Partition of unity systems and B-splines

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Abstract

This paper presents the basic principles of partition of unity systems and B-splines. Analysis of these systems is performed using Fourier analysis, multi-resolution analysis, and wavelet analysis.

Contents

1 Background: harmonic analysis

1.1 Families of functions

This paper is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition 1.2 page 2). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition 1.1 page 2). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition 1.1. *Let X and Y be sets.*

> *The space Y^X represents the set of all functions with* DOMAIN X *and* RANGE Y *such that*
> $$Y^X \triangleq \{\mathsf{f}(x)|\mathsf{f}(x) : X \to Y\}$$

Definition 1.2. *Let $\mathbb{R}$ be the set of real numbers, $\mathscr{B}$ the set of* BOREL SETS *on $\mathbb{R}$, and μ the standard* BOREL MEASURE *on $\mathscr{B}$. Let $\mathbb{R}^{\mathbb{R}}$ be as in Definition 1.1 page 2.*

The space of ***Lebesgue square-integrable functions*** $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ *(or* $L^2_{\mathbb{R}}$*) is defined as*

$$L^2_{\mathbb{R}} \triangleq L^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \mathsf{d}\mu < \infty \right\}.$$

The ***standard inner product*** $\langle \triangle \mid \triangledown \rangle$ *on* $L^2_{\mathbb{R}}$ *is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\,\mathsf{d}x.$$

The ***standard norm*** $\|\cdot\|$ *on* $L^2_{\mathbb{R}}$ *is defined as* $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$.

Definition 1.3. [1] *Let X be a set.*

The ***indicator function*** $\mathbb{1} \in \{0,1\}^{2^X}$ *is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \qquad \forall x \in X,\ A \in 2^X \\ 0 & \text{for } x \notin A \qquad \forall x \in X,\ A \in 2^X \end{cases}$$

The indicator function $\mathbb{1}$ *is also called the* ***characteristic function***.

1.2 Trigonometric functions

1.2.1 Definitions

Lemma 1.1. [2] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and* $\frac{\mathsf{d}}{\mathsf{dx}} \in \boldsymbol{C}^{\boldsymbol{C}}$ *the differentiation operator.* $\frac{\mathsf{d}}{\mathsf{dx}}^2 \mathsf{f} + \mathsf{f} = 0 \iff$

$$\left\{ \begin{aligned} \mathsf{f}(x) &= \underbrace{[\mathsf{f}](0) \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0) \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \\ &= \left(\mathsf{f}(0) + \left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)x\right) - \left(\frac{\mathsf{f}(0)}{2!}x^2 + \frac{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)}{3!}x^3\right) + \left(\frac{\mathsf{f}(0)}{4!}x^4 + \frac{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)}{5!}x^5\right) \cdots \end{aligned} \right\}$$

Definition 1.4. [3] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and* $\frac{\mathsf{d}}{\mathsf{dx}} \in \boldsymbol{C}^{\boldsymbol{C}}$ *the differentiation operator.*

The ***cosine*** *function* $\cos(x)$ *is the function* $\mathsf{f} \in \boldsymbol{C}$ *that satisfies the following conditions:*

$$\underbrace{\frac{\mathsf{d}}{\mathsf{dx}}^2 \mathsf{f} + \mathsf{f} = 0}_{\text{2nd order homogeneous differential equation}} \qquad \underbrace{\mathsf{f}(0) = 1}_{\text{1st initial condition}} \qquad \underbrace{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0) = 0}_{\text{2nd initial condition}}$$

The ***sine*** *function* $\sin(x)$ *is the function* $\mathsf{g} \in \boldsymbol{C}$ *that satisfies the following conditions:*

$$\underbrace{\frac{\mathsf{d}}{\mathsf{dx}}^2 \mathsf{g} + \mathsf{g} = 0}_{\text{2nd order homogeneous differential equation}} \qquad \underbrace{\mathsf{g}(0) = 0}_{\text{1st initial condition}} \qquad \underbrace{\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{g}\right](0) = 1}_{\text{2nd initial condition}}$$

Theorem 1.1. [4]

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[1] Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) page 22, de la Vallée-Poussin (1915) page 440
[2] Rosenlicht (1968) page 156, Liouville (1839)
[3] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[4] Rosenlicht (1968) page 157

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} &= 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots && \forall x\in\mathbb{R}\\
\sin(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots && \forall x\in\mathbb{R}
\end{aligned}$$

Proposition 1.1. [5] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathsf{d}}{\mathsf{dx}}\in\boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $\mathsf{f}'(0)\triangleq\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0)$.*

$$\underbrace{\frac{\mathsf{d}}{\mathsf{dx}}^2\mathsf{f}+\mathsf{f}=0}_{\text{2ND ORDER HOMOGENEOUS DIFFERENTIAL EQUATION}} \iff \mathsf{f}(x)=\mathsf{f}(0)\cos(x)+\mathsf{f}'(0)\sin(x) \qquad \forall\mathsf{f}\in\boldsymbol{C},\forall x\in\mathbb{R}$$

Theorem 1.2. [6] *Let $\frac{\mathsf{d}}{\mathsf{dx}}\in\boldsymbol{C}^{\boldsymbol{C}}$ be the differentiation operator.*

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{dx}}\cos(x) &= -\sin(x) && \forall x\in\mathbb{R}\\
\frac{\mathsf{d}}{\mathsf{dx}}\sin(x) &= \cos(x) && \forall x\in\mathbb{R}
\end{aligned}$$

1.2.2 The complex exponential

Definition 1.5. *The function $\mathsf{f}\in\mathbb{C}^{\mathbb{C}}$ is the **exponential function** $\exp(ix)\triangleq\mathsf{f}(x)$ if*

1. $\frac{\mathsf{d}}{\mathsf{dx}}^2\mathsf{f}+\mathsf{f} = 0$ *(second order homogeneous differential equation)* *and*
2. $\mathsf{f}(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{f}\right](0) = i$ *(second initial condition).*

Theorem 1.3 (Euler's identity). [7]

$$e^{ix}=\cos(x)+i\sin(x) \qquad \forall x\in\mathbb{R}$$

Corollary 1.1.

$$e^{ix}=\sum_{n\in\mathbb{W}}\frac{(ix)^n}{n!} \qquad \forall x\in\mathbb{R}$$

Corollary 1.2. [8]

$$e^{i\pi}+1=0$$

The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any LTI operator (Theorem 1.4 page 5).
- The exponential generates a continuous point spectrum for the differential operator.

[5] Rosenlicht (1968) page 157. The general solution for the *non-homogeneous* equation $\frac{\mathsf{d}}{\mathsf{dx}}^2\mathsf{f}(x)+\mathsf{f}(x)=\mathsf{g}(x)$ with initial conditions $\mathsf{f}(a)=1$ and $\mathsf{f}'(a)=\rho$ is

$$\mathsf{f}(x)=\cos(x)+\rho\sin(x)+\int_a^x\mathsf{g}(y)\sin(x-y)\,\mathsf{d}y.$$

This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992) page 371, Liouville (1839). Volterra equation references: Pedersen (2000) page 99, Lalescu (1908), Lalescu (1911).

[6] Rosenlicht (1968) page 157

[7] Euler (1748), Bottazzini (1986) page 12

[8] Euler (1748), Euler (1988), http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

Theorem 1.4. [9] *Let* **L** *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*

$$\check{\mathsf{h}}(s) \triangleq \left\langle \mathsf{h}(t,\omega) \,|\, e^{st} \right\rangle \qquad \text{(LAPLACE TRANSFORM).}$$

$$\left.\begin{array}{ll} 1. & \mathbf{L} \textit{ is linear and} \\ 2. & \mathbf{L} \textit{ is time-invariant} \end{array}\right\} \implies \mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{\text{eigenvalue}} \; \underbrace{e^{st}}_{\text{eigenvector}}$$

1.2.3 Trigonometric Identities

Corollary 1.3 (Euler formulas). [10]

$$\begin{array}{rclclr} \cos(x) &=& \Re\left(e^{ix}\right) &=& \dfrac{e^{ix}+e^{-ix}}{2} & \forall x\in\mathbb{R} \\ \sin(x) &=& \Im\left(e^{ix}\right) &=& \dfrac{e^{ix}-e^{-ix}}{2i} & \forall x\in\mathbb{R} \end{array}$$

Theorem 1.5. [11]

$$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

Theorem 1.6 (shift identities).

$$\begin{array}{rclr|rclr} \cos\left(x+\frac{\pi}{2}\right) &=& -\sin x & \forall x\in\mathbb{R} & \sin\left(x+\frac{\pi}{2}\right) &=& +\cos x & \forall x\in\mathbb{R} \\ \cos\left(x-\frac{\pi}{2}\right) &=& +\sin x & \forall x\in\mathbb{R} & \sin\left(x-\frac{\pi}{2}\right) &=& -\cos x & \forall x\in\mathbb{R} \end{array}$$

Theorem 1.7 (product identities).

$$\begin{array}{rcrlclr} \cos x \cos y &=& & \frac{1}{2}\cos(x-y) &+& \frac{1}{2}\cos(x+y) & \forall x,y\in\mathbb{R} \\ \cos x \sin y &=& - & \frac{1}{2}\sin(x-y) &+& \frac{1}{2}\sin(x+y) & \forall x,y\in\mathbb{R} \\ \sin x \cos y &=& & \frac{1}{2}\sin(x-y) &+& \frac{1}{2}\sin(x+y) & \forall x,y\in\mathbb{R} \\ \sin x \sin y &=& & \frac{1}{2}\cos(x-y) &-& \frac{1}{2}\cos(x+y) & \forall x,y\in\mathbb{R} \end{array}$$

Theorem 1.8 (double angle formulas). [12]

$$\begin{array}{rclr} \cos(x+y) &=& \cos x \cos y - \sin x \sin y & \forall x,y\in\mathbb{R} \\ \sin(x+y) &=& \sin x \cos y + \cos x \sin y & \forall x,y\in\mathbb{R} \\ \tan(x+y) &=& \dfrac{\tan x + \tan y}{1 - \tan x \tan y} & \forall x,y\in\mathbb{R} \end{array}$$

Theorem 1.9 (squared identities).

$$\begin{array}{rclr} \cos^2 x &=& \frac{1}{2}(1+\cos 2x) & \forall x\in\mathbb{R} \\ \sin^2 x &=& \frac{1}{2}(1-\cos 2x) & \forall x\in\mathbb{R} \\ \cos^2 x + \sin^2 x &=& 1 & \forall x\in\mathbb{R} \end{array}$$

[9] Mallat (1999) page 2, …page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf
[10] Euler (1748), Bottazzini (1986) page 12
[11] Rudin (1987) page 1
[12] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

1.3 Fourier Series

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition 1.6. [13]

*The **Fourier Series operator** $\hat{\mathbf{F}} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{R}}$ is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{f}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{f} \text{ is periodic with period } \tau\right\}$$

Theorem 1.10. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

*The **inverse Fourier Series** operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{x}}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall \left(\!\left(\tilde{\mathsf{x}}_n\right)\!\right) \in \boldsymbol{\ell}^2_{\mathbb{R}}$$

Theorem 1.11.

*The **Fourier Series adjoint** operator $\hat{\mathbf{F}}^*$ is given by*

$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathsf{x}(x) \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} \text{ Definition 1.6 page 6} \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x) \left\langle e^{-i\frac{2\pi}{\tau}nx} \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} \mathrm{d}x && \text{by additivity property of } \langle \triangle \,|\, \triangledown \rangle \\
&= \int_0^\tau \mathsf{x}(x) \frac{1}{\sqrt{\tau}} \left\langle \tilde{\mathsf{y}}(n) \,|\, e^{-i\frac{2\pi}{\tau}nx} \right\rangle^*_{\mathbb{Z}} \mathrm{d}x && \text{by property of } \langle \triangle \,|\, \triangledown \rangle \\
&= \int_0^\tau \mathsf{x}(x) \left[\hat{\mathbf{F}}^{-1}\tilde{\mathsf{y}}(n)\right]^* \mathrm{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} \text{ page 6} \\
&= \left\langle \mathsf{x}(x) \,|\, \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^*}\tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{R}}
\end{aligned}$$

✏

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary* (Corollary 1.4 page 6).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary 1.5 page 7).

Corollary 1.4. *Let $\mathbf{I}$ be the identity operator and let $\hat{\mathbf{F}}$ be the Fourier Series operator with adjoint $\hat{\mathbf{F}}^*$.*

$$\hat{\mathbf{F}}\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^*\hat{\mathbf{F}} = \mathbf{I} \qquad (\hat{\mathbf{F}} \text{ is unitary...and thus also normal and isometric})$$

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$ (Theorem 1.11 (page 6)). ✏

[13] Katznelson (2004) page 3

Corollary 1.5. *Let $\hat{\mathbf{F}}$ be the Fourier series operator, $\hat{\mathbf{F}}^*$ be its adjoint, and $\hat{\mathbf{F}}^{-1}$ be its inverse.*

$$\begin{array}{rclcll}
\mathscr{R}(\hat{\mathbf{F}}) &=& \mathscr{R}(\hat{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
\|\hat{\mathbf{F}}\| &=& \|\hat{\mathbf{F}}^{-1}\| &=& 1 & \text{(UNITARY)}\\
\langle \hat{\mathbf{F}}\boldsymbol{x} \,|\, \hat{\mathbf{F}}\boldsymbol{y}\rangle &=& \langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \,|\, \hat{\mathbf{F}}^{-1}\boldsymbol{y}\rangle &=& \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle & \text{(PARSEVAL'S EQUATION)}\\
\|\hat{\mathbf{F}}\boldsymbol{x}\| &=& \|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\| &=& \|\boldsymbol{x}\| & \text{(PLANCHEREL'S FORMULA)}\\
\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\| &=& \|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\| &=& \|\boldsymbol{x} - \boldsymbol{y}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary 1.4 page 6) and from the properties of unitary operators. ✏

Theorem 1.12.

The set

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i\frac{2\pi}{\tau}nx} \,\middle|\, n\in\mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathsf{f}(x)$ *with support in* $[0 : \tau]$.

1.4 Fourier Transform

Definition 1.7. [14]

The ***Fourier Transform*** *operator* $\tilde{\mathbf{F}}$ *is defined as*[15]

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

This definition of the Fourier Transform is also called the ***unitary Fourier Transform***.

Remark 1.1 (Fourier transform scaling factor). [16] *If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as*

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq A \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x \quad \text{and} \quad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B \int_{\mathbb{R}} \mathsf{F}(\omega) e^{i\omega x}\, \mathrm{d}\omega \quad ,$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x}\, \mathrm{d}x$ *(using oscillatory frequency free variable f) or*
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x}\, \mathrm{d}x$ *(using angular frequency free variable ω).*

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int \cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not UNITARY *(see Theorem 1.14 page 8)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the* ADJOINT *of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses* AND *$\tilde{\mathbf{F}}$ is* UNITARY*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.*

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[14] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965) page 144, Knapp (2005) pages 374–375, Fourier (1822), Fourier (1878) page 336?

[16] Greenhoe (2013) page 274 ⟨Remark F.1⟩, Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005) pages 374–375

Theorem 1.13 (Inverse Fourier transform). [17] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition 1.7 page 7). *The inverse* $\tilde{\mathbf{F}}^{-1}$ *of* $\tilde{\mathbf{F}}$ *is*

$$\boxed{\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\,\mathsf{d}\omega \qquad \forall \tilde{\mathsf{f}} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}}$$

Theorem 1.14. *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with inverse* $\tilde{\mathbf{F}}^{-1}$ *and adjoint* $\tilde{\mathbf{F}}^*$.

$$\boxed{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}}$$

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \mathsf{g} \right\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathsf{d}x \,|\, \mathsf{g}(\omega) \right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 7} \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,\left\langle e^{-i\omega x} \,|\, \mathsf{g}(\omega) \right\rangle\,\mathsf{d}x && \text{by \textit{additive property} of } \langle \triangle \,|\, \nabla \rangle \\
&= \int_{\mathbb{R}} \mathsf{f}(x)\,\frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle^*\,\mathsf{d}x && \text{by \textit{conjugate symmetric property} of } \langle \triangle \,|\, \nabla \rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle \right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \\
&= \left\langle \mathsf{f} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\mathsf{g} \right\rangle && \text{by Theorem 1.13 page 8}
\end{aligned}$$

✏

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* (Corollary 1.6—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem 1.15 page 8).

Corollary 1.6. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

$$\boxed{\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})}$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem 1.14 page 8). ✏

Theorem 1.15. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $\left(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \nabla \rangle\right)$. *Let* $\mathscr{R}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

$$\boxed{\begin{array}{rcrcll}
\mathscr{R}(\mathbf{F}\tau) &=& \mathscr{R}(\tilde{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| &=& |||\tilde{\mathbf{F}}^{-1}||| &=& 1 & \text{(UNITARY)} \\
\left\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g} \right\rangle &=& \left\langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g} \right\rangle &=& \langle \mathsf{f} \,|\, \mathsf{g} \rangle & \text{(PARSEVAL'S EQUATION)} \\
\left\|\tilde{\mathbf{F}}\mathsf{f}\right\| &=& \left\|\tilde{\mathbf{F}}^{-1}\mathsf{f}\right\| &=& \|\mathsf{f}\| & \text{(PLANCHEREL'S FORMULA)} \\
\left\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\right\| &=& \left\|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\right\| &=& \|\mathsf{f} - \mathsf{g}\| & \text{(ISOMETRIC)}
\end{array}}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary 1.6 page 8) and from the properties of unitary operators. ✏

[17] Chorin and Hald (2009) page 13

Theorem 1.16 (Shift relations). [18] *Let $\tilde{\mathbf{F}}$ be the Fourier transform operator.*

$$\boxed{\begin{array}{rcl} \tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) &=& e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \\ \left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) &=& \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-v) \end{array}}$$

Theorem 1.17 (Complex conjugate). [19] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

$$\boxed{\tilde{\mathbf{F}}\mathsf{f}^*(-x) = \left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* \qquad \forall \mathsf{f}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}}$$

Definition 1.8. [20] *The* ***convolution operation*** *is defined as*

$$\left[\mathsf{f}\star\mathsf{g}\right](x) \triangleq \mathsf{f}(x)\star\mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem 1.18 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem 1.18 (convolution theorem). [21] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $\star$ the convolution operator.*

$$\boxed{\begin{array}{rcll} \underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} &=& \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} & \forall \mathsf{f},\mathsf{g}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \\ \underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} &=& \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} & \forall \mathsf{f},\mathsf{g}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}. \end{array}}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega) &= \tilde{\mathbf{F}}\left[\int_{u\in\mathbb{R}}\mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u\right](\omega) && \text{by def. of } \star \text{ (Definition 1.8 page 9)}\\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)\left[\tilde{\mathbf{F}}\mathsf{g}(x-u)\right](\omega)\,\mathrm{d}u \\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega)\,\mathrm{d}u && \text{by Theorem 1.16 page 9}\\
&= \sqrt{2\pi}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right)}_{\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)}\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)\\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition 1.7 page 7)}\\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \tilde{\mathbf{F}}\left[\left(\tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathsf{f}(x)\right)\mathsf{g}(x)\right](\omega) && \text{by definition of operator inverse}\\
&= \tilde{\mathbf{F}}\left[\left(\frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)e^{ivx}\,\mathrm{d}v\right)\mathsf{g}(x)\right](\omega) && \text{by Theorem 1.13 page 8}\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\left(e^{ivx}\,\mathsf{g}(x)\right)\right](\omega,v)\,\mathrm{d}v\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-v)\,\mathrm{d}v && \text{by Theorem 1.16 page 9}\\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by def. of } \star \text{ (Definition 1.8 page 9)}
\end{aligned}$$

[18] Greenhoe (2013) page 276 ⟨Theorem F.4⟩
[19] Greenhoe (2013) page 276 ⟨Theorem F.5⟩
[20] Bachman (1964) page 6
[21] Greenhoe (2013) pages 277–278 ⟨Theorem F.6⟩, Greenhoe (2014) ⟨Theorem 2.31⟩

1.5 Z-transform

Definition 1.9. [22] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers. A function f in X^Y is a **sequence** over X if $Y = \mathbb{Z}$.*
A sequence may be denoted in the form $(\!(x_n)\!)_{n\in\mathbb{Z}}$ or simply as $(\!(x_n)\!)$.

Definition 1.10. [23] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a field.*

*The **space of all absolutely square summable sequences** $\ell^2_{\mathbb{F}}$ over $\mathbb{F}$ is defined as*

$$\ell^2_{\mathbb{F}} \triangleq \left\{ (\!(x_n)\!)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}$$

The space $\ell^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\ell^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\ell^2_{\mathbb{R}}$ is isomorphic to $\boldsymbol{L}^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition 1.11. *The **convolution** operation $\star$ is defined as*

$$(\!(x_n)\!) \star (\!(y_n)\!) \triangleq \left(\!\!\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m} \right)\!\!\right)_{n\in\mathbb{Z}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

Definition 1.12. [24]

*The **z-transform** $\mathbf{Z}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$[\mathbf{Z}\,(\!(x_n)\!)](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\text{Laurent series}} \qquad \forall (\!(x_n)\!) \in \ell^2_{\mathbb{R}}$$

Proposition 1.2. [25] *Let $\star$ be the* CONVOLUTION OPERATOR (Definition 1.11 page 10).

$$(\!(x_n)\!) \star (\!(y_n)\!) = (\!(y_n)\!) \star (\!(x_n)\!) \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

Theorem 1.19. [26] *Let $\star$ be the convolution operator* (Definition 1.11 page 10).

$$\underbrace{\mathbf{Z}\left((\!(x_n)\!) \star (\!(y_n)\!)\right)}_{\text{sequence convolution}} = \underbrace{\left(\mathbf{Z}\,(\!(x_n)\!)\right)\left(\mathbf{Z}\,(\!(y_n)\!)\right)}_{\text{series multiplication}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

1.6 Discrete Time Fourier Transform

Definition 1.13. *The **discrete-time Fourier transform** $\check{\mathbf{F}}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$[\check{\mathbf{F}}\,(\!(x_n)\!)](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

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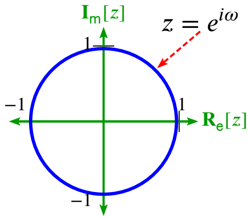
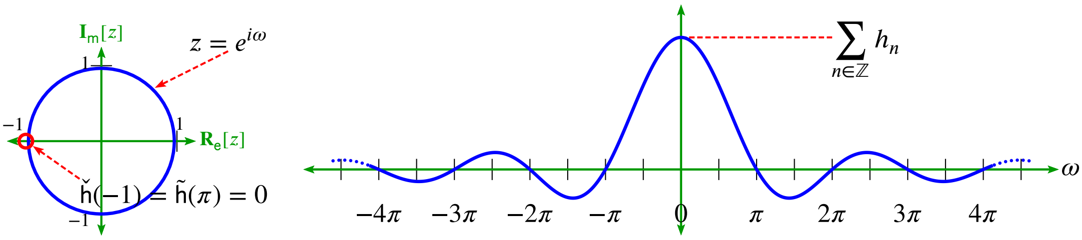
[22] Bromwich (1908) page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[23] Kubrusly (2011) page 347 ⟨Example 5.K⟩
[24] *Laurent series*: Abramovich and Aliprantis (2002) page 49
[25] Greenhoe (2013) page 344 ⟨Proposition J.1⟩
[26] Greenhoe (2013) pages 344–345 ⟨Theorem J.1⟩



If we compare the definition of the *Discrete Time Fourier Transform* (Definition 1.13 page 10) to the definition of the Z-transform (Definition 1.12 page 10), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.

