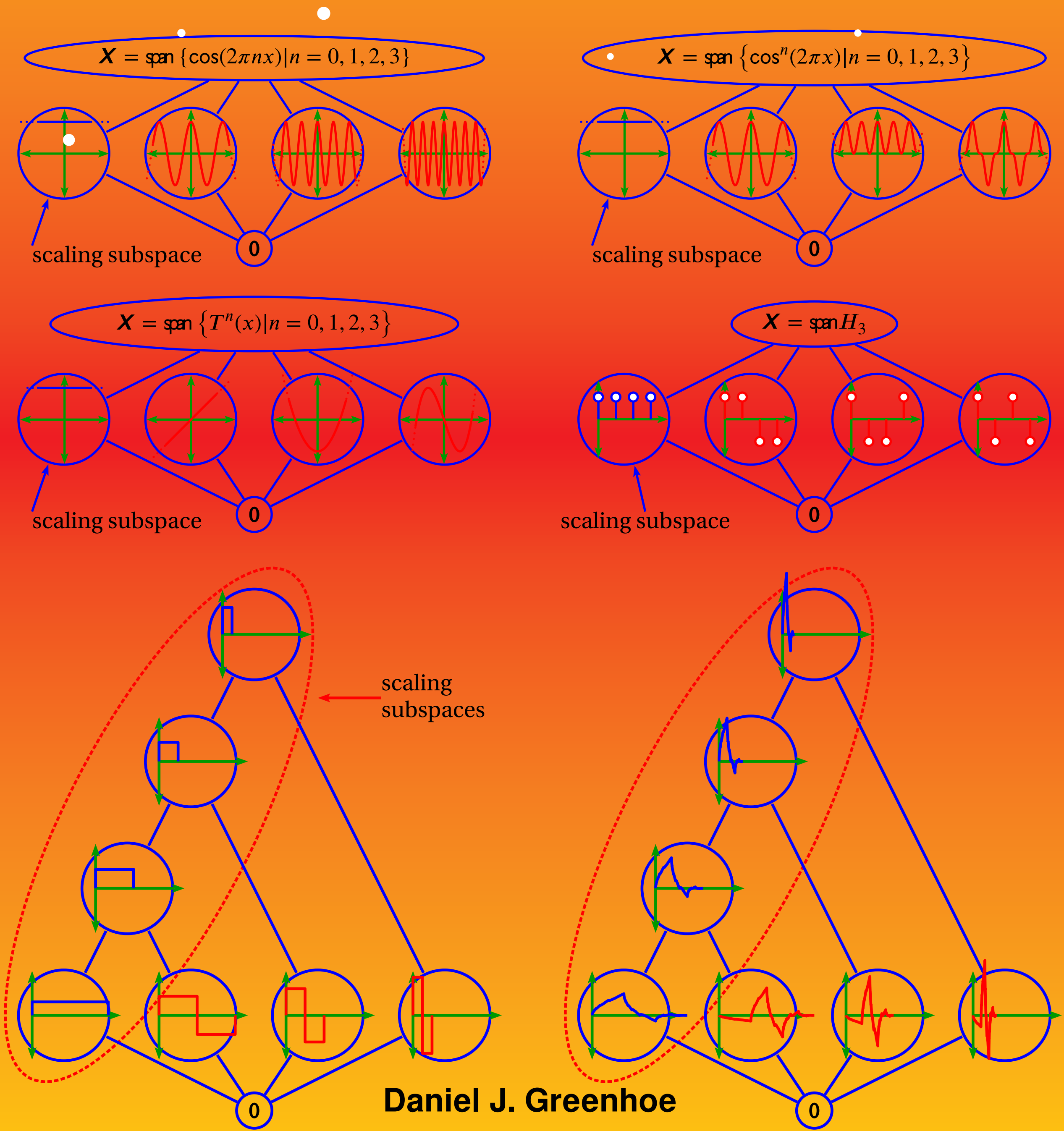
A Book Concerning Transforms

VERSION 0.10



Daniel J. Greenhoe

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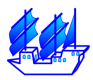
The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]



[1] Paine (2000) page 63 ⟨Golden Hind⟩

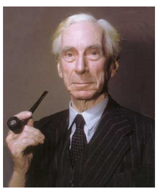
❝Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?❞

❝Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.❞

Alfred Edward Housman, English poet (1859–1936) [2]

❝The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.❞

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



❝As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.❞

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

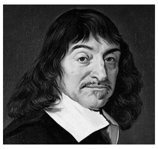
[2] quote: Housman (1936) page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961) page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967) page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.❞



❝Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.❞

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

	symbol	description	
numbers:			
	$\mathbb{Z}$	integers	$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
	$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \ldots$

...continued on next page...

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$
$\mathbb{Z}^{-}$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_{o}$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_{e}$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^{+}$	positive real numbers	$(0, \infty)$
$\mathbb{R}^{-}$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^{*}$	extended real numbers	$\mathbb{R}^{*} \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
Ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \triangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}(®)$	domain of a relation ®	
$\mathcal{I}(®)$	image of a relation ®	
$\mathcal{R}(®)$	range of a relation ®	
$\mathcal{N}(®)$	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\not\subseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^{c}	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_{A}(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	

...continued on next page...

symbol	description	
0	"false" condition	
$\neg$	logical NOT operation	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\le$	reflexive ordering relation	"less than or equal to"
$\ge$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_{\mathsf{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_{\mathsf{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_{\mathsf{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		

…continued on next page…

symbol	description
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition 4.2 page 16)
$\hat{\mathbf{F}}$	*Fourier Series* operator (Definition 7.1 page 51)
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition 6.1 page 41)
$\mathbf{Z}$	*Z-Transform* operator (Definition 5.4 page 28)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\check{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

CHAPTER 1

ANALYSES AND TRANSFORMS

❝*The analytical equations, unknown to the ancient geometers, which Descartes was the first to introduce into the study of curves and surfaces, . . . they extend to all general phenomena. There cannot be a language more universal and more simple, more free from errors and from obscurities, . . . mathematical analysis is as extensive as nature itself; it defines all perceptible relations, measures times, spaces, forces, temperatures ; this difficult science is formed slowly, but it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them.*❞

Joseph Fourier (1768–1830) [1]

1.1 Abstract spaces

The **abstract space** was introduced by Maurice Fréchet in his 1906 Ph.D. thesis.[2] An *abstract space* in mathematics does not really have a rigorous definition; but in general it is a set together with some other unifying structure. Examples of spaces include *topological space*s, *metric space*s, and *linear space*s (*vector space*s).

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[1] quote: Fourier (1878) pages 7–8 ⟨Preliminary Discourse⟩
image: `http://en.wikipedia.org/wiki/File:Fourier2.jpg`, public domain

[2] Fréchet (1906), Fréchet (1928). *"A collection of these abstract elements will be called an abstract set. If to this set there is added some rule of association of these elements, or some relation between them, the set will be called an abstract space."*—Maurice Fréchet

1.2 Lattice of subspaces

An abstract space can be decomposed into one or more *subspace*s. Roughly speaking, a subspace of an abstract space is simply a subset the abstract space that has the same properties of that abstract space. The subspaces can be ordered under the ordering relation $\subseteq$ (subset or equal to relation) to form a *lattice*.

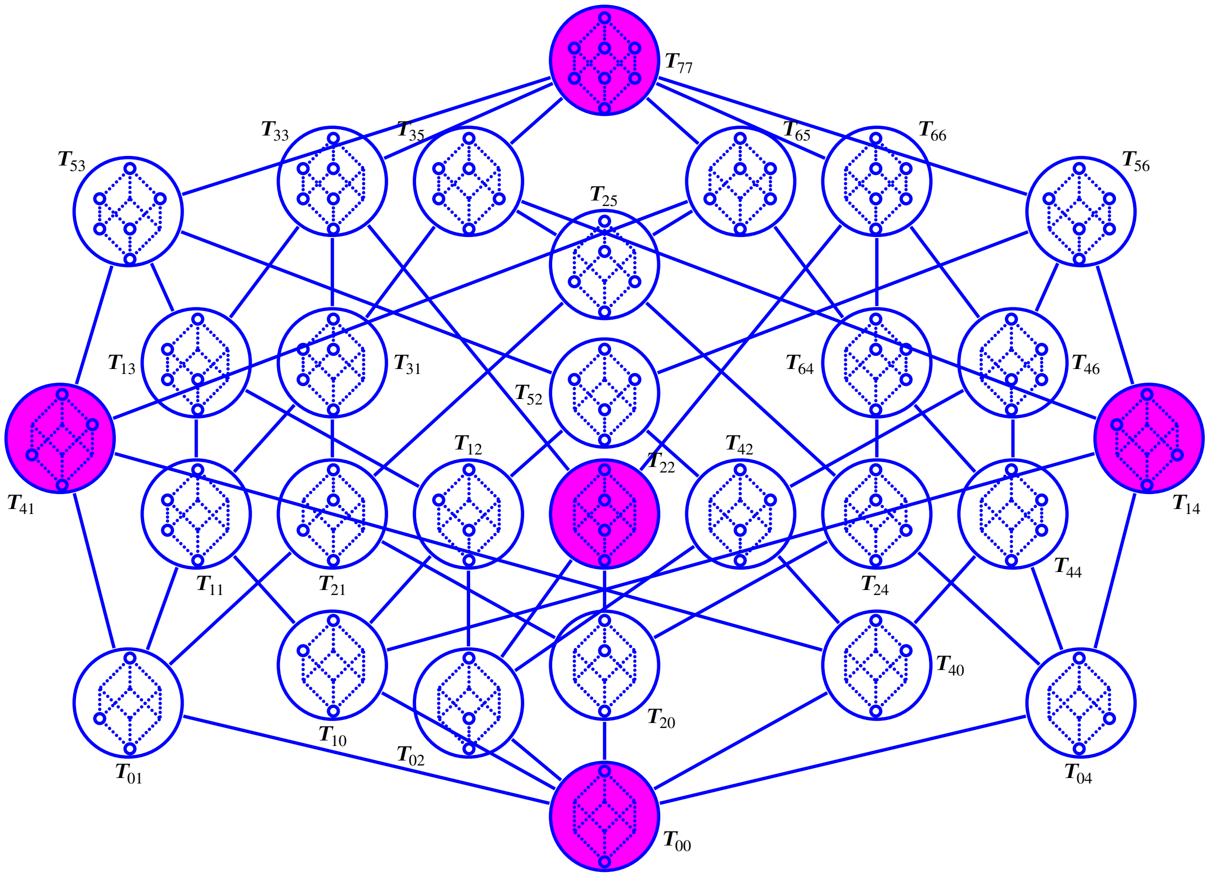
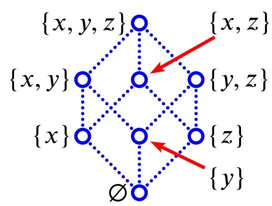


Figure 1.1: lattice of topologies on $X \triangleq \{x, y, z\}$ (Example 1.1 page 2)

Example 1.1. [3]The power set 2^X is a *topology* on the set X. But there are also 28 other topologies on $\{x, y, z\}$, and these are all *subspaces* of $2^{\{x,y,z\}}$. Let a given topology in $\mathcal{T}(\{x, y, z\})$ be represented by a Hasse diagram as illustrated to the right, where a circle present means the indicated set is in the topology, and a circle absent means the indicated set is not in the[4] topology. The lattice of the 29 topologies $(\mathcal{T}(\{x, y, z\}), \cup, \cap; \subseteq)$ is illustrated in Figure 1.1 (page 2). The lattice of these 29 topologies is *non-distributive* (it contains the *N5 lattice*). The five topologies illustrated by red shaded nodes are also *algebras of sets*.



Example 1.2. The power set 2^X is an *algebra of sets* on the set X. But there are also 14 other algebras of sets on $\{w, x, y, z\}$, and these are all *subspaces* of $2^{\{w,x,y,z\}}$. The *lattice of algebras of sets* on $\{w, x, y, z\}$ is illustrated in Figure 1.2 (page 3).

A *linear subspace* is a subspace of a *linear space* (*vector space*). Linear subspaces have some special properties: Every linear subspace contains the additive identity zero vector, and every linear subspace is *convex*.

[4] Isham (1999) page 44, Isham (1989) page 1516, Steiner (1966) page 386

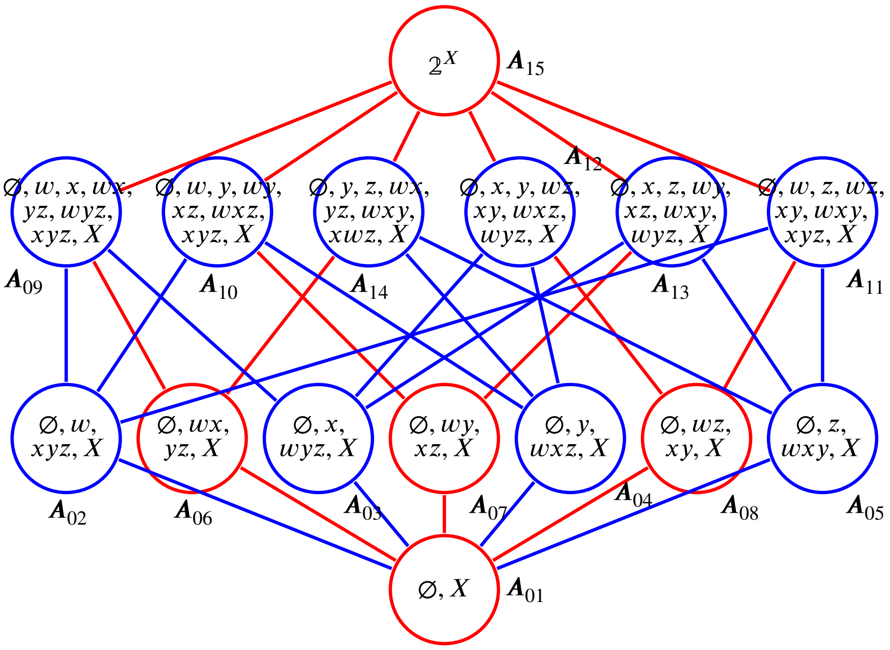
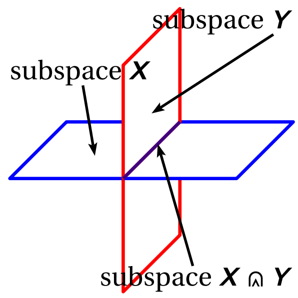


Figure 1.2: lattice of *algebras of sets* on $\{w, x, y, z\}$ (Example 1.2 page 2)

Example 1.3. The 3-dimensional Euclidean space $\mathbb{R}^3$ contains the 2-dimensional xy-plane and xz-plane subspaces, which in turn both contain the 1-dimensional x-axis subspace. These subspaces are illustrated in the figure to the right and in Figure 1.3 (page 4).



1.3 Analyses

An **analysis** of a space $\boldsymbol{X}$ is any lattice of subspaces of $\boldsymbol{X}$. The partial or complete reconstruction of $\boldsymbol{X}$ from this set is a **synthesis**.[5]

Example 1.4. The lattices of subspaces illustrated in Figure 1.4 (page 4) are all *analyses* of $\mathbb{R}^3$.

1.4 Transform

Definition 1.1. *A* ***transform*** *on a space* $\boldsymbol{X}$ *is a sequence of projection operators that induces an* ANALYSIS *on* $\boldsymbol{X}$.

Section 1.3 defined an **analysis** of a space $\boldsymbol{X}$ as is any lattice of subspaces of $\boldsymbol{X}$. In like manner, an **analysis** of a function $\mathsf{f}(x)$ with respect to a transform $\mathbf{T}$ is simply the transform $\mathbf{T}$ of f ($\mathbf{T}\mathsf{f}$). Such

[5]The word *analysis* comes from the Greek word ἀνάλυσις, meaning "dissolution" (Perschbacher (1990) page 23 ⟨entry 359⟩), which in turn means "the resolution or separation into component parts" (Black et al. (2009), http://dictionary.reference.com/browse/dissolution)

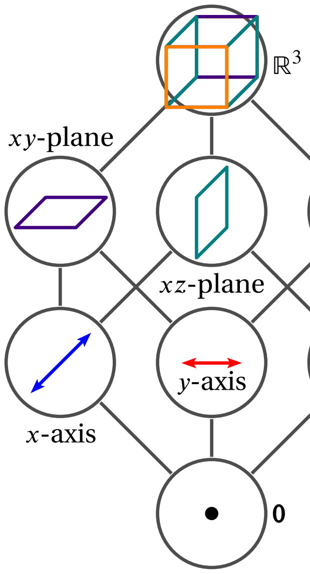


Figure 1.3: lattice of subspaces of $\mathbb{R}^3$ (Example 1.3 page 3)

linearly ordered analysis of $\mathbb{R}^3$ M-3 analysis of $\mathbb{R}^3$ wavelet-like analysis of $\mathbb{R}^3$

Figure 1.4: some anayses of $\mathbb{R}^3$ (Example 1.4 page 3)

an analysis or transform is often represented as the sequence of coefficients $\langle\!\langle \lambda_n \rangle\!\rangle$ multiplying the basis vectors $\langle\!\langle \psi_n(x) \rangle\!\rangle$ such that

$$\mathsf{f}(x) = \mathbf{T}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \lambda_n \psi_n(x)$$

Example 1.5. A *Fourier analysis* is a sequence of subspaces with sinusoidal bases. Examples of subspaces in a Fourier analysis include $\boldsymbol{V}_1 = \operatorname{span}\left\{e^{ix}\right\}$, $\boldsymbol{V}_{2.3} = \operatorname{span}\left\{e^{i2.3x}\right\}$, $\boldsymbol{V}_{\sqrt{2}} = \operatorname{span}\left\{e^{i\sqrt{2}x}\right\}$, etc. A **transform** is a set of *projection operators* that maps a family of functions (e.g. $\boldsymbol{L}^2_{\mathbb{R}}$) into an analysis. The *Fourier transform*" for Fourier Analysis is (Definition 4.2 page 16)

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x$$

1.5 Properties of subspace order structures

The ordered set of all linear subspaces of a *Hilbert space* is an *orthomodular lattice*. Orthomodular lattices (and hence Hilbert subspaces) have some special properties (next theorem). One is that they satisfy *de Morgan's law*.

Theorem 1.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an algebraic structure.*

THM

$$\left.\begin{array}{l}\boldsymbol{L} \text{ is an ORTHOMODULAR} \\ \text{LATTICE}\end{array}\right\} \implies \left\{\begin{array}{llcll} 1. & (x \vee y)^{\perp} & = & x^{\perp} \wedge y^{\perp} & \forall x,y \in X \quad (\text{DE MORGAN}) \quad \text{and} \\ 2. & (x \wedge y)^{\perp} & = & x^{\perp} \vee y^{\perp} & \forall x,y \in X \quad (\text{DE MORGAN}) \quad \text{and} \\ 3. & (z^{\perp} \wedge y^{\perp})^{\perp} \vee x & = & (x \vee y) \vee z & \forall x,y,z \in X \quad \text{and} \\ 4. & x \wedge (x \vee y) & = & x & \forall x,y \in X \quad \text{and} \\ 5. & x \vee (y \wedge y^{\perp}) & = & x & \forall x,y \in X. \end{array}\right.$$

[6] Beran (1985) pages 30–33, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Beran (1976) pages 251–252

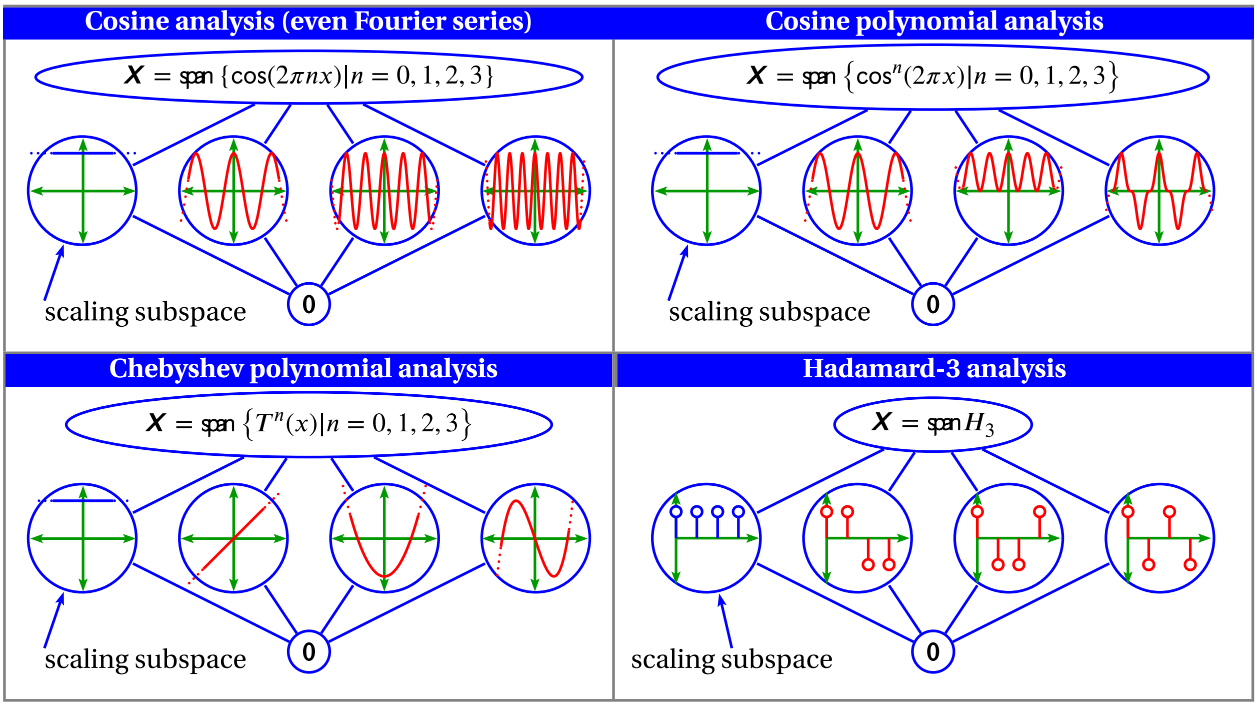
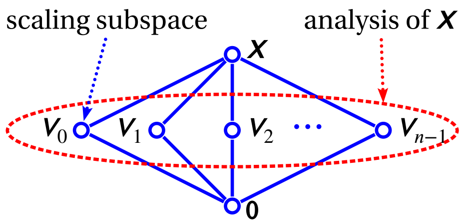
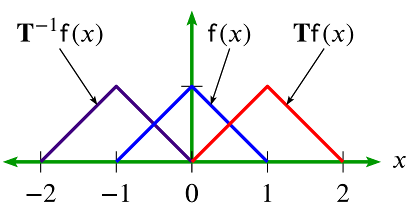


Figure 1.5: some common transforms

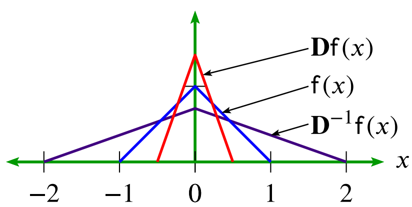
Most transforms have a very simple M-n order structure, as illustrated to the right and in Figure 1.5 page 5. The M-n lattices for $n \geq 3$ are *modular* but not *distributive*. Analyses typically have one subspace that is a *scaling* subspace; and this subspace is often simply a family of constants (as is the case with *Fourier Analysis*). There is one noteable exception to this—MRA induced *wavelet analysis*.



1.6 Operator inducing analyses



An *analysis* is often defined in terms of a small number (e.g. 2) operators. Two such operators are the *translation operator* and the *dilation operator* (Definition I.3 page 188).



Example 1.6. In *Fourier analysis*, continuous dilations (Definition I.3 page 188) of the *complex exponential* form a *basis* (Definition G.7 page 154) for the *space of square integrable functions* $L^2_{\mathbb{R}}$ (Definition B.1 page 69) such that $L^2_{\mathbb{R}} = \text{span}\{\mathbf{D}_\omega e^{ix} | \omega \in \mathbb{R}\}$.

Example 1.7. In *Fourier series analysis* (Theorem 7.1 page 52), discrete dilations of the complex exponential form a basis for $L^2_{\mathbb{R}}(0:2\pi)$ such that $L^2_{\mathbb{R}}(0:2\pi) = \text{span}\left\{\mathbf{D}_j e^{ix} \middle| j \in \mathbb{Z}\right\}$.

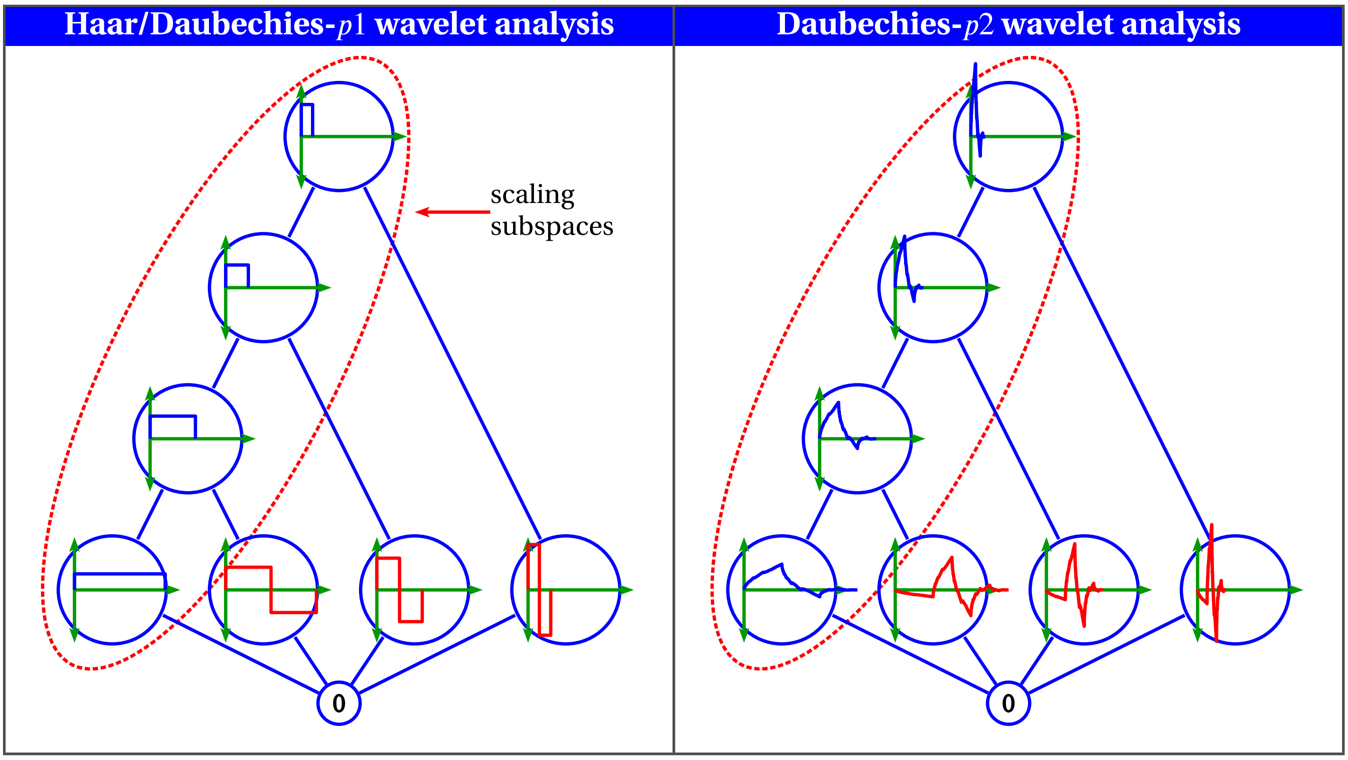
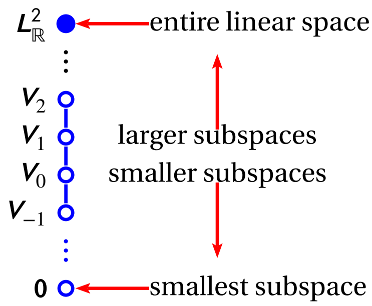


Figure 1.6: some wavelet transforms

1.7 Wavelet analyses

The term "wavelet" comes from the French word "*ondelette*", meaning "small wave". And in essence, wavelets are "small waves" (as opposed to the "long waves" of Fourier analysis) that form a basis for the Hilbert space $L^2_{\mathbb{R}}$.[8]

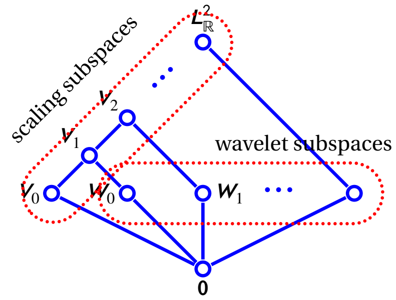
A special characteristic of wavelet analysis is that there is not just one scaling subspace, (as is with the case of Fourier and several other analyses), but an entire sequence of scaling subspaces (Figure 1.6 page 6). These scaling subspaces are *linearly ordered* with respect to the ordering relation $\subseteq$. In wavelet theory, this structure is called a *multiresolution analysis*, or *MRA*. The MRA was introduced by Stéphane G. Mallat in 1989. The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan, and later as the *Gaussian Pyramid* by Burt and Adelson in the 1980s in the West.[9]



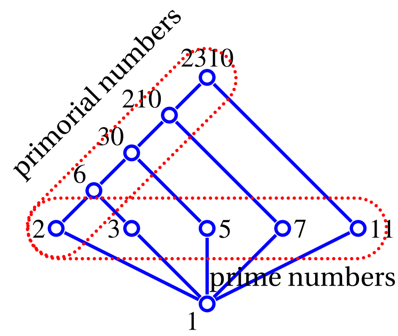
The MRA has become the dominate wavelet construction method. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA.[11]

[8] Strang and Nguyen (1996) page ix Atkinson and Han (2009) page 191
[10] Mallat (1989) page 70 Iijima (1959) Burt and Adelson (1983) Adelson and Burt (1981) Lindeberg (1993) Alvarez et al. (1993) Guichard et al. (2012) Weickert (1999) ⟨historical survey⟩
[11] Lemarié (1990), Mallat (1999) page 240

A second special characteristic of wavelet analysis is that it's order structure with respect to the $\subseteq$ relation is not a simple M-n lattice (as is with the case of Fourier and several other analyses). Rather, it is a lattice of the form illustrated to the right and in Figure 1.6 (page 6). This lattice is *non-complemented*, *non-distributive*, *non-modular*, and *non-Boolean* (Proposition **??** page **??**).[12]



In the world of mathematical structures, the order structure of wavelet analyses is quite rare, but not completely unique. One example of a system with similar structure is the set of *Primorial*[14] *numbers* together with the | ("divides") ordering relation[13] as illustrated to the right.



The basis sequence of most transform are fixed with no design freedom For example, the Fourier Transform uses the complex exponential, Taylor Expansion uses monomials of the form $(x - a)^n$. However, there are an infinite number of wavelet basis sequences—lots and lots of design freedom. For information regarding designing wavelet basis sequences, see Greenhoe (2013).

However, one arguable disadvantage is that wavelets do not support a **convolution theorem**—a theorem enjoyed by the Fourier transforms, Laplace Transform, and Z Transform. These other transforms induce a convolution theorem because they are defined in terms of an exponential (e.g. $e^{-i\omega t}$, $e^{-i\omega n}$, e^{-st}, z^{-n}), and exponentials sport the property $a^{x+y} = a^x a^y$.

[12] Greenhoe (2013) page 72 ⟨Section 2.4.3 Order structure⟩
[14] Sloane (2014) ⟨http://oeis.org/A002110⟩, Greenhoe (2013) page 30

CHAPTER 2

TAYLOR EXPANSIONS (TRANSFORMS)

2.1 Introduction

For modeling real-world processes above the quantum level, measurements are *continuous* in time—that is, the first derivative of a function over time representing the measurement *exists*.

But even for "simple" physical systems, it is not just the first derivative that matters. For example, the classical "vibrating string" vertical displacement $\mathsf{u}(x,t)$ wave equation can be described as

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Not only do physical systems demonstrate heavy dependence on the derivatives of their measurement functions, but also commonly exhibit *oscillation*, as demonstrated by sunspot activity over the last 300 years or earthquake activity (Figure 2.1 page 10).

In fact, derivatives and oscillations are fundamentally linked as demonstrated by the fact that all solutions of homogeneous second order differential equations are linear combinations of sine and cosine functions (Theorem C.3 page 78):

$$\left\{\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \mathsf{f}'(0)\sin(x)\right\} \qquad \forall \mathsf{f}\in\boldsymbol{C}, \forall x\in\mathbb{R}$$

Derivatives are calculated *locally* about a point. Oscillations are observed *globally* over a range, and analyzed (decomposed) by projecting the function onto a sequence of basis functions—sinusoids in the case of Fourier Transform family. Projection is accomplished using inner products, and often these are calculated using *integration*. Note that derivatives and integrals are also fundamentally linked as demonstrated by the *Fundamental Theorem of Calculus*...which shows that integration can be calculated using anti-differentiation:

$$\int_a^b \mathsf{f}(x)\,\mathsf{d}x = \mathsf{F}(b) - \mathsf{F}(a) \qquad \text{where } \mathsf{F}(x) \text{ is the } \textit{antiderivative} \text{ of } \mathsf{f}(x).$$

Brook Taylor showed that for *analytic* functions,[1] knowledge of the derivatives of a function at a location $x = a$ allows you to determine (predict) arbitrarily closely all the points $\mathsf{f}(x)$ in the vicinity

[1] *analytic* functions: Functions for which all their derivatives exist.

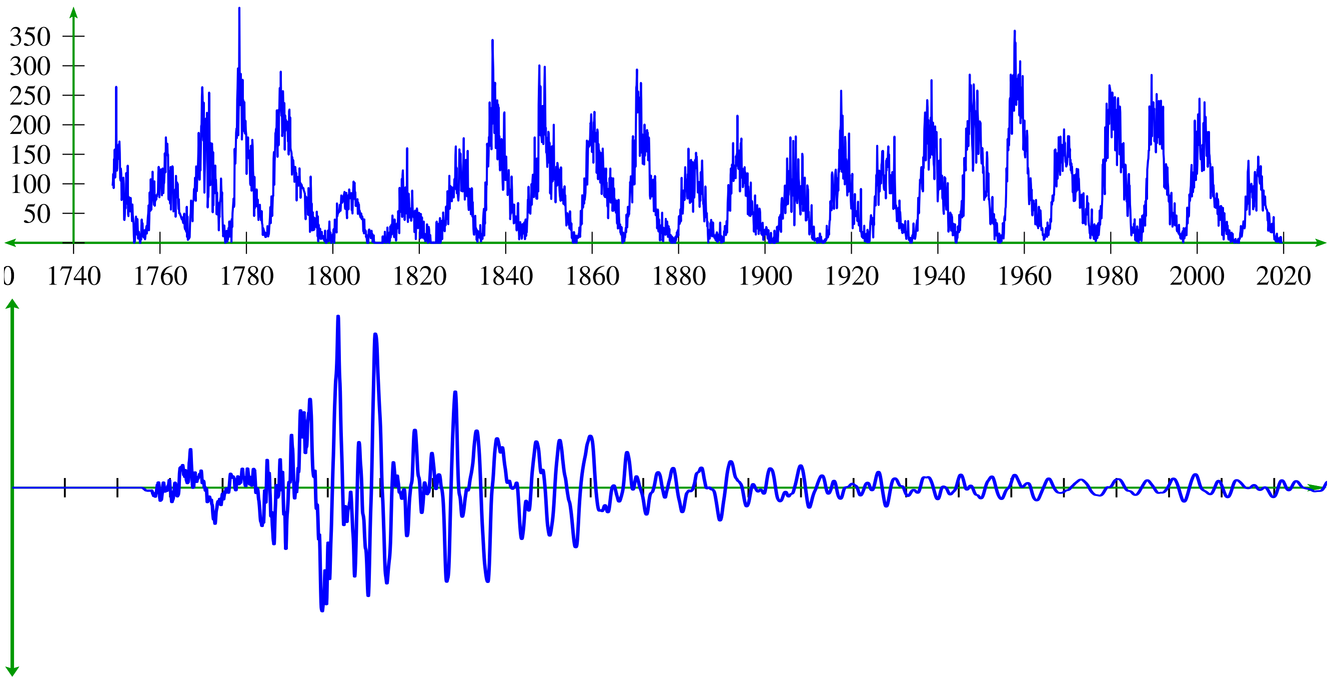


Figure 2.1: Sunspot and earthquake measurements

of $x = a$:[2]

$$f(x) = f(a) + \frac{1}{1!}f'(a)[x-a] + \frac{1}{2!}f''(x)[x-a]^2 + \frac{1}{3!}f'''(x)[x-a]^3 + \cdots$$

On the other hand, the *Fourier Transform* is a kind of counter-part of the Taylor expansion:[3]

	Taylor coefficients	Fourier coefficients
	Depend on derivatives $\frac{d^n}{dx^n}f(x)$	Depend on integrals $\int_{x\in\mathbb{R}} f(x)e^{-i\omega x}\,dx$
	Behavior in the vicinity of a point.	Behavior over the entire function.
	Demonstrate trends locally.	Demonstrate trends globally, such as oscillations.
	Admits *analytic* functions only.	Admits *non-analytic* functions as well.
	Function must be *continuous*.	Function can be *discontinuous*.

2.2 Taylor Expansion

Theorem 2.1 (Taylor Series). [4] *Let $\boldsymbol{C}$ be the space of all* ANALYTIC *functions and $\frac{d}{dx}$ in $\boldsymbol{C}^{\boldsymbol{C}}$ the* DIFFERENTIATION OPERATOR.

THM

*A **Taylor Series** about the point $x = a$ of a function $f(x) \in \boldsymbol{C}^{\boldsymbol{C}}$ is*

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{\left[\frac{d}{dx}^n f\right](a)}{n!}}_{\textit{coefficient}} \underbrace{(x-a)^n}_{\textit{basis function}} \qquad \forall a\in\mathbb{R}, f\in\boldsymbol{C}$$

*A **Maclaurin Series** is a* TAYLOR SERIES *about the point $a = 0$.*

[2] Robinson (1982) page 886
[3] Robinson (1982) page 886
[4] Flanigan (1983) page 221 ⟨Theorem 15⟩, Strichartz (1995) page 281, Sohrab (2003) page 317 ⟨Theorem 8.4.9⟩, Taylor (1715), Maclaurin (1742)

CHAPTER 3

LAPLACE TRANSFORM

3.1 Definition

Definition 3.1. *Let* $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF

*The **Laplace Transform** operator* $\mathbf{L}$ *is here defined as*

$$[\mathbf{L}\mathsf{f}](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x\in\mathbb{R}} \mathsf{f}(x) e^{-sx}\, \mathrm{d}x \qquad \forall \mathsf{f} \in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

In Definition 3.1, the scaling factor $\frac{1}{\sqrt{2\pi}}$ is not normally found in most definitions of the Laplace Transform. However it is included here to make the operator $\mathbf{L}$ more directly compatible with the *Unitary Fourier Transform* operator $\tilde{\mathbf{F}}$ (Definition 4.2 page 16).

3.2 Shift relations

Theorem 3.1 (Shift relations)**.** *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* *(Definition 3.1 page 11)*.

THM

$$\begin{aligned} \mathbf{L}[\mathsf{f}(x-y)](s) &= e^{-sy}\, [\mathbf{L}\mathsf{f}(x)](s) \\ \left[\mathbf{L}\left(e^{rx}\mathsf{g}(x)\right)\right](s) &= \left[\mathbf{L}\mathsf{g}(x)\right](s-r) \end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{L}[\mathsf{f}(x-y)](s) &= \frac{1}{\sqrt{2\pi}} \int_{x\in\mathbb{R}} \mathsf{f}(x-y) e^{-sx}\, \mathrm{d}x && \text{by definition of } \mathbf{L} && \text{(Definition 3.1 page 11)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{u\in\mathbb{R}} \mathsf{f}(u) e^{-s(y+u)}\, \mathrm{d}u && \text{where } u \triangleq x-y && \implies x = y+u \\
&= e^{-sy} \frac{1}{\sqrt{2\pi}} \int_{u\in\mathbb{R}} \mathsf{f}(u) e^{-su}\, \mathrm{d}u && \text{by property of exponents} && a^{x+y} = a^x a^y \\
&= e^{-sy} \frac{1}{\sqrt{2\pi}} \int_{x\in\mathbb{R}} \mathsf{f}(x) e^{-sx}\, \mathrm{d}u && \text{by change of variable} && u \to x \\
&= e^{-sy}\, [\mathbf{L}\mathsf{f}(x)](s) && \text{by definition of } \mathbf{L} && \text{(Definition 3.1 page 11)}
\end{aligned}$$

$$\begin{aligned}
\left[\mathbf{L}(e^{rx}\mathsf{g}(x))\right](s) &= \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}} e^{rx}\mathsf{g}(x)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{L} && \text{(Definition 3.1 page 11)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}} \mathsf{g}(x)e^{-(s-r)x}\,\mathrm{d}x && \text{by property of exponents} && a^{x+y}=a^x a^y\\
&= \left[\mathbf{L}\mathsf{g}(x)\right](s-r) && \text{by definition of } \mathbf{L} && \text{(Definition 3.1 page 11)}
\end{aligned}$$

3.3 Convolution relations

Definition 3.2. [1]

DEF *The **convolution operation** is defined as*

$$\left[\mathsf{f}\star\mathsf{g}\right](x) \triangleq \mathsf{f}(x)\star\mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem 3.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "s domain" and vice-versa.

Theorem 3.2 (convolution theorem)**.** *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition 3.1 page 11) *and* $\star$ *the convolution operator* (Definition 4.3 page 19).

THM

$$\underbrace{\mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\,\left[\mathbf{L}\mathsf{g}\right](s)}_{\text{multiplication in "s domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\mathbf{L}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s)\star\left[\mathbf{L}\mathsf{g}\right](s)}_{\text{convolution in "s domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF:

$$\begin{aligned}
\mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](s) &= \mathbf{L}\left[\int_{u\in\mathbb{R}}\mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u\right](s) && \text{by definition of } \star && \text{(Definition 4.3 page 19)}\\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)\left[\mathbf{L}\mathsf{g}(x-u)\right](s)\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-su}\left[\mathbf{L}\mathsf{g}(x)\right](s)\,\mathrm{d}u && \text{by \textit{Fourier shift theorem}} && \text{(Theorem 4.4 page 18)}\\
&= \sqrt{2\pi}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-su}\,\mathrm{d}u\right)}_{[\mathbf{L}\mathsf{f}](s)}\left[\mathbf{L}\mathsf{g}\right](s)\\
&= \sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s) && \text{by definition of } \mathbf{L} && \text{(Definition 4.2 page 16)}\\
\mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s) &= \mathbf{L}\left[\left(\mathbf{L}^{-1}\mathbf{L}\mathsf{f}(x)\right)\mathsf{g}(x)\right](s) && \text{by def. of operator inverse} && \text{(Definition F.3 page 120)}\\
&= \mathbf{L}\left[\left(\frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}[\mathbf{L}\mathsf{f}(x)](v)e^{sxv}\,\mathrm{d}v\right)\mathsf{g}(x)\right](s) && \text{by Theorem 4.1 page 17}\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}[\mathbf{L}\mathsf{f}(x)](v)\left[\mathbf{L}\left(e^{sxv}\,\mathsf{g}(x)\right)\right](s,v)\,\mathrm{d}v\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}[\mathbf{L}\mathsf{f}(x)](v)\left[\mathbf{L}\mathsf{g}(x)\right](s-v)\,\mathrm{d}v && \text{by Theorem 4.4 page 18}\\
&= \frac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s)\star\left[\mathbf{L}\mathsf{g}\right](s) && \text{by definition of } \star && \text{(Definition 4.3 page 19)}
\end{aligned}$$

[1] Bachman (1964) page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩

3.4 Calculus relations

Theorem 3.3. *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition 3.1 page 11).

THM
$$\left\{\lim_{t\to-\infty} \mathsf{x}(t) = 0\right\} \implies \left\{\mathbf{L}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right] = s[\mathbf{L}\mathsf{x}](s)\right\}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{L}\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right]} &\triangleq \int_{t\in\mathbb{R}} \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{x}(t)\right]}_{\mathsf{d}v}\underbrace{e^{-st}}_{u}\,\mathsf{d}t && \text{by definition of } \mathbf{L}\\
&= \underbrace{e^{-st}}_{u}\underbrace{\mathsf{x}(t)}_{v}\Big|_{t=-\infty}^{t=+\infty} - \int_{t\in\mathbb{R}} \underbrace{\mathsf{x}(t)}_{v}\underbrace{(-s)e^{-st}\,\mathsf{d}t}_{\mathsf{d}u} && \text{by \textit{Integration by Parts}}\\
&= \cancelto{0}{e^{-s\infty}}\mathsf{x}(\infty) - e^{s\infty}\cancelto{0}{\mathsf{x}(-\infty)} - (-s)\underbrace{\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathsf{d}t}_{\text{\textit{Laplace Transform} of } \mathsf{x}(t)} && \text{by left hypothesis}\\
&= \boxed{s[\mathbf{L}\mathsf{x}](s)}
\end{aligned}$$

✏

Theorem 3.4. *Let* $\mathbf{L}$ *be the* LAPLACE TRANSFORM *operator* (Definition 3.1 page 11).

THM
$$\mathbf{L}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u = \frac{1}{s}[\mathbf{L}\mathsf{x}](s)$$

✎PROOF:

1. Define the *Heaviside function* $\mathsf{h}(t)$ as $\mathsf{h}(t) \triangleq \begin{cases} 1 & \text{for } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$

2. Remainder of proof...

$$\begin{aligned}
\boxed{\mathbf{L}\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u} &\triangleq \int_{t=-\infty}^{t=+\infty}\left[\int_{u=-\infty}^{u=t} \mathsf{x}(u)\,\mathsf{d}u\right]e^{-st}\,\mathsf{d}t && \text{by definition of } \mathbf{L}\\
&= \int_{t=-\infty}^{t=+\infty}\left[\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)\mathsf{h}(t-u)\,\mathsf{d}u\right]e^{-st}\,\mathsf{d}t && \left(\text{by definition of \textit{Heaviside function} definition 1}\right)\\
&= \int_{v=-\infty}^{v=+\infty}\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)\mathsf{h}(v)e^{-s(u+v)}\,\mathsf{d}u\,\mathsf{d}v && \left(\text{where } v \triangleq t-u \implies t = u+v\right)\\
&= \left[\int_{v=-\infty}^{v=+\infty} \mathsf{h}(v)e^{-sv}\,\mathsf{d}v\right]\underbrace{\left[\int_{u=-\infty}^{u=+\infty} \mathsf{x}(u)e^{-su}\,\mathsf{d}u\right]}_{\text{\textit{Laplace Transform} of } \mathsf{x}(t)}\\
&= \left[\int_{v=0}^{v=\infty} e^{-sv}\,\mathsf{d}v\right][\mathbf{L}\mathsf{x}](s) && \left(\text{by definition of \textit{Heaviside function} definition 1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{1}{-s}e^{-sv}\right|_{v=0}^{v=\infty}[\mathbf{L}\mathrm{x}](s) && \text{by \textit{Fundamental Theorem of Calculus}}\\
&= \boxed{\frac{1}{s}[\mathbf{L}\mathrm{x}](s)}
\end{aligned}
$$

✏

CHAPTER 4

FOURIER TRANSFORM

❝ *Up to this point we have supposed that the function whose development is required in a series of sines of multiple arcs can be developed in a series arranged according to powers of the variable x, ... We can extend the same results to any functions, even to those which are discontinuous and entirely arbitrary. ... even entirely arbitrary functions may be developed in series of sines of multiple arcs.* ❞

Joseph Fourier (1768–1830) [1]

4.1 Introduction

Historically, before the Fourier Transform was the Taylor Expansion (transform). The Taylor Expansion demonstrates that for **analytic** functions knowledge of the derivatives of a function at a location $x = a$ allows you to determine (predict) arbitrarily closely all the points $\mathsf{f}(x)$ in the vicinity of $x = a$ (Chapter 2 page 9). But analytic functions are by definition functions for which all their derivatives exist. Thus, if a function is *discontinuous*, it is simply not a candidate for a Taylor Expansion. And some 300 years ago, mathematician giants of the day were fairly content with this.

But then in came an engineer named Joseph Fourier whose day job was working as a governor of lower Egypt under Napolean. He claimed that, rather than expansion based on derivatives, one could expand based on integrals over sinusoids, and that this would work not just for analytic functions, but for **discontinuous** ones as well![2]

Needless to say, this did not go over too well initially in the mathematical community. But over time (on the order of 200 or so years), the Fourier Transform has in many ways won the day.


[3]

[1] quote: Fourier (1878) page 184,186 ⟨§219,220⟩
image: `http://en.wikipedia.org/wiki/File:Fourier2.jpg`, public domain

[2] Robinson (1982) page 886

[3] Caricature of Legendre (left) and Fourier (right), 1820, by Julien-Léopold Boilly (1796–1874). "*Album de 73*

4.2 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on $\mathbb{R}$, and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \middle| \int_{\mathbb{R}} |\mathsf{f}|^2 \, \mathsf{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \mid \triangledown \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \, \mathsf{d}\mu$ such that

$$\langle \mathsf{f} \mid \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, \mathsf{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \mid \triangledown \rangle \right)$ is a *Hilbert space*.

Definition 4.1. *Let κ be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF *The function κ is the* ***Fourier kernel*** *if* $\quad \kappa(x,\omega) \triangleq e^{i\omega x} \quad \forall x,\omega\in\mathbb{R}$

Definition 4.2. [4] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

DEF *The* ***Fourier Transform*** *operator $\tilde{\mathbf{F}}$ is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \, \mathsf{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

This definition of the Fourier Transform is also called the ***unitary Fourier Transform***.

Remark 4.1 (**Fourier transform scaling factor**). [5] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A \int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x} \, \mathsf{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B \int_{\mathbb{R}} \mathsf{F}(\omega)e^{i\omega x} \, \mathsf{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \, \mathsf{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x} \, \mathsf{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x} \, \mathsf{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int \cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x} \, \mathsf{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem 4.2 page 17)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

Portraits-Charge Aquarelle's des Membres de I'Institute (watercolor portrait #29). Biliotheque de l'Institut de France." Public domain. https://en.wikipedia.org/wiki/File:Legendre_and_Fourier_(1820).jpg

[4] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965) page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

[5] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005b) pages 374–375

4.3 Operator properties

Theorem 4.1 (Inverse Fourier transform). [6] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition 4.2 page 16). *The inverse* $\tilde{\mathbf{F}}^{-1}$ *of* $\tilde{\mathbf{F}}$ *is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\, \mathrm{d}\omega \qquad \forall \tilde{\mathsf{f}} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem 4.2. *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with inverse* $\tilde{\mathbf{F}}^{-1}$ *and adjoint* $\tilde{\mathbf{F}}^*$.

THM
$$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \mathsf{g}\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \,\middle|\, \mathsf{g}(\omega)\right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 16}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\, \left\langle e^{-i\omega x} \,|\, \mathsf{g}(\omega)\right\rangle\, \mathrm{d}x && \text{by \textit{additive property} of } \langle \triangle \,|\, \triangledown\rangle\\
&= \int_{\mathbb{R}} \mathsf{f}(x)\, \frac{1}{\sqrt{2\pi}} \left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x}\right\rangle^*\, \mathrm{d}x && \text{by \textit{conjugate symmetric property} of } \langle \triangle \,|\, \triangledown\rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \frac{1}{\sqrt{2\pi}} \left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x}\right\rangle\right\rangle && \text{by definition of } \langle \triangle \,|\, \triangledown\rangle\\
&= \Big\langle \mathsf{f} \,\Big|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\mathsf{g} \Big\rangle && \text{by Theorem 4.1 page 17}
\end{aligned}$$

✏

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* [7] (Corollary 4.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem 4.3 page 17).

Corollary 4.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

COR
$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem 4.2 page 17). ✏

Theorem 4.3. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \triangledown\rangle)$. *Let* $\boldsymbol{\mathcal{R}}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

THM
$$\begin{array}{rclcll}
\boldsymbol{\mathcal{R}}(\mathbf{F}\tau) &=& \boldsymbol{\mathcal{R}}(\tilde{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| &=& |||\tilde{\mathbf{F}}^{-1}||| &=& 1 & \text{(UNITARY)}\\
\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g}\rangle &=& \langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g}\rangle &=& \langle \mathsf{f} \,|\, \mathsf{g}\rangle & \text{(PARSEVAL'S EQUATION)}\\
\|\tilde{\mathbf{F}}\mathsf{f}\| &=& \|\tilde{\mathbf{F}}^{-1}\mathsf{f}\| &=& \|\mathsf{f}\| & \text{(PLANCHEREL'S FORMULA)}\\
\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\| &=& \|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\| &=& \|\mathsf{f} - \mathsf{g}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary 4.1 page 17) and from the properties of unitary operators (Theorem F.26 page 144). ✏

[6] Chorin and Hald (2009) page 13
[7] *unitary operators*: Definition F.14 page 143

4.4 Shift relations

Theorem 4.4 (Shift relations). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition 4.2 page 16).

T
H
M

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega)\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r)
\end{aligned}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition 3.1 page 11).

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-y)](\omega) &= \mathbf{L}[\mathsf{f}(x-y)](s)\big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition 3.1 page 11)}\\
&= e^{-sy}\,[\mathbf{L}\mathsf{f}(x)](s)\big|_{s=i\omega} && \text{by \textit{Laplace shift relation}} && \text{(Theorem 3.1 page 11)}\\
&= e^{-i\omega y}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 4.2 page 16)}\\
\left[\tilde{\mathbf{F}}\left(e^{irx}\mathsf{g}(x)\right)\right](\omega) &= \left[\mathbf{L}\left(e^{irx}\mathsf{g}(x)\right)\right](s)\big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition 3.1 page 11)}\\
&= \left[\left[\mathbf{L}\mathsf{g}(x)\right](s-r)\right]\big|_{s=i\omega} && \text{by \textit{Laplace shift relation}} && \text{(Theorem 3.1 page 11)}\\
&= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-r) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 4.2 page 16)}
\end{aligned}$$

✏

Theorem 4.5 (Complex conjugate). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

T
H
M

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{f}^*(-x) &= -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \forall \mathsf{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)}\\
\mathsf{f} \text{ is real} \implies \tilde{\mathsf{f}}(-\omega) &= \left[\tilde{\mathsf{f}}(\omega)\right]^* && \forall\omega\in\mathbb{R} && \text{REALITY CONDITION}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(-x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 4.2 page 16)}\\
&= \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(u)e^{i\omega u}(-1)\,\mathrm{d}u && \text{where } u\triangleq -x \implies \mathrm{d}x = -\,\mathrm{d}u\\
&= -\left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right]^*\\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 4.2 page 16)}\\
\tilde{\mathsf{f}}(-\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i(-\omega)x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 4.2 page 16)}\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(x)e^{-i\omega x}\,\mathrm{d}x\right]^*\\
&= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]^* && \text{by } \mathsf{f} \text{ is real hypothesis}\\
&\triangleq \tilde{\mathsf{f}}^*(\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition 4.2 page 16)}
\end{aligned}$$

✏

4.5 Convolution relations

Definition 4.3. [8]

DEF *The **convolution operation** is defined as*

$$\left[\mathsf{f} \star \mathsf{g}\right](x) \triangleq \mathsf{f}(x) \star \mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem 5.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem 4.6 (convolution theorem). [9] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator* (Definition 4.2 page 16) *and $\star$ the convolution operator* (Definition 4.3 page 19).

THM

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition 3.1 page 11).

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega) &= \mathbf{L}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](s)\Big|_{s=i\omega} && \text{by definition of } \mathbf{L} && \text{(Definition 3.1 page 11)}\\
&= \sqrt{2\pi}[\mathbf{L}\mathsf{f}](s)\left[\mathbf{L}\mathsf{g}\right](s)\Big|_{s=i\omega} && \text{by } \textit{Laplace convolution} \text{ result} && \text{(Theorem 3.2 page 12)}\\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)\\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \mathbf{L}[\mathsf{f}(x)\mathsf{g}(x)](s)|_{s=i\omega}\\
&= \frac{1}{\sqrt{2\pi}}[\mathbf{L}\mathsf{f}](s)\star\left[\mathbf{L}\mathsf{g}\right](s)\Big|_{s=i\omega}\\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)
\end{aligned}$$

✏

4.6 Calculus relations

Theorem 4.7. *Let $\tilde{\mathbf{F}}$ be the* FOURIER TRANSFORM *operator* (Definition 4.2 page 16).

THM

$$\left\{\lim_{t\to-\infty}\mathsf{x}(t)=0\right\} \implies \left\{\tilde{\mathbf{F}}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t)\right]=i\omega\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)\right\}$$

✎PROOF: Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition 3.1 page 11).

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t)\right] &\triangleq \mathbf{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t)\right](s)\Big|_{s=i\omega} && \text{by definitions of } \mathbf{L} \text{ and } \tilde{\mathbf{F}} && \text{(Definition 3.1 page 11)}\\
&= s[\mathbf{L}\mathsf{x}(t)](s)|_{s=i\omega} && \text{by Theorem 3.3 page 13}\\
&= i\omega\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)
\end{aligned}$$

✏

[8] Bachman (1964) page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩
[9] Bracewell (1978) page 110

Theorem 4.8. *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator* (Definition 4.2 page 16).

THM
$$\tilde{\mathbf{F}} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\, \mathsf{d}u = \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}\right](\omega)$$

Let $\mathbf{L}$ be the *Laplace Transform* operator (Definition 3.1 page 11). ✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\, \mathsf{d}u &\triangleq \mathbf{L} \int_{u=-\infty}^{u=t} \mathsf{x}(u)\, \mathsf{d}u \bigg|_{s=i\omega} \\
&= \frac{1}{s}[\mathbf{L}\mathsf{x}(t)](s)\Big|_{s=i\omega} && \text{by Theorem 3.4 page 13} \\
&= \frac{1}{i\omega}\left[\tilde{\mathbf{F}}\mathsf{x}(t)\right](\omega)
\end{aligned}$$

✏

4.7 Real valued functions

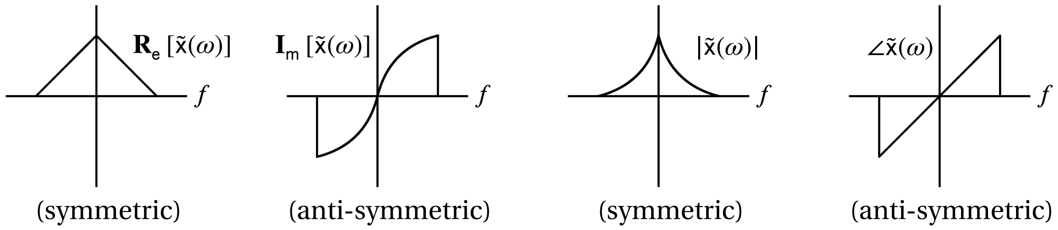


Figure 4.1: Fourier transform components of real-valued signal

Theorem 4.9. *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM
$$\left\{\begin{array}{l} \mathsf{f}(x) \text{ is REAL-VALUED} \\ (\mathsf{f} \in \mathbb{R}^{\mathbb{R}}) \end{array}\right\} \implies \left\{\begin{array}{rcll}
\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & (\text{HERMITIAN SYMMETRIC}) \\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] & (\text{SYMMETRIC}) \\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] & (\text{ANTI-SYMMETRIC}) \\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| & (\text{SYMMETRIC}) \\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & (\text{ANTI-SYMMETRIC}).
\end{array}\right\}$$

✎PROOF:

$$\begin{array}{lclclclcl}
\tilde{\mathsf{f}}(\omega) &\triangleq& [\tilde{\mathbf{F}}\mathsf{f}(x)](\omega) &\triangleq& \left\langle \mathsf{f}(x) \,|\, e^{i\omega x}\right\rangle &=& \left\langle \mathsf{f}(x) \,|\, e^{i(-\omega)x}\right\rangle^* &\triangleq& \tilde{\mathsf{f}}^*(-\omega) \\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{I_m}\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}^*(-\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| \\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}^*(-\omega) &=& -\angle\tilde{\mathsf{f}}(-\omega)
\end{array}$$

✏

4.8 Moment properties

Definition 4.4. [10]

DEF
The quantity M_n *is the* n**th moment** *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *if*
$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\, \mathsf{d}x \qquad \text{for } n \in \mathbb{W}.$$

[10] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83

Lemma 4.1. [11] *Let* M_n *be the* nTH MOMENT (Definition 4.4 page 20) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition 4.2 page 16) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition B.1 page 69).

LEM

$$\begin{aligned}
\mathsf{M}_n &= \sqrt{2\pi}(i)^n \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}\\
\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} &= \frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\sqrt{2\pi}(i)^n \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 4.2 page 16)}\\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega x}\right] \mathsf{d}x\Big|_{\omega=0}\\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathsf{d}x\Big|_{\omega=0}\\
&= (-i^2)^n \int_{\mathbb{R}} \mathsf{f}(x) x^n \,\mathsf{d}x\\
&= \int_{\mathbb{R}} \mathsf{f}(x) x^n \,\mathsf{d}x\\
&\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \quad \text{(Definition 4.4 page 20)}
\end{aligned}$$

✏

Lemma 4.2. [12] *Let* M_n *be the* nTH MOMENT (Definition 4.4 page 20) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition 4.2 page 16) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition B.1 page 69).

LEM

$$\mathsf{M}_n = 0 \quad\iff\quad \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
0 &= \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by left hypothesis}\\
&= \sqrt{2\pi}(-i)^{-n}\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \text{by Lemma 4.1 page 21}\\
\Longrightarrow &\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} = 0
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \text{by right hypothesis}\\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\Big|_{\omega=0} && \text{by definition of } \tilde{\mathsf{f}}(\omega)\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega x}\,\mathsf{d}x\Big|_{\omega=0}\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathsf{d}x\Big|_{\omega=0}\\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) x^n \,\mathsf{d}x\\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by definition of } \langle \cdot \,|\, \cdot \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{(Definition B.1 page 69)}
\end{aligned}$$

[11] Goswami and Chan (1999) pages 38–39
[12] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

Lemma 4.3 (Strang-Fix condition). [13] *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* M_n *the* nTH MOMENT (Definition 4.4 page 20) *of* $\mathsf{f}(x)$. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR (Definition I.3 page 188).

