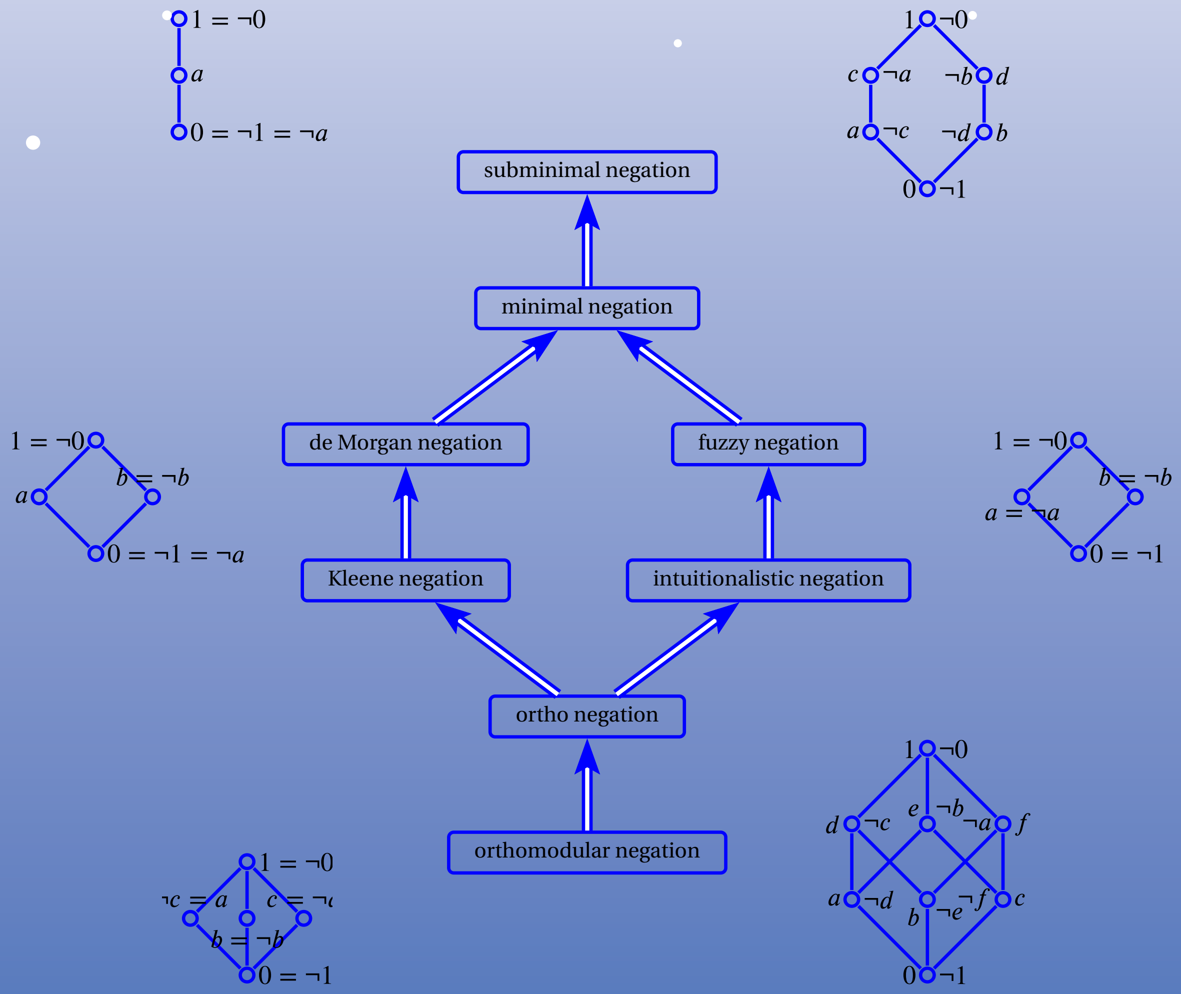
Negation, Implication and Logic

VERSION 0.52



Daniel J. Greenhoe

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The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]

[1] Paine (2000) page 63 ⟨Golden Hind⟩

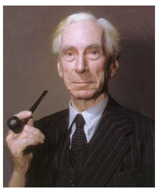
❝Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?❞

❝Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.❞

Alfred Edward Housman, English poet (1859–1936) [2]

❝The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.❞

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



❝As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.❞

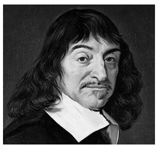
Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

[2] quote: Housman (1936) page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961) page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967) page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

"rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit."



"Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters."

René Descartes (1596–1650), French philosopher and mathematician [5]

"In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished."

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \ldots$
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$

…continued on next page…

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$
$\mathbb{Z}^{-}$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_{\mathsf{o}}$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_{\mathsf{e}}$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^{+}$	positive real numbers	$(0, \infty)$
$\mathbb{R}^{-}$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^{*}$	extended real numbers	$\mathbb{R}^{*} \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
⊘	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \triangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\rightarrow$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\nsubseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^{c}	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	
0	"false" condition	

...continued on next page...

symbol	description	
$\neg$	logical NOT operation	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\le$	reflexive ordering relation	"less than or equal to"
$\ge$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_{\mathsf{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_{\mathsf{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_{\mathsf{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		
$\tilde{\mathbf{F}}$	*Fourier Transform* operator	

...continued on next page...

symbol	description
$\hat{\mathbf{F}}$	*Fourier Series* operator
$\check{\mathbf{F}}$	*Discrete Time Fourier Series* operator
$\mathbf{Z}$	*Z-Transform* operator
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\check{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\check{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

CHAPTER 1

NEGATION

> ❝ *When we say not-being, we speak, I think, not of something that is the opposite of being, but only of something different. ...Then when we are told that the negative signifies the opposite, we shall not admit it; we shall admit only that the particle "not" indicates something different from the words to which it is prefixed, or rather from the things denoted by the words that follow the negative.* ❞
>
> Plato's the *Sophist* (circa 360 B.C.) [1]

> ❝ *Clearly, then, it is a principle of this kind that is the most certain of all principles.... Let us next state what this principle is. "It is impossible for the same attribute at once to belong and not to belong to the same thing and in the same relation"; ...This is the most certain of all principles,...for it is impossible for anyone to suppose that the same thing is and is not,...it is by nature the starting-point of all the other axioms as well.* ❞
>
> Aristotle (384BC–322BC), Greek philosopher [2]

1.1 Definitions

Definition 1.1. [3] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1 \,;\, \leq)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133)*.

DEF: *A* FUNCTION $\neg \in X^X$ *is a* ***subminimal negation*** *on* $\mathbf{L}$ *if*

$$x \leq y \implies \neg y \leq \neg x \qquad \forall x,y \in X \qquad \text{(ANTITONE)}$$ [4]

[1] Plato (circa 360 B.C.) ⟨257b–257c⟩, Horn (2001) page 5

[2] Aristotle page 4.1005b

[3] Dunn (1996) pages 4–6, Dunn (1999) pages 24–26 ⟨2 The Kite of Negations⟩

[4] The *antitone* property may also be referred to as *antitonic*, *order-reversing*, or *contrapositive*.

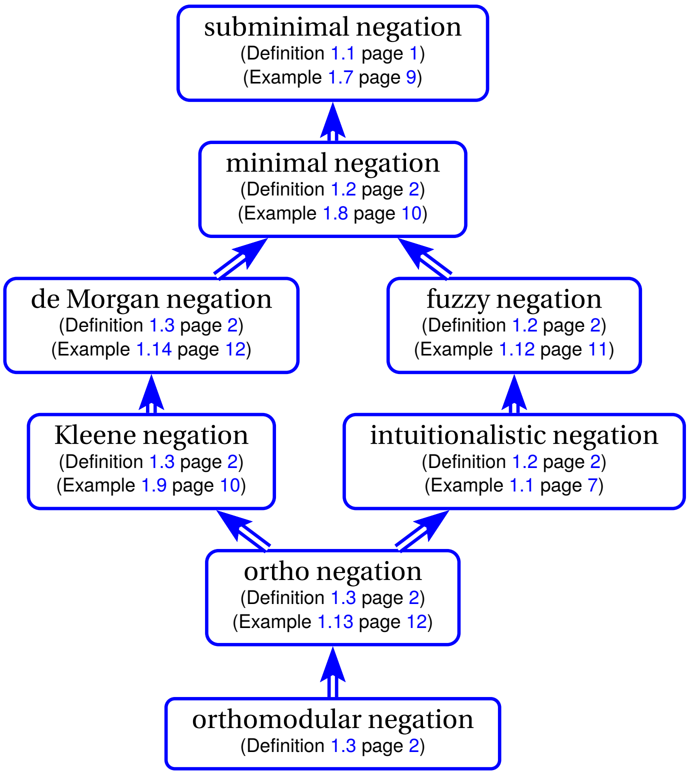


Figure 1.1: lattice of negations

Remark 1.1. [5] In the context of natural language, D. Devidi argues that, *subminimal negation* (Definition 1.1 page 1) is "difficult to take seriously as" a negation. He essentially gives this example: Let $x \triangleq$"p is a fish" and $y \triangleq$"p has gills". Suppose "p is a fish" implies "p has gills" ($x \le y$). Now let $p \triangleq$ "many dogs". Then the *antitone* property and $x \le y$ tells us ($\implies$) that "Not many dogs have gills" implies that "Not many dogs are fish".

Definition 1.2. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133).*

DEF

A FUNCTION $\neg \in X^X$ *is a* ***negation****, or* ***minimal negation****, on* $\boldsymbol{L}$ *if*

1. $x \le y \implies \neg y \le \neg x \quad \forall x,y \in X$ (ANTITONE) *and*
2. $x \le \neg\neg x \quad \forall x \in X$ (WEAK DOUBLE NEGATION).

A MINIMAL NEGATION $\neg$ *is an* ***intuitionistic negation*** *if*

3. $x \wedge \neg x = 0 \quad \forall x,y \in X$ (NON-CONTRADICTION).

A MINIMAL NEGATION $\neg$ *is a* ***fuzzy negation*** *if*

4. $\neg 1 = 0$ (BOUNDARY CONDITION).

Definition 1.3. [7] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133).*

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[5] Devidi (2010) page 511, Devidi (2006) page 568

[6] Dunn (1996) pages 4–6, Dunn (1999) pages 24–26 ⟨2 The Kite of Negations⟩, Troelstra and van Dalen (1988) page 4 ⟨1.6 Intuitionism. (b)⟩, de Vries (2007) page 11 ⟨Definition 16⟩, Gottwald (1999) page 21 ⟨Definition 3.3⟩, Novák et al. (1999) page 50 ⟨Definition 2.26⟩, Nguyen and Walker (2006) pages 98–99 ⟨5.4 Negations⟩, Höhle (1978) ⟨???⟩, Bellman and Giertz (1973) pages 155–156 ⟨(N1) $\neg 0 = 1$ and $\neg 1 = 0$, (N3) $\neg\neg x = x$⟩

[7] Dunn (1999) pages 24–26 ⟨2 The Kite of Negations⟩, Jenei (2003) page 283, Kalmbach (1983) page 22, Lidl and Pilz (1998) page 90, Husimi (1937)

DEF

A MINIMAL NEGATION $\neg$ *is a* ***de Morgan negation*** *if*

5. $x = \neg\neg x \quad \forall x \in X$ (INVOLUTORY).

A DE MORGAN NEGATION $\neg$ *is a* ***Kleene negation*** *if*

6. $x \wedge \neg x \leq y \vee \neg y \quad \forall x, y \in X$ (KLEENE CONDITION).

A DE MORGAN NEGATION $\neg$ *is an* ***ortho negation*** *if*

7. $x \wedge \neg x = 0 \quad \forall x, y \in X$ (NON-CONTRADICTION).

A DE MORGAN NEGATION $\neg$ *is an* ***orthomodular negation*** *if*

8. $x \wedge \neg x = 0 \quad \forall x, y \in X$ (NON-CONTRADICTION) *and*
9. $x \leq y \implies x \vee (y \wedge \neg x) = y \quad \forall x, y \in X$ (ORTHOMODULAR).

Remark 1.2. [8] The *Kleene condition* is basically a weakened form of the *non-contradiction* and *excluded middle* properties because $\underbrace{x \wedge \neg x = 0}_{\text{non-contradiction}} \leq \underbrace{1 = y \vee \neg y}_{\text{excluded middle}}$.

Definition 1.4. [9]

DEF

A MINIMAL NEGATION $\neg \in X^X$ *is* ***strict*** *($\neg$ is a* ***strict negation****) if*

1. $x \lneq y \implies \neg y \lneq \neg x \quad \forall x, y \in X$ (STRICTLY ANTITONE) *and*
2. $\neg$ *is* CONTINUOUS

A STRICT NEGATION $\neg$ *is* ***strong*** *($\neg$ is a* ***strong negation****) if*

3. $\neg\neg x = x \quad \forall x \in X$ (INVOLUTORY).

Definition 1.5. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, \neg, 0, 1 ; \leq)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133) with a function* $\neg$ *in* X^X.

DEF

If $\neg$ *is a* MINIMAL NEGATION, *then* $\mathbf{L}$ *is a* ***lattice with negation****.*

1.2 Properties of negations

Lemma 1.1. [10] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *(Definition E.1 page 133).*

LEM

$$\underbrace{x \leq y \implies \neg y \leq \neg x\}}_{\text{ANTITONE}} \implies \begin{cases} \neg x \vee \neg y \leq \neg(x \wedge y) & \forall x, y \in X \quad \text{(CONJUNCTIVE DE MORGAN INEQ.) and} \\ \neg(x \vee y) \leq \neg x \wedge \neg y & \forall x, y \in X \quad \text{(DISJUNCTIVE DE MORGAN INEQ.) and} \end{cases}$$

✎PROOF:

1. Proof that *antitone* $\implies$ *conjunctive de Morgan*:

$$\begin{aligned} & x \wedge y \leq x \text{ and } x \wedge y \leq y && \text{by definition of } \wedge \\ \implies & \neg(x \wedge y) \geq \neg x \text{ and } \neg(x \wedge y) \geq \neg y && \text{by } \textit{antitone} \\ \implies & \neg(x \wedge y) \geq \neg x \vee \neg y && \text{by definition of } \vee \end{aligned}$$

[8] Cattaneo and Ciucci (2009) page 78

[9] Fodor and Yager (2000), pages 127–128, Bellman and Giertz (1973)

[10] Beran (1985) page 31 ⟨Theorem 1.2 Proof⟩, Fáy (1967) page 268 ⟨Lemma 1 Proof⟩, de Vries (2007) page 12 ⟨Theorem 18⟩

2. Proof that *antitone* $\implies$ *disjunctive de Morgan*:

$$\begin{aligned}
& x \le x \vee y \text{ and } y \le x \vee y && \text{by definition of } \vee\\
\implies & \neg x \ge \neg(x \vee y) \text{ and } \neg y \ge \neg(x \vee y) && \text{by } \textit{antitone}\\
\implies & \neg x \wedge \neg y \ge \neg(x \vee y) && \text{by definition of } \wedge\\
\implies & \neg(x \vee y) \le \neg x \wedge \neg y
\end{aligned}$$

Lemma 1.2. [11] *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ (Definition D.3 page 117).

If $x = (\neg\neg x)$ *for all* $x \in X$ (INVOLUTORY), *then*

$$\underbrace{\{x \le y \implies \neg y \le \neg x\}}_{\text{ANTITONE}} \iff \underbrace{\left\{\begin{array}{lcll} \neg(x \vee y) & = & \neg x \wedge \neg y \quad \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN) and}\\ \neg(x \wedge y) & = & \neg x \vee \neg y \quad \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)} \end{array}\right.}_{\text{DE MORGAN}}$$

✎PROOF:

1. Proof that *antitone* $\implies$ *de Morgan* equalities:

   (a) Proof that $\neg(\neg x \wedge \neg y) \ge x \vee y$:

   $$\begin{aligned}
   \neg(\neg x \wedge \neg y) &\ge \neg\neg x \vee \neg\neg y && \text{by Lemma 1.1}\\
   &= x \vee y && \text{by } \textit{involutory} \text{ property (Definition 1.5 page 3)}
   \end{aligned}$$

   (b) Proof that $\neg(\neg x \vee \neg y) \le x \wedge y$:

   $$\begin{aligned}
   \neg(\neg x \vee \neg y) &\le \neg\neg x \wedge \neg\neg y && \text{by Lemma 1.1}\\
   &= x \wedge y && \text{by } \textit{involutory} \text{ property (Definition 1.5 page 3)}
   \end{aligned}$$

   (c) Proof that $\neg(x \wedge y) = \neg x \vee \neg y$:

   $$\begin{aligned}
   \neg(x \wedge y) &\ge \neg x \vee \neg y && \text{by Lemma 1.1}\\
   \neg(x \wedge y) &= \neg[\neg\neg x \wedge \neg\neg y] && \text{by } \textit{involutory} \text{ property (Definition 1.5 page 3)}\\
   &\le \neg x \vee \neg y && \text{by item (1b)}
   \end{aligned}$$

   (d) Proof that $\neg(x \vee y) = \neg x \wedge \neg y$:

   $$\begin{aligned}
   \neg(x \vee y) &\ge \neg x \wedge \neg y && \text{by Lemma 1.1}\\
   \neg(x \vee y) &= \neg[\neg\neg x \vee \neg\neg y] && \text{by } \textit{involutory} \text{ property (Definition 1.5 page 3)}\\
   &\le \neg x \wedge \neg y && \text{by item (1a)}
   \end{aligned}$$

2. Proof that *antitone* $\impliedby$ *de Morgan*:

$$\begin{aligned}
x \le y \implies \neg y &= \neg(x \vee y) && \text{because } x \le y\\
&= \neg x \wedge \neg y && \text{by } \textit{de Morgan}\\
&\le \neg x && \text{by definition of } \wedge
\end{aligned}$$

[11] Beran (1985) pages 30–31 ⟨Theorem 1.2⟩, Fáy (1967) page 268 ⟨Lemma 1⟩, Nakano and Romberger (1971) ⟨cf Beran 1985⟩

Lemma 1.3. *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ (Definition D.3 page 117).

LEM $\left\{\begin{array}{lll} 1. & x \leq \neg\neg x \quad \forall x \in X & \text{(WEAK DOUBLE NEGATION)} \quad and \\ 2. & \neg 1 = 0 & \text{(BOUNDARY CONDITION)} \end{array}\right\} \implies \left\{ \neg 0 = 1 \quad \text{(BOUNDARY CONDITION)} \right\}$

✎PROOF:

$$\begin{aligned} \neg 0 &= \neg\neg 1 && \text{by } \textit{boundary condition} \text{ hypothesis (2)} \\ &\geq 1 && \text{by } \textit{weak double negation} \text{ hypothesis (1)} \\ \implies \neg 0 &= 1 && \text{by } \textit{upper bound} \text{ property (Definition E.1 page 133)} \end{aligned}$$

✏

Lemma 1.4. *Let* $\neg \in X^X$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ (Definition D.3 page 117).

LEM $\left\{ (x \wedge \neg x = 0 \quad \forall x \in X \quad \text{(NON-CONTRADICTION)} \right\} \implies \left\{ \neg 1 = 0 \quad \text{(BOUNDARY CONDITION)} \right\}$

✎PROOF:

$$\begin{aligned} 0 &= 1 \wedge \neg 1 && \text{by } \textit{non-contradiction} \text{ hypothesis} \\ &= \neg 1 && \text{by definition of g.u.b. } 1 \text{ and } \wedge \end{aligned}$$

✏

Lemma 1.5. [12] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ (Definition E.1 page 133).

LEM $\left\{\begin{array}{llll} (A). & \neg \text{ is BIJECTIVE} & & and \\ (B). & x \leq y \implies \neg y \leq \neg x & \forall x,y \in X & \text{(ANTITONE)} \end{array}\right\} \implies \underbrace{\left\{\begin{array}{lll} (1). & \neg 0 = 1 & and \\ (2). & \neg 1 = 0 & \end{array}\right\}}_{\text{BOUNDARY CONDITIONS}}$

✎PROOF:

1. Proof that $\neg 0 = 1$:

$$\begin{aligned} & x \leq 1 && \forall x \in X && \text{by definition of l.u.b. } 1 \\ \implies & \neg 1 \leq \neg x && \forall x \in X && \text{by } \textit{antitone} \text{ hypothesis} \\ \implies & \neg 1 \leq y && \forall y \in X && \text{by } \textit{bijective} \text{ hypothesis} \\ \implies & \neg 1 = 0 && && \text{by definition of g.l.b. } 0 \end{aligned}$$

2. Proof that $\neg 0 = 1$:

$$\begin{aligned} & 0 \leq x && \forall x \in X && \text{by definition of g.l.b. } 0 \\ \implies & \neg x \leq \neg 0 && \forall x \in X && \text{by } \textit{antitone} \text{ hypothesis} \\ \implies & \neg x \leq y && \forall y \in X && \text{by } \textit{bijective} \text{ hypothesis} \\ \implies & \neg 0 = 1 && && \text{by definition of l.u.b. } 1 \end{aligned}$$

✏

[12] Varadarajan (1985) page 42

Theorem 1.1. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition E.1 page 133).*

THM $\left\{\begin{array}{l}\neg \text{ is an} \\ \text{INTUITIONISTIC NEGATION}\end{array}\right\} \implies \left\{ \neg 1 = 0 \quad \text{(BOUNDARY CONDITION)} \right\}$

✎PROOF: This follows directly from Definition 1.5 (page 3) and Lemma 1.4 (page 5). ✏

Theorem 1.2. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition E.1 page 133).*

THM $\left\{\begin{array}{l}\neg \text{ is a} \\ \text{FUZZY NEGATION}\end{array}\right\} \implies \left\{ \neg 0 = 1 \quad \text{(BOUNDARY CONDITION)} \right\}$

✎PROOF: This follows directly from Definition 1.2 (page 2) and Lemma 1.3 (page 5). ✏

Theorem 1.3. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition E.1 page 133).*

THM $\left\{\begin{array}{l}\neg \text{ is a} \\ \text{minimal} \\ \text{negation}\end{array}\right\} \implies \left\{\begin{array}{lllll}\neg x \vee \neg y & \leq & \neg(x \wedge y) & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN INEQUALITY) and} \\ \neg(x \vee y) & \leq & \neg x \wedge \neg y & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN INEQUALITY)}\end{array}\right\}$

✎PROOF: This follows directly from Definition 1.5 (page 3) and Lemma 1.1 (page 3). ✏

Theorem 1.4. *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition E.1 page 133).*

THM $\left\{\begin{array}{l}\neg \text{ is a} \\ \text{de Morgan negation}\end{array}\right\} \implies \left\{\begin{array}{lllll}\neg(x \vee y) & = & \neg x \wedge \neg y & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN) and} \\ \neg(x \wedge y) & = & \neg x \vee \neg y & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)}\end{array}\right.$

✎PROOF: This follows directly from Definition 1.5 (page 3) and Lemma 1.2 (page 4). ✏

Theorem 1.5. [13] *Let* $\neg \in X^X$ *be a function on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition E.1 page 133).*

THM $\left\{\begin{array}{l}\neg \text{ is an} \\ \text{ortho negation}\end{array}\right\} \implies \left\{\begin{array}{rrclll}1. & \neg 0 & = & 1 & & \text{(BOUNDARY CONDITION) and} \\ 2. & \neg 1 & = & 0 & & \text{(BOUNDARY CONDITION) and} \\ 3. & \neg(x \vee y) & = & \neg x \wedge \neg y & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN) and} \\ 4. & \neg(x \wedge y) & = & \neg x \vee \neg y & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN) and} \\ 5. & x \vee \neg x & = & 1 & \forall x \in X & \text{(EXCLUDED MIDDLE) and} \\ 6. & x \wedge \neg x & \leq & y \vee \neg y & \forall x,y \in X & \text{(KLEENE CONDITION).}\end{array}\right\}$

✎PROOF:

1. Proof for $0 = \neg 1$ boundary condition: by Lemma 1.4 (page 5)
2. Proof for boundary conditions:

$$\begin{aligned} 1 &= \neg\neg 1 && \text{by } \textit{involutory} \text{ property} \\ &= \neg 0 && \text{by previous result} \end{aligned}$$

[13] Beran (1985) pages 30–31, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Cohen (1989) page 37 ⟨3B.13. Theorem⟩

3. Proof for *de Morgan* properties:
   (a) By Definition 1.5 (page 3), *ortho negation* is *involutory* and *antitone*.
   (b) Therefore by Lemma 1.2 (page 4), *de Morgan* properties hold.

4. Proof for *excluded middle* property:

$$\begin{aligned}
x \vee \neg x &= (x \vee \neg x)^{\neg\neg} && \text{by } \textit{involutory} \text{ property of } \textit{ortho negation} \text{ (Definition 1.5 page 3)}\\
&= \neg(x\neg \wedge x^{\neg\neg}) && \text{by } \textit{disjunctive de Morgan} \text{ property}\\
&= \neg(\neg x \wedge x) && \text{by } \textit{involutory} \text{ property of } \textit{ortho negation} \text{ (Definition 1.5 page 3)}\\
&= \neg(x \wedge \neg x) && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Definition D.3 page 117)}\\
&= \neg 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{ortho negation} \text{ (Definition 1.5 page 3)}\\
&= 1 && \text{by } \textit{boundary condition} \text{ (item (2) page 6) of } \textit{minimal negation}
\end{aligned}$$

5. Proof for *Kleene condition*:

$$\begin{aligned}
x \wedge \neg x &= 0 && \text{by } \textit{non-contradiction} \text{ property (Definition 1.5 page 3)}\\
&\le 1 && \text{by definition of 0 and 1}\\
&= y \vee \neg y && \text{by } \textit{excluded middle} \text{ property (item (4) page 7)}
\end{aligned}$$

1.3 Examples

Example 1.1 (discrete negation). [14] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ be a *bounded lattice* (Definition E.1 page 133) with a function $\neg \in X^X$.

EX The function $\neg x$ defined as

$$\neg x \triangleq \begin{cases} 1 & \text{for } x = 0\\ 0 & \text{otherwise}\end{cases}$$

is an *intuitionistic negation* (Definition 1.2 page 2) and a *fuzzy negation* (Definition 1.2 page 2).

✎PROOF: To be an *intuitionistic negation*, $\neg x$ must be *antitone*, have *weak double negation*, and have the *non-contradiction* property (Definition 1.2 page 2). To be a *fuzzy negation*, $\neg x$ must be *antitone*, have *weak double negation*, and have the *boundary condition* $\neg 1 = 0$.

$$\left\{\begin{array}{lclll} \neg y \le \neg x & \iff & 1 \le 1 & \text{for } 0 = x = y\\ \neg y \le \neg x & \iff & 0 \le 1 & \text{for } 0 = x \lneq y\\ \neg y \le \neg x & \iff & 0 \le 0 & \text{for } 0 \ne x \le y\end{array}\right\} \implies \neg x \text{ is } \textit{antitone}$$

$$\left\{\begin{array}{lllllllll} \neg\neg x & = & \neg 1 & = & 0 & \ge & 0 & = & x & \text{for } x = 0\\ \neg\neg x & = & \neg 0 & = & 1 & \ge & x & = & x & \text{for } x \ne 0\end{array}\right\} \implies \neg x \text{ has } \textit{weak double negation}$$

$$\left\{\begin{array}{lllllll} x \wedge \neg x & = & x \wedge 1 & = & 0 \wedge 0 & = & 0 & \text{for } x = 0\\ x \wedge \neg x & = & x \wedge 0 & = & x \wedge 0 & = & 0 & \text{for } x \ne 0\end{array}\right\} \implies \neg x \text{ has } \textit{non-contradiction} \text{ property}$$

$$\neg 1 = 0 \implies \neg x \text{ has the } \textit{boundary condition} \text{ property}$$

[14] Fodor and Yager (2000) page 128, Yager (1980) pages 256–257, Yager (1979) ⟨cf Fodor⟩

Example 1.2 (dual discrete negation). [15] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1\,;\, \leq)$ be a *bounded lattice* (Definition E.1 page 133) with a function $\neg \in X^X$.

EX The function $\neg x$ defined as

$$\neg x \triangleq \begin{cases} 0 & \text{for } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

is a *subminimal negation* (Definition 1.1 page 1) but it is *not* a *minimal negation* (Definition 1.2 page 2) (and not any other negation defined here).

✎PROOF: To be an *subminimal negation*, $\neg x$ must be *antitone* (Definition 1.1 page 1). To be a *minimal negation*, $\neg x$ must be *antitone* and have *weak double negation* (Definition 1.2 page 2).

$$\left\{\begin{array}{lcll} \neg y \leq \neg x & \iff & 0 \leq 0 & \text{for } x = y = 1 \\ \neg y \leq \neg x & \iff & 0 \leq 1 & \text{for } x \lneq y = 1 \\ \neg y \leq \neg x & \iff & 1 \leq 1 & \text{for } x \leq y \neq 1 \end{array}\right\} \implies \neg x \text{ is } \textit{antitone}$$

$$\left\{\begin{array}{lllllllll} \neg\neg x & = & \neg 0 & = & 1 & \geq & x & \text{for } x = 1 \\ \neg\neg x & = & \neg 1 & = & 0 & \leq & x & \text{for } x \neq 1 \end{array}\right\} \implies \neg x \text{ does } \textit{not} \text{ have } \textit{weak double negation}$$

⇨

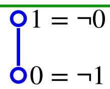
Example 1.3. [16] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ be a *bounded lattice*

EX The function $\neg x$ is an *intuitionistic negation* (Definition 1.2 page 2) if

$$\neg x \triangleq \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 1.4.

EX The function $\neg$ illustrated to the right is an *ortho negation* (Definition 1.3 page 2).



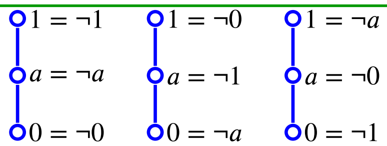
✎PROOF:

1. Proof that $\neg$ is *antitone*: $0 \;\leq\; 1 \;\implies\; \neg 1 \;=\; 0 \;\leq\; x \;=\; \neg 0 \;\implies\; \neg$ is *antitone* over $(0, 1)$
2. Proof that $\neg$ is *involutory*: $1 = \neg 0 = \neg\neg 1$
3. Proof that $\neg$ has the *non-contradiction* property: $\begin{array}{lllllllll} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 \\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 \end{array}$

⇨

Example 1.5.

EX The functions $\neg$ illustrated to the right are *not* any negation defined here. In particular, they are *not antitone*.

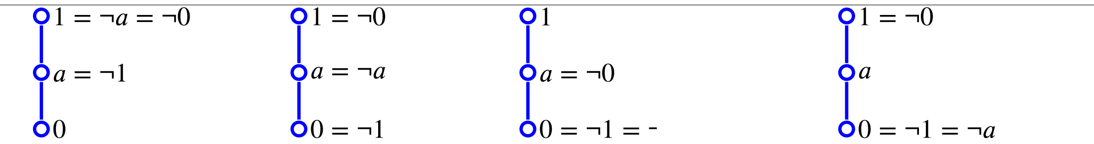


(a) (b) (c)

✎PROOF:

1. Proof that (a) is *not antitone*: $a \leq 1 \implies \neg 1 = 1 \nleq a = \neg a$
2. Proof that (b) is *not antitone*: $a \leq 1 \implies \neg 1 = a \nleq 0 = \neg a$
3. Proof that (c) is *not antitone*: $0 \leq a \implies \neg a = 1 \nleq a = \neg 0$

[15] Fodor and Yager (2000) page 128, Ovchinnikov (1983) page 235 ⟨Example 4⟩
[16] Fodor and Yager (2000) page 128

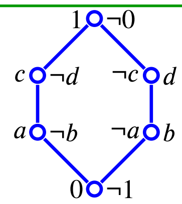


(A) *minimal negation* (Example 1.8 page 10) (B) *Kleene negation* (Example 1.9 page 10) (C) *intuitionistic negation* (Example 1.10 page 10) (D) *fuzzy* and *intuitionistic* negation (Example 1.11 page 11)

Figure 1.2: negations on $\boldsymbol{L}_3$

Example 1.6.

EX The function $\neg$ as illustrated to the right is *not* a *subminimal negation* (it is *not antitone*) and so is *not* any negation defined here. Note however that the problem is *not* the O_6 *lattice*—it is possible to define a negation on an O_6 *lattice* (Example 1.16 page 13).

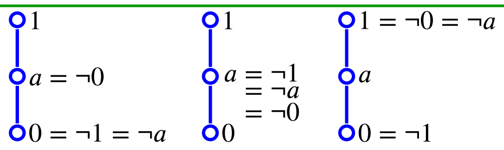


✎PROOF: Proof that $\neg$ is *not antitone*: $a \leq c \implies \neg c = d \nleq b = \neg a$

Remark 1.3. The concept of a *complement* (Definition H.1 page 165) and the concept of a *negation* are fundamentally different. A *complement* is a *relation* (Definition B.1 page 73) on a lattice $\boldsymbol{L}$ and a *negation* is a *function* (Definition B.8 page 85). In Example 1.6 (page 9), b and d are both complements of a, but yet $\neg$ is *not* a negation. In the right side lattice of Example 1.16 (page 13), both b and d are complements of a (and so the lattice is *multipy complemented*), but yet only d is equal to the negation of a ($d = \neg a$). It can also be said that complementation is a property *of* a lattice, whereas negation is a function defined *on* a lattice.

Example 1.7.

EX Each of the functions $\neg$ illustrated to the right is a *subminimal negation* (Definition 1.1 page 1); *none* of them is a *minimal negation* (each fails to have *weak double negation*).



(a) (b) (c)

✎PROOF:

1. Proof that (a) $\neg$ is *antitone*:
$$\begin{array}{lllllllll} a \leq 1 & \implies & \neg 1 & = & 0 & \leq & 0 & = & \neg a & \implies & \neg \text{ is antitone over } (a,1)\\ 0 \leq 1 & \implies & \neg 1 & = & 0 & \leq & a & = & \neg 0 & \implies & \neg \text{ is antitone over } (0,1)\\ 0 \leq a & \implies & \neg a & = & 0 & \leq & a & = & \neg 0 & \implies & \neg \text{ is antitone over } (0,a)\end{array}$$

2. Proof that (a) $\neg$ *fails* to have *weak double negation*:
$1 \nleq a = \neg 0 = \neg\neg 1$

3. Proof that (b) $\neg$ is *antitone*:
$$\begin{array}{lllllllll} a \leq 1 & \implies & \neg 1 & = & a & \leq & a & = & \neg a & \implies & \neg \text{ is antitone over } (a,1)\\ 0 \leq 1 & \implies & \neg 1 & = & a & \leq & a & = & \neg 0 & \implies & \neg \text{ is antitone over } (0,1)\\ 0 \leq a & \implies & \neg a & = & a & \leq & a & = & \neg 0 & \implies & \neg \text{ is antitone over } (0,a)\end{array}$$

4. Proof that (b) $\neg$ *fails* to have *weak double negation*: $1 \nleq a = \neg a = \neg\neg 1$

5. (c) is a special case of the *dual discrete negation* (Example 1.2 page 8).

Example 1.8. The function ¬ illustrated in Figure 1.2 page 9 (A) is a **minimal negation** (Definition 1.2 page 2); it is *not* an *intuitionistic negation* (it does not have the *non-contradiction* property), it is *not* a *de Morgan negation* (it is not *involutory*), and it is *not* a *fuzzy negation* ($\neg 1 \neq 0$).

✎PROOF:

1. Proof that ¬ is *antitone*: $$\begin{array}{ccccccccccl} a & \le & 1 & \implies & \neg 1 & = & a & \le & 1 & = & \neg a \implies \neg \text{ is } \textit{antitone} \text{ over } (a,1) \\ 0 & \le & 1 & \implies & \neg 1 & = & a & \le & 1 & = & \neg 0 \implies \neg \text{ is } \textit{antitone} \text{ over } (0,1) \\ 0 & \le & a & \implies & \neg a & = & 1 & \le & 1 & = & \neg 0 \implies \neg \text{ is } \textit{antitone} \text{ over } (0,a) \end{array}$$

2. Proof that ¬ is a *weak double negation* (and so is a *minimal negation*, but is *not* a *de Morgan negation*): $$\begin{array}{ccccccccl} 1 & = & 1 & = & \neg a & = & \neg\neg 1 & \implies & \neg \text{ is } \textit{involutory} \text{ at } 1 \\ a & = & a & = & \neg 1 & = & \neg\neg a & \implies & \neg \text{ is } \textit{involutory} \text{ at } a \\ 0 & \le & a & = & \neg 1 & = & 0^{\neg\neg} & \implies & \neg \text{ is a } \textit{weak double negation} \text{ at } 0 \end{array}$$

3. Proof that ¬ does *not* have the *non-contradiction* property (and so is not an *intuitionistic negation*): $$1 \wedge \neg 1 = 1 \wedge a = a \neq 0$$

4. Proof that ¬ is not a *fuzzy negation*: $\neg 1 = a \neq 0$

✎

Example 1.9 (Łukasiewicz 3-valued logic/Kleene 3-valued logic/RM_3 logic). [17] The function ¬ illustrated in Figure 1.2 page 9 (B) is a **Kleene negation** (Definition 1.3 page 2), and is also a *fuzzy negation* (Definition 1.2 page 2); but it is *not* an *ortho negation* and is *not* an *intuitionistic negation* (it does not have the *non-contradiction* property).

✎PROOF:

1. Proof that ¬ is *antitone*: $$\begin{array}{ccccccccccl} a & \le & 1 & \implies & \neg 1 & = & 0 & \le & a & = & \neg a \implies \neg \text{ is } \textit{antitone} \text{ over } (a,1) \\ 0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & 1 & = & \neg 0 \implies \neg \text{ is } \textit{antitone} \text{ over } (0,1) \\ 0 & \le & a & \implies & \neg a & = & a & \le & 1 & = & \neg 0 \implies \neg \text{ is } \textit{antitone} \text{ over } (0,a) \end{array}$$

2. Proof that ¬ is *involutory* (and so is a *de Morgan negation*): $$\begin{array}{cccccccl} 1 & = & \neg 0 & = & \neg\neg 1 & \implies & \neg \text{ is } \textit{involutory} \text{ at } 1 \\ a & = & \neg a & = & \neg\neg a & \implies & \neg \text{ is } \textit{involutory} \text{ at } a \\ 0 & = & \neg 0 & = & 0^{\neg\neg} & \implies & \neg \text{ is } \textit{involutory} \text{ at } 0 \end{array}$$

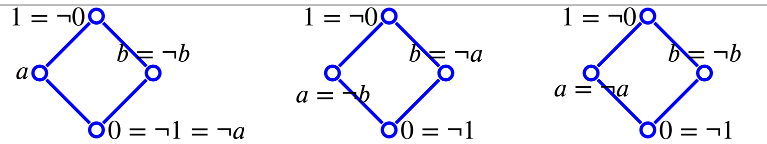
3. Proof that ¬ does *not* have the *non-contradiction* property (and so is not an *ortho negation*): $$x \wedge \neg x = x \wedge x = x \neq 0$$

4. Proof that ¬ satisfies the *Kleene condition* (and so is a *Kleene negation*): $$\begin{array}{ccccccccccccccccc} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 & \le & a & = & a & \vee & a & = & a \vee \neg a \\ 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0 & \le & 1 & = & 0 & \vee & 1 & = & 0 \vee \neg 0 \\ a & \wedge & \neg a & = & 1 & \wedge & a & = & a & \le & 1 & = & 1 & \vee & 0 & = & 1 \vee \neg 1 \\ a & \wedge & \neg a & = & 1 & \wedge & a & = & a & \le & 1 & = & 0 & \vee & 1 & = & 0 \vee \neg 0 \\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 & \le & 1 & = & 1 & \vee & 0 & = & 1 \vee \neg 1 \\ 0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0 & \le & a & = & a & \vee & a & = & a \vee \neg a \end{array}$$

✎

Example 1.10. The function ¬ illustrated in Figure 1.2 page 9 (C) an **intuitionistic negation** (Definition 1.2 page 2); but it is *not* a *fuzzy negation* ($1 \neq \neg 0$), and it is *not* a *de Morgan negation* (it is not *involutory*).

[17] Łukasiewicz (1920), Avron (1991) pages 277–278, Kleene (1938) page 153, Kleene (1952) pages 332–339 (§64. The 3-valued logic), Sobociński (1952)



(A) *fuzzy negation* (Example 1.12 page 11) (B) *ortho negation* (Example 1.13 page 12) (C) *de Morgan negation* (Example 1.14 page 12)

Figure 1.3: negations on $\boldsymbol{M}_2$

✎PROOF:

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{lllllllllll} a & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & 0 & = & \neg a & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (a,1)\\ 0 & \le & 1 & \Longrightarrow & \neg 1 & = & 0 & \le & a & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,1)\\ 0 & \le & a & \Longrightarrow & \neg a & = & 0 & \le & a & = & \neg 0 & \Longrightarrow & \neg \text{ is } \textit{antitone} \text{ at } (0,a) \end{array}$$
2. Proof that $\neg$ has *weak double negation* property (and so is a *minimal negation*, but *not* a *de Morgan negation*):
$$\begin{array}{lllllllll} 1 & \le & a & = & \neg 0 & = & \neg\neg 1 & \Longrightarrow & \neg \text{ has } \textit{weak double negation} \text{ at } 1\\ a & = & \neg 0 & = & \neg\neg a & & & \Longrightarrow & \neg \text{ has } \textit{weak double negation} \text{ at } a\\ 0 & = & \neg a & = & 0^{\neg\neg} & & & \Longrightarrow & \neg \text{ is } \textit{involutory} \text{ at } 0 \end{array}$$
3. Proof that $\neg$ has the *non-contradiction* property (and so is an *intuitionistic negation*):
$$\begin{array}{lllllllll} 1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0\\ a & \wedge & \neg a & = & a & \wedge & 0 & = & 0\\ 0 & \wedge & \neg 0 & = & 0 & \wedge & a & = & 0 \end{array}$$
4. Proof that $\neg$ is *not* a *fuzzy negation*: $\neg 1 \ne 0$

✏

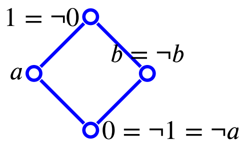
Example 1.11 (Heyting 3-valued logic/Jaśkowski's first matrix). [18] The function $\neg$ illustrated in Figure 1.2 page 9 (D) is an **intuitionistic negation** (Definition 1.2 page 2), and is also a **fuzzy negation** (Definition 1.2 page 2), but it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF: This is simply a special case of the *discrete negation* (Example 1.1 page 7). ✏

Remark 1.4. There is only one linearly ordered (Definition C.4 page 103) 3-element lattice ($\boldsymbol{L}_3$) that is a *fuzzy negation* (Example 1.11 page 11). However, this lattice is also an *intuitionistic negation*. There are no $\boldsymbol{L}_3$ lattices that are *fuzzy* but yet not *intuitionistic*. In fact, there are only three linearly ordered 3-element lattices with with $1 = \neg 0$ and $0 = \neg 1$. Of these three, only one is both *fuzzy* and *intuitionistic* (Example 1.11 page 11), one is *Kleene* but not *fuzzy* (Example 1.9 page 10), and one is *subminimal* but not *fuzzy* (Example 1.7 page 9). It can be claimed that the "simplist" *fuzzy negation* that is not *de Morgan* and not *intuitionistic* is the $\boldsymbol{M}_2$ lattice of Example 1.12 (next).

Example 1.12. The function $\neg$ illustrated in Figure 1.3 page 11 (A) is a **fuzzy negation** (Definition 1.2 page 2). It is not an *intuitionistic negation* (it does not have the *non-contradiction* property) and it is *not* a *de Morgan negation* (it is not *involutory*).

✎PROOF: Note that



(Example 1.12 page 11)

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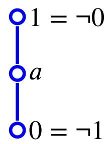
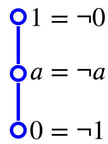


fuzzy and *intuitionistic* (Example 1.11 page 11)

+



Kleene negation (Example 1.9 page 10)

[18] Karpenko (2006) page 45, Johnstone (1982) page 9 ⟨§1.12⟩, Heyting (1930a), Heyting (1930b), Heyting (1930c), Heyting (1930d), Jaskowski (1936), Mancosu (1998)

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{lllllllllllll}
a & \le & 1 & \implies & \neg 1 & = & 0 & \le & 0 & = & \neg a & \implies & \neg \text{ is } \textit{antitone} \text{ at } (a,1)\\
0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ at } (0,1)\\
0 & \le & a & \implies & \neg a & = & 0 & \le & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ at } (0,a)\\
b & \le & 1 & \implies & \neg 1 & = & 0 & \le & b & = & \neg b & \implies & \neg \text{ is } \textit{antitone} \text{ at } (b,1)\\
0 & \le & b & \implies & \neg b & = & b & \le & 1 & = & \neg 0 & \implies & \neg \text{ is } \textit{antitone} \text{ at } (0,b)
\end{array}$$

2. Proof that $\neg$ has *weak double negation* property (and so is a *minimal negation*, but *not* a *de Morgan negation*):
$$\begin{array}{lllllllll}
1 & = & \neg 0 & = & \neg\neg 1 & & & \implies & \neg \text{ is } \textit{involutory} \text{ at } 1\\
a & \le & 1 & = & \neg 0 & = & \neg\neg a & \implies & \neg \text{ has } \textit{weak double negation} \text{ at } a\\
0 & = & \neg 1 & = & 0^{\neg\neg} & & & \implies & \neg \text{ is } \textit{involutory} \text{ at } 0\\
b & = & \neg b & = & \neg\neg b & = & & \implies & \neg \text{ is } \textit{involutory} \text{ at } b
\end{array}$$

3. Proof that $\neg$ does *not* have the *non-contradiction* property (and so is *not* an *intuitionistic negation*):
$b \wedge \neg b = b \wedge b = b \neq 0$

4. Proof that $\neg$ is has *boundary conditions* (and so is a *fuzzy negation*): $\neg 1 = 0, \quad \neg 0 = 1$

Example 1.13. [19] The function $\neg$ illustrated in Figure 1.3 page 11 (B) is an *ortho negation* (Definition 1.3 page 2).

PROOF:

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{lllllllllll}
a & \le & 1 & \implies & \neg 1 & = & 0 & \le & b & = & \neg a\\
0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & 1 & = & \neg 0\\
0 & \le & a & \implies & \neg a & = & b & \le & 1 & = & \neg 0\\
b & \le & 1 & \implies & \neg 1 & = & 0 & \le & a & = & \neg b\\
0 & \le & b & \implies & \neg b & = & a & \le & 1 & = & \neg 0
\end{array}$$

2. Proof that $\neg$ is *involutory* (and so is a *de Morgan negation*):
$$\begin{array}{lllll}
1 & = & \neg 0 & = & \neg\neg 1\\
a & = & \neg a & = & \neg\neg a\\
b & = & \neg b & = & \neg\neg b\\
0 & = & \neg 0 & = & 0^{\neg\neg}
\end{array}$$

3. Proof that $\neg$ is has the *non-contradiction* property (and so is an *ortho negation*):
$$\begin{array}{lllllllll}
1 & \wedge & \neg 1 & = & 1 & \wedge & 0 & = & 0\\
a & \wedge & \neg a & = & a & \wedge & b & = & 0\\
b & \wedge & \neg b & = & b & \wedge & a & = & 0\\
0 & \wedge & \neg 0 & = & 0 & \wedge & 1 & = & 0
\end{array}$$

Example 1.14 (BN_4). [20] The function $\neg$ illustrated in Figure 1.3 page 11 (C) is a **de Morgan negation** (Definition 1.3 page 2), but it is *not* a *Kleene negation* and not an *ortho negation* (it does *not* satisfy the *Kleene condition*).

PROOF:

[19] Belnap (1977) page 13, Restall (2000) page 177 ⟨Example 8.44⟩, Pavičić and Megill (2008) page 28 ⟨Definition 2, *classical implication*⟩

[20] Cignoli (1975) page 270, Restall (2000) page 171 ⟨Example 8.39⟩, de Vries (2007) pages 15–16 ⟨Example 26⟩, Dunn (1976), Belnap (1977)

1. Proof that $\neg$ is *antitone*:
$$\begin{array}{ccccccccccc} a & \le & 1 & \implies & \neg 1 & = & 0 & \le & b & = & \neg a \\ 0 & \le & 1 & \implies & \neg 1 & = & 0 & \le & 1 & = & \neg 0 \\ 0 & \le & a & \implies & \neg a & = & a & \le & 1 & = & \neg 0 \\ b & \le & 1 & \implies & \neg 1 & = & 0 & \le & b & = & \neg b \\ 0 & \le & b & \implies & \neg b & = & b & \le & 1 & = & \neg 0 \end{array}$$

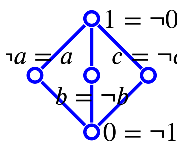
2. Proof that $\neg$ is *involutory* (and so is a *de Morgan negation*):
$$\begin{array}{ccccc} 1 & = & \neg 0 & = & \neg\neg 1 \\ a & = & \neg a & = & \neg\neg a \\ b & = & \neg b & = & \neg\neg b \\ 0 & = & \neg 0 & = & 0^{\neg\neg} \end{array}$$

3. Proof that $\neg$ does *not* have the *non-contradiction* property (and so is *not* an *ortho negation*):
$$\begin{array}{ccccccccc} a & \wedge & \neg a & = & a & \wedge & a & = & a \neq 0 \\ b & \wedge & \neg b & = & b & \wedge & b & = & b \neq 0 \end{array}$$

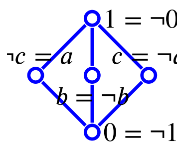
4. Proof that $\neg$ does *not* satisfy the *Kleene condition* (and so is a *de Morgan negation*):
$$a \wedge \neg a = a \wedge a = a \not\le b \wedge \neg b = b$$

Example 1.15.

EX

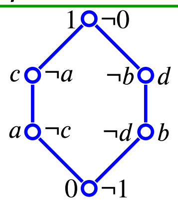


The function $\neg$ illustrated to the left is a *de Morgan negation* (Definition 1.3 page 2), but it is *not* a *Kleene negation* and not an *ortho negation* (it does *not* satisfy the *Kleene condition*). The *negation* illustrated to the right is a *Kleene negation* (Definition 1.3 page 2), but it is *not* an *ortho negation* (it does *not* have the *non-contradiction* property).

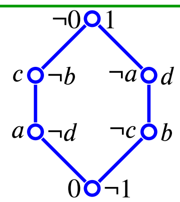


Example 1.16.

EX

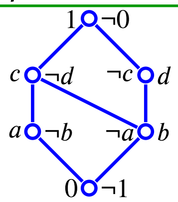


The function $\neg$ illustrated to the left is a *de Morgan negation* (Definition 1.3 page 2); it is *not* a *Kleene negation* (it does not satisfy the Kleene condition). The *negation* illustrated to the right is an *ortho negation* (Definition 1.3 page 2).

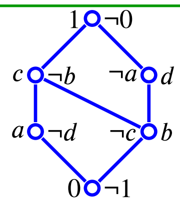


Example 1.17.

EX



The function $\neg$ illustrated to the left is *not antitone* and therefore is not a *negation* (Definition 1.2 page 2). The function $\neg$ illustrated to the right is a *Kleene negation* (Definition 1.3 page 2); it is *not* an *ortho negation* (it does not have the *non-contradiction* property).



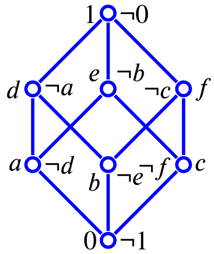
✎PROOF:

1. Proof that left $\neg$ is *not antitone*: $a \le c$ but $\neg c \not\le \neg a$.

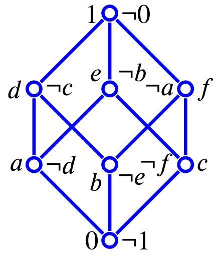
2. Proof that right $\neg$ satisfies the *Kleene condition*:
$$x \wedge \neg x = \begin{cases} b & \text{for } x = b \\ 0 & \text{otherwise} \end{cases} \quad \forall x \in X \qquad \text{and} \qquad y \wedge \neg y = \begin{cases} c & \text{for } y = c \\ 0 & \text{otherwise} \end{cases} \quad \forall y \in X$$
$$\implies \quad x \wedge \neg x \le y \vee \neg y \qquad \forall x, y \in X$$

3. Proof that right $\neg$ does not have the *non-contradiction* property: $b \wedge \neg b = b \wedge c = b \neq 0$

Example 1.18.



The lattices illustrated to the left and right are *Boolean* (Definition I.1 page 171). The function ¬ illustrated to the left is a *Kleene negation* (Definition 1.3 page 2), but it is *not* an *ortho negation* (it does *not* have the *non-contradiction* property). The *negation* illustrated to the right is an *ortho negation* (Definition 1.3 page 2).



✎PROOF:

1. Proof that left side negation does *not* have *non-contradiction* property (and so is *not* an *ortho negation*):
   $a \wedge \neg a = a \wedge d = a \neq 0$

2. Proof that left side negation does *not* satisfy *Kleene condition* (and so is *not* a *Kleene negation*):
   $a \wedge \neg a = a \wedge d = a \not\leq f = c \vee f = c \vee \neg c$

CHAPTER 2

IMPLICATION

In this document, *implication* is defined as in Definition 3.1 (next).

Definition 2.1. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \le)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133).*

DEF

The function $\to$ *in* X^X *is an* ***implication*** *on* $\mathbf{L}$ *if*

1. $\{x \le y\} \implies x \to y \ \ge\ x \vee y \quad \forall x,y \in X$ (WEAK ENTAILMENT) *and*
2. $x \wedge (x \to y) \ \le\ \neg x \vee y \quad \forall x,y \in X$ (WEAK MODUS PONENS)

Proposition 2.1. *Let* $\to$ *be an* IMPLICATION *(Definition 3.1 page 22) on a* BOUNDED LATTICE $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \le)$ *(Definition E.1 page 133).*

PRP

$$\{x \le y\} \iff \{x \to y \ge x \vee y\} \qquad \forall x,y \in X$$

✎PROOF:

1. Proof for $\implies$ case: by *weak entailment* property of *implications* (Definition 3.1 page 22).
2. Proof for $\impliedby$ case:

$y \ge x \wedge (x \to y)$	by right hypothesis
$\ge x \wedge (x \vee y)$	by *modus ponens* property of $\to$ (Definition 3.1 page 22)
$= x$	by *absorptive* property of *lattices* (Definition D.3 page 117)

✏

Remark 2.1. [1] Let $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \le)$ be a *bounded lattice* (Definition E.1 page 133). In the context of *ortho lattices*, a more common (and stronger) definition of *implication* $\to$ might be

1. $x \le y \implies x \to y \ =\ 1 \quad \forall x,y \in X$ (*entailment / strong entailment*) and
2. $x \wedge (x \to y) \ \le\ y \quad \forall x,y \in X$ (*modus ponens / strong modus ponens*)

This definition yields a result stronger than that of Proposition 3.1 (page 22):

$$\{x \le y\} \iff \{x \to y = 1\} \qquad \forall x,y \in X$$

[1] Hardegree (1979) page 59 ⟨(E),(MP),(E*)⟩, Kalmbach (1973) page 498, Kalmbach (1983) pages 238–239 ⟨Chapter 4 §15⟩, Pavičić and Megill (2008) page 24, Xu et al. (2003) page 27 ⟨Definition 2.1.1⟩, Xu (1999) page 25, Jun et al. (1998) page 54

The *Heyting 3-valued logic* (Example 3.6 page 30) and *Sasaki hook logic* (Example 3.9 page 31) have both *strong entailment* and *strong modus ponens*. However, for non-ortho logics in general, these two properties seem inappropriate to serve as a definition for *implication*. For example, the *Kleene 3-valued logic* (Example 3.3 page 28), RM_3 *logic* (Example 3.5 page 29), and BN_4 *logic* (Example 3.10 page 31) do not have the *strong entailment* property; and the *Kleene 3-valued logic*, *Łukasiewicz 3-valued logic* (Example 3.4 page 29), and BN_4 *logic* do not have the *strong modus ponens* property.

✎PROOF:

1. Proof for $\implies$ case: by *entailment* property of *implications* (Definition 3.1 page 22).
2. Proof for $\impliedby$ case:

$$\begin{aligned}
x \to y = 1 &\implies x \wedge 1 \le y && \text{by } \textit{modus ponens} \text{ property (Definition 3.1 page 22)}\\
&\implies x \le y && \text{by definition of 1 } (\textit{least upper bound}) \text{ (Definition C.21 page 114)}
\end{aligned}$$

✏

Example 2.1. [2] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ be a *lattice with negation* (Definition 1.5 page 3).

EX

If $\boldsymbol{L}$ is an **orthomodular lattice** (Definition 1.3 page 2), then the functions listed below are all examples of valid *implication* functions (Definition 3.1 page 22) on $\boldsymbol{L}$. If $\boldsymbol{L}$ is an **ortho lattice**, then 1–5 are *implication* relations.

1. $x \overset{c}{\to} y \triangleq \neg x \vee y \quad \forall x,y \in X$ (*classical implication / material implication / horseshoe*)
2. $x \overset{s}{\to} y \triangleq \neg x \vee (x \wedge y) \quad \forall x,y \in X$ (*Sasaki hook / quantum implication*)
3. $x \overset{d}{\to} y \triangleq y \vee (\neg x \wedge \neg y) \quad \forall x,y \in X$ (*Dishkant implication*)
4. $x \overset{k}{\to} y \triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (x \wedge (\neg x \vee y)) \quad \forall x,y \in X$ (*Kalmbach implication*)
5. $x \overset{n}{\to} y \triangleq (\neg x \wedge y) \vee (x \wedge y) \vee ((\neg x \vee y) \wedge \neg y) \quad \forall x,y \in X$ (*non-tollens implication*)
6. $x \overset{r}{\to} y \triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) \quad \forall x,y \in X$ (*relevance implication*)

Moreover, if $\boldsymbol{L}$ is a **Boolean lattice**, then all of these implications are equivalent to $\overset{c}{\to}$, and all of them have *strong entailment* and *strong modus ponens*.

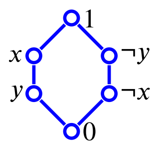
Note that $\forall x,y \in X$, $x \overset{d}{\to} y = \neg y \overset{s}{\to} \neg x$ and $x \overset{n}{\to} y = \neg y \overset{k}{\to} \neg x$. The values for the 6 implications on an *orthocomplemented* O_6 *lattice* (Definition J.2 page 196) are listed in Example 3.11 (page 31).

✎PROOF:

1. Proofs for the *classical implication* $\overset{c}{\to}$:
   (a) Proof that on an *ortho lattice*, $\overset{c}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{c}{\to} y &\triangleq \neg x \vee y && \text{by definition of } \overset{c}{\to}\\
&\ge \neg y \vee y && \text{by } x \le y \text{ and } \textit{antitone} \text{ property of } \neg \text{ (Definition 1.3 page 2)}\\
&= 1 && \text{by } \textit{excluded middle} \text{ property of } \neg \text{ (Theorem 1.5 page 6)}\\
&\implies \textit{strong entailment} && \text{by definition of } \textit{strong entailment}\\
x \wedge (\neg x \vee y) &\le \neg x \vee y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\implies \textit{weak modus ponens} && \text{by definition of } \textit{weak modus ponens}
\end{aligned}$$

Note that in general for an *ortho lattice*, the bound cannot be tightened to *strong modus ponens* because, for example in the O_6 *lattice* (Definition J.2 page 196) illustrated to the right



$$x \wedge (\neg x \vee y) = x \wedge 1 = x \not\le y \implies \textit{not strong modus ponens}$$

[2] Kalmbach (1973) page 499, Kalmbach (1974), Mittelstaedt (1970) ⟨Sasaki hook⟩, Finch (1970) page 102 ⟨Sasaki hook (1.1)⟩, Kalmbach (1983) page 239 ⟨Chapter 4 §15, 3. THEOREM⟩

(b) Proof that on a *Boolean lattice*, $\overset{\varsigma}{\to}$ is an *implication*:

$$\begin{aligned}
x \wedge (\neg x \vee y) &= (x \wedge \neg x) \vee (x \wedge y) && \text{by \textit{distributive} property of \textit{Boolean lattices} (Definition I.1 page 171)}\\
&= 1 \vee (x \wedge y) && \text{by \textit{excluded middle} property of \textit{Boolean lattices}}\\
&= x \wedge y && \text{by definition of } 1\\
&\leq y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\implies \textit{strong modus ponens} && \text{by definition of \textit{strong modus ponens}}
\end{aligned}$$

2. Proofs for *Sasaki implication* $\overset{s}{\to}$:

(a) Proof that on an *ortho lattice*, $\overset{s}{\to}$ is an *implication*:

$$\begin{aligned}
x \leq y \implies x \overset{s}{\to} y &\triangleq \neg x \vee (x \wedge y) && \text{by definition of } \overset{k}{\to}\\
&= \neg x \vee x && \text{by } x \leq y \text{ hypothesis}\\
&= 1 && \text{by \textit{excluded middle} prop. of \textit{ortho negation} (Theorem 1.5 page 6)}\\
&\implies \textit{strong entailment} && \text{by definition of \textit{strong entailment}}\\
x \wedge (x \overset{s}{\to} y) &\triangleq x \wedge [\neg x \vee (x \wedge y)] && \text{by definition of } \overset{s}{\to}\\
&\leq [\neg x \vee (x \wedge y)] && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\leq \neg x \vee y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(b) Proof that on a *Boolean lattice*, $\overset{s}{\to} = \overset{\varsigma}{\to}$:

$$\begin{aligned}
x \overset{s}{\to} y &\triangleq \neg x \vee (x \wedge y) && \text{by definition of } \overset{s}{\to}\\
&= \neg x \vee y && \text{by Lemma I.2 (page 177)}\\
&= x \overset{\varsigma}{\to} y && \text{by definition of } \overset{\varsigma}{\to}
\end{aligned}$$

3. Proofs for *Dishkant implication* $\overset{d}{\to}$:

(a) Proof that $x \overset{d}{\to} y \equiv \neg y \overset{s}{\to} \neg x$:

$$\begin{aligned}
x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= y \vee (\neg y \wedge \neg x) && \text{by \textit{commutative} property of \textit{lattices} (Theorem D.3 page 118)}\\
&= \neg\neg y \vee (\neg y \wedge \neg x) && \text{by \textit{involutory} prop. of \textit{ortho negations} (Definition 1.3 page 2)}\\
&\triangleq \neg y \overset{s}{\to} \neg x && \text{by definition of } \overset{s}{\to}
\end{aligned}$$

(b) Proof that on an *ortho lattice*, $\overset{d}{\to}$ is an *implication*:

$$\begin{aligned}
x \leq y \implies x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= y \vee \neg y && \text{by } x \leq y \text{ hypothesis and \textit{antitone} property (Definition 1.3 page 2)}\\
&= 1 && \text{by \textit{excluded middle} prop. of \textit{ortho negation} (Theorem 1.5 page 6)}\\
&\implies \textit{strong entailment} && \text{by definition of \textit{strong entailment}}\\
x \wedge (x \overset{d}{\to} y) &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= y \vee \neg x && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(c) Proof that on a *Boolean lattice*, $\overset{d}{\to} = \overset{\varsigma}{\to}$:

$$\begin{aligned}
x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to}\\
&= \neg x \vee y && \text{by Lemma I.2 (page 177)}\\
&= x \overset{\varsigma}{\to} y && \text{by definition of } \overset{\varsigma}{\to}
\end{aligned}$$

4. Proofs for the *Kalmbach implication* $\overset{k}{\rightarrow}$:

   (a) Proof that on an *ortho lattice*, $\overset{k}{\rightarrow}$ is an *implication*:

$$
\begin{aligned}
x \le y \implies x \overset{k}{\rightarrow} y
&\triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \overset{k}{\rightarrow}\\
&= (\neg x \wedge y) \vee (\neg y) \vee [x \wedge (\neg x \vee y)] && \text{by \textit{antitone} property (Definition 1.3 page 2)}\\
&= (\neg x \wedge y) \vee \neg y \vee [x \wedge (1)] && \\
&= (\neg x \wedge y) \vee (x \vee \neg y) && \text{by definition of 1 (Definition C.21 page 114)}\\
&= \neg\neg(\neg x \wedge y) \vee (x \vee \neg y) && \text{by \textit{involutory} property (Definition 1.3 page 2)}\\
&= \neg(\neg\neg x \vee \neg y) \vee (x \vee \neg y) && \text{by \textit{de Morgan} property (Theorem 1.5 page 6)}\\
&= \neg(x \vee \neg y) \vee (x \vee \neg y) && \text{by \textit{involutory} property (Definition 1.3 page 2)}\\
&= 1 && \text{by \textit{excluded middle} property (Theorem 1.5 page 6)}\\
&\implies \textit{strong entailment}
\end{aligned}
$$

$$
\begin{aligned}
x \wedge (x \overset{k}{\rightarrow} y) &\triangleq x \wedge [(\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)]] && \text{by definition of } \overset{k}{\rightarrow}\\
&\le (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\le (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (\neg x \vee y) && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\le y \vee (\neg x \wedge \neg y) \vee \neg x \vee y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&= y \vee \neg x \vee (\neg x \wedge \neg y) && \text{by \textit{idempotent} p. (Theorem D.3 page 118)}\\
&\le y \vee \neg x \vee \neg x && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&= \neg x \vee y && \text{by \textit{idempotent} p. (Theorem D.3 page 118)}\\
&\implies \textit{weak modus ponens}
\end{aligned}
$$

   (b) Proof that on a *Boolean lattice*, $\overset{k}{\rightarrow} = \overset{c}{\rightarrow}$:

$$
\begin{aligned}
x \overset{k}{\rightarrow} y &\triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \overset{k}{\rightarrow}\\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [(x \wedge \neg x) \vee (x \wedge y)] && \text{by \textit{distributive} property (Definition I.1 page 171)}\\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [(0) \vee (x \wedge y)] && \text{by \textit{non-contradiction} property}\\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (x \wedge y) && \text{by \textit{bounded} property (Definition E.1 page 133)}\\
&= \neg x \wedge (y \vee \neg y) \vee (x \wedge y) && \text{by \textit{distributive} property (Definition I.1 page 171)}\\
&= \neg x \wedge 1 \vee (x \wedge y) && \text{by \textit{excluded middle} property}\\
&= \neg x \vee (x \wedge y) && \text{by definition of 1 (Definition C.21 page 114)}\\
&= \neg x \vee y && \text{by Lemma I.2 (page 177)}\\
&\triangleq x \overset{c}{\rightarrow} y && \text{by definition of } \overset{c}{\rightarrow}
\end{aligned}
$$

5. Proofs for the *non-tollens implication* $\overset{n}{\rightarrow}$:

   (a) Proof that $x \overset{n}{\rightarrow} y \equiv \neg y \overset{k}{\rightarrow} \neg x$:

$$
\begin{aligned}
x \overset{n}{\rightarrow} y &\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee [(\neg x \vee y) \wedge \neg y] && \text{by definition of } \overset{n}{\rightarrow}\\
&= (y \wedge \neg x) \vee (y \wedge x) \vee [\neg y \wedge (y \vee \neg x)] && \\
&= (\neg\neg y \wedge \neg x) \vee (\neg\neg y \wedge \neg\neg x) \vee [\neg y \wedge (\neg\neg y \vee \neg x)] && \\
&\triangleq \neg y \overset{k}{\rightarrow} \neg x && \text{by definition of } \overset{k}{\rightarrow}
\end{aligned}
$$

(b) Proof that on an *ortho lattice*, $\overset{n}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{n}{\to} y & \\
\equiv \neg y \overset{k}{\to} \neg x & \quad \text{by item (5a) page 25} \\
= 1 & \quad \text{by item (4a) page 25} \\
\implies \textit{strong entailment} &
\end{aligned}$$

$$\begin{aligned}
x \wedge (x \overset{n}{\to} y) &= x \wedge (\neg y \overset{k}{\to} \neg x) && \text{by item (5a) page 25} \\
&\le \neg\neg y \vee \neg x && \text{by item (4a) page 25} \\
&= y \vee \neg x && \text{by } \textit{involutory} \text{ property of } \neg \text{ (Definition 1.3 page 2)} \\
&= \neg x \vee y && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Definition D.3 page 117)} \\
&\implies \textit{weak modus ponens}
\end{aligned}$$

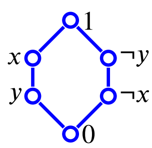
(c) Proof that on a *Boolean lattice*, $\overset{n}{\to} = \overset{c}{\to}$:

$$\begin{aligned}
x \overset{n}{\to} y &= \neg y \overset{k}{\to} \neg x && \text{by item (5a) page 25} \\
&= \neg\neg y \vee \neg x && \text{by item (4b) page 25} \\
&= y \vee \neg x && \text{by } \textit{involutory} \text{ property of } \neg \text{ (Definition 1.3 page 2)} \\
&= \neg x \vee y && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Definition D.3 page 117)} \\
&\triangleq x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}$$

6. Proofs for the *relevance implication* $\overset{r}{\to}$:

(a) Proof that on an *ortho lattice*, $\overset{r}{\to}$ does *not* have *weak entailment*:
In the *ortho lattice* to the right…

$$\begin{aligned}
x \le y \implies x \overset{r}{\to} y & \\
\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) & \quad \text{by definition of } \overset{r}{\to} \\
= 0 \vee x \vee \neg y & \\
= x \vee \neg y & \\
\ne x \vee y &
\end{aligned}$$



(b) Proof that on an *orthomodular lattice*, $\overset{r}{\to}$ *does* have *strong entailment*:

$$\begin{aligned}
x \le y \implies x \overset{r}{\to} y & \\
\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) & \quad \text{by definition of } \overset{r}{\to} \\
= (\neg x \wedge y) \vee x \vee (\neg x \wedge \neg y) & \quad \text{by } x \le y \text{ hypothesis} \\
= (\neg x \wedge y) \vee x \vee \neg y & \quad \text{by } x \le y \text{ and } \textit{antitone} \text{ property (Definition 1.3 page 2)} \\
= y \vee \neg y & \quad \text{by } \textit{orthomodular identity} \text{ (Definition J.3 page 205)} \\
= 1 & \quad \text{by } \textit{excluded middle} \text{ property of } \neg \text{ (Theorem 1.5 page 6)}
\end{aligned}$$

(c) Proof that on an *ortho lattice*, $\overset{r}{\to}$ *does* have *weak modus ponens*:

$$\begin{aligned}
x \wedge (x \overset{r}{\to} y) &\triangleq x \wedge [(\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y)] && \text{by definition of } \overset{r}{\to} \\
&\le [(\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y)] && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\le \neg x \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\le \neg x \vee y \vee (\neg x \wedge \neg y) && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\le \neg x \vee y && \text{by } \textit{absorption} \text{ property (Theorem D.3 page 118)} \\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(d) Proof that on a *Boolean lattice*, $\overset{r}{\to} = \overset{c}{\to}$:

$$
\begin{aligned}
x \overset{r}{\to} y &\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{r}{\to} \\
&= [\neg x \wedge (y \vee \neg y)] \vee (x \wedge y) && \text{by \textit{distributive} property (Definition I.1 page 171)} \\
&= [\neg x \wedge 1] \vee (x \wedge y) && \text{by \textit{excluded middle} property of } \neg \text{ (Theorem 1.5 page 6)} \\
&= \neg x \vee (x \wedge y) && \text{by definition of 1 and } \wedge \text{ (Definition C.22 page 114)} \\
&= \neg x \vee y && \text{by property of \textit{Boolean lattices} (Lemma I.2 page 177)} \\
&\triangleq x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}
$$

✏

CHAPTER 3

LOGIC

❝*I dare say that this is the last effort of the human mind, and when this project shall have been carried out, all that men will have to do will be to be happy, since they will have an instrument that will serve to exalt the intellect not less than the telescope serves to perfect their vision.*❞

Gottfried Leibniz (1646–1716), German mathematician, sharing his thoughts regarding *mathematical logic.* [1]

❝*I cannot forget or omit to record this day last week. I was sleeping as usual for the night at St. Michael's Hamlet. As I awoke in the morning, the sun was shining brightly into my room. There was a consciousness on my mind that I was the discoverer of the true logic of the future. For a few minutes I felt a delight such as one can seldom hope to feel. But it would not last long— I remembered only too soon how unworthy and weak an instrument I was for accomplishing so great a work, and how hardly could I expect to do it.*❞

William Stanley Jevons (1835–1882), English economist and logician [2]

3.1 Implications

Arguably a logic is not a logic without the inclusion of an *implication* function $\rightarrow$. The mathematical structure *logic* is formally defined in Definition 3.2 (page 27). But before defining a logic, this text offers a very general definition (a "weak" definition) of implication that can be used in defining a very wide class of logics—including *non-Boolean* ones. For *Boolean* logics, the *classical implication* function $x \overset{c}{\rightarrow} y$ (Example 3.1 page 23) is arguably adequate. Two key properties of *classical implication*

[1] quote: Padoa (1912) page 21
Cajori (1993) ⟨paragraph 541⟩
image: http://en.wikipedia.org/wiki/Gottfried_Leibniz, public domain

[2] image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Jevons.html
quote: Jevons (1886) page 219 ⟨1866 March 28 entry⟩

on a *Boolean* logic are *entailment* and *modus ponens*. The following definition exploits weakened versions of these two properties to define implication. Note that the definition is at this time probably not standard in the literature. But without it, it is difficult to offer a complete definition of a logic.

