(alpha) - sin(alpha) ;
       1 + cos(alpha) + sin(alpha) ;
       1 - cos(alpha) + sin(alpha) ;
       1 - cos(alpha) - sin(alpha) ;
      ];
%  h =  (sqrt(2)/4) * [\  Burrus ?
%       1 - cos(alpha) + sin(alpha) ;
%       1 + cos(alpha) + sin(alpha) ;
%       1 + cos(alpha) - sin(alpha) ;
%       1 - cos(alpha) - sin(alpha) ;
%      ];
endfunction

%-----------------------------------
% Write data to file
%-----------------------------------
function data2file(h,g,rR,rH,filename,comment)
  data  = fopen(filename,"w");
  fprintf(data,'%%=========================================================================\n');
  fprintf(data,'%% Daniel J. Greenhoe \n');
  fprintf(data,'%% file: %s \n',filename);
  fprintf(data,'%% %s \n',comment);
  fprintf(data,'%% %s \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
  fprintf(data,'%%=========================================================================\n\n');

  p=length(h)/2;
  fprintf(data ,'scaling coefficients\n' );
  fprintf(data ,'-------------------------\n' );
  for n=1:length(h)
    fprintf(data ,'h_{%02d} = %13.10f  \n',n-1,h(n) );
  endfor

  fprintf(data ,'\n' );
  fprintf(data ,'wavelet coefficients\n');
  fprintf(data ,'-------------------------\n' );
  for n=1:length(g)
    fprintf(data ,'g_{%02d} = %13.10f  \n',n-1,g(n) );
  endfor

  fprintf(data ,'\n' );
  fprintf(data ,'roots of Q(z)\n' );
  fprintf(data ,'-------------------------\n' );
  for n=1:p-1
    fprintf(data ,'r_{%02d} = %13.10f  %13.10fi \n',n-1, real(rR(n)), imag(rR(n)) );
  endfor

  fprintf(data ,'\n' );
  fprintf(data ,'roots of Q(z^-1)\n' );
```


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```
  fprintf(data ,'-------------------------\n');
  for n=p:length(rR)
    fprintf(data ,'r_{%02d} = %13.10f  %13.10fi \n',n-1, real(rR(n)), imag(rR(n)) );
  endfor

  fprintf(data ,'\n');
  fprintf(data ,'roots of H(z)\n');
  fprintf(data ,'-------------------------\n');
  for n=1:length(rH)
    fprintf(data ,'r_{%02d} = %13.10f  %13.10fi \n',n-1, real(rH(n)), imag(rH(n)) );
    if(n==p-1) fprintf(data,'\n'); endif
  endfor

  fprintf(data ,'\n' );
  fprintf(data ,'LaTeX drawing commands   \n');
  fprintf(data ,'-------------------------\n');
  for n=1:p-1
    fprintf(data ,'  \\put( %15.10f, %15.10f ) {\\circle {15}} \n',real(rR(n))*100, imag(rR(n))*100 );
  endfor
  fprintf(data,"\n");
  for n=p:length(rR)
    fprintf(data ,'  \\put( %15.10f, %15.10f ) {\\circle*{15}} \n',real(rR(n))*100, imag(rR(n))*100 );
  endfor

  fflush(data);
  fclose(data);
  printf('output file name = \"%s\" \n',filename);
endfunction

%-----------------------------------------------------------------------
% Write data to plot file
% <filename> can be used by the command '\fileplot' in a TeX environment
% to generate a plot of (x,y).
% The command '\fileplot' is available in the PSTricks package of TeX.
% Reference: http://www.ctan.org/pkg/pstricks
%-----------------------------------------------------------------------
function data2plotfile(x,y,filename,comment)
  data  = fopen(filename,'w');
  fprintf(data,'%%=====================================================================\n');
  fprintf(data,'%% Daniel J. Greenhoe \n');
  fprintf(data,'%% file: %s \n',filename);
  fprintf(data,'%% number of data points = %d\n',length(x));
  fprintf(data,'%% %s \n',comment);
  fprintf(data,'%% %s \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
  fprintf(data,'%% The command \\fileplot{%s} may be used in a LaTeX environment for plotting
          data.\n',filename);
  fprintf(data,'%% \\fileplot is available in the LaTeX PSTricks package.\n');
  fprintf(data,'%% Reference: http://www.ctan.org/pkg/pstricks\n');
  fprintf(data,'%%=====================================================================\n');

  fprintf(data ,'[\n');
  for n=1:length(x)
    fprintf(data ,'(%13.10f,%13.10f)\n',x(n),y(n));
  endfor
  fprintf(data,']\n');
  fprintf(data,'%%---end of file---' );
  fflush(data);
  fclose(data);
  printf("data written to \'%s\' \n",filename);
endfunction

%-----------------------------------------------------------------------
% Demonstration of Daubechies-p wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%-----------------------------------------------------------------------
function demo_Dp(p,N,iterations)
                                    % calculate coefficient sequences
                                    % ------------------------
  [h,rQQ,rH] = gen_Dp(p);           % Daubechies-p
  g   = h2g_coefs(h);               % generate wavelet coefficients g(n)
  d   = round(N/length(h));         % density=round(samples/unit(1))
  phi = gen_phi(h,  iterations,d);  % generate phi(x) from h(n)
  psi = gen_psi(phi,g,d);           % generate psi(x) from g(n)
  M   = length(phi);

                                    % Generate output data files
```