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "time"}} \iff \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "frequency"}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by definition of } \tilde{\mathsf{f}}(\omega) \quad \text{(Definition 4.2 page 16)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k)\bar{\delta}_k && \text{by } \textit{PSF} \quad \text{(Theorem I.2 page 196)} \\
&= \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[(-i)^n \bar{\delta}_k \mathsf{M}_n\right] e^{-i2\pi kx} && \text{by definition of } \bar{\delta} \quad \text{(Definition G.12 page 160)} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis} \\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } \textit{PSF} \quad \text{(Theorem I.2 page 196)}
\end{aligned}$$

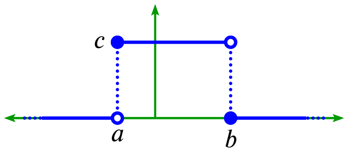
[13] Jawerth and Sweldens (1994) pages 16–17, Sweldens and Piessens (1993) page 2, Vidakovic (1999) page 83, Mallat (1999) pages 241–243, Fix and Strang (1969)

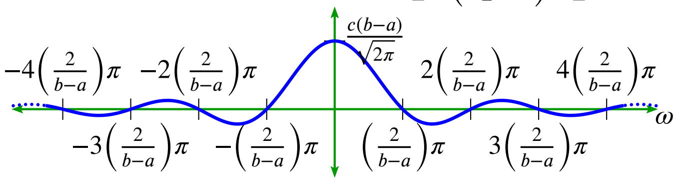
4.9 Examples

Example 4.1 (rectangular pulse). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right]$$





✎PROOF:

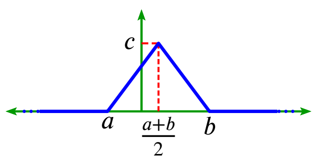
$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} \quad \text{(Theorem 4.4 page 18)} \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by definition of } \mathsf{f}(x) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} \quad \text{(Definition I.2 page 188)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 4.2 page 16)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} ce^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \quad \text{(Definition I.2 page 188)} \\
&= \frac{c}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\right)\omega}\frac{1}{-i\omega}e^{-i\omega x}\Big|_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \\
&= \frac{2c}{\sqrt{2\pi}\omega}e^{-i\left(\frac{a+b}{2}\right)\omega}\left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i}\right] \\
&= \frac{c(b-a)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right] && \text{by } \textit{Euler formulas} \quad \text{(Corollary C.2 page 81)}
\end{aligned}$$

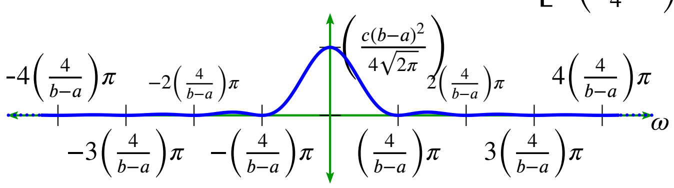
✏

Example 4.2 (triangle). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \left\{\begin{array}{ll} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{array}\right\} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$





✎PROOF:

$$\tilde{\mathsf{f}}(\omega) = \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) \qquad \text{by definition of } \tilde{\mathsf{f}}(\omega)$$

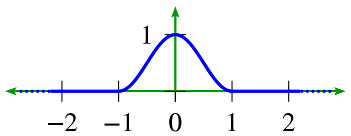
$$\begin{aligned}
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x-\frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} && \text{(Theorem 4.4 page 18)}\\
&= \tilde{\mathbf{F}}\left[c\left(1-\frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) && \text{by definition of } \mathsf{f}(x)\\
&= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\star\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega)\\
&= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] && \text{by } \textit{convolution theorem} && \text{(Theorem 5.2 page 30)}\\
&= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2\\
&= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 && \text{by } \textit{Rectangular pulse} \text{ ex.} && \text{Example 4.1 page 23}\\
&= \frac{c(b-a)^2}{4\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2
\end{aligned}$$

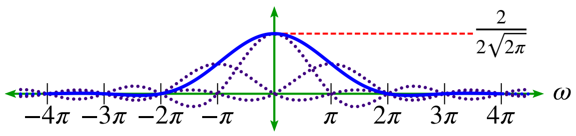
✏

Example 4.3. Let a function f be defined in terms of the cosine function (Definition C.1 page 75) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases}\cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x|\le 1\\ 0 & \text{otherwise}\end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$





✎**Proof**: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition I.2 page 188) on a set A.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition 4.2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathsf{f}(x)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\cos^2\left(\frac{\pi}{2}x\right)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \text{ (Definition I.2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left[\frac{e^{i\frac{\pi}{2}x}+e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\,\mathrm{d}x && \text{by Corollary C.2 page 81}\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1}\left[2+e^{i\pi x}+e^{-i\pi x}\right]e^{-i\omega x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1}2e^{-i\omega x}+e^{-i(\omega+\pi)x}+e^{-i(\omega-\pi)x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\left[2\frac{e^{-i\omega x}}{-i\omega}+\frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}\\
&= \frac{1}{2\sqrt{2\pi}}\left[2\frac{e^{-i\omega}-e^{+i\omega}}{-2i\omega}+\frac{e^{-i(\omega+\pi)}-e^{+i(\omega+\pi)}}{-2i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)}-e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}
\end{aligned}$$

$$= \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$

✏

CHAPTER 5

Z TRANSFORM

5.1 Convolution operator

Definition 5.1. [1] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF *A function f in X^Y is a **sequence** over X if $Y = \mathbb{Z}$. A sequence may be denoted in the form $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ or simply as $\left(\!\left(x_n\right)\!\right)$.*

Definition 5.2. [2] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a* FIELD *(Definition A.5 page 68).*

DEF *The **space of all absolutely square summable sequences** $\boldsymbol{\ell}^2_{\mathbb{F}}$ over $\mathbb{F}$ is defined as*

$$\boldsymbol{\ell}^2_{\mathbb{F}} \triangleq \left\{ \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}$$

The space $\boldsymbol{\ell}^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is isomorphic to $\boldsymbol{L}^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition 5.3.

DEF *The **convolution** operation $\star$ is defined as*

$$\left(\!\left(x_n\right)\!\right) \star \left(\!\left(y_n\right)\!\right) \triangleq \left(\!\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m} \right)\!\right)_{n\in\mathbb{Z}} \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}, \left(\!\left(y_n\right)\!\right)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$$

Proposition 5.1. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition 5.3 page 27).*

PRP

$$\left(\!\left(x_n\right)\!\right) \star \left(\!\left(y_n\right)\!\right) = \left(\!\left(y_n\right)\!\right) \star \left(\!\left(x_n\right)\!\right) \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}, \left(\!\left(y_n\right)\!\right)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

[1] Bromwich (1908) page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31

[2] Kubrusly (2011) page 347 ⟨Example 5.K⟩

✎PROOF:

$$\begin{aligned}
[x \star y](n) &\triangleq \sum_{m\in\mathbb{Z}} x_m y_{n-m} && \text{by Definition 5.3 page 27}\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{where } k \triangleq n-m \implies m = n-k\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{by \textit{commutativity} of addition}\\
&= \sum_{m\in\mathbb{Z}} x_{n-m} y_m && \text{by change of variables}\\
&= \sum_{m\in\mathbb{Z}} y_m x_{n-m} && \text{by commutative property of the field over } \mathbb{C}\\
&\triangleq (y \star x)_n && \text{by Definition 5.3 page 27}
\end{aligned}$$

✑

Proposition 5.2. *Let $\star$ be the* CONVOLUTION OPERATOR (Definition 5.3 page 27). *Let $\ell^2_{\mathbb{R}}$ be the set of* ABSOLUTELY SUMMABLE *sequences* (Definition 5.2 page 27).

PRP
$$\left\{\begin{array}{lllll} \text{(A).} & \mathsf{x}(n) & \in & \ell^2_{\mathbb{R}} & \text{and}\\ \text{(B).} & \mathsf{y}(n) & \in & \ell^2_{\mathbb{R}} & \end{array}\right\} \implies \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] = \mathsf{x}[-n] \star \mathsf{y}(n) \right\}$$

✎PROOF:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \implies k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by \textit{absolutely summable} hypothesis} && \text{(Definition 5.2 page 27)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \implies \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of \textit{convolution} } \star && \text{(Definition 5.3 page 27)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n] &&
\end{aligned}$$

✑

5.2 Z-transform

Definition 5.4. [3]

DEF
*The **z-transform** $\mathbf{Z}$ of $(x_n)_{n\in\mathbb{Z}}$ is defined as*

$$[\mathbf{Z}(x_n)](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\text{Laurent series}} \qquad \forall (x_n)\in\ell^2_{\mathbb{R}}$$

Theorem 5.1. *Let $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ be the* Z-TRANSFORM *of* $\mathsf{x}[n]$.

THM
$$\left\{ \check{\mathsf{x}}(z) \triangleq \mathbf{Z}(\mathsf{x}[n]) \right\} \implies \left\{\begin{array}{lrcll}
\text{(1).} & \mathbf{Z}(\alpha\mathsf{x}[n]) &=& \alpha\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(2).} & \mathbf{Z}(\mathsf{x}[n-k]) &=& z^{-k}\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(3).} & \mathbf{Z}(\mathsf{x}[-n]) &=& \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(4).} & \mathbf{Z}(\mathsf{x}^*[n]) &=& \check{\mathsf{x}}^*(z^*) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(5).} & \mathbf{Z}(\mathsf{x}^*[-n]) &=& \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}}
\end{array}\right\}$$

[3] *Laurent series*: Abramovich and Aliprantis (2002) page 49

✎PROOF:

$$\begin{aligned}
\alpha\mathbb{Z}\breve{\mathsf{x}}(z) &\triangleq \alpha\mathbf{Z}\left(\!\left(\mathsf{x}[n]\right)\!\right) && \text{by definition of } \breve{\mathsf{x}}(z)\\
&\triangleq \alpha\sum_{n\in\mathbb{Z}}\mathsf{x}[n]z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator}\\
&\triangleq \sum_{n\in\mathbb{Z}}(\alpha\mathsf{x}[n])z^{-n} && \text{by \textit{distributive} property}\\
&\triangleq \mathbf{Z}\left(\!\left(\alpha\mathsf{x}[n]\right)\!\right) && \text{by definition of } \mathbf{Z} \text{ operator}\\
z^{-k}\breve{\mathsf{x}}(z) &= z^{-k}\mathbf{Z}\left(\!\left(\mathsf{x}[n]\right)\!\right) && \text{by definition of } \breve{\mathsf{x}}(z) && \text{(left hypothesis)}\\
&\triangleq z^{-k}\sum_{n=-\infty}^{n=+\infty}\mathsf{x}[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
&= \sum_{n=-\infty}^{n=+\infty}\mathsf{x}[n]z^{-n-k}\\
&= \sum_{m-k=-\infty}^{m-k=+\infty}\mathsf{x}[m-k]z^{-m} && \text{where } m \triangleq n+k && \implies n=m-k\\
&= \sum_{m=-\infty}^{m=+\infty}\mathsf{x}[m-k]z^{-m}\\
&= \sum_{n=-\infty}^{n=+\infty}\mathsf{x}[n-k]z^{-n} && \text{where } n \triangleq m\\
&\triangleq \mathbf{Z}\left(\!\left(\mathsf{x}[n-k]\right)\!\right) && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
\mathbf{Z}\left(\!\left(\mathsf{x}^*[n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}}\mathsf{x}^*[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
&\triangleq \left(\sum_{n\in\mathbb{Z}}\mathsf{x}[n](z^*)^{-n}\right)^* && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
&\triangleq \breve{\mathsf{x}}^*(z^*) && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
\mathbf{Z}\left(\!\left(\mathsf{x}[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}}\mathsf{x}[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
&= \sum_{-m\in\mathbb{Z}}\mathsf{x}[m]z^{m} && \text{where } m \triangleq -n && \implies n=-m\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition 5.2 page 27)}\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition 5.2 page 27)}\\
&\triangleq \breve{\mathsf{x}}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
\mathbf{Z}\left(\!\left(\mathsf{x}^*[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}}\mathsf{x}^*[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
&= \sum_{-m\in\mathbb{Z}}\mathsf{x}^*[m]z^{m} && \text{where } m \triangleq -n && \implies n=-m\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}^*[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition 5.2 page 27)}\\
&= \sum_{m\in\mathbb{Z}}\mathsf{x}^*[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition 5.2 page 27)}\\
&= \left(\sum_{m\in\mathbb{Z}}\mathsf{x}[m]\left(\frac{1}{z^*}\right)^{-m}\right)^* && \text{by \textit{absolutely summable} property} && \text{(Definition 5.2 page 27)}
\end{aligned}$$

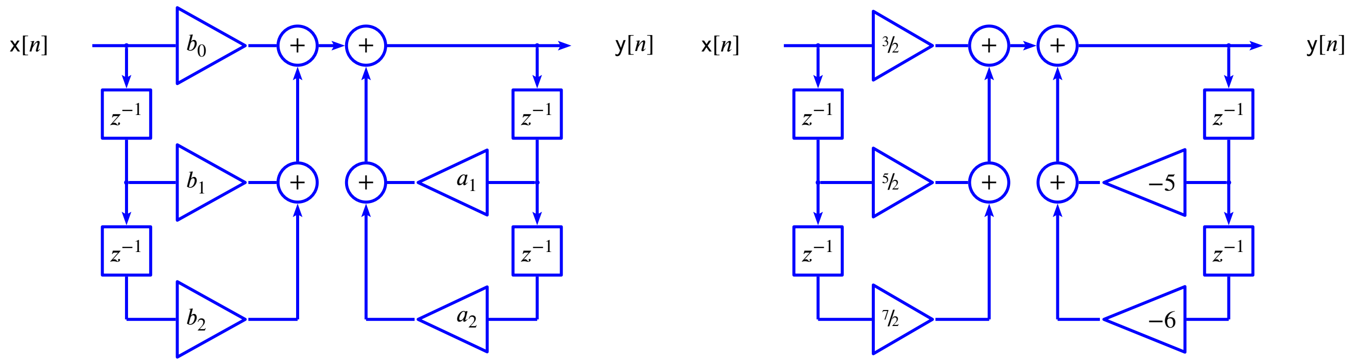


Figure 5.1: Direct form 1 order 2 IIR filters

$$\triangleq \check{x}^*\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition 5.4 page 28)}$$

✏

Theorem 5.2 (convolution theorem). *Let* $\star$ *be the convolution operator* (Definition 5.3 page 27).

THM

$$\underbrace{\mathbf{Z}\left(\left(\!\left(x_n\right)\!\right) \star \left(\!\left(y_n\right)\!\right)\right)}_{\textit{sequence convolution}} = \underbrace{\left(\mathbf{Z}\left(\!\left(x_n\right)\!\right)\right)\left(\mathbf{Z}\left(\!\left(y_n\right)\!\right)\right)}_{\textit{series multiplication}} \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}, \left(\!\left(y_n\right)\!\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x \star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by definition of } \star && \text{(Definition 5.3 page 27)}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n} &&\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k \triangleq n-m && \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right] &&&&\\
&\triangleq \left[\mathbf{Z}\left(\!\left(x_n\right)\!\right)\right]\left[\mathbf{Z}\left(\!\left(y_n\right)\!\right)\right] && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)}
\end{aligned}$$

✏

5.3 From z-domain back to time-domain

$$\check{y}(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) - a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Example 5.1. See Figure 5.1 (page 30)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2\left(z^2+5z+6\right)} = \frac{\left(\tfrac{3}{2}z^2+\tfrac{5}{2}z+\tfrac{7}{2}\right)}{z^2+5z+6} = \frac{\left(\tfrac{3}{2}+\tfrac{5}{2}z^{-1}+\tfrac{7}{2}z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

5.4 Zero locations

The system property of *minimum phase* is defined in Definition 5.5 (next) and illustrated in Figure 5.2 (page 31).

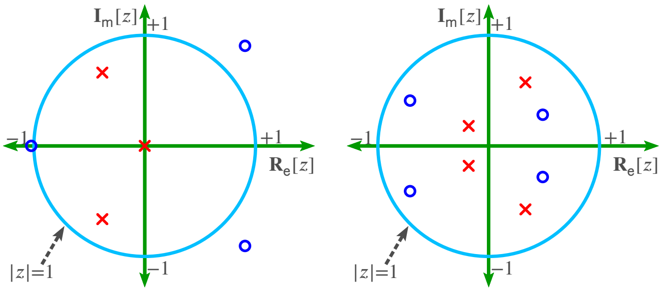


Figure 5.2: Minimum Phase filter

Definition 5.5. [4] *Let* $\check{x}(z) \triangleq \mathbf{Z}\,(\!(x_n)\!)$ *be the* Z TRANSFORM (Definition 5.4 page 28) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $(\!(z_n)\!)_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\check{x}(z)$.

DEF *The sequence* $(\!(x_n)\!)$ *is* ***minimum phase*** *if*

$$\underbrace{|z_n| < 1 \qquad \forall n \in \mathbb{Z}}_{\check{x}(z) \text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem 5.3 (Robinson's Energy Delay Theorem). [5] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM

$$\left\{\begin{array}{ll} \mathsf{p} & \text{is MINIMUM PHASE} \quad \text{and} \\ \mathsf{q} & \text{is NOT } \textit{minimum phase} \end{array}\right\} \implies \underbrace{\sum_{n=0}^{m-1} |a_n|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{p}(z)} \geq \underbrace{\sum_{n=0}^{m-1} |b_n|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{q}(z)} \qquad \forall 0 \leq m \leq N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle (Figure 5.3 page 32).

Example 5.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure 5.3 (page 32).

[4] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36
[5] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976) pages 52–53

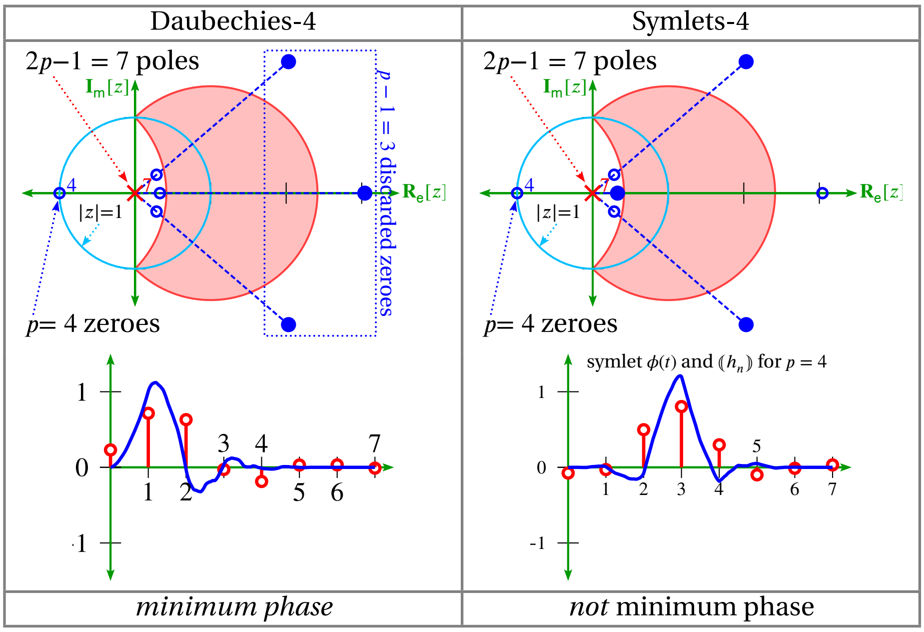


Figure 5.3: Daubechies-4 and Symlet-4 scaling functions pole-zero plots

5.5 Pole locations

Definition 5.6.

DEF	*A filter (or system or operator)* **H** *is* ***causal*** *if its current output does not depend on future inputs.*

Definition 5.7.

DEF	*A filter (or system or operator)* **H** *is* ***time-invariant*** *if the mapping it performs does not change with time.*

Definition 5.8.

DEF	*An operation* **H** *is* ***linear*** *if any output* y_n *can be described as a linear combination of inputs* x_n *as in* $\mathsf{y}_n = \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\mathsf{x}(n-m)$.

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem 5.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Example 5.3. Stable/unstable filters are illustrated in Figure 5.4 (page 33).

True or False? This filter has no poles:

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$

$$\mathsf{H}(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$

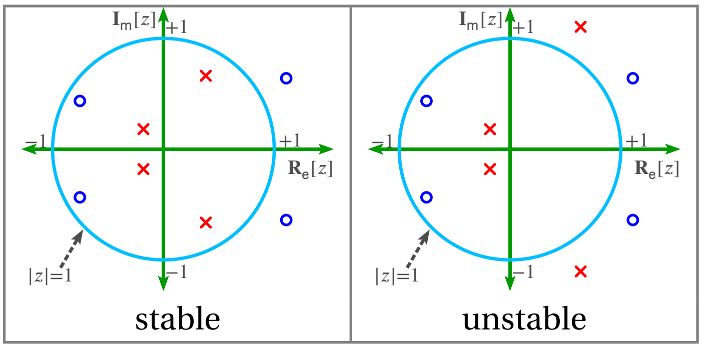


Figure 5.4: Pole-zero plot stable/unstable causal LTI filters (Example 5.3 page 32)

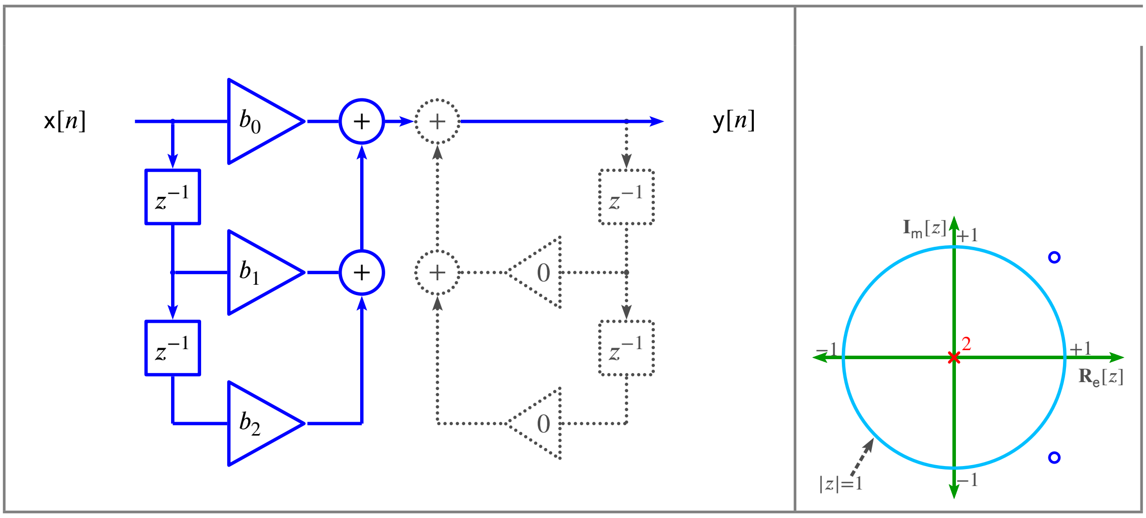


Figure 5.5: FIR filters

5.6 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

Proposition 5.3.

PRP	$\left\{\begin{array}{l}\text{ZEROS } \textit{and} \text{ POLES}\\ \textit{occur in} \text{ CONJUGATE PAIRS}\end{array}\right\} \implies \left\{\begin{array}{l}\text{COEFFICIENTS}\\ \textit{are} \text{ REAL.}\end{array}\right\}$

✎PROOF:

$$\begin{aligned}
\left(z-p_1\right)\left(z-p_1^*\right) &= [z-(a+ib)][z-(a-ib)]\\
&= z^2+[-a+ib-ib-a]z-[ib]^2
\end{aligned}$$

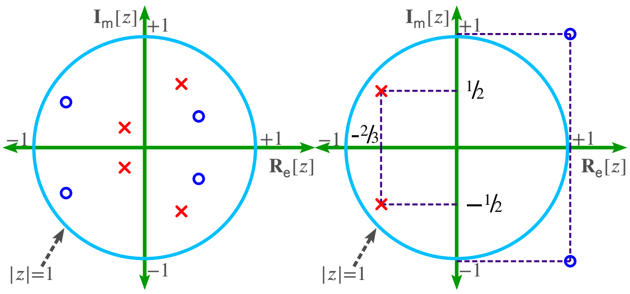


Figure 5.6: Conjugate pair structure yielding real coefficients

$$= z^2 - 2az + b^2$$

✏

Example 5.4. See Figure 5.6 (page 33).

$$\begin{aligned}
H(z) &= G\frac{[z-z_1][z-z_2]}{[z-p_1][z-p_2]} = G\frac{[z-(1+i)][z-(1-i)]}{[z-(-2/3+i1/2)][z-(-2/3-i1/2)]} \\
&= G\frac{z^2 - z[(1-i)+(1+i)] + (1-i)(1+i)}{z^2 - z[(-2/3+i1/2)+(-2/3+i1/2)] + (-2/3+i1/2)(-2/3+i1/2)} \\
&= G\frac{z^2-2z+2}{z^2 - 4/3z + (4/3+1/4)} = G\frac{z^2-2z+2}{z^2-4/3z+19/12}
\end{aligned}$$

5.7 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma 5.1. *A causal LTI operator* **H** *can be expressed as a rational expression* $\check{h}(z)$.

$$\begin{aligned}
\check{h}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \frac{\sum_{n=0}^{N} b_n z^{-n}}{1 + \sum_{n=1}^{N} a_n z^{-n}}
\end{aligned}$$

A filter operation $\check{h}(z)$ can be expressed as a product of its roots (poles and zeros).

$$\begin{aligned}
\check{h}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \alpha\frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}
\end{aligned}$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with $\times$ and zeros with $\bigcirc$. An example is shown in Figure 5.7 page 35. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

5.8 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{x}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{y}(z)$ is a *high-pass filter*.

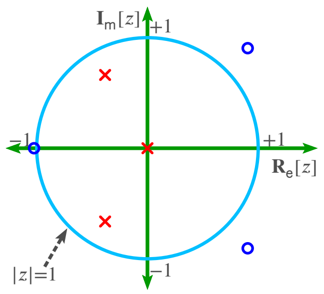


Figure 5.7: Pole-zero plot for rational expression with real coefficients

Definition 5.9. [6] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES (Definition 5.1 page 27) *in* $\ell^2_{\mathbb{R}}$ (Definition 5.2 page 27).

DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter***.
Any triple $\left((\!(x_n)\!), (\!(y_n)\!), N\right)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition***.

Theorem 5.5 (CQF theorem). [7] *Let* $\breve{y}(\omega)$ *and* $\breve{x}(\omega)$ *be the* DTFT*s* (Definition 6.1 page 41) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$, *respectively, in* $\ell^2_{\mathbb{R}}$ (Definition 5.2 page 27).

THM

$$\begin{array}{llcll}
\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{\text{(1) CQF in "time"}} & \iff \breve{y}(z) & = & \pm(-1)^N z^{-N}\breve{x}^*\left(\frac{-1}{z^*}\right) & \text{(2) CQF in "z-domain"}\\
 & \iff \breve{y}(\omega) & = & \pm(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi) & \text{(3) CQF in "frequency"}\\
 & \iff x_n & = & \pm(-1)^N(-1)^n y^*_{N-n} & \text{(4) "reversed" CQF in "time"}\\
 & \iff \breve{x}(z) & = & \pm z^{-N}\breve{y}^*\left(\frac{-1}{z^*}\right) & \text{(5) "reversed" CQF in "z-domain"}\\
 & \iff \breve{x}(\omega) & = & \pm e^{-i\omega N}\breve{y}^*(\omega+\pi) & \text{(6) "reversed" CQF in "frequency"}
\end{array}$$

$\forall N\in\mathbb{Z}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{y}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-transform} \quad \text{(Definition 5.4 page 28)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm\sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies \quad n = N-m\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N}\left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*
\end{aligned}$$

[6] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

[7] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

$$= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) \qquad \text{by definition of } z\text{-transform} \quad \text{(Definition 5.4 page 28)}$$

2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\check{y}(z) &= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by (2)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-transform (Definition 5.4 page 28)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m^* \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-transform (Definition 5.4 page 28)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x_m^* z^{-(N-m)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^n x_{N-n}^* z^{-n} && \text{where } n = N-m \implies m \triangleq N-n\\
\implies x_n &= \pm(-1)^n x_{N-n}^*
\end{aligned}$$

3. Proof that (1) $\implies$ (3):

$$\begin{aligned}
\check{y}(\omega) &\triangleq \check{x}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition 6.1 page 41)}\\
&= \left[\pm(-1)^N z^{-N} \check{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)}\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i\pi} e^{i\omega}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i(\omega+\pi)}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition 6.1 page 41)}
\end{aligned}$$

4. Proof that (1) $\implies$ (6):

$$\begin{aligned}
\check{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT \text{ (Definition 6.1 page 41)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x_{N-n}^*}_{CQF} e^{-i\omega n} && \text{by (1)}\\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x_m^* e^{-i\omega(N-m)} && \text{where } m \triangleq N-n \implies n = N-m\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} (-1)^m x_m^* e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} e^{i\pi m} x_m^* e^{i\omega m}\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} x_m^* e^{i(\omega+\pi)m}\\
&= \pm(-1)^N e^{-i\omega N} \left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^*\\
&= \pm(-1)^N e^{-i\omega N} \check{x}^*(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition 6.1 page 41)}
\end{aligned}$$

5. Proof that (1) $\Longleftarrow$ (3):

$$\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{y}(\omega)e^{i\omega n}\,\mathrm{d}\omega && \text{by inverse DTFT} && \text{(Theorem 6.3 page 47)}\\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\breve{x}^*(\omega+\pi)}_{\text{right hypothesis}}e^{i\omega n}\,\mathrm{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{x}^*(\omega+\pi)e^{i\omega(n-N)}\,\mathrm{d}\omega && \text{by right hypothesis}\\
&= \pm(-1)^N\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{i(v-\pi)(n-N)}\,\mathrm{d}v && \text{where } v \triangleq \omega+\pi \implies && \omega = v-\pi\\
&= \pm(-1)^N e^{-i\pi(n-N)}\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{iv(n-N)}\,\mathrm{d}v\\
&= \pm(-1)^N\underbrace{(-1)^N}_{e^{i\pi N}}\underbrace{(-1)^n}_{e^{-i\pi n}}\left[\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}(v)e^{iv(N-n)}\,\mathrm{d}v\right]^*\\
&= \pm(-1)^n x^*_{N-n} && \text{by inverse DTFT} && \text{(Theorem 6.3 page 47)}
\end{aligned}$$

6. Proof that (1) $\Longleftrightarrow$ (4):

$$\begin{aligned}
y_n = \pm(-1)^n x^*_{N-n} &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x^*_{N-n}\\
&\iff \pm(-1)^n y_n = x^*_{N-n}\\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x^*_{N-n}\right)^*\\
&\iff \pm(-1)^n y^*_n = x_{N-n}\\
&\iff x_{N-n} = \pm(-1)^n y^*_n\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m} && \text{where } m \triangleq N-n \implies && n = N-m\\
&\iff x_m = \pm(-1)^{N-m} y^*_{N-m}\\
&\iff x_m = \pm(-1)^N(-1)^m y^*_{N-m}\\
&\iff x_n = \pm(-1)^N(-1)^n y^*_{N-n} && \text{by change of free variables}
\end{aligned}$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

✏

Theorem 5.6. [8] *Let $\breve{y}(\omega)$ and $\breve{x}(\omega)$ be the* DTFT*s (Definition 6.1 page 41) of the sequences $(\!(y_n)\!)_{n\in\mathbb{Z}}$ and $(\!(x_n)\!)_{n\in\mathbb{Z}}$, respectively, in $\ell^2_{\mathbb{R}}$ (Definition 5.2 page 27).*

THM

Let $y_n = \pm(-1)^n x^*_{N-n}$ (CQF CONDITION, Definition 5.9 page 35). Then

$$\left\{\begin{array}{lrcrcl}
\text{(A)} & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} = 0 & \iff & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} & = 0 & \text{(B)}\\
& & \iff & \displaystyle\sum_{k\in\mathbb{Z}}(-1)^k k^n x_k & = 0 & \text{(C)}\\
& & \iff & \displaystyle\sum_{k\in\mathbb{Z}} k^n y_k & = 0 & \text{(D)}
\end{array}\right\} \quad \forall n\in\mathbb{W}$$

✎PROOF:

[8] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

1. Proof that (A) $\implies$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{y}}(\omega)\bigg|_{\omega=0} && \text{by (A)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\breve{\mathsf{x}}^*(\omega+\pi)\bigg|_{\omega=0} && \text{by } \textit{CQF theorem} \quad \text{(Theorem 5.5 page 35)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n}\binom{n}{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^\ell \left[e^{-i\omega N}\right]\cdot\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{\mathsf{x}}^*(\omega+\pi)\right]\bigg|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma B.2 page 71)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n}\binom{n}{\ell}-iN^\ell e^{-i\omega N}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{\mathsf{x}}^*(\omega+\pi)\right]\bigg|_{\omega=0}\\
&= (\pm)(-1)^N e^{-i0N}\sum_{\ell=0}^{n}\binom{n}{\ell}-iN^\ell\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{\mathsf{x}}^*(\omega+\pi)\right]\bigg|_{\omega=0}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{\mathsf{x}}^{(0)}(\pi) &= 0\\
&\implies & \breve{\mathsf{x}}^{(1)}(\pi) &= 0\\
&\implies & \breve{\mathsf{x}}^{(2)}(\pi) &= 0\\
&\implies & \breve{\mathsf{x}}^{(3)}(\pi) &= 0\\
&\implies & \breve{\mathsf{x}}^{(4)}(\pi) &= 0\\
&\ \vdots & &\vdots\\
&\implies & \breve{\mathsf{x}}^{(n)}(\pi) &= 0 \quad \text{for } n=0,1,2,\ldots
\end{aligned}$$

2. Proof that (A) $\impliedby$ (B):

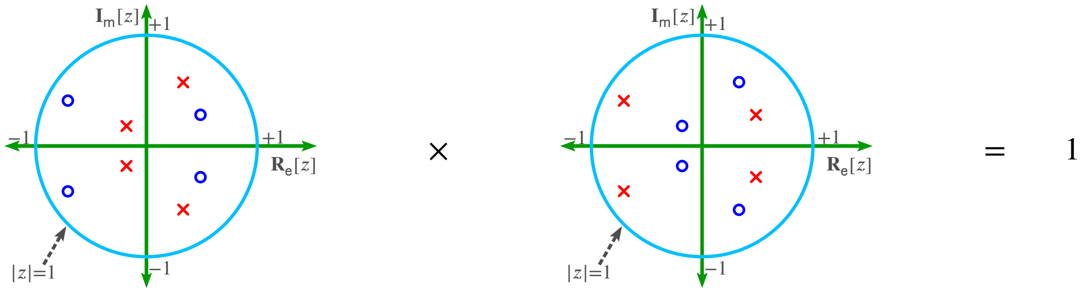
$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=\pi} && \text{by (B)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n (\pm)e^{-i\omega N}\breve{\mathsf{y}}^*(\omega+\pi)\bigg|_{\omega=\pi} && \text{by } \textit{CQF theorem} \quad \text{(Theorem 5.5 page 35)}\\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^\ell \left[e^{-i\omega N}\right]\cdot\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{\mathsf{y}}^*(\omega+\pi)\right]\bigg|_{\omega=\pi} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma B.2 page 71)}\\
&= (\pm)\sum_{\ell=0}^{n}\binom{n}{\ell}(-iN)^\ell e^{-i\omega N}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{\mathsf{y}}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}\\
&= (\pm)e^{-i\pi N}\sum_{\ell=0}^{n}\binom{n}{\ell}-iN^\ell\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{\mathsf{y}}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}\\
&= (\pm)(-1)^N\sum_{\ell=0}^{n}\binom{n}{\ell}-iN^\ell\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^{n-\ell}\left[\breve{\mathsf{y}}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{\mathsf{y}}^{(0)}(0) &= 0\\
&\implies & \breve{\mathsf{y}}^{(1)}(0) &= 0\\
&\implies & \breve{\mathsf{y}}^{(2)}(0) &= 0\\
&\implies & \breve{\mathsf{y}}^{(3)}(0) &= 0\\
&\implies & \breve{\mathsf{y}}^{(4)}(0) &= 0\\
&\ \vdots & &\vdots\\
&\implies & \breve{\mathsf{y}}^{(n)}(0) &= 0\\
&\implies & \breve{\mathsf{y}}^{(n)}(0) &= 0 \quad \text{for } n=0,1,2,\ldots
\end{aligned}$$

3. Proof that (B) $\iff$ (C): by Theorem 6.5 page 49

4. Proof that (A) $\iff$ (D): by Theorem 6.5 page 49

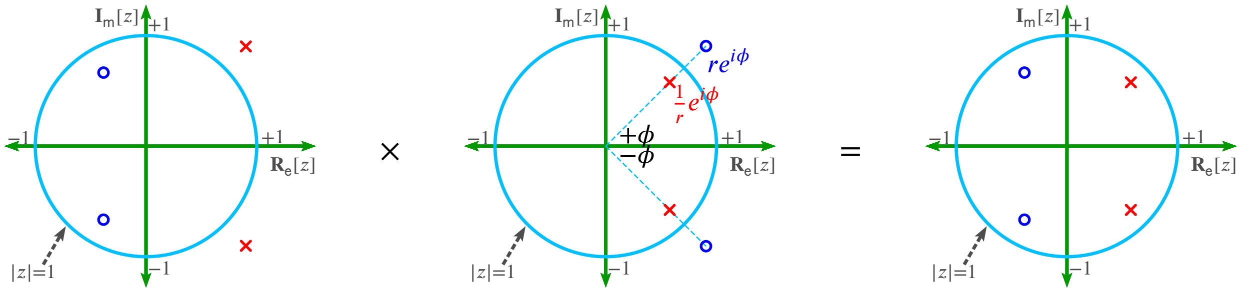
5. Proof that (CQF) $\not\Leftarrow$ (A): Here is a counterexample: $\breve{\mathsf{y}}(\omega)=0$.



$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

5.9 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.



unstable · all-pass · stable!

$$\begin{aligned}
|A(z)|_{z=e^{i\omega}} &= \frac{1}{r}\left|\frac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|e^{i\phi}\left(\frac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|-z\left(\frac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\frac{1}{e^{-iv}}\left(\frac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right| \\
&= 1
\end{aligned}$$

$$\begin{aligned}
&= \left|\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|z\left(\frac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\frac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
&= \left|\frac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right|
\end{aligned}$$

CHAPTER 6

DISCRETE TIME FOURIER TRANSFORM

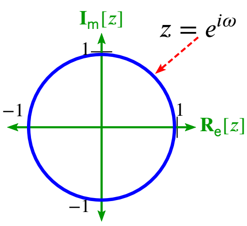
6.1 Definition

Definition 6.1.

DEF *The* ***discrete-time Fourier transform*** $\breve{\mathbf{F}}$ *of* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *is defined as*

$$\left[\breve{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition 6.1 page 41) to the definition of the Z-transform (Definition 5.4 page 28), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.



6.2 Properties

Proposition 6.1 (DTFT periodicity). *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 41) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP
$$\underbrace{\breve{x}(\omega) = \breve{x}(\omega + 2\pi n)}_{\text{PERIODIC } \textit{with period } 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\breve{x}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} & &= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi nm}} \\
&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} & &= \breve{x}(\omega)
\end{aligned}$$

✏

Theorem 6.1. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 41) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM
$$\left\{ \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x[n])\!) \right\} \implies \left\{ \begin{array}{llcll} (1). & \breve{\mathbf{F}}(\!(x[-n])\!) & = & \tilde{x}(-\omega) & \text{and} \\ (2). & \breve{\mathbf{F}}(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m} \\
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x^*[n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{i\omega n}\right)^* && \text{by \textit{distributive} property of } *\text{-\textbf{algebras}} && \text{(Definition E.3 page 114)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{-i(-\omega)n}\right)^* \\
&\triangleq \tilde{x}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x^*[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
&= \left(\sum_{n\in\mathbb{Z}} x[-n]e^{i\omega n}\right)^* && \text{by \textit{distributive} property of } *\text{-\textbf{algebras}} && \text{(Definition E.3 page 114)} \\
&= \left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n = -m \\
&\triangleq \tilde{x}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

⇰

Theorem 6.2. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 41) of a sequence* $(\!(x[n])\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM
$$\left\{ \begin{array}{lll} (1). & \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x[n])\!) & \text{and} \\ (2). & (\!(x[n])\!) \text{ is REAL-VALUED} & \end{array} \right\} \implies \left\{ \begin{array}{llclcll} (1). & \breve{\mathbf{F}}(\!(x[-n])\!) & = & \tilde{x}(-\omega) & & & \text{and} \\ (2). & \breve{\mathbf{F}}(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & = & \tilde{x}(\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & = & \tilde{x}(-\omega) & \end{array} \right\}$$

✎PROOF:

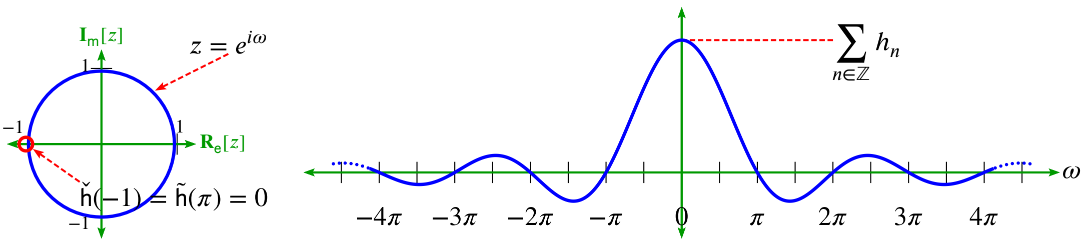
$$\begin{aligned}
\breve{\mathbf{F}}(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}
\end{aligned}$$

$$\triangleq \tilde{\mathsf{x}}(-\omega) \qquad \text{by left hypothesis}$$

$$\begin{aligned}
\boxed{\tilde{\mathsf{x}}^*(-\omega)} &= \boxed{\breve{\mathbf{F}}\,\langle\!\langle \mathsf{x}^*[n] \rangle\!\rangle} && \text{by Theorem 6.1 page 42}\\
&= \breve{\mathbf{F}}\,\langle\!\langle \mathsf{x}[n] \rangle\!\rangle && \text{by \textit{real-valued} hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(\omega)} && \text{by definition of } \tilde{\mathsf{x}}(\omega) \quad \text{(Definition 6.1 page 41)}
\end{aligned}$$

$$\begin{aligned}
\boxed{\tilde{\mathsf{x}}^*(\omega)} &= \boxed{\breve{\mathbf{F}}\,\langle\!\langle \mathsf{x}^*[-n] \rangle\!\rangle} && \text{by Theorem 6.1 page 42}\\
&= \breve{\mathbf{F}}\,\langle\!\langle \mathsf{x}[-n] \rangle\!\rangle && \text{by \textit{real-valued} hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(-\omega)} && \text{by result (1)}
\end{aligned}$$

✏



Proposition 6.2. *Let* $\breve{\mathsf{x}}(z)$ *be the* Z-TRANSFORM *(Definition 5.4 page 28) and* $\breve{\mathsf{x}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition 6.1 page 41) of* $\langle\!\langle x_n \rangle\!\rangle$.

PRP

$$\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall \langle\!\langle x_n \rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}},\, c\in\mathbb{R}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{\mathsf{x}}(z)\Big|_{z=1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \text{by definition of } \breve{\mathsf{x}}(z) \text{ (Definition 5.4 page 28)}\\
&= \sum_{n\in\mathbb{Z}} x_n && \text{because } z^n = 1 \text{ for all } n\in\mathbb{Z}\\
&= c && \text{by hypothesis (1)}
\end{aligned}$$

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} && \text{by definition of } \breve{\mathsf{x}}(\omega) \quad \text{(Definition 6.1 page 41)}\\
&= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} &&\\
&= \breve{\mathsf{x}}(z)\Big|_{z=1} && \text{by definition of } \breve{\mathsf{x}}(z) \quad \text{(Definition 5.4 page 28)}\\
&= c && \text{by hypothesis (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=0} \\
&= \breve{\mathsf{x}}(\omega) && \text{by definition of } \breve{\mathsf{x}}(\omega) && \text{(Definition 6.1 page 41)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

Proposition 6.3. *If the coefficients are **real**, then the magnitude response (MR) is **symmetric**.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathsf{h}}(-\omega)\right| &\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{i\omega m}\right| && = \left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}^*[m]e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } real} && = \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right| \\
&\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\tilde{\mathsf{h}}(\omega)\right|
\end{aligned}$$

Proposition 6.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}}(-1)^n x_n = c}_{\text{(1) in "time"}} \iff \underbrace{\breve{\mathsf{x}}(z)|_{z=-1} = c}_{\text{(2) in "z domain"}} \iff \underbrace{\breve{\mathsf{x}}(\omega)|_{\omega=\pi} = c}_{\text{(3) in "frequency"}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{\mathsf{x}}(z)|_{z=-1} &= \left.\sum_{n\in\mathbb{Z}} x_n z^{-n}\right|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \sum_{n\in\mathbb{Z}} z^{-n} x_n \Big|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n \Big|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\implies$ (4):

   (a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

   (b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

(e) Solutions to these equations give

$$\sum_{n\in\mathbb{Z}} x_{2n} \triangleq A = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)$$

$$\sum_{n\in\mathbb{Z}} x_{2n+1} \triangleq B = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1} \\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1} \\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)} \\
&= c
\end{aligned}$$

✏

Lemma 6.1. *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* DTFT (Definition 6.1 page 41) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

LEM

$$\underbrace{(x_n \in \mathbb{R})_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{\mathsf{x}}(\omega)|^2 = |\breve{\mathsf{x}}(-\omega)|^2}_{\text{EVEN}} \qquad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
|\breve{\mathsf{x}}(\omega)|^2 &= |\breve{\mathsf{x}}(z)|^2\Big|_{z=e^{i\omega}} \\
&= \breve{\mathsf{x}}(z)\breve{\mathsf{x}}^*(z)\Big|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^* (z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right] \\
&= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2 \sum_{n\in\mathbb{Z}} \sum_{m>n} x_n x_m \cos[\omega(m-n)]
\end{aligned}$$

Since cos is real and even, then $|\breve{x}(\omega)|^2$ must also be real and even. ✏

Theorem 6.3 (inverse DTFT). [2] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM (Definition 6.1 page 41) *of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\tilde{x}^{-1}$ be the inverse of $\breve{x}$.*

THM

$$\underbrace{\left\{ \breve{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\breve{x}(\omega) \triangleq \breve{\mathbf{F}}(x_n)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi} \int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega) e^{i\omega n}\, d\omega \quad \forall\alpha\in\mathbb{R} \right\}}_{(x_n) = \breve{\mathbf{F}}^{-1}\breve{\mathbf{F}}(x_n)} \quad \forall (x_n)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega) e^{i\omega n}\, d\omega &= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \right]}_{\breve{x}(\omega)} e^{i\omega n}\, d\omega && \text{by definition of } \breve{x}(\omega) \\
&= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\, d\omega \\
&= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\, d\omega \\
&= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} x_m \left[2\pi \bar{\delta}_{m-n} \right] \\
&= x_n
\end{aligned}$$

✏

Theorem 6.4 (orthonormal quadrature conditions). [3] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM (Definition 6.1 page 41) *of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\bar{\delta}_n$ be the* KRONECKER DELTA FUNCTION *at n* (Definition G.12 page 160).

THM

$$\begin{array}{rclcrcll}
\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} &=& 0 & \iff & \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) &=& 0 & \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}} \\
\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} &=& \bar{\delta}_n & \iff & |\breve{x}(\omega)|^2 + |\breve{x}(\omega+\pi)|^2 &=& 2 & \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}}
\end{array}$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩
[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

1. Proof that $2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)$:

$$\begin{aligned}
&2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} \\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n} \\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n}\left(1+e^{i\pi n}\right) \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n} \\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n \\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_m z^{m} + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_m e^{+i(\omega+\pi)m} \\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^* \\
&\triangleq \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)
\end{aligned}$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 0 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n} \\
&= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis} \\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)} \\
&= 2 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

✏

6.3 Derivatives

Theorem 6.5. [4] *Let* $\breve{\mathsf{x}}(\omega)$ *be the* DTFT (Definition 6.1 page 41) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

THM

$$\text{(A)}\quad \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=0} = 0 \iff \sum_{k\in\mathbb{Z}} k^n x_k = 0 \quad \text{(B)} \quad \forall n\in\mathbb{W}$$

$$\text{(C)}\quad \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=\pi} = 0 \iff \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 \quad \text{(D)} \quad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=0} && \text{by hypothesis (A)}\\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \breve{\mathsf{x}}(\omega) \text{ (Definition 6.1 page 41)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=0} &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \tilde{g}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k\\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

[4] Vidakovic (1999) pages 82–83, Mallat (1999) pages 241–242

3. Proof that $(C) \implies (D)$:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{\mathrm{x}}(\omega)\bigg|_{\omega=\pi} && \text{by hypothesis (C)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=\pi} && \text{by definition of } \check{\mathrm{x}} \text{ (Definition 6.1 page 41)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k
\end{aligned}$$

4. Proof that $(C) \impliedby (D)$:

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{\mathrm{x}}(\omega)\bigg|_{\omega=\pi} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=\pi} && \text{by definition of } \check{\mathrm{x}} \text{ (Definition 6.1 page 41)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k\\
&= 0 && \text{by hypothesis (D)}
\end{aligned}$$

✎

CHAPTER 7

FOURIER SERIES

> ❝ ...et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur.❞


MÉMOIRE
PROPAGATION DE LA CHALEUR
DANS LES CORPS SOLIDES.

> ❝ ...and the innovation of the subject, together with its importance, convinced the committee to crown this work. By observing however that the way in which the author arrives at his equations is not free from difficulties, and the analysis of which, to integrate them, still leaves something to be desired, either relative to generality, or even on the side of rigour.❞

A competition awards committee consisting of the mathematical giants Lagrange, Laplace, Legendre, and others, commenting on Fourier's 1807 landmark paper *Dissertation on the propagation of heat in solid bodies* that introduced the *Fourier Series*. [1]

7.1 Definition

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition 7.1. [2]

DEF *The **Fourier Series operator** $\hat{\mathbf{F}} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{R}}$ is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{f}(x) e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{d}x \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{f} \text{ is periodic with period } \tau\right\}$$

[1] quote: Lagrange et al. (1812b) page 374, Lagrange et al. (1812a) page 112, Kahane (2008) page 199
translation: assisted by Google Translate, Castanedo (2005) ⟨chapter 2 footnote 5⟩
paper: Fourier (1807)

[2] Katznelson (2004) page 3

7.2 Inverse Fourier Series operator

Theorem 7.1. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

THM *The **inverse Fourier Series** operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{x}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}} \tilde{x}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall\left(\!\left(\tilde{x}_n\right)\!\right)\in\ell^2_{\mathbb{R}}$$

✎PROOF: The proof of the pointwise convergence of the Fourier Series is notoriously difficult. It was conjectured in 1913 by Nokolai Luzin that the Fourier Series for all square summable periodic functions are pointwise convergent: Luzin (1913)

Fifty-three years later (1966) at a conference in Moscow, Lennart Axel Edvard Carleson presented one of the most spectacular results ever in mathematics; he demonstrated that the Luzin conjecture is indeed correct. Carleson formally published his result that same year: Carleson (1966)

Carleson's proof is expounded upon in Reyna's (2002) 175 page book: de Reyna (2002)

Interestingly enough, Carleson started out trying to disprove Luzin's conjecture. Carleson said this in an interview published in 2001:[3] “*Well, the problem of course presents itself already when you are a student and I was thinking of the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I suddenly realized why there should be a counter-example and how you should produce it. I thought I really understood what was the background and then to my amazement I could prove that this "correct" counter-example couldn't exist and therefore I suddenly realized that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. The most important aspect in solving a mathematical problem is the conviction of what is the true result! Then it took like 2 or 3 years using the technique that had been developed during the past 20 years or so. It is actually a problem related to analytic functions basically even though it doesn't look that way.*”

For now, if you just want some intuitive justification for the Fourier Series, and you can somehow imagine that the Dirichlet kernel generates a *comb function* of *Dirac delta* functions, then perhaps what follows may help (or not). It is certainly not mathematically rigorous and is by no means a real proof (but at least it is less than 175 pages).

$$\begin{aligned}
\left[\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathrm{x}\right](x) &= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathrm{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}\mathrm{x}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 7.1 page 51)}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathrm{x}(u)e^{-i\frac{2\pi}{\tau}nu}\,\mathrm{d}u\right]e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \hat{\mathbf{F}}^{-1}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathrm{x}(u)e^{-i\frac{2\pi}{\tau}nu}e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathrm{x}(u)e^{i\frac{2\pi}{\tau}n(x-u)}\,\mathrm{d}u
\end{aligned}$$

[3] Carleson and Engquist (2001)

$$
\begin{aligned}
&= \int_0^\tau \mathsf{x}(u)\underbrace{\frac{1}{\tau}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi}{\tau}n(x-u)}}_{\lim_{N\to\infty}\mathsf{D}_n(x)}\,\mathsf{d}u \\
&= \int_0^\tau \mathsf{x}(u)\left[\sum_{n\in\mathbb{Z}}\delta(x-u-n\tau)\right]\mathsf{d}u \\
&= \sum_{n\in\mathbb{Z}}\int_{u=0}^{u=\tau}\mathsf{x}(u)\delta(x-u-n\tau)\,\mathsf{d}u \\
&= \sum_{n\in\mathbb{Z}}\int_{v-n\tau=0}^{v-n\tau=\tau}\mathsf{x}(v-n\tau)\delta(x-v)\,\mathsf{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}}\int_{v=n\tau}^{v=(n+1)\tau}\mathsf{x}(v-n\tau)\delta(x-v)\,\mathsf{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}}\int_{v=n\tau}^{v=(n+1)\tau}\mathsf{x}(v)\delta(x-v)\,\mathsf{d}v && \text{because } \mathsf{x} \text{ is periodic with period } \tau \\
&= \int_{\mathbb{R}}\mathsf{x}(v)\delta(x-v)\,\mathsf{d}v \\
&= \mathsf{x}(x) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} && \text{(Definition F.3 page 120)}
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](n) &= \hat{\mathbf{F}}\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right]e^{-i\frac{2\pi}{\tau}nx}\,\mathsf{d}x && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 7.1 page 51)} \\
&= \frac{1}{\tau}\int_0^\tau\left[\sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}(k-n)x}\right]\mathsf{d}x \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\left[\frac{1}{\tau}\int_0^\tau e^{i\frac{2\pi}{\tau}(k-n)x}\,\mathsf{d}x\right] \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\frac{1}{\tau}\left[\frac{1}{i\frac{2\pi}{\tau}(k-n)}e^{i\frac{2\pi}{\tau}(k-n)x}\right]_0^\tau \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\frac{1}{i2\pi(k-n)}\left[e^{i2\pi(k-n)}-1\right] \\
&= \sum_{k\in\mathbb{Z}}\tilde{\mathsf{x}}(k)\,\bar{\delta}(k-n)\,\lim_{x\to 0}\left[\frac{e^{i2\pi x}-1}{i2\pi x}\right] \\
&= \tilde{\mathsf{x}}(n)\left.\frac{\frac{\mathsf{d}}{\mathsf{d}x}\left(e^{i2\pi x}-1\right)}{\frac{\mathsf{d}}{\mathsf{d}x}(i2\pi x)}\right|_{x=0} && \text{by \textit{l'Hôpital's rule}} \\
&= \tilde{\mathsf{x}}(n)\left.\frac{i2\pi e^{i2\pi x}}{i2\pi}\right|_{x=0} \\
&= \tilde{\mathsf{x}}(n) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} && \text{(Definition F.3 page 120)}
\end{aligned}
$$

⇰

Theorem 7.2.

THM *The* ***Fourier Series adjoint*** *operator* $\hat{\mathbf{F}}^*$ *is given by*

$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathsf{x}(x) \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{x}(x) e^{-i\frac{2\pi}{\tau}nx}\, \mathrm{d}x \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} && \text{(Definition 7.1 page 51)} \\
&= \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{x}(x) \left\langle e^{-i\frac{2\pi}{\tau}nx} \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} \mathrm{d}x && \text{by additivity property of } \langle \triangle \,|\, \triangledown \rangle \\
&= \int_0^\tau \mathsf{x}(x) \frac{1}{\sqrt{\tau}} \left\langle \tilde{\mathsf{y}}(n) \,|\, e^{-i\frac{2\pi}{\tau}nx} \right\rangle_{\mathbb{Z}}^* \mathrm{d}x && \text{by property of } \langle \triangle \,|\, \triangledown \rangle \\
&= \int_0^\tau \mathsf{x}(x) \left[\hat{\mathbf{F}}^{-1} \tilde{\mathsf{y}}(n)\right]^* \mathrm{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem 7.1 page 52)} \\
&= \left\langle \mathsf{x}(x) \,|\, \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^*} \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{R}}
\end{aligned}$$

✏

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary* [4] (Corollary 7.1 page 54).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary 7.2 page 54).

Corollary 7.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\hat{\mathbf{F}}$ *be the Fourier Series operator with adjoint* $\hat{\mathbf{F}}^*$.

COR
$$\left\{ \hat{\mathbf{F}}\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^*\hat{\mathbf{F}} = \mathbf{I} \right\} \qquad \left(\hat{\mathbf{F}} \textit{ is } \textbf{\textit{unitary}} \textit{ ...and thus also } \text{NORMAL} \textit{ and } \text{ISOMETRIC} \right)$$

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$ (Theorem 7.2 page 53). ✏

Corollary 7.2. *Let* $\hat{\mathbf{F}}$ *be the Fourier series operator with adjoint* $\hat{\mathbf{F}}^*$ *and inverse* $\hat{\mathbf{F}}^{-1}$.

COR
$$\begin{array}{rclcll}
\mathcal{R}(\hat{\mathbf{F}}) &=& \mathcal{R}(\hat{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
\left|\left|\left|\hat{\mathbf{F}}\right|\right|\right| &=& \left|\left|\left|\hat{\mathbf{F}}^{-1}\right|\right|\right| &=& 1 & (\text{UNITARY}) \\
\left\langle \hat{\mathbf{F}}\boldsymbol{x} \,|\, \hat{\mathbf{F}}\boldsymbol{y} \right\rangle &=& \left\langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \,|\, \hat{\mathbf{F}}^{-1}\boldsymbol{y} \right\rangle &=& \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle & (\text{PARSEVAL'S EQUATION}) \\
\left\|\hat{\mathbf{F}}\boldsymbol{x}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\right\| &=& \|\boldsymbol{x}\| & (\text{PLANCHEREL'S FORMULA}) \\
\left\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\right\| &=& \left\|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\right\| &=& \|\boldsymbol{x} - \boldsymbol{y}\| & (\text{ISOMETRIC})
\end{array}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary 7.1 page 54) and from the properties of unitary operators (Theorem F.26 page 144). ✏

7.3 Fourier series for compactly supported functions

Theorem 7.3.

THM
The set

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i\frac{2\pi}{\tau}nx} \middle|\, n \in \mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathsf{f}(x)$ *with support in* $[0 : \tau]$.

[4] *unitary operators*: Definition F.14 page 143

CHAPTER 8

FAST WAVELET TRANSFORM (FWT)

The Fast Wavelet Transform can be computed using simple discrete filter operations (as a conjugate mirror filter).

Definition 8.1 (Wavelet Transform). *Let the wavelet transform* $\mathbf{W} : \{f : \mathbb{R} \to \mathbb{C}\} \to \{w : \mathbb{Z}^2 \to \mathbb{C}\}$ *be defined as* [1]

DEF
$$[\mathbf{W}\mathsf{f}](j,n) \triangleq \langle \mathsf{f}(x) \,|\, \psi_{k,n}(x) \rangle$$

Definition 8.2. *The following relations are defined as described below:*

DEF

scaling coefficients	$v_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$v_j(n)$	$\triangleq$	$\langle f(x) \,\|\, \phi_{j,n}(x) \rangle$
wavelet coefficients	$w_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$w_j(n)$	$\triangleq$	$\langle f(x) \,\|\, \psi_{j,n}(x) \rangle$
scaling filter coefficients	$\bar{\mathsf{h}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{h}}(n)$	$\triangleq$	$\mathsf{h}(-n)$
wavelet filter coefficients	$\bar{\mathsf{g}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{g}}(n)$	$\triangleq$	$\mathsf{g}(-n)$

The scaling and wavelet filter coefficients at scale j are equal to the filtered and downsampled (Theorem **??** page **??**) scaling filter coefficients at scale $j+1$:[2]

- The convolution (Definition 5.3 page 27) of $v_{j+1}(n)$ with $\bar{\mathsf{h}}(n)$ and then downsampling by 2 produces $v_j(n)$.
- The convolution of $v_{j+1}(n)$ with $\bar{\mathsf{g}}(n)$ and then downsampling by 2 produces $w_j(n)$.

This is formally stated and proved in the next theorem.

---

[1] Notice that this definition is similar to the definition of transforms of other analysis systems:

Laplace Transform	$\mathcal{L}f(s)$	$\triangleq$	$\langle \mathsf{f}(x) \,\|\, e^{sx} \rangle$	$\triangleq$	$\int_x \mathsf{f}(x)e^{-sx}\,\mathsf{d}x$
Continuous Fourier Transform	$\mathcal{F}f(\omega)$	$\triangleq$	$\langle \mathsf{f}(x) \,\|\, e^{i\omega x} \rangle$	$\triangleq$	$\int_x \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x$
Fourier Series Transform	$\mathcal{F}_s f(k)$	$\triangleq$	$\langle \mathsf{f}(x) \,\|\, e^{i\frac{2\pi}{T}kx} \rangle$	$\triangleq$	$\int_x \mathsf{f}(x)e^{-i\frac{2\pi}{T}kx}\,\mathsf{d}x$
Z-Transform	$\mathcal{Z}f(z)$	$\triangleq$	$\langle \mathsf{f}(x) \,\|\, z^n \rangle$	$\triangleq$	$\sum_n \mathsf{f}(x)z^{-n}$
Discrete Fourier Transform	$\mathcal{F}_d f(k)$	$\triangleq$	$\langle f(n) \,\|\, e^{i\frac{2\pi}{N}kn} \rangle$	$\triangleq$	$\sum_n \mathsf{f}(x)e^{-i\frac{2\pi}{N}kn}$

[2] Mallat (1999) page 257, Burrus et al. (1998) page 35

Theorem 8.1.

THM

$$\begin{aligned} v_j(n) &= [\bar{\mathsf{h}} \star v_{j+1}](2n) \\ w_j(n) &= [\bar{\mathsf{g}} \star v_{j+1}](2n) \end{aligned}$$

✎PROOF:

$$\begin{aligned}
v_j(n) &= \langle \mathsf{f}(x) \,|\, \phi_{j,n}(x) \rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\phi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\sqrt{2}\sum_m h(m)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sum_m h(j)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \,|\, \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \,|\, \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m h(m) v_{j+1}(2n+m) \\
&= \sum_p h(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p-2n \\
&= \sum_p \bar{\mathsf{h}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{h}} \star v_{j+1}](2n)
\end{aligned}$$

$$\begin{aligned}
w_j(n) &= \langle \mathsf{f}(x) \,|\, \psi_{j,n}(x) \rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\psi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sqrt{2^j}\sqrt{2}\sum_m \mathsf{g}(j)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \sum_m \mathsf{g}(m)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \,|\, \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \,|\, \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m \mathsf{g}(m) v_{j+1}(2n+m) \\
&= \sum_p \mathsf{g}(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p-2n \\
&= \sum_p \bar{\mathsf{g}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{g}} \star v_{j+1}](2n)
\end{aligned}$$

✑

These filtering and downsampling operations are equivalent to the operations performed by a filter bank. Therefore, a filter bank can be used to implement a *Fast Wavelet Transform* (*FWT*), as illustrated in Figure 8.1 (page 57).

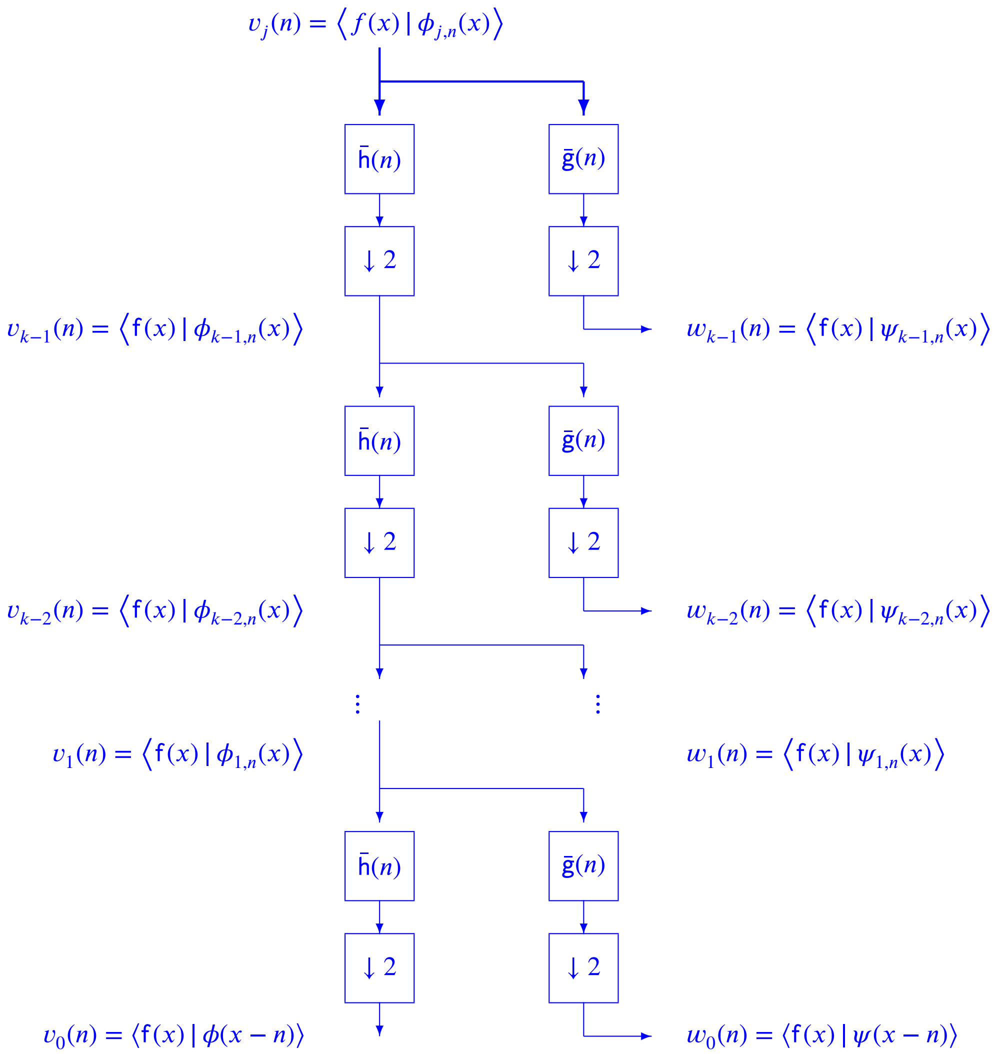


Figure 8.1: k-Stage Fast Wavelet Transform

CHAPTER 9

KL EXPANSION (TRANSFORM)

9.1 Definitions

Definition 9.1. *Let* $\mathsf{x}(t)$ *be random processes with continuous* AUTO-CORRELATION $\mathsf{R}_{\mathsf{xx}}(t,u)$ *(Definition J.2 page 201) or discrete* AUTO-CORRELATION $\mathsf{R}_{\mathsf{xx}}(n,m)$ *(Definition K.2 page 203).*

DEF *The **auto-correlation operator** $\mathbf{R}$ of $\mathsf{x}(t)$ is defined as*

$$\underbrace{\mathbf{R}\mathsf{f} \triangleq \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{f}(u)\,\mathrm{d}u}_{\text{(continuous case)}} \qquad or \qquad \underbrace{\mathbf{R}\mathsf{f} \triangleq \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{xx}}(n,m)\mathsf{f}(m)}_{\text{(discrete case)}}$$

Definition 9.2. *Let* $\mathsf{x}(t)$ *be a* RANDOM PROCESS *with* AUTO-CORRELATION $\mathsf{R}_{\mathsf{xx}}(\tau)$ *(Definition J.2 page 201).*

DEF *A* RANDOM PROCESS $\mathsf{x}(t)$ *is **white** if* $\mathsf{R}_{\mathsf{xx}}(\tau) = \delta(\tau)$

If a random process $\mathsf{x}(t)$ is *white* (Definition 9.2 page 59) and the set $\Psi = \{\psi_1(t), \psi_2(t), \ldots, \psi_N(t)\}$ is **any** set of orthonormal basis functions, then the innerproducts $\langle n(t)\,|\,\psi_n(t)\rangle$ and $\langle n(t)\,|\,\psi_m(t)\rangle$ are *uncorrelated* for $m \neq n$. However, if $\mathsf{x}(t)$ is **colored** (not white), then the innerproducts are not in general uncorrelated. But if the elements of Ψ are chosen to be the eigenfunctions of $\mathbf{R}$ such that $\mathbf{R}\psi_n = \lambda_n\psi_n$, then by Theorem J.1 (page 202), the set $\{\psi_n(t)\}$ are *orthogonal* and the innerproducts **are** *uncorrelated* even though $\mathsf{x}(t)$ is not white. This criterion is called the Karhunen-Loève criterion for $\mathsf{x}(t)$.