Definition 3.1. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133).*

DEF

The function $\rightarrow$ *in* X^X *is an* ***implication*** *on* $\boldsymbol{L}$ *if*

1. $\{x \leq y\} \implies x \rightarrow y \;\geq\; x \vee y \quad \forall x,y \in X$ (WEAK ENTAILMENT) *and*
2. $x \wedge (x \rightarrow y) \;\leq\; \neg x \vee y \quad \forall x,y \in X$ (WEAK MODUS PONENS)

Proposition 3.1. *Let* $\rightarrow$ *be an* IMPLICATION *(Definition 3.1 page 22) on a* BOUNDED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *(Definition E.1 page 133).*

PRP

$$\{x \leq y\} \iff \{x \rightarrow y \geq x \vee y\} \qquad \forall x,y \in X$$

✎PROOF:

1. Proof for $\implies$ case: by *weak entailment* property of *implications* (Definition 3.1 page 22).
2. Proof for $\impliedby$ case:

$$\begin{aligned} y &\geq x \wedge (x \rightarrow y) && \text{by right hypothesis} \\ &\geq x \wedge (x \vee y) && \text{by } \textit{modus ponens} \text{ property of } \rightarrow \text{ (Definition 3.1 page 22)} \\ &= x && \text{by } \textit{absorptive} \text{ property of } \textit{lattices} \text{ (Definition D.3 page 117)} \end{aligned}$$

✏

Remark 3.1. [3] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ be a *bounded lattice* (Definition E.1 page 133). In the context of *ortho lattices*, a more common (and stronger) definition of *implication* $\rightarrow$ might be

1. $x \leq y \implies x \rightarrow y \;=\; 1 \quad \forall x,y \in X$ (*entailment / strong entailment*) and
2. $x \wedge (x \rightarrow y) \;\leq\; y \quad \forall x,y \in X$ (*modus ponens / strong modus ponens*)

This definition yields a result stronger than that of Proposition 3.1 (page 22):

$$\{x \leq y\} \iff \{x \rightarrow y = 1\} \qquad \forall x,y \in X$$

The *Heyting 3-valued logic* (Example 3.6 page 30) and *Sasaki hook logic* (Example 3.9 page 31) have both *strong entailment* and *strong modus ponens*. However, for non-ortho logics in general, these two properties seem inappropriate to serve as a definition for *implication*. For example, the *Kleene 3-valued logic* (Example 3.3 page 28), RM_3 *logic* (Example 3.5 page 29), and BN_4 *logic* (Example 3.10 page 31) do not have the *strong entailment* property; and the *Kleene 3-valued logic*, *Łukasiewicz 3-valued logic* (Example 3.4 page 29), and BN_4 *logic* do not have the *strong modus ponens* property.

✎PROOF:

1. Proof for $\implies$ case: by *entailment* property of *implications* (Definition 3.1 page 22).
2. Proof for $\impliedby$ case:

$$\begin{aligned} x \rightarrow y = 1 &\implies x \wedge 1 \leq y && \text{by } \textit{modus ponens} \text{ property (Definition 3.1 page 22)} \\ &\implies x \leq y && \text{by definition of 1 (} \textit{least upper bound} \text{) (Definition C.21 page 114)} \end{aligned}$$

[3] Hardegree (1979) page 59 ⟨(E),(MP),(E*)⟩, Kalmbach (1973) page 498, Kalmbach (1983) pages 238–239 ⟨Chapter 4 §15⟩, Pavičić and Megill (2008) page 24, Xu et al. (2003) page 27 ⟨Definition 2.1.1⟩, Xu (1999) page 25, Jun et al. (1998) page 54

Example 3.1. [4] Let $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1 ; \leq)$ be a *lattice with negation* (Definition 1.5 page 3).

If $\boldsymbol{L}$ is an **orthomodular lattice** (Definition 1.3 page 2), then the functions listed below are all examples of valid *implication* functions (Definition 3.1 page 22) on $\boldsymbol{L}$. If $\boldsymbol{L}$ is an **ortho lattice**, then 1–5 are *implication* relations.

1. $x \overset{c}{\rightarrow} y \triangleq \neg x \vee y \quad \forall x,y \in X$ (*classical implication / material implication / horseshoe*)
2. $x \overset{s}{\rightarrow} y \triangleq \neg x \vee (x \wedge y) \quad \forall x,y \in X$ (*Sasaki hook / quantum implication*)
3. $x \overset{d}{\rightarrow} y \triangleq y \vee (\neg x \wedge \neg y) \quad \forall x,y \in X$ (*Dishkant implication*)
4. $x \overset{k}{\rightarrow} y \triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (x \wedge (\neg x \vee y)) \quad \forall x,y \in X$ (*Kalmbach implication*)
5. $x \overset{n}{\rightarrow} y \triangleq (\neg x \wedge y) \vee (x \wedge y) \vee ((\neg x \vee y) \wedge \neg y) \quad \forall x,y \in X$ (*non-tollens implication*)
6. $x \overset{r}{\rightarrow} y \triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) \quad \forall x,y \in X$ (*relevance implication*)

Moreover, if $\boldsymbol{L}$ is a **Boolean lattice**, then all of these implications are equivalent to $\overset{c}{\rightarrow}$, and all of them have *strong entailment* and *strong modus ponens*.

Note that $\forall x,y \in X$, $x \overset{d}{\rightarrow} y = \neg y \overset{s}{\rightarrow} \neg x$ and $x \overset{n}{\rightarrow} y = \neg y \overset{k}{\rightarrow} \neg x$. The values for the 6 implications on an *orthocomplemented* O_6 *lattice* (Definition J.2 page 196) are listed in Example 3.11 (page 31).

✎Proof:

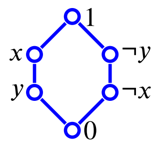
1. Proofs for the *classical implication* $\overset{c}{\rightarrow}$:

 (a) Proof that on an *ortho lattice*, $\overset{c}{\rightarrow}$ is an *implication*:

$$\begin{aligned}
x \leq y \implies x \overset{c}{\rightarrow} y &\triangleq \neg x \vee y && \text{by definition of } \overset{c}{\rightarrow} \\
&\geq \neg y \vee y && \text{by } x \leq y \text{ and } \textit{antitone} \text{ prop. of } \neg \text{ (Definition 1.3 page 2)} \\
&= 1 && \text{by } \textit{excluded middle} \text{ prop. of } \neg \text{ (Theorem 1.5 page 6)} \\
&\implies \textit{strong entailment} && \text{by definition of } \textit{strong entailment} \\
x \wedge (\neg x \vee y) &\leq \neg x \vee y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\implies \textit{weak modus ponens} && \text{by definition of } \textit{weak modus ponens}
\end{aligned}$$

 Note that in general for an *ortho lattice*, the bound cannot be tightened to *strong modus ponens* because, for example in the O_6 *lattice* (Definition J.2 page 196) illustrated to the right

$$x \wedge (\neg x \vee y) = x \wedge 1 = x \not\leq y \implies \textit{not strong modus ponens}$$



 (b) Proof that on a *Boolean lattice*, $\overset{c}{\rightarrow}$ is an *implication*:

$$\begin{aligned}
x \wedge (\neg x \vee y) &= (x \wedge \neg x) \vee (x \wedge y) && \text{by } \textit{distributive} \text{ prop. of Boolean lat. (Definition I.1 page 171)} \\
&= 1 \vee (x \wedge y) && \text{by } \textit{excluded middle} \text{ property of } \textit{Boolean lattices} \\
&= x \wedge y && \text{by definition of 1} \\
&\leq y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\implies \textit{strong modus ponens} && \text{by definition of } \textit{strong modus ponens}
\end{aligned}$$

2. Proofs for *Sasaki implication* $\overset{s}{\rightarrow}$:

[4] Kalmbach (1973) page 499, Kalmbach (1974), Mittelstaedt (1970) ⟨Sasaki hook⟩, Finch (1970) page 102 ⟨Sasaki hook (1.1)⟩, Kalmbach (1983) page 239 ⟨Chapter 4 §15, 3. Theorem⟩

(a) Proof that on an *ortho lattice*, $\overset{s}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{s}{\to} y & && \\
&\triangleq \neg x \vee (x \wedge y) && \text{by definition of } \overset{k}{\to} \\
&= \neg x \vee x && \text{by } x \le y \text{ hypothesis} \\
&= 1 && \text{by } \textit{excluded middle} \text{ prop. of ortho neg. (Theorem 1.5 page 6)} \\
&\implies \textit{strong entailment} && \text{by definition of } \textit{strong entailment} \\
x \wedge (x \overset{s}{\to} y) &\triangleq x \wedge [\neg x \vee (x \wedge y)] && \text{by definition of } \overset{s}{\to} \\
&\le [\neg x \vee (x \wedge y)] && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\le \neg x \vee y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\implies \textit{weak modus ponens} &&
\end{aligned}$$

(b) Proof that on a *Boolean lattice*, $\overset{s}{\to} = \overset{c}{\to}$:

$$\begin{aligned}
x \overset{s}{\to} y &\triangleq \neg x \vee (x \wedge y) && \text{by definition of } \overset{s}{\to} \\
&= \neg x \vee y && \text{by Lemma I.2 (page 177)} \\
&= x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}$$

3. Proofs for *Dishkant implication* $\overset{d}{\to}$:

(a) Proof that $x \overset{d}{\to} y \equiv \neg y \overset{s}{\to} \neg x$:

$$\begin{aligned}
x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to} \\
&= y \vee (\neg y \wedge \neg x) && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Theorem D.3 page 118)} \\
&= \neg\neg y \vee (\neg y \wedge \neg x) && \text{by } \textit{involutory} \text{ property of } \textit{ortho negations} \text{ (Definition 1.3 page 2)} \\
&\triangleq \neg y \overset{s}{\to} \neg x && \text{by definition of } \overset{s}{\to}
\end{aligned}$$

(b) Proof that on an *ortho lattice*, $\overset{d}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{d}{\to} y & && \\
&\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to} \\
&= y \vee \neg y && \text{by } x \le y \text{ hypoth. and } \textit{antitone} \text{ prop. (Definition 1.3 page 2)} \\
&= 1 && \text{by } \textit{excluded middle} \text{ prop. of ortho neg. (Theorem 1.5 page 6)} \\
&\implies \textit{strong entailment} && \text{by definition of } \textit{strong entailment} \\
x \wedge (x \overset{d}{\to} y) &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to} \\
&= y \vee \neg x && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\implies \textit{weak modus ponens} &&
\end{aligned}$$

(c) Proof that on a *Boolean lattice*, $\overset{d}{\to} = \overset{c}{\to}$:

$$\begin{aligned}
x \overset{d}{\to} y &\triangleq y \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{d}{\to} \\
&= \neg x \vee y && \text{by Lemma I.2 (page 177)} \\
&= x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}$$

4. Proofs for the *Kalmbach implication* $\overset{k}{\to}$:

(a) Proof that on an *ortho lattice*, $\overset{k}{\to}$ is an *implication*:

$$
\begin{aligned}
x \le y \implies x \overset{k}{\to} y
&\triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \overset{k}{\to} \\
&= (\neg x \wedge y) \vee (\neg y) \vee [x \wedge (\neg x \vee y)] && \text{by \textit{antitone} property (Definition 1.3 page 2)} \\
&= (\neg x \wedge y) \vee \neg y \vee [x \wedge (1)] && \\
&= (\neg x \wedge y) \vee (x \vee \neg y) && \text{by definition of } 1 \text{ (Definition C.21 page 114)} \\
&= \neg\neg(\neg x \wedge y) \vee (x \vee \neg y) && \text{by \textit{involutory} property (Definition 1.3 page 2)} \\
&= \neg(\neg\neg x \vee \neg y) \vee (x \vee \neg y) && \text{by \textit{de Morgan} property (Theorem 1.5 page 6)} \\
&= \neg(x \vee \neg y) \vee (x \vee \neg y) && \text{by \textit{involutory} property (Definition 1.3 page 2)} \\
&= 1 && \text{by \textit{excluded middle} property (Theorem 1.5 page 6)} \\
&\implies \textit{strong entailment}
\end{aligned}
$$

$$
\begin{aligned}
x \wedge (x \overset{k}{\to} y) &\triangleq x \wedge [(\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)]] && \text{by definition of } \overset{k}{\to} \\
&\le (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\le (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (\neg x \vee y) && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&\le y \vee (\neg x \wedge \neg y) \vee \neg x \vee y && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&= y \vee \neg x \vee (\neg x \wedge \neg y) && \text{by \textit{idempotent} p. (Theorem D.3 page 118)} \\
&\le y \vee \neg x \vee \neg x && \text{by definition of } \wedge \text{ (Definition C.22 page 114)} \\
&= \neg x \vee y && \text{by \textit{idempotent} p. (Theorem D.3 page 118)} \\
&\implies \textit{weak modus ponens}
\end{aligned}
$$

(b) Proof that on a *Boolean lattice*, $\overset{k}{\to} = \overset{c}{\to}$:

$$
\begin{aligned}
x \overset{k}{\to} y &\triangleq (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [x \wedge (\neg x \vee y)] && \text{by definition of } \overset{k}{\to} \\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [(x \wedge \neg x) \vee (x \wedge y)] && \text{by \textit{distributive} property (Definition I.1 page 171)} \\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee [(0) \vee (x \wedge y)] && \text{by \textit{non-contradiction} property} \\
&= (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (x \wedge y) && \text{by \textit{bounded} property (Definition E.1 page 133)} \\
&= \neg x \wedge (y \vee \neg y) \vee (x \wedge y) && \text{by \textit{distributive} property (Definition I.1 page 171)} \\
&= \neg x \wedge 1 \vee (x \wedge y) && \text{by \textit{excluded middle} property} \\
&= \neg x \vee (x \wedge y) && \text{by definition of } 1 \text{ (Definition C.21 page 114)} \\
&= \neg x \vee y && \text{by Lemma I.2 (page 177)} \\
&\triangleq x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}
$$

5. Proofs for the *non-tollens implication* $\overset{n}{\to}$:

(a) Proof that $x \overset{n}{\to} y \equiv \neg y \overset{k}{\to} \neg x$:

$$
\begin{aligned}
x \overset{n}{\to} y &\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee [(\neg x \vee y) \wedge \neg y] && \text{by definition of } \overset{n}{\to} \\
&= (y \wedge \neg x) \vee (y \wedge x) \vee [\neg y \wedge (y \vee \neg x)] && \\
&= (\neg\neg y \wedge \neg x) \vee (\neg\neg y \wedge \neg\neg x) \vee [\neg y \wedge (\neg\neg y \vee \neg x)] && \\
&\triangleq \neg y \overset{k}{\to} \neg x && \text{by definition of } \overset{k}{\to}
\end{aligned}
$$

(b) Proof that on an *ortho lattice*, $\overset{n}{\to}$ is an *implication*:

$$\begin{aligned}
x \le y \implies x \overset{n}{\to} y & \\
&\equiv \neg y \overset{k}{\to} \neg x && \text{by item (5a) page 25}\\
&= 1 && \text{by item (4a) page 25}\\
&\implies \textit{strong entailment}\\
x \wedge (x \overset{n}{\to} y) &= x \wedge (\neg y \overset{k}{\to} \neg x) && \text{by item (5a) page 25}\\
&\le \neg\neg y \vee \neg x && \text{by item (4a) page 25}\\
&= y \vee \neg x && \text{by \textit{involutory} property of } \neg \text{ (Definition 1.3 page 2)}\\
&= \neg x \vee y && \text{by \textit{commutative} property of \textit{lattices} (Definition D.3 page 117)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

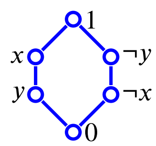
(c) Proof that on a *Boolean lattice*, $\overset{n}{\to} = \overset{\varsigma}{\to}$:

$$\begin{aligned}
x \overset{n}{\to} y &= \neg y \overset{k}{\to} \neg x && \text{by item (5a) page 25}\\
&= \neg\neg y \vee \neg x && \text{by item (4b) page 25}\\
&= y \vee \neg x && \text{by \textit{involutory} property of } \neg \text{ (Definition 1.3 page 2)}\\
&= \neg x \vee y && \text{by \textit{commutative} property of \textit{lattices} (Definition D.3 page 117)}\\
&\triangleq x \overset{\varsigma}{\to} y && \text{by definition of } \overset{\varsigma}{\to}
\end{aligned}$$

6. Proofs for the *relevance implication* $\overset{r}{\to}$:

(a) Proof that on an *ortho lattice*, $\overset{r}{\to}$ does *not* have *weak entailment*:
In the *ortho lattice* to the right...

$$\begin{aligned}
x \le y \implies x \overset{r}{\to} y & \\
&\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{r}{\to}\\
&= 0 \vee x \vee \neg y\\
&= x \vee \neg y\\
&\ne x \vee y
\end{aligned}$$



(b) Proof that on an *orthomodular lattice*, $\overset{r}{\to}$ *does* have *strong entailment*:

$$\begin{aligned}
x \le y \implies x \overset{r}{\to} y & \\
&\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{r}{\to}\\
&= (\neg x \wedge y) \vee x \vee (\neg x \wedge \neg y) && \text{by } x \le y \text{ hypothesis}\\
&= (\neg x \wedge y) \vee x \vee \neg y && \text{by } x \le y \text{ and \textit{antitone} property (Definition 1.3 page 2)}\\
&= y \vee \neg y && \text{by \textit{orthomodular identity} (Definition J.3 page 205)}\\
&= 1 && \text{by \textit{excluded middle} property of } \neg \text{ (Theorem 1.5 page 6)}
\end{aligned}$$

(c) Proof that on an *ortho lattice*, $\overset{r}{\to}$ *does* have *weak modus ponens*:

$$\begin{aligned}
x \wedge (x \overset{r}{\to} y) &\triangleq x \wedge [(\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y)] && \text{by definition of } \overset{r}{\to}\\
&\le [(\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y)] && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\le \neg x \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\le \neg x \vee y \vee (\neg x \wedge \neg y) && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\le \neg x \vee y && \text{by \textit{absorption} property (Theorem D.3 page 118)}\\
&\implies \textit{weak modus ponens}
\end{aligned}$$

(d) Proof that on a *Boolean lattice*, $\overset{r}{\to} = \overset{c}{\to}$:

$$
\begin{aligned}
x \overset{r}{\to} y &\triangleq (\neg x \wedge y) \vee (x \wedge y) \vee (\neg x \wedge \neg y) && \text{by definition of } \overset{r}{\to} \\
&= [\neg x \wedge (y \vee \neg y)] \vee (x \wedge y) && \text{by } \textit{distributive} \text{ property (Definition I.1 page 171)} \\
&= [\neg x \wedge 1] \vee (x \wedge y) && \text{by } \textit{excluded middle} \text{ property of } \neg \text{ (Theorem 1.5 page 6)} \\
&= \neg x \vee (x \wedge y) && \text{by definition of 1 and } \wedge \text{ (Definition C.22 page 114)} \\
&= \neg x \vee y && \text{by property of } \textit{Boolean lattices} \text{ (Lemma I.2 page 177)} \\
&\triangleq x \overset{c}{\to} y && \text{by definition of } \overset{c}{\to}
\end{aligned}
$$

✏

3.2 Logics

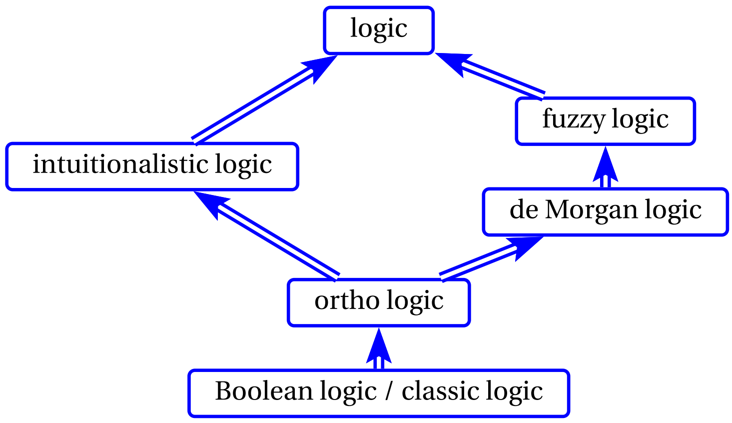


Figure 3.1: lattice of logics

Definition 3.2. [5] *Let* $\to$ *be an* IMPLICATION *(Definition 3.1 page 22) defined on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1\,;\, \le)$ *(Definition 1.5 page 3).*

DEF

$(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *is a* ***logic***	*if* $\neg$ *is a* MINIMAL NEGATION.
$(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *is a* ***fuzzy logic***	*if* $\neg$ *is a* FUZZY NEGATION.
$(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *is an* ***intuitionalistic logic***	*if* $\neg$ *is an* INTUITIONALISTIC NEGATION.
$(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *is a* ***de Morgan logic***	*if* $\neg$ *is a* DE MORGAN NEGATION.
$(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *is a* ***Kleene logic***	*if* $\neg$ *is a* KLEENE NEGATION.
$(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *is an* ***ortho logic***	*if* $\neg$ *is an* ORTHO NEGATION.
$(X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *is a* ***Boolean logic***	*if* $\neg$ *is an* ORTHO NEGATION *and* $\boldsymbol{L}$ *is* BOOLEAN.

Definition 3.3. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1\,;\, \le, \to)$ *be a* LOGIC *(Definition 3.2 page 27).*

DEF

The function $\leftrightarrow$ *in* X^X *is an* ***equivalence*** *on* $\boldsymbol{L}$ *if*

$$x \leftrightarrow y \triangleq (x \to y) \wedge (y \to x) \quad \forall x,y \in X$$

Example 3.2 (Aristotelian logic/classical logic). [7]

---

[5] Straßburger (2005) page 136 ⟨Definition 2.1⟩, de Vries (2007) page 11 ⟨Definition 16⟩

[6] Novák et al. (1999) page 18

[7] Novák et al. (1999) pages 17–18 ⟨EXAMPLE 2.1⟩

EX The *classical bi-variate logic* is defined below. It is a 2 element *Boolean logic* (Definition 3.2 page 27). with $\boldsymbol{L} \triangleq (\{1,0\}, \wedge, \neg, 0, 1, \leq; \vee)$ and a *classical implication* $\rightarrow$ with *strong entailment* and *strong modus ponens*. The value 1 represents "*true*" and 0 represents "*false*".

$1 = \neg 0$
$0 = \neg 1$

$$x \rightarrow y \triangleq \left\{ \begin{array}{ll} 1 & \forall x \leq y \\ y & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{c|cc} \rightarrow & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 1 \end{array} \quad \forall x,y \in X \right\} = \neg x \vee y$$

✎PROOF:

1. Proof that $\neg$ is an *ortho negation*: by Definition 1.3 (page 2)
2. Proof that $\rightarrow$ is an *implication* with *strong entailment* and *strong modus ponens*:
   (a) $\boldsymbol{L}$ is *Boolean* and therefore is *orthocomplemented*.
   (b) $\rightarrow$ is equivalent to the *classical implication* $\overset{c}{\rightarrow}$ (Example 3.1 page 23).
   (c) By Example 3.1 (page 23), $\rightarrow$ has *strong entailment* and *strong modus ponens*.

✏

The *classical logic* (previous example) can be generalized in several ways. Arguably one of the simplest of these is the 3-valued logic due to Kleene (next example).

Example 3.3 (Kleene 3-valued logic). [8]

EX The *Kleene 3-valued logic* $(X, \vee, \wedge, \neg, 0, 1; \leq, \rightarrow)$ is defined below. The function $\neg$ is a *Kleene negation* (Definition 1.3 page 2, Example 1.9 page 10) defined on a 3 element *linearly ordered lattice* (Definition C.4 page 103). The function $\rightarrow$ is the *classical implication* $x \rightarrow y \triangleq \neg x \vee y$. The values 1 represents "*true*", 0 represents "*false*", and n represents "*neutral*" or "*undecided*".

$1 = \neg 0$
$n = \neg n$
$0 = \neg 1$

$$x \rightarrow y \triangleq \left\{ \neg x \vee y \quad \forall x \in X \right\} = \left\{ \begin{array}{c|ccc} \rightarrow & 1 & n & 0 \\ \hline 1 & 1 & n & 0 \\ n & 1 & n & n \\ 0 & 1 & 1 & 1 \end{array} \quad \forall x,y \in X \right\}$$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 1.9 (page 10)
2. Proof that $\rightarrow$ is an *implication*: This follows directly from the definition of $\rightarrow$ and the definition of an *implication* (Definition 3.1 page 22).
3. Proof that $\rightarrow$ does not have *strong entailment*: $n \rightarrow n = n = n \vee n \neq 1$.
4. Proof that $\rightarrow$ does not have *strong modus ponens*: $n \rightarrow 0 = n = \neg n \vee 0 \nleq 0$.

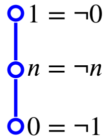
✏

A lattice and negation alone do not uniquely define a logic. Łukasiewicz also introduced a 3-valued logic with identical lattice structure to Kleene, but with a different implication relation (next example). Historically, Łukasiewicz's logic was introduced before Kleene's.

[8] Kleene (1938) page 153, Kleene (1952) pages 332–339 (§64. The 3-valued logic), Avron (1991) page 277

Example 3.4 (Łukasiewicz 3-valued logic). [9]

The *Łukasiewicz 3-valued logic* $(X, \vee, \wedge, \neg, 0, 1; \leq, \rightarrow)$ is defined to the right and below. The function $\neg$ is a *Kleene negation* (Definition 1.3 page 2) defined on a 3 element *linearly ordered lattice* (Definition C.4 page 103). The implication has *strong entailment* but *weak modus ponens*. In the implication table below, values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded.



$$x \rightarrow y \triangleq \begin{cases} 1 & \forall x \leq y \\ \neg x \vee y & \text{otherwise} \end{cases} = \left\{ \begin{array}{c|ccc} \rightarrow & 1 & n & 0 \\ \hline 1 & 1 & n & 0 \\ n & 1 & \boxed{1} & n \\ 0 & 1 & 1 & 1 \end{array} \quad \forall x,y \in X \right\} = \begin{cases} 1 & \text{for } x = y = n \\ \neg x \vee y & \text{otherwise} \end{cases}$$

✎PROOF:

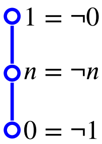
1. Proof that $\neg$ is a *Kleene negation*: see Example 1.9 (page 10)
2. Proof that $\rightarrow$ is an *implication*: This follows directly from the definition of $\rightarrow$ and the definition of an *implication* (Definition 3.1 page 22).
3. Proof that $\rightarrow$ does not have *strong modus ponens*: $n \rightarrow 0 = n = \neg n \vee 0 \nleq 0$.

✏

Example 3.5 (RM_3 logic). [10]

EX

The *RM_3 logic* $(X, \vee, \wedge, \neg, 0, 1; \leq, \rightarrow)$ is defined below. The function $\neg$ is a *Kleene negation* (Definition 1.3 page 2) defined on a 3 element *linearly ordered lattice* (Definition C.4 page 103). The implication function has *weak entailment* by *strong modus ponens*. In the implication table below, values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded.



$$x \rightarrow y \triangleq \begin{cases} 1 & \forall x < y \\ n & \forall x = y \\ 0 & \forall x > y \end{cases} = \left\{ \begin{array}{c|ccc} \rightarrow & 1 & n & 0 \\ \hline 1 & 1 & \boxed{0} & 0 \\ n & 1 & n & \boxed{0} \\ 0 & 1 & 1 & 1 \end{array} \quad \forall x,y \in X \right\}$$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 1.9 (page 10)
2. Proof that $\rightarrow$ is an *implication*: This follows directly from the definition of $\rightarrow$ and the definition of an *implication* (Definition 3.1 page 22).
3. Proof that $\rightarrow$ does not have *strong entailment*: $n \rightarrow n = n = n \vee n \neq 1$.

✏

In a 3-valued logic, the negation does not necessarily have to be as in the previous three examples. The next example offers a different negation.

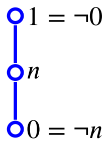
[9] Łukasiewicz (1920) page 17 ⟨II. The principles of consequence⟩, Avron (1991) page 277 ⟨Łukasiewicz.⟩

[10] Avron (1991) pages 277–278
Sobociński (1952)

Example 3.6 (Heyting 3-valued logic/Jaśkowski's first matrix). [11]

EX

The *Heyting 3-valued logic* $(X, \vee, \wedge, \neg, 0, 1; \leq, \rightarrow)$ is defined below. The negation $\neg$ is both *intuitionistic* and *fuzzy* (Definition 1.2 page 2), and is defined on a 3 element *linearly ordered lattice* (Definition C.4 page 103). The implication function has both *strong entailment* and *strong modus ponens*. In the implication table below, values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded.



$$x \rightarrow y \triangleq \left\{\begin{array}{ll} 1 & \forall x \leq y \\ y & \text{otherwise} \end{array}\right\} = \left\{\begin{array}{c|ccc} \rightarrow & 1 & n & 0 \\ \hline 1 & 1 & n & 0 \\ n & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array} \quad \forall x,y \in X\right\}$$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 1.11 (page 11)

2. Proof that $\rightarrow$ is an *implication*: by definition of *implication* (Definition 3.1 page 22)

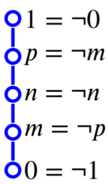
✏

Of course it is possible to generalize to more than 3 values (next example).

Example 3.7 (Łukasiewicz 5-valued logic). [12]

EX

The *Łukasiewicz 5-valued logic* $(X, \vee, \wedge, \neg, 0, 1; \leq, \rightarrow)$ is defined below. The implication function has *strong entailment* but *weak modus ponens*. In the implication table below, values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded.



$$x \rightarrow y \triangleq \left\{\begin{array}{c|ccccc} \rightarrow & 1 & p & n & m & 0 \\ \hline 1 & 1 & p & n & m & 0 \\ p & 1 & 1 & n & m & m \\ n & 1 & 1 & 1 & m & n \\ m & 1 & 1 & 1 & 1 & p \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right. \quad \forall x,y \in X$$

✎PROOF:

✏

All the previous examples in this section are *linearly ordered*. The following examples employ logics that are not.

Example 3.8 (Boolean 4-valued logic). [13]

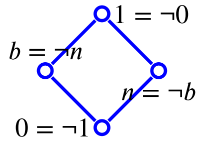
---

[11] Karpenko (2006) page 45, Johnstone (1982) page 9 ⟨§1.12⟩, Heyting (1930a), Heyting (1930b), Heyting (1930c), Heyting (1930d), Jaskowski (1936), Mancosu (1998)

[12] Xu et al. (2003) page 29 ⟨Example 2.1.3⟩
Jun et al. (1998) page 54 ⟨Example 2.2⟩

[13] Belnap (1977) page 13, Restall (2000) page 177 ⟨Example 8.44⟩, Pavičić and Megill (2008) page 28 ⟨Definition 2, *classical implication*⟩, Mittelstaedt (1970), Finch (1970) page 102 ⟨(1.1)⟩, Smets (2006) page 270

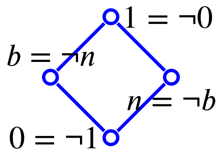
EX The *Boolean 4-valued logic* is defined below. The negation function ¬ is an *ortho negation* (Example 1.13 page 12) defined on an M_2 *lattice*. The value 1 represents "*true*", 0 represents "*false*", and m and n represent some intermediate values.



$$x \to y \triangleq \neg x \vee y = \left\{\begin{array}{c|cccc} \to & 1 & b & n & 0 \\ \hline 1 & 1 & b & n & 0 \\ b & 1 & 1 & n & n \\ n & 1 & b & 1 & b \\ 0 & 1 & 1 & 1 & 1 \end{array}\right. \qquad \forall x,y \in X$$

Example 3.9 (Sasaki hook / quantum implication). [14]

EX The *Sasaki hook logic* $(X, \vee, \wedge, \neg, 0, 1\,;\, \leq, \to)$ is defined below. The order structure and negation are the same as in Example 3.8 (page 30).

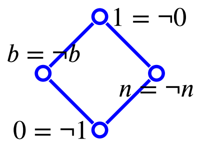


$$x \to y \triangleq \neg x \vee (x \wedge y) = \left\{\begin{array}{c|cccc} \to & 1 & b & n & 0 \\ \hline 1 & 1 & b & n & 0 \\ b & 1 & 1 & n & n \\ n & 1 & b & 1 & b \\ 0 & 1 & 1 & 1 & 1 \end{array}\right. \qquad \forall x,y \in X$$

All the previous examples in this section are *distributive*; the previous example was *Boolean*. The next example is *non-distributive*, and *de Morgan* (but *non-Boolean*). Note for a given order structure, the method of negation may not be unique; in the previous and following examples both have identical lattices, but are negated differently.

Example 3.10 (BN_4 logic). [15]

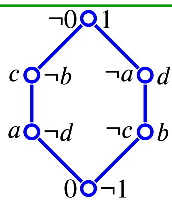
EX The BN_4 *logic* is defined below. The function ¬ is a *de Morgan negation* (Example 1.14 page 12) defined on a 4 element M_2 *lattice*. The value 1 represents "*true*", 0 represents "*false*", b represents "*both*" (both true and false), and n represents "*neither*". In the implication table below, the values that differ from those of the *classical implication* $\overset{c}{\to}$ are shaded.



$$x \to y \triangleq \left\{\begin{array}{c|cccc} \to & 1 & n & b & 0 \\ \hline 1 & 1 & n & 0 & 0 \\ n & 1 & 1 & n & n \\ b & 1 & n & b & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right. \qquad \forall x,y \in X$$

Example 3.11.

EX The tables that follow are the 6 implications defined in Example 3.1 (page 23) on the O_6 *lattice with ortho negation* (Definition 1.3 page 2), or the O_6 *orthocomplemented lattice* (Definition J.2 page 196), illustrated to the right. In the tables, the values that differ from those of the *classical implication* $\overset{c}{\to}$ are shaded.



$\overset{c}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	c	1	a	a
c	1	d	1	b	1	b
b	1	1	c	1	c	c
a	1	d	1	d	1	d
0	1	1	1	1	1	1

$\overset{s}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	c	c
a	1	1	1	d	1	d
0	1	1	1	1	1	1

$\overset{d}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	c	1	a	a
c	1	d	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

[14] Pavičić and Megill (2008) page 28 ⟨Definition 2⟩, Mittelstaedt (1970), Finch (1970) page 102 ⟨(1.1)⟩, Smets (2006) page 270

[15] Restall (2000) page 171 ⟨Example 8.39⟩

$\overset{k}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

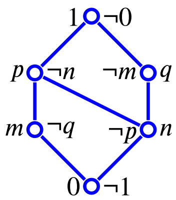
$\overset{n}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

$\overset{r}{\to}$	1	d	c	b	a	0
1	1	d	c	b	a	0
d	1	1	a	1	a	a
c	1	b	1	b	1	b
b	1	1	c	1	a	c
a	1	d	1	b	1	d
0	1	1	1	1	1	1

Example 3.12. [16]

EX

A 6 element logic is defined below. The function $\neg$ is a *Kleene negation* (Example 1.17 page 13). The implication has *strong entailment* but *weak modus ponens*. In the implication table below, the values that differ from those of the *classical implication* $\overset{c}{\to}$ are shaded.



$x \to y \triangleq$

$\to$	1	p	q	m	n	0
1	1	p	q	m	n	0
p	1	1	q	p	q	n
q	1	p	1	m	p	m
m	1	1	q	1	q	q
n	1	1	1	p	1	p
0	1	1	1	1	1	1

$\forall x,y \in X$

✎PROOF:

1. Proof that $\neg$ is a *Kleene negation*: see Example 1.17 (page 13)
2. Proof that $\to$ is an *implication*: This follows directly from the definition of $\to$ and the definition of an *implication* (Definition 3.1 page 22).
3. Proof that $\to$ does not have *strong modus ponens*:

$$\begin{array}{rclclclclcl}
\neg p \wedge (p \to m) & = & n \wedge p & = & n & \leq & p & = & \neg p \vee m & \not\leq & m \\
\neg n \wedge (n \to m) & = & n \wedge p & = & n & \leq & p & = & \neg p \vee m & \not\leq & m \\
\neg p \wedge (p \to 0) & = & n \wedge n & = & n & \leq & n & = & \neg p \vee 0 & \not\leq & 0 \\
\neg n \wedge (n \to 0) & = & p \wedge n & = & n & \leq & p & = & \neg n \vee 0 & \not\leq & 0
\end{array}$$

✏

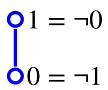
For an example of an 8-valued logic, see Kamide (2013). For examples of 16-valued logics, see Shramko and Wansing (2005).

3.3 Classical two-valued logic

Definition 3.4 (Aristotelian logic/classical logic). [17]

DEF

The ***classical 2-value logic*** *is a 2 element* LATTICE WITH ORTHO NEGATION *(Definition 1.3 page 2)* $(\{1,0\}, \vee, \wedge, \neg, 0, 1\,;\, \leq, \overset{c}{\to})$ *as illustrated below with values* 1 *representing* "TRUE", 0 *representing* "FALSE", *and with an implication connective* $\Longrightarrow$ *as specified below:*



$$x \Longrightarrow y \triangleq \left\{\begin{array}{ll} 1 & \forall x \leq y \\ y & \text{otherwise} \end{array}\right\} = \left\{\begin{array}{c|cc} \Longrightarrow & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 1 \end{array} \quad \forall x,y \in X \right\} = \neg x \vee y$$

[16] Xu et al. (2003) pages 29–30 ⟨Example 2.1.4⟩

[17] Novák et al. (1999) pages 17–18 ⟨EXAMPLE 2.1⟩

Theorem 3.1.

THM

If $(\{1,0\},\ \vee,\ \wedge,\ \neg,\ 0,\ 1\ ;\ \le,\ \overset{\neg}{\to})$ *is the* CLASSICAL 2-VALUE LOGIC *(Definition 3.4 page 32), then the* ***logical OR*** $\vee$, ***logical AND*** $\wedge$, *and* ***logical equivalence*** $\iff$ *operations are defined as follows:*

$\vee$	1	0
1	1	1
0	1	0

$\wedge$	1	0
1	1	0
0	0	0

$\iff$	1	0
1	1	0
0	0	1

✎PROOF:

1. Proof for *logical OR* operation $\vee$: This follows from the *lattice* (Definition D.3 page 117) properties of $\boldsymbol{L}_2$.
2. Proof for *logical AND* operation $\wedge$: This follows from the *lattice* (Definition D.3 page 117) properties of $\boldsymbol{L}_2$.
3. Proof for *logical if and only if* operation $\iff$: This follows from the definition of $\implies$ (Definition 3.4 page 32) and Definition 3.3 (page 27).

✏

One of the most useful facts concerning propositional logic systems is that they form a *Boolean algebra* (next theorem). Because they are a Boolean algebra, a number of useful properties automatically follow (next theorem) from the properties of Boolean algebras (Theorem I.2 page 176).

Theorem 3.2 (Boolean algebra properties). [18] *Let* $\{0,\ 1\}$ *be the set of logical properties* FALSE *and* TRUE *(Axiom ?? page ??). Let* $\vee$ *be the* LOGICAL OR *and* $\wedge$ *the* LOGICAL AND *operations (Definition 3.1 page 33). Let* $\implies$ *be the* LOGICAL IMPLIES *relation (Definition ?? page ??).*

THM

$(\{0,\ 1\},\ \vee,\ \wedge\ ;\ \implies)$ *is a* BOOLEAN ALGEBRA. *In particular for all* $x, y, z \in \{0,\ 1\}$,

property with emphasizing $\vee$	dual property emphasizing $\wedge$	property name
$x \vee x = x$	$x \wedge x = x$	(IDEMPOTENT)
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	(COMMUTATIVE)
$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(ASSOCIATIVE)
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	(ABSORPTIVE)
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(DISTRIBUTIVE)
$x \vee 0 = x$	$x \wedge 1 = x$	(IDENTITY)
$x \vee 1 = 1$	$x \wedge 0 = 0$	(BOUNDED)
$x \vee x' = 1$	$x \wedge x' = 0$	(COMPLEMENTED)[19]
$(x')' = x$		(UNIQUELY COMP.)
$(x \vee y)' = x' \wedge y'$	$(x \wedge y)' = x' \vee y'$	(DE MORGAN'S LAWS)

✎PROOF: This follows directly from the fact that the *classical 2-valued logic* (Definition 3.4 page 32) is a *Boolean algebra* (Definition I.1 page 171) and from Theorem I.2 (page 176). ✏

Definition 3.5 (additional logic operations). [20] *Let* $(\{0,\ 1\},\ \implies,\ \vee,\ \wedge,\ \neg,\ 0,\ 1)$ *be a propositional logic system. Let* $x' \triangleq \neg x$ *and* $y' \triangleq \neg y$. *The following table defines additional operations on* $\{0,\ 1\}$ *in*

[18] MacLane and Birkhoff (1999) page 488, Givant and Halmos (2009) page 10, Müller (1909) pages 20–21, Schröder (1890), Whitehead (1898) pages 35–37, Peano (1889) page 88

[19] The property $x \vee x' = 1$ is also called the *law of the excluded middle*.
The property $x \wedge x' = 0$ is also called *non-contradiction* or *explosion*.
References: Renedo et al. (2003) page 71
Restall (2004) pages 73–75
Restall (2001) pages 1–3

[20] Givant and Halmos (2009) page 32 ⟨disjunction, conjunction, negation⟩, Shiva (1998) page 83 ⟨inhibit, transfer⟩, Whitesitt (1995) pages 68–69 ⟨*Sheffer stroke functions* $\downarrow=\uparrow$, $|=\downarrow$⟩, Quine (1979) pages 45–48 ⟨joint denial $\downarrow$, alternate denial $|$⟩, Bernstein (1934) page 876 ⟨implication $\supset$⟩

terms of $\vee$, $\wedge$, *and* $\neg$.

name	symbol	definition			
joint denial	$\downarrow$	$x \downarrow y$	$\triangleq$	$x' \wedge y'$	$\forall x,y \in \{0,1\}$
inhibit x	$\ominus$	$x \ominus y$	$\triangleq$	$x' \wedge y$	$\forall x,y \in \{0,1\}$
inhibit y	$-$	$x - y$	$\triangleq$	$x \wedge y'$	$\forall x,y \in \{0,1\}$
complete disjunction	$\oplus$	$x \oplus y$	$\triangleq$	$(x' \wedge y) \vee (x \wedge y')$	$\forall x,y \in \{0,1\}$
alternative denial	$\mid$	$x \mid y$	$\triangleq$	$x' \vee y'$	$\forall x,y \in \{0,1\}$

There are a total of $2^4 = 16$ possible binary operations on the set of relations $\{0,\ 1\} \times \{0,\ 1\} \to \{0,\ 1\}$. The following table summarizes these 16 operations.[21]

logic operations									
		$(x,y) =$							
name and symbol		11	10	01	00	**operation in terms of $\vee$, $\wedge$, and $\neg$**			
zero	0	0	0	0	0	0	$=$	$x \wedge x'$	$\forall x \in \{0,1\}$
joint denial	$\downarrow$	0	0	0	1	$x \downarrow y$	$=$	$x' \wedge y'$	$\forall x,y \in \{0,1\}$
inhibit x	$\ominus$	0	0	1	0	$x \ominus y$	$=$	$x' \wedge y$	$\forall x,y \in \{0,1\}$
complement x	◶	0	0	1	1	x◶y	$=$	x'	$\forall x,y \in \{0,1\}$
inhibit y	$-$	0	1	0	0	$x - y$	$=$	$x \wedge y'$	$\forall x,y \in \{0,1\}$
complement y	◷	0	1	0	1	x◷y	$=$	y'	$\forall x,y \in \{0,1\}$
complete disjunction	$\oplus$	0	1	1	0	$x \oplus y$	$=$	$(x' \wedge y) \vee (x \wedge y')$	$\forall x,y \in \{0,1\}$
alternative denial	$\mid$	0	1	1	1	$x \mid y$	$=$	$x' \vee y'$	$\forall x,y \in \{0,1\}$
conjunction	$\wedge$	1	0	0	0	$x \wedge y$	$=$	$x \wedge y$	$\forall x,y \in \{0,1\}$
equivalence	$\Leftrightarrow$	1	0	0	1	$x \Leftrightarrow y$	$=$	$(x \wedge y) \vee (x' \wedge y')$	$\forall x,y \in \{0,1\}$
transfer y	$\Vdash$	1	0	1	0	$x \Vdash y$	$=$	y	$\forall x,y \in \{0,1\}$
implication	$\Rightarrow$	1	0	1	1	$x \Rightarrow y$	$=$	$x' \vee y$	$\forall x,y \in \{0,1\}$
transfer x	$\dashV$	1	1	0	0	$x \dashV y$	$=$	x	$\forall x,y \in \{0,1\}$
implied by	$\Leftarrow$	1	1	0	1	$x \Leftarrow y$	$=$	$x \vee y'$	$\forall x,y \in \{0,1\}$
disjunction	$\vee$	1	1	1	0	$x \vee y$	$=$	$x \vee y$	$\forall x,y \in \{0,1\}$
identity	1	1	1	1	1	1	$=$	$x \vee x'$	$\forall x \in \{0,1\}$

The 16 logic operations of propositional logic can all be represented using the logic operations of *disjunction* $\vee$, *conjunction* $\wedge$, and *negation* $\neg$. Using these representations, all 16 operations can be generalized to *Boolean algebras* using the equivalent Boolean algebra/lattice operations of *join*, *meet*, and *complement*.[22]

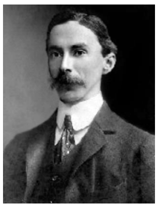
In addition to Boolean algebras, the 16 operations can also have equivalent operations on *algebra of sets* where the logic operations essentially define the set operations as in

$$
\begin{aligned}
A \cup B &= \{x \in X | (x \in A) \vee (x \in B)\} \\
A \cap B &= \{x \in X | (x \in A) \wedge (x \in B)\} \\
A \setminus B &= \{x \in X | (x \in A) \ominus (x \in B)\} \\
A \triangle B &= \{x \in X | (x \in A) \oplus (x \in B)\} \\
A^c &= \{x \in X | \neg(x \in A)\}
\end{aligned}
$$

[21] Shiva (1998) page 83
[22] Givant and Halmos (2009), page 32

Computer science also makes use of some of the 16 logic operations, where *disjunction* becomes *OR*, and *conjunction* becomes *AND*. So, there are four fields (Boolean algebra, logic, set theory, computer science) that all use essentially the same operations, but sometimes call them by different names. The following table attempts to identify to these terms across the four fields:[23]

terminology								
	Boolean algebra		logic		algebra of sets		computer science	
0000	0	**bottom**	0	*false*	∅	**empty set**	0	**zero**
0001	↓	**rejection**	↓	*joint denial*	↓	**rejection**	↓	**nor**
0010	⊖	**inhibit** x	⊖	*inhibit* x	⊖	**inhibit** x	⊖	**inhibit** x
0011	⦶	**complement** x	⦶	*negation* x	c_x	**complement** x	⦶	**not** x
0100	−	**exception**	−	*inhibit* y	\	**difference**	−	**difference**
0101	⦶	**complement** y	⦶	*negation* y	c_y	**complement** y	⦶	**not** y
0110	△	**Boolean addition**	⊕	*complete disjunction*	△	**symmetric difference**	⊕	**exclusive-or**
0111	|	**Sheffer stroke**	|	*alternate denial*	|	**Sheffer stroke**	|	**nand**
1000	∧	**meet**	∧	*conjuction*	∩	**intersection**	∧	**and**
1001	⇔	**biconditional**	⇔	*equivalence*	⇔	**equivalence**	⇔	**equivalence**
1010	⊫	**projection** y	⊫	*transfer* y	⊫	**projection** y	⊫	**projection** y
1011	⇒	**implication**	⇒	*implication*	⇒	**implication**	⇒	**implication**
1100	⫣	**projection** x	⫣	*transfer* x	⫣	**projection** x	⫣	**projection** x
1101	÷	**adjunction**	⇐	*implied by*	÷	**adjunction**	÷	**adjunction**
1110	∨	**join**	∨	*disjunction*	∪	**union**	∨	**or**
1111	1	**top**	1	*true*	X	**universal set**	1	**one**



“*I spent September in extending his [Peano's] methods to the logic of relations.... The time was one of intellectual intoxication. My sensations resembled those one has after climbing a mountain in a mist, when, on reaching the summit, the mist suddenly clears, and the country becomes visible for forty miles in every direction....Suddenly, in the space of a few weeks, I discovered what appeared to be definitive answers to the problems which had baffled me for years. And in the course of discovering these answers, I was introducing a new mathematical technique, by which regions formerly abandoned to the vaguenesses of philosophers were conquered for the precision of exact formulae. Intellectually, the month of September 1900 was the highest point of my life. I went about saying to myself that now at last I had done something worth doing, and I had the feeling that I must be careful not to be run over in the street before I had written it down.*”

Bertrand Russell (1872–1970), British mathematician, [24]

[23] http://groups.google.com/group/sci.math/browse_thread/thread/c1e9a7beb9a82311

[24] quote: Russell (1951) pages 217–218
image: http://en.wikipedia.org/wiki/File:Russell1907-2.jpg, public domain

APPENDIX A
SET STRUCTURES

A.1 General set structures

Similar to the definition of a *relation* on a set X as being any subset of the *Cartesian product* $X \times X$ (Definition B.1 page 73), a *set structure* on a set X is simply any subset of the *power set* 2^X (next) of the set X.

Definition A.1.

DEF *The **power set** 2^X on a set X is defined as*
$$2^X \triangleq \{A | A \subseteq X\} \qquad \text{(the set of all subsets of } X\text{)}$$

Definition A.2. [1] *Let 2^X be the* POWER SET *(Definition A.1 page 37) of a set X.*

DEF *A set $\mathcal{S}(X)$ is a **set structure** on X if* $\mathcal{S}(X) \subseteq 2^X$.
A SET STRUCTURE *$\mathcal{Q}(X)$ is a **paving** on X if* $\varnothing \in \mathcal{Q}(X)$.

Definition A.3. [2] *Let $\mathcal{Q}(X)$ be a* PAVING *(Definition A.2 page 37) on a set X. Let Y be a set containing the element* 0.

DEF *A function $\mathsf{m} \in Y^{\mathcal{Q}(X)}$ is a **set function** if*
$$\mathsf{m}(\varnothing) = 0.$$

A.2 Operations on the power set

A.2.1 Standard operations

Definition A.4. [3] *Let 2^X be a set. Let $|X|$ be a function in the function space* $[0 : +\infty]^X$ *(Definition B.8 page 85).*

---

[1] Molchanov (2005) page 389, Pap (1995) page 7, Hahn and Rosenthal (1948) page 254

[2] Pap (1995) page 8 ⟨Definition 2.3: extended real-valued set function⟩, Halmos (1950) page 30 ⟨§7. MEASURE ON RINGS⟩, Hahn and Rosenthal (1948), Choquet (1954)

[3] Tao (2011) page 12 ⟨Example 3.6⟩, Tao (2010) page 7 ⟨Example 1.1.14⟩

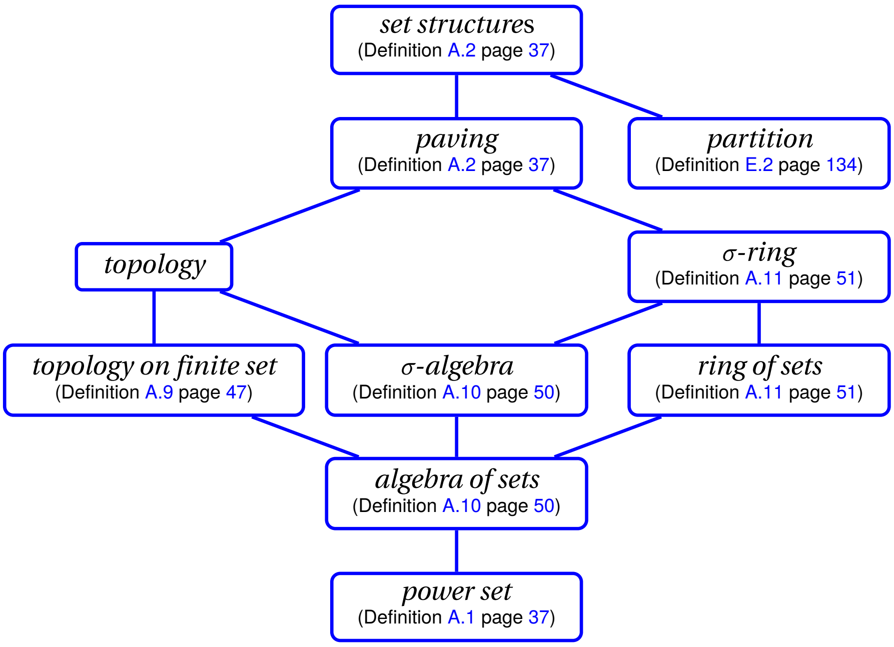


Figure A.1: some standard set structures

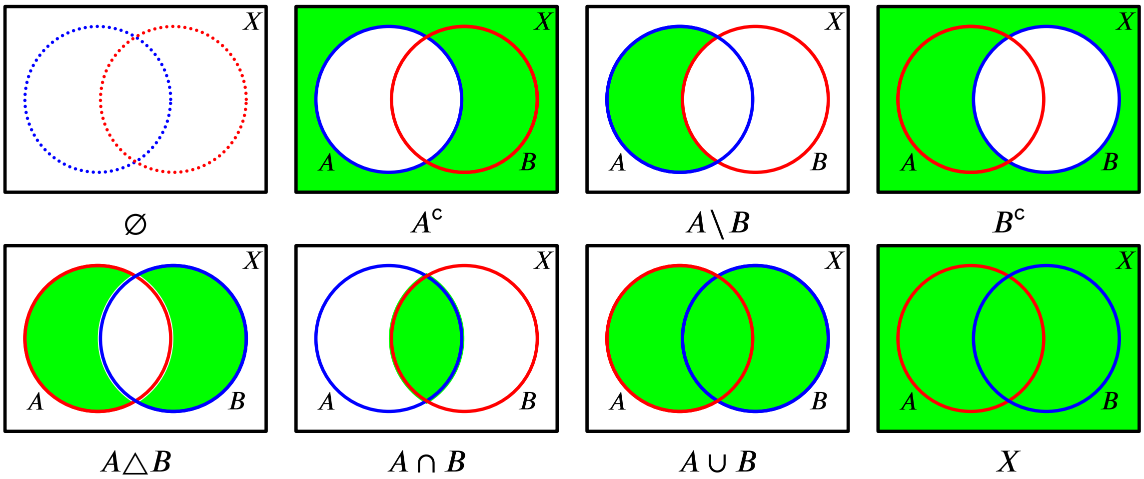


Figure A.2: Venn diagrams for standard set operations (Definition A.5 page 38)

DEF

$|X|$ *is the* ***cardinality*** *or* ***order*** *of* X *if*

$$|X| \triangleq \begin{cases} \text{number of elements in } X & \text{if } X \text{ is FINITE} \\ +\infty & \text{otherwise} \end{cases}$$

Definition A.5 (next) introduces seven standard set operations: two *nullary* operations, one *unary* operation, and four *binary operations* (Definition B.9 page 86).

Definition A.5. [4] *Let* 2^X *be the* POWER SET *(Definition A.1 page 37) on a set* X. *Let* $\neg$ *represent the* LOGICAL NOT *operation,* $\vee$ *represent the* LOGICAL OR *operation,* $\wedge$ *represent the* LOGICAL AND *operation (Definition 3.2 page 27), and* $\oplus$ *represent the* LOGICAL EXCLUSIVE-OR *operation (Definition 3.5 page 33).*

[4] Aliprantis and Burkinshaw (1998) pages 2–4

DEF

name/symbol		arity	definition	domain
emptyset	$\varnothing$	0	$\varnothing \triangleq \{x \in X \mid x \neq x\}$	
universal set	X	0	$X \triangleq \{x \in X \mid x = x\}$	
complement	c	1	$A^c \triangleq \{x \in X \mid \neg(x \in A)\}$	$\forall A \in 2^X$
union	$\cup$	2	$A \cup B \triangleq \{x \in X \mid (x \in A) \vee (x \in B)\}$	$\forall A,B \in 2^X$
intersection	$\cap$	2	$A \cap B \triangleq \{x \in X \mid (x \in A) \wedge (x \in B)\}$	$\forall A,B \in 2^X$
difference	$\setminus$	2	$A \setminus B \triangleq \{x \in X \mid (x \in A) \wedge \neg(x \in B)\}$	$\forall A,B \in 2^X$
symmetric difference	$\triangle$	2	$A \triangle B \triangleq \{x \in X \mid (x \in A) \oplus (x \in B)\}$	$\forall A,B \in 2^X$

With regards to the standard seven set operations only, Theorem A.1 (next) expresses each of the set operations in terms of pairs of other operations.

Theorem A.1.

THM

$$
\begin{array}{rcllll}
X &=& \varnothing^c & & & \\
\varnothing &=& X^c = \left(A \cup A^c\right)^c &= A \cap A^c &= A \setminus A &= A \triangle A \\
X &=& A \cup A^c &= \left(A \cap A^c\right)^c & & \\
A^c &=& X \setminus A &= X \triangle A & & \\
A \cup B &=& \left(A^c \cap B^c\right)^c &= (A \triangle B) \triangle (A \cap B) &= (A \setminus B) \triangle B & \\
A \cap B &=& \left(A^c \cup B^c\right)^c &= (A \cup B) \triangle A \triangle B &= A \setminus (A \setminus B) & \\
A \setminus B &=& \left(A^c \cup B\right)^c &= A \cap B^c &= (A \cup B) \triangle B &= (A \triangle B) \cap A \\
A \triangle B &=& \left[\left(A^c \cup B\right)^c\right] \cup \left[\left(A \cup B^c\right)^c\right] & &= \left[\left(A^c \cap B^c\right)^c\right] \cap (A \cap B)^c & \\
&=& (A \setminus B) \cup (B \setminus A) & & &
\end{array}
$$

Proposition A.1. *Let X be a set and 2^X the power set of X. Let $\boldsymbol{R} \subseteq X$ such that $\boldsymbol{R}$ is closed with respect to the set symmetric difference operator $\triangle$.*

PRP

$(\boldsymbol{R}, \triangle)$ *is a* GROUP. *In particular,*

1. $\varnothing \triangle A = A \triangle \varnothing = A \quad \forall A \in \boldsymbol{R}$ *($\varnothing$ is the* IDENTITY *element)*
2. $A \triangle A = \varnothing \quad \forall A \in \boldsymbol{R}$ *(A is the* INVERSE *of A)*
3. $A \triangle (B \triangle C) = (A \triangle B) \triangle C \quad \forall A,B,C \in \boldsymbol{R}$ *(*ASSOCIATIVE*)*

✎PROOF:

Proof that $\varnothing$ is the *identity* element:

1a. Proof that $\varnothing \in \boldsymbol{R}$:

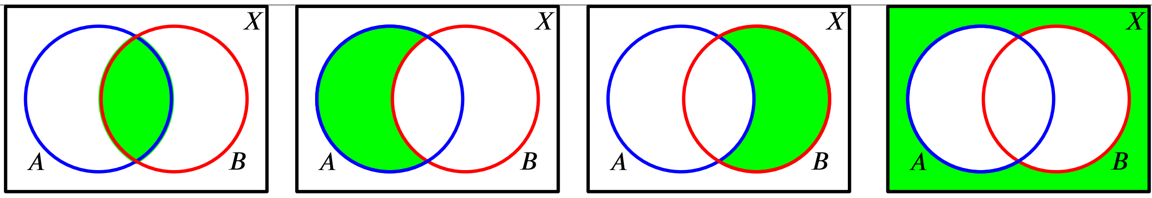
$$
\begin{aligned}
\varnothing &= A \triangle A && \triangle \text{ closed with respect to } \boldsymbol{R} \\
&\in \boldsymbol{R}
\end{aligned}
$$

1b. Proof that $\varnothing \triangle A = A$:

$$
\begin{aligned}
\varnothing \triangle A &= \{x \in X \mid (x \in \varnothing) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 38} \\
&= \{x \in X \mid (x \in \{x \in X \mid x \neq x\}) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 38} \\
&= \{x \in X \mid 0 \oplus (x \in A)\} \\
&= \{x \in X \mid (x \in A)\} && \text{by definition of } \oplus \text{ (Definition 3.1 page 33)} \\
&= A
\end{aligned}
$$

1c. Proof that $A \triangle \varnothing = A$:

$$
\begin{aligned}
A \triangle \varnothing &= \{x \in X \mid (x \in A) \oplus (x \in \varnothing)\} && \text{by definition of } \triangle \text{ page 38} \\
&= \{x \in X \mid (x \in A) \oplus (x \in \{x \in X \mid x \neq x\})\} && \text{by definition of } \triangle \text{ page 38} \\
&= \{x \in X \mid (x \in A) \oplus 0\} \\
&= \{x \in X \mid (x \in A)\} && \text{by definition of } \oplus \text{ (Definition 3.1 page 33)} \\
&= A
\end{aligned}
$$



$A \cap B$ (in both A and B) | $A \cap B^c$ (in A but not in B) | $A^c \cap B$ (not in A but in B) | $A^c \cap B^c$ (neither in A nor B)

Figure A.3: The partition of a set X into 4 regions by subsets A and B

2. Proof that $A \triangle A$:

$$\begin{aligned}
A \triangle A &= \{x \in X | (x \in A) \oplus (x \in A)\} && \text{by definition of } \triangle \text{ page 38}\\
&= \{x \in X | 0\} && \text{by definition of } \triangle \text{ page 38}\\
&= \varnothing && \text{by definition of } \triangle \text{ page 38}
\end{aligned}$$

3. Proof that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$:

$$\begin{aligned}
A \triangle (B \triangle C) &= \{x \in X | (x \in A) \oplus [x \in (B \triangle C)]\} && \text{by definition of } \triangle \text{ page 38}\\
&= \{x \in X | (x \in A) \oplus [(x \in B) \oplus (x \in C)]\} && \text{by definition of } \triangle \text{ page 38}\\
&= \{x \in X | [(x \in A) \oplus (x \in B)] \oplus (x \in C)\}\\
&= (A \triangle B) \triangle C
\end{aligned}$$

A.2.2 Non-standard operations

Two subsets A and B of a set X that are intersecting but yet one is not contained in the other, partition the set X into four regions, as illustrated in Figure A.3 (page 40). Because there are four regions, the number of ways we can select one or more of them is $2^4 = 16$. Therefore, a binary operator on sets A and B can likewise result in one of $2^4 = 16$ possibilities. Definition A.6 (page 40) presents 7 set operations. Therefore, there should be an additional $16 - 7 = 9$ operations. Definition A.6 (next definition) attempts to define these additional operations. Some definitions are adapted from logic (Table 3.3 page 34). But in general these definitions are non-standard definitions with respect to set theory. The 16 set operations under the inclusion relation $\subseteq$ form a lattice; this lattice is illustrated by a *Hasse diagram* in Figure A.4 (page 41).

Definition A.6. [5] *Let 2^X be the power set on a set X. For any sets $A, B \in 2^X$, let $AB \triangleq (A \cap B)$.*

[5] standard ops: Aliprantis and Burkinshaw (1998) pages 2–4

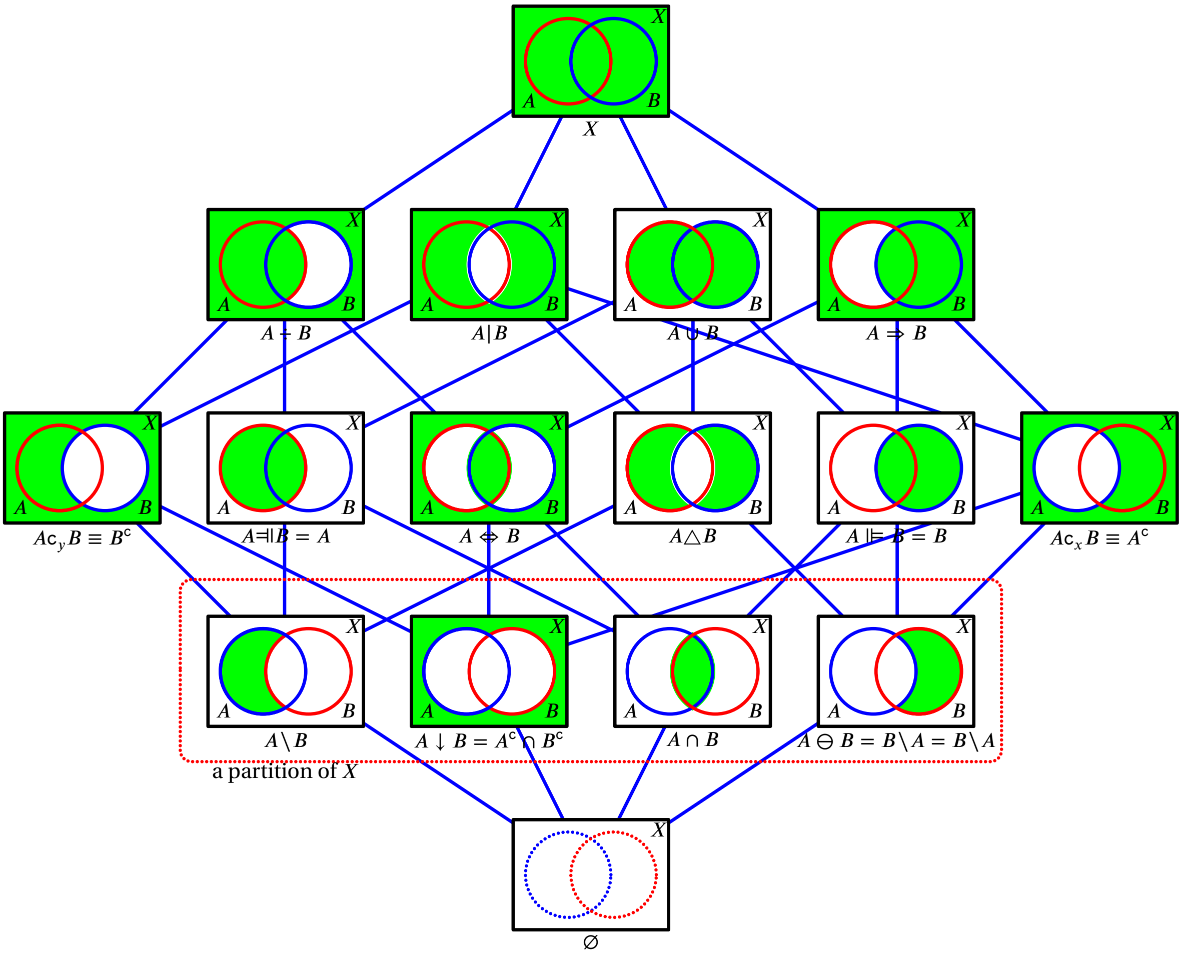


Figure A.4: lattice of set operations

DEF

name/symbol		*arity*	*definition*	*domain*
empty set	$\oslash$	2	$A \oslash B \triangleq \varnothing$	$\forall A,B\in 2^X$
rejection	$\downarrow$	2	$A \downarrow B \triangleq A^c B^c$	$\forall A,B\in 2^X$
inhibit *x*	$\ominus$	2	$A \ominus B \triangleq A^c B$	$\forall A,B\in 2^X$
complement *x*	c_x	2	$A\, c_x\, B \triangleq A^c B \cup A^c B^c$	$\forall A,B\in 2^X$
difference	$\setminus$	2	$A \setminus B \triangleq A B^c$	$\forall A,B\in 2^X$
complement *y*	c_y	2	$A\, c_y\, B \triangleq A B^c \cup A^c B^c$	$\forall A,B\in 2^X$
symmetric difference	$\triangle$	2	$A \triangle B \triangleq A B^c \cup A^c B$	$\forall A,B\in 2^X$
Sheffer stroke	$\mid$	2	$A \mid B \triangleq A B^c \cup A^c B \cup A^c B^c$	$\forall A,B\in 2^X$
intersection	$\cap$	2	$A \cap B \triangleq A B \cup$	$\forall A,B\in 2^X$
equivalence	$\Leftrightarrow$	2	$A \Leftrightarrow B \triangleq A B \cup A^c B^c$	$\forall A,B\in 2^X$
projection *y*	$\Vdash$	2	$A \Vdash B \triangleq A B \cup A^c B$	$\forall A,B\in 2^X$
implication	$\Rightarrow$	2	$A \Rightarrow B \triangleq A B \cup A^c B \cup A^c B^c$	$\forall A,B\in 2^X$
projection *x*	$\dashv\vert$	2	$A \dashv\vert B \triangleq A B \cup A B^c$	$\forall A,B\in 2^X$
adjunction	$\div$	2	$A \div B \triangleq A B \cup A B^c \cup A^c B^c$	$\forall A,B\in 2^X$
union	$\cup$	2	$A \cup B \triangleq A B \cup A B^c \cup A^c B$	$\forall A,B\in 2^X$
universal set	$\otimes$	2	$A \otimes B \triangleq A B \cup A B^c \cup A^c B \cup A^c B^c$	$\forall A,B\in 2^X$

A.2.3 Generated operations

Definition A.5 (page 38) defines set operations in terms of logical operations. However, it is also possible to express set operations in terms of two or more other set operations. When all the set operations can be expressed in terms of a set of operations, then that set of operations is *functionally complete* (next definition, but see also Definition I.3 page 182).