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```
                                        % ------------------------
  x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
  data2file(h,g,rQQ,rH,sprintf('d%d.dat',p),     sprintf('data for Daubechies-%d wavelets',p) );
  data2plotfile(x,phi,sprintf('d%d_phi.dat',p),sprintf('plot file for Daubechies-%d scaling
            function',p));
  data2plotfile(x,psi,sprintf('d%d_psi.dat',p),sprintf('plot file for Daubechies-%d wavelet
            function',p));
  plot(x,phi,x,psi);

endfunction

%-----------------------------------------------------------------------------
% Demonstration of Symmlet-p wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%-----------------------------------------------------------------------------
function demo_Symmlet_p(p,N,iterations)
                                        % Generate scaling coefficients {h_n}
                                        % ------------------------
  [h,rQQ,rH] = gen_Sp(p);               % Symmlet-p
  g   = h2g_coefs(h);                   % generate wavelet coefficients g(n)
  d   = round(N/length(h));             % density=round(samples/unit(1))
  phi = gen_phi(h,  iterations,d);      % generate phi(x) from h(n)
  psi = gen_psi(phi,g,d);               % generate psi(x) from g(n)
  M   = length(phi);

                                        % Generate output data files
                                        % ------------------------
  x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
  data2file(h,g,rQQ,rH, sprintf("s%d.dat",p),     sprintf("Symmlet-%d data file",p));
  data2plotfile(x,phi,  sprintf('s%d_phi.dat',p),sprintf('plot file for Symmlet-%d scaling
            function',p));
  data2plotfile(x,psi,  sprintf('s%d_psi.dat',p),sprintf('plot file for Symmlet-%d wavelet
            function',p));
  plot(x,phi,x,psi);
endfunction

%-----------------------------------------------------------------------------
% Demonstration of Daubechies-p class wavelet with R(y)!=0
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%-----------------------------------------------------------------------------
function demo_Ry_p(p,N,iterations)
                                        % R(y)
                                        % ------------------------
 R=[  3 0 5 0 7 0 3 0 1 0];                          % R(y)
  ptitle=sprintf("p=%d      R(y)=3y^9 + 5y^7 + 7y^5 + 3y^3 + y      Daniel J. Greenhoe",p);% title

                                        % Generate scaling coefficients {h_n}
                                        % ------------------------
  [n,h,rQQ,rH]= gen_Rp(p,R);            % Ry(p)
  g   = h2g_coefs(h);                   % generate wavelet coefficients g(n)
  d   = round(N/length(h));             % density=round(samples/unit(1))
  phi = gen_phi(h,  iterations,d);      % generate phi(x) from h(n)
  psi = gen_psi(phi,g,d);               % generate psi(x) from g(n)

                                        % Generate output
                                        % -------------------------
  data2file(h,g,rQQ,rH, sprintf("R%d.dat",p), sprintf("R(y)-%d data file",p) );   pause(delay_sec);
endfunction