9.2 Properties

Theorem 9.1. *Let* $\mathbf{R}$ *be an* AUTO-CORRELATION *operator.*

THM

$$\left\{\begin{array}{lll} (A). & \langle \mathsf{x}\,|\,\mathsf{y}\rangle \triangleq \displaystyle\int_{t\in\mathbb{R}} \mathsf{x}(t)\mathsf{y}^*(t)\,\mathrm{d}t & \text{OR} \\ (B). & \langle \mathsf{x}\,|\,\mathsf{y}\rangle \triangleq \displaystyle\sum_{n\in\mathbb{Z}} \mathsf{x}(n)\mathsf{y}^*(n) & \end{array}\right\} \implies \left\{\begin{array}{llll} (1). & \langle \mathbf{R}\mathsf{x}\,|\,\mathsf{x}\rangle \geq 0 & \text{(NON-NEGATIVE)} & and \\ (2) & \langle \mathbf{R}\mathsf{x}\,|\,\mathsf{y}\rangle = \langle \mathsf{x}\,|\,\mathbf{R}\mathsf{y}\rangle & \text{(SELF-ADJOINT)} & \end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{R}$ is *non-negative* under hypothesis (A):

$$
\begin{aligned}
\langle \mathbf{R}\mathsf{y} \mid \mathsf{y} \rangle &= \left\langle \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{y}(u)\, \mathsf{d}u \mid \mathsf{y}(t) \right\rangle && \text{by definition of } \mathbf{R} && \text{(Definition 9.1 page 59)}\\
&= \left\langle \int_{u\in\mathbb{R}} \mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right]\mathsf{y}(u)\, \mathsf{d}u \mid \mathsf{y}(t) \right\rangle && \text{by definition of } \mathsf{R}_{\mathsf{xx}}(t,u) && \text{(Definition J.2 page 201)}\\
&= \mathsf{E}\left[\left\langle \int_{u\in\mathbb{R}} \mathsf{x}(t)\mathsf{x}^*(u)\mathsf{y}(u)\, \mathsf{d}u \mid \mathsf{y}(t) \right\rangle\right] && \text{by } \textit{linearity} \text{ of } \langle \triangle \mid \nabla \rangle \text{ and } \int\\
&= \mathsf{E}\left[\int_{u\in\mathbb{R}} \mathsf{x}^*(u)\mathsf{y}(u)\, \mathsf{d}u \,\langle \mathsf{x}(t) \mid \mathsf{y}(t) \rangle\right] && \text{by } \textit{additivity} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \mathsf{E}\left[\langle \mathsf{y}(u) \mid \mathsf{x}(u) \rangle \,\langle \mathsf{x}(t) \mid \mathsf{y}(t) \rangle\right] && \text{by local definition of } \langle \triangle \mid \nabla \rangle\\
&= \mathsf{E}\left[\langle \mathsf{x}(u) \mid \mathsf{y}(u) \rangle^* \,\langle \mathsf{x}(t) \mid \mathsf{y}(t) \rangle\right] && \text{by } \textit{conjugate symmetry} \text{ prop.}\\
&= \mathsf{E}\left|\langle \mathsf{x}(t) \mid \mathsf{y}(t) \rangle\right|^2 && \text{by definition of } |\cdot| && \text{(Definition A.4 page 68)}\\
&\geq 0 && \text{by } \textit{strictly positive} \text{ property of norms}
\end{aligned}
$$

2. Proof that $\mathbf{R}$ is *self-adjoint* under hypothesis (A):

$$
\begin{aligned}
\langle [\mathbf{R}\mathsf{x}](t) \mid \mathsf{y} \rangle &= \left\langle \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{x}(u)\, \mathsf{d}u \mid \mathsf{y}(t) \right\rangle && \text{by definition of } \mathbf{R} && \text{(Definition 9.1 page 59)}\\
&= \int_{u\in\mathbb{R}} \mathsf{x}(u)\,\langle \mathsf{R}_{\mathsf{xx}}(t,u) \mid \mathsf{y}(t) \rangle\; \mathsf{d}u && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \int_{u\in\mathbb{R}} \mathsf{x}(u)\,\langle \mathsf{y}(t) \mid \mathsf{R}_{\mathsf{xx}}(t,u) \rangle^*\; \mathsf{d}u && \text{by } \textit{conjugate symmetry} \text{ prop.}\\
&= \langle \mathsf{x}(u) \mid \langle \mathsf{y}(t) \mid \mathsf{R}_{\mathsf{xx}}(t,u) \rangle \rangle && \text{by local definition of } \langle \triangle \mid \nabla \rangle\\
&= \left\langle \mathsf{x}(u) \mid \int_{t\in\mathbb{R}} \mathsf{y}(t)\mathsf{R}^*_{\mathsf{xx}}(t,u)\, \mathsf{d}t \right\rangle\\
&= \left\langle \mathsf{x}(u) \mid \int_{t\in\mathbb{R}} \mathsf{y}(t)\mathsf{R}_{\mathsf{xx}}(u,t)\, \mathsf{d}t \right\rangle && \text{by property of } \mathsf{R}_{\mathsf{xx}} && \text{(Theorem J.1 page 202)}\\
&= \left\langle \mathsf{x}(u) \mid \underbrace{\mathbf{R}}_{\mathbf{R}^*}\mathsf{y} \right\rangle && \text{by definition of } \mathbf{R} && \text{(Definition 9.1 page 59)}\\
\Longrightarrow \mathbf{R} &= \mathbf{R}^* && \text{by definition of } \textit{adjoint } \mathbf{R}^* && \text{(Definition F.8 page 129)}\\
\Longrightarrow \mathbf{R} &\text{ is } \textit{self-adjoint} && \text{by definition of } \textit{self-adjoint} && \text{(Definition F.11 page 137)}
\end{aligned}
$$

3. Proofs under hypothesis (B): substitute $\sum_{n\in\mathbb{Z}}$ operator for $\int_{t\in\mathbb{R}} \mathsf{d}t$ operator in above proofs.

✏

Theorem 9.2. [1] *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be the eigenvalues and $(\psi_n)_{n\in\mathbb{Z}}$ be the eigenfunctions of operator $\mathbf{R}$ such that $\mathbf{R}\psi_n = \lambda_n\psi_n$.*

THM

(1).	$\lambda_n \in \mathbb{R}$	(REAL-VALUED)
(2).	$\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0$	(ORTHOGONAL)
(3).	$\left\|\psi_n(t)\right\|^2 > 0 \implies \lambda_n \geq 0$	(NON-NEGATIVE)
(4).	$\left\|\psi_n(t)\right\|^2 > 0, \langle \mathbf{R}\mathsf{f} \mid \mathsf{f} \rangle > 0 \implies \lambda_n > 0$	($\mathbf{R}$ POSITIVE DEFINITE $\implies \lambda_n$ POSITIVE)

✎PROOF:

[1] Keener (1988) pages 114–119

1. Proof that eigenvalues are *real-valued*: Because $\mathbf{R}$ is *self-adjoint*, its eigenvalues are real (Theorem F.18 page 137).

2. Proof that eigenfunctions associated with distinct eigenvalues are orthogonal: Because $\mathbf{R}$ is *self-adjoint*, this property follows (Theorem F.18 page 137).

3. Proof that eigenvalues are *non-negative*:

$$\begin{aligned}
0 &\le \langle \mathbf{R}\psi_n \mid \psi_n \rangle && \text{by definition of } \textit{non-negative definite} \\
&= \langle \lambda_n \psi_n \mid \psi_n \rangle && \text{by hypothesis} \\
&= \lambda_n \langle \psi_n \mid \psi_n \rangle && \text{by definition of inner-products} \\
&= \lambda_n \|\psi_n\|^2 && \text{by definition of norm induced by inner-product}
\end{aligned}$$

4. Proof that eigenvalues are *positive* if $\mathbf{R}$ is *positive definite*:

$$\begin{aligned}
0 &< \langle \mathbf{R}\psi_n \mid \psi_n \rangle && \text{by definition of } \textit{positive definite} \\
&= \langle \lambda_n \psi_n \mid \psi_n \rangle && \text{by hypothesis} \\
&= \lambda_n \langle \psi_n \mid \psi_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \nabla \rangle \\
&= \lambda_n \|\psi_n\|^2 && \text{by } \textit{induced norm} \text{ theorem}
\end{aligned}$$

✏

Theorem 9.3 (Karhunen-Loève Expansion). [2] *Let* $\mathbf{R}$ *be the* AUTO-CORRELATION OPERATOR *(Definition 9.1 page 59) of a* RANDOM PROCESS $\mathsf{x}(t)$. *Let* $(\lambda_n)_{n\in\mathbb{Z}}$ *be the eigenvalues of* $\mathbf{R}$ *and* $(\psi_n)_{n\in\mathbb{Z}}$ *are the eigenfunctions of* $\mathbf{R}$ *such that* $\mathbf{R}\psi_n = \lambda_n\psi_n$.

THM

$$\underbrace{\|\psi_n(t)\| = 1}_{\{\psi_n(t)\} \text{ are NORMALIZED}} \implies \underbrace{\mathsf{E}\left(\left|\mathsf{x}(t) - \sum_{n\in\mathbb{Z}} \langle \mathsf{x}(t) \mid \psi_n(t) \rangle\, \psi_n(t)\right|^2\right) = 0}_{\text{CONVERGENCE IN PROBABILITY}} \qquad (\{\psi_n(t)\} \textit{ is a BASIS for } \mathsf{x}(t))$$

✎PROOF:

1. Define $\dot{x}_n \triangleq \langle \mathsf{x}(t) \mid \psi_n(t) \rangle$

2. Define $\mathbf{R}\mathsf{x}(t) \triangleq \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{x}(u)\, \mathsf{d}u$

3. lemma: $\mathsf{E}[\mathsf{x}(t)\mathsf{x}(t)] = \sum_{n\in\mathbb{Z}} \lambda_n |\psi_n(t)|^2$. Proof:

$$\mathsf{E}[\mathsf{x}(t)\mathsf{x}(t)] = \sum_{n\in\mathbb{Z}} \lambda_n |\psi_n(t)|^2 \qquad \begin{array}{lll} \text{by} & \textit{non-negative} \text{ property} & \text{(Theorem 9.1 page 59)} \\ \text{and} & \textit{Mercer's Theorem} & \text{(Theorem L.4 page 212)} \end{array}$$

[2] Keener (1988) pages 114–119

4. lemma:

$$
\begin{aligned}
&\mathsf{E}\left[\mathsf{x}(t)\left(\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right)^*\right] \\
&\triangleq \mathsf{E}\left[\mathsf{x}(t)\left(\sum_{n\in\mathbb{Z}}\int_{u\in\mathbb{R}}\mathsf{x}(u)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\right)^*\right] && \text{by definition of } \dot{x} && \text{(definition 1 page 61)} \\
&= \sum_{n\in\mathbb{Z}}\left(\int_{u\in\mathbb{R}}\mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right]\psi_n(u)\,\mathsf{d}u\right)\psi_n^*(t) && \text{by } \textit{linearity} \\
&\triangleq \sum_{n\in\mathbb{Z}}\left(\int_{u\in\mathbb{R}}\mathsf{R}_{\mathsf{xx}}(t,u)\psi_n(u)\,\mathsf{d}u\right)\psi_n^*(t) && \text{by definition of } \mathsf{R}_{\mathsf{xx}}(t,u) && \text{(Definition J.2 page 201)} \\
&\triangleq \sum_{n\in\mathbb{Z}}\left(\mathbf{R}\psi_n(t)\psi_n^*(t)\right) && \text{by definition of } \mathbf{R} && \text{(definition 2 page 61)} \\
&= \sum_{n\in\mathbb{Z}}\lambda_n\psi_n(t)\psi_n^*(t) && \text{by property of } \textit{eigen-system} \\
&= \sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2
\end{aligned}
$$

5. lemma:

$$
\begin{aligned}
&\mathsf{E}\left[\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\left(\sum_{m\in\mathbb{Z}}\dot{x}_m\psi_m(t)\right)^*\right] \\
&\triangleq \mathsf{E}\left[\sum_{n\in\mathbb{Z}}\int_{u\in\mathbb{R}}\mathsf{x}(u)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\left(\sum_{m\in\mathbb{Z}}\int_v\mathsf{x}(v)\psi_m^*(v)\,\mathsf{d}v\psi_m(t)\right)^*\right] && \text{by definition of } \dot{x} \text{ (definition 1 page 61)} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\int_v\mathsf{E}\left[\mathsf{x}(u)\mathsf{x}^*(v)\right]\psi_m(v)\,\mathsf{d}v\right)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\psi_m^*(t) && \text{by } \textit{linearity} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\int_v\mathsf{R}_{\mathsf{xx}}(u,v)\psi_m(v)\,\mathsf{d}v\right)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\psi_m^*(t) && \text{by definition of } \mathsf{R}_{\mathsf{xx}}(t,u) \text{ (Definition J.2 page 201)} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\mathbf{R}\psi_m(u)\right)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\psi_m^*(t) && \text{by definition of } \mathbf{R} \text{ (definition 2 page 61)} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\lambda_m\psi_m(u)\right)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\psi_m^*(t) && \text{by property of } \textit{eigen-system} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\left(\int_{u\in\mathbb{R}}\psi_m(u)\psi_n^*(u)\,\mathsf{d}u\right)\psi_n(t)\psi_m^*(t) && \text{by } \textit{linearity} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\left\|\psi(t)\right\|^2\bar{\delta}_{mn}\psi_n(t)\psi_m^*(t) && \text{by } \textit{orthogonal} \text{ property (Theorem 9.2 page 60)} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\bar{\delta}_{mn}\psi_n(t)\psi_m^*(t) && \text{by } \textit{normalized} \text{ hypothesis} \\
&= \sum_{n\in\mathbb{Z}}\lambda_n\psi_n(t)\psi_n^*(t) \quad \text{by definition of } \textit{Kronecker delta } \bar{\delta} && \text{(Definition G.12 page 160)} \\
&= \sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2
\end{aligned}
$$

6. Proof that $\{\psi_n(t)\}$ is a *basis* for $\mathsf{x}(t)$:

$$
\mathsf{E}\left(\left|\mathsf{x}(t)-\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right|^2\right)
$$

$$
\begin{aligned}
&= \mathsf{E}\left(\left[\mathsf{x}(t) - \sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\right]\left[\mathsf{x}(t) - \sum_{m\in\mathbb{Z}} \dot{x}_m\psi_m(t)\right]^*\right) \\
&= \mathsf{E}\left(\mathsf{x}(t)\mathsf{x}^*(t) - \mathsf{x}(t)\left[\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\right]^* - \mathsf{x}^*(t)\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t) + \left[\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\right]\left[\sum_{m\in\mathbb{Z}} \dot{x}_m\psi_m(t)\right]^*\right) \\
&= \mathsf{E}\left(\mathsf{x}(t)\mathsf{x}^*(t)\right) - \mathsf{E}\left[\mathsf{x}(t)\left[\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\right]^*\right] - \mathsf{E}\left[\mathsf{x}^*(t)\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\right] + \mathsf{E}\left[\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\left[\sum_{m\in\mathbb{Z}} \dot{x}_m\psi_m(t)\right]^*\right] \\
&\qquad \text{by } \textit{linearity} \text{ of } \mathsf{E} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} \lambda_n|\psi_n(t)|^2}_{\text{by (3) lemma}} - \underbrace{\sum_{n\in\mathbb{Z}} \lambda_n|\psi_n(t)|^2}_{\text{by (4) lemma}} - \underbrace{\left[\sum_{n\in\mathbb{Z}} \lambda_n|\psi_n(t)|^2\right]^*}_{\text{by (4) lemma}} + \underbrace{\sum_{n\in\mathbb{Z}} \lambda_n|\psi_n(t)|^2}_{\text{by (5) lemma}} \\
&= 0
\end{aligned}
$$

✏

9.3 Quasi-basis

The *auto-correlation operator* **R** (Definition 9.1 page 59) in the discrete case can be approximated using a *correlation matrix*. In the *zero-mean* case, this becomes

$$
\mathbf{R} \triangleq \begin{bmatrix} \mathsf{E}[\mathsf{y}_1\mathsf{y}_1] & \mathsf{E}[\mathsf{y}_1\mathsf{y}_2] & \cdots & \mathsf{E}[\mathsf{y}_1\mathsf{y}_n] \\ \mathsf{E}[\mathsf{y}_2\mathsf{y}_1] & \mathsf{E}[\mathsf{y}_2\mathsf{y}_2] & & \mathsf{E}[\mathsf{y}_2\mathsf{y}_n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{E}[\mathsf{y}_n\mathsf{y}_1] & \mathsf{E}[\mathsf{y}_n\mathsf{y}_2] & \cdots & \mathsf{E}[\mathsf{y}_n\mathsf{y}_n] \end{bmatrix}
$$

The eigen-vectors (and hence a quasi-basis) for **R** can be found using a *Cholesky Decomposition*.

Proposition 9.1. [3]

PRP | *The* AUTO-CORRELATION MATRIX **R** *is* ***Toeplitz***.

Remark 9.1. For more information about the properties of **Toeplitz matrices**, see

1. Grenander and Szegö (1958),
2. Widom (1965),
3. Gray (1971),
4. Smylie et al. (1973) page 408 ⟨§"B. PROPERTIES OF THE TOEPLITZ MATRIX"⟩,
5. GRENANDER AND SZEGÖ (1984),
6. HAYKIN AND KESLER (1979),
7. HAYKIN AND KESLER (1983),
8. S. LAWRENCE MARPLE (1987) PAGES 80–92 ⟨§"3.8 THE TOEPLITZ MATRIX"⟩,
9. BÖTTCHER AND SILBERMANN (1999) ⟨ISBN:9780387985701⟩,
10. GRAY (2006),
11. S. LAWRENCE MARPLE (2019) PAGES 80–93 ⟨§"3.8 THE TOEPLITZ MATRIX"⟩.

[3] See Clarkson (1993) page 131 ⟨§"Appendix 3A — Positive Semi-Definite Form of the Autocorrelation Matrix"⟩

APPENDIX A

ALGEBRAIC STRUCTURES

❝*In this book, learned reader, you have the rules of algebra…It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one.…this most abstruse and unsurpassed treasury of the entire arithmetic being brought to light and, as in a theater, exposed to the sight of all…*❞

Gerolamo Cardano (1501–1576), Italian mathematician, physician, and astrologer [1]

A set together with one or more operations forms several standard mathematical structures:

group $\supseteq$ *ring* $\supseteq$ *commutative ring* $\supseteq$ *integral domain* $\supseteq$ *field*

Definition A.1. [2] *Let X be a set and $\diamond : X \times X \to X$ be an operation on X.*

DEF

*The pair $(X, \diamond)$ is a **group** if*

1. $\exists e \in X$ such that $e \diamond x = x \diamond e = x$ $\quad \forall x \in X$ (IDENTITY *element*) *and*
2. $\exists (-x) \in X$ such that $(-x) \diamond x = x \diamond (-x) = e$ $\quad \forall x \in X$ (INVERSE *element*) *and*
3. $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ $\quad \forall x,y,z \in X$ (ASSOCIATIVE)

Definition A.2. [3] *Let $+ : X \times X \to X$ and $* : X \times X \to X$ be operations on a set X. Furthermore, let the operation $*$ also be represented by juxtapostion as in $a * b \equiv ab$.*

DEF

*The triple $(X, +, *)$ is a **ring** if*

1. $(X, +)$ is a group. (*additive group*) *and*
2. $x(yz) = (xy)z$ $\quad \forall x,y,z \in X$ (ASSOCIATIVE *with respect to* $*$) *and*
3. $x(y+z) = (xy) + (xz)$ $\quad \forall x,y,z \in X$ ($*$ *is* LEFT DISTRIBUTIVE *over* $+$) *and*
4. $(x+y)z = (xz) + (yz)$ $\quad \forall x,y,z \in X$ ($*$ *is* RIGHT DISTRIBUTIVE *over* $+$).

Definition A.3. [4]

[1] quote: Cardano (1545) page 1
image: http://en.wikipedia.org/wiki/Image:Cardano.jpg

[2] Durbin (2000) page 29

[3] Durbin (2000) pages 114–115

[4] Durbin (2000) page 118

DEF

A triple $(X, +, *)$ *is a* ***commutative ring*** *if*

1.	$(X, +, *)$ *is a* RING		*and*	
2.	$xy = yx$	$\forall x,y \in X$	(COMMUTATIVE).	

Definition A.4. [5] *Let* R *be a* COMMUTATIVE RING *(Definition A.3 page 67).*

DEF

A function $|\cdot|$ *in* $\mathbb{R}^{\mathbb{R}}$ *is an* ***absolute value*** *(or* ***modulus****) if*

1.	$\lvert x\rvert \geq 0$	$x \in \mathbb{R}$	(NON-NEGATIVE)	*and*
2.	$\lvert x\rvert = 0 \iff x = 0$	$x \in \mathbb{R}$	(NONDEGENERATE)	*and*
3.	$\lvert xy\rvert = \lvert x\rvert \cdot \lvert y\rvert$	$x,y \in \mathbb{R}$	(HOMOGENEOUS / SUBMULTIPLICATIVE)	*and*
4.	$\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$	$x,y \in \mathbb{R}$	(SUBADDITIVE / TRIANGLE INEQUALITY)	

Definition A.5. [6]

DEF

The structure $F \triangleq (X, +, \cdot, 0, 1)$ *is a* ***field*** *if*

1.	$(X, +, *)$ *is a ring*		*(ring)*	*and*
2.	$xy = yx$	$\forall x,y \in X$	*(commutative with respect to* $*$*)*	*and*
3.	$(X \setminus \{0\}, *)$ *is a group*		*(group with respect to* $*$*).*	

Definition A.6. [7] *Let* $\boldsymbol{V} = (F, +, \cdot)$ *be a vector space and* $\otimes : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ *be a vector-vector multiplication operator.*
An ***algebra*** *is any pair* $(\boldsymbol{V}, \otimes)$ *that satisfies (*$\otimes$ *is represented by juxtaposition)*

DEF

1.	$(\boldsymbol{ux})\boldsymbol{y} = \boldsymbol{u}(\boldsymbol{xy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$	(ASSOCIATIVE)	*and*
2.	$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}) = (\boldsymbol{ux}) + (\boldsymbol{uy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$	(LEFT DISTRIBUTIVE)	*and*
3.	$(\boldsymbol{u}+\boldsymbol{x})\boldsymbol{y} = (\boldsymbol{uy}) + (\boldsymbol{xy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$	(RIGHT DISTRIBUTIVE)	*and*
4.	$\alpha(\boldsymbol{xy}) = (\alpha\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x}(\alpha\boldsymbol{y})$	$\forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$ *and* $\alpha \in F$	(SCALAR COMMUTATIVE)	.

[5] Cohn (2002) page 312
[6] Durbin (2000) page 123, Weber (1893)
[7] Abramovich and Aliprantis (2002) page 3, Michel and Herget (1993) page 56

APPENDIX B

CALCULUS

Definition B.1. *Let* $\mathbb{R}$ *be the set of real numbers,* $\mathscr{B}$ *the set of* BOREL SETS *on* $\mathbb{R}$*, and* μ *the standard* BOREL MEASURE *on* $\mathscr{B}$*. Let* $\mathbb{R}^{\mathbb{R}}$ *be as in Definition* *I.1* *page* *187*.

DEF

The ***space of Lebesgue square-integrable functions*** $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ *(or* $L^2_{\mathbb{R}}$*) is defined as*

$$L^2_{\mathbb{R}} \triangleq L^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \mathrm{d}\mu < \infty \right\}.$$

The ***standard inner product*** $\langle \triangle \mid \triangledown \rangle$ *on* $L^2_{\mathbb{R}}$ *is defined as*

$$\langle \mathsf{f}(x) \mid \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\,\mathrm{d}x.$$

The ***standard norm*** $\|\cdot\|$ *on* $L^2_{\mathbb{R}}$ *is defined as* $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \mid \mathsf{f}(x) \rangle^{\frac{1}{2}}$

Definition B.2. *Let* $\mathsf{f}(x)$ *be a* FUNCTION *in* $\mathbb{R}^{\mathbb{R}}$.

DEF

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon\to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition B.1.

PRP

$$\left\{ \begin{array}{lll} (1). & \mathsf{f}(x) \text{ is CONTINUOUS} & \textit{and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC about a point } a} & \end{array} \right\} \implies \left\{ \begin{array}{llll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) \quad (\text{ANTI-SYMMETRIC about } a) \\ (2). & \mathsf{f}'(a) & = & 0 \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{f}'(a+x) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\
&= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\
&= -\mathsf{f}(a-x)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\
&= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result}
\end{aligned}$$

$$= 0$$

✏

Lemma B.1.

$$\mathsf{f}(x) \text{ is INVERTIBLE} \implies \left\{ \frac{\mathrm{d}}{\mathrm{d}y}\mathsf{f}^{-1}(y) = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$$

✎PROOF:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to 0} \frac{\mathsf{f}^{-1}(y+\varepsilon) - \mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathrm{d}}{\mathrm{d}y} && \text{(Definition B.2 page 69)} \\
&= \lim_{\delta\to 0} \left. \frac{1}{\left[\frac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]} \right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1} \\
&\triangleq \left. \frac{1}{\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x)} \right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathrm{d}}{\mathrm{d}x} && \text{(Definition B.2 page 69)} \\
&= \frac{1}{\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✏

Theorem B.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

$$\int_{[0:1)^n} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n} x_k\right) \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathsf{f}(k) \qquad \forall n \in \mathbb{N}$$

✎PROOF: Proof by induction:

1. Base case …proof for $n = 1$ case:

$$\begin{aligned}
\int_{[0:1)} \mathsf{f}^{(1)}(x)\, \mathrm{d}x &= \mathsf{f}(1) - \mathsf{f}(0) && \text{by } \textit{Fundamental theorem of calculus} \\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1) + (-1)^{1+0}\binom{1}{0}\mathsf{f}(0) \\
&= \sum_{k=0}^{1} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

2. Induction step …proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_{n+1} \\
&= \int_{[0:1)^n} \left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right) \mathrm{d}x_{n+1}\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n && \text{by \textit{Fundamental theorem of calculus}} \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(1 + \sum_{k=1}^{n} x_k\right) - \mathsf{f}^{(n)}\left(0 + \sum_{k=1}^{n} x_k\right)\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1) - \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by induction hypothesis} \\
&= \sum_{m=1}^{m=n+1} (-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m) + \sum_{k=0}^{n}(-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{where } m \triangleq k+1 \implies k = m-1 \\
&= \left[\mathsf{f}(n+1) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right] + \left[(-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right] \\
&= \mathsf{f}(n+1) + (-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\text{use \textit{Stifel formula}}}\mathsf{f}(k) \\
&= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1) + (-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0) + \sum_{k=1}^{n}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by \textit{Stifel formula}} \\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)
\end{aligned}$$

✏

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma B.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots$, *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example B.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\big[\mathsf{f}(x)\mathsf{g}(x)\big] = \mathsf{f}'''(x)\mathsf{g}(x) + 3\mathsf{f}''(x)\mathsf{g}'(x) + 3\mathsf{f}'(x)\mathsf{g}''(x) + \mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem B.2 (Leibniz integration rule). [3]

THM
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathsf{a}(x)}^{\mathsf{b}(x)} \mathsf{g}(t)\, \mathrm{d}t = \mathsf{g}[\mathsf{b}(x)]\mathsf{b}'(x) - \mathsf{g}[\mathsf{a}(x)]\mathsf{a}'(x)$$

[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)

[3] Flanders (1973) page 615 ⟨(1.1)⟩ Talvila (2001), Knapp (2005b) page 389 ⟨Chapter VII⟩, ? page 422 ⟨Leibniz Rule. Theorem 1.⟩, http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

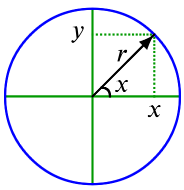
APPENDIX C

TRIGONOMETRIC FUNCTIONS

C.1 Definition Candidates

There are several ways of defining the sine and cosine functions, including the following:[1]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[2]



$$\begin{aligned}\cos x &\triangleq \frac{x}{r}\\ \sin x &\triangleq \frac{y}{r}\end{aligned}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[3]

$$\cos x \triangleq \mathbf{R}_{\mathrm{e}} e^{ix} \qquad\qquad \sin x \triangleq \mathbf{I}_{\mathrm{m}}\left(e^{ix}\right)$$

3. **Polynomial:** Let $\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of *Taylor expansions* such that[4]

$$\begin{aligned}\cos(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\ \sin(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

[1] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means "bay" or "inlet"—thus resulting in the Latin translation *sinus*, which also means "bay" or "inlet". Reference: Boyer and Merzbach (1991) page 252

[2] Abramowitz and Stegun (1972) page 78

[3] Euler (1748)

[4] Rosenlicht (1968) page 157, Abramowitz and Stegun (1972) page 74

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \prod_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of a product of factors such that[5]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1-\left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[6]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2-(n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-(n\pi)^2}\right)}_{\cot(x)}\sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{\mathsf{d}}{\mathsf{d}x}$ such that

$$\begin{array}{lllllll}
\cos(x) & \triangleq & \mathsf{f}(x) & \text{where} & \underbrace{\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}+\mathsf{f}=0}_{\text{differential equation}} & \underbrace{\mathsf{f}(0)=1}_{\text{1st initial condition}} & \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)=0}_{\text{2nd initial condition}} \\
\sin(x) & \triangleq & \mathsf{g}(x) & \text{where} & \underbrace{\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{g}+\mathsf{g}=0}_{\text{differential equation}} & \underbrace{\mathsf{g}(0)=0}_{\text{1st initial condition}} & \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{g}\right](0)=1}_{\text{2nd initial condition}}
\end{array}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[7]

$$\begin{array}{lllll}
\cos(x) & \triangleq & \mathsf{f}^{-1}(x) \quad \text{where} \quad \mathsf{f}(x) & \triangleq & \underbrace{\int_x^1 \sqrt{\frac{1}{1-y^2}}\,\mathsf{d}y}_{\arccos(x)} \\
\sin(x) & \triangleq & \mathsf{g}^{-1}(x) \quad \text{where} \quad \mathsf{g}(x) & \triangleq & \underbrace{\int_0^x \sqrt{\frac{1}{1-y^2}}\,\mathsf{d}y}_{\arcsin(x)}
\end{array}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{\mathsf{d}}{\mathsf{d}x}$ (Definition C.1 page 75). Support for such an approach includes the following:

- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{\mathsf{d}}{\mathsf{d}x}$ (Theorem C.1 page 76).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem C.3 page 78).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem C.4 page 79).

[5] Abramowitz and Stegun (1972) page 75
[6] Abramowitz and Stegun (1972) page 75
[7] Abramowitz and Stegun (1972) page 79

- The complex exponential function is a solution of a second order homogeneous differential equation (Definition C.4 page 80).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section C.6 page 88).

C.2 Definitions

Definition C.1. [8] *Let C be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{d}{dx} \in C^C$ the differentiation operator.*

DEF: *The function $f \in \mathbb{C}^{\mathbb{C}}$ is the **cosine** function $\cos(x) \triangleq f(x)$ if*

1. $\frac{d^2}{dx^2}f + f = 0$ *(second order homogeneous differential equation)* *and*
2. $f(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{d}{dx}f\right](0) = 0$ *(second initial condition).*

Definition C.2. [9] *Let C and $\frac{d}{dx} \in C^C$ be defined as in definition of* $\cos(x)$ (Definition C.1 page 75).

DEF: *The function $f \in \mathbb{C}^{\mathbb{C}}$ is the **sine** function $\sin(x) \triangleq f(x)$ if*

1. $\frac{d^2}{dx^2}f + f = 0$ *(second order homogeneous differential equation)* *and*
2. $f(0) = 0$ *(first initial condition)* *and*
3. $\left[\frac{d}{dx}f\right](0) = 1$ *(second initial condition).*

Definition C.3. [10]

DEF: *Let π ("pi") be defined as the element in $\mathbb{R}$ such that*

(1). $\cos\left(\frac{\pi}{2}\right) = 0$ *and*

(2). $\pi > 0$ *and*

(3). *π is the **smallest** of all elements in $\mathbb{R}$ that satisfies (1) and (2).*

C.3 Basic properties

Lemma C.1. [11] *Let C be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{d}{dx} \in C^C$ the differentiation operator.*

LEM:

$$\left\{\frac{d^2}{dx^2}f + f = 0\right\} \iff \left\{\begin{aligned} f(x) &= \underbrace{[f](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{d}{dx}f\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \\ &= \left(f(0) + \left[\frac{d}{dx}f\right](0)x\right) - \left(\frac{f(0)}{2!}x^2 + \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3\right) + \left(\frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5\right)\cdots \end{aligned}\right\}$$

[8] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[10] Rosenlicht (1968) page 158
[11] Rosenlicht (1968) page 156, Liouville (1839)

✎PROOF: Let $\mathsf{f}'(x) \triangleq \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x)$.

$$\begin{aligned}
\mathsf{f}'''(x) &= -\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](x) \\
\mathsf{f}^{(4)}(x) &= -\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](x) && = -\left[\tfrac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}\right](x) = \mathsf{f}(x)
\end{aligned}$$

1. Proof that $\left[\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}\right](x) + \mathsf{f}(x) = 0 \implies \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\mathsf{f}(x) &= \sum_{n=0}^{\infty}\frac{\mathsf{f}^{(n)}(0)}{n!}x^n && \text{by Taylor expansion} \\
&= \mathsf{f}(0) + \left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)x - \frac{\left[\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}\right](0)}{2!}x^2 - \frac{\mathsf{f}^3(0)}{3!}x^3 + \frac{\mathsf{f}^4(0)}{4!}x^4 + \frac{\mathsf{f}^5(0)}{5!}x^5 - \cdots \\
&= \mathsf{f}(0) + \left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)x - \frac{\mathsf{f}(0)}{2!}x^2 - \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{3!}x^3 + \frac{\mathsf{f}(0)}{4!}x^4 + \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{5!}x^5 - \cdots \\
&= \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]
\end{aligned}$$

2. Proof that $\left[\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}\right](x) + \mathsf{f}(x) = 0 \impliedby \mathsf{f}(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned}
\left[\tfrac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}\right](x) &= \tfrac{\mathsf{d}}{\mathsf{d}x}\tfrac{\mathsf{d}}{\mathsf{d}x}[\mathsf{f}(x)] \\
&= \tfrac{\mathsf{d}}{\mathsf{d}x}\tfrac{\mathsf{d}}{\mathsf{d}x}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right] && \text{by right hypothesis} \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)\mathsf{f}(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n-1}\right] \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{\mathsf{f}(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{(2n-1)!}x^{2n-1}\right] \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{\mathsf{f}(0)}{(2n)!}x^{2n} + \frac{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)}{(2n+1)!}x^{2n+1}\right] \\
&= -\mathsf{f}(x) && \text{by right hypothesis}
\end{aligned}$$

✎

Theorem C.1 (Taylor series for cosine/sine). [12]

THM

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && \forall x \in \mathbb{R} \\
\sin(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \forall x \in \mathbb{R}
\end{aligned}$$

[12] Rosenlicht (1968) page 157

✎PROOF:

$$\cos(x) = \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}}$$ by Lemma C.1 page 75

$$= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$ by cos initial conditions (Definition C.1 page 75)

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\sin(x) = \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}}$$ by Lemma C.1 page 75

$$= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$ by sin initial conditions (Definition C.2 page 75)

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

✏

Theorem C.2. [13]

THM

$$\begin{array}{rcl|rcll} \cos(0) &=& 1 & \cos(-x) &=& \cos(x) & \forall x\in\mathbb{R}\\ \sin(0) &=& 0 & \sin(-x) &=& -\sin(x) & \forall x\in\mathbb{R}\end{array}$$

✎PROOF:

$$\cos(0) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=0}$$ by *Taylor series for cosine* (Theorem C.1 page 76)

$$= 1$$

$$\sin(0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big|_{x=0}$$ by *Taylor series for sine* (Theorem C.1 page 76)

$$= 0$$

$$\cos(-x) = 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \cdots$$ by *Taylor series for cosine* (Theorem C.1 page 76)

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \cos(x)$$ by *Taylor series for cosine* (Theorem C.1 page 76)

$$\sin(-x) = (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots$$ by *Taylor series for sine* (Theorem C.1 page 76)

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \sin(x)$$ by *Taylor series for sine* (Theorem C.1 page 76)

✏

Lemma C.2. [14]

LEM

$$\begin{array}{rcl|l} \cos(1) &>& 0 & x\in(0:2) \implies \sin(x) > 0\\ \cos(2) &<& 0 & \end{array}$$

[13] Rosenlicht (1968) page 157

[14] Rosenlicht (1968) page 158

✎PROOF:

$$\begin{aligned}
\cos(1) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=1} && \text{by *Taylor series for cosine*} && \text{(Theorem C.1 page 76)}\\
&= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots\\
&> 0\\
\cos(2) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=2} && \text{by *Taylor series for cosine*} && \text{(Theorem C.1 page 76)}\\
&= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots\\
&< 0
\end{aligned}$$

$$\begin{aligned}
x \in (0:2) &\implies \text{each term in the sequence } \left(\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots\right) \text{ is } > 0\\
&\implies \sin(x) > 0
\end{aligned}$$

✏

Proposition C.1. *Let π be defined as in Definition C.3 (page 75).*

PRP
(A). *The value π **exists** in $\mathbb{R}$.*
(B). $2 < \pi < 4$.

✎PROOF:

$$\begin{aligned}
&\cos(1) > 0 && \text{by Lemma C.2 page 77}\\
&\cos(2) < 0 && \text{by Lemma C.2 page 77}\\
&\implies 1 < \frac{\pi}{2} < 2\\
&\implies 2 < \pi < 4
\end{aligned}$$

✏

Theorem C.3. [15] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $\mathsf{f}'(0) \triangleq \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)$.*

THM
$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \mathsf{f}'(0)\sin(x)\right\} \qquad \forall \mathsf{f}\in\boldsymbol{C}, \forall x\in\mathbb{R}$$

✎PROOF:

1. Proof that $\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}\right](x) = -\mathsf{f}(x) \implies \mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sin(x)$:

$$\begin{aligned}
\mathsf{f}(x) &= \mathsf{f}(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by left hypothesis and Lemma C.1 page 75}\\
&= \mathsf{f}(0)\cos x + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sin x && \text{by definitions of cos and sin (Definition C.1 page 75, Definition C.2 page 75)}
\end{aligned}$$

[15] Rosenlicht (1968) page 157. The general solution for the *non-homogeneous* equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x) + \mathsf{f}(x) = \mathsf{g}(x)$ with initial conditions $\mathsf{f}(a) = 1$ and $\mathsf{f}'(a) = \rho$ is $\mathsf{f}(x) = \cos(x) + \rho\sin(x) + \int_a^x \mathsf{g}(y)\sin(x-y)\,\mathrm{d}y$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992) page 371, Liouville (1839). Volterra equation references: Pedersen (2000) page 99, Lalescu (1908), Lalescu (1911)

2. Proof that $\frac{d^2}{dx^2}f = -f \impliedby f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$\begin{aligned}
f(x) &= f(0)\cos x + \left[\tfrac{d}{dx}f\right](0)\sin x && \text{by right hypothesis}\\
&= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\tfrac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)}\\
\implies \quad & \frac{d^2}{dx^2}f + f = 0 && \text{by Lemma C.1 page 75}
\end{aligned}$$

✏

Theorem C.4. [16] *Let $\frac{d}{dx} \in C^C$ be the differentiation operator.*

THM	$\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x\in\mathbb{R}$	$\frac{d}{dx}\sin(x) = \cos(x) \quad \forall x\in\mathbb{R}$	$\cos^2(x) + \sin^2(x) = 1 \quad \forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} && \text{by *Taylor series* (Theorem C.1 page 76)}\\
&= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}\\
&= -\sin(x) && \text{by *Taylor series* (Theorem C.1 page 76)}\\
\frac{d}{dx}\sin(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by *Taylor series* (Theorem C.1 page 76)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= \cos(x) && \text{by *Taylor series* (Theorem C.1 page 76)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left[\cos^2(x) + \sin^2(x)\right] &= -2\cos(x)\sin(x) + 2\sin(x)\cos(x)\\
&= 0\\
&\implies \cos^2(x) + \sin^2(x) \text{ is *constant*}\\
&\implies \cos^2(x) + \sin^2(x)\\
&= \cos^2(0) + \sin^2(0)\\
&= 1 + 0 = 1 && \text{by Theorem C.2 page 77}
\end{aligned}$$

✏

Proposition C.2.

PRP	$\sin\left(\frac{\pi}{2}\right) = 1$

[16] Rosenlicht (1968) page 157

✎PROOF:

$$\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2)+0} && \\
&= \pm\sqrt{\sin^2(\pi/2)+\cos^2(\pi/2)} && \text{by definition of } \pi \quad \text{(Definition C.3 page 75)}\\
&= \pm\sqrt{1} && \text{by Theorem C.4 page 79}\\
&= \pm 1 && \\
&= 1 && \text{by Lemma C.2 page 77}
\end{aligned}$$

✏

C.4 The complex exponential

Definition C.4.

DEF *The function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq \mathsf{f}(x)$ *if*

1. $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} = 0$ *(second order homogeneous differential equation)* *and*
2. $\mathsf{f}(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) = i$ *(second initial condition).*

Theorem C.5 (Euler's identity). [17]

THM $$e^{ix} = \cos(x) + i\sin(x) \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\exp(ix) &= \mathsf{f}(0)\cos(x) + \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)\sin(x) && \text{by Theorem C.3 page 78}\\
&= \cos(x) + i\sin(x) && \text{by Definition C.4 page 80}
\end{aligned}$$

✏

Proposition C.3.

PRP $$e^{-i\pi/2} = -i \quad \Big| \quad e^{i\pi/2} = i$$

✎PROOF:

$$\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem C.5 page 80)}\\
&= 0 + i && \text{by Theorem C.2 (page 77) and Proposition C.2 (page 79)}\\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem C.5 page 80)}\\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem C.2 page 77}\\
&= 0 - i && \text{by Theorem C.2 (page 77) and Proposition C.2 (page 79)}
\end{aligned}$$

✏

Corollary C.1.

COR $$e^{ix} = \sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x\in\mathbb{R}$$

[17] Euler (1748), Bottazzini (1986) page 12

✎PROOF:

$$\begin{aligned}
\boxed{e^{ix}} &= \cos(x) + i\sin(x) && \text{by *Euler's identity*} && \text{(Theorem C.5 page 80)}\\
&= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i\underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by *Taylor series*} && \text{(Theorem C.1 page 76)}\\
&= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} && = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!}\\
&= \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} && = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}
\end{aligned}$$

✏

Corollary C.2 (Euler formulas). [18]

COR

$$\cos(x) = \mathbf{R_e}\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \sin(x) = \mathbf{I_m}\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{R_e}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}+\left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} && \text{by definition of } \Re && \text{(Definition E.5 page 115)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} && \text{by *Euler's identity*} && \text{(Theorem C.5 page 80)}\\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} && = \frac{\cos(x)}{2} + \frac{\cos(x)}{2} && = \boxed{\cos(x)}\\
\boxed{\mathbf{I_m}\left(e^{ix}\right)} &\triangleq \frac{e^{ix}-\left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} && \text{by definition of } \Im && \text{(Definition E.5 page 115)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} && \text{by *Euler's identity*} && \text{(Theorem C.5 page 80)}\\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} && = \frac{i\sin(x)}{2i} + \frac{i\sin(x)}{2i} && = \boxed{\sin(x)}
\end{aligned}$$

✏

Theorem C.6. [19]

THM

$$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

✎PROOF:

$$\begin{aligned}
e^{\alpha}\, e^{\beta} &= \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) && \text{by Corollary C.1 page 80}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{n!}{n!} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}
\end{aligned}$$

[18] Euler (1748), Bottazzini (1986) page 12
[19] Rudin (1987) page 1

$$\begin{aligned}
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\,\alpha^k\,\beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \alpha^k\,\beta^{n-k} \\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the \textit{Binomial Theorem}} \\
&= e^{\alpha+\beta} && \text{by Corollary C.1 page 80}
\end{aligned}$$

✏

C.5 Trigonometric Identities

Theorem C.7 (shift identities).

THM				
$\cos\left(x+\frac{\pi}{2}\right) = -\sin x \quad \forall x\in\mathbb{R}$		$\sin\left(x+\frac{\pi}{2}\right) = \cos x \quad \forall x\in\mathbb{R}$		
$\cos\left(x-\frac{\pi}{2}\right) = \sin x \quad \forall x\in\mathbb{R}$		$\sin\left(x-\frac{\pi}{2}\right) = -\cos x \quad \forall x\in\mathbb{R}$		

✎PROOF:

$$\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)}+e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary C.2 page 81)} \\
&= \frac{e^{ix}e^{i\frac{\pi}{2}}+e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta}=e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem C.6 page 81)} \\
&= \frac{e^{ix}(i)+e^{-ix}(-i)}{2} && \text{by Proposition C.3 page 80} \\
&= \frac{e^{ix}-e^{-ix}}{-2i} \\
&= -\sin x && \text{by \textit{Euler formulas}} && \text{(Corollary C.2 page 81)} \\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)}+e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary C.2 page 81)} \\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}}+e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta}=e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem C.6 page 81)} \\
&= \frac{e^{ix}(-i)+e^{-ix}(i)}{2} && \text{by Proposition C.3 page 80} \\
&= \frac{e^{ix}-e^{-ix}}{2i} \\
&= \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary C.2 page 81)} \\
\sin\left(x+\frac{\pi}{2}\right) &= \cos\left(\left[x+\frac{\pi}{2}\right]-\frac{\pi}{2}\right) && \text{by previous result} \\
&= \cos(x) \\
\sin\left(x-\frac{\pi}{2}\right) &= -\cos\left(\left[x-\frac{\pi}{2}\right]+\frac{\pi}{2}\right) && \text{by previous result} \\
&= -\cos(x)
\end{aligned}$$

✏

Theorem C.8 (product identities).

THM

$$
\begin{array}{lrcrcrl}
\text{(A).} & \cos x\cos y &=& \tfrac{1}{2}\cos(x-y) &+& \tfrac{1}{2}\cos(x+y) & \forall x,y\in\mathbb{R}\\
\text{(B).} & \cos x\sin y &=& -\tfrac{1}{2}\sin(x-y) &+& \tfrac{1}{2}\sin(x+y) & \forall x,y\in\mathbb{R}\\
\text{(C).} & \sin x\cos y &=& \tfrac{1}{2}\sin(x-y) &+& \tfrac{1}{2}\sin(x+y) & \forall x,y\in\mathbb{R}\\
\text{(D).} & \sin x\sin y &=& \tfrac{1}{2}\cos(x-y) &-& \tfrac{1}{2}\cos(x+y) & \forall x,y\in\mathbb{R}
\end{array}
$$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary C.2 page 81) (algebraic method requiring *complex number system* $\mathbb{C}$):

$$
\begin{aligned}
\cos x\cos y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right) && \text{by Euler formulas (Corollary C.2 page 81)}\\
&= \frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(-x+y)}+e^{i(-x-y)}}{4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{4}+\frac{e^{i(x-y)}+e^{-i(x-y)}}{4}\\
&= \frac{2\cos(x+y)}{4}+\frac{2\cos(x-y)}{4} && \text{by Euler formulas (Corollary C.2 page 81)}\\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)
\end{aligned}
$$

2. Proof for (A) using *Volterra integral equation* (Theorem C.3 page 78) (differential equation method requiring only *real number system* $\mathbb{R}$):

$$
\begin{aligned}
&\mathsf{f}(x) \triangleq \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)\\
\implies\ & \tfrac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) = -\tfrac{1}{2}\sin(x-y)-\tfrac{1}{2}\sin(x+y) && \text{by Theorem C.4 page 79}\\
\implies\ & \tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x) = -\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y) && \text{by Theorem C.4 page 79}\\
\implies\ & \tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x)+\mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property}\\
\implies\ & \underbrace{\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)}_{\mathsf{f}(x)} = \underbrace{[\tfrac{1}{2}\cos(0-y)+\tfrac{1}{2}\cos(0+y)]}_{\mathsf{f}''(0)}\cos(x)+\underbrace{[-\tfrac{1}{2}\sin(0-y)-\tfrac{1}{2}\sin(0+y)]}_{\mathsf{f}'(0)}\sin(x)\\
\implies\ & \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y) = \cos y\cos x+0\sin(x)\\
\implies\ & \cos x\cos y = \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)
\end{aligned}
$$

3. Proof for (B) using *Euler formulas* (Corollary C.2 page 81):

$$
\begin{aligned}
\sin x\sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary C.2 page 81}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary C.2 page 81}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}
$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x \sin y &= \cos(x)\cos\left(y - \frac{\pi}{2}\right) && \text{by } \textit{shift identities} && \text{(Theorem C.7 page 82)}\\
&= \frac{1}{2}\cos\left(x + y - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(x - y + \frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y) - \frac{1}{2}\sin(x-y) && \text{by } \textit{shift identities} && \text{(Theorem C.7 page 82)}\\
\sin x \cos y &= \cos y \sin x\\
&= \frac{1}{2}\sin(y+x) - \frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y) && \text{by Theorem C.2 page 77}
\end{aligned}$$

⇨

Proposition C.4.

PRP				
	(A). $\cos(\pi) = -1$	(C). $\cos(2\pi) = 1$	(E). $e^{i\pi} = -1$
	(B). $\sin(\pi) = 0$	(D). $\sin(2\pi) = 0$	(F). $e^{i2\pi} = 0$

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1 + 1 + \cos(\pi)\\
&= -1 + 2[{}^1\!/\!_2\cos({}^\pi\!/\!_2 - {}^\pi\!/\!_2) + {}^1\!/\!_2\cos({}^\pi\!/\!_2 + {}^\pi\!/\!_2)] && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem C.2 page 77)}\\
&= -1 + 2\cos({}^\pi\!/\!_2)\cos({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem C.8 page 82)}\\
&= -1 + 2(0)(0) && \text{by definition of } \pi && \text{(Definition C.3 page 75)}\\
&= -1\\
\sin(\pi) &= 0 + \sin(\pi)\\
&= 2[-{}^1\!/\!_2\sin({}^\pi\!/\!_2 - {}^\pi\!/\!_2) + {}^1\!/\!_2\sin({}^\pi\!/\!_2 + {}^\pi\!/\!_2)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem C.2 page 77)}\\
&= 2\cos({}^\pi\!/\!_2)\sin({}^\pi\!/\!_2) && \text{by } \textit{product identities} && \text{(Theorem C.8 page 82)}\\
&= 2(0)\sin({}^\pi\!/\!_2) && \text{by definition of } \pi && \text{(Definition C.3 page 75)}\\
&= 0\\
\cos(2\pi) &= 1 + \cos(2\pi) - 1\\
&= 2[{}^1\!/\!_2\cos(\pi - \pi) + {}^1\!/\!_2\cos(\pi + \pi)] - 1 && \text{by } \cos(0) = 1 \text{ result} && \text{(Theorem C.2 page 77)}\\
&= 2\cos(\pi)\cos(\pi) - 1 && \text{by } \textit{product identities} && \text{(Theorem C.8 page 82)}\\
&= 2(-1)(-1) - 1 && \text{by (A)}\\
&= 1\\
\sin(2\pi) &= 0 + \sin(2\pi)\\
&= 2[{}^1\!/\!_2\sin(\pi - \pi) + {}^1\!/\!_2\sin(\pi + \pi)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem C.2 page 77)}\\
&= 2\sin(\pi)\cos(\pi) && \text{by } \textit{product identities} && \text{(Theorem C.8 page 82)}\\
&= 2(0)(-1) && \text{by (A) and (B)}\\
&= 0\\
e^{i\pi} &= \cos(\pi) + i\sin(\pi) && \text{by } \textit{Euler's identity} && \text{(Theorem C.5 page 80)}\\
&= -1 + 0 && \text{by (A) and (B)}\\
&= -1\\
e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) && \text{by } \textit{Euler's identity} && \text{(Theorem C.5 page 80)}\\
&= 1 + 0 && \text{by (C) and (D)}\\
&= 1
\end{aligned}$$

⇨

Theorem C.9 (double angle formulas). [20]

THM

$$\begin{array}{llcll}
\text{(A).} & \cos(x+y) & = & \cos x \cos y - \sin x \sin y & \forall x,y\in\mathbb{R}\\
\text{(B).} & \sin(x+y) & = & \sin x \cos y + \cos x \sin y & \forall x,y\in\mathbb{R}\\
\text{(C).} & \tan(x+y) & = & \dfrac{\tan x + \tan y}{1-\tan x \tan y} & \forall x,y\in\mathbb{R}
\end{array}$$

✎PROOF:

1. Proof for (A) using *product identities* (Theorem C.8 page 82).

$$\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x+y)}_{\cos(x+y)} + \underbrace{\frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x-y)}_{0}\\
&= \left[\frac{1}{2}\cos(x-y)+\frac{1}{2}\cos(x+y)\right] - \left[\frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)\right]\\
&= \cos x\cos y - \sin x \sin y && \text{by Theorem C.8 page 82}
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem C.3 page 78):

$$\begin{aligned}
\mathsf{f}(x) \triangleq \cos(x+y) &\implies \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) = -\sin(x+y) && \text{by Theorem C.4 page 79}\\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x) = -\cos(x+y) && \text{by Theorem C.4 page 79}\\
&\implies \frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f}(x) + \mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property}\\
&\implies \cos(x+y) = \cos y \cos x - \sin y \sin x && \text{by Theorem C.3 page 78}\\
&\implies \cos(x+y) = \cos x \cos y - \sin x \sin y && \text{by \textit{commutative} property}
\end{aligned}$$

3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x - \frac{\pi}{2} + y\right) && \text{by \textit{shift identities} (Theorem C.7 page 82)}\\
&= \cos\left(x-\frac{\pi}{2}\right)\cos(y) - \sin\left(x-\frac{\pi}{2}\right)\sin(y) && \text{by (A)}\\
&= \sin(x)\cos(y) + \cos(x)\sin(y) && \text{by \textit{shift identities} (Theorem C.7 page 82)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)}\\
&= \frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} && \text{by (A)}\\
&= \left(\frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}\right)\left(\frac{\cos x\cos y}{\cos x \cos y}\right)\\
&= \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}
\end{aligned}$$

✏

Theorem C.10 (trigonometric periodicity).

THM

$$\begin{array}{llclll|llclll}
\text{(A).} & \cos(x+M\pi) & = & (-1)^M\cos(x) & \forall x\in\mathbb{R}, & M\in\mathbb{Z} & \text{(D).} & \cos(x+2M\pi) & = & \cos(x) & \forall x\in\mathbb{R}, & M\in\mathbb{Z}\\
\text{(B).} & \sin(x+M\pi) & = & (-1)^M\sin(x) & \forall x\in\mathbb{R}, & M\in\mathbb{Z} & \text{(E).} & \sin(x+2M\pi) & = & \sin(x) & \forall x\in\mathbb{R}, & M\in\mathbb{Z}\\
\text{(C).} & e^{i(x+M\pi)} & = & (-1)^M e^{ix} & \forall x\in\mathbb{R}, & M\in\mathbb{Z} & \text{(F).} & e^{i(x+2M\pi)} & = & e^{ix} & \forall x\in\mathbb{R}, & M\in\mathbb{Z}
\end{array}$$

[20] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

✎PROOF:

1. Proof for (A):
   (a) $M = 0$ case: $\cos(x + 0\pi) = \cos(x) = (-1)^0 \cos(x)$
   (b) Proof for $M > 0$ cases (by induction):
      i. Base case $M = 1$:

$$\begin{aligned}
\cos(x+\pi) &= \cos x \cos\pi - \sin x \sin\pi && \text{by } \textit{double angle formulas} && \text{(Theorem C.9 page 85)}\\
&= \cos x(-1) - \sin x(0) && \text{by } \cos\pi = -1 \text{ result} && \text{(Proposition C.4 page 84)}\\
&= (-1)^1 \cos x
\end{aligned}$$

      ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\cos(x+[M+1]\pi) &= \cos([x+\pi]+M\pi)\\
&= (-1)^M \cos(x+\pi) && \text{by induction hypothesis } (M \text{ case})\\
&= (-1)^M(-1)\cos(x) && \text{by base case (item 1(b)i page 86)}\\
&= (-1)^{M+1}\cos(x)\\
&\implies M+1 \text{ case}
\end{aligned}$$

   (c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\cos(x+M\pi) &\triangleq \cos(x-N\pi) && \text{by definition of } N\\
&= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by } \textit{double angle formulas} && \text{(Theorem C.9 page 85)}\\
&= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem C.2 page 77}\\
&= \cos(x)\cos(0+N\pi) + \sin(x)\sin(0+N\pi)\\
&= \cos(x)(-1)^N\cos(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item 1b page 86)}\\
&= (-1)^N\cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem C.2 page 77)}\\
&\triangleq (-1)^{-M}\cos(x) && \text{by definition of } N\\
&= (-1)^M\cos(x)
\end{aligned}$$

   (d) Proof using complex exponential:

$$\begin{aligned}
\cos(x+M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by } \textit{Euler formulas} && \text{(Corollary C.2 page 81)}\\
&= e^{iM\pi}\left[\frac{e^{ix}+e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem C.6 page 81)}\\
&= \left(e^{i\pi}\right)^M \cos x && \text{by } \textit{Euler formulas} && \text{(Corollary C.2 page 81)}\\
&= (-1)^M \cos x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition C.4 page 84)}
\end{aligned}$$

2. Proof for (B):
   (a) $M = 0$ case: $\sin(x + 0\pi) = \sin(x) = (-1)^0 \sin(x)$
   (b) Proof for $M > 0$ cases (by induction):
      i. Base case $M = 1$:

$$\begin{aligned}
\sin(x+\pi) &= \sin x \cos\pi + \cos x \sin\pi && \text{by } \textit{double angle formulas} && \text{(Theorem C.9 page 85)}\\
&= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} && \text{(Proposition C.4 page 84)}\\
&= (-1)^1 \sin x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\sin(x+[M+1]\pi) &= \sin([x+\pi]+M\pi) \\
&= (-1)^M \sin(x+\pi) && \text{by induction hypothesis (M case)} \\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 86)} \\
&= (-1)^{M+1}\sin(x) \\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M<0$ cases: Let $N \triangleq -M \ldots \implies N>0$.

$$\begin{aligned}
\sin(x+M\pi) &\triangleq \sin(x-N\pi) && \text{by definition of } N \\
&= \sin(x)\sin(-N\pi)-\sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem C.9 page 85)} \\
&= \sin(x)\sin(N\pi)+\sin(x)\sin(N\pi) && \text{by Theorem C.2 page 77} \\
&= \sin(x)\sin(0+N\pi)+\sin(x)\sin(0+N\pi) \\
&= \sin(x)(-1)^N\sin(0)+\sin(x)(-1)^N\sin(0) && \text{by } M\geq 0 \text{ results} && \text{(item (2b) page 86)} \\
&= (-1)^N\sin(x) && \text{by } \sin(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem C.2 page 77)} \\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N \\
&= (-1)^M\sin(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)}-e^{-i(x+M\pi)}}{2i} && \text{by \textit{Euler formulas}} && \text{(Corollary C.2 page 81)} \\
&= e^{iM\pi}\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta}=e^\alpha e^\beta \text{ result} && \text{(Theorem C.6 page 81)} \\
&= \left(e^{i\pi}\right)^M \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary C.2 page 81)} \\
&= (-1)^M \sin x && \text{by } e^{i\pi}=-1 \text{ result} && \text{(Proposition C.4 page 84)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta}=e^\alpha e^\beta \text{ result} && \text{(Theorem C.6 page 81)} \\
&= \left(e^{i\pi}\right)^M\left(e^{ix}\right) \\
&= (-1)^M e^{ix} && \text{by } e^{i\pi}=-1 \text{ result} && \text{(Proposition C.4 page 84)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):
$$\begin{array}{lclcll}
\cos(i(x+2M\pi)) &=& (-1)^{2M}\cos(ix) &=& \cos(ix) & \text{by (A)} \\
\sin(i(x+2M\pi)) &=& (-1)^{2M}\sin(ix) &=& \sin(ix) & \text{by (B)} \\
e^{i(x+2M\pi)} &=& (-1)^{2M}e^{ix} &=& e^{ix} & \text{by (C)}
\end{array}$$

✏

Theorem C.11 (half-angle formulas/squared identities).

THM
(A). $\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$
(B). $\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$
(C). $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x)+\tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem C.8 page 82)} \\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem C.2 page 77)} \\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x)-\tfrac{1}{2}\cos(x+x) && \text{by \textit{product identities}} && \text{(Theorem C.8 page 82)} \\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem C.2 page 77)} \\
\cos^2 x+\sin^2 x &= \tfrac{1}{2}[1+\cos(2x)]+\tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)} \\
& && \text{note: see also} && \text{Theorem C.4 page 79}
\end{aligned}$$

✏

C.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

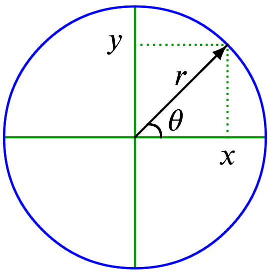
$$
\begin{aligned}
\langle \cos(x) \mid \sin(x) \rangle &= \int_{-\pi}^{+\pi} \cos(x)\sin(x)\, dx \\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x-x)\, dx + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x+x)\, dx && \text{by Theorem C.8 page 82} \\
&= \cancelto{0}{\frac{1}{2}\int_{-\pi}^{+\pi} \sin(0)\, dx} + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(2x)\, dx \\
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi} \cos(2x) \\
&= -\frac{1}{4}[\cos(2\pi) - \cos(-2\pi)] \\
&= 0
\end{aligned}
$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

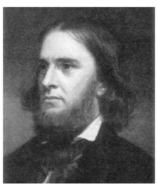
We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$



$$
\begin{array}{rcl@{\qquad}rcl}
\cos\theta & \triangleq & \frac{x}{r} & \sec\theta & \triangleq & \frac{r}{x} \\
\sin\theta & \triangleq & \frac{y}{r} & \csc\theta & \triangleq & \frac{r}{y} \\
\tan\theta & \triangleq & \frac{y}{x} & \cot\theta & \triangleq & \frac{x}{y}
\end{array}
$$

C.7 The power of the exponential



❝Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.❞

Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [21]

[21] quote: Kasner and Newman (1940) page 104
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

"Young man, in mathematics you don't understand things. You just get used to them."

John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [22]

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary C.3. [23]

COR
$$e^{i\pi} + 1 = 0$$

✎PROOF:

$$\begin{aligned}
e^{ix}\Big|_{x=\pi} &= [\cos x + i \sin x]_{x=\pi} && \text{by Euler's identity (Theorem C.5 page 80)}\\
&= -1 + i \cdot 0 && \text{by Proposition C.4 page 84}\\
&= -1
\end{aligned}$$

⇨

There are many transforms available, several of them integral transforms $[\mathbf{A}\mathsf{f}](s) \triangleq \int_t \mathsf{f}(s)\kappa(t,s)\,\mathrm{d}s$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:

① The *Laplace Transform* with kernel $\kappa(t, s) \triangleq e^{st}$
② The *Fourier Transform* with kernel $\kappa(t, \omega) \triangleq e^{i\omega t}$.

Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem C.12 page 89).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem C.12. [24] *Let* $\mathbf{L}$ *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*

$$\check{\mathsf{h}}(s) \triangleq \left\langle \mathsf{h}(t,\omega) \,|\, e^{st} \right\rangle \qquad \text{(LAPLACE TRANSFORM).}$$

---

[22] quote: Zukav (1980) page 208
image: http://en.wikipedia.org/wiki/John_von_Neumann
The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."