Proposition 1.3. [27] *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}[\langle\!\langle x_n \rangle\!\rangle](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 1.13 page 10) of a sequence* $\langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

$$\boxed{\underbrace{\breve{x}(\omega) = \breve{x}(\omega + 2\pi n)}_{\text{PERIODIC with period } 2\pi} \qquad \forall n \in \mathbb{Z}}$$



Proposition 1.4. [28] *Let* $\breve{x}(z)$ *be the* Z-TRANSFORM *(Definition 1.12 page 10) and* $\breve{x}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 1.13 page 10) of* $\langle\!\langle x_n \rangle\!\rangle$.

$$\boxed{\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\breve{x}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{x}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall \langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}, c\in\mathbb{R}}$$

Proposition 1.5. [29]

$$\boxed{\begin{array}{l}\underbrace{\sum_{n\in\mathbb{Z}} (-1)^n x_n = c}_{\text{(1) in "time"}} \iff \underbrace{\breve{x}(z)|_{z=-1} = c}_{\text{(2) in "z domain"}} \iff \underbrace{\breve{x}(\omega)|_{\omega=\pi} = c}_{\text{(3) in "frequency"}} \\ \iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}} \\ \forall c\in\mathbb{R}, \langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}}, \langle\!\langle y_n \rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}\end{array}}$$

Lemma 1.2. [30] *Let* $\tilde{f}(\omega)$ *be the* DTFT *(Definition 1.13 page 10) of a sequence* $\langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}}$.

$$\boxed{\underbrace{\langle\!\langle x_n \in \mathbb{R} \rangle\!\rangle_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{x}(\omega)|^2 = |\breve{x}(-\omega)|^2}_{\text{EVEN}} \qquad \forall \langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}}$$

Theorem 1.20 (inverse DTFT). [31] *Let* $\breve{x}(\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 1.13 page 10) of a sequence* $\langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. *Let* $\tilde{x}^{-1}$ *be the inverse of* $\breve{x}$.

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[27] Greenhoe (2013) pages 348–349 ⟨Proposition J.2⟩
[28] Greenhoe (2013) pages 349–350 ⟨Proposition J.3⟩
[29] Chui (1992) page 123
[30] Greenhoe (2013) pages 352–353 ⟨Lemma J.2⟩
[31] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩

$$\underbrace{\left\{\breve{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right\}}_{\breve{x}(\omega)\triangleq\breve{\mathbf{F}}(\!(x_n)\!)} \implies \underbrace{\left\{x_n = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega)e^{i\omega n}\,\mathrm{d}\omega \quad \forall\alpha\in\mathbb{R}\right\}}_{(\!(x_n)\!)=\breve{\mathbf{F}}^{-1}\breve{\mathbf{F}}(\!(x_n)\!)} \quad \forall(\!(x_n)\!)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}$$

Theorem 1.21 (orthonormal quadrature conditions). [32] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM (Definition 1.13 page 10) *of a sequence $(\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\bar{\delta}_n$ be the* KRONECKER DELTA FUNCTION *at n* (Definition 6.1 page 48).

$$\begin{array}{rclcrcll}
\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} &=& 0 &\iff& \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) &=& 0 & \forall n\in\mathbb{Z}, \forall(\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}}\\
\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} &=& \bar{\delta}_n &\iff& |\breve{x}(\omega)|^2 + |\breve{x}(\omega+\pi)|^2 &=& 2 & \forall n\in\mathbb{Z}, \forall(\!(x_n)\!),(\!(y_n)\!)\in\ell^2_{\mathbb{R}}
\end{array}$$

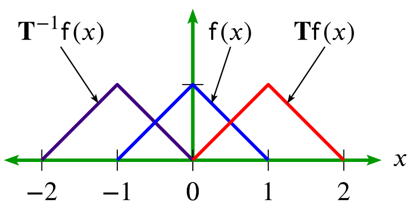
2 Background: transversal operators

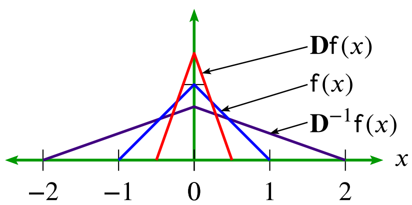
2.1 Definitions

Much of B-spline and wavelet theory can be constructed with the help of the **translation operator** $\mathbf{T}$ and the **dilation operator** $\mathbf{D}$ (next).

Definition 2.1. [33]

1. $\mathbf{T}$ *is the* ***translation operator*** *on* $\mathbb{C}^{\mathbb{C}}$ *defined as*
$$\mathbf{T}_\tau \mathsf{f}(x) \triangleq \mathsf{f}(x-\tau) \quad \textit{and} \quad \mathbf{T} \triangleq \mathbf{T}_1 \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$
2. $\mathbf{D}$ *is the* ***dilation operator*** *on* $\mathbb{C}^{\mathbb{C}}$ *defined as*
$$\mathbf{D}_\alpha \mathsf{f}(x) \triangleq \mathsf{f}(\alpha x) \quad \textit{and} \quad \mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2 \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$





[32] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

[33] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2000) page 473, Packer (2004) page 260, zay (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a) page 639, Dai and Lu (1996) page 81, Dai and Larson (1998) page 2, Greenhoe (2013) page 2

2.2 Properties

2.2.1 Algebraic properties

Proposition 2.1. [34] *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 2.1 page 12).*

$$\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x+1) \qquad \forall \mathsf{f}\in\mathbb{R}^{\mathbb{R}} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) \text{ is PERIODIC with period } 1\right)$$

In a linear space, every operator has an *inverse*. Although the inverse always exists as a relation, it may not exist as a function or as an operator. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exists as operators, as demonstrated by Proposition 2.2 (next).

Proposition 2.2. [35] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 2.1 page 12.*

$\mathbf{T}$ *has an inverse* $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(translation operator inverse).}$$

$\mathbf{D}$ *has an inverse* $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(dilation operator inverse).}$$

Proposition 2.3. [36] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 2.1 page 12. Let* $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^j x - n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

2.2.2 Linear space properties

Definition 2.2. [37] *Let* $+$ *be an addition operator on a tuple* $(x_n)_m^N$.

The ***summation*** *of* (x_n) *from index* m *to index* N *with respect to* $+$ *is*

$$\sum_{n=m}^{N} x_n \triangleq \begin{cases} 0 & \text{for } N < m \\ \left(\sum_{n=m}^{N-1} x_n\right) + x_N & \text{for } N \ge m \end{cases}$$

An infinite summation $\sum_{n=1}^{\infty}\phi_n$ is meaningless outside some topological space (e.g. metric space, normed space, etc.). The sum $\sum_{n=1}^{\infty}\phi_n$ is an abbreviation for $\lim_{N\to\infty}\sum_{n=1}^{N}\phi_n$ (the limit of partial sums). And the concept of limit is also itself meaningless outside of a topological space.

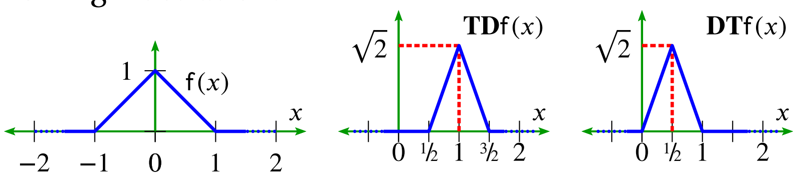
Definition 2.3. [38] *Let* (X, T) *be a topological space and* lim *be the limit induced by the topology* T.

$$\sum_{n=1}^{\infty} x_n \triangleq \sum_{n\in\mathbb{N}} x_n \triangleq \lim_{N\to\infty}\sum_{n=1}^{N} x_n$$

$$\sum_{n=-\infty}^{\infty} x_n \triangleq \sum_{n\in\mathbb{Z}} x_n \triangleq \lim_{N\to\infty}\left(\sum_{n=0}^{N} x_n\right) + \left(\lim_{N\to-\infty}\sum_{n=-1}^{N} x_n\right)$$

[34] Greenhoe (2013) page 3
[35] Greenhoe (2013) page 3
[36] Greenhoe (2013) page 4
[37] Berberian (1961) page 8 ⟨Definition I.3.1⟩, Fourier (1820) page 280 ⟨"∑" notation⟩
[38] Klauder (2010) page 4, Kubrusly (2001) page 43, Bachman and Narici (1966) pages 3–4

In general the operators $\mathbf{T}$ and $\mathbf{D}$ are *noncommutative* ($\mathbf{TD} \neq \mathbf{DT}$), as demonstrated by Proposition 2.5 and by the following illustration.



Proposition 2.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12.*

$$\boxed{\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}\mathsf{g}\right] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z},\ \mathsf{f}\in\mathbb{C}^{\mathbb{C}}}$$

Proposition 2.5 (commutator relation)**.** [39] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12.*

$$\boxed{\begin{array}{rcll}\mathbf{D}^j\mathbf{T}^n &=& \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j & \forall j,n\in\mathbb{Z}\\ \mathbf{T}^n\mathbf{D}^j &=& \mathbf{D}^j\mathbf{T}^{2^j n} & \forall n,j\in\mathbb{Z}\end{array}}$$

2.2.3 Inner-product space properties

In an inner product space, every operator has an *adjoint* and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator $\mathbf{U}$ coincide, then $\mathbf{U}$ is said to be *unitary*. And in this case, $\mathbf{U}$ has several nice properties (see Proposition 2.9 and Theorem 2.1 page 15). Proposition 2.6 (next) gives the adjoints of $\mathbf{D}$ and $\mathbf{T}$, and Proposition 2.7 (page 14) demonstrates that both $\mathbf{D}$ and $\mathbf{T}$ are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ and the *rotation matrix operator*.

Proposition 2.6. *Let* $\mathbf{T}$ *be the translation operator* (Definition 2.1 page 12) *with adjoint* $\mathbf{T}^*$ *and* $\mathbf{D}$ *the dilation operator with adjoint* $\mathbf{D}^*$.

$$\boxed{\begin{array}{rcll}\mathbf{T}^*\mathsf{f}(x) &=& \mathsf{f}(x+1) & \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \textit{(translation operator adjoint)}\\ \mathbf{D}^*\mathsf{f}(x) &=& \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) & \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \textit{(dilation operator adjoint)}\end{array}}$$

Proposition 2.7. [40] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 2.2 page 13.*

$$\boxed{\begin{array}{l}\mathbf{T}\ \textit{is}\ \text{UNITARY}\ \textit{in}\ \boldsymbol{L}^2_{\mathbb{R}} \quad (\mathbf{T}^{-1} = \mathbf{T}^*\ \textit{in}\ \boldsymbol{L}^2_{\mathbb{R}}).\\ \mathbf{D}\ \textit{is}\ \text{UNITARY}\ \textit{in}\ \boldsymbol{L}^2_{\mathbb{R}} \quad (\mathbf{D}^{-1} = \mathbf{D}^*\ \textit{in}\ \boldsymbol{L}^2_{\mathbb{R}}).\end{array}}$$

2.2.4 Normed linear space properties

Proposition 2.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR (Definition 2.1 page 12).

$$\boxed{\left\{\begin{array}{lll}\text{(1).} & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) & \textit{and}\\ \text{(2).} & \mathsf{f}(x)\ \textit{is}\ \text{CONTINUOUS} & \end{array}\right\} \iff \left\{\mathsf{f}(x)\ \textit{is a}\ \text{CONSTANT}\right\} \qquad \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}}$$

Note that in Proposition 2.8, it is not possible to remove the *continuous* constraint outright (next two counterexamples).

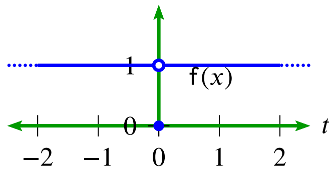
[39] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a) page 641, Goodman et al. (1993b) page 110

[40] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

Counterexample 2.1. *Let* $\mathsf{f}(x)$ *be a function in* $\mathbb{R}^{\mathbb{R}}$.

Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{otherwise.} \end{cases}$

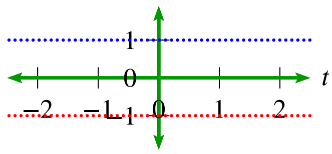
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, *but* $\mathsf{f}(x)$ *is* NOT CONSTANT.



Counterexample 2.2. *Let* $\mathsf{f}(x)$ *be a function in* $\mathbb{R}^{\mathbb{R}}$. *Let* $\mathbb{Q}$ *be the set of* RATIONAL NUMBERS *and* $\mathbb{R}\backslash\mathbb{Q}$ *the set of* IRRATIONAL NUMBERS.

Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ -1 & \text{for } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, *but* $\mathsf{f}(x)$ *is* NOT CONSTANT.



Proposition 2.9 (Operator norm)**.** *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 2.2 page 13. Let* $\mathbf{T}^{*}$ *and* $\mathbf{D}^{*}$ *be as in Proposition 2.6 page 14. Let* $\|\cdot\|$ *and* $\langle\triangle\,|\,\triangledown\rangle$ *be as in Definition 1.2 page 2. Let* $|||\cdot|||$ *be the operator norm induced by* $\|\cdot\|$.

$$\boxed{|||\mathbf{T}||| = |||\mathbf{D}||| = |||\mathbf{T}^{*}||| = |||\mathbf{D}^{*}||| = |||\mathbf{T}^{-1}||| = |||\mathbf{D}^{-1}||| = 1}$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* and from properties of unitary operators. ⇛

Theorem 2.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 2.2 page 13. Let* $\|\cdot\|$ *and* $\langle\triangle\,|\,\triangledown\rangle$ *be as in Definition 1.2 page 2.*

1.	$\|\mathbf{T}\mathsf{f}\|$	$=$	$\|\mathbf{D}\mathsf{f}\|$	$=$	$\|\mathsf{f}\|$	$\forall \mathsf{f}\in \boldsymbol{L}^2_{\mathbb{R}}$	(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}\mathsf{f} - \mathbf{T}\mathsf{g}\|$	$=$	$\|\mathbf{D}\mathsf{f} - \mathbf{D}\mathsf{g}\|$	$=$	$\|\mathsf{f} - \mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in \boldsymbol{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
3.	$\left\|\mathbf{T}^{-1}\mathsf{f} - \mathbf{T}^{-1}\mathsf{g}\right\|$	$=$	$\left\|\mathbf{D}^{-1}\mathsf{f} - \mathbf{D}^{-1}\mathsf{g}\right\|$	$=$	$\|\mathsf{f} - \mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in \boldsymbol{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
4.	$\langle \mathbf{T}\mathsf{f}\,|\,\mathbf{T}\mathsf{g}\rangle$	$=$	$\langle \mathbf{D}\mathsf{f}\,|\,\mathbf{D}\mathsf{g}\rangle$	$=$	$\langle \mathsf{f}\,|\,\mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in \boldsymbol{L}^2_{\mathbb{R}}$	(SURJECTIVE)
5.	$\left\langle \mathbf{T}^{-1}\mathsf{f}\,|\,\mathbf{T}^{-1}\mathsf{g}\right\rangle$	$=$	$\left\langle \mathbf{D}^{-1}\mathsf{f}\,|\,\mathbf{D}^{-1}\mathsf{g}\right\rangle$	$=$	$\langle \mathsf{f}\,|\,\mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in \boldsymbol{L}^2_{\mathbb{R}}$	(SURJECTIVE)

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition 2.7 page 14) and from properties of unitary operators. ⇛

Proposition 2.10. *Let* $\mathbf{T}$ *be as in Definition 2.1 page 12. Let* $\mathbf{A}^{*}$ *be the adjoint of an operator* $\mathbf{A}$.

$$\boxed{\left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^{*} \qquad \left(\text{The operator } \left[\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right] \text{ is SELF-ADJOINT}\right)}$$

2.2.5 Fourier transform properties

Proposition 2.11. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12. Let* $\mathbf{B}$ *be the* TWO-SIDED LAPLACE TRANSFORM *defined as*

$$[\mathbf{B}\mathsf{f}](s) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-sx}\, \mathsf{d}x\,.$$

$$\begin{array}{llcllll}
1. & \mathbf{BT}^n & = & e^{-sn}\mathbf{B} & & & \forall n\in\mathbb{Z} \\
2. & \mathbf{BD}^j & = & \mathbf{D}^{-j}\mathbf{B} & & & \forall j\in\mathbb{Z} \\
3. & \mathbf{DB} & = & \mathbf{BD}^{-1} & & & \forall n\in\mathbb{Z} \\
4. & \mathbf{BD}^{-1}\mathbf{B}^{-1} & = & \mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} & = & \mathbf{D} & \forall n\in\mathbb{Z} \quad (\mathbf{D}^{-1} \text{ is SIMILAR to } \mathbf{D}) \\
5. & \mathbf{DBD} & = & \mathbf{D}^{-1}\mathbf{BD}^{-1} & = & \mathbf{B} & \forall n\in\mathbb{Z}
\end{array}$$

Corollary 2.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 1.7 page 7) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition 1.2 page 2).*

$$\begin{array}{llclcl}
1. & \tilde{\mathbf{F}}\mathbf{T}^n & = & e^{-i\omega n}\tilde{\mathbf{F}} & & \\
2. & \tilde{\mathbf{F}}\mathbf{D}^j & = & \mathbf{D}^{-j}\tilde{\mathbf{F}} & & \\
3. & \mathbf{D}\tilde{\mathbf{F}} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1} & & \\
4. & \mathbf{D} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} & = & \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}} \\
5. & \tilde{\mathbf{F}} & = & \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} & = & \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}
\end{array}$$

✎PROOF: These results follow directly from Proposition 2.11 page 15. ✏

Proposition 2.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 2.1 page 12. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 1.7 page 7) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition 1.2 page 2).*

$$\boxed{\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = \frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{\mathsf{f}}\left(\frac{\omega}{2^j}\right)}$$

Proposition 2.13. *Let* $\mathbf{T}$ *be the translation operator (Definition 2.1 page 12). Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 1.7 page 7) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT *(Definition 1.13 page 10) of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition 1.10 page 10).*

$$\boxed{\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) \qquad \forall (\!(a_n)\!)\in\boldsymbol{\ell}^2_{\mathbb{R}}, \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}}$$

Theorem 2.2 (Poisson Summation Formula—PSF). [41] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition 1.7 page 7) of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

$$\underbrace{\sum_{n\in\mathbb{Z}}\mathbf{T}^n_\tau\mathsf{f}(x) = \sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

where $\mathbf{S} \in \boldsymbol{\ell}^{2\,\boldsymbol{L}^2_{\mathbb{R}}}_{\mathbb{R}}$ *is the* SAMPLING OPERATOR *defined as*

$$[\mathbf{S}\mathsf{f}(x)](n) \triangleq \mathsf{f}\left(\frac{2\pi}{\tau}n\right) \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)},\ \tau\in\mathbb{R}^+$$

Theorem 2.3 (Inverse Poisson Summation Formula—IPSF). [42] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition 1.7 page 7) of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

$$\underbrace{\sum_{n\in\mathbb{Z}}\mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega-\frac{2\pi}{\tau}n\right)}_{\text{summation in "frequency"}} = \underbrace{\frac{\tau}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\text{summation in "time"}}$$

[41] Andrews et al. (2001) page 624, Knapp (2005) page 389, Lasser (1996) page 254, Rudin (1987) pages 194–195, Folland (1992) page 337, Greenhoe (2013) pages 12–13 ⟨Theorem 1.2⟩
[42] Greenhoe (2013) pages 14–15 ⟨Theorem 1.3⟩, Gauss (1900) page 88,

Remark 2.1. *The left hand side of the* POISSON SUMMATION FORMULA (Theorem 2.2 page 16) *is very similar to the* ZAK TRANSFORM $\mathbf{Z}$*:* [43]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark 2.2. *A generalization of the* POISSON SUMMATION FORMULA (Theorem 2.2 page 16) *is the* ***Selberg Trace Formula***.[44]

Lemma 2.1. [45] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T}\right)$ *be a topological linear space. Let* $\operatorname{span} A$ *be the* SPAN *of a set* A*. Let* $\tilde{\mathsf{f}}(\omega)$ *and* $\tilde{\mathsf{g}}(\omega)$ *be the* FOURIER TRANSFORMS (Definition 1.7 page 7) *of the functions* $\mathsf{f}(x)$ *and* $\mathsf{g}(x)$*, respectively, in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition 1.2 page 2)*. Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT (Definition 1.13 page 10) *of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ (Definition 1.10 page 10)*.*

$$\left\{\begin{array}{ll} (1). & \left\{\mathbf{T}^n\mathsf{f} \,|\, n\in\mathbb{Z}\right\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega} \quad \text{and} \\ (2). & \left\{\mathbf{T}^n\mathsf{g} \,|\, n\in\mathbb{Z}\right\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega} \end{array}\right\} \implies \left\{\begin{array}{l} \exists\, (\!(a_n)\!)_{n\in\mathbb{Z}} \quad \text{such that} \\ \tilde{\mathsf{f}}(\omega) = \breve{\mathsf{a}}(\omega)\tilde{\mathsf{g}}(\omega) \end{array}\right\}$$

Theorem 2.4 (Battle-Lemarié orthogonalization). [46] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition 1.7 page 7) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$*.*

$$\left\{\begin{array}{ll} 1. & \left\{\mathbf{T}^n\mathsf{g} \,|\, n\in\mathbb{Z}\right\} \text{ is a RIESZ BASIS for } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{and} \\ 2. & \tilde{\mathsf{f}}(\omega) \triangleq \dfrac{\tilde{\mathsf{g}}(\omega)}{\sqrt{2\pi\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2}} \end{array}\right\} \implies \left\{\begin{array}{l} \left\{\mathbf{T}^n\mathsf{f} \,|\, n\in\mathbb{Z}\right\} \\ \text{is an ORTHONORMAL BASIS for } \boldsymbol{L}^2_{\mathbb{R}} \end{array}\right\}$$

3 Background: MRA-wavelet analysis

3.1 Orientation

In Fourier analysis, *continuous* dilations (Definition 2.1 page 12) of the *complex exponential* form a *basis* for the *space of square integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ such that

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}_\omega e^{ix} | \omega\in\mathbb{R}\right\}.$$

In Fourier series analysis , *discrete* dilations of the complex exponential form a basis for $\boldsymbol{L}^2_{\mathbb{R}}(0:2\pi)$ such that

$$\boldsymbol{L}^2_{\mathbb{R}}(0:2\pi) = \operatorname{span}\left\{\mathbf{D}_j e^{ix} \,\middle|\, j\in\mathbb{Z}\right\}.$$

In Wavelet analysis, for some *mother wavelet* (Definition 3.5 page 21) $\psi(x)$,

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}_\omega\mathbf{T}_\tau\psi(x) | \omega,\tau\in\mathbb{R}\right\}.$$

However, the ranges of parameters ω and τ can be much reduced to the countable set $\mathbb{Z}$ resulting in a *dyadic* wavelet basis such that for some mother wavelet $\psi(x)$,

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\psi(x) | j,n\in\mathbb{Z}\right\}.$$

[43] Janssen (1988) page 24, Zayed (1996) page 482
[44] Lax (2002) page 349, Selberg (1956), Terras (1999)
[45] Daubechies (1992) page 140, Greenhoe (2013) pages 22–23 ⟨Lemma 1.1⟩
[46] Wojtaszczyk (1997) page 25 ⟨Remark 2.4⟩, Vidakovic (1999) page 71, Mallat (1989) page 72, Mallat (1999) page 225, Daubechies (1992) page 140 ⟨(5.3.3)⟩, Greenhoe (2013) pages 23–24 ⟨Theorem 1.7⟩

Wavelets that are both *dyadic* and *compactly supported* have the attractive feature that they can be easily implemented in hardware or software by use of the *Fast Wavelet Transform*.