Definition A.7. [6] *Let S be a set structure.*

DEF

A set of operations Φ is ***functionally complete*** *in S if*

$\cup, \cap, c, \varnothing$, *and X*

can all be expressed in terms of elements of Φ.

Example A.1. Here are some examples of *functionally complete* sets:

EX

- $\{\downarrow\}$ (*rejection*)
- $\{\mid\}$ (*Sheffer stroke*)
- $\{\div, \varnothing\}$ (*adjunction* and $\varnothing$)
- $\{\setminus, X\}$ (*set difference* and X)
- $\{\cup, c\}$ (*union* and *complement*)
- $\{\cap, c\}$ (*intersection* and *complement*)
- $\{\triangle, \cap, X\}$ (*symmetric difference*, *intersection*, and X)
- $\{\triangle, \cup, X\}$ (*symmetric difference*, *union*, and X)
- $\{\triangle, \setminus, c\}$ (*symmetric difference*, *set difference*, and *complement*)

The five theorems that follow demonstrate which operations can be generated by sets of generating operations:

- 2 generators, $\binom{7}{2} = 21$ possibilities, Proposition A.2 page 43
- 3 generators, $\binom{7}{3} = 35$ possibilities, Proposition A.3 page 43
- 4 generators, $\binom{7}{4} = 35$ possibilities, Proposition A.4 page 44
- 5 generators, $\binom{7}{5} = 21$ possibilities, Proposition A.5 page 45
- 6 generators, $\binom{7}{6} = 7$ possibilities, Proposition A.6 page 45

[6] Whitesitt (1995) page 69

Starting with any two subsets A and B and using all the operations of a *functionally complete* set of operations, an *algebra of sets* (Definition A.10 page 50) is produced. Thus, a *functionally complete* set of set operations induces an *algebra of sets*. Other less powerful sets of operations generate fewer operations and induce only a *ring of sets* (Definition A.11 page 51). And some sets of operations, such as $\{\cup, \cap\}$, generate no set operations but themselves.

Proposition A.2 (2 generators). *The following table demonstrates the "standard" operations generated by sets of 2 operations.*

generators			generated operations							induced set structure
1.	$\varnothing$	X	$\varnothing$	X						
2.	$\varnothing$	c	$\varnothing$	X	c					
3.	$\varnothing$	$\cup$	$\varnothing$			$\cup$				
4.	$\varnothing$	$\cap$	$\varnothing$				$\cap$			
5.	$\varnothing$	$\setminus$	$\varnothing$					$\setminus$		
6.	$\varnothing$	$\triangle$	$\varnothing$						$\triangle$	
7.	X	c	$\varnothing$	X	c					
8.	X	$\cup$		X		$\cup$				
9.	X	$\cap$		X			$\cap$			
10.	X	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
11.	X	$\triangle$	$\varnothing$	X	c				$\triangle$	
12.	c	$\cup$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
13.	c	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
14.	c	$\setminus$	$\varnothing$	X	c			$\setminus$		
15.	c	$\triangle$	$\varnothing$	X	c				$\triangle$	
16.	$\cup$	$\cap$				$\cup$	$\cap$			
17.	$\cup$	$\setminus$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
18.	$\cup$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
19.	$\cap$	$\setminus$	$\varnothing$				$\cap$	$\setminus$		
20.	$\cap$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
21.	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*

Proposition A.3 (3 generators). *The following table demonstrates the "standard" operations generated by sets of 3 operations.*

	generators			generated operations							induced set structure
1.	$\varnothing$	X	c	$\varnothing$	X	c					
2.	$\varnothing$	X	$\cup$	$\varnothing$	X		$\cup$				
3.	$\varnothing$	X	$\cap$	$\varnothing$	X			$\cap$			
4.	$\varnothing$	X	$\setminus$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
5.	$\varnothing$	X	$\triangle$	$\varnothing$	X	c				$\triangle$	
6.	$\varnothing$	c	$\cup$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
7.	$\varnothing$	c	$\cap$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*
8.	$\varnothing$	c	$\setminus$	$\varnothing$	X	c			$\setminus$		
9.	$\varnothing$	c	$\triangle$	$\varnothing$	X	c				$\triangle$	
10.	$\varnothing$	$\cup$	$\cap$	$\varnothing$			$\cup$	$\cap$			
11.	$\varnothing$	$\cup$	$\setminus$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
12.	$\varnothing$	$\cup$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
13.	$\varnothing$	$\cap$	$\setminus$	$\varnothing$				$\cap$	$\setminus$		
14.	$\varnothing$	$\cap$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
15.	$\varnothing$	$\setminus$	$\triangle$	$\varnothing$			$\cup$	$\cap$	$\setminus$	$\triangle$	*ring of sets*
16.	X	c	$\cup$	$\varnothing$	X	c	$\cup$	$\cap$	$\setminus$	$\triangle$	*algebra of sets*

17.	X	c	∩	∅	X	c	∪	∩	\	△	*algebra of sets*
18.	X	c	\	∅	X	c	∪	∩	\	△	*algebra of sets*
19.	X	c	△	∅	X	c				△	
20.	X	∪	∩		X		∪	∩			
21.	X	∪	\	∅	X	c	∪	∩	\	△	*algebra of sets*
22.	X	∪	△	∅	X	c	∪	∩	\	△	*algebra of sets*
23.	X	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
24.	X	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
25.	X	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
26.	c	∪	∩	∅	X	c	∪	∩	\	△	*algebra of sets*
27.	c	∪	\	∅	X	c	∪	∩	\	△	*algebra of sets*
28.	c	∪	△	∅	X	c	∪	∩	\	△	*algebra of sets*
29.	c	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
30.	c	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
31.	c	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
32.	∪	∩	\	∅			∪	∩	\	△	*ring of sets*
33.	∪	∩	△	∅			∪	∩	\	△	*ring of sets*
34.	∪	\	△	∅			∪	∩	\	△	*ring of sets*
35.	∩	\	△	∅			∪	∩	\	△	*ring of sets*

Proposition A.4 (4 generators). *The following table demonstrates the "standard" operations generated by sets of 4 operations.*

	generators				generated operations							induced set structure
1.	∅	X	c	∪	∅	X	c	∪	∩	\	△	*algebra of sets*
2.	∅	X	c	∩	∅	X	c	∪	∩	\	△	*algebra of sets*
3.	∅	X	c	\	∅	X	c	∪	∩	\	△	*algebra of sets*
4.	∅	X	c	△	∅	X	c				△	
5.	∅	X	∪	∩	∅	X		∪	∩			*pre-topology*
6.	∅	X	∪	\	∅	X	c	∪	∩	\	△	*algebra of sets*
7.	∅	X	∪	△	∅	X	c	∪	∩	\	△	*algebra of sets*
8.	∅	X	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
9.	∅	X	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
10.	∅	X	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
11.	∅	c	∪	∩	∅	X	c	∪	∩	\	△	*algebra of sets*
12.	∅	c	∪	\	∅	X	c	∪	∩	\	△	*algebra of sets*
13.	∅	c	∪	△	∅	X	c	∪	∩	\	△	*algebra of sets*
14.	∅	c	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
15.	∅	c	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
16.	∅	c	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
17.	∅	∪	∩	\	∅			∪	∩	\	△	*ring of sets*
18.	∅	∪	∩	△	∅			∪	∩	\	△	*ring of sets*
19.	∅	∪	\	△	∅			∪	∩	\	△	*ring of sets*
20.	∅	∩	\	△	∅			∪	∩	\	△	*ring of sets*
21.	X	c	∪	∩	∅	X	c	∪	∩	\	△	*algebra of sets*
22.	X	c	∪	\	∅	X	c	∪	∩	\	△	*algebra of sets*
23.	X	c	∪	△	∅	X	c	∪	∩	\	△	*algebra of sets*
24.	X	c	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
25.	X	c	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
26.	X	c	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
27.	X	∪	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*

28.	X	∪	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
29.	X	∪	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
30.	X	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
31.	c	∪	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
32.	c	∪	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
33.	c	∪	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
34.	c	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
35.	∪	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*

Proposition A.5 (5 generators). *The following table demonstrates the "standard" operations generated by sets of 5 operations.*

	generators					generated operations							induced set structure
1.	∅	X	c	∪	∩	∅	X	c	∪	∩	\	△	*algebra of sets*
2.	∅	X	c	∪	\	∅	X	c	∪	∩	\	△	*algebra of sets*
3.	∅	X	c	∪	△	∅	X	c	∪	∩	\	△	*algebra of sets*
4.	∅	X	c	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
5.	∅	X	c	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
6.	∅	X	c	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
7.	∅	X	∪	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
8.	∅	X	∪	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
9.	∅	X	∪	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
10.	∅	X	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
11.	∅	c	∪	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
12.	∅	c	∪	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
13.	∅	c	∪	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
14.	∅	c	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
15.	∅	∪	∩	\	△	∅			∪	∩	\	△	*ring of sets*
16.	X	c	∪	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
17.	X	c	∪	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
18.	X	c	∪	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
19.	X	c	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
20.	X	∪	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
21.	c	∪	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*

Proposition A.6 (6 genererators). *The following table demonstrates the "standard" operations generated by sets of 6 operations.*

	generators						generated operations							induced set structure
1.	∅	X	c	∪	∩	\	∅	X	c	∪	∩	\	△	*algebra of sets*
2.	∅	X	c	∪	∩	△	∅	X	c	∪	∩	\	△	*algebra of sets*
3.	∅	X	c	∪	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
4.	∅	X	c	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
5.	∅	X	∪	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
6.	∅	c	∪	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*
7.	X	c	∪	∩	\	△	∅	X	c	∪	∩	\	△	*algebra of sets*

A.2.4 Set multiplication

The *Cartesian product* operation $\times$ (next definition) is a kind of *set multiplication* operation.

Definition A.8. [7] *Let X and Y be sets, and let (x,y) be an* ORDERED PAIR.

DEF *The **Cartesian product** $X \times Y$ of X and Y is*
$$X \times Y \triangleq \{(x,y) \,|\, (x \in X) \text{ and } (y \in Y)\}$$

Theorem A.2 (next theorem) demonstrates how this set operation interacts with certain other set operations. The Cartesian product is of critical importance in general because, for example, relations (Definition B.1 page 73) and functions (Definition B.8 page 85) are subsets of Cartesian products.

Theorem A.2. [8] *Let X, Y, Z be sets.*

THM
$$\begin{array}{rcll}
X \times (Y \cup Z) &=& (X \times Y) \cup (X \times Z) & (\times \text{ distributes over } \cup)\\
X \times (Y \cap Z) &=& (X \times Y) \cap (X \times Z) & (\times \text{ distributes over } \cap)\\
X \times (Y \setminus Z) &=& (X \times Y) \setminus (X \times Z) & (\times \text{ distributes over } \setminus)\\
(X \times Y) \cap (Y \times X) &=& (X \cap Y) \times (Y \cap X) & \\
(X \times X) \cap (Y \times Y) &=& (X \cap Y) \times (X \cap Y) &
\end{array}$$

✎PROOF:

$$\begin{aligned}
X \times (Y \cup Z) &= \{(a,b) \,|\, (a \in X) \wedge (b \in Y \cup Z)\} \\
&= \{(a,b) \,|\, (a \in X) \wedge [(b \in Y) \vee (b \cup Z)]\} && \text{by Definition A.5}\\
&= \{(a,b) \,|\, [(a \in X) \wedge (b \in Y)] \vee [(a \in X) \wedge (b \in Z)]\} && \text{by Theorem 3.2}\\
&= \underbrace{\{(a,b) \,|\, [(a \in X) \wedge (b \in Y)]\}}_{X \times Y} \cup \underbrace{\{(a,b) \,|\, [(a \in X) \wedge (b \in Z)]\}}_{X \times Z} && \text{by Definition A.5}\\
&= (X \times Y) \cup (X \times Z)
\end{aligned}$$

$$\begin{aligned}
X \times (Y \cap Z) &= \{(a,b) \,|\, (a \in X) \wedge (b \in Y \cap Z)\} \\
&= \{(a,b) \,|\, (a \in X) \wedge [(b \in Y) \wedge (b \cup Z)]\} && \text{by Definition A.5}\\
&= \{(a,b) \,|\, [(a \in X) \wedge (b \in Y)] \wedge [(a \in X) \wedge (b \in Z)]\} \\
&= \underbrace{\{(a,b) \,|\, [(a \in X) \wedge (b \in Y)]\}}_{X \times Y} \cap \underbrace{\{(a,b) \,|\, [(a \in X) \wedge (b \in Z)]\}}_{X \times Z} && \text{by Definition A.5}\\
&= (X \times Y) \cap (X \times Z)
\end{aligned}$$

$$\begin{aligned}
X \times (Y \setminus Z) &= \{(a,b) \,|\, (a \in X) \wedge (b \in Y \setminus Z)\} \\
&= \left\{(a,b) \,|\, (a \in X) \wedge (b \in Y \cap Z^{c})\right\} \\
&= \left\{(a,b) \,|\, (a \in X) \wedge \left[(b \in Y) \wedge (b \in Z^{c})\right]\right\} && \text{by Definition A.5}\\
&= \left\{(a,b) \,|\, [(a \in X) \wedge (b \in Y)] \wedge \left[(a \in X) \wedge (b \in Z^{c})\right]\right\} \\
&= \underbrace{\{(a,b) \,|\, [(a \in X) \wedge (b \in Y)]\}}_{X \times Y} \cap \underbrace{\left\{(a,b) \,|\, \left[(a \in X) \wedge (b \in Z^{c})\right]\right\}}_{X \times Z^{c}} && \text{by Definition A.5}\\
&= (X \times Y) \cap (X \times Z^{c}) \\
&\neq (X \times Y) \setminus (X \times Z)
\end{aligned}$$

[7] Halmos (1960) page 24
G. Frege, 2007 August 25, http://groups.google.com/group/sci.logic/msg/3b3294f5ac3a76f0
[8] Menini and Oystaeyen (2004) page 50, Halmos (1960) page 25

$$\begin{aligned}
(X\times Y)\cap(Y\times X) &= \{(a,b)\,|(a\in X)\wedge(b\in Y)\}\cap\{(a,b)\,|(a\in Y)\wedge(b\in X)\} \\
&= \{(a,b)\,|[(a\in X)\wedge(b\in Y)]\wedge[(a\in Y)\wedge(b\in X)]\} && \text{by Definition A.5} \\
&= \{(a,b)\,|[(a\in X)\wedge(a\in Y)]\wedge[(b\in Y)\wedge(b\in X)]\} \\
&= \{(a,b)\,|(a\in X\cap Y)\wedge(b\in Y\cap X)\} \\
&= (X\cap Y)\times(Y\cap X)
\end{aligned}$$

$$\begin{aligned}
(X\times X)\cap(Y\times Y) &= \{(a,b)\,|(a\in X)\wedge(b\in X)\}\cap\{(a,b)\,|(a\in Y)\wedge(b\in Y)\} \\
&= \{(a,b)\,|[(a\in X)\wedge(b\in X)]\ \wedge\ [(a\in Y)\wedge(b\in Y)]\} && \text{by Definition A.5} \\
&= \{(a,b)\,|[(a\in X)\wedge(a\in Y)]\ \wedge\ [(b\in X)\wedge(b\in Y)]\} \\
&= \{(a,b)\,|(a\in X\cap Y)\ \wedge\ (b\in X\cap Y)\} \\
&= (X\cap Y)\times(X\cap Y)
\end{aligned}$$

A.3 Standard set structures

Set structures are typically designed to satisfy some special properties— such as being closed with respect to certain set operations. Examples of commonly occurring set structures include

- *power set* (Definition A.1 page 37)
- *topologies* (Definition A.9 page 47)
- *algebra of sets* (Definition A.10 page 50)
- *ring of sets* (Definition A.11 page 51)
- *partitions* (Definition A.12 page 53)

A.3.1 Topologies

Definition A.9. [9] *Let Γ be a set with an arbitrary (possibly uncountable) number of elements. Let 2^X be the* POWER SET *of a set X.*

DEF

*A family of sets $\boldsymbol{T}\subseteq 2^X$ is a **topology** on a set X if*

1. $\varnothing\in\boldsymbol{T}$ *($\varnothing$ is in $\boldsymbol{T}$)* *and*
2. $X\in\boldsymbol{T}$ *(X is in $\boldsymbol{T}$)* *and*
3. $U,V\in\boldsymbol{T} \implies U\cap V\in\boldsymbol{T}$ *(the intersection of a finite number of open sets is open)* *and*
4. $\{U_\gamma|\gamma\in\Gamma\}\subseteq\boldsymbol{T} \implies \bigcup_{\gamma\in\Gamma}U_\gamma\in\boldsymbol{T}$ *(the union of an arbitrary number of open sets is open).*

*A **topological space** is the pair $(X,\boldsymbol{T})$. An **open set** is any member of $\boldsymbol{T}$.*
*A **closed set** is any set D such that D^c is* OPEN.
The set of topologies on a set X is denoted $\mathcal{T}(X)$. That is,

$$\mathcal{T}(X)\triangleq\left\{\boldsymbol{T}\subseteq 2^X|\boldsymbol{T}\text{ is a **topology**}\right\}.$$

If X is FINITE, *then $\boldsymbol{T}$ is a **topology on a finite set**, and (4.) can be replaced by*

$$U,V\in\boldsymbol{T} \quad\implies\quad U\cup V\in\boldsymbol{T}.$$

Example A.2. [10] Let $\mathcal{T}(X)$ be the set of topologies on a set X and 2^X the *power set* (Definition A.1 page 37)

[9] Munkres (2000) page 76, Riesz (1909), Hausdorff (1914), Tietze (1923) ⟨cited by Thron page 18⟩, Hausdorff (1937) page 258

[10] Munkres (2000) page 77, Kubrusly (2011) page 107 ⟨Example 3.J⟩, Steen and Seebach (1978) pages 42–43 ⟨II.4⟩, DiBenedetto (2002) page 18

on X.

EX			
$\{\varnothing,\ X\}$	is a *topology* in	$\mathcal{T}(X)$	(*indiscrete topology* or *trivial topology*)
2^X	is a *topology* in	$\mathcal{T}(X)$	(*discrete topology*)

Example A.3. [11] There are four topologies on the set $X \triangleq \{x, y\}$:

topologies on $\{x, y\}$	corresponding closed sets
$T_0 = \{\varnothing, X\}$	$\{\varnothing, X\}$
$T_1 = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y\}, X\}$
$T_2 = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x\}, X\}$
$T_3 = \{\varnothing, \{x\}, \{y\}, X\}$	$\{\varnothing, \{x\}, \{y\}, X\}$

The topologies (X, T_1) and (X, T_2), as well as their corresponding closed set topological spaces, are all *Serpiński spaces*.

Example A.4. There are a total of 29 *topologies* (Definition A.9 page 47) on the set $X \triangleq \{x, y, z\}$:

topologies on $\{x, y, z\}$	corresponding closed sets
$T_{00} = \{\varnothing, X\}$	$\{\varnothing, X\}$
$T_{01} = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y,z\}, X\}$
$T_{02} = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x,z\}\ X\}$
$T_{04} = \{\varnothing, \{z\}, X\}$	$\{\varnothing, \{x,y\}, X\}$
$T_{10} = \{\varnothing, \{x,y\}, X\}$	$\{\varnothing, \{z\}, X\}$
$T_{20} = \{\varnothing, \{x,z\}, X\}$	$\{\varnothing, \{y\}, X\}$
$T_{40} = \{\varnothing, \{y,z\}, X\}$	$\{\varnothing, \{x\}, X\}$
$T_{11} = \{\varnothing, \{x\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{y,z\}, X\}$
$T_{21} = \{\varnothing, \{x\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}\ \{y,z\}, X\}$
$T_{41} = \{\varnothing, \{x\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y,z\}, X\}$
$T_{12} = \{\varnothing, \{y\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,z\}\ X\}$
$T_{22} = \{\varnothing, \{y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,z\}, X\}$
$T_{42} = \{\varnothing, \{y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,z\}, X\}$
$T_{14} = \{\varnothing, \{z\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,y\}, X\}$
$T_{24} = \{\varnothing, \{z\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,y\}, X\}$
$T_{44} = \{\varnothing, \{z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,y\}, X\}$
$T_{31} = \{\varnothing, \{x\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$
$T_{52} = \{\varnothing, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$
$T_{64} = \{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$
$T_{13} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{25} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,y\}, \{y,z\}, X\}$
$T_{46} = \{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,y\}, \{x,z\}, X\}$
$T_{33} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{53} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$T_{35} = \{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$
$T_{65} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$
$T_{56} = \{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$
$T_{66} = \{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$
$T_{77} = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$

Theorem A.3. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge\,;\, \leq)$ *be a* LATTICE.

THM: T *is a* TOPOLOGY $\implies$ $(T, \cup, \cap\,;\, \subseteq)$ *is a* DISTRIBUTIVE LATTICE

✎PROOF:

1. By Proposition A.15 (page 60), $(S, \subseteq)$ is an *ordered set.*

[11] Isham (1999) page 44, Isham (1989) page 1515

2. By Proposition A.16 (page 61), $\cup$ is *least upper bound* operation on $(\boldsymbol{S}, \subseteq)$. and $\cap$ is *greatest lower bound* operation on $(\boldsymbol{S}, \subseteq)$.

3. Therefore, by Definition D.3 (page 117), $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is a lattice.

4. By Theorem D.3 (page 118), $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is *idempotent*, *commutative*, *associative*, and *absorptive*.

5. Proof that $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is *distributive*:

   (a) Proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:

$$\begin{aligned}
&A \cap (B \cup C) \\
&= \{x \in X | x \in A \wedge x \in (B \cup C)\} && \text{by definition of } \cap \text{ (Definition A.5 page 38)} \\
&= \{x \in X | x \in A \wedge x \in \{x \in X | x \in B \vee x \in C\}\} && \text{by definition of } \cup \text{ (Definition A.5 page 38)} \\
&= \{x \in X | x \in A \wedge (x \in B \vee x \in C)\} \\
&= \{x \in X | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} && \text{by Theorem 3.2 page 33} \\
&= \{x \in X | x \in A \wedge x \in B\} \cup \{x \in X | x \in A \wedge x \in C\} && \text{by definition of } \cup \text{ (Definition A.5 page 38)} \\
&= (A \cap B) \cup (A \cap C) && \text{by definition of } \cap \text{ (Definition A.5 page 38)}
\end{aligned}$$

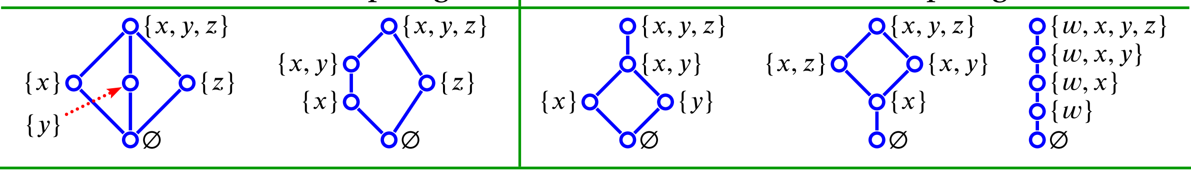
   (b) Proof that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:
   This follows from the fact that $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is a lattice (item (3) page 49), that $\cap$ distributes over $\cup$ (item (5) page 49), and by Theorem G.1 (page 146).

⇨

Example A.5. There are five unlabeled lattices on a five element set (Proposition D.2 page 123). Of these five, three are *distributive* (Proposition G.3 page 163). The following illustrates that the distributive lattices are isomorphic to topologies, while the non-distributive lattices are not.

EX

non-distributive/ *not* topologies	*distributive*/ *are* topologies



✎PROOF:

1. The first two lattices are non-distributive by *Birkhoff distributivity criterion* (Theorem G.2 page 150).

   (a) This lattice is not a topology because, for example,
   $\{x\} \vee \{y\} = \{x, y, z\} \neq \{x, y\} = \{x\} \cup \{y\}$.
   That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.

   (b) This lattice is not a topology because, for example,
   $\{x\} \vee \{y\} = \{y\} \neq \{x, y\} = \{x\} \cup \{y\}$

2. The last three lattices are distributive by *Birkhoff distributivity criterion* (Theorem G.2 page 150).

   (a) This lattice is the topology $\boldsymbol{T}_{13}$ of Example A.4 (page 48). On the set $\{x, y, z\}$, there are a total of three topologies that have this order structure (see Example A.4):

$$\begin{aligned}
\boldsymbol{T}_{13} &= \{ \varnothing, \{x\}, \{y\}, \{x, y\}, \{x, y, z\} \} \\
\boldsymbol{T}_{25} &= \{ \varnothing, \{x\}, \{z\}, \{x, z\}, \{x, y, z\} \} \\
\boldsymbol{T}_{46} &= \{ \varnothing, \{y\}, \{z\}, \{y, z\}, \{x, y, z\} \}
\end{aligned}$$

(b) This lattice is the topology T_{31} of Example A.4 (page 48). On the set $\{x, y, z\}$, there are a total of three topologies that have this order structure (see Example A.4):

$$\begin{array}{lcllllll} T_{31} &=& \{ & \emptyset, & \{x\}, & \{x,y\}, & \{x,z\}, & \{x,y,z\} \ \} \\ T_{52} &=& \{ & \emptyset, & \{y\}, & \{x,y\}, & \{y,z\}, & \{x,y,z\} \ \} \\ T_{64} &=& \{ & \emptyset, & \{z\}, & \{x,z\}, & \{y,z\}, & \{x,y,z\} \ \} \end{array}$$

(c) This lattice is a topology by Definition A.9 (page 47).

A.3.2 Algebras of sets

Definition A.10. [12] *Let X be a set with* POWER SET 2^X (Definition A.1 page 37).

D E F

*$\mathbf{A} \subseteq 2^X$ is an **algebra of sets** on X if*

1. $A \in \mathbf{A} \implies A^c \in \mathbf{A}$ *(closed under complement operation)* *and*
2. $A, B \in \mathbf{A} \implies A \cap B \in \mathbf{A}$ *(closed under $\cap$)*

The set of all algebra of sets on a set X is denoted $\mathcal{A}(X)$ such that

$$\mathcal{A}(X) \triangleq \{\mathbf{A} \subseteq 2^X \mid \mathbf{A} \text{ is an algebra of sets}\}.$$

An ALGEBRA OF SETS *$\mathbf{A}$ on X is a **σ-algebra** on X if*

3. $\{A_n \mid n \in \mathbb{Z}\} \subseteq \mathbf{A} \implies \bigcup_{n\in\mathbb{Z}} A_n \in \mathbf{A}$ *(closed under countable union operations).*

On every set X with at least 2 elements, there are always two particular algebras of sets: the *smallest algebra* and the *largest algebra*, as demonstrated by Example A.6 (next).

Example A.6. [13] Let $\mathcal{A}(X)$ be the set of *algebras of sets* (Definition A.10 page 50) on a set X and 2^X the *power set* (Definition A.1 page 37) on X.

E X

$$\begin{array}{lcll} \{\emptyset, X\} &\in& \mathcal{A}(X) & \text{(smallest algebra)} \\ 2^X &\in& \mathcal{A}(X) & \text{(largest algebra)} \end{array}$$

Isomorphically, all *algebras of sets* are *boolean algebras* (Definition I.1 page 171) and all boolean algebras are algebras of sets (next theorem).

Theorem A.4 (Stone Representation Theorem). [14] *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ be a* LATTICE.

T H M

$$\mathbf{L} \text{ is BOOLEAN} \iff \left\{ \begin{array}{l} \mathbf{L} \text{ is isomorphic to } (\mathbf{A}, \cup, \cap, \emptyset, X; \subseteq) \\ \text{for some ALGEBRA OF SETS (Definition A.10 page 50) } \mathbf{A} \end{array} \right\}$$

✎PROOF:

1. Proof that *algebra of sets* $\implies$ *Boolean algebra*:

   (a) Proof that $\mathbf{S}$ is closed under $\cup$ and $\cap$: by hypothesis.

   (b) By item (1b) and by Theorem A.6 (page 57), $\mathbf{L}$ is a *distributive* lattice.

[12] Aliprantis and Burkinshaw (1998) page 95, Aliprantis and Burkinshaw (1998) page 151, Halmos (1950) page 21, Hausdorff (1937) page 91

[13] Stroock (1999) page 33, Aliprantis and Burkinshaw (1998) pages 95–96

[14] Levy (2002) page 257, Grätzer (2003) page 85, Joshi (1989) page 224, Saliĭ (1988) page 32 ⟨"Stone's Theorem"⟩, Stone (1936)

(c) By item (1b) and properties of *lattices* (Theorem D.3 page 118), $\mathbf{L}$ is *idempotent*, *commutative*, *associative*, and *absorptive*.

(d) Proof that $\mathbf{L}$ has *identity*:

$$\begin{aligned}
A \cup \varnothing &= \{x \in X | (x \in A) \vee (x \in \varnothing)\} && \text{by definition of } \cup \text{ Definition A.5 page 38}\\
&= \{x \in X | x \in A\} && \text{by definition of } \varnothing \text{ Definition A.5 page 38}\\
&= A\\
A \cap X &= \{x \in X | (x \in A) \wedge (x \in X)\} && \text{by definition of } \cap \text{ Definition A.5 page 38}\\
&= \{x \in X | x \in A\} && \text{by definition of } \varnothing \text{ Definition A.5 page 38}\\
&= A
\end{aligned}$$

(e) Proof that $\mathbf{L}$ is *complemented*: by hypothesis.

(f) Because $\mathbf{L}$ is *commutative* (item (1c) page 51), *distributive* (item (1b) page 50), has *identity* (item (1d) page 51), and is *complemented* (item (1e) page 51), and by the definition of *Boolean algebras* (Definition I.1 page 171), $\mathbf{L}$ is a *Boolean algebra*.

2. Proof that *Boolean algebra* $\implies$ *algebra of sets*: not included at this time.

⇒

A.3.3 Rings of sets

A *ring of sets* (next definition) is a family of subsets that is closed under an "addition-like" set union operator $\cup$ and "subtraction-like" set difference operator $\setminus$. Using these two operations, it is not difficult to show that a ring of sets is also closed under a "multiplication-like" set intersection operator $\cap$. Because of this, a ring of sets behaves like an *algebraic ring*. Note however that a ring of sets is not necessarily a *topology* (Definition A.9 page 47) because it does not necessarily include X itself.

Definition A.11. [15] *Let X be a set with* POWER SET 2^X *(Definition A.1 page 37).*

DEF

$\mathbf{R} \subseteq 2^X$ is a ***ring of sets*** *on X if*

1. $A, B \in \mathbf{R} \implies A \cup B$ *(closed under $\cup$)* *and*
2. $A, B \in \mathbf{R} \implies A \setminus B \in \mathbf{R}$ *(closed under $\setminus$)*

The set of all rings of sets on a set X is denoted $\mathcal{R}(X)$ such that

$$\mathcal{R}(X) \triangleq \left\{\mathbf{R} \subseteq 2^X | \mathbf{R} \text{ is a ring of sets}\right\}.$$

A RING OF SETS *$\mathbf{R}$ on X is a* ***σ-ring*** *on X if*

3. $\left\{A_n | n \in \mathbb{Z}\right\} \subseteq \mathbf{R} \implies \bigcup_{n \in \mathbb{Z}} A_n \in \mathbf{R}$ *(closed under countable union operations).*

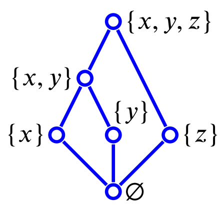
Example A.7. Table A.7 (page 52) lists some *rings of sets* on a finite set X.

Example A.8. Let $X \triangleq \{x, y, z\}$ be a set and $\mathbf{R}$ be the family of sets

$\mathbf{R} \triangleq \{\varnothing, X, \{x\}, \{y\}, \{z\}, \{x, y\}\}$.

Note that $(\mathbf{R}, \subseteq, \cup, \cap)$ is a lattice as illustrated in the figure to the right. However, $\mathbf{R}$ is *not* a ring of sets on X because, for example,

$\{x, y, z\} \setminus \{x\} = \{y, z\} \notin \mathbf{R}$.



[15] Berezansky et al. (1996) page 4, Halmos (1950) page 19, Hausdorff (1937) page 90

rings $\mathcal{R}(X)$ on a set X

$$\mathcal{R}(\varnothing) = \left\{ \boldsymbol{R}_1 = \{\varnothing\} \right\}$$

$$\mathcal{R}(\{x\}) = \left\{ \begin{array}{lcllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\} & \} \end{array} \right\}$$

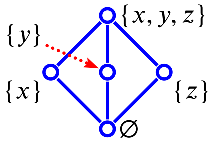
$$\mathcal{R}(\{x,y\}) = \left\{ \begin{array}{lcllllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\}, & & & \} \\ \boldsymbol{R}_3 & = & \{ & \varnothing, & & \{y\}, & & \} \\ \boldsymbol{R}_4 & = & \{ & \varnothing, & & & \{x,y\} & \} \\ \boldsymbol{R}_5 & = & \{ & \varnothing, & \{x\}, & \{y\}, & \{x,y\} & \} \end{array} \right\}$$

$$\mathcal{R}(\{x,y,z\}) = \left\{ \begin{array}{lcllllllllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & & & & & & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\}, & & & & & & & \} \\ \boldsymbol{R}_3 & = & \{ & \varnothing, & & \{y\}, & & & & & & \} \\ \boldsymbol{R}_4 & = & \{ & \varnothing, & & & \{z\}, & & & & & \} \\ \boldsymbol{R}_5 & = & \{ & \varnothing, & & & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_6 & = & \{ & \varnothing, & & & & & \{x,z\}, & & & \} \\ \boldsymbol{R}_7 & = & \{ & \varnothing, & & & & & & \{y,z\}, & & \} \\ \boldsymbol{R}_8 & = & \{ & \varnothing, & \{x\}, & \{y\}, & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_9 & = & \{ & \varnothing, & \{x\}, & & \{z\}, & & \{x,z\}, & & & \} \\ \boldsymbol{R}_{10} & = & \{ & \varnothing, & & \{y\}, & \{z\}, & & & \{y,z\}, & & \} \\ \boldsymbol{R}_{11} & = & \{ & \varnothing, & & & & & & & X & \} \\ \boldsymbol{R}_{12} & = & \{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X & \} \\ \boldsymbol{R}_{13} & = & \{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X & \} \\ \boldsymbol{R}_{14} & = & \{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X & \} \\ \boldsymbol{R}_{15} & = & \{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X & \} \end{array} \right\}$$

Table A.7: some *rings of sets* on a finite set X (Example A.7 page 51)

Example A.9. Let $X \triangleq \{x, y, z\}$ be a set and $\boldsymbol{R}$ be the family of sets $\boldsymbol{R} \triangleq \{\emptyset, X, \{x\}, \{y\}, \{z\}\}$. Note that $(\boldsymbol{T}, \subseteq) \cup \cap$ is a lattice as illustrated in the figure to the right. However, $\boldsymbol{R}$ is *not* a ring of sets on X because, for example,

$\{x, y, z\} \setminus \{x\} = \{y, z\} \notin \boldsymbol{R}$.



Proposition A.7. [16] *Let $\mathcal{R}(X)$ be the set of* RINGS OF SETS *(Definition A.11 page 51) on a set X.*

PRP: $$\left\{\begin{array}{l}\boldsymbol{R}_1 \text{ and } \boldsymbol{R}_2\\ \text{are rings of sets}\end{array}\right\} \implies \left\{\begin{array}{l}(\boldsymbol{R}_1 \cap \boldsymbol{R}_2)\\ \text{is a ring of sets}\end{array}\right\}$$

A.3.4 Partitions

The following definition is a special case of *partition* defined on lattices (Definition E.2 page 134).

Definition A.12. [17]

DEF: *A* SET STRUCTURE $\left\{P_n \in 2^X \,\middle|\, n=1,2,\ldots,N\right\}$ *is a **partition** of the set X if*

1. $P_n \neq \emptyset \quad \forall n \in \{1,2,\ldots,N\}$ NON-EMPTY *and*
2. $P_n \cap P_m = \emptyset \quad \forall n \neq m$ MUTUALLY EXCLUSIVE *and*
3. $\displaystyle\bigcup_{n\in\mathbb{Z}} P_n = X$

Example A.10. Let $A, B \subseteq X$, as illustrated in Figure A.3 (page 40). There are a total of 15 partitions of X induced by A and B (Proposition A.11 page 55). Here are 5 of these partitions:

EX:
1. $\{X\}$ (1 region)
2. $\{A, A^c\}$ (2 regions)
3. $\{A \cup B, A^c \cap B^c\}$ (2 regions)
4. $\{A \cap B, A \triangle B, A^c \cap B^c\}$ (3 regions)
5. $\{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$ (4 regions) [see also Figure A.3 page 40 and Figure A.4 page 41]

Proposition A.8. [18] *Let $\mathcal{P}(X)$ be the set of partitions on a set X.*

PRP: *The relation $\trianglelefteq \in 2^{\mathbb{PP}}$ defined as*

$$\boldsymbol{P} \trianglelefteq \boldsymbol{Q} \quad \overset{\text{def}}{\iff} \quad \forall B \in \boldsymbol{Q}, \exists A \in \boldsymbol{P} \quad \text{such that} \quad B \subseteq A$$

is an ordering relation on $\mathcal{P}(X)$.

Example A.11. Table A.8 (page 54) lists some partitions $\boldsymbol{P}(X)$ on a finite set X.

A.4 Numbers of set structures

Proposition A.9. [19]

PRP: *The **number of topologies** t_n on a finite set X_n with n elements is*

n	0	1	2	3	4	5	6	7	8
t_n	1	1	4	29	355	6942	209,527	9,535,241	642,779,354

n	9	10
t_n	63,260,289,423	8,977,053,873,043

[16] Kolmogorov and Fomin (1975) page 32, Bartle (2001) page 318
[17] Munkres (2000) page 23, Rota (1964) page 498, Halmos (1950) page 31
[18] Roman (2008) page 111, Comtet (1974) page 220, Grätzer (2007) page 697
[19] Sloane (2014) ⟨http://oeis.org/A000798⟩, Brown and Watson (1996) page 31, Comtet (1974) page 229, Comtet (1966), Chatterji (1967) page 7, Evans et al. (1967), Krishnamurthy (1966) page 157

partitions $\mathcal{P}(X)$ on a set X		
$\mathcal{P}(\varnothing)$	=	$\boldsymbol{P}_1 = \varnothing$
$\mathcal{P}(\{x\})$	=	$\boldsymbol{P}_1 = \{\{x\}\}$
$\mathcal{P}(\{x,y\})$	=	$\boldsymbol{P}_1 = \{\{x\}, \{y\}\}$
		$\boldsymbol{P}_2 = \{\{x,y\}\}$
$\mathcal{P}(\{x,y,z\})$	=	$\boldsymbol{P}_1 = \{\{x,y,z\}\}$
		$\boldsymbol{P}_2 = \{\{x\}, \{y,z\}\}$
		$\boldsymbol{P}_3 = \{\{y\}, \{x,z\}\}$
		$\boldsymbol{P}_4 = \{\{z\}, \{x,y\}\}$
		$\boldsymbol{P}_5 = \{\{x\}, \{y\}, \{z\}\}$
$\mathcal{P}(\{w,x,y,z\})$	=	$\boldsymbol{P}_1 = \{X\}$
		$\boldsymbol{P}_2 = \{\{w\}, \{x,y,z\}\}$
		$\boldsymbol{P}_3 = \{\{x\}, \{w,y,z\}\}$
		$\boldsymbol{P}_4 = \{\{y\}, \{w,x,z\}\}$
		$\boldsymbol{P}_5 = \{\{z\}, \{w,x,y\}\}$
		$\boldsymbol{P}_6 = \{\{w,x\}, \{y,z\}\}$
		$\boldsymbol{P}_7 = \{\{w,y\}, \{x,z\}\}$
		$\boldsymbol{P}_8 = \{\{w,z\}, \{x,y\}\}$
		$\boldsymbol{P}_9 = \{\{w\}, \{x\}\ \{y,z\}\}$
		$\boldsymbol{P}_{10} = \{\{w\}, \{y\}, \{x,z\}\}$
		$\boldsymbol{P}_{11} = \{\{w\}, \{z\}, \{x,y\}\}$
		$\boldsymbol{P}_{12} = \{\{x\}, \{y\}, \{w,z\}\}$
		$\boldsymbol{P}_{13} = \{\{x\}, \{z\}, \{w,y\}\}$
		$\boldsymbol{P}_{14} = \{\{y\}, \{z\}, \{w,x\}\}$
		$\boldsymbol{P}_{15} = \{\{w\}, \{x\}, \{y\}, \{z\}\}$

Table A.8: some partitions $\boldsymbol{P}(X)$ on a finite set X (Example A.11 page 53)

Proposition A.10. [20] *Let t_n be the number of topologies on a finite set with n elements.*

PRP

$$\begin{array}{rcll}
\lim_{n\to\infty} \frac{t_n}{2^{\frac{n^2}{4}}} &=& \infty & & \text{(lower bound)}\\
\lim_{n\to\infty} \frac{t_n}{2^{\left(\frac{1}{2}+\epsilon\right)n^2}} &=& 0 & \forall\epsilon>0 & \text{(upper bound)}\\
t_n &>& nt_{n-1} & & \text{(rate of growth)}
\end{array}$$

Similar to the amazing relationship between e, π, i, 1, and 0 given by $e^{i\pi} + 1 = 0$, we find another relationship between e and the number of partitions, rings of sets, and algebras of sets (Theorem A.5 page 56).

Definition A.13. [21]

DEF

*The **Bell numbers** are the elements of the sequence $(\!(B_n)\!)_{n\in\mathbb{W}}$ defined as the solution to the following equation:*

$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n.$$

*The Bell numbers are also called the **exponential numbers**.*

Proposition A.11. [22] *Let $(\!(B_n)\!)_{n\in\mathbb{W}}$ be the sequence of Bell numbers. Then $(\!(B_n)\!)$ has the following values:*

PRP

n	0	1	2	3	4	5	6	7	8	9	10	11
B_n	1	1	2	5	15	52	203	877	4140	21,147	115,975	678,570

✎PROOF: By Definition A.13 (page 55), the sequence $(\!(B_n)\!)$ is the solution to

$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n.$$

Let $\mathsf{f}^{(n)}(x)$ be the nth derivative of a function $\mathsf{f} : \mathbb{R} \to \mathbb{R}$. The Maclaurin expansion of $\mathsf{f}(x)$ is

$$\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\mathsf{f}^n(0)}{n!}x^n$$

Let $\mathsf{f}(x) \triangleq e^{e^x}$. Then

$$\begin{aligned}
\mathsf{f}^{(0)}(0) &= \mathsf{f}^{(0)}(x)\big|_{x=0}\\
&= e^{e^0}\\
&= e\\
\mathsf{f}^{(1)}(0) &= \mathsf{f}^{(1)}(x)\big|_{x=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}e^{e^x}\bigg|_{x=0}\\
&= e^{e^x}e^x\Big|_{x=0}\\
&= e\\
\mathsf{f}^{(2)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}^{(1)}(x)\bigg|_{x=0}
\end{aligned}$$

[20] Chatterji (1967) pages 6–7, Kleitman and Rothschild (1970)
[21] Comtet (1974) pages 210–211, Rota (1964) page 499, Bell (1934) page 417, d'Ocagne (1887) page 371
[22] Sloane (2014) ⟨http://oeis.org/A000110⟩

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x} e^x \Big|_{x=0} \\
&= \left(e^{e^x} e^x\right) e^x + e^{e^x} e^x \Big|_{x=0} \\
&= e^{e^x}\left(e^{2x} + e^x\right)\Big|_{x=0} \\
&= 2e \\
\mathsf{f}^{(3)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(2)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x} e^x \left(e^{2x} + e^x\right) + e^{e^x}\left(2e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x}\left(e^{3x} + 3e^{2x} + e^x\right)\Big|_{x=0} \\
&= 5e \\
\mathsf{f}^{(4)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(3)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{3x} + 3e^{2x} + e^x\right)\Big|_{x=0} \\
&= \left(e^{e^x} e^x\right)\left(e^{3x} + 3e^{2x} + e^x\right) + e^{e^x}\left(3e^{3x} + 6e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x}\left(e^{4x} + 6e^{3x} + 7e^{2x} + e^x\right)\Big|_{x=0} \\
&= 15e \\
\mathsf{f}^{(5)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(4)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{4x} + 6e^{3x} + 7e^{2x} + e^x\right)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{e^x} e^x\right)\left(e^{4x} + 6e^{3x} + 7e^{2x} + e^x\right) + e^{e^x}\left(4e^{4x} + 18e^{3x} + 14e^{2x} + e^x\right)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{5x} + 10e^{4x} + 25e^{3x} + 15e^{2x} + e^x\right)\Big|_{x=0} \\
&= 52e \\
\mathsf{f}^{(6)}(0) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathsf{f}^{(5)}(x)\Big|_{x=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} e^{e^x}\left(e^{5x} + 10e^{4x} + 25e^{3x} + 15e^{2x} + e^x\right)\Big|_{x=0} \\
&= \left(e^{e^x} e^x\right)\left(e^{5x} + 10e^{4x} + 25e^{3x} + 15e^{2x} + e^x\right) + e^{e^x}\left(5e^{5x} + 40e^{4x} + 75e^{3x} + 30e^{2x} + e^x\right)\Big|_{x=0} \\
&= e^{e^x}\left(e^{6x} + 15e^{5x} + 65e^{4x} + 90e^{3x} + 31e^{2x} + e^x\right)\Big|_{x=0} \\
&= 203e
\end{aligned}
$$

Thus, e^{e^x} has Maclaurin expansion

$$e^{e^x} = e\left(1 + x + \frac{2}{2}x^2 + \frac{5}{3!}x^3 + \frac{15}{4!}x^4 + \frac{52}{5!}x^5 + \frac{203}{6!}x^6 + \cdots\right) = e\sum_{n=0}^{\infty} \frac{B_n}{n!}x^n$$

⇨

Theorem A.5. [23] *Let X_n be a finite set with n elements. Let $⦅B_n⦆_{n\in\mathbb{W}}$ be the sequence of Bell numbers.*

[23] http://groups.google.com/group/sci.math/browse_thread/thread/70a73e734b69a6ec/

THM

The number of PARTITIONS *on* X_n *is* B_n.
The number of RINGS OF SETS *on* X_n *is* B_{n+1}.
The number of ALGEBRAS OF SETS *on* X_n *is* B_n.

A.5 Operations on set structures

Proposition A.12.

PRP

closed under	*partition*	*ring of sets*	*algebra of sets*	*topology*
$\emptyset$		✓	✓	✓
X	✓		✓	✓
c			✓	
$\cup$		✓	✓	✓
$\cap$		✓	✓	✓
$\triangle$		✓	✓	
$\setminus$		✓	✓	

✎PROOF:

1. Proof for closure in a *topology*: Definition A.9 (page 47)
2. Proof for closure in a *ring of sets*: Definition A.11 (page 51) and Theorem A.14 (page 59)
3. Proof for closure in an *algebra of sets*: Definition A.10 (page 50) and Theorem A.13 (page 57)

⇨

Theorem A.6. *Let* T *be a* SET STRUCTURE *(Definition A.2 page 37) on a set* X.

THM

T *is a* ***topology*** $\implies \forall A, B, C \in T$

property with emphasis on $\cup$	dual property with emphasis on $\cap$	property name
$A \cup A = A$	$A \cap A = A$	(IDEMPOTENT)
$A \cup B = B \cup A$	$A \cap B = B \cap A$	(COMMUTATIVE)
$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	(ASSOCIATIVE)
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	(ABSORPTIVE)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(DISTRIBUTIVE)

✎PROOF:

1. By Definition A.9 (page 47), T is a *topology*.
2. By Theorem A.4 (page 50), $(T, \cup, \cap; \subseteq)$ is a *distributive lattice*.
3. The properties listed are all properties of *distributive lattice*s, as provided by Theorem D.3 (page 118), Definition G.2 (page 146), and Theorem G.1 (page 146).

⇨

Proposition A.13. *Let* A *be a* SET STRUCTURE *(Definition A.2 page 37) on a set* X.

PRP

$$\left\{\begin{array}{l}\boldsymbol{A} \text{ is an} \\ \textbf{\textit{algebra of sets}}\end{array}\right\} \implies \left\{\begin{array}{lllll}1. & \varnothing & \in \boldsymbol{A} & & (\boldsymbol{A} \text{ includes the } \cup \text{ identity element})\\ 2. & X & \in \boldsymbol{A} & & (\boldsymbol{A} \text{ includes the } \cap \text{ identity element})\\ 3. & A^{c} & \in \boldsymbol{A} & \forall A \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } c)\\ 4. & A \cup B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \cup)\\ 5. & A \cap B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \cap)\\ 6. & A \setminus B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \setminus)\\ 7. & A \triangle B & \in \boldsymbol{A} & \forall A, B \in \boldsymbol{A} & (\boldsymbol{A} \text{ is closed under } \triangle)\end{array}\right\}$$

✎PROOF:

$$\begin{aligned} \varnothing &= A \cap A^{c} \\ X &= c\varnothing \\ A \cup B &= c(A^{c} \cap B^{c}) && \text{by } \textit{de Morgan's Law} \text{ (Theorem A.8 page 58)} \\ A \setminus B &= A \cap B^{c} \\ A \triangle B &= (A \setminus B^{c}) \cup (B \setminus A) \end{aligned}$$

$(\boldsymbol{A}, \cup, \setminus)$ is a ring of sets because $\cup$ and $\setminus$ are closed in $\boldsymbol{A}$ (as shown above). ✏

Theorem A.7. [24] *Let* $\boldsymbol{A}$ *be a* SET STRUCTURE (Definition A.2 page 37) *on a set* X.

THM

$\boldsymbol{A}$ *is an* ***algebra of sets*** $\implies \forall A, B, C \in \boldsymbol{A}$

property emphasizing $\cup$	dual property emphasizing $\cap$	property name
$A \cup A = A$	$A \cap A = A$	(IDEMPOTENT)
$A \cup B = B \cup A$	$A \cap B = B \cap A$	(COMMUTATIVE)
$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	(ASSOCIATIVE)
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	(ABSORPTIVE)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(DISTRIBUTIVE)
$A \cup \varnothing = A$	$A \cap X = A$	(IDENTITY)
$A \cup X = X$	$A \cap \varnothing = \varnothing$	(BOUNDED)
$A \cup A^{c} = X$	$A \cap A^{c} = \varnothing$	(COMPLEMENTED)
$(A^{c})^{c} = A$		(UNIQUELY COMPLEMENTED)
$(A \cup B)^{c} = A^{c} \cap B^{c}$	$(A \cap B)^{c} = A^{c} \cup B^{c}$	(DE MORGAN)

✎PROOF:

1. By Definition A.10 (page 50), $\boldsymbol{S}$ is an *algebra of sets*.
2. By the *Stone Representation Theorem* (Theorem A.4 page 50), $(\boldsymbol{S}, \cup, \cap, \varnothing, X; \subseteq)$ is a *Boolean algebra*.
3. The properties listed are all properties of *Boolean algebras* (Theorem I.2 page 176).

✏

Theorem A.8. [25] *Let* $\boldsymbol{A}$ *be an* ALGEBRA OF SETS (Definition A.10 page 50) *on a set* X.

THM

$\boldsymbol{A}$ *is an* ***algebra of sets*** $\implies \forall A_1, A_2, \ldots, A_N, B \in \boldsymbol{A}$ *and* $\forall N \in \mathbb{N}$

property emphasizing $\cup$	dual property emphasizing $\cap$	property name
$\left(\bigcup_{n=1}^{N} A_n\right)^{c} = \bigcap_{n=1}^{N} A^{c}{}_n$	$\left(\bigcap_{n=1}^{N} A_n\right)^{c} = \bigcup_{n=1}^{N} A^{c}{}_n$	(DE MORGAN)
$\left(\bigcup_{n=1}^{N} A_n\right) \cap B = \bigcup_{n=1}^{N} (A_n \cap B)$	$\left(\bigcap_{n=1}^{N} A_n\right) \cup B = \bigcap_{n=1}^{N} (A_n \cup B)$	(DISTRIBUTIVE *with respect to* $\cup$ *and* $\cap$)
$\left(\bigcup_{n=1}^{N} A_n\right) \setminus B = \bigcup_{n=1}^{N} (A_n \setminus B)$	$\left(\bigcap_{n=1}^{N} A_n\right) \setminus B = \bigcap_{n=1}^{N} (A_n \setminus B)$	(DISTRIBUTIVE *with respect to* $\setminus$ *and* $\cap$)

[24] Dieudonné (1969) pages 3–4, Copson (1968) page 9

[25] Michel and Herget (1993) page 12, Aliprantis and Burkinshaw (1998) page 4, Vaidyanathaswamy (1960) pages 3–4

✎PROOF:

1. By Theorem A.4 (page 50), the lattice $(X, \cup, \cap; \subseteq)$ is *Boolean*.
2. The first four properties are true any Boolean system Theorem I.4 (page 177).
3. Proof for the remaining two:

$$\begin{aligned}
\left(\bigcap_{n=1}^{N} A_n\right)\setminus B &= \left(\bigcap_{n=1}^{N} A_n\right)\cap B^c && \text{by Theorem A.1 page 39}\\
&= \bigcap_{n=1}^{N}(A_n \cap B^c) && \text{by previous result}\\
&= \bigcap_{n=1}^{N}(A_n \setminus B) && \text{by Theorem A.1 page 39}
\end{aligned}$$

$$\begin{aligned}
\left(\bigcup_{n=1}^{N} A_n\right)\setminus B &= \left(\bigcup_{n=1}^{N} A_n\right)\cap B^c && \text{by Theorem A.1 page 39}\\
&= \bigcup_{n=1}^{N}(A_n \cap B^c) && \text{by previous result}\\
&= \bigcup_{n=1}^{N}(A_n \setminus B) && \text{by Theorem A.1 page 39}
\end{aligned}$$

⇒

Proposition A.14. [26] *Let $\mathbf{R}$ be a* SET STRUCTURE *(Definition A.2 page 37) on a set X.*

PRP

$$\left\{\begin{array}{l}\mathbf{R} \text{ is a}\\ \textbf{ring of sets}\\ \text{on } X\end{array}\right\} \implies \left\{\begin{array}{llllll}
1. & \varnothing & \in \mathbf{R} & & (\mathbf{R} \text{ includes the } \cup \text{ identity element}) & \text{and}\\
2. & A\cup B & \in \mathbf{R} & \forall A,B\in\mathbf{R} & (\mathbf{R} \text{ is closed under } \cup) & \text{and}\\
3. & A\cap B & \in \mathbf{R} & \forall A,B\in\mathbf{R} & (\mathbf{R} \text{ is closed under } \cap) & \text{and}\\
4. & A\setminus B & \in \mathbf{R} & \forall A,B\in\mathbf{R} & (\mathbf{R} \text{ is closed under } \setminus) & \text{and}\\
5. & A\triangle B & \in \mathbf{R} & \forall A,B\in\mathbf{R} & (\mathbf{R} \text{ is closed under } \triangle) &
\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
A\triangle B &= (A\setminus B)\cup(B\setminus A)\\
A\cap B &= (A\cup B)\setminus(A\triangle B)\\
A\setminus A &= \varnothing
\end{aligned}$$

⇒

Theorem A.9. [27] *Let $\mathbf{R}$ be a* SET STRUCTURE *(Definition A.2 page 37) on a set X.*

THM

If $\mathbf{R}$ is an ***ring of sets*** *on X, then $(\mathbf{R}, \triangle, \cap)$ is an* ALGEBRAIC RING*; in particular,*

$$\begin{array}{lllll|lllll}
A\triangle\varnothing & = & A & \forall A\in\mathbf{R} & & A\cap\varnothing & = & \varnothing & \forall A\in\mathbf{R}\\
A\triangle X & = & A^c & \forall A\in\mathbf{R} & & A\cap X & = & A & \forall A\in\mathbf{R}\\
A\triangle\varnothing & = & A & \forall A\in\mathbf{R} & & A\cap A & = & A & \forall A\in\mathbf{R}
\end{array}$$

$$A\cap(B\triangle C) = (A\cap B)\triangle(A\cap C) \qquad \forall A,B,C\in\mathbf{R}$$

properties emphasizing $\triangle$ | *properties emphasizing* $\cap$

[26] Berezansky et al. (1996) page 4, Halmos (1950) pages 19–20

[27] Vaidyanathaswamy (1960) pages 17–18, Kelley and Srinivasan (1988) page 22, Wilker (1982) page 211, Vaidyanathaswamy (1960) page 19

✎PROOF:

1. Proof that $(\boldsymbol{R}, \cup, \setminus)$ is an *algebraic ring*: by Theorem A.9 (page 59)
2. Proof that a ring of sets is equivalent to $(\boldsymbol{R}, \cup, \setminus)$: This is proven simply by noting that $\cup$ and $\setminus$ (the two operations in a ring of sets $(\boldsymbol{R}, \cup, \setminus)$) can be expressed in terms of $\triangle$ and $\cap$ (the two operations in the algebraic ring $(\boldsymbol{R}, \triangle, \cap)$) and vice-versa. And this is demonstrated by Theorem A.1 (page 39).

1. Proof that $(\boldsymbol{S}, \triangle)$ is a group: see Proposition A.1 (page 39).

2. Proof that $A \cap (B \cap C) = (A \cap B) \cap C$:

$$\begin{aligned}
A \cap (B \cap C) &= \{x \in X | (x \in A) \wedge [(x \in B) \wedge (x \in C)]\} && \text{by definition of } \cap \text{ page 38} \\
&= \{x \in X | [(x \in A) \wedge (x \in B)] \wedge (x \in C)\} \\
&= (A \cap B) \cap C && \text{by definition of } \cap \text{ page 38}
\end{aligned}$$

3. Proof that $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$:

$$\begin{aligned}
A \cap (B \triangle C) &= \{x \in X | (x \in A) \wedge [(x \in B) \oplus (x \in C)]\} && \text{by definition of } \cap, \triangle \text{ page 38} \\
&= \{x \in X | [(x \in A) \wedge (x \in B)] \oplus [(x \in A) \wedge (x \in C)]\} \\
&= (A \cap B) \triangle (A \cap C) && \text{by definition of } \cap, \triangle \text{ page 38}
\end{aligned}$$

4. Proof that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$:

$$\begin{aligned}
(A \triangle B) \cap C &= \{x \in X | [(x \in A) \oplus (x \in B)] \wedge (x \in C)\} && \text{by definition of } \cap, \triangle \text{ page 38} \\
&= \{x \in X | [(x \in A) \wedge (x \in C)] \oplus [(x \in B) \wedge (x \in C)]\} \\
&= (A \cap C) \triangle (B \cap C) && \text{by definition of } \cap, \triangle \text{ page 38}
\end{aligned}$$

✏

A.6 Lattices of set structures

A.6.1 Ordering relations

The *set inclusion* relation $\subseteq$ (Definition A.14 page 60) is an *order relation* (Definition C.2 page 102) on set structures, as demonstrated by Proposition A.15 (next proposition).

Definition A.14. *Let S be a* SET STRUCTURE *(Definition A.2 page 37) on a set X.*

DEF *The relation $\subseteq \in 2^{SS}$ is defined as*

$$A \subseteq B \quad \text{if} \quad x \in A \implies x \in B \qquad \forall x \in X$$

Proposition A.15 (order properties). *Let S be a* SET STRUCTURE *(Definition A.2 page 37) on a set X.*

PRP *The pair $(S, \subseteq)$ is an* ORDERED SET. *In particular,*

$$\begin{array}{lllllllll}
A \subseteq A & & & & \forall A \in S & \text{(REFLEXIVE)} & \text{and} \\
A \subseteq B & \text{and} & B \subseteq C & \implies & A \subseteq C & \forall A,B,C \in S & \text{(TRANSITIVE)} & \text{and} \\
A \subseteq B & \text{and} & B \subseteq A & \implies & A = B & \forall A,B \in S & \text{(ANTI-SYMMETRIC).}
\end{array}$$

✎PROOF: By Definition C.2 (page 102), a relation is an *order relation* if it is *reflexive*, *transitive*, and *anti-symmetric*.

1. Proof that $\subseteq$ is *reflexive* on 2^X:

$$\begin{aligned} x \in A &\implies x \in A \\ &\implies A \subseteq A \end{aligned}$$

2. Proof that $\subseteq$ is *transitive* on 2^X:

$$\begin{aligned} x \in A &\implies x \in B && \text{by first left hypothesis} \\ &\implies x \in C && \text{by second left hypothesis} \\ &\implies A \subseteq C \end{aligned}$$

3. Proof that $\subseteq$ is *anti-symmetric* on 2^X:

$$\begin{aligned} A \subseteq B &\implies (x \in A \implies x \in B) \\ B \subseteq A &\implies (x \in B \implies x \in A) \\ A \subseteq B \text{ and } B \subseteq A &\implies (x \in A \iff x \in B) \\ &\implies A = B \end{aligned}$$

⇨

In a set structure that is *closed* under the *union* operation $\cup$ and *intersection* operation $\cap$, the *greatest lower bound* of any two elements A and B is simply $A \cap B$ and *least upper bound* is simply $A \cup B$ (Proposition A.16 page 61). However, this may not be true for a set structure that is *not* closed under these operations (Example A.12 page 62).

Proposition A.16. *Let S be a* SET STRUCTURE *(Definition A.2 page 37) on a set X.*

PRP

If** S is closed under $\cup$ and $\cap$ **then

$A \cup B$	*is the* LEAST UPPER BOUND	*of A and B in $(S, \subseteq)$*	$(\cup = \vee)$	*and*
$A \cap B$	*is the* GREATEST LOWER BOUND	*of A and B in $(S, \subseteq)$*	$(\cap = \wedge)$.	

✎PROOF:

1. Proof that $A \cup B$ is the least upper bound:

$$\begin{aligned} A &= \{x \in X | x \in A\} \\ &\subseteq \{x \in X | x \in A \text{ or } x \in B\} \\ &= A \cup B && \text{by Definition A.5 page 38} \\ B &= \{x \in X | x \in B\} \\ &\subseteq \{x \in X | x \in A \text{ or } x \in B\} \\ &= A \cup B && \text{by Definition A.5 page 38} \end{aligned}$$

$$\begin{aligned} A \subseteq C \text{ and } B \subseteq C &\implies \left\{x \in A \text{ and } y \in B \implies x, y \in C\right\} \\ &\implies \{x \in A \text{ or } x \in B \implies x \in C\} \\ &\implies \{x \in A \cup B \implies x \in C\} \\ &\implies A \cup B \subseteq C \end{aligned}$$

2. Proof that $A \cap B$ is the greatest lower bound:

$$\begin{aligned}
A \cap B &= \left\{x \in X \mid x \in A \text{ and } x \in B\right\} && \text{by Definition A.5 page 38}\\
&\subseteq \{x \in X \mid x \in A\}\\
&= A\\
A \cap B &= \left\{x \in X \mid x \in A \text{ and } x \in B\right\} && \text{by Definition A.5 page 38}\\
&\subseteq \{x \in X \mid x \in B\}\\
&= B\\
C \subseteq A \text{ and } C \subseteq B &\implies \left\{x \in C \implies x \in A \quad \text{and} \quad x \in C \implies x \in B\right\}\\
&\implies \{x \in C \implies x \in A \text{ or } x \in B\}\\
&\implies \{x \in C \implies x \in A \cap B\}\\
&\implies C \subseteq A \cap B
\end{aligned}$$

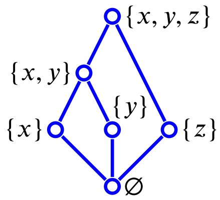
Example A.12. The set structure

$\boldsymbol{S} \triangleq \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,y,z\}\}$

ordered by the set inclusion relation $\subseteq$ is illustrated by the Hasse diagram to the right. Note that

$\{x\} \vee \{z\} = \{x,y,z\} \neq \{x,z\} = \{x\} \cup \{z\}$.

That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.

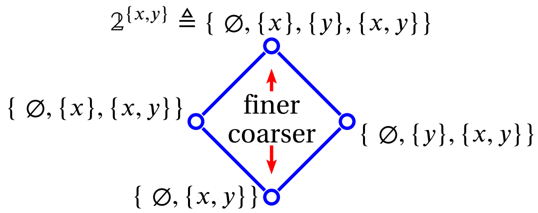


A.6.2 Lattices of topologies

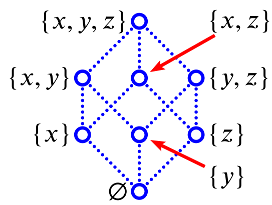
Example A.13. [28] Example A.3 (page 48) lists the four topologies on the set $X \triangleq \{x,y\}$. The lattice of these topologies

$\left(\{\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{T}_4\}, \cup, \cap; \subseteq\right)$

is illustrated by the *Hasse diagram* to the right.



Example A.14. [29] Let a given topology in $\mathcal{T}(\{x,y,z\})$ be represented by a Hasse diagram as illustrated to the right, where a circle present means the indicated set is in the topology, and a circle absent means the indicated set is not in the topology. Example A.4 (page 48) lists the 29 topologies $\mathcal{T}(\{x,y,z\})$. The lattice of these 29 topologies $(\mathcal{T}(\{x,y,z\}), \cup, \cap; \subseteq)$ is illustrated in Figure A.5 (page 63). The five topologies $\boldsymbol{T}_1$, $\boldsymbol{T}_{41}$, $\boldsymbol{T}_{22}$, $\boldsymbol{T}_{14}$, and $\boldsymbol{T}_{77}$ are also *algebras of sets* (Definition A.10 page 50); these five sets are shaded in Figure A.5.



Theorem A.10. [30] *Let $\mathcal{T}(X)$ be the **lattice of topologies** on a set X with $|X|$ elements.*

THM				
	$\lvert X \rvert \leq 2$	$\implies$	$\mathcal{T}(X)$ *is* DISTRIBUTIVE	
	$\lvert X \rvert \geq 3$	$\implies$	$\mathcal{T}(X)$ *is* NOT MODULAR *(and not distributive)*	

[28] Isham (1999) page 44, Isham (1989) page 1515

[29] Isham (1999) page 44, Isham (1989) page 1516, Steiner (1966) page 386

[30] Steiner (1966) page 384

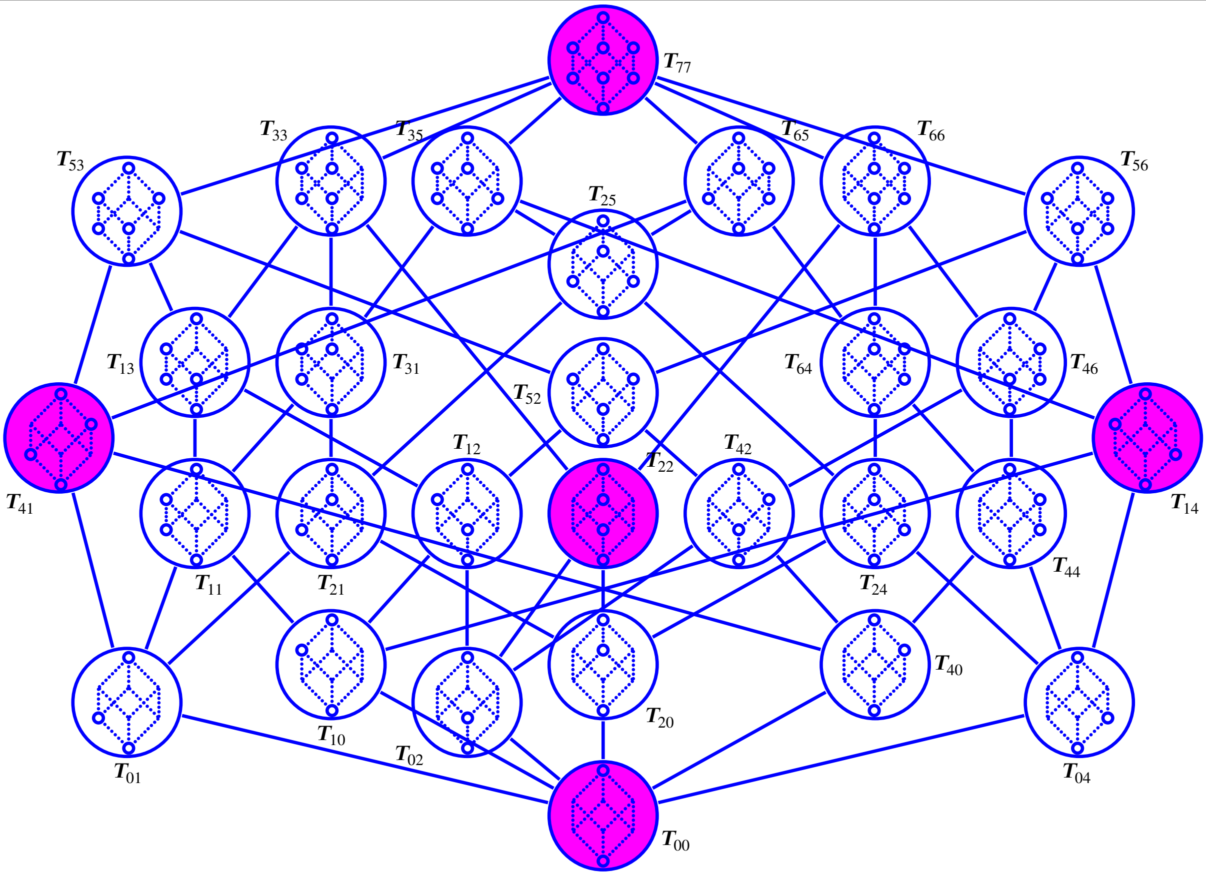


Figure A.5: Lattice of *topologies* on $X \triangleq \{x, y, z\}$ (see Example A.14 page 62)

Theorem A.11. [31] *Let $\mathcal{T}(X)$ be the **lattice of topologies** on a set X.*

THM	$\mathcal{T}(X)$ *is* SELF-DUAL $\iff$ $\lvert X \rvert \le 3$

Theorem A.12. [32]

THM	*Every lattice of topologies is complemented.*

Theorem A.13. [33]

THM	*Every* TOPOLOGY (Definition A.9 page 47) *except the* DISCRETE TOPOLOGY *and* INDISCRETE TOPOLOGY (Example A.2 page 47) *in the **lattice of topologies** on a set X has at least* $\lvert X \rvert - 1$ COMPLEMENTS.

Example A.15. Example A.4 (page 48) lists the 29 topologies on a set $X \triangleq \{x, y, z\}$. By Theorem A.13 (page 63), with the exception of T_{00} (the indiscrete topology) and T_{77} (the discrete topology), each of those topologies has exactly $\lvert X \rvert - 1 = 3 - 1 = 2$ complements. Table A.9 (page 64) lists the 29 topologies on $\{x, y, z\}$ along with their respective complements.