[23] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

[24] Mallat (1999) page 2, …page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

THM
$$\left\{\begin{array}{ll}1. & \mathbf{L} \text{ is LINEAR and}\\ 2. & \mathbf{L} \text{ is TIME-INVARIANT}\end{array}\right\} \implies \left\{\mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{\text{eigenvalue}} \; \underbrace{e^{st}}_{\text{eigenvector}}\right\}$$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \mid \mathsf{h}((t;u),s)\rangle && \text{by linear hypothesis}\\
&= \langle e^{su} \mid \mathsf{h}((t-u),s)\rangle && \text{by time-invariance hypothesis}\\
&= \left\langle e^{s(t-v)} \mid \mathsf{h}(v,s)\right\rangle && \text{let } v = t-u \implies u = t-v\\
&= e^{st}\ \langle e^{-sv} \mid \mathsf{h}(v,s)\rangle && \text{by additivity of } \langle \triangle \mid \nabla\rangle\\
&= \langle \mathsf{h}(v,s) \mid e^{-sv}\rangle^*\ e^{st} && \text{by conjugate symmetry of } \langle \triangle \mid \nabla\rangle\\
&= \left\langle \mathsf{h}(v,s) \mid e^{(-s)v}\right\rangle^*\ e^{st} &&\\
&= \check{\mathsf{h}}^*(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}$$

✏

APPENDIX D

TRIGONOMETRIC POLYNOMIALS

"I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives."

Charles Hermite (1822 – 1901), French mathematician, in an 1893 letter to Stieltjes, in response to the "pathological" everywhere continuous but nowhere differentiable *Weierstrass functions* $\mathsf{f}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$.[1]

D.1 Trigonometric expansion

Theorem D.1 (DeMoivre's Theorem).

THM $\left(re^{ix}\right)^n = r^n(\cos nx + i\sin nx) \qquad \forall r,x \in \mathbb{R}$

✎PROOF:

$$\begin{aligned}\left(re^{ix}\right)^n &= r^n e^{inx} \\ &= r^n\left(\cos nx + i\sin nx\right) && \text{by Euler's identity (Theorem C.5 page 80)}\end{aligned}$$

✏

The cosine with argument nx can be expanded as a polynomial in $\cos(x)$ (next).

Theorem D.2 (trigonometric expansion). [2]

---

[1] quote: Hermite (1893)
translation: Lakatos (1976) page 19
image: http://www-groups.dcs.sx-and.ac.uk/~history/PictDisplay/Hermite.html

[2] Rivlin (1974) page 3 ⟨(1.8)⟩

THM

$$\cos(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

$$\sin(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\sin x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(nx) &= \Re\left(\cos nx + i\sin nx\right) \\
&= \Re\left(e^{inx}\right) \\
&= \Re\left[\left(e^{ix}\right)^n\right] \\
&= \Re\left[(\cos x + i\sin x)^n\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n}\binom{n}{k}(\cos x)^{n-k}(i\sin x)^k\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n} i^k\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \Re\left[\sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + i\sum_{k\in\{1,5,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right. \\
&\quad \left. - \sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + -i\sum_{k\in\{3,7,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x - \sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x \\
&= \sum_{k\in\{0,2,\ldots,n\}}\binom{n}{k}(-1)^{\frac{k}{2}}\cos^{n-k}x\sin^k x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\sin^{2k}x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x(1-\cos^2 x)^k \\
&= \left[\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\right]\left[\sum_{m=0}^{k}\binom{k}{m}(-1)^m\cos^{2m}x\right] \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}x
\end{aligned}$$

$$\begin{aligned}
\sin(nx) &= \cos\left(nx - \frac{\pi}{2}\right) \\
&= \cos\left(n\left[x - \frac{\pi}{2n}\right]\right) \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}\left(n\left[x - \frac{\pi}{2n}\right]\right)
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(nx - \frac{\pi}{2} \right)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \sin^{n-2(k-m)} (nx)$$

✏

Example D.1.

EX	$\cos 5x$	$= 16\cos^5 x - 20\cos^3 x + 5\cos x$
	$\sin 5x$	$= 16\sin^5 x - 20\sin^3 x + 5\sin x.$

✎PROOF:

1. Proof using *DeMoivre's Theorem* (Theorem D.1 page 91):

$$\begin{aligned}
&\cos 5x + i\sin 5x \\
&= e^{i5x} \\
&= \left(e^{ix}\right)^5 \\
&= (\cos x + i\sin x)^5 \\
&= \sum_{k=0}^{5} \binom{5}{k} [\cos x]^{5-k} [i\sin x]^k \\
&= \binom{5}{0}[\cos x]^{5-0}[i\sin x]^0 + \binom{5}{1}[\cos x]^{5-1}[i\sin x]^1 + \binom{5}{2}[\cos x]^{5-2}[i\sin x]^2 + \\
&\binom{5}{3}[\cos x]^{5-3}[i\sin x]^3 + \binom{5}{4}[\cos x]^{5-4}[i\sin x]^4 + \binom{5}{5}[\cos x]^{5-5}[i\sin x]^5 \\
&= 1\cos^5 x + i5\cos^4 x\sin x - 10\cos^3 x\sin^2 x - i10\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i1\sin^5 x \\
&= \left[\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\right] + i\left[5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)(1-\cos^2 x)\right] + \\
&i\left[5(1-\sin^2 x)(1-\sin^2 x)\sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)\right] + \\
&i\left[5(1 - 2\sin^2 x + \sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5(\cos x - 2\cos^3 x + \cos^5 x)\right] + \\
&i\left[5(\sin x - 2\sin^3 x + \sin^5 x) - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \underbrace{\left[16\cos^5 x - 20\cos^3 x + 5\cos x\right]}_{\cos 5x} + i\underbrace{\left[16\sin^5 x - 20\sin^3 x + 5\sin x\right]}_{\sin 5x}
\end{aligned}$$

2. Proof using trigonometric expansion (Theorem D.2 page 91):

$$\begin{aligned}
\cos 5x &= \sum_{k=0}^{\lfloor \frac{5}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{2} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{5-2(k-m)} \\
&= (-1)^0 \binom{5}{0}\binom{0}{0}\cos^5 x + (-1)^1 \binom{5}{2}\binom{1}{0}\cos^3 x + (-1)^2 \binom{5}{2}\binom{1}{1}\cos^5 x + \\
&(-1)^2 \binom{5}{4}\binom{2}{0}\cos^1 x + (-1)^3 \binom{5}{4}\binom{2}{1}\cos^3 x + (-1)^4 \binom{5}{4}\binom{2}{2}\cos^5 x
\end{aligned}$$

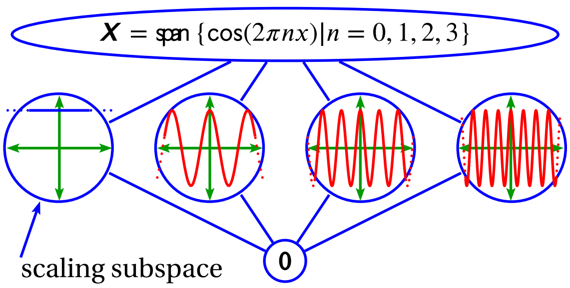


Figure D.1: Lattice of harmonic cosines $\{\cos(nx) | n = 0, 1, 2, \ldots\}$

$$
\begin{aligned}
&= +(1)(1)\cos^5 x - (10)(1)\cos^3 x + (10)(1)\cos^5 x + (5)(1)\cos x - (5)(2)\cos^3 x + (5)(1)\cos^5 x \\
&= +(1 + 10 + 5)\cos^5 x + (-10 - 10)\cos^3 x + 5\cos x \\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}
$$

✏

Example D.2. [3]

n	$\cos nx$		polynomial in $\cos x$	n	$\cos nx$		polynomial in $\cos x$
0	$\cos 0x$	$=$	1	4	$\cos 4x$	$=$	$8\cos^4 x - 8\cos^2 x + 1$
1	$\cos 1x$	$=$	$\cos^1 x$	5	$\cos 5x$	$=$	$16\cos^5 x - 20\cos^3 x + 5\cos x$
2	$\cos 2x$	$=$	$2\cos^2 x - 1$	6	$\cos 6x$	$=$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
3	$\cos 3x$	$=$	$4\cos^3 x - 3\cos x$	7	$\cos 7x$	$=$	$64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

✎PROOF:

$$
\begin{aligned}
\cos 2x &= \sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{2-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0}\cos^2 x + (-1)^1 \binom{3}{2}\binom{1}{0}\cos^0 x + (-1)^2 \binom{3}{2}\binom{1}{1}\cos^2 x \\
&= +(1)(1)\cos^2 x - (1)(1) + (1)(1)\cos^2 x \\
&= 2\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 3x &= \sum_{k=0}^{\lfloor \frac{3}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{3-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0}\cos^3 x + (-1)^1 \binom{3}{2}\binom{1}{0}\cos^1 x + (-1)^2 \binom{3}{2}\binom{1}{1}\cos^3 x \\
&= +\binom{3}{0}\binom{0}{0}\cos^3 x - \binom{3}{2}\binom{1}{0}\cos^1 x + \binom{3}{2}\binom{1}{1}\cos^3 x \\
&= +(1)(1)\cos^3 x - (3)(1)\cos^1 x + (3)(1)\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}
$$

$$
\cos 4x = \sum_{k=0}^{\lfloor \frac{4}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{4}{2k} \binom{k}{m} (\cos x)^{4-2(k-m)}
$$

[3] Abramowitz and Stegun (1972) page 795, Guillemin (1957) page 593 ⟨(21)⟩, Sloane (2014) ⟨http://oeis.org/A039991⟩, Sloane (2014) ⟨http://oeis.org/A028297⟩

$$
\begin{aligned}
&= \sum_{k=0}^{2}\sum_{m=0}^{k}(-1)^{k+m}\binom{4}{2k}\binom{k}{m}(\cos x)^{4-2(k-m)}\\
&= (-1)^{0+0}\binom{4}{2\cdot 0}\binom{0}{0}(\cos x)^{4-2(0-0)} + (-1)^{1+0}\binom{4}{2\cdot 1}\binom{1}{0}(\cos x)^{4-2(1-0)}\\
&\quad + (-1)^{1+1}\binom{4}{2\cdot 1}\binom{1}{1}(\cos x)^{4-2(1-1)} + (-1)^{2+0}\binom{4}{2\cdot 2}\binom{2}{0}(\cos x)^{4-2(2-0)}\\
&\quad + (-1)^{2+1}\binom{4}{2\cdot 2}\binom{2}{1}(\cos x)^{4-2(2-1)} + (-1)^{2+2}\binom{4}{2\cdot 2}\binom{2}{2}(\cos x)^{4-2(2-2)}\\
&= (1)(1)\cos^4 x - (6)(1)\cos^2 x + (6)(1)\cos^4 x + (1)(1)\cos^0 x - (1)(2)\cos^2 x + (1)(1)\cos^4 x\\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}
$$

$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ see Example D.1 page 93

$$
\begin{aligned}
\cos 6x &= \sum_{k=0}^{\left\lfloor\frac{6}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{6}{2k}\binom{k}{m}(\cos x)^{6-2(k-m)}\\
&= (-1)^0\binom{6}{0}\binom{0}{0}\cos^6 x + (-1)^1\binom{6}{2}\binom{1}{0}\cos^4 x + (-1)^2\binom{6}{2}\binom{1}{1}\cos^6 x + (-1)^2\binom{6}{4}\binom{2}{0}\cos^2 x +\\
&(-1)^3\binom{6}{4}\binom{2}{1}\cos^4 x + (-1)^4\binom{6}{4}\binom{2}{2}\cos^6 x + (-1)^3\binom{6}{6}\binom{3}{0}\cos^0 x + (-1)^4\binom{6}{6}\binom{3}{1}\cos^2 x +\\
&(-1)^5\binom{6}{6}\binom{3}{2}\cos^4 x + (-1)^6\binom{6}{6}\binom{3}{3}\cos^6 x\\
&= +(1)(1)\cos^6 x - (15)(1)\cos^4 x + (15)(1)\cos^6 x + (15)(1)\cos^2 x - (15)(2)\cos^4 x + (15)(1)\cos^6 x\\
&\quad - (1)(1)\cos^0 x + (1)(3)\cos^2 x - (1)(3)\cos^4 x + (1)(1)\cos^6 x\\
&= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 7x &= \sum_{k=0}^{\left\lfloor\frac{7}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)}\\
&= \sum_{k=0}^{3}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{7-2(k-m)}\\
&= (-1)^0\binom{7}{0}\binom{0}{0}\cos^7 x + (-1)^1\binom{7}{2}\binom{1}{0}\cos^5 x + (-1)^2\binom{7}{2}\binom{1}{1}\cos^7 x + (-1)^2\binom{7}{4}\binom{2}{0}\cos^3 x\\
&\quad + (-1)^3\binom{7}{4}\binom{2}{1}\cos^5 x + (-1)^4\binom{7}{4}\binom{2}{2}\cos^7 x + (-1)^3\binom{7}{6}\binom{3}{0}\cos^1 x + (-1)^4\binom{7}{6}\binom{3}{1}\cos^3 x\\
&\quad + (-1)^5\binom{7}{6}\binom{3}{2}\cos^5 x + (-1)^6\binom{7}{6}\binom{3}{3}\cos^7 x\\
&= (1)(1)\cos^7 x - (21)(1)\cos^5 x + (21)(1)\cos^7 x + (35)(1)\cos^3 x\\
&\quad - (35)(2)\cos^5 x + (35)(1)\cos^7 x - (7)(1)\cos^1 x + (7)(3)\cos^3 x\\
&\quad - (7)(3)\cos^5 x + (7)(1)\cos^7 x\\
&= (1+21+35+7)\cos^7 x - (21+70+21)\cos^5 x + (35+21)\cos^3 x - (7)\cos^1 x\\
&= 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x
\end{aligned}
$$

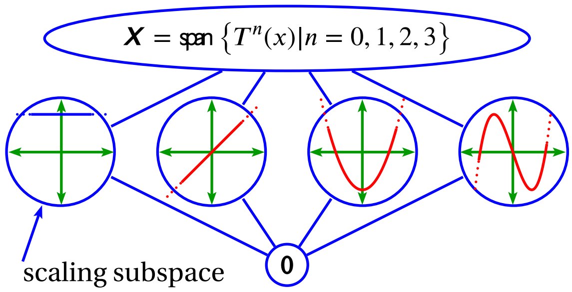


Figure D.2: Lattice of Chebyshev polynomials $\{T_n(x)|n = 0, 1, 2, 3\}$

Note: Trigonometric expansion of $\cos(nx)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[4]

```
trigexpand(cos(2*x));
trigexpand(cos(3*x));
trigexpand(cos(4*x));
trigexpand(cos(5*x));
trigexpand(cos(6*x));
trigexpand(cos(7*x));
```

Definition D.1.

DEF *The* n*th* ***Chebyshev polynomial of the first kind*** *is defined as*

$$T_n(x) \triangleq \cos nx \qquad \text{where} \qquad \cos x \triangleq x$$

Theorem D.3. [5] *Let* $T_n(x)$ *be a* CHEBYSHEV POLYNOMIAL *with* $n \in \mathbb{W}$.

THM

$$\begin{array}{lcl} n \text{ is EVEN} & \Longrightarrow & T_n(x) \text{ is EVEN.} \\ n \text{ is ODD} & \Longrightarrow & T_n(x) \text{ is ODD.} \end{array}$$

Example D.3. Let $T_n(x)$ be a *Chebyshev polynomial* with $n \in \mathbb{W}$.

EX

$T_0(x) = 1$	$T_4(x) = 8x^4 - 8x^2 + 1$
$T_1(x) = x$	$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_2(x) = 2x^2 - 1$	$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
$T_3(x) = 4x^3 - 3x$	

✎PROOF: Proof of these equations follows directly from Example D.2 (page 94).

D.2 Trigonometric reduction

Theorem D.2 (page 91) showed that $\cos nx$ can be expressed as a polynomial in $\cos x$. Conversely, Theorem D.4 (next) shows that a polynomial in $\cos x$ can be expressed as a linear combination of $(\cos nx)_{n\in\mathbb{Z}}$.

Theorem D.4 (trigonometric reduction)**.**

[4] maxima pages 157–158 ⟨10.5 Trigonometric Functions⟩

[5] Rivlin (1974) page 5 ⟨(1.13)⟩, Süli and Mayers (2003) page 242 ⟨Lemma 8.2⟩, Davidson and Donsig (2010) page 222 ⟨exercise 10.7.A(a)⟩

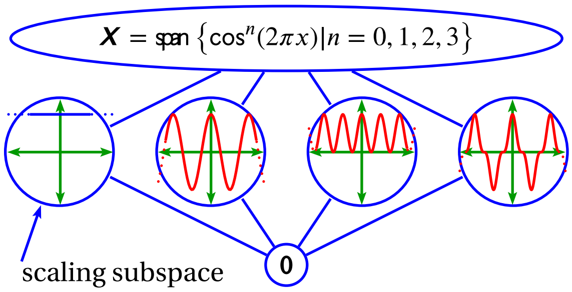


Figure D.3: Lattice of exponential cosines $\left\{\cos^n x|n = 0,1,2,3\right\}$

THM

$$\begin{aligned}
\cos^n x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases}
\frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & \text{for } n \text{ even} \\
\frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & \text{for } n \text{ odd}
\end{cases}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\cos^n x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n \\
&= \mathbf{R_e}\left[\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)x}e^{-ikx}\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\cos[(n-2k)x] + i\sin[(n-2k)x]\right)\right] \\
&= \mathbf{R_e}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] + i\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sin[(n-2k)x]\right] \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\
&= \begin{cases}
\frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & : \ n \text{ even} \\
\frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & : \ n \text{ odd}
\end{cases}
\end{aligned}$$

⇒

Example D.4. [6]

[6] Abramowitz and Stegun (1972) page 795, Sloane (2014) ⟨http://oeis.org/A100257⟩, Sloane (2014) ⟨http://oeis.org/A008314⟩

EX

n	$\cos^n x$		trigonometric reduction	n	$\cos^n x$		trigonometric reduction
0	$\cos^0 x$	$=$	1	4	$\cos^4 x$	$=$	$\dfrac{\cos 4x + 4\cos 2x + 3}{2^3}$
1	$\cos^1 x$	$=$	$\cos x$	5	$\cos^5 x$	$=$	$\dfrac{\cos 5x + 5\cos 3x + 10\cos x}{2^4}$
2	$\cos^2 x$	$=$	$\dfrac{\cos 2x + 1}{2}$	6	$\cos^6 x$	$=$	$\dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{2^5}$
3	$\cos^3 x$	$=$	$\dfrac{\cos 3x + 3\cos x}{2^2}$	7	$\cos^7 x$	$=$	$\dfrac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{2^6}$

✎PROOF:

$$\begin{aligned}
\cos^0 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=0} \\
&= \frac{1}{2^0}\sum_{k=0}^{0}\binom{0}{k}\cos[(0-2k)x] \\
&= \binom{0}{0}\cos[(0-2\cdot 0)x] \\
&= 1 \\
\cos^1 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=1} \\
&= \frac{1}{2^1}\sum_{k=0}^{1}\binom{1}{k}\cos[(1-2k)x] \\
&= \frac{1}{2}\left[\binom{1}{0}\cos[(1-2\cdot 0)x] + \binom{1}{1}\cos[(1-2\cdot 1)x]\right] \\
&= \frac{1}{2}\left[1\cos x + 1\cos(-x)\right] \\
&= \frac{1}{2}(\cos x + \cos x) \\
&= \cos x \\
\cos^2 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=2} \\
&= \frac{1}{2^2}\sum_{k=0}^{2}\binom{2}{k}\cos([2-2k]x) \\
&= \frac{1}{2^2}\left[\binom{2}{0}\cos([2-2\cdot 0]x) + \binom{2}{1}\cos([2-2\cdot 1]x) + \binom{2}{2}\cos([2-2\cdot 2]x) +\right] \\
&= \frac{1}{2^2}\left[1\cos(2x) + 2\cos(0x) + 1\cos(-2x)\right] \\
&= \frac{1}{2^2}\left[\cos(2x) + 2 + \cos(2x)\right] \\
&= \frac{1}{2}\left[\cos(2x) + 1\right] \\
\cos^3 x &= \left.\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\right|_{n=3} \\
&= \frac{1}{2^3}\sum_{k=0}^{3}\binom{3}{k}\cos([3-2k]x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^3}\left[1\cos(3x) + 3\cos(1x) + 3\cos(-1x) + 1\cos(-3x)\right] \\
&= \frac{1}{2^3}\left[\cos(3x) + 3\cos(x) + 3\cos(x) + \cos(3x)\right] \\
&= \frac{1}{2^2}\left[\cos(3x) + 3\cos(x)\right]
\end{aligned}$$

$$\begin{aligned}
\cos^4 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=4} \\
&= \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}\cos([4-2k]x) \\
&= \frac{1}{2^4}\left[1\cos(4x) + 4\cos(2x) + 6\cos(0x) + 4\cos(-2x) + 1\cos(-4x)\right] \\
&= \frac{1}{2^3}\left[\cos(4x) + 4\cos(2x) + 3\right]
\end{aligned}$$

$$\begin{aligned}
\cos^5 x &= \frac{1}{2^{5-1}}\sum_{k=0}^{\lfloor\frac{5}{2}\rfloor}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\sum_{k=0}^{2}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\left[\binom{5}{0}\cos 5x + \binom{5}{1}\cos 3x + \binom{5}{2}\cos x\right] \\
&= \frac{1}{16}\left[\cos 5x + 5\cos 3x + 10\cos x\right]
\end{aligned}$$

$$\begin{aligned}
\cos^6 x &= \frac{1}{2^6}\binom{6}{\frac{6}{2}} + \frac{1}{2^{6-1}}\sum_{k=0}^{\frac{6}{2}-1}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{2^6}\binom{6}{3} + \frac{1}{2^5}\sum_{k=0}^{2}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{64}20 + \frac{1}{32}\left[\binom{6}{0}\cos 6x + \binom{6}{1}\cos 4x\binom{6}{2}\cos 2x\right] \\
&= \frac{1}{32}\left[\cos 6x + 6\cos 4x + 15\cos 2x + 10\right]
\end{aligned}$$

$$\begin{aligned}
\cos^7 x &= \frac{1}{2^{7-1}}\sum_{k=0}^{\lfloor\frac{7}{2}\rfloor}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\sum_{k=0}^{2}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\left[\binom{7}{0}\cos 7x + \binom{7}{1}\cos 5x + \binom{7}{2}\cos 3x + \binom{7}{3}\cos x\right] \\
&= \frac{1}{64}\left[\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right]
\end{aligned}$$

Note: Trigonometric reduction of $\cos^n(x)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[7]

```
trigreduce((cos(x))^2);
trigreduce((cos(x))^3);
trigreduce((cos(x))^4);
trigreduce((cos(x))^5);
trigreduce((cos(x))^6);
trigreduce((cos(x))^7);
```

7 http://maxima.sourceforge.net/docs/manual/en/maxima_15.html
maxima page 158 ⟨10.5 Trigonometric Functions⟩

✏

D.3 Spectral Factorization

Theorem D.5 (Fejér-Riesz spectral factorization). [8] *Let* $[0,\infty) \subsetneq \mathbb{R}$ *and*

$$\mathsf{p}\left(e^{ix}\right) \triangleq \sum_{n=-N}^{N} a_n e^{inx} \qquad \textit{(Laurent trigonometric polynomial order } 2N\textit{)}$$

$$\mathsf{q}\left(e^{ix}\right) \triangleq \sum_{n=1}^{N} b_n e^{inx} \qquad \textit{(standard trigonometric polynomial order } N\textit{)}$$

THM $\mathsf{p}\left(e^{ix}\right) \in [0,\infty) \quad \forall x \in [0,2\pi] \implies \begin{cases} \exists (\!(b_n)\!)_{n\in\mathbb{Z}} & \text{such that} \\ \mathsf{p}\left(e^{ix}\right) = \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right) & \forall x \in \mathbb{R} \end{cases}$

✎PROOF:

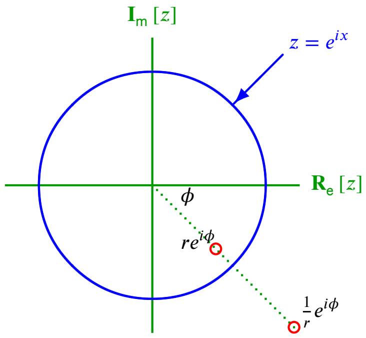
1. Proof that $a_n = a^*_{-n}$ ($(\!(a_n)\!)_{n\in\mathbb{Z}}$ is *Hermitian symmetric*):
   Let $a_n \triangleq r_n e^{i\phi_n}$, $r_n, \phi_n \in \mathbb{R}$. Then

$$\begin{aligned}
\mathsf{p}\left(e^{inx}\right) &\triangleq \sum_{n=-N}^{N} a_n e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{i\phi_n} e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{inx+\phi_n} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\sum_{n=-N}^{N} r_n \sin(nx+\phi_n) \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\underbrace{\left[r_0\sin(0x+\phi_0) + \sum_{n=1}^{N} r_n\sin(nx+\phi_n) + \sum_{n=1}^{N} r_{-n}\sin(-nx+\phi_{-n})\right]}_{\text{imaginary part must equal 0 because } p(x)\in\mathbb{R}} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i\underbrace{\left[r_0\sin(\phi_0) + \sum_{n=1}^{N} r_n\sin(nx+\phi_n) - \sum_{n=1}^{N} r_{-n}\sin(nx-\phi_{-n})\right]}_{\implies r_n = r_{-n},\ \phi_n = -\phi_{-n} \quad \implies a_n = a^*_{-n},\ a_0\in\mathbb{R}}
\end{aligned}$$

2. Because the coefficients $(\!(c_n)\!)_{n\in\mathbb{Z}}$ are *Hermitian symmetric*, the zeros of $P(z)$ occur in *conjugate reciprical pairs*. This means that if $\sigma \in \mathbb{C}$ is a zero of $P(z)$ ($P(\sigma)=0$), then $\frac{1}{\sigma^*}$ is also a zero of $P(z)$ ($P\left(\frac{1}{\sigma^*}\right) = 0$). In the complex z plane, this relationship means zeros are reflected across the unit circle such that

$$\frac{1}{\sigma^*} = \frac{1}{(re^{i\phi})^*} = \frac{1}{r}\frac{1}{e^{-i\phi}} = \frac{1}{r}e^{i\phi}$$

[8] Pinsky (2002) pages 330–331



3. Because the zeros of $\mathsf{p}(z)$ occur in conjugate recipricol pairs, $\mathsf{p}\left(e^{ix}\right)$ can be factored:

$$
\begin{aligned}
\mathsf{p}\left(e^{ix}\right) &= \mathsf{p}(z)|_{z=e^{ix}} \\
&= z^{-N} C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z - \frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N} z^{-1}\left(z - \frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(1 - \frac{1}{\sigma_n^*}z^{-1}\right)\bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z^{-1} - \sigma_n^*\right)\left(-\frac{1}{\sigma_n^*}\right)\bigg|_{z=e^{ix}} \\
&= \left[C \prod_{n=1}^{N}\left(-\frac{1}{\sigma_n^*}\right)\right]\left[\prod_{n=1}^{N}(z-\sigma_n)\right]\left[\prod_{n=1}^{N}\left(\frac{1}{z^*} - \sigma_n\right)\right]^*\bigg|_{z=e^{ix}} \\
&= \left[C_2 \prod_{n=1}^{N}(z-\sigma_n)\right]\left[C_2 \prod_{n=1}^{N}\left(\frac{1}{z^*} - \sigma_n\right)\right]^*\bigg|_{z=e^{ix}} \\
&= \mathsf{q}(z)\mathsf{q}^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{ix}} \\
&= \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right)
\end{aligned}
$$

D.4 Dirichlet Kernel

“Dirichlet alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous proof is. Rather we learn it first from him. When Gauss says he has proved something it is clear; when Cauchy says it, one can wager as much pro as con; when Dirichlet says it, it is certain.”

Carl Gustav Jacob Jacobi (1804–1851), Jewish-German mathematician [9]

[9] quote: Schubring (2005) page 558
image: http://en.wikipedia.org/wiki/File:Carl_Jacobi.jpg, public domain

The *Dirichlet Kernel* is critical in proving what is not immediately obvious in examining the Fourier Series—that for a broad class of periodic functions, a function can be recovered from (with uniform convergence) its Fourier Series analysis.

Definition D.2. [10]

DEF

*The **Dirichlet Kernel** $\mathsf{D}_n \in \mathbb{R}^{\mathbb{W}}$ with period τ is defined as*

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}kx}$$

Proposition D.1. [11] *Let D_n be the* DIRICHLET KERNEL *with period τ* (Definition D.2 page 102).

PRP

$$\mathsf{D}_n(x) = \frac{1}{\tau}\frac{\sin\left(\frac{\pi}{\tau}[2n+1]x\right)}{\sin\left(\frac{\pi}{\tau}x\right)}$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &\triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \mathsf{D}_n \quad \text{(Definition D.2 page 102)}\\
&= \frac{1}{\tau}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}(k-n)x} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}kx} = \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\sum_{k=0}^{2n}\left(e^{i\frac{2\pi}{\tau}x}\right)^k\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\frac{1-\left(e^{i\frac{2\pi}{\tau}x}\right)^{2n+1}}{1-e^{i\frac{2\pi}{\tau}x}} && \text{by \textit{geometric series}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\frac{1-e^{i\frac{2\pi}{\tau}(2n+1)x}}{1-e^{i\frac{2\pi}{\tau}x}} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(\frac{e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{i\frac{\pi}{\tau}x}}\right)\frac{e^{-i\frac{\pi}{\tau}(2n+1)x}-e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{-i\frac{\pi}{\tau}x}-e^{i\frac{\pi}{\tau}x}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(e^{i\frac{2\pi n}{\tau}x}\right)\frac{-2i\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{-2i\sin\left[\frac{\pi}{\tau}x\right]} = \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}x\right]}
\end{aligned}$$

✏

Proposition D.2. [12] *Let D_n be the* DIRICHLET KERNEL *with period τ* (Definition D.2 page 102).

PRP

$$\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x = 1$$

✎PROOF:

$$\begin{aligned}
\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x &\triangleq \int_0^\tau \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \mathsf{D}_n \text{ (Definition D.2 page 102)}\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right) + i\sin\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x
\end{aligned}$$

[10] Katznelson (2004) page 14, Heil (2011) pages 443–444, Folland (1992) pages 33–34
[11] Katznelson (2004) page 14, Heil (2011) page 444, Folland (1992) page 34
[12] Bruckner et al. (1997) pages 620–621

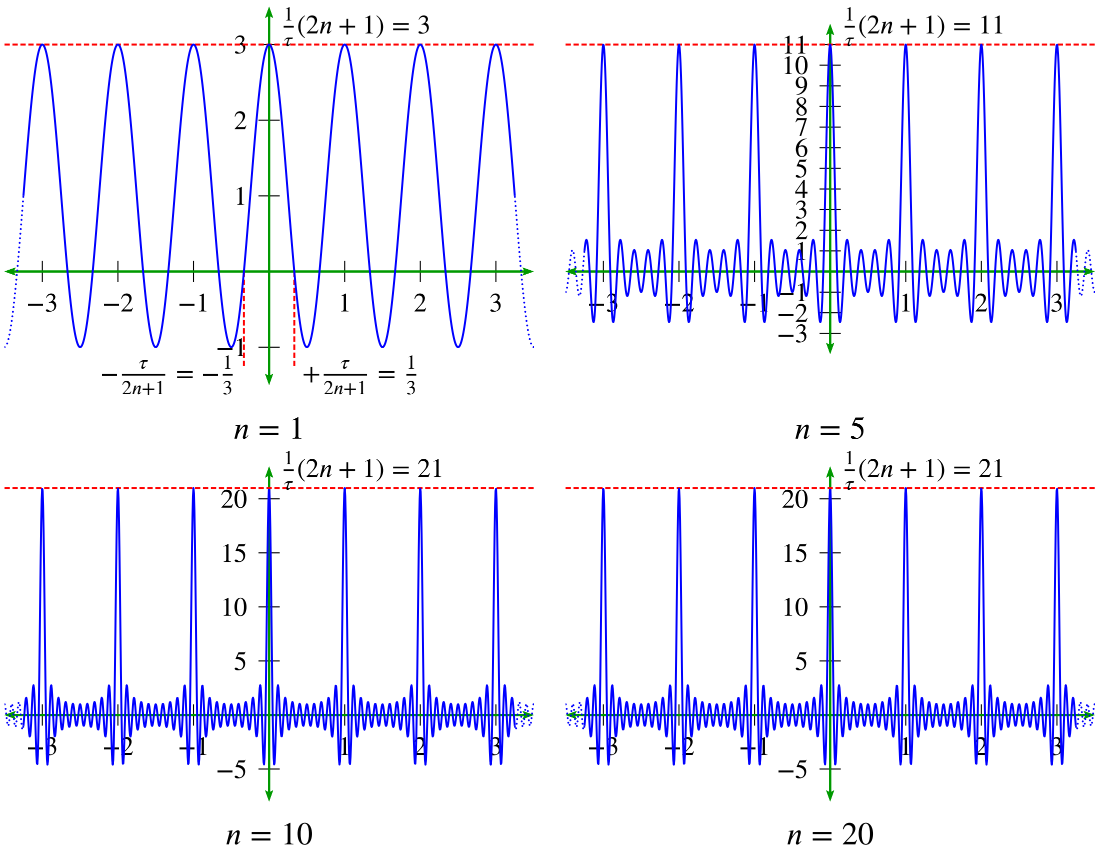


Figure D.4: D_n function for $N = 1, 5, 10, 20$. $\mathsf{D}_n \to$ comb. (See Proposition D.1 page 102).

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x \\
&= \frac{1}{\tau}\sum_{k=-n}^{n} \left.\frac{\sin\left(\frac{2\pi}{\tau}nx\right)}{\frac{2\pi}{\tau}n}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left[\frac{\sin\left(\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n} - \frac{\sin\left(-\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n}\right] \\
&= \frac{1}{\tau}\frac{\tau}{2}\sum_{k=-n}^{n}\left[\frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi n)}{\pi n}\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n)}{\pi n}\right]_{k=0} \\
&= 1
\end{aligned}
$$

Proposition D.3. *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition D.2 page 102). *Let* w_N *(the "*WIDTH*" of* $\mathsf{D}_n(x)$*) be the distance between the two points where the center pulse of* $\mathsf{D}_n(x)$ *intersects the x axis.*

PRP

$$
\begin{aligned}
\mathsf{D}_n(0) &= \frac{1}{\tau}(2n+1) \\
w_n &= \frac{2\tau}{2n+1}
\end{aligned}
$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(0) &= \mathsf{D}_n(x)\big|_{t=0} \\
&= \left.\frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by Proposition D.1 page 102} \\
&= \left.\frac{1}{\tau}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\frac{\mathrm{d}}{\mathrm{d}x}\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \left.\frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{\cos\left[\frac{\pi}{\tau}(2n+1)x\right]}{\cos\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{1}{1} \\
&= \frac{1}{\tau}(2n+1)
\end{aligned}$$

The center pulse of kernel $\mathsf{D}_n(x)$ intersects the x axis at

$$t = \pm\frac{\tau}{(2n+1)}$$

which implies

$$w_n = \frac{\tau}{2n+1} + \frac{\tau}{2n+1} = \frac{2\tau}{(2n+1)}.$$

✏

Proposition D.4. [13] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition D.2 page 102).

PRP: $\mathsf{D}_n(x) = \mathsf{D}_n(-x)$ (D_n *is an* EVEN *function*)

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]} && \text{by Proposition D.1 page 102} \\
&= \frac{1}{\tau}\frac{-\sin\left[-\frac{\pi}{\tau}(2n+1)x\right]}{-\sin\left[-\frac{\pi}{\tau}t\right]} && \text{because } \sin x \text{ is an } \textit{odd} \text{ function} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)(-x)\right]}{\sin\left[\frac{\pi}{\tau}(-x)\right]} \\
&= \mathsf{D}_n(-x) && \text{by Proposition D.1 page 102}
\end{aligned}$$

✏

[13] Bruckner et al. (1997) pages 620–621

D.5 Trigonometric summations

Theorem D.6 (Lagrange trigonometric identities). [14]

THM

$$\sum_{n=0}^{N-1}\cos(nx) = \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)+\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)+\cos\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=0}^{N-1}\mathfrak{R}e^{inx} = \mathfrak{R}\sum_{n=0}^{N-1}e^{inx} = \mathfrak{R}\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \mathfrak{R}\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \mathfrak{R}\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-i\frac{1}{2}x}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \mathfrak{R}\left[\left(e^{i\frac{1}{2}(N-1)x}\right)\left(\frac{-i\frac{1}{2}\sin\left(\frac{1}{2}Nx\right)}{-i\frac{1}{2}\sin\left(\frac{1}{2}x\right)}\right)\right] \\
&= \cos\left(\frac{1}{2}(N-1)x\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{-\frac{1}{2}\sin\left(-\frac{1}{2}x\right)+\frac{1}{2}\sin\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by } \textit{product identities} \quad \text{(Theorem C.8 page 82)} \\
&= \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=0}^{N-1}\mathfrak{I}e^{inx} = \mathfrak{I}\sum_{n=0}^{N-1}e^{inx} = \mathfrak{I}\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \mathfrak{I}\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by } \textit{geometric series} \\
&= \mathfrak{I}\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-ix/2}-e^{i\frac{1}{2}x}}\right)\right] \\
&= \mathfrak{I}\left[\left(e^{i(N-1)x/2}\right)\left(\frac{-\frac{1}{2}i\sin\left(\frac{1}{2}Nx\right)}{-\frac{1}{2}i\sin\left(\frac{1}{2}x\right)}\right)\right]
\end{aligned}$$

[14] Muniz (1953) page 140 ⟨"Lagrange's Trigonometric Identities"⟩, Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (14), (13)⟩

$$= \sin\left(\frac{(N-1)x}{2}\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right)$$

$$= \frac{\frac{1}{2}\cos\left(-\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} \qquad \text{by } \textit{product identities} \qquad \text{(Theorem C.8 page 82)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}$$

Note that these results (summed with indices from $n = 0$ to $n = N-1$) are compatible with Muniz (1953) page 140 (summed with indices from $n = 1$ to $n = N$) as demonstrated next:

$$\sum_{n=0}^{N-1}\cos(nx) = \sum_{n=1}^{N}\cos(nx) + [\cos(0x) - \cos(Nx)]$$

$$= \left[-\frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + [\cos(0x) - \cos(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \left(1 - \frac{1}{2}\right) + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\cos(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\left[\sin\left(\left[\frac{1}{2}-N\right]x\right) + \sin\left[\left(\frac{1}{2}+N\right)x\right]\right]}{2\sin\left(\frac{1}{2}x\right)} \qquad \text{by Theorem C.8 page 82}$$

$$= \frac{1}{2} + \frac{\sin\left(\frac{1}{2}[2N-1]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \sum_{n=1}^{N}\sin(nx) + [\sin(0x) - \sin(Nx)]$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} + [0 - \sin(Nx)] \qquad \text{by Muniz (1953) page 140}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\sin(Nx)}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - \left[\cos\left(\left[\frac{1}{2}-N\right]x\right) - \cos\left(\left[\frac{1}{2}+N\right]x\right)\right]}{2\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \Longrightarrow \text{above result}$$

✏

Theorem D.7. [15]

[15] Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (16) and (17)⟩

THM

$$\sum_{n=0}^{N-1}\cos(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]-\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]+\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}\sum_{n=0}^{N-1}\cos(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)-\sin(nx)\sin(y)] && \text{by *double angle formulas* (Theorem C.9 page 85)}\\ &= \cos(y)\sum_{n=0}^{N-1}\cos(nx)-\sin(y)\sum_{n=0}^{N-1}\sin(nx)\\ \sum_{n=0}^{N-1}\sin(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)+\sin(nx)\sin(y)] && \text{by *double angle formulas* (Theorem C.9 page 85)}\\ &= \cos(y)\sum_{n=0}^{N-1}\cos(nx)+\sin(y)\sum_{n=0}^{N-1}\sin(nx)\end{aligned}$$

✏

Corollary D.1 (Summation around unit circle).

THM

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)=0 \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

$$\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right)=\frac{N}{2} \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

✎PROOF:

$$\begin{aligned}&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\\ &= \cos(\theta)\sum_{n=0}^{N-1}\cos\left(\frac{2nM\pi}{N}\right)-\sin(\theta)\sum_{n=0}^{N-1}\sin\left(\frac{2nM\pi}{N}\right) && \text{by Theorem C.9 page 85}\\ &= \cos(\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{2M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] && \text{by Theorem D.6 page 105}\\ &= \cos(\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{\cos\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]\\ &= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{M\pi}{N}\right)}{\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)\right] && \text{by *trigonometric periodicity* (Theorem C.10 page 85)}\\ &= \cos(\theta)[0]-\sin(\theta)[0]\\ &= 0\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1}\cos\left(\theta-\frac{\pi}{2}+\frac{2nM\pi}{N}\right) && \text{by \textit{shift identities}} \quad \text{(Theorem C.7 page 82)}\\
&= \sum_{n=0}^{N-1}\cos\left(\phi+\frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta-\frac{\pi}{2}\\
&= 0 && \text{by previous result}
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)\\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(\left[\theta+\frac{2nM\pi}{N}\right]-\left[\theta+\frac{2nM\pi}{N}\right]\right)+\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(\left[\theta+\frac{2nM\pi}{N}\right]+\left[\theta+\frac{2nM\pi}{N}\right]\right) && \text{by Theorem C.8 page 82}\\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\cancelto{0}{\sin(0)}+\frac{1}{2}\sum_{n=0}^{N-1}\sin\left(2\theta+\frac{4nM\pi}{N}\right)\\
&= \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)+\frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right) && \text{by Theorem C.9 page 85}\\
&= \cos(2\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{4M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{4M\pi}{N}\right)}\right] && \text{by Theorem D.6 page 105}\\
&= \cos(2\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{2M\pi}{N}-4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]-\sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)-\frac{\cos\left(\frac{2M\pi}{N}-4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{2M\pi}{N}\right)}{\sin\left(\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity}} \quad \text{(Theorem C.10 page 85)}\\
&= \cos(\theta)[0]-\sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left[1+\cos\left(2\theta+\frac{4nM\pi}{N}\right)\right] && \text{by Theorem C.11 page 87}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}\left[1+\cos(2\theta)\cos\left(\frac{4nM\pi}{N}\right)-\sin(2\theta)\sin\left(\frac{4nM\pi}{N}\right)\right] && \text{by Theorem C.9 page 85}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}1+\frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\cos\left(\frac{4nM\pi}{N}\right)-\frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)\\
&= \left[\frac{1}{2}\sum_{n=0}^{N-1}1\right]+\frac{1}{2}\cos(2\theta)0-\frac{1}{2}\sin(2\theta)0 && \text{by previous results}\\
&= \frac{N}{2}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1}\cos^2\left(\theta-\frac{\pi}{2}+\frac{2nM\pi}{N}\right) && \text{by *shift identities* (Theorem C.7 page 82)}\\
&= \sum_{n=0}^{N-1}\cos^2\left(\phi+\frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta-\frac{\pi}{2}\\
&= \frac{N}{2} && \text{by previous result}
\end{aligned}$$

✏

D.6 Summability Kernels

Definition D.3. [16] *Let $(\!(\kappa_n)\!)_{n\in\mathbb{Z}}$ be a sequence of* CONTINUOUS 2π PERIODIC *functions.*

DEF

The sequence $(\!(\kappa_n)\!)_{n\in\mathbb{Z}}$ is a ***summability kernel*** *if*

1. $\frac{1}{2\pi}\int_0^{2\pi}\kappa_n(x)\,dx = 1 \quad \forall n\in\mathbb{Z}$ *and*
2. $\frac{1}{2\pi}\int_0^{2\pi}|\kappa_n(x)|\,dx \in \mathbb{R} \quad \forall n\in\mathbb{Z}$ *and*
3. $\lim_{n\to\infty}\int_\delta^{2\pi-\delta}|\kappa_n(x)|\,dx = 0 \quad \forall n\in\mathbb{Z}, 0<\delta<\pi$

Theorem D.8. [17] *Let $(\!(\kappa_n)\!)_{n\in\mathbb{Z}}$ be a sequence. Let $\mathbb{T}$ be the quotient $\mathbb{R}/2\pi\mathbb{Z}$.*

THM

$$\left.\begin{array}{ll}1.\ \mathsf{f}\in L^1(\mathbb{T}) & \textit{and}\\ 2.\ (\!(\kappa_n)\!) \text{ is a summability kernel} & \end{array}\right\} \implies \mathsf{f}(x) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbb{T}}\kappa_n(x)\mathsf{f}(x-x)\,dx$$

The *Dirichlet kernel* (Definition D.2 page 102) is *not* a *summability kernel*. Examples of kernels that *are* summability kernels include

1. *Fejér's kernel* (Definition D.4 page 109)
2. *de la Vallée Poussin kernel* (Definition D.5 page 111)
3. *Jackson kernel* (Definition D.6 page 111)
4. *Poisson kernel* (Definition D.7 page 111 .)

Definition D.4. [18]

DEF

Fejér's kernel *K_n is defined as*

$$\mathsf{K}_n(x) \triangleq \sum_{k=-n}^{k=n}\left(1-\frac{|k|}{n+1}\right)e^{ikx}$$

Proposition D.5. [19] *Let K_n be Fejér's kernel (Definition D.4 page 109).*

PRP

$$\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$$

[16] Cerdà (2010) page 56, Katznelson (2004) page 10, de Reyna (2002) page 21, Walnut (2002) pages 40–41, Heil (2011) page 440, Istrăţescu (1987) page 309

[17] Katznelson (2004) page 11

[18] Katznelson (2004) page 12

[19] Katznelson (2004) page 12, Heil (2011) page 448

✎PROOF:

1. Lemma: Proof that $\sin^2\frac{x}{2} \equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right)$:

$$\begin{aligned}
\sin^2\frac{x}{2} &\equiv \left(\frac{e^{-i\frac{x}{2}} - e^{+i\frac{x}{2}}}{2i}\right)^2 && \text{by Euler Formulas (Corollary C.2 page 81)}\\
&\equiv \frac{-1}{4}\left(e^{-2i\frac{x}{2}} - 2e^{-i\frac{x}{2}}e^{i\frac{x}{2}} + e^{2i\frac{x}{2}}\right)\\
&\equiv \frac{-1}{4}\left(e^{-ix} - 2 + e^{ix}\right) \;:
\end{aligned}$$

2. Lemma:

$$2|k| - |k+1| - |k-1| = \begin{cases} -2 & \text{for } k = 0\\ 0 & \text{for } k \in \mathbb{Z}\backslash 0\end{cases}$$

3. Proof that $\mathsf{K}_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2$:

$$\begin{aligned}
&-4(n+1)\left(\sin\frac{1}{2}x\right)^2 \mathsf{K}_n(x)\\
&= -4(n+1)\left(\frac{-1}{4}\right)\left(e^{-ix} - 2 + e^{ix}\right)\mathsf{K}_n(x) && \text{by item (1)}\\
&= (n+1)\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}\left(1 - \frac{|k|}{n+1}\right)e^{ikx} && \text{by Definition D.4}\\
&= (n+1)\frac{1}{n+1}\left(e^{-ix} - 2 + e^{ix}\right)\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= e^{-ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}e^{ix}\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\\
&= \sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k-1)x} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n}^{k=n}(n+1-|k|)e^{i(k+1)x}\\
&= \sum_{k=-n-1}^{k=n-1}(n+1-|k+1|)e^{ikx} - 2\sum_{k=-n}^{k=n}(n+1-|k|)e^{ikx}\sum_{k=-n+1}^{k=n+1}(n+1-|k-1|)e^{ikx}\\
&= \underbrace{e^{-i(n+1)x}}_{k=-n-1} + \underbrace{2e^{-inx}}_{k=-n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad \underbrace{-2e^{-inx}}_{k=-n} + \underbrace{-2e^{inx}}_{k=n} - 2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad \underbrace{e^{i(n+1)x}}_{k=n+1} + \underbrace{2e^{inx}}_{k=n} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}\\
&= e^{-i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k+1|)e^{ikx} +\\
&\qquad -2\sum_{k=-n+1}^{k=n-1}(n+1-|k|)e^{ikx} +\\
&\qquad e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1}(n+1-|k-1|)e^{ikx}
\end{aligned}$$

$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} [(n+1-|k+1|) - 2(n+1-|k|) + (n+1-|k-1|)]\, e^{ikx}$$
$$= e^{-i(n+1)x} + e^{i(n+1)x} + \sum_{k=-n+1}^{k=n-1} (2|k| - |k+1| - |k-1|)\, e^{ikx}$$
$$= e^{-i(n+1)x} + e^{i(n+1)x} - 2 \qquad \text{by item (2)}$$
$$= -4\left(\sin\frac{n+1}{2}x\right)^2 \qquad \text{by item (1)}$$

✏

Definition D.5. [20] *Let* K_n *be* FEJÉR'S KERNEL *(Definition D.4 page 109).*

DEF: *The* ***de la Vallée Poussin kernel*** V_n *is defined as*
$$\mathsf{V}_n(x) \triangleq 2K_{2n+1}(x) - K_n(x)$$

Definition D.6. [21] *Let* K_n *be* FEJÉR'S KERNEL *(Definition D.4 page 109).*

DEF: *The* ***Jackson kernel*** J_n *is defined as*
$$\mathsf{J}_n(x) \triangleq \left\|\mathsf{K}_n\right\|^{-2} \mathsf{K}_n^2(x)$$

Definition D.7. [22]

DEF: *The* ***Poisson kernel*** P *is defined as*
$$P(r,x) \triangleq \sum_{k\in\mathbb{Z}} r^{|k|} e^{ikx}$$

[20] Katznelson (2004) page 16
[21] Katznelson (2004) page 17
[22] Katznelson (2004) page 16

APPENDIX E

NORMED ALGEBRAS

E.1 Algebras

All *linear spaces* are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition E.1. [3] *Let* $\boldsymbol{A}$ *be an* ALGEBRA.

DEF *An algebra* $\boldsymbol{A}$ *is* ***unital*** *if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition E.2. [4] *Let* $\boldsymbol{A}$ *be an* UNITAL ALGEBRA *(Definition E.1 page 113) with unit* e.

DEF

$$\begin{array}{llcll}
\textit{The } \textbf{\textit{spectrum}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \sigma(x) & \triangleq & \{\lambda \in \mathbb{C} | \lambda e - x \text{ is not invertible}\}. & \\
\textit{The } \textbf{\textit{resolvent}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \rho_x(\lambda) & \triangleq & (\lambda e - x)^{-1} & \forall \lambda \notin \sigma(x). \\
\textit{The } \textbf{\textit{spectral radius}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & r(x) & \triangleq & \sup\{|\lambda| | \lambda \in \sigma(x)\}. &
\end{array}$$

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[1] Fuchs (1995) page 2

[2] Hazewinkel (2000) page v

[3] Folland (1995) page 1

[4] Folland (1995) pages 3–4

E.2 Star-Algebras

Definition E.3. [5] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF

*The pair $(\boldsymbol{A}, *)$ is a **$*$-algebra**, or **star-algebra**, if*

1.	$(x+y)^*$	$=$	$x^* + y^*$	$\forall x,y\in\boldsymbol{A}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha x)^*$	$=$	$\bar{\alpha} x^*$	$\forall x\in\boldsymbol{A}, \alpha\in\mathbb{C}$	(CONJUGATE LINEAR)	*and*
3.	$(xy)^*$	$=$	$y^* x^*$	$\forall x,y\in\boldsymbol{A}$	(ANTIAUTOMORPHIC)	*and*
4.	x^{**}	$=$	x	$\forall x\in\boldsymbol{A}$	(INVOLUTORY)	

*The operator $*$ is called an **involution** on the algebra $\boldsymbol{A}$.*

Proposition E.1. [6] *Let $(\boldsymbol{A}, *)$ be an* UNITAL $*$-ALGEBRA.

PRP

$$x \text{ is invertible} \implies \begin{cases} 1.\ x^* \text{ is INVERTIBLE} & \forall x\in\boldsymbol{A} \quad \text{and} \\ 2.\ \left(x^*\right)^{-1} = \left(x^{-1}\right)^* & \forall x\in\boldsymbol{A} \end{cases}$$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= \left(x\,e^*\right)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \left(x^*\,e^{**}\right)^{*} && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \left(x^*\,e\right)^{*} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \left(x^*\right)^{*} && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)}\\
e^*\,x &= \left(e^*\,x\right)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \left(e^{**}\,x^*\right)^{*} && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \left(e\,x^*\right)^{*} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \left(x^*\right)^{*} && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)}
\end{aligned}$$

2. Proof that $\left(x^*\right)^{-1} = \left(x^{-1}\right)^*$:

$$\begin{aligned}
\left(x^{-1}\right)^*\left(x^*\right) &= \left[x\left(x^{-1}\right)\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition E.3 page 114)}\\
&= e^*\\
&= e && \text{by item (1) page 114}\\
\left(x^*\right)\left(x^{-1}\right)^* &= \left[x^{-1}\,x\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition E.3 page 114)}\\
&= e^*\\
&= e && \text{by item (1) page 114}
\end{aligned}$$

✏

Definition E.4. [7] *Let $(\boldsymbol{A}, \|\cdot\|)$ be a* $*$-ALGEBRA *(Definition E.3 page 114).*

DEF

- *An element $x \in \boldsymbol{A}$ is **hermitian** or **self-adjoint** if $x^* = x$.*
- *An element $x \in \boldsymbol{A}$ is **normal** if $xx^* = x^*x$.*
- *An element $x \in \boldsymbol{A}$ is a **projection** if $xx = x$ (INVOLUTORY) and $x^* = x$ (HERMITIAN).*

[5] Rickart (1960) page 178, Gelfand and Naimark (1964), page 241
[6] Folland (1995) page 5
[7] Rickart (1960) page 178, Gelfand and Naimark (1964), page 242

Theorem E.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (*Definition E.3 page 114*).

THM

$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are HERMITIAN}} \implies \left\{\begin{array}{lll} x+y &=& (x+y)^* \quad (x+y \text{ is self adjoint}) \\ x^* &=& (x^*)^* \quad (x^* \text{ is self adjoint}) \\ \underbrace{xy = (xy)^*}_{(xy) \text{ is HERMITIAN}} & \iff & \underbrace{xy = yx}_{\text{commutative}} \end{array}\right.$$

✎PROOF:

$$\begin{aligned} (x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition E.3 page 114)} \\ &= x + y && \text{by left hypothesis} \end{aligned}$$

$$(x^*)^* = x \qquad \text{by } \textit{involutory} \text{ property of } * \qquad \text{(Definition E.3 page 114)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned} xy &= (xy)^* && \text{by left hypothesis} \\ &= y^* x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition E.3 page 114)} \\ &= yx && \text{by left hypothesis} \end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned} (xy)^* &= (yx)^* && \text{by left hypothesis} \\ &= x^* y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition E.3 page 114)} \\ &= xy && \text{by left hypothesis} \end{aligned}$$

✏

Definition E.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA (*Definition E.3 page 114*).

DEF

*The **real part** of* x *is defined as* $\mathbf{R}_{\mathrm{e}} x \triangleq \frac{1}{2}\left(x + x^*\right)$

*The **imaginary part** of* x *is defined as* $\mathbf{I}_{\mathrm{m}} x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem E.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA (*Definition E.3 page 114*).

THM

$$\begin{array}{llll} \mathbf{R}_{\mathrm{e}} x &=& \left(\mathbf{R}_{\mathrm{e}} x\right)^* \quad \forall x \in \boldsymbol{A} & (\mathbf{R}_{\mathrm{e}} x \text{ is HERMITIAN}) \\ \mathbf{I}_{\mathrm{m}} x &=& \left(\mathbf{I}_{\mathrm{m}} x\right)^* \quad \forall x \in \boldsymbol{A} & (\mathbf{I}_{\mathrm{m}} x \text{ is HERMITIAN}) \end{array}$$

✎PROOF:

$$\begin{aligned} (\mathbf{R}_{\mathrm{e}} x)^* &= \left(\frac{1}{2}(x + x^*)\right)^* && \text{by definition of } \mathfrak{R} && \text{(Definition E.5 page 115)} \\ &= \frac{1}{2}(x^* + x^{**}) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition E.3 page 114)} \\ &= \frac{1}{2}(x^* + x) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)} \\ &= \mathbf{R}_{\mathrm{e}} x && \text{by definition of } \mathfrak{R} && \text{(Definition E.5 page 115)} \\ (\mathbf{I}_{\mathrm{m}} x)^* &= \left(\frac{1}{2i}(x - x^*)\right)^* && \text{by definition of } \mathfrak{I} && \text{(Definition E.5 page 115)} \end{aligned}$$

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[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998) page 42

$$\begin{aligned}
&= \frac{1}{2i}\left(x^* - x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \frac{1}{2i}\left(x^* - x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition E.3 page 114)}\\
&= \mathbf{I_m}x && \text{by definition of } \Im && \text{(Definition E.5 page 115)}
\end{aligned}$$

✏

Theorem E.3 (Hermitian representation). [11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition E.3 page 114).*

THM
$$a = x + iy \quad\iff\quad x = \mathbf{R_e}a \quad \textit{and} \quad y = \mathbf{I_m}a$$

✎PROOF:

Proof that $a = x + iy \implies x = \mathbf{R_e}a$ and $y = \mathbf{I_m}a$:

$$\begin{aligned}
& & a &= x + iy && \text{by left hypothesis}\\
&\implies & a^* &= (x+iy)^* && \text{by definition of } \textit{adjoint} && \text{(Definition E.4 page 114)}\\
& & &= x^* - iy^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition E.3 page 114)}\\
& & &= x - iy && \text{by Theorem E.2 page 115}\\
&\implies & x &= a - iy && \text{by solving for } x \text{ in } a = x + iy \text{ equation}\\
& & x &= a^* + iy && \text{by solving for } x \text{ in } a^* = x - iy \text{ equation}\\
&\implies & x + x &= a + a^* && \text{by adding previous 2 equations}\\
&\implies & 2x &= a + a^* && \text{by solving for } x \text{ in previous equation}\\
&\implies & x &= \frac{1}{2}\left(a + a^*\right)\\
& & &= \mathbf{R_e}a && \text{by definition of } \Re && \text{(Definition E.5 page 115)}
\end{aligned}$$

$$\begin{aligned}
& & iy &= a - x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation}\\
& & iy &= -a^* + x && \text{by solving for } iy \text{ in } a = x + iy \text{ equation}\\
&\implies & iy + iy &= a - a^* && \text{by adding previous 2 equations}\\
&\implies & y &= \frac{1}{2i}\left(a - a^*\right) && \text{by solving for } iy \text{ in previous equations}\\
& & &= \mathbf{I_m}a && \text{by definition of } \Im && \text{(Definition E.5 page 115)}
\end{aligned}$$

Proof that $a = x + iy \impliedby x = \mathbf{R_e}a$ and $y = \mathbf{I_m}a$:

$$\begin{aligned}
x + iy &= \mathbf{R_e}a + i\,\mathbf{I_m}a && \text{by right hypothesis}\\
&= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\mathbf{R_e}a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\mathbf{I_m}a} && \text{by definition of } \Re \text{ and } \Im && \text{(Definition E.5 page 115)}\\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \cancelto{0}{\left(\frac{1}{2}a^* - \frac{1}{2}a^*\right)}\\
&= a
\end{aligned}$$

✏

[11] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Neumark (1943b) page 7

E.3 Normed Algebras

Definition E.6. [12] *Let A be an algebra.*

DEF

*The pair $(A, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in A \qquad \text{(multiplicative condition)}$$

*A normed algebra $(A, \|\cdot\|)$ is a **Banach algebra** if $(A, \|\cdot\|)$ is also a Banach space.*

Proposition E.2.

PRP

$(A, \|\cdot\|)$ *is a normed algebra* $\implies$ *multiplication is **continuous** in* $(A, \|\cdot\|)$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x - y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(A, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 117)}\\
&= \|z(x - y)\| \\
&\le \|z\|\,\|x - y\| && \text{by definition of \textit{normed algebra}} && \text{(Definition E.6 page 117)}\\
&\triangleq \|z\|\,\delta && \text{by definition of } \delta && \text{(item (2) page 117)}\\
&\le \epsilon && \text{for some value of } \delta > 0
\end{aligned}$$

✏

Theorem E.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l} (A, \|\cdot\|) \text{ is a Banach algebra} \\ \text{every nonzero } x \in A \text{ is invertible} \end{array}\right\} \implies A \equiv \mathbb{C} \quad (A \text{ is isomorphic to } \mathbb{C})$$

E.4 C* Algebras

Definition E.7. [14]

DEF

*The triple $(A, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(A, \|\cdot\|)$ *is a Banach algebra* and
2. $(A, *)$ *is a $*$-algebra* and
3. $\|x^* x\| = \|x\|^2 \quad \forall x \in A$.

*A C^* **algebra** $(A, \|\cdot\|, *)$ is also called a **C star algebra**.*

[12] Rickart (1960) page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩
[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩
[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem E.5. [15] *Let A be an algebra.*

THM: $(A, \|\cdot\|, *)$ *is a* C^* ***algebra*** $\implies$ $\|x^*\| = \|x\|$

✎PROOF:

$$\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\ \|x\|^2 \\
&= \frac{1}{\|x\|}\ \|x^* x\| && \text{by definition of } C^*\text{-}\textit{algebra} && \text{(Definition E.7 page 117)} \\
&\le \frac{1}{\|x\|}\ \|x^*\|\,\|x\| && \text{by definition of } \textit{normed algebra} && \text{(Definition E.6 page 117)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by } \textit{involution} \text{ property of } * && \text{(Definition E.3 page 114)}
\end{aligned}$$

✏

[15] Folland (1995) page 1, Gelfand and Neumark (1943b) page 4, Gelfand and Neumark (1943a)

APPENDIX F

OPERATORS ON LINEAR SPACES

❝*And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.*❞
Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

F.1 Operators on linear spaces

F.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition F.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD (Definition A.5 page 68). *Let* X *be a set, let* $+$ *be an* OPERATOR (Definition F.2 page 120) *in* X^{X^2}, *and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991) page 41, Halmos (1948) pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

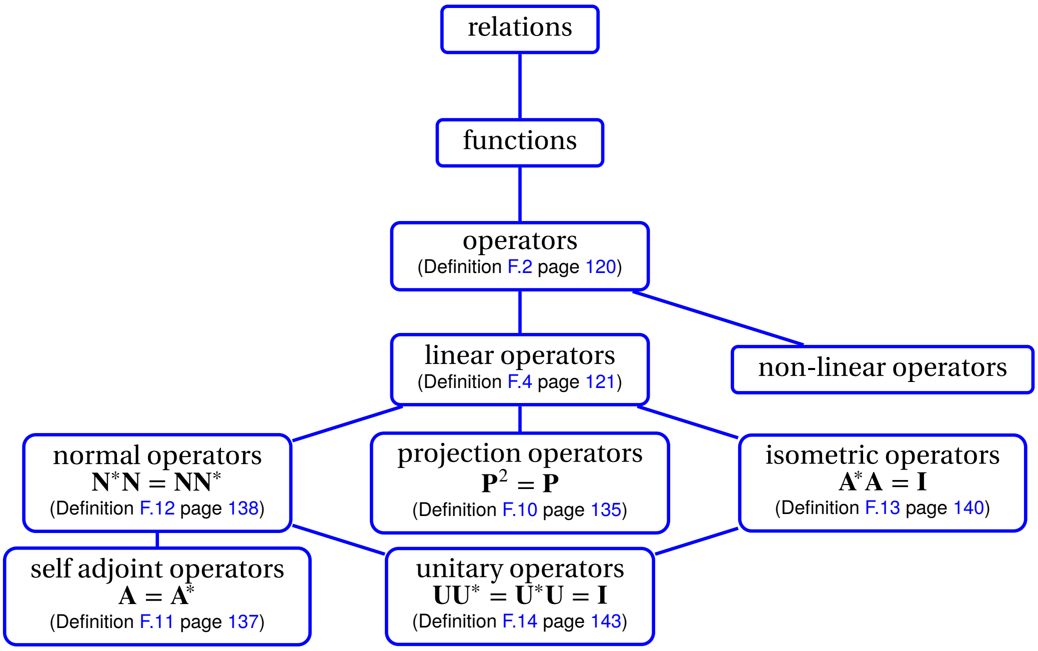


Figure F.1: Some operator types

DEF

The structure $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1.	$\exists \mathbb{0} \in X$ such that	$\boldsymbol{x} + \mathbb{0}$	$=$	$\boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	(+ IDENTITY)	*
2.	$\exists \boldsymbol{y} \in X$ such that	$\boldsymbol{x} + \boldsymbol{y}$	$=$	$\mathbb{0}$	$\forall \boldsymbol{x} \in X$	(+ INVERSE)	
3.		$(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$	$=$	$\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$	$\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$	(+ *is* ASSOCIATIVE)	
4.		$\boldsymbol{x} + \boldsymbol{y}$	$=$	$\boldsymbol{y} + \boldsymbol{x}$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(+ *is* COMMUTATIVE)	
5.		$1 \cdot \boldsymbol{x}$	$=$	$\boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	($\cdot$ IDENTITY)	
6.		$\alpha \cdot (\beta \cdot \boldsymbol{x})$	$=$	$(\alpha \cdot \beta) \cdot \boldsymbol{x}$	$\forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$	($\cdot$ ASSOCIATES *with* $\cdot$)	
7.		$\alpha \cdot (\boldsymbol{x} + \boldsymbol{y})$	$=$	$(\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y})$	$\forall \alpha \in S$ *and* $\boldsymbol{x}, \boldsymbol{y} \in X$	($\cdot$ DISTRIBUTES *over* +)	
8.		$(\alpha + \beta) \cdot \boldsymbol{x}$	$=$	$(\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x})$	$\forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$	($\cdot$ PSEUDO-DISTRIBUTES *over* +)	

The set X *is called the* ***underlying set****. The elements of* X *are called* ***vectors****. The elements of* $\mathbb{F}$ *are called* ***scalars****. A linear space is also called a* ***vector space****. If* $\mathbb{F} \triangleq \mathbb{R}$*, then* $\boldsymbol{\Omega}$ *is a* ***real linear space****. If* $\mathbb{F} \triangleq \mathbb{C}$*, then* $\boldsymbol{\Omega}$ *is a* ***complex linear space****.*

Definition F.2. [3]

DEF

A function $\mathbf{A}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *is an* ***operator*** *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *if*
$\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both* LINEAR SPACES *(Definition F.1 page 119).*

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}\boldsymbol{x} = \mathbf{B}\boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{X}\boldsymbol{Y}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}\boldsymbol{x} = \mathbf{I}$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

Definition F.3. [4] *Let* $\boldsymbol{X}^{\boldsymbol{X}}$ *be the set of all operators with from a* LINEAR SPACE $\boldsymbol{X}$ *to* $\boldsymbol{X}$*. Let* $\mathbf{I}$ *be an operator in* $\boldsymbol{X}^{\boldsymbol{X}}$*. Let* $①(\boldsymbol{X})$ *be the* IDENTITY ELEMENT *in* $\boldsymbol{X}^{\boldsymbol{X}}$*.*

DEF

$\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\quad \mathbf{I} = ①(\boldsymbol{X})$.