In 1989, Stéphane G. Mallat introduced the *Multiresolution Analysis* (MRA, Definition 3.1 page 18) method for wavelet construction. The MRA has since become the dominate wavelet construction method. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA.[47]

3.2 Multiresolution analysis

3.2.1 Definition

A multiresolution analysis provides "coarse" approximations of a function in a linear space $\boldsymbol{L}^2_{\mathbb{R}}$ at multiple "scales" or "resolutions". Key to this process is a sequence of *scaling functions*. Most traditional transforms feature a single *scaling function* $\phi(x)$ set equal to one ($\phi(x) = 1$). This allows for convenient representation of the most basic functions, such as constants.[48] A multiresolution system, on the other hand, uses a generalized form of the scaling concept:[49]

1. Instead of the scaling function simply being set *equal to unity* ($\phi(x) = 1$), a multiresolution analysis (Definition 3.1 page 18) is often constructed in such a way that the scaling function $\phi(x)$ forms a *partition of unity* such that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x) = 1$.
2. Instead of there being *just one* scaling function, there is an entire sequence of scaling functions $\left(\!\left(\mathbf{D}^j\phi(x)\right)\!\right)_{j\in\mathbb{Z}}$, each corresponding to a different "*resolution*".

Definition 3.1. [50] *Let $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ be a sequence of subspaces on $\boldsymbol{L}^2_{\mathbb{R}}$. Let A^- be the* CLOSURE *of a set A. The sequence $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **multiresolution analysis** on $\boldsymbol{L}^2_{\mathbb{R}}$ if*

1. $\boldsymbol{V}_j = \boldsymbol{V}_j^-$ $\quad \forall j\in\mathbb{Z}$ (CLOSED) *and*
2. $\boldsymbol{V}_j \subset \boldsymbol{V}_{j+1}$ $\quad \forall j\in\mathbb{Z}$ (LINEARLY ORDERED) *and*
3. $\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}$ (DENSE *in* $\boldsymbol{L}^2_{\mathbb{R}}$) *and*
4. $\mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{D}\mathsf{f} \in \boldsymbol{V}_{j+1}$ $\quad \forall j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ (SELF-SIMILAR) *and*
5. $\exists\phi$ such that $\left\{\mathbf{T}^n\phi | n\in\mathbb{Z}\right\}$ *is a* RIESZ BASIS *for* $\boldsymbol{V}_0$.

A MULTIRESOLUTION ANALYSIS *is also called an **MRA**. An element $\boldsymbol{V}_j$ of $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **scaling subspace** of the space $\boldsymbol{L}^2_{\mathbb{R}}$. The pair $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right)\right)$ is a **multiresolution analysis space**, or **MRA space**. The function ϕ is the **scaling function** of the* MRA SPACE.

[47] Lemarié (1990), Mallat (1999) page 240

[48] Jawerth and Sweldens (1994) page 8

[49] The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan, and later as the *Gaussian Pyramid* by Burt and Adelson in the 1980s in the West. Mallat (1989) page 70, Iijima (1959), Burt and Adelson (1983), Adelson and Burt (1981), Lindeberg (1993), Alvarez et al. (1993), Guichard et al. (2012), Weickert (1999) ⟨historical survey⟩

[50] Hernández and Weiss (1996) page 44, Mallat (1999) page 221 ⟨Definition 7.1⟩ , Mallat (1989) page 70, Meyer (1992) page 21 ⟨Definition 2.2.1⟩, Christensen (2003) page 284 ⟨Definition 13.1.1⟩, Bachman et al. (2000) pages 451–452 ⟨Definition 7.7.6⟩, Walnut (2002) pages 300–301 ⟨Definition 10.16⟩, Daubechies (1992) pages 129–140 ⟨Riesz basis: page 139⟩

The traditional definition of the *MRA* also includes the following:

6. $\mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{T}^n\mathsf{f} \in \boldsymbol{V}_j \quad \forall n,j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ (*translation invariant*)
7. $\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \{\mathbb{0}\}$ (*greatest lower bound* is **0**)

However, it can be shown that these follow from the *MRA* as defined in Definition 3.1.[51]

The MRA (Definition 3.1 page 18) is more than just an interesting mathematical toy. Under some very "reasonable" conditions (next proposition), as $j \to \infty$, the *scaling subspace* $\boldsymbol{V}_j$ is *dense* in $\boldsymbol{L}^2_{\mathbb{R}}$ …meaning that with the MRA we can represent any "reasonable" function to within an arbitrary accuracy.

Proposition 3.1. [52]

$$\boxed{\left\{\begin{array}{lll} \text{(1).} & (\!(\mathbf{T}^n\phi)\!) \textit{ is a } \text{RIESZ SEQUENCE} & \textit{and} \\ \text{(2).} & \tilde{\phi}(\omega) \textit{ is } \text{CONTINUOUS} \textit{ at } 0 & \textit{and} \\ \text{(3).} & \tilde{\phi}(0) \neq 0 & \end{array}\right\} \implies \left\{ \overline{\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)} = \boldsymbol{L}^2_{\mathbb{R}} \quad (\text{DENSE } \textit{in } \boldsymbol{L}^2_{\mathbb{R}}) \right\}}$$

3.2.2 Dilation equation

The scaling function $\phi(x)$ (Definition 3.1 page 18) exhibits a kind of *self-similar* property. By Definition 3.1 page 18, the dilation $\mathbf{D}\mathsf{f}$ of each vector f in $\boldsymbol{V}_0$ is in $\boldsymbol{V}_1$. If $\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_0$, then $\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_1$, $\{\mathbf{D}^2\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_2$, …; and in general $\{\mathbf{D}^j\mathbf{T}^m\phi| j\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_j$. Also, if ϕ is in $\boldsymbol{V}_0$, then it is also in $\boldsymbol{V}_1$ (because $\boldsymbol{V}_0 \subset \boldsymbol{V}_1$). And because ϕ is in $\boldsymbol{V}_1$ and because $\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_1$, ϕ is a linear combination of the elements in $\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$. That is, ϕ can be represented as a linear combination of translated and dilated versions of itself. The resulting equation is called the *dilation equation* (Definition 3.2, next).[53]

Definition 3.2. [54] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!)\right)$ *be a* MULTIRESOLUTION ANALYSIS SPACE *with scaling function* ϕ *(Definition 3.1 page 18). Let* $(\!(h_n)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE *(Definition 1.9 page 10) in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition 1.10 page 10). The equation*

$$\phi(x) = \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n\phi(x) \qquad \forall x\in\mathbb{R}$$

is called the ***dilation equation****. It is also called the* ***refinement equation****,* ***two-scale difference equation****, and* ***two-scale relation****.*

Theorem 3.1 (dilation equation). [55] *Let an* MRA SPACE *and* SCALING FUNCTION *be as defined in Definition 3.1 page 18.*

$$\boxed{\left\{\begin{array}{l} \left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!)\right) \textit{ is an } \text{MRA SPACE} \\ \textit{with } \text{SCALING FUNCTION } \phi \end{array}\right\} \implies \underbrace{\left\{\begin{array}{ll} \exists (\!(h_n)\!)_{n\in\mathbb{Z}} & \text{such that} \\ \phi(x) = \sum_{n\in\mathbb{Z}} h_n\mathbf{D}\mathbf{T}^n\phi(x) & \forall x\in\mathbb{R} \end{array}\right\}}_{\text{DILATION EQUATION IN "TIME"}}}$$

Lemma 3.1. [56] *Let* $\phi(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *(Definition 1.2 page 2). Let* $\tilde{\phi}(\omega)$ *be the* FOURIER TRANSFORM *of*

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[51] Hernández and Weiss (1996) page 45 ⟨Theorem 1.6⟩, Wojtaszczyk (1997) pages 19–28 ⟨Proposition 2.14⟩, Pinsky (2002) pages 313–314 ⟨Lemma 6.4.28⟩, Greenhoe (2013) pages 32–35 ⟨Propositions 2.1, 2.2⟩

[52] Wojtaszczyk (1997) pages 28–31 ⟨Proposition 2.15⟩, Greenhoe (2013) pages 35–37 ⟨Proposition 2.3⟩

[53] The property of *translation invariance* is of particular significance in the theory of *normed linear spaces* (a Hilbert space is a complete normed linear space equipped with an inner product).

[54] Jawerth and Sweldens (1994) page 7

[55] Greenhoe (2013) page 39 ⟨Theorem 2.1⟩

[56] Mallat (1999) page 228, Greenhoe (2013) pages 39–41 ⟨Lemma 2.1⟩

$\phi(x)$. Let $\breve{\mathsf{h}}(\omega)$ be the Discrete time Fourier transform *of a sequence $(h_n)_{n\in\mathbb{Z}}$.*

$$\text{(A)}\quad \phi(x) = \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n\phi(x) \quad \forall x\in\mathbb{R} \iff \tilde{\phi}(\omega) = \frac{\sqrt{2}}{2}\,\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \quad \forall\omega\in\mathbb{R} \quad (1)$$

$$\iff \tilde{\phi}(\omega) = \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\,\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) \quad \forall n\in\mathbb{N},\omega\in\mathbb{R} \quad (2)$$

Definition 3.3 (next) formally defines the coefficients that appear in Theorem 3.1 (page 19).

Definition 3.3. *Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!)\right)$ be a multiresolution analysis space with scaling function ϕ. Let $(h_n)_{n\in\mathbb{Z}}$ be a sequence of coefficients such that $\phi = \sum_{n\in\mathbb{Z}} h_n\mathbf{D}\mathbf{T}^n\phi$. A **multiresolution system** is the tuple $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$. The sequence $(h_n)_{n\in\mathbb{Z}}$ is the **scaling coefficient sequence**. A multiresolution system is also called an **MRA system**. An* MRA system *is an **orthonormal MRA system** if $\left\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\right\}$ is* orthonormal.

Definition 3.4. *Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ be a multiresolution system, and $\mathbf{D}$ the dilation operator. The **normalization coefficient at resolution** n is the quantity*

$$\left\|\mathbf{D}^j\phi\right\|.$$

Theorem 3.2. [57] *Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ be an* MRA system *(Definition 3.3 page 20). Let $\operatorname{span} A$ be the* linear span *of a set A.*

$$\underbrace{\operatorname{span}\left\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\right\} = \boldsymbol{V}_0}_{\left\{\mathbf{T}^n\phi|\,n\in\mathbb{Z}\right\}\text{ is a BASIS for }\boldsymbol{V}_0} \implies \underbrace{\operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi\big|\, n\in\mathbb{Z}\right\} = \boldsymbol{V}_j \quad \forall j\in\mathbb{W}}_{\left\{\mathbf{D}^j\mathbf{T}^n\phi|\,n\in\mathbb{Z}\right\}\text{ is a BASIS for }\boldsymbol{V}_j}$$

3.2.3 Necessary Conditions

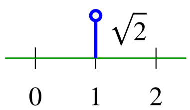
Theorem 3.3 (admissibility condition). [58] *Let $\breve{\mathsf{h}}(z)$ be the* Z-transform *(Definition 1.12 page 10) and $\breve{\mathsf{h}}(\omega)$ the* discrete-time Fourier transform *(Definition 1.13 page 10) of a sequence $(h_n)_{n\in\mathbb{Z}}$.*

$$\left\{\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)\text{ is an MRA SYSTEM (Definition 3.3 page 20)}\right\}$$

$$\begin{array}{c}\Longrightarrow\\ \not\Longleftarrow\end{array} \underbrace{\left\{\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}\right\}}_{\text{(1) ADMISSIBILITY in “time”}} \iff \underbrace{\left\{\breve{\mathsf{h}}(z)\Big|_{z=1} = \sqrt{2}\right\}}_{\text{(2) ADMISSIBILITY in “z domain”}} \iff \underbrace{\left\{\breve{\mathsf{h}}(\omega)\Big|_{\omega=0} = \sqrt{2}\right\}}_{\text{(3) ADMISSIBILITY in “frequency”}}$$

Counterexample 3.1. *Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ be an* MRA system *(Definition 3.3 page 20).*

$$\left\{ (\!(h_n)\!) \triangleq \sqrt{2}\bar{\delta}_{n-1} \triangleq \begin{cases}\sqrt{2} & \text{for } n=1\\ 0 & \text{otherwise.}\end{cases} \right\} \implies \{\phi(x) = 0\}$$



which means

$$\left\{\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}\right\} \;\not\Longrightarrow\; \left\{\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)\text{ is an MRA system for }\boldsymbol{L}^2_{\mathbb{R}}.\right\}$$

[57] Greenhoe (2013) page 43 ⟨Theorem 2.2⟩
[58] Greenhoe (2013) pages 36–37 ⟨Theorem 2.3⟩

✎PROOF:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n \phi(x) && \text{by } \textit{dilation equation} \text{ (Theorem 3.1 page 19)}\\
&= \sum_{n\in\mathbb{Z}} h_n \phi(2x-n) && \text{by definitions of } \mathbf{D} \text{ and } \mathbf{T} \text{ (Definition 2.1 page 12)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\sqrt{2}\bar{\delta}_{n-1}}_{(\!(h_n)\!)} \phi(2x-n) && \text{by definitions of } (\!(h_n)\!)\\
&= \sqrt{2}\phi(2x-1) && \text{by definition of } \phi(x)\\
\implies \phi(x) &= 0
\end{aligned}$$

This implies $\phi(x)=0$, which implies that $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ is *not* an *MRA system* for $\boldsymbol{L}^2_{\mathbb{R}}$ because

$$\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^{-} = \left(\bigcup_{j\in\mathbb{Z}} \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi|_{n\in\mathbb{Z}}\right\}\right)^{-} \neq \boldsymbol{L}^2_{\mathbb{R}}$$

(the *least upper bound* is *not* $\boldsymbol{L}^2_{\mathbb{R}}$). ✏

Theorem 3.4 (Quadrature condition in "time"). [59] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 3.3 page 20).*

$$\boxed{\sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h^*_k \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \langle \phi \,|\, \mathbf{T}^n\phi\rangle \qquad \forall n\in\mathbb{Z}}$$

3.2.4 Sufficient conditions

Theorem 3.5 (next) gives a set of *sufficient* conditions on the *scaling function* (Definition 3.1 page 18) ϕ to generate an *MRA*.

Theorem 3.5. [60] *Let an* MRA *be defined as in Definition 3.1 page 18. Let* $\boldsymbol{V}_j \triangleq \operatorname{span}\{\mathbf{T}\phi(x)|\, n\in\mathbb{Z}\}$.

$$\boxed{\left\{\begin{array}{lll}
(1). & (\!(\mathbf{T}^n\phi)\!) \text{ is a RIESZ SEQUENCE} & \text{and}\\
(2). & \exists (\!(h_n)\!) \text{ such that } \phi(x) = \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n\phi(x) & \text{and}\\
(3). & \tilde{\phi}(\omega) \text{ is CONTINUOUS at } 0 & \text{and}\\
(4). & \tilde{\phi}(0) \neq 0 &
\end{array}\right\} \implies \left\{(\!(\boldsymbol{V}_j)\!)_{j\in\mathbb{Z}} \text{ is an MRA}\right\}}$$

3.3 Wavelet analysis

3.3.1 Definition

The term "wavelet" comes from the French word "*ondelette*", meaning "small wave". And in essence, wavelets are "small waves" (as opposed to the "long waves" of Fourier analysis) that form a basis for the Hilbert space $\boldsymbol{L}^2_{\mathbb{R}}$.[61]

Definition 3.5. [62] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 2.1 page 12. A function* $\psi(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ *is a* ***wavelet function*** *for* $\boldsymbol{L}^2_{\mathbb{R}}$ *if*

[59] Greenhoe (2013) page 48 ⟨Theorem 2.4⟩
[60] Wojtaszczyk (1997) page 28 ⟨Theorem 2.13⟩, Pinsky (2002) page 313 ⟨Theorem 6.4.27⟩, Greenhoe (2013) pages 49–50 ⟨Theorem 2.6⟩
[61] Strang and Nguyen (1996) page ix, Atkinson and Han (2009) page 191
[62] Wojtaszczyk (1997) page 17 ⟨Definition 2.1⟩, Greenhoe (2013) page 50 ⟨Definition 2.4⟩

$\left\{\mathbf{D}^j\mathbf{T}^n\psi|_{j,n\in\mathbb{Z}}\right\}$ *is a* RIESZ BASIS *for* $\boldsymbol{L}^2_{\mathbb{R}}$.
In this case, ψ *is also called the* ***mother wavelet*** *of the basis* $\left\{\mathbf{D}^j\mathbf{T}^n\psi|_{j,n\in\mathbb{Z}}\right\}$. *The sequence of subspaces* $\left(\!\left(\boldsymbol{W}_j\right)\!\right)_{j\in\mathbb{Z}}$ *is the* ***wavelet analysis*** *induced by* ψ, *where each subspace* $\boldsymbol{W}_j$ *is defined as*

$$\boldsymbol{W}_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\psi\,\middle|\, n\in\mathbb{Z}\right\}.$$

A *wavelet analysis* $\left(\!\left(\boldsymbol{W}_j\right)\!\right)$ is often constructed from a *multiresolution anaysis* (Definition 3.1 page 18) $\left(\!\left(\boldsymbol{V}_j\right)\!\right)$ under the relationship

$$\boldsymbol{V}_{j+1} = \boldsymbol{V}_j \dotplus \boldsymbol{W}_j, \qquad \text{where } \dotplus \text{ is subspace addition (\textit{Minkowski addition}).}$$

By this relationship alone, $\left(\!\left(\boldsymbol{W}_j\right)\!\right)$ is in no way uniquely defined in terms of a multiresolution analysis $\left(\!\left(\boldsymbol{V}_j\right)\!\right)$. In general there are many possible complements of a subspace $\boldsymbol{V}_j$. To uniquely define such a wavelet subspace, one or more additional constraints are required. One of the most common additional constraints is *orthogonality*, such that $\boldsymbol{V}_j$ and $\boldsymbol{W}_j$ are orthogonal to each other.

3.3.2 Dilation equation

Suppose $\left(\mathbf{T}^n\psi\right)_{n\in\mathbb{Z}}$ is a basis for $\boldsymbol{W}_0$. By Definition 3.5 page 21, the wavelet subspace $\boldsymbol{W}_0$ is contained in the scaling subspace $\boldsymbol{V}_1$. By Definition 3.1 page 18, the sequence $\left(\mathbf{D}\mathbf{T}^n\phi\right)_{n\in\mathbb{Z}}$ is a basis for $\boldsymbol{V}_1$. Because $\boldsymbol{W}_0$ is contained in $\boldsymbol{V}_1$, the sequence $\left(\mathbf{D}\mathbf{T}^n\phi\right)_{n\in\mathbb{Z}}$ is also a basis for $\boldsymbol{W}_0$.

Theorem 3.6. [63] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *be a multiresolution system and* $\left(\!\left(\boldsymbol{W}_j\right)\!\right)_{j\in\mathbb{Z}}$ *a wavelet analysis with respect to* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *and with wavelet function* ψ.

$$\boxed{\exists \left(\!\left(g_n\right)\!\right)_{n\in\mathbb{Z}} \quad \text{such that} \quad \psi = \sum_{n\in\mathbb{Z}} g_n\mathbf{D}\mathbf{T}^n\phi}$$

A *wavelet system* (next definition) consists of two subspace sequences:

- A **multiresolution analysis** $\left(\!\left(\boldsymbol{V}_j\right)\!\right)$ (Definition 3.1 page 18) provides "coarse" approximations of a function in $\boldsymbol{L}^2_{\mathbb{R}}$ at different "scales" or resolutions.
- A **wavelet analysis** $\left(\!\left(\boldsymbol{W}_j\right)\!\right)$ provides the "detail" of the function missing from the approximation provided by a given scaling subspace (Definition 3.5 page 21).

Definition 3.6. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *be a multiresolution system (Definition 3.1 page 18) and* $\left(\!\left(\boldsymbol{W}_j\right)\!\right)_{j\in\mathbb{Z}}$ *a wavelet analysis (Definition 3.5 page 21) with respect to* $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$. *Let* $\left(\!\left(g_n\right)\!\right)_{n\in\mathbb{Z}}$ *be a sequence of coefficients such that* $\psi = \sum_{n\in\mathbb{Z}} g_n\mathbf{D}\mathbf{T}^n\phi$.

> *A* ***wavelet system*** *is the tuple*
> $$\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$$
> *and the sequence* $\left(\!\left(g_n\right)\!\right)_{n\in\mathbb{Z}}$ *is the* ***wavelet coefficient sequence***.

3.3.3 Necessary conditions

Theorem 3.7 (quadrature conditions in "time"). [64] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system (Definition 3.6 page 22).*

[63] Greenhoe (2013) page 51 ⟨Theorem 2.6⟩
[64] Greenhoe (2013) pages 55–56 ⟨Theorem 2.9⟩

$$\begin{array}{lrcll}
1. & \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle & = & \langle \phi \,|\, \mathbf{T}^n \phi \rangle & \forall n\in\mathbb{Z} \\
2. & \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle & = & \langle \psi \,|\, \mathbf{T}^n \psi \rangle & \forall n\in\mathbb{Z} \\
3. & \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle & = & \langle \phi \,|\, \mathbf{T}^n \psi \rangle & \forall n\in\mathbb{Z}
\end{array}$$

Proposition 3.2. [65] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system. Let* $\tilde{\phi}(\omega)$ *and* $\tilde{\psi}(\omega)$ *be the* FOURIER TRANSFORMS *of* $\phi(x)$ *and* $\psi(x)$*, respectively. Let* $\breve{g}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *of* $(\!(g_n)\!)$*.*

$$\boxed{\tilde{\psi}(\omega) = \frac{\sqrt{2}}{2}\,\breve{g}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)}$$

3.3.4 Sufficient condition

In this text, an often used sufficient condition for designing the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 3.6 page 22) is the *conjugate quadrature filter condition.* It expresses the sequence $(\!(g_n)\!)$ in terms of the *scaling coefficient sequence* (Definition 3.3 page 20) and a "shift" integer N as $g_n = \pm(-1)^n h^*_{N-n}$.

Theorem 3.8. [66] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.6 page 22). Let* $\breve{g}(\omega)$ *be the* DTFT *(Definition 1.13 page 10) and* $\breve{g}(z)$ *the* Z-TRANSFORM *(Definition 1.12 page 10) of* $(\!(g_n)\!)$*.*

$$\begin{array}{lcrcll}
\underbrace{g_n = \pm(-1)^n h^*_{N-n},\; N\in\mathbb{Z}}_{\text{CONJUGATE QUADRATURE FILTER}} & \iff & \breve{g}(\omega) & = & \pm(-1)^N e^{-i\omega N}\breve{h}^*(\omega+\pi)\Big|_{\omega=\pi} & (1) \\
 & \implies & \sum_{n\in\mathbb{Z}}(-1)^n g_n & = & \sqrt{2} & (2) \\
 & \iff & \breve{g}(z)\Big|_{z=-1} & = & \sqrt{2} & (3) \\
 & \iff & \breve{g}(\omega)\Big|_{\omega=\pi} & = & \sqrt{2} & (4)
\end{array}$$

3.4 Support size

The *support* of a function is what it's non-zero part "sits" on. If the support of the scaling coefficients $(\!(h_n)\!)$ goes from say $[0,3]$ in $\mathbb{Z}$, what is the support of the scaling function $\phi(x)$? The answer is $[0,3]$ in $\mathbb{R}$—essentially the same as the support of $(\!(h_n)\!)$ except that the two functions have different domains ($\mathbb{Z}$ versus $\mathbb{R}$). This concept is defined in Definition 3.7 (next definition) and proven in Theorem 3.9 (next theorem).