Theorem A.14. [34]

THM	$\mathcal{T}(X)$ *is a topology of sets* $\implies$ $\begin{cases} \mathcal{T}(X) \text{ is atomic.} \\ \mathcal{T}(X) \text{ is anti-atomic.} \end{cases}$

[31] Steiner (1966) page 385
[32] van Rooij (1968), Steiner (1966) page 397, Gaifman (1961), Hartmanis (1958)
[33] Hartmanis (1958), Schnare (1968) page 56, Watson (1994), Brown and Watson (1996) page 32
[34] Larson and Andima (1975) page 179, Frölich (1964), Vaidyanathaswamy (1960), Vaidyanathaswamy (1947)

topologies on $\{x, y, z\}$	1st complement	2nd compl.
$T_{00} = \{\emptyset, X\}$	T_{77}	
$T_{01} = \{\emptyset, \{x\}, X\}$	T_{56}	T_{66}
$T_{02} = \{\emptyset, \{y\}, X\}$	T_{65}	T_{35}
$T_{04} = \{\emptyset, \{z\}, X\}$	T_{53}	T_{33}
$T_{10} = \{\emptyset, \{x,y\}, X\}$	T_{65}	T_{66}
$T_{20} = \{\emptyset, \{x,z\}, X\}$	T_{53}	T_{56}
$T_{40} = \{\emptyset, \{y,z\}, X\}$	T_{33}	T_{35}
$T_{11} = \{\emptyset, \{x\}, \{x,y\}, X\}$	T_{64}	T_{46}
$T_{21} = \{\emptyset, \{x\}, \{x,z\}, X\}$	T_{52}	T_{46}
$T_{41} = \{\emptyset, \{x\}, \{y,z\}, X\}$	T_{22}	T_{14}
$T_{12} = \{\emptyset, \{y\}, \{x,y\}, X\}$	T_{64}	T_{25}
$T_{22} = \{\emptyset, \{y\}, \{x,z\}, X\}$	T_{41}	T_{14}
$T_{42} = \{\emptyset, \{y\}, \{y,z\}, X\}$	T_{31}	T_{25}
$T_{14} = \{\emptyset, \{z\}, \{x,y\}, X\}$	T_{41}	T_{22}
$T_{24} = \{\emptyset, \{z\}, \{x,z\}, X\}$	T_{52}	T_{13}
$T_{44} = \{\emptyset, \{z\}, \{y,z\}, X\}$	T_{31}	T_{13}
$T_{31} = \{\emptyset, \{x\}, \{x,y\}, \{x,z\}, X\}$	T_{42}	T_{44}
$T_{52} = \{\emptyset, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{21}	T_{24}
$T_{64} = \{\emptyset, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{11}	T_{12}
$T_{13} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, X\}$	T_{24}	T_{44}
$T_{25} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, X\}$	T_{12}	T_{42}
$T_{46} = \{\emptyset, \{y\}, \{z\}, \{y,z\}, X\}$	T_{11}	T_{21}
$T_{33} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{04}	T_{40}
$T_{53} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	T_{04}	T_{20}
$T_{35} = \{\emptyset, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	T_{02}	T_{40}
$T_{65} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{02}	T_{10}
$T_{56} = \{\emptyset, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	T_{01}	T_{20}
$T_{66} = \{\emptyset, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{01}	T_{10}
$T_{77} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	T_{00}	

Table A.9: the 29 topologies on a set $\{x, y, z\}$ along with their respective complements (Example A.15 page 63)

Theorem A.15. [35] *Let $\mathcal{T}(X)$ be the lattice of topologies on a set X and let $n \triangleq |X|$.*

THM

$\mathcal{T}(X)$ *contains* $2^n - 2$ *atoms*	*for finite* X.
$\mathcal{T}(X)$ *contains* $2^{|X|}$ *atoms*	*for infinite* X.
$\mathcal{T}(X)$ *contains* $n(n-1)$ *anti-atoms*	*for finite* X.
$\mathcal{T}(X)$ *contains* $2^{2^{|X|}}$ *anti-atoms*	*for infinite* X.

A.6.3 Lattices of algebra of sets

Example A.16. The following table lists some algebras of sets on a finite set X. Lattices of algebras of sets are illustrated in Figure A.8 (page 67) and Figure A.6 (page 66).

algebra of sets $\mathcal{A}(X)$ on a set X

$$\mathcal{A}(\varnothing) = \left\{ \boldsymbol{A}_1 = \{\varnothing\} \right\}$$

$$\mathcal{A}(\{x\}) = \left\{ \boldsymbol{A}_1 = \{\varnothing, \{x\}\} \right\}$$

$$\mathcal{A}(\{x,y\}) = \left\{ \begin{array}{lcl} \boldsymbol{A}_1 &=& \{\varnothing, X\} \\ \boldsymbol{A}_2 &=& \{\varnothing, \{x\}, \{y\}, X\} \end{array} \right\}$$

$$\mathcal{A}(\{x,y,z\}) = \left\{ \begin{array}{lclllllllll} \boldsymbol{A}_1 &=& \{ & \varnothing, & & & & & & & X \} \\ \boldsymbol{A}_2 &=& \{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X \} \\ \boldsymbol{A}_3 &=& \{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X \} \\ \boldsymbol{A}_4 &=& \{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X \} \\ \boldsymbol{A}_5 &=& \{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X \} \end{array} \right\}$$

$$\mathcal{A}(\{w,x,y,z\}) =$$

$$\left\{ \begin{array}{lclllll} \boldsymbol{A}_1 &=& \{ & \varnothing, & & & & X \} \\ \boldsymbol{A}_2 &=& \{ & \varnothing, & \{w\}, & & \{x,y,z\}, & X \} \\ \boldsymbol{A}_3 &=& \{ & \varnothing, & \{x\}, & & \{w,y,z\}, & X \} \\ \boldsymbol{A}_4 &=& \{ & \varnothing, & \{y\}, & & \{w,x,z\}, & X \} \\ \boldsymbol{A}_5 &=& \{ & \varnothing, & \{z\}, & & \{w,x,y\}, & X \} \\ \boldsymbol{A}_6 &=& \{ & \varnothing, & & \{w,x\},\{y,z\}, & & X \} \\ \boldsymbol{A}_7 &=& \{ & \varnothing, & & \{w,y\},\{x,z\}, & & X \} \\ \boldsymbol{A}_8 &=& \{ & \varnothing, & & \{w,z\},\{x,y\}, & & X \} \\ \boldsymbol{A}_9 &=& \{ & \varnothing, & \{w\},\{x\}, & \{w,x\},\{y,z\}, & \{w,y,z\},\{x,y,z\}, & X \} \\ \boldsymbol{A}_{10} &=& \{ & \varnothing, & \{w\},\{y\}, & \{w,y\},\{x,z\}, & \{w,x,z\},\{x,y,z\}, & X \} \\ \boldsymbol{A}_{11} &=& \{ & \varnothing, & \{w\},\{z\}, & \{w,z\},\{x,y\}, & \{w,x,y\},\{x,y,z\}, & X \} \\ \boldsymbol{A}_{12} &=& \{ & \varnothing, & \{x\},\{y\}, & \{w,z\},\{x,y\}, & \{w,x,z\},\{w,y,z\}, & X \} \\ \boldsymbol{A}_{13} &=& \{ & \varnothing, & \{x\},\{z\}, & \{w,y\},\{x,z\}, & \{w,x,y\},\{w,y,z\}, & X \} \\ \boldsymbol{A}_{14} &=& \{ & \varnothing, & \{y\},\{z\}, & \{w,x\},\{y,z\}, & \{w,x,y\},\{w,x,z\}, & X \} \\ \boldsymbol{A}_{15} &=& 2^X \end{array} \right\}$$

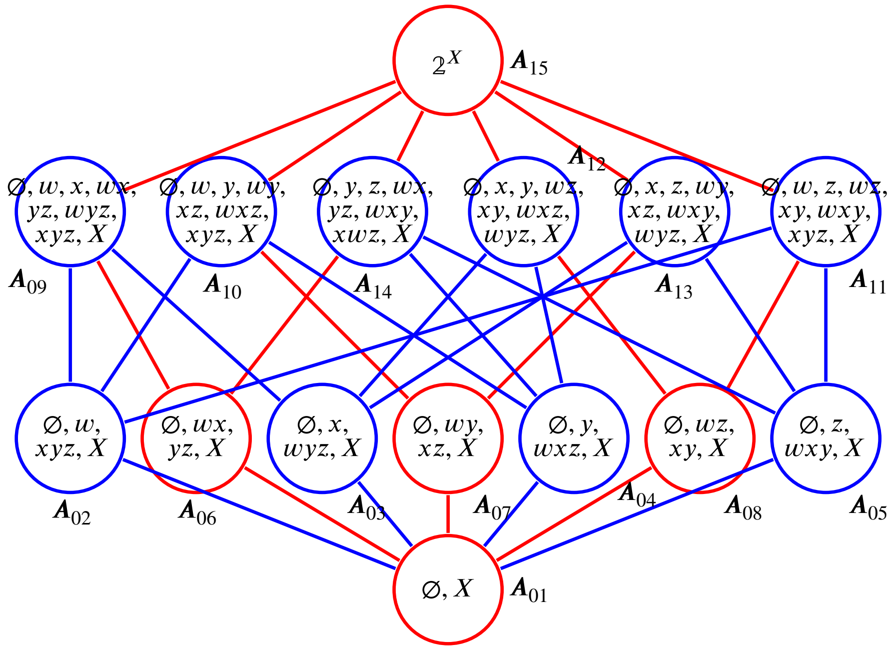


Figure A.6: lattice of *algebras of sets* on $\{w, x, y, z\}$ (Example A.16 page 65)

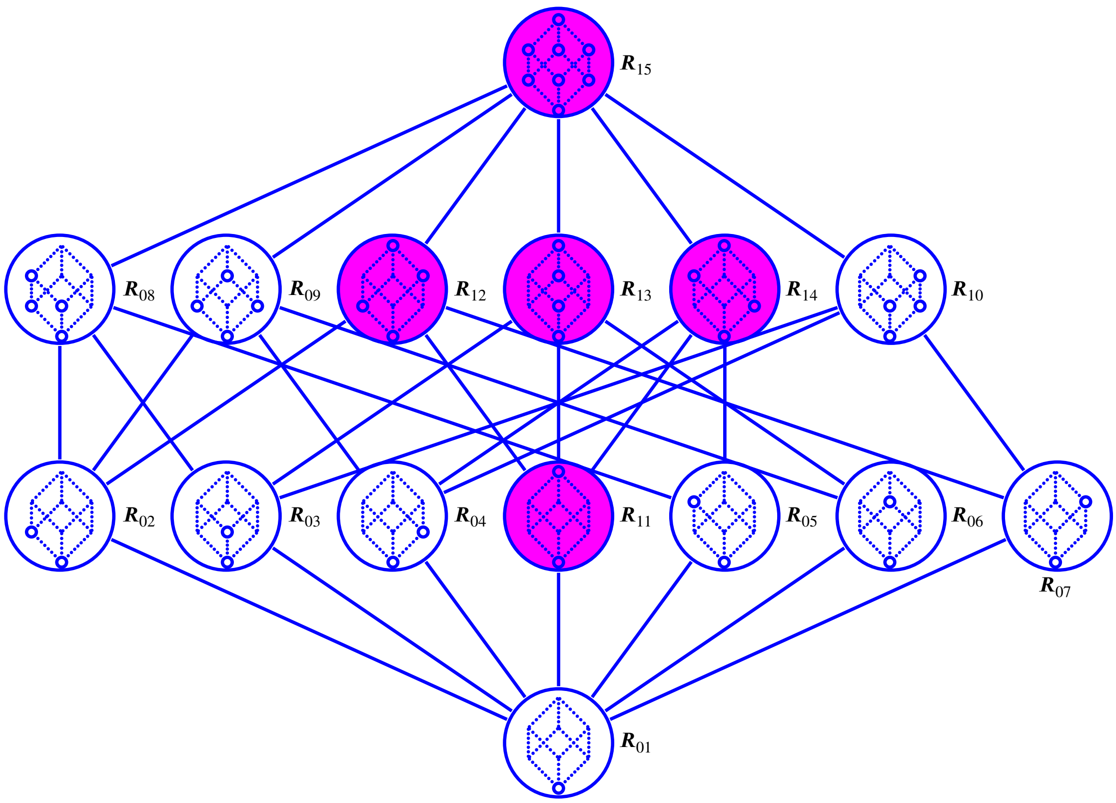


Figure A.7: Lattice of rings of sets on $X \triangleq \{x, y, z\}$ (Example A.17 page 67)

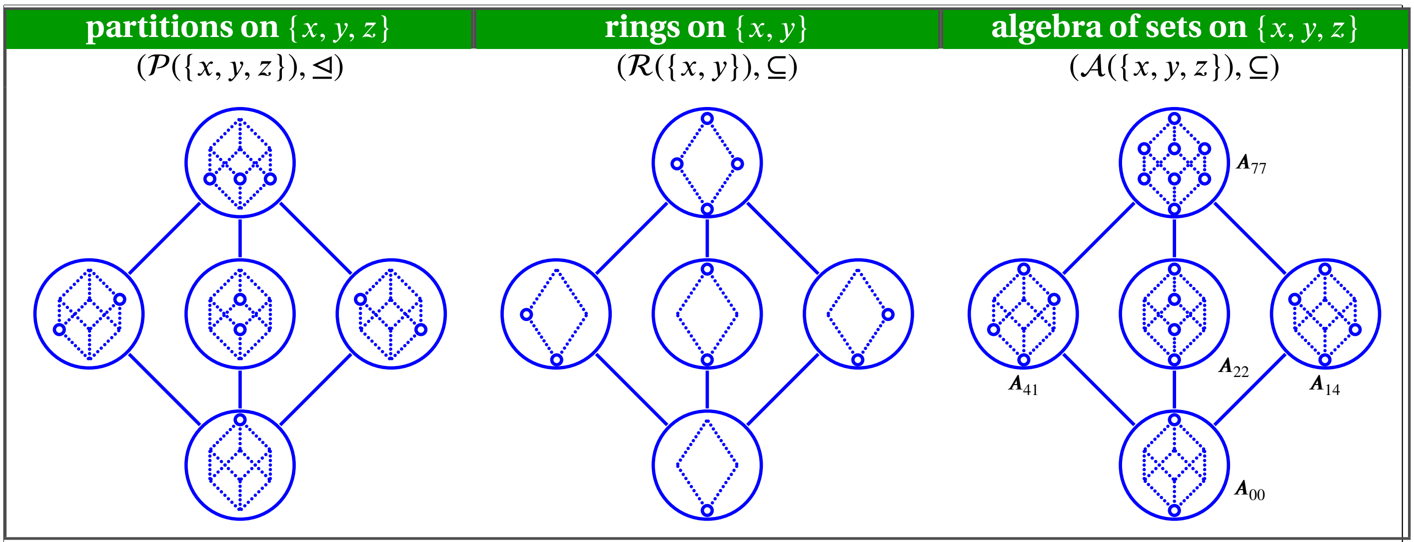


Figure A.8: Lattices of set structures (see Example A.18 (page 67), Example A.7 (page 51), and Example A.16 (page 65))

A.6.4 Lattices of rings of sets

Example A.17. There are a total of **15** rings of sets on the set $X \triangleq \{x, y, z\}$. These rings of sets are listed in Example A.7 (page 51) and illustrated in Figure A.7 (page 66). The five rings containing X ($\boldsymbol{R}_{11}$–$\boldsymbol{R}_{15}$) are also *algebras of sets* (Proposition A.18 page 69), and thus also *Boolean algebras* (Theorem A.4 page 50). The five algebras of sets are shaded Figure A.7.

A.6.5 Lattices of partitions of sets

Example A.18. There are a total of **5** partitions of sets on the set $X \triangleq \{x, y, z\}$. These sets are listed in Example A.11 (page 53) and illustrated in Figure A.8 (page 67).

Example A.19. There are a total of **15** partitions of sets on the set $X \triangleq \{w, x, y, z\}$. These sets are listed in Example A.11 (page 53) and illustrated in Figure A.9 (page 68).

In 1946, Philip Whitman proposed an amazing conjecture—that all finite lattices are isomorphic to a lattice of partitions. A proof for this was published some 30 years later by Pavel Pudlák and Jiří Tůma (next theorem).

Theorem A.16. [36] *Let* $\boldsymbol{L}$ *be a lattice.*

THM			
THM	$\boldsymbol{L}$ *is* FINITE	$\Longrightarrow$	$\boldsymbol{L}$ *is isomorphic to a* LATTICE OF PARTITIONS

Example A.20. There are five unlabeled lattices on a five element set as stated in Proposition D.2 (page 123) and illustrated in Example D.11 (page 124). All of these lattices are isomorphic to a lattice of partitions (Theorem A.16 page 67), as illustrated next.

[35] Larson and Andima (1975) page 179, Frölich (1964)
[36] Pudlák and Tůma (1980) ⟨improved proof⟩, Pudlák and Tůma (1977) ⟨proof⟩, Whitman (1946) ⟨conjecture⟩, Saliĭ (1988) page vii ⟨list of lattice theory breakthroughs⟩

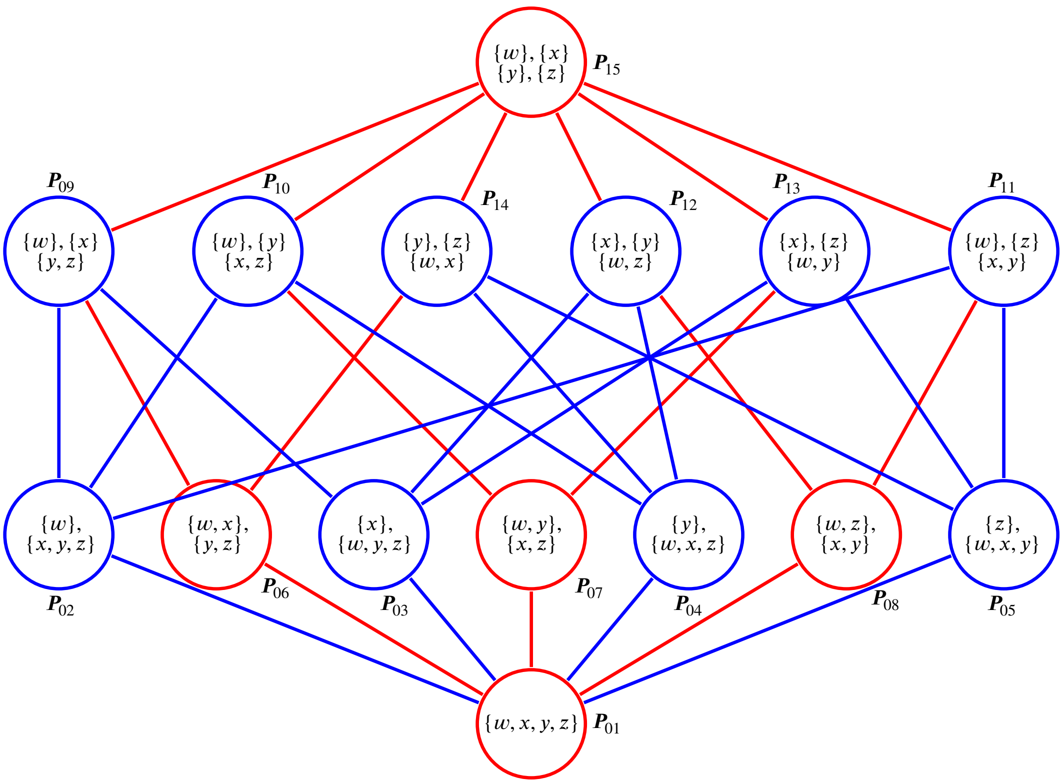
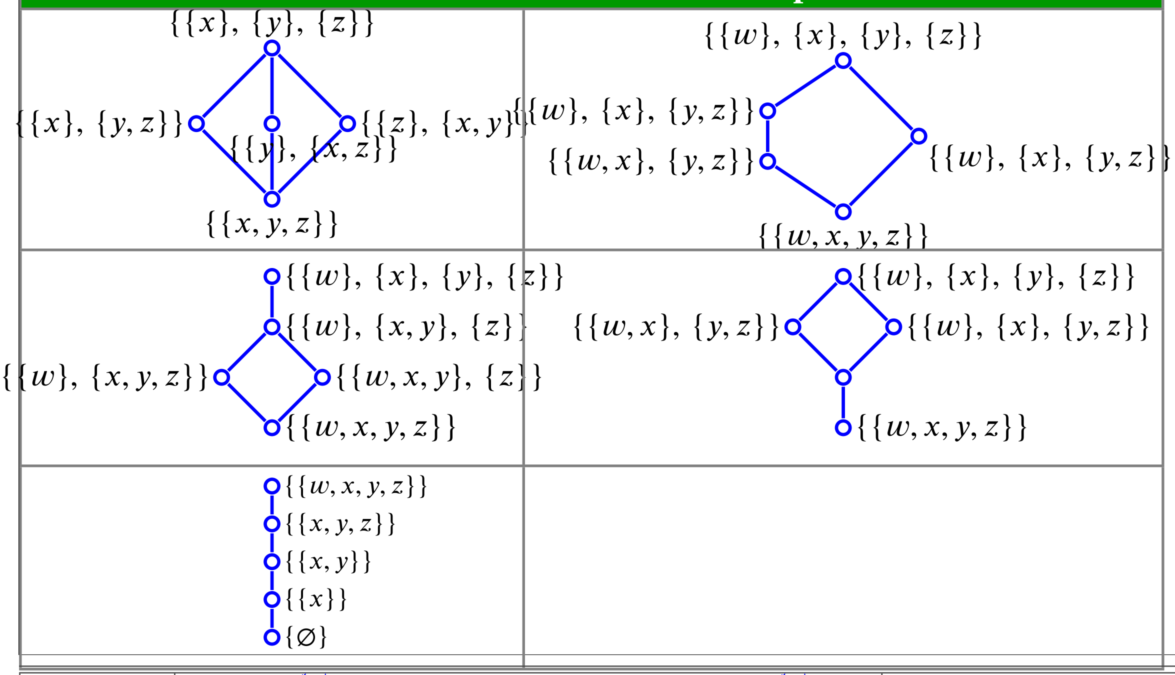


Figure A.9: Lattice of partitions of sets on $X \triangleq \{w,x,y,z\}$ (Example A.19 page 67)

lattices on 5 element set as lattices of partitions



A.7 Relationships between set structures

Proposition A.17. [37]

PRP
$$\left\{\begin{array}{l} \boldsymbol{R} \text{ is a \textbf{ring of sets}} \\ \text{on a set } X \end{array}\right\} \implies \left\{\begin{array}{l} \boldsymbol{R} \cup X \text{ is an \textbf{algebra of sets}} \\ \text{on } X \end{array}\right\}$$

Theorem A.17. *Let X be a set.*

THM
$$\left\{\begin{array}{l} \boldsymbol{A} \text{ is an \textbf{algebra of sets}} \\ \text{on } X \end{array}\right\} \iff \left\{\begin{array}{lll} 1. & \boldsymbol{A} \text{ is a \textbf{topology} on } X & \text{and} \\ 2. & \boldsymbol{A} \text{ is a \textbf{ring of sets} on } X & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} \boldsymbol{A} \text{ is an algebra of sets on } X &\implies \boldsymbol{A} \text{ is closed under } \cup, \cap, \mathsf{c}, \setminus, \varnothing, X && \text{by Theorem A.12 page 57} \\ &\implies \left\{\begin{array}{ll} 1. & \boldsymbol{A} \text{ is a topology on } X \\ & \text{AND} \\ 2. & \boldsymbol{A} \text{ is a ring of sets on } X \end{array}\right\} \end{aligned}$$

$$\begin{aligned} \left\{\begin{array}{ll} 1. & \boldsymbol{A} \text{ is a topology on } X \\ & \text{AND} \\ 2. & \boldsymbol{A} \text{ is a ring of sets on } X \end{array}\right\} &\implies \boldsymbol{A} \text{ is closed under } \mathsf{c} \text{ and } \cap && \text{by Theorem A.12 page 57} \\ &\implies \boldsymbol{A} \text{ is a ring of sets} \end{aligned}$$

✏

Corollary A.1. *Let X be a set and 2^X the power set of X.*

COR
$$\left\{\boldsymbol{A} \subseteq 2^X \middle| \boldsymbol{A} \text{ is an algebra of sets on } X\right\} = \left\{\boldsymbol{T} \subseteq 2^X \middle| \boldsymbol{T} \text{ is a topology on } X\right\} \cap \left\{\boldsymbol{R} \subseteq 2^X \middle| \boldsymbol{R} \text{ is a ring of sets on } X\right\}$$

✎PROOF:

$$\begin{aligned} &\{\boldsymbol{T} | \boldsymbol{T} \text{ is a topology}\} \cap \{\boldsymbol{R} | \boldsymbol{R} \text{ is a ring of sets}\} \\ &= \{\boldsymbol{Y} | \boldsymbol{Y} \text{ is a topology AND a ring of sets}\} && \text{by Definition A.5 page 38} \\ &= \{\boldsymbol{Y} | \boldsymbol{Y} \text{ is an algebra of sets}\} && \text{by Theorem A.17 page 69} \\ &= \{\boldsymbol{A} | \boldsymbol{A} \text{ is an algebra of sets}\} && \text{by change of variable} \end{aligned}$$

✏

Example A.21. Note that the *intersection* of the lattice of topologies on $\{x, y, z\}$ (Figure A.5 page 63) *and* the lattice of rings of sets on $\{x, y, z\}$ (Figure A.7 page 66) is *equal to* the lattice of algebras of sets on $\{x, y, z\}$ (Figure A.8 page 67).

Proposition A.18. *Let $\mathcal{R}(X)$ be the set of* RINGS OF SETS *(Definition A.11 page 51) and $\mathcal{A}(X)$ the set of* ALGEBRAS OF SETS *(Definition A.10 page 50) on a set X.*

PRP
$$\left\{\begin{array}{lll} 1. & \boldsymbol{R} \text{ is a \textbf{ring of sets}} & \text{and} \\ 2. & X \in \boldsymbol{R} & \end{array}\right\} \iff \left\{ \boldsymbol{R} \text{ is an \textbf{algebra of sets}} \right\}$$

✎PROOF:

$$\begin{aligned} A^{\mathsf{c}} &= X \setminus A && \text{by Theorem A.1 page 39} \\ A \cap B &= A \setminus (A \setminus B) && \text{by Theorem A.1 page 39} \end{aligned}$$

[37] Berezansky et al. (1996) page 4, Halmos (1950) page 21

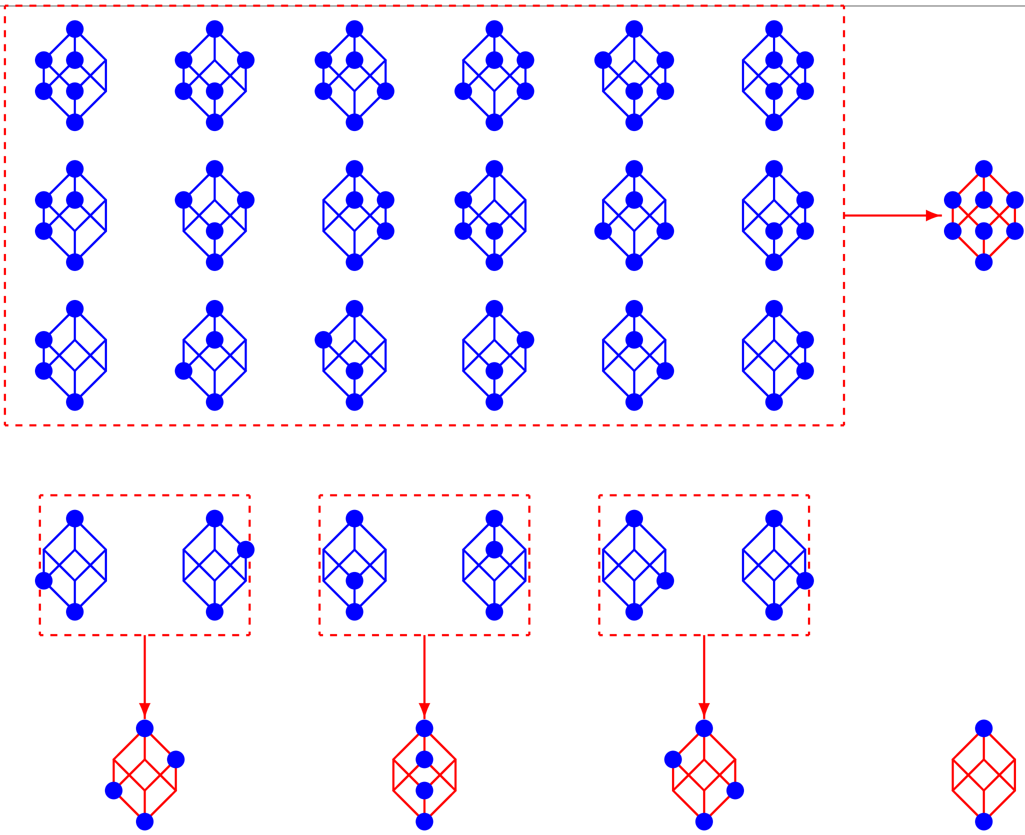


Figure A.10: Algebras of sets generated by topologies on the set $X \triangleq \{x, y, z\}$ (see Example A.23 page 70)

Therefore, $(\boldsymbol{R} \cup X)$ is closed under c and $\cap$, and thus by the definition of algebras of sets (Definition A.10 page 50), $(\boldsymbol{R} \cup X)$ is an algebra of sets. ⇨

Definition A.15. [38]

D E F *The **Borel set** $\mathrm{B}(X, \boldsymbol{T})$ generated by the topological space $(X, \boldsymbol{T})$ is the σ-algebra generated by the topology $\boldsymbol{T}$.*

Example A.22. Suppose we have a dice with the standard six possible outcomes X. Suppose also we construct the following topology $\boldsymbol{T}$ on X, and this in turn generates the following Borel set (σ-algebra) B on X:

E X

$$X = \{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}$$

$$\boldsymbol{T} = \left\{ \underbrace{\{\ \}}_{\varnothing}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}}_{\Omega}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\}}_{\text{first four}}, \underbrace{\{\text{⚃},\text{⚄},\text{⚅}\}}_{\text{last three}}, \underbrace{\{\text{⚃}\}}_{\{1234\}\cap\{456\}}, \right\}$$

$$B = \left\{ \underbrace{\{\ \}}_{\varnothing}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}}_{\Omega}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\}}_{\text{first four}}, \underbrace{\{\text{⚃},\text{⚄},\text{⚅}\}}_{\text{last three}}, \underbrace{\{\text{⚃}\}}_{\{1234\}\cap\{456\}}, \right.$$
$$\left. \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚄},\text{⚅}\}}_{\{4\}}, \underbrace{\{\text{⚄},\text{⚅}\}}_{(\{4\})\cap\{456\}}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂}\}}_{\{1234\}\cap\{4\}}, \right\}$$

Example A.23. There are a total of 29 *topologies* on the set $X \triangleq \{x, y, z\}$; and of these, 5 are also *algebras of sets*, 24 are not. Figure A.10 (page 70) illustrates the 24 topologies on the set $\{x, y, z\}$ that

[38] Aliprantis and Burkinshaw (1998) page 97

are *not* algebras of sets and the 5 algebras of sets that they generate.

A.8 Literature

Literature survey:

1. Origin of the symbols ∪ and ∩:
   Peano (1888a)
   Peano (1888b)
2. There is some difference in the definition of *ring of sets*:
   (a) *ring of sets* defined as closed under $\triangle, \cap$:
   Stone (1936) page 38
   Kolmogorov and Fomin (1975) page 31
   Kolmogorov and Fomin (1999) page 20
   Constantinescu (1984) page 155
   (b) *ring of sets* defined as closed under $\cup, \setminus$ (compatible definition):
   Wilker (1982) page 211
   Kelley and Srinivasan (1988) page 21
   Aliprantis and Burkinshaw (1998) page 96
   Haaser and Sullivan (1991) page 2
   Hewitt and Ross (1994) page 118
   (c) *ring of sets* defined as closed under $\cup, \setminus, \emptyset$ (compatible definition):
   Rao (2004) page 15
   (d) *ring of sets* defined as closed under $\cup, \cap$ (incompatible definition):
   Hausdorff (1927) ⟨???,p.77?⟩
   Hausdorff (1937) page 90
   Birkhoff (1937) page 443
   Erdös and Tarski (1943) page 315
   MacLane and Birkhoff (1999) page 485
3. Relationship to lattices (order theory):
   Stone (1936)
4. More references dealing with set structures ...
   Vaidyanathaswamy (1947)
   Bagley (1955)
   Hartmanis (1958)
   Vaidyanathaswamy (1960)
   Gaifman (1961)
   Gaifman (1966)
   Steiner (1966)
   van Rooij (1968)
   Schnare (1968)
   Rayburn (1969)
   Larson and Andima (1975)
   Pudlák and Tůma (1980)
   Brown and Watson (1991)
   Watson (1994)
   Brown and Watson (1996)
5. Partitions
   Deza and Deza (2006) page 142
   Day (1981)
   Rota (1964)
6. Distributive and modular properties in lattice of topologies

(a) Remark that “It can be shewn easily that the lattice of topologies is not distributive.”
Vaidyanathaswamy (1947)
Vaidyanathaswamy (1960) page 134

(b) Proof that the lattice of $T1$ topologies is not modular:
Bagley (1955)

(c) Proof that the lattice of topologies on any set with 3 or more elements is not modular (and thus also not distributive):
Steiner (1966) page 384

7. Complements in lattice of topologies:

(a) Proof that every lattice of topologies over a *finite* set is complemented:
Hartmanis (1958)

(b) Proof that every lattice of topologies over a *countably infinite* set is complemented:
Gaifman (1961)

(c) Proof that every lattice of topologies over a *any arbitrary* set is complemented:
Steiner (1966) page 397

(d) van Rooij (1968)

(e) Every topology in $\hat{\Sigma}(X)$ has at least 2 complements for $|X| \geq 3$:
Hartmanis (1958)

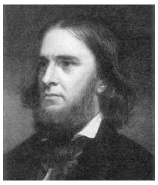
(f) Every topology in $\hat{\Sigma}(X)$ has at least $|X| - 1$ complements for $|X| \geq 2$:
Schnare (1968)

(g) A large number of topologies in $\hat{\Sigma}(X)$ have at least $2^{|X|}$ complements for $|X| \geq 4$:
Brown and Watson (1996)

APPENDIX B

RELATIONS AND FUNCTIONS

B.1 Relations



> *"A dual relative term, such as "lover," "benefactor," "servant," is a common name signifying a pair of objects. Of the two members of the pair, a determinate one is generally the first, and the other the second; so that if the order is reversed, the pair is not considered as remaining the same."*
>
> Charles Sanders Peirce (1839–1914), American mathematician and logician [1]

B.1.1 Definition and examples

A relation on the sets X and Y is any subset of the Cartesian product $X \times Y$ (next definition). Alternatively, a relation is a generalization of a *function* (Definition B.8 page 85) in the sense that both are subsets of a Cartesian product, but the relation allows mapping from a single element in its domain to two different elements in its range, whereas functions do not— a single element in a function's domain may map to one and only one element in its range. The set of all relations in $X \times Y$ is denoted 2^{XY}, which is suitable since the number of relations in $X \times Y$ when X and Y are finite is $2^{|X|\cdot|Y|}$ (Proposition B.1 page 74). Examples include the following:

- Example B.2 page 74 Relations in the Cartesian product $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$
- Example B.20 page 87 Functions in the Cartesian product $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$
- Example B.21 page 87 Functions in the Cartesian product $\{x, y, z\} \times \{x, y, z\}$
- Example B.18 page 86 discrete examples
- Example B.19 page 86 continuous examples

Definition B.1. [2] *Let X and Y be sets.*

[1] quote: Peirce (1883a) page 187
image: `http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html`

[2] Maddux (2006) page 4, Halmos (1960) page 26

DEF

*A **relation** $\circledR : X \to Y$ is any subset of $X \times Y$. That is,*

$$\circledR \subseteq X \times Y$$

A pair $(x, y) \in \circledR$ is alternatively denoted $x \circledR y$.

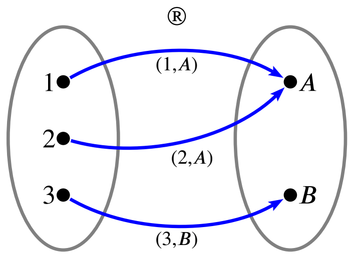
The set of all relations that are subsets of $X \times Y$ is denoted $\mathbb{2}^{XY}$; that is,

$$\mathbb{2}^{XY} \triangleq \{\circledR | \circledR \subseteq (X \times Y)\}.$$

Example B.1.

Let
$$\begin{aligned} X &\triangleq \{1,\ 2,\ 3\} \\ Y &\triangleq \{A,\ B\} \\ \circledR &\triangleq \{(1, A),\ (2, A),\ (3, B)\} \end{aligned}$$

The sets X and Y and the relation $\circledR$ are illustrated to the right.



Proposition B.1. *Let $\mathbb{2}^{XY}$ be the set of all relations from a set X to a set Y. Let $|\cdot|$ be the counting measure for sets.*

PRP

$$\underbrace{\left|\mathbb{2}^{XY}\right|}_{\text{number of possible relations in } X \times Y} = 2^{|X \times Y|} = 2^{|X|\cdot|Y|}$$

✎PROOF:

1. Let X be a finite set with m elements.

2. Let Y be a finite set with n elements.

3. Then the number of elements in $X \times Y$ is mn.

4. A relation is any subset of $X \times Y$, which may (represent this with a 1) or may not (represent this with a 0) contain a given element of $X \times Y$.

5. Therefore, the number of possible relations is $2^{mn} = 2^{|X|\cdot|Y|}$.

✏

Example B.2 (next) lists all of the 64 possible relations in the Cartesian product $\{x_1,\ x_2,\ x_3\} \times \{y_1,\ y_2\}$. Eight of these 64 relations are also functions. These eight functions are listed in Example B.20 (page 87). Of these eight functions, six are *surjective*. These six surjective functions are listed in Example B.27 (page 90).

Example B.2. Let $X \triangleq \{x_1,\ x_2,\ x_3\}$ and $Y \triangleq \{y_1,\ y_2\}$. Let $\mathbb{2}^{XY}$ be the set of all relations in $X \times Y$. There are a total of $\left|\mathbb{2}^{XY}\right| = 2^{|X|\cdot|Y|} = 2^{3\times 2} = 64$ possible relations. These are listed below. Of these 64 relations, only 8 are *functions*, as listed in Example B.20 (page 87).

relations in $\{x_1,\ x_2,\ x_3\} \times \{y_1,\ y_2\}$						
$\circledR_1$	$=$	$\varnothing$				
$\circledR_2$	$=$	$\{$	(x_1, y_1),			$\}$
$\circledR_3$	$=$	$\{$		(x_1, y_2)		$\}$
$\circledR_4$	$=$	$\{$	(x_1, y_1),	(x_1, y_2)		$\}$
$\circledR_5$	$=$	$\{$			(x_2, y_1)	$\}$

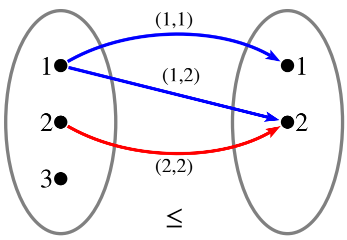
$$
\begin{array}{lcl cccccc l}
\circledR_{6} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & & & \} \\
\circledR_{7} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & & & \} \\
\circledR_{8} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & & & & \} \\
\circledR_{9} & = & \{ & & & & (x_2,y_2) & & & \} \\
\circledR_{10} & = & \{ & (x_1,y_1), & & & (x_2,y_2) & & & \} \\
\circledR_{11} & = & \{ & & (x_1,y_2) & & (x_2,y_2) & & & \} \\
\circledR_{12} & = & \{ & (x_1,y_1), & (x_1,y_2) & & (x_2,y_2) & & & \} \\
\circledR_{13} & = & \{ & & & (x_2,y_1) & (x_2,y_2) & & & \} \\
\circledR_{14} & = & \{ & (x_1,y_1), & & (x_2,y_1) & (x_2,y_2) & & & \} \\
\circledR_{15} & = & \{ & & (x_1,y_2), & (x_2,y_1) & (x_2,y_2) & & & \} \\
\circledR_{16} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & (x_2,y_2) & & & \} \\
\circledR_{17} & = & \{ & & & & & (x_3,y_1) & & \} \\
\circledR_{18} & = & \{ & (x_1,y_1), & & & & (x_3,y_1) & & \} \\
\circledR_{19} & = & \{ & & (x_1,y_2) & & & (x_3,y_1) & & \} \\
\circledR_{20} & = & \{ & (x_1,y_1), & (x_1,y_2) & & & (x_3,y_1) & & \} \\
\circledR_{21} & = & \{ & & & (x_2,y_1) & & (x_3,y_1) & & \} \\
\circledR_{22} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & (x_3,y_1) & & \} \\
\circledR_{23} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & (x_3,y_1) & & \} \\
\circledR_{24} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & & (x_3,y_1) & & \} \\
\circledR_{25} & = & \{ & & & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{26} & = & \{ & (x_1,y_1), & & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{27} & = & \{ & & (x_1,y_2) & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{28} & = & \{ & (x_1,y_1), & (x_1,y_2) & & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{29} & = & \{ & & & (x_2,y_1) & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{30} & = & \{ & (x_1,y_1), & & (x_2,y_1) & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{31} & = & \{ & & (x_1,y_2), & (x_2,y_1) & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{32} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & (x_2,y_2) & (x_3,y_1) & & \} \\
\circledR_{33} & = & \{ & & & & & & (x_3,y_2) & \} \\
\circledR_{34} & = & \{ & (x_1,y_1), & & & & & (x_3,y_2) & \} \\
\circledR_{35} & = & \{ & & (x_1,y_2) & & & & (x_3,y_2) & \} \\
\circledR_{36} & = & \{ & (x_1,y_1), & (x_1,y_2) & & & & (x_3,y_2) & \} \\
\circledR_{37} & = & \{ & & & (x_2,y_1) & & & (x_3,y_2) & \} \\
\circledR_{38} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & & (x_3,y_2) & \} \\
\circledR_{39} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & & (x_3,y_2) & \} \\
\circledR_{40} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & & & (x_3,y_2) & \} \\
\circledR_{41} & = & \{ & & & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{42} & = & \{ & (x_1,y_1), & & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{43} & = & \{ & & (x_1,y_2) & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{44} & = & \{ & (x_1,y_1), & (x_1,y_2) & & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{45} & = & \{ & & & (x_2,y_1) & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{46} & = & \{ & (x_1,y_1), & & (x_2,y_1) & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{47} & = & \{ & & (x_1,y_2), & (x_2,y_1) & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{48} & = & \{ & (x_1,y_1), & (x_1,y_2), & (x_2,y_1), & (x_2,y_2) & & (x_3,y_2) & \} \\
\circledR_{49} & = & \{ & & & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{50} & = & \{ & (x_1,y_1), & & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{51} & = & \{ & & (x_1,y_2) & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{52} & = & \{ & (x_1,y_1), & (x_1,y_2) & & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{53} & = & \{ & & & (x_2,y_1) & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{54} & = & \{ & (x_1,y_1), & & (x_2,y_1) & & (x_3,y_1) & (x_3,y_2) & \} \\
\circledR_{55} & = & \{ & & (x_1,y_2), & (x_2,y_1) & & (x_3,y_1) & (x_3,y_2) & \}
\end{array}
$$

$®_{56}$	$=$	$\{$	(x_1, y_1),	(x_1, y_2),	(x_2, y_1),		(x_3, y_1)	(x_3, y_2)	$\}$
$®_{57}$	$=$	$\{$				(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$
$®_{58}$	$=$	$\{$	(x_1, y_1),			(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$
$®_{59}$	$=$	$\{$		(x_1, y_2)		(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$
$®_{60}$	$=$	$\{$	(x_1, y_1),	(x_1, y_2)		(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$
$®_{61}$	$=$	$\{$			(x_2, y_1)	(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$
$®_{62}$	$=$	$\{$	(x_1, y_1),		(x_2, y_1)	(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$
$®_{63}$	$=$	$\{$		(x_1, y_2),	(x_2, y_1)	(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$
$®_{64}$	$=$	$\{$	(x_1, y_1),	(x_1, y_2),	(x_2, y_1),	(x_2, y_2)	(x_3, y_1)	(x_3, y_2)	$\}$

Example B.3.

Let $X \triangleq \{1, 2, 3\}$, $Y \triangleq \{1, 2\}$, and 2^{XY} the set of all of the $2^{3\times 2} = 64$ relations in $X \times Y$. Furthermore, let $x_1 \triangleq 1$, $x_2 \triangleq 2$, $x_3 \triangleq 3$, $y_1 \triangleq 1$, and $y_2 \triangleq 2$. Then the following common relations are the relations of Example B.2 (page 74):

$$\begin{array}{|ccll|}
\hline
\le & \equiv & \{(1,1)\,,\,(1,2)\,,\,(2,2)\} & \equiv\ ®_{12} \\
\ge & \equiv & \{(1,1)\,,\,(2,1)\,,\,(2,2)\,,\,(3,1)\,,\,(3,2)\} & \equiv\ ®_{62} \\
< & \equiv & \{(1,2)\} & \equiv\ ®_{3} \\
> & \equiv & \{(2,1)\,,\,(3,1)\,,\,(3,2)\} & \equiv\ ®_{53} \\
= & \equiv & \{(1,1)\,,\,(2,2)\} & \equiv\ ®_{10} \\
\hline
\end{array}$$



B.1.2 Calculus of Relations

Proposition B.2. [3] *Let 2^{XY} be the set of all relations in $X \times Y$.*

$$\boxed{\varnothing \in 2^{XY} \qquad (\varnothing \text{ is a relation})}$$

✎PROOF:

$$\begin{aligned}
&\varnothing \subseteq X \times Y \\
&\implies \varnothing \text{ is a relation.} && \text{by definition of relation Definition B.1 page 73}
\end{aligned}$$

✏

Proposition B.3. [4] *Let 2^{XY} be the set of all relations from the sets X to the set Y.*

P R P
$$\boxed{\left.\begin{array}{lll} ® \in 2^{XY} & (® \text{ is a relation}) & \text{and} \\ Ⓢ \subseteq ® & (Ⓢ \text{ is a subset of } ®) & \end{array}\right\} \implies Ⓢ \in 2^{XY} \quad (Ⓢ \text{ is a relation})}$$

✎PROOF:

$$\begin{aligned}
Ⓢ &\subseteq ® && \text{by right hypothesis} \\
&\subseteq X \times Y && \text{by definition of relation Definition B.1 page 73} \\
&\implies \varnothing \text{ is a relation.} && \text{by definition of relation Definition B.1 page 73}
\end{aligned}$$

[3] Suppes (1972) page 58
[4] Suppes (1972) page 58

A function does not always have an inverse that is also a function. But unlike functions, *every* relation has an inverse that is also a relation. Note that since all functions are relations, every function *does* have an inverse that is at least a relation, and in some cases this inverse is also a function.

Definition B.2. [5] *Let $\circledR$ be a relation in $\mathbb{2}^{XY}$.*

DEF
$\circledR^{-1}$ *is the* ***inverse*** *of relation* $\circledR$ *if*

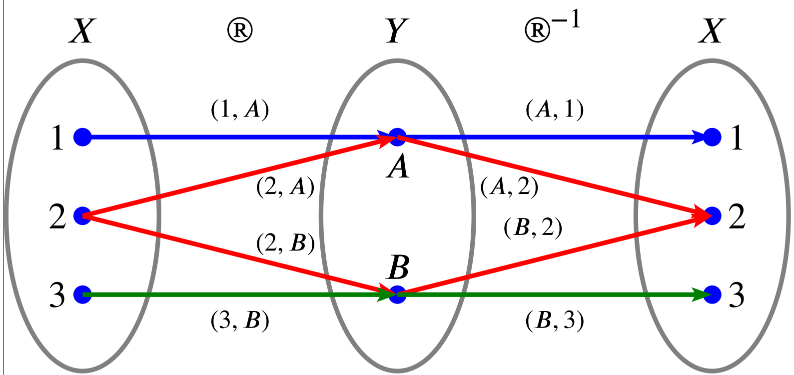
$$\circledR^{-1} \triangleq \{(y,x) \in Y \times X \mid (x,y) \in \circledR\}$$

The inverse relation $\circledR^{-1}$ *is also called the* ***converse*** *of* $\circledR$.

Example B.4.

Let	X	$\triangleq$	$\{1,\ 2,\ 3\}$
and	Y	$\triangleq$	$\{A,\ B\}$
and	$\circledR$	$\triangleq$	$\{(1,A),\ (2,A),\ (2,B),\ (3,B)\}$.
Then	$\circledR^{-1}$	$=$	$\{(A,1),\ (A,2),\ (B,2),\ (B,3)\}$.

The sets X and Y and the relations $\circledR$ and $\circledR^{-1}$ are illustrated below.



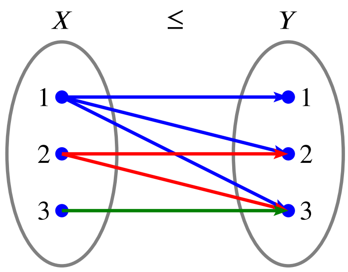
Example B.5.

Let $X \triangleq \{1,2,3\}$. Then the "less than or equal to" relation $\le$ in $\mathbb{2}^{XX}$ is

$$\le \equiv \{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3)\}$$

and it's inverse $\le^{-1}$ is equivalent to the "greater than or equal to" relation $\ge$:

$$\le^{-1} \equiv \{(1,1),\ (2,1),\ (3,1),\ (2,2),\ (3,2),\ (3,3)\} \equiv \ge.$$



Example B.6.

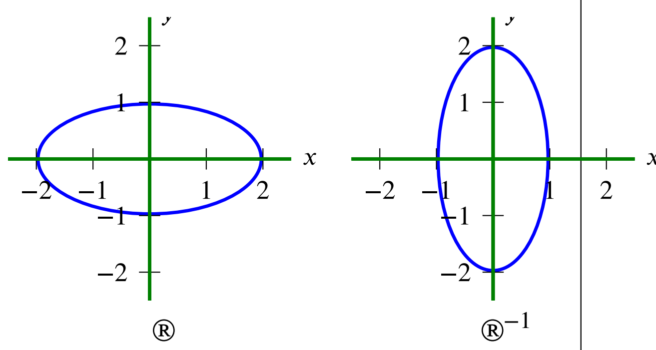
Let $\circledR$ be the *ellipse* relation in $\mathbb{2}^{\mathbb{R}\mathbb{R}}$ such that

$$\circledR \triangleq \left\{(x,y) \in \mathbb{R}^2 \middle| \frac{x^2}{2^2} + \frac{y^2}{1^2} = 1\right\}.$$

Then the inverse relation $\circledR^{-1}$ is

$$\circledR^{-1} = \left\{(x,y) \in \mathbb{R}^2 \middle| \frac{x^2}{2^2} + \frac{2^2}{2^2} = 1\right\}.$$

Both of these relations are illustrated to the right.



Example B.7. Let $\mathbf{I} \in X^X$ be an identity function, and $\mathsf{f}, \mathsf{f}^{-1} \in X^X$ be functions.

f^{-1} is the **inverse** of f if $\mathsf{f}\mathsf{f}^{-1} = \mathsf{f}^{-1}\mathsf{f} = \mathbf{I}$.

[5] Suppes (1972) page 61 ⟨Defintion 6, inverse="converse"⟩, Kelley (1955) page 7, Peirce (1883a) page 188 ⟨inverse="converse"⟩

Theorem B.1. [6] *Let ® be a relation with inverse* $®^{-1}$.

THM

$$\left(®^{-1}\right)^{-1} = ®$$

✎PROOF:

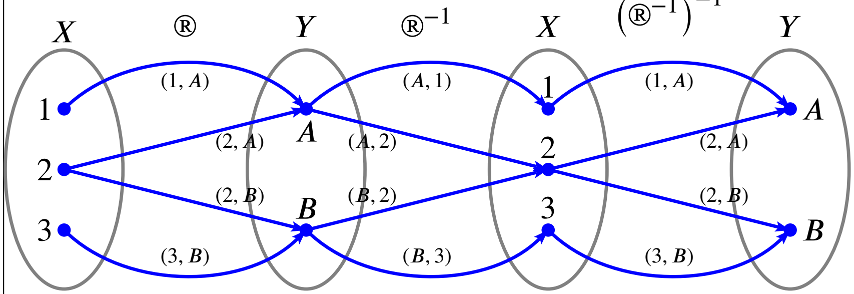
$$\begin{aligned}
\left(®^{-1}\right)^{-1} &= \underbrace{\{(x,y) \mid (y,x) \in ®\}}_{®^{-1}}{}^{-1} && \text{by definition of } ®^{-1} \text{ (Definition B.2 page 77)}\\
&= \{(x,y) \mid (y,x) \in \{(y,x) \mid (x,y) \in ®\}\} && \text{by definition of } ®^{-1} \text{ (Definition B.2 page 77)}\\
&= \{(x,y) \mid (x,y) \in ®\}\\
&= ®
\end{aligned}$$

⇰

Example B.8.

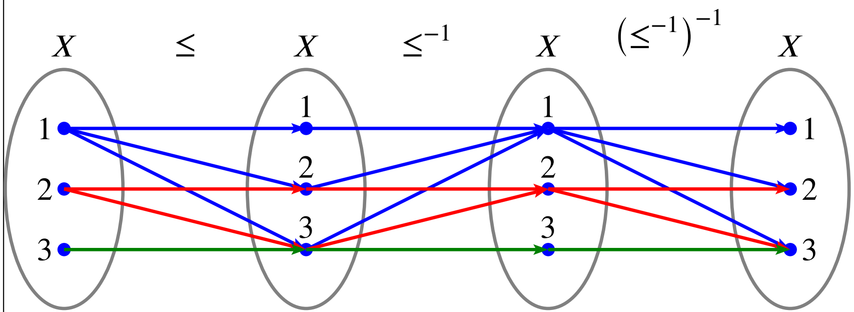
Let	X	$\triangleq$	$\{1,\ 2,\ 3\}$
and	Y	$\triangleq$	$\{A,\ B\}$
and	$®$	$\triangleq$	$\{(1,A),\ (2,A),\ (2,B),\ (3,B)\}$.
Then	$®^{-1}$	$=$	$\{(A,1),\ (A,2),\ (B,2),\ (B,3)\}$
and	$\left(®^{-1}\right)^{-1}$	$=$	$\{(1,A),\ (2,A),\ (2,B),\ (3,B)\} = ®$.

The sets X and Y and the relations $®$, $®^{-1}$, and $\left(®^{-1}\right)^{-1}$ are illustrated below.



Example B.9. Let $X \triangleq \{1,2,3\}$. Let $\le \in 2^{XX}$ be the "less than or equal to" relation in 2^{XX}.

$$\begin{aligned}
\left(\le^{-1}\right)^{-1} &\triangleq \left(\{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3)\}^{-1}\right)^{-1}\\
&= (\{(1,1),\ (2,1),\ (3,1),\ (2,2),\ (3,2),\ (3,3)\})^{-1}\\
&= (\{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3)\})\\
&\triangleq \le
\end{aligned}$$



Definition B.3. [7] *Let* $® \in 2^{XY}$ *and* $Ⓢ \in 2^{YZ}$ *be relations. Let* $\wedge$ *be the logical and function.*

DEF *The composition function* $\circ$ *on relations* $®$ *and* $Ⓢ$ *is defined as*

$$Ⓢ \circ ® \triangleq \{(x,z) \mid \exists y \quad \text{such that} \quad (x,y) \in ® \ \wedge\ (y,z) \in Ⓢ\}$$

[6] Kelley (1955) page 8, Peirce (1883a) page 188
[7] Kelley (1955) pages 7–8, Fuhrmann (2012) page 2

Theorem B.2. [8] *Let X, Y, and Z be sets.*

THM	$(\circledR \circ \circledS)^{-1} = \left(\circledS^{-1}\right) \circ \left(\circledR^{-1}\right)$	$\forall \circledR \in 2^{WX}, \circledS \in 2^{XY}$		(IDEMPOTENT)
	$\circled{Q} \circ (\circledS \circ \circledR) = (\circled{Q} \circ \circledS) \circ \circledR$	$\forall \circledR \in 2^{WX}, \circledS \in 2^{XY}, \circled{Q} \in 2^{YZ}$		(ASSOCIATIVE)

✎PROOF:

$$
\begin{aligned}
(\circledR \circ \circledS)^{-1} &= \left\{(x,z) \,|\, \exists y \text{ such that } (x,y) \in \circledR \text{ and } (y,z) \in \circledS\right\}^{-1} && \text{by definition of } \circ \text{ (page 78)}\\
&= \left\{(z,x) \,|\, (x,z) \in \left\{(x,z) \,|\, \exists y \text{ such that } (x,y) \in \circledR \text{ and } (y,z) \in \circledS\right\}\right\} && \text{by definition of } \circledR^{-1} \text{ (page 77)}\\
&= \left\{(z,x) \,|\, \exists y \text{ such that } (x,y) \in \circledR \text{ and } (y,z) \in \circledS\right\} &&\\
&= \left\{(z,x) \,|\, \exists y \text{ such that } (y,x) \in \circledR^{-1} \text{ and } (z,y) \in \circledS^{-1}\right\} && \text{by definition of } \circledR^{-1} \text{ (page 77)}\\
&= \left(\circledS^{-1}\right) \circ \left(\circledR^{-1}\right) && \text{by definition of } \circ \text{ (page 78)}
\end{aligned}
$$

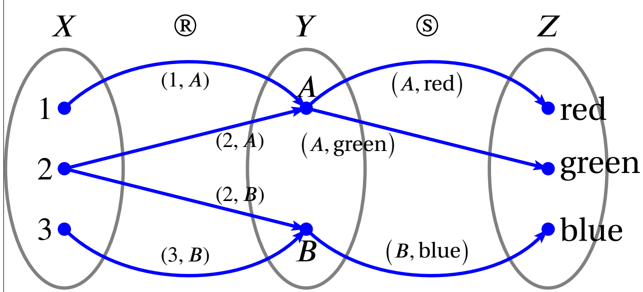
$$
\begin{aligned}
&\circled{Q} \circ (\circledS \circ \circledR)\\
&= \left\{(w,z) \,|\, \exists y \text{ such that } (w,y) \in (\circledS \circ \circledR) \text{ and } (y,z) \in \circled{Q}\right\}\\
&\qquad \text{by definition of } \circ \text{ (page 78)}\\
&= \left\{(w,z) \,|\, \exists y \text{ such that } (w,y) \in \left\{(w,y) \,|\, \exists x \text{ such that } (w,x) \in \circledR \text{ and } (x,y) \in \circledS\right\} \text{ and } (y,z) \in \circled{Q}\right\}\\
&\qquad \text{by definition of } \circ \text{ (page 78)}\\
&= \left\{(w,z) \,|\, \exists x,y \text{ such that } (w,x) \in \circledR \text{ and } (x,y) \in \circledS \text{ and } (y,z) \in \circled{Q}\right\}\\
&= \left\{(w,z) \,|\, \exists x \text{ such that } (w,x) \in \circledR \text{ and } (x,z) \in \left\{(x,z) \,|\, \exists y \text{ such that } (x,y) \in \circledS \text{ and } (y,z) \in \circled{Q}\right\}\right\}\\
&= \left\{(w,z) \,|\, \exists x \text{ such that } (w,x) \in \circledR \text{ and } (x,z) \in (\circledS \circ \circled{Q})\right\}\\
&\qquad \text{by definition of } \circ \text{ (page 78)}\\
&= (\circled{Q} \circ \circledS) \circ \circledR\\
&\qquad \text{by definition of } \circ \text{ (page 78)}
\end{aligned}
$$

⇒

Example B.10.

Let	X	$\triangleq$	$\{1, 2, 3\}$
and	Y	$\triangleq$	$\{A, B\}$
and	Z	$\triangleq$	$\{\text{red}, \text{green}, \text{blue}\}$
and	$\circledR$	$\triangleq$	$\{(1,A), (2,A), (2,B), (3,B)\}$.
and	$\circledS$	$\triangleq$	$\{(A,\text{red}), (A,\text{green}), (B,\text{blue})\}$.
Then	$\circledR \circ \circledS$	$=$	$\{(1,\text{red}), (1,\text{green}), (2,\text{green}), (2,\text{blue}), (3,\text{blue})\}$.
and	$(\circledR \circ \circledS)^{-1}$	$=$	$\{(\text{red},1), (\text{green},1), (\text{green},2), (\text{blue},2), (\text{blue},3)\}$.
		$=$	$\circledS^{-1} \circ \circledR^{-1}$

The quanitities are illustrated below.



[8] Kelley (1955) page 8

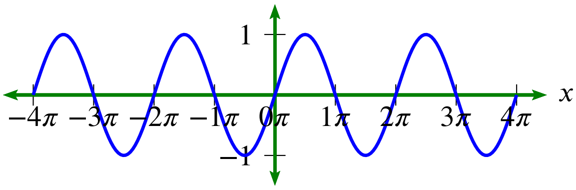
Definition B.4. [9] *Let ® $\in 2^{XY}$ be a relation.*

DEF

*The **domain** of ® is*	$\mathcal{D}(®)$	$\triangleq$	$\{x \in X \mid \exists y$ such that $(x,y) \in ®\}$.
*The **image set** of ® is*	$\mathcal{I}(®)$	$\triangleq$	$\{y \in Y \mid \exists x$ such that $(x,y) \in ®\}$.
*The **null space** of ® is*	$\mathcal{N}(®)$	$\triangleq$	$\{x \in X \mid (x,0) \in ®\}$.
*The **range** of ® is*	*any set* $\mathcal{R}(®)$ *such that* $\mathcal{I}(®) \subseteq \mathcal{R}(®)$		

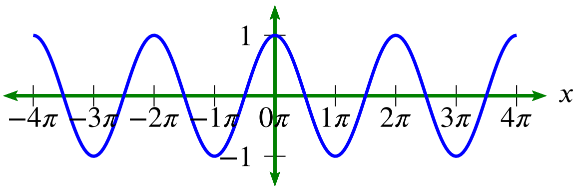
Example B.11. Let ® $\triangleq \sin x$. Then ...

$$\begin{aligned}
\mathcal{D}(®) &= \mathbb{R} \\
\mathcal{I}(®) &= -1 \le y \le 1 \\
\mathcal{N}(®) &= \{n\pi \mid n \in \mathbb{Z}\}. \\
\mathcal{R}(®) &= \mathbb{R}
\end{aligned}$$



Example B.12. Let ® $\triangleq \cos x$. Then ...

$$\begin{aligned}
\mathcal{D}(®) &= \mathbb{R} \\
\mathcal{I}(®) &= -1 \le y \le 1 \\
\mathcal{N}(®) &= \left\{\left(n + \tfrac{1}{2}\right)\pi \mid n \in \mathbb{Z}\right\}. \\
\mathcal{R}(®) &= \mathbb{R}
\end{aligned}$$



Example B.13. (Rudin, 1991)99 Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be linear functions and Y^X be the set of all functions from $\boldsymbol{X}$ to $\boldsymbol{Y}$. Let f be an function in Y^X.

The **domain** of f is	$\mathcal{D}(\mathrm{f})$	$\triangleq$	$\boldsymbol{X}$
The **range** of f is	$\mathcal{I}(\mathrm{f})$	$\triangleq$	$\{\boldsymbol{y} \in \boldsymbol{Y} \mid \exists \boldsymbol{x} \in \boldsymbol{X}$ such that $\boldsymbol{y} = \mathrm{f}\boldsymbol{x}\}$
The **null space** of f is	$\mathcal{N}(\mathrm{f})$	$\triangleq$	$\{\boldsymbol{x} \in \boldsymbol{X} \mid \mathrm{f}\boldsymbol{x} = 0\}$

Theorem B.3. [10] *Let $\mathcal{D}(®)$ be the domain of a relation ® and $\mathcal{I}(®)$ the image of ®.*

THM

$$\begin{aligned}
\mathcal{D}\left(\bigcup_{i\in I} ®_i\right) &= \bigcup_{i\in I} \mathcal{D}(®_i) & \qquad \mathcal{I}\left(\bigcup_{i\in I} ®_i\right) &= \bigcup_{i\in I} \mathcal{I}(®_i) \\
\mathcal{D}\left(\bigcap_{i\in I} ®_i\right) &\subseteq \bigcap_{i\in I} \mathcal{D}(®_i) & \qquad \mathcal{I}\left(\bigcap_{i\in I} ®_i\right) &\subseteq \bigcap_{i\in I} \mathcal{I}(®_i) \\
\mathcal{D}(® \setminus Ⓢ) &\supseteq \mathcal{D}(®) \setminus \mathcal{D}(Ⓢ) & \qquad \mathcal{I}(® \setminus Ⓢ) &\supseteq \mathcal{I}(®) \setminus \mathcal{I}(Ⓢ)
\end{aligned}$$

✎PROOF:

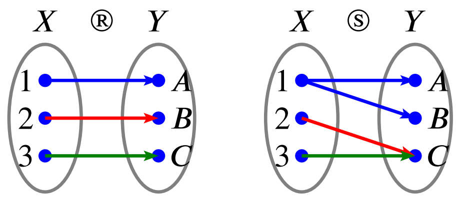
$$\begin{aligned}
\mathcal{D}\left(\bigcup_{i\in I} ®_i\right) &= \left\{x \mid \exists y \text{ such that } (x,y) \in \bigcup_{i\in I} ®_i\right\} && \text{by Definition B.4 page 80} \\
&= \left\{x \mid \exists y \text{ such that } (x,y) \in \left\{(x,y) \mid \bigvee_i (x,y) \in ®_i\right\}\right\} && \text{by Definition A.5 page 38} \\
&= \left\{x \mid \exists y \text{ such that } \bigvee_i (x,y) \in ®_i\right\} \\
&= \left\{x \mid \bigvee_i \left[\exists y \text{ such that } (x,y) \in ®_i\right]\right\} \\
&= \bigcup_i \{x \mid \exists y \text{ such that } (x,y) \in ®_i\} && \text{by Definition A.5 page 38} \\
&= \bigcup_i \mathcal{D}(®_i) && \text{by Definition B.4 page 80}
\end{aligned}$$

[9] Munkres (2000) page 16, Kelley (1955) page 7
[10] Suppes (1972) pages 60–61

$$
\begin{aligned}
\mathcal{D}\left(\bigcap_{i\in I}\circledR_i\right) &= \left\{x|\exists y \quad \text{such that} \quad (x,y)\in\bigcap_{i\in I}\circledR_i\right\} && \text{by Definition B.4 page 80}\\
&= \left\{x|\exists y \quad \text{such that} \quad (x,y)\in\left\{(x,y)\,|\,\bigwedge_i (x,y)\in\circledR_i\right\}\right\} && \text{by Definition A.5 page 38}\\
&= \left\{x|\exists y \quad \text{such that} \quad \bigwedge_i (x,y)\in\circledR_i\right\}\\
&= \left\{x|\bigwedge_i \left[\exists y \quad \text{such that} \quad (x,y)\in\circledR_i\right]\right\}\\
&= \bigcap_i \left\{x|\exists y \quad \text{such that} \quad (x,y)\in\circledR_i\right\} && \text{by Definition A.5 page 38}\\
&= \bigcap_i \mathcal{D}(\circledR_i) && \text{by Definition B.4 page 80}
\end{aligned}
$$

Example B.14.

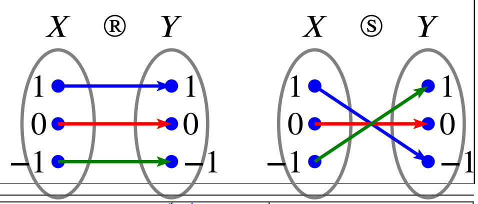
$$
\begin{array}{llll}
\text{Let} & X & \triangleq & \{1,2,3\}\\
\text{and} & Y & \triangleq & \{A,B,C\}\\
\text{and} & \circledR & \triangleq & \{(1,A),\,(2,B),\,(3,C)\}\\
\text{and} & \circledS & \triangleq & \{(1,A),\,(1,B),\,(2,C),\,(3,C)\}.
\end{array}
$$



$$
\begin{aligned}
\mathcal{D}(\circledR\cup\circledS) &= \mathcal{D}(\{(1,A),\,(2,B),\,(3,C)\}\cup\{(1,A),\,(1,B),\,(2,C),\,(3,C)\}).\\
&= \mathcal{D}\{(1,A),\,(1,B),\,(2,B),\,(2,C),\,(3,C)\}\\
&= \{1,\;2,\;3\}\\
&= \{1,\;2,\;3\}\cup\{1,\;2\,3\}\\
&= \mathcal{D}\circledR\cup\mathcal{D}\circledS\\
\mathcal{D}(\circledR\cap\circledS) &= \{(1,A),\,(3,C)\}\\
&= \{1,\;3\}\\
&\subseteq \{1,\;2,\;3\}\cap\{1,\;2\,3\}\\
&= \mathcal{D}\circledR\cap\mathcal{D}\circledS\\
\mathcal{I}(\circledR\cup\circledS) &= \mathcal{I}(\{(1,A),\,(2,B),\,(3,C)\}\cup\{(1,A),\,(1,B),\,(2,C),\,(3,C)\}).\\
&= \mathcal{I}\{(1,A),\,(1,B),\,(2,B),\,(2,C),\,(3,C)\}\\
&= \{A,\;B,\;C\}\\
&= \{A,\;B,\;C\}\cup\{A,\;B\,C\}\\
&= \mathcal{I}\circledR\cup\mathcal{I}\circledS\\
\mathcal{I}(\circledR\cap\circledS) &= \{(1,A),\,(3,C)\}\\
&= \{A,\;C\}\\
&\subseteq \{A,\;B,\;C\}\cap\{A,\;B\,C\}\\
&= \mathcal{I}\circledR\cap\mathcal{I}\circledS
\end{aligned}
$$

Example B.15.