[3] Heil (2011) page 42
[4] Michel and Hergert (1993) page 411

F.1.2 Linear operators

Definition F.4. [5] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be linear spaces.*

DEF

An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *is* ***linear*** *if*

$$\begin{array}{llcll} 1. & \mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x}+\mathbf{L}\boldsymbol{y} & \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X} \quad (\text{ADDITIVE}) \quad \textit{and} \\ 2. & \mathbf{L}(\alpha\boldsymbol{x}) & = & \alpha\mathbf{L}\boldsymbol{x} & \forall \boldsymbol{x}\in\boldsymbol{X},\ \forall\alpha\in\mathbb{F} \quad (\text{HOMOGENEOUS}). \end{array}$$

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ *such that*

$$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \left\{\mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}} | \mathbf{L} \text{ is linear}\right\}.$$

Theorem F.1. [6] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$*.*

THM

$$\{\mathbf{L} \text{ is LINEAR}\} \implies \left\{\begin{array}{llcll} 1. & \mathbf{L}\mathbb{0} & = & \mathbb{0} & \textit{and} \\ 2. & \mathbf{L}(-\boldsymbol{x}) & = & -(\mathbf{L}\boldsymbol{x}) & \forall \boldsymbol{x}\in\boldsymbol{X} \quad \textit{and} \\ 3. & \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} & \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X} \quad \textit{and} \\ 4. & \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) & = & \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}\boldsymbol{x}_n\right) & \boldsymbol{x}_n\in\boldsymbol{X},\alpha_n\in\mathbb{F} \end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned} \mathbf{L}\mathbb{0} &= \mathbf{L}(0\cdot\mathbb{0}) && \text{by additive identity property} \\ &= 0\cdot(\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \quad \text{(Definition F.4 page 121)} \\ &= \mathbb{0} && \text{by } \textit{additive identity} \text{ property} \end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned} \mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1\cdot\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \\ &= -1\cdot(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \quad \text{(Definition F.4 page 121)} \\ &= -(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}$:

$$\begin{aligned} \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x}+(-\boldsymbol{y})) && \text{by } \textit{additive inverse} \text{ property} \\ &= \mathbf{L}(\boldsymbol{x})+\mathbf{L}(-\boldsymbol{y}) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \quad \text{(Definition F.4 page 121)} \\ &= \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} && \text{by item (2)} \end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) = \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}\boldsymbol{x}_n\right)$:

   (a) Proof for $N = 1$:

$$\begin{aligned} \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_1\boldsymbol{x}_1\right) && \text{by } N=1 \text{ hypothesis} \\ &= \alpha_1\left(\mathbf{L}\boldsymbol{x}_1\right) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \quad \text{(Definition F.4 page 121)} \end{aligned}$$

[5] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[6] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

(b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1}\alpha_n\boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1}+\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right)\\
&= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1}\right)+\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \quad \text{(Definition F.4 page 121)}\\
&= \alpha_{N+1}\mathbf{L}\left(\boldsymbol{x}_{N+1}\right)+\sum_{n=1}^{N}\mathbf{L}\left(\alpha_n\boldsymbol{x}_n\right) && \text{by left } N+1 \text{ hypothesis}\\
&= \sum_{n=1}^{N+1}\mathbf{L}\left(\alpha_n\boldsymbol{x}_n\right)
\end{aligned}$$

⇰

Theorem F.2. [7] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathcal{N}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{X}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathcal{I}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{Y}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	

✎PROOF:

1. Proof that $\mathcal{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0}\in\mathcal{N}(\mathbf{L}) \implies \mathcal{N}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{N}(\mathbf{L})\triangleq\{\boldsymbol{x}\in\boldsymbol{X}|\mathbf{L}\boldsymbol{x}=\mathbb{0}\}\subseteq\boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{N}(\mathbf{L}) \implies \mathbb{0}=\mathbf{L}(\boldsymbol{x}+\boldsymbol{y})=\mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\boldsymbol{X} \implies \mathbb{0}=\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
2. Proof that $\mathcal{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0}\in\mathcal{I}(\mathbf{L}) \implies \mathcal{I}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{I}(\mathbf{L})\triangleq\{\boldsymbol{y}\in\boldsymbol{Y}|\exists\boldsymbol{x}\in\boldsymbol{X}$ such that $\boldsymbol{y}=\mathbf{L}\boldsymbol{x}\}\subseteq\boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{v}\in\boldsymbol{X}$ such that $\mathbf{L}\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{x}\in\boldsymbol{X}$ such that $\boldsymbol{y}=\mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y}=\alpha\mathbf{L}\boldsymbol{x}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$

⇰

Example F.1. [8] Let $C([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX
$C([a:b],\mathbb{R})$ is a linear space.

Theorem F.3. [9] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of a linear operator $\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM		
$\mathbf{L}\boldsymbol{x}=\mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y}\in\mathcal{N}(\mathbf{L})$
$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathcal{N}(\mathbf{L})=\{\mathbb{0}\}$

[7] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩
[8] Eidelman et al. (2004) page 3
[9] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

✎PROOF:

1. Proof that $\mathbf{L}x = \mathbf{L}y \implies x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned} \mathbf{L}(x-y) &= \mathbf{L}x - \mathbf{L}y && \text{by Theorem F.1 page 121} \\ &= \mathbb{0} && \text{by left hypothesis} \\ &\implies \quad x - y \in \mathcal{N}(\mathbf{L}) && \text{by definition of } \textit{null space} \end{aligned}$$

2. Proof that $\mathbf{L}x = \mathbf{L}y \impliedby x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned} \mathbf{L}y &= \mathbf{L}y + \mathbb{0} && \text{by definition of linear space (Definition F.1 page 119)} \\ &= \mathbf{L}y + \mathbf{L}(x-y) && \text{by right hypothesis} \\ &= \mathbf{L}y + (\mathbf{L}x - \mathbf{L}y) && \text{by Theorem F.1 page 121} \\ &= (\mathbf{L}y - \mathbf{L}y) + \mathbf{L}x && \text{by \textit{associative} and \textit{commutative} properties (Definition F.1 page 119)} \\ &= \mathbf{L}x \end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned} \mathbf{L} \text{ is } \textit{injective} &\iff \{(\mathbf{L}x = \mathbf{L}y \iff x = y) \quad \forall x, y \in X\} \\ &\iff \left\{\left[\mathbf{L}x - \mathbf{L}y = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\} \\ &\iff \left\{\left[\mathbf{L}(x-y) = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\} \\ &\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\} \end{aligned}$$

✏

Theorem F.4. [10] *Let W, X, Y, and Z be linear spaces over a field $\mathbb{F}$.*

THM					
1.	$\mathbf{L}(\mathbf{MN})$	$=$ $(\mathbf{LM})\mathbf{N}$	$\forall \mathbf{L} \in \mathcal{L}(Z,W), \mathbf{M} \in \mathcal{L}(Y,Z), \mathbf{N} \in \mathcal{L}(X,Y)$	(ASSOCIATIVE)
2.	$\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})$	$=$ $(\mathbf{LM}) \dotplus (\mathbf{LN})$	$\forall \mathbf{L} \in \mathcal{L}(Y,Z), \mathbf{M} \in \mathcal{L}(X,Y), \mathbf{N} \in \mathcal{L}(X,Y)$	(LEFT DISTRIBUTIVE)
3.	$(\mathbf{L} \dotplus \mathbf{M})\mathbf{N}$	$=$ $(\mathbf{LN}) \dotplus (\mathbf{MN})$	$\forall \mathbf{L} \in \mathcal{L}(Y,Z), \mathbf{M} \in \mathcal{L}(Y,Z), \mathbf{N} \in \mathcal{L}(X,Y)$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{LM})$	$=$ $(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L} \in \mathcal{L}(Y,Z), \mathbf{M} \in \mathcal{L}(X,Y), \alpha \in \mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{MN}) = (\mathbf{LM})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}(\mathbf{M} \dotplus \mathbf{N}) = (\mathbf{LM}) \dotplus (\mathbf{LN})$:

$$\begin{aligned} [\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})]x &= \mathbf{L}[(\mathbf{M} \dotplus \mathbf{N})x] \\ &= \mathbf{L}[(\mathbf{M}x) \dotplus (\mathbf{N}x)] \\ &= [\mathbf{L}(\mathbf{M}x)] \dotplus [\mathbf{L}(\mathbf{N}x)] && \text{by \textit{additive} property Definition F.4 page 121} \\ &= [(\mathbf{LM})x] \dotplus [(\mathbf{LN})x] \end{aligned}$$

3. Proof that $(\mathbf{L} \dotplus \mathbf{M})\mathbf{N} = (\mathbf{LN}) \dotplus (\mathbf{MN})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{LM}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{LM}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned} [\alpha(\mathbf{LM})]x &= \alpha[(\mathbf{LM})x] \\ &= \mathbf{L}[\alpha(\mathbf{M}x)] && \text{by \textit{homogeneous} property Definition F.4 page 121} \\ &= \mathbf{L}[(\alpha\mathbf{M})x] \\ &= [\mathbf{L}(\alpha\mathbf{M})]x \end{aligned}$$

[10] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

Theorem F.5 (Fundamental theorem of linear equations). Michel and Herget (1993) page 99 *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ (Definition ?? page ??).*

THM
$$\dim \mathcal{I}(\mathbf{L}) + \dim \mathcal{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1, 2, \ldots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for $\mathcal{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \mathbb{0} \right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ⇨

F.2 Operators on Normed linear spaces

F.2.1 Operator norm

Definition F.5. [11] *Let $\boldsymbol{V} = (X, \mathbb{F}, \dot{+}, \cdot)$ be a linear space and $\mathbb{F}$ be a field with absolute value function $|\cdot| \in \mathbb{R}^{\mathbb{F}}$ (Definition A.4 page 68).*

DEF

*A **norm** is any functional $\|\cdot\|$ in $\mathbb{R}^X$ that satisfies*

1.	$\|\boldsymbol{x}\| \geq 0$		$\forall \boldsymbol{x} \in X$	(STRICTLY POSITIVE)	*and*
2.	$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbb{0}$		$\forall \boldsymbol{x} \in X$	(NONDEGENERATE)	*and*
3.	$\|a\boldsymbol{x}\| = |a| \, \|\boldsymbol{x}\|$		$\forall \boldsymbol{x} \in X,\ a \in \mathbb{C}$	(HOMOGENEOUS)	*and*
4.	$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$		$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(SUBADDITIVE/*triangle inquality*).	

*A **normed linear space** is the pair $(\boldsymbol{V}, \|\cdot\|)$.*

[11] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

Definition F.6. [12] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [13]

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} \qquad \forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition F.1 (next) shows that the functional defined in Definition F.6 (previous) is a *norm* (Definition F.5 page 124).

Proposition F.1. [14] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ and $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1. $|||\mathbf{A}||| \geq 0$ $\quad \forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ (NON-NEGATIVE) *and*
2. $|||\mathbf{A}||| = 0 \iff \mathbf{A} \mathrel{\mathring{=}} \mathbb{0}$ $\quad \forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ (NONDEGENERATE) *and*
3. $|||\alpha\mathbf{A}||| = |\alpha|\, |||\mathbf{A}|||$ $\quad \forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$ (HOMOGENEOUS) *and*
4. $\left|\left|\left|\mathbf{A} \mathrel{\mathring{+}} \mathbf{B}\right|\right|\right| \leq |||\mathbf{A}||| + |||\mathbf{B}|||$ $\quad \forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ (SUBADDITIVE).

*Moreover, $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \neq \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition F.6 page 125)}\\
&> 0
\end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \mathrel{\mathring{=}} \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition F.6 page 125)}\\
&= \sup_{x\in\boldsymbol{X}} \{\|\mathbb{0}x\| \mid \|x\| \leq 1\}\\
&= 0
\end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha|\, |||\mathbf{A}|||$:

$$\begin{aligned}
|||\alpha\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\alpha\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition F.6 page 125)}\\
&= \sup_{x\in\boldsymbol{X}} \{|\alpha|\, \|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition F.6 page 125)}\\
&= |\alpha| \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of sup}\\
&= |\alpha|\ |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition F.6 page 125)}
\end{aligned}$$

[12] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225

[13] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990) page 290

[14] Rudin (1991) page 93

4. Proof that $\| \mathbf{A} \dot{+} \mathbf{B} \| \le \|\mathbf{A}\| + \|\mathbf{B}\|$:

$$\begin{aligned}
\|\mathbf{A} \dot{+} \mathbf{B}\| &\triangleq \sup_{x \in \boldsymbol{X}} \left\{ \|(\mathbf{A} \dot{+} \mathbf{B})\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\} && \text{by definition of } \|\cdot\| \text{ (Definition F.6 page 125)} \\
&= \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \\
&\le \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x}\| + \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} && \text{by definition of } \|\cdot\| \text{ (Definition F.6 page 125)} \\
&\le \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} + \sup_{x \in \boldsymbol{X}} \{ \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \\
&\triangleq \|\mathbf{A}\| + \|\mathbf{B}\| && \text{by definition of } \|\cdot\| \text{ (Definition F.6 page 125)}
\end{aligned}$$

⇒

Lemma F.1. *Let* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), \|\cdot\|)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

LEM

$$\|\mathbf{L}\| = \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} \qquad \forall \boldsymbol{x} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF: [15]

1. Proof that $\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \ge \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \}$:

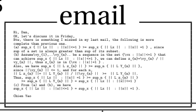
$$\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \ge \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ \boldsymbol{y} \in X \mid \begin{array}{ll} 1. & \|\mathbf{L}\boldsymbol{y}\| = \sup_{\boldsymbol{x} \in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \text{ and} \\ 2. & 0 < \|\boldsymbol{y}\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \le \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \}$:

$$\begin{aligned}
\sup_{\boldsymbol{x} \in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} &= \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of set } Y \\
&= \frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \|\mathbf{L}\boldsymbol{y}\| \\
&= \|\boldsymbol{y}\| \left\| \frac{1}{\|\boldsymbol{y}\|} \mathbf{L}\boldsymbol{y} \right\| && \text{by homogeneous property (page 124)} \\
&= \|\boldsymbol{y}\| \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| && \text{by homogeneous property (page 121)} \\
&\le \|\boldsymbol{y}\| \sup_{\boldsymbol{y} \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \right\} && \text{by definition of supremum} \\
&= \|\boldsymbol{y}\| \sup_{\boldsymbol{y} \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \text{ for all } \boldsymbol{y} \in Y \\
&\le \sup_{\boldsymbol{y} \in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } 0 < \|\boldsymbol{y}\| \le 1 \\
&\le \sup_{\boldsymbol{x} \in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} && \text{because } \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in X \quad \forall \boldsymbol{y} \in Y
\end{aligned}$$

[15] email

Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

4. By (1) and (3),

$$\sup_{x \in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} = \sup_{x \in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$$

✏

Proposition F.2. [16] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{X}), |||\cdot|||)$.

PRP: $|||\mathbf{I}||| = 1$

✎PROOF:

$$\begin{aligned} |||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition F.6 page 125)} \\ &= \sup\{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition F.3 page 120)} \\ &= 1 \end{aligned}$$

✏

Theorem F.6. [17] *Let* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM:

$$\begin{array}{rcll} \|\mathbf{L}\boldsymbol{x}\| & \le & |||\mathbf{L}|||\ \|\boldsymbol{x}\| & \forall \mathbf{L} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}),\ \boldsymbol{x} \in \boldsymbol{X} \\ |||\mathbf{K}\mathbf{L}||| & \le & |||\mathbf{K}|||\ |||\mathbf{L}||| & \forall \mathbf{K}, \mathbf{L} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}) \end{array}$$

✎PROOF:

1. Proof that $\|\mathbf{L}\boldsymbol{x}\| \le |||\mathbf{L}|||\ \|\boldsymbol{x}\|$:

$$\begin{aligned} \|\mathbf{L}\boldsymbol{x}\| &= \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \|\mathbf{L}\boldsymbol{x}\| \\ &= \|\boldsymbol{x}\| \left\| \frac{1}{\|\boldsymbol{x}\|} \mathbf{L}\boldsymbol{x} \right\| && \text{by property of norms} \\ &= \|\boldsymbol{x}\| \left\| \mathbf{L} \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right\| && \text{by property of linear operators} \\ &\triangleq \|\boldsymbol{x}\|\ \|\mathbf{L}\boldsymbol{y}\| && \text{where } y \triangleq \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \\ &\le \|\boldsymbol{x}\| \sup_{y} \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of supremum} \\ &= \|\boldsymbol{x}\| \sup_{y} \{\|\mathbf{L}\boldsymbol{y}\| \mid \|\boldsymbol{y}\| = 1\} && \text{because } \|\boldsymbol{y}\| = \left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right\| = \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|} = 1 \\ &\triangleq \|\boldsymbol{x}\|\ \|\mathbf{L}\| && \text{by definition of operator norm} \end{aligned}$$

[16] Michel and Herget (1993) page 410
[17] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

2. Proof that $|||\mathbf{KL}||| \le |||\mathbf{K}|||\ |||\mathbf{L}|||$:

$$\begin{aligned}
|||\mathbf{KL}||| &\triangleq \sup_{x\in X} \{\|(\mathbf{KL})x\| \mid \|x\| \le 1\} && \text{by Definition F.6 page 125 } (|||\cdot|||)\\
&= \sup_{x\in X} \{\|\mathbf{K}(\mathbf{L}x)\| \mid \|x\| \le 1\} \\
&\le \sup_{x\in X} \{|||\mathbf{K}|||\ \|\mathbf{L}x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&\le \sup_{x\in X} \{|||\mathbf{K}|||\ |||\mathbf{L}|||\ \|x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&= \sup_{x\in X} \{|||\mathbf{K}|||\ |||\mathbf{L}|||\ 1 \mid \|x\| \le 1\} && \text{by definition of sup}\\
&= |||\mathbf{K}|||\ |||\mathbf{L}||| && \text{by definition of sup}
\end{aligned}$$

✏

F.2.2 Bounded linear operators

Definition F.7. [18] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be a normed space of linear operators.*

DEF

An operator $\mathbf{B}$ *is* ***bounded*** *if* $|||\mathbf{B}||| < \infty$.
The quantity $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *is the set of all* ***bounded linear operators*** *on* $(\boldsymbol{X}, \boldsymbol{Y})$ *such that*

$$\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \{\mathbf{L} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}) \mid |||\mathbf{L}||| < \infty\}.$$

Theorem F.7. [19] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the set of linear operators over normed linear spaces $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ and $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.*

THM

The following conditions are all EQUIVALENT*:*

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $x_0 \in X$ $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \iff$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point $x \in X$)* $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \iff$
3. $|||\mathbf{L}||| < \infty$ *($\mathbf{L}$ is* BOUNDED*)* $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \iff$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}x\| \le M\,\|x\| \quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), x\in X$

✎PROOF:

1. Proof that 1 $\implies$ 2:

$$\begin{aligned}
\epsilon &> \left\|\mathbf{L}x - \mathbf{L}x_0\right\| && \text{by hypothesis 1}\\
&= \left\|\mathbf{L}(x - x_0)\right\| && \text{by linearity (Definition F.4 page 121)}\\
&= \left\|\mathbf{L}(x + y - x_0 - y)\right\| \\
&= \left\|\mathbf{L}(x + y) - \mathbf{L}(x_0 + y)\right\| && \text{by linearity (Definition F.4 page 121)}\\
&\implies \mathbf{L} \text{ is continuous at point } x + y \\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}$$

2. Proof that 2 $\implies$ 1: obvious:

[18] Rudin (1991) pages 92–93
[19] Aliprantis and Burkinshaw (1998) page 227

3. Proof that $4 \implies 2$:[20]

$$\begin{aligned}
|||\mathbf{L}x||| \le M\,\|x\| &\implies |||\mathbf{L}(x-y)||| \le M\,\|x-y\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le M\,\|x-y\| && \text{by linearity of } \mathbf{L} \text{ (Definition F.4 page 121)}\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le \epsilon \text{ whenever } M\,\|x-y\| < \epsilon\\
&\implies |||\mathbf{L}x-\mathbf{L}y||| \le \epsilon \text{ whenever } \|x-y\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that $3 \implies 4$:

$$\begin{aligned}
\|\mathbf{L}x\| &\le \underbrace{|||\mathbf{L}|||}_{M}\ \|x\| && \text{by Theorem F.6 page 127}\\
&= M\ \|x\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that $1 \implies 3$:[21]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \infty\\
&\implies \exists\,(x_n) \quad \text{such that} \quad \|x_n\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\|\mathbf{L}x_n\| \mid \|x_n\| \le 1\right\} = \infty\\
&\implies \|x_n\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \|\mathbf{L}x_n\|\\
&\implies \|x_n\| = 1 \text{ and } \|\mathbf{L}x_n\| \ge n\\
&\implies \frac{1}{n}\|x_n\| = \frac{1}{n} \text{ and } \frac{1}{n}\|\mathbf{L}x_n\| \ge 1\\
&\implies \left\|\frac{x_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

✎

F.2.3 Adjoints on normed linear spaces

Definition F.8. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *of $\boldsymbol{X}$.*

DEF: $\mathbf{B}^*$ *is the* ***adjoint*** *of an operator* $\mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *if*

$$\mathsf{f}(\mathbf{B}x) = \left[\mathbf{B}^*\mathsf{f}\right](x) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ x \in \boldsymbol{X}$$

Theorem F.8. [22] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *$\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM:

$$\begin{aligned}
(\mathbf{A} \dotplus \mathbf{B})^* &= \mathbf{A}^* \dotplus \mathbf{B}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\lambda\mathbf{A})^* &= \lambda\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\mathbf{A}\mathbf{B})^* &= \mathbf{B}^*\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})
\end{aligned}$$

[20] Bollobás (1999) page 29
[21] Aliprantis and Burkinshaw (1998) page 227
[22] Bollobás (1999) page 156

✎PROOF:

$$\begin{aligned}
\left[\mathbf{A} \overset{\circ}{+} \mathbf{B}\right]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \overset{\circ}{+} \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition F.8 page 129)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition F.4 page 121)}\\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} &&\\
&= \mathbf{A}^{*}\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^{*}\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition F.8 page 129)}\\
&= \left[\mathbf{A}^{*} + \mathbf{B}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{linear functional} &&
\end{aligned}$$

$$\begin{aligned}
[\lambda\mathbf{A}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition F.8 page 129)}\\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} &&\\
&= \left[\lambda\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition F.8 page 129)}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{A}\mathbf{B}]^{*}\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition F.8 page 129)}\\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of } \textit{linear operators} && \text{(Definition F.4 page 121)}\\
&= \left[\mathbf{A}^{*}\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition F.8 page 129)}\\
&= \mathbf{B}^{*}\left[\mathbf{A}^{*}\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition F.8 page 129)}\\
&= \left[\mathbf{B}^{*}\mathbf{A}^{*}\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition F.8 page 129)}
\end{aligned}$$

⇨

Theorem F.9. [23] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^{*}$ be the adjoint of an operator $\mathbf{B}$.*

THM

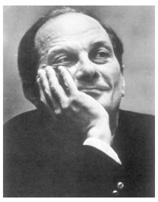
$$|||\mathbf{B}||| = \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| \qquad \forall \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF:

$$\begin{aligned}
|||\mathbf{B}||| &\triangleq \sup\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by Definition F.6 page 125}\\
&\overset{?}{=} \sup\left\{|\mathsf{g}(\mathbf{B}\boldsymbol{x}; \boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x}; \mathbf{B}^{*}\boldsymbol{y}^{*})| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&\triangleq \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&= \sup\left\{\left\|\mathbf{B}^{*}\boldsymbol{y}^{*}\right\| \mid \left\|\boldsymbol{y}^{*}\right\| \le 1\right\} &&\\
&\triangleq \left|\left|\left|\mathbf{B}^{*}\right|\right|\right| && \text{by Definition F.6 page 125}
\end{aligned}$$

⇨

F.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞

Stanislaus M. Ulam (1909–1984), Polish mathematician [24]

[23] Rudin (1991) page 98

Theorem F.10 (Mazur-Ulam theorem). [25] *Let $\phi \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be a function on normed linear spaces $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. Let $\mathbf{I} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$ be the identity operator on $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.*

THM

$$\left.\begin{array}{ll} 1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} \qquad \text{and} \\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}}_{\text{isometric}} \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$
   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$
2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$
3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned}
\left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\
&= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis} \\
&= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1}
\end{aligned}$$

4. Proof that $\mathrm{g}\boldsymbol{z} = \boldsymbol{z}$:

$$\begin{aligned}
2\lambda &= 2\sup\{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\} && \text{by definition of } \lambda \text{ item (2)} \\
&\leq 2\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\| && \text{by definition of sup} \\
&= \|2\boldsymbol{z} - 2\mathrm{g}\boldsymbol{z}\| \\
&= \|\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}\boldsymbol{z}\| && \text{by definition of } \psi \text{ item (1)} \\
&= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}^{-1}\mathrm{g}\boldsymbol{z}\right\| && \text{by item (3)} \\
&= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| && \text{by definition of } \mathrm{g}^{-1} \\
&= \left\|\psi\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| \\
&= \left\|\mathrm{g}^{*}\boldsymbol{z} - \boldsymbol{z}\right\| \\
&\leq \lambda && \text{by definition of } \lambda \text{ item (2)} \\
&\implies 2\lambda \leq \lambda \\
&\implies \lambda = 0 \\
&\implies \mathrm{g}\boldsymbol{z} = \boldsymbol{z}
\end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) = \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}$:

$$\begin{aligned}
\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) &= \\
&= \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}
\end{aligned}$$

[24] quote: Ulam (1991) page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html
[25] Oikhberg and Rosenthal (2007) page 598, Väisälä (2003) page 634, Giles (2000) page 11, Dunford and Schwartz (1957) page 91, Mazur and Ulam (1932)

6. Proof that $\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}$:

$$\begin{aligned}\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) &= \\ &= [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}\end{aligned}$$

✑

Theorem F.11 (Neumann Expansion Theorem). [26] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{ll}1. & \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) \quad (\mathbf{A} \text{ is bounded}) \\ 2. & |||\mathbf{A}||| < 1\end{array}\right\} \implies \left\{\begin{array}{lrcl}1. & (\mathbf{I}-\mathbf{A})^{-1} & & \text{exists} \\ 2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \le & \dfrac{1}{1-|||\mathbf{A}|||} \\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \displaystyle\sum_{n=0}^{\infty}\mathbf{A}^n \\ & \multicolumn{3}{r}{\text{with uniform convergence}}\end{array}\right.$$

F.3 Operators on Inner product spaces

F.3.1 General Results

Definition F.9. [27] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space.*

DEF

A function $\langle \triangle \mid \triangledown \rangle \in \mathbb{F}^{X\times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

$$\begin{array}{lrcll l l}1. & \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle & \ge & 0 & \forall \boldsymbol{x}\in X & \text{(non-negative)} & \text{and}\\ 2. & \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle & = & 0 \iff \boldsymbol{x} = \mathbb{0} & \forall \boldsymbol{x}\in X & \text{(nondegenerate)} & \text{and}\\ 3. & \langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle & = & \alpha\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle & \forall \boldsymbol{x},\boldsymbol{y}\in X,\ \forall\alpha\in\mathbb{C} & \text{(homogeneous)} & \text{and}\\ 4. & \langle \boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{u}\rangle & = & \langle \boldsymbol{x} \mid \boldsymbol{u}\rangle + \langle \boldsymbol{y} \mid \boldsymbol{u}\rangle & \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X & \text{(additive)} & \text{and}\\ 5. & \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle & = & \langle \boldsymbol{y} \mid \boldsymbol{x}\rangle^* & \forall \boldsymbol{x},\boldsymbol{y}\in X & \text{(conjugate symmetric).} & \end{array}$$

An inner product is also called a ***scalar product***.
An ***inner product space*** *is the pair* $(\boldsymbol{\Omega}, \langle \triangle \mid \triangledown \rangle)$.

Theorem F.12. [28] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM

$$\begin{array}{rclcrclc}\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle & = & 0 & \forall \boldsymbol{x}\in X & \iff & \mathbf{B}\boldsymbol{x} & = & \mathbb{0} \quad \forall \boldsymbol{x}\in X\\ \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle & = & \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle & \forall \boldsymbol{x}\in X & \iff & \mathbf{A} & = & \mathbf{B}\end{array}$$

✎PROOF:

[26] Michel and Herget (1993) page 415
[27] Haaser and Sullivan (1991) page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72
[28] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

1. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x+\mathbf{B}x) \mid (x+\mathbf{B}x)\rangle + i\,\langle \mathbf{B}(x+i\mathbf{B}x) \mid (x+i\mathbf{B}x)\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}x+\mathbf{B}^2x) \mid x+\mathbf{B}x\right\rangle\right\} + i\left\{\left\langle \mathbf{B}x+i\mathbf{B}^2x) \mid x+i\mathbf{B}x\right\rangle\right\} && \text{by Definition F.4 page 121}\\
&= \left\{\langle \mathbf{B}x \mid x\rangle + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + \left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\} && \text{by Definition F.9 page 132}\\
&\quad + i\left\{\langle \mathbf{B}x \mid x\rangle - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2\left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle\right\} + \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle - \left\langle \mathbf{B}^2x \mid x\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle\\
&= 2\,\|\mathbf{B}x\|^2\\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition F.5 page 124}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products.

3. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \implies \mathbf{A} \stackrel{\circ}{=} \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \mid x\rangle - \langle \mathbf{B}x \mid x\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}x - \mathbf{B}x \mid x\rangle && \text{by \textit{additivity} property of } \langle \triangle \mid \nabla\rangle \text{ (Definition F.9 page 132)}\\
&= \langle (\mathbf{A}-\mathbf{B})x \mid x\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A}-\mathbf{B})x &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \impliedby \mathbf{A} \stackrel{\circ}{=} \mathbf{B}$:

$$\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \qquad \text{by } \mathbf{A} \stackrel{\circ}{=} \mathbf{B} \text{ hypothesis}$$

✎

F.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition F.3 page 133). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem F.13 page 134).
- Both support decomposition into "real" and "imaginary" parts (Theorem E.3 page 116).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *null space* of an operator (Theorem F.14 page 135).

Proposition F.3. [29] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *(Definition F.7 page 128) on a* HILBERT SPACE $\boldsymbol{H}$.

PRP	
	*An operator $\mathbf{B}^*$ is the **adjoint** of $\mathbf{B} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ if* $\langle \mathbf{B}x \mid y\rangle = \langle x \mid \mathbf{B}^*y\rangle \quad \forall x, y \in \boldsymbol{H}$.

✎PROOF:

[29] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000) page 182, von Neumann (1929) page 49, Stone (1932) page 41

1. For fixed y, $\mathsf{f}(\boldsymbol{x}) \triangleq \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is a *functional* in $\mathbb{F}^{\boldsymbol{X}}$.
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$
\begin{aligned}
\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle &\triangleq \mathsf{f}(\mathbf{B}\boldsymbol{x}) \\
&\triangleq \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{operator adjoint} && \text{(Definition F.8 page 129)} \\
&= \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle
\end{aligned}
$$

⇨

Example F.2.

EX

In matrix algebra ("linear algebra")

- The inner product operation $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is represented by $\boldsymbol{y}^H \boldsymbol{x}$.
- The linear operator is represented as a matrix $\boldsymbol{A}$.
- The operation of $\boldsymbol{A}$ on a vector $\boldsymbol{x}$ is represented as $\boldsymbol{A}\boldsymbol{x}$.
- The adjoint of matrix $\boldsymbol{A}$ is the Hermitian matrix $\boldsymbol{A}^H$.

✎PROOF:

$$\langle \boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H \boldsymbol{A}\boldsymbol{x} = [(\boldsymbol{A}\boldsymbol{x})^H \boldsymbol{y}]^H = [\boldsymbol{x}^H \boldsymbol{A}^H \boldsymbol{y}]^H = (\boldsymbol{A}^H \boldsymbol{y})^H \boldsymbol{x} \triangleq \left\langle \boldsymbol{x} \mid \boldsymbol{A}^H \boldsymbol{y} \right\rangle$$

⇨

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition E.3 page 114). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[30]

Theorem F.13 (operator star-algebra). [31] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM

The pair $(\boldsymbol{H}, *)$ *is a* $*$-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \stackrel{\circ}{+} \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(INVOLUTARY)	

✎PROOF:

$$
\begin{aligned}
\left\langle \boldsymbol{x} \mid (\mathbf{A} \stackrel{\circ}{+} \mathbf{B})^* \boldsymbol{y} \right\rangle &= \left\langle (\mathbf{A} \stackrel{\circ}{+} \mathbf{B})\boldsymbol{x} \mid \boldsymbol{y} \right\rangle && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)} \\
&= \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition F.9 page 132)} \\
&= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of operator addition} \\
&= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} + \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition F.9 page 132)} \\
&= \langle \boldsymbol{x} \mid (\mathbf{A}^* + \mathbf{B}^*)\boldsymbol{y} \rangle && \text{by definition of operator addition}
\end{aligned}
$$

$$
\begin{aligned}
\langle \boldsymbol{x} \mid (\alpha\mathbf{A})^* \boldsymbol{y} \rangle &= \langle (\alpha\mathbf{A})\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)} \\
&= \langle \alpha(\mathbf{A}\boldsymbol{x}) \mid \boldsymbol{y} \rangle && \text{by definition of scalar multiplication} \\
&= \alpha \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition F.9 page 132)} \\
&= \alpha \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)} \\
&= \langle \boldsymbol{x} \mid \alpha^*\mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition F.9 page 132)}
\end{aligned}
$$

[30] Sakai (1998) page 1
[31] Halmos (1998) pages 39–40, Rudin (1991) page 311

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, (\mathbf{AB})^* \boldsymbol{y} \rangle &= \langle (\mathbf{AB})\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)} \\
&= \langle \mathbf{A}(\mathbf{B}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle && \text{by definition of operator multiplication} \\
&= \langle (\mathbf{B}\boldsymbol{x}) \,|\, \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{B}^* \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{A}^{**} \boldsymbol{y} \rangle &= \langle \mathbf{A}^* \boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)} \\
&= \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle^* && \text{by definition of inner product} && \text{(Definition F.9 page 132)} \\
&= \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* && \text{by definition of adjoint} && \text{(Proposition F.3 page 133)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A} \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition F.9 page 132)}
\end{aligned}$$

✏

Theorem F.14. [32] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\begin{aligned}
\mathcal{N}(\mathbf{A}) &= \mathcal{I}(\mathbf{A}^*)^\perp \\
\mathcal{N}(\mathbf{A}^*) &= \mathcal{I}(\mathbf{A})^\perp
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathcal{I}(\mathbf{A}^*)^\perp &= \left\{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}^*) \right\} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} \\
&= \{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathbf{A}^* && \text{(Proposition F.3 page 133)} \\
&= \{ y \in \boldsymbol{H} \,|\, \mathbf{A} \boldsymbol{y} = 0 \} \\
&= \mathcal{N}(\mathbf{A}) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\mathbf{A})^\perp &= \{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}) \} \\
&= \{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \mathbf{A} \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathcal{I} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A}^* \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition F.3 page 133)} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \mathbf{A}^* \boldsymbol{y} = 0 \right\} \\
&= \mathcal{N}(\mathbf{A}^*) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

✏

F.4 Special Classes of Operators

F.4.1 Projection operators

Definition F.10. [33] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF

$\mathbf{P}$ *is a* ***projection*** *operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[32] Rudin (1991) page 312

[33] Rudin (1991) page 133 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 6, Halmos (1958) page 73 ⟨§41. Projections⟩

Theorem F.15. [34] *Let* $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{P}$ *be a bounded linear operator in* $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *with* NULL SPACE $\mathcal{N}(\mathbf{P})$ *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

$$\left.\begin{array}{llll} 1. & \mathbf{P}^2 = \mathbf{P} & (\mathbf{P} \text{ is a projection operator}) & \text{and} \\ 2. & \boldsymbol{\Omega} = \boldsymbol{X} \dotplus \boldsymbol{Y} & (\boldsymbol{Y} \text{ compliments } \boldsymbol{X} \text{ in } \boldsymbol{\Omega}) & \text{and} \\ 3. & \mathbf{P}\boldsymbol{\Omega} = \boldsymbol{X} & (\mathbf{P} \text{ projects onto } \boldsymbol{X}) & \end{array}\right\} \implies \left\{\begin{array}{lll} 1. & \mathcal{I}(\mathbf{P}) = \boldsymbol{X} & \text{and} \\ 2. & \mathcal{N}(\mathbf{P}) = \boldsymbol{Y} & \text{and} \\ 3. & \boldsymbol{\Omega} = \mathcal{I}(\mathbf{P}) \dotplus \mathcal{N}(\mathbf{P}) & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\ &= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\ &= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\ &= \boldsymbol{\Omega}_1 \end{aligned}$$

$$\begin{aligned} \mathcal{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\} \\ &= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_2 \end{aligned}$$

✏

Theorem F.16. [35] *Let* $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{P}$ *be a bounded linear operator in* $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.

THM

$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P} \text{ is a projection operator}} \iff \underbrace{(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})}_{(\mathbf{I} - \mathbf{P}) \text{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} (\mathbf{I} - \mathbf{P})^2 &= (\mathbf{I} - \mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I}(\mathbf{I} - \mathbf{P}) + (-\mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2 \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} && \text{by left hypothesis} \\ &= \mathbf{I} - \mathbf{P} \end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} \mathbf{P}^2 &= \underbrace{\mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P}^2}_{(\mathbf{I} - \mathbf{P})^2} - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P})^2 - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P}) - (\mathbf{I} - \mathbf{P} - \mathbf{P}) && \text{by right hypothesis} \\ &= \mathbf{P} \end{aligned}$$

✏

[34] Michel and Herget (1993) pages 120–121

[35] Michel and Herget (1993) page 121

Theorem F.17. [36] *Let $\boldsymbol{H}$ be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathcal{N}(\mathbf{P})$, *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

1. $\mathbf{P}^* = \mathbf{P}$ ($\mathbf{P}$ *is* SELF-ADJOINT) $\iff$
2. $\mathbf{P}^*\mathbf{P} = \mathbf{P}\mathbf{P}^*$ ($\mathbf{P}$ *is* NORMAL) $\iff$
3. $\mathcal{I}(\mathbf{P}) = \mathcal{N}(\mathbf{P})^\perp$ $\iff$
4. $\langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle = \|\mathbf{P}\boldsymbol{x}\|^2 \quad \forall x \in X$

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\implies$ (2):

$$\begin{aligned} \mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)} \\ &= \mathbf{P}\mathbf{P}^* && \text{by Theorem F.13 page 134} \end{aligned}$$

Proof that (1) $\implies$ (3):

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathcal{N}(\mathbf{P}^*)^\perp && \text{by Theorem F.14 page 135} \\ &= \mathcal{N}(\mathbf{P})^\perp && \text{by (1)} \end{aligned}$$

Proof that (3) $\implies$ (4):

Proof that (4) $\implies$ (1):

✏

F.4.2 Self Adjoint Operators

Definition F.11. [37] *Let* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be* ***self-adjoint*** *or* ***hermitian*** *if* $\mathbf{B} \stackrel{\circ}{=} \mathbf{B}^*$.

Example F.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}.$$

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}} \mathsf{f}(u)\, \mathsf{d}u$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

Theorem F.18. [38] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}$.

THM

$$\left\{ \underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S} \text{ is self adjoint}} \right\} \implies \left\{ \begin{array}{lll} 1. & \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \in \mathbb{R} & \text{(the hermitian quadratic form of } \mathbf{S} \text{ is REAL-VALUED)} \\ 2. & \lambda_n \in \mathbb{R} & \text{(eigenvalues of } \mathbf{S} \text{ are REAL-VALUED)} \\ 3. & \lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0 & \text{(eigenvectors are ORTHOGONAL)} \end{array} \right\}$$

[36] Rudin (1991) page 314

[37] Historical works regarding self-adjoint operators: von Neumann (1929) page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932) page 50 ⟨"self-adjoint transformations"⟩

[38] Lax (2002) pages 315–316, Keener (1988) pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩, Bertero and Boccacci (1998) page 225 ⟨§"9.2 SVD of a matrix ...If all eigenvectors are normalized..."⟩

✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \in \mathbb{R}$:

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition F.9 page 132}
\end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned}
\lambda_n \|\psi_n\|^2 &= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition} \\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition F.9 page 132} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_n \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \lambda_n^* \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition F.9 page 132} \\
&= \lambda_n^* \|\psi_n\|^2 && \text{by definition}
\end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition F.9 page 132} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_m \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \lambda_m^* \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition F.9 page 132} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

F.4.3 Normal Operators

Definition F.12. [39] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{N}^*$ be the adjoint of an operator $\mathbf{N} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF: $\mathbf{N}$ *is* ***normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

Theorem F.19. [40] *Let $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE *$\boldsymbol{H}$. Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$.*

THM:
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \text{ is normal}} \iff \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\| \qquad \forall \boldsymbol{x} \in \boldsymbol{H}$$

[39] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969) page 167, Frobenius (1878), Frobenius (1968) page 391

[40] Rudin (1991) pages 312–313

✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition F.3 page 133 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (}\mathbf{N}\text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition F.3 page 133 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition F.3 page 133 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem F.13 page 134 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis } (\|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by Proposition F.3 page 133 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem F.20. [41] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same null space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \left\{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{(definition of } \mathcal{N})\\
&= \left\{\boldsymbol{x} | \, \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition F.5 page 124)}\\
&= \left\{\boldsymbol{x} | \, \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} &&\\
&= \left\{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition F.5 page 124)}\\
&= \mathcal{N}(\mathbf{N}) && \text{(definition of } \mathcal{N})
\end{aligned}$$

✏

Theorem F.21. [42] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m\rangle = 0}_{\text{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. [Rudin] claims both to be true.(Rudin, 1991)313

[41] Rudin (1991) pages 312–313
[42] Rudin (1991) pages 312–313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\quad\ = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\quad\ = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem F.13 page 134} \\
&\quad\ = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem F.13 page 134} \\
&\quad\ = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\Longrightarrow \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\iff \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n\psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition F.9 page 132} \\
&= \langle \mathbf{N}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{N}^*\psi_m \rangle && \text{by Proposition F.3 page 133 (definition of adjoint)} \\
&= \langle \psi_n \,|\, \lambda_m^*\psi_m \rangle && \text{by (4.)} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition F.9 page 132}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

F.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition F.13. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES *(Definition F.5 page 124).*

DEF *An operator* $\mathbf{M} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is* ***isometric*** *if*

$$\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \qquad \forall x \in X.$$

Theorem F.22. [43] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM

$$\underbrace{\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall x \in X}_{\text{isometric in length}} \iff \underbrace{\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall x, y \in X}_{\text{isometric in distance}}$$

✎PROOF:

1. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \implies \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| &= \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\| && \text{by definition of linear operators (Definition F.4 page 121)} \\
&= \|\mathbf{M}\boldsymbol{u}\| && \text{let } \boldsymbol{u} \triangleq \boldsymbol{x} - \boldsymbol{y} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by left hypothesis}
\end{aligned}$$

[43] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

2. Proof that $\|\mathbf{M}x\| = \|x\| \impliedby \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$:

$$\begin{aligned}
\|\mathbf{M}x\| &= \|\mathbf{M}(x - \mathbb{0})\| \\
&= \|\mathbf{M}x - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition F.4 page 121)} \\
&= \|x - \mathbb{0}\| && \text{by right hypothesis} \\
&= \|x\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition F.13 page 140) in the more general normed linear spaces, while Theorem F.22 (page 140) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}x\| = \|x\| \iff \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$ for all $x, y \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem F.23 (next) demonstrates two additional equivalent properties.

Theorem F.23. [44] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.*

THM *The following conditions are all **equivalent**:*

1.	$\mathbf{M}^*\mathbf{M}$	$=$	$\mathbf{I}$			$\iff$
2.	$\langle \mathbf{M}x \mid \mathbf{M}y\rangle$	$=$	$\langle x \mid y\rangle$	$\forall x,y \in X$	($\mathbf{M}$ *is surjective*)	$\iff$
3.	$\|\mathbf{M}x - \mathbf{M}y\|$	$=$	$\|x - y\|$	$\forall x,y \in X$	(*isometric in distance*)	$\iff$
4.	$\|\mathbf{M}x\|$	$=$	$\|x\|$	$\forall x \in X$	(*isometric in length*)	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}x \mid \mathbf{M}y\rangle &= \langle x \mid \mathbf{M}^*\mathbf{M}y\rangle && \text{by Proposition F.3 page 133 (definition of adjoint)} \\
&= \langle x \mid \mathbf{I}y\rangle && \text{by (1)} \\
&= \langle x \mid y\rangle && \text{by Definition F.3 page 120 (definition of } \mathbf{I}\text{)}
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}x\| &= \sqrt{\langle \mathbf{M}x \mid \mathbf{M}x\rangle} && \text{by definition of } \|\cdot\| \\
&= \sqrt{\langle x \mid x\rangle} && \text{by right hypothesis} \\
&= \|x\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}x \mid \mathbf{M}y\rangle &= \|\mathbf{M}x + \mathbf{M}y\|^2 - \|\mathbf{M}x - \mathbf{M}y\|^2 + i\,\|\mathbf{M}x + i\mathbf{M}y\|^2 - i\,\|\mathbf{M}x - i\mathbf{M}y\|^2 && \text{by polarization id.} \\
&= \|\mathbf{M}(x + y)\|^2 - \|\mathbf{M}(x - y)\|^2 + i\,\|\mathbf{M}(x + iy)\|^2 - i\,\|\mathbf{M}(x - iy)\|^2 && \text{by Definition F.4} \\
&= \|x + y\|^2 - \|x - y\|^2 + i\,\|x + iy\|^2 - i\,\|x - iy\|^2 && \text{by left hypothesis}
\end{aligned}$$

4. Proof that (3) $\iff$ (4): by Theorem F.22 page 140

[44] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

5. Proof that (4) $\implies$ (1):

$$
\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition F.3 page 133 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem F.13 page 134 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y=0\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by Definition F.3 page 120 (definition of } \mathbf{I})\\
\implies \qquad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}
$$

✏

Theorem F.24. [45] *Let $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X},\boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle}$.*

THM

$$
\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M}\text{ is isometric}} \qquad \implies \qquad \left\{\begin{array}{rcll} |||\mathbf{M}||| &=& 1 & \text{(UNIT LENGTH)} \quad \textit{and}\\ |\lambda| &=& 1 & \forall\lambda\in\Lambda \end{array}\right.
$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \implies |||\mathbf{M}||| = 1$:

$$
\begin{aligned}
|||\mathbf{M}||| &= \sup_{\boldsymbol{x}\in X} \{\|\mathbf{M}\boldsymbol{x}\| \,|\, \|\boldsymbol{x}\| = 1\} && \text{by Definition F.6 page 125}\\
&= \sup_{\boldsymbol{x}\in X} \{\|\boldsymbol{x}\| \,|\, \|\boldsymbol{x}\| = 1\} && \text{by Theorem F.23 page 141}\\
&= \sup_{\boldsymbol{x}\in X} \{1\}\\
&= 1
\end{aligned}
$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x},\lambda)$ be an eigenvector-eigenvalue pair.

$$
\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|}\ \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem F.23 page 141}\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|}\ |\lambda|\ \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}
$$

✏

Example F.4 (One sided shift operator). [46] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ $(0,1,2,\ldots)$ and shift operators defined as

1. $\mathbf{S}_r\,(\!(x_0, x_1, x_2, \ldots)\!) \triangleq (\!(0, x_0, x_1, x_2, \ldots)\!)$ (right shift operator)
2. $\mathbf{S}_l\,(\!(x_0, x_1, x_2, \ldots)\!) \triangleq (\!(x_1, x_2, x_3, \ldots)\!)$ (left shift operator)

EX

1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

[45] Michel and Herget (1993) page 432
[46] Michel and Herget (1993) page 441

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r (\!(x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_0, y_1, y_2, \ldots)\!) \rangle &= \langle (\!(0, x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_0, y_1, y_2, \ldots)\!) \rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\; y_n^* \\
&= \sum_{n=0}^{\infty} x_n\; y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\; y_{n+1}^* \\
&= \langle (\!(x_0, x_1, x_2, \ldots)\!) \,|\, (\!(y_1, y_2, y_3, \ldots)\!) \rangle \\
&= \left\langle (\!(x_0, x_1, x_2, \ldots)\!) \,\middle|\, \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} (\!(y_0, y_1, y_2, \ldots)\!) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✐

F.4.5 Unitary operators

Definition F.14. [47] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF: *The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} \;=\; \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition F.4. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP:
$$\left.\begin{array}{l} \mathbf{U} \text{ is UNITARY} \quad \text{and} \\ \mathbf{V} \text{ is UNITARY} \end{array}\right\} \implies (\mathbf{UV}) \text{ is UNITARY.}$$

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\big(\mathbf{V}^*\mathbf{U}^*\big) && \text{by Theorem F.8 page 129} \\
&= \mathbf{U}\big(\mathbf{VV}^*\big)\mathbf{U}^* && \text{by associative property} \\
&= \mathbf{UIU}^* && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition F.14 page 143)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition F.14 page 143)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \big(\mathbf{V}^*\mathbf{U}^*\big)(\mathbf{UV}) && \text{by Theorem F.8 page 129} \\
&= \mathbf{V}^*\big(\mathbf{U}^*\mathbf{U}\big)\mathbf{V} && \text{by associative property} \\
&= \mathbf{V}^*\mathbf{IV} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition F.14 page 143)} \\
&= \mathbf{I} && \text{by definition of } \textit{unitary} \text{ operators} && \text{(Definition F.14 page 143)}
\end{aligned}$$

[47] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

Theorem F.25. [48] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{I}(\mathbf{U})$ be the* IMAGE SET *of* $\mathbf{U}$.

THM *If* $\mathbf{U}$ *is a* ***bounded linear operator*** ($\mathbf{U} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H})$), *then the following conditions are* ***equivalent***:

1. $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ (UNITARY) $\iff$
2. $\langle \mathbf{U}\boldsymbol{x} \mid \mathbf{U}\boldsymbol{y}\rangle = \langle \mathbf{U}^*\boldsymbol{x} \mid \mathbf{U}^*\boldsymbol{y}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ *and* $\mathcal{I}(\mathbf{U}) = X$ (SURJECTIVE) $\iff$
3. $\|\mathbf{U}\boldsymbol{x} - \mathbf{U}\boldsymbol{y}\| = \left\|\mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\right\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN DISTANCE) $\iff$
4. $\|\mathbf{U}\boldsymbol{x}\| = \|\boldsymbol{x}\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN LENGTH)

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) $\langle \mathbf{U}\boldsymbol{x} \mid \mathbf{U}\boldsymbol{y}\rangle = \langle \mathbf{U}^*\boldsymbol{x} \mid \mathbf{U}^*\boldsymbol{y}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ by Theorem F.23 (page 141).
   (b) Proof that $\mathcal{I}(\mathbf{U}) = X$:

$$\begin{aligned}
X &\supseteq \mathcal{I}(\mathbf{U}) && \text{because } \mathbf{U} \in X^X \\
&\supseteq \mathcal{I}(\mathbf{U}\mathbf{U}^*) \\
&= \mathcal{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}) \\
&= X && \text{by Definition F.3 page 120 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\iff$ (3) $\iff$ (4): by Theorem F.23 page 141.
3. Proof that (3) $\implies$ (1):
   (a) Proof that $\|\mathbf{U}\boldsymbol{x} - \mathbf{U}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \implies \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem F.23 page 141
   (b) Proof that $\left\|\mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\right\| = \|\boldsymbol{x} - \boldsymbol{y}\| \implies \mathbf{U}\mathbf{U}^* = \mathbf{I}$:

$$\begin{aligned}
\left\|\mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\right\| = \|\boldsymbol{x} - \boldsymbol{y}\| \implies && \mathbf{U}^{**}\mathbf{U}^* = \mathbf{I} && \text{by Theorem F.23 page 141} \\
&& \mathbf{U}\mathbf{U}^* = \mathbf{I} && \text{by Theorem F.13 page 134}
\end{aligned}$$

Theorem F.26. *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathbf{U}$ *be a bounded linear operator in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathcal{N}(\mathbf{U})$ *the* NULL SPACE *of* $\mathbf{U}$, *and* $\mathcal{I}(\mathbf{U})$ *the* IMAGE SET *of* $\mathbf{U}$.

THM

$$\underbrace{\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U} \text{ is unitary}} \implies \left\{\begin{array}{lllll}
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \textit{and} \\
\mathcal{I}(\mathbf{U}) & = & \mathcal{I}(\mathbf{U}^*) & = X & \textit{and} \\
\mathcal{N}(\mathbf{U}) & = & \mathcal{N}(\mathbf{U}^*) & = \{\mathbb{0}\} & \textit{and} \\
|||\mathbf{U}||| & = & |||\mathbf{U}^*||| & = 1 & (\text{UNIT LENGTH})
\end{array}\right\}$$

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both *isometric* and *normal*:

$$\begin{array}{llllll}
\mathbf{U}^*\mathbf{U} & & & = \mathbf{I} & \implies & \mathbf{U} \text{ is } \textit{isometric} \\
\mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = \mathbf{I} & \implies & \mathbf{U}^* \text{ is } \textit{isometric} \\
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \implies & \mathbf{U}^{-1} \text{ is } \textit{isometric}
\end{array}$$

$$\begin{array}{llllll}
\mathbf{U}^*\mathbf{U} & = & \mathbf{U}\mathbf{U}^* & = & \implies & \mathbf{U} \text{ is } \textit{normal} \\
\mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & \implies & \mathbf{U}^* \text{ is } \textit{normal} \\
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \implies & \mathbf{U}^{-1} \text{ is } \textit{normal}
\end{array}$$

[48] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem F.25 page 144.

3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}^{-1})$:

$$\begin{aligned}
\mathcal{N}(\mathbf{U}^*) &= \mathcal{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both } \textit{normal} \text{ and by Theorem F.21 page 139}\\
&= \mathcal{I}(\mathbf{U})^\perp && \text{by Theorem F.14 page 135}\\
&= X^\perp && \text{by above result}\\
&= \{\mathbb{0}\}
\end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem F.24 page 142.

✎

Example F.5 (Rotation matrix). [49]

EX
$$\underbrace{\left\{ \mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}}_{\textbf{rotation matrix } \mathbf{R}_\theta \,:\, \mathbb{R}^2 \to \mathbb{R}^2} \implies \left\{ \begin{array}{llcll} (1). & \mathbf{R}^{-1}{}_\theta & = & \mathbf{R}_{-\theta} & \text{and} \\ (2). & \mathbf{R}^*{}_\theta & = & \mathbf{R}^{-1}{}_\theta & (\mathbf{R} \text{ is unitary}) \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R}\\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of } \textit{Hermetian transpose} \text{ operator } H\\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{by Theorem C.2 page 77}\\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R}\\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✎

Example F.6. [50] Let $\mathbf{A}$ and $\mathbf{B}$ be matrix operators.

EX
$$\underbrace{\begin{array}{c|c} \mathbf{A} \triangleq \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} & \mathbf{B} \triangleq \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \mathbf{A} \text{ is a } \textit{rotation operator}. & \mathbf{B} \text{ is a } \textit{reflection operator}. \end{array}}_{\text{Both } \mathbf{A} \text{ and } \mathbf{B} \text{ are } \textbf{unitary}.}$$

Example F.7. Examples of *Fredholm integral operators* include

EX

1.	**Fourier Transform**	$[\tilde{\mathbf{F}}\mathrm{x}](f)$	$= \int_{t\in\mathbb{R}} \mathrm{x}(t)e^{-i2\pi ft}\,\mathrm{d}t$	$\kappa(t,f)$	$= e^{-i2\pi ft}$
2.	**Inverse Fourier Transform**	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathrm{x}}](t)$	$= \int_{f\in\mathbb{R}} \tilde{\mathrm{x}}(f)e^{i2\pi ft}\,\mathrm{d}f$	$\kappa(f,t)$	$= e^{i2\pi ft}$
3.	**Laplace operator**	$[\mathbf{L}\mathrm{x}](s)$	$= \int_{t\in\mathbb{R}} \mathrm{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t,s)$	$= e^{-st}$

Example F.8 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathrm{f}(x) \triangleq \mathrm{f}(x-1) \qquad \forall \mathrm{f} \in \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

[49] Noble and Daniel (1988) page 311
[50] Gel'fand (1963) page 4, Gelfand et al. (2018) page 4

EX

1.	$\mathbf{T}^{-1}\mathsf{f}(x)$	$=$ $\mathsf{f}(x+1)$	$\forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(inverse translation operator)
2.	$\mathbf{T}^*$	$=$ $\mathbf{T}^{-1}$		($\mathbf{T}$ is invertible)
3.	$\mathbf{T}^*\mathbf{T}$	$=$ $\mathbf{T}\mathbf{T}^* = \mathbf{I}$		($\mathbf{T}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$
$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned}
\langle \mathbf{T}\mathsf{f}(x)\,|\,\mathsf{g}(x)\rangle &= \langle \mathsf{f}(x-1)\,|\,\mathsf{g}(x)\rangle && \text{by definition of } \mathbf{T}\\
&= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathrm{d}x\\
&= \langle \mathsf{f}(x)\,|\,\mathsf{g}(x+1)\rangle\\
&= \left\langle \mathsf{f}(x)\,\middle|\,\underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example F.9 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T}\in\boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX

1.	$\mathbf{D}^{-1}\mathsf{f}(x)$	$=$ $\frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$	$\forall\mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(inverse dilation operator)
2.	$\mathbf{D}^*$	$=$ $\mathbf{D}^{-1}$		($\mathbf{D}$ is invertible)
3.	$\mathbf{D}^*\mathbf{D}$	$=$ $\mathbf{D}\mathbf{D}^* = \mathbf{I}$		($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x)\,|\,\mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x)\,\middle|\,\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\,\mathrm{d}u\\
&= \left\langle \mathsf{f}(x)\,\middle|\,\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x)\,\middle|\,\underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

Example F.10 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq (\!(x_{n-1})\!)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq \left(\!\!\left(x_{n+1}\right)\!\!\right)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\,(\!(x_n)\!) \,|\, (\!(y_n)\!) \rangle &= \langle (\!(x_{n-1})\!) \,|\, (\!(y_n)\!) \rangle && \text{by definition of } \mathbf{D} \\
&= \sum_n \boldsymbol{x}_{n-1}\, \boldsymbol{y}_n^* \\
&= \sum_n \boldsymbol{x}_n\, \boldsymbol{y}_{n+1}^* \\
&= \left\langle (\!(x_n)\!) \,|\, \left(\!\!\left(y_{n+1}\right)\!\!\right) \right\rangle \\
&= \left\langle (\!(x_n)\!) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\,(\!(y_n)\!) \right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary. ✏

Example F.11 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator (Theorem 7.1 page 52)

$$[\tilde{\mathbf{F}}\mathrm{x}](f) \triangleq \int_t \mathrm{x}(t)\underbrace{e^{-i2\pi f t}}_{\kappa(t,f)}\,\mathrm{d}t \qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathrm{x}}\right](t) \triangleq \int_f \tilde{\mathrm{x}}(f)\underbrace{e^{i2\pi f t}}_{\kappa^*(t,f)}\,\mathrm{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathrm{x} \,|\, \tilde{\mathrm{y}} \right\rangle &= \left\langle \int_t \mathrm{x}(t)e^{-i2\pi f t}\,\mathrm{d}t \,|\, \tilde{\mathrm{y}}(f) \right\rangle \\
&= \int_t \mathrm{x}(t) \left\langle e^{-i2\pi f t} \,|\, \tilde{\mathrm{y}}(f) \right\rangle \mathrm{d}t \\
&= \int_t \mathrm{x}(t) \int_f e^{-i2\pi f t}\tilde{\mathrm{y}}^*(f)\,\mathrm{d}f\,\mathrm{d}t \\
&= \int_t \mathrm{x}(t) \left[\int_f e^{i2\pi f t}\tilde{\mathrm{y}}(f)\,\mathrm{d}f\right]^* \mathrm{d}t \\
&= \left\langle \mathrm{x}(t) \,|\, \int_f \tilde{\mathrm{y}}(f)e^{i2\pi f t}\,\mathrm{d}f \right\rangle \\
&= \left\langle \mathrm{x} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathrm{y}} \right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✏

F.5 Operator order

Definition F.15. [51] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF: $\mathbf{P}$ *is **positive** if* $\langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle \geq 0 \; \forall \boldsymbol{x} \in \boldsymbol{X}$.
This condition is denoted $\mathbf{P} \geq 0$.

Theorem F.27. [52]

THM:
$$\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \implies \left\{ \begin{array}{lllll} (\mathbf{P}+\mathbf{Q}) & \geq & 0 & & ((\mathbf{P}+\mathbf{Q}) \text{ is positive}) \\ \mathbf{A}^*\mathbf{P}\mathbf{A} & \geq & 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{X}) & (\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive}) \\ \mathbf{A}^*\mathbf{A} & \geq & 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{X}) & (\mathbf{A}^*\mathbf{A} \text{ is positive}) \end{array} \right.$$

✎PROOF:

$$\begin{aligned}
\langle (\mathbf{P}+\mathbf{Q})\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle + \langle \mathbf{Q}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by additive property of } \langle \triangle \mid \nabla \rangle \text{ (Definition F.9 page 132)} \\
&\geq \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \mathbf{P}\mathbf{A}\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} \rangle && \text{by definition of adjoint (Proposition F.3 page 133)} \\
&= \langle \mathbf{P}\boldsymbol{y} \mid \boldsymbol{y} \rangle && \text{where } \boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition F.3 page 120)} \\
&\geq 0 && \text{by non-negative property of } \langle \triangle \mid \nabla \rangle \text{ (Definition F.9 page 132)} \\
&\implies \mathbf{I} \text{ is positive} \\
\langle \mathbf{A}^*\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition F.3 page 120)} \\
&\geq 0 && \text{by two previous results}
\end{aligned}$$

✑

Definition F.16. [53] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be* BOUNDED *operators.*

DEF: $\mathbf{A} \geq \mathbf{B}$ *("$\mathbf{A}$ is greater than or equal to $\mathbf{B}$") if*
$\mathbf{A} - \mathbf{B} \geq 0$ *("($\mathbf{A} - \mathbf{B}$) is positive")*

[51] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩
[52] Michel and Herget (1993) page 429
[53] Michel and Herget (1993) page 429

APPENDIX G

LINEAR COMBINATIONS

G.1 Linear combinations in linear spaces

A *linear space* (Definition F.1 page 119) in general is not equipped with a *topology*. Without a topology, it is not possible to determine whether an *infinite sum* of vectors converges. Therefore in this section (dealing with linear spaces), all definitions related to sums of vectors will be valid for *finite* sums only (finite "N").

Definition G.1. [1] *Let* $\left\{\boldsymbol{x}_n \in X \middle| n=1,2,\ldots,N\right\}$ *be a set of vectors in a* LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

DEF: *A vector* $\boldsymbol{x} \in X$ *is a* ***linear combination*** *of the vectors in* $\left\{\boldsymbol{x}_n\right\}$ *if there exists* $\left\{\alpha_n \in \mathbb{F} \middle| n=1,2,\ldots,N\right\}$ *such that*
$$\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n .$$

Definition G.2. [2] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space and* Y *be a subset of* X.