Definition 3.7. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system. Let* X^- *represent the closure of a set* X *in* $\boldsymbol{L}^2_{\mathbb{R}}$*,* $\vee X$ *the* LEAST UPPER BOUND *of an ordered set* $(X, \le)$*,* $\wedge X$ *the* GREATEST LOWER BOUND *of an ordered set* $(X, \le)$*, and*

$$\begin{array}{rcll}
\lfloor x \rfloor & \triangleq & \bigvee \{n\in\mathbb{Z} | n \le x\} \quad \forall x\in\mathbb{R} & (\text{FLOOR of } x) \\
\lceil x \rceil & \triangleq & \bigwedge \{n\in\mathbb{Z} | n \ge x\} \quad \forall x\in\mathbb{R} & (\text{CEILING of } x).
\end{array}$$

The ***support*** $\mathrm{S}\mathsf{f}$ *of a function* $\mathsf{f} \in Y^X$ *is defined as*

$$\mathrm{S}\mathsf{f} \triangleq \begin{cases} \{x\in\mathbb{R} | \mathsf{f}(x) \ne 0\}^- & \text{for } X = \mathbb{R} \quad (\text{domain of } \mathsf{f} \text{ is } \mathbb{R}) \\ \left\{x\in\mathbb{R} | \mathsf{f}(\lfloor x \rfloor) \ne 0 \text{ and } \mathsf{f}(\lceil x \rceil) \ne 0\right\}^- & \text{for } X = \mathbb{Z} \quad (\text{domain of } \mathsf{f} \text{ is } \mathbb{Z}) \end{cases}$$

[65] Greenhoe (2013) page 56 ⟨Proposition 2.7⟩
[66] Greenhoe (2013) pages 58–59 ⟨Theorem 2.11⟩

Theorem 3.9 (support size). [67] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system. Let* $\operatorname{supp} f$ *be the support of a function* f (Definition 3.7 page 23).

$$\boxed{\operatorname{supp}\phi = \operatorname{supp} h}$$

4 Background: binomial relations

4.1 Factorials

Definition 4.1 (factorial).

*The **factorial** $n!$ is defined as*

$$n! \triangleq \begin{cases} n(n-1)(n-2)\cdots 1 & \text{for } n\in\mathbb{Z},\ n\ge 1 \\ 1 & \text{for } n\in\mathbb{Z},\ n=0 \\ 0 & \text{for } n\in\mathbb{Z},\ n\le -1 \end{cases}$$

Definition 4.2. [68] *The quantities "x **to the** m **falling**", "x **to the** m **rising**", "x **to the** m **central**" are defined as follows:*

$$x^{\underline{m}} \triangleq \begin{cases} \underbrace{x(x-1)(x-2)\cdots(x-m+1)}_{m\text{ factors}} & \forall x\in\mathbb{C}, m\in\mathbb{N} \\ 1 & \forall x\in\mathbb{C}, m=0 \end{cases} \qquad \text{("}x\text{ to the }m\text{ falling")}$$

$$x^{\overline{m}} \triangleq \begin{cases} \underbrace{x(x+1)(x+2)\cdots(x+m-1)}_{m\text{ factors}} & \forall x\in\mathbb{C}, m\in\mathbb{N} \\ 1 & \forall x\in\mathbb{C}, m=0 \end{cases} \qquad \text{("}x\text{ to the }m\text{ rising")}$$

$$x^{\overline{\underline{m}}} \triangleq \begin{cases} \underbrace{x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1\right)}_{m\text{ factors}} & \forall x\in\mathbb{C}, m\in\mathbb{N} \\ 1 & \forall x\in\mathbb{C}, m=0 \end{cases} \qquad \text{("}x\text{ to the }m\text{ central")}$$

The rising and central expressions may be expressed in terms of the falling expression (next).

Proposition 4.1. [69]

$$x^{\overline{m}} = (-1)^m x^{\underline{m}} \qquad\qquad x^{\overline{\underline{m}}} = x\left(x+\frac{m}{2}-1\right)^{\underline{(m-1)}}$$

✎PROOF:

$$\begin{aligned}
(-1)^m(-x)^{\underline{m}} &= (-1)^m[(-x)(-x-1)(-x-2)\cdots(-x-m+1)] && \text{by Definition 4.2 page 24}\\
&= (-1)^m(-1)^m[(x)(x+1)(x+2)\cdots(x+m-1)] \\
&= x^{\overline{m}} && \text{by Definition 4.2 page 24}
\end{aligned}$$

$$x\left(x+\frac{m}{2}-1\right)^{\underline{(m-1)}} = x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-1-1\right)\cdots\left(x+\frac{m}{2}-1-(m-1)+1\right) \qquad \text{by Definition 4.2 page 24}$$

[67] Mallat (1999) pages 243–244, Greenhoe (2013) pages 60–61 ⟨Theorem 2.12⟩

[68] Graham et al. (1994) pages 47–48 ⟨equations (2.43), (2.44)⟩, Knuth (1992b) page 414 ⟨(2.11), (2.12)⟩, Aigner (2007) page 10, Steffensen (1950) page 8 ⟨descending, ascending, and central factorials⟩, Steffensen (1927) page 8 ⟨descending, ascending, and central factorials⟩

[69] Steffensen (1950) page 8 ⟨(3)⟩

$$
\begin{aligned}
&= x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1\right)\\
&= x^{\overline{m}}
\end{aligned}
$$

✏

4.2 Binomial identities

Definition 4.3 (Binomial coefficient). [70] *Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{Z}$ the set of integers. Let $x^{\underline{m}}$ represent "x to the m falling" (Definition 4.2). Let $n!$ represent "n factorial" (Definition 4.1).*

> *The **binomial coefficient** $\binom{x}{k}$ is defined as*
> $$\binom{x}{k} \triangleq \begin{cases} \dfrac{x^{\underline{k}}}{k!} & \forall x\in\mathbb{C} \quad k\in\mathbb{W} \quad (k=0,1,2,3,\ldots)\\ 0 & \forall x\in\mathbb{C} \quad k\in\mathbb{Z}^- \quad (k=-1,-2,-3,\ldots)\end{cases}$$
> *The value x is called the **upper index** and the value k is called the **lower index**.*

Proposition 4.2. *Let $\binom{n}{k}$ be the* BINOMIAL COEFFICIENT *(Definition 4.3 page 25).*

1. $\binom{x}{0} = 1 \quad \forall x\in\mathbb{C}$	2. $\binom{n}{n} = 1 \quad \forall n\in\mathbb{W}$
3. $\binom{x}{1} = x \quad \forall x\in\mathbb{C}$	4. $\binom{x}{k} = 0 \quad \forall x\in\mathbb{C},\ x<k$

✎PROOF:

1. Proof that $\binom{x}{0}=1$:

$$
\begin{aligned}
\binom{x}{0} &= \frac{x^{\underline{0}}}{0!} && \text{by Definition 4.3 page 25}\\
&= \frac{x^{\underline{0}}}{1} && \text{by Definition 4.1 page 24}\\
&= 1 && \text{by Definition 4.2 page 24}
\end{aligned}
$$

2. Proof that $\binom{n}{n}=1$:

$$
\begin{aligned}
\binom{n}{n} &= \frac{n^{\underline{n}}}{n!} && \text{by Definition 4.3 page 25}\\
&= \frac{n(n-1)\cdots(n-n+1)}{n!} && \text{by Definition 4.2 page 24}\\
&= \frac{n(n-1)\cdots(1)}{n(n-1)\cdots(1)} && \text{by Definition 4.1 page 24}\\
&= 1
\end{aligned}
$$

[70] Graham et al. (1994) page 154 ⟨equation (5.1)⟩, Aigner (2007) page 10 ⟨(1)⟩, Coolidge (1949) pages 149–150, Stifel (1544)

3. Proof that $\binom{x}{1} = x$:

$$\begin{aligned}
\binom{x}{1} &= \frac{x^{\underline{1}}}{1!} && \text{by Definition 4.3 page 25}\\
&= \frac{x^{\underline{1}}}{1} && \text{by Definition 4.1 page 24}\\
&= x && \text{by Definition 4.2 page 24}
\end{aligned}$$

4. Proof that $\binom{x}{k} = 0,\ \forall x < k$:

$$\begin{aligned}
\binom{x}{k} &= \frac{x^{\underline{k}}}{k!} && \text{by Definition 4.3 page 25}\\
&= \frac{x(x-1)\cdots(0)\cdots(x-k+1)}{k!} && \text{by Definition 4.2 page 24}\\
&= 0
\end{aligned}$$

✏

Theorem 4.1. [71] *Let $\binom{n}{k}$ be the* BINOMIAL COEFFICIENT *(Definition 4.3 page 25).*

1.	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\forall n,k\in\mathbb{Z},\ n\geq k\geq 0$	(FACTORIAL EXPANSION)
2.	$\binom{n}{k} = \binom{n}{n-k}$	$\forall n,k\in\mathbb{Z},\ n\geq 0$	(SYMMETRY)
3.	$\binom{n+x+1}{n} = \binom{n+x}{n} + \binom{n+x}{n-1}$	$\forall n\in\mathbb{Z},\ x\in\mathbb{C}$	(PASCAL'S RULE)
4.	$\binom{x+1}{k+1} = \binom{x}{k+1} + \binom{x}{k}$	$\forall k\in\mathbb{Z},\ x\in\mathbb{C}$	(PASCAL'S IDENTITY / STIFEL FORMULA)
5.	$\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}$	$\forall k,m\in\mathbb{Z},\ x\in\mathbb{C}$	(TRINOMIAL REVISION)
6.	$\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1}$	$\forall k\in\mathbb{Z},\ x\in\mathbb{C}$	(ABSORPTION IDENTITY)
7.	$\binom{x}{k} = (-1)^k\binom{k-x-1}{k}$	$\forall k\in\mathbb{Z},\ x\in\mathbb{C}$	(UPPER NEGATION)
8.	$\binom{x}{k} = \binom{x-2}{k-2} + 2\binom{x-2}{k-1} + \binom{x-2}{k}$	$\forall k\in\mathbb{Z},\ x\in\mathbb{C}$	(SECOND-ORDER PASCAL'S IDENTITY)
9.	$\binom{x-1}{k-1}\binom{x}{k+1}\binom{x+1}{k} = \binom{x-1}{k}\binom{x}{k-1}\binom{x+1}{k+1}$	$\forall k\in\mathbb{Z},\ x\in\mathbb{C}$	(HEXAGON IDENTITY)

✎PROOF:

[71] Graham et al. (1994) page 174 ⟨Table 174⟩, Gallier (2010) page 221, Gross (2008) page 227 ⟨Table 4.1.2⟩, Coolidge (1949) pages 149–150, Stifel (1544), Balakrishnan (1996) page 43 ⟨Pascal's Rule⟩, Harris et al. (2008) page 143 ⟨*hexagon identity*, (2.15)⟩, Ferland (2009) page 216 ⟨*second-order pascal identity*⟩

1. Proof for *factorial expansion*:

$$\begin{aligned}
\binom{n}{k} &\triangleq \frac{n^{\underline{k}}}{k!} && \forall n,k \in \mathbb{Z},\ n \ge k \ge 0 && \text{by Definition 4.3}\\
&= \frac{n(n-1)(n-2)\cdot(n-k+1)}{k!} && \forall n,k \in \mathbb{Z},\ n \ge k \ge 0 && \text{by Definition 4.2}\\
&= \frac{n(n-1)(n-2)\cdot(n-k+1)(n-k)(n-k-1)\cdots 1}{k!(n-k)!} && \forall n,k \in \mathbb{Z},\ n \ge k \ge 0 && \text{by Definition 4.2}\\
&= \frac{n!}{k!(n-k)!} && \forall n,k \in \mathbb{Z},\ n \ge k \ge 0 && \text{by Definition 4.1}
\end{aligned}$$

2. Proof for *symmetry* property:

   (a) Proof for $n,k \in \mathbb{Z},\ n \ge k \ge 0$: (use item (1) page 27)

$$\begin{aligned}
\binom{n}{n-k} &= \frac{n!}{(n-k)!(n-(n-k)!} && \forall n,k \in \mathbb{Z},\ n \ge k \ge 0 && \text{by item (1) page 27}\\
&= \frac{n!}{k!(n-k)!} && \forall n,k \in \mathbb{Z},\ n \ge k \ge 0 &&\\
&= \binom{n}{k} && \forall n,k \in \mathbb{Z},\ n \ge k \ge 0 && \text{by item (1) page 27}
\end{aligned}$$

   (b) Proof for $n,k \in \mathbb{Z},\ n \ge 0 > k$:

$$\begin{aligned}
\binom{n}{n-k} &= \frac{n^{\underline{n-k}}}{(n-k)!} && \forall n,k \in \mathbb{Z},\ n \ge 0 > k && \text{by Definition 4.3}\\
&= \frac{n(n-1)(n-2)\cdots 0 \cdots (n-n+k+1)}{(n-k)!} && \forall n,k \in \mathbb{Z},\ n \ge 0 > k && \text{by Definition 4.2}\\
&= 0 && &&\\
&= \binom{n}{k} && \forall n,k \in \mathbb{Z},\ n \ge 0 > k && \text{by Definition 4.3}
\end{aligned}$$

   (c) Proof for $n,k \in \mathbb{Z},\ n \ge 0 > k$:

$$\begin{aligned}
\binom{n}{k} &= \frac{n^{\underline{k}}}{k!} && \forall n,k \in \mathbb{Z},\ k > n \ge 0 && \text{by Definition 4.3 page 25}\\
&= \frac{n(n-1)(n-2)\cdots 0 \cdots (n-k+1)}{(n-k)!} && \forall n,k \in \mathbb{Z},\ k > n \ge 0 && \text{by Definition 4.2 page 24}\\
&= 0 && &&\\
&= \binom{n}{n-k} && \forall n,k \in \mathbb{Z},\ k > n \ge 0 && \text{by Definition 4.3 page 25}
\end{aligned}$$

3. Proof for *Pascal's Rule*:

   (a) Proof for $n < 0,\ x \in \mathbb{C}$:

$$\begin{aligned}
\binom{n+x}{n} + \binom{n+x}{n-1} &= 0 + 0 && \text{by Definition 4.3 page 25}\\
&= \binom{n+x+1}{n} && \text{by Definition 4.3 page 25}
\end{aligned}$$

   (b) Proof for $n = 0,\ x \in \mathbb{C}$:

$$\begin{aligned}
\binom{n+x}{n} + \binom{n+x}{n-1} &= \binom{n+x}{0} + \binom{n+x}{-1} && \text{by } n = 0 \text{ hypothesis}\\
&= 1 + 0 && \text{by Definition 4.3 page 25}\\
&= \binom{n+x+1}{0} && \text{by Definition 4.3 page 25}\\
&= \binom{n+x+1}{n} && \text{by } n = 0 \text{ hypothesis}
\end{aligned}$$

(c) Proof for $n > 0$, $x \in \mathbb{C}$:

$$
\begin{aligned}
&\binom{n+x}{n} + \binom{n+x}{n-1} \\
&\triangleq \frac{n + x^{\underline{n}}}{n!} + \frac{n + x^{\underline{n-1}}}{(n-1)!} && \text{by Definition 4.3 page 25} \\
&\triangleq \frac{(n+x)(n+x-1)\cdots(n+x-n+1)}{n!} \\
&\quad + \frac{(n+x)(n+x-1)\cdots(n+x-n+1+1)}{(n-1)!} && \text{by Definition 4.2 page 24} \\
&= \frac{[(n+x)(n+x-1)\cdots(x+1)] + [(n+x)(n+x-1)\cdots(x+2)n]}{n!} \\
&= \frac{[(x+1)+n][(n+x)(n+x-1)\cdots(x+2)]}{n!} \\
&= \frac{(n+x+1)(n+x)(n+x-1)\cdots(x+2)}{n!} \\
&\triangleq \frac{(n+x+1)^{\underline{n}}}{n!} && \text{by Definition 4.2 page 24} \\
&\triangleq \binom{n+x+1}{n} && \text{by Definition 4.3 page 25}
\end{aligned}
$$

4. Proof for *Pascal's Identity*:

$$
\begin{aligned}
\binom{x+1}{k+1} &= \binom{k+y+1}{k+1} && \text{where } y \triangleq x-k \implies x = y+k \\
&= \binom{y+k}{k+1} + \binom{y+k}{k} && \text{by Pascal's Rule (item (3))} \\
&= \binom{x}{k+1} + \binom{x}{k} && \text{by definition of } m
\end{aligned}
$$

5. Proof for *Trinomial revision*:

(a) Proof for $k < 0$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= \binom{x}{m}0 && \text{by } k<0 \text{ hypothesis and Definition 4.3 page 25} \\
&= \cancelto{0}{\binom{x}{k}}\binom{x-k}{m-k} && \text{by } k<0 \text{ hypothesis and Definition 4.3 page 25}
\end{aligned}
$$

(b) Proof for $k \geq 0$, $m < 0$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= 0\binom{m}{k} && \text{by } m<0 \text{ hypothesis and Definition 4.3 page 25} \\
&= \binom{x}{k}\cancelto{0}{\binom{x-k}{m-k}} && \text{by } k \geq 0, m<0 \text{ hypothesis and Definition 4.3 page 25}
\end{aligned}
$$

(c) Proof for $m < k$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= \binom{x}{m}0 && \text{by Proposition 4.2 page 25} \\
&= \binom{x}{k}\cancelto{0}{\binom{x-k}{m-k}} && \text{by } m<k \text{ hypothesis and Definition 4.3 page 25}
\end{aligned}
$$

(d) Proof for remaining cases:

$$
\begin{aligned}
&\binom{x}{m}\binom{m}{k} \\
&= \frac{x^{\underline{m}}}{m!}\frac{m^{\underline{k}}}{k!} && \text{by Definition 4.3} \\
&= \frac{x(x-1)\cdots(x-m+1)}{m!}\frac{m(m-1)\cdots(m-k+1)}{k!} && \text{by Definition 4.2} \\
&= \frac{x(x-1)\cdots(x-m+1)}{(m-k)!}\frac{1}{k!} \\
&= \frac{x(x-1)\cdots(x-k+1)}{k!}\frac{(x-k)(x-k-1)\cdots(x-m+1)}{(m-k)!} \\
&= \frac{x(x-1)\cdots(x-k+1)}{k!}\frac{(x-k)(x-k-1)\cdots((x-k)-(m-k)+1)}{(m-k)!} \\
&\triangleq \frac{x^{\underline{k}}}{k!}\frac{(x-k)^{\underline{m-k}}}{(m-k)!} && \text{by Definition 4.2} \\
&\triangleq \binom{x}{k}\binom{x-k}{m-k} && \text{by Definition 4.3}
\end{aligned}
$$

6. Proof for *Absorption identity*:

$$
\begin{aligned}
\frac{x}{k}\binom{x-1}{k-1} &= \frac{1}{k}\binom{x}{1}\binom{n-1}{k-1} && \text{by Proposition 4.2 page 25} \\
&= \frac{1}{k}\binom{x}{k}\binom{k}{1} && \text{by Trinomial revision (item (5))} \\
&= \frac{1}{k}\binom{x}{k}k && \text{by Proposition 4.2 page 25} \\
&= \binom{x}{k}
\end{aligned}
$$

7. Proof for *Upper Negation*:

$$
\begin{aligned}
&(-1)^k\binom{k-x-1}{k} \\
&\triangleq (-1)^k\frac{(k-x-1)^{\underline{k}}}{k!} && \text{by Definition 4.3 page 25} \\
&\triangleq (-1)^k\frac{(k-x-1)(k-x-2)(k-x-3)\cdots(k-x-1-k+1)}{k!} && \text{by Definition 4.2 page 24} \\
&= (-1)^k\frac{(k-x-1)(k-x-2)(k-x-3)\cdots(-x)}{k!} \\
&= (-1)^k(-1)^k\frac{(x)(x-1)\cdots(x(x-k+3)(x-k+2)(x-k+1)}{k!} \\
&\triangleq \frac{x^{\underline{k}}}{k!} && \text{by Definition 4.2 page 24} \\
&\triangleq \binom{x}{k} && \text{by Definition 4.3 page 25}
\end{aligned}
$$

8. Proof for *2nd Order Pascal's Identity*:

$$
\begin{aligned}
&\binom{n-2}{k-2}+2\binom{n-2}{k-1}+\binom{n-2}{k} \\
&\triangleq \frac{(x-2)^{\underline{(k-2)}}}{(k-2)!}+\frac{(x-2)^{\underline{(k-1)}}}{(k-1)!}+\frac{(x-2)^{\underline{k}}}{k!} \\
&\triangleq \frac{(x-2)(x-1)\cdots(x-k+2+1)}{(k-2)!}+2\frac{(x-2)(x-1)\cdots(x-k+1+1)}{(k-1)!}+\frac{(x-2)\cdots(x-k+1)}{k!} \\
&= \frac{(x-2)\cdots(x-2-k+2+1)k(k-1)+2(x-2)\cdots(x-2-k+1+1)k+(x-2)\cdots(x-k-1)}{k!}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(x-2)(x-1)\cdots(x-k+1)k(k-1)+2(x-2)(x-1)\cdots(x-k)k+(n-2)(n-1)\cdots(x-k-1)}{k!} \\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+2(x-k)k+(x-k)(x-k-1)]}{k!} \\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+2(x-k)k-(x-k)k+(x-k)(x-1)]}{k!} \\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+(x-k)k+(x-k)(x-1)]}{k!} \\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)]\left[k^2-k+kx-k^2+x^2-x-kx+k\right]}{k!} \\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)]\left[x^2-x\right]}{k!} \\
&= \frac{x(x-1)(x-2)(x-1)\cdots(x-k+1)}{k!} \\
&\triangleq \frac{n^{\underline{k}}}{k!} \\
&\triangleq \binom{n}{k}
\end{aligned}
$$

9. Proof for *Hexagon Identity*:

$$
\begin{aligned}
&\binom{x-1}{k-1}\binom{x}{k+1}\binom{x+1}{k} \\
&\triangleq \left[\frac{(x-1)^{\underline{(k-1)}}}{(k-1)!}\right]\left[\frac{x^{\underline{(k+1)}}}{(k+1)!}\right]\left[\frac{(x+1)^{\underline{k}}}{k!}\right] \\
&\triangleq \left[\frac{(x-1)\cdots(x-1-k+1+1)}{(k-1)!}\right]\left[\frac{x(x-1)\cdots(x-k-1+1)}{(k+1)!}\right]\left[\frac{(x+1)(x)\cdots(x+1-k+1)}{k!}\right] \\
&= \left[\frac{(x-1)\cdots(x-k+2)(x-k+1)}{(k-1)!}\right]\left[\frac{x(x-1)\cdots(x-k)}{(k+1)!}\right]\left[\frac{(x+1)(x)(x-1)\cdots(x-k+2)}{k!}\right] \\
&= \left[\frac{(x)(x-1)\cdots(x-k+2)}{(k-1)!}\right]\left[\frac{(x+1)x(x-1)\cdots(x-k)(x-k+1)}{(k+1)!}\right]\left[\frac{(x-1)\cdots(x-k)}{k!}\right] \\
&\triangleq \left[\frac{x^{\underline{(k-1)}}}{(k-1)!}\right]\left[\frac{(x+1)^{\underline{(k+1)}}}{(k+1)!}\right]\left[\frac{(x-1)^{\underline{k}}}{k!}\right] \\
&\triangleq \binom{x}{k-1}\binom{x+1}{k+1}\binom{x-1}{k}
\end{aligned}
$$

✐

From Pascal's Recursion we can construct *Pascal's Triangle*:[72]

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

[72] Pascal (1655), Granville (1992), Granville (1997), Edwards (2002), Hall and Knight (1894) pages 320–321 ⟨article 393⟩

4.3 Binomial summations

Theorem 4.2. [73] *Let* $\langle\!\langle x_n \rangle\!\rangle_1^N$ *and* $\langle\!\langle y_n \rangle\!\rangle_1^N$ *be sequences over a ring* $(\mathbb{X}, +, \times)$.

$$\boxed{\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) = \sum_{n=0}^{p+q} \underbrace{\left(\sum_{k=\max(0,n-q)}^{\min(n,p)} x_k y_{n-k}\right)}_{\textit{Cauchy product}}}$$

✎PROOF:

1.

$$\begin{aligned}
\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) &= \sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m} \\
&= \sum_{n=0}^{p}\sum_{k=n}^{q+n} x_n y_{k-n} && k = n+m \qquad m = k-n \\
&\vdots \\
&= \sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)
\end{aligned}$$

2. Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	y_0	y_1	y_2	y_3	$\cdots$	y_q
x_0	x_0y_0	x_0y_1	x_0y_2	x_0y_3	$\cdots$	x_0y_q
x_1	x_1y_0	x_1y_1	x_1y_2	x_1y_3	$\cdots$	x_1y_q
x_2	x_2y_0	x_2y_1	x_2y_2	x_2y_3	$\cdots$	x_2y_q
x_3	x_3y_0	x_3y_1	x_3y_2	x_3y_3	$\cdots$	x_3y_q
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_p	x_py_0	x_py_1	x_py_2	x_py_3	$\cdots$	x_py_q

(a) The expression $\sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

(b) If we switched the order of summation to $\sum_{m=0}^{q}\sum_{n=0}^{p} x_n y_m z^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

(c) However the final result expression $\sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)$ is equivalent to adding *diagonally* starting from the upper left corner and proceding to the lower right.