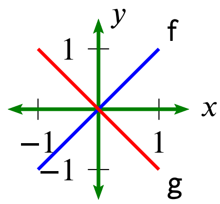
$$
\begin{array}{llll}
\text{Let} & X & \triangleq & \{-1,0,1\}\\
\text{and} & Y & \triangleq & \{-1,0,1\}\\
\text{and} & \circledR & \triangleq & \{(-1,-1),\,(0,0),\,(1,1)\}\\
\text{and} & \circledS & \triangleq & \{(-1,1),\,(0,0),\,(1,-1)\}.
\end{array}
$$



$$\begin{aligned}
\mathcal{D}(\circledR \cup \circledS) &= \mathcal{D}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cup \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \mathcal{D}\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\,,\,(-1,1)\,,\,(1,-1)\} \\
&= \{-1,\ 0,\ 1\} \\
&= \{-1,\ 0,\ 1\} \cup \{-1,\ 0\,1\} \\
&= \mathcal{D}\circledR \cup \mathcal{D}\circledS \\
\mathcal{D}(\circledR \cap \circledS) &= \mathcal{D}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cap \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \mathcal{D}\{(0,0)\} \\
&= \{0\} \\
&\subseteq \{-1,\ 0,\ 1\} \cap \{-1,\ 0\,1\} \\
&= \mathcal{D}\circledR \cap \mathcal{D}\circledS \\
\mathcal{I}(\circledR \cup \circledS) &= \mathcal{I}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cup \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \mathcal{I}\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\,,\,(-1,1)\,,\,(1,-1)\} \\
&= \{-1,\ 0,\ 1\} \\
&= \{-1,\ 0,\ 1\} \cup \{-1,\ 0\,1\} \\
&= \mathcal{I}\circledR \cup \mathcal{D}\circledS \\
\mathcal{I}(\circledR \cap \circledS) &= \mathcal{I}(\{(-1,-1)\,,\,(0,0)\,,\,(1,1)\} \cap \{(-1,1)\,,\,(0,0)\,,\,(1,-1)\}). \\
&= \mathcal{I}\{(0,0)\} \\
&= \{0\} \\
&\subseteq \{-1,\ 0,\ 1\} \cap \{-1,\ 0\,1\} \\
&= \mathcal{I}\circledR \cap \mathcal{I}\circledS
\end{aligned}$$

Example B.16.

$$\begin{array}{lcll}
\text{Let} & \mathsf{f}(x) & \triangleq & x \\
\text{and} & \mathsf{g}(x) & \triangleq & -x.
\end{array}$$



$$\begin{aligned}
\mathcal{D}(\mathsf{f} \cup \mathsf{g}) &= \mathcal{D}\left(\left\{(x,y) \in \mathbb{R}^2 | y = x\right\} \cup \left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
&= \mathcal{D}\left\{(x,y) \in \mathbb{R}^2 | y = x \text{ or } y = -x\right\} \\
&= \mathbb{R} \\
&= \mathbb{R} \cup \mathbb{R} \\
&= \left(\mathcal{D}\left\{(x,y) \in \mathbb{R}^2 | y = x\right\}\right) \cup \left(\mathcal{D}\left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
\mathcal{D}(\mathsf{f} \cap \mathsf{g}) &= \mathcal{D}\left(\left\{(x,y) \in \mathbb{R}^2 | y = x\right\} \cap \left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right) \\
&= \mathcal{D}\left\{(x,y) \in \mathbb{R}^2 | y = x \text{ and } y = -x\right\} \\
&= \mathcal{D}\{(0,0)\} \\
&= \{0\} \\
&\subseteq \mathbb{R} \\
&= \mathbb{R} \cap \mathbb{R} \\
&= \left(\mathcal{D}\left\{(x,y) \in \mathbb{R}^2 | y = x\right\}\right) \cap \left(\mathcal{D}\left\{(x,y) \in \mathbb{R}^2 | y = -x\right\}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\mathsf{f}\cup\mathsf{g}) &= \mathcal{I}\left(\left\{(x,y)\in\mathbb{R}^2|y=x\right\}\cup\left\{(x,y)\in\mathbb{R}^2|y=-x\right\}\right)\\
&= \mathcal{I}\left\{(x,y)\in\mathbb{R}^2|y=x \text{ or } y=-x\right\}\\
&= \mathbb{R}\\
&= \mathbb{R}\cup\mathbb{R}\\
&= \left(\mathcal{I}\left\{(x,y)\in\mathbb{R}^2|y=x\right\}\right)\cup\left(\mathcal{I}\left\{(x,y)\in\mathbb{R}^2|y=-x\right\}\right)\\
\mathcal{I}(\mathsf{f}\cap\mathsf{g}) &= \mathcal{I}\left(\left\{(x,y)\in\mathbb{R}^2|y=x\right\}\cap\left\{(x,y)\in\mathbb{R}^2|y=-x\right\}\right)\\
&= \mathcal{I}\left\{(x,y)\in\mathbb{R}^2|y=x \text{ and } y=-x\right\}\\
&= \mathcal{I}\{(0,0)\}\\
&= \{0\}\\
&\subseteq \mathbb{R}\\
&= \mathbb{R}\cap\mathbb{R}\\
&= \left(\mathcal{I}\left\{(x,y)\in\mathbb{R}^2|y=x\right\}\right)\cap\left(\mathcal{I}\left\{(x,y)\in\mathbb{R}^2|y=-x\right\}\right)
\end{aligned}$$

Definition B.5. [11] *Let ® be a relation in* 2^{XY}.

DEF

$$\begin{array}{rclll}
®(A) & \triangleq & \{y\in Y|\exists x\in A \text{ such that } (x,y)\in ®\} & \forall A\in 2^X & (\textit{image of } A \textit{ under } ®)\\
®^{-1}(B) & \triangleq & \{x\in X|\exists y\in B \text{ such that } (x,y)\in ®\} & \forall B\in 2^Y & (\textit{image of } B \textit{ under } ®^{-1})
\end{array}$$

Theorem B.4. [12]

THM

$$\begin{array}{rcl}
®(\varnothing) & = & \varnothing\\
®\left(\bigcup_{i\in I} A_i\right) & = & \bigcup_{i\in I} ®(A_i)\\
®\left(\bigcap_{i\in I} A_i\right) & \subseteq & \bigcap_{i\in I} ®(A_i)
\end{array}$$

✎PROOF:

$$\begin{aligned}
®(\varnothing) &= \{y\in Y|\exists x\in\varnothing \text{ such that } (x,y)\in ®\} && \text{by Definition B.5 page 83}\\
&= \varnothing
\end{aligned}$$

$$\begin{aligned}
®\left(\bigcup_{i\in I} A_i\right) &= \left\{y\in Y|\exists x\in\bigcup_{i\in I} A_i \text{ such that } (x,y)\in ®\right\} && \text{by Definition B.5 page 83}\\
&= \left\{y\in Y|\exists x\in\left\{x\in X|\bigvee_{i\in I} x\in A_i\right\} \text{ such that } (x,y)\in ®\right\} && \text{by Definition A.5 page 38}\\
&= \left\{y\in Y|\exists x\in X \text{ such that } \left[\bigvee_{i\in I} x\in A_i\right]\wedge(x,y)\in ®\right\}\\
&= \left\{y\in Y|\exists x\in X \text{ such that } \bigvee_{i\in I}\left[x\in A_i\wedge(x,y)\in ®\right]\right\}\\
&= \left\{y\in Y|\bigvee_{i\in I}\left[\exists x\in X \text{ such that } x\in A_i\wedge(x,y)\in ®\right]\right\}\\
&= \bigcup_{i\in I}\left\{y\in Y|\exists x\in X \text{ such that } x\in A_i\wedge(x,y)\in ®\right\} && \text{by Definition A.5 page 38}\\
&= \bigcup_{i\in I} ®(A_i) && \text{by Definition B.5 page 83}
\end{aligned}$$

[11] Kelley (1955) page 8
[12] Kelley (1955) page 8

$$
\begin{aligned}
®\left(\bigcap_{i\in I} A_i\right) &= \left\{ y \in Y | \exists x \in \bigcap_{i\in I} A_i \quad \text{such that} \quad (x,y) \in ® \right\} && \text{by Definition B.5 page 83}\\
&= \left\{ y \in Y | \exists x \in \left\{ x \in X | \bigwedge_{i\in I} x \in A_i \right\} \quad \text{such that} \quad (x,y) \in ® \right\} && \text{by Definition A.5 page 38}\\
&= \left\{ y \in Y | \exists x \in X \quad \text{such that} \quad \left[\bigwedge_{i\in I} x \in A_i\right] \wedge (x,y) \in ® \right\}\\
&= \left\{ y \in Y | \exists x \in X \quad \text{such that} \quad \bigwedge_{i\in I} \left[x \in A_i \wedge (x,y) \in ®\right] \right\}\\
&\subseteq \left\{ y \in Y | \bigwedge_{i\in I} \left[\exists x \in X \quad \text{such that} \quad x \in A_i \wedge (x,y) \in ®\right] \right\}\\
&= \bigcap_{i\in I} \left\{ y \in Y | \exists x \in X \quad \text{such that} \quad x \in A_i \wedge (x,y) \in ® \right\} && \text{by Definition A.5 page 38}\\
&= \bigcap_{i\in I} ®(A_i) && \text{by Definition B.5 page 83}
\end{aligned}
$$

⊫

Definition B.6 (next) provides some properties associated with special types of relations. Relations can be defined based on their properties. For example, *equivalence relations* are *reflexive*, *symmetric*, and *transitive*; whereas *order relations* are (Definition C.2 page 102) are *reflexive*, *anti-symmetric*, and *transitive*.

Definition B.6. [13] *Let X be a set and ® a relation in 2^{XX}.*

DEF

® is ***reflexive***	*if* $x®x$	$\forall x \in X$
® is ***irreflexive***	*if* $(x,x) \notin ®$	$\forall x \in X$
® is ***symmetric***	*if* $x®y \implies y®x$	$\forall x,y \in X$
® is ***asymmetric***	*if* $x®y \implies (y,x) \notin ®$	$\forall x,y \in X$
® is ***anti-symmetric***	*if* $x®y$ *and* $y®x \implies x = y$	$\forall x,y \in X$
® is ***transitive***	*if* $x®y$ *and* $y®z \implies x®z$	$\forall x,y,z \in X$
® is ***connected***	*if* $x \neq y \implies x®y$ *or* $y®x$	$\forall x,y,z \in X$
® is ***strongly connected***	*if* $x®y$ *or* $y®x$	$\forall x,y,z \in X$

Definition B.7. [14]

DEF

*The **identity element** ①(X) with respect to ® $\in 2^{XX}$ is defined as*

$$①(X) \triangleq \{(x,x) \,|\, (x,x) \in ®\}.$$

The identity element ①(X) may also be denoted as simply ①.

Proposition B.4. *Let ① be the identity element in 2^{XX} with respect to the composition function $\circ$.*

PRP

$$① \circ ® = ® \circ ① = ® \qquad \forall ® \in 2^{XX}$$

Example B.17. (Michel and Herget, 1993)411 Let $\boldsymbol{X}$ be a linear space and X^X the set of all functions from $\boldsymbol{X}$ to $\boldsymbol{X}$ (Definition B.8 page 85). Let $\mathbf{I}$ be an function in X^X. $\mathbf{I}$ is an **identity function** in X^X if $\mathbf{I}\boldsymbol{x} = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \boldsymbol{X}$.

Theorem B.5. [15] *Let ® be a relation in 2^{XX}. Let ① be the identity element in 2^{XX} with respect to composition.*

[13] Suppes (1972) page 69 ⟨Defintion 10–Definition 17⟩, Kelley (1955) page 9
[14] Kelley (1955) page 9
[15] Kelley (1955) page 9

THM

$$\begin{array}{lcrcl}
® \text{ is reflexive} & \iff & ① & \subseteq & ® \\
® \text{ is symmetric} & \iff & ® & = & ®^{-1} \\
® \text{ is anti-symmetric} & \iff & ® \cap ®^{-1} & = & \varnothing \\
® \text{ is transitive} & \iff & ® \circ ® & \subseteq & ® \\
® \text{ is transitive and reflexive} & \implies & ® \circ ® & = & ®
\end{array}$$

✎PROOF:

$$\begin{array}{rcll}
® \text{ is reflexive} & \iff & (x,x) \in ® \quad \forall x \in X & \text{by Definition B.6 page 84} \\
 & \iff & ① \subseteq ® & \text{by Definition B.7 page 84} \\
® \text{ is symmetric} & \iff & [(x,y) \in ® \implies (y,x) \in ®] & \text{by Definition B.6 page 84} \\
 & \iff & ® = ®^{-1} & \text{by Definition B.2 page 77} \\
® \text{ is anti-symmetric} & \iff & [(x,y) \in ® \implies (y,x) \notin ®] & \text{by Definition B.6 page 84} \\
 & \iff & ® \bigcap ®^{-1} = \varnothing & \text{by Definition B.2 page 77} \\
® \text{ is transitive} & \iff & [(x,y),(y,z) \in ® \implies (x,z) \in ®] & \text{by Definition B.6 page 84} \\
 & \iff & ® \circ ® \subseteq ® & \text{by Definition B.3 page 78} \\
® \text{ is transitive and reflexive} & \iff & [® \circ ® \subseteq ® \text{ and } ① \subseteq ®] & \text{by previous results} \\
 & \implies & [® \circ ® \subseteq ® \text{ and } ® = ① \circ ® \subseteq ® \circ ®] & \text{by definition of ① page 84} \\
 & \iff & [® \circ ® \subseteq ® \text{ and } ® \subseteq ® \circ ®] & \\
 & \implies & ® \circ ® = ® &
\end{array}$$

✏

B.2 Functions

The function is a special case of the relation in that while both are subsets of a Cartesian product, an element in the domain of a function can only map to *one* element in the range (Definition B.8—next definition). The set of all functions in the Cartesian product $X \times Y$ is denoted Y^X; this is suitable because the number of functions in $X \times Y$ for finite X and Y is $|Y|^{|X|}$ (Proposition B.5 page 86). The fact that not all functions are relations is demonstrated in Example B.18 (page 86) (discrete cases) and Example B.19 (page 86) (continuous cases).

B.2.1 Definition and examples

Definition B.8. [16] *Let X and Y be sets. Let $\wedge$ be the "logical and" operation* (Definition 3.1 page 33).

DEF

A relation $\mathsf{f} \in 2^{XY}$ is a ***function*** *if*

$$\left(x, y_1\right) \in \mathsf{f} \;\wedge\; \left(x, y_2\right) \in \mathsf{f} \implies y_1 = y_2 \qquad \text{(for each } x\text{, there is only one } \mathsf{f}(x)\text{)}$$

The set of all functions in 2^{XY} is denoted

$$Y^X \triangleq \left\{\mathsf{f} \in 2^{XY} | \mathsf{f} \text{ is a function}\right\}.$$

A function may also be referred to as a ***correspondence***, ***transformation***, *or* ***map***.

As indicated in Definition B.8 (previous definition), functions customarily come disguised in different names depending on the context in which they are found. This is particularly true with respect

[16] Suppes (1972) page 86, Kelley (1955) page 10, Bourbaki (1939), Bottazzini (1986) page 7

to *vector spaces*, as illustrated next:

① *function*: maps from a field to a field
② *functional*: maps from a vector space to a field
③ *function*: maps from a vector space to a vector space

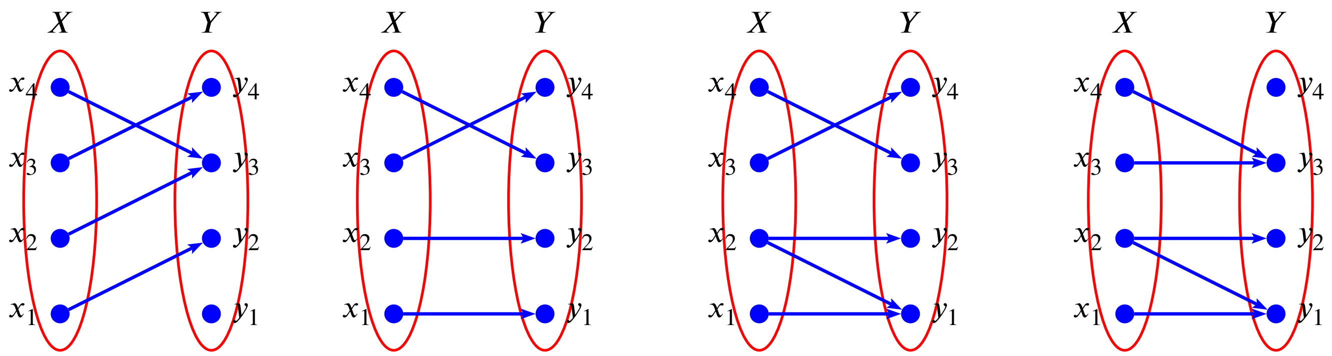
However, no matter what name is used, a function is still a function as long as it satisfies Definition B.8.

Definition B.9. [17]

DEF

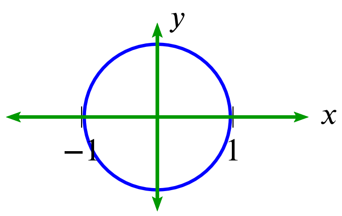
A function $\mathsf{f} \in Y^{X^n}$	*is said to have* ***arity*** n.
A function $\mathsf{f} \in Y^{X^3}$	*is said to be* ***ternary***.
A function $\mathsf{f} \in Y^{X^2}$	*is said to be* ***binary***.
A function $\mathsf{f} \in Y^{X^1} \triangleq Y^X$	*is said to be* ***unary***.
A function $\mathsf{f} \in Y^{X^0} \triangleq Y$	*is said to be* ***nullary***.

Example B.18. The figure below illustrates two discrete examples of relations that *are* functions and two that are *not*.



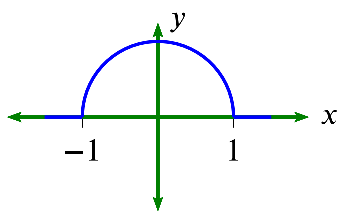
two *relations* in 2^{XY} that *are functions* two *relations* in 2^{XY} that are *not functions*

Example B.19. [18] The figures below illustrates one example of a continuous relation that is *not* a function and one that *is*.



$$\left\{(x,y) \in X \times Y | x^2 + y^2 = 1\right\}$$

(a relation that is *not* a function)



$$\left\{(x,y) \in X \times Y | \begin{array}{ll} y = \sqrt{1-x^2} & \text{for } -1 < x < 1 \\ y = 0 & \text{otherwise} \end{array}\right\}$$

(a relation that *is* a function)

Proposition B.5. [19] *Let Y^X be the set of all functions from a set X to a set Y. Let $|\cdot|$ be the counting measure for sets.*

PRP

$$\underbrace{\left|Y^X\right|}_{\text{number of possible functions in } X \times Y} = |Y|^{|X|}$$

---

[17] Burris and Sankappanavar (2000) pages 25–26

[18] Apostol (1975) page 34

[19] Comtet (1974) page 4

✎PROOF: Let $X \triangleq \{x_1, x_2, \ldots, x_m\}$.
Let $Y \triangleq \{y_1, y_2, \ldots, y_n\}$.
Then x_1 can map to exactly one of the n elements in set Y: y_1, y_2, ..., or y_n.
Likewise, x_2 can also map to one of the n elements in set Y.
So, the total number of possible functions in Y^X is

$$n^m = |Y|^{|X|}$$

✏

Example B.20. Let $X \triangleq \{x_1, x_2, x_3\}$ and $Y \triangleq \{y_1, y_2\}$. There are a total of $|\mathbb{R}| = 2^{|X|\cdot|Y|} = 2^{3\times 2} = 64$ possible relations on $X \times Y$, as listed in Example B.2 (page 74). Let $\mathbb{F} \triangleq (\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \ldots)$ be the set of all **functions** from X to Y. There are a total of $|\mathbb{F}| = |Y|^{|X|} = 2^3 = 8$ possible functions. These 8 functions are listed below. Of these 8 functions, 6 are *surjective*, as listed in Example B.27 (page 90).

functions on $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$	
$\boldsymbol{F}_1 = \{ (x_1, y_1), (x_2, y_1), (x_3, y_1) \}$	$\boldsymbol{F}_5 = \{ (x_1, y_1), (x_2, y_1), (x_3, y_2) \}$
$\boldsymbol{F}_2 = \{ (x_1, y_2), (x_2, y_1), (x_3, y_1) \}$	$\boldsymbol{F}_6 = \{ (x_1, y_2), (x_2, y_1), (x_3, y_2) \}$
$\boldsymbol{F}_3 = \{ (x_1, y_1), (x_2, y_2), (x_3, y_1) \}$	$\boldsymbol{F}_7 = \{ (x_1, y_1), (x_2, y_2), (x_3, y_2) \}$
$\boldsymbol{F}_4 = \{ (x_1, y_2), (x_2, y_2), (x_3, y_1) \}$	$\boldsymbol{F}_8 = \{ (x_1, y_2), (x_2, y_2), (x_3, y_2) \}$

Example B.21. Let $X \triangleq \{x, y, z\}$ There are a total of $|\mathbb{R}| = 2^{|X\times X|} = 2^{|X|\cdot|X|} = 2^{3\times 3} = 2^9 = 512$ possible relations on X^2. Of these 512 relations, only 27 are **functions**. These 27 functions are listed below. Of these 27 functions, only 7 are *surjective* functions, as listed in Example B.28 (page 91).

functions on $\{x, y, z\} \times \{x, y, z\}$	
$\boldsymbol{F}_1 = \{ (x,x), (y,x), (z,x) \}$	$\boldsymbol{F}_{15} = \{ (x,z), (y,y), (z,y) \}$
$\boldsymbol{F}_2 = \{ (x,y), (y,x), (z,x) \}$	$\boldsymbol{F}_{16} = \{ (x,x), (y,z), (z,y) \}$
$\boldsymbol{F}_3 = \{ (x,z), (y,x), (z,x) \}$	$\boldsymbol{F}_{17} = \{ (x,y), (y,z), (z,y) \}$
$\boldsymbol{F}_4 = \{ (x,x), (y,y), (z,x) \}$	$\boldsymbol{F}_{18} = \{ (x,z), (y,z), (z,y) \}$
$\boldsymbol{F}_5 = \{ (x,y), (y,y), (z,x) \}$	$\boldsymbol{F}_{19} = \{ (x,x), (y,x), (z,z) \}$
$\boldsymbol{F}_6 = \{ (x,z), (y,y), (z,x) \}$	$\boldsymbol{F}_{20} = \{ (x,y), (y,x), (z,z) \}$
$\boldsymbol{F}_7 = \{ (x,x), (y,z), (z,x) \}$	$\boldsymbol{F}_{21} = \{ (x,z), (y,x), (z,z) \}$
$\boldsymbol{F}_8 = \{ (x,y), (y,z), (z,x) \}$	$\boldsymbol{F}_{22} = \{ (x,x), (y,y), (z,z) \}$
$\boldsymbol{F}_9 = \{ (x,z), (y,z), (z,x) \}$	$\boldsymbol{F}_{23} = \{ (x,y), (y,y), (z,z) \}$
$\boldsymbol{F}_{10} = \{ (x,x), (y,x), (z,y) \}$	$\boldsymbol{F}_{24} = \{ (x,z), (y,y), (z,z) \}$
$\boldsymbol{F}_{11} = \{ (x,y), (y,x), (z,y) \}$	$\boldsymbol{F}_{25} = \{ (x,x), (y,z), (z,z) \}$
$\boldsymbol{F}_{12} = \{ (x,z), (y,x), (z,y) \}$	$\boldsymbol{F}_{26} = \{ (x,y), (y,z), (z,z) \}$
$\boldsymbol{F}_{13} = \{ (x,x), (y,y), (z,y) \}$	$\boldsymbol{F}_{27} = \{ (x,z), (y,z), (z,z) \}$
$\boldsymbol{F}_{14} = \{ (x,y), (y,y), (z,y) \}$	

Definition B.10. [20] *Let Y^X be the set of functions from a set X to a set Y.*

DEF
Functions $\mathsf{f} \in Y^X$ *and* $\mathsf{g} \in Y^X$ *are* ***equal*** *if*
$$\mathsf{f}(x) = \mathsf{g}(x) \qquad \forall x \in X$$
This is denoted as $\mathsf{f} \mathrel{\mathring{=}} \mathsf{g}$.

[20] Berberian (1961) page 73

B.2.2 Properties of functions

Theorem B.6. [21] *Let* f *be a* FUNCTION *(Definition B.8 page 85) in* Y^X *with inverse relation* f^{-1} *in* 2^{XY}.

THM

1.		$\mathsf{f}(\varnothing) = \varnothing$		$\forall \mathsf{f}\in Y^X$	
2.		$\mathsf{f}^{-1}(\varnothing) = \varnothing$		$\forall \mathsf{f}\in Y^X$	
3.	$A \subseteq B \implies$	$\mathsf{f}(A) \subseteq \mathsf{f}(B)$		$\forall \mathsf{f}\in Y^X, A,B\in 2^X$	(ISOTONE)
4.	$A \subseteq B \implies$	$\mathsf{f}^{-1}(A) \subseteq \mathsf{f}^{-1}(B)$		$\forall \mathsf{f}\in Y^X, A,B\in 2^Y$	(ISOTONE)

✎PROOF:

1. Proof that $\mathsf{f}(\varnothing) = \varnothing$:

$$\begin{aligned}
\mathsf{f}(\varnothing) &= \{y \in Y | \exists x \in \varnothing \quad \text{such that} \quad (x,y) \in \mathsf{f}\} && \text{by Definition B.5 page 83}\\
&= \varnothing && \text{by definition of } \varnothing \text{ page ??}
\end{aligned}$$

2. Proof that $A \subseteq B \implies \mathsf{f}(A) \subseteq \mathsf{f}(B)$:

$$\begin{aligned}
\mathsf{f}(A) &= \{y \in Y | \exists x \in A \quad \text{such that} \quad (x,y) \in \mathsf{f}\} && \text{by Definition B.5 page 83}\\
&\subseteq \{y \in Y | \exists x \in B \quad \text{such that} \quad (x,y) \in \mathsf{f}\} && \text{by left hypothesis}\\
&= \mathsf{f}(B) && \text{by Definition B.5 page 83}
\end{aligned}$$

3. Proof that $\mathsf{f}^{-1}(\varnothing) = \varnothing$:

$$\begin{aligned}
\mathsf{f}^{-1}(\varnothing) &= \{x \in X | \exists y \in \varnothing \quad \text{such that} \quad (x,y) \in \mathsf{f}\} && \text{by Definition B.5 page 83}\\
&= \varnothing && \text{by definition of } \varnothing \text{ page ??}
\end{aligned}$$

4. Proof that $A \subseteq B \implies \mathsf{f}^{-1}(A) \subseteq \mathsf{f}^{-1}(B)$:

$$\begin{aligned}
\mathsf{f}^{-1}(A) &= \left\{x \in X | \exists y \in A \quad \text{such that} \quad (x,y) \in \mathsf{f}^{-1}\right\} && \text{by Definition B.5 page 83}\\
&\subseteq \{x \in X | \exists y \in B \quad \text{such that} \quad (x,y) \in \mathsf{f}\} && \text{by left hypothesis}\\
&= \mathsf{f}^{-1}(B) && \text{by Definition B.5 page 83}
\end{aligned}$$

✏

B.2.3 Types of functions

In general, a function $\mathsf{f} \in Y^X$ can be described as "*into*" because f maps each element of X *into* Y such that $\mathsf{f}(X) \subseteq Y$. However there are some common more restrictive special types of functions. These are defined in Definition B.11 (next defintion).

Definition B.11. [22] *Let* $\mathsf{f} \in Y^X$.

DEF

f *is* ***surjective***	*(also called* ***onto****)*	*iff* $\mathsf{f}(X) = Y$
f *is* ***injective***	*(also called* ***one-to-one****)*	*iff* $\mathsf{f}(x_n) = \mathsf{f}(x_m) \implies x_n = x_m$
f *is* ***bijective***	*(also called* ***one-to-one and onto****)*	*iff* f *is both surjective and injective.*

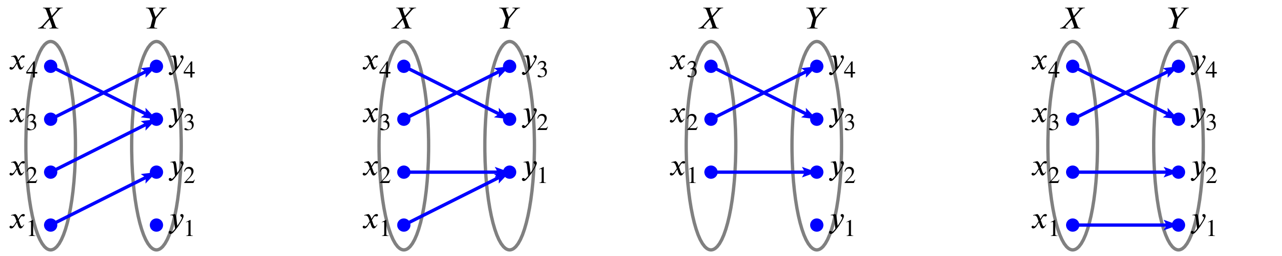
We also define the following sets of functions:

$$\begin{aligned}
\mathcal{S}_{\mathrm{j}}(X,Y) &\triangleq \left\{\mathsf{f} \in Y^X | \mathsf{f} \text{ is surjective}\right\} && \text{(the set of all surjective functions in } Y^X\text{)}\\
\mathcal{I}_{\mathrm{j}}(X,Y) &\triangleq \left\{\mathsf{f} \in Y^X | \mathsf{f} \text{ is injective}\right\} && \text{(the set of all injective functions in } Y^X\text{)}\\
\mathcal{B}_{\mathrm{j}}(X,Y) &\triangleq \left\{\mathsf{f} \in Y^X | \mathsf{f} \text{ is bijective}\right\} && \text{(the set of all bijective functions in } Y^X\text{)}
\end{aligned}$$

[21] Davis (2005) pages 6–8, Vaidyanathaswamy (1960) page 10

[22] Michel and Herget (1993) pages 14–15, Fuhrmann (2012) page 2, Comtet (1974) page 5

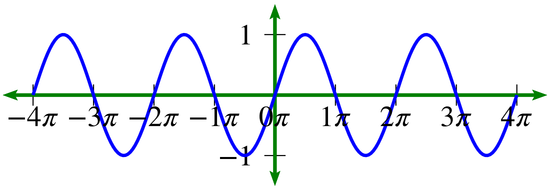
The types described in Definition B.11 are illustrated below:



“*into*”	“*onto*”	“*one-to-one*”	“*one-to-one and onto*”
(arbitrary function in Y^X)	*surjective*	*injective*	*bijective*

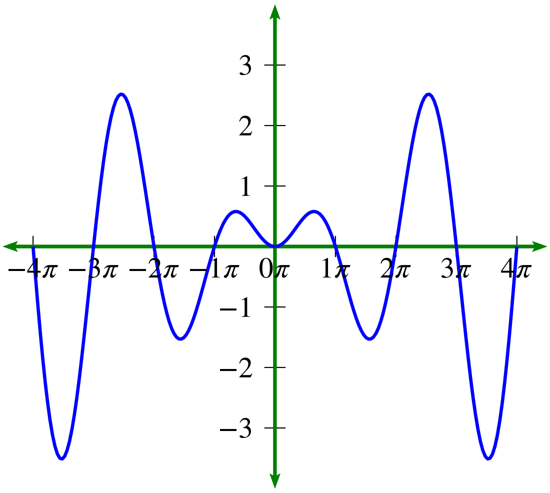
Example B.22.

In the set $\mathbb{R}^{\mathbb{R}}$, the function $\sin x$ is *not injective, not surjective*, and *not bijective*.



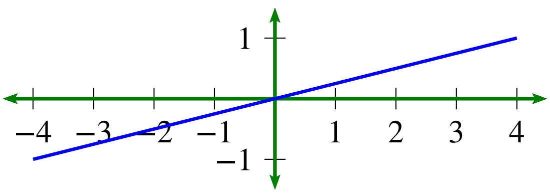
Example B.23.

In the set $\mathbb{R}^{\mathbb{R}}$, the function $x \sin x$ is *surjective*, but *not injective* and *not bijective*.

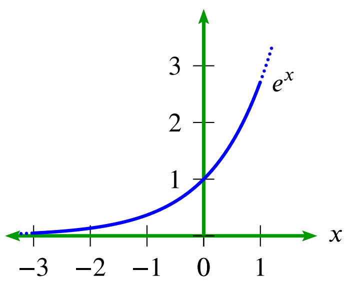


Example B.24.

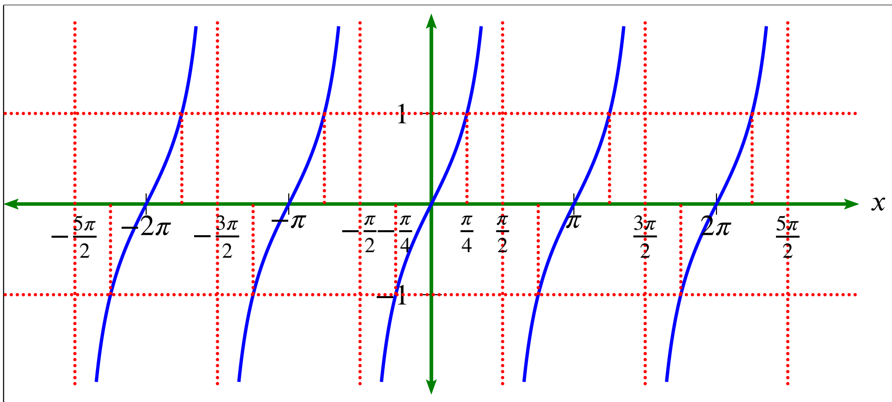
In the set $\mathbb{R}^{\mathbb{R}}$, the function $y = \frac{1}{4}x$ is *injective, surjective*, and *bijective*.



Example B.25. In the set $\mathbb{R}^{\mathbb{R}}$, the function e^x is *injective*, but *not surjective* and *not bijective*.



Example B.26. In the set $\mathbb{R}^{\mathbb{R}}$, the function $\tan x$ is *not injective, not surjective* (it's range does not include $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc.) and *not bijective*.



Theorem B.7. [23]

THM

$$\begin{array}{lllcll}
\mathsf{f} \textit{ and } \mathsf{g} & \textit{are} & \textit{surjective} & \implies & \mathsf{g}\circ\mathsf{f} & \textit{is surjective}\\
\mathsf{g}\circ\mathsf{f} & \textit{is} & \textit{surjective} & \implies & \mathsf{g} & \textit{is surjective}\\
\mathsf{f} \textit{ and } \mathsf{g} & \textit{are} & \textit{injective} & \implies & \mathsf{g}\circ\mathsf{f} & \textit{is injective}\\
\mathsf{g}\circ\mathsf{f} & \textit{is} & \textit{injective} & \implies & \mathsf{f} & \textit{is injective}
\end{array}$$

✎PROOF:

$$\begin{array}{rcll}
\mathsf{f},\mathsf{g} \text{ are surjective} & \implies & \mathsf{f}(X)=Y, \text{ and } \mathsf{g}(Y)=Z & \text{by definition of surjective page 88}\\
& \implies & \mathsf{g}\circ\mathsf{f}(X)=\mathsf{g}(Y)=Z & \\
& \implies & \mathsf{g}\circ\mathsf{f} \text{ is surjective} & \text{by definition of surjective page 88}
\end{array}$$

$$\begin{array}{rcll}
\mathsf{g}\circ\mathsf{f} \text{ is surjective} & \implies & \mathsf{g}\circ\mathsf{f}(X)=Z & \text{by definition of surjective page 88}\\
& \implies & \mathsf{g}(\mathsf{f}(X))=Z & \\
& \implies & \mathsf{g}(Y)=Z & \text{because } \mathsf{f}(X)\subseteq Y \text{ and by isotone property page 88}\\
& \implies & \mathsf{g} \text{ is surjective} & \text{by definition of surjective page 88}
\end{array}$$

$$\begin{array}{rcll}
\mathsf{g}\circ\mathsf{f}(x_1)=\mathsf{g}\circ\mathsf{f}(x_2) & \implies & \mathsf{g}(\mathsf{f}(x_1))=\mathsf{g}(\mathsf{f}(x_2)) & \\
& \implies & \mathsf{f}(x_1)=\mathsf{f}(x_2) & \text{because } \mathsf{g} \text{ is injective}\\
& \implies & x_1=x_2 & \text{because } \mathsf{f} \text{ is injective}\\
& \implies & \mathsf{g}\circ\mathsf{f} \text{ is injective} &
\end{array}$$

$$\begin{array}{rcll}
\mathsf{f}(x_1)=\mathsf{f}(x_2) & \implies & \mathsf{g}(\mathsf{f}(x_1))=\mathsf{g}(\mathsf{f}(x_2)) & \\
& \implies & \mathsf{g}\circ\mathsf{f}(x_1)=\mathsf{g}\circ\mathsf{f}(x_2) & \\
& \implies & x_1=x_2 & \text{because } \mathsf{g}\circ\mathsf{f} \text{ is injective}\\
& \implies & \mathsf{f} \text{ is injective} &
\end{array}$$

✏

Theorem B.8 (Bernstein-Cantor-Schröder Theorem). [24]

$$\left(\exists\mathsf{f}\in\mathcal{I}_{\mathrm{j}}(X,Y)\right) \textit{ and } \left(\exists\mathsf{g}\in\mathcal{I}_{\mathrm{j}}(Y,X)\right) \implies \exists\mathsf{h}\in\mathcal{B}_{\mathrm{j}}(X,Y)$$

Example B.27. Let $X \triangleq \{x_1, x_2, x_3\}$ and $Y \triangleq \{y_1, y_2\}$. There are a total of $|\mathbb{R}| = 2^{3\times 2} = 64$ possible relations, as listed in Example B.2 (page 74). There are a total of $|\mathbb{F}| = 2^3 = 8$ possible functions, as listed in Example B.20 (page 87). Let $\mathbb{S} \triangleq (\!(S_1, S_2, S_3, \ldots)\!)$ be the set of all **surjective** functions from

[23] Durbin (2000) pages 16–17

[24] Schröder (2003) page 116, Nievergelt (2002) page 213, Suppes (1972) page 95, Fraenkel (1953) pages 102–103 ⟨???⟩

X to Y. There are a total of $|\mathbb{S}| = 6$ possible surjective functions, as listed next:

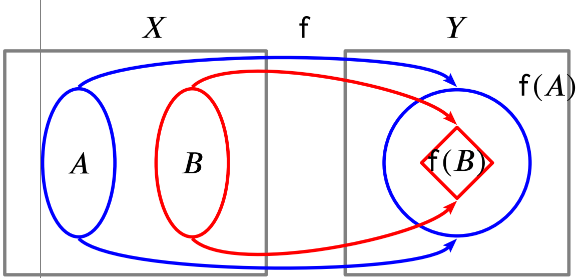
surjective functions on $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$	
$S_1 = \{(x_1,y_2),(x_2,y_1),(x_3,y_1)\}$	$S_4 = \{(x_1,y_1),(x_2,y_1),(x_3,y_2)\}$
$S_2 = \{(x_1,y_1),(x_2,y_2),(x_3,y_1)\}$	$S_5 = \{(x_1,y_2),(x_2,y_1),(x_3,y_2)\}$
$S_3 = \{(x_1,y_2),(x_2,y_2),(x_3,y_1)\}$	$S_6 = \{(x_1,y_1),(x_2,y_2),(x_3,y_2)\}$

Example B.28. Let $X \triangleq \{x, y, z\}$ There are a total of $|\mathbb{R}| = 2^{|X\times X|} = 2^{|X|\cdot|X|} = 2^{3\times 3} = 2^9 = 512$ possible relations on $X \times X$. Of these 512 relations, only 27 are **functions**. These 27 functions are listed in Example B.21 (page 87). Of these 27 functions, only 7 are *surjective* functions, as listed below. Actually, in the case of a function mapping from a finite set onto the same finite set, The set $\mathbb{S}$ of surjective functions is equal to the set of injective functions and the set of bijective functions.

surjective functions on $\{x, y, z\} \times \{x, y, z\}$	
$S_1 = \{(x,z),(y,x),(z,x)\}$	$S_5 = \{(x,x),(y,z),(z,y)\}$
$S_2 = \{(x,z),(y,y),(z,x)\}$	$S_6 = \{(x,y),(y,x),(z,z)\}$
$S_3 = \{(x,y),(y,z),(z,x)\}$	$S_7 = \{(x,x),(y,y),(z,z)\}$
$S_4 = \{(x,z),(y,x),(z,y)\}$	

B.2.4 Image relations

Consider two subsets A and B of the domain of a function f. What is the relationship between the image under f of their union and the union of their images under f? Are they equal? Is one a subset of the other? What is the relationship between the image of their intersection under f and the intersection of their images f? Theorem B.9 (next theorem) answers these questions.



Theorem B.9. [25] *Let* f *be a function in* Y^X.

THM

$$\mathsf{f}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathsf{f}(A_i) \qquad \forall \mathsf{f}\in Y^X, A_i\in 2^X \qquad \text{(additive)}$$

$$\mathsf{f}\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} \mathsf{f}(A_i) \qquad \forall \mathsf{f}\in Y^X, A_i\in 2^X$$

✎PROOF:

[25] Davis (2005) pages 6–7, Vaidyanathaswamy (1960) page 10

1. Proof that $\mathsf{f}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathsf{f}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}\left(\bigcup_{i\in I} A_i\right) &= \left\{ y\in Y | \exists x \in \bigcup_{i\in I} A_i \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\} && \text{by Definition B.5 page 83}\\
&= \bigcup_{i\in I} \left\{ y\in Y | \exists x \in A_i \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\}\\
&= \bigcup_{i\in I} \mathsf{f}\left(A_i\right) && \text{by Definition B.5 page 83}
\end{aligned}$$

2. Proof that $\mathsf{f}\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} \mathsf{f}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}\left(\bigcap_{i\in I} A_i\right) &= \left\{ y\in Y | \exists x \in \bigcap_{i\in I} A_i \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\} && \text{by Definition B.5 page 83}\\
&= \left\{ y\in Y | \exists x \quad \text{such that} \quad \bigwedge_{i\in I} [x\in A_i] \text{ and } (x,y)\in\mathsf{f}] \right\} && \text{by Definition A.5 page 38}\\
&\subseteq \left\{ y\in Y | \bigwedge_{i\in I} [\exists x\in A_i \quad \text{such that} \quad (x,y)\in\mathsf{f}] \right\}\\
&= \bigcap_{i\in I} \left\{ y\in Y | \exists x \in A_i \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\} && \text{by Definition A.5 page 38}\\
&= \bigcap_{i\in I} \mathsf{f}(A_i) && \text{by Definition B.5 page 83}
\end{aligned}$$

⇒

Theorem B.10. [26] *Let* $\mathsf{f}^{-1}\in X^Y$ *be the inverse of a function* $\mathsf{f}\in Y^X$.

THM

$$\begin{aligned}
\mathsf{f}^{-1}(Y) &= X && \forall \mathsf{f}\in Y^X\\
\mathsf{f}^{-1}(A^{\mathsf{c}}) &= \mathsf{c}\left[\mathsf{f}^{-1}(A)\right] && \forall \mathsf{f}\in Y^X, A\in 2^Y\\
\mathsf{f}^{-1}\left(\bigcup_{i\in I} A_i\right) &= \bigcup_{i\in I} \mathsf{f}^{-1}\left(A_i\right) && \forall \mathsf{f}\in Y^X, A_i\in 2^Y\\
\mathsf{f}^{-1}\left(\bigcap_{i\in I} A_i\right) &= \bigcap_{i\in I} \mathsf{f}^{-1}\left(A_i\right) && \forall \mathsf{f}\in Y^X, A_i\in 2^Y
\end{aligned}$$

✎PROOF:

1. Proof that $\mathsf{f}^{-1}(A^{\mathsf{c}}) = \mathsf{c}\left[\mathsf{f}^{-1}(A)\right]$:

$$\begin{aligned}
\mathsf{c}\left[\mathsf{f}^{-1}(Y)\right] &= \mathsf{c}\left\{ x\in X | \exists y \in A \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\} && \text{by Definition B.5 page 83}\\
&= \left\{ x\in X | \neg\{\exists y \in A \quad \text{such that} \quad (x,y)\in\mathsf{f} \}\right\} && \text{by Definition A.5 page 38}\\
&= \left\{ x\in X | \nexists y \in A \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\} && \text{by Definition A.5 page 38}\\
&= \left\{ x\in X | \exists y \in A^{\mathsf{c}} \quad \text{such that} \quad (x,y)\in\mathsf{f} \right\}\\
&= \mathsf{f}^{-1}(A^{\mathsf{c}}) && \text{by Definition B.5 page 83}
\end{aligned}$$

[26] Davis (2005) pages 7–8, Vaidyanathaswamy (1960) page 10

2. Proof that $\mathsf{f}^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mathsf{f}^{-1}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}^{-1}\left(\bigcup_{i\in I} A_i\right) &= \left\{x \in X \mid \exists y \in \bigcup_{i\in I} A_i \quad \text{such that} \quad (x,y) \in \mathsf{f}\right\} && \text{by Definition B.5 page 83}\\
&= \left\{x \in X \mid \bigvee_{i\in I} \{\exists y \in A_i \quad \text{such that} \quad (x,y) \in \mathsf{f}\}\right\}\\
&= \bigcup_{i\in I} \{\exists x \in X \mid y \in A_i \quad \text{such that} \quad (x,y) \in \mathsf{f}\} && \text{by Definition A.5 page 38}\\
&= \bigcup_{i\in I} \mathsf{f}^{-1}(A_i) && \text{by Definition B.5 page 83}
\end{aligned}$$

3. Proof that $\mathsf{f}^{-1}(Y) = X$:

$$\begin{aligned}
\mathsf{f}^{-1}(Y) &= \mathsf{f}^{-1}(\mathcal{I}X \cup Y \setminus \mathcal{I}X)\\
&= \mathsf{f}^{-1}(\mathcal{I}X) \cup \mathsf{f}^{-1}(Y \setminus \mathcal{I}X) && \text{by item 4}\\
&= X \cup \varnothing && \text{by Definition B.4 page 80}\\
&= X
\end{aligned}$$

4. Proof that $\mathsf{f}^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} \mathsf{f}^{-1}\left(A_i\right)$:

$$\begin{aligned}
\mathsf{f}^{-1}\left(\bigcap_{i\in I} A_i\right) &= \left\{x \in X \mid \exists y \in \bigcap_{i\in I} A_i \quad \text{such that} \quad (x,y) \in \mathsf{f}\right\} && \text{by Definition B.5 page 83}\\
&= \left\{x \in X \mid \exists y \quad \text{such that} \quad \left\{y \in \bigwedge_{i\in I} A_i \text{ and } (x,y) \in \mathsf{f}\right\}\right\} && \text{by Definition A.5 page 38}\\
&= \left\{x \in X \mid \bigwedge_{i\in I} [\exists y \in A_i \quad \text{such that} \quad (x,y) \in \mathsf{f}]\right\} && \text{by definition of function page 85}\\
&= \bigcap_{i\in I} \{x \in X \mid \exists y \in A_i \quad \text{such that} \quad (x,y) \in \mathsf{f}\} && \text{by Definition A.5 page 38}\\
&= \bigcap_{i\in I} \mathsf{f}^{-1}\left(A_i\right) && \text{by Definition B.5 page 83}
\end{aligned}$$

5. Proof that $\mathsf{f}^{-1}(Y \setminus A) = X \setminus \mathsf{f}^{-1}(A)$:

$$\begin{aligned}
\mathsf{f}^{-1}(Y \setminus A) &= \mathsf{f}^{-1}(Y \cap A^{\mathsf{c}})\\
&= \mathsf{f}^{-1}(Y) \cap \mathsf{f}^{-1}(A^{\mathsf{c}}) && \text{by 6.}\\
&= X \cap \mathsf{f}^{-1}(A^{\mathsf{c}}) && \text{by 5.}\\
&= X \cap \mathsf{c}\left[\mathsf{f}^{-1}(A)\right] && \text{by 3.}\\
&= X \setminus \mathsf{f}^{-1}(A) && \text{by Definition A.5 page 38}
\end{aligned}$$

✎

B.2.5 Indicator functions

By the *axiom of extension*, a set is uniquely defined by the elements that are in that set. Thus, we are often interested in the Boolean result of whether an element is in a set A, or is not in A, but exclude the possibility of both being true. That a statement is either true or false but definitely not both is called *the law of the excluded middle* and is a fundamental property of all *Boolean algebras*

$(\{1,0\}, \vee, \wedge)$. [27] The *indicator function* (next definition) is a convenient "indicator" of whether or not a particular element is in a set, and has several interesting properties (Theorem B.11 page 94).

Definition B.12. [28] *Let X be a set.*

DEF

*The **indicator function** $\mathbb{1} \in \{0,1\}^{2^X}$ is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \qquad \forall x \in X,\ A \in 2^X \\ 0 & \text{for } x \notin A \qquad \forall x \in X,\ A \in 2^X \end{cases}$$

*The indicator function $\mathbb{1}$ is also called the **characteristic function**.*

Theorem B.11. [29] *Let $\mathbb{1}$ be the* INDICATOR FUNCTION *(Definition B.12 page 94). Let $x \vee y$ represent the maximum of $\{x, y\}$.*

THM

$$\begin{array}{rcl@{\qquad}rcl} \mathbb{1}_\varnothing &=& 0 & \mathbb{1}_X &=& 1 \\ \mathbb{1}_{A\cup B} &=& \mathbb{1}_A \vee \mathbb{1}_B & \mathbb{1}_{A\cap B} &=& \mathbb{1}_A \mathbb{1}_B \\ \mathbb{1}_{A \triangle B} &=& \mathbb{1}_A \mathbb{1}_B & \mathbb{1}_{A \setminus B} &=& \mathbb{1}_A\left(1 - \mathbb{1}_B\right) \\ \mathbb{1}_{A^c} &=& 1 - \mathbb{1}_A & & & \end{array}$$

✎PROOF:

$$\begin{aligned} \mathbb{1}_{A\cup B}(x) &\triangleq \begin{cases} 1 & \text{for } x \in A \cup B \quad \forall x \in X \\ 0 & \text{for } x \notin A \cup B \quad \forall x \in X \end{cases} && \text{by Definition B.12} \\ &= \begin{cases} 1 & \text{for } x \in A \vee x \in B \quad \forall x \in X \\ 0 & \text{otherwise} \end{cases} && \text{by Definition A.5 page 38} \\ &= \left\{ \begin{array}{ll} 1 & \text{for } x \in A \\ 0 & \text{otherwise} \end{array} \right\} \vee \left\{ \begin{array}{ll} 1 & \text{for } x \in B \\ 0 & \text{otherwise} \end{array} \right\} \\ &= \mathbb{1}_A(x) \vee \mathbb{1}_B(x) && \text{by Definition B.12} \end{aligned}$$

$$\begin{aligned} \mathbb{1}_{A\cap B}(x) &\triangleq \begin{cases} 1 & \text{for } x \in A \cap B \quad \forall x \in X \\ 0 & \text{for } x \notin A \cap B \quad \forall x \in X \end{cases} && \text{by Definition B.12} \\ &= \begin{cases} 1 & \text{for } x \in A \wedge x \in B \quad \forall x \in X \\ 0 & \text{otherwise} \end{cases} && \text{by Definition A.5 page 38} \\ &= \left\{ \begin{array}{ll} 1 & \text{for } x \in A \\ 0 & \text{otherwise} \end{array} \right\} \wedge \left\{ \begin{array}{ll} 1 & \text{for } x \in B \\ 0 & \text{otherwise} \end{array} \right\} \\ &= \mathbb{1}_A(x) \wedge \mathbb{1}_B(x) \\ &= \mathbb{1}_A \mathbb{1}_B && \text{by Definition B.12} \end{aligned}$$

$$\begin{aligned} \mathbb{1}_{A^c}(x) &= \begin{cases} 1 & \text{for } x \in A^c \quad \forall x \in X \\ 0 & \text{for } x \notin A^c \quad \forall x \in X \end{cases} && \text{by Definition B.12} \\ &= \begin{cases} 1 & \text{for } x \notin A \quad \forall x \in X \\ 0 & \text{for } x \in A \quad \forall x \in X \end{cases} \\ &= 1 - \mathbb{1}_A \end{aligned}$$

$$\begin{aligned} \mathbb{1}_{A \setminus B} &= \mathbb{1}_{A \cap B^c} \\ &= \mathbb{1}_A \mathbb{1}_{B^c} \\ &= \mathbb{1}_A\left(1 - \mathbb{1}_B\right) \end{aligned}$$

[27] excluded middle: Theorem 3.2 page 33

[28] ? page 104, Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) page 22, de la Vallée-Poussin (1915) page 440

[29] Aliprantis and Burkinshaw (1998) page 126, Hausdorff (1937) pages 22–23

$$\begin{aligned}
\mathbb{1}_{A\triangle B} &= \mathbb{1}_{(A\setminus B^c)\cup(B\setminus A^c)} \\
&= \left(\mathbb{1}_{A\setminus B^c}\right) \vee \left(\mathbb{1}_{B\setminus A^c}\right) \\
&= \left[\mathbb{1}_A\left(1-\mathbb{1}_{B^c}\right)\right] \vee \left[\mathbb{1}_B\left(1-\mathbb{1}_{A^c}\right)\right] \\
&= \left[\mathbb{1}_A\left(1-1+\mathbb{1}_B\right)\right] \vee \left[\mathbb{1}_B\left(1-1+\mathbb{1}_A\right)\right] \\
&= \left[\mathbb{1}_A\mathbb{1}_B\right] \vee \left[\mathbb{1}_B\mathbb{1}_A\right] \\
&= \mathbb{1}_A\mathbb{1}_B
\end{aligned}$$

$$\begin{aligned}
\mathbb{1}_{\varnothing} &= \mathbb{1}_{A\setminus A} \\
&= \mathbb{1}_A\left(1-\mathbb{1}_A\right) \\
&= \mathbb{1}_A - \mathbb{1}_A\mathbb{1}_A \\
&= \mathbb{1}_A - \mathbb{1}_A \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{1}_X &= \mathbb{1}_{A\cup A^c} \\
&= \mathbb{1}_A \vee \mathbb{1}_{A^c} \\
&= \mathbb{1}_A \vee \left(1-\mathbb{1}_A\right) \\
&= 1
\end{aligned}$$

⇨

B.2.6 Calculus of functions

Definition B.13. [30] *Let Y^X be the set of all functions from a set X to a set Y.*

DEF

$$\begin{array}{rcllr}
[-\mathsf{f}](x) & \triangleq & -[\mathsf{f}(x)] & \forall x\in X, \mathsf{f}\in Y^X & \text{(NEGATION)} \\
\left[\mathsf{f} \dot{+} \mathsf{g}\right](x) & \triangleq & \mathsf{f}(x)+\mathsf{g}(x) & \forall x\in X\ \mathsf{f},\mathsf{g}\in Y^X & \text{(FUNCTION ADDITION)} \\
\left[\mathsf{f} - \mathsf{g}\right](x) & \triangleq & \mathsf{f}(x)+\left[-\mathsf{g}\right](x) & \forall x\in X\ \mathsf{f},\mathsf{g}\in Y^X & \text{(FUNCTION SUBTRACTION)} \\
\left[\mathsf{g}\mathsf{f}\right](x) & \triangleq & \mathsf{g}[\mathsf{f}(x)] & \forall x\in X\ \mathsf{f},\mathsf{g}\in Y^X & \text{(FUNCTION MULTIPLICATION)} \\
[\alpha\mathsf{f}](x) & \triangleq & \alpha[\mathsf{f}(x)] & \forall x\in X, \alpha\in \boldsymbol{Y}\ \mathsf{f}\in Y^X & \text{(SCALAR MULTIPLICATION)}
\end{array}$$

Definition B.14. *Let f be a function in X^X with inverse relation f^{-1} and let $\mathbf{I}$ be the identity function in X^X.*

DEF

$$\mathsf{f}^n \triangleq \begin{cases} \mathbf{I} & \text{for } n=0 \\ \prod_1^n \mathsf{f} & \text{for } n\in\mathbb{N} \\ \left(\mathsf{f}^{-1}\right)^n & \text{for } n\in\mathbb{Z}^- \end{cases}$$

Theorem B.12. [31] *Let X, Y, and Z be sets.*

THM

$$\begin{array}{lrcllr}
1. & (\mathsf{f}\mathsf{g})^{-1} & = & \left(\mathsf{g}^{-1}\right)\left(\mathsf{f}^{-1}\right) & \forall \mathsf{f}\in Y^X, \mathsf{g}\in Z^Y & \text{(IDEMPOTENT)} \\
2. & \mathsf{h}(\mathsf{g}\mathsf{f}) & = & (\mathsf{h}\mathsf{g})\mathsf{f} & \forall \mathsf{f}\in X^W, \mathsf{g}\in Y^X, \mathsf{h}\in Z^Y & \text{(ASSOCIATIVE)} \\
3. & \left(\mathsf{f} \dot{+} \mathsf{g}\right)\mathsf{h} & = & (\mathsf{f}\mathsf{h}) \dot{+} (\mathsf{g}\mathsf{h}) & \forall \mathsf{f},\mathsf{g}\in Y^X, \mathsf{h}\in Z^Y & \text{(RIGHT DISTRIBUTIVE)} \\
4. & \alpha(\mathsf{f}\mathsf{g}) & = & (\alpha\mathsf{f})\mathsf{g} & \forall \mathsf{f}\in Y^X, \mathsf{g}\in Z^Y & \text{(HOMOGENOUS)}
\end{array}$$

✎PROOF:

[30] Michel and Herget (1993) page 409, Cayley (1858), Riesz (1913), Hilbert et al. (1927) page 6
[31] Kelley (1955) page 8, Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

1. Proof of the *idempotent* property:

   (a) Note that $\mathsf{f}\mathsf{g} = \mathsf{f} \circ \mathsf{g}$, where $\circ$ is the *composition function* (Definition B.3 page 78).

   (b) The result follows from Theorem B.2 (page 79), where it is demonstrated to be true for the more general case of *relations*.

2. Proof of the *associative* property: This result follows from Theorem B.2 (page 79), where it is demonstrated to be true for the more general case of *relations*.

3. Proof of the *right distributive* property:

$$\begin{aligned}
\left[(\mathsf{f} \mathbin{\mathring{+}} \mathsf{g})\mathsf{h}\right]x &= (\mathsf{f} \mathbin{\mathring{+}} \mathsf{g})(\mathsf{h}x) && \text{by Definition B.13 page 95}\\
&= [\mathsf{f}(\mathsf{h}x)] \mathbin{\mathring{+}} [\mathsf{g}(\mathsf{h}x)] && \text{by Definition B.13 page 95}\\
&= [(\mathsf{f}\mathsf{h})x] \mathbin{\mathring{+}} [(\mathsf{g}\mathsf{h})x] && \text{by Definition B.13 page 95}
\end{aligned}$$

4. Proof of the *homogeneous* property:

$$\begin{aligned}
\left[\alpha[\mathsf{f}\mathsf{g}]\right](x) &= \alpha\left[[\mathsf{f}\mathsf{g}](x)\right] && \text{by Definition B.13 page 95}\\
&= \alpha\left[\mathsf{f}\left[\mathsf{g}(x)\right]\right] && \text{by Definition B.13 page 95}\\
&= [\alpha\mathsf{f}]\left[\mathsf{g}(x)\right] && \text{by Definition B.13 page 95}\\
&= \left[[\alpha\mathsf{f}]\mathsf{g}\right](x) && \text{by Definition B.13 page 95}
\end{aligned}$$

⇨

Theorem B.13. *Let $\mathcal{A} \triangleq X^X$ be the set of functions on X^X.*

THM

1. $(\mathcal{A}, \mathring{+})$ *is an additive group.*
2. $(\mathcal{A}, \mathring{+}, \cdot)$ *is a ring.*
3. $(\mathcal{A}, \mathring{+})$ *is a linear space.*
4. $(\mathcal{A}, \mathring{+}, \cdot)$ *is an algebra.*

✎PROOF:

1. additive group:

1.	$\mathsf{f} \mathbin{\mathring{+}} \mathbb{0}$	$=$	$\mathbb{0} + \mathsf{f} = \mathsf{f}$	$\forall \mathsf{f} \in \mathcal{A}$	($\mathbb{0} \in \mathcal{A}$ is the identity element)
2.	$\mathsf{f} \mathbin{\mathring{+}} (-\mathsf{f})$	$=$	$(-\mathsf{f}) + \mathsf{f} = \mathbb{0}$	$\forall \mathsf{f} \in \mathcal{A}$	($(-\mathsf{f})$ is the inverse of f)
3.	$(\mathsf{f} \mathbin{\mathring{+}} \mathsf{g}) + \mathsf{h}$	$=$	$\mathsf{f} \mathbin{\mathring{+}} (\mathsf{g} + \mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h} \in \mathcal{A}$	($(\mathcal{A}, \cdot)$ is associative)

2. ring:

1.	$(\mathcal{A}, +, *)$ is a group with respect to $(\mathcal{A}, +)$				(additive group)
2.	$\mathsf{f}(\mathsf{g}\mathsf{h})$	$=$	$(\mathsf{f}\mathsf{g})\mathsf{h}$	$\forall \mathsf{f},\mathsf{g},\mathsf{h} \in \mathcal{A}$	(associative with respect to $*$)
3.	$\mathsf{f}(\mathsf{g} + \mathsf{h})$	$=$	$(\mathsf{f}\mathsf{g}) + (\mathsf{f}\mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h} \in \mathcal{A}$	($*$ is left distributive over $+$)
4.	$(\mathsf{f} \mathbin{\mathring{+}} \mathsf{g})\mathsf{h}$	$=$	$(\mathsf{f}\mathsf{h}) + (\mathsf{g}\mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h} \in \mathcal{A}$	($*$ is right distributive over $+$).

3. linear space:

1.	$(\mathsf{f} \mathbin{\mathring{+}} \mathsf{g}) \mathbin{\mathring{+}} \mathsf{h}$	$=$	$\mathsf{f} \mathbin{\mathring{+}} (\mathsf{g} \mathbin{\mathring{+}} \mathsf{h})$	$\forall \mathsf{f},\mathsf{g},\mathsf{h} \in \mathcal{A}$	($\mathring{+}$ is *associative*)
2.	$\mathsf{f} \mathbin{\mathring{+}} \mathsf{g}$	$=$	$\mathsf{g} \mathbin{\mathring{+}} \mathsf{f}$	$\forall \mathsf{f},\mathsf{g} \in \mathcal{A}$	($\mathring{+}$ is *commutative*)
3.	$\exists \mathbb{0} \in X$ such that $\mathsf{f} \mathbin{\mathring{+}} \mathbb{0}$	$=$	f	$\forall \mathsf{f} \in X\mathcal{A}$	($\mathring{+}$ identity)
4.	$\exists \mathsf{g} \in X$ such that $\mathsf{f} \mathbin{\mathring{+}} \mathsf{g}$	$=$	$\mathbb{0}$	$\forall \mathsf{f} \in \mathcal{A}$	($\mathring{+}$ inverse)
5.	$\alpha \otimes (\mathsf{f} \mathbin{\mathring{+}} \mathsf{g})$	$=$	$(\alpha \otimes \mathsf{f}) \mathbin{\mathring{+}} (\alpha \otimes \mathsf{g})$	$\forall \alpha \in S$ and $\mathsf{f},\mathsf{g} \in \mathcal{A}$	($\otimes$ *distributes* over $\mathring{+}$)
6.	$(\alpha + \beta) \otimes \mathsf{f}$	$=$	$(\alpha \otimes \mathsf{f}) \mathbin{\mathring{+}} (\beta \otimes \mathsf{f})$	$\forall \alpha,\beta \in S$ and $\mathsf{f} \in \mathcal{A}$	($\otimes$ pseudo-distributes over $+$)
7.	$\alpha(\beta \otimes \mathsf{f})$	$=$	$(\alpha \cdot \beta) \otimes \mathsf{f}$	$\forall \alpha,\beta \in S$ and $\mathsf{f} \in \mathcal{A}$	($\cdot$ associates with $\otimes$)
8.	$1 \otimes \mathsf{f}$	$=$	f	$\forall \mathsf{f} \in \mathcal{A}$	($\otimes$ identity)

4. algebra:

1.	$(\mathsf{fg})\mathsf{h}$	$=$	$\mathsf{f}(\mathsf{gh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	(*associative*)
2.	$\mathsf{f}(\mathsf{g} \mathring{+} \mathsf{h})$	$=$	$(\mathsf{fg}) + (\mathsf{fh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	(*left distributive*)
3.	$(\mathsf{f} \mathring{+} \mathsf{g})\mathsf{h}$	$=$	$(\mathsf{fh}) + (\mathsf{gh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	(*right distributive*)
4.	$\alpha(\mathsf{gh})$	$=$	$(\alpha\mathsf{g})\mathsf{h} = \mathsf{g}(\alpha\mathsf{h})$	$\forall \mathsf{g,h} \in \mathcal{A}$ and $\alpha \in \mathbb{F}$	(scalar *commutative*)

⇨

Theorem B.14. *Let* $\mathcal{A} \triangleq \left\{\mathsf{f} \in X^X \middle| \exists \mathsf{f}^{-1} \text{ such that } \mathsf{f}^{-1}\mathsf{f} \mathrel{\mathring{=}} \mathsf{f}\mathsf{f}^{-1} \mathrel{\mathring{=}} \mathbf{I}\right\}$ *be the set of invertible functions on* X^X.

THM $(\mathcal{A}, \cdot)$ *is a (multiplicative) group.*

✎PROOF:

1. multiplicative group:

1.	$\mathsf{f}\mathbf{I} = \mathbf{I}\mathsf{f} = \mathsf{f}$	$\forall \mathsf{f} \in \mathcal{A}$	($\mathbf{I} \in \mathcal{A}$ is the identity element)
2.	$\mathsf{f}^{-1}\mathsf{f} = \mathsf{f}\mathsf{f}^{-1} = \mathbf{I}$	$\forall \mathsf{f} \in \mathcal{A}$	(f^{-1} is the inverse of f)
3.	$(\mathsf{fg})\mathsf{h} = \mathsf{f}(\mathsf{gh})$	$\forall \mathsf{f,g,h} \in \mathcal{A}$	($(\mathcal{A}, \cdot)$ is associative)

2. field:

1.	$(X, +, *)$ is a ring		(ring)
2.	$\boldsymbol{xy} = \boldsymbol{yx}$	$\forall \boldsymbol{x,y} \in X$	(commutative with respect to $*$)
3.	$(X\backslash\{0\}, *)$ is a group		(group with respect to $*$).

⇨

Theorem B.15. *Let* $\mathcal{D}(\mathsf{f})$ *be the domain of an function* f *and* $\mathcal{I}(\mathsf{f})$ *the image of* f.

THM

$$\begin{array}{rclcrcl}
\mathcal{D}\left(\bigcup_{i\in I}\mathsf{f}_i\right) & = & \bigcup_{i\in I}\mathcal{D}(\mathsf{f}_i) & \quad & \mathcal{I}\left(\bigcup_{i\in I}\mathsf{f}_i\right) & = & \bigcup_{i\in I}\mathcal{I}(\mathsf{f}_i)\\
\mathcal{D}\left(\bigcap_{i\in I}\mathsf{f}_i\right) & \subseteq & \bigcap_{i\in I}\mathcal{D}(\mathsf{f}_i) & & \mathcal{I}\left(\bigcap_{i\in I}\mathsf{f}_i\right) & \subseteq & \bigcap_{i\in I}\mathcal{I}(\mathsf{f}_i)\\
\mathcal{D}(\mathsf{f}\backslash\mathsf{g}) & \supseteq & \mathcal{D}(\mathsf{f})\backslash\mathcal{D}(\mathsf{g}) & & \mathcal{I}(\mathsf{f}\backslash\mathsf{g}) & \supseteq & \mathcal{I}(\mathsf{f})\backslash\mathcal{I}(\mathsf{g})
\end{array}$$

✎PROOF: These results follow from Theorem B.3 (page 80). ⇨

Definition B.15. [32] *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be linear spaces over a field* $\mathbb{F}$ *and with dual spaces*

$$\begin{array}{rcll}
\boldsymbol{X}^* & \triangleq & \left\{\mathsf{f}(\boldsymbol{x};\boldsymbol{x}^*) \in \mathbb{F}^{\boldsymbol{X}} \middle| \boldsymbol{x}^* \in \boldsymbol{X}^*\right\} & \text{(set of functionals with parameter } \boldsymbol{x}^* \text{ from } \boldsymbol{X} \text{ to } \mathbb{F})\\
\boldsymbol{Y}^* & \triangleq & \left\{\mathsf{g}(\boldsymbol{y};\boldsymbol{y}^*) \in \mathbb{F}^{\boldsymbol{Y}} \middle| \boldsymbol{y}^* \in \boldsymbol{Y}^*\right\}. & \text{(set of functionals with parameter } \boldsymbol{y}^* \text{ from } \boldsymbol{Y} \text{ to } \mathbb{F})
\end{array}$$

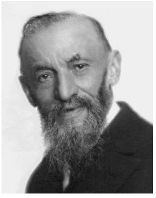
Let $\mathsf{f} \in Y^X$ *be a function.*

DEF *A function* f^* *in* $\boldsymbol{X}^{*\boldsymbol{Y}^*}$ *is the* ***conjugate*** *of the function* f *if*

$$\mathsf{g}(\mathsf{f}\boldsymbol{x};\boldsymbol{y}^*) = \mathsf{f}(\boldsymbol{x};\mathsf{f}^*\boldsymbol{y}^*) \qquad \forall \boldsymbol{x} \in \boldsymbol{X}, \mathsf{f} \in \boldsymbol{X}^*, \mathsf{g} \in \boldsymbol{Y}^*$$

[32] Michel and Herget (1993) page 420, Giles (2000) page 171

B.3 Tempered Distributions



❝ *I am sure that something must be found. There must exist a notion of generalized functions which are to functions what the real numbers are to the rationals.*❞
Giuseppe Peano (1858–1932), Italian mathematician[33]

Definition B.16. [34]
DEF
*A **test function** is any function ϕ that satisfies*
1. $\phi \in \mathbb{C}^{\mathbb{R}}$
2. ϕ *is* INFINITELY DIFFERENTIABLE.

*The set of all test functions is denoted $\mathbb{C}^{\infty}(\mathbb{R})$. A test function ϕ belongs to the **Schwartz class** $\mathcal{S}$ if, for some set of constants $\{C_{n,k} | n, k \in \mathbb{W}\}$,*
$$(1+|x|)^n \left|\phi^{(k)}\right| \le C_{n,k} \qquad \forall n,k \in \mathbb{W},\ \forall x \in \mathbb{R}$$

Definition B.17. [35] *Let $\mathcal{S}$ be the* SCHWARTZ CLASS *of functions (Definition B.16).*
DEF
$\mathsf{d}[\cdot]$ *is a **tempered distribution** if*
1. $\mathsf{d}\left[\alpha_1\phi_1 + \alpha_2\phi_2\right] = \mathsf{d}\left[\alpha_1\phi_1\right] + \mathsf{d}\left[\alpha_2\phi_2\right] \qquad \forall \phi_1,\phi_2 \in \mathcal{S},\, \alpha_1,\alpha_2 \in \mathbb{R}$ *(LINEAR)* *and*
2. $\lim\limits_{n\to\infty} \phi_n = \phi \implies \lim\limits_{n\to\infty} \mathsf{d}\left[\phi_n\right] = \mathsf{d}[\phi] \qquad \forall \phi_1,\phi_2 \in \mathcal{S}$ *(CONTINUOUS)*

Definition B.18. [36] *Let $\mathcal{S}$ be the* SCHWARTZ CLASS *of functions (Definition B.16).*
DEF
*Two tempered distributions d_1 and d_2 are **equal** if*
$$\mathsf{d}\left[\phi_1\right] = \mathsf{d}\left[\phi_2\right] \qquad \forall \phi_1, \phi_2 \in \mathcal{S}$$

Theorem B.16 (next) demonstrates that all continuous and what we might call "well behaved" functions generate a tempered distribution.