%-----------------------------------------------------------------------------
% Demonstration of Pollen length 4 wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%  alpha:      parameter alpha
%  p4_phi.dat and p4_psi.dat can be used with LaTeX's \plotfile command.
%  \pltfile is available in the pstricks package
%   Reference: http://www.ctan.org/pkg/pstricks
%-----------------------------------------------------------------------------
function demo_pollen4a(alpha,N,iterations)
                                        % calculate coefficient sequences
                                        % ------------------------
  h   = gen_pollen4(alpha);      % Pollen length-4
  g   = h2g_coefs(h);                   % generate wavelet coefficients g(n)
  d   = round(N/length(h));             % density=round(samples/unit(1))
  phi = gen_phi(h,  iterations,d);      % generate phi(x) from h(n)
```


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```
  psi = gen_psi(phi,g,d);                % generate psi(x) from g(n)
  M   = length(phi);

                                         % Generate output data files
                                         % ------------------------
  x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
  %data2file(h,g,rQQ,rH,'p4.dat',   sprintf('data for Pollen-4 alpha=%13.10f wavelets',alpha));
  data2plotfile(x,phi,'p4_phi.dat',sprintf('plot file for Pollen-4 alpha=%13.10f scaling
            function',alpha));
  data2plotfile(x,psi,'p4_psi.dat',sprintf('plot file for Pollen-4 alpha=%13.10f wavelet
            function',alpha));
  plot(x,phi,x,psi);

endfunction

%-----------------------------------------------------------------------------
% Demonstration of Pollen length 4 wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%  nalpha:     number of alpha values
%  a:          starting alpha value
%  b:          ending alpha value
%  p4_phi.dat and p4_psi.dat can be used with GNU Plot's 'splot' function
%  Reference: http://www.gnuplot.info
%-----------------------------------------------------------------------------
function demo_pollen4(a,b,nalpha,N,iterations)
  filename1='p4_psi.dat';
  filename2='p4_phi.dat';
  data1  = fopen(filename1,'w');
  data2  = fopen(filename2,'w');
  fprintf(data1,'#=========================================================================\n');
  fprintf(data1,'# Daniel J. Greenhoe \n');
  fprintf(data1,'# file: %s \n',filename1);
  fprintf(data1,'# %s \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
  fprintf(data1,'# The command splot{%s} may be used in a GNU Plot environment for plotting
            data.\n',filename1);
  fprintf(data1,'# Reference: http://www.gnuplot.info\n');
  fprintf(data1,'#=========================================================================\n');

  fprintf(data2,'#=========================================================================\n');
  fprintf(data2,'# Daniel J. Greenhoe \n');
  fprintf(data2,'# file: %s \n',filename2);
  fprintf(data2,'# %s \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
  fprintf(data2,'# The command splot{%s} may be used in a GNU Plot environment for plotting
            data.\n',filename2);
  fprintf(data2,'# Reference: http://www.gnuplot.info\n');
  fprintf(data2,'#=========================================================================\n');

  for i = 0:nalpha-1
                                         % calculate coefficient sequences
                                         % ------------------------
    alpha = a + i/(nalpha-1)*(b-a)       % calculate alpha
    h     = gen_pollen4(alpha);          % Pollen length-4
    g     = h2g_coefs(h);                % generate wavelet coefficients g(n)
    d     = round(N/length(h));          % density=round(samples/unit(1))
    phi   = gen_phi(h,  iterations,d);   % generate phi(x) from h(n)
    psi   = gen_psi(phi,g,d);            % generate psi(x) from g(n)
    M     = length(phi);

                                         % Generate output data files
                                         % ------------------------
    x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
    y1 = psi;
    y2 = phi;

    for n=1:length(x)
      fprintf(data1 ,'%13.10f  %13.10f  %13.10f\n',alpha,x(n),y1(n));
    endfor
    for n=1:length(x)
      fprintf(data2 ,'%13.10f  %13.10f  %13.10f\n',alpha,x(n),y2(n));
    endfor
    fprintf(data1 ,'\n');
    fprintf(data2 ,'\n');

  endfor
  fprintf(data1,'##---end of file---' );
  fprintf(data2,'##---end of file---' );
```


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```
  fflush(data1);
  fflush(data2);
  fclose(data1);
  fclose(data2);
  printf("data written to \'%s\' \n",filename1);
  printf("data written to \'%s\' \n",filename2);

endfunction

%=====================================
% Main
%=====================================
                                      % parameters
                                      %-------------------------------------
N = 1024;                             % number of data points
iterations = 16;                      % number of iterations