DEF: *The* ***linear span*** *of* Y *is defined as* $\operatorname{span} Y \triangleq \left\{ \sum_{\gamma\in\Gamma} \alpha_\gamma \boldsymbol{y}_\gamma \middle| \alpha_\gamma \in \mathbb{F},\ \boldsymbol{y}_\gamma \in Y \right\}$.
The set Y ***spans*** *a set* A *if* $A \subseteq \operatorname{span} Y$.

Proposition G.1. [3] *Let* $\left\{\boldsymbol{x}_n \in X \middle| n=1,2,\ldots,N\right\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

PRP:
1. $\operatorname{span}\left\{\boldsymbol{x}_n\right\}$ *is a* LINEAR SPACE (Definition F.1 page 119) *and*
2. $\operatorname{span}\left\{\boldsymbol{x}_n\right\}$ *is a* LINEAR SUBSPACE *of* $\boldsymbol{L}$.

Definition G.3. [4] *Let* $\boldsymbol{L} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a* LINEAR SPACE.

DEF: *The set* $Y \triangleq \left\{\boldsymbol{x}_n \in X \middle| n=1,2,\ldots,N\right\}$ *is* ***linearly independent*** *in* $\boldsymbol{L}$ *if*
$$\left\{\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = 0\right\} \implies \left\{\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0\right\}.$$
The set Y *is* ***linearly dependent*** *in* $\boldsymbol{L}$ *if* Y *is not linearly independent in* $\boldsymbol{L}$.

[1] Berberian (1961) page 11 ⟨Definition I.4.1⟩, Kubrusly (2001) page 46
[2] Michel and Herget (1993) page 86 ⟨3.3.7 Definition⟩, Kurdila and Zabarankin (2005) page 44, Searcóid (2002) page 71 ⟨Definition 3.2.5—more general definition⟩
[3] Kubrusly (2001) page 46
[4] Bachman and Narici (1966) pages 3–4, Christensen (2003) page 2, Heil (2011) page 156 ⟨Definition 5.7⟩

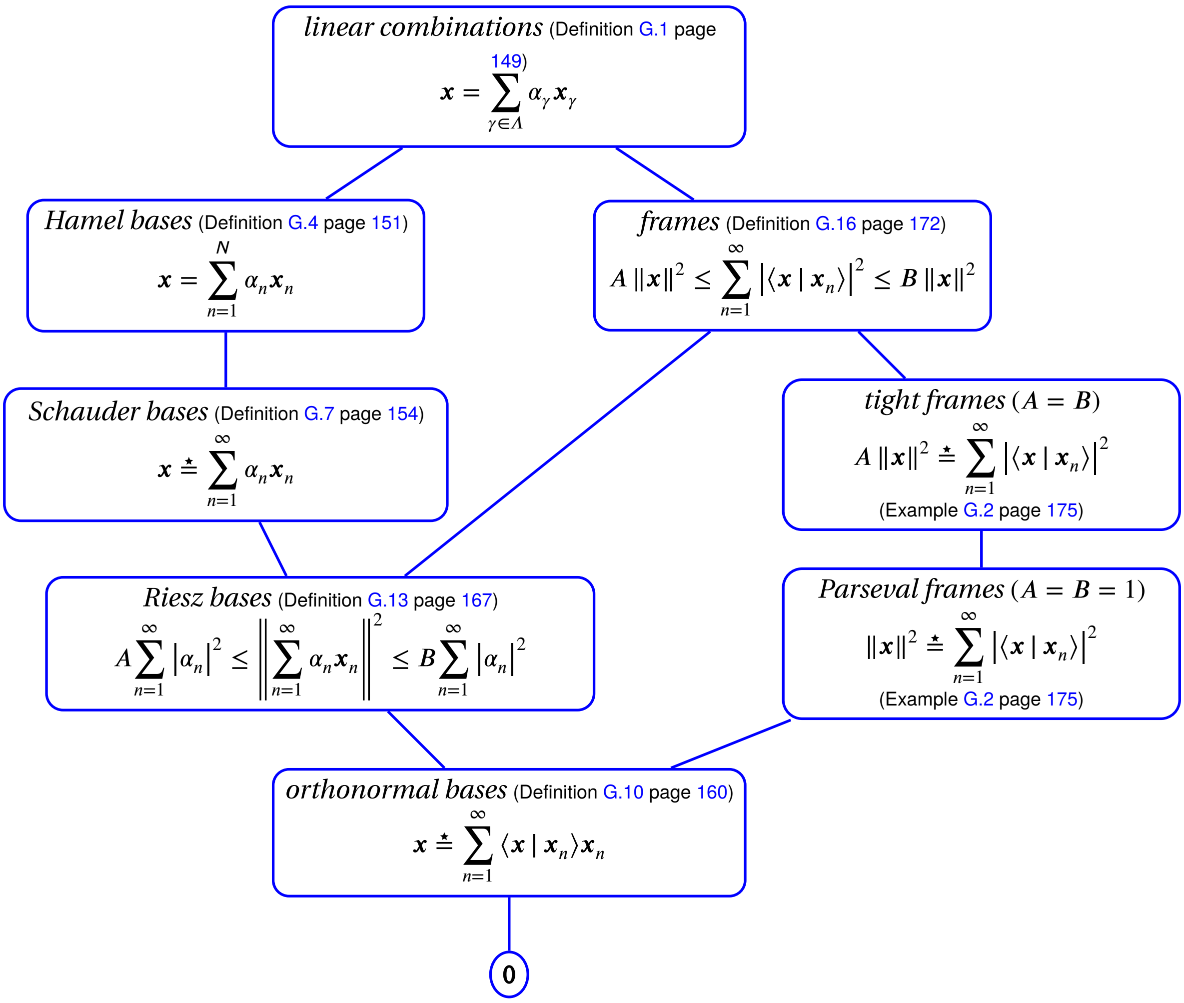


Figure G.1: Lattice of *linear combinations*

Definition G.4. [5] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF *The set* $\{\boldsymbol{x}_n\}$ *is a* ***Hamel basis*** *for* $\boldsymbol{L}$ *if*

1. $\{\boldsymbol{x}_n\}$ SPANS $\boldsymbol{L}$ (Definition G.2 page 149) *and*
2. $\{\boldsymbol{x}_n\}$ *is* LINEARLY INDEPENDENT *in* $\boldsymbol{L}$ (Definition G.1 page 149) .

A HAMEL BASIS *is also called a* ***linear basis***.

Definition G.5. [6] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* $\boldsymbol{x}$ *be a* VECTOR *in* $\boldsymbol{L}$ *and* $Y \triangleq \{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in* $\boldsymbol{L}$.

DEF *The expression* $\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$ *is the* ***expansion*** *of* $\boldsymbol{x}$ *on* Y *in* $\boldsymbol{L}$ *if* $\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$.
In this case, the sequence $(\!(\alpha_n)\!)_{n=1}^{N}$ *is the* ***coordinates*** *of* $\boldsymbol{x}$ *with respect to* Y *in* $\boldsymbol{L}$.
If $\alpha_N \neq 0$, *then* N *is the* ***dimension*** $\dim \boldsymbol{L}$ *of* $\boldsymbol{L}$.

Theorem G.1. [7] *Let* $\{\boldsymbol{x}_n \mid n=1,2,\ldots,N\}$ *be a* HAMEL BASIS (Definition G.4 page 151) *for a* LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

THM
$$\left\{\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = \sum_{n=1}^{N} \beta_n \boldsymbol{x}_n\right\} \implies \underbrace{\alpha_n = \beta_n \quad \forall n=1,2,\ldots,N}_{\text{coordinates of } \boldsymbol{x} \text{ are UNIQUE}} \qquad \forall \boldsymbol{x} \in X$$

✎PROOF:

$$\begin{aligned}
\mathbb{0} &= \boldsymbol{x} - \boldsymbol{x} \\
&= \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n - \sum_{n=1}^{N} \beta_n \boldsymbol{x}_n \\
&= \sum_{n=1}^{N} (\alpha_n - \beta_n) \boldsymbol{x}_n \\
&\implies \{\boldsymbol{x}_n\} \text{ is } \textit{linearly dependent} \text{ if } (\alpha_n - \beta_n) \neq 0 \qquad \forall n = 1,2,\ldots,N \\
&\implies (\alpha_n - \beta_n) = 0 \qquad \forall n = 1,2,\ldots,N \qquad \text{(because } \{\boldsymbol{x}_n\} \text{ is a } \textit{basis} \text{ and therefore must be } \textit{linearly independent}\text{)} \\
&\implies \alpha_n = \beta_n \text{ for } n = 1,2,\ldots,N
\end{aligned}$$

✏

Theorem G.2. [8] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

THM
$$\left\{\begin{array}{ll} 1. & \{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \quad \textit{and} \\ 2. & \{\boldsymbol{y}_n \in X \mid n=1,2,\ldots,M\} \text{ is a set of LINEARLY INDEPENDENT vectors in } \boldsymbol{L} \end{array}\right\}$$
$$\implies \left\{\begin{array}{lll} 1. & M \leq N & \textit{and} \\ 2. & M = N \implies \{\boldsymbol{y}_n \mid n=1,2,\ldots,M\} \text{ is a BASIS for } \boldsymbol{L} & \textit{and} \\ 3. & M \neq N \implies \{\boldsymbol{y}_n \mid n=1,2,\ldots,M\} \text{ is NOT a basis for } \boldsymbol{L} & \end{array}\right\}$$

✎PROOF:

[5] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[6] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[7] Michel and Herget (1993) pages 89–90 ⟨Theorem 3.3.25⟩

[8] Michel and Herget (1993) pages 90–91 ⟨Theorem 3.3.26⟩

1. Proof that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is a *basis* for $\boldsymbol{L}$:

   (a) Proof that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ *spans* $\boldsymbol{L}$:

   i. Because $\{\boldsymbol{x}_n | n=1,2,\ldots,N\}$ is a *basis* for $\boldsymbol{L}$, there exists $\beta \in \mathbb{F}$ and $\{\alpha_n \in \mathbb{F} | n=1,2,\ldots,N\}$ such that $\beta \boldsymbol{y}_1 + \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = 0$.

   ii. Select an n such that $\alpha_n \neq 0$ and renumber (if necessary) the above indices such that $\boldsymbol{x}_n = -\frac{\beta}{\alpha_n}\boldsymbol{y}_1 + \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n}\boldsymbol{x}_n$.

   iii. Then, for any $\boldsymbol{y} \in X$, we can write

$$
\begin{aligned}
\boldsymbol{y} &= \sum_{n=1}^{N} \gamma_{n\in\mathbb{Z}} \boldsymbol{x}_n \\
&= \left(\sum_{n=1}^{N-1} \gamma_{n\in\mathbb{Z}} \boldsymbol{x}_n\right) + \gamma_{n\in\mathbb{Z}}\left(-\frac{\beta}{\alpha_n}\boldsymbol{y}_1 - \sum_{n=1}^{N-1}\frac{\alpha_n}{\alpha_n}\boldsymbol{x}_n\right) \\
&= -\frac{\beta\gamma_n}{\alpha_n}\boldsymbol{y}_1 + \sum_{n=1}^{N-1}\left(\gamma_n - \frac{\alpha_n\gamma_n}{\alpha_n}\right)\boldsymbol{x}_n \\
&= \delta \boldsymbol{y}_1 + \sum_{n=1}^{N-1} \delta_{n\in\mathbb{Z}} \boldsymbol{x}_n
\end{aligned}
$$

   iv. This implies that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ *spans* $\boldsymbol{L}$:

   (b) Proof that $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is *linearly independent*:

   i. If $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is *linearly dependent*, then there exists $\{\epsilon, \epsilon_1, \ldots, \epsilon_{N-1}\}$ such that $\epsilon \boldsymbol{y}_1 + \left(\sum_{n=1}^{N-1} \epsilon_{n\in\mathbb{Z}} \boldsymbol{x}_n\right) + 0\boldsymbol{x}_n = 0$.

   ii. item (1(b)i) implies that the coordinate of $\boldsymbol{y}_1$ associated with $\boldsymbol{x}_n$ *is* 0. $\boldsymbol{y}_1 = -\left(\sum_{n=1}^{N-1} \frac{\epsilon_n}{\epsilon}\boldsymbol{x}_n\right) + 0\boldsymbol{x}_n = 0$.

   iii. item (1(a)i) implies that the coordinate of $\boldsymbol{y}_1$ associated with $\boldsymbol{x}_n$ is *not* 0. $\boldsymbol{y}_1 = -\sum_{n=1}^{N} \frac{\alpha_n}{\beta}\boldsymbol{x}_n$.

   iv. This implies that item (1(b)i) (that the set is linearly dependent) is *false* because item (1(b)ii) and item (1(b)iii) *contradict* each other.

   v. This implies $\{\boldsymbol{y}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ is *linearly independent*.

2. Proof that $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-2}\}$ is a *basis*: Repeat item (1).

3. Suppose $m = n$. Proof that $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_M\}$ is a *basis*: Repeat item (1) $M-1$ times.

4. Proof that $M \not> N$:

   (a) Suppose that $M = N+1$.

   (b) Then because $\{\boldsymbol{y}_n | n=1,2,\ldots,N\}$ is a *basis*, there exists $\{\zeta_n | n=1,2,\ldots,N+1\}$ such that $\sum_{n=1}^{N+1} \zeta_{n\in\mathbb{Z}} \boldsymbol{y}_{n\in\mathbb{Z}} = 0$.

   (c) This implies that $\{\boldsymbol{y}_n | n=1,2,\ldots,N+1\}$ is *linearly dependent*.

(d) This implies that $\left\{\boldsymbol{y}_n | n=1,2,\ldots,N+1\right\}$ is *not* a basis.

(e) This implies that $M \not> N$.

5. Proof that $M \neq N \implies \left\{\boldsymbol{y}_n | n=1,2,\ldots,M\right\}$ is *not* a basis for $\boldsymbol{L}$:

   (a) Proof that $M > N \implies \left\{\boldsymbol{y}_n | n=1,2,\ldots,M\right\}$ is *not* a basis for $\boldsymbol{L}$: same as in item (4).

   (b) Proof that $M < N \implies \left\{\boldsymbol{y}_n | n=1,2,\ldots,M\right\}$ is *not* a basis for $\boldsymbol{L}$:

      i. Suppose $m = N - 1$.

      ii. Then $\left\{\boldsymbol{y}_n | n=1,2,\ldots,N-1\right\}$ is a *basis* and there exists λ such that
      $$\sum_{n=1}^{N} \lambda_{n\in\mathbb{Z}} \boldsymbol{y}_{n\in\mathbb{Z}} = 0.$$

      iii. This implies that $\left\{\boldsymbol{y}_n \middle| n=1,2,\ldots,N\right\}$ is *linearly dependent* and is *not* a basis.

      iv. But this contradicts item (3), therefore $M \neq N - 1$.

      v. Because $M = N$ yields a basis but $M = N - 1$ does not, $M < N - 1$ also does not yield a basis.

✏

Corollary G.1. [9] *Let $\boldsymbol{L} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ be a linear space.*

COR
$$\underbrace{\left\{\begin{array}{ll} 1. & \left\{\boldsymbol{x}_n \in X \middle| n=1,2,\ldots,N\right\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \quad \text{and} \\ 2. & \left\{\boldsymbol{y}_n \in X | n=1,2,\ldots,M\right\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \end{array}\right\} \implies \{N = M\}}_{\text{(all Hamel bases for } \boldsymbol{L} \text{ have the same number of vectors)}}$$

✎PROOF: This follows from Theorem G.2 (page 151). ✏

G.2 Bases in topological linear spaces

A linear space supports the concept of the *span* of a set of vectors (Definition G.2 page 149). In a topological linear space $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T}\right)$, a set A is said to be *total* in $\boldsymbol{\Omega}$ if the span of A is *dense* in $\boldsymbol{\Omega}$. In this case, A is said to be a *total set* or a *complete set*. However, this use of "complete" in a "*complete set*" is not equivalent to the use of "complete" in a "*complete metric space*" .[10] In this text, except for these comments and Definition G.6, "complete" refers to the metric space definition only.

If a set is both *total* and *linearly independent* (Definition G.3 page 149) in $\boldsymbol{\Omega}$, then that set is a *Hamel basis* (Definition G.4 page 151) for $\boldsymbol{\Omega}$.

Definition G.6. [11] *Let A^- be the* CLOSURE *of a A in a* TOPOLOGICAL LINEAR SPACE $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T}\right)$*. Let* $\operatorname{span} A$ *be the* SPAN (Definition G.2 page 149) *of a set A.*

DEF
*A set of vectors A is **total** (or **complete** or **fundamental**) in $\boldsymbol{\Omega}$ if*
$$(\operatorname{span} A)^- = \boldsymbol{\Omega} \qquad \text{(SPAN of } A \text{ is DENSE in } \boldsymbol{\Omega}).$$

[9] Kubrusly (2001) page 52 ⟨Theorem 2.7⟩, Michel and Herget (1993) page 91 ⟨Theorem 3.3.31⟩

[10] Haaser and Sullivan (1991) pages 296–297 ⟨6·Orthogonal Bases⟩, Rynne and Youngson (2008) page 78 ⟨Remark 3.50⟩, Heil (2011) page 21 ⟨Remark 1.26⟩

[11] Young (2001) page 19 ⟨Definition 1.5.1⟩, Sohrab (2003) page 362 ⟨Definition 9.2.3⟩, Gupta (1998) page 134 ⟨Definition 2.4⟩, Bachman and Narici (1966) pages 149–153 ⟨Definition 9.3, Theorems 9.9 and 9.10⟩

G.3 Schauder bases in Banach spaces

Definition G.7. [12] *Let* $\boldsymbol{B} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *be a* BANACH SPACE. *Let* $\stackrel{\star}{=}$ *represent* STRONG CONVERGENCE *in* $\boldsymbol{B}$.

DEF

The countable set $\left\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\right\}$ *is a **Schauder basis** for* $\boldsymbol{B}$ *if for each* $\boldsymbol{x} \in X$

1. $\exists \left(\alpha_n \in \mathbb{F}\right)_{n\in\mathbb{N}}$ such that $\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ (STRONG CONVERGENCE *in* $\boldsymbol{B}$) *and*
2. $\left\{\sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \stackrel{\star}{=} \sum_{n=1}^{\infty} \beta_n \boldsymbol{x}_n\right\} \implies \left\{(\!(\alpha_n)\!) = (\!(\beta_n)\!)\right\}$ (COEFFICIENT FUNCTIONALS *are* UNIQUE)

In this case, $\sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ *is the **expansion** of* $\boldsymbol{x}$ *on* $\left\{\boldsymbol{x}_n \mid n\in\mathbb{N}\right\}$ *and the elements of* $(\!(\alpha_n)\!)$ *are the **coefficient functionals** associated with the basis* $\{\boldsymbol{x}_n\}$. *Coefficient functionals are also called **coordinate functionals**.*

In a Banach space, the existence of a Schauder basis implies that the space is *separable* (Theorem G.3 page 154). The question of whether the converse is also true was posed by Banach himself in 1932,[13] and became know as "*The basis problem*". This remained an open question for many years. The question was finally answered some 41 years later in 1973 by Per Enflo (University of California at Berkley), with the answer being "no". Enflo constructed a counterexample in which a separable Banach space does *not* have a Schauder basis.[14] Life is simpler in Hilbert spaces where the converse *is* true: a Hilbert space has a Schauder basis *if and only if* it is separable (Theorem G.11 page 167).

Theorem G.3. [15] *Let* $\boldsymbol{B} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *be a* BANACH SPACE. *Let* $\mathbb{Q}$ *be the field of rational numbers.*

THM

$$\left\{\begin{array}{ll} 1. & \boldsymbol{B} \text{ has a SCHAUDER BASIS} \quad \text{and} \\ 2. & \mathbb{Q} \text{ is DENSE in } \mathbb{F}. \end{array}\right\} \implies \left\{\ \boldsymbol{B} \text{ is SEPARABLE}\ \right\}$$

✎PROOF:

1. lemma:

$$\begin{aligned}
\left|\left\{\boldsymbol{x} \mid \exists\, (\!(\alpha_n \mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x} - \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right\| = 0\right\}\right| &= |\,\mathbb{Q}\times\mathbb{N}\,| \\
&= |\,\mathbb{Z}\times\mathbb{Z}\,| \\
&= |\,\mathbb{Z}\,| \\
&= \textit{countably infinite}
\end{aligned}$$

[12] Carothers (2005) pages 24–25, Christensen (2003) pages 46–49 ⟨Definition 3.1.1 and page 49⟩, Young (2001) page 19 ⟨Section 6⟩, Singer (1970) page 17, Schauder (1927), Schauder (1928)
[13] Banach (1932a) page 111
[14] Enflo (1973), Lindenstrauss and Tzafriri (1977) pages 84–95 ⟨Section 2.d⟩
[15] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Giles (2000) page 17, Heil (2011) page 21 ⟨Theorem 1.27⟩

2. remainder of proof:

$\boldsymbol{B}$ has a *Schauder basis* $(\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{N}}$

$$\Longrightarrow \text{ for every } \boldsymbol{x}\in\boldsymbol{B}, \text{ there exists } (\!(\alpha_n\in\mathbb{F})\!)_{n\in\mathbb{N}} \text{ such that } \boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n \qquad \text{by Definition G.7 page 154}$$

$$\Longrightarrow \text{ for every } \boldsymbol{x}\in\boldsymbol{B}, \text{ there exists } (\!(\alpha_n\in\mathbb{F})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0$$

$$\Longrightarrow \text{ for every } \boldsymbol{x}\in\boldsymbol{B}, \text{ there exists } (\!(\alpha_n\in\mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0 \quad \text{because } \mathbb{Q}^- = \mathbb{F}$$

$$\Longrightarrow \boldsymbol{B} = \left\{\boldsymbol{x}\,|\,\exists\,(\!(\alpha_n\mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0\right\}$$

$$\Longrightarrow \boldsymbol{B} = \left\{\boldsymbol{x}\,|\,\exists\,(\!(\alpha_n\mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \boldsymbol{x} = \lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\}^-$$

$$\Longrightarrow \boldsymbol{B} \text{ is } \textit{separable} \qquad \text{by (1) lemma page 154}$$

✏

Definition G.8. [16] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

DEF
$\{\boldsymbol{x}_n\}$ *is **equivalent** to* $\{\boldsymbol{y}_n\}$
if there exists a BOUNDED INVERTIBLE *operator* $\mathbf{R}$ *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *such that* $\quad \mathbf{R}\boldsymbol{x}_n = \boldsymbol{y}_n \qquad \forall n\in\mathbb{Z}$

Theorem G.4. [17] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

THM
$$\left\{\{\boldsymbol{x}_n\} \text{ is EQUIVALENT to } \{\boldsymbol{y}_n\}\right\} \iff \left\{\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n \text{ is CONVERGENT} \iff \sum_{n=1}^{\infty}\alpha_n\boldsymbol{y}_n \text{ is CONVERGENT}\right\}$$

Lemma G.1. [18] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$ *be a topological linear space. Let* $\operatorname{span} A$ *be the* SPAN *of a set* A (Definition G.2 page 149). *Let* $\tilde{\mathsf{f}}(\omega)$ *and* $\tilde{\mathsf{g}}(\omega)$ *be the* FOURIER TRANSFORMS (Definition 4.2 page 16) *of the functions* $\mathsf{f}(x)$ *and* $\mathsf{g}(x)$*, respectively, in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition B.1 page 69). *Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT (Definition 6.1 page 41) *of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ (Definition 5.2 page 27).

LEM
$$\left\{\begin{array}{ll}(1). & \{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega} \text{ and}\\ (2). & \{\mathbf{T}^n\mathsf{g}\,|\,n\in\mathbb{Z}\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\exists\,(\!(a_n)\!)_{n\in\mathbb{Z}} \quad \text{such that}\\ \tilde{\mathsf{f}}(\omega) = \breve{\mathsf{a}}(\omega)\tilde{\mathsf{g}}(\omega)\end{array}\right\}$$

✎PROOF: Let V_0' be the space spanned by $\{\mathbf{T}^n\phi|\,n\in\mathbb{Z}\}$.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 4.2 page 16)}\\
&= \tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}}a_n\mathbf{T}\mathsf{g} && \text{by (2)}\\
&= \sum_{n\in\mathbb{Z}}a_n\tilde{\mathbf{F}}\mathbf{T}\mathsf{g}
\end{aligned}$$

[16] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[17] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[18] Daubechies (1992) page 140

$$= \underbrace{\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}}_{\breve{a}(\omega)} \underbrace{\tilde{\mathbf{F}}\mathsf{g}}_{\tilde{\mathsf{g}}(\omega)}$$ by Corollary I.1 page 195

$$= \breve{a}(\omega)\tilde{\mathsf{g}}(\omega)$$ by definition of $\breve{\mathbf{F}}$ and $\tilde{\mathbf{F}}$ by (Definition 6.1 page 41, Definition 4.2 page 16)

$$\begin{aligned}
V_0 &\triangleq \left\{ \mathsf{f}(x) | \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathbf{F}}\mathsf{f}(x) = \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \breve{\mathsf{b}}(\omega)\tilde{\mathsf{g}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \breve{\mathsf{b}}(\omega)\breve{a}(\omega)\tilde{\mathsf{f}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \breve{\mathsf{c}}(\omega)\tilde{\mathsf{f}}(\omega) \right\} && \text{where } \breve{\mathsf{c}}(\omega) \triangleq \breve{\mathsf{b}}(\omega)\breve{a}(\omega) \\
&= \left\{ \mathsf{f}(x) | \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{c}_n \mathsf{f}(x-n) \right\} \\
&\triangleq V_0'
\end{aligned}$$

G.4 Linear combinations in inner product spaces

Definition G.9. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle | \triangledown \rangle)$ *be an* INNER PRODUCT SPACE *(Definition F.9 page 132).*

DEF: *Two vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* X *are* ***orthogonal*** *if*

$$\langle \mathbf{x} | \mathbf{y} \rangle = \begin{cases} 0 & \text{for } \mathbf{x} \neq \mathbf{y} \\ c \in \mathbb{F}\backslash 0 & \text{for } \mathbf{x} = \mathbf{y} \end{cases}$$

In an *inner product space*, *orthogonality* is a special case of *linear independence*; or alternatively, linear independence is a generalization of orthogonality (next theorem).

Theorem G.5. [19] *Let* $\{\mathbf{x}_n \in X | n=1,2,\ldots,N\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition F.9 page 132)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle | \triangledown \rangle)$.

THM: $$\left\{ \{\mathbf{x}_n\} \text{ is ORTHOGONAL (Definition G.9 page 156)} \right\} \implies \left\{ \{\mathbf{x}_n\} \text{ is LINEARLY INDEPENDENT (Definition G.1 page 149)} \right\}$$

✎PROOF:

1. Proof using *Pythagorean theorem* :
   Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence with at least one nonzero element.

[19] Aliprantis and Burkinshaw (1998) page 283 ⟨Corollary 32.8⟩, Kubrusly (2001) page 352 ⟨Proposition 5.34⟩

$$\begin{aligned}
\left\| \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \right\|^2 &= \sum_{n=1}^{N} \left\| \alpha_n \boldsymbol{x}_n \right\|^2 && \text{by left hypoth. and } \textit{Pythagorean Theorem} \\
&= \sum_{n=1}^{N} |\alpha_n|^2 \left\| \boldsymbol{x}_n \right\|^2 && \text{by definition of } \|\cdot\| && \text{(Definition F.5 page 124)} \\
&> 0
\end{aligned}$$

$$\begin{aligned}
&\implies \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \neq 0 \\
&\implies (\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{N}} \text{ is } \textit{linearly independent} && \text{by definition of } \textit{linear independence} && \text{(Definition G.3 page 149)}
\end{aligned}$$

2. Alternative proof:

$$\begin{aligned}
\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = \mathbb{0} &\implies \left\langle \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \,\middle|\, \boldsymbol{x}_m \right\rangle = \langle \mathbb{0} \,|\, \boldsymbol{x}_m \rangle \\
&\implies \sum_{n=1}^{N} \alpha_n \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_m \rangle = 0 \\
&\implies \sum_{n=1}^{N} \alpha_n \bar{\delta}(k-m) = 0 \\
&\implies \alpha_m = 0 \qquad \text{for } m = 1, 2, \ldots, N
\end{aligned}$$

✏

Theorem G.6 (Bessel's Equality). [20] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition F.9 page 132)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \triangledown \rangle\right)$ *and with* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}$.

T H M

$$\left\{ \{\boldsymbol{x}_n\} \text{ is ORTHONORMAL (Definition G.9 page 156)} \right\} \implies \left\{ \underbrace{\left\| \boldsymbol{x} - \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2}_{\text{approximation error}} = \|\boldsymbol{x}\|^2 - \sum_{n=1}^{N} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 \quad \forall \boldsymbol{x} \in X \right\}$$

✎PROOF:

$$\begin{aligned}
&\left\| \boldsymbol{x} - \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 \\
&= \|\boldsymbol{x}\|^2 + \left\| \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 - 2\Re \left\langle \boldsymbol{x} \,\middle|\, \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\rangle && \text{by } \textit{polar identity} \\
&= \|\boldsymbol{x}\|^2 + \left\| \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right)^* \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right] && \text{by property of } \langle \triangle \,|\, \triangledown \rangle && \text{(Definition F.9 page 132)} \\
&= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N} \left\| \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right)^* \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right] && \text{by } \textit{Pythagorean Theorem}
\end{aligned}$$

[20] Bachman et al. (2000) page 103, Pedersen (2000) pages 38–39

$$
\begin{aligned}
&= \|x\|^2 + \sum_{n=1}^{N} \|\langle x \,|\, x_n\rangle\, x_n\|^2 - 2\Re\left(\sum_{n=1}^{N} \langle x \,|\, x_n\rangle^* \,\langle x \,|\, x_n\rangle\right) \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \underbrace{\|x_n\|^2}_{1} - 2\Re\left(\sum_{n=1}^{N} \langle x \,|\, x_n\rangle^* \,\langle x \,|\, x_n\rangle\right) && \text{by property of } \|\cdot\| && \text{(Definition F.5 page 124)} \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \cdot 1 - 2\Re\left(\sum_{n=1}^{N} \langle x \,|\, x_n\rangle^* \,\langle x \,|\, x_n\rangle\right) && \text{by def. of } \textit{orthonormality} && \text{(Definition G.9 page 156)} \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 - 2\Re \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 - 2 \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 && \text{because } |\cdot| \text{ is real} \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2
\end{aligned}
$$

✏

Theorem G.7 (Bessel's inequality). [21] *Let* $\left\{ x_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition F.9 page 132)* $\left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times}),\, \langle \triangle \,|\, \nabla \rangle\right)$ *and with* $\|x\| \triangleq \sqrt{\langle x \,|\, x\rangle}$.

THM
$$\left\{ \begin{array}{c} \{x_n\} \text{ is ORTHONORMAL} \\ \text{(Definition G.9 page 156)} \end{array} \right\} \implies \left\{ \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \;\leq\; \|x\|^2 \quad \forall x \in X \right\}$$

✎PROOF:

$$
\begin{aligned}
0 &\leq \left\| x - \sum_{n=1}^{N} \langle x \,|\, x_n\rangle\, x_n \right\|^2 && \text{by definition of } \|\cdot\| && \text{(Definition F.5 page 124)} \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 && \text{by } \textit{Bessel's Equality} && \text{(Theorem G.6 page 157)}
\end{aligned}
$$

✏

The *Best Approximation Theorem* (next) shows that

- the best sequence for representing a vector is the sequence of projections of the vector onto the sequence of basis functions
- the error of the projection is orthogonal to the projection.

Theorem G.8 (Best Approximation Theorem). [22] *Let* $\left\{ x_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition F.9 page 132)* $\left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times}),\, \langle \triangle \,|\, \nabla \rangle\right)$ *and with* $\|x\| \triangleq \sqrt{\langle x \,|\, x\rangle}$.

[21] Giles (2000) pages 54–55 ⟨3.13 Bessel's inequality⟩, Bollobás (1999) page 147, Aliprantis and Burkinshaw (1998) page 284

[22] Walter and Shen (2001) pages 3–4, Pedersen (2000) page 39, Edwards (1995) pages 94–100, Weyl (1940)

THM $\left\{\begin{array}{c}\{x_n\} \textit{ is}\\ \text{ORTHONORMAL}\\ \textit{(Definition G.9 page 156)}\end{array}\right\} \implies \left\{\begin{array}{lll}1. & \arg\min\limits_{(\alpha_n)_{n=1}^N}\left\|x-\sum_{n=1}^N \alpha_n x_n\right\| = \underbrace{(\langle x\,|\,x_n\rangle)_{n=1}^N}_{\text{best } \alpha_n = \langle x\,|\,x_n\rangle} & \forall x\in X \quad \textit{and}\\ 2. & \underbrace{\left(\sum_{n=1}^N \langle x\,|\,x_n\rangle x_n\right)}_{\text{approximation}} \perp \underbrace{\left(x-\sum_{n=1}^N \langle x\,|\,x_n\rangle x_n\right)}_{\text{approximation error}} & \forall x\in X\end{array}\right\}$

✎PROOF:

1. Proof that $(\langle x\,|\,x_n\rangle)$ is the best sequence:

$$\begin{aligned}
&\left\|x-\sum_{n=1}^N \alpha_n x_n\right\|^2\\
&= \|x\|^2 - 2\Re\left\langle x\,\Big|\,\sum_{n=1}^N \alpha_n x_n\right\rangle + \left\|\sum_{n=1}^N \alpha_n x_n\right\|^2\\
&= \|x\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^* \langle x\,|\,x_n\rangle\right) + \sum_{n=1}^N \|\alpha_n x_n\|^2 && \text{by } \textit{Pythagorean Theorem}\\
&= \|x\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^* \langle x\,|\,x_n\rangle\right) + \sum_{n=1}^N |\alpha_n|^2 + \underbrace{\left[\sum_{n=1}^N |\langle x\,|\,x_n\rangle|^2 - \sum_{n=1}^N |\langle x\,|\,x_n\rangle|^2\right]}_{0}\\
&= \left[\|x\|^2 - \sum_{n=1}^N |\langle x\,|\,x_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle x\,|\,x_n\rangle|^2 - 2\mathbf{R_e}\left[\alpha_n^*\langle x\,|\,x_n\rangle\right] + |\alpha_n|^2\right]\\
&= \left[\|x\|^2 - \sum_{n=1}^N |\langle x\,|\,x_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle x\,|\,x_n\rangle|^2 - \alpha_n^*\langle x\,|\,x_n\rangle - \alpha_n\langle x\,|\,x_n\rangle^* + |\alpha_n|^2\right]\\
&= \left\|x-\sum_{n=1}^N \langle x\,|\,x_n\rangle x_n\right\|^2 + \sum_{n=1}^N |\langle x\,|\,x_n\rangle - \alpha_n|^2 && \text{by } \textit{Bessel's Equality} \quad \text{(Theorem G.6 page 157)}\\
&\geq \left\|x-\sum_{n=1}^N \langle x\,|\,x_n\rangle x_n\right\|^2
\end{aligned}$$

2. Proof that the approximation and approximation error are orthogonal:

$$\begin{aligned}
\left\langle \sum_{n=1}^N \langle x\,|\,x_n\rangle x_n \,\Big|\, x-\sum_{n=1}^N \langle x\,|\,x_n\rangle x_n\right\rangle &= \left\langle \sum_{n=1}^N \langle x\,|\,x_n\rangle x_n \,\Big|\, x\right\rangle - \left\langle \sum_{n=1}^N \langle x\,|\,x_n\rangle x_n \,\Big|\, \sum_{n=1}^N \langle x\,|\,x_n\rangle x_n\right\rangle\\
&= \sum_{n=1}^N \langle x\,|\,x_n\rangle^* \langle x\,|\,x_n\rangle - \sum_{n=1}^N\sum_{m=1}^N \langle x\,|\,x_n\rangle \langle x\,|\,x_m\rangle^* \langle x_n\,|\,x_m\rangle\\
&= \sum_{n=1}^N |\langle x\,|\,x_n\rangle|^2 - \sum_{n=1}^N\sum_{m=1}^N \langle x\,|\,x_n\rangle \langle x\,|\,x_m\rangle^* \bar{\delta}_{nm}\\
&= \sum_{n=1}^N |\langle x\,|\,x_n\rangle|^2 - \sum_{n=1}^N |\langle x\,|\,x_n\rangle|^2\\
&= 0
\end{aligned}$$

G.5 Orthonormal bases in Hilbert spaces

Definition G.10. *Let* $\{x_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in an* INNER PRODUCT SPACE (Definition F.9 page 132) $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

DEF

The set $\{x_n\}$ *is an* ***orthogonal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{x_n\}$ *is* ORTHOGONAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.

The set $\{x_n\}$ *is an* ***orthonormal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{x_n\}$ *is* ORTHONORMAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.

Definition G.11. [23] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a Hilbert space.*

DEF

Suppose there exists a set $\{x_n \in X \mid n\in\mathbb{N}\}$ *such that* $x \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n$.

Then the quantities $\langle x \mid x_n \rangle$ *are called the* ***Fourier coefficients*** *of* x *and the sum* $\sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n$ *is called the* ***Fourier expansion*** *of* x *or the* ***Fourier series*** *for* x.

Definition G.12.

DEF

The ***Kronecker delta function*** $\bar{\delta}_n$ *is defined as*
$$\bar{\delta}_n \triangleq \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \quad \text{and} \quad \forall n\in\mathbb{Z}$$

Lemma G.2 (Perfect reconstruction). *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

LEM

$$\left\{ \begin{array}{ll} (1). & (x_n) \text{ is a BASIS for } \boldsymbol{H} \quad \text{and} \\ (2). & (x_n) \text{ is ORTHONORMAL} \end{array} \right\} \implies \overbrace{x \stackrel{\star}{=} \sum_{n=1}^{\infty} \underbrace{\langle x \mid x_n \rangle}_{\text{Fourier coefficient}} x_n}^{\text{Fourier expansion}} \quad \forall x\in X$$

✎PROOF:

$$\begin{aligned}
\langle x \mid x_n \rangle &= \left\langle \sum_{m\in\mathbb{Z}} \alpha_m x_m \mid x_n \right\rangle && \text{by left hypothesis (1)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \langle x_m \mid x_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition F.9 page 132)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \bar{\delta}_{n-m} && \text{by left hypothesis (2)} && \text{(Definition G.9 page 156)} \\
&= \alpha_n
\end{aligned}$$

Proposition G.2. [24] *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

[23] Fabian et al. (2010) page 27 ⟨Theorem 1.55⟩, Young (2001) page 6, Young (1980) page 6
[24] Han et al. (2007) pages 93–94 ⟨Proposition 3.11⟩

PRP

$$\underbrace{\|\boldsymbol{x}\|^2 \doteq \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2}_{\text{PARSEVAL FRAME}} \iff \underbrace{\boldsymbol{x} \doteq \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \, \boldsymbol{x}_n}_{\text{FOURIER EXPANSION (Definition G.11 page 160)}} \qquad \forall x \in X$$

✎PROOF:

1. Proof that *Parseval frame* $\Longleftarrow$ *Fourier expansion*

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &\triangleq \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \|\cdot\| \\
&= \left\langle \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \, \boldsymbol{x} \,\middle|\, \boldsymbol{x}_n \right\rangle && \text{by right hypothesis} \\
&\doteq \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&\doteq \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle^* && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&\doteq \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by property of } \mathbb{C} \quad \text{(Definition E.7 page 117)}
\end{aligned}$$

2. Proof that *Parseval frame* $\Longrightarrow$ *Fourier expansion*

   (a) Let $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ be the *standard othornormal basis* such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.

   (b) Let $\mathbf{M}$ be an operator in $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{x} \triangleq \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \, \boldsymbol{e}_n$.

   (c) lemma: $\mathbf{M}$ is *isometric*. Proof:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \, \boldsymbol{e}_n \right\|^2 && \text{by definition of } \mathbf{M} \quad \text{(item (2b) page 161)} \\
&= \sum_{n=1}^{\infty} \| \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \, \boldsymbol{e}_n \|^2 && \text{by \textit{Pythagorean Theorem}} \\
&= \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 \, \|\boldsymbol{e}_n\|^2 && \text{by \textit{homogeneous} property of } \|\cdot\| \quad \text{(Definition F.5 page 124)} \\
&= \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by definition of \textit{orthonormal}} \quad \text{(Definition G.9 page 156)} \\
&= \|\boldsymbol{x}\|^2 && \text{by Parseval frame hypothesis} \\
\Longrightarrow \quad & \mathbf{M} \text{ is \textit{isometric}} && \text{by definition of \textit{isometric}} \quad \text{(Definition F.13 page 140)}
\end{aligned}$$

   (d) Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthornormal basis* for $\boldsymbol{H}$.

(e) Proof for *Fourier expansion*:

$$
\begin{aligned}
\boldsymbol{x} &= \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by } \textit{Fourier expansion} \text{ (Proposition G.3 page 164)} \\
&= \sum_{n=1}^{\infty} \langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by (2c) lemma page 161 and Theorem F.23 page 141} \\
&= \sum_{n=1}^{\infty} \left\langle \sum_{m=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_m \rangle \boldsymbol{e}_m \,\middle|\, \sum_{k=1}^{\infty} \langle \boldsymbol{u}_n \,|\, \boldsymbol{x}_k \rangle \boldsymbol{e}_k \right\rangle \boldsymbol{u}_n && \text{by item (2b) page 161} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_m \rangle \sum_{k=1}^{\infty} \langle \boldsymbol{u}_n \,|\, \boldsymbol{x}_k \rangle^* \langle \boldsymbol{e}_m \,|\, \boldsymbol{e}_k \rangle \boldsymbol{u}_n && \text{by Definition F.9 page 132} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_m \rangle \langle \boldsymbol{u}_n \,|\, \boldsymbol{x}_m \rangle^* \boldsymbol{u}_n && \text{by item (2a) page 161 and Definition G.9 page 156} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_m \rangle \langle \boldsymbol{x}_m \,|\, \boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by Definition F.9 page 132} \\
&= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_m \rangle \sum_{n=1}^{\infty} \langle \boldsymbol{x}_m \,|\, \boldsymbol{u}_n \rangle \boldsymbol{u}_n \\
&= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_m \rangle \boldsymbol{x}_m && \text{by item (2d) page 161}
\end{aligned}
$$

✏

When is a set of orthonormal vectors in a Hilbert space $\boldsymbol{H}$ *total*? Theorem G.9 (next) offers some help.

Theorem G.9 (The Fourier Series Theorem). [25] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n\in\mathbb{N} \right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle\right)$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}$.

THM

(A) $\left\{\boldsymbol{x}_n\right\}$ *is* ORTHONORMAL *in* $\boldsymbol{H}$ $\implies$

$$
\left\{
\begin{array}{llrcll}
 & (1). & \left(\operatorname{span}\left\{\boldsymbol{x}_n\right\}\right)^{-} & = & \boldsymbol{H} & & (\{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{H}) \\
\iff & (2). & \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{y} \,|\, \boldsymbol{x}_n \rangle^* & \forall \boldsymbol{x},\boldsymbol{y}\in X & (\text{GENERALIZED PARSEVAL'S IDENTITY}) \\
\iff & (3). & \|\boldsymbol{x}\|^2 & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle\right|^2 & \forall \boldsymbol{x}\in X & (\text{PARSEVAL'S IDENTITY}) \\
\iff & (4). & \boldsymbol{x} & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n & \forall \boldsymbol{x}\in X & (\text{FOURIER SERIES EXPANSION})
\end{array}
\right\}
$$

✎PROOF:

[25] Bachman and Narici (1966) pages 149–155 ⟨Theorem 9.12⟩, Kubrusly (2001) pages 360–363 ⟨Theorem 5.48⟩, Aliprantis and Burkinshaw (1998) pages 298–299 ⟨Theorem 34.2⟩, Christensen (2003) page 57 ⟨Theorem 3.4.2⟩, Berberian (1961) pages 52–53 ⟨Theorem II§8.3⟩, Heil (2011) pages 34–35 ⟨Theorem 1.50⟩, Bracewell (1978) page 112 ⟨Rayleigh's theorem⟩

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle x \,|\, y\rangle &\doteq \left\langle \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle\, x_n \,\middle|\, \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle\, x_m \right\rangle && \text{by (A) and (1)} \\
&\doteq \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \left\langle x_n \,\middle|\, \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle\, x_m \right\rangle && \text{by property of } \langle\cdot\,|\,\cdot\rangle && \text{(Definition F.9 page 132)} \\
&\doteq \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle^* \langle x_n \,|\, x_m\rangle && \text{by property of } \langle\cdot\,|\,\cdot\rangle && \text{(Definition F.9 page 132)} \\
&\doteq \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle^* \bar{\delta}_{mn} && \text{by (A)} \\
&\doteq \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \langle y \,|\, x_n\rangle^* && \text{by definition of } \bar{\delta}_n && \text{(Definition G.12 page 160)}
\end{aligned}$$

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\|x\|^2 &\triangleq \langle x \,|\, x\rangle && \text{by definition of \textit{induced norm}} \\
&= \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \langle x \,|\, x_n\rangle^* && \text{by (2)} \\
&= \sum_{n=1}^{\infty} \left|\langle x \,|\, x_n\rangle\right|^2
\end{aligned}$$

3. Proof that (3) $\iff$ (4) *not* using (A): by Proposition G.2 page 160
4. Proof that (3) $\implies$ (1) (proof by contradiction):
   (a) Suppose $\{x_n\}$ is *not total*.
   (b) Then there must exist a vector y in H such that the set $B \triangleq \{x_n\} \cup y$ is *orthonormal*.
   (c) Then $1 = \|y\|^2 \neq \sum_{n=1}^{\infty} \left|\langle y \,|\, x_n\rangle\right|^2 = 0$.
   (d) But this contradicts (3), and so $\{x_n\}$ must be *total* and (3) $\implies$ (1).
5. Extraneous proof that (3) $\implies$ (4) (this proof is not really necessary here):

$$\begin{aligned}
\left\| x - \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle\, x_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x \,|\, x_n\rangle|^2 && \text{by \textit{Bessel's Equality}} && \text{(Theorem G.6 page 157)} \\
&= 0 && \text{by (3)} \\
\implies \quad x &\doteq \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle\, x_n && \text{by definition of } \doteq
\end{aligned}$$

6. Extraneous proof that (A) $\implies$ (4) (this proof is not really necessary here)
   (a) The sequence $\sum_{n=1}^{N} \left|\langle x \,|\, x_n\rangle\right|^2$ is *monotonically increasing* in n.
   (b) By Bessel's inequality (page 158), the sequence is upper bounded by $\|x\|^2$:

$$\sum_{n=1}^{N} \left|\langle x \,|\, x_n\rangle\right|^2 \leq \|x\|^2$$

(c) Because this sequence is both monotonically increasing and bounded in n, it must equal its bound in the limit as n approaches infinity:

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 = \|\boldsymbol{x}\|^2 \tag{G.1}$$

(d) If we combine this result with *Bessel's Equality* (Theorem G.6 page 157) we have

$$\begin{aligned}
\lim_{N\to\infty}\left\|\boldsymbol{x} - \sum_{n=1}^{N}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_n\right\|^2 &= \|\boldsymbol{x}\|^2 - \lim_{N\to\infty}\sum_{n=1}^{N}|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle|^2 && \text{by Bessel's Equality (Theorem G.6 page 157)}\\
&= \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 && \text{by equation (G.1) page 164}\\
&= 0
\end{aligned}$$

✏

Proposition G.3 (Fourier expansion). *Let* $\left\{\boldsymbol{x}_n \in X \,\middle|\, n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla\rangle)$.

PRP

$$\underbrace{\{\boldsymbol{x}_n\} \text{ is an ORTHONORMAL BASIS for } \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n}_{(1)} \iff \underbrace{\alpha_n = \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle}_{(2)}\right\}$$

✎PROOF:

1. Proof that (1) $\implies$ (2): by Lemma G.2 page 160
2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\left\|\boldsymbol{x} - \sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_{n\in\mathbb{Z}}\right\|^2 &= \left\|\boldsymbol{x} - \sum_{n=1}^{\infty}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_{n\in\mathbb{Z}}\right\|^2 && \text{by right hypothesis}\\
&= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{\infty}|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle|^2 && \text{by Bessel's equality (Theorem G.6 page 157)}\\
&= 0 && \text{by Parseval's Identity (Theorem G.9 page 162)}\\
&\stackrel{\text{def}}{\iff} \boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty}\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_n && \text{by definition of strong convergence}
\end{aligned}$$

✏

Proposition G.4 (Riesz-Fischer Theorem). [26] *Let* $\left\{\boldsymbol{x}_n \in X \,\middle|\, n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla\rangle)$.

PRP

$$\underbrace{\{\boldsymbol{x}_n\} \text{ is an ORTHONORMAL BASIS for } \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{\sum_{n=1}^{\infty}|\alpha_n|^2 < \infty}_{(1)} \iff \underbrace{\exists \boldsymbol{x}\in\boldsymbol{H} \text{ such that } \alpha_n = \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle}_{(2)}\right\}$$

✎PROOF:

[26] Young (2001) page 6

1. Proof that (1) $\Longrightarrow$ (2):

   (a) If (1) is true, then let $\boldsymbol{x} \triangleq \sum_{n\in\mathbb{N}} \alpha_n \boldsymbol{x}_n$.

   (b) Then

$$
\begin{aligned}
\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle &= \left\langle \sum_{m\in\mathbb{N}} \alpha_m \boldsymbol{x}_m \,\middle|\, \boldsymbol{x}_n \right\rangle && \text{by definition of } \boldsymbol{x} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \langle \boldsymbol{x}_m \,|\, \boldsymbol{x}_n \rangle && \text{by \textit{homogeneous} property of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition F.9 page 132)} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \bar{\delta}_{mn} && \text{by (A)} \\
&= \sum_{m\in\mathbb{N}} \alpha_n && \text{by definition of } \bar{\delta} && \text{(Definition G.12 page 160)}
\end{aligned}
$$

2. Proof that (1) $\Longleftarrow$ (2):

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} |\alpha_n|^2 &= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by (2)} \\
&\le \|\boldsymbol{x}\|^2 && \text{by \textit{Bessel's Inequality}} && \text{(Theorem G.7 page 158)} \\
&\le \infty
\end{aligned}
$$

✏

Theorem G.10. [27]

THM *All* SEPARABLE HILBERT SPACES *are* ISOMORPHIC. *That is,*

$$
\left\{ \begin{array}{l} \boldsymbol{X} \text{ is a separable} \\ \text{Hilbert space} \quad \text{and} \\ \boldsymbol{Y} \text{ is a separable} \\ \text{Hilbert space} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lllll} \multicolumn{5}{l}{\text{there is a BIJECTIVE operator } \mathbf{M} \in \boldsymbol{Y}^{\boldsymbol{X}} \text{ such that}} \\ (1). & \boldsymbol{y} & = & \mathbf{M}\boldsymbol{x} & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} \quad \text{and} \\ (2). & \|\mathbf{M}\boldsymbol{x}\| & = & \|\boldsymbol{x}\| & \forall x\in\boldsymbol{X} \quad \text{and} \\ (3). & \langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y} \rangle & = & \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} \end{array} \right\}
$$

✎PROOF:

1. Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{ \boldsymbol{x}_n | n\in\mathbb{N} \}$.
   Let $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{ \boldsymbol{y}_n | n\in\mathbb{N} \}$.

2. Proof that there exists *bijective* operator $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$:

   (a) Let $\mathbf{M}$ be defined such that $\boldsymbol{y}_n \triangleq \mathbf{M}\boldsymbol{x}_n$.

   (b) Thus $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$.

   (c) Because $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$, $\mathbf{M}$ has an inverse operator $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$ such that $\boldsymbol{x}_n = \mathbf{M}^{-1}\boldsymbol{y}_n$.

3. Proof that $\mathbf{M}$ and $\mathbf{M}^{-1}$ are *bijective* operators between $\boldsymbol{X}$ and $\boldsymbol{Y}$:

[27] Young (2001) page 6

(a) Proof that $\mathbf{M}$ maps $\boldsymbol{X}$ *into* $\boldsymbol{Y}$:

$$\begin{aligned}
\boldsymbol{x} \in \boldsymbol{X} \iff \boldsymbol{x} &\stackrel{\star}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 162)}\\
&\implies \exists \boldsymbol{y} \in \boldsymbol{Y} \quad \text{such that} \quad \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle = \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition G.4 page 164)}\\
&\implies \\
\boldsymbol{y} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 162)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{y}_n && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition G.4 page 164)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n && \text{by definition of } \mathbf{M} && \text{(item (2a) page 165)}\\
&= \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by prop. of linear ops.} && \text{(Theorem F.1 page 121)}\\
&= \mathbf{M}\boldsymbol{x} && \text{by definition of } \boldsymbol{x}
\end{aligned}$$

(b) Proof that $\mathbf{M}^{-1}$ maps $\boldsymbol{Y}$ *into* $\boldsymbol{X}$:

$$\begin{aligned}
\boldsymbol{y} \in \boldsymbol{Y} \iff \boldsymbol{y} &\stackrel{\star}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 162)}\\
&\implies \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle = \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition G.4 page 164)}\\
&\implies \\
\boldsymbol{x} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 162)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{x}_n && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition G.4 page 164)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \mathbf{M}^{-1}\boldsymbol{y}_n && \text{by definition of } \mathbf{M}^{-1} && \text{(item (2c) page 165)}\\
&= \mathbf{M}^{-1} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by prop. of \textit{linear ops.}} && \text{(Theorem F.1 page 121)}\\
&= \mathbf{M}^{-1}\boldsymbol{y} && \text{by definition of } \boldsymbol{y}
\end{aligned}$$

4. Proof for (2):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 162)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n \right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem F.1 page 121)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{y}_n \right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2a) page 165)}\\
&= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition G.4 page 164)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition G.4 page 164)}\\
&= \|\boldsymbol{x}\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 162)}
\end{aligned}$$

5. Proof for (3): by (2) and Theorem F.23 page 141

⇨

Theorem G.11. [28] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has a* SCHAUDER BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

Theorem G.12. [29] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has an* ORTHONORMAL BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

G.6 Riesz bases in Hilbert spaces

Definition G.13. [30] *Let $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+},$

DEF: $\{\boldsymbol{x}_n\}$ *is a* ***Riesz basis*** *for $\boldsymbol{H}$ if $\{\boldsymbol{x}_n\}$ is* EQUIVALENT *(Definition G.8 page 155) to some* ORTHONORMAL BASIS *(Definition G.10 page 160) in $\boldsymbol{H}$.*

Definition G.14. [31] *Let $(\boldsymbol{x}_n \in X)_{n\in\mathbb{N}}$ be a sequence of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

DEF: *The sequence $(\boldsymbol{x}_n)$ is a* ***Riesz sequence*** *for $\boldsymbol{H}$ if*

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 \qquad \forall(\alpha_n)\in\ell^2_{\mathbb{F}}.$$

Definition G.15. *Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ be an* INNER PRODUCT SPACE *(Definition F.9 page 132).*

DEF: *The sequences $(\boldsymbol{x}_n \in X)_{n\in\mathbb{Z}}$ and $(\boldsymbol{y}_n \in X)_{n\in\mathbb{Z}}$ are* ***biorthogonal*** *with respect to each other in $\boldsymbol{X}$ if* $\langle \boldsymbol{x}_n \mid \boldsymbol{y}_m \rangle = \bar{\delta}_{nm}$

Lemma G.3. [32] *Let $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ be a sequence in a* HILBERT SPACE $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$. *Let $\{\boldsymbol{y}_n \mid n\in\mathbb{N}\}$ be a sequence in a* HILBERT SPACE $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$. *Let*

LEM:

$$\left\{\begin{array}{lll}
(i). & \{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{X} & \text{and}\\
(ii). & \text{There exists } A>0 \text{ such that } A\sum_{n\in C}|a_n|^2 \le \left\|\sum_{n\in C}a_n\boldsymbol{x}_n\right\|^2 \quad \text{for finite } C & \text{and}\\
(iii). & \text{There exists } B>0 \text{ such that } \left\|\sum_{n=1}^{\infty}b_n\boldsymbol{y}_n\right\|^2 \le B\sum_{n=1}^{\infty}|b_n|^2 \quad \forall(b_n)_{n\in\mathbb{N}}\in\ell^2_{\mathbb{F}} &
\end{array}\right\} \implies$$

$$\left\{\begin{array}{lll}
(1). & \mathbf{R}^\circ \text{ is a linear bounded operator that maps from } \operatorname{span}\{\boldsymbol{x}_n\} \text{ to } \operatorname{span}\{\boldsymbol{y}_n\} & \\
 & \text{where } \mathbf{R}^\circ \sum_{n\in C}c_n\boldsymbol{x}_n \triangleq \sum_{n\in C}c_n\boldsymbol{y}_n, \text{ for some sequence } (c_n) \text{ and finite set } C & \text{and}\\
(2). & \mathbf{R} \text{ has a unique extension to a bounded operator } \mathbf{R} \text{ that maps from } \boldsymbol{X} \text{ to } \boldsymbol{Y} & \text{and}\\
(3). & |||\mathbf{R}^\circ||| \le \frac{B}{A} & \text{and}\\
(4). & |||\mathbf{R}||| \le \frac{B}{A} &
\end{array}\right\}$$

[28] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Berberian (1961) page 53 ⟨Theorem II§8.3⟩
[29] Kubrusly (2001) page 357 ⟨Proposition 5.43⟩
[30] Young (2001) page 27 ⟨Definition 1.8.2⟩, Christensen (2003) page 63 ⟨Definition 3.6.1⟩, Heil (2011) page 196 ⟨Definition 7.9⟩
[31] Christensen (2003) pages 66–68 ⟨page 68 and (3.24) on page 66⟩, Wojtaszczyk (1997) page 20 ⟨Definition 2.6⟩
[32] Christensen (2003) pages 65–66 ⟨Lemma 3.6.5⟩

Theorem G.13. [33] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n\in\mathbb{N} \right\}$ *be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$.

THM

$$\left\{ \begin{array}{l} \{\boldsymbol{x}_n\} \text{ is a RIESZ BASIS} \\ \text{for } \boldsymbol{H} \end{array} \right\} \iff \left\{ \begin{array}{lll} (1). & \{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{H} & \text{and} \\ (2). & \exists A,B\in\mathbb{R}^+ \text{ such that } & \forall (\alpha_n)\in\ell^2_{\mathbb{F}}, \\ & A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 & \end{array} \right\}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:
   (a) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (1): all bases for $\boldsymbol{H}$ are *total* in $\boldsymbol{H}$.
   (b) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (2):
      i. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthonormal basis* for $\boldsymbol{H}$.
      ii. Let $\mathbf{R}$ be a *bounded bijective* operator such that $\boldsymbol{x}_n = \mathbf{R}\boldsymbol{u}_n$.
      iii. Proof for upper bound B:

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \left\|\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii))} \\
&= \left\|\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem F.1 page 121} \\
&\le |||\mathbf{R}|||^2 \left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem F.6 page 127} \\
&= |||\mathbf{R}|||^2 \sum_{n=1}^{\infty}\left\|\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by \textit{Pythagorean Theorem}} \\
&= |||\mathbf{R}|||^2 \sum_{n=1}^{\infty}|\alpha|^2\left\|\boldsymbol{u}_n\right\|^2 && \text{by \textit{homogeneous} property of norms} && \text{(Definition F.5 page 124)} \\
&= \underbrace{|||\mathbf{R}|||^2}_{B} \sum_{n=1}^{\infty}|\alpha|^2 && \text{by definition of \textit{orthonormality}} && \text{(Definition G.9 page 156)}
\end{aligned}$$

      iv. Proof for lower bound A:

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \frac{|||\mathbf{R}^{-1}|||^2}{|||\mathbf{R}^{-1}|||^2}\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{because } |||\mathbf{R}^{-1}||| > 0 && \text{(Proposition F.1 page 125)} \\
&\ge \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{by Theorem F.6 page 127} \\
&= \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii) page 168)} \\
&= \frac{1}{|||\mathbf{R}^{-1}|||^2}\left\|\mathbf{R}^{-1}\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem F.1 page 121)}
\end{aligned}$$

[33] Young (2001) page 27 ⟨Theorem 1.8.9⟩, Christensen (2003) page 66 ⟨Theorem 3.6.6⟩, Heil (2011) pages 197–198 ⟨Theorem 7.13⟩, Christensen (2008) pages 61–62 ⟨Theorem 3.3.7⟩

$$= \frac{1}{\|\mathbf{R}^{-1}\|^2}\left\|\sum_{n=1}^{\infty}\alpha_n \boldsymbol{u}_n\right\|^2 \qquad \text{by definition of inverse op.} \qquad \text{(Definition F.3 page 120)}$$

$$= \frac{1}{\|\mathbf{R}^{-1}\|^2}\sum_{n=1}^{\infty}\|\alpha_n \boldsymbol{u}_n\|^2 \qquad \text{by } \textit{Pythagorean Theorem}$$

$$= \frac{1}{\|\mathbf{R}^{-1}\|^2}\sum_{n=1}^{\infty}|\alpha_n|^2\,\|\boldsymbol{u}_n\|^2 \qquad \text{by } \|\cdot\| \textit{ homogeneous} \text{ prop.} \qquad \text{(Definition F.5 page 124)}$$

$$= \underbrace{\frac{1}{\|\mathbf{R}^{-1}\|^2}}_{A}\sum_{n=1}^{\infty}|\alpha_n|^2 \qquad \text{by def. of } \textit{orthonormality} \qquad \text{(Definition G.9 page 156)}$$

2. Proof for ($\Longrightarrow$) case:
   (a) Let $\{\boldsymbol{u}_n | n\in\mathbb{N}\}$ be an *orthonormal basis* for $\boldsymbol{H}$.
   (b) Using (2) and Lemma G.3 (page 167), construct an bounded extension operator $\mathbf{R}$ such that $\mathbf{R}\boldsymbol{u}_n = \boldsymbol{x}_n$ for all $n \in \mathbb{N}$.
   (c) Using (2) and Lemma G.3 (page 167), construct an bounded extension operator $\mathbf{S}$ such that $\mathbf{S}\boldsymbol{x}_n = \boldsymbol{u}_n$ for all $n \in \mathbb{N}$.
   (d) Then, $\mathbf{R}\mathbf{V}\boldsymbol{x} = \mathbf{V}\mathbf{R}\boldsymbol{x} \implies \mathbf{V} = \mathbf{R}^{-1}$, and so $\mathbf{R}$ is a bounded invertible operator
   (e) and $\{\boldsymbol{x}_n\}$ is a *Riesz sequence*.

✏

Theorem G.14. [34] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a* SEPARABLE HILBERT SPACE.

T H M

$$\left\{\begin{array}{l}(\boldsymbol{x}_n \in \boldsymbol{H})_{n\in\mathbb{Z}} \text{ is a} \\ \text{RIESZ BASIS for } \boldsymbol{H}\end{array}\right\} \implies \left\{\begin{array}{l}\text{There exists } (\boldsymbol{y}_n \in \boldsymbol{H})_{n\in\mathbb{Z}} \text{ such that} \\ (1).\quad (\boldsymbol{x}_n) \text{ and } (\boldsymbol{y}_n) \text{ are BIORTHOGONAL} \quad \text{and} \\ (2).\quad (\boldsymbol{y}_n) \text{ is also a RIESZ BASIS for } \boldsymbol{H} \quad \text{and} \\ (3).\quad \exists B>A>0 \quad \text{such that} \\ \quad A\sum_{n=1}^{\infty}|a_n|^2 \le \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{x}_n\right\|^2 = \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{y}_n\right\|^2 \le B\sum_{n=1}^{\infty}|a_n|^2 \\ \quad \forall (a_n)_{n\in\mathbb{N}} \in \boldsymbol{\ell}^2_{\mathbb{F}}\end{array}\right\}$$

✎PROOF:

1. Proof for (1):
   (a) Let $\boldsymbol{e}_n$ be the *unit vector* in $\boldsymbol{H}$ such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.
   (b) Let $\mathbf{M}$ be an operator on $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{e}_n = \boldsymbol{x}_n$.
   (c) Note that $\mathbf{M}$ is *isometric*, and as such $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall \boldsymbol{x}\in\boldsymbol{H}$.
   (d) Let $\boldsymbol{y}_n \triangleq (\mathbf{M}^{-1})^*$.
   (e) Then,

$$\begin{aligned}\langle \boldsymbol{y}_n \mid \boldsymbol{x}_m\rangle &= \left\langle (\mathbf{M}^{-1})^*\boldsymbol{e}_n \mid \mathbf{M}\boldsymbol{e}_m\right\rangle \\ &= \left\langle \boldsymbol{e}_n \mid \mathbf{M}^{-1}\mathbf{M}\boldsymbol{e}_m\right\rangle \\ &= \langle \boldsymbol{e}_n \mid \boldsymbol{e}_m\rangle \\ &= \bar{\delta}_{nm} \\ &\implies \{\boldsymbol{x}_n\} \text{ and } \{\boldsymbol{y}_n\} \text{ are } \textit{biorthogonal} \qquad \text{by Definition G.9 page 156}\end{aligned}$$

[34] Wojtaszczyk (1997) page 20 ⟨Lemma 2.7(a)⟩

2. Proof for (3):

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{y}_n\right\| &= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \left(\mathbf{M}^{-1}\right)^* \boldsymbol{e}_n\right\| && \text{by definition of } \boldsymbol{y}_n && \text{(Proposition 1d page 169)}\\
&= \left\|\left(\mathbf{M}^{-1}\right)^* \sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{by property of } \textit{linear ops.}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \left(\mathbf{M}^{-1}\right)^* \text{ is } \textit{isometric} && \text{(Definition F.13 page 140)}\\
&= \left\|\mathbf{M}\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \mathbf{M} \text{ is } \textit{isometric} && \text{(Definition F.13 page 140)}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \mathbf{M}\boldsymbol{e}_n\right\| && \text{by property of } \textit{linear operators}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{x}_n\right\| && \text{by definition of } \mathbf{M}\\
&\implies \left\{\boldsymbol{y}_n\right\} \text{ is a } \textit{Riesz basis} && \text{by left hypothesis}
\end{aligned}$$

3. Proof for (2): by (3) and definition of *Riesz basis* (Definition G.13 page 167)

Proposition G.5. [35] *Let* $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

P R P

$$\left\{\begin{array}{l} \left\{\boldsymbol{x}_n\right\} \textit{ is a } \text{RIESZ BASIS } \textit{for } \boldsymbol{H} \textit{ with}\\ A\displaystyle\sum_{n=1}^{\infty}|a_n|^2 \le \left\|\sum_{n=1}^{\infty} a_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}|a_n|^2\\ \forall\left\{a_n\right\}\in\ell^2_{\mathbb{F}}\end{array}\right\} \implies \left\{\begin{array}{l} \left\{\boldsymbol{x}_n\right\} \textit{ is a } \text{FRAME } \textit{for } \boldsymbol{H} \textit{ with}\\ \underbrace{\dfrac{1}{B}\|\boldsymbol{x}\|^2 \le \displaystyle\sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \le \frac{1}{A}\|\boldsymbol{x}\|^2}_{\text{STABILITY CONDITION}}\\ \forall \boldsymbol{x}\in\boldsymbol{H}\end{array}\right\}$$

✎PROOF:

1. Let $\left\{\boldsymbol{y}_n \middle| n\in\mathbb{N}\right\}$ be a *Riesz basis* that is *biorthogonal* to $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ (Theorem G.14 page 169).