Theorem B.16. [37] *Let f be a function in $\mathbb{C}^{\mathbb{R}}$. Let T_f be defined as*
$$\mathsf{T}_\mathsf{f}[\phi] \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\phi(x)\, \mathrm{d}x.$$
THM
$$\left.\begin{array}{lll} 1. & \mathsf{f} \text{ is CONTINUOUS} & \textit{and} \\ 2. & \exists n, M \text{ such that } |\mathsf{f}(x)| \le M(1+|x|)^n & \forall x \in \mathbb{R} \end{array}\right\} \implies \mathsf{T}_\mathsf{f}[\phi] \text{ is a tempered distribution.}$$

✎PROOF:

1. Proof that T_f is *linear*:

$$\begin{aligned}
\mathsf{T}_\mathsf{f}\left[\phi_1+\phi_2\right] &= \int_{\mathbb{R}} \mathsf{f}(x)\big(\phi_1(x)+\phi_2(x)\big)\, \mathrm{d}x && \text{by definition of } \mathsf{T}_\mathsf{f} \\
&= \int_{\mathbb{R}} \mathsf{f}(x)\phi_1(x)\, \mathrm{d}x + \int_{\mathbb{R}} \mathsf{f}(x)\phi_2(x)\, \mathrm{d}x && \text{by linearity of } \int \\
&= \mathsf{T}_\mathsf{f}\left[\phi_1\right] + \mathsf{T}_\mathsf{f}\left[\phi_2\right] && \text{by definition of } \mathsf{T}_\mathsf{f}
\end{aligned}$$

[33] quote: Duistermaat and Kolk (2010) page ix
image http://en.wikipedia.org/wiki/File:Giuseppe_Peano.jpg, public domain
[34] Vretblad (2003) page 200
[35] Vretblad (2003) pages 203–204 ⟨Definition 8.3⟩
[36] Vretblad (2003) page 206
[37] Vretblad (2003) page 204

2. Proof that $\mathsf{T_f}$ is *cotinuous*:

$$\begin{aligned}
\lim_{n\to\infty} \left|\mathsf{T_f}[\phi_n] - \mathsf{T_f}[\phi]\right| &= \lim_{n\to\infty} \left|\int_{\mathbb{R}} \mathsf{f}(x)\phi_n(x)\,\mathsf{d}x - \int_{\mathbb{R}} \mathsf{f}(x)\phi(x)\,\mathsf{d}x\right| && \text{by definition of } \mathsf{T_f}\\
&= \lim_{n\to\infty} \left|\int_{\mathbb{R}} \mathsf{f}(x)\big(\phi_n(x) - \phi(x)\,\mathsf{d}x\big)\right| && \text{by linearity of } \int\\
&\le \lim_{n\to\infty} \int_{\mathbb{R}} M(1+|x|)^m \left|\phi_n(x) - \phi(x)\right|\,\mathsf{d}x\\
&= \int_{\mathbb{R}} M(1+|x|)^{m+2} \left|\phi_n(x) - \phi(x)\right| \frac{1}{(1+|x|)^2}\,\mathsf{d}x\\
&\le \lim_{n\to\infty} \max_x \left\{M(1+|x|)^{m+2}\left|\phi_n(x) - \phi(x)\right|\right\} \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\,\mathsf{d}x\\
&= 0
\end{aligned}$$

✏

Definition B.19. [38]

DEF *The **Dirac delta distribution** $\delta \in \mathbb{C}^{\mathbb{R}}$ is defined as*
$$\delta[\phi] \triangleq \phi(0)$$

One could argue that a tempered distribution d behaves *as if* it satisfies the following relation:
$$\mathsf{d}[\phi] \backsimeq \int_{\mathbb{R}} \mathsf{d}(x)\phi(x)\,\mathsf{d}x.$$
This is not technically correct because in general d is not a function that can be evaluated at a given point x (and hence the here undefined relation "$\backsimeq$"). But despite this failure, the notation is still very useful in that distributions do behave "as if" they are defined by the above integral relation.

Using this notation, the Dirac delta distribution looks likes this:
$$\delta[\phi] \triangleq \phi(0) \backsimeq \int_{\mathbb{R}} \delta(x)\phi(x)\,\mathsf{d}x$$

We could also define another "scaled" and "translated" distribution δ_{ab} such that
$$\delta_{ab}[\phi] \triangleq b\phi(ab) \backsimeq \int_{\mathbb{R}} \delta\left(\frac{x}{b} - a\right)\phi(x)\,\mathsf{d}x$$
because

$$\begin{aligned}
\int_{\mathbb{R}} \delta\left(\frac{x}{b} - a\right)\phi(x)\,\mathsf{d}x &= \int_{\mathbb{R}} \delta(u-a)\phi(ub)b\,\mathsf{d}u && \text{where } u = \frac{x}{b}\\
&= b\int_{\mathbb{R}} \delta(u-a)\phi(ub)\,\mathsf{d}u\\
&= b\phi(ab)
\end{aligned}$$

B.4 Literature

Literature survey:

[38] Vretblad (2003) page 205 ⟨Example 8.13⟩, Friedlander and Joshi (1998) page 8

1. Reference books:
   Maddux (2006)
   Suppes (1972) ⟨0486616304⟩Chapter 3: *Relations and Functions*
   Kelley (1955) pages 6–13

2. Pioneering papers on relations:
   de Morgan (1864a)
      de Morgan (1864b)
   Peirce (1883a)
      Peirce (1883c)
      Peirce (1883b)
   Schröder (1895)

3. Axiomization of calculus of relations:
   Tarski (1941)

4. Historically oriented presentations:
   Maddux (1991)
   Pratt (1992) pages 248–254

5. Theory of Distributions
   Vretblad (2003)
   Hömander (2003) ⟨Referenced by Vretblad(2003) as a standard work.⟩
   Knapp (2005)

6. Miscellaneous:
   Peirce (1870a)
      Peirce (1870b)
      Peirce (1870c)

APPENDIX C

ORDER

Equivalence relations require *symmetry* ($x \asymp y \iff y \asymp x$). However another very important type of relation, the *order relation*, actually requires *anti-symmetry*. This chapter presents some useful structures regarding order relations. Ordering relations on a set allow us to *compare* some pairs of elements in a set and determine whether or not one element is *less than* another. In this case, we say that those two elements are *comparable*; otherwise, they are *incomparable*. A set together with an order relation is called an *ordered set*, a *partially ordered set*, or a *poset* (Definition C.2 page 102).

C.1 Preordered sets

Definition C.1. [1] *Let X be a set.*

DEF

A relation $\sqsubseteq$ is a ***preorder relation*** *on X if*

1. $x \sqsubseteq x \qquad \forall x \in X \qquad$ (REFLEXIVE) *and*
2. $x \sqsubseteq y$ *and* $y \sqsubseteq z \implies x \sqsubseteq z \qquad \forall x,y,z \in X \qquad$ (TRANSITIVE)

A ***preordered set*** *is the pair* $(X, \sqsubseteq)$.

Example C.1. [2]

EX

$\sqsubseteq$ is a *preorder relation* on the set of *positive integers* $\mathbb{N}$ if

$$n \sqsubseteq m \iff (p \text{ is a prime factor of } n \implies p \text{ is a prime factor of } m)$$

[1] Schröder (2003) page 115, Brown and Watson (1991) page 317

[2] Shen and Vereshchagin (2002) page 43

C.2 Order relations

Definition C.2. [3] *Let X be a set. Let 2^{XX} be the set of all relations on X.*

DEF

*A relation $\le$ is an **order relation** in 2^{XX} if*

1. $x \le x$ $\quad \forall x \in X$ (REFLEXIVE) and — preorder
2. $x \le y$ and $y \le z \implies x \le z$ $\quad \forall x,y,z \in X$ (TRANSITIVE) and — preorder
3. $x \le y$ and $y \le x \implies x = y$ $\quad \forall x,y \in X$ (ANTI-SYMMETRIC)

*An **ordered set** is the pair $(X, \le)$. The set X is called the **base set** of $(X, \le)$. If $x \le y$ or $y \le x$, then elements x and y are said to be **comparable**, denoted $x \sim y$. Otherwise they are **incomparable**, denoted $x \| y$. The relation $\lneq$ is the relation $\le \setminus =$ ("less than but not equal to"), where $\setminus$ is the SET DIFFERENCE operator, and $=$ is the equality relation. An order relation is also called a **partial order relation**. An ordered set is also called a **partially ordered set** or **poset**.*

The familiar relations $\ge$, $<$, and $>$ (next) can be defined in terms of the order relation $\le$ (Definition C.2—previous).

Definition C.3. [4] *Let $(X, \le)$ be an ordered set.*

DEF

The relations $\ge$, $<$, $> \in 2^{XX}$ are defined as follows:

$$x \ge y \overset{\text{def}}{\iff} y \le x \qquad \forall x,y \in X$$
$$x \lneq y \overset{\text{def}}{\iff} x \le y \text{ and } x \ne y \qquad \forall x,y \in X$$
$$x \gneq y \overset{\text{def}}{\iff} x \ge y \text{ and } x \ne y \qquad \forall x,y \in X$$

*The relation $\ge$ is called the **dual** of $\le$.*

Theorem C.1. [5] *Let X be a set.*

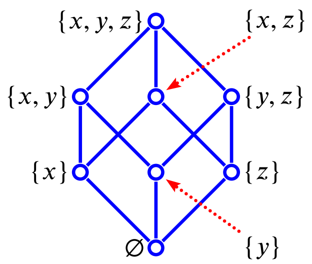
THM

$(X, \le)$ *is an ordered set* $\iff$ $(X, \ge)$ *is an ordered set*

Example C.2.

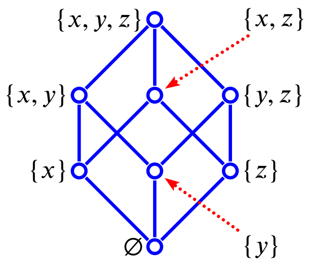
EX

order relation		dual order relation	
$\le$	(integer less than or equal to)	$\ge$	(integer greater than or equal to)
$\subseteq$	(subset)	$\supseteq$	(super set)
$\mid$	(divides)		(divided by)
$\implies$	(implies)	$\impliedby$	(implied by)



Example C.3. The Hasse diagram to the left illustrates the ordered set $\left(2^{\{x,y,z\}}, \subseteq\right)$ and the Hasse diagram to the right illustrates its dual $\left(2^{\{x,y,z\}}, \supseteq\right)$.

[3] MacLane and Birkhoff (1999) page 470, Beran (1985) page 1, Korselt (1894) page 156 ⟨I, II, (1)⟩, Dedekind (1900) page 373 ⟨I–III⟩
[4] Peirce (1880b) page 2
[5] Grätzer (1998) page 3



C.3 Linearly ordered sets

In an ordered set we can say that some element is less than or equal to some other element. That is, we can say that these two elements are *comparable*—we can *compare* them to see which one is lesser or equal to the other. But it is very possible that there are two elements that are not comparable, or *incomparable*. That is, we cannot say that one element is less than the other—it is simply not possible to compare them because their ordered pair is not an element of the order relation.

For example, in the ordered set $(2^{\{x,y,z\}}, \subseteq)$ of Example C.9, we can say that $\{x\} \subseteq \{x,z\}$ (we can compare these two sets with respect to the order relation $\subseteq$), but we cannot say $\{y\} \subseteq \{x,z\}$, nor can we say $\{x,z\} \subseteq \{y\}$. Rather, these two elements $\{y\}$ and $\{x,z\}$ are simply *incomparable*.

However, there are some ordered sets in which every element is comparable with every other element; and in this special case we say that this ordered set is a *totally ordered* set or is *linearly ordered* (next definition).

Definition C.4. [6]

DEF *A relation $\le$ is a **linear order relation** on X if*

1. $\le$ *is an* ORDER RELATION (*Definition C.2 page 102*) *and*
2. $x \le y$ or $y \le x \quad \forall x,y \in X$ (COMPARABLE).

*A **linearly ordered set** is the pair $(X, \le)$.*
*A linearly ordered set is also called a **totally ordered set**, a **fully ordered set**, and a **chain**.*

Definition C.5 (poset product). [7]

DEF *The **product** $P \times Q$ of ordered pairs $P \triangleq (X, \precsim)$ and $Q \triangleq (Y, \trianglelefteq)$ is the ordered pair $(X \times Y, \le)$ where*

$$(x_1, y_1) \le (x_2, y_2) \quad \overset{\text{def}}{\iff} \quad x_1 \precsim x_2 \text{ and } y_1 \trianglelefteq y_2 \qquad \forall x_1,x_2 \in X;\ y_1,y_2 \in Y$$

C.4 Representation

Definition C.6. [8]

DEF *y **covers** x in the ordered set $(X, \le)$ if*

1. $x \le y$ (*y is greater than x*) *and*
2. $(x \le z \le y) \implies (z = x \text{ or } z = y)$ (*there is no element between x and y*).

The case in which y covers x is denoted
$x \prec y$.

[6] MacLane and Birkhoff (1999) page 470, Ore (1935) page 410
[7] Birkhoff (1948) page 7, MacLane and Birkhoff (1967) page 489
[8] Birkhoff (1933a) page 445

Example C.4. Let $(\{x, y, z\}, \leq)$ be an ordered set with cover relation $\prec$.

EX $\{x < y < z\} \implies \left\{\begin{array}{lll} y & \text{covers} & x \\ z & \text{covers} & y \\ z & \text{does } \textbf{not} \text{ cover} & x \end{array}\right\}$

An ordered set can be represented in four ways:

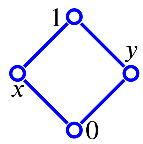
1. Hasse diagram
2. tables
3. set of ordered pairs of order relations
4. set of ordered pairs of cover relations

Definition C.7. *Let $(X, \leq)$ be an ordered pair.*

DEF *A diagram is a* ***Hasse diagram*** *of $(X, \leq)$ if it satisfies the following criteria:*
- *Each element in X is represented by a dot or small circle.*
- *For each $x, y \in X$, if $x \prec y$, then y appears at a higher position than x and a line connects x and y.*

Example C.5. Here are three ways of representing the ordered set $\left(2^{\{x,y\}}, \subseteq\right)$;

1. **Hasse diagrams**: If two elements are comparable, then the lesser of the two is drawn lower on the page than the other with a line connecting them.

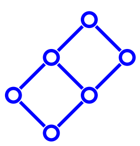


2. Sets of ordered pairs specifying *order relations* (Definition C.2 page 102):

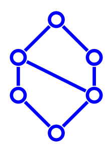
$$\subseteq = \left\{\begin{array}{ccccc} (\varnothing, \varnothing), & (\{x\}, \{x\}), & (\{y\}, \{y\}), & (\{x,y\}, \{x,y\}), & \\ (\varnothing, \{x\}), & (\varnothing, \{y\}), & (\varnothing, \{x,y\}), & (\{x\}, \{x,y\}), & (\{y\}, \{x,y\}) \end{array}\right\}$$

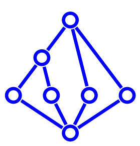
3. Sets of ordered pairs specifying *covering relations*:

$$\prec = \{\ (\varnothing, \{x\}),\ \ (\varnothing, \{y\}),\ \ (\{x\}, \{x,y\}), (\{y\}, \{x,y\})\ \}$$

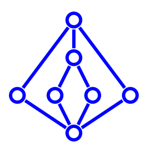


Example C.6. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.

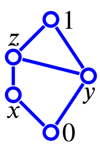




Example C.7. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.

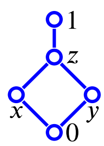


Example C.8. The Hasse diagrams to the left and right represent *equivalent* ordered sets.



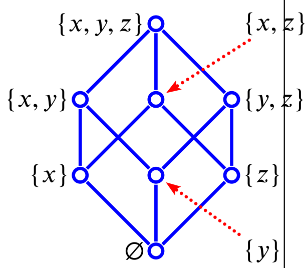
In particular, the line extending from 1 to y in the diagram to the left is redundant because other lines already indicate that $z \leq 1$ and $y \leq z$; and thus by the *transitive* property (Definition C.2 page 102), these two relations imply $1 \leq y$. A more concise explanation is that both have the same convering relation:

$$\prec = \{(z,1), (x,z), (0,x), (y,z), (0,y)\}$$

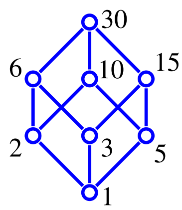


C.5 Examples

Examples of order relations include the following:



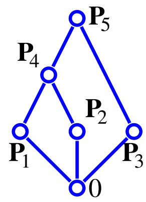
Example C.9 (Set inclusion order relation). [9]Let X be a set, 2^X the power set of X, and $\subseteq$ the set inclusion relation. Then, $\subseteq$ is an *order relation* on the set 2^X and the pair $(2^X, \subseteq)$ is an ordered set. The ordered set $(2^{\{x,y,z\}}, \subseteq)$ is illustrated to the right by its *Hasse diagram*.



Example C.10 (Integer divides order relation). [10]Let $|$ be the "divides" relation on the set $\mathbb{N}$ of positive integers such that $n|m$ represents m divides n. Then $|$ is an *order relation* on $\mathbb{N}$ and the pair $(\mathbb{N}, |)$ is an *ordered set*. The ordered set $(\{n \in \mathbb{N} | n|2 \text{ or } n|3 \text{ or } n|5\}, |)$ is illustrated by a *Hasse diagram* to the right.

Example C.11 (Operator order relation). [11] Let $\boldsymbol{X}$ be an inner-product space. We can define the order relation $\lessapprox$ on the linear operators $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3 \ldots \in X^X$ as follows:

$$\mathbf{L}_1 \lessapprox \mathbf{L}_2 \quad \overset{\text{def}}{\iff} \quad \langle \mathbf{L}_2 \boldsymbol{x} - \mathbf{L}_1 \boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X}$$



Example C.12 (Projection operator order relation). [12]Let $(\!(\boldsymbol{V}_n)\!)$ be a sequence of subspaces in a Hilbert space $\boldsymbol{X}$. We can define a projection operator $\mathbf{P}_n$ for every subspace $\boldsymbol{V}_n \subseteq \boldsymbol{X}$ in a subspace lattice such that

$$\boldsymbol{V}_n = \mathbf{P}_n \boldsymbol{X} \qquad \forall n \in \mathbb{Z}.$$

Each projection operator $\mathbf{P}_n$ in the lattice "projects" the range space $\boldsymbol{X}$ onto a subspace $\boldsymbol{V}_n$. We can define an order relation on the projection operators as follows:

$$\mathbf{P}_1 \leq \mathbf{P}_2 \quad \overset{\text{def}}{\iff} \quad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1$$

---

[9] Menini and Oystaeyen (2004) pages 56–57

[10] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩

[11] Michel and Herget (1993) page 429, Pedersen (2000) page 87

[12] Isham (1999) pages 21–22, Dunford and Schwartz (1957) page 481, ? page 72

Example C.13 (Integer order relation). Let $\le$ be the standard order relation on the set of integers $\mathbb{Z}$. Then the ordered pair $(\mathbb{Z}, \le)$ is a totally ordered set. The totally ordered set $(\{1, 2, 3, 4\}, \le)$ is illustrated to the right. Other familiar examples of totally ordered sets include the pair $(\mathbb{Q}, \le)$ (where $\mathbb{Q}$ is the set of rational numbers) and $(\mathbb{R}, \le)$ (where $\mathbb{R}$ is the set of real numbers).

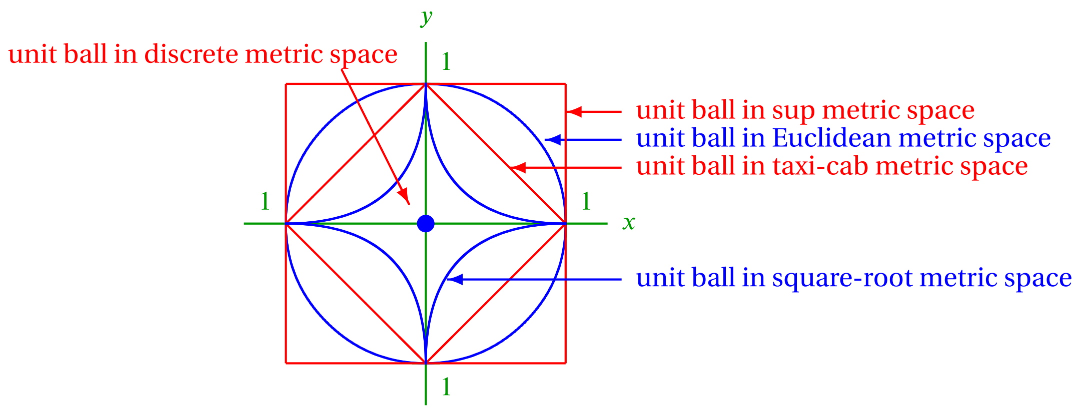
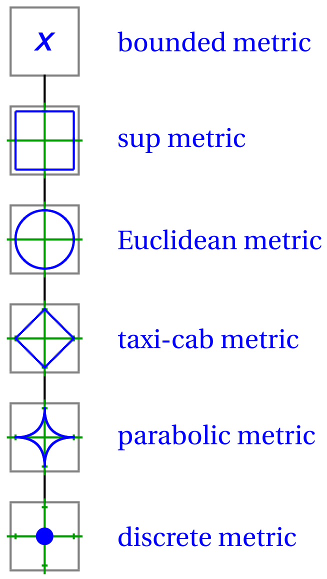


Figure C.1: Balls on the set $\mathbb{R}^2$ using different metrics



Example C.14 (Metric order relation). [13]Let d_n be a metric on the set X and B_n be the unit ball centered at "0" in the metric space (X, d_n). Define an order relation $\le$ on the set of metric spaces $\{(X, \mathsf{d}_n) \mid n = 1, 2, \ldots\}$ such that

$$(X, \mathsf{d}_n) \le (X, \mathsf{d}_m) \qquad \Longleftrightarrow \qquad \mathsf{B}_n \subseteq \mathsf{B}_m.$$

The the tuple $(\{(X, \mathsf{d}_n) \mid n = 1, 2, \ldots\}, \le)$ is an ordered set. The ordered set of several common metric spaces is a *totally ordered* set, as illustrated to the right and with associated unit balls illustrated in Figure C.1 (page 106).

Example C.15 (Coordinatewise order relation). [14] Let $(X, \le)$ be an ordered set. Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

E
X

The **coordinatewise order relation** $\precsim$ on the Cartesian product X^n is defined for all $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \precsim \boldsymbol{y} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \{x_1 \le y_1 \text{ and } x_2 \le y_2 \text{ and } \ldots \text{ and } x_n \le y_n\}$$

Example C.16 (Lexicographical order relation). [15] Let $(X, \le)$ be an ordered set. Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

[13] Michel and Herget (1993) page 354, Giles (1987) page 29
[14] Shen and Vereshchagin (2002) page 43
[15] Shen and Vereshchagin (2002) page 44, Halmos (1960) page 58, Hausdorff (1937) page 54

EX

The **lexicographical order relation** $\precsim$ on the Cartesian product X^n is defined forall $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \precsim \boldsymbol{y} \overset{\text{def}}{\iff} \left\{ \begin{array}{llll} (x_1 < y_1 & &) & \text{or} \\ (x_2 < y_2 & \text{and} \quad x_1 = y_1 &) & \text{or} \\ (x_3 < y_3 & \text{and} \quad (x_1, x_2) = (y_1, y_2) &) & \text{or} \\ (\ldots & \ldots \quad \ldots &) & \text{or} \\ (x_{n-1} < y_{n-1} & \text{and} \quad (x_1, x_2, \ldots, x_{n-2}) = (y_1, y_2, \ldots, y_{n-2}) &) & \text{or} \\ (x_n \le y_n & \text{and} \quad (x_1, x_2, \ldots, x_{n-1}) = (y_1, y_2, \ldots, y_{n-1}) &) & \end{array} \right\}$$

The lexicographical order relation is also called the **dictionary order relation** or **alphabetic order relation**.

Definition C.8.

DEF

*An ordered set is **labeled** if the labels on the elements are significant.*
*An ordered set is **unlabeled** if the labels on the elements are not significant.*

Proposition C.1. [16] *Let X_n be a finite set with order $n = |X_n|$. Let P_n be the number of labeled ordered sets on X_n and p_n the number of unlabeled ordered sets.*

PRP

n	0	1	2	3	4	5	6	7	8	9
P_n	1	1	3	19	219	4231	130,023	6,129,859	431,723,379	44,511,042,511
p_n	1	1	2	5	16	63	318	2045	16,999	183,231

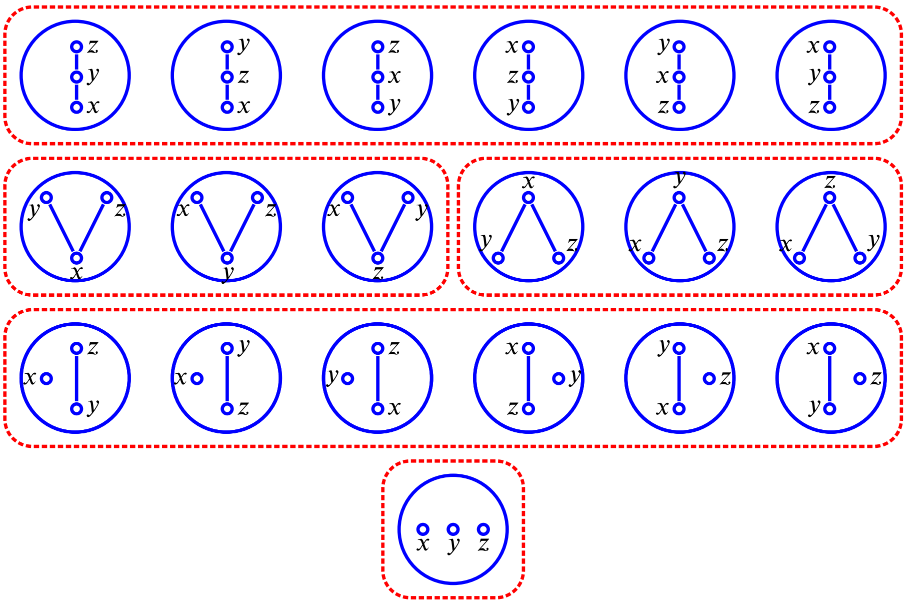


Figure C.2: All possible orderings of the set $\{x, y, z\}$ (Example C.17 page 107).

Example C.17. Proposition C.1 (page 107) indicates that there are exactly 19 labeled order relations on the set $\{x, y, z\}$ and 5 unlabeled order relations.

EX

The 19 labeled order relations on $\{x, y, z,\}$ are represented here using three methods:

1. Hasse diagrams: Figure C.2 page 107
2. order relations: Table C.2 page 108
3. covering relations: Table C.3 page 108

In each of these three methods, the 19 *labeled* order relations are arranged into 5 groups, each group representing one of the 5 *unlabeled* order relations.

[16] Sloane (2014) ⟨http://oeis.org/A001035⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Comtet (1974) page 60, Brinkmann and McKay (2002)

labeled order relations on $\{x, y, z\}$		
$\le_1$	$=$	$\{ (x,x), (y,y), (z,z) \}$
$\le_2$	$=$	$\{ (x,x), (y,y), (z,z), (y,z) \}$
$\le_3$	$=$	$\{ (x,x), (y,y), (z,z), (z,y) \}$
$\le_4$	$=$	$\{ (x,x), (y,y), (z,z), (x,z) \}$
$\le_5$	$=$	$\{ (x,x), (y,y), (z,z), (z,x) \}$
$\le_6$	$=$	$\{ (x,x), (y,y), (z,z), (x,y) \}$
$\le_7$	$=$	$\{ (x,x), (y,y), (z,z), (y,x) \}$
$\le_8$	$=$	$\{ (x,x), (y,y), (z,z), (x,y), (x,z) \}$
$\le_9$	$=$	$\{ (x,x), (y,y), (z,z), (x,y), (y,z) \}$
$\le_{10}$	$=$	$\{ (x,x), (y,y), (z,z), (z,x), (z,y) \}$
$\le_{11}$	$=$	$\{ (x,x), (y,y), (z,z), (y,x), (z,x) \}$
$\le_{12}$	$=$	$\{ (x,x), (y,y), (z,z), (x,y), (z,y) \}$
$\le_{13}$	$=$	$\{ (x,x), (y,y), (z,z), (x,z), (y,z) \}$
$\le_{14}$	$=$	$\{ (x,x), (y,y), (z,z), (x,y), (y,z), (x,z) \}$
$\le_{15}$	$=$	$\{ (x,x), (y,y), (z,z), (x,z), (x,y), (z,y) \}$
$\le_{16}$	$=$	$\{ (x,x), (y,y), (z,z), (y,x), (y,z), (x,z) \}$
$\le_{17}$	$=$	$\{ (x,x), (y,y), (z,z), (y,z), (y,x), (z,x) \}$
$\le_{18}$	$=$	$\{ (x,x), (y,y), (z,z), (z,x), (z,y), (x,y) \}$
$\le_{19}$	$=$	$\{ (x,x), (y,y), (z,z), (z,y), (z,x), (y,x) \}$

Table C.2: labeled order relations on $\{x, y, z\}$

labeled cover relations on $\{x, y, z\}$					
$\prec_1$	$=$	$\varnothing$	$\prec_{11}$	$=$	$\{ (y,x), (z,x) \}$
$\prec_2$	$=$	$\{ (y,z) \}$	$\prec_{12}$	$=$	$\{ (x,y), (z,y) \}$
$\prec_3$	$=$	$\{ (z,y) \}$	$\prec_{13}$	$=$	$\{ (x,z), (y,z) \}$
$\prec_4$	$=$	$\{ (x,z) \}$	$\prec_{14}$	$=$	$\{ (x,y), (y,z) \}$
$\prec_5$	$=$	$\{ (z,x) \}$	$\prec_{15}$	$=$	$\{ (x,z), (x,y) \}$
$\prec_6$	$=$	$\{ (x,y) \}$	$\prec_{16}$	$=$	$\{ (y,x), (y,z) \}$
$\prec_7$	$=$	$\{ (y,x) \}$	$\prec_{17}$	$=$	$\{ (y,z), (y,x) \}$
$\prec_8$	$=$	$\{ (x,y), (x,z) \}$	$\prec_{18}$	$=$	$\{ (z,x), (z,y) \}$
$\prec_9$	$=$	$\{ (x,y), (y,z) \}$	$\prec_{19}$	$=$	$\{ (z,y), (z,x) \}$
$\prec_{10}$	$=$	$\{ (z,x), (z,y) \}$			

Table C.3: labeled cover relations on $\{x, y, z\}$

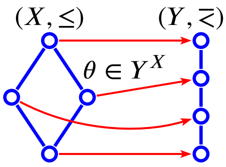
C.6 Functions on ordered sets

Definition C.9. [17] *Let $(X, \leq)$ and $(Y, \precsim)$ be ordered sets.*

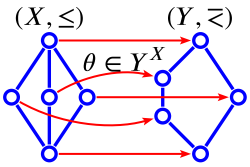
DEF: *A function $\theta \in Y^X$ is **order preserving** with respect to $\leq$ and $\precsim$ if*

$$x \leq y \quad \Longrightarrow \quad \theta(x) \precsim \theta(y) \qquad \forall x, y \in X.$$

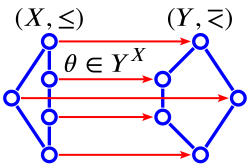
Example C.18. [18] In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} is *not* order preserving. This example also illustrates the fact that that order preserving does not imply *isomorphic*.



Example C.19. In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} is *not* order preserving. Like Example C.18 (page 109), this example also illustrates the fact that that order preserving does not imply *isomorphic*.



Example C.20. In the diagram to the right, the function $\theta \in Y^X$ is *order preserving* with respect to $\leq$ and $\precsim$. Note that θ^{-1} *is also* order preserving. In this case, θ is an *isomorphism* and the ordered sets $(X, \leq)$ and $(Y, \precsim)$ are *isomorphic*.



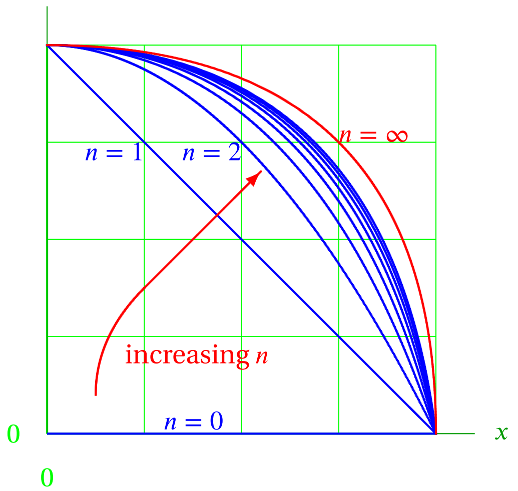
Example C.21. [19]

EX: The function $\mathsf{f}(x) \triangleq \dfrac{x}{1-x^2}$ in $\mathbb{R}^{(-1:1)}$ is *bijective* and *order preserving*.

Theorem C.2 (Pointwise ordering relation). [20] *Let X be a set, $(Y, \leq)$ an ordered set, and $\mathsf{f}, \mathsf{g} \in Y^X$.*

THM: $\mathsf{f}(x) \leq \mathsf{g}(x) \forall x \in X \quad \Longrightarrow \quad \left(Y^X, \precsim\right)$ *is an ordered set.*
In this case we say f *is "dominated by"* g *in* X*, or we say* g *"dominates"* f *in* X.

Example C.22 (Pointwise ordering relation). [21] Let $\mathsf{f} \precsim \mathsf{g}$ represent that $\mathsf{f}(x) \leq \mathsf{g}(x)$ for all $0 \leq x \leq 1$ (we say f is "dominated by" g in the region $[0,1]$, or we say g "dominates" f in the region $[0,1]$). The pair $\left(\left\{\mathsf{f}_n(x) = 1 - x^n | n \in \mathbb{N}\right\}, \precsim\right)$ is a totally ordered set.



[17] Burris and Sankappanavar (2000) page 10
[18] Burris and Sankappanavar (2000) page 10
[19] Munkres (2000) page 25 ⟨Example 1§3.9⟩
[20] Shen and Vereshchagin (2002) page 43, Giles (2000) page 252
[21] Shen and Vereshchagin (2002) page 43, Giles (2000) page 252, Aliprantis and Burkinshaw (2006) page 2

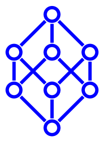
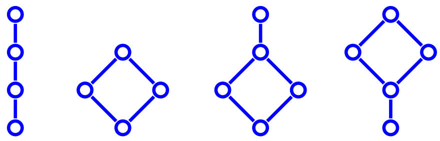
C.7 Decomposition

C.7.1 Subposets

Definition C.10. [22]

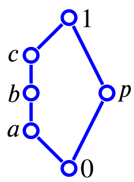
DEF *The tupple $(Y, \precsim)$ is a **subposet** of the ordered set $(X, \leq)$ if*
1. $Y \subseteq X$ *(Y is a subset of X)* *and*
2. $\precsim = \leq \cap Y^2$ *($\precsim$ is the relation $\leq$ restricted to $Y \times Y$)*

Example C.23.

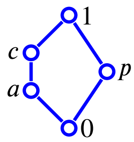
Subposets of  include 

Example C.24. Let

$$(X, \leq) \triangleq \Big(\{0, a, b, c, p, 1\}, \quad \Big\{ (0,0), (a,a), (b,b), (c,c), (p,p), (1,1), (0,a), (0,b), (0,c), (0,p), (0,1), (a,b), (a,c), (a,1), (p,1), (b,c), (b,1), (c,1), (p,1) \Big\}\Big)$$



$$(Y, \precsim) \triangleq \Big(\{0, a, c, p, 1\}, \quad \Big\{ (0,0), (a,a), (c,c), (p,p), (1,1), (0,a), (0,c), (0,p), (0,1), (a,c), (a,1), (p,1), (c,1), (p,1) \Big\}\Big).$$

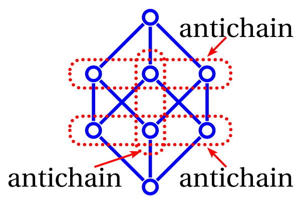


Then $(Y, \precsim)$ is a subposet of $(X, \leq)$ because $Y \subseteq X$ and $\precsim = (\leq \cap Y^2)$.

A *chain* is an ordered set in which every pair of elements is *comparable* (Definition C.4 page 103). An *antichain* is just the opposite—it is an ordered set in which *no* pair of elements is comparable (next definition).

Definition C.11. [23]

DEF *The subposet $(A, \leq)$ in the ordered set $(X, \leq)$ is an **antichain** if*
$a \| b \qquad \forall a, b \in A$
(all elements in A are INCOMPARABLE*).*



Definition C.12. [24]

DEF
- *The **length** of a chain $(C, \leq)$ equals $|C| - 1$.*
- *The **length** of a poset $(X, \leq)$ is the length of the longest chain in the ordered set.*
- *The **width** of a poset $(X, \leq)$ is number of elements in the largest antichain in the ordered set.*

Theorem C.3 (Dilworth's theorem). [25] *Let $(X, \leq)$ be an ordered set with width n.*

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[22] Grätzer (2003) page 2
[23] Grätzer (2003) page 2
[24] Grätzer (2003) page 2, Birkhoff (1967) page 5
[25] Dilworth (1950a) page 161, Dilworth (1950b), Farley (1997) page 4

THM $\left\{\begin{array}{l}\text{WIDTH } n \text{ of } (X, \leq) \\ \text{is FINITE}\end{array}\right\} \implies \left\{\begin{array}{ll}1. & \text{there exists a PARTITION of } (X, \leq) \text{ into } n \text{ chains} \quad \text{and} \\ 2. & \text{there does not exist any PARTITION} \\ & \text{of } (X, \leq) \text{ into less than } n \text{ chains}\end{array}\right\}$

C.7.2 Operations on posets

Definition C.13. [26] *Let X and Y be disjoint sets. Let $\mathbf{P} \triangleq (X, \precsim)$ and $\mathbf{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **direct sum** of $\mathbf{P}$ and $\mathbf{Q}$ is defined as*

$$\mathbf{P} + \mathbf{Q} \triangleq (X \cup Y, \leq)$$

where $x \leq y$ if

1. $x, y \in X$ and $x \precsim y$ or
2. $x, y \in Y$ and $x \trianglelefteq y$

*The direct sum operation is also called the **disjoint union**. The notation $n\mathbf{P}$ is defined as*

$$n\mathbf{P} \triangleq \underbrace{\mathbf{P} + \mathbf{P} + \cdots + \mathbf{P}}_{n-1 \text{ "+" operations}}.$$

Definition C.14. [27] *Let X and Y be disjoint sets. Let $\mathbf{P} \triangleq (X, \precsim)$ and $\mathbf{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **direct product** of $\mathbf{P}$ and $\mathbf{Q}$ is defined as*

$$\mathbf{P} \times \mathbf{Q} \triangleq (X \times Y, \leq)$$

where $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \precsim x_2$ and $y_1 \precsim y_2$.

*The direct product operation is also called the **cartesian product**. The order relation $\leq$ is called a **coordinate wise order relation**. The notation $\mathbf{P}^n$ is defined as*

$$\mathbf{P}^n \triangleq \underbrace{\mathbf{P} \times \mathbf{P} \times \cdots \times \mathbf{P}}_{n-1 \text{ "}\times\text{" operations}}.$$

Definition C.15. [28] *Let X and Y be disjoint sets. Let $\mathbf{P} \triangleq (X, \precsim)$ and $\mathbf{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **ordinal sum** of $\mathbf{P}$ and $\mathbf{Q}$ is defined as*

$$\mathbf{P} \oplus \mathbf{Q} \triangleq (X \cup Y, \leq)$$

where $x \leq y$ if

1. $x, y \in X$ and $x \precsim y$ or
2. $x, y \in Y$ and $x \trianglelefteq y$ or
3. $x \in X$ and $y \in Y$.

Definition C.16. [29] *Let X and Y be disjoint sets. Let $\mathbf{P} \triangleq (X, \precsim)$ and $\mathbf{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF *The **ordinal product** of $\mathbf{P}$ and $\mathbf{Q}$ is defined as*

$$\mathbf{P} \otimes \mathbf{Q} \triangleq (X \times Y, \leq)$$

where $(x_1, y_1) \leq (x_2, y_2)$ if

1. $x_1 \neq x_2$ and $x_1 \precsim x_2$ or
2. $x_1 = x_2$ and $y_1 \trianglelefteq y_2$

*The order relation $\leq$ is called a **lexicographical** order relation, **dictionary order relation**, or **alphabetic order relation**.*

[26] Stanley (1997) page 100
[27] Stanley (1997) pages 100–101, Shen and Vereshchagin (2002) page 43
[28] Stanley (1997) page 100
[29] Stanley (1997) page 101, Shen and Vereshchagin (2002) page 44, Halmos (1960) page 58, Hausdorff (1937) page 54

Definition C.17. [30] *Let $\boldsymbol{P} \triangleq (X, \leq)$ be an ordered set. Let $\geq$ be the dual order relation of $\leq$.*

DEF: *The **dual** of $\boldsymbol{P}$ is defined as*
$$\boldsymbol{P}^* \triangleq (X, \geq)$$

Definition C.18. [31] *Let X and Y be disjoint sets. Let $\boldsymbol{P} \triangleq (X, \precsim)$ and $\boldsymbol{Q} \triangleq (Y, \trianglelefteq)$ be ordered sets on X and Y.*

DEF: *The **ordinal product** of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as*
$$\boldsymbol{Q}^{\boldsymbol{P}} \triangleq \left(\left\{ \mathsf{f} \in Y^X | \mathsf{f} \text{ is ORDER PRESERVING} \right\}, \leq \right)$$
where $\mathsf{f} \leq \mathsf{g}$ iff $\mathsf{f}(x) \leq \mathsf{g}(x) \quad \forall x \in X$.
*The order relation $\leq$ is called a **pointwise order relation*** (Example C.22 page 109).

Theorem C.4 (cardinal arithmetic). [32] *Let $\boldsymbol{P} \triangleq (X, \leq)$ be an ordered set.*

THM:

1.	$\boldsymbol{P} + \boldsymbol{Q}$	$=$	$\boldsymbol{Q} + \boldsymbol{P}$	*commutative*
2.	$\boldsymbol{P} \times \boldsymbol{Q}$	$=$	$\boldsymbol{Q} \times \boldsymbol{P}$	*commutative*
3.	$(\boldsymbol{P} + \boldsymbol{Q}) + (\mathbb{R}, \leq)$	$=$	$\boldsymbol{P} + (\boldsymbol{Q} + (\mathbb{R}, \leq))$	*associative*
4.	$(\boldsymbol{P} \times \boldsymbol{Q}) \times (\mathbb{R}, \leq)$	$=$	$\boldsymbol{P} \times (\boldsymbol{Q} \times (\mathbb{R}, \leq))$	*associative*
5.	$\boldsymbol{P} \times (\boldsymbol{Q} + (\mathbb{R}, \leq))$	$=$	$(\boldsymbol{P} \times \boldsymbol{Q}) + (\boldsymbol{P} \times (\mathbb{R}, \leq))$	*distributive*
6.	$(\mathbb{R}, \leq)^{\boldsymbol{P}+\boldsymbol{Q}}$	$=$	$(\mathbb{R}, \leq)^{\boldsymbol{P}} \times (\mathbb{R}, \leq)^{\boldsymbol{Q}}$	
7.	$\left(\boldsymbol{P}^{\boldsymbol{Q}}\right)^{(\mathbb{R}, \leq)}$	$=$	$\boldsymbol{P}^{\boldsymbol{Q} \times (\mathbb{R}, \leq)}$	

C.7.3 Primitive subposets

Definition C.19.

DEF: *The ordered set $\boldsymbol{L}_1$ is defined as $(\{x\}, \leq)$, for some value x.*

The $\boldsymbol{L}_1$ ordered set is illustrated by the Hasse diagram to the right.

Definition C.20.

DEF: *The ordered set $\mathbb{2}$ is defined as $\mathbb{2} \triangleq \mathbb{1}^2$.*

The $\mathbb{2}$ ordered set is illustrated by the Hasse diagram to the right.

C.7.4 Decomposition examples

Example C.25. Figure C.3 (page 113) illustrates the four ordered set operations $+$, $\times$, $\oplus$, and $\otimes$.

Example C.26. [33]The ordered set $n\mathbb{1}$ is the *anti-chain* with n elements. The ordered set $4\mathbb{1}$ is illustrated to the right.

[30] Stanley (1997) page 101
[31] Stanley (1997) page 101
[32] Stanley (1997) page 102
[33] Stanley (1997) page 100

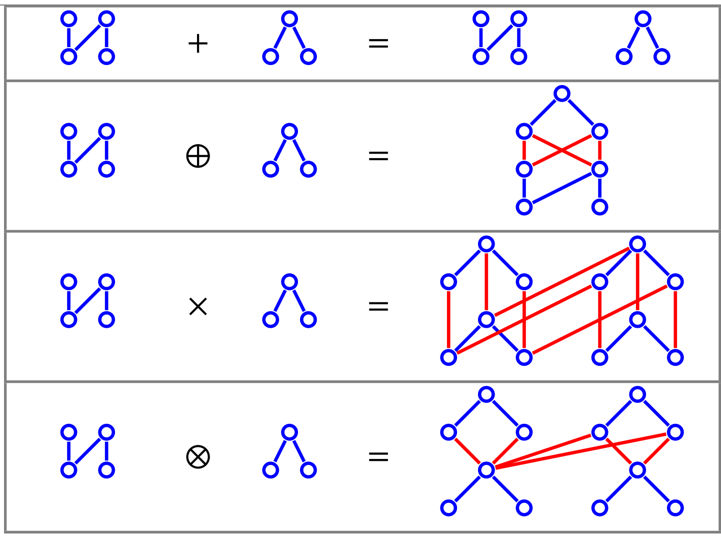
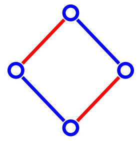
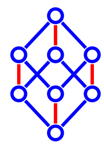


Figure C.3: Operations on ordered sets (Example C.25 page 112)

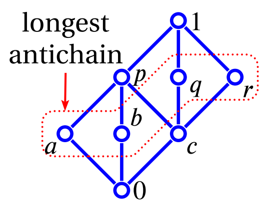
Example C.27. The ordered set $\mathbb{1}^n$ is the *chain* with n elements. The ordered set $\mathbb{1}^4$ is illustrated to the right.



Example C.28. The ordered set $\mathbb{2}^2$ is the 4 element *Boolean algebra* illustrated to the right.



Example C.29. The ordered set $\mathbb{2}^3$ is the 8 element *Boolean algebra* illustrated to the right.



Example C.30. [34]The longest *antichain* (Definition C.11 page 110) in the figure to the right has 4 elements giving this ordered set a *width* (Definition C.12 page 110) of 4. The longest chain also has 4 elements, giving the ordered set a *length* (Definition C.12 page 110) of 3. By *Dilworth's theorem* (Theorem C.3 page 110), the smallest *partition* consists of four *chains* (Definition C.4 page 103). One such *partition* is

$$\{\{0, a, p, 1\}, \{b\}, \{c, q\}, \{r\}\}.$$

C.8 Bounds on ordered sets

In an *ordered set* (Definition C.2 page 102), a pair of elements $\{x, y\}$ may not be *comparable*. Despite this, we may still be able to find elements that *are* comparable to both x and y and are "*greater*" than both of them. Such a greater element is called an *upper bound* of x and y. There may be many elements that are upper bounds of x and y. But if one of these upper bounds is comparable with and is smaller than all the other upper bounds, than this "smallest" of the "greater" elements is called the *least upper bound* (*lub*) of x and y, and is denoted $x \vee y$ (Definition C.21 page 114). Likewise,

[34] Farley (1997) page 4

we may also be able to find elements that are comparable to $\{x, y\}$ and are "*lesser*" than both of them. Such a lesser element is called a *lower bound* of x and y. If one of these lower bounds is comparable with and is larger than all the other lower bounds, than this "largest" of the "lesser" elements is called the *greatest lower bound* (*glb*) of $\{x, y\}$ and is denoted $x \wedge y$ (Definition C.22 page 114). If every pair of elements in an ordered set has both a least upper bound and a greatest lower bound in the ordered set, then that ordered set is a *lattice* (Definition D.3 page 117).

Definition C.21. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

*For any set $A \in 2^X$, c is an **upper bound** of A in $(X, \leq)$ if*

1. $x \leq c \quad \forall x \in A.$

*An element b is the **least upper bound**, or **lub**, of A in $(X, \leq)$ if*

2. *b and c are* UPPER BOUNDS *of A* $\implies b \leq c.$

*The least upper bound of the set A is denoted $\bigvee A$. It is also called the **supremum** of A, which is denoted $\sup A$. The **join** $x \vee y$ of x and y is defined as $x \vee y \triangleq \bigvee\{x, y\}$.*

Definition C.22. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

*For any set $A \in 2^X$, p is a **lower bound** of A in $(X, \leq)$ if*

1. $p \leq x \quad \forall x \in A.$

*An element a is the **greatest lower bound**, or **glb**, of A in $(X, \leq)$ if*

2. *a and p are* LOWER BOUNDS *of A* $\implies p \leq a.$

*The greatest lower bound of the set A is denoted $\bigwedge A$. It is also called the **infimum** of A, which is denoted $\inf A$. The **meet** $x \wedge y$ of x and y is defined as $x \wedge y \triangleq \bigwedge\{x, y\}$.*

Definition C.23 (least upper bound property). [35] *Let X be a set. Let $\sup A$ be the supremum (least upper bound) of a set A.*

DEF

*A set X satisfies the **least upper bound property** if*

$$\left.\begin{array}{lll} 1. & A \subseteq X & \text{and} \\ 2. & A \neq \varnothing & \text{and} \\ 3. & \exists b \in X \text{ such that } \forall a \in A,\ a \leq b & (A \text{ is bounded above in } X) \end{array}\right\} \implies \exists \sup A \in X$$

*A set X that satisfies the least upper bound property is also said to be **complete**.*

Proposition C.2. *Let $(X, \vee, \wedge; \leq)$ be an* ORDERED SET *(Definition C.2 page 102).*

PRP

$$x \leq y \iff \left\{\begin{array}{llll} 1. & x \wedge y = x & \text{and} \\ 2. & x \vee y = y \end{array}\right\} \quad \forall x,y \in X$$

Proposition C.3. *Let 2^X be the* POWER SET *of a set X.*

PRP

$$A \subseteq B \implies \left\{\begin{array}{llll} 1. & \bigvee A \leq \bigvee B & \text{and} \\ 2. & \bigwedge A \leq \bigwedge B \end{array}\right\} \quad \forall A,B \in 2^X$$

[35] Pugh (2002) page 13, Rudin (1976) page 4

APPENDIX D

LATTICES

D.1 Semi-lattices

Definition C.21 (page 114) defined the least upper bound $\vee$ of pairs of elements in terms of an ordering relation $\le$. However, the converse development is also possible— we can first define a binary operation $\circledvee$ with a handful of "least upper bound like properties", and then define an ordering relation $\lessapprox$ in terms of $\circledvee$ (Definition D.1 page 115). In fact, Theorem D.1 (page 115) shows that under Definition D.1, $(X, \lessapprox)$ *is* a partially ordered set and $\circledvee$ is a least upper bound on that ordered set.

The same development is performed with regards to a greatest lower bound $\circledwedge$ with the result that $(X, \lessapprox)$ *is* a partially ordered set and $\circledwedge$ is a greatest lower bound on that ordered set (Theorem D.2 page 116).

Definition D.1. [1] *Let* $\circledvee, \lessapprox \colon X^2 \to X$ *be binary operators on a set* X.

DEF *The algebraic structure* $(X, \lessapprox, \circledvee)$ *is a **join semilattice** if*

1.	$x \circledvee x = x$	$\forall x \in X$	(IDEMPOTENT)	*and*
2.	$x \circledvee y = y \circledvee x$	$\forall x,y \in X$	(COMMUTATIVE)	*and*
3.	$(x \circledvee y) \circledvee z = x \circledvee (y \circledvee z)$	$\forall x,y,z \in X$	(ASSOCIATIVE).	

Definition D.2. [2] *Let* $\circledwedge, \lessapprox \colon X^2 \to X$ *be binary operators on a set* X.

DEF *The algebraic structure* $(X, \lessapprox, \circledwedge)$ *is a **meet semilattice** if*

1.	$x \circledwedge x = x$	$\forall x \in X$	(IDEMPOTENT)	*and*
2.	$x \circledwedge y = y \circledwedge x$	$\forall x,y \in X$	(COMMUTATIVE)	*and*
3.	$(x \circledwedge y) \circledwedge z = x \circledwedge (y \circledwedge z)$	$\forall x,y,z \in X$	(ASSOCIATIVE).	

Theorem D.1. [3] *Let* $\circledvee, \lessapprox \colon X^2 \to X$ *be binary operators over a set* X.

THM $\left\{ (X, \lessapprox, \circledvee) \text{ is a JOIN SEMILATTICE} \right\} \implies \left\{ \begin{array}{lll} 1. & (X, \lessapprox) \text{ is a PARTIALLY ORDERED SET} & \text{and} \\ 2. & x \circledvee y \text{ is a LEAST UPPER BOUND of } x \text{ and } y & \forall x, y \in X. \end{array} \right\}$

✎PROOF: In order for $(X, \le)$ to be an ordered set, $\le$ must be, according to Definition C.2 (page 102), *reflexive*, *antisymmetric*, and *transitive*;

[1] MacLane and Birkhoff (1999) page 475, Birkhoff (1967) page 22
[2] MacLane and Birkhoff (1999) page 475
[3] MacLane and Birkhoff (1999) page 475

Proof that $\leq$ is reflexive:

$$\begin{aligned} & x = x \ovee x && \text{by idempotent hypothesis} \\ \iff & x \leq x && \text{by definition of } \leq \\ \implies & \leq \text{ is reflexive} \end{aligned}$$

Proof that $\leq$ is antisymmetric:

$$\begin{aligned} x \leq y \text{ and } y \leq x \iff & x \ovee y = y \text{ and } y \ovee x = x && \text{by definition of } \leq \\ \implies & x \ovee y = y \text{ and } x \ovee y = x && \text{by commutative hypothesis} \\ \implies & x = y \\ \implies & \leq \text{ is antisymmetric} \end{aligned}$$

Proof that $\leq$ is transitive:

$$\begin{aligned} x \leq y \text{ and } y \leq z \iff & x \ovee y = y \text{ and } y \ovee z = z && \text{by definition of } \leq \\ \implies & (x \ovee y) \ovee z = z \\ \iff & x \ovee (y \ovee z) = z && \text{by associative hypothesis} \\ \implies & x \ovee z = z \\ \iff & x \leq z \\ \iff & \leq \text{ is transitive} \end{aligned}$$

Proof that $x \ovee y$ is a lub of x and y:

$$\begin{aligned} x \ovee y = y \iff & x \leq y && \text{by definition of } \leq \\ \iff & x \vee y = y && \text{by definition of } \vee \\ \implies & x \ovee y = x \vee y \\ \implies & x \ovee y \text{ is the lub of } x \text{ and } y \end{aligned}$$

Theorem D.2. [4] *Let* $\owedge, \precsim \colon X^2 \to X$ *be binary operators over a set* X.

$$\text{THM}\quad \left\{ \begin{array}{l} (X, \precsim, \owedge) \textit{ is a} \\ \text{MEET SEMILATTICE} \end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & (X, \precsim) \textit{ is a } \text{PARTIALLY ORDERED SET} & \textit{and} \\ 2. & x \owedge y \textit{ is a } \text{GREATEST LOWER BOUND } \textit{of } x \textit{ and } y & \forall x, y \in X. \end{array} \right\}$$

✎PROOF: In order for $(X, \leq)$ to be an ordered set, $\leq$ must be, according to Definition C.2 (page 102), *reflexive*, *antisymmetric*, and *transitive*;

Proof that $\leq$ is reflexive:

$$\begin{aligned} & x = x \owedge x && \text{by idempotent hypothesis} \\ \iff & x \leq x && \text{by definition of } \leq \\ \implies & \leq \text{ is reflexive} \end{aligned}$$

Proof that $\leq$ is antisymmetric:

$$\begin{aligned} x \leq y \text{ and } y \leq x \iff & x \owedge y = x \text{ and } y \owedge x = y && \text{by definition of } \leq \\ \implies & x \owedge y = x \text{ and } x \owedge y = y && \text{by commutative hypothesis} \\ \implies & x = y \\ \implies & \leq \text{ is antisymmetric} \end{aligned}$$

[4] MacLane and Birkhoff (1999) page 475

Proof that $\leq$ is transitive:

$$\begin{aligned}
x \leq y \text{ and } y \leq z &\iff x \owedge y = x \text{ and } y \owedge z = y && \text{by definition of } \leq\\
&\implies x \owedge (y \owedge z) = x\\
&\iff (x \owedge y) \owedge z = x && \text{by associative hypothesis}\\
&\implies x \owedge z = x\\
&\iff x \leq z\\
&\iff \leq \text{ is transitive}
\end{aligned}$$

Proof that $x \owedge y$ is a glb of x and y:

$$\begin{aligned}
x \owedge y = x &\iff x \leq y && \text{by definition of } \leq\\
&\iff x \wedge y = x && \text{by definition of } \wedge\\
&\implies x \owedge y = x \wedge y\\
&\implies x \owedge y \text{ is the glb of } x \text{ and } y
\end{aligned}$$

D.2 Lattices

An *ordered set* is a set together with the additional structure of an ordering relation (Definition C.2 page 102). However, this amount of structure tends to be insufficient to ensure "well-behaved" mathematical systems. This situation is greatly remedied if every pair of elements in an ordered set (partially or linearly ordered) has both a *least upper bound* and a *greatest lower bound* (Definition C.22 page 114) in the ordered set; in this case, that ordered set is a *lattice* (next definition). Gian-Carlo Rota (1932–1999) illustrates the advantage of lattices over simple ordered sets by pointing out that the *ordered set* of partitions of an integer "is fraught with pathological properties", while the *lattice* of partitions of a set "remains to this day rich in pleasant surprises".[5] Further examples of lattices follow in Section D.3 (page 122).

Definition D.3. [6]

DEF

An algebraic structure $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *is a* ***lattice*** *if*

1. $(X, \leq)$ *is an ordered set* *and*
2. $x, y \in X \implies x \vee y \in X$ *and*
3. $x, y \in X \implies x \wedge y \in X$

The algebraic structure $\mathbf{L}^* \triangleq (X, \ovee, \owedge; \geq)$ *is the* ***dual*** *lattice of* $\mathbf{L}$*, where* $\ovee$ *and* $\owedge$ *are determined by* $\geq$*. The* LATTICE $\mathbf{L}$ *is* ***linear*** *if* $(X, \leq)$ *is a* CHAIN *(Definition C.4 page 103).*

Definition D.3 (previous) characterizes lattices in terms of *order properties*. Under this definition, lattices have an equivalent characterization in terms of *algebraic properties*. In particular, all lattices have four basic algebraic properties: all lattices are *idempotent*, *commutative*, *associative*, and *absorptive*. Conversely, any structure that possesses these four properties *is* a lattice. These results are demonstrated by Theorem D.3 (next). However, note that the four properties are not *independent*, as it is possible to prove that any structure $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ that is *commutative*, *associative*, and *absorptive*, is also *idempotent* (Theorem D.8 page 126). Thus, when proving that $\mathbf{L}$ is a lattice, it is only necessary to prove that it is *commutative*, *associative*, and *absorptive*.

[5] Rota (1997) page 1440 ⟨Introduction⟩, Rota (1964) page 498 ⟨partitions of a set⟩

[6] MacLane and Birkhoff (1999) page 473, Birkhoff (1948) page 16, Ore (1935), Birkhoff (1933a) page 442, Maeda and Maeda (1970) page 1

Theorem D.3. [7]

THM

$(X, \vee, \wedge ; \leq)$ *is a* LATTICE $\iff$

$$\left\{\begin{array}{rcl|rcl|lll} x \vee x &=& x & x \wedge x &=& x & \forall x\in X & (\text{IDEMPOTENT}) & \text{and}\\ x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \forall x,y\in X & (\text{COMMUTATIVE}) & \text{and}\\ (x \vee y) \vee z &=& x \vee (y \vee z) & (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & \forall x,y,z\in X & (\text{ASSOCIATIVE}) & \text{and}\\ x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \forall x,y\in X & (\text{ABSORPTIVE}). & \end{array}\right\}$$

✎PROOF:

1. Proof that $(X, \vee, \wedge ; \leq)$ is a lattice $\implies$ 4 properties:
   These follow directly from the definitions of least upper bound $\vee$ and greatest lower bound $\wedge$. For the absorptive property,

$$\begin{aligned} x \leq y &\implies x \vee (x \wedge y) = x \vee x = x\\ y \leq x &\implies x \vee (x \wedge y) = x \vee y = x\\ x \leq y &\implies x \wedge (x \vee y) = x \wedge y = x\\ y \leq x &\implies x \wedge (x \vee y) = x \wedge x = x \end{aligned}$$

2. Proof that $(X, \vee, \wedge ; \leq)$ is a lattice $\impliedby$ 4 properties:

   According to Definition D.3 (page 117), in order for $(X, \vee, \wedge ; \leq)$ to be a lattice, $(X, \vee, \wedge ; \leq)$ must be an ordered set, $x \vee y$ must be the least upper bound for any $x, y \in X$ and $x \wedge y$ must be the greatest lower bound for any $x, y \in X$.

   (a) By Theorem D.1 (page 115), $(X, \vee, \wedge ; \leq)$ is an ordered set.
   (b) By Theorem D.1 (page 115), $x \vee y$ is the least upper bound for any $x, y \in X$.
   (c) Proof that $x \wedge y$ is the greatest lower bound for any $x, y \in X$: To prove this, we must show that $x \leq y \iff x \wedge y = x$.

   Proof that $x \leq y \implies x \wedge y = x$:

$$\begin{aligned} x &= x \wedge (x \vee y) && \text{by absorptive hypothesis}\\ &= x \wedge y && \text{by } x \leq y \text{ hypothesis and definition of } \leq \end{aligned}$$

   Proof that $x \leq y \impliedby x \wedge y = x$:

$$\begin{aligned} y &= y \vee (y \wedge x) && \text{by absorptive hypothesis}\\ &= y \vee (x \wedge y) && \text{by commutative hypothesis}\\ &= y \vee x && \text{by } x \wedge y = x \text{ hypothesis}\\ &= x \vee y && \text{by commutative hypothesis}\\ \implies\ & x \leq y && \text{by definition of } \leq \end{aligned}$$

⇨

[7] MacLane and Birkhoff (1999) pages 473–475 ⟨LEMMA 1, THEOREM 4⟩, Burris and Sankappanavar (1981) pages 4–7, Birkhoff (1938) pages 795–796, Ore (1935) page 409 ⟨(α)⟩, Birkhoff (1933a) page 442, Dedekind (1900) pages 371–372 ⟨(1)–(4)⟩. Peirce (1880b) credits Boole and Jevons with the *commutative* property: Peirce (1880b) page 33 ⟨"(5)"⟩. Peirce (1880b) credits Boole and Jevons with the *associative* property. Peirce (1880b) credits Jevons (1864) with the *idempotent* property: Jevons (1864) page 41

$$\begin{array}{rcll} A + A &=& A & \text{"Law of Unity"}\\ AA &=& A & \text{"Law of Simplicity"} \end{array}$$

Lemma D.1. [8] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition D.3 page 117)*.

LEM
$$x \leq y \quad \Longleftrightarrow \quad x = x \wedge y \qquad \forall x,y \in \mathbf{L}$$

✎PROOF:

1. Proof for $\Longrightarrow$ case: by left hypothesis and definition of $\wedge$ (Definition C.22 page 114).
2. Proof for $\Longleftarrow$ case: by right hypothesis and definition of $\wedge$ (Definition C.22 page 114).

✎

The identities of Theorem D.3 (page 118) occur in pairs that are *duals* of each other. That is, for each identity, if you swap the join and meet operations, you will have the other identity in the pair. Thus, the characterization of lattices provided by Theorem D.3 (page 118) is called *self-dual*. And because of this, lattices support the *principle of duality* (next theorem).

Theorem D.4 (Principle of duality). [9] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

THM
$$\left\{\begin{array}{l}\phi \text{ is an identity on } \mathbf{L} \text{ in terms} \\ \text{of the operations } \vee \text{ and } \wedge\end{array}\right\} \quad \Longrightarrow \quad \mathbf{T}\phi \text{ is also an identity on } \mathbf{L}$$
where the operator $\mathbf{T}$ *performs the following mapping on the operations of* ϕ*:*
$$\vee \to \wedge, \qquad \wedge \to \vee$$

✎PROOF: For each of the identities in Theorem D.3 (page 118), the operator $\mathbf{T}$ produces another identity that is also in the set of identities:

$$\begin{array}{rclclclclcl}
\mathbf{T}(1a) &=& \mathbf{T}[x \vee y &=& y \vee x] &=& [x \wedge y &=& y \wedge x] &=& (1b) \\
\mathbf{T}(1b) &=& \mathbf{T}[x \wedge y &=& y \wedge x] &=& [x \vee y &=& y \vee x] &=& (1a) \\
\mathbf{T}(2a) &=& \mathbf{T}[x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& [x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& (2b) \\
\mathbf{T}(2b) &=& \mathbf{T}[x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& [x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& (2a)
\end{array}$$

Therefore, if the statement ϕ is consistent with regards to the lattice $\mathbf{L}$, then $\mathbf{T}\phi$ is also consistent with regards to the lattice $\mathbf{L}$. ✎

Proposition D.1 (Monotony laws). [10] *Let* $(X, \vee, \wedge; \leq)$ *be a lattice.*

PRP
$$\left.\begin{array}{lcll} a &\leq& b & \text{and} \\ x &\leq& y. & \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{lcll} a \wedge x &\leq& b \wedge y & \text{and} \\ a \vee x &\leq& b \vee y. & \end{array}\right.$$

[8] Holland (1970) page ???
[9] Padmanabhan and Rudeanu (2008) pages 7–8, Beran (1985) pages 29–30
[10] Givant and Halmos (2009) page 39, Doner and Tarski (1969) pages 97–99

✎PROOF:

$$\begin{array}{lll}
1. & (a \wedge x) \le a & \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition C.22 page 114} \\
 & \le b & \text{by left hypothesis} \\
2. & (a \wedge x) \le x & \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition C.22 page 114} \\
 & \le y & \text{by left hypothesis} \\
3. & (a \wedge x) = \underbrace{(a \wedge x)}_{\le b} \wedge \underbrace{(a \wedge x)}_{\le y} & \text{by } \textit{idempotent} \text{ property Theorem D.3 page 118} \\
 & \le b \wedge y & \text{by 1 and 2} \\
4. & (a \vee x) = \underbrace{(a \vee x)}_{\le b} \vee \underbrace{(a \vee x)}_{\le y} & \text{by } \textit{idempotent} \text{ property Theorem D.3 page 118} \\
 & \le b \vee y & \text{by 1 and 2}
\end{array}$$

✏

Minimax inequality. Suppose we arrange a finite sequence of values into m groups of n elements per group. This could be represented as an $m \times n$ matrix. Suppose now we find the minimum value in each row, and the maximum value in each column. We can call the maximum of all the minimum row values the *maximin*, and the minimum of all the maximum column values the *minimax*. Now, which is greater, the maximin or the minimax? The *minimax inequality* demonstrates that the maximin is always less than or equal to the minimax. The minimax inequality is illustrated below and stated formerly in Theorem D.5 (page 120).

$$\underbrace{\bigvee_1^m \left\{ \begin{array}{l}
\bigwedge_1^n \{\, x_{11} \quad x_{12} \quad \cdots \quad x_{1n} \,\} \\ \hline
\bigwedge_1^n \{\, x_{21} \quad x_{22} \quad \cdots \quad x_{2n} \,\} \\ \hline
\bigwedge_1^n \{\, \vdots \quad \ddots \quad \ddots \quad \vdots \,\} \\ \hline
\bigwedge_1^n \{\, x_{m1} \quad x_{m2} \quad \cdots \quad x_{mn} \,\}
\end{array} \right\}}_{\text{maximin}}
\le
\underbrace{\bigwedge_1^n \left\{ \begin{array}{c|c|c|c}
\bigvee_1^m & \bigvee_1^m & \bigvee_1^m & \bigvee_1^m \\
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{array} \right\}}_{\text{minimax}}$$

Theorem D.5 (Minimax inequality). [11] *Let* $(X, \vee, \wedge; \le)$ *be a lattice.*

THM

$$\underbrace{\bigvee_{i=1}^m \bigwedge_{j=1}^n x_{ij}}_{\textit{maxmini: largest of the smallest}} \le \underbrace{\bigwedge_{j=1}^n \bigvee_{i=1}^m x_{ij}}_{\textit{minimax: smallest of the largest}} \qquad \forall x_{ij} \in X$$

[11] Birkhoff (1948) pages 19–20

✎PROOF:

$$\underbrace{\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{smallest for any given } i} \le x_{ij} \le \underbrace{\left(\bigvee_{k=1}^{n} x_{kj}\right)}_{\text{largest for any given } j} \qquad \forall i,j$$

$$\implies \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{largest amoung all } i\text{s of the smallest values}} \le \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{k=1}^{m} x_{kj}\right)}_{\text{smallest amoung all } j\text{s of the largest values}}$$

$$\implies \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{j=1}^{n} x_{ij}\right)}_{\text{maxmini}} \le \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{i=1}^{m} x_{ij}\right)}_{\text{minimax}} \qquad \text{(change of variables)}$$

✏

Distributive inequalities. Special cases of the minimax inequality include three distributive *inequalities* (next theorem). If for some lattice any *one* of these inequalities is an *equality*, then *all three* are *equalities* (Theorem G.1 page 146); and in this case, the lattice is a called a *distributive* lattice (Definition G.2 page 145).

Theorem D.6 (distributive inequalities). [12]

THM

$(X, \vee, \wedge; \le)$ *is a lattice* $\implies$ *for all* $x,y,z \in X$

$$\begin{array}{rcll}
x \wedge (y \vee z) & \ge & (x \wedge y) \vee (x \wedge z) & \text{(JOIN SUPER-DISTRIBUTIVE) } \textit{and} \\
x \vee (y \wedge z) & \le & (x \vee y) \wedge (x \vee z) & \text{(MEET SUB-DISTRIBUTIVE) } \textit{and} \\
(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \le & (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \text{(MEDIAN INEQUALITY).}
\end{array}$$

✎PROOF:

1. Proof that $\wedge$ sub-distributes over $\vee$:

$$\begin{aligned}
(x \wedge y) \vee (x \wedge z) &\le (x \vee x) \wedge (y \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem D.5 page 120)} \\
&= x \wedge (y \vee z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem D.3 page 118)}
\end{aligned}$$

$$\bigvee\left\{\begin{array}{c} \bigwedge\left\{\begin{array}{cc} x & y \end{array}\right\} \\ \hline \bigwedge\left\{\begin{array}{cc} x & z \end{array}\right\} \end{array}\right\} \le \bigwedge\left\{\begin{array}{c|c} \bigvee & \bigvee \\ x & y \\ x & z \end{array}\right\}$$

2. Proof that $\vee$ super-distributes over $\wedge$:

$$\begin{aligned}
x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem D.3 page 118)} \\
&\le (x \vee y) \wedge (x \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem D.5 page 120)}
\end{aligned}$$

$$\bigvee\left\{\begin{array}{c} \bigwedge\left\{\begin{array}{cc} x & x \end{array}\right\} \\ \hline \bigwedge\left\{\begin{array}{cc} y & z \end{array}\right\} \end{array}\right\} \le \bigwedge\left\{\begin{array}{c|c} \bigvee & \bigvee \\ x & x \\ y & z \end{array}\right\}$$

3. Proof that of median inequality: by *minimax inequality* (Theorem D.5 page 120)

✏

[12] Davey and Priestley (2002) page 85, Grätzer (2003) page 38, Birkhoff (1933a) page 444, Korselt (1894) page 157, Müller-Olm (1997) page 13 ⟨terminology⟩

Modular inequalities. Besides the distributive property, another consequence of the minimax inequality is the *modularity inequality* (next theorem). A lattice in which this inequality becomes equality is said to be *modular* (Definition F.3 page 136).