                                      % demos
                                      %-------------------------------------
%for p=1:12
%  demo_Dp(p,N,iterations);
%endfor
%demo_Dp(16,N,iterations);

%demo_Symmlet_p( 4,N,iterations);
%demo_Symmlet_p( 8,N,iterations);
%demo_Symmlet_p(12,N,iterations);
%demo_Symmlet_p(16,N,iterations);
%demo_Ry_p(3);
%demo_pollen4a(pi/6,16,iterations)%
%demo_pollen4a(pi/4,16,iterations)%
%demo_pollen4a(pi/2,16,iterations)%
%demo_pollen4a(pi,16,iterations)%
%demo_pollen4(0,pi,256,256,iterations)%
demo_pollen4(0,pi,32,32,iterations)%
%function demo_pollen4(a,b,nalpha,N,iterations)

%=====================================
% End Processing
%=====================================
```

Y.2 B-spline polynomial calculation and plotting

The polynomials for *B-spline*s as demonstrated in Example O.3 (page 338)–Example O.9 (page 342) can be calculated using the free and open source software package *Maxima* along with the following script file:

```
/*=========================================================================*
 * Daniel J. Greenhoe                                                      *
 * Maxima script file                                                      *
 * To execute this script, start Maxima in a command window                *
 * in the subdirectory containing this file (e.g. c:\math\maxima\)         *
 * and then after the (%i...) prompt enter                                 *
 *   batchload("bspline.max")$                                             *
 * Data produced will be written to the file "bsplineout.txt".             *
 * reference: http://maxima.sourceforge.net/documentation.html             *
 *=========================================================================*/
/*-------------------------------------
 * initialize script
 *-------------------------------------*/
reset();
kill(all);
load(orthopoly);
display2d:false; /* 2-dimensional display */
writefile("bsplineout.txt");
/*-------------------------------------
 * n = B-spline order parameter
 * may be set to any value in {1,2,3,...}
```


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```
 *-------------------------------------*/
n:2;
print("=====================================================================");
print("Daniel J. Greenhoe");
print("Output file for nth order B-spline Nn(x) calculation, n=",n," .");
print("Output produced using Maxima running the script file bspline.max");
print("=====================================================================");
Nnx:(1/n!)*sum((-1)^k*binomial(n+1,k)*(x-k)^n*unit_step(x-k),k,0,n+1);
print("-------------------------------------------------------");
print("              n+1      k (n+1)       n                    ");
print(" n! Nn(x) = SUM (-1)  (    ) (x-k)   step(x-k) ,n=",n);
print("              k=0       ( k )                            ");
print("              ",n+1,"      k (",n+1,")      ",n);
print(n,"! Nn(x) = SUM (-1)  (    ) (x-k)   step(x-k)");
print("              k=0       ( k )");
print("            = ",expand(n!*Nnx));
print("-------------------------------------------------------");
assume(x<=0);     print(n!,"N(x)= ",expand(n!*Nnx)," for x<=0");  forget(x<=0);
for i:0 thru n step 1 do(
  assume(x>i,x<(i+1)),
  print(n!,"N(x)= ",expand(n!*Nnx)," for ",i,"<x<",i+1),
  tex(expand(n!*Nnx),"djg.tex"),/*write output in TeX format to file "djg.tex"*/
  forget(x>i,x<(i+1))
  );
assume(x>(n+1)); print(n!,"N(x)= ",expand(n!*Nnx)," for x>",n+1); forget(x>(n+1));
print("-------------------------------------------------------");
print(" values at some specific points x:                      ");
print("-------------------------------------------------------");
y:Nnx,x=(n+1)/2;print("N(",(n+1)/2,")= ",y," (center value)");
y:Nnx,x=(n+2)/2;print("N(",(n+2)/2,")= ",y);
y:Nnx,x=(n+3)/2;print("N(",(n+3)/2,")= ",y);
/*-------------------------------------
 * close output file
 *-------------------------------------*/
closefile();
```

Once the polynomial expressions for a *B-spline* have been calculated, they can be plotted within a LaTeX environment using the pst-plot package along with a LaTeX source file such as the following:[2]