(d) Upper limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:

$$\left.\begin{array}{rcl} k &\le& n \\ k &\le& p \end{array}\right\} \implies k \le \min(n,p)$$

(e) Lower limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:

$$\left.\begin{array}{rcl} k &\ge& 0 \\ k &\ge& n-q \end{array}\right\} \implies k \ge \max(0, n-q)$$

⇨

[73] Apostol (1975) page 237

Theorem 4.3. [74] *Let* $\binom{n}{k}$ *be the* BINOMIAL COEFFICIENT *(Definition 4.3 page 25).*

$$\begin{array}{rcll}
\sum_{k=0}^{n}\binom{n}{k} &=& 2^n & \text{(row sum)}\\
\sum_{k=m}^{n}\binom{k}{m} &=& \binom{n+1}{m+1} & \text{(upper sum / column sum)}\\
\sum_{k=0}^{n}\binom{m+k}{k} &=& \binom{n+m+1}{n} & \text{(parallel summation formula/ southeast diagonal)}\\
\sum_{k=0}^{m}\binom{n-k}{m-k} &=& \binom{n+1}{m} & \text{(northwest diagonal)}\\
\sum_{j=0}^{n}\binom{m}{j}\binom{n}{k-j} &=& \binom{m+n}{k} & \text{(Vandermonde's convolution)}\\
\sum_{i=-j}^{n-j}\binom{m}{j+i}\binom{n}{k-i} &=& \binom{m+n}{j+k} & \text{(alternate Vandermonde's convolution)}\\
\sum_{k=0}^{n}\binom{n}{k}^2 &=& \binom{2n}{n} &
\end{array}$$

✎PROOF:

1. Proof for *row sum* relation:

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}x^k &= \left.\sum_{k=0}^{n}\binom{n}{k}x^k\right|_{x=1}\\
&= (1+x)^n|_{x=1} && \text{by Binomial Theorem}\\
&= (1+1)^n\\
&= 2^n
\end{aligned}$$

2. Proof for *upper sum* relation (proof by induction):

   (a) Proof for $(n,m) = (0,0)$ case:
   $$\sum_{0}^{0}\binom{k}{m} = \binom{0}{0} = 1 = \binom{0+1}{0+1}$$

   (b) Proof for $(n,m) = (1,0)$ case:
   $$\sum_{0}^{1}\binom{k}{m} = \binom{1}{0} + \binom{1}{1} = 2 = \binom{1+1}{0+1}$$

   (c) Proof for $(n,m) = (1,1)$ case:
   $$\sum_{0}^{1}\binom{k}{m} = \binom{1}{1} = 1 = \binom{1+1}{1+1}$$

[74] Graham et al. (1994) page 169 ⟨Table 169⟩, Gallier (2010) pages 218–223, Gross (2008) page 227 ⟨Table 4.1.2⟩, Harris et al. (2008) pages 137–142, Knuth (1992a), Vandermonde (1772), Zhū (1303)

(d) Proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
\sum_{k=m}^{n+1}\binom{k}{m} &= \binom{n+1}{m} + \sum_{k=m}^{n} \\
&= \binom{n+1}{m} + \binom{n+1}{m+1} && \text{by left hypothesis} \\
&= \binom{n+2}{m+1} && \text{by Pascal's recursion (Theorem 4.1 page 26)}
\end{aligned}$$

3. Proof for *Parallel summation formula* (Proof by induction):

(a) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n}$ is true for $n=0$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{m+k}{k}\right|_{n=0} &= \binom{m+0}{0} \\
&= \frac{(m+0)!}{(m-0)!\,0!} && \text{by Definition 4.3 page 25} \\
&= \frac{(m+1)!}{(m+1-0)!\,0!} \\
&= \binom{m+1}{0} && \text{by Definition 4.3 page 25} \\
&= \left.\binom{n+m+1}{n}\right|_{n=0}
\end{aligned}$$

(b) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n}$ is true for $n=1$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{m+k}{k}\right|_{n=1} &= \binom{m+0}{0} + \binom{m+1}{1} \\
&= \binom{m+1}{0} + \binom{m+1}{1} \\
&= \binom{m+1+1}{1} && \text{by Pascal's Rule page 26} \\
&= \left.\binom{n+m+1}{n}\right|_{n=1}
\end{aligned}$$

(c) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n} \implies \sum_{k=0}^{n+1}\binom{m+k}{k} = \binom{(n+1)+m+1}{n+1}$:

$$\begin{aligned}
\sum_{k=0}^{n+1}\binom{m+k}{k} &= \binom{m}{0} + \sum_{k=1}^{n+1}\binom{m+k}{k} \\
&= \binom{m}{0} + \sum_{k=0}^{n}\binom{m+k+1}{k+1} \\
&= \binom{m}{0} + \sum_{k=0}^{n}\binom{m+k}{k} - \binom{m}{0} + \binom{m+n+1}{n+1} \\
&= \binom{n+m+1}{n} + \binom{m+n+1}{n+1} && \text{by left hypothesis} \\
&= \binom{n+m+2}{n+1} && \text{by Pascal's Rule page 26} \\
&= \binom{(n+1)+m+1}{(n+1)}
\end{aligned}$$

4. Proof for *Vandermonde's convolution*:

$$\begin{aligned}
\sum_{k=0}^{m+n}\binom{m+n}{k}x^k &= (1+x)^{m+n} && \text{by Binomial Theorem)}\\
&= \left[\sum_{k=0}^{m}\binom{m}{k}x^k\right]\left[\sum_{j=0}^{n}\binom{n}{j}x^j\right] && \text{by Binomial Theorem)}\\
&= \sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}x^kx^j\\
&= \sum_{k=0}^{m+n}\left[\sum_{j=0}^{n}\binom{m}{j}\binom{n}{k-j}\right]x^k && \text{by Theorem 4.2 page 31}\\
\implies \binom{m+n}{k} &= \sum_{j=0}^{n}\binom{m}{j}\binom{n}{k-j}
\end{aligned}$$

5. Proof for *alternate Vandermonde's convolution*:

$$\begin{aligned}
\binom{m+n}{j+k} &= \binom{m+n}{u} && \text{where } u \triangleq j+k \implies k = u-j\\
&= \sum_{v=0}^{n}\binom{m}{v}\binom{n}{u-v}\\
&= \sum_{v=0}^{n}\binom{m}{v}\binom{n}{j+k-v}\\
&= \sum_{i+j=0}^{i+j=n}\binom{m}{j+i}\binom{n}{k-i} && \text{where } i \triangleq v-j \implies v = i+j\\
&= \sum_{i=-j}^{i=n-j}\binom{m}{j+i}\binom{n}{k-i}
\end{aligned}$$

6. Proof that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$:

$$\begin{aligned}
\binom{2n}{n} &= \binom{n+n}{n}\\
&= \sum_{k=0}n\binom{n}{k}\binom{n}{n-k} && \text{by Vandermonde's convolution (item (4) page 34)}\\
&= \sum_{k=0}n\binom{n}{k}\binom{n}{k} && \text{by item (2)}\\
&= \sum_{k=0}^{n}\binom{n}{k}^2
\end{aligned}$$

✏

Theorem 4.4. [75]

$$\boxed{\sum_{k=1}^{n}\frac{1}{k+1} < \ln(n+1) < \sum_{k=1}^{n}\frac{1}{k}}$$

[75] Rivlin (1969) page 60

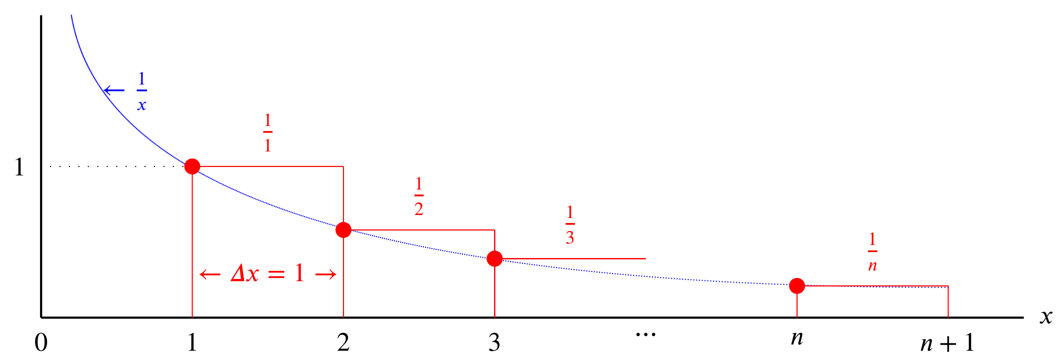


Figure 1: $\ln(n+1)$

✎PROOF: The summations are simply lower and upper bounds of the integral of $\frac{1}{x}$ in the range $[1,\ n+1]$. This is illustrated in Figure 1.

1. Proof that $\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k}$:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k} &> \int_{1}^{n+1} \frac{1}{x}\,\mathsf{d}x \\ &= \ln x \Big|_{1}^{n+1} \\ &= \ln(n+1) - \cancelto{0}{\ln(1)} \\ &= \ln(n+1)\end{aligned}$$

2. Proof that $\sum_{k=1}^{n} \frac{1}{k+1} < \ln(n+1)$:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k+1} &< \int_{1}^{n+1} \frac{1}{x}\,\mathsf{d}x \\ &= \ln(n+1) - \cancelto{0}{\ln(1)} \\ &= \ln(n+1)\end{aligned}$$

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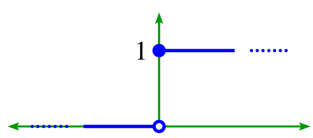
5 B-splines

5.1 Definition

Definition 5.1. *Let X be a set.*

*The **step function** $\sigma \in \mathbb{R}^{\mathbb{R}}$ is defined as*

$$\sigma(x) \triangleq \mathbb{1}_{[0:\infty)}(x) \qquad \forall x \in \mathbb{R}.$$



Definition 5.2. [76] *Let $\mathbb{1}$ be the* SET INDICATOR *function* (Definition 1.3 page 3). *Let $\mathsf{f}(x) \star \mathsf{g}(x)$ represent the* CONVOLUTION *operation* (Definition 1.8 page 9).

*The **nth order cardinal B-spline** N_n for $n \in \mathbb{W}$ is defined as*

$$\mathsf{N}_n(x) \triangleq \begin{cases} \mathbb{1}_{[0:1]}(x) & \text{for } n = 0 \\ \mathsf{N}_{n-1}(x) \star \mathsf{N}_1(x) & \text{for } n \in \mathbb{W}\backslash 0 \end{cases}$$

Lemma 5.1. [77]

[76] Chui (1992) page 85 ⟨(4.2.1)⟩, Christensen (2008) page 140, Chui (1988) page 1
[77] Christensen (2008) page 140, Chui (1992) page 85 ⟨(4.2.1)⟩, Chui (1988) page 1

$$\boxed{\mathsf{N}_n(x) = \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau \qquad \forall n\in\mathbb{W}\backslash 0}$$

✎PROOF:

$$\begin{aligned}
\mathsf{N}_n(x) &\triangleq \int_{\mathbb{R}} \mathsf{N}_{n-1}(x-\tau)\mathsf{N}_1(\tau)\,\mathrm{d}\tau && \text{by definition of } \mathsf{N}_n \text{ (Definition 5.2 page 35)}\\
&= \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau && \text{by definition of } \mathsf{N}_1 \text{ (Definition 5.2 page 35)}
\end{aligned}$$

✏

Lemma 5.2. [78] *Let* $\mathbb{1}$ *be the* SET INDICATOR *function (Definition 1.3 page 3). Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition 5.1 page 35).*

$$\boxed{\begin{aligned}
\mathsf{N}_0(x) &= \sigma(x)-\sigma(x-1) && \forall x\in\mathbb{R}\\
&= \begin{cases} 1 & \text{for } x\in[0:1)\\ 0 & \text{for } x\in\mathbb{R}\backslash[0:1)\end{cases}\\
\mathsf{N}_1(x) &= x\sigma(x)-2(x-1)\sigma(x-1)+(x-2)\sigma(x-2) && \forall x\in\mathbb{R}\\
&= \begin{cases} x & \text{for } x\in[0:1]\\ -x+2 & \text{for } x\in[1:2]\\ 0 & \text{for } x\in\mathbb{R}\backslash[0:2]\end{cases}\\
\mathsf{N}_2(x) &= \tfrac{1}{2}x^2\sigma(x)+\left[-\tfrac{3}{2}x^2+3x-\tfrac{3}{2}\right]\sigma(x-1)+\left[\tfrac{3}{2}x^2-6x+6\right]\sigma(x-2)\\
&\quad+\left[-\tfrac{1}{2}x^2+3x-\tfrac{9}{2}\right]\sigma(x-3) && \forall x\in\mathbb{R}\\
&= \begin{cases} \tfrac{1}{2}x^2 & \text{for } x\in[0:1]\\ -x^2+3x-\tfrac{3}{2} & \text{for } x\in[1:2]\\ \tfrac{1}{2}x^2-3x+\tfrac{9}{2} & \text{for } x\in[2:3]\\ 0 & \text{for } x\in\mathbb{R}\backslash[0:3]\end{cases}
\end{aligned}}$$

✎PROOF:

$$\begin{aligned}
\mathsf{N}_0(x) &= \mathbb{1}_{[0:1]}(x) && \text{by definition of } \mathsf{N}_n \text{ (page 35)}\\
\mathsf{N}_1(x) &= \int_0^1 \mathsf{N}_0(x-\tau)\,\mathrm{d}\tau && \text{by Lemma 5.1 page 35}\\
&= \int_0^1 \mathbb{1}_{[0:1]}(x-\tau)\,\mathrm{d}\tau && \text{by definition of } \mathsf{N}_1 \text{ (page 35)}\\
&= \int_{x-u=0}^{x-u=1} \mathbb{1}_{[0:1]}(u)(-1)\,\mathrm{d}u && \text{where } u\triangleq x-\tau \implies \tau = x-u\\
&= \int_{u=x-1}^{u=t} \mathbb{1}_{[0:1]}(u)\,\mathrm{d}u\\
&= u\sigma(u)-(u-1)\sigma(u-1)+a\big|_{u=x-1}^{u=t}\\
&= \underbrace{\{x\sigma(x)-(x-1)\sigma(x-1)+a\}}_{u=t}\\
&\quad-\underbrace{\{(x-1)\sigma(x-1)-(x-2)\sigma(x-2)+a\}}_{u=x-1}
\end{aligned}$$

[78] Christensen (2008) page 148 ⟨Exercise 6.2⟩, Christensen (2010) page 212 ⟨Exercise 10.2⟩, Schumaker (2007) page 136 ⟨Table 1⟩

$$
\begin{aligned}
&= x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)\\
&= \begin{cases} t & \text{for } x \in [0:1] \\ -x+2 & \text{for } x \in [1:2] \\ 0 & \text{for } x \in \mathbb{R}\backslash[0:2] \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{N}_2(x)\\
&= \int_0^1 \mathsf{N}_1(x-\tau)\,\mathrm{d}\tau && \text{by Lemma 5.1 page 35}\\
&= \int_0^1 (x-\tau)\sigma(x-\tau) - 2(x-\tau-1)\sigma(x-\tau-1) + (x-\tau-2)\sigma(x-\tau-2)\,\mathrm{d}\tau && \text{by result for } \mathsf{N}_2\\
&= \int_{x-u=0}^{x-u=1} u\sigma(u) - 2(u-1)\sigma(u-1) + (u-2)\sigma(u-2)(-1)\,\mathrm{d}u && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \int_{u=x-1}^{u=t} u\sigma(u)\,\mathrm{d}u + \int_{u=x-1}^{u=t} (-2u+2)\sigma(u-1)\,\mathrm{d}u + \int_{u=x-1}^{u=t} (u-2)\sigma(u-2)\,\mathrm{d}u\\
&= \left[\frac{1}{2}+a\right]u^2\sigma(u) + \left[-u^2+2u+b\right]\sigma(u-1) + \left[\frac{1}{2}u^2-2u+c\right]\sigma(u-2)\bigg|_{u=x-1}^{u=t}\\
&= \underbrace{\left\{\left[\tfrac{1}{2}x^2+a\right]\sigma(x) + \left[-x^2+2x+b\right]\sigma(x-1) + \left[\tfrac{1}{2}x^2-2x+c\right]\sigma(x-2)\right\}}_{u=t}\\
&\quad - \underbrace{\left\{\left[\tfrac{1}{2}(x-1)^2+a\right]\sigma(x-1) + \left[-(x-1)^2+2(x-1)+b\right]\sigma(x-2) + \left[\tfrac{1}{2}(x-1)^2-2(x-1)+c\right]\sigma(x-3)\right\}}_{u=x-1}\\
&= \left[\frac{1}{2}x^2+a\right]\sigma(x) + \left[-x^2+2x+b-\frac{1}{2}x^2+x-\frac{1}{2}-a\right]\sigma(x-1)\\
&\qquad + \left[\frac{1}{2}x^2-2x+c+x^2-2x+1-2x+2-b\right]\sigma(x-2) + \left[-\frac{1}{2}x^2+x-\frac{1}{2}+2x-2-c\right]\sigma(x-3)\\
&= \left[\frac{1}{2}x^2+a\right]\sigma(x) + \left[-\frac{3}{2}x^2+3t-\frac{1}{2}+b-a\right]\sigma(x-1) + \left[\frac{3}{2}x^2-6t+3+c-b\right]\sigma(x-2)\\
&\qquad + \left[-\frac{1}{2}x^2+3t-\frac{5}{2}-c\right]\sigma(x-3)\\
&= \begin{cases} \frac{1}{2}x^2+a & \text{for } x \in [0:1] \\ -x^2+3x-\frac{1}{2}+b & \text{for } x \in [1:2] \\ \frac{1}{2}x^2-3x+\frac{5}{2}+c & \text{for } x \in [2:3] \\ 0 & \text{for } x \in \mathbb{R}\backslash[0:3] \end{cases}
\end{aligned}
$$

The B-spline $N_3(x)$ is continuous. Therefore, at each point n where $\sigma(x-n)$ jumps from 0 to 1, the factor $\mathsf{f}_n(x)$ in $\mathsf{f}_n(x)\sigma(x-n)$ must be 0. We can use this to compute the boundary conditions a, b, and c:

$$
\begin{aligned}
\frac{1}{2}x^2+a\bigg|_{t=0} &= 0 &&\implies 0+a=0 &&\implies a=0\\
-\frac{3}{2}x^2+3x-\frac{1}{2}+b-a\bigg|_{t=1} &= 0 &&\implies -\frac{3}{2}+3-\frac{1}{2}+b-0=0 &&\implies b=-1\\
\frac{3}{2}x^2-6x+3+c-b\bigg|_{t=2} &= 0 &&\implies \frac{12}{2}-12+3+c+1=0 &&\implies c=2\\
-\frac{1}{2}x^2+3x-\frac{5}{2}-c\bigg|_{t=3} &= 0 &&\implies -\frac{9}{2}+9-\frac{5}{2}-c=0 &&\implies c=2
\end{aligned}
$$

⇨

5.2 Properties

Theorem 5.1. [79] *Let* $\mathbb{1}$ *be the* SET INDICATOR *function* (Definition 1.3 page 3). *Let* $\sigma(x)$ *be the* STEP FUNCTION (Definition 5.1 page 35).

$$\boxed{\mathsf{N}_n(x) = \frac{1}{(n)!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n\sigma(x-k) \qquad \forall n\in\mathbb{W}\backslash 0}$$

✎PROOF: Proof by induction:

1. Proof for $n=1$ case:

$$\begin{aligned}
\mathsf{N}_1(x) &= x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2) && \text{by Lemma 5.2 page 36}\\
&= \frac{1}{(2-1)!}\sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)^{2-1}\sigma(x-k)
\end{aligned}$$

2. Proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\mathsf{N}_{n+1}(x)\\
&= \int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma 5.1 page 35}\\
&= \int_0^1 \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-\tau-k)^{n-1}\sigma(x-\tau-k) && \text{by left hypothesis}\\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{-1}{n}\right)(x-\tau-k)^n\sigma(x-\tau-k)\Big|_0^1\\
&= \frac{1}{(n)!}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}(x-\tau-k)^n\sigma(x-\tau-k)\Big|_0^1\\
&= \underbrace{\left\{\frac{1}{(n)!}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}(x-k-1)^n\sigma(x-k-1)\right\}}_{\tau=1} - \underbrace{\left\{\frac{1}{(n)!}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}(x-k)^n\sigma(x-k)\right\}}_{\tau=0}\\
&= \left\{\frac{1}{(n)!}\sum_{m=1}^{n}(-1)^m\binom{n}{m-1}(x-m)^n\sigma(x-m)\right\} - \left\{\frac{1}{(n)!}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}(x-k)^n\sigma(x-k)\right\}\\
&\quad \text{where } m\triangleq k+1 \implies k=m-1\\
&= \left\{\frac{1}{(n)!}\sum_{m=1}^{n}(-1)^m\left\{\binom{n+1}{m}-\binom{n}{m}\right\}(x-m)^n\sigma(x-m)\right\} + \left\{\frac{1}{(n)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n\sigma(x-k)\right\}\\
&\quad \text{by Stifel's formula Theorem 4.1 page 26}\\
&= \left\{\frac{1}{(n)!}\sum_{m=1}^{n}(-1)^m\binom{n+1}{m}(x-m)^n\sigma(x-m)\right\} + \left\{\frac{1}{(n)!}(-1)^0\binom{n+1}{0}(x-0)^n\sigma(x-0)\right\}\\
&= \frac{1}{(n)!}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}(x-k)^n\sigma(x-k)
\end{aligned}$$

✏

[79] Christensen (2008) page 142 ⟨Theorem 6.1.3⟩, Chui (1992) page 84 ⟨(4.1.12)⟩

Lemma 5.3. [80]

$$\boxed{\tfrac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_n(x) = \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1) \qquad \forall n \in \mathbb{W}\backslash\{1,2\},\ \forall x \in \mathbb{R}}$$

✎PROOF:

1. Proof using Fundamental Theorem of Calculus (FTC):

$$\begin{aligned}
\tfrac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_n(x) &= \tfrac{\mathsf{d}}{\mathsf{dx}}\int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathsf{d}\tau && \text{by Lemma 5.1 page 35}\\
&= \tfrac{\mathsf{d}}{\mathsf{dx}}\int_{x-u=0}^{x-u=1} \mathsf{N}_{n-1}(u)(-1)\,\mathsf{d}u && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \tfrac{\mathsf{d}}{\mathsf{dx}}\int_{u=x-1}^{u=x} \mathsf{N}_{n-1}(u)\,\mathsf{d}u\\
&= \left\{\tfrac{\mathsf{d}}{\mathsf{dx}}\int \mathsf{N}_{n-1}(u)\,\mathsf{d}u\bigg|_{u=x}\right\} - \left\{\tfrac{\mathsf{d}}{\mathsf{dx}}\int \mathsf{N}_{n-1}(u)\,\mathsf{d}u\bigg|_{u=x-1}\right\} && \text{by FTC[81]}\\
&= \left\{\mathsf{N}_{n-1}(x)\tfrac{\mathsf{d}}{\mathsf{dx}}(x)\right\} - \left\{\mathsf{N}_{n-1}(x-1)\tfrac{\mathsf{d}}{\mathsf{dx}}(x-1)\right\} && \text{by \textit{Chain Rule}[82]}\\
&= \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)
\end{aligned}$$

2. Proof by induction:

   (a) Proof for $n=2$ case:

$$\begin{aligned}
&\mathsf{N}_1(x) - \mathsf{N}_1(x-1)\\
&= \underbrace{x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)}_{\mathsf{N}_1(x)}\\
&\qquad - \underbrace{[(x-1)\sigma(x-1) - 2(x-2)\sigma(x-2) + (x-3)\sigma(x-3)]}_{\mathsf{N}_1(x-1)} && \text{by Lemma 5.2 page 36}\\
&= x\sigma(x) + [-2x+2-x+1]\sigma(x-1) + [x-2+2x-4]\sigma(x-2) + [-x+3]\sigma(x-3)\\
&= x\sigma(x) + [-3x+3]\sigma(x-1) + [3x-6]\sigma(x-2) + [-x+3]\sigma(x-3)\\
&= \tfrac{\mathsf{d}}{\mathsf{dx}}\left\{\begin{array}{l}\tfrac{1}{2}x^2\sigma(x) + \left[-\tfrac{3}{2}x^2+3x-\tfrac{1}{2}\right]\sigma(x-1) + \left[\tfrac{3}{2}x^2-6x+3\right]\sigma(x-2)\\ \qquad +\left[-\tfrac{1}{2}x^2+3x-\tfrac{5}{2}\right]\sigma(x-3)\end{array}\right\}\\
&= \tfrac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_2(x) \qquad \text{by Lemma 5.2 page 36}
\end{aligned}$$

   (b) Proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
\tfrac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_{n+1}(x) &= \tfrac{\mathsf{d}}{\mathsf{dx}}\int_0^1 \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by Lemma 5.1 page 35}\\
&= \int_0^1 \tfrac{\partial}{\partial\mathsf{x}}\mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{see note later}\\
&= \int_0^1 \left[\mathsf{N}_{n-1}(x-\tau) - \mathsf{N}_{n-1}(x-1-\tau)\right]\mathsf{d}\tau && \text{by left hypothesis}\\
&= \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathsf{d}\tau - \int_0^1 \mathsf{N}_{n-1}(x-1-\tau)\,\mathsf{d}\tau\\
&= \mathsf{N}_n(x) - \mathsf{N}_n(x-1) && \text{by Lemma 5.1 page 35}
\end{aligned}$$

[80] Höllig (2003) page 25 ⟨(3.2)⟩, Schumaker (2007) page 121 ⟨Theorem 4.16⟩
[81] Hijab (2011) page 163 ⟨Theorem 4.4.3⟩
[82] Hijab (2011) pages 73–74 ⟨Theorem 3.1.2⟩

Note: For information about differentiation of an integral, see Flanders (1973), Talvila (2001), Knapp (2005) page 389 ⟨Chapter VII⟩

Theorem 5.2. [83] *Let* $\operatorname{supp} \mathsf{f}$ *be the* SUPPORT *of a function* f.

$$\begin{array}{llcllll}
1. & \mathsf{N}_n(x) & \geq & 0 & \forall n \in \mathbb{W}, & \forall x \in \mathbb{R} & \text{(POSITIVE)} \\
2. & \operatorname{supp} \mathsf{N}_n(x) & = & [0:n+1] & \forall n \in \mathbb{W} & & \text{(COMPACT SUPPORT)} \\
3. & \int_{\mathbb{R}} \mathsf{N}_n(x)\,\mathrm{d}x & = & 1 & \forall n \in \mathbb{W} & & \text{(UNIT AREA)} \\
4. & \sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) & = & 1 & \forall n \in \mathbb{W}\backslash 0 & & \text{(PARTITION OF UNITY)} \\
5. & \mathsf{N}_n(x) & = & \frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1) & \forall n \in \mathbb{W}\backslash\{1\}, \forall x \in \mathbb{R} & & \\
6. & \mathsf{N}_n\left(\frac{n+1}{2}+x\right) & = & \mathsf{N}_n\left(\frac{n+1}{2}-x\right) & \forall n \in \mathbb{W} & \forall x \in \mathbb{R} & \text{(SYMMETRIC)}
\end{array}$$

✎PROOF:

1. Proof that $\operatorname{supp} \mathsf{N}_n(x) \geq 0$ (proof by induction):
   (a) Proof that $\mathsf{N}_0(x) \geq 0$: by Definition 5.2 page 35
   (b) Proof that $\mathsf{N}_n \geq 0 \implies \mathsf{N}_{n+1} \geq 0$:

$$\begin{aligned}
&\mathsf{N}_{n+1}(x) \\
&= \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma 5.1 page 35} \\
&\geq 0 && \text{by left hypothesis}
\end{aligned}$$

2. Proof that $\operatorname{supp} \mathsf{N}_n(x) = [0:n]$ (proof by induction):
   (a) Proof that $\operatorname{supp} \mathsf{N}_0 = [0:1]$: by Definition 5.2 page 35
   (b) Proof that $\operatorname{supp} \mathsf{N}_n = [0:n] \implies \operatorname{supp} \mathsf{N}_{n+1} = [0:n+1]$:

$$\begin{aligned}
\operatorname{supp} \mathsf{N}_{n+1}(x) &= \operatorname{supp} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma 5.1 page 35} \\
&= \left\{x \in \mathbb{R} | x-\tau \in [0:n] \text{ for some } \tau \in [0:1]\right\} && \text{by left hypothesis} \\
&= [0:n+1]
\end{aligned}$$

3. Proof that $\int_{\mathbb{R}} \mathsf{N}_n(x)\,\mathrm{d}x = 1$ (proof by induction):
   (a) Proof that $\int_{\mathbb{R}} \mathsf{N}_1(x) = 1$:

$$\int_{\mathbb{R}} \mathsf{N}_0(x)\,\mathrm{d}x = 0 \qquad \text{by definition of } \mathsf{N}_1 \text{ (Definition 5.2 page 35)}$$

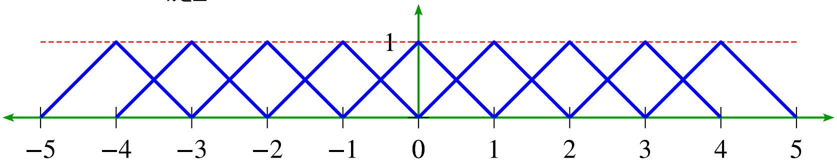
[83] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972)

(b) Proof that $\int_{\mathbb{R}} \mathsf{N}_n(x) = 1 \implies \int_{\mathbb{R}} \mathsf{N}_{n+1} = 1$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{N}_{n+1}(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau\,\mathrm{d}x && \text{by Lemma 5.1 page 35}\\
&= \int_0^1\int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&= \int_0^1\int_{\mathbb{R}} \mathsf{N}_n(u)\,\mathrm{d}u\,\mathrm{d}\tau && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \int_0^1 1\,\mathrm{d}\tau && \text{by left hypothesis}\\
&= 1
\end{aligned}$$

4. Proof that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1$ for $n \in \mathbb{W}\backslash 0$ (proof by induction):

(a) Proof that $\sum_{k\in\mathbb{Z}} \mathsf{N}_1(x-k) = 1$:



(b) Proof that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1 \implies \sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) = 1$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) &= \sum_{k\in\mathbb{Z}} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-k-\tau)\,\mathrm{d}\tau && \text{by Lemma 5.1 page 35}\\
&= \sum_{k\in\mathbb{Z}} \int_{x-u=0}^{x-u=1} \mathsf{N}_n(u-k)(-1)\,\mathrm{d}u && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \sum_{k\in\mathbb{Z}} \int_{u=x-1}^{u=t} \mathsf{N}_n(u-k)\,\mathrm{d}u\\
&= \int_{u=x-1}^{u=t} \left(\sum_{k\in\mathbb{Z}} \mathsf{N}_n(u-k)\right)\mathrm{d}u\\
&= \int_{u=x-1}^{u=t} 1\,\mathrm{d}\tau && \text{by left hypothesis}\\
&= 1
\end{aligned}$$

5. Proof for recursion equation (proof by induction):

(a) Proof for $n = 1$ case:

$$\begin{aligned}
\frac{x}{1}\mathsf{N}_0(x) + \frac{1+1-x}{1}\mathsf{N}_0(x-1) &= \frac{x}{1}\underbrace{[\sigma(x)-\sigma(x-1)]}_{\mathsf{N}_0(x)} + \frac{1+1-x}{1}\underbrace{[\sigma(x-1)-\sigma(x-2)]}_{\mathsf{N}_0(x-1)}\\
&= x\sigma(x) + [-x-x+2]\sigma(x-1) + [x-2]\sigma(x-2)\\
&= \mathsf{N}_1(x) \qquad \text{by Lemma 5.2 page 36}
\end{aligned}$$

(b) Proof that n case $\implies$ $n+1$ case:

$$
\begin{aligned}
&\frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1) + c_1 \\
&= \int \frac{\mathsf{d}}{\mathsf{d}x}\left\{ \frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1)\right\} \,\mathsf{d}x \\
&= \int \underbrace{\frac{1}{n+1}\mathsf{N}_n(x) + \frac{x}{n+1}\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_n(x)}_{\frac{\mathsf{d}}{\mathsf{d}x}\frac{x}{n+1}\mathsf{N}_n(x)} + \underbrace{\frac{-1}{n+1}\mathsf{N}_n(x-1) + \frac{n+2-x}{n}\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_n(x-1)}_{\frac{\mathsf{d}}{\mathsf{d}x}\frac{n+2-x}{n+1}\mathsf{N}_n(x-1)} \,\mathsf{d}x \\
&\qquad \text{by } \textit{product rule} \\
&= \int \frac{1}{n+1}\underbrace{\left[\frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1)\right]}_{\text{by } n \text{ hypothesis}} + \frac{x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)\right]}_{\text{by Lemma 5.3 page 39}} \\
&\qquad - \underbrace{\left[\frac{x-1}{n^2+n}\mathsf{N}_{n-1}(x-1) + \frac{n-x+2}{n(n+1)}\mathsf{N}_{n-1}(x-2)\right]}_{\text{by } n \text{ hypothesis}} \\
&\qquad + \frac{n+2-x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x-1) - \mathsf{N}_{n-1}(x-2)\right]}_{\text{by Lemma 5.3 page 39}} \,\mathsf{d}x \\
&= \int \left[\frac{x}{n(n+1)} + \frac{x}{n+1}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n-x+1}{n(n+1)} - \frac{x-1}{n(n+1)} + \frac{n+2-2x}{n+1}\right]\mathsf{N}_{n-1}(x-1) \\
&\qquad + \left[\frac{-n-2+x}{n(n+1)} + \frac{-n-2+x}{n+1}\right]\mathsf{N}_{n-1}(x-2) \,\mathsf{d}x \\
&= \int \left[\frac{x+nx}{n(n+1)}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x+n(n+2-2x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-1) \\
&\qquad + \left[\frac{-n-2+x+n(-n-2+x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-2) \,\mathsf{d}x \\
&= \int \left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x}{n}\right]\mathsf{N}_{n-1}(x-1) + \left[\frac{-n-2+x}{n}\right]\mathsf{N}_{n-1}(x-2) \,\mathsf{d}x \\
&= \int \underbrace{\left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+1-x}{n}\right]\mathsf{N}_{n-1}(x-1)}_{\mathsf{N}_n(x)} \\
&\qquad - \underbrace{\left[\frac{x-1}{n}\right]\mathsf{N}_{n-1}(x-1) - \left[\frac{n+2-x}{n}\right]\mathsf{N}_{n-1}(x-2)}_{\mathsf{N}_{n-1}(x-1)} \,\mathsf{d}x \\
&= \int \mathsf{N}_n(x) - \mathsf{N}_n(x-1) \,\mathsf{d}x \qquad \text{by } n \text{ hypothesis} \\
&= \int \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_{n+1}(x) \,\mathsf{d}x \qquad \text{by Lemma 5.3 page 39} \\
&= \mathsf{N}_{n+1}(x) + c_2
\end{aligned}
$$

Proof that $c_1 = c_2$: By item (2) (page 40), $\mathsf{N}_n(x) = 0$ for $x < 0$. Therefore, $c_1 = c_2$.

6. Proof for symmetric equation (proof by induction):
Note that it is true for $\mathsf{N}_0(x)$. Then here is the proof that $n-1$ case $\implies$ n case ...

$$
\begin{aligned}
&\mathsf{N}_n\left(\frac{n+1}{2} + x\right) \\
&= \frac{\frac{n+1}{2} + x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2} + x\right) + \frac{n+1-\left(\frac{n+1}{2} + x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2} + x - 1\right) \qquad \text{by item (5) page 41}
\end{aligned}
$$

$$= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x+\frac{1}{2}\right]\right)+\frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x-\frac{1}{2}\right]\right)$$

$$= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x+\frac{1}{2}\right]\right)+\frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x-\frac{1}{2}\right]\right) \qquad \text{by left hypothesis}$$

$$= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\left[\frac{n+1}{2}-x\right]-1\right)+\frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}-x\right)$$

$$= \mathsf{N}_n\left(\frac{n+1}{2}-x\right) \qquad \text{by item (5) page 41}$$

✏

Theorem 5.3. [84] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

$$\boxed{\int_{[0:1]^n}\mathsf{f}^{(n)}\left(\sum_{k=1}^{n}x_k\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n\in\mathbb{N}}$$

✎PROOF: Proof by induction:

1. Proof for $n=1$ case:

$$\begin{aligned}\int_{[0:1]}\mathsf{f}^{(1)}(x)\,\mathrm{d}x &= \mathsf{f}(x)|_0^1\\ &= \mathsf{f}(1)-\mathsf{f}(0)\\ &= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1)+(-1)^{1+0}\binom{1}{0}\mathsf{f}(0)\\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)\end{aligned}$$

2. Proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}&\int_{[0:1]^{n+1}}\mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n}x_k\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_{n+1}\\
&= \int_{[0:1]^n}\left\{\mathsf{f}^{(n)}\left(x_{n+1}+\sum_{k=1}^{n}x_k\right)\Bigg|_0^1\right\}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n\\
&= \int_{[0:1]^n}\mathsf{f}^{(n)}\left(1+\sum_{k=1}^{n}x_k\right)-\mathsf{f}^{(n)}\left(0+\sum_{k=1}^{n}x_k\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n\\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1)-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \text{by } n \text{ case hypothesis}\\
&= \sum_{k=1}^{n+1}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)+\sum_{k=0}^{n}(-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)\\
&= \left\{\mathsf{f}(n+1)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right\}+\left\{(-1)^{n+1}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right\}\\
&= \mathsf{f}(n+1)+(-1)^{n+1}\mathsf{f}(0)+\sum_{k=1}^{n}(-1)^{n-k+1}\left[\binom{n}{k-1}+\binom{n}{k}\right]\mathsf{f}(k)\end{aligned}$$

[84] Chui (1992) page 86 ⟨item (ii)⟩

$$= \mathsf{f}(n+1) + (-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) \qquad \text{by Pascal's Recursion Theorem 4.1 page 26}$$

$$= \sum_{k=0}^{n+1} (-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)$$

✏

Theorem 5.4. [85] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$.

1. $$\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_n(x)\,\mathrm{d}x = \int_{[0:1]^{n+1}} \mathsf{f}(x_0 + x_1 + \cdots + x_n)\,\mathrm{d}x_0\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$
2. $$\int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathrm{d}x = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k)$$

✎PROOF:

1. Proof for (1) (proof by induction):
   (a) Proof for $n = 0$ case:
   $$\int_{\mathbb{R}} \mathsf{N}_0(x)\mathsf{f}(x)\,\mathrm{d}x = \int_{[0:1]} \mathsf{f}(x)\,\mathrm{d}x$$
   (b) Proof that N_n case $\implies$ N_{n+1} case:
   $$\begin{aligned}
   &\int_{\mathbb{R}} \mathsf{N}_{n+1}(x)\mathsf{f}(x)\,\mathrm{d}x \\
   &= \int_{\mathbb{R}} \left(\int_{[0:1]} \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau\right)\mathsf{f}(x)\,\mathrm{d}x && \text{by Lemma 5.1 page 35} \\
   &= \int_{[0:1]}\int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\mathsf{f}(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
   &= \int_{[0:1]}\int_{\mathbb{R}} \mathsf{N}_n(u)\mathsf{f}(u+\tau)\,\mathrm{d}u\,\mathrm{d}\tau && \text{where } u \triangleq x-\tau \implies x = u+\tau \\
   &= \int_{[0:1]}\int_{[0:1]^n} \mathsf{f}(u_0+u_1+\cdots+u_n+\tau)\,\mathrm{d}u_0\,\mathrm{d}u_1\cdots\mathrm{d}u_n\,\mathrm{d}\tau && \text{by left hypothesis} \\
   &= \int_{[0:1]^{n+1}} \mathsf{f}(u_0+u_1+\cdots+u_n)\,\mathrm{d}u_0\,\mathrm{d}u_1\cdots\mathrm{d}u_n\,\mathrm{d}\tau \\
   &= \int_{[0:1]^{n+1}} \mathsf{f}(x_0+x_1+\cdots+x_n+x_{n+1})\,\mathrm{d}x_0\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,\mathrm{d}x_{n+1} && \text{by change of variables}
   \end{aligned}$$
2. Proof for (2):
   $$\begin{aligned}
   \int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathrm{d}x &= \int_{[0:1]^{n+1}} \mathsf{f}^{(n)}\left(\sum_{k=0}^{n} x_k\right)\mathrm{d}x_0\,\mathrm{d}x_1\cdots\mathrm{d}x_n && \text{by item 1} \\
   &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by Theorem 5.3 page 43}
   \end{aligned}$$

✏

[85] Chui (1992) page 85 ⟨(4.2.2), (4.2.3)⟩, Christensen (2008) page 140 ⟨Theorem 6.1.1⟩

Theorem 5.5. [86] *Let $\tilde{\mathbf{F}}$ be the* FOURIER TRANSFORM *operator* (Definition 1.7 page 7).

$$\tilde{\mathbf{F}}\mathsf{N}_n(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1} = \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\underbrace{\left(\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\right)}_{\operatorname{sinc}\left(\frac{\omega}{2}\right)}^{n+1}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{N}_n(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} \text{ Definition 1.7 page 7}\\
&= \frac{1}{\sqrt{2\pi}}\int_{[0:1]^{n+1}} e^{-i\omega(x_0+x_1+\cdots x_n)}\,\mathrm{d}x_0\,\mathrm{d}x_1\,\cdots,\,\mathrm{d}x_n && \text{by Theorem 5.4 page 44}\\
&= \frac{1}{\sqrt{2\pi}}\prod_{k=0}^{n}\left(\int_{[0:1]} e^{-i\omega x_k}\,\mathrm{d}x_k\right) && \text{by Theorem 5.4 page 44}\\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{[0:1]} e^{-i\omega x}\,\mathrm{d}x\right)^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-i\omega x}}{-i\omega}\bigg|_0^1\right)^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}}{i\omega}\right)\right]^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{2i\sin\left(\frac{\omega}{2}\right)}{\frac{2i\omega}{2}}\right)\right]^{n+1} && \text{by \textit{Euler formulas} (Corollary 1.3 page 5)}\\
&= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\right)^{n+1}
\end{aligned}$$

✏

5.3 Spline function spaces

Definition 5.3. [87] *Let $\mathsf{N}_n(x)$ be an* nTH ORDER CARDINAL B-SPLINE (Definition 5.2 page 35).

*The **space of all splines of order n** is denoted $\boldsymbol{S}^n(a\mathbb{Z})$ and is defined as*
$$\boldsymbol{S}^n(a\mathbb{Z}) \triangleq \operatorname{span}\left\{\mathbf{T}^m\mathsf{N}_n(ax)|_{m\in\mathbb{Z}}\right\}.$$

Theorem 5.6. [88] *Let $\boldsymbol{S}^n(\mathbb{Z})$ be the* SPACE OF ALL SPLINES OF ORDER N (Definition 5.3 page 45).

[86] Christensen (2008) page 142 ⟨Corollary 6.1.2⟩
[87] Wojtaszczyk (1997) page 52 ⟨Definition 3.5⟩
[88] Wojtaszczyk (1997) page 55 ⟨Theorem 3.11⟩

$$\underbrace{\left\{ \mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{T}^n \mathsf{N}_n(x-k) = \sum_{k\in\mathbb{Z}} \beta_k \mathbf{T}^n \mathsf{N}_n(x-k) \right\} \implies \left\{ (\!(\alpha_k)\!)_{k\in\mathbb{Z}} = (\!(\beta_k)\!)_{k\in\mathbb{Z}} \right\}}_{\textit{coefficients are } \textsc{unique}}$$

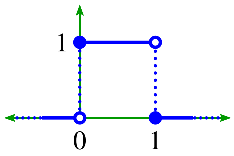
Lemma 5.4. [89] *Let $\boldsymbol{S}^n(\mathbb{Z})$ be the* SPACE OF ALL SPLINES OF ORDER N *(Definition 5.3 page 45).*

For each $n \in \mathbb{W}$,
$(\!(\mathbf{T}^n \mathsf{N}_n(x))\!)_{n\in\mathbb{Z}}$ is a RIESZ BASIS *in $\boldsymbol{L}^2_{\mathbb{R}}$.*

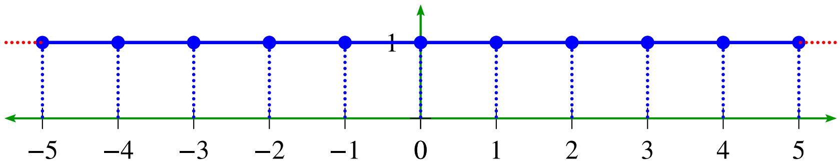
5.4 Examples

Example 5.1 (Square pulse).

The B-Spline $\mathsf{N}_0(x)$ is calculated in Lemma 5.2 page 36 and illustrated to the right.