Theorem D.7 (Modular inequality). [13] *Let* $(X, \vee, \wedge\,;\, \leq)$ *be a* LATTICE *(Definition D.3 page 117).*

THM $$x \leq y \quad\Longrightarrow\quad x \vee (y \wedge z) \leq y \wedge (x \vee z)$$

✎PROOF:

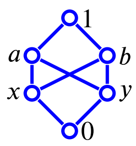
$$\begin{aligned}
x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
&\leq (x \vee y) \wedge (x \vee z) && \text{by the } \textit{minimax inequality} \text{ (Theorem D.5 page 120)}\\
&= y \wedge (x \vee z) && \text{by left hypothesis}
\end{aligned}$$

$$\bigvee\left\{ \begin{array}{c} \bigwedge\left\{\; x \quad x \;\right\} \\ \hline \bigwedge\left\{\; y \quad z \;\right\} \end{array} \right\} \quad\leq\quad \bigwedge\left\{ \begin{array}{c|c} \bigvee & \bigvee \\ x & x \\ y & z \end{array} \right\}$$

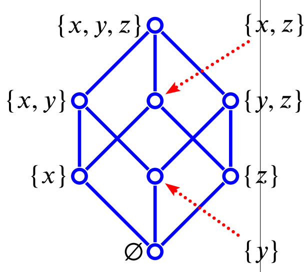
✏

D.3 Examples

Example D.1. [14]the ordered set illustrated to the right is **not** a lattice because, for example, while x and y have *upper bounds* a, b, and 1, x and y have no *least upper bound*. Obviously 1 is not the least upper bound because $a \leq 1$ and $b \leq 1$. And neither a nor b is a least upper bound because $a \nleq b$ and $b \nleq a$; rather, a and b are incomparable ($a \| b$). Note that if we remove either or both of the two lines crossing the center, the ordered set becomes a lattice.



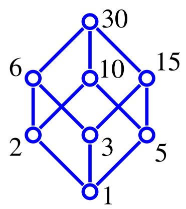
Example D.2 (Discrete lattice). Let 2^A be the power set of a set A, $\subseteq$ the set inclusion relation, $\cup$ the set union operation, and $\cap$ the set intersection operation. Then the tupple $\left(2^{\{x,y,z\}}, \cup, \cap\,;\, \subseteq\right)$ is a lattice.



Examples of least upper bounds	Examples of greatest lower bounds
$\{x\} \cup \{z\} = \{x, z\}$	$\{x\} \cap \{z\} = \varnothing$
$\{x, y\} \cup \{y\} = \{x, y\}$	$\{x, y\} \cap \{y\} = \{y\}$
$\{x, z\} \cup \{y, z\} = \{x, y, z\}$	$\{x, z\} \cap \{y, z\} = \{z\}$

Example D.3 (Integer factor lattice). [15]For any pair of natural numbers $n, m \in \mathbb{N}$, let $n|m$ represent the relation "m divides n", $\operatorname{lcm}(n,m)$ the *least common multiple* of n and m, and $\gcd(n,m)$ the *greatest common divisor* of n and m.

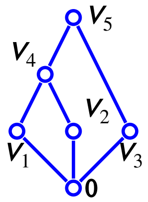
EX $\left(\{1, 2, 3, 5, 6, 10, 15, 30\}, \gcd, \operatorname{lcm}\,;\, |\right)$ is a lattice.



[13] Birkhoff (1948) page 19, Burris and Sankappanavar (1981) page 11, Dedekind (1900) page 374
[14] Oxley (2006) page 54, Farley (1997) page 3, Farley (1996) page 5, Birkhoff (1967) pages 15–16
[15] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩

Example D.4 (Linear lattice). Let $\le$ be the standard counting ordering relation on the set of integers; and for any pair of integers $n, m \in \mathbb{N}$, let $\max(n, m)$ be the maximum of n and m, and $\min(n, m)$ be the minimum of n and m. Then the tupple $(\{1, 2, 3, 4\}, \max, \min; \le)$ is a lattice.

Example D.5 (Subspace lattices). [16]Let $(\boldsymbol{V}_n)$ be a sequence of subspaces, $\subseteq$ be the set inclusion relation, $+$ the subspace addition operator, and $\cap$ the set intersection operator. Then the tuple $(\{\boldsymbol{V}_n\}, +, \cap; \subseteq)$ is a lattice.

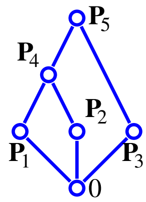


Example D.6 (Projection operator lattices). [17]Let $(\mathbf{P}_n)$ be a sequence of projection operators in a Hilbert space X.

EX

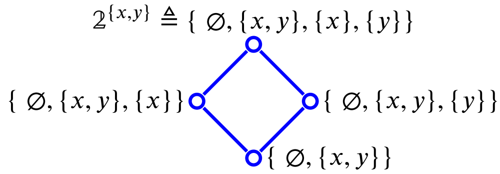
$(\{\mathbf{P}_n\}, \vee, \wedge; \le)$ is a lattice

$$\text{where}\quad \begin{array}{rcl} \mathbf{P}_1 \le \mathbf{P}_2 & \stackrel{\text{def}}{\iff} & \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1 \\ \mathbf{P}_1 \vee \mathbf{P}_2 & = & \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 \\ \mathbf{P}_1 \wedge \mathbf{P}_2 & = & \mathbf{P}_1\mathbf{P}_2 \end{array}$$



Example D.7 (Lattice of a single topology). [18] Let X be a set, τ a topology on X, $\subseteq$ the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $(\tau, \cup, \cap; \subseteq)$ is a lattice.

Example D.8 (Lattice of topologies). [19]Let X be a set and $\{\tau_1, \tau_2, \tau_3, \ldots\}$ all the possible topologies on X. Let $\subseteq$ be the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $(\{(X, \tau_n)\}, \cup, \cap; \subseteq)$ is a lattice.



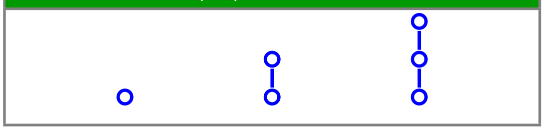
Proposition D.2. [20] *Let X_n be a finite set with order $n = |X_n|$. Let L_n be the number of labeled lattices on X_n, l_n the number of unlabeled lattices, and p_n the number of unlabeled posets.*

PRP

n	0	1	2	3	4	5	6	7	8	9	10
L_n	1	1	2	6	36	380	6390	157962	5396888	243, 179, 064	13, 938, 711, 210
l_n	1	1	1	1	2	5	15	53	222	1078	5994
p_n	1	1	2	5	16	63	318	2045	16, 999	183, 231	2, 567, 284

Example D.9 (lattices on 1–3 element sets). [21]There is only one unlabeled lattice for finite sets with 3 or fewer elements (Proposition D.2 page 123). Thus, these lattices are all linearly ordered. These 3 lattices are illustrated to the right.

lattices on 1, 2, and 3 element sets



[16] Isham (1999) pages 21–22

[17] Isham (1999) pages 21–22, Dunford and Schwartz (1957) pages 481–482

[18] Burris and Sankappanavar (1981) page 9, Birkhoff (1936a) page 161

[19] Isham (1999) page 44, Isham (1989) page 1515

[20] Sloane (2014) ⟨http://oeis.org/A055512⟩, Sloane (2014) ⟨http://oeis.org/A006966⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Heitzig and Reinhold (2002)

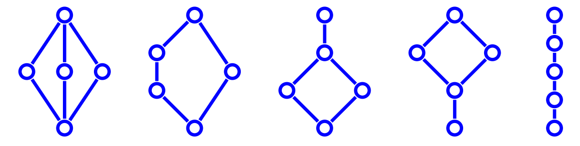
[21] Kyuno (1979) page 412, Stanley (1997) page 102

Example D.10 (lattices on a 4 element set). [22]There are 2 unlabeled lattices on a 4 element set (Proposition D.2 page 123). These are illustrated to the right.

lattices on 4 element sets

Example D.11 (lattices on a 5 element set). [23]There are 5 unlabeled lattices on a 5 element set (Proposition D.2 page 123). These are illustrated to the right.

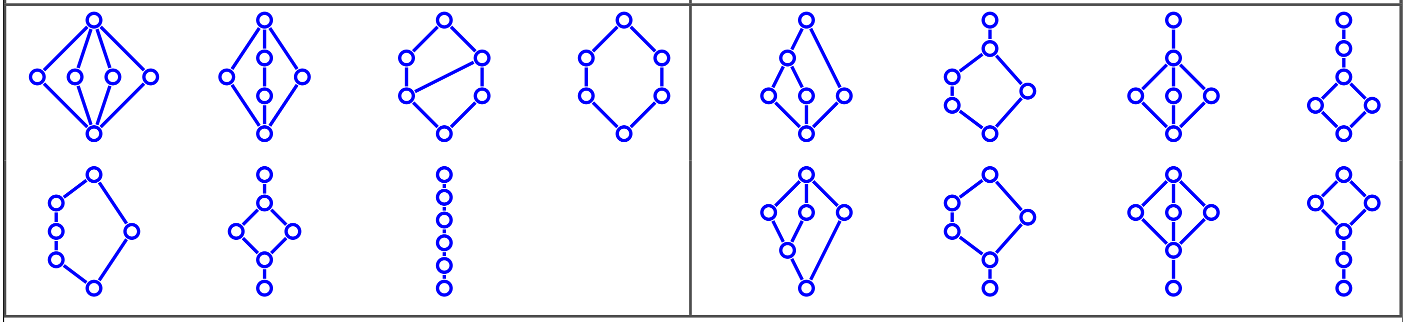
lattices on 5 element sets



Example D.12 (lattices on a 6 element set). [24] There are 15 *unlabeled lattices* on a 6 element set (Proposition D.2 page 123). These are illustrated in the following table. Notice that the lattices in the second row are simply generated from the 5 element lattices (Example D.11 page 124) with a "head" or "tail" added to each one.

lattices on 6 element sets

self-dual | *non-self dual*



Example D.13 (lattices on a 7 element set). [25] There are 53 unlabeled lattices on a 7 element set (Proposition D.2 page 123). These are illustrated in Figure D.1 (page 125).

Example D.14 (lattices on 8 element sets). There are 222 unlabeled lattices on a 8 element set (Proposition D.2 page 123). See Kyuno's paper[26] for Hasse diagrams of all 222 lattices.

D.4 Characterizations

Theorem D.3 (page 118)gave eight equations in three variables and two operators that are true of all lattices. But the converse is also true: that is, if the eight equations of Theorem D.3 are true for all values of the underlying set, then that set together with the two operators are a lattice.

That is, the eight equations in three variables of Theorem D.3 *characterize* lattices, or serve as an *equational basis* for lattices.[27] And this is not the only system of equations that characterize a lattice. There are other systems that use fewer equations in more variables. Here are some examples of lattice characterizations:

[22] Kyuno (1979) page 412, Stanley (1997) page 102
[23] Kyuno (1979) page 413, Stanley (1997) page 102
[24] Kyuno (1979) page 413, Stanley (1997) page 102
[25] Kyuno (1979) pages 413–414
[26] Kyuno (1979) pages 415–421
[27] McKenzie (1970) page 24, Tarski (1966)

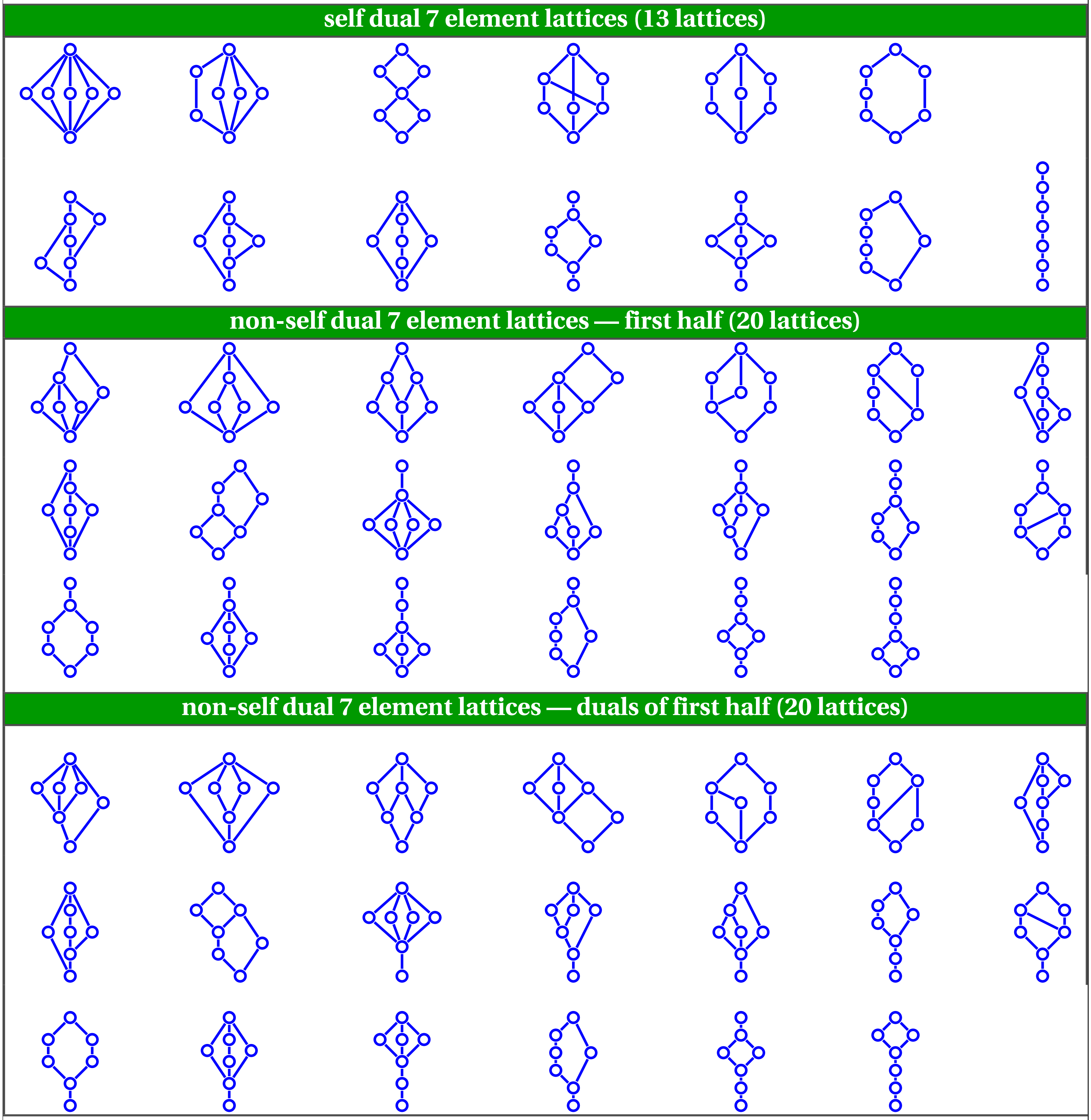


Figure D.1: The 53 unlabeled lattices on a 7 element set (Example D.13 page 124)

- 8 equations in 3 variables — Theorem D.3 — page 118
- 6 equations in 3 variables — Theorem D.8 — page 126
- 2 equations in 5 variables — Theorem D.9 — page 126
- 1 equation in 8 variables with length 29 — Theorem D.10 — page 126
- 1 equation in 7 variables with length 79 — Theorem D.10 — page 126

Since these characterizations are equivalent to the definition of the lattice, we could in fact change things around and essentially make any of these characterizations into the definition, and make the definition into a theorem.[28]

Theorem D.3 (page 118) gave 4 necessary and sufficient pairs of properties for a structure $(X, \vee, \wedge; \leq)$ to be a *lattice*. However, these 4 pairs are actually *overly* sufficient (they are not *independent*), as demonstrated next.

Theorem D.8. [29]

THM

$(X, \vee, \wedge; \leq)$ *is a lattice* $\iff$

$$\left\{\begin{array}{rcl|rcl|lll}
x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \forall x,y \in X & (\text{COMMUTATIVE}) & \textit{and} \\
(x \vee y) \vee z &=& x \vee (y \vee z) & (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & \forall x,y,z \in X & (\text{ASSOCIATIVE}) & \textit{and} \\
x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \forall x,y \in X & (\text{ABSORPTIVE}) &
\end{array}\right\}$$

✎PROOF: Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$.

1. Proof that $\boldsymbol{L}$ is a *lattice* $\implies$ 3 properties: by Theorem D.3 page 118
2. Proof that $\boldsymbol{L}$ is a *lattice* $\impliedby$ 3 properties:
   (a) Proof that 3 properties $\implies$ $\boldsymbol{L}$ is *idempotent*:

$$\begin{aligned}
x \vee x &= x \vee [x \wedge (x \vee y)] && \text{by } \textit{absorptive} \text{ property} \\
&= x \vee [x \wedge z] && \text{where } z \triangleq x \vee y \\
&= x && \text{by } \textit{absorptive} \text{ property} \\
x \wedge x &= x \wedge [x \vee (x \wedge y)] && \text{by } \textit{absorptive} \text{ property} \\
&= x \wedge [x \vee z] && \text{where } z \triangleq x \wedge y \\
&= x && \text{by } \textit{absorptive} \text{ property}
\end{aligned}$$

   (b) By Theorem D.3 page 118 and because $\boldsymbol{L}$ is *commutative*, *associative*, *absorptive*, and *idempotent* with respect to $\vee$ and $\wedge$, $\boldsymbol{L}$ is a *lattice*.

⇨

Theorem D.9 (Lattice characterization in 2 equations and 5 variables). [30] *Let X be a set and $\vee$ and $\wedge$ be two binary operators on X.*

THM

$(X, \leq, \vee, \wedge)$ *is a lattice* ***if and only if***

$$\begin{array}{rcll}
x &=& (x \wedge y) \vee x & \forall x,y \in X \quad \textit{and} \\
[(x \wedge y) \wedge z \vee u] \vee w &=& [(y \wedge z) \wedge x \vee w] \vee (y \vee u) \wedge u & \forall x,y,z,u,w \in X
\end{array}$$

Theorem D.10 (Lattice characterizations in 1 equation). [31] *Let X be a set and $\vee$ and $\wedge$ be two binary*

[28] Burris and Sankappanavar (1981) pages 6–7,
[29] Padmanabhan and Rudeanu (2008) page 8, Beran (1985) page 5, McKenzie (1970) page 24
[30] Tamura (1975) page 137
[31] McCune et al. (2003b) page 2, McCune et al. (2003a), McCune and Padmanabhan (1996) page 144, http://www.cs.unm.edu/%7Everoff/LT/

operators on X.

THM *The following four statements are all equivalent:*

1. $(X, \vee, \wedge; \leq)$ *is a* ***lattice***
2. $(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x$ $\quad \forall x,y,z,u,v,w,s,t \in X$ *(1 equation, 8 variables, length 29)*
3. $(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (((w \vee x) \wedge (s \vee x)) \vee t) = x$ $\quad \forall x,y,z,u,v,w,s,t \in X$ *(1 equation, 8 variables, length 29)*
4. $(((x \wedge y) \vee (y \wedge (x \vee y))) \wedge z) \vee (((x \wedge (((x_1 \wedge y) \vee (y \wedge x_2)) \vee y)) \vee (((y \wedge (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)) \vee (u \wedge (y \vee (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)))) \wedge (x \vee (((x_1 \wedge y) \vee (y \wedge x_2)) \vee y)))) \wedge (((x \wedge y) \vee (y \wedge (x \vee y))) \vee z)) = y$ $\quad \forall x,y,z,x_1,x_2,x_3,u \in X$ *(1 equation, 7 variables, length 79)*

D.5 Functions on lattices

D.5.1 Isomorphisms

Lattices and *ordered set* (Definition C.2 page 102) are examples of mathematical *order structures*. Often we are interested in similarities between two lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ with respect to order. Similarities between lattices can be described by defining a function θ that maps from the first lattice to the second. The degree of similarity can be roughly described in terms of the mapping θ as follows:

1. If there exists a mapping that is *bijective* then the number of elements in $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ is the same. However, their order structure may still be very different.

2. Lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are more similar if there exists a mapping that is *bijective* and *order preserving* (Definition C.9 page 109). Despite having this property however, their order structure may still be remarkably different, as illustrated by Example C.18 (page 109) and Example C.19 (page 109).

3. Lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are essentially identical (except possibly for their labeling) if there exists a mapping θ that is not only *bijective* and *order preserving*, but whose *inverse* (Definition B.2 page 77) is *also bijective* (Theorem D.11 page 127). In this case, the lattices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are *isomorphic* and the mapping θ is an *isomorphism*. An isomorphism between $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ implies that the two lattices have an identical order structure. In particular, the isomorphism θ preserves joins and meets (next definition).

Definition D.4. *Let* $\boldsymbol{L}_1 \triangleq (X, \vee, \wedge; \leq)$ *and* $\boldsymbol{L}_2 \triangleq (Y, \varovee, \varowedge; \precsim)$ *be lattices.*

DEF $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_2$ *are* ***algebraically isomorphic****, or simply* ***isomorphic****, if there exists a function* $\theta \in Y^X$ *such that*

1. $\theta(x \vee y) = \theta(x) \varovee \theta(y) \quad \forall x, y \in X$ (PRESERVES JOINS) *and*
2. $\theta(x \wedge y) = \theta(x) \varowedge \theta(y) \quad \forall x, y \in X$ (PRESERVES MEETS).

In this case, the function θ *is said to be an* ***isomorphism*** *from* $\boldsymbol{L}_1$ *to* $\boldsymbol{L}_2$*, and the isomorphic relationship between* $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_2$ *is denoted as*

$\boldsymbol{L}_1 \equiv \boldsymbol{L}_2$.

Theorem D.11. [32] *Let* $(X, \vee, \wedge; \leq)$ *and* $(Y, \varovee, \varowedge; \precsim)$ *be lattices and* $\theta \in Y^X$ *be a* BIJECTIVE *function with inverse* $\theta^{-1} \in X^Y$*. Let* $(X, \vee, \wedge; \leq) \equiv (Y, \varovee, \varowedge; \precsim)$ *represent the condition that the two lattices*

[32] Burris and Sankappanavar (2000) page 10

are ISOMORPHIC.

THM

$$\underbrace{\left.\begin{array}{lcll} x_1 \le x_2 & \Longrightarrow & \theta(x_1) \lesssim \theta(x_2) & \forall x_1,x_2 \in X \\ y_1 \lesssim y_2 & \Longrightarrow & \theta^{-1}(y_1) \lesssim \theta^{-1}(y_2) & \forall y_1,y_2 \in Y \end{array}\right\}}_{\theta \text{ and } \theta^{-1} \text{ are ORDER PRESERVING with respect to } \le \text{ and } \lesssim \text{[33]}} \iff \underbrace{(X, \vee, \wedge; \le) \equiv (Y, \varovee, \varowedge; \lesssim)}_{\text{isomorphic}}$$

✎PROOF: Let $\theta \in Y^X$ be the isomorphism between lattices $(X, \vee, \wedge; \le)$ and $(Y, \varovee, \varowedge; \lesssim)$.

1. Proof that *order preserving* $\Longrightarrow$ *preserves joins*:

   (a) Proof that $\theta(x_1 \vee x_2) \ogreaterthan \theta(x_1) \varovee \theta(x_2)$:

   i. Note that

$$\begin{aligned} x_1 &\le x_1 \vee x_2 \\ x_2 &\le x_1 \vee x_2. \end{aligned}$$

   ii. Because θ is *order preserving*

$$\begin{aligned} \theta(x_1) &\lesssim \theta(x_1 \vee x_2) \\ \theta(x_2) &\lesssim \theta(x_1 \vee x_2). \end{aligned}$$

   iii. We can then finish the proof of item (1a):

$$\begin{aligned} \theta(x_1) \varovee \theta(x_2) &\lesssim \underbrace{\theta(x_1 \vee x_2)}_{x_1 \le x_1 \vee x_2} \varovee \underbrace{\theta(x_1 \vee x_2)}_{x_2 \le x_1 \vee x_2} && \text{by *order preserving* hypothesis} \\ &= \theta(x_1 \vee x_2) && \text{by *idempotent* property page 118} \end{aligned}$$

   (b) Proof that $\theta(x_1 \vee x_2) \lesssim \theta(x_1) \varovee \theta(x_2)$:

   i. Just as in item (1a), note that $\theta^{-1}(y_1) \vee \theta^{-1}(y_2) \le \theta^{-1}(y_1 \varovee y_2)$:

$$\begin{aligned} \theta^{-1}(y_1) \vee \theta^{-1}(y_2) &\le \underbrace{\theta^{-1}(y_1 \varovee y_2)}_{y_1 \lesssim y_1 \varovee y_2} \vee \underbrace{\theta^{-1}(y_1 \varovee y_2)}_{y_2 \lesssim y_1 \varovee y_2} && \text{by *order preserving* hypothesis} \\ &= \theta^{-1}(y_1 \varovee y_2) && \text{by *idempotent* property page 118} \end{aligned}$$

   ii. Because θ is *order preserving*

$$\begin{aligned} \theta\left[\theta^{-1}(y_1) \vee \theta^{-1}(y_2)\right] &\lesssim \theta\theta^{-1}(y_1 \varovee y_2) && \text{by item (1(b)i) page 128} \\ &= y_1 \varovee y_2 && \text{by definition of inverse function } \theta^{-1} \end{aligned}$$

   iii. Let $u_1 \triangleq \theta(x_1)$ and $u_2 \triangleq \theta(x_2)$.

   iv. We can then finish the proof of item (1b):

$$\begin{aligned} \theta(x_1 \vee x_2) &= \theta\left[\theta^{-1}\theta(x_1) \vee \theta^{-1}\theta(x_2)\right] && \text{by definition of inverse function } \theta^{-1} \\ &= \theta\left[\theta^{-1}(u_1) \vee \theta^{-1}(u_2)\right] && \text{by definition of } u_1, u_2, \text{ item (1(b)iii)} \\ &\lesssim u_1 \varovee u_2 && \text{by item (1(b)ii)} \\ &= \theta(x_1) \varovee \theta(x_2) && \text{by definition of } u_1, u_2, \text{ item (1(b)iii)} \end{aligned}$$

   (c) And so, combining item (1a) and item (1b), we have

$$\left.\begin{array}{lcll} \theta(x_1 \vee x_2) & \ogreaterthan & \theta(x_1) \varovee \theta(x_2) & \text{(item (1a) page 128)} \quad \text{and} \\ \theta(x_1 \vee x_2) & \lesssim & \theta(x_1) \varovee \theta(x_2) & \text{(item (1b) page 128)} \end{array}\right\} \Longrightarrow \theta(x_1 \vee x_2) = \theta(x_1) \varovee \theta(x_2)$$

[33] *order preserving*: Definition C.9 page 109

2. Proof that *order preserving* $\implies$ *preserves meets*:

   (a) Proof that $\theta(x_1 \wedge x_2) \lessapprox \theta(x_1) \owedge \theta(x_2)$:

$$\theta(x_1) \owedge \theta(x_2) \otimes \underbrace{\theta(x_1 \wedge x_2)}_{x_1 \geq x_1 \wedge x_2} \owedge \underbrace{\theta(x_1 \wedge x_2)}_{x_2 \geq x_1 \wedge x_2} \qquad \text{by } \textit{order preserving} \text{ hypothesis}$$
$$= \theta(x_1 \wedge x_2) \qquad \text{by } \textit{idempotent} \text{ property page 118}$$

   (b) Proof that $\theta(x_1 \wedge x_2) \otimes \theta(x_1) \owedge \theta(x_2)$:

      i. Just as in item (2a), note that $\theta^{-1}(y_1) \wedge \theta^{-1}(y_2) \geq \theta^{-1}(y_1 \owedge y_2)$:

$$\theta^{-1}(y_1) \wedge \theta^{-1}(y_2) \geq \underbrace{\theta^{-1}(y_1 \owedge y_2)}_{y_1 \otimes y_1 \owedge y_2} \owedge \underbrace{\theta^{-1}(y_1 \owedge y_2)}_{y_2 \otimes y_1 \owedge y_2} \qquad \text{by } \textit{order preserving} \text{ hypothesis}$$
$$= \theta^{-1}(y_1 \owedge y_2) \qquad \text{by } \textit{idempotent} \text{ property page 118}$$

      ii. Because θ is *order preserving*

$$\theta\left[\theta^{-1}(y_1) \wedge \theta^{-1}(y_2)\right] \otimes \theta\theta^{-1}(y_1 \owedge y_2) \qquad \text{by item (2(b)i)}$$
$$= y_1 \owedge y_2$$

      iii. Let $v_1 \triangleq \theta(x_1)$ and $v_2 \triangleq \theta(x_2)$.

      iv. We can then finish the proof of item (2a):

$$\begin{aligned}
\theta(x_1 \wedge x_2) &= \theta\left[\theta^{-1}\theta(x_1) \wedge \theta^{-1}\theta(x_2)\right] \\
&= \theta\left[\theta^{-1}(v_1) \wedge \theta^{-1}(v_2)\right] && \text{by item (2(b)iii)} \\
&\otimes v_1 \owedge v_2 && \text{by item (2(b)ii)} \\
&= \theta(x_1) \owedge \theta(x_2) && \text{by item (2(b)iii)}
\end{aligned}$$

   (c) And so, combining item (2a) and item (2b), we have

$$\left.\begin{array}{lcll}
\theta(x_1 \wedge x_2) & \lessapprox & \theta(x_1) \owedge \theta(x_2) & \text{(item (2a) page 129)} \quad \text{and} \\
\theta(x_1 \wedge x_2) & \otimes & \theta(x_1) \owedge \theta(x_2) & \text{(item (2b) page 129)}
\end{array}\right\} \implies \theta(x_1 \wedge x_2) = \theta(x_1) \owedge \theta(x_2)$$

3. Proof that *order preserving* $\impliedby$ *isomorphic*:

$$\begin{aligned}
x \leq y &\implies \theta(y) = \theta(x \vee y) = \theta(x) \ovee \theta(y) && \text{by right hypothesis} \\
&\implies \theta(x) \lessapprox \theta(y) \\
x \leq y &\implies \theta(x) = \theta(x \wedge y) = \theta(x) \owedge \theta(y) && \text{by right hypothesis} \\
&\implies \theta(x) \lessapprox \theta(y)
\end{aligned}$$

✎

Example D.15. Let $\boldsymbol{L} \equiv \boldsymbol{M}$ represent the condition that a lattice $\boldsymbol{L}$ and a lattice $\boldsymbol{M}$ are *isomorphic*.

EX
$$\left(2^{\{x,y,z\}}, \cup, \cap; \subseteq\right) \equiv \left(\{1,2,3,5,6,10,15,30\}, \text{lcm}, \text{gcd}; |\right)$$
with isomorphism
$$\theta(A) = 5^{\mathbb{1}_A(z)} \cdot 3^{\mathbb{1}_A(y)} \cdot 2^{\mathbb{1}_A(x)} \qquad \forall A \in 2^{\{a,b,c\}}$$

Explicit cases are listed below and illustrated in Example C.9 (page 105) and Example C.10 (page 105).

$$\begin{aligned}
\theta(\varnothing) &= 5^0 \cdot 3^0 \cdot 2^0 &&= 1 & \theta(\{z\}) &= 5^1 \cdot 3^0 \cdot 2^0 &&= 5 \\
\theta(\{x\}) &= 5^0 \cdot 3^0 \cdot 2^1 &&= 2 & \theta(\{x,z\}) &= 5^1 \cdot 3^0 \cdot 2^1 &&= 10 \\
\theta(\{y\}) &= 5^0 \cdot 3^1 \cdot 2^0 &&= 3 & \theta(\{y,z\}) &= 5^1 \cdot 3^1 \cdot 2^0 &&= 15 \\
\theta(\{x,y\}) &= 5^0 \cdot 3^1 \cdot 2^1 &&= 6 & \theta(\{x,y,z\}) &= 5^1 \cdot 3^1 \cdot 2^1 &&= 30
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\theta(A \cup B) &= 5^{\mathbb{1}_{A\cup B}(a)} \cdot 3^{\mathbb{1}_{A\cup B}(b)} \cdot 2^{\mathbb{1}_{A\cup B}(c)} \\
&= 5^{\mathbb{1}_A(a)\vee \mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_A(b)\vee \mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_A(c)\vee \mathbb{1}_B(c)} && \text{by Theorem B.11 page 94} \\
&= \operatorname{lcm}\left(5^{\mathbb{1}_A(a)},\ 5^{\mathbb{1}_B(a)}\right) \cdot \operatorname{lcm}\left(3^{\mathbb{1}_A(b)},\ 3^{\mathbb{1}_B(b)}\right) \cdot \operatorname{lcm}\left(2^{\mathbb{1}_A(c)},\ 2^{\mathbb{1}_B(c)}\right) \\
&= \operatorname{lcm}\left(5^{\mathbb{1}_A(a)} \cdot 3^{\mathbb{1}_A(b)} \cdot 2^{\mathbb{1}_A(c)},\ 5^{\mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_B(c)}\right) \\
&= \operatorname{lcm}(\theta(A),\ \theta(B))
\end{aligned}$$

$$\begin{aligned}
\theta(A \cap B) &= 5^{\mathbb{1}_{A\cap B}(a)} \cdot 3^{\mathbb{1}_{A\cap B}(b)} \cdot 2^{\mathbb{1}_{A\cap B}(c)} \\
&= 5^{\mathbb{1}_A(a)\wedge \mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_A(b)\wedge \mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_A(c)\wedge \mathbb{1}_B(c)} && \text{by Theorem B.11 page 94} \\
&= \gcd\left(5^{\mathbb{1}_A(a)},\ 5^{\mathbb{1}_B(a)}\right) \cdot \gcd\left(3^{\mathbb{1}_A(b)},\ 3^{\mathbb{1}_B(b)}\right) \cdot \gcd\left(2^{\mathbb{1}_A(c)},\ 2^{\mathbb{1}_B(c)}\right) \\
&= \gcd\left(5^{\mathbb{1}_A(a)} \cdot 3^{\mathbb{1}_A(b)} \cdot 2^{\mathbb{1}_A(c)},\ 5^{\mathbb{1}_B(a)} \cdot 3^{\mathbb{1}_B(b)} \cdot 2^{\mathbb{1}_B(c)}\right) \\
&= \gcd(\theta(A),\ \theta(B))
\end{aligned}$$

✏

D.5.2 Metrics

Definition D.5. [34] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

DEF

A function $\mathsf{v} \in \mathbb{R}^X$ *is a* ***subvaluation*** *if*

1. $\mathsf{v}(x) \geq 0 \qquad \forall x \in X \qquad$ *and*
2. $\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) \leq \mathsf{v}(x) + \mathsf{v}(y) \qquad \forall x,y \in X$

A subvaluation v *is* ***isotone*** *if* $x \leq y \implies \mathsf{v}(x) \leq \mathsf{v}(y)$.
A subvaluation v *is* ***positive*** *if* $x < y \implies \mathsf{v}(x) < \mathsf{v}(y)$.

Definition D.6. [35] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

DEF

A function $\mathsf{v} \in \mathbb{R}^X$ *is a* ***valuation*** *if*

1. $\mathsf{v}(x) \geq 0 \qquad \forall x \in X \qquad$ *and*
2. $\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) = \mathsf{v}(x) + \mathsf{v}(y) \qquad \forall x,y \in X \qquad$ *and*
3. $x \leq y \implies \mathsf{v}(x) \leq \mathsf{v}(y) \qquad \forall x,y \in X \qquad$ (ISOTONE).

Proposition D.3 (lattice subvaluation metric). [36] *Let* $\mathbf{L}$ *be a lattice.*

PRP

$\left\{ \mathsf{v} \text{ is a positive SUBVALUATION on } \mathbf{L} \right\} \implies \left\{ \mathsf{d}(x,y) = 2\mathsf{v}(x \vee y) - \mathsf{v}(x) - \mathsf{v}(y) \text{ is a metric.} \right\}$

Proposition D.4 (lattice valuation metric). [37] *Let* $\mathbf{L}$ *be a lattice.*

PRP

$\left\{ \mathsf{v} \text{ is a positive VALUATION on } \mathbf{L} \right\} \implies \left\{ \mathsf{d}(x,y) = \mathsf{v}(x) + \mathsf{v}(y) - 2\mathsf{v}(x \wedge y) \text{ is a metric.} \right\}$

[34] Deza and Deza (2006) page 143
[35] Deza and Deza (2006) page 143, Istrăţescu (1987) page 127 ⟨differs from Deza⟩, Birkhoff (1948) page 74 ⟨not compatible with Deza⟩
[36] Deza and Deza (2006) page 143
[37] Deza and Deza (2006) page 143

D.5.3 Lattice products

Theorem D.12 (lattice product). [38] *Let* $(X \times Y, \leq)$ *be the* POSET PRODUCT[39] *of* $(X, \precsim)$ *and* $(Y, \trianglelefteq)$.

THM
$$\left.\begin{array}{lll} (X, \varovee, \varowedge; \precsim) & \text{is a lattice} & \text{and} \\ (Y, \veebar, \barwedge; \trianglelefteq) & \text{is a lattice} & \end{array}\right\} \implies (X \times Y, \vee, \wedge; \leq) \text{ is also a lattice}$$

D.6 Literature

Literature survey:

1. Early lattice theory concepts:
   Dedekind (1900)
   Ore (1935)
2. Garrett Birkhoff's contribution:
   (a) The modern concept of the lattice was introduced by Garrett Birkhoff in 1933:
   Birkhoff (1933a)
   Birkhoff (1933b)
   (b) However, Birkhoff came to realize that the concept of the lattice had actually already been published in 1900 by Richard Dedekind. Birkhoff later remarked in an interview "My ideas about lattice theory developed gradually …It was my father who, when he told Ore at Yale about what I was doing some time in 1933, found out from Ore that my lattices coincided with Dedekind's Dualgruppen …I was lucky to have gone beyond Dedekind before I discovered his work. It would have been quite discouraging if I had discovered all my results anticipated by Dedekind."[40]
   (c) Birkhoff wrote a book in 1940 called *Lattice Theory*. There are basically three editions:
   Birkhoff (1940)
   Birkhoff (1948)
   Birkhoff (1967) With regards to his *Lattice Theory* book and another book entitled *A Survey of Modern Algebra* written with Saunders MacLane, Birkhoff remarked, "Morse had told me that no one under 30 should write a book. So I thought it over and wrote two!"[41]
3. Standard text books of lattice theory:
   Birkhoff (1967)
   Grätzer (1998)
   Crawley and Dilworth (1973)
4. Characterizations / equational bases:
   (a) General discussion:
   Tarski (1966)
   Baker (1969)
   McKenzie (1970)
   McKenzie (1972)
   Pigozzi (1975)
   Taylor (1979)
      Taylor (2008)
   Jipsen and Rose (1992) pages 115–127 ⟨Chapter 5⟩
   Padmanabhan and Rudeanu (2008)
   (b) Characterizations for lattices:
   Kalman (1968)
   Tamura (1975)
   Sobociński (1979)

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[38] MacLane and Birkhoff (1967) page 489
[39] *poset product*: Definition C.5 page 103
[40] Albers and Alexanderson (1985) page 4
[41] Albers and Alexanderson (1985) page 4

(c) Specific characterizations:
Padmanabhan (1969) ⟨2 equations in 7 variables⟩
McCune and Padmanabhan (1996) page 144 ⟨1 equation, 7 variables, length 79⟩
McCune et al. (2003a) ⟨1 equation, 8 variables, length 29⟩
McCune et al. (2003b) ⟨1 equation, 8 variables, length 29⟩

5. Lattice drawing program:
Ralph Freese, http://www.math.hawaii.edu/~ralph/LatDraw/

APPENDIX E

BOUNDED LATTICES

Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice. By the definition of a *lattice* (Definition D.3 page 117), the *upper bound* $(x \vee y)$ and *lower bound* $(x \wedge y)$ of any two elements in X is also in X. But what about the upper and lower bounds of the entire set X ($\bigvee X$ and $\bigwedge X$)[1]? If both of these are in X, then the lattice $\boldsymbol{L}$ is said to be *bounded* (next definition). All *finite* lattices are bounded (next proposition). However, not all lattices are bounded—for example, the lattice $(\mathbb{Z}, \le)$ (the lattice of integers with the standard integer ordering relation) is *unbounded*. Proposition E.2 (page 133) gives two properties of bounded lattices. Boundedness is one of the "*classic 10*" properties (Theorem I.2 page 176) of *Boolean algebras* (Definition I.1 page 171). Conversely, a bounded and complemented lattice that satisfies the conditions $1' = 0$ and *Elkan's law* is a *Boolean algebra* (Proposition I.4 page 187).

Definition E.1. *Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice. Let $\bigvee X$ be the least upper bound of $(X, \le)$ and let $\bigwedge X$ be the greatest lower bound of $(X, \le)$.*

DEF

$\boldsymbol{L}$ *is* ***upper bounded***	*if* $\left(\bigvee X\right) \in X$.	
$\boldsymbol{L}$ *is* ***lower bounded***	*if* $\left(\bigwedge X\right) \in X$.	
$\boldsymbol{L}$ *is* ***bounded***	*if* $\boldsymbol{L}$ *is both upper and lower bounded.*	

A BOUNDED *lattice is optionally denoted* $(X, \vee, \wedge, 0, 1; \le)$, *where* $0 \triangleq \bigwedge X$ *and* $1 \triangleq \bigvee X$.

Proposition E.1. *Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice.*

PRP

$$\boldsymbol{L} \text{ is FINITE} \implies \boldsymbol{L} \text{ is BOUNDED}$$

Proposition E.2. *Let $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice with $\bigvee X \triangleq 1$ and $\bigwedge X \triangleq 0$.*

PRP

$$\left\{\begin{array}{l}\boldsymbol{L} \text{ is BOUNDED} \\ \text{(Definition E.1 page 133)}\end{array}\right\} \implies \left\{\begin{array}{lllll} x \vee 1 &=& 1 & \forall x \in X & \text{(UPPER BOUNDED)} \quad and \\ x \wedge 0 &=& 0 & \forall x \in X & \text{(LOWER BOUNDED)} \quad and \\ x \vee 0 &=& x & \forall x \in X & \text{(JOIN-IDENTITY)} \quad and \\ x \wedge 1 &=& x & \forall x \in X & \text{(MEET-IDENTITY)} \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} x \vee 1 &= x \vee \left(\bigvee X\right) && \text{by definition of } 1 \text{ (Definition E.1 page 133)} \\ &= \bigvee X && \text{because } x \in X \end{aligned}$$

[1] $\bigvee X$: Definition C.21 page 114, $\bigwedge X$:Definition C.22 (page 114)

$$
\begin{aligned}
&= 1 && \text{by definition of } 1 \text{ (Definition E.1 page 133)}\\
x \wedge 0 &= x \wedge \left(\bigwedge X\right) && \text{by definition of } 0 \text{ (Definition E.1 page 133)}\\
&= \bigwedge X && \text{because } x \in X\\
&= 0 && \text{by definition of } 0 \text{ (Definition E.1 page 133)}\\
\boxed{x} &= \bigvee \{x\}\\
&\le \bigvee \{x, 0\} && \text{because } \{x\} \subseteq \{0, x\} \text{ and } \textit{isotone} \text{ property (Proposition C.3 page 114)}\\
&= \boxed{x \vee 0} && \text{by definition of } \vee \text{ (Definition C.21 page 114)}\\
&= x \vee \left(\bigwedge X\right) && \text{by definition of } 0 \text{ (Definition E.1 page 133)}\\
&\le x \vee \left(\bigwedge \{x\}\right) && \text{because } \{x\} \subseteq X \text{ and } \textit{isotone} \text{ property (Proposition C.3 page 114)}\\
&\le x \vee \left(\bigwedge \{x, x\}\right) && \text{by definition of } \{\cdot\}\\
&= x \vee (x \wedge x) && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&= \boxed{x} && \text{by } \textit{absorptive} \text{ property of lattices (Theorem D.3 page 118)}\\
&= x \wedge (x \vee x) && \text{by } \textit{absorptive} \text{ property of lattices (Theorem D.3 page 118)}\\
&\triangleq x \wedge \left(\bigvee \{x, x\}\right) && \text{by definition of } \vee \text{ (Definition C.21 page 114)}\\
&\triangleq x \wedge \left(\bigvee \{x\}\right) && \text{by definition of set } \{\cdot\}\\
&\le x \wedge \left(\bigvee X\right) && \text{because } \{x\} \subseteq \{x, 1\} \text{ and by } \textit{isotone} \text{ property of } \bigwedge \text{ (Proposition C.3 page 114)}\\
&= \boxed{x \wedge 1} && \text{by definition of } 1 \text{ (Definition E.1 page 133)}\\
&= \bigwedge \{x, 1\} && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
&\le \bigwedge \{x\} && \text{because } \{x\} \subseteq \{x, 1\} \text{ and by } \textit{isotone} \text{ property of } \bigwedge \text{ (Proposition C.3 page 114)}\\
&= \boxed{x}
\end{aligned}
$$

✏

Definition E.2. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133).*

DEF

A set $\{x_1, x_2, \ldots\}$ *is a* ***partition*** *of an element* $y \in X$ *if*

1.	$x_n \ne 0$	$\forall n$	NON-EMPTY	*and*
2.	$x_n \wedge x_m = 0$	$\forall n \ne m$	MUTUALLY EXCLUSIVE	*and*
3.	$\bigvee_n x_n = 1$			

Definition E.3. [2] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133).*

DEF

The ***height*** $\mathsf{h}(x)$ *of a point* $x \in \mathbf{L}$ *is the* LEAST UPPER BOUND *of the* LENGTHS *(Definition C.12 page 110) of all the* CHAINS *that have* 0 *and in which* x *is the* LEAST UPPER BOUND. *The* ***height*** $\mathsf{h}(\mathbf{L})$ *of the lattice* $\mathbf{L}$ *is defined as*

$$\mathsf{h}(\mathbf{L}) \triangleq \mathsf{h}(1).$$

[2] Birkhoff (1967) page 5

APPENDIX F

MODULAR LATTICES

F.1 Modular relation

Definition F.1. [1] *Let* $(X, \vee, \wedge; \le)$ *be a lattice. Let* 2^{XX} *be the set of all* RELATIONS *in* X^2.

DEF

The ***modularity*** *relation* Ⓜ $\in 2^{XX}$ *and the* ***dual modularity*** *relation* $Ⓜ^* \in 2^{XX}$ *are defined as*

$$x Ⓜ y \overset{def}{\iff} \left\{(x,y) \in X^2 | a \le y \implies y \wedge (x \vee a) = (y \wedge x) \vee a \quad \forall a \in X\right\}$$

$$x Ⓜ^* y \overset{def}{\iff} \left\{(x,y) \in X^2 | a \ge y \implies y \vee (x \wedge a) = (y \vee x) \wedge a \quad \forall a \in X\right\}.$$

A pair $(x, y) \in$ Ⓜ *is alternatively denoted as* (x, y) Ⓜ, *and is called a* ***modular*** *pair. A pair* $(x, y) \in Ⓜ^*$ *is alternatively denoted as* $(x, y) Ⓜ^*$, *and is called a* ***dual modular*** *pair. A pair* (x, y) *that is* NOT *a modular pair* ($(x, y) \notin$ Ⓜ) *is denoted* $x \not{Ⓜ} y$. *A pair* (x, y) *that is* NOT *a dual modular pair is denoted* $x \not{Ⓜ}^* y$.

Proposition F.1. [2] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ *be a lattice.*

PRP

$$\left\{x Ⓜ y \iff x Ⓜ^* y\right\} \quad \forall x,y \in X$$

✎PROOF:

$$\begin{aligned}
x Ⓜ y &\iff \{a \le y \implies y \wedge (x \vee a) = (y \wedge x) \vee a \quad \forall a \in X\} && \text{by definition of Ⓜ (Definition F.1 page 135)}\\
&\iff \{a \ge y \implies a \wedge (x \vee y) = (a \wedge x) \vee y \quad \forall a \in X\} && \text{by definition of } \ge \text{ (Definition C.3 page 102)}\\
&\iff \{a \ge y \implies (a \wedge x) \vee y = a \wedge (x \vee y) \quad \forall a \in X\} && \text{by \textit{symmetric} property of } = \text{ (Definition ?? page ??)}\\
&\iff \{a \ge y \implies y \vee (x \wedge a) = (y \vee x) \wedge a \quad \forall a \in X\} && \text{by \textit{commutative} prop. of lat. (Theorem D.3 page 118)}\\
&\iff x Ⓜ^* y && \text{by definition of } Ⓜ^* \text{ (Definition F.1 page 135)}
\end{aligned}$$

✏

Proposition F.2. [3] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ *be a lattice.*

[1] Stern (1999) page 11, Maeda and Maeda (1970) page 1 ⟨Definition (1.1)⟩, Maeda (1966) page 248
[2] Maeda and Maeda (1970) page 1 ⟨Lemma (1.2)⟩
[3] Maeda and Maeda (1970) page 1

PRP

$$\underbrace{\left.\begin{array}{l} x \le y \quad or \\ y \le x \end{array}\right\}}_{x,\, y \text{ are COMPARABLE}} \implies \left\{\begin{array}{llll} x & Ⓜ & y & and \\ y & Ⓜ & x & and \\ x & Ⓜ^* & y & and \\ y & Ⓜ^* & x. & \end{array}\right.$$

✎PROOF:

$$\begin{aligned}
x \le y &\implies \{a \le y \implies y \wedge (x \vee a) = x \vee a = (y \wedge x) \vee a \quad \forall a \in X\} \\
&\iff xⓂy \qquad \text{by definition of Ⓜ (Definition F.1 page 135)} \\
x \le y &\implies \{a \le x \implies x \wedge (y \vee a) = x = x \vee a = (x \wedge y) \vee a \quad \forall a \in X\} \\
&\iff yⓂx \qquad \text{by definition of Ⓜ (Definition F.1 page 135)} \\
x \le y &\implies xⓂ^*y \qquad \text{because } x \le y \implies xⓂy \text{ and by Proposition F.1 page 135} \\
x \le y &\implies yⓂ^*x \qquad \text{because } x \le y \implies yⓂx \text{ and by Proposition F.1 page 135}
\end{aligned}$$

✏

Proposition F.3. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \le)$ *be a lattice.*

PRP

x	Ⓜ	x	$\forall x \in X$	(Ⓜ *is* REFLEXIVE)
x	$Ⓜ^*$	x	$\forall x \in X$	($Ⓜ^*$ *is* REFLEXIVE)

✎PROOF: Because $x \le x$ and by Proposition F.2 (page 135). ✏

F.2 Semimodular lattices

Definition F.2. [4]

DEF *A lattice* $(X, \vee, \wedge; \le)$ *is **semimodular** if*

$$xⓂy \implies yⓂx$$

*A semimodular lattice is also called **M-symmetric**.*

F.3 Modular lattices

Modular lattices are a generalization of the distributive lattice in the sense that all distributive lattices are modular, but not equivalent because not all modular lattices are distributive (Theorem G.5 page 161).

Definition F.3. [5]

DEF *A lattice* $(X, \vee, \wedge; \le)$ *is **modular** if*

$$xⓂy \qquad \forall x, y \in X.$$

F.3.1 Characterizations

This section describes some characterizations of modular lattices— that is, sets of properties that are equivalent to the definition of modular lattices (Definition F.3 page 136):

[4] Maeda and Maeda (1970) page 3 ⟨Definition (1.7)⟩

[5] Birkhoff (1967) page 82, Maeda and Maeda (1970) page 3 ⟨Definition (1.7)⟩

Ore 1935	(order characterization)	Theorem F.1	page 137
N5 lattice	(order characterization)	Theorem F.2	page 138
Riecan 1957	(algebraic characterization)	Theorem F.3	page 140

Alternatively, any of the sets of properties listed in this section could be used as the definition of modular lattices and the definition would in turn become a theorem/proposition.

Order characterizations

Theorem F.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

THM

$$\begin{array}{lclcl}
\boldsymbol{L} \text{ is MODULAR} & \iff & \{x \leq y \implies x \vee (z \wedge y) = (x \vee z) \wedge y\} & & \forall x,y,z \in X \\
& \iff & x \vee [(x \vee y) \wedge z] = (x \vee y) \wedge (x \vee z) & & \forall x,y,z \in X \\
& \iff & x \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee (x \wedge z) & & \forall x,y,z \in X
\end{array}$$

✎PROOF:

1. Proof that $\boldsymbol{L}$ is *modular* $\iff \{x \leq y \implies x \vee (z \wedge y) = (x \vee z) \wedge y\}$:

$$\begin{array}{rll}
\{\boldsymbol{L} \text{ is } modular\} & \iff \{x \leq y \implies y \wedge (z \vee x) = (y \wedge z) \vee x \quad \forall x,y,z \in X\} & \text{by Definition F.3 page 136} \\
& \iff \{a \leq y \implies y \wedge (x \vee a) = (y \wedge x) \vee a \quad \forall x,y,a \in X\} & \text{by change of variables} \\
& \iff \{x \text{Ⓜ} y \quad \forall x,y \in X\} & \text{by Definition F.1 page 135}
\end{array}$$

2. Proof that $\boldsymbol{L}$ is *modular* $\iff x \vee [(x \vee y) \wedge z] = (x \vee y) \wedge (x \vee z)$:

   (a) Proof that $\boldsymbol{L}$ is *modular* $\implies x \vee [(x \vee y) \wedge z] = (x \vee y) \wedge (x \vee z)$:
   First note that $x \leq x \vee y$.

$$\begin{aligned}
x \vee [(x \vee y) \wedge z] &= x \vee (u \wedge z)|_{u \triangleq x \vee y} && \text{by substitution } u \triangleq x \vee y \\
&= u \wedge (x \vee z)|_{u \triangleq x \vee y} && \text{by modularity hypothesis} \\
&= (x \vee y) \wedge (x \vee z) && \text{because } u \triangleq x \vee y
\end{aligned}$$

   (b) Proof that $\boldsymbol{L}$ is *modular* $\impliedby x \vee [(x \vee y) \wedge z] = (x \vee y) \wedge (x \vee z)$:

$$\begin{aligned}
x \leq y \implies x \vee (y \wedge z) &= x \vee (y \wedge z) && \text{by right hypothesis and } x \leq y \\
&= x \vee (z \wedge y) && \text{by \textit{commutative} property Theorem D.3 page 118} \\
&= x \vee [z \wedge (x \vee y)] && \text{because } x \leq y \\
&= x \vee [(x \vee y) \wedge z] && \text{by \textit{commutative} property Theorem D.3 page 118} \\
&= (x \vee y) \wedge (x \vee z) && \text{by right hypothesis} \\
&= y \wedge (x \vee z) && \text{because } x \leq y
\end{aligned}$$

3. Proof that $\boldsymbol{L}$ is modular $\iff \{y \leq x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\}$:

$$\begin{array}{rll}
\boldsymbol{L} \text{ is modular} & \iff \underbrace{\{x \leq y \implies x \vee (y \wedge z) = y \wedge (x \vee z)\}}_{\text{modularity definition (Definition F.3 page 136)}} & \text{by definition of modular page 136} \\
& \iff \{y \leq x \implies y \vee (x \wedge z) = x \wedge (y \vee z)\} & \text{by change of variables: } x \leftrightarrow y \\
& \iff \underbrace{\{y \leq x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\}}_{\text{dual of Definition F.3}} & \text{by \textit{reflexive} property of = (Definition ?? page ??)}
\end{array}$$

[6] Padmanabhan and Rudeanu (2008) page 39, Ore (1935) page 413 ⟨(2)⟩

4. Proof that $\{y \le x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\} \iff \{x \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee (x \wedge z)\}$:

   (a) Proof that $\{y \le x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\} \implies \{x \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee (x \wedge z)\}$:
   First note that $x \wedge y \le x$.

$$\begin{aligned}
x \wedge [(x \wedge y) \vee z] &= x \wedge (u \vee z)|_{u \triangleq x \wedge y} && \text{by substitution } u \triangleq x \wedge y\\
&= u \vee (x \wedge z)|_{u \triangleq x \wedge y} && \text{by left hypothesis}\\
&= (x \wedge y) \vee (x \wedge z) && \text{because } u \triangleq x \wedge y
\end{aligned}$$

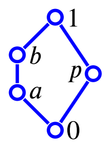
   (b) Proof that $\{y \le x \implies x \wedge (y \vee z) = y \vee (x \wedge z)\} \impliedby \{x \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee (x \wedge z)\}$:

$$\begin{aligned}
y \le x \implies x \wedge (y \vee z) &= x \wedge (z \vee y) && \text{by } \textit{commutative} \text{ property Theorem D.3 page 118}\\
&= x \wedge [z \vee (x \wedge y)] && \text{because } y \le x\\
&= x \wedge [(x \wedge y) \vee z] && \text{by } \textit{commutative} \text{ property Theorem D.3 page 118}\\
&= (x \wedge y) \vee (x \wedge z) && \text{by right hypothesis}\\
&= y \vee (x \wedge z) && \text{because } y \le x
\end{aligned}$$

⇨

Definition F.4 (N5 lattice/pentagon). [7]

DEF *The* ***N5 lattice*** *is the ordered set* $(\{0, a, b, p, 1\}, \le)$ *with cover relation* $\lessdot = \{(0, a), (a, b), (b, 1), (p, 1), (0, p)\}$.
The N5 lattice is also called the ***pentagon***.



Lemma F.1. [8]

LEM *The* $N5$ *lattice (pentagon lattice) is* NON-MODULAR.

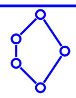
✎PROOF:

$$\begin{aligned}
x \le y \implies y \wedge (z \vee x) &= y \wedge b && \text{by Definition C.21 page 114 (lub)}\\
&= y && \text{by Definition C.22 page 114 (glb)}\\
&\ne x\\
&= x \vee a && \text{by Definition C.21 page 114 (lub)}\\
&= (y \wedge z) \vee x && \text{by Definition C.21 page 114 (lub)}
\end{aligned}$$

⇨

Theorem F.2. [9] *Let* $\boldsymbol{L}$ *be a* LATTICE *(Definition D.3 page 117).*

THM $\boldsymbol{L}$ *is* MODULAR $\iff$ $\boldsymbol{L}$ *does* NOT *contain* $N5$ *as a sublattice.*



✎PROOF:

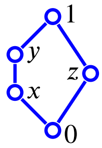
1. Proof that $\boldsymbol{L}$ is modular $\implies$ $\boldsymbol{L}$ does *not* contain $N5$:
   This is because $N5$ is a non-modular lattice. Proof: Lemma F.1 page 138

[7] Beran (1985) pages 12–13, Dedekind (1900) pages 391–392 ⟨(44) and (45)⟩
[8] Burris and Sankappanavar (1981) page 11
[9] Burris and Sankappanavar (1981) page 11, Grätzer (1971) page 70, Dedekind (1900) ⟨cf Stern 1999 page 10⟩

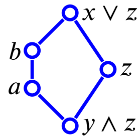
2. Proof that $\boldsymbol{L}$ does not contain $N5 \implies \boldsymbol{L}$ is modular:

   (a) In what follows, we will prove the equivalent contrapositive statement:
   $N5 \in L \qquad \Longleftarrow \qquad L$ is not modular
   (every non-modular lattice *must* contain $N5$).

   (b) We will show that for any choice of $x, y \in L$ such that $x \le y$ and under the following definitions, all non-modular lattices contain the $N5$ lattice illustrated below:



$$a \triangleq x \vee (y \wedge z)$$
$$b \triangleq y \wedge (x \vee z)$$



   (c) Proofs for comparable elements:

$$\begin{aligned} b &= y \wedge (x \vee z) && \text{by definition of } b \text{ in item (2b)} \\ &\le x \vee z && \text{by definition of } \wedge \text{ page 114} \end{aligned}$$

$$\begin{aligned} a &= x \vee (y \wedge z) && \text{by definition of } a \text{ in item (2b)} \\ &\le y \wedge (x \vee z) && \text{by modularity inequality Theorem D.7} \\ &= b && \text{by definition of } b \text{ in item (2b)} \end{aligned}$$

$$\begin{aligned} y \wedge z &\le x \vee (y \wedge z) && \text{by definition of } \vee \text{ page 114} \\ &= a && \text{by definition of } a \text{ in item (2b)} \end{aligned}$$

$$z \le x \vee z \qquad \text{by definition of } \wedge \text{ page 114}$$

$$y \wedge z \le z \qquad \text{by definition of } \wedge \text{ page 114}$$

   (d) Proofs for noncomparable elements:

$$\begin{aligned} a \vee z &= [x \vee (y \wedge z)] \vee z && \text{by definition of } a \\ &= z \vee [x \vee (y \wedge z)] && \text{by \textit{commutative property} of lattices (page 118)} \\ &= [z \vee x] \vee (y \wedge z) && \text{by \textit{associative property} of lattices (page 118)} \\ &= [x \vee z] \vee (y \wedge z) && \text{by \textit{commutative property} of lattices (page 118)} \\ &= x \vee [z \vee (y \wedge z)] && \text{by \textit{associative property} of lattices (page 118)} \\ &= x \vee z && \text{by \textit{absorptive property} of lattices (page 118)} \end{aligned}$$

$$\begin{aligned} b \vee z &= (b \vee a) \vee z && \text{by previous result} \\ &= b \vee (a \vee z) && \text{by \textit{associative property} of lattices (page 118)} \\ &= b \vee (x \vee z) && \text{by previous result} \\ &= x \vee z && \text{by previous result} \end{aligned}$$

$$\begin{aligned} a \wedge z &= (a \wedge b) \wedge z && \text{by previous result} \\ &= a \wedge (b \wedge z) && \text{by \textit{associative property} of lattices (page 118)} \\ &= a \wedge (y \wedge z) && \text{by previous result} \\ &= y \wedge z && \text{by previous result} \end{aligned}$$

$$\begin{aligned} b \wedge z &= [y \wedge (x \vee z)] \vee z && \text{by definition of } a \\ &= z \wedge [y \wedge (x \vee z)] && \text{by \textit{commutative property} of lattices (page 118)} \end{aligned}$$

$$\begin{aligned}
&= [z \wedge y] \wedge (x \vee z) && \text{by } \textit{associative property} \text{ of lattices (page 118)}\\
&= [y \wedge z] \wedge (x \vee z) && \text{by } \textit{commutative property} \text{ of lattices (page 118)}\\
&= y \wedge [z \wedge (x \vee z)] && \text{by } \textit{associative property} \text{ of lattices (page 118)}\\
&= y \wedge z && \text{by } \textit{absorptive property} \text{ of lattices (page 118)}
\end{aligned}$$

(e) Thus, *all* non-modular lattices *must* contain an $N5$ sublattice. That is,

$$\mathbf{L} \text{ is a non-modular lattice} \implies \mathbf{L} \text{ contains an } N5 \text{ sublattice.}$$

And this implies (by the contrapostive of the statement)

$$\mathbf{L} \text{ does } \textit{not} \text{ contain an } N5 \text{ sublattice} \implies \mathbf{L} \text{ is modular lattice.}$$

✏

Algebraic characterizations

Theorem F.3. [10] *Let* $\mathbf{A} \triangleq (X, \vee, \wedge\,; \leq)$ *be an algebraic structure.*

THM
$$\left\{\begin{array}{lcll}
(x \wedge y) \vee (x \wedge z) & = & [(z \wedge x) \vee y] \wedge x & \forall x,y,z \in X \quad \text{and}\\
[x \vee (y \vee z)] \wedge z & = & z & \forall x,y,z \in X
\end{array}\right\} \iff \left\{\begin{array}{l}\mathbf{A} \textit{ is a}\\ \textbf{\textit{modular lattice}}\end{array}\right\}$$

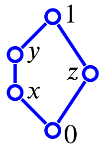
F.3.2 Special cases

Theorem F.4. [11] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1\,; \leq)$ *be a bounded lattice.*

THM
$$\left\{\begin{array}{lll}
1. & \mathbf{L} \text{ is COMPLEMENTED} & \text{and}\\
2. & \mathbf{L} \text{ is ATOMIC} & \text{and}\\
3. & \mathbf{L} \textit{ does NOT contain an N5 lattice} & \\
 & \textit{with elements } 0 \textit{ and } 1 &
\end{array}\right\} \implies \left\{\begin{array}{lll}
1. & \mathbf{L} \textit{ does not contain} & \\
 & \textit{any N5 sublattice} & \text{and}\\
2. & \mathbf{L} \text{ is MODULAR} &
\end{array}\right\}$$

F.4 Examples

Example F.1. The lattice illustrated to the right is the *N5 lattice* (Definition F.4 page 138). The N5 lattice has a total of $5 \times 5 = 25$ pairs of elements of the form (x, y) where $x, y \in X$. Of these 25, *all* are modular pairs *except* for the pair (z, y). That is, $z \not\!\!\text{Ⓜ} y$. Therefore, the N5 lattice is *non-semimodular* (and *non-modular*).



✎PROOF:

1. Five are of the form (x, x) and are therefore modular pairs by the *reflexive* property and Proposition F.3 page 136:
   1Ⓜ1, yⓂy, xⓂx, zⓂz, 0Ⓜ0.

---

[10] Padmanabhan and Rudeanu (2008) pages 42–43, Riečan (1957)

[11] Saliĭ (1988) page 27, Dilworth (1982) pages 333–353 ⟨cf Stern 1999⟩, Stern (1999) page 11, McLaughlin (1956)

2. Of the remaining 20, 16 more are modular pairs simply because they are *comparable* and by Proposition F.2 (page 135):

$$\begin{array}{llllllll} 1\textcircled{M}y & 1\textcircled{M}x & 1\textcircled{M}0 & y\textcircled{M}x & y\textcircled{M}0 & x\textcircled{M}0 & 1\textcircled{M}z & z\textcircled{M}0 \\ y\textcircled{M}1 & x\textcircled{M}1 & 0\textcircled{M}1 & x\textcircled{M}y & 0\textcircled{M}y & 0\textcircled{M}x & z\textcircled{M}1 & 0\textcircled{M}z \end{array}$$

3. Of the remaining 4, 3 are modular pairs and 1 is a nonmodular pair:

$$\begin{array}{ll} y\textcircled{M}z & x\textcircled{M}z \\ z\not\textcircled{M}y & z\textcircled{M}x \end{array}$$

$$\begin{array}{rlllllll} x \le y & \Longrightarrow & y \wedge (z \vee x) = y \wedge 1 & = y & \ne x & = 0 \vee x & = (y \wedge z) \vee x & \Longrightarrow z\not\textcircled{M}y \\ 0 \le z & \Longrightarrow & z \wedge (y \vee 0) = z \wedge y & = 0 & & = 0 \vee 0 & = (z \wedge y) \vee 0 & \Longrightarrow y\textcircled{M}z \\ 0 \le z & \Longrightarrow & z \wedge (x \vee 0) = z \wedge x & = 0 & & = 0 \vee 0 & = (z \wedge x) \vee 0 & \Longrightarrow x\textcircled{M}z \\ 0 \le x & \Longrightarrow & x \wedge (z \vee 0) = x \wedge z & = 0 & & = 0 \vee 0 & = (x \wedge z) \vee 0 & \Longrightarrow z\textcircled{M}x \end{array}$$

⇨

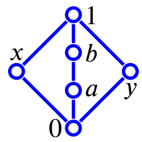
Example F.2. Of the non-comparable pairs in the lattice illustrated to the right, the following are *modular* pairs:

$x\textcircled{M}y$, $y\textcircled{M}x$, $x\textcircled{M}a$, $a\textcircled{M}x$, $y\textcircled{M}a$, $a\textcircled{M}y$, $b\textcircled{M}x$, $b\textcircled{M}y$

and the remaining non-comparable pairs are *non-modular*:

$x\not\textcircled{M}b$, $y\not\textcircled{M}b$.

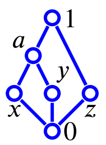
Therefore, although the Hasse diagram shown is horizontally and vertically symmetric, the lattice itself is *not M-symmetric* (not semimodular), and thus also not modular and not distributive.



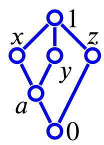
✎PROOF:

$$\begin{array}{llllll} y(x+0) = yx & = yx + 0 & & & & \Longrightarrow x\textcircled{M}y \\ x(y+0) = xy & = xy + 0 & & & & \Longrightarrow y\textcircled{M}x \\ a(x+0) = ax & = ax + 0 & & & & \Longrightarrow x\textcircled{M}a \\ x(a+0) = xa & = xa + 0 & & & & \Longrightarrow a\textcircled{M}x \\ a(y+0) = ay & = ay + 0 & & & & \Longrightarrow y\textcircled{M}a \\ y(a+0) = ya & = ya + 0 & & & & \Longrightarrow a\textcircled{M}y \\ b(x+a) = b1 & = b & \ne a & = 0 + a & = bx + a & \Longrightarrow x\not\textcircled{M}b \\ x(b+0) = xb & = xb + 0 & & & & \Longrightarrow b\textcircled{M}x \\ b(y+a) = b1 & = b & \ne a & = 0 + a & = by + a & \Longrightarrow y\not\textcircled{M}b \\ y(b+0) = yb & = yb + 0 & & & & \Longrightarrow b\textcircled{M}y \end{array}$$

⇨



Example F.3. The lattices illustrated to the right and left are duals of each other. Both are *non-modular* and both are *non-semimodular*.



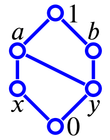
✎PROOF:

Left hand side lattice:

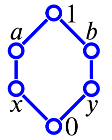
$$\begin{array}{llllll} a(z+x) = a1 & = a & \ne x & = 0 + x & = az + x & \Longrightarrow z\not\textcircled{M}a \\ z(a+0) = za & = za + 0 & & & & \Longrightarrow a\textcircled{M}z \end{array}$$

Right hand side lattice:

$$\begin{array}{llllll} z(x+0) = zx & = zx + 0 & & & & \Longrightarrow x\textcircled{M}z \\ x(z+a) = x1 & = x & \ne a & = 0 + a & = xz + a & \Longrightarrow z\not\textcircled{M}x \end{array}$$



Example F.4. The lattice illustrated to the left is *modular*. The lattice illustrated to the right is *non-modular* and *non-semimodular*.



✎PROOF:

1. Proof that the left hand side is *modular*: because it does not contain the N5 lattice and by Theorem F.2 (page 138).
2. Proof that the right hand side is *non-modular* and *non-semimodular*:

$$\begin{array}{llllll}
x(b+y) = xb & = 0 & & = 0 + y & = xb + y & \implies b \textcircled{M} x \\
b(x+y) = b1 & = b & \neq y & = 0 + y & = bx + y & \implies x \not\textcircled{M} b \\
y(a+x) = ya & = 0 & & = 0 + x & = ya + x & \implies a \textcircled{M} y \\
a(y+x) = a1 & = a & \neq x & = 0 + x & = ay + x & \implies y \not\textcircled{M} a
\end{array}$$

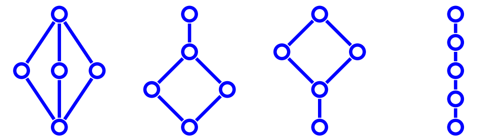
Proposition F.4. [12] *Let X_n be a finite set with order $n = |X_n|$. Let l_n be the number of unlabeled lattices on X_n, d_n the number of unlabeled distributive lattices on X_n, and m_n the number of unlabeled modular lattices on X_n.*

PRP	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
	l_n	1	1	1	1	2	5	15	53	222	1078	5994	37622	262,776	2,018,305
	m_n	1	1	1	1	2	4	8	16	34	72	157	343		

Example F.5 (modularity in 5 element sets). There are a total of five unlabeled lattices on a five element set (Proposition D.2 page 123); and of these five, four are modular, and three of the five are *distributive* (Example G.2 page 163).