2. Let $\boldsymbol{x} \triangleq \sum_{n=1}^{\infty} a_n \boldsymbol{y}_n$.

3. lemma:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 &= \sum_{n=1}^{\infty}\left|\left\langle \sum_{m=1}^{\infty} a_n \boldsymbol{y}_m \middle| \boldsymbol{x}_n\right\rangle\right|^2 && \text{by definition of } \boldsymbol{x} && \text{(item (2) page 170)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \langle \boldsymbol{y}_m \mid \boldsymbol{x}_n\rangle\right|^2 && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition F.9 page 132)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \bar{\delta}_{mn}\right|^2 && \text{by definition of } \textit{biorthogonal} && \text{(Definition G.15 page 167)}\\
&= \sum_{n=1}^{\infty}|a_n|^2 && \text{by definition of } \bar{\delta} && \text{(Definition G.12 page 160)}
\end{aligned}$$

[35] Igari (1996) page 220 ⟨Lemma 9.8⟩, Wojtaszczyk (1997) pages 20–21 ⟨Lemma 2.7(a)⟩

4. Then

$$
\begin{array}{rlllll}
& A\sum_{n=1}^{\infty}|a_n|^2 & \le & \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{x}_n\right\|^2 & \le B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 170)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le & \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{y}_n\right\|^2 & \le B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 170)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le & \|\boldsymbol{x}\|^2 & \le B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \boldsymbol{x} \text{ (item (2) page 170)}\\
\implies & A\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \le & \|\boldsymbol{x}\|^2 & \le B\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \text{by (3) lemma}\\
\implies & \frac{1}{B}\|\boldsymbol{x}\|^2 & \le & \sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \le \frac{1}{A}\|\boldsymbol{x}\|^2 &
\end{array}
$$

✏

Theorem G.15 (Battle-Lemarié orthogonalization). [36] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition 4.2 page 16) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM

$$
\left\{
\begin{array}{ll}
1. & \{\mathbf{T}^n\mathsf{g}\,|\,n\in\mathbb{Z}\} \text{ is a RIESZ BASIS for } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{and}\\
2. & \tilde{\mathsf{f}}(\omega) \triangleq \dfrac{\tilde{\mathsf{g}}(\omega)}{\sqrt{2\pi\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2}}
\end{array}
\right\}
\implies
\left\{
\begin{array}{l}
\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}\\
\text{is an ORTHONORMAL BASIS for } \boldsymbol{L}^2_{\mathbb{R}}
\end{array}
\right\}
$$

✎PROOF:

1. Proof that $\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}$ is orthonormal:

$$
\begin{aligned}
\tilde{S}_{\phi\phi}(\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem H.1 page 179}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{2\pi\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi(n-m))|^2}}\right|^2 && \text{by left hypothesis}\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{1}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2|\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= \frac{1}{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= 1\\
\implies \{\mathsf{f}_n\,|\,n\in\mathbb{Z}\} &\text{ is orthonormal} && \text{by Theorem H.3 page 185}
\end{aligned}
$$

[36] Wojtaszczyk (1997) page 25 ⟨Remark 2.4⟩, Vidakovic (1999) page 71, Mallat (1989) page 72, Mallat (1999) page 225, Daubechies (1992) page 140 ⟨(5.3.3)⟩

2. Proof that $\{\mathbf{T}^n \mathsf{f} \mid n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_0$: by Lemma G.1 page 155.

⇨

G.7 Frames in Hilbert spaces

Definition G.16. [37] *Let* $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

DEF

The set $\{\boldsymbol{x}_n\}$ *is a* ***frame*** *for* $\boldsymbol{H}$ *if* (STABILITY CONDITION)

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \qquad A\,\|\boldsymbol{x}\|^2 \le \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle\right|^2 \le B\,\|\boldsymbol{x}\|^2 \qquad \forall x\in X.$$

The quantities A and B are ***frame bounds****.*
The quantity A' is the ***optimal lower frame bound*** *if*
$A' = \sup\{A \in \mathbb{R}^+ \mid A \text{ is a lower frame bound}\}$.
The quantity B' is the ***optimal upper frame bound*** *if*
$B' = \inf\{B \in \mathbb{R}^+ \mid B \text{ is an upper frame bound}\}$.
A frame is a ***tight frame*** *if $A = B$.*
A frame is a ***normalized tight frame*** *(or a* ***Parseval frame****) if $A = B = 1$.*
A frame $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ *is an* ***exact frame*** *if for some $m \in \mathbb{Z}$,* $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\} \setminus \{\boldsymbol{x}_m\}$ *is* NOT *a frame.*

A frame is a *Parseval frame* (Definition G.16) if it satisfies *Parseval's Identity* (Theorem G.9 page 162). All orthonormal bases are Parseval frames (Theorem G.9 page 162); but not all Parseval frames are orthonormal bases.

Definition G.17. *Let* $\{\boldsymbol{x}_n\}$ *be a* ***frame*** *(Definition G.16 page 172) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$. *Let* $\mathbf{S}$ *be an* OPERATOR *on* $\boldsymbol{H}$.

DEF

$\mathbf{S}$ *is a* ***frame operator*** *for* $\{\boldsymbol{x}_n\}$ *if* $\qquad \mathbf{S}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \qquad \forall \mathsf{f}\in\boldsymbol{H}.$

Theorem G.16. [38] *Let* $\mathbf{S}$ *be a* FRAME OPERATOR *(Definition G.17 page 172) of a* FRAME $\{\boldsymbol{x}_n\}$ *(Definition G.16 page 172) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

THM

(1). $\mathbf{S}$ *is* INVERTIBLE. *and*

(2). $\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \rangle \boldsymbol{x}_n = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n \rangle \mathbf{S}^{-1}\boldsymbol{x}_n \quad \forall \mathsf{f} \in \boldsymbol{H}$

Theorem G.17. [39] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

THM

$\{\boldsymbol{x}_n\}$ *is a* FRAME *for* $\operatorname{span}\{\boldsymbol{x}_n\}$.

✎PROOF:

---

[37] Young (2001) pages 154–155, Christensen (2003) page 88 ⟨Definitions 5.1.1, 5.1.2⟩, Heil (2011) pages 204–205 ⟨Definition 8.2⟩, Jørgensen et al. (2008) page 267 ⟨Definition 12.22⟩, Duffin and Schaeffer (1952) page 343, Daubechies et al. (1986) page 1272

[38] Christensen (2008) pages 100–102 ⟨Theorem 5.1.7⟩

[39] Christensen (2003) page 3

1. Upper bound: Proof that there exists B such that $\sum_{n=1}^{N} \left|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle\right|^2 \le B \left\|\boldsymbol{x}\right\|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

$$
\begin{aligned}
\sum_{n=1}^{N} \left|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle\right|^2 &\le \sum_{n=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by \textit{Cauchy-Schwarz inequality}} \\
&= \underbrace{\left\{ \sum_{n=1}^{N} \left\|\boldsymbol{x}_n\right\|^2 \right\}}_{B} \left\|\boldsymbol{x}\right\|^2
\end{aligned}
$$

2. Lower bound: Proof that there exists A such that $A \left\|\boldsymbol{x}\right\|^2 \le \sum_{n=1}^{N} \left|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle\right|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

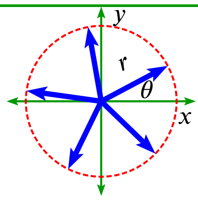
$$
\begin{aligned}
\sum_{n=1}^{N} \left|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle\right|^2 &= \sum_{n=1}^{N} \left|\left\langle \boldsymbol{x}_n \,\middle|\, \frac{\boldsymbol{x}}{\left\|\boldsymbol{x}\right\|} \right\rangle\right|^2 \left\|\boldsymbol{x}\right\|^2 \\
&\ge \underbrace{\left(\inf_{\boldsymbol{y}} \left\{ \sum_{n=1}^{N} \left|\langle \boldsymbol{x}_n \,|\, \boldsymbol{y} \rangle\right|^2 \,\middle|\, \left\|\boldsymbol{y}\right\| = 1 \right\} \right)}_{A} \left\|\boldsymbol{x}\right\|^2
\end{aligned}
$$

✏

Example G.1. Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \triangledown \rangle\right)$ be an inner product space with $\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \,\middle|\, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1 x_2 + y_1 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition G.17 page 172) with *inverse* $\mathbf{S}^{-1}$.

EX

Let $N \in \{3, 4, 5, \ldots\}$, $\theta \in \mathbb{R}$, and $r \in \mathbb{R}^+$ ($r > 0$).
Let $\boldsymbol{x}_n \triangleq r \begin{bmatrix} \cos(\theta + 2n\pi/N) \\ \sin(\theta + 2n\pi/N) \end{bmatrix} \quad \forall n \in \{0, 1, \ldots, N-1\}$.
Then, $(\!(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})\!)$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{Nr^2}{2}$.



Moreover, $\mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{S}^{-1} = \frac{2}{Nr^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\boldsymbol{v} = \frac{2}{Nr^2} \sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \quad \forall \boldsymbol{v} \in \mathbb{R}^2$.

✎PROOF:

1. Proof that $(\!(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})\!)$ is a *tight frame* with *frame bound* $A = \frac{Nr^2}{2}$: Let $\boldsymbol{v} \triangleq (x, y) \in \mathbb{R}^2$.

$$
\begin{aligned}
\sum_{n=0}^{N-1} \left|\langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle\right|^2 &\triangleq \sum_{n=0}^{N-1} \left| \boldsymbol{v}^{\mathbf{H}} r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \right|^2 && \text{by definitions of } \boldsymbol{v} \text{ of } \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle \\
&\triangleq \sum_{n=0}^{N-1} r^2 \left| x\cos\left(\theta + \frac{2n\pi}{N}\right) + y\sin\left(\theta + \frac{2n\pi}{N}\right) \right|^2 && \text{by definition of } \boldsymbol{y}^{\mathbf{H}}\boldsymbol{x} \text{ operation} \\
&= r^2 x^2 \sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 y^2 \sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 xy \sum_{n=0}^{N-1} \cos\left(\theta + \frac{2n\pi}{N}\right) \sin\left(\theta + \frac{2n\pi}{N}\right) \\
&= r^2 x^2 \frac{N}{2} + r^2 y^2 \frac{N}{2} + r^2 xy 0 && \text{by Corollary D.1 page 107} \\
&= (x^2 + y^2) \frac{Nr^2}{2} = \left(\frac{Nr^2}{2}\right) \boldsymbol{v}^{\mathbf{H}} \boldsymbol{v} \triangleq \underbrace{\left(\frac{Nr^2}{2}\right)}_{A} \left\|\boldsymbol{v}\right\|^2 && \text{by definition of } \left\|\boldsymbol{v}\right\|
\end{aligned}
$$

2. Proof that $\mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

(a) Let $\boldsymbol{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) lemma: $\mathbf{S}\boldsymbol{e}_1 = \frac{Nr^2}{2}\boldsymbol{e}_1$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_1 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\cos\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \cos^2\left(\theta + \frac{2n\pi}{N}\right) \\ \cos\left(\theta + \frac{2n\pi}{N}\right)\sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} N/2 \\ 0 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_1 && \text{by \textit{Summation around unit circle} (Corollary D.1 page 107)}
\end{aligned}$$

(c) lemma: $\mathbf{S}\boldsymbol{e}_2 = \frac{Nr^2}{2}\boldsymbol{e}_2$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_2 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_2 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\sin\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \sin\left(\theta + \frac{2n\pi}{N}\right)\cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin^2\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} 0 \\ N/2 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_2 && \text{by \textit{Summation around unit circle} (Corollary D.1 page 107)}
\end{aligned}$$

(d) Complete the proof of item (2) using *Eigendecomposition* $\mathbf{S} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \frac{Nr^2}{2}\boldsymbol{e}_1 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_1 \text{ is an \textit{eigenvector} of } \mathbf{S} \text{ with \textit{eigenvalue} } \frac{Nr^2}{2} \\
\mathbf{S}\boldsymbol{e}_2 &= \frac{Nr^2}{2}\boldsymbol{e}_2 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_2 \text{ is an \textit{eigenvector} of } \mathbf{S} \text{ with \textit{eigenvalue} } \frac{Nr^2}{2}
\end{aligned}$$

$$\mathbf{S} = \overbrace{\underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}^{-1}}_{\mathbf{Q}^{-1}}}^{\textit{Eigendecomposition} \text{ of } \mathbf{S}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Proof that $\mathbf{S}^{-1} = \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$\begin{aligned}
\mathbf{S}\mathbf{S}^{-1} &= \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)} \\
\mathbf{S}^{-1}\mathbf{S} &= \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)}
\end{aligned}$$

4. Proof that $\boldsymbol{v} = \frac{2}{Nr^2}\sum_{n=0}^{N-1} (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n = \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,\middle|\, \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n && \text{by item (3)} \\
&= \frac{2}{Nr^2}\sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \frac{2}{Nr^2}\sum_{n=0}^{N-1} (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n && \text{by definition of } \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle
\end{aligned}$$

✏

Example G.2 (Peace Frame/Mercedes Frame). [40] Let $(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be an inner product space with $\langle (x_1, y_1) \mid (x_2, y_2) \rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition G.17 page 172) with *inverse* $\mathbf{S}^{-1}$.

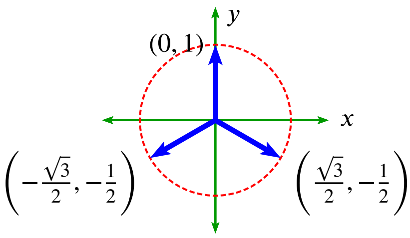
EX

Let $\boldsymbol{x}_1 \triangleq \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\boldsymbol{x}_2 \triangleq \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}$, and $\boldsymbol{x}_3 \triangleq \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$.

Then, $(\!(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\!)$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{3}{2}$.

Moreover, $\mathbf{S} = \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \mathbf{S}^{-1} = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

and $\quad \boldsymbol{v} = \frac{2}{3}\sum_{n=1}^{3} \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \triangleq \frac{2}{3}\sum_{n=1}^{3} (\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_n) \boldsymbol{x}_n \qquad \forall \boldsymbol{v} \in \mathbb{R}^2.$



✎PROOF:

1. This frame is simply a special case of the frame presented in Example G.1 (page 173) with $r = 1$, $N = 3$, and $\theta = \pi/2$.

2. Let's give it a try! Let $\boldsymbol{v} \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned}
\sum_{n=1}^{3} \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1} \boldsymbol{x}_n \right\rangle \boldsymbol{x}_n &= \frac{2}{3} \sum_{n=1}^{3} (\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_n) \boldsymbol{x}_n && \text{by Example G.1 page 173} \\
&= (\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_1)\boldsymbol{x}_1 + (\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_2)\boldsymbol{x}_2 + (\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_3)\boldsymbol{x}_3 \\
&= \frac{2}{3}\left(\left(\boldsymbol{v}^{\mathbf{H}} \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}} \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix} \right) \boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}} \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \right) \boldsymbol{x}_3 \right) \\
&= \frac{2}{3} \cdot \frac{1}{2}\left(\left(\boldsymbol{v}^{\mathbf{H}} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}} \begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix} \right) \boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}} \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \right) \boldsymbol{x}_3 \right) \\
&= \frac{1}{3}\left((2)\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \left(-\sqrt{3} - 1\right)\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix} + \left(\sqrt{3} - 1\right)\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \right) \\
&= \frac{1}{6}\begin{bmatrix} 2(0) & + & (-\sqrt{3} - 1)(-\sqrt{3}) & + & (\sqrt{3} - 1)(\sqrt{3}) \\ 2(2) & + & (-\sqrt{3} - 1)(-1) & + & (\sqrt{3} - 1)(-1) \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 0 & + & (3 + \sqrt{3}) & + & (3 - \sqrt{3}) \\ 4 & + & (1 + \sqrt{3}) & + & (1 - \sqrt{3}) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \triangleq \boldsymbol{v}
\end{aligned}$$

✏

In Example G.1 (page 173) and Example G.2 (page 175), the frame operator $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ were computed. In general however, it is not always necessary or even possible to compute these, as illustrated in Example G.3 (next).

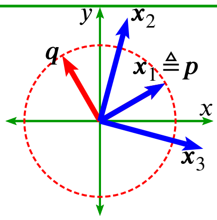
Example G.3. [41] Let $(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be an inner product space with $\langle (x_1, y_1) \mid (x_2, y_2) \rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition G.17 page 172) with *inverse* $\mathbf{S}^{-1}$.

[40] Heil (2011) pages 204–205 ⟨$r = 1$ case⟩, Byrne (2005) page 80 ⟨$r = 1$ case⟩, Han et al. (2007) page 91 ⟨Example 3.9, $r = \sqrt{2/3}$ case⟩

[41] Christensen (2003) pages 7–8 ⟨?⟩

EX

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be *orthonormal* vectors in $\boldsymbol{X} \triangleq \operatorname{span}\{\boldsymbol{p}, \boldsymbol{q}\}$.
Let $\boldsymbol{x}_1 \triangleq \boldsymbol{p}$, $\boldsymbol{x}_2 \triangleq \boldsymbol{p}+\boldsymbol{q}$, and $\boldsymbol{x}_3 \triangleq \boldsymbol{p}-\boldsymbol{q}$.
Then, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, , \boldsymbol{x}_3\}$ is a **frame** for $\boldsymbol{X}$ with *frame bounds* $A=0$ and $B=5$.



Moreover,
$\mathbf{S}^{-1}\boldsymbol{x}_1 = \tfrac{1}{3}\boldsymbol{p}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_2 = \tfrac{1}{3}\boldsymbol{p} + \tfrac{1}{2}\boldsymbol{q}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_3 = \tfrac{1}{3}\boldsymbol{p} - \tfrac{1}{2}\boldsymbol{q}$.

✎PROOF:

1. Proof that $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is a *frame* with *frame bounds* $A=0$ and $B=5$:

$$\begin{aligned}
\sum_{n=1}^{3} \left|\langle \boldsymbol{v} \,|\, \boldsymbol{x}_n\rangle\right|^2 &\triangleq |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle|^2 && \text{by definitions of } \boldsymbol{x}_1, \boldsymbol{x}_2, \text{ and } \boldsymbol{x}_3\\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 && \text{by } \textit{additivity} \text{ of } \langle \triangle \,|\, \nabla\rangle \text{ (Definition F.9 page 132)}\\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right)\\
&\quad + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 - \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right)\\
&= 3|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + 2|\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2\\
&\boxed{\leq} 3\,\|\boldsymbol{v}\|\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{v}\|\,\|\boldsymbol{q}\| && \text{by } \textit{CS Inequality}\\
&= \|\boldsymbol{v}\|\,(3\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{q}\|)\\
&= \boxed{5}\,\|\boldsymbol{v}\| && \text{by } \textit{orthonormality} \text{ of } \boldsymbol{p} \text{ and } \boldsymbol{q}
\end{aligned}$$

2. lemma: $\mathbf{S}\boldsymbol{p} = 3\boldsymbol{p}$, $\mathbf{S}\boldsymbol{q} = 2\boldsymbol{q}$, $\mathbf{S}^{-1}\boldsymbol{p} = \frac{1}{3}\boldsymbol{p}$, and $\mathbf{S}^{-1}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{p} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{p} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n\\
&= \langle \boldsymbol{p} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{p} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\,(\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{p} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\,(\boldsymbol{p}-\boldsymbol{q})\\
&= (1)\boldsymbol{p} + (1+0)(\boldsymbol{p}+\boldsymbol{q}) + (1-0)(\boldsymbol{p}-\boldsymbol{q})\\
&= 3\boldsymbol{p}\\
\implies \mathbf{S}^{-1}\boldsymbol{p} &= \frac{1}{3}\boldsymbol{p}\\
\mathbf{S}\boldsymbol{q} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{q} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n\\
&= \langle \boldsymbol{q} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{q} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\,(\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{q} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\,(\boldsymbol{p}-\boldsymbol{q})\\
&= (0)\boldsymbol{q} + (0+1)(\boldsymbol{p}+\boldsymbol{q}) + (0-1)(\boldsymbol{p}-\boldsymbol{q})\\
&= 2\boldsymbol{q}\\
\implies \mathbf{S}^{-1}\boldsymbol{q} &= \frac{1}{2}\boldsymbol{q}
\end{aligned}$$

3. Remark: Without knowing $\boldsymbol{p}$ and $\boldsymbol{q}$, from (2) lemma it follows that it is not possible to compute $\mathbf{S}$ or $\mathbf{S}^{-1}$ explicitly.

4. Proof that $\mathbf{S}^{-1}\boldsymbol{x}_1 = \tfrac{1}{3}\boldsymbol{p}$, $\mathbf{S}^{-1}\boldsymbol{x}_2 = \tfrac{1}{3}\boldsymbol{p} + \tfrac{1}{2}\boldsymbol{q}$ and $\mathbf{S}^{-1}\boldsymbol{x}_3 = \tfrac{1}{3}\boldsymbol{p} - \tfrac{1}{2}\boldsymbol{q}$:

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_1 &\triangleq \mathbf{S}^{-1}\boldsymbol{p} && \text{by definition of } \boldsymbol{x}_1\\
&= \frac{1}{3}\boldsymbol{p} && \text{by (2) lemma}\\
\mathbf{S}^{-1}\boldsymbol{x}_2 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2\\
&= \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_3 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2 \\
&= \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

5. Check that $\boldsymbol{v} = \sum_n \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=1}^{3} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n \\
&= \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q}) \\
&= \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q}) \\
&= \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \left(\frac{1}{3}-\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{q} + \left(\frac{1}{2}-\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{p} + \left(\frac{1}{2}+\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q} \\
&= \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}
\end{aligned}$$

✏

APPENDIX H

POWER SPECTRUM FUNCTIONS

H.1 Correlation

Definition H.1 and Definition H.2 define four quantities. In this document, the quantities' notation and terminology are similar to those used in the study of *random processes*.

Definition H.1. [1] *Let* $\langle \triangle \mid \nabla \rangle$ *be the* STANDARD INNER PRODUCT *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition B.1 page 69).

$$\begin{array}{lllll} \mathsf{R}_{\mathsf{fg}}(n) & \triangleq & \langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{g}(x) \rangle, & n \in \mathbb{Z}; \quad \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{cross-correlation function} \text{ of } \mathsf{f} \text{ and } \mathsf{g}. \\ \mathsf{R}_{\mathsf{ff}}(n) & \triangleq & \langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{f}(x) \rangle, & n \in \mathbb{Z}; \quad \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{autocorrelation function} \text{ of } \mathsf{f}. \end{array}$$

Definition H.2. [2] *Let* $\mathsf{R}_{\mathsf{fg}}(n)$ *and* $\mathsf{R}_{\mathsf{ff}}(n)$ *be the sequences defined in Definition H.1 page 179. Let* $\mathbf{Z}\left(\!\left(x_n\right)\!\right)$ *be the* Z-TRANSFORM (Definition 5.4 page 28) *of a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$.

$$\begin{array}{lllll} \check{\mathsf{S}}_{\mathsf{fg}}(z) & \triangleq & \mathbf{Z}\left[\mathsf{R}_{\mathsf{fg}}(n)\right], & \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{complex cross-power spectrum} \text{ of } \mathsf{f} \text{ and } \mathsf{g}. \\ \check{\mathsf{S}}_{\mathsf{ff}}(z) & \triangleq & \mathbf{Z}\left[\mathsf{R}_{\mathsf{ff}}(n)\right], & \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{complex auto-power spectrum} \text{ of } \mathsf{f}. \end{array}$$

H.2 Power Spectrum

Definition H.3. [3] *Let* $\check{\mathsf{S}}_{\mathsf{fg}}(z)$ *and* $\check{\mathsf{S}}_{\mathsf{ff}}(z)$ *be the functions defined in Definition H.2 page 179.*

$$\begin{array}{lllll} \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) & \triangleq & \check{\mathsf{S}}_{\mathsf{fg}}\left(e^{i\omega}\right), & \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{cross-power spectrum} \text{ of } \mathsf{f} \text{ and } \mathsf{g}. \\ \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) & \triangleq & \check{\mathsf{S}}_{\mathsf{ff}}\left(e^{i\omega}\right), & \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{F}}, & \text{is the } \textbf{auto-power spectrum} \text{ of } \mathsf{f}. \end{array}$$

Theorem H.1. [4] *Let* $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ *and* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be defined as in Definition H.3 (page 179). Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition 4.2 page 16) *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{F}}$.

$$\begin{array}{llll} \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) & = & 2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n) & \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{\mathbb{F}} \\ \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) & = & 2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 & \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{F}} \end{array}$$

[1] Chui (1992) page 134, Papoulis (1991) pages 294–332 ⟨(10-29), (10-169)⟩
[2] Chui (1992) page 134, Papoulis (1991) page 334 ⟨(10-178)⟩
[3] Chui (1992) page 134, Papoulis (1991) page 333 ⟨(10-179)⟩
[4] Chui (1992) page 135

✎PROOF: Let $z \triangleq e^{i\omega}$.

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) &\triangleq \check{\mathsf{S}}_{\mathsf{fg}}(z) && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition H.3 page 179)}\\
&= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n) z^{-n} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition H.2 page 179)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\, z^{-n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition H.3 page 179)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathbf{F}}[\mathsf{f}(x)] \,|\, \tilde{\mathbf{F}}\big[\mathsf{g}(x-n)\big] \right\rangle z^{-n} && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} && \text{(Theorem 4.3 page 17)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathsf{f}}(v) \,|\, e^{-ivn}\tilde{\mathsf{g}}(v) \right\rangle z^{-n} && \text{by \textit{shift relation}} && \text{(Theorem 4.4 page 18)}\\
&= \sum_{n\in\mathbb{Z}} \sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v)e^{ivu}\,\mathrm{d}v\right]_{u=n} z^{-n} && \text{by definition of } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition B.1 page 69)}\\
&= \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \left[\tilde{\mathbf{F}}^{-1}\left(\sqrt{2\pi}\tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v)\right)\right]_{u=n} e^{-i\omega n} && \text{by Theorem 4.1 page 17}\\
&= 2\pi\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n) && \text{by \textit{IPSF} with } \tau=1 && \text{(Theorem I.3 page 197)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)\Big|_{\mathsf{g}=\mathsf{f}} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) && \text{(Definition H.3 page 179)}\\
&= 2\pi\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n)\Bigg|_{\mathsf{g}\triangleq\mathsf{f}} && \text{by previous result}\\
&= 2\pi\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{f}}^*(\omega+2\pi n)\\
&= 2\pi\sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{because } |z|^2 \triangleq zz^* \quad \forall z\in\mathbb{C}
\end{aligned}$$

✏

Proposition H.1. *Let $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition H.3 (page 179).*

PRP: $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) \;\geq\; 0$ (NON-NEGATIVE)

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= 2\pi\sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem H.1 page 179}\\
&\geq 0 && \text{because } |z|\geq 0 \quad \forall z\in\mathbb{C}
\end{aligned}$$

✏

Proposition H.2. *Let $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ and $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition H.3 (page 179).*

PRP:
$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) && \text{(PERIODIC \textit{with period} } 2\pi)\\
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) && \text{(PERIODIC \textit{with period} } 2\pi)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(\omega+2\pi) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\omega+2\pi+2\pi n)\tilde{g}^*(\omega+2\pi+2\pi n) && \text{by Theorem H.1 page 179}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}[\omega+2\pi(n+1)]\tilde{g}^*[\omega+2\pi(n+1)]\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{f}[\omega+2\pi m]\tilde{g}^*[\omega+2\pi m] && \text{where } m\triangleq n+1\\
&= \tilde{S}_{fg}(\omega) && \text{by Theorem H.1 page 179}\\
\tilde{S}_{ff}(\omega+2\pi) &= \tilde{S}_{fg}(\omega+2\pi)\Big|_{g=f}\\
&= \tilde{S}_{fg}(\omega)\Big|_{g=f} && \text{by previous result}\\
&= \tilde{S}_{ff}(\omega)
\end{aligned}$$

✏

Proposition H.3. *Let $\tilde{S}_{fg}(\omega)$ and $\tilde{S}_{ff}(\omega)$ be defined as in Definition H.3 (page 179).*

PRP

f, g *are real*	$\Longrightarrow$	$\tilde{S}_{fg}(-\omega)$	$=$	$\tilde{S}_{gf}(\omega)$	
f *is real*	$\Longrightarrow$	$\tilde{S}_{ff}(-\omega)$	$=$	$\tilde{S}_{ff}(\omega)$	(SYMMETRIC *about* 0)
f, g *are real*	$\Longrightarrow$	$\tilde{S}_{fg}(\pi-\omega)$	$=$	$\tilde{S}_{gf}(\pi+\omega)$	
f *is real*	$\Longrightarrow$	$\tilde{S}_{ff}(\pi-\omega)$	$=$	$\tilde{S}_{ff}(\pi+\omega)$	(SYMMETRIC *about* π)

✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(-\omega+2\pi n)\tilde{g}^*(-\omega+2\pi n) && \text{by Theorem H.1 page 179}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\omega-2\pi n)\tilde{g}(\omega-2\pi n) && \text{by hypothesis and Theorem 4.5 page 18}\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\omega+2\pi m)\tilde{f}^*(\omega+2\pi m) && \text{where } m\triangleq -n\\
&= \tilde{S}_{gf}(\omega) && \text{by Theorem H.1 page 179}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{fg}(\pi-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\pi-\omega+2\pi n)\tilde{g}^*(\pi-\omega+2\pi n) && \text{by Theorem H.1 page 179}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(-\pi+\omega-2\pi n)\tilde{g}(-\pi+\omega-2\pi n) && \text{by hypothesis and Theorem 4.5 page 18}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega-2\pi-2\pi n)\tilde{g}(\pi+\omega-2\pi-2\pi n)\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega+2\pi(-n-1))\tilde{g}(\pi+\omega+2\pi(-n-1))\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\pi+\omega+2\pi m)\tilde{f}^*(\pi+\omega+2\pi m) && \text{where } m\triangleq -n-1\\
&= \tilde{S}_{gf}(\pi+\omega) && \text{by Theorem H.1 page 179}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{ff}(-\omega) &= \tilde{S}_{fg}(-\omega)\Big|_{g\triangleq f}\\
&= \tilde{S}_{gf}(+\omega)\Big|_{g\triangleq f} && \text{by previous result}\\
&= \tilde{S}_{ff}(+\omega) && \text{by definition of g } (g\triangleq f)
\end{aligned}$$

$$\tilde{S}_{ff}(\pi-\omega) = \tilde{S}_{fg}(\pi-\omega)\Big|_{g\triangleq f}$$

$$= \tilde{S}_{gf}(\pi+\omega)\big|_{g \triangleq f} \qquad \text{by previous result}$$
$$= \tilde{S}_{ff}(\pi+\omega) \qquad \text{by definition of } g \ (g \triangleq f)$$

⇒

Proposition H.4. *Let* $\tilde{S}_{ff}(\omega)$ *be the* AUTO-POWER SPECTRUM (Definition H.3 page 179) *of a function* $f(x) \in L^2_{\mathbb{R}}$ *and* $\tilde{S}'_{ff}(\omega) \triangleq \frac{d}{d\omega}\tilde{S}_{ff}(\omega)$ (Definition B.2 page 69).

PRP

$$\left\{\begin{array}{ll} (a). & f \text{ is REAL and} \\ (b). & \tilde{S}_{ff}(\omega) \text{ is CONTINUOUS at } \omega = 0 \end{array}\right\} \implies \left\{\begin{array}{l} (1).\ \tilde{S}'_{ff}(0) = 0 \quad \text{and} \\ (2).\ \underbrace{\tilde{S}'_{ff}(\omega) = -\tilde{S}'_{ff}(-\omega) \quad \forall \omega \in \mathbb{R}}_{\text{ANTI-SYMMETRIC about } 0} \end{array}\right\}$$

$$\left\{\begin{array}{ll} (c). & f \text{ is REAL and} \\ (d). & \tilde{S}_{ff}(\omega) \text{ is CONTINUOUS at } \omega = \pi \end{array}\right\} \implies \left\{\begin{array}{l} (3).\ \tilde{S}'_{ff}(\pi) = 0 \quad \text{and} \\ (4).\ \underbrace{\tilde{S}'_{ff}(\pi+\omega) = -\tilde{S}'_{ff}(\pi-\omega) \quad \forall \omega \in \mathbb{R}}_{\text{ANTI-SYMMETRIC about } \pi} \end{array}\right\}$$

✎PROOF: This follows from Proposition H.3 (page 181) and Proposition B.1 (page 69). ⇒

Theorem H.2 (next) is a major result and provides strong motivation for bothering with *power spectrum* functions in the first place. In particular, the *auto-power spectrum* being *bounded* provides a necessary and sufficient condition for a sequence of functions $(\!(\phi(x-n))\!)_{n\in\mathbb{Z}}$ to be a *Riesz basis* (Definition G.13 page 167) for the *span* $\operatorname{span}(\!(\phi(x-n))\!)$ of the sequence.

Theorem H.2. [5] *Let* $\tilde{S}_{ff}(\omega)$ *be defined as in Definition H.3 (page 179). Let* $\|\cdot\|$ *be defined as in Definition B.1 (page 69). Let* $0 < A < B$.

THM

$$\underbrace{\left\{A\sum_{n\in\mathbb{N}}|a_n|^2 \le \left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 \le B\sum_{n\in\mathbb{N}}|\alpha_n|^2 \quad \forall (\!(a_n)\!) \in \ell^2_{\mathbb{F}}\right\}}_{(\!(\phi(x-n))\!) \text{ is a RIESZ BASIS for } \operatorname{span}(\!(\phi(x-n))\!) \text{ (Theorem G.13 page 168)}} \iff \left\{A \le \tilde{S}_{\phi\phi}(\omega) \le B\right\}$$

✎PROOF:

1. lemma:

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 &= \left\|\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 && \text{because } \tilde{\mathbf{F}} \text{ is } \textit{unitary} \text{ (Theorem 4.2 page 17)} \\
&= \left\|\breve{a}(\omega)\tilde{\phi}(\omega)\right\|^2 && \text{by Proposition I.13 page 195} \\
&= \int_{\mathbb{R}} \left|\breve{a}(\omega)\tilde{\phi}(\omega)\right|^2 d\omega && \text{by definition of } \|\cdot\| \\
&= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \left|\breve{a}(\omega+2\pi n)\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega \\
&= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |\breve{a}(\omega+2\pi n)|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega \\
&= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |\breve{a}(\omega)|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega && \text{by Proposition 6.1 page 41} \\
&= \int_0^{2\pi} |\breve{a}(\omega)|^2 \frac{1}{2\pi} 2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega+2\pi n)\right|^2 d\omega
\end{aligned}$$

[5] Wojtaszczyk (1997) pages 22–23 ⟨Proposition 2.8⟩, Igari (1996) page 219 ⟨Lemma 9.6⟩, Pinsky (2002) page 306 ⟨Theorem 6.4.8⟩

$$= \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega)\, d\omega \qquad \text{by definition of } \tilde{S}_{\phi\phi}(\omega) \text{ (Theorem H.1 page 179)}$$

2. lemma:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2\, d\omega &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right|^2 d\omega && \text{by def. of } DTFT \text{ (Definition 6.1 page 41)}\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\left[\sum_{m\in\mathbb{Z}} a_m e^{-i\omega m}\right]^* d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\left[\sum_{m\in\mathbb{Z}} a_m^* e^{i\omega m}\right] d\omega\\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n a_m^* \int_0^{2\pi} e^{-i\omega(n-m)}\, d\omega\\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n a_m^* 2\pi\bar{\delta}_{nm}\\
&= \sum_{n\in\mathbb{Z}} |a_n|^2 && \text{by definition of } \bar{\delta} \text{ (Definition G.12 page 160)}
\end{aligned}$$

3. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\boxed{A\sum_{n\in\mathbb{Z}} |a_n|^2} &= \frac{A}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2\, d\omega && \text{by (2) lemma page 183}\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 A\, d\omega\\
&\boxed{\le} \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega)\, d\omega && \text{by right hypothesis}\\
&= \boxed{\left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2} && \text{by (1) lemma page 182}\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega)\, d\omega && \text{by (1) lemma page 182}\\
&\boxed{\le} \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 B\, d\omega && \text{by right hypothesis}\\
&= \frac{B}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2\, d\omega\\
&= \boxed{B\sum_{n\in\mathbb{Z}} |a_n|^2} && \text{by (2) lemma page 183}
\end{aligned}$$

4. Proof for ($\Longrightarrow$) case:

(a) Let $Y \triangleq \left\{\omega \in [0 : 2\pi] | \tilde{S}_{\phi\phi}(\omega) > \alpha\right\}$
and $X \triangleq \left\{\omega \in [0 : 2\pi] | \tilde{S}_{\phi\phi}(\omega) < \alpha\right\}$

(b) Let $\mathbb{1}_{A(x)}$ be the *set indicator* (Definition I.2 page 188) of a set A.
Let $(b_n)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem 6.3 page 47) of $\mathbb{1}_Y(\omega)$ such that

$$\mathbb{1}_Y(\omega) \triangleq \sum_{n\in\mathbb{N}} b_n e^{-i\omega n} \triangleq \tilde{b}(\omega).$$

Let $(a_n)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem 6.3 page 47) of $\mathbb{1}_X(\omega)$ such that

$$\mathbb{1}_X(\omega) \triangleq \sum_{n\in\mathbb{N}} a_n e^{-i\omega n} \triangleq \breve{a}(\omega).$$

(c) Proof that $\alpha \le B$:
Let $\mu(A)$ be the *measure* of a set A.

$$
\begin{aligned}
\boxed{B}\sum_{n\in\mathbb{Z}}|b_n|^2 &\ge \left\|\sum_{n\in\mathbb{Z}} b_n\phi(x-n)\right\|^2 && \text{by left hypothesis}\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\tilde{\mathsf{b}}(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by (1) lemma page 182}\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\mathbb{1}_Y(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 183)}\\
&= \frac{1}{2\pi}\int_Y |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 183)}\\
&\boxed{\ge} \frac{\alpha}{2\pi}\mu(Y) && \text{by definition of } Y && \text{(item (4a) page 183)}\\
&= \int_0^{2\pi} \left|\mathbb{1}_Y(\omega)\right|^2\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 183)}\\
&= \int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} b_n e^{-i\omega n}\right|^2\,\mathrm{d}\omega && \text{by definition of } (\!(b_n)\!) && \text{(item (4b) page 183)}\\
&= \int_0^{2\pi} \left|\tilde{\mathsf{b}}(\omega)\right|^2\,\mathrm{d}\omega && \text{by definition of } \tilde{\mathsf{b}}(\omega) && \text{(item (4b) page 183)}\\
&= \boxed{\alpha}\sum_{n\in\mathbb{Z}}|b_n|^2 && \text{by (2) lemma page 183}
\end{aligned}
$$

(d) Proof that $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$:
  (i). $\tilde{\mathsf{S}}_{\phi\phi}(\omega) > \alpha$ whenever $\omega \in Y$ (item (4a) page 183).
  (ii). But even then, $\alpha \le B$ (item (4c) page 184).
  (iii). So, $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$.

(e) Proof that $A \le \alpha$:
Let $\mu(A)$ be the *measure* of a set A.

$$
\begin{aligned}
\boxed{A}\sum_{n\in\mathbb{Z}}|a_n|^2 &\le \left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|^2 && \text{by left hypothesis}\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by (1) lemma page 182}\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\mathbb{1}_X(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition I.2 page 188)}\\
&= \frac{1}{2\pi}\int_X |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition I.2 page 188)}\\
&\boxed{\le} \frac{\alpha}{2\pi}\mu(X) && \text{by definition of } X && \text{(item (4a) page 183)}\\
&= \int_0^{2\pi} \left|\mathbb{1}_X(\omega)\right|^2\,\mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition I.2 page 188)}\\
&= \int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right|^2\,\mathrm{d}\omega && \text{by definition of } (\!(a_n)\!) && \text{((2) lemma page 183)}\\
&= \int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2\,\mathrm{d}\omega && \text{by definition of } \breve{\mathsf{a}}(\omega) && \text{((2) lemma page 183)}\\
&= \boxed{\alpha}\sum_{n\in\mathbb{Z}}|a_n|^2 && \text{by (2) lemma page 183}
\end{aligned}
$$

(f) Proof that $A \le \tilde{S}_{\phi\phi}(\omega)$:

(i). $\tilde{S}_{\phi\phi}(\omega) < \alpha$ whenever $\omega \in X$ (item (4a) page 183).

(ii). But even then, $A \le \alpha$ (item (4e) page 184).

(iii). So, $A \le \tilde{S}_{\phi\phi}(\omega)$.

✏

In the case that f and g are *orthonormal*, the spectral density relations simplify considerably (next).

Theorem H.3. [6] *Let* $\tilde{S}_{ff}$ *and* $\tilde{S}_{fg}$ *be the* SPECTRAL DENSITY FUNCTIONS *(Definition H.3 page 179).*

THM

$$\begin{array}{rcll c rcl l}
\langle f(x) \,|\, f(x-n)\rangle &=& \bar{\delta}_n & (\{f(x-n)\}\text{ is ORTHONORMAL}) & \iff & \tilde{S}_{ff}(\omega) &=& 1 & \forall f \in L^2_{\mathbb{F}} \\
\langle f(x) \,|\, g(x-n)\rangle &=& 0 & (f(x)\text{ is ORTHOGONAL to }\{g(x-n)\}) & \iff & \tilde{S}_{fg}(\omega) &=& 0 & \forall f,g \in L^2_{\mathbb{F}}
\end{array}$$

✎PROOF:

1. Proof that $\langle f(x) \,|\, f(x-n)\rangle = \bar{\delta}_n \iff \tilde{S}_{ff}(\omega) = 1$: This follows directly from Theorem H.2 (page 182) with $A = B = 1$ (by *Parseval's Identity* Theorem G.9 page 162 since $\{\mathbf{T}^n f\}$ is *orthonormal*)

2. Alternate proof that $\langle f(x) \,|\, f(x-n)\rangle = \bar{\delta}_n \implies \tilde{S}_{ff}(\omega) = 1$:

$$\begin{aligned}
\tilde{S}_{ff}(\omega) &= \sum_{n\in\mathbb{Z}} R_{ff}(n) e^{-i\omega n} && \text{by definition of } \tilde{S}_{fg} && \text{(Definition H.3 page 179)} \\
&= \sum_{n\in\mathbb{Z}} \langle f(x) \,|\, f(x-n)\rangle\, e^{-i\omega n} && \text{by definition of } R_{ff} && \text{(Definition H.1 page 179)} \\
&= \sum_{n\in\mathbb{Z}} \bar{\delta}_n e^{-i\omega n} && \text{by left hypothesis} \\
&= 1 && \text{by definition of } \bar{\delta} && \text{(Definition G.12 page 160)}
\end{aligned}$$

3. Alternate proof that $\langle f(x) \,|\, f(x-n)\rangle = \bar{\delta}_n \impliedby \tilde{S}_{ff}(\omega) = 1$:

$$\begin{aligned}
&\langle f(x) \,|\, f(x-n)\rangle \\
&= \left\langle \tilde{\mathbf{F}} f(x) \,|\, \tilde{\mathbf{F}} f(x-n) \right\rangle && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} && \text{(Theorem 4.3 page 17)} \\
&= \left\langle \tilde{f}(\omega) \,|\, e^{-i\omega n} \tilde{f}(\omega) \right\rangle && \text{by \textit{shift property} of } \tilde{\mathbf{F}} && \text{(Theorem 4.4 page 18)} \\
&= \int_{\mathbb{R}} \tilde{f}(\omega) e^{i\omega n} \tilde{f}^*(\omega)\, d\omega && \text{by definition of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition B.1 page 69)} \\
&= \int_{\mathbb{R}} \left|\tilde{f}(\omega)\right|^2 e^{i\omega n}\, d\omega \\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \left|\tilde{f}(\omega)\right|^2 e^{i\omega n}\, d\omega \\
&= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \left|\tilde{f}(u+2\pi n)\right|^2 e^{i(u+2\pi n)n}\, du && \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{f}(u+2\pi n)\right|^2 \right] e^{iun} \cancelto{1}{e^{i2\pi nn}}\, du \\
&= \frac{1}{2\pi} \int_0^{2\pi} \tilde{S}_{ff}(\omega) e^{iun}\, du && \text{by Theorem H.1 page 179} \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{iun}\, du && \text{by right hypothesis} \\
&= \bar{\delta}_n && \text{by definition of } \bar{\delta} && \text{(Definition G.12 page 160)}
\end{aligned}$$

[6] Hernández and Weiss (1996) page 50 ⟨PROPOSITION 2.1.11⟩, Wojtaszczyk (1997) PAGE 23 ⟨COROLLARY 2.9⟩, Igari (1996) PAGES 214–215 ⟨LEMMA 9.2⟩, Pinsky (2002) PAGE 306 ⟨COROLLARY 6.4.9⟩

4. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle = 0 \implies \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) &= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n) e^{-i\omega n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition H.3 page 179)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\, e^{-i\omega n} && \text{by definition of } \mathsf{R}_{\mathsf{fg}} && \text{(Definition H.1 page 179)}\\
&= \sum_{n\in\mathbb{Z}} 0 e^{-i\omega n} && \text{by left hypothesis}\\
&= 0
\end{aligned}$$

5. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle = 0 \impliedby \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

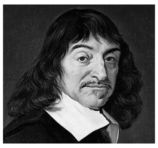
$$\begin{aligned}
&\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\\
&= \left\langle \tilde{\mathbf{F}}\mathsf{f}(x) \,|\, \tilde{\mathbf{F}}\mathsf{g}(x-n)\right\rangle && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} && \text{(Theorem 4.3 page 17)}\\
&= \left\langle \tilde{\mathsf{f}}(\omega) \,|\, e^{-i\omega n}\tilde{\mathsf{g}}(\omega)\right\rangle && \text{by \textit{shift property} of } \tilde{\mathbf{F}} && \text{(Theorem 4.4 page 18)}\\
&= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) e^{i\omega n} \tilde{\mathsf{g}}^*(\omega)\, \mathrm{d}\omega && \text{by definition of } \langle \triangle \,|\, \triangledown\rangle && \text{(Definition B.1 page 69)}\\
&= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) \tilde{\mathsf{g}}^*(\omega) e^{i\omega n}\, \mathrm{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \tilde{\mathsf{f}}(\omega) \tilde{\mathsf{g}}^*(\omega) e^{i\omega n}\, \mathrm{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{0}^{2\pi} \tilde{\mathsf{f}}(u+2\pi n) \tilde{\mathsf{g}}^*(u+2\pi n) e^{i(u+2\pi n)n}\, \mathrm{d}u && \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n\\
&= \frac{1}{2\pi}\int_{0}^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(u+2\pi n) \tilde{\mathsf{g}}^*(u+2\pi n)\right] e^{iun} \cancelto{1}{e^{i2\pi nn}}\, \mathrm{d}u\\
&= \frac{1}{2\pi}\int_{0}^{2\pi} \tilde{\mathsf{S}}_{\mathsf{fg}}(u) e^{iun}\, \mathrm{d}u && \text{by Theorem H.1 page 179}\\
&= \frac{1}{2\pi}\int_{0}^{2\pi} 0 \cdot e^{iun}\, \mathrm{d}u && \text{by right hypothesis}\\
&= 0
\end{aligned}$$

✏

APPENDIX I

TRANSVERSAL OPERATORS

❝Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:❞



❝I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them.❞

René Descartes, philosopher and mathematician (1596–1650) [1]

I.1 Families of Functions

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition B.1 page 69). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition I.1 page 187). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition I.1. *Let X and Y be sets.*

DEF *The space Y^X represents the set of all functions with* DOMAIN X *and* RANGE Y *such that*

$$Y^X \triangleq \{\mathsf{f}(x)|\mathsf{f}(x) : X \to Y\}$$

[1] quote: Descartes (1637b)
translation: Descartes (1637c) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

Definition I.2. [2] *Let X be a set.*

DEF

*The **indicator function** $\mathbb{1} \in \{0,1\}^{2^X}$ is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \quad \forall x \in X,\ A \in 2^X \\ 0 & \text{for } x \notin A \quad \forall x \in X,\ A \in 2^X \end{cases}$$

*The indicator function $\mathbb{1}$ is also called the **characteristic function**.*

I.2 Definitions and algebraic properties

Much of the wavelet theory developed in this text is constructed using the **translation operator T** and the **dilation operator D** (next).

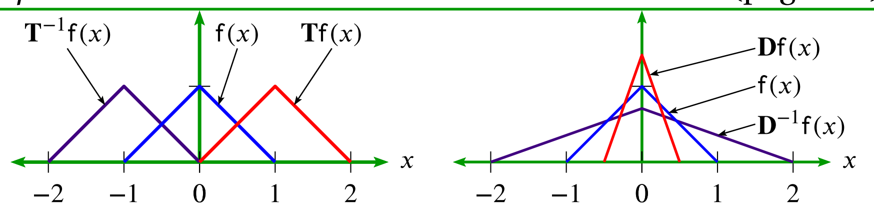
Definition I.3. [3]

DEF

$$\begin{array}{llll} \mathbf{T}_\tau \textit{ is a \textbf{translation operator} on } \mathbb{C}^{\mathbb{C}} \textit{ if} & \mathbf{T}_\tau \mathsf{f}(x) \triangleq \mathsf{f}(x-\tau) & \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}} & . \\ \mathbf{D}_\alpha \textit{ is a \textbf{dilation operator} on } \mathbb{C}^{\mathbb{C}} \textit{ if} & \mathbf{D}_\alpha \mathsf{f}(x) \triangleq \mathsf{f}(\alpha x) & \forall \mathsf{f} \in \mathbb{C}^{\mathbb{C}} & . \\ \textit{Moreover, } \mathbf{T} \triangleq \mathbf{T}_1 \quad \textit{and} \quad \mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2 & & & . \end{array}$$

Example I.1. Let **T** and **D** be defined as in Definition I.3 (page 188).

EX



Proposition I.1. *Let $\mathbf{T}_\tau$ be a* TRANSLATION OPERATOR *(Definition I.3 page 188).*

PRP

$$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) \qquad \forall \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) \textit{ is } \text{PERIODIC} \textit{ with period } \tau \right)$$

✎PROOF:

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x - n\tau + \tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition I.3 page 188)} \\ &= \sum_{m\in\mathbb{Z}} \mathsf{f}(x - m\tau) && \text{where } m \triangleq n-1 && \implies n = m+1 \\ &= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m \mathsf{f}(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition I.3 page 188)} \end{aligned}$$

⇨

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exist as operators, as demonstrated next.

[2] Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) page 22, de la Vallée-Poussin (1915) page 440

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2000) page 473, Packer (2004) page 260, zay (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a) page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996) page 81, Dai and Larson (1998) page 2

Proposition I.2 (transversal operator inverses). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition I.3 page 188.*

PRP

$\mathbf{T}$ *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall\mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(translation operator inverse)}.$$

$\mathbf{D}$ *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall\mathsf{f}\in\mathbb{C}^{\mathbb{C}} \qquad \text{(dilation operator inverse)}.$$

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} \quad \text{(Definition I.3 page 188)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} \quad \text{(Definition I.3 page 188)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\implies \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} \quad \text{(Definition I.3 page 188)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} \quad \text{(Definition I.3 page 188)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\implies \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

✏

Proposition I.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition I.3 page 188.*
Let $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP

$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^jx-n\right) \qquad \forall j,n\in\mathbb{Z},\mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

I.3 Linear space properties

Proposition I.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition I.3 page 188.*

PRP

$$\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}\mathsf{g}\right] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z},\mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

✎PROOF:

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n\left[\mathsf{f}(x)\mathsf{g}(x)\right] &= 2^{j/2}\mathsf{f}\left(2^jx-n\right)\mathsf{g}\left(2^jx-n\right) && \text{by Proposition I.3 page 189}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^jx-n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^jx-n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition I.3 page 189}
\end{aligned}$$

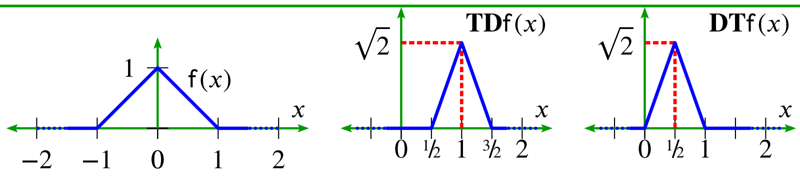
⇒

In general the operators $\mathbf{T}$ and $\mathbf{D}$ are *noncommutative* ($\mathbf{TD} \neq \mathbf{DT}$), as demonstrated by Counterexample I.1 (next) and Proposition I.5 (page 190).

Counterexample I.1.

CNT

As illustrated to the right, it is **not** always true that $\mathbf{TD} = \mathbf{DT}$:



Proposition I.5 (commutator relation). [4] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition I.3 page 188.*

PRP

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j && \forall j,n\in\mathbb{Z}\\
\mathbf{T}^n\mathbf{D}^j &= \mathbf{D}^j\mathbf{T}^{2^j n} && \forall n,j\in\mathbb{Z}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^{2^j n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - 2^j n) && \text{by Proposition I.4 page 189}\\
&= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 68)}\\
&= \mathbf{T}^n 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition I.3 page 188)}\\
&= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition I.3 page 188)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - n) && \text{by Proposition I.4 page 189}\\
&= 2^{j/2}\,\mathsf{f}\left(2^j\left[x-2^{-j/2}n\right]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 68)}\\
&= \mathbf{T}^{2^{-j/2}n}2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition I.3 page 188)}\\
&= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition I.3 page 188)}
\end{aligned}$$

⇒

I.4 Inner product space properties

In an inner product space, every operator has an *adjoint* (Proposition F.3 page 133) and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator $\mathbf{U}$ coincide, then $\mathbf{U}$ is said to be *unitary* (Definition F.14 page 143). And in this case, $\mathbf{U}$ has several nice properties (see Proposition I.9 and Theorem I.1 page 193). Proposition I.6 (next) gives the adjoints of $\mathbf{D}$ and $\mathbf{T}$, and Proposition I.7 (page 191) demonstrates that both $\mathbf{D}$ and $\mathbf{T}$ are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ (Corollary 4.1 page 17) and the *rotation matrix operator* (Example F.5 page 145).

Proposition I.6. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition I.3 page 188) with* ADJOINT $\mathbf{T}^*$ *and* $\mathbf{D}$ *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$ *(Definition F.8 page 129).*

PRP

$$\begin{aligned}
\mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(TRANSLATION OPERATOR ADJOINT)}\\
\mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(DILATION OPERATOR ADJOINT)}
\end{aligned}$$

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a) page 641, Goodman et al. (1993b) page 110

✎PROOF:

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^* && \text{(Definition F.8 page 129)} \\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} && \text{(Definition I.3 page 188)} \\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1 \\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^* && \text{(Definition F.8 page 129)} \\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition I.3 page 188)} \\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\, \mathrm{d}u && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^* \mathrm{d}x && \text{where } x = 2u \\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}$$

✏

Proposition I.7. [5] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition I.3 (page 188). Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition I.2 (page 189).*

PRP:
$\mathbf{T}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $(\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $L^2_{\mathbb{R}})$.
$\mathbf{D}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $(\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $L^2_{\mathbb{R}})$.

✎PROOF:

$$\begin{aligned}
\mathbf{T}^{-1} &= \mathbf{T}^* && \text{by Proposition I.2 page 189 and Proposition I.6 page 190} \\
&\implies \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition F.14 page 143)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}^{-1} &= \mathbf{D}^* && \text{by Proposition I.2 page 189 and Proposition I.6 page 190} \\
&\implies \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition F.14 page 143)}
\end{aligned}$$

✏

I.5 Normed linear space properties

Proposition I.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR *(Definition I.3 page 188).*

PRP:
$$\left\{\begin{array}{ll} (1). & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) \quad \textit{and} \\ (2). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f} \in L^2_{\mathbb{R}}$$

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

✎PROOF:

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition I.3 page 188)}\\
&= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis}
\end{aligned}$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(x+h)\| &= \|\mathsf{f}(x) - \mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis}\\
&= \|0\|\\
&= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|\\
&\le \varepsilon\\
&\Longrightarrow \forall h > 0,\ \exists \varepsilon \quad \text{such that} \quad \|\mathsf{f}(x) - \mathsf{f}(x+h)\| < \varepsilon\\
&\overset{\text{def}}{\Longleftrightarrow} \mathsf{f}(x) \text{ is } \textit{continuous}
\end{aligned}$$

3. Proof that (1,2) $\Longrightarrow$ *constant* property:

   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \neq \mathsf{f}(y)$.

   (b) Let $(\!(x_n)\!)_{n\in\mathbb{N}}$ be a sequence with limit x and $(\!(y_n)\!)_{n\in\mathbb{N}}$ a sequence with limit y

   (c) Then

$$\begin{aligned}
0 &< \|\mathsf{f}(x) - \mathsf{f}(y)\| && \text{by assumption in item (3a) page 192}\\
&= \lim_{n\to\infty} \left\|\mathsf{f}(x_n) - \mathsf{f}(y_n)\right\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 192}\\
&= \lim_{n\to\infty} \left\|\mathsf{f}(2^m x_n) - \mathsf{f}(2^\ell y_n)\right\| \quad \forall m,\ell \in \mathbb{Z} && \text{by (1)}\\
&= 0
\end{aligned}$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

✏

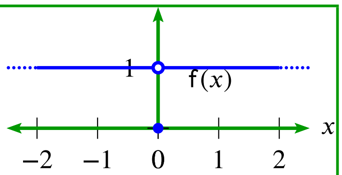
Remark I.1.

REM In Proposition I.8 page 191, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample I.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x = 0\\ 1 & \text{otherwise.}\end{cases}$

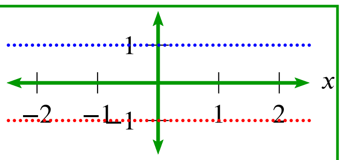
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Counterexample I.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\backslash\mathbb{Q}$ the set of *irrational numbers*.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x \in \mathbb{Q}\\ -1 & \text{for } x \in \mathbb{R}\backslash\mathbb{Q}.\end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Proposition I.9 (Operator norm). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition I.3 page 188. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition I.2 page 189. Let* $\mathbf{T}^*$ *and* $\mathbf{D}^*$ *be as in Proposition I.6 page 190. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition B.1 page 69. Let* $|||\cdot|||$ *be the operator norm* (Definition F.6 page 125) *induced by* $\|\cdot\|$.

PRP:
$$|||\mathbf{T}||| = |||\mathbf{D}||| = |||\mathbf{T}^*||| = |||\mathbf{D}^*||| = |||\mathbf{T}^{-1}||| = |||\mathbf{D}^{-1}||| = 1$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition I.7 page 191) and from Theorem F.25 page 144 and Theorem F.26 page 144. ✏

Theorem I.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition I.3 page 188.*
Let $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition I.2 page 189. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition B.1 page 69.*

THM:

1.	$\|\mathbf{T}\mathsf{f}\|$	$=$	$\|\mathbf{D}\mathsf{f}\|$	$=$	$\|\mathsf{f}\|$	$\forall \mathsf{f}\in\mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}\mathsf{f}-\mathbf{T}\mathsf{g}\|$	$=$	$\|\mathbf{D}\mathsf{f}-\mathbf{D}\mathsf{g}\|$	$=$	$\|\mathsf{f}-\mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
3.	$\|\mathbf{T}^{-1}\mathsf{f}-\mathbf{T}^{-1}\mathsf{g}\|$	$=$	$\|\mathbf{D}^{-1}\mathsf{f}-\mathbf{D}^{-1}\mathsf{g}\|$	$=$	$\|\mathsf{f}-\mathsf{g}\|$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
4.	$\langle \mathbf{T}\mathsf{f} \mid \mathbf{T}\mathsf{g}\rangle$	$=$	$\langle \mathbf{D}\mathsf{f} \mid \mathbf{D}\mathsf{g}\rangle$	$=$	$\langle \mathsf{f} \mid \mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(SURJECTIVE)
5.	$\langle \mathbf{T}^{-1}\mathsf{f} \mid \mathbf{T}^{-1}\mathsf{g}\rangle$	$=$	$\langle \mathbf{D}^{-1}\mathsf{f} \mid \mathbf{D}^{-1}\mathsf{g}\rangle$	$=$	$\langle \mathsf{f} \mid \mathsf{g}\rangle$	$\forall \mathsf{f},\mathsf{g}\in\mathcal{L}^2_{\mathbb{R}}$	(SURJECTIVE)

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition I.7 page 191) and from Theorem F.25 page 144 and Theorem F.26 page 144. ✏

Proposition I.10. *Let* $\mathbf{T}$ *be as in Definition I.3 page 188. Let* $\mathbf{A}^*$ *be the* ADJOINT (Definition F.8 page 129) *of an operator* $\mathbf{A}$. *Let the property "*SELF ADJOINT*" be defined as in Definition F.11 (page 137).*

PRP:
$$\left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)^* \iff \left(\text{The operator } \left[\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right] \text{ is SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)\mathsf{f}(x) \,\middle|\, \mathsf{g}(x)\right\rangle &= \left\langle \sum_{n\in\mathbb{Z}}\mathsf{f}(x-n) \,\middle|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition I.3 page 188)}\\
&= \left\langle \sum_{n\in\mathbb{Z}}\mathsf{f}(x+n) \,\middle|\, \mathsf{g}(x)\right\rangle && \text{by \textit{commutative} property} && \text{(Definition A.5 page 68)}\\
&= \sum_{n\in\mathbb{Z}}\langle \mathsf{f}(x+n) \mid \mathsf{g}(x)\rangle && \text{by \textit{additive} property of } \langle \triangle \mid \nabla \rangle\\
&= \sum_{n\in\mathbb{Z}}\langle \mathsf{f}(u) \mid \mathsf{g}(u-n)\rangle && \text{where } u \triangleq x+n\\
&= \left\langle \mathsf{f}(u) \,\middle|\, \sum_{n\in\mathbb{Z}}\mathsf{g}(u-n)\right\rangle && \text{by \textit{additive} property of } \langle \triangle \mid \nabla \rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \sum_{n\in\mathbb{Z}}\mathsf{g}(x-n)\right\rangle && \text{by change of variable: } u \to x\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \sum_{n\in\mathbb{Z}}\mathbf{T}^n\mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition I.3 page 188)}\\
\iff \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) &= \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right)^* && \text{by definition of \textit{adjoint}} && \text{(Proposition F.3 page 133)}\\
\iff \left(\sum_{n\in\mathbb{Z}}\mathbf{T}^n\right) &\text{ is \textit{self-adjoint}} && \text{by definition of \textit{self-adjoint}} && \text{(Definition F.11 page 137)}
\end{aligned}$$

✏

I.6 Fourier transform properties

Proposition I.11. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition* I.3 *page* 188.
Let $\mathbf{B}$ *be the* TWO-SIDED LAPLACE TRANSFORM *defined as* $[\mathbf{B}\mathsf{f}](s) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x$.