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% N_3(x) B-spline
% nominal unit = 10mm
%=========================================================================
\begin{pspicture}(-1,-0.5)(5,1)
  %-------------------------------------
  % parameters
  %-------------------------------------
  \psset{plotpoints=64,labelsep=1pt}%
  %-------------------------------------
  % axes
  %-------------------------------------
  \psaxes[linewidth=0.75pt,linecolor=axis,yAxis=false,ticks=x,labels=x]{<->}(0,0)(-1,0)(5,1)% x axis
  \psaxes[linewidth=0.75pt,linecolor=axis,xAxis=false,ticks=x,labels=x]{->}(0,0)(-1,0)(5,1)%  y axis
  %-------------------------------------
  % annotation
  %-------------------------------------
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](2,0)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.667)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](1,0)(1,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](3,0)(3,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.1667)(3,0.1667)%
  \uput[180](0,0.667){$\sfrac{2}{3}$}%
  \uput[180](0,0.1667){$\sfrac{1}{6}$}%
  %-------------------------------------
  % function plot
  %-------------------------------------
  \psplot{0}{1}{+1 x 3 exp mul                                    6 div}% for 0<=x<=1
```

[2]For help with PostScript®math operators, see Adobe (1999) pages 508–509 ⟨Arithmetic and Math Operators⟩.

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```
  \psplot{1}{2}{-3 x 3 exp mul +12 x 2 exp mul add -12 x mul add  +4 add 6 div}% for 1<=x<=2
  \psplot{2}{3}{+3 x 3 exp mul -24 x 2 exp mul add +60 x mul add -44 add 6 div}% for 2<=x<=3
  \psplot{3}{4}{-1 x 3 exp mul +12 x 2 exp mul add -48 x mul add +64 add 6 div}% for 3<=x<=4
  \psline(0,0)(-0.5,0)\psline[linestyle=dotted](-0.5,0)(-0.75,0)%              % for x<=0
  \psline(4,0)(4.5,0)\psline[linestyle=dotted](4.5,0)(4.75,0)%                 % for x>=4
\end{pspicture}%
```

Alternatively, one can plot $N_3(x)$ more or less directly from the equation given in Theorem O.1 (page 339) without first calculating the polynomial expressions:

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% N_3(x) B-spline
% nominal unit = 10mm
%==========================================================================
\begin{pspicture}(-1,-0.5)(5,1)
  %-------------------------------------
  % parameters
  %-------------------------------------
  \psset{plotpoints=64,labelsep=1pt}%
  %-------------------------------------
  % axes
  %-------------------------------------
  \psaxes[linewidth=0.75pt,linecolor=axis,yAxis=false,ticks=x,labels=x]{<->}(0,0)(-1,0)(5,1)% x axis
  \psaxes[linewidth=0.75pt,linecolor=axis,xAxis=false,ticks=x,labels=x]{->}(0,0)(-1,0)(5,1)%  y axis
  %-------------------------------------
  % annotation
  %-------------------------------------
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](2,0)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.667)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](1,0)(1,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](3,0)(3,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.1667)(3,0.1667)%
  \uput[180](0,0.667){$\sfrac{2}{3}$}%
  \uput[180](0,0.1667){$\sfrac{1}{6}$}%
  %-----------------------------------------------------------------------
  % for n=3
  %        1  n+1        (n+1)                    1   4          ( 4 )
  % N_n(x)--- SUM (-1)^k (    ) (x-k)^n s(x-k) = --- SUM  (-1)^k (    ) (x-k)^3 s(x-k)
  %        n! k=1        ( k )                    3! k=1         ( k )
  %
  % where s(x) = 0 for x<0 and 1 for x>=0  (step function)
  %-----------------------------------------------------------------------
  \psplot{0}{1}  {1 x 0 sub 3 exp mul   6 div}%
  \psplot{1}{2}  {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 div}%
  \psplot{2}{3}  {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   6 div}%
  \psplot{3}{4}  {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   4 x 3 sub
            3 exp mul sub   6 div}%
  \psplot{4}{4.5}{1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   4 x 3 sub
            3 exp mul sub   1 x 4 sub 3 exp mul add 6 div}%
  %    N_3(x) = [(4choose0)(x-0)^3    - (4choose1)(x-1)^3       + (4choose2)(x-2)^3       -
            (4choose3)(x-3)^3       + (4choose4)(x-4)^3       ]/3!
  %           = [         1 (x-0)^3    -         4 (x-1)^3       +         6 (x-2)^3       -         4
            (x-3)^3       +         1 (x-4)^3       ]/6
  \psline(0,0)(-0.5,0)%
  \psline[linestyle=dotted](-0.5,0)(-0.75,0)%
  \psline[linestyle=dotted](4.5,0)(4.75,0)%
\end{pspicture}%
```

The LaTeX source listed above can be made to output a pdf file with tight borders using the preview package and[3]