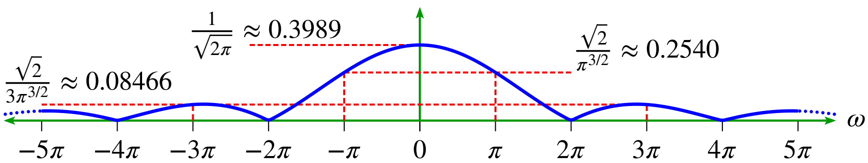
The B-spline $\mathsf{N}_0(x)$ forms a PARTITION OF UNITY *(Theorem 5.2 page 40), as illustrated to the right.*



Here is the Fourier transform $\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)$ of $\mathsf{N}_0(x)$:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{f}(x) &\triangleq \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i\omega x}\,\mathsf{d}x && \text{by definition of } \tilde{\mathbf{F}} \text{ page 7}\\
&= \frac{1}{-i\omega}\frac{1}{\sqrt{2\pi}} e^{-i\omega x}\Big|_0^1 \\
&= \frac{1}{-i\omega}\frac{1}{\sqrt{2\pi}} \left(e^{-i\omega\frac{1}{2}} - e^{i\omega\frac{1}{2}}\right) e^{-i\omega\frac{1}{2}} \\
&= \frac{1}{-i\omega}\frac{1}{\sqrt{2\pi}} \left[-2i\sin\left(\frac{\omega}{2}\right)\right] e^{-i\omega\frac{1}{2}} && \text{by Corollary 1.3 page 5}\\
&= \frac{2}{2}\frac{1}{\sqrt{2\pi}} \frac{\sin\left(\frac{\omega}{2}\right)}{\omega\frac{1}{2}} e^{-i\omega\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} e^{-i\omega\frac{1}{2}}
\end{aligned}$$

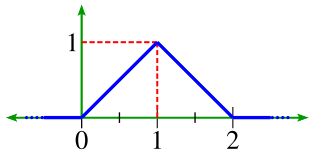
Note that $\tilde{\mathbf{F}}\mathsf{N}_0(0) = \frac{1}{\sqrt{2\pi}}$, which agrees with the result demonstrated in Theorem 5.5 page 45.



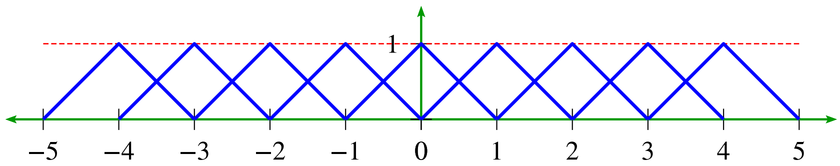
[89] Wojtaszczyk (1997) page 56 ⟨Proposition 3.12⟩

Example 5.2. [90]

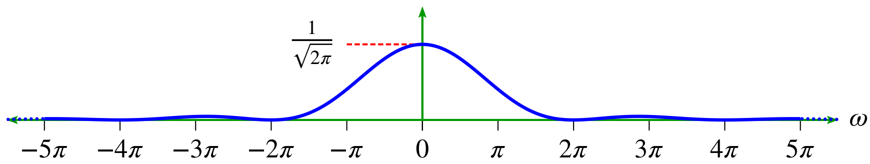
The B-Spline $\mathsf{N}_1(x)$ *is calculated in Lemma 5.2 page 36 and illustrated to the right.*



B-spline $\mathsf{N}_1(x)$ *forms a* PARTITION OF UNITY *(Theorem 5.2 page 40), as illustrated to the right.*

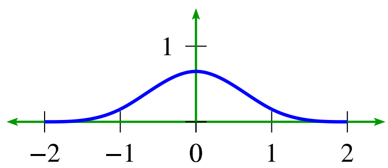


The Fourier transform $\left[\tilde{\mathbf{F}}\mathsf{N}_1\right](\omega)$ *of the function* $\mathsf{N}_1(x)$ *is illustrated to the right.*

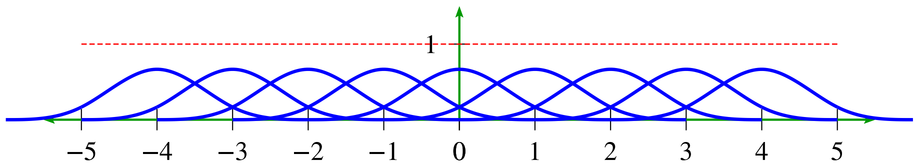


Example 5.3 (centered cubic B-spline). [91] *Let a function* f *be the* CENTERED CUBIC B-SPLINE *defined as follows:*

$$\mathsf{f}(x) \triangleq \begin{cases} \frac{2}{3} - \frac{1}{2}|x|^2(2-|x|) & \text{for } |x| < 1 \\ \frac{1}{6}(2-|x|)^3 & \text{for } 1 \le |x| < 2 \\ 0 & \text{otherwise} \end{cases}$$



Then f *forms a* PARTITION OF UNITY *because* $\sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) = 1$.



✎PROOF: Note that the function $\mathsf{h}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)$ is periodic with period 1 (Proposition 2.1 page 13). So it is only necessary to examine a single interval of length one. Here we use the interval $[0:1]$. In this interval, there are four functions contributing to the sum $\sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)$ (see previous illustration).

$$\begin{aligned}
\sum_{n=-1}^{n=2} \mathsf{f}(x-n) &= \underbrace{\frac{1}{6}(2-|x+1|)^3}_{\mathsf{f}(x+1)} + \underbrace{\frac{2}{3} - \frac{1}{2}|x|^2(2-|x|)}_{\mathsf{f}(x)} + \underbrace{\frac{2}{3} - \frac{1}{2}|x-1|^2(2-|x-1|)}_{\mathsf{f}(x-1)} + \underbrace{\frac{1}{6}(2-|x-2|)^3}_{\mathsf{f}(x-2)} \\
&= \underbrace{\frac{1}{6}(2-(x+1))^3}_{\mathsf{f}(x+1)} + \underbrace{\frac{2}{3} - \frac{1}{2}x^2(2-x)}_{\mathsf{f}(x)} + \underbrace{\frac{2}{3} - \frac{1}{2}(1-x)^2(2-(1-x))}_{\mathsf{f}(x-1)} + \underbrace{\frac{1}{6}(2-(2-x))^3}_{\mathsf{f}(x-2)} \\
&= \underbrace{\frac{1}{6}\left(-x^3+3x^2-3x+1\right)}_{\mathsf{f}(x+1)} + \underbrace{\frac{2}{3} - \frac{1}{2}\left(-x^3+2x^2\right)}_{\mathsf{f}(x)} + \underbrace{\frac{2}{3} - \frac{1}{2}\left(x^2-2x+1\right)(x+1)}_{\mathsf{f}(x-1)} + \underbrace{\frac{1}{6}x^3}_{\mathsf{f}(x-2)} \\
&= \underbrace{\frac{1}{6}\left(-x^3+3x^2-3x+1\right)}_{\mathsf{f}(x+1)} + \underbrace{\frac{2}{3} - \frac{1}{2}\left(-x^3+2x^2\right)}_{\mathsf{f}(x)} + \underbrace{\frac{2}{3} - \frac{1}{2}\left(x^3-x^2-x+1\right)}_{\mathsf{f}(x-1)} + \underbrace{\frac{1}{6}x^3}_{\mathsf{f}(x-2)} \\
&= x^3\left(-\frac{1}{6}+\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) + x^2\left(\frac{3}{6}-\frac{2}{2}+\frac{1}{2}\right) + x\left(-\frac{3}{6}+\frac{1}{2}\right) + \left(\frac{1}{6}+\frac{2}{3}+\frac{2}{3}-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

[90] Christensen (2008) pages 146–147 ⟨Corollary 6.2.1⟩

[91] Christensen (2008) page 146 ⟨Corollary 6.2.1⟩, Bankman (2008) page 479, de Boor (2001)

6 Partition of unity

6.1 Motivation

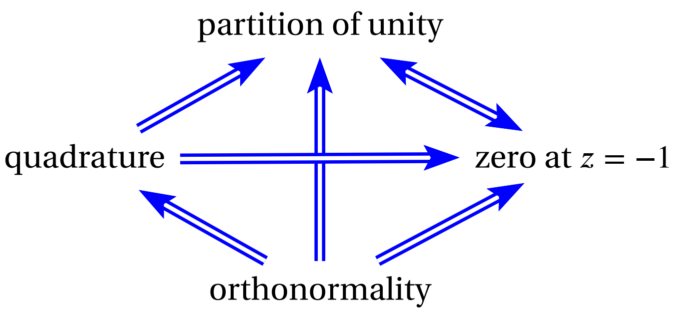


Figure 2: Implications of scaling function properties

A very common property of scaling functions (Definition 3.1 page 18) is the *partition of unity* property (Definition 6.2 page 49). The partition of unity is a kind of generalization of *orthonormality*; that is, *all* orthonormal scaling functions form a partition of unity. But the partition of unity property is not just a consequence of orthonormality, but also a generalization of orthonormality, in that if you remove the orthonormality constraint, the partition of unity is still a reasonable constraint in and of itself.

There are two reasons why the partition of unity property is a reasonable constraint on its own:

- Without a partition of unity, it is difficult to represent a function as simple as a constant.[92]
- For a multiresolution system $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$, the partition of unity property is equivalent to $\sum_{n\in\mathbb{Z}}(-1)^n h_n = 0$ (Theorem 6.2 page 51). As viewed from the perspective of discrete time signal processing, this implies that the scaling coefficients form a "*lowpass filter*"; lowpass filters provide a kind of "coarse approximation" of a function. And that is what the scaling function is "supposed" to do—to provide a coarse approximation at some resolution or "scale" (Definition 3.1 page 18).

6.2 Definition and results

Definition 6.1.

*The **Kronecker delta function** $\bar{\delta}_n$ is defined as*

$$\bar{\delta}_n \triangleq \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0. \end{cases} \quad \text{and} \qquad \forall n\in\mathbb{Z}$$

[92] Jawerth and Sweldens (1994) page 8

Definition 6.2. [93]

A function $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ *forms a* ***partition of unity*** *if*

$$\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) = 1 \qquad \forall x \in \mathbb{R}.$$

Theorem 6.1. [94] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be a multiresolution system* (Definition 3.3 page 20). *Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* FOURIER TRANSFORM (Definition 1.7 page 7) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION.

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f} = c}_{\text{PARTITION OF UNITY in "time"}} \qquad \Longleftrightarrow \qquad \underbrace{\left[\tilde{\mathbf{F}}\mathsf{f}\right](2\pi n) = \bar{\delta}_n}_{\text{PARTITION OF UNITY in "frequency"}}$$

✎PROOF: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
c &= \sum_{m\in\mathbb{Z}} \mathbf{T}^m \mathsf{f}(x) && \text{by left hypothesis} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m) && \text{by definition of } \mathbf{T} \text{ (Definition 2.1 page 12)} \\
&= \sqrt{2\pi} \sum_{m\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi m) e^{i2\pi m x} && \text{by } PSF \text{ (Theorem 2.2 page 16)} \\
&= \underbrace{\sqrt{2\pi}\,\tilde{\mathsf{f}}(2\pi n) e^{i2\pi n x}}_{\text{real and constant for } n=0} + \underbrace{\sqrt{2\pi} \sum_{m\in\mathbb{Z}\setminus n} \tilde{\mathsf{f}}(2\pi m) e^{i2\pi m x}}_{\text{complex and non-constant}} \\
\Longrightarrow\ & \sqrt{2\pi}\tilde{\mathsf{f}}(2\pi n) = c\bar{\delta}_n && \text{because } c \text{ is real and constant for all } t
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) && \text{by definition of } \mathbf{T} \text{ (Definition 2.1 page 12)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi n) e^{-i2\pi n x} && \text{by } PSF \text{ (Theorem 2.2 page 16)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \frac{c}{\sqrt{2\pi}} \bar{\delta}_n e^{-i2\pi n x} && \text{by right hypothesis} \\
&= \sqrt{2\pi} \frac{c}{\sqrt{2\pi}} e^{-i2\pi 0 x} && \text{by definition of } \bar{\delta}_n \text{ (Definition 6.1 page 48)} \\
&= c
\end{aligned}$$

✏

Corollary 6.1.

$$\left\{\begin{array}{l} \exists \mathsf{g} \in \boldsymbol{L}^2_{\mathbb{R}} \textit{ such that} \\ \mathsf{f}(x) = \mathbb{1}_{[-1:1)}(x) \star \mathsf{g}(x) \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} \mathsf{f}(x) \textit{ generates} \\ \textit{a} \text{ PARTITION OF UNITY} \end{array}\right\}$$

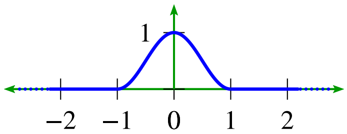
[93] Kelley (1955) page 171, Munkres (2000) page 225, Jänich (1984) page 116, Willard (1970) page 152 ⟨item 20C⟩, Willard (2004) page 152 ⟨item 20C⟩

[94] Jawerth and Sweldens (1994) page 8

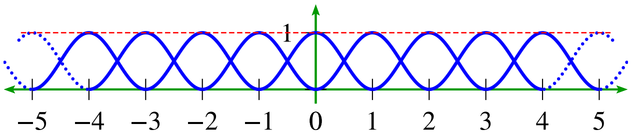
Example 6.1. *All* B-SPLINES *form a partition of unity. All B-splines of order* $n = 1$ *or greater can be generated by convolution with a* PULSE *function, similar to that specified in Corollary 6.1 (page 49).*

Example 6.2. *Let a function* f *be defined in terms of the cosine function* (Definition 1.4 page 3) *as follows:*

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

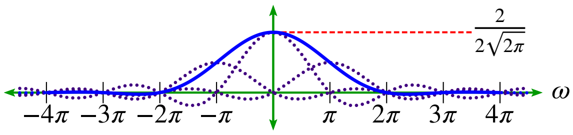


Then f *forms a* PARTITION OF UNITY:



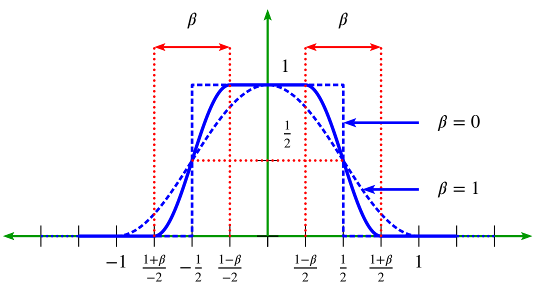
Note that $\tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)}\right]$

and so $\tilde{\mathsf{f}}(2\pi n) = \frac{1}{\sqrt{2\pi}}\bar{\delta}_n$:

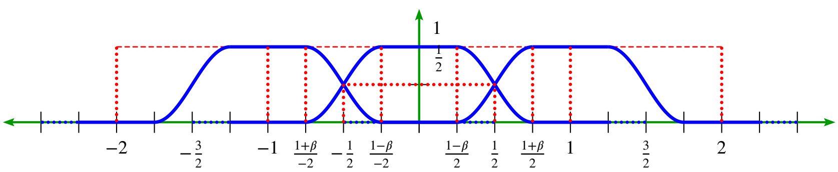


Example 6.3 (raised cosine). [95] *Let a function* f *be defined in terms of the cosine function* (Definition 1.4 page 3) *as follows: Let*

$$\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } 0 \le |x| < \frac{1-\beta}{2} \\ \frac{1}{2}\left\{1 + \cos\left[\frac{\pi}{\beta}\left(|x| - \frac{1-\beta}{2}\right)\right]\right\} & \text{for } \frac{1-\beta}{2} \le |x| < \frac{1+\beta}{2} \\ 0 & \text{otherwise} \end{cases}$$



Then f *forms a* PARTITION OF UNITY:



6.3 Scaling functions with partition of unity

The Z transform (Definition 1.12 page 10) of a sequence $(\!(h_n)\!)$ with sum $\sum_{n\in\mathbb{Z}}(-1)^n h_n = 0$ has a zero at $z = -1$. Somewhat surprisingly, the *partition of unity* and *zero at* $z = -1$ properties are actually equivalent (next theorem).

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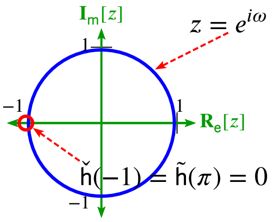
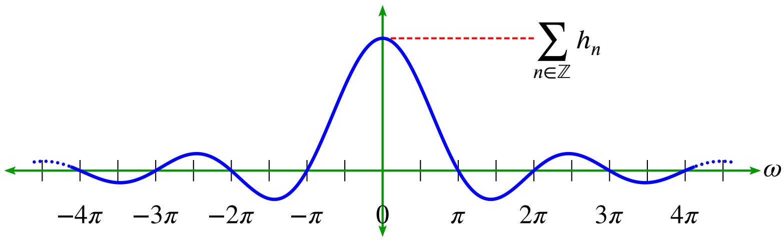
[95] Proakis (2001) pages 560–561





Theorem 6.2. [96] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be a multiresolution system* (Definition 3.3 page 20). *Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* FOURIER TRANSFORM (Definition 1.7 page 7) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION.

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi = c \quad \text{for some } c \in \mathbb{R}\backslash 0}_{\text{(1) PARTITION OF UNITY}} \iff \underbrace{\sum_{n\in\mathbb{Z}} (-1)^n h_n = 0}_{\text{(2) ZERO AT } z=-1} \iff \underbrace{\sum_{n\in\mathbb{Z}} h_{2n} = \sum_{n\in\mathbb{Z}} h_{2n+1} = \frac{\sqrt{2}}{2}}_{\text{(3) sum of even = sum of odd = } \frac{\sqrt{2}}{2}}$$

✎PROOF: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

1. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \left[\sum_{m\in\mathbb{Z}} h_m \mathbf{D}\mathbf{T}^m\phi\right] && \text{by dilation equ. (Theorem 3.1 page 19)} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathbf{D}\mathbf{T}^m\phi \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{n\in\mathbb{Z}} \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^m\phi && \text{by Proposition 2.5 page 14} \\
&= \mathbf{D}\sum_{m\in\mathbb{Z}} h_m \sum_{n\in\mathbb{Z}} \mathbf{T}^{2n}\mathbf{T}^m\phi \\
&= \mathbf{D}\sum_{m\in\mathbb{Z}} h_m \left[\sqrt{\frac{2\pi}{2}}\hat{\mathbf{F}}^{-1}\mathbf{S}_2\tilde{\mathbf{F}}(\mathbf{T}^m\phi)\right] && \text{by PSF (Theorem 2.2 page 16)} \\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} h_m \hat{\mathbf{F}}^{-1}\mathbf{S}_2 e^{-i\omega m}\tilde{\mathbf{F}}\phi && \text{by Corollary 2.1 page 16} \\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} h_m \hat{\mathbf{F}}^{-1} e^{-i\frac{2\pi}{2}km}\mathbf{S}_2\tilde{\mathbf{F}}\phi && \text{by definition of } \mathbf{S} \text{ (Theorem 2.2 page 16)} \\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} h_m \hat{\mathbf{F}}^{-1}(-1)^{km}\mathbf{S}_2\tilde{\mathbf{F}}\phi \\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} h_m \left[\frac{\sqrt{2}}{2}\sum_{k\in\mathbb{Z}}(-1)^{km}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\frac{2\pi}{2}kx}\right] && \text{by def. of } \hat{\mathbf{F}}^{-1} \text{ (Theorem 1.10 page 6)} \\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}h_m \\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}h_m \\
&\quad + \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}h_m
\end{aligned}$$

[96] Jawerth and Sweldens (1994) page 8, Chui (1992) page 123

$$= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}}h_m}_{\sqrt{2}}$$

$$+\frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}}(-1)^m h_m}_{0}$$

$$= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}$$ by Theorem 3.3 (page 20) and right hyp.

$$= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e}\tilde{\phi}\left(\frac{2\pi}{2}k\right)e^{i\pi kx}$$ by definitions of $\tilde{\mathbf{F}}$ and $\mathbf{S}_2$

$$= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}}\tilde{\phi}(2\pi k)e^{i2\pi kx}$$ by definition of $\mathbb{Z}_e$

$$= \frac{1}{\sqrt{2}}\mathbf{D}\left\{\sqrt{2\pi}\sum_{k\in\mathbb{Z}}\tilde{\phi}(2\pi k)e^{i2\pi kx}\right\}$$

$$= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n\in\mathbb{Z}}\phi(x+n)$$ by PSF (Theorem 2.2 page 16)

$$= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n}\mathbf{T}^n\phi$$ by definition of $\mathbf{T}$ (Definition 2.1 page 12)

The above equation sequence demonstrates that

$$\mathbf{D}\sum_n\mathbf{T}^n\phi = \sqrt{2}\sum_n\mathbf{T}^n\phi$$

(essentially that $\sum_n\mathbf{T}^n\phi$ is equal to it's own dilation). This implies that $\sum_n\mathbf{T}^n\phi$ is a constant (Proposition 2.8 page 14).

2. Proof that (1) $\Longrightarrow$ (2):

$$c = \sum_{n\in\mathbb{Z}}\mathbf{T}^n\phi$$ by left hypothesis

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\phi$$ by PSF (Theorem 2.2 page 16)

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\underbrace{\sqrt{2}\left(\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}}h_n e^{-i\omega n}\right)(\mathbf{D}^{-1}\tilde{\mathbf{F}}\phi)}_{\tilde{\mathbf{F}}\phi}$$ by Lemma 3.1 page 19

$$= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}}h_n e^{-i\omega n}\right)(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi)$$ by Corollary 2.1 page 16

$$= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}h_n e^{-i\frac{\omega}{2}n}\right)(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi)$$ by Proposition 2.2 page 13

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}h_n e^{-i\frac{2\pi k}{2}n}\right)(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi)$$ by def. of $\mathbf{S}$ (Theorem 2.2 page 16)

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}h_n(-1)^{kn}\right)(\mathbf{S}\mathbf{D}^{-1}\mathbf{F}\phi)$$

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}h_n(-1)^{kn}\right)\left(\mathbf{S}\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{\omega}{2}\right)\right)$$ by def. of $\mathbf{S}$ (Theorem 2.2 page 16)

$$= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}h_n(-1)^{kn}\right)\left(\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{2\pi k}{2}\right)\right)$$

$$
\begin{aligned}
&= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} h_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} && \text{by Theorem 1.10 page 6}\\
&= \sqrt{\pi}\sum_{k \text{ even}}\sum_{n\in\mathbb{Z}} h_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx}\\
&\quad + \sqrt{\pi}\sum_{k \text{ odd}}\sum_{n\in\mathbb{Z}} h_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx}\\
&= \sqrt{\pi}\sum_{k \text{ even}}\left(\cancelto{\sqrt{2}}{\sum_{n\in\mathbb{Z}} h_n}\right)\tilde{\phi}(\pi k)e^{i2\pi kx}\\
&\quad + \sqrt{\pi}\sum_{k \text{ odd}}\left(\sum_{n\in\mathbb{Z}} h_n(-1)^n\right)\tilde{\phi}(\pi k)e^{i2\pi kx}\\
&= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sqrt{2}\,\tilde{\phi}(\pi 2k)e^{i2\pi 2kx}\\
&\quad + \sqrt{\pi}\sum_{k\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} h_n(-1)^n\right)\tilde{\phi}(\pi[2k+1])e^{i2\pi[2k+1]x} && \text{by Theorem 3.3 page 20}\\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\tilde{\phi}(0) + \sqrt{\pi}e^{i2\pi x}\sum_{n\in\mathbb{Z}} h_n(-1)^n \sum_{k\in\mathbb{Z}}\tilde{\phi}(\pi[2k+1])e^{i4\pi kx} && \text{by left hyp. and Theorem 6.1 page 49}\\
\implies &\qquad \left(\sum_{n\in\mathbb{Z}} h_n(-1)^n\right) = 0 && \text{because the right side must equal } c
\end{aligned}
$$

3. Proof that (2) $\implies$ (3):

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}_e} h_n = \sum_{n\in\mathbb{Z}_o} h_n &= \frac{1}{2}\sum_{n\in\mathbb{Z}} h_n && \text{by (2) and Proposition 1.5 page 11}\\
&= \frac{\sqrt{2}}{2} && \text{by \textit{admissibility condition} (Theorem 3.3 page 20)}
\end{aligned}
$$

4. Proof that (2) $\impliedby$ (3):

$$
\begin{aligned}
\frac{\sqrt{2}}{2} &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n h_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n h_n}_{\text{odd terms}} && \text{by (3)}\\
\implies & \sum_{n\in\mathbb{Z}}(-1)^n h_n = 0 && \text{by Proposition 1.5 page 11}
\end{aligned}
$$

✎

Proposition 6.1.