PRP

1.	$\mathbf{BT}^n$	$=$	$e^{-sn}\mathbf{B}$			$\forall n\in\mathbb{Z}$	
2.	$\mathbf{BD}^j$	$=$	$\mathbf{D}^{-j}\mathbf{B}$			$\forall j\in\mathbb{Z}$	
3.	$\mathbf{DB}$	$=$	$\mathbf{BD}^{-1}$			$\forall n\in\mathbb{Z}$	
4.	$\mathbf{BD}^{-1}\mathbf{B}^{-1}$	$=$	$\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B}$	$=$	$\mathbf{D}$	$\forall n\in\mathbb{Z}$	($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5.	$\mathbf{DBD}$	$=$	$\mathbf{D}^{-1}\mathbf{BD}^{-1}$	$=$	$\mathbf{B}$	$\forall n\in\mathbb{Z}$	

✎PROOF:

$$\begin{aligned}
\mathbf{BT}^n\mathsf{f}(x) &= \mathbf{B}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition I.3 page 188)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathrm{d}u && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right]\\
&= e^{-sn}\,\mathbf{B}\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}$$

$$\begin{aligned}
\mathbf{BD}^j\mathsf{f}(x) &= \mathbf{B}\left[2^{j/2}\,\mathsf{f}\left(2^jx\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition I.3 page 188)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}\left(2^jx\right)\right]e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathrm{d}u && \text{let } u \triangleq 2^jx \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathrm{d}u\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right] && \text{by Proposition I.6 page 190 and} && \text{Proposition I.7 page 191}\\
&= \mathbf{D}^{-j}\,\mathbf{B}\,\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}$$

$$\begin{aligned}
\mathbf{DB}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x\right] && \text{by definition of } \mathbf{B}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathrm{d}x && \text{by definition of } \mathbf{D} && \text{(Definition I.3 page 188)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{u}{2}\right)\right]e^{-su}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\,e^{-su}\,\mathrm{d}u && \text{by Proposition I.6 page 190 and} && \text{Proposition I.7 page 191}\\
&= \mathbf{BD}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{BD} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition F.3 page 120)}\\
\mathbf{BD}^{-1}\mathbf{B}^{-1} &= \mathbf{DBB}^{-1} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition F.3 page 120)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{DBD} &= \mathbf{DD}^{-1}\mathbf{B} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse} \quad \text{(Definition F.3 page 120)}\\
\mathbf{D}^{-1}\mathbf{BD}^{-1} &= \mathbf{D}^{-1}\mathbf{DB} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse} \quad \text{(Definition F.3 page 120)}
\end{aligned}$$

✏

Corollary I.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition I.3 page 188. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 4.2 page 16) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition B.1 page 69).*

COR

1. $\tilde{\mathbf{F}}\mathbf{T}^n = e^{-i\omega n}\tilde{\mathbf{F}}$
2. $\tilde{\mathbf{F}}\mathbf{D}^j = \mathbf{D}^{-j}\tilde{\mathbf{F}}$
3. $\mathbf{D}\tilde{\mathbf{F}} = \tilde{\mathbf{F}}\mathbf{D}^{-1}$
4. $\mathbf{D} = \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} = \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}}$
5. $\tilde{\mathbf{F}} = \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} = \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}$

✎PROOF: These results follow directly from Proposition I.11 page 194 with $\tilde{\mathbf{F}} = \mathbf{B}|_{s=i\omega}$. ✏

Proposition I.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition I.3 page 188. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 4.2 page 16) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition B.1 page 69).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = \frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{\mathsf{f}}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n\mathsf{f}(x) && \text{by Corollary I.1 page 195 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}\mathsf{f}(x) && \text{by Corollary I.1 page 195 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathsf{f}}(\omega)\\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{\mathsf{f}}\left(2^{-j}\omega\right) && \text{by Proposition I.2 page 189}
\end{aligned}$$

✏

Proposition I.13. *Let* $\mathbf{T}$ *be the translation operator (Definition I.3 page 188). Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition 4.2 page 16) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT *(Definition 6.1 page 41) of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition 5.2 page 27).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) \qquad \forall (\!(a_n)\!)\in\boldsymbol{\ell}^2_{\mathbb{R}}, \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}^n\phi(x)\\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\tilde{\mathbf{F}}\phi(x) && \text{by Corollary I.1 page 195}\\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega)\\
&= \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) && \text{by definition of } \textit{DTFT} \text{ (Definition 6.1 page 41)}
\end{aligned}$$

✏

Definition I.4. *Let* $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition B.1 page 69). Let* $\ell^2_{\mathbb{R}}$ *be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition B.1 page 69).*

DEF: $\mathbf{S}$ *is the* ***sampling operator*** *in* $\ell^{2\,L^2_{\mathbb{R}}}_{\mathbb{R}}$ *if* $[\mathbf{S}\mathsf{f}(x)](n) \triangleq \mathsf{f}\left(\frac{2\pi}{\tau}n\right) \qquad \forall \mathsf{f}\in L^2_{(\mathbb{R},\mathscr{B},\mu)},\ \tau\in\mathbb{R}^+$

Theorem I.2 (Poisson Summation Formula—PSF).[6] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition 4.2 page 16) of a function* $\mathsf{f}(x)\in L^2_{\mathbb{R}}$*. Let* $\mathbf{S}$ *be the* SAMPLING OPERATOR *(Definition I.4 page 196).*

THM:
$$\underbrace{\sum_{n\in\mathbb{Z}}\mathbf{T}^n_\tau\mathsf{f}(x) = \sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(x) \triangleq \sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)$ then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:

   Note that $\mathsf{h}(x)$ is *periodic* with period τ. Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of PSF (this theorem—Theorem I.2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau) && \text{by (1) lemma page 196}\\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau\left(\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}[\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition 7.1 page 51)}\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau\mathsf{f}(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right]\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau}\mathsf{f}(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,\mathrm{d}u\right] && \text{where } u\triangleq x+n\tau \implies x=u-n\tau\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\cancel{e^{i2\pi kn}}^{\,1}\int_{u=n\tau}^{u=(n+1)\tau}\mathsf{f}(u)e^{-i\frac{2\pi}{\tau}ku}\,\mathrm{d}u\right]\\
&= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,\mathrm{d}u\right]}_{[\tilde{\mathbf{F}}\mathsf{f}]\left(\frac{2\pi}{\tau}k\right)} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 7.1 page 52)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]\left(\frac{2\pi}{\tau}k\right)\right] && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 4.2 page 16)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \mathbf{S} \quad \text{(Definition I.4 page 196)}\\
&= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 7.1 page 52)}
\end{aligned}$$

✏

[6] Andrews et al. (2001) page 624, Knapp (2005b) page 389, Lasser (1996) page 254, Rudin (1987) pages 194–195, Folland (1992) page 337

Theorem I.3 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{f}(\omega)$ *be the* FOURIER TRANSFORM (*Definition 4.2 page 16*) *of a function* $f(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_{2\pi/\tau}\tilde{f}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega - \frac{2\pi}{\tau}n\right)}_{\text{summation in “frequency”}} \quad = \quad \underbrace{\frac{\tau}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} f(n\tau)e^{-i\omega n\tau}}_{\text{summation in “time”}}$$

✎PROOF:

1. lemma: If $h(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right)$, then $h \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}h$. Proof:
   Note that $h(\omega)$ is periodic with period $2\pi/T$:

$$h\left(\omega + \frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau} + \frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + (n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right) \triangleq h(\omega)$$

   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}h$.

2. Proof of IPSF (this theorem—Theorem I.3):

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right) \\
&= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}} \sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right) && \text{by (1) lemma page 197} \\
&= \hat{\mathbf{F}}^{-1} \underbrace{\left[\sqrt{\frac{\tau}{2\pi}} \int_0^{\frac{2\pi}{\tau}} \sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right) e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\, d\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right)\right]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition 7.1 page 51)} \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \sum_{n\in\mathbb{Z}} \int_0^{\frac{2\pi}{\tau}} \tilde{f}\left(\omega + \frac{2\pi}{\tau}n\right) e^{-i\omega Tk}\, d\omega\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \sum_{n\in\mathbb{Z}} \int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)} \tilde{f}(u) e^{-i\left(u - \frac{2\pi}{\tau}n\right)Tk}\, du\right] && \text{where } u \triangleq \omega + \frac{2\pi}{\tau}n \implies \quad \omega = u - \frac{2\pi}{\tau}n \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \sum_{n\in\mathbb{Z}} \cancelto{1}{e^{i2\pi nk}} \int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)} \tilde{f}(u) e^{-iu\tau k}\, du\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}} \int_{\mathbb{R}} \tilde{f}(u) e^{-iu\tau k}\, du\right] \\
&= \sqrt{\tau}\, \hat{\mathbf{F}}^{-1} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(u) e^{iu(-\tau k)}\, du\right]}_{\left[\tilde{\mathbf{F}}^{-1}\tilde{f}\right](-k\tau)} \\
&= \sqrt{\tau}\, \hat{\mathbf{F}}^{-1}\left[\left[\tilde{\mathbf{F}}^{-1}\tilde{f}\right](-k\tau)\right] && \text{by value of } \tilde{\mathbf{F}}^{-1} \quad \text{(Theorem 4.1 page 17)} \\
&= \sqrt{\tau}\, \hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\, \tilde{f} && \text{by definition of } \mathbf{S} \quad \text{(Definition I.4 page 196)} \\
&= \sqrt{\tau}\, \hat{\mathbf{F}}^{-1}\mathbf{S}f(x) && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition 4.2 page 16)} \\
&= \sqrt{\tau}\, \hat{\mathbf{F}}^{-1}f(-k\tau) && \text{by definition of } \mathbf{S} \quad \text{(Definition I.4 page 196)} \\
&= \sqrt{\tau}\, \frac{1}{\sqrt{\frac{2\pi}{\tau}}} \sum_{k\in\mathbb{Z}} f(-k\tau) e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} && \text{by definition of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem 7.1 page 52)}
\end{aligned}$$

[7] Gauss (1900) page 88

$$= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}} \sum_{k\in\mathbb{Z}} \mathsf{f}(-k\tau)e^{ik\tau\omega} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem 7.1 page 52)}$$

$$= \frac{\tau}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \mathsf{f}(m\tau)e^{-i\omega m\tau} \qquad \text{let } m \triangleq -k$$

✏

Remark I.2. The left hand side of the *Poisson Summation Formula* (Theorem I.2 page 196) is very similar to the *Zak Transform* $\mathbf{Z}$: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark I.3. A generalization of the *Poisson Summation Formula* (Theorem I.2 page 196) is the **Selberg Trace Formula**.[9]

I.7 Examples

Example I.2 (linear functions). [10] Let $\mathbf{T}$ be the *translation operator* (Definition I.3 page 188). Let $\mathcal{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX
1. $\{x,\ \mathbf{T}x\}$ is a *basis* for $\mathcal{L}(\mathbb{C},\mathbb{C})$ and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x \qquad \forall \mathsf{f}\in\mathcal{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition I.3 page 188}\\
&= (ax+b)|_{x=1}\, x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of f}\\
&= (a+b)x - b(x-1)\\
&= ax + bx - bx + b\\
&= ax+b\\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of f}
\end{aligned}$$

✏

Example I.3 (Cardinal Series). Let $\mathbf{T}$ be the *translation operator* (Definition I.3 page 188). The *Paley-Wiener* class of functions $\boldsymbol{PW}^2_\sigma$ are those functions which are "*bandlimited*" with respect to their Fourier transform (Definition 4.2 page 16). The cardinal series forms an orthogonal basis for such a space. The *Fourier coefficients* (Definition G.11 page 160) for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of $\mathsf{f}(x)$ taken at regular intervals. In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \,|\, \mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\,\mathrm{d}x \triangleq \mathsf{f}(n)$$

EX
1. $\left\{ \mathbf{T}^n \frac{\sin(\pi x)}{\pi x} \middle| n\in\mathbb{N} \right\}$ is a *basis* for $\boldsymbol{PW}^2_\sigma$ and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin(\pi x)}{\pi x}}_{\textit{Cardinal series}} \qquad \forall \mathsf{f}\in\boldsymbol{PW}^2_\sigma,\ \sigma\le\frac{1}{2}$

[8] Janssen (1988) page 24, Zayed (1996) page 482
[9] Lax (2002) page 349, Selberg (1956), Terras (1999)
[10] Higgins (1996) page 2

Example I.4 (Fourier Series).

EX

1. $\left\{\mathbf{D}_n e^{ix} \middle| n\in\mathbb{Z}\right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix} \quad \forall x\in(0:2\pi), \mathsf{f}\in\boldsymbol{L}(0:2\pi)$ where
3. $\alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x)\mathbf{D}_n e^{-ix}\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}(0:2\pi)$

✎PROOF: See Theorem 7.1 page 52. ✏

Example I.5 (Fourier Transform). [11]

EX

1. $\left\{\mathbf{D}_\omega e^{ix} | \omega\in\mathbb{R}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)\mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}_\omega e^{-ix}\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example I.6 (Gabor Transform). [12]

EX

1. $\left\{\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{ix}\right) \middle| \tau,\omega\in\mathbb{R}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \int_{\mathbb{R}} \mathsf{G}(\tau,\omega)\, \mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\mathsf{G}(\tau,\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{-ix}\right)\, \mathrm{d}x \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example I.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX

1. $\left\{\mathbf{D}^k\mathbf{T}^n\psi(x) \middle| k,n\in\mathbb{Z}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha_{k,n}\mathbf{D}^k\mathbf{T}^n\psi(x) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\alpha_n \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}^k\mathbf{T}^n\psi^*(x)\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

[11] cross reference: Definition 4.2 page 16

[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

APPENDIX J

CONTINUOUS RANDOM PROCESSES

J.1 Definitions

Definition J.1. [1] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE.

DEF	*The function* $\mathsf{x} : \Omega \to \mathbb{R}$ *is a* ***random variable****.* *The function* $\mathsf{y} : \mathbb{R} \times \Omega \to \mathbb{R}$ *is a* ***random process****.*

The random process $\mathsf{x}(t, \omega)$, where t commonly represents time and $\omega \in \Omega$ is an outcome of an experiment, can take on more specialized forms depending on whether t and ω are fixed or allowed to vary. These forms are illustrated in Figure J.1 page 201[2] and Figure J.2 page 202.

$\mathsf{x}(t,\omega)$	fixed t	variable t
fixed ω	number	time function
variable ω	random variable	random process

Figure J.1: Specialized forms of a random process $\mathsf{x}(t, \omega)$

Definition J.2. [3] *Let* $\mathsf{x}(t)$ *and* $\mathsf{y}(t)$ *be random processes.*

DEF	*The* ***mean***	$\mu_{\mathsf{x}}(t)$	*of* $\mathsf{x}(t)$ *is*	$\mu_{\mathsf{x}}(t)$	$\triangleq \mathsf{E}[\mathsf{x}(t)]$
	The ***cross-correlation***	$\mathsf{R}_{\mathsf{xy}}(t)$	*of* $\mathsf{x}(t)$ *and* $\mathsf{y}(t)$ *is*	$\mathsf{R}_{\mathsf{xy}}(t,u)$	$\triangleq \mathsf{E}\left[\mathsf{x}(t)\mathsf{y}^*(u)\right]$
	The ***auto-correlation function***	$\mathsf{R}_{\mathsf{xx}}(t)$	*of* $\mathsf{x}(t)$ *is*	$\mathsf{R}_{\mathsf{xx}}(t,u)$	$\triangleq \mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right]$

Remark J.1. [4] The equation $\int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{f}(u)\,\mathsf{d}u$ is a *Fredholm integral equation of the first kind* and $\mathsf{R}_{\mathsf{xx}}(t,u)$ is the *kernel* of the equation.

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[1] Papoulis (1991) page 63, Papoulis (1991) page 285

[2] Papoulis (1991) pages 285–286

[3] Papoulis (1984) page 216 ⟨$R_{xy}(t_1,t_2) = E\left\{\mathsf{x}(t_1)\mathsf{y}^*(t_2)\right\}$ (9-35)⟩,

[4] Fredholm (1900), Fredholm (1903) page 365, Michel and Herget (1993) page 97, Keener (1988) page 101

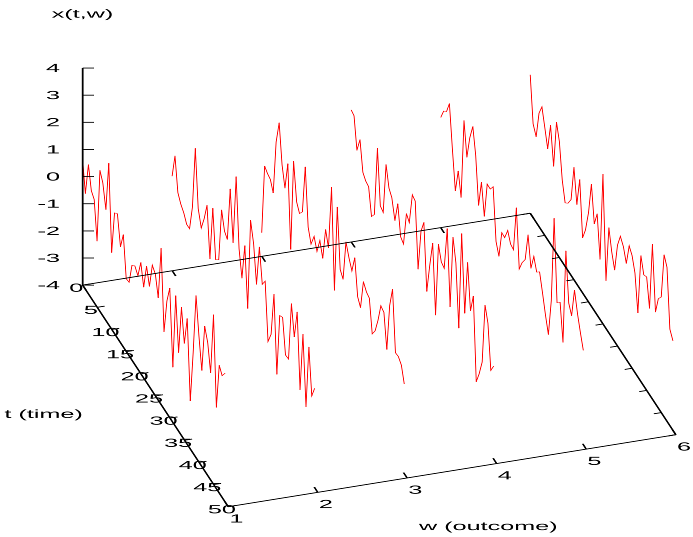


Figure J.2: Example of a random process $\mathsf{x}(t,\omega)$

J.2 Properties

Theorem J.1. *Let* $\mathsf{x}(t)$ *and* $\mathsf{y}(t)$ *be random processes with cross-correlation* $\mathsf{R}_{\mathsf{xy}}(t,u)$ *and let* $\mathsf{R}_{\mathsf{xx}}(t,u)$ *be the auto-correlation of* $\mathsf{x}(t)$.

THM

$$\begin{aligned}\mathsf{R}_{\mathsf{xx}}(t,u) &= \mathsf{R}^*_{\mathsf{xx}}(u,t) \quad \text{(CONJUGATE SYMMETRIC)}\\ \mathsf{R}_{\mathsf{xy}}(t,u) &= \mathsf{R}^*_{\mathsf{yx}}(u,t)\end{aligned}$$

✎PROOF:

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right] = \mathsf{E}\left[\mathsf{x}^*(u)\mathsf{x}(t)\right] = \left(\mathsf{E}\left[\mathsf{x}(u)\mathsf{x}^*(t)\right]\right)^* \triangleq \mathsf{R}^*_{\mathsf{xx}}(u,t)$$

$$\mathsf{R}_{\mathsf{xy}}(t,u) \triangleq \mathsf{E}\left[\mathsf{x}(t)\mathsf{y}^*(u)\right] = \mathsf{E}\left[\mathsf{y}^*(u)\mathsf{x}(t)\right] = \left(\mathsf{E}\left[\mathsf{y}(u)\mathsf{x}^*(t)\right]\right)^* \triangleq \mathsf{R}^*_{\mathsf{yx}}(u,t)$$

✏

APPENDIX K

RANDOM SEQUENCES

❝*We are quite in danger of sending highly trained and highly intelligent young men out into the world with tables of erroneous numbers under their arms, and with a dense fog in the place where their brains ought to be. In this century, of course, they will be working on guided missiles and advising the medical profession on the control of disease, and there is no limit to the extent to which they could impede every sort of national effort.*❞

Ronald A. Fisher, (1890–1962), Statistician, at a lecture in 1958 at Michigan State University [1]

K.1 Definitions

Definition K.1.

DEF *A **random sequence** is a* SEQUENCE *over a* PROBABILITY SPACE *(Definition ?? page ??).*

Definition K.2. [2] *Let* $\mathsf{x}(n)$ *and* $\mathsf{y}(n)$ *be* RANDOM SEQUENCES.

DEF

*The **mean***	$\mu_{\mathsf{x}}(n)$	*of* $\mathsf{x}(n)$ *is*	$\mu_{\mathsf{x}}(n)$	$\triangleq$	$\mathsf{E}[\mathsf{x}(n)]$
*The **variance***	$\sigma^2_{\mathsf{x}}(n)$	*of* $\mathsf{x}(n)$ *is*	$\sigma^2_{\mathsf{x}}(n)$	$\triangleq$	$\mathsf{E}\left(\left[\mathsf{x}(n) - \mu_{\mathsf{x}}(n)\right]^2\right)$
*The **cross-correlation***	$\mathsf{R}_{\mathsf{xy}}(n,m)$	*of* $\mathsf{x}(n)$ *and* $\mathsf{y}(n)$ *is*	$\mathsf{R}_{\mathsf{xy}}(n,m)$	$\triangleq$	$\mathsf{E}\left[\mathsf{x}(n+m)\mathsf{y}^*(n)\right]$
*The **auto-correlation***	$\mathsf{R}_{\mathsf{xx}}(n,m)$	*of* $\mathsf{x}(n)$ *is*	$\mathsf{R}_{\mathsf{xx}}(n,m)$	$\triangleq$	$\mathsf{R}_{\mathsf{xy}}(n,m)\big\vert_{\mathsf{y}=\mathsf{x}}$

[1] quote: Yates and Mather (1963) page 107. image: http://www.genetics.org/content/154/4/1419

[2] Papoulis (1984) page 263 ⟨$R_{xy}(m) = E\left\{\mathsf{x}(m)\mathsf{y}^*(0)\right\}$⟩, Wilks (1963) page 77 ⟨§3.4 "Moments of two-dimensional random variables"⟩, Cadzow (1987) page 341 ⟨$r_{xy}(m) = E\left[\mathsf{x}(m)\mathsf{y}^*(0)\right]$⟩, MatLab (2018b) ⟨$R_{xy}(m) = E\left\{x_{n+m}y^*_n\right\}$⟩, MatLab (2018a) ⟨$R_{xy}(m) = E\left\{x_{n+m}y^*_n\right\}$⟩

K.2 Properties

Theorem K.1.

THM
$$\begin{aligned} \mathsf{R}_{xx}(n,m) &= \mathsf{R}^*_{xx}(n+m,-m) \\ \mathsf{R}_{xy}(n,m) &= \mathsf{R}^*_{yx}(n+m,-m) \end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{xy}(n,m) &\triangleq \mathsf{E}\left[\mathsf{x}(n+m)\mathsf{y}^*(n)\right] && \text{by definition of } \mathsf{R}_{xy}(n,m) && \text{(Definition K.2 page 203)}\\
&= \mathsf{E}\left[\mathsf{y}^*(n)\mathsf{x}(n+m)\right] && \text{by } \textit{commutative} \text{ property of } (\mathbb{C},+,\cdot,0,1) && \text{(Definition A.5 page 68)}\\
&= \left(\mathsf{E}\left[\mathsf{y}(n)\mathsf{x}^*(n+m)\right]\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\textbf{algebras} && \text{(Definition E.3 page 114)}\\
&= \left(\mathsf{E}\left[\mathsf{y}(n+m-m)\mathsf{x}^*(n+m)\right]\right)^* && \text{by } \textit{additive identity} \text{ property of } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.5 page 68)}\\
&\triangleq \mathsf{R}^*_{yx}(n+m,-m) && \text{by definition of } \mathsf{R}_{xy}(n,m) && \text{(Definition K.2 page 203)}
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{xx}(n,m) &= \mathsf{R}_{xy}(n,m)\big|_{\mathsf{y}=\mathsf{x}} && \text{by } \mathsf{y}=\mathsf{x} \text{ constraint}\\
&= \mathsf{R}^*_{xy}(n+m,-m)\big|_{\mathsf{y}=\mathsf{x}} && \text{by previous result}\\
&= \mathsf{R}^*_{xx}(n+m,-m) && \text{by } \mathsf{y}=\mathsf{x} \text{ constraint}
\end{aligned}$$

✏

K.3 Wide Sense Stationary processes

Definition K.3. *Let* $\mathsf{x}(n)$ *be a* RANDOM SEQUENCE *with* MEAN $\mu_{\mathsf{x}}(n)$ *and* VARIANCE $\sigma^2(n)$ *(Definition K.2 page 203).*

DEF
$\mathsf{x}(n)$ *is **wide sense stationary** (WSS) if*

1. $\mu_{\mathsf{x}}(n)$ *is* CONSTANT *with respect to* n (STATIONARY IN THE 1ST MOMENT) *and*
2. $\sigma^2_{\mathsf{x}}(n)$ *is* CONSTANT *with respect to* n (STATIONARY IN THE 2ND MOMENT)

Definition K.4. [3] *Let* $\mathsf{x}(n)$ *be a* RANDOM SEQUENCE *with statistics* $\mu_{\mathsf{x}}(n)$, $\sigma^2_{\mathsf{x}}(n)$, $\mathsf{R}_{xx}(n,m)$, *and* $\mathsf{R}_{xy}(n,m)$ *(Definition K.2 page 203).*

DEF
$\{$ x *and* y *are* WIDE SENSE STATIONARY $\}$ $\Longrightarrow$

(1).	*The **mean***	μ_{x}	*of* $\mathsf{x}(n)$ *is*	μ_{x}	$\triangleq \mathsf{E}[\mathsf{x}(0)]$
(2).	*The **variance***	σ^2_{x}	*of* $\mathsf{x}(n)$ *is*	σ^2_{x}	$\triangleq \mathsf{E}\left(\left[\mathsf{x}(0)-\mu_{\mathsf{x}}\right]^2\right)$
(4).	*The **cross-correlation***	$\mathsf{R}_{xy}(m)$	*of* $\mathsf{x}(n)$ *and* $\mathsf{y}(n)$ *is*	$\mathsf{R}_{xy}(m)$	$\triangleq \mathsf{E}\left[\mathsf{x}(m)\mathsf{y}^*(0)\right]$
(3).	*The **auto-correlation***	$\mathsf{R}_{xx}(m)$	*of* $\mathsf{x}(n)$ *is*	$\mathsf{R}_{xx}(m)$	$\triangleq \mathsf{R}_{xy}(m)\big\vert_{\mathsf{y}=\mathsf{x}}$

Remark K.1. The $\mathsf{R}_{xy}(n,m)$ of Definition K.2 (page 203) and the $\mathsf{R}_{xy}(m)$ of Definition K.4 (page 204) (etc.) are examples of *function overload*—that is, functions that use the same mnemonic but are distinguished by different domains. Perhaps a more common example of function overload is the "+" mnemonic. Traditionally it is used with domain of the natural numbers $\mathbb{N}$ as in $3+2$. Later it was extended for domain real numbers $\mathbb{R}$ as in $\sqrt{3}+\sqrt{2}$, or even complex numbers $\mathbb{C}$ as in

[3] Papoulis (1984) page 263 ⟨"$R_{xy}(\tau) = E\{\mathsf{x}(t+\tau)\mathsf{y}^*(t)\}$"⟩, Cadzow (1987) page 341 ⟨$r_{xy}(n) = E\left[\mathsf{x}(k+n)\mathsf{y}^*(k)\right]$ (10.41)⟩

$\left(\sqrt{3}+i\sqrt{2}\right)+(e+i\pi)$. And it was even more dramatically extended for use with domain $\mathbb{R}^N \times \mathbb{R}^M$ in "linear algebra" as in

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Remark K.2. [4] The definition for $\mathsf{R}_{xy}(m)$ can be defined with the conjugate $*$ on either x or y, or on neither or both; and moreover x may either lead or lag y. In total, there are $2\times2\times2=8$ different ways to define $\mathsf{R}_{xy}(m)$.[5] and $\mathsf{R}_{xx}(m)$ involve complex numbers. This may seem curious when typical ADCs provide real-valued sequences. Note however that complex-valued sequences often come up in signal processing due to some common system architectures:

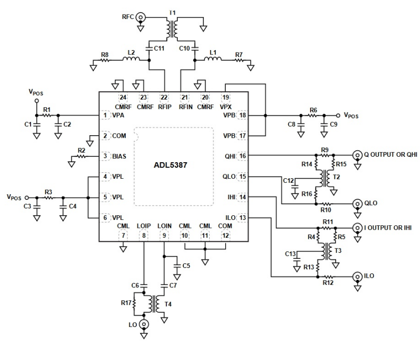
1. The presence of an *FFT* operator in the signal processing path
2. The *complex envelope* $\mathsf{x}_l(t)$ of a modulated *narrowband* communications signal $\mathsf{x}(t)$.
3. Communications channel processing involving phase discrimination (e.g. PSK and QAM).

In the case of a narrowband signal $\mathsf{x}(t)$ modulated by a sinusoid at center frequence f_c, we have three canonical forms. These can be shown to be equivalent:

$$\begin{aligned}
\mathsf{x}(t) &\triangleq \underbrace{\boxed{\mathsf{a}(t)\cos\left[2\pi f_c t+\phi(t)\right]}}_{\text{amplitude-phase form}} && \text{amplitude and phase form} \\
&= \underbrace{\mathsf{a}(t)\cos[\phi(t)]}_{\mathsf{p}(t)}\cos[2\pi f_c t] - \underbrace{\mathsf{a}(t)\sin[\phi(t)]}_{\mathsf{q}(t)}\sin[2\pi f_c t] && \text{by \textit{double angle formulas}} \quad \text{(Theorem C.9 page 85)} \\
&= \underbrace{\boxed{\mathsf{p}(t)\cos[2\pi f_c t]-\mathsf{q}(t)\sin[2\pi f_c t]}}_{\text{quadrature form}} && \text{quadrature form} \\
&= \mathbf{R}_e\left([\mathsf{p}(t)+i\mathsf{q}(t)][\cos(2\pi f_c t)+i\sin(2\pi f_c t)]\right) && \text{by definitions of } \mathbf{R}_e \\
&= \underbrace{\boxed{\mathbf{R}_e\left[\mathsf{x}_l(t)e^{i2\pi f_c t}\right]}}_{\text{complex envelope form}} && \text{by \textit{Euler's identity}} \quad \text{(Theorem C.5 page 80)}
\end{aligned}$$

Note that in these equivalent forms, the *complex envelope* $\mathsf{x}_l(t)$ is conveniently represented as a *complex-valued* function in terms of the *quadrature component* $\mathsf{p}(t)$ and the *inphase component* $\mathsf{q}(t)$ such that $\mathsf{x}_l(t)=\mathsf{p}(t)+i\mathsf{q}(t)$.

In practice (with real hardware), you will likely first have access to the quadrature components $\mathsf{p}(t)$ and $\mathsf{q}(t)$. Take for example the *Analog Devices ADL5387 Quadrature Demodulator* and evaluation board, as illustrated to the right.[6]Note that *quadrature component* $\mathsf{p}(t)$ is available at connector "Q OUTPUT" and *inphase component* $\mathsf{q}(t)$ is available at connector "I OUTPUT".



Proposition K.1. *Let* $\mathsf{y}(n)$ *be a* RANDOM SEQUENCE, $\mathsf{x}(n)$ *a* RANDOM SEQUENCE *with* AUTO-CORRELATION

[4] S. Lawrence Marple (1987) pages 51–53 ⟨"Appendix 2.A Source of Complex-Valued Signals"⟩, S. Lawrence Marple (2019) pages 48–50 ⟨§"2.12 Extra: Source of Complex-Valued Signals"⟩, Greenhoe (2019b) ⟨Chapter 2: Narrowband Signals⟩

[5] Greenhoe (2019a)

[6] diagram copied from Devices (2016)

$\mathsf{R_{xx}}(n,m)$, *and* $\mathsf{R_{xy}}$ *the* CROSS-CORRELATION *of* x *and* y.

PRP

$$\left\{\begin{array}{l}\mathsf{x} \text{ and } \mathsf{y} \text{ are} \\ \text{WIDE SENSE STATIONARY} \\ \text{(WSS) (Definition ?? page ??)}\end{array}\right\} \implies \left\{\begin{array}{rcll}\mathsf{R_{xx}}(n,m) &=& \mathsf{R_{xx}}(m) & \forall n\in\mathbb{Z} \\ \mathsf{R_{xy}}(n,m) &=& \mathsf{R_{xy}}(m) & \forall n\in\mathbb{Z} \\ \text{(Definition K.2 page 203)} && \text{(Definition K.4 page 204)} &\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R_{xy}}(n,m) &\triangleq \mathsf{E}\left[\mathsf{x}[n+m]\mathsf{y}^*[n]\right] && \text{by definition of } \mathsf{R_{xy}}(n,m) && \text{(Definition K.2 page 203)}\\
&= \mathsf{E}\left[\mathsf{x}[n-n+m]\mathsf{y}^*[n-n]\right] && \text{by \textit{wide sense stationary} hypothesis}\\
&= \mathsf{E}\left[\mathsf{x}[m]\mathsf{y}^*[0]\right]\\
&\triangleq \mathsf{R_{xy}}(m) && \text{by definition of } \mathsf{R_{xy}}(m) && \text{(Definition K.4 page 204)}\\
\mathsf{R_{xx}}(n,m) &= \mathsf{R_{xy}}(n,m)\big|_{\mathsf{y}=\mathsf{x}}\\
&= \mathsf{R_{xy}}(m)\big|_{\mathsf{y}=\mathsf{x}} && \text{by previous result}\\
&= \mathsf{R_{xx}}(m)
\end{aligned}$$

✏

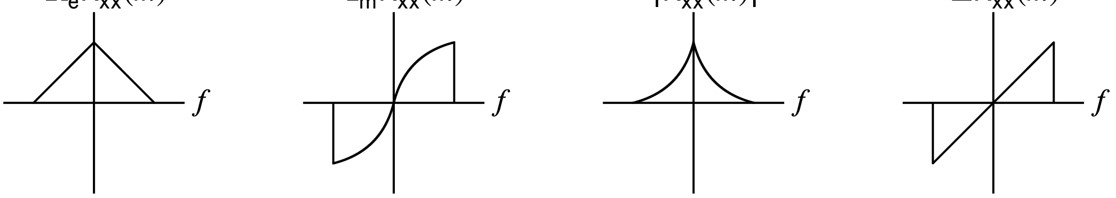


Figure K.1: *auto-correlation* $\mathsf{R_{xx}}(m)$

Corollary K.1. *Let* $\mathsf{x}(n)$ *be a* RANDOM SEQUENCE *with* AUTO-CORRELATION $\mathsf{R_{xx}}(n,m)$, $\mathsf{y}(n)$ *a* RANDOM SEQUENCE *with* AUTO-CORRELATION $\mathsf{R_{yy}}(n,m)$, *and* $\mathsf{R_{xy}}(n,m)$ *the* CROSS-CORRELATION *of* x *and* y. *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathsf{x}(n)$ *and output* $\mathsf{y}(n)$.

COR

$$\left\{\begin{array}{lll}\text{(A).} & \mathsf{x} \text{ is WSS} & \text{and}\\ \text{(B).} & \mathsf{y} \text{ is WSS} & \text{and}\\ \text{(C).} & \mathbf{S} \text{ is LTI} & \end{array}\right\} \implies \left\{\begin{array}{lrcl ll}
\text{(1).} & \mathsf{R_{xy}}(m) &=& \mathsf{R^*_{yx}}(-m) & & \text{and}\\
\text{(2).} & \mathsf{R_{xx}}(m) &=& \mathsf{R^*_{xx}}(-m) & \text{(CONJUGATE SYMMETRIC)} & \text{and}\\
\text{(3).} & \mathbf{R_e}\mathsf{R_{xx}}(m) &=& \mathbf{R_e}\mathsf{R_{xx}}(-m) & \text{(SYMMETRIC)} & \text{and}\\
\text{(4).} & \mathbf{I_m}\mathsf{R_{xx}}(m) &=& -\mathbf{I_m}\mathsf{R_{xx}}(-m) & \text{(ANTI-SYMMETRIC)} & \text{and}\\
\text{(5).} & \left|\mathsf{R_{xx}}(m)\right| &=& \left|\mathsf{R_{xx}}(-m)\right| & \text{(SYMMETRIC)} & \text{and}\\
\text{(6).} & \angle\mathsf{R_{xx}}(m) &=& -\angle\mathsf{R_{xx}}(-m) & \text{(ANTI-SYMMETRIC)} &
\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R_{xy}}(m) &= \mathsf{R_{xy}}(n,m) && \text{by Proposition K.1 page 205} && \text{and hypotheses (A),(B)}\\
&= \mathsf{R^*_{yx}}(n+m,-m) && \text{by Theorem K.1 page 204} && \text{and hypothesis (B)}\\
&= \mathsf{R^*_{yx}}(-m) && \text{by Proposition K.1 page 205} && \text{and hypothesis (A)}\\
\mathsf{R_{xx}}(m) &= \mathsf{R_{xx}}(n,m) && \text{by Proposition K.1 page 205} && \text{and hypothesis (A)}\\
&= \mathsf{R^*_{xx}}(n+m,-m) && \text{by Theorem K.1 page 204} && \text{and hypothesis (B)}\\
&= \mathsf{R^*_{xx}}(-m) && \text{by Proposition K.1 page 205} && \text{and hypothesis (A)}
\end{aligned}$$

✏

K.4 Spectral density

Definition K.5. *Let* $\mathsf{x}(n)$ *and* $\mathsf{y}(n)$ *be* WIDE SENSE STATIONARY RANDOM SEQUENCES *with auto-correlation* $\mathsf{R}_{\mathsf{xx}}(m)$ *and cross-correlation* $\mathsf{R}_{\mathsf{xy}}(m)$. *Let* $\mathbf{Z}$ *be the* Z-TRANSFORM OPERATOR *(Definition 5.4 page 28).*

DEF

The ***z-domain cross spectral density*** ***(CSD)*** $\check{\mathsf{S}}_{\mathsf{xy}}(z)$ *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is*

$$\check{\mathsf{S}}_{\mathsf{xy}}(z) \triangleq \mathbf{Z}\mathsf{R}_{\mathsf{xy}}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(m) z^{-m}$$

The ***z-domain power spectral density*** ***(PSD)*** $\check{\mathsf{S}}_{\mathsf{xx}}(z)$ *of* $\mathbf{x}$ *is* $\quad \check{\mathsf{S}}_{\mathsf{xx}}(z) \triangleq \check{\mathsf{S}}_{\mathsf{xy}}(z)\big|_{\mathsf{y}(n)=\mathsf{x}(n)}$

Definition K.6. *Let* $\mathsf{x}(n)$ *and* $\mathsf{y}(n)$ *be* WIDE SENSE STATIONARY RANDOM SEQUENCES *with auto-correlation* $\mathsf{R}_{\mathsf{xx}}(m)$ *and cross-correlation* $\mathsf{R}_{\mathsf{xy}}(m)$. *Let* $\check{\mathbf{F}}$ *be the* DISCRETE TIME FOURIER TRANSFORM *(DTFT) operator (Definition 6.1 page 41).*

DEF

The ***auto-spectral density*** $\check{\mathsf{S}}_{\mathsf{xx}}(z)$ *of* $\mathbf{x}$ *is* $\quad \tilde{\mathsf{S}}_{\mathsf{xx}}(z) \triangleq \check{\mathbf{F}}\mathsf{R}_{\mathsf{xx}}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xx}}(m) e^{-i\omega m}$

The ***cross spectral density*** ***(CSD)*** $\check{\mathsf{S}}_{\mathsf{xy}}(z)$ *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is* $\quad \tilde{\mathsf{S}}_{\mathsf{xy}}(z) \triangleq \check{\mathbf{F}}\mathsf{R}_{\mathsf{xy}}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(m) e^{-i\omega m}$

The ***auto-spectral density*** *is also called* ***power spectral density*** ***(PSD)***.

Theorem K.2. *Let* $\mathbf{S}$ *be a system with* IMPULSE RESPONSE $\mathsf{h}(n)$, INPUT $\mathsf{x}(n)$, *and* OUTPUT $\mathsf{y}(n)$.

THM

$$\left\{ \mathsf{x} \text{ and } \mathsf{y} \text{ are WIDE SENSE STATIONARY} \right\} \implies \left\{ \begin{array}{llcll} (1). & \check{\mathsf{S}}_{\mathsf{xx}}(z) & = & \check{\mathsf{S}}^*_{\mathsf{xx}}\left(\frac{1}{z^*}\right) & \text{and} \\ (2). & \check{\mathsf{S}}_{\mathsf{yx}}(z) & = & \check{\mathsf{S}}^*_{\mathsf{xy}}\left(\frac{1}{z^*}\right) & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\check{\mathsf{S}}_{\mathsf{yx}}(z) &\triangleq \mathbf{Z}\mathsf{R}_{\mathsf{yx}}(m) && \text{by definition of } \check{\mathsf{S}}_{\mathsf{xy}}(z) && \text{(Definition K.6 page 207)} \\
&\triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{yx}}(m) z^{-m} && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)} \\
&\triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}^*_{\mathsf{xy}}(-m) z^{-m} && \text{by Corollary K.1 page 206} \\
&= \left[\sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(-m)(z^*)^{-m}\right]^* && \text{by \textit{antiautomorphic} property of *-algebras} && \text{(Definition E.3 page 114)} \\
&= \left[\sum_{-p\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(p)(z^*)^{p}\right]^* && \text{where } p \triangleq -m && \implies m = -p \\
&= \left[\sum_{p\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(p)(z^*)^{p}\right]^* && \text{by \textit{absolutely summable} property} && \text{(Definition 5.2 page 27)} \\
&= \left[\sum_{p\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(p)\left(\frac{1}{z^*}\right)^{-p}\right]^* \\
&= \check{\mathsf{S}}^*_{\mathsf{xy}}\left(\frac{1}{z^*}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition 5.4 page 28)} \\
\check{\mathsf{S}}_{\mathsf{xx}}(z) &= \check{\mathsf{S}}_{\mathsf{xy}}(z)\big|_{\mathsf{y}=\mathsf{x}} \\
&= \check{\mathsf{S}}^*_{\mathsf{yx}}(z)\big|_{\mathsf{y}=\mathsf{x}} \\
&= \check{\mathsf{S}}^*_{\mathsf{xy}}\left(\frac{1}{z^*}\right)\bigg|_{\mathsf{y}=\mathsf{x}} && \text{by (2)—previous result}
\end{aligned}$$

$$= \check{S}^*_{xx}\left(\frac{1}{z^*}\right)$$

✏

Corollary K.2. *Let* S *be a system with* IMPULSE RESPONSE $h(n)$, INPUT $x(n)$, *and* OUTPUT $y(n)$.

COR

$$\left\{\begin{array}{lll} (A). & h \text{ is LTI} & \text{and} \\ (B). & x \text{ and } y \text{ are WSS} & \end{array}\right\} \implies \left\{\begin{array}{llllll} (1). & \tilde{S}^*_{xy}(\omega) & = & \tilde{S}_{yx}(\omega) & \text{(CONJUGATE-SYMMETRIC)} & \text{and} \\ (2). & \tilde{S}^*_{xx}(\omega) & = & \tilde{S}_{xx}(\omega) & \text{(CONJUGATE SYMMETRIC)} & \text{and} \\ (3). & \tilde{S}_{xx}(\omega) & \in & \mathbb{R} & \text{(REAL-VALUED)} & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\tilde{S}^*_{xy}(\omega) &= \check{S}^*_{xy}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
&= \check{S}^{**}_{yx}\left(\frac{1}{z^*}\right)\Big|_{z=e^{i\omega}} && \text{by Theorem K.2 page 207} \\
&= \check{S}_{yx}\left(\frac{1}{z^*}\right)\Big|_{z=e^{i\omega}} && \text{by } involutory \text{ property of } *\textbf{-algebras} && \text{(Definition E.3 page 114)} \\
&= \check{S}_{yx}\left(\frac{1}{e^{i\omega *}}\right) \\
&= \check{S}_{yx}\left(e^{i\omega}\right) \\
&= \tilde{S}_{yx}(\omega) && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
\tilde{S}^*_{xx}(\omega) &= \check{S}^*_{xx}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
&= \check{S}^{**}_{xx}\left(\frac{1}{z^*}\right)\Big|_{z=e^{i\omega}} && \text{by Theorem K.2 page 207} \\
&= \check{S}_{xx}\left(\frac{1}{z^*}\right)\Big|_{z=e^{i\omega}} && \text{by } involutory \text{ property of } *\textbf{-algebras} && \text{(Definition E.3 page 114)} \\
&= \check{S}_{xx}\left(\frac{1}{e^{i\omega *}}\right) \\
&= \check{S}_{xx}\left(e^{i\omega}\right) \\
&= \tilde{S}_{xx}(\omega) && \text{by definition of } DTFT && \text{(Definition 6.1 page 41)} \\
&\implies \tilde{S}_{xx}(\omega) \text{ is } real\text{-}valued \\
\tilde{S}^*_{xx}(\omega) &= \tilde{S}^*_{xy}(\omega)\Big|_{y=x} \\
&= \tilde{S}_{yx}(\omega)\Big|_{y=x} && \text{by previous result} \\
&= \tilde{S}_{xx}(\omega)
\end{aligned}$$

✏

K.5 Spectral Power

The term "*spectral power*" is a bit of an oxymoron because "spectral" deals with leaving the time-domain for the frequency-domain, howbeit the concept of power is solidly founded on the concept of time in that power = energy per time.

However, *Parseval's Theorem* (Proposition G.2 page 160) demonstrates that power in time can also be calculated in frequency. So, it makes some sense to speak of the term "spectral power". Moreover, one way to estimate this power is to average the Fourier Transforms of the product $|x(n)|^2 = x(n)x^*(n)$...that is, to use an estimate of the auto-spectral density $\tilde{S}_{xx}(\omega)$. Thus, an alternate name for *auto-spectral density* is **power spectral density** (PSD).

APPENDIX L

SPECTRAL THEORY

L.1 Operator Spectrum

Definition L.1. [1] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator over the linear spaces* $\boldsymbol{X} = (X, F, \oplus, \otimes)$ *and* $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$. *Let* $\mathcal{N}(\mathbf{A})$ *be the* NULL SPACE *of* $\mathbf{A}$.

DEF

An ***eigenvalue*** *of* $\mathbf{A}$ *is any value* λ *such that there exists* $\mathbf{x}$ *such that* $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.
The ***eigenspace*** $\boldsymbol{H}_\lambda$ *of* $\mathbf{A}$ *at eigenvalue* λ *is* $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$.
An ***eigenvector*** *of* $\mathbf{A}$ *associated with eigenvalue* λ *is any element of* $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$.

Example L.1. [2] Let $\mathbf{D}$ be the differntial operator.

EX

The set $\left\{e^{\lambda x} | \lambda \in \mathbb{C}\right\}$ are the eigenvectors of $\mathbf{D}$.

$$\begin{array}{rcll} \rho(\mathbf{D}) & = & \emptyset & \text{($\mathbf{D}$ has no non-spectral points whatsoever)} \\ \sigma_\mathsf{p}(\mathbf{D}) & = & \sigma(\mathbf{D}) & \text{(the spectrum of $\mathbf{D}$ is all eigenvalues)} \\ \sigma_\mathsf{c}(\mathbf{D}) & = & \emptyset & \text{($\mathbf{D}$ has no continuous spectrum)} \\ \sigma_\mathsf{r}(\mathbf{D}) & = & \emptyset & \text{($\mathbf{D}$ has no resolvent spectrum)} \end{array}$$

✎PROOF:

$$\begin{aligned} (\mathbf{D} - \lambda\mathbf{I})e^{\lambda x} &= \mathbf{D}e^{\lambda x} - \lambda\mathbf{I}e^{\lambda x} \\ &= \lambda e^{\lambda x} - \lambda e^{\lambda x} \\ &= 0 && \forall \lambda \in \mathbb{C} \end{aligned}$$

This theorem and proof needs more work and investigation to prove/disprove its claims. ✏

Definition L.2. [3] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator over the linear spaces* $\boldsymbol{X} = (X, F, \oplus, \otimes)$ *and* $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$.

[1] Bollobás (1999) page 168, Descartes (1637a), Descartes (1954), Cayley (1858), Hilbert (1904) page 67, Hilbert (1912),
[2] Pedersen (2000) page 79
[3] Michel and Herget (1993) page 439

quantity		$\mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\}$ ($x = \mathbb{0}$ is the only solution)	$\overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X}$ (dense)	$(\mathbf{A}-\lambda\mathbf{I})^{-1} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ (continuous/bounded)
$\rho(\mathbf{A})$	(resolvent set)	1	1	1
$\sigma_p(\mathbf{A})$	(point spectrum)	0		
$\sigma_r(\mathbf{A})$	(residual spectrum)	1	0	
$\sigma_c(\mathbf{A})$	(continuous spectrum)	1	1	0

Table L.1: Spectrum of an operator $\mathbf{A}$

DEF

*The **resolvent set** $\rho(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\rho(\mathbf{A}) \triangleq \left\{ \lambda \in F \;\middle|\; \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \text{(no non-zero eigenvectors)} \quad \text{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X} & \text{(the range is dense in } \boldsymbol{X}\text{).} \quad \text{and} \\ 3. & (\mathbf{A}-\lambda\mathbf{I})^{-1} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y}) & \text{(inverse is continuous/bounded).} \end{array} \right\}$$

*The **spectrum** $\sigma(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma(\mathbf{A}) \triangleq F \backslash \rho(\mathbf{A}).$$

Definition L.3. [4] *Let $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be an operator over the linear spaces $\boldsymbol{X} = (X, F, \oplus, \otimes)$ and $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$.*

DEF

*The **point spectrum** $\sigma_p(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

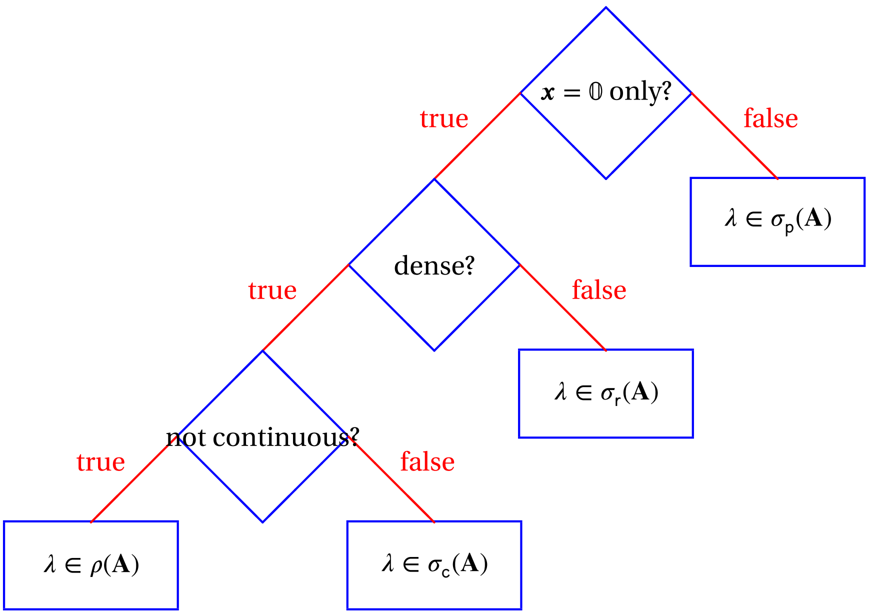
$$\sigma_p(\mathbf{A}) \triangleq \left\{ \lambda \in F \;\middle|\; 1.\ \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) \supsetneq \{\mathbb{0}\} \quad \text{(has non-zero eigenvector)} \right\}$$

*The **residual spectrum** $\sigma_r(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma_r(\mathbf{A}) \triangleq \left\{ \lambda \in F \;\middle|\; \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \text{(no non-zero eigenvectors)} \quad \text{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} \neq \boldsymbol{X} & \text{(not dense in } \boldsymbol{X}\text{—has gaps).} \end{array} \right\}$$

*The **continuous spectrum** $\sigma_c(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma_c(\mathbf{A}) \triangleq \left\{ \lambda \in F \;\middle|\; \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \text{(no non-zero eigenvectors)} \quad \text{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X} & \text{(dense in } \boldsymbol{X}\text{).} \quad \text{and} \\ 3. & (\mathbf{A}-\lambda\mathbf{I})^{-1} \notin \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y}) & \text{(not continuous / not bounded)} \end{array} \right\}$$



The spectral components' definitions are illustrated in the figure to the left and summarized in Table L.1 (page 210). Let a family of operators $\mathbf{B}(\lambda)$ be defined with respect to an operator $\mathbf{A}$ such that $\mathbf{B}(\lambda) \triangleq (\mathbf{A} - \lambda\mathbf{I})$. Normally, we might expect a "normal" or "regular" or even "mundane" operator $\mathbf{B}(\lambda)$ to have the properties

1. $\mathbf{B}(\lambda)\boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \mathbb{0}$
2. $\mathbf{B}(\lambda)\boldsymbol{x}$ spans virtually all of $\boldsymbol{X}$ as we vary $\boldsymbol{x}$
3. $\mathbf{B}^{-1}(\lambda)$ is continuous.

After all, these are the properties that we would have if $\mathbf{B}(\lambda)$ were simply an affine operator in the

[4] Bollobás (1999) page 168, Hilbert (1906) pages 169–172

field of real numbers— such as $[\mathbf{B}(\lambda)](x) \triangleq [\lambda](x) = \lambda x$ which is 0 if and only if $x = 0$, has range $\mathcal{R}(\lambda) = \mathbb{R}$, and its inverse $\lambda^{-1}x$ is continuous.

If for some λ the operator $\mathbf{B}(\lambda)$ does have all these "regular" properties, then that λ part of the *resolvent set* of $\mathbf{A}$ and λ is called *regular*. However if for some λ the operator $\mathbf{B}(\lambda)$ fails any of these conditions, then that λ part of the *spectrum* of $\mathbf{A}$. And which conditions it fails determines which component of the spectrum it is in.

Theorem L.1. [5] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator.*

THM
$$\sigma(\mathbf{A}) = \sigma_{\mathsf{p}}(\mathbf{A}) \cup \sigma_{\mathsf{c}}(\mathbf{A}) \cup \sigma_{\mathsf{r}}(\mathbf{A})$$

Theorem L.2 (Spectral Theorem). [6] *Let* $\mathbf{N} \in Y^X$ *be an operator.*

THM
$$\left.\begin{array}{ll}\text{(A).} & \underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \text{ is NORMAL}} \\ \text{(B).} & \mathbf{N} \text{ is COMPACT}\end{array}\right\} \implies \left\{\begin{array}{lrcl}\text{(1).} & \mathbf{N} &=& \sum_n \lambda_n \mathbf{P}_n \\ \text{(2).} & \sum_n \mathbf{P}_n &=& \mathbf{I} \\ \text{(3).} & \mathbf{P}_n\mathbf{P}_m &=& \bar{\delta}_{n-m}\mathbf{P}_n \\ \text{(4).} & \dim(\boldsymbol{H}_n) &<& \infty \\ \text{(5).} & \multicolumn{3}{l}{\left|\left\{\lambda_n | \lambda_n \neq 0\right\}\right| \text{ is COUNTABLY INFINITE}}\end{array}\right.$$

where

$$\begin{array}{rcll}(\!(\lambda_n)\!)_{n\in\mathbb{Z}} &\triangleq& \sigma_{\mathsf{p}}(\mathbf{N}) & \text{(eigenvalues of } \mathbf{N}) \\ \boldsymbol{H}_n &\triangleq& \mathcal{N}(\mathbf{N} - \lambda_n \mathbf{I}) & (\lambda_n \text{ is the eigenspace of } \mathbf{N} \text{ at } \lambda_n \text{ in } \boldsymbol{Y}) \\ \boldsymbol{H}_n &=& \mathbf{P}_n \boldsymbol{Y} & (\mathbf{P}_n \text{ is the projection operator that generates } \boldsymbol{H}_n)\end{array}$$

L.2 Fredholm kernels

Definition L.4. [7]

DEF
*A **Fredholm operator** $\mathbf{K}$ is defined as*

$$[\mathbf{K}\mathsf{f}](t) \triangleq \underbrace{\int_a^b \underbrace{\kappa(t,s)}_{\text{kernel}} \mathsf{f}(s) \,\mathsf{d}s}_{\text{Fredholm integral equation of the first kind [8]}} \qquad \forall \mathsf{f} \in \boldsymbol{L}_2([a,b])$$

Example L.2. Examples of Fredholm operators include

1.	Fourier Transform	$[\tilde{\mathbf{F}}\mathsf{x}](f)$	$=$	$\int_t \mathsf{x}(t)e^{-i2\pi ft}\,\mathsf{d}t$	$\kappa(t,f)$	$=$	$e^{-i2\pi ft}$
2.	Inverse Fourier Transform	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t)$	$=$	$\int_f \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathsf{d}f$	$\kappa(f,t)$	$=$	$e^{i2\pi ft}$
3.	Laplace operator	$[\mathbf{L}\mathsf{x}](s)$	$=$	$\int_t \mathsf{x}(t)e^{-st}\,\mathsf{d}t$	$\kappa(t,s)$	$=$	e^{-st}
4.	autocorrelation operator	$[\mathbf{R}\mathsf{x}](t)$	$=$	$\int_s R(t,s)\mathsf{x}(s)\,\mathsf{d}s$	$\kappa(t,s)$	$=$	$R(t,s)$

Theorem L.3. *Let* $\mathbf{K}$ *be a Fredholm operator with kernel* $\kappa(t,s)$ *and adjoint* $\mathbf{K}^*$.

THM
$$[\mathbf{K}\mathsf{f}](t) = \int_A \kappa(t,s)\mathsf{f}(s)\,\mathsf{d}s \iff \left[\mathbf{K}^*\mathsf{f}\right](t) = \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathsf{d}s$$

[5] Michel and Herget (1993) page 440

[6] Michel and Herget (1993) page 457, Bollobás (1999) page 200, Hilbert (1906), Hilbert (1912), von Neumann (1929), de Witt (1659)

[7] Michel and Herget (1993) page 425

[8] The equation $\int_u \kappa(t,s)\mathsf{f}(s)\,\mathsf{d}s$ is a **Fredholm integral equation of the first kind** and $\kappa(t,u)$ is the **kernel** of the equation. References: Fredholm (1900), Fredholm (1903) page 365, Michel and Herget (1993) page 97, Keener (1988) page 101

✎PROOF:

$$
\begin{aligned}
[\mathbf{K}\mathsf{f}](t) &= \int_A \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s \\
\iff \langle [\mathbf{K}\mathsf{f}](t) \mid \mathsf{g}(t)\rangle &= \left\langle \int_s \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s \mid \mathsf{g}(t) \right\rangle && \text{by left hypothesis} \\
&= \int_s \mathsf{f}(s)\,\langle \kappa(t,s) \mid \mathsf{g}(t)\rangle\,\mathrm{d}s && \text{by additivity property of } \langle \triangle \mid \nabla \rangle \\
&= \int_s \mathsf{f}(s)\,\langle \mathsf{g}(t) \mid \kappa(t,s)\rangle^*\,\mathrm{d}s && \text{by conjugate symmetry property of } \langle \triangle \mid \nabla \rangle \\
&= \langle \mathsf{f}(s) \mid \langle \mathsf{g}(t) \mid \kappa(t,s)\rangle\rangle && \text{by local definition of } \langle \triangle \mid \nabla \rangle \\
&= \left\langle \mathsf{f}(s) \mid \underbrace{\int_t \kappa^*(t,s)\mathsf{g}(t)\,\mathrm{d}t}_{[\mathbf{K}^*\mathsf{g}](s)} \right\rangle && \text{by local definition of } \langle \triangle \mid \nabla \rangle \\
\iff [\mathbf{K}^*\mathsf{g}](s) &= \int_A \kappa^*(t,s)\mathsf{g}(t)\,\mathrm{d}t && \text{by right hypothesis} \\
\iff [\mathbf{K}^*\mathsf{g}](\sigma) &= \int_A \kappa^*(\tau,\sigma)\mathsf{g}(\tau)\,\mathrm{d}\tau && \text{by change of variable: } \tau = t,\ \sigma = s \\
\iff [\mathbf{K}^*\mathsf{f}](t) &= \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathrm{d}s && \text{by change of variable: } t = \sigma,\ s = \tau,\ \mathsf{f} = \mathsf{g}
\end{aligned}
$$

✏

Corollary L.1. [9] *Let* $\mathbf{K}$ *be an Fredholm operator with kernel* $\kappa(t,s)$ *and adjoint* $\mathbf{K}^*$.

COR

$$\underbrace{\mathbf{K} = \mathbf{K}^*}_{\mathbf{K}\text{ is self-adjoint}} \iff \underbrace{\kappa(t,s) = \kappa^*(s,t)}_{\text{kernel is conjugate symmetric}}$$

✎PROOF:

$$
\begin{aligned}
\mathbf{K} = \mathbf{K}^* &\iff \int_A \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s = \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathrm{d}s && \text{by Theorem L.3 page 211} \\
&\iff \kappa(t,s) = \kappa^*(s,t)
\end{aligned}
$$

✏

Theorem L.4 (Mercer's Theorem). [10] *Let* $\mathbf{K}$ *be an Fredholm operator with kernel* $\kappa(t,s)$ *and eigensystem* $\left(\!\left((\lambda_n, \phi_n(t))\right)\!\right)_{n\in\mathbb{Z}}$.

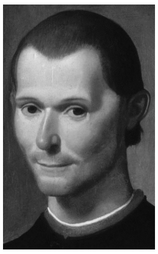
THM

$$
\left\{
\begin{array}{ll}
\text{(A).} & \underbrace{\int_a^b \int_a^b \kappa(t,s)\mathsf{f}(t)\mathsf{f}^*(s)\,\mathrm{d}t \geq 0}_{\textit{positive}} \quad \textit{and} \\
\text{(B).} & \kappa(t,s) \textit{ is } \text{CONTINUOUS} \textit{ on} \\
 & [a:b] \times [a:b]
\end{array}
\right\}
\implies
\left\{
\begin{array}{ll}
\text{(1).} & \kappa(t,s) = \sum_n \lambda_n \phi_n(t)\phi_n^*(s) \quad \textit{and} \\
\text{(2).} & \kappa(t,s) \text{ CONVERGES ABSOLUTELY} \\
 & \textit{and } \text{UNIFORMLY } \textit{on} \\
 & [a:b] \times [a:b]
\end{array}
\right\}
$$

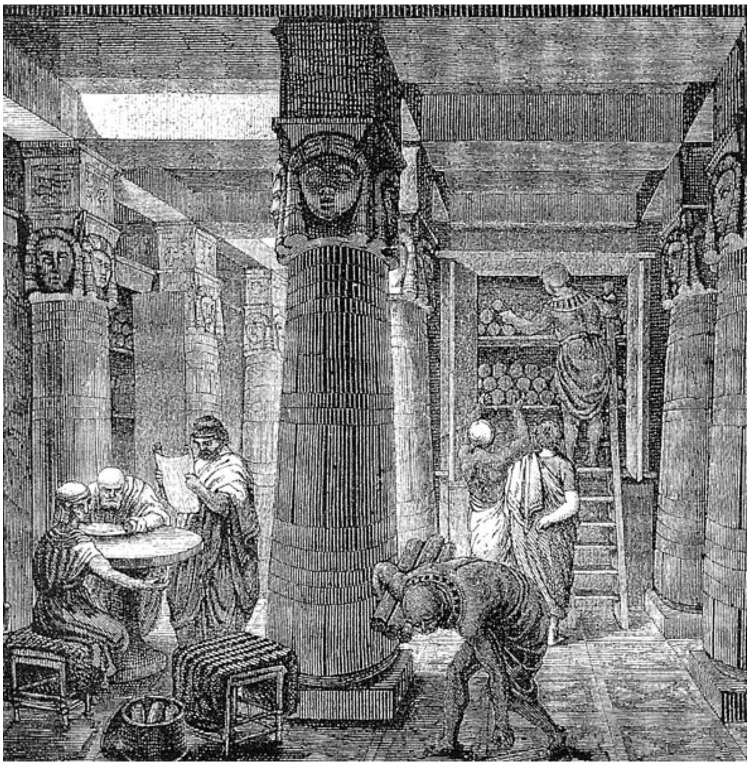
[9] Michel and Herget (1993) page 430
[10] Gohberg et al. (2003) page 198, Courant and Hilbert (1930) pages 138–140, Mercer (1909) page 439

Back Matter

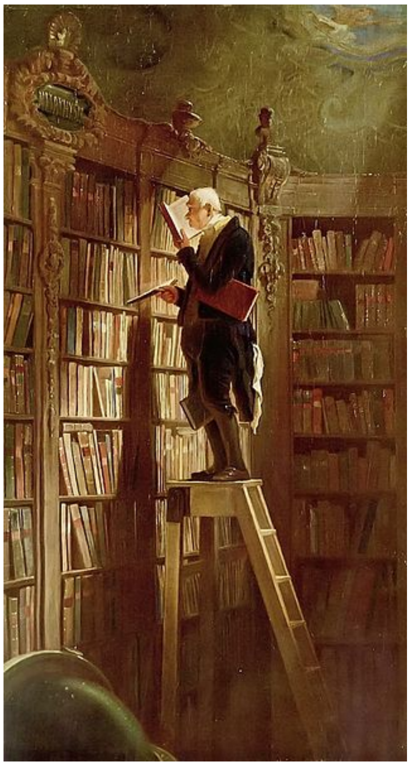
❝*It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.*❞
Niels Henrik Abel (1802–1829), Norwegian mathematician [11]



❝*When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.*❞
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [12]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850

[13]

[11] quote: Simmons (2007) page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

[12] quote: Machiavelli (1961) page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

[13] http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

❝*To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.*❞
Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [14]

[14] quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

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