```
\usepackage[active,tightpage]{preview}%
\PreviewBorder=0pt%
\PreviewEnvironment{pspicture}%
```

[3] http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput

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Y.3 Auto-power spectrum plot

The plot provided in Figure O.3 (page 353) was generated using C and LaTeX code very similar to that displayed in this section:

1. C code for generating data points. This outputs to stdout (standard output terminal), but can be redirected into a file such as Snn_bspline1.dat, Snn_bspline2.dat, Snn_bspline3.dat, etc. In a *Windows* command environment ("DOS" window), this can be performed with something like this:

   C:\ mycprogram.exe > Snn_bspline1.dat

```c
int bspline_Sdat(void){
  const double n=3;       /* order of B-spline                          */
  const long M=1024;      /* approximate number of data points          */
  const long N=1000000;   /* number of iterations per data point        */
  double w=0,s=0;
  long k=0;
  printf("[\n");                                /* LaTeX \fileplot support */
  for(w=-8.0;w<=8.0;w+=16./(double)M){
    s=0;
    for(k=1; k<=N; k++){
      s += pow(1.0/(2*k-w/PI),2.*n+2.);         /* 1st summation           */
      s += pow(1.0/(2*k+w/PI),2.*n+2.);         /* 2nd summation           */
      }
    s*=pow(sin(w/2.)/(PI/2.),2.*n+2.);          /* scaling factor          */
    if(w==0) s+=1;                              /* 1st term: w=0 case      */
    else     s+=pow(sin(w/2.)/(w/2.),2.*n+2.);  /* 1st term: w!=0 case     */
    printf("(%lf, %lf)\n",w,s);                 /* print one data point    */
    }
  printf("]\n");                                /* LaTeX \fileplot support */
  return 0;
  }
```

2. The files produced by item (1) can be plotted within LaTeX using the pst-plot package.

```latex
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% auto-power spectrum plots for B-splines of order n=0,1,2,3
% nominal xunit = 10mm
% nominal yunit = 20mm
%==========================================================================
\begin{pspicture}(-8.5,-0.25)(9,1.25)%
  %-------------------------------------
  % axes
  %-------------------------------------
  \psaxes[linecolor=axis,linewidth=0.75pt,yAxis=false,ticks=none,labels=none]{<->}(0,0)(-8.5,0)(8.5,1.25)% x-axis
  \psaxes[linecolor=axis,linewidth=0.75pt,xAxis=false,ticks=none,labels=none]{->}(0,0)(-8.5,0)(8.5,1.25)% y-axis
  %-------------------------------------
  % y-axis guide lines
  %-------------------------------------
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-8.5,0.333333)(8.5,0.333333)% min line for n=1 B-spline
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-8.5,0.133333)(8.5,0.133333)% min line for n=2 B-spline
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-8.5,0.053968)(8.5,0.053968)% min line for n=3 B-spline
  %-------------------------------------
  % x-axis labeling
  %-------------------------------------
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](3.1415927,0)(3.1415927,1)%
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-3.1415927,0)(-3.1415927,1)%
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](6.283185,0)(6.283185,1)%
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-6.283185,0)(-6.283185,1)%
  \uput{2pt}[0](8.5,0){$\omega$}%
  \uput{2pt}[-90](0,0){$0$}%
  \uput{2pt}[-90](-6.2831853071795864769252867665 59,0){$-2\pi$}%
  \uput{2pt}[-90](-3.1415926535897932384626433832795,0){$-\pi$}%
  \uput{2pt}[-90](3.1415926535897932384626433832795,0){$\pi$}%
```


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```
  \uput{2pt}[-90](6.28318530717958647692528676655 9,0){$2\pi$}%
  %-------------------------------------
  % data plot
  %-------------------------------------
  \psline(-8,1)(8,1)%                                          order n=0 B-spline data plot
  \fileplot{../bsplines/Snn_bspline1.dat}% order n=1 B-spline data plot
  \fileplot{../bsplines/Snn_bspline2.dat}% order n=2 B-spline data plot
  \fileplot{../bsplines/Snn_bspline3.dat}% order n=3 B-spline data plot
  %-------------------------------------
  % data plot lables
  %-------------------------------------
  \uput{2pt}[90](1.9,1){$n=0$}%
  \uput{2pt}[90](3.1415297,0.333333){$n=1$}%
  \uput{2pt}[90](3.1415297,0.133333){$n=2$}%
  \uput{2pt}[-45](4.5,0.335){$n=3$}%
  %-------------------------------------
  % y-axis labels
  %-------------------------------------
  \uput*{2mm}[180](0,1){$1$}%
  \uput*{2mm}[180](0,0.333333){$\sfrac{1}{3}\approx0.33$}%   min value for n=1 B-spline
  \uput*{2mm}[180](0,0.133333){$\sfrac{2}{15}\approx0.13$}%  min value for n=2 B-spline
  \uput*{2mm}[0](0,0.053968){$\sfrac{17}{315}\approx0.054$}% min value for n=3 B-spline
\end{pspicture}%
```