$\boxed{\phi(x) \textit{ generates a } \text{PARTITION OF UNITY} \quad \not\Longrightarrow \quad \phi(x) \textit{ generates an } \text{MRA} \textit{ system.}}$

6.4 Spline wavelet systems

Theorem 6.3. [97] *Let $S^n(\mathbb{Z})$ be the* SPACE OF ALL SPLINES OF ORDER N *(Definition 5.3 page 45).*

$\boxed{\begin{array}{l}\textit{For each } n \in \mathbb{W},\\ S^n(2^k\mathbb{Z}) \textit{ is a } \text{MULTIRESOLUTION ANALYSIS } \textit{(an MRA).}\end{array}}$

[97] Wojtaszczyk (1997) page 57 ⟨Theorem 3.13⟩

Theorem 6.4 (B-spline wavelet coefficients). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM (Definition 3.3 page 20). *Let* $\mathsf{N}_n(x)$ *be a* nTH ORDER B-SPLINE.

$$\underbrace{\phi(x) \triangleq \mathsf{N}_n(x)}_{\text{(1) B-spline scaling function}} \implies (\!(h_k)\!) = \begin{cases} \dfrac{\sqrt{2}}{2^{n+1}}\dbinom{n}{k} & \text{for } k = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \quad \text{(2) scaling sequence in "time"}$$

$$\iff \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} = \frac{\sqrt{2}}{2^n}\left(1 + z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} \quad \text{(3) scaling sequence in "z domain"}$$

$$\iff \tilde{\mathsf{h}}(\omega) = 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} \quad \text{(4) scaling sequence in "frequency"}$$

✎PROOF:

1. Proof that (1) $\implies$ (3): By Theorem 6.3 page 53 we know that $\mathsf{N}_n(x)$ is a *scaling function* (Definition 3.1 page 18). So then we know that we can use Lemma 3.1 page 19.

$$\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \sqrt{2}\frac{\tilde{\phi}(2\omega)}{\tilde{\phi}(\omega)} && \text{by Lemma 3.1 page 19} \\
&= \sqrt{2}\frac{\tilde{\mathsf{N}}_n(2\omega)}{\tilde{\mathsf{N}}_n(\omega)} && \text{by (1)} \\
&= \sqrt{2}\frac{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i2\omega}}{2i\omega}\right)^{n+1}}{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1}} && \text{by Theorem 5.5 page 45} \\
&= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{1-z^{-2}}{1-z^{-1}}\right)^{n+1}\Bigg|_{z=e^{i\omega}} \\
&= \frac{\sqrt{2}}{2^{n+1}}\left[\left(\frac{1-z^{-2}}{1-z^{-1}}\right)\left(\frac{1+z^{-1}}{1+z^{-1}}\right)\right]^{n+1}\Bigg|_{z=e^{i\omega}} \\
&= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{\left(1-z^{-2}\right)\left(1+z^{-1}\right)}{1-z^{-2}}\right)^{n+1}\Bigg|_{z=e^{i\omega}} \\
&= \frac{\sqrt{2}}{2^n}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}}
\end{aligned}$$

2. Proof that (3) $\iff$ (2):

$$\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \frac{\sqrt{2}}{2^n}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} && \text{by (3)} \\
&= \frac{\sqrt{2}}{2^n}\left(\sum_{k=0}^{n+1}\binom{n}{k}z^{-k}\right)\Bigg|_{z\triangleq e^{i\omega}} && \text{by \textit{binomial theorem}} \\
\iff \quad h_k &= \frac{\sqrt{2}}{2^{n+1}}\binom{n}{k} && \text{by definition of \textit{Z transform} (Definition 1.12 page 10)}
\end{aligned}$$

3. Proof that (3) $\implies$ (4):

$$\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} && \text{by definition of } \textit{DTFT} \text{ (Definition 1.13 page 10)}\\
&= \frac{\sqrt{2}}{2^n}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} && \text{by (3)}\\
&= \frac{\sqrt{2}}{2^n}\left(1+e^{-i\omega}\right)^{n+1} && \text{by definition of } z\\
&= \frac{\sqrt{2}}{2^n}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^n}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1}\\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1}
\end{aligned}$$

4. Proof that (3) $\Longleftarrow$ (4):

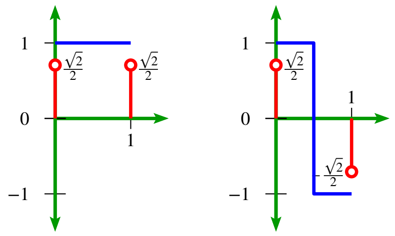
$$\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \check{\mathsf{h}}\left(e^{i\omega}\right)\\
&= \tilde{\mathsf{h}}(\omega)\\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} && \text{by (4)}\\
&= \frac{\sqrt{2}}{2^n}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^n}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^n}\left(1+e^{-i\omega}\right)^{n+1}\\
&= \frac{\sqrt{2}}{2^n}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}}
\end{aligned}$$

6.5 Examples

Example 6.4 (2 coefficient case/Haar wavelet system/order 0 B-spline wavelet system). [98]
Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be an* ORTHOGONAL *wavelet system with two non-zero scaling coefficients.*

$$\left\{\begin{array}{lll}
1.\ \operatorname{supp}\phi(x) = [0:1] & \text{(Theorem 3.9 page 24)} & \text{and}\\
2.\ \text{ADMISSIBILITY CONDITION} & \text{(Theorem 3.3 page 20)} & \text{and}\\
3.\ \text{PARTITION OF UNITY} & \text{(Theorem 6.2 page 51)} & \text{and}\\
4.\ g_n = \pm(-1)^n h^*_{N-n}\ \forall n\in\mathbb{Z} & \text{(Theorem 3.8 page 23)} &
\end{array}\right\}
\implies
\left\{\begin{array}{c|c|c}
n & h_n & g_n\\
\hline
0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\
1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\
\text{other} & 0 & 0
\end{array}\right\}$$

[98] Haar (1910), Wojtaszczyk (1997) pages 14–15 ⟨"Sources and comments"⟩



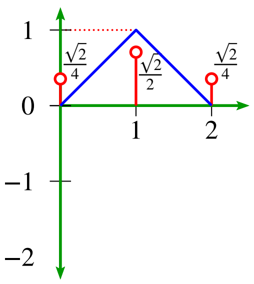
✎PROOF:

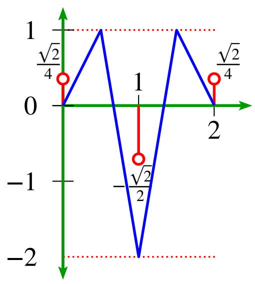
1. Proof that (1) $\implies$ that only h_0 and h_1 are non-zero: by Theorem 3.9 page 24.
2. Proof for values of h_0 and h_1:
   (a) Method 1: Under the constraint of two non-zero scaling coefficients, a scaling function design is fully constrained using the *admissibility equation* (Theorem 3.3 page 20) and the *partition of unity* constraint (Definition 6.2 page 49). The partition of unity formed by $\phi(x)$ is illustrated in Example 5.1 page 46.
   Here are the equations:
   $$\begin{array}{rcllll} h_0 + h_1 &=& \sqrt{2} & \text{(admissibility equation} & \text{Theorem 3.3} & \text{page 20)} \\ h_0 - h_1 &=& 0 & \text{(partition of unity/zero at } -1 & \text{Theorem 6.2} & \text{page 51)} \end{array}$$
   Here are the calculations for the coefficients:
   $$\begin{array}{rcll} (h_0 + h_1) + (h_0 - h_1) = 2h_0 &=& \sqrt{2} & \text{(add two equations together)} \\ (h_0 + h_1) - (h_0 - h_1) = 2h_1 &=& \sqrt{2} & \text{(subtract second from first)} \end{array}$$
   $$g_0 = h_1$$
   $$g_1 = -h_0$$
   (b) Method 2: By Theorem 6.4 page 54.
3. Note: h_0 and h_1 can also be produced using other systems of equations including the following:
   (a) Admissibility condition and *orthonormality*
   (b) *Daubechies-p*1 wavelets computed using spectral techniques
4. Proof for values of g_0 and g_1: by (4) and Theorem 3.8 page 23.

✏

Example 6.5 (order 1 B-spline wavelet system). [99] *The following figures illustrate scaling and wavelet coefficients and functions for the* B-SPLINE B_2, *or* TENT FUNCTION. *The partition of unity formed by the scaling function* $\phi(x)$ *is illustrated in Example* 5.2 *page* 47.

n	h_n	g_n
0	$\left(\frac{\sqrt{2}}{4}\right)$	$\left(\frac{\sqrt{2}}{4}\right)$
1	$2\left(\frac{\sqrt{2}}{4}\right)$	$-2\left(\frac{\sqrt{2}}{4}\right)$
2	$\left(\frac{\sqrt{2}}{4}\right)$	$\left(\frac{\sqrt{2}}{4}\right)$





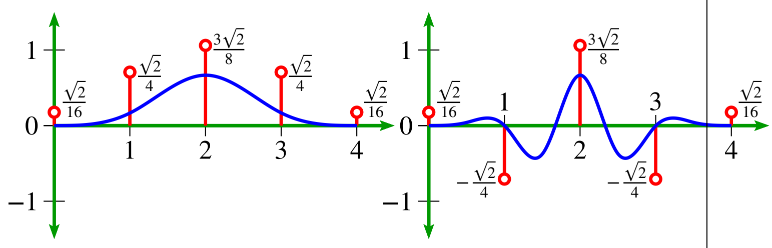
[99] Strang (1989) page 616, Daubechies (1992) pages 146–148 ⟨§5.4⟩

✎PROOF: These results follow from Theorem 6.4 page 54.

$$\left(\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}\right)$$
✏

Example 6.6 (order 3 B-spline wavelet system). [100] *The following figures illustrate scaling and wavelet coefficients and functions for a* B-SPLINE.

n	h_n	g_n
0	$\left(\frac{\sqrt{2}}{16}\right)$	$\left(\frac{\sqrt{2}}{16}\right)$
1	$4\left(\frac{\sqrt{2}}{16}\right)$	$-4\left(\frac{\sqrt{2}}{16}\right)$
2	$6\left(\frac{\sqrt{2}}{16}\right)$	$6\left(\frac{\sqrt{2}}{16}\right)$
3	$4\left(\frac{\sqrt{2}}{16}\right)$	$-4\left(\frac{\sqrt{2}}{16}\right)$
4	$\left(\frac{\sqrt{2}}{16}\right)$	$\left(\frac{\sqrt{2}}{16}\right)$



✎PROOF: These results follow from Theorem 6.4 page 54.

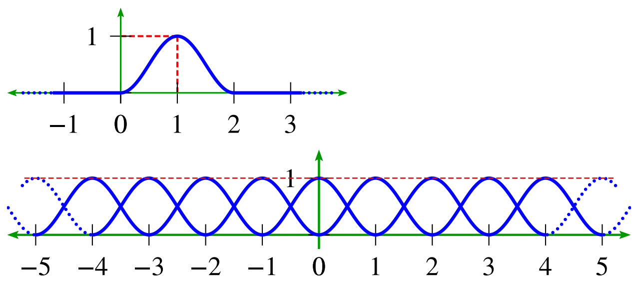
$$\left(\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}\right)$$
✏

Not all functions that form a *partition of unity* are a bases for an *MRA*. Counterexample 6.1 (next) and Counterexample 6.2 (page 60) provide two counterexamples.

Counterexample 6.1. *Let a function* f *be defined in terms of the sine function* (Definition 1.4 page 3) *as follows:*

$$\phi(x) \triangleq \begin{cases} \sin^2\left(\frac{\pi}{2}x\right) & \text{for } x \in [0:2] \\ 0 & \text{otherwise} \end{cases}$$

Then $\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x = 1$ *and* ϕ *forms a* PARTITION OF UNITY, ***but*** $\left\{\mathbf{T}^n\phi \mid n\in\mathbb{Z}\right\}$ *does **not** generate an* MRA.



✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.3 page 3) on a set A.

1. Proof that $\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x = 1$:

$$\begin{aligned} \int_{\mathbb{R}} \phi(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x)\,\mathrm{d}x && \text{by definition of } \phi(x) \\ &= \int_0^2 \sin^2\left(\frac{\pi}{2}x\right)\mathrm{d}x && \text{by definition of } \mathbb{1}_{A(x)} \text{ (Definition 1.3 page 3)} \end{aligned}$$

[100] Strang (1989) page 616

$$\begin{aligned}
&= \int_0^2 \frac{1}{2}[1-\cos(\pi x)]\,\mathrm{d}x && \text{by Theorem 1.9 page 5}\\
&= \frac{1}{2}\left[x - \frac{1}{\pi}\sin(\pi x)\right]_0^2\\
&= \frac{1}{2}[2-0-0-0]\\
&= 1
\end{aligned}$$

2. Proof that $\phi(x)$ forms a *partition of unity*:

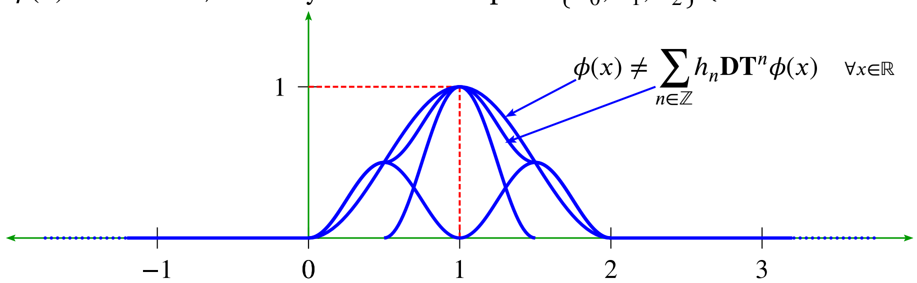
$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x) && \text{by definition of } \phi(x)\\
&= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{because } \sin^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 2\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^{m-1}\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{where } m \triangleq n+1 \implies n = m-1\\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m+1)\right)\mathbb{1}_{[0:2)}(x-m+1) && \text{by definition of } \mathbf{T} \text{ (Definition 2.1 page 12)}\\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m)+\frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-m)\\
&= \sum_{m\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-m)\right)\mathbb{1}_{[-1:1)}(x-m) && \text{by Theorem 1.9 page 5}\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x) && \text{by definition of } \mathbf{T} \text{ (Definition 2.1 page 12)}\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x) && \text{because } \cos^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 1\\
&= 1 && \text{by Example 6.2 page 50}
\end{aligned}$$

3. Proof that $\phi(x) \notin \operatorname{span}\{\mathbf{DT}^n\phi(x)|_{n\in\mathbb{Z}}\}$ (and so does not generate an *MRA*):

   (a) Note that the *support* (Definition 3.7 page 23) of ϕ is $\operatorname{supp}\phi = [0:2]$.

   (b) Therefore, the *support* of $(\!(h_n)\!)$ is $\operatorname{supp}(\!(h_n)\!) = \{0,1,2\}$ (Theorem 3.9 page 24).

   (c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\{h_0, h_1, h_2\}$ (the rest would be 0).



Here would be the values of $\{h_1, h_2, h_3\}$:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} h_n\mathbf{DT}^n\phi(x)\\
&= \sum_{n\in\mathbb{Z}} h_n\mathbf{DT}^n\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x)\\
&= \sum_{n\in\mathbb{Z}} h_n\sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n)\\
&= \sum_{n=0}^{2} h_n\sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 3.9}
\end{aligned}$$

(d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$ (see the diagram in item (3c) page 58):

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot\frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} h_n\sin^2\left(\frac{\pi}{2}(1-n)\right)\mathbb{1}_{[0:2]}(1-n) \\
&= h_0\sin^2\left(\frac{\pi}{2}(1-0)\right)\mathbb{1}_{[0:2]}(1-0) + h_1\sin^2\left(\frac{\pi}{2}(1-1)\right)\mathbb{1}_{[0:2]}(1-1) \\
&\qquad + h_2\sin^2\left(\frac{\pi}{2}(1-2)\right)\mathbb{1}_{[0:2]}(1-2) \\
&= h_0\cdot 1\cdot 1 + h_1\cdot 0\cdot 1 + h_2(-1)\cdot 0 \\
&= h_0
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot 1 &= \frac{\sqrt{2}}{2}(1)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{2}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi(1) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} h_n\sin^2\left(\frac{\pi}{2}(2-n)\right)\mathbb{1}_{[0:2]}(2-n) \\
&= h_0\sin^2\left(\frac{\pi}{2}(2-0)\right)\mathbb{1}_{[0:2]}(2-0) + h_1\sin^2\left(\frac{\pi}{2}(2-1)\right)\mathbb{1}_{[0:2]}(2-1) \\
&\qquad + h_2\sin^2\left(\frac{\pi}{2}(2-2)\right)\mathbb{1}_{[0:2]}(2-2) \\
&= h_0\cdot 0\cdot 1 + h_1\cdot 1\cdot 1 + h_2\cdot 0\cdot 1 \\
&= h_1
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot\frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{-\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{3\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{3}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} h_n\sin^2\left(\frac{\pi}{2}(3-n)\right)\mathbb{1}_{[0:2]}(3-n) \\
&= h_0\sin^2\left(\frac{\pi}{2}(3-0)\right)\mathbb{1}_{[0:2]}(3-0) + h_1\sin^2\left(\frac{\pi}{2}(3-1)\right)\mathbb{1}_{[0:2]}(3-1) \\
&\qquad + h_2\sin^2\left(\frac{\pi}{2}(3-2)\right)\mathbb{1}_{[0:2]}(3-2) \\
&= h_0\cdot(-1)\cdot 0 + h_1\cdot 0\cdot 1 + h_2 1\cdot 1 \\
&= h_2
\end{aligned}$$

(e) These values for $(\!(h_0, h_1, h_2)\!)$ are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 3.1 page 19). In particular,

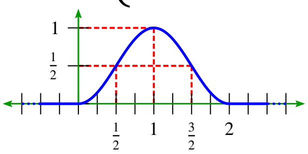
$$\phi(x) \neq \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n\phi(x)$$

(see the diagram in item (3c) page 58)

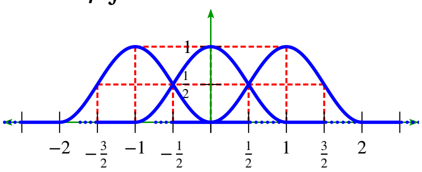
✏

Counterexample 6.2 (raised sine). [101] *Let a function* f *be defined in terms of a shifted cosine function* (Definition 1.4 page 3) *as follows:*

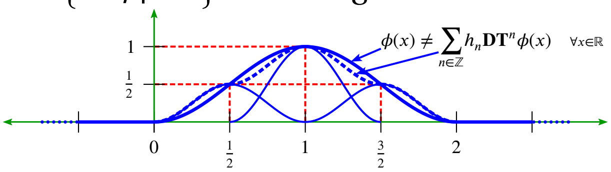
$$\phi(x) \triangleq \begin{cases} \frac{1}{2}\left\{1+\cos[\pi(|x-1|)]\right\} & \text{for } 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$



Then ϕ *forms a* PARTITION OF UNITY*:*



but $\left\{\mathbf{T}^n\phi \,\middle|\, n\in\mathbb{Z}\right\}$ *does **not** generate an* MRA.



✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.3 page 3) on a set A.

1. Proof that $\phi(x)$ forms a *partition of unity*:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x+1) && \text{by Proposition 2.1 page 13} \\
&= \sum_{n\in\mathbb{Z}} \phi(x+1-n) && \text{by Definition 2.1 page 12} \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2}\{1+\cos[\pi(|x-1+1-n|)]\}\mathbb{1}_{[0:2)}(x+1-n) && \text{by definition of } \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2}\{1+\cos[\pi(|x-n|)]\}\mathbb{1}_{[-1:1)}(x-n) && \text{by Definition 1.3 page 3} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{\frac{1}{2}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n|-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_{[-1:1)}(x-n)\bigg|_{\beta=1}}_{\textit{raised cosine} \text{ (Example 6.3 page 50) with } \beta=1} \\
&= 1 && \text{by Example 6.3 page 50}
\end{aligned}$$

2. Proof that $\phi(x) \notin \operatorname{span}\left\{\mathbf{DT}^n\phi(x) \,\middle|\, n\in\mathbb{Z}\right\}$ (and so does not generate an *MRA*):
   (a) Note that the *support* (Definition 3.7 page 23) of ϕ is $\operatorname{supp}\phi = [0:2]$.
   (b) Therefore, the *support* of $(\!(h_n)\!)$ is $\operatorname{supp}(\!(h_n)\!) = \{0,1,2\}$ (Theorem 3.9 page 24).

[101] Proakis (2001) pages 560–561

(c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\{h_0, h_1, h_2\}$ (the rest would be 0). Here would be the values of $\{h_1, h_2, h_3\}$:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n \ \frac{1}{2}\left\{\ 1+\cos[\pi(|x-1|)]\right\} \mathbb{1}_{[0:2]}(x) && \text{by definition of } \phi(x) \\
&= \sum_{n\in\mathbb{Z}} h_n \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[\pi(|2x-1-n|)]\right\} \mathbb{1}_{[0:2]}(2x-n) && \text{by Definition 2.1 page 12} \\
&= \sum_{n=0}^{2} h_n \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[\pi(|2x-1-n|)]\right\} \mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 3.9}
\end{aligned}$$

(d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$ (see the diagram in item (3c) page 58):

$$\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} h_n \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[\pi(|2x-1-n|)]\right\} \mathbb{1}_{[0:2]}(2x-n)\Big|_{x=\frac{1}{2}} \\
&= h_0 \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[1-1-0]\right\} \\
&= h_0\sqrt{2} \\
&\implies h_0 = \frac{\sqrt{2}}{4}
\end{aligned}$$

$$\begin{aligned}
1 &= \sum_{n=0}^{2} h_n \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[\pi(|2x-1-n|)]\right\} \mathbb{1}_{[0:2]}(2x-n)\Big|_{x=1} \\
&= h_1 \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[2-1-1]\right\} \\
&= h_1\sqrt{2} \\
&\implies h_1 = \frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} h_n \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[\pi(|2x-1-n|)]\right\} \mathbb{1}_{[0:2]}(2x-n)\Big|_{x=\frac{3}{2}} \\
&= h_2 \ \frac{\sqrt{2}}{2}\left\{\ 1+\cos[1-1-0]\right\} \\
&= h_2\sqrt{2} \\
&\implies h_2 = \frac{\sqrt{2}}{4}
\end{aligned}$$

(e) These values for $(\!(h_0, h_1, h_2)\!)$ are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 3.1 page 19). In particular (see diagram),

$$\phi(x) \neq \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n \phi(x)\,.$$

⇛

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Shì Jié Zhū. *Jade Mirror of the Four Unknowns (Chinese: 四元玉鑑, pinyin: Sì Yúan Yù Jiàn)*. 1303. ISBN 9787538269239. URL http://www.amazon.com/dp/7538269231/. author: 朱世傑 (pinyin: Zhū Shì Jié); Many many thanks to Po-Ning Chen (Chinese: 陳伯寧, pinyin: Chén Bó Nińg) for his consultation with regards to the translation of the book title: "Originally, '鑑' is a basin or container made by metals. It can then be used as a mirror after pouring water into it, and hence is extended to indicate a book that reflects faithfully about a subject. '玉' is a precious stone and hence can be translated as Jade. The combination of the two characters is therefore translated as 'The Jade Mirror'."—Po-Ning Chen (陳伯寧), 2012 September 12).