3. The LaTeX source listed in item (2) can be made to output a pdf file with tight borders using the preview package and[4]

```
\usepackage[active,tightpage]{preview}%
\PreviewBorder=0pt%
\PreviewEnvironment{pspicture}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% XeLaTeX file
% generate tight pdf graphics file for inclusion in a document
% nominal unit = 10mm
% nominal font size = \scriptsize
%=========================================================================
\input{shelltop.tex}%
\begin{document}%
  %\gsize%
  %\scriptsize%
  %\psset{unit=1\latunit}%
  \psset{xunit=1.0\latunit}%
  \psset{yunit=2.0\latunit}%
  \input{../bsplines/Snn_bspline.tex}%
\end{document}%
```

4. The source listed in item (3) uses a file called ../common/math/graphics/pdfs/shelltop.tex:

```
%==========================================================================
% Daniel J. Greenhoe
% XeLaTeX file
% preamble packages for shell files to
% generate tight pdf graphics file for inclusion in a document
%=========================================================================
\documentclass{article}%
%-------------------------------------
% Style Packages
%-------------------------------------
  \usepackage{../../../sty/packages}%
  \usepackage{../../../sty/fonts}%
  \usepackage{../../../sty/dan}%
  \usepackage{../../../sty/colors_rgb}%
%\usepackage{../../../sty/colors_cmyk}%
%\usepackage{../../../sty/colors_gray}%
  \usepackage{../../../sty/math}%
  \usepackage{../../../sty/wavelets}%
%\usepackage{../../../sty/language}%
  \usepackage{../../../sty/ocs}%
  \usepackage{../../../sty/defaults}  % default values
```

[4] http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput

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```
 \usepackage{../../../sty/switches}  %
%-------------------------------------
% color space
%-------------------------------------
%\selectcolormodel{cmyk}%
 \selectcolormodel{rgb}%
%-------------------------------------
% redefine some commands
%-------------------------------------
\renewcommand{\struct}[1]{{#1}}% place here and in the index and under index structures! heading
\renewcommand{\structe}[1]{\emph{#1}}% place here with emphasis and in the index and under index structures! heading
\renewcommand{\structb}[1]{\textbf{#1}}% place here with emphasis and in the index and under index structures! heading
\renewcommand{\structd}[1]{\textbf{#1}}% place here with emphasis and in the index and under index structures! heading
%-------------------------------------
% preview mode
% reference:
          http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput
%-------------------------------------
\usepackage[active,tightpage]{preview}%
\PreviewBorder=0pt%
\PreviewEnvironment{pspicture}%
```


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Back Matter

❝It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.❞
Niels Henrik Abel (1802–1829), Norwegian mathematician [5]

❝When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.❞
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [6]

ancient library of Alexandria

The Book Worm by *Carl Spitzweg*, circa 1850 [7]

[5] quote: Simmons (2007) page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

[6] quote: Machiavelli (1961) page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

[7] http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

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❝To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.❞

Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [8]

[8] quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

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