EX

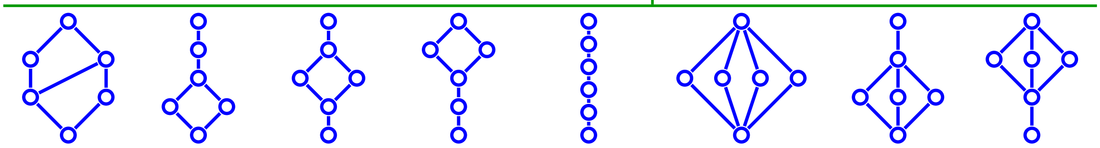
modular lattices on 5 element sets



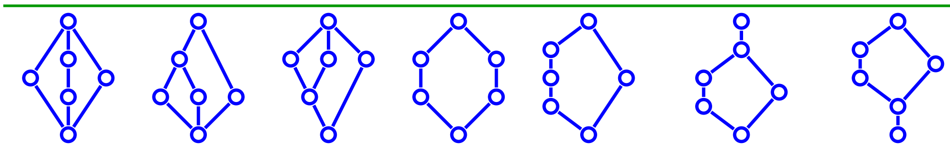
Example F.6 (modularity in 6 element sets). There are a total of 15 unlabeled lattices on a six element set (Proposition D.2 page 123 and Example D.12 page 124); and of these 15, eight are modular, and five of the eight are distributive (Proposition G.3 page 163). There are no six element non-modular lattices that are also *semimodular*.

EX

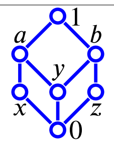
modular and distributive lattices | modular but not distributive



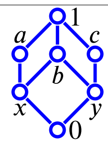
non-semimodular lattices (and non-modular and non-distributive)



[12] l_n: 💻 Sloane (2014) ⟨http://oeis.org/A006966⟩ | m_n: 💻 Sloane (2014) ⟨http://oeis.org/A006981⟩ | l_n: 📘 Heitzig and Reinhold (2002)



Example F.7. The lattices illustrated to the left and right are duals of each other. Both are *non-modular*. The left hand side lattice is also *non-semimodular*, however the right hand side lattice is *semimodular*.[13]



✎PROOF:

Proof for lattice on left hand side:

$$\begin{array}{llllll}
y(a+0)=ya & & & =ya+0 & & \implies a\textcircled{M}y\\
a(y+x)=aa & =a & =y+x & =ay+x & & \implies y\textcircled{M}a\\
b(a+z)=b1 & =b & =y+z & =ba+z & & \implies a\textcircled{M}b\\
a(b+x)=a1 & =a & =y+x & =ab+x & & \implies b\textcircled{M}a\\
b(x+z)=b1 & =b & \neq z & =0+z & bx+z & \implies x\not\textcircled{M}b\\
x(b+0)=xb & & = & =xb+0 & & \implies b\textcircled{M}x
\end{array}$$

Proof for lattice on right hand side:

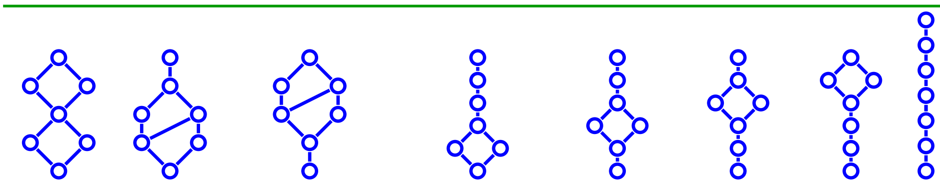
$$\begin{array}{llllll}
c(x+y)=cb & =y & =0+y & =cx+y & & \implies x\textcircled{M}c\\
x(c+0)=xc & =xc+0 & & & & \implies c\textcircled{M}x\\
b(a+x)=ba & =x & =x+x & =ba+x & \text{and} & \\
b(a+y)=b1 & =b & =x+y & =ba+y & & \implies a\textcircled{M}b\\
a(b+x)=ab & =1 & =1+x & =ab+x & & \implies b\textcircled{M}a\\
c(a+y)=c1 & =c & \neq y & =0+y & =ca+y & \implies a\not\textcircled{M}c\\
a(c+x)=a1 & =a & \neq x & =0+x & =ac+x & \implies c\not\textcircled{M}a\\
c(x+y)=cb & =y & & =0+y & =cx+y & \implies x\textcircled{M}c\\
x(c+0)=xc & =xc+0 & & & & \implies c\textcircled{M}x\\
\vdots & & & & &
\end{array}$$

✏

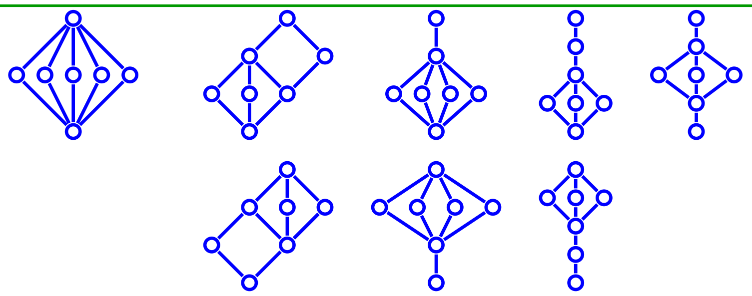
Example F.8 (modular lattices on 7 element sets). There are a total of 53 unlabeled lattices on a seven element set (Example D.13 page 124). Of these 53, 16 are modular, and 8 of these 16 are distributive (Proposition G.3 page 163).

EX

modular (and distributive) lattices on 7 element sets



modular but non-distributive lattices on 7 element sets



13 Maeda and Maeda (1970) page 5 ⟨Exercise 1.1⟩

APPENDIX G

DISTRIBUTIVE LATTICES

G.1 Distributivity relation

Definition G.1. [1] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE (Definition D.3 page 117). *Let* 2^{XXX} *be the set of all* RELATIONS *in* X^3.

DEF *The **distributivity** relation* $ⓓ \in 2^{XXX}$ *and the **dual distributivity** relation* $ⓓ^* \in 2^{XXX}$ *are defined as*

$$\begin{array}{rcll} ⓓ & \triangleq & \left\{(x,y,z) \in X^3 | x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\right\} & \text{(each } (x,y,z) \text{ is DISJUNCTIVE DISTRIBUTIVE) and} \\ ⓓ^* & \triangleq & \left\{(x,y,z) \in X^3 | x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\right\} & \text{(each } (x,y,z) \text{ is CONJUNCTIVE DISTRIBUTIVE).} \end{array}$$

A triple $(x,y,z) \in ⓓ$ *is alternatively denoted as* $(x,y,z)ⓓ$, *and is called a **distributive** triple. A triple* $(x,y,z) \in ⓓ^*$ *is alternatively denoted as* $(x,y,z)ⓓ^*$, *and is called a **dual distributive** triple. A set* $\{x,y,z\} \subseteq X$ *is **distributive** in* $\mathbf{L}$ *if each of the possible* $3! = 6$ *triples* [(x,y,z), (z,x,y), ...] *constructed from the set is* DISTRIBUTIVE *in* $\mathbf{L}$.

G.2 Distributive Lattices

G.2.1 Definition

This section introduces *distributive lattices*. Theorem D.6 (page 121) demonstrates that *all* lattices $(X, \vee, \wedge; \leq)$ satisfy the following *distributive inequalities*:

$$\begin{array}{rcll} x \wedge (y \vee z) & \geq & (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X \quad \text{(join super-distributive)} \quad \text{and} \\ x \vee (y \wedge z) & \leq & (x \vee y) \wedge (x \vee z) & \forall x,y,z \in X \quad \text{(meet sub distributive).} \quad \text{and} \\ (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \leq & (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \forall x,y,z \in X \quad \text{(median inequality).} \end{array}$$

Theorem G.1 (page 146) demonstrates that when *one* of these inequalities is equality, then *all three* of them are equalities. And in this case, the lattice is defined to be *distributive* (next definition).

[1] Maeda and Maeda (1970) page 15 ⟨Definition 4.1⟩, Foulis (1962) page 67, von Neumann (1960) page 32 ⟨Definition 5.1⟩, Davis (1955) page 314 ⟨*disjunctive distributive* and *conjunctive distributive* f.⟩

Definition G.2. [2]

DEF *A lattice* $(X, \vee, \wedge; \leq)$ *is* ***distributive*** *if*

$$(x, y, z) \in \mathbb{D} \quad \forall x, y, z \in X$$

Are all lattices *distributive*? The answer is "no". Lemma G.1 (page 148) and Lemma G.2 (page 149) demonstrate two lattices that are *not* distributive: the N5 lattice (Definition F.4 page 138) and the M3 lattice (Definition G.3 page 149).

G.2.2 Characterizations

This section describes some characterizations (equational bases) of distributive lattices both in terms of lattices (order characterizations) and in terms of abstract algebraic structures (algebraic characterizations).

- Order characterizations (first assuming a structure is a lattice):
  - Median property 1894 Theorem G.1 page 146
  - Birkhoff distributivity criterion 1934 Theorem G.2 page 150
  - Cancellation property 1934 Theorem G.3 page 153
- Algebraic characterizations (first assuming nothing):
  - Birkhoff 1946 Proposition G.1 page 156
  - Birkhoff 1948 Proposition G.2 page 156
  - Sholander 1951 Theorem G.4 page 156

Alternatively, any of the sets of properties listed in this section could be used as the definition of distributive lattices and the definition would in turn become a theorem/proposition.

In addition, if a lattice is *uniquely complemented* and satisfies any one of a number of *Huntington properties*, then it is also *distributive* (Theorem H.2 page 167), and hence also a *Boolean algebra* (Definition I.1 page 171).

Order characterizations

By the definition given in Definition G.2 (page 146), a lattice is *distributive* if the meet operation $\wedge$ distributes over the join operation $\vee$. And in view that the properties of lattices are self-dual, it is perhaps not surprising that the dual of the identity of Definition G.2 is also true for any distributive lattice— that is, the join operation $\vee$ distributes over the meet operation $\wedge$ (next theorem). But besides these two identities that are duals of each other, there is another identity that is not only equivalent to the first two, but is a dual of itself. This is called the *median property*,[3] and is given by (3) in Theorem G.1 (next theorem).

Theorem G.1. [4] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition D.3 page 117).*

[2] Burris and Sankappanavar (1981) page 10, Birkhoff (1948) page 133, Ore (1935) page 414 ⟨*arithmetic axiom*⟩, Birkhoff (1933a) page 453, Balbes and Dwinger (1975) page 48 ⟨Definition II.5.1⟩

[3] *median property*: see also Literature item 5 page 168

[4] Dilworth (1984) page 237, Burris and Sankappanavar (1981) page 10, Ore (1935) page 416 ⟨(7),(8), Theorem 3⟩, Ore (1940) ⟨cf Gratzer 2003 page 159⟩, Schröder (1890) page 286 ⟨cf Birkhoff(1948)p.133⟩, Korselt (1894) ⟨cf Birkhoff(1948)p.133⟩

THM *L* is DISTRIBUTIVE (Definition G.2 page 146)

$$\iff x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x,y,z \in X \qquad \text{(DISJUNCTIVE DISTRIBUTIVE)}$$
$$\iff x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x,y,z \in X \qquad \text{(CONJUNCTIVE DISTRIBUTIVE)}$$
$$\iff (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \quad \forall x,y,z \in X \qquad \text{(MEDIAN PROPERTY)}$$

✎PROOF: Let the join operation $\vee$ be represented by $+$, the meet operation $\wedge$ be represented by juxtaposition, and let meet take algebraic precedence over join $(+)$.

1. Proof that *distributive* $\iff$ *disjunctive distributive*:

$$\begin{aligned}
\{\boldsymbol{L} \text{ is } \textit{distributive}\} &\iff \{x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x,y,z \in X\} && \text{by Definition G.2 page 146}\\
&\iff \{(x,y,z) \in \mathbb{D} \quad \forall x,y,z \in X\} && \text{by Definition G.1 page 145}
\end{aligned}$$

2. Proof that *disjunctive distributive* $\implies$ *conjunctive distributive*:

$$\begin{aligned}
x + (yz) &= \underbrace{[x + (xy)]}_{\text{expand } x \text{ wrt } y} + (yz) && \text{by \textit{absorptive} property of lattices page 118}\\
&= x + [(xy) + (yz)] && \text{by \textit{associative} property of lattices page 118}\\
&= x + [(yx) + (yz)] && \text{by \textit{commutative} property of lattices page 118}\\
&= x + [y(x + z)] && \text{by left hypothesis}\\
&= \underbrace{[x(x + z)]}_{\text{expand } x \text{ wrt } z} + [y(x + z)] && \text{by \textit{absorptive} property of lattices page 118}\\
&= [(x + z)x] + [(x + z)y] && \text{by \textit{commutative} property of lattices page 118}\\
&= (x + z)(x + y) && \text{by left hypothesis}\\
&= (x + y)(x + z) && \text{by \textit{commutative} property of lattices page 118}
\end{aligned}$$

3. Proof that *conjunctive distributive* $\implies$ *disjunctive distributive*:

$$\begin{aligned}
x(y + z) &= \underbrace{[x(x + y)]}_{\text{expand } x \text{ wrt } y} (y + z) && \text{by \textit{absorptive} property of lattices page 118}\\
&= x[(x + y)(y + z)] && \text{by \textit{associative} property of lattices page 118}\\
&= x[(y + x)(y + z)] && \text{by \textit{commutative} property of lattices page 118}\\
&= x[y + (xz)] && \text{by right hypothesis}\\
&= \underbrace{[x + (xz)]}_{\text{expand } x \text{ wrt } z} [y + (xz)] && \text{by \textit{absorptive} property of lattices page 118}\\
&= [(xz) + x][(xz) + y] && \text{by \textit{commutative} property of lattices page 118}\\
&= (xz) + (xy) && \text{by left hypothesis}\\
&= (xy) + (xz) && \text{by \textit{commutative} property of lattices page 118}
\end{aligned}$$

4. Proof that *disjunctive distributive* $\implies$ *median property*:

$$\begin{aligned}
&(x + y)(x + z)(y + z)\\
&= (x + y)[(x + z)y + (x + z)z] && \text{by \textit{disjunctive distributive} hypothesis}\\
&= (x + y)[y(x + z) + z(x + z)] && \text{by \textit{commutative} property (Theorem D.3 page 118)}\\
&= (x + y)(yx + yz + zx + zz) && \text{by \textit{disjunctive distributive} hypothesis}\\
&= (x + y)(xy + xz + yz + z) && \text{by Theorem D.3 page 118}\\
&= (x + y)xy + (x + y)xz + (x + y)yz + (x + y)z && \text{by \textit{disjunctive distributive} hypothesis}
\end{aligned}$$

$$\begin{aligned}
&= xy(x+y) + xz(x+y) + yz(x+y) + z(x+y) && \text{by \textit{commutative} property (Theorem D.3 page 118)}\\
&= xyx + xyy + xzx + xzy + yzx + yzy + zx + zy && \text{by \textit{disjunctive distributive} hypothesis}\\
&= xy + xy + xz + xyz + xyz + yz + xz + yz && \text{by Theorem D.3 page 118}\\
&= xy + xyz + xz + yz && \text{by \textit{idempotent} property (Theorem D.3 page 118)}\\
&= (xy)(xy) + xyz + xz + yz && \text{by \textit{idempotent} property (Theorem D.3 page 118)}\\
&= (xy)(xy+z) + xz + yz && \text{by \textit{disjunctive distributive} hypothesis}\\
&= xy + xz + yz && \text{by \textit{absorptive} property (Theorem D.3 page 118)}
\end{aligned}$$

5. Proof that *median property* $\implies$ *disjunctive distributive*:

   (a) Proof that $\boldsymbol{L}$ is *modular*:

$$\begin{aligned}
y \le x \implies x(y+z) &= x(x+z)(y+z) && \text{by \textit{absorptive} property (Theorem D.3 page 118)}\\
&= (x+y)(x+z)(y+z) && \text{by } y \le x \text{ hypothesis}\\
&= xy + xz + yz && \text{by \textit{median property} hypothesis}\\
&= y + xz + yz && \text{by } y \le x \text{ hypothesis}\\
&= y + xz && \text{by \textit{absorptive} property (Theorem D.3 page 118)}\\
&\implies \boldsymbol{L} \text{ is \textit{modular}}
\end{aligned}$$

   (b) Proof that $a + ab = a$:

$$\begin{aligned}
&ab \le a && \text{by definition of } \wedge \text{ Definition C.22 page 114}\\
&\implies a + ab = a && \text{by definition of } \vee \text{ Definition C.21 page 114}
\end{aligned}$$

   (c) Proof that *median property* $\implies$ *disjunctive distributive*:

$$\begin{aligned}
x(y+z) &= xx(y+z) && \text{by \textit{idempotent} property (Theorem D.3 page 118)}\\
&= \underbrace{x(x+y)}_{x}\underbrace{x(x+z)}_{x}(y+z) && \text{by \textit{absorptive} property (Theorem D.3 page 118)}\\
&= x[(x+y)(x+z)(y+z)] && \text{by Theorem D.3 page 118}\\
&= x(xy + \underbrace{xz + yz}_{z'}) && \text{by \textit{median property} hypothesis}\\
&= x(xy) + x(xz + \underbrace{yz}_{z''}) && \text{by item (5a) and by Theorem F.1 page 137}\\
&= x(xy) + x(xz) + x(yz) && \text{by item (5a) and by Theorem F.1 page 137}\\
&= xy + xz + xyz && \text{by Theorem D.3 page 118}\\
&= xy + xz && \text{by item (5b)}
\end{aligned}$$

✏

Lemma G.1. [5]

LEM *The $N5$ lattice is* NON-DISTRIBUTIVE

[5] Burris and Sankappanavar (1981) page 11

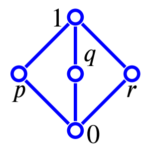
✎PROOF:

$$
\begin{aligned}
y \wedge (x \vee z) &= y \wedge b && \text{by Definition C.21 page 114 (lub)}\\
&= y && \text{by Definition C.22 page 114 (glb)}\\
&= y \vee a && \text{by Definition C.21 page 114 (lub)}\\
&= y \vee (y \wedge z) && \text{by Definition C.22 page 114 (glb)}\\
&\neq x \vee (y \wedge z) && \text{because } x \neq y\\
&= (y \wedge x) \vee (y \wedge z) && \text{by Definition C.22 page 114 (glb)}
\end{aligned}
$$

✏

Definition G.3 (M3 lattice/diamond). [6]

DEF *The **M3 lattice** is the ordered set $(\{0, p, q, r, 1\}, \leq)$ with covering relation*
$\lessdot = \{(p, 1), (q, 1), (r, 1), (0, p), (0, q), (0, r)\}$.
*The M3 lattice is also called the **diamond**,*
and is illustrated by the Hasse diagram to the right.



Remark G.1. The M3 lattice is isomorphic to the lattices

- $(\mathcal{P}(\{x, y, z\}), \leq)$[7] where $\mathcal{P}(\{x, y, z\})$ is the set of *partitions* on $\{x, y, z\}$ and with $\leq$ defined as in Proposition A.8 (page 53)
- $(\mathcal{R}(\{x, y\}), \subseteq)$ where $\mathcal{R}(\{x, y\})$ is the set of *rings of sets* on $\{x, y\}$
- $(\mathcal{A}(\{x, y, z\}), \subseteq)$ where $\mathcal{A}(\{x, y, z\})$ is the set of *algebras of sets* on $\{x, y, z\}$.

See Example A.11 (page 53), Example A.7 (page 51), Example A.16 (page 65), and Figure A.8 (page 67).

Lemma G.2. [8]

LEM $\left\{ \begin{array}{l} \textit{L is an } \textbf{\textit{M3 lattice}} \\ \text{(Definition G.3 page 149)} \end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & L \text{ is NOT } \textit{distributive} & \text{(Definition G.2 page 146)} \quad \textit{and} \\ 2. & L \text{ IS } \textit{modular} & \text{(Definition F.3 page 136)} \end{array} \right\}$

✎PROOF:

1. Proof that $M3$ is non-distributive:

$$
\begin{aligned}
x \wedge (a \vee c) &= x \wedge y && \text{by def. of l.u.b. page 114}\\
&= x && \text{by def. of g.l.b. page 114}\\
&\neq b\\
&= b \vee b && \text{by Theorem D.3 page 118 (idempotent property)}\\
&= \underbrace{(x \wedge a)}_{b} \vee \underbrace{(x \wedge c)}_{b} && \text{by def. of g.l.b. page 114}
\end{aligned}
$$

2. Proof that $M3$ is modular: (proof by exhaustion)

$$
\begin{aligned}
x \vee (y \wedge a) &= x \vee a\\
&= y\\
&= y \wedge y
\end{aligned}
\qquad
\begin{aligned}
&= y \wedge (x \vee a)\\
x \vee (y \wedge b) &= x \vee b\\
&= x\\
&= y \wedge x
\end{aligned}
\qquad
\begin{aligned}
&= y \wedge (x \vee b)\\
x \vee (y \wedge c) &= x \vee c\\
&= y\\
&= y \wedge y
\end{aligned}
$$

---

[6] Beran (1985) pages 12–13, Korselt (1894) page 157 $\langle p_1 \equiv x,\ p_2 \equiv y,\ p_3 \equiv z,\ g \equiv 1,\ 0 \equiv 0 \rangle$

[7] Saliĭ (1988) page 22

[8] Birkhoff (1948) page 6, Burris and Sankappanavar (1981) page 11, Korselt (1894) page 157 ⟨cf Salii1988 p. 37⟩

$$\begin{aligned}
&= y \wedge (x \vee c)\\
a \vee (y \wedge x) &= a \vee x\\
&= y\\
&= y \wedge y\\
&= y \wedge (a \vee x)\\
a \vee (y \wedge b) &= a \vee b\\
&= a\\
&= y \wedge a\\
&= y \wedge (a \vee b)\\
a \vee (y \wedge c) &= a \vee c\\
&= y\\
&= y \wedge y\\
&= y \wedge (a \vee c)\\
c \vee (y \wedge a) &= c \vee a\\
&= y\\
&= y \wedge y\\
&= y \wedge (c \vee a)\\
c \vee (y \wedge x) &= c \vee x\\
&= y\\
&= y \wedge y\\
&= y \wedge (c \vee x)\\
c \vee (y \wedge b) &= c \vee b\\
&= c\\
&= y \wedge c\\
&= y \wedge (c \vee b)
\end{aligned}$$

$$\begin{aligned}
b \vee (y \wedge a) &= b \vee a\\
&= a\\
&= y \wedge a\\
&= y \wedge (b \vee a)\\
b \vee (y \wedge x) &= b \vee x\\
&= x\\
&= y \wedge x\\
&= y \wedge (b \vee x)\\
b \vee (y \wedge c) &= b \vee c\\
&= c\\
&= y \wedge c\\
&= y \wedge (b \vee c)\\
b \vee (a \wedge x) &= b \vee b\\
&= b\\
&= a \wedge x\\
&= a \wedge (b \vee x)\\
b \vee (a \wedge y) &= b \vee a\\
&= a\\
&= a \wedge y\\
&= a \wedge (b \vee y)\\
b \vee (a \wedge c) &= b \vee b\\
&= b\\
&= a \wedge c\\
&= a \wedge (b \vee c)
\end{aligned}$$

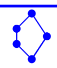
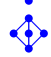
$$\begin{aligned}
b \vee (x \wedge a) &= b \vee b\\
&= b\\
&= x \wedge a\\
&= x \wedge (b \vee a)\\
b \vee (x \wedge c) &= b \vee b\\
&= b\\
&= x \wedge c\\
&= x \wedge (b \vee c)\\
b \vee (x \wedge y) &= b \vee x\\
&= x\\
&= x \wedge y\\
&= x \wedge (b \vee y)\\
b \vee (c \wedge x) &= b \vee b\\
&= b\\
&= c \wedge x\\
&= c \wedge (b \vee x)\\
b \vee (c \wedge y) &= b \vee c\\
&= c\\
&= c \wedge y\\
&= c \wedge (b \vee y)\\
b \vee (c \wedge a) &= b \vee b\\
&= b\\
&= c \wedge a\\
&= c \wedge (b \vee a)
\end{aligned}$$

✏

The *Birkhoff distributivity criterion* (next) demonstrates that a lattice is distributive *if and only if* it does not contain either the N5 or M3 lattices. If a lattice does contain either of these, it is *not* distributive. If a lattice is distributive, it does *not* contain either the N5 or M3 lattices. There was a similar theorem for *modular* lattices and the N5 lattice (Theorem F.2 page 138).

Theorem G.2 (Birkhoff distributivity criterion). [9] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE.

THM $\quad$ $\mathbf{L}$ *is* DISTRIBUTIVE $\iff$ $\begin{cases} \mathbf{L} \textit{ does } \textbf{not} \textit{ contain } N5 \textit{ as a sublattice} & \text{and}\\ \mathbf{L} \textit{ does } \textbf{not} \textit{ contain } M3 \textit{ as a sublattice} \end{cases}$

✎PROOF:

1. Proof that $\mathbf{L}$ is distributive $\implies$ $\mathbf{L}$ does *not* contain $N5$:
   This follows directly from Lemma G.1 (page 148).

[9] Burris and Sankappanavar (1981) page 12, Birkhoff (1948) page 134, Birkhoff and Hall (1934)

2. Proof that $\mathbf{L}$ is distributive $\implies$ $\mathbf{L}$ does *not* contain $M3$:
   This follows directly from Lemma G.2 (page 149).

3. Proof that $\mathbf{L}$ is distributive $\impliedby$ $N5 \notin L$ and $M3 \notin L$:

   (a) Proof that this statement is equivalent to [10]

$$(\mathbf{L} \text{ is nondistributive}) \wedge (N5 \notin L) \implies (M3 \in L):$$

   Let $P \equiv Q$ denote that statement P is equivalent to statement Q. Then ...

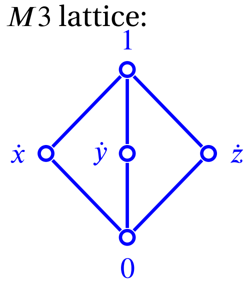
$$\begin{array}{ll}
(\mathbf{L} \text{ is distributive}) \impliedby (N5 \notin L) \wedge (M3 \notin L) & \\
\equiv (\mathbf{L} \text{ is nondistributive}) \implies (N5 \in L) \vee (M3 \in L) & \text{contrapositive} \\
\equiv \neg(\mathbf{L} \text{ is nondistributive}) \vee [(N5 \in L) \vee (M3 \in L)] & \text{by definition of } \implies \text{ (Definition 3.1 page 33)} \\
\equiv \big[\neg(\mathbf{L} \text{ is nondistributive}) \vee (N5 \in L)\big] \vee (M3 \in L) & \text{by associative property (Theorem 3.2 page 33)} \\
\equiv \neg\neg\big[\neg(\mathbf{L} \text{ is nondistributive}) \vee \neg(N5 \notin L)\big] \vee (M3 \in L) & \text{by involutary property (Theorem 3.2 page 33)} \\
\equiv \neg\big[(\mathbf{L} \text{ is nondistributive}) \wedge (N5 \notin L)\big] \vee (M3 \in L) & \text{by de Morgan's law (Theorem 3.2 page 33)} \\
\equiv (\mathbf{L} \text{ is nondistributive}) \wedge (N5 \notin L) \implies (M3 \in L) & \text{by definition of } \implies \text{ (Definition 3.1 page 33)}
\end{array}$$

   (b) Proof that $\mathbf{L}$ is *not* distributive and $N5 \notin L \implies M3 \in L$:

      i. Because $N5 \notin L$ and by Theorem F.2 (page 138), $\mathbf{L}$ is modular (so we can use the modularity property of Definition F.3 page 136).

      ii. We will show that the five values defined below form an $M3$ lattice:

$$\begin{aligned}
b &\triangleq (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \\
a &\triangleq (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \\
\dot{x} &\triangleq (x \wedge b) \vee a \\
\dot{y} &\triangleq (y \wedge b) \vee a \\
\dot{z} &\triangleq (z \wedge b) \vee a
\end{aligned}$$



      iii. Proof that $a \leq b$:

$$\begin{array}{rll}
a &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \text{by definition of } a \text{ (item (3(b)ii))} \\
&= (x \wedge y \wedge x) \vee (x \wedge z \wedge z) \vee (y \wedge z \wedge z) & \text{by } \textit{idempotent property} \text{ of lattices (page 118)} \\
&\leq (x \vee x \vee y) \wedge (y \vee z \vee z) \wedge (x \vee z \vee z) & \text{by minimax inequality Theorem D.5 page 120} \\
&= (x \vee y) \wedge (y \vee z) \wedge (x \vee z) & \text{by } \textit{idempotent property} \text{ of lattices (page 118)} \\
&= (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \text{by } \textit{commutative property} \text{ of lattices (page 118)} \\
&= b & \text{by definition of } b \text{ (item (3(b)ii))}
\end{array}$$

$$\bigvee\left\{\begin{array}{c}
\bigwedge\left\{\begin{array}{ccc} x & y & x \end{array}\right\} \\ \hline
\bigwedge\left\{\begin{array}{ccc} x & z & z \end{array}\right\} \\ \hline
\bigwedge\left\{\begin{array}{ccc} y & z & z \end{array}\right\}
\end{array}\right\}
\leq
\bigwedge\left\{\begin{array}{c|c|c}
\bigvee & \bigvee & \bigvee \\
x & y & x \\
x & z & z \\
y & z & z
\end{array}\right\}$$

      iv. Proof that $a \leq \dot{x} \leq \dot{y} \leq \dot{z} \leq b$:

         A. By item (3(b)iii), $a \leq b$.
         B. By definition of $\wedge$, $(x \wedge b)$ must be less than or equal to b.
         C. By definition of $\vee$, $(x \wedge b) \vee a$ must be greater than or equal to a.
         D. By definition of $\dot{x}$ (item (3(b)ii)), $a \leq \dot{x} \leq b$.

[10] Many many thanks to University of Waterloo Professor Emeritus Stanley Burris for his brilliant help with the logical structure of this proof.

E. The proofs for $a \le \dot{y} \le b$ and $a \le \dot{z} \le b$ are essentially identical to that of $a \le \dot{x} \le b$.

v. Proof that $\dot{x} \wedge \dot{y} = \dot{x} \wedge \dot{z} = \dot{y} \wedge \dot{z} = a$:

$$\begin{aligned}
\dot{x} \wedge \dot{y} &= [\underbrace{(x \wedge b) \vee a}_{\dot{x}}] \wedge \dot{y} && \text{by definition of } \dot{x} \text{ item (3(b)ii)}\\
&= [(x \wedge b)\underline{\wedge \dot{y}}]\underline{\vee a} && \text{by modularity page 136}\\
&= [(x \wedge b) \wedge (\underbrace{(y \wedge b) \vee a}_{\dot{y}})] \vee a && \text{by definition of } \dot{y} \text{ item (3(b)ii)}\\
&= [(x \wedge b) \wedge (y\underline{\vee a})\underline{\wedge b}] \vee a && \text{by modularity page 136}\\
&= [(x \wedge b) \wedge (y \vee a)] \vee a && \text{by idempotent property page 118}\\
&= \left[\left(x \wedge \underbrace{[(x \vee y) \wedge (x \vee z) \wedge (y \vee z)]}_{b}\right) \wedge \right.\\
&\qquad \left.\left(y \vee \underbrace{[(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)]}_{a}\right)\right] \vee a && \text{by definitions of } a \text{ and } b \text{ item (3(b)ii)}\\
&= [(x \wedge (y \vee z)) \wedge (y \vee (x \wedge z))] \vee a && \text{by absorption property page 118}\\
&= \left[x \wedge \left(\underline{y \vee}\left(\underline{(y \vee z) \wedge}(x \wedge z)\right)\right)\right] \vee a && \text{by modularity page 136}\\
&= [x \wedge (y \vee (x \wedge z))] \vee a && \text{because } (x \wedge z) \le (y \vee z)\\
&= \left[\underline{(x \wedge z) \vee}(\underline{x \wedge} y)\right] \vee a && \text{by modularity page 136}\\
&= [(x \wedge z) \vee (x \wedge y)] \vee \underbrace{[(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)]}_{a} && \text{by definition of } a \text{ item (3(b)ii)}\\
&= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) && \text{by idempotent property page 118}\\
&= a && \text{by definition of } a \text{ item (3(b)ii)}
\end{aligned}$$

vi. To prove that $\dot{x} \wedge \dot{z} = a$, simply replace $\dot{y}$ with $\dot{z}$ and y with z in item (3(b)v).

vii. To prove that $\dot{y} \wedge \dot{z} = a$, simply replace $\dot{x}$ with $\dot{z}$ and x with z in item (3(b)v).

viii. Proof that $\dot{x} \vee \dot{y} = b$:

$$\begin{aligned}
\dot{x} \vee \dot{y} &= [\underbrace{(x \wedge b) \vee a}_{\dot{x}}] \vee \dot{y} && \text{by definition of } \dot{x} \text{ item (3(b)ii)}\\
&= [(x\underline{\vee a})\underline{\wedge b}] \vee \dot{y} && \text{by modularity page 136}\\
&= [(x \vee a)\underline{\vee \dot{y}}]\underline{\wedge b} && \text{by modularity page 136}\\
&= [(x \vee a) \vee (\underbrace{(y \wedge b) \vee a}_{\dot{y}})] \wedge b && \text{by definition of } \dot{y} \text{ item (3(b)ii)}\\
&= [(x \vee a) \vee (y \wedge b)] \wedge b && \text{by idempotent property page 118}\\
&= \left[\left(x \vee \underbrace{[(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)]}_{a}\right) \vee \right.\\
&\qquad \left.\left(y \wedge \underbrace{[(x \vee y) \wedge (x \vee z) \wedge (y \vee z)]}_{b}\right)\right] \wedge b && \text{by definitions of } a \text{ and } b \text{ item (3(b)ii)}\\
&= [(x \vee (y \wedge z)) \vee (y \wedge (x \vee z))] \wedge b && \text{by absorption property page 118}\\
&= [x \vee (y \wedge z) \vee (y \wedge (x \vee z))] \wedge b && \text{by associative property page 118}
\end{aligned}$$

$$
\begin{aligned}
&= \left[x \vee \left(\underline{y \wedge}\left[\underline{(y \wedge z) \vee}(x \vee z)\right]\right)\right] \wedge b && \text{by modularity page 136} \\
&= [x \vee (y \wedge (x \vee z))] \wedge b && \text{by Definition C.21 and Definition C.22} \\
&= \left[\underline{(x \vee z) \wedge}\left(\underline{x \vee} y\right)\right] \wedge b && \text{by modularity page 136} \\
&= [(x \vee z) \wedge (x \vee y)] \wedge \underbrace{[(x \vee z) \wedge (x \vee y) \wedge (y \vee z)]}_{b} && \text{by definition of } b \text{ item (3(b)ii)} \\
&= (x \vee z) \wedge (x \vee y) \wedge (y \vee z) && \text{by idempotent property page 118} \\
&= b && \text{by definition of } b \text{ item (3(b)ii)}
\end{aligned}
$$

ix. To prove that $\dot{x} \vee \dot{z} = b$, simply replace $\dot{y}$ with $\dot{z}$ and y with z in item (3(b)viii).

x. To prove that $\dot{y} \vee \dot{z} = b$, simply replace $\dot{x}$ with $\dot{z}$ and x with z in item (3(b)viii).

✏

Theorem G.3 (cancellation criterion). [11] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE.

THM

$$
\mathbf{L} \text{ is DISTRIBUTIVE} \iff \underbrace{\left\{\left\{\begin{array}{lllll} x \vee z & = & y \vee z & \forall x,y,z \in X \quad \text{and} & (1) \\ x \wedge z & = & y \wedge z & \forall x,y,z \in X & (2) \end{array}\right\} \implies x = y\right\}}_{\text{CANCELLATION } property}
$$

✎PROOF:

1. Proof that *distributive* property $\implies$ *cancellation* property:

$$
\begin{aligned}
x &= x(x+z) && \text{by \textit{absorbtive} property (Theorem D.3 page 118)} \\
&= x(y+z) && \text{by (1)} \\
&= xy + xz && \text{by \textit{distributive} hypothesis} \\
&= xy + yz && \text{by (2)} \\
&= yx + yz && \text{by \textit{commutative} property (Theorem D.3 page 118)} \\
&= y(x+z) && \text{by \textit{distributive} hypothesis} \\
&= y(y+z) && \text{by (1)} \\
&= y && \text{by \textit{absorbtive} property (Theorem D.3 page 118)}
\end{aligned}
$$

2. Proof that *distributive* property $\impliedby$ *cancellation* property:

   (a) Define

$$
\begin{aligned}
a &\triangleq x(y+z) \\
b &\triangleq y(x+z) \\
c &\triangleq z(x+y) \\
d &\triangleq (x+y)(x+z)(y+z)
\end{aligned}
$$

[11] Blyth (2005) pages 67–68, Birkhoff and Hall (1934)

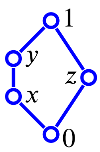
(b) Proof that $ab = xy$, $ac = xz$, and $bc = yz$:

$$\begin{aligned}
ab &= [x(y+z)][y(x+z)] && \text{by item (2a)}\\
&= [x(x+z)][y(y+z)] && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= xy && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
ac &= [x(y+z)][z(x+y)] && \text{by item (2a)}\\
&= [x(x+y)][z(z+y)] && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= xz && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
bc &= [y(x+z)][z(x+y)] && \text{by item (2a)}\\
&= [y(y+x)][z(z+x)] && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= yz && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}
\end{aligned}$$

(c) Proof of some inequalities:

$$\begin{aligned}
a &= x(y+z) && \text{by item (2a)}\\
&\le (x+y)(y+z) && \text{by definition of } \vee\\
&\le (x+y)[(x+y)+z] && \text{by definition of } \vee\\
&= x+y && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
a &= x(y+z) && \text{by item (2a)}\\
&= x(z+y) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&\le (x+z)(z+y) && \text{by definition of } \vee\\
&\le (x+z)[(x+z)+y] && \text{by definition of } \vee\\
&= x+z && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
b &= y(x+z) && \text{by item (2a)}\\
&\le (x+y)(x+z) && \text{by definition of } \vee\\
&\le (x+y)[(x+y)+z] && \text{by definition of } \vee\\
&= x+y && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
c &= z(x+y) && \text{by item (2a)}\\
&\le (x+z)(x+y) && \text{by definition of } \vee\\
&\le (x+z)[(x+z)+y] && \text{by definition of } \vee\\
&= x+z && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}
\end{aligned}$$

(d) Proof that $\boldsymbol{L}$ is *modular*:

i. Consider the following $N5$ lattice: 

ii. For the $N5$ lattice, the *cancellation* property does not hold because
$$\begin{aligned}
1 &= x+z = y+z = 1 \quad \text{and}\\
0 &= xz \;\;= yz \;\;= 0,
\end{aligned}$$
but yet $x \neq y$.

iii. Because $N5$ does *not* support the *cancellation* property and by the hypothesis that $\boldsymbol{L}$ *does* support the cancellation property, $\boldsymbol{L}$ therefore does *not* contain $N5$.

iv. Because $\boldsymbol{L}$ does not contain $N5$ and by Theorem F.2 (page 138), $\boldsymbol{L}$ is *modular*.

(e) Proof that $a + b = a + c = b + c = d$:

$$\begin{aligned}
a + b &= a + y(x + z) && \text{by definition of } c \text{ (item (2a) page 153)}\\
&= (a + y)(x + z) && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= [x(y + z) + y](x + z) && \text{by definition of } a \text{ (item (2a) page 153)}\\
&= [y + x(y + z)](x + z) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= (y + x)(y + z)(x + z) && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= (x + y)(x + z)(y + z) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= d && \text{by definition of } d \text{ (item (2a) page 153)}\\
a + c &= a + z(x + y) && \text{by definition of } c \text{ (item (2a) page 153)}\\
&= (a + z)(x + y) && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= [x(y + z) + z](x + y) && \text{by definition of } a \text{ (item (2a) page 153)}\\
&= [z + x(y + z)](x + y) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= (z + x)(y + z)(x + y) && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= (x + y)(x + z)(y + z) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= d && \text{by definition of } d \text{ (item (2a) page 153)}\\
b + c &= b + z(x + y) && \text{by definition of } c \text{ (item (2a) page 153)}\\
&= (b + z)(x + y) && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= [y(x + z) + z](x + y) && \text{by definition of } a \text{ (item (2a) page 153)}\\
&= [z + y(x + z)](x + y) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= (z + y)(x + z)(x + y) && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= (x + y)(x + z)(y + z) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= d && \text{by definition of } d \text{ (item (2a) page 153)}
\end{aligned}$$

(f) Proof that $(a + yz) + c = (b + xz) + c$ and $(a + yz)c = (b + xz)c$:

$$\begin{aligned}
(a + yz) + c &= (a + bc) + c && \text{by item (2b)}\\
&= a + (c + cb) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= a + c && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
&= d && \text{by item (2e)}\\
&= b + c && \text{by item (2e)}\\
&= b + (c + ca) && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
&= (b + ac) + c && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= (b + xz) + c && \text{by item (2b)}\\
(a + yz)c &= c(a + yz) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= c(a + bc) && \text{by item (2b)}\\
&= (bc + a)c && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= bc + ac && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= ac + bc && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= (ac + b)c && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= (b + ac)c && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= (b + xz)c && \text{by item (2b)}
\end{aligned}$$

(g) Proof that $a + yz = b + xz$: by item (2f) and *cancellation hypothesis*.

(h) Proof that $a + yz = d$:

$$\begin{aligned}
a + yz &= (a + yz) + (a + yz) && \text{by } \textit{idempotent} \text{ property (Theorem D.3 page 118)}\\
&= (a + yz) + (b + xz) && \text{by item (2g)}\\
&= (a + bc) + (b + ac) && \text{by item (2b)}\\
&= (a + ac) + (b + bc) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= a + b && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
&= d && \text{by item (2e)}
\end{aligned}$$

(i) Proof that $z(x + y) = zx + zy$ (*distributivity*):

$$\begin{aligned}
z(x + y) &= c && \text{by item (2a)}\\
&= c(c + a) && \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}\\
&= c(a + c) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= cd && \text{by item (2e)}\\
&= c(a + yz) && \text{by item (2h)}\\
&= c(a + bc) && \text{by item (2b)}\\
&= (bc + a)c && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}\\
&= bc + ac && \text{by } \textit{modularity}\text{: item (2c) and item (2d)}\\
&= yz + xz && \text{by item (2b)}\\
&= zx + zy && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)}
\end{aligned}$$

✏

Algebraic characterizations

Proposition G.1. [12] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\left\{\begin{array}{l}\boldsymbol{A} \text{ is a}\\ \textbf{distributive lattice}\end{array}\right\} \iff \left\{\begin{array}{llcll}
1. & x \wedge x & = & x & \forall x \in X \quad \text{and}\\
2. & x \vee 1 & = & 1 \vee x = 1 & \forall x \in X \quad \text{and}\\
3. & x \wedge 1 & = & 1 \wedge x = x & \forall x \in X \quad \text{and}\\
4. & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X \quad \text{and}\\
5. & (y \vee z) \wedge x & = & (y \wedge x) \vee (z \wedge x) & \forall x,y,z \in X
\end{array}\right\}$$

Proposition G.2. [13] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\left\{\begin{array}{l}\boldsymbol{A} \text{ is a}\\ \textbf{distributive lattice}\end{array}\right\} \iff \left\{\begin{array}{llcll}
1. & x \wedge x & = & x & \forall x \in X \quad \text{and}\\
2. & x \vee y & = & y \vee x & \forall x,y \in X \quad \text{and}\\
3. & x \wedge y & = & y \wedge x & \forall x,y \in X \quad \text{and}\\
4. & x \wedge (y \wedge z) & = & (x \wedge y) \wedge z & \forall x,y,z \in X \quad \text{and}\\
5. & x \wedge (x \vee y) & = & x & \forall x,y \in X \quad \text{and}\\
6. & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X.
\end{array}\right\}$$

Theorem G.4. [14] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

THM

$$\left\{\begin{array}{l}\boldsymbol{A} \text{ is a}\\ \textbf{distributive lattice}\end{array}\right\} \iff \left\{\begin{array}{llcll}
1. & x \wedge (x \vee y) & = & x & \forall x,y \in X \quad \text{and}\\
2. & x \wedge (y \vee z) & = & (z \wedge x) \vee (y \wedge x) & \forall x,y,z \in X
\end{array}\right\}$$

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12 Birkhoff (1948) pages 135–136, Birkhoff and Birkhoff (1946) ⟨???⟩
13 Padmanabhan and Rudeanu (2008) page 58, Birkhoff (1948) pages 134–135 ⟨Ex.6⟩
14 Padmanabhan and Rudeanu (2008) page 59, Sholander (1951) page 28 ⟨P1, P2⟩

✎PROOF:

1. Proof that $xx = x$ (*meet idempotent* property):

$$\begin{aligned} xx &= x[x(x+x)] && \text{by 1} \\ &= x(xx+xx) && \text{by 2} \\ &= xxx+xxx && \text{by 2} \\ &= xxx(x+x)+xxx(x+x) && \text{by 1} \\ &= xx(xx+xx)+xx(xx+xx) && \text{by 2} \\ &= xx+xx && \text{by 1} \\ &= x(x+x) && \text{by 2} \\ &= x && \text{by 1} \end{aligned}$$

2. Proof that $x + x = x$ (*join idempotent* property):

$$\begin{aligned} x+x &= xx+xx && \text{by } \textit{meet idempotent} \text{ property (item (1) page 157)} \\ &= x(x+x) && \text{by 2} \\ &= x && \text{by 1} \end{aligned}$$

3. Proof that $xy = yx$ (*meet commutative* property):

$$\begin{aligned} xy &= xy+xy && \text{by } \textit{join idempotent} \text{ property (item (2) page 157)} \\ &= y(x+x) && \text{by 2} \\ &= yx && \text{by } \textit{join idempotent} \text{ property (item (2) page 157)} \end{aligned}$$

4. Proof that $x(y+z) = xy + xz$ (*conjunctive distributive* property):

$$\begin{aligned} x(y+z) &= yx+zx && \text{by 2} \\ &= xy+xz && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)} \end{aligned}$$

5. Proof that $x + xy = x$ (*join absorptive* property):

$$\begin{aligned} x &= x(x+y) && \text{by 1} \\ &= yx+xx && \text{by 2} \\ &= yx+x && \text{by } \textit{meet idempotent} \text{ property (item (1) page 157)} \\ &= (yx+x)(yx+x) && \text{by } \textit{meet idempotent} \text{ property (item (1) page 157)} \\ &= x(yx+x)+yx(yx+x) && \text{by 2} \\ &= x(yx+x)+yx && \text{by 1} \\ &= [xx+(yx)x]+yx && \text{by 2} \\ &= x(yx+x)+yx && \text{by 2} \\ &= x(yx+xx)+yx && \text{by } \textit{meet idempotent} \text{ property (item (1) page 157)} \\ &= x[x(x+y)]+yx && \text{by 2} \\ &= xx+yx && \text{by 1} \\ &= x+yx && \text{by } \textit{meet idempotent} \text{ property (item (1) page 157)} \\ &= x+xy && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)} \end{aligned}$$

6. Proof that $x+y=y+x$ (*join commutative* property):

$$\begin{aligned}
x+y &= (x+y)(x+y) && \text{by } \textit{meet idempotent} \text{ property (item (2) page 157)}\\
&= y(x+y)+x(x+y) && \text{by 2}\\
&= y(x+y)+x && \text{by 1}\\
&= (yy+xy)+x && \text{by 2}\\
&= (y+xy)+x && \text{by } \textit{meet idempotent} \text{ property (item (2) page 157)}\\
&= (y+yx)+x && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)}\\
&= y+x && \text{by } \textit{join absorptive} \text{ property (item (5) page 157)}
\end{aligned}$$

7. Proof that $(x+y)+z=x+(y+z)$ (*join associative* property):

   (a) Let $P \triangleq (x+y)+z$ and $Q \triangleq x+(y+z)$

   (b) Proof that $Px=x$, $Py=y$, and $Pz=z$:

$$\begin{aligned}
Px &= [(x+y)+z]x && \text{by definition of } P \text{ (item (7a) page 158)}\\
&= x[(x+y)+z] && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)}\\
&= x(x+y)+xz && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 157)}\\
&= x+xz && \text{by 1}\\
&= x && \text{by } \textit{join absorptive} \text{ property (item (5) page 157)}\\
Py &= [(x+y)+z]y && \text{by definition of } P \text{ (item (7a) page 158)}\\
&= y[(x+y)+z] && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)}\\
&= y(x+y)+yz && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 157)}\\
&= y(y+x)+yz && \text{by } \textit{join commutative} \text{ property (item (6) page 158)}\\
&= y+yz && \text{by 1}\\
&= y && \text{by } \textit{join absorptive} \text{ property (item (5) page 157)}\\
Pz &= [(x+y)+z]z && \text{by definition of } P \text{ (item (7a) page 158)}\\
&= z[(x+y)+z] && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)}\\
&= z[z+(x+y)] && \text{by } \textit{join commutative} \text{ property (item (6) page 158)}\\
&= z && \text{by 1}
\end{aligned}$$

   (c) Proof that $Qx=x$, $Qy=y$, and $Qz=z$:

$$\begin{aligned}
Qx &= [x+(y+z)]x && \text{by definition of } Q \text{ (item (7a) page 158)}\\
&= x[x+(y+z)] && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)}\\
&= x && \text{by 1}\\
Qy &= [x+(y+z)]y && \text{by definition of } Q \text{ (item (7a) page 158)}\\
&= y[x+(y+z)] && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)}\\
&= yx+y(y+z) && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 157)}\\
&= yx+y && \text{by 2}\\
&= y+yx && \text{by } \textit{join commutative} \text{ property (item (6) page 158)}\\
&= y && \text{by } \textit{join absorptive} \text{ property (item (5) page 157)}\\
Qz &= [x+(y+z)]z && \text{by definition of } Q \text{ (item (7a) page 158)}\\
&= z[x+(y+z)] && \text{by } \textit{meet commutative} \text{ property (item (3) page 157)}\\
&= zx+z(y+z) && \text{by } \textit{conjunctive distributive} \text{ property (item (4) page 157)}\\
&= z(z+y)+zx && \text{by } \textit{join commutative} \text{ property (item (6) page 158)}\\
&= z+zx && \text{by 1}\\
&= z+zx && \text{by 1}\\
&= z && \text{by } \textit{join absorptive} \text{ property (item (5) page 157)}
\end{aligned}$$

(d) Proof that $(x+y)+z = x+(y+z)$:

$$
\begin{aligned}
(x+y)+z &= Qx + (Qy + Qz) && \text{by item (7c)}\\
&= Qx + Q(y+z) && \text{by \textit{conjunctive distributive} property (item (4) page 157)}\\
&= Q[x+(y+z)] && \text{by \textit{conjunctive distributive} property (item (4) page 157)}\\
&= QP && \text{by definition of } Q \text{ (item (7a) page 158)}\\
&= PQ && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= PQ && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= P[x+(y+z)] && \text{by definition of } Q \text{ (item (7a) page 158)}\\
&= Px + P(y+z) && \text{by \textit{conjunctive distributive} property (item (4) page 157)}\\
&= Px + (Py + Pz) && \text{by \textit{conjunctive distributive} property (item (4) page 157)}\\
&= x + (y+z) && \text{by item (7b)}
\end{aligned}
$$

8. Proof that $x + yz = (x+y)(x+z)$ (*disjunctive distributive* property):

$$
\begin{aligned}
(x+y)(x+z) &= (x+y)x + (x+y)z && \text{by \textit{conjunctive distributive} property (item (4) page 157)}\\
&= x(x+y) + z(x+y) && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= x + z(x+y) && \text{by 1}\\
&= x + (zx + zy) && \text{by \textit{conjunctive distributive} property (item (4) page 157)}\\
&= x + (xz + yz) && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= (x + xz) + yz && \text{by \textit{join associatiave} property (item (7) page 158)}\\
&= x + yz && \text{by \textit{join absorptive} property (item (5) page 157)}
\end{aligned}
$$

9. Proof that $(xy)z = x(yz)$ (*meet associative* property):

(a) Let $P \triangleq (xy)z$ and $Q \triangleq x(yz)$

(b) Proof that $P + x = x$, $P + y = y$, and $P + z = z$:

$$
\begin{aligned}
P + x &= (xy)z + x && \text{by definition of } P \text{ (item (9a) page 159)}\\
&= x + (xy)z && \text{by \textit{join commutative} property (item (6) page 158)}\\
&= [x + (xy)][x+z] && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= x[x+z] && \text{by 1}\\
&= x && \text{by 1}\\
P + y &= (xy)z + y && \text{by definition of } P \text{ (item (9a) page 159)}\\
&= y + (xy)z && \text{by \textit{join commutative} property (item (6) page 158)}\\
&= y + (yx)z && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= [y + (yx)][y+z] && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= y[y+z] && \text{by 1}\\
&= y && \text{by 1}\\
P + z &= (xy)z + z && \text{by definition of } P \text{ (item (9a) page 159)}\\
&= z + (xy)z && \text{by \textit{join commutative} property (item (6) page 158)}\\
&= z + z(yx) && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= z && \text{by 1}
\end{aligned}
$$

(c) Proof that $Q+x=x$, $Q+y=y$, and $Q+z=z$:

$$\begin{aligned}
Q+x &= x(yz)+x && \text{by definition of } Q \text{ (item (9a) page 159)}\\
&= x+x(yz) && \text{by \textit{join commutative} property (item (6) page 158)}\\
&= x && \text{by 1}\\
Q+y &= x(yz)+y && \text{by definition of } Q \text{ (item (9a) page 159)}\\
&= y+x(yz) && \text{by \textit{join commutative} property (item (6) page 158)}\\
&= (y+x)(y+yz) && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= (y+x)y && \text{by 1}\\
&= y(y+x) && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= y && \text{by 1}\\
Q+z &= x(yz)+z && \text{by definition of } Q \text{ (item (9a) page 159)}\\
&= z+x(yz) && \text{by \textit{join commutative} property (item (6) page 158)}\\
&= (z+x)(z+yz) && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= (z+x)(z+zy) && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= (z+x)z && \text{by 1}\\
&= z(z+x) && \text{by \textit{meet commutative} property (item (3) page 157)}\\
&= z && \text{by 1}
\end{aligned}$$

(d) Proof that $(xy)z = x(yz)$:

$$\begin{aligned}
(xy)z &= [(Q+x)(Q+y)](Q+z) && \text{by item (9c)}\\
&= (Q+xy)(Q+z) && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= Q+(xy)z && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= Q+P && \text{by definition of } P \text{ (item (9a) page 159)}\\
&= P+Q && \text{by \textit{join commutative} property (item (6) page 158)}\\
&= P+x(yz) && \text{by definition of } Q \text{ (item (9a) page 159)}\\
&= (P+x)(P+yz) && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= (P+x)[(P+y)(P+z)] && \text{by \textit{disjunctive distributive} property (item (8) page 159)}\\
&= x(yz) && \text{by item (9b)}
\end{aligned}$$

10. Proof that $\boldsymbol{A}$ is a *distributive* lattice:

(a) Proof that $\boldsymbol{A}$ is a lattice:

i. $\boldsymbol{A}$ is *idempotent* by item (1) and item (2).

ii. $\boldsymbol{A}$ is *commutative* by item (3) and item (6).

iii. $\boldsymbol{A}$ is *associative* by item (9) and item (7).

iv. $\boldsymbol{A}$ is *absorptive* by 1 and item (5).

v. Because $\boldsymbol{A}$ is *idempotent*, *commutative*, *associative*, and *absorptive*, then by Theorem D.3 (page 118), $\boldsymbol{A}$ is a *lattice*.

(b) Proof that $\boldsymbol{A}$ is *distributive*: by item (4) and Definition G.2 (page 146).

✏

G.2.3 Properties

Distributive lattices are a special case of modular lattices. That is, all distributive lattices are modular, but not all modular lattices are distributive (next theorem). An example is the M3 lattice— it is modular, but yet it is not *distributive* (Lemma G.2 page 149).

Theorem G.5. [15] *Let* $(X, \vee, \wedge; \leq)$ *be a lattice.*

THM $(X, \vee, \wedge; \leq)$ *is* DISTRIBUTIVE $\begin{array}{c}\Longrightarrow\\ \not\Longleftarrow\end{array}$ $(X, \vee, \wedge; \leq)$ *is* MODULAR.

✎PROOF:

1. Proof that distributivity $\Longrightarrow$ modularity:

$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) && \text{by distributive hypothesis} \\ &= y \wedge (x \vee z) && \text{by } x \leq y \text{ hypothesis} \end{aligned}$$

2. Proof that distributivity $\not\Longleftarrow$ modularity:
   By Lemma G.2 page 149, the $M3$ lattice is modular, but yet it is *non-distributive*.

✏

Theorem G.6 (Birkhoff's Theorem). [16] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice. Let* 2^X *be the power set of some set* X.

THM $\left\{ \mathbf{L} \text{ is DISTRIBUTIVE} \right\} \Longrightarrow \left\{ \mathbf{L} \text{ is isomorphic to a sublattice of } (2^X, \cup, \cap; \subseteq) \text{ for some set } X. \right\}$

Theorem G.7. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

THM $$\left\{ \mathbf{L} \text{ is DISTRIBUTIVE} \right\} \Longrightarrow \left\{ \underbrace{\left(\bigwedge_{n=1}^{N} x_n\right) \vee y = \bigwedge_{n=1}^{N} (x_n \vee y)}_{\textit{tautology}} \quad \underbrace{\left(\bigvee_{n=1}^{N} x_n\right) \wedge y = \bigvee_{n=1}^{N} (x_n \wedge y)}_{\textit{dual}} \right\}$$

✎PROOF:

1. Proof that $\left(\bigwedge_{n=1}^{N} x_n\right) \vee y = \bigvee_{n=1}^{N} (x_n \vee y)$ (by induction):

   Proof for $N = 1$ case:

$$\begin{aligned} \left(\bigwedge_{n=1}^{N=1} x_n\right) \vee y &= x_1 \vee y && \text{by definition of } \wedge \\ &= \bigwedge_{n=1}^{N=1} (x_n \vee y) && \text{by definition of } \wedge \end{aligned}$$

   Proof for $N = 2$ case:

$$\begin{aligned} \left(\bigwedge_{n=1}^{N=2} x_n\right) \vee y &= (x_1 \vee y) \wedge (x_2 \vee y) && \text{by Theorem G.1 page 146} \\ &= \bigwedge_{n=1}^{N=2} (x_n \vee y) && \text{by definition of } \wedge \end{aligned}$$

[15] Birkhoff (1948) page 134, Burris and Sankappanavar (1981) page 11
[16] Saliĭ (1988) page 24

Proof that (N case) $\implies$ ($N+1$ case):

$$\begin{aligned}
\left(\bigwedge_{n=1}^{N+1} x_n\right) \vee y &= \left[\left(\bigwedge_{n=1}^{N} x_n\right) \wedge x_{N+1}\right] \vee y && \text{by definition of } \wedge \\
&= \left[\left(\bigwedge_{n=1}^{N} x_n\right) \vee y\right] \wedge (x_{N+1} \vee y) && \text{by Theorem G.1 page 146} \\
&= \left[\bigwedge_{n=1}^{N} (x_n \vee y)\right] \wedge (x_{N+1} \vee y) && \text{by left hypothesis} \\
&= \bigwedge_{n=1}^{N+1} (x_n \vee y) && \text{by definition of } \wedge
\end{aligned}$$

2. Proof that $\left(\bigvee_{n=1}^{N} x_n\right) \wedge y = \bigwedge_{n=1}^{N}(x_n \wedge y)$: by *principle of duality* (Theorem D.4 page 119).

Theorem G.8. [17] *Let* $(X, \vee, \wedge;\, \leq)$ *be a lattice.*

THM

$$\underbrace{(X, \leq)}_{\text{ordered set}} \text{ is LINEARLY ORDERED } \implies \underbrace{(X, \vee, \wedge;\, \leq)}_{\text{lattice}} \text{ is DISTRIBUTIVE}$$

✎PROOF:

$$\begin{array}{llllll}
x \leq y \leq z \implies & x \wedge (y \vee z) & = x \wedge z & = x & = x \vee x & = (x \wedge y) \vee (x \wedge z) \\
x \leq z \leq y \implies & x \wedge (y \vee z) & = x \wedge y & = x & = x \vee x & = (x \wedge y) \vee (x \wedge z) \\
z \leq x \leq y \implies & x \wedge (y \vee z) & = x \wedge y & = x & = x \vee z & = (x \wedge y) \vee (x \wedge z) \\
y \leq z \leq x \implies & x \wedge (y \vee z) & = x \wedge z & = z & = y \vee z & = (x \wedge y) \vee (x \wedge z) \\
y \leq x \leq z \implies & x \wedge (y \vee z) & = x \wedge z & = x & = y \vee x & = (x \wedge y) \vee (x \wedge z) \\
z \leq y \leq x \implies & x \wedge (y \vee z) & = x \wedge y & = y & = y \vee z & = (x \wedge y) \vee (x \wedge z)
\end{array}$$

Theorem G.9. [18] *Let* $Y^X \triangleq \{\mathsf{f} : X \to Y\}$ *(the set of all functions from the set* X *to the set* Y*).*

THM

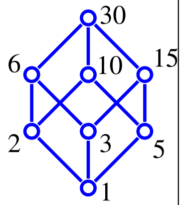
$$(Y, \varovee, \varowedge;\, \lessapprox) \text{ is a distributive lattice} \implies \left(Y^X, \vee, \wedge;\, \leq\right) \text{ is a distributive lattice}$$
$$\text{where } \mathsf{f} \leq \mathsf{g} \iff \mathsf{f}(x) \lessapprox \mathsf{g}(x) \quad \forall x \in X$$

✎PROOF:

$$\begin{aligned}
\left[\mathsf{f} \wedge (\mathsf{g} \vee \mathsf{h})\right](x) &= \mathsf{f}(x) \varowedge (\mathsf{g}(x) \varovee \mathsf{h}(x)) \\
&= (\mathsf{f}(x) \varowedge \mathsf{g}(x)) \varovee (\mathsf{f}(x) \varowedge \mathsf{h}(x)) && \text{because } (Y, \varovee, \varowedge;\, \lessapprox) \text{ is distributive} \\
&= \left[\mathsf{f} \wedge \mathsf{g}\right](x) \vee [\mathsf{f} \wedge \mathsf{h}](x) && \text{because } (Y, \varovee, \varowedge;\, \lessapprox) \text{ is distributive}
\end{aligned}$$

17 MacLane and Birkhoff (1999) page 484
18 MacLane and Birkhoff (1999) page 484

G.2.4 Examples



Example G.1. [19]For any pair of natural numbers $n, m \in \mathbb{N}$, let $n|m$ represent the relation "m divides n", $\operatorname{lcm}(n, m)$ the least common multiple of n and m, and $\gcd(n, m)$ the greatest common divisor of n and m.

EX $(\mathbb{N}, \gcd, \operatorname{lcm}\,;\, |)$ is a *distributive* lattice.

✎PROOF:

1. For all $m \in \mathbb{N}$, m can be analyzed as a product of prime factors such that
$$m = 2^{\mathsf{e}(1)}3^{\mathsf{e}(2)}5^{\mathsf{e}(3)}7^{\mathsf{e}(4)} \cdots p_k^{\mathsf{e}(k)}$$
where $\mathsf{e}(n)$ is a function $\mathsf{e} : \mathbb{N} \to \mathbb{W}$ expressing the number of prime factors p_n in m. For example,
$$84 = 2^2 3^1 7^1 \qquad \Longrightarrow \qquad \mathsf{e}(1) = 2,\ \mathsf{e}(2) = 1,\ \mathsf{e}(3) = 0,\ \mathsf{e}(4) = 1,\ \mathsf{e}(5) = 0,\ \mathsf{e}(6) = 0, \ldots$$
2. Because $\mathbb{W}$ is a chain and by Theorem G.8 page 162, $(\mathbb{W}, \vee, \wedge\,;\, \leq)$ is a distributive lattice where $\leq$ is the standard ordering on $\mathbb{W}$ and $\vee$ and $\wedge$ are defined in terms of $\leq$.
3. Let $\mathbb{W}^{\mathbb{N}}$ represent the set of all functions $\mathsf{e} : \mathbb{N} \to \mathbb{W}$. By Theorem G.9 page 162, $(\mathbb{W}^{\mathbb{N}}, \varovee, \varowedge\,;\, \precsim)$ is also a distributive lattice where $\precsim$ is defined in terms of $\leq$ as
$$\mathsf{e} \precsim \mathsf{f} \qquad \Longleftrightarrow \qquad \mathsf{e}(n) \leq \mathsf{f}(n) \quad \forall n \in \mathbb{N}.$$
4. Again by Theorem G.9 page 162, $(\mathbb{N}, \gcd, \operatorname{lcm}\,;\, |)$ is a distributive lattice because $m|k$ if $e_m(n) \precsim e_k(n)$.

✏

Proposition G.3. [20] *Let X_n be a finite set with order $n = |X_n|$. Let l_n be the number of unlabeled lattices on X_n, m_n the number of unlabeled modular lattices on X_n, and d_n the number of unlabeled distributive lattices on X_n.*

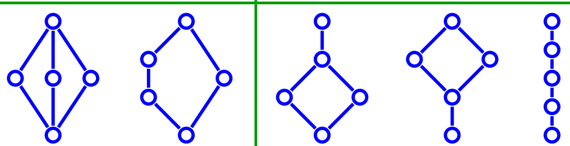
THM

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
l_n	1	1	1	1	2	5	15	53	222	1078	5994	37622			
m_n	1	1	1	1	2	4	8	16	34	72	157	343			
d_n	1	1	1	1	2	3	5	8	15	26	47	82	151	269	494

Example G.2. [21] There are a total of five unlabeled lattices on a five element set; and of these five, three are distributive (Proposition G.3 page 163). Example D.11 (page 124) illustrated all five of the unlabeled lattices, Example F.5 (page 142) illustrated the 4 modular lattices, and the following table illustrates the 3 distributive lattices. Note that none of these lattices are *complemented* (none are *Boolean* (Definition I.1 page 171)).

EX

non-distributive	*distributive*



[19] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩

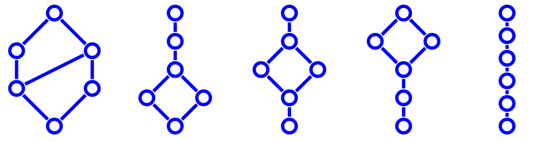
[20] l_n: Sloane (2014) ⟨http://oeis.org/A006966⟩ | m_n: Sloane (2014) ⟨http://oeis.org/A006981⟩ | d_n: Sloane (2014) ⟨http://oeis.org/A006982⟩ | l_n: Heitzig and Reinhold (2002) | m_n: Thakare et al. (2002)? | d_n: Erné et al. (2002) page 17

[21] Erné et al. (2002) pages 4–5

Example G.3. [22] There are a total of 15 unlabeled lattices on a six element set; and of these 15, five are distributive (Proposition G.3 page 163). Example D.12 (page 124) illustrated all 15 of the unlabeled lattices, Example F.6 (page 142) illustrated the 8 modular lattices, and the following illustrates the 5 distributive lattices.
Note that none of these lattices are *complemented* (none are *Boolean* (Definition I.1 page 171)).

EX

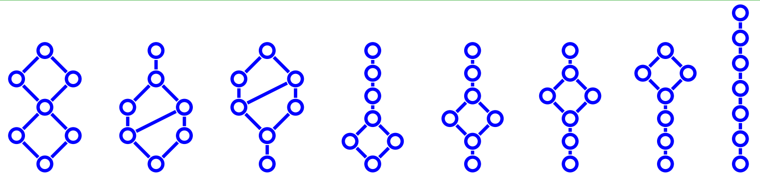
distributive lattices on 6 element sets



Example G.4. [23] There are a total of 53 unlabeled lattices on a seven element set; and of these, 8 are *distributive* (Proposition G.3 page 163). Example D.13 (page 124) illustrated all 53 of the unlabeled lattices, Example F.8 (page 143) illustrated the 16 *modular* lattices, and the following illustrates the 8 distributive lattices. Note that none of these lattices are *complemented* (none are *Boolean* (Definition I.1 page 171)).

EX

distributive lattices on 7 element sets



[22] Erné et al. (2002) pages 4–5
[23] Erné et al. (2002) pages 4–5

APPENDIX H

COMPLEMENTED LATTICES

H.1 Definitions

Definition H.1. [1] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE *(Definition* E.1 *page* 133*).*

D E F

An element $x' \in X$ *is a* ***complement*** *of an element* x *in* $\mathbf{L}$ *if*

1. $x \wedge x' = 0$ (NON-CONTRADICTION) *and*
2. $x \vee x' = 1$ (EXCLUDED MIDDLE).

An element x' *in* $\mathbf{L}$ *is the* UNIQUE COMPLEMENT *of* x *in* $\mathbf{L}$ *if* x' *is a* COMPLEMENT *of* x *and* y' *is a* COMPLEMENT *of* $x \implies x' = y'$. $\mathbf{L}$ *is* ***complemented*** *if every element in* X *has a complement in* X. $\mathbf{L}$ *is* ***uniquely complemented*** *if every element in* X *has a unique complement in* X. *A complemented lattice that is* NOT *uniquely complemented is* ***multiply complemented****. A* ***complemented lattice*** *is optionally denoted* $(X, \vee, \wedge, 0, 1; \leq)$.

Definition H.1 (previous) introduced the concept of a *complement* of a lattice. Definition H.2 (next) introduces the concept of a *relative complement* in an *interval* (Definition ?? page ??).

Definition H.2. [2] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

D E F

An element $y \in X$ *is a* ***relative complement*** *of* x *in* $[a, b]$ *with respect to* $\mathbf{L}$ *if*

1. $x \vee y = b$ *and*
2. $x \wedge y = a$.

A lattice $\mathbf{L}$ *is* ***relatively complemented*** *if every element in every closed interval* $[a, b]$ *in* $\mathbf{L}$ *has a complement in* $[a, b]$.

H.2 Examples

Example H.1. [3] The lattice $\left(2^{\{x,y,z\}}, \cup, \cap; \subseteq\right)$ of Example D.2 page 122 is a complemented lattice. The "lattice complement" of each element A is simply the "set complement" $A^c \triangleq 2^{\{x,y,z\}} \setminus A$:

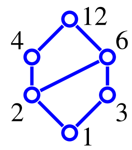
[1] Stern (1999) page 9, Birkhoff (1948) page 23
[2] Birkhoff (1948) page 23
[3] ? page 72

EX

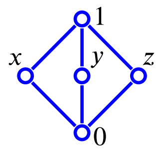
A^c	$A \cup A^c$	$A \cap A^c$
$\mathsf{c}\varnothing = \{x, y, z\}$	$\varnothing \cup \{x, y, z\} = \{x, y, z\}$	$\varnothing \cap \{x, y, z\} = \varnothing$
$\mathsf{c}\{x\} = \{y, z\}$	$\{x\} \cup \{y, z\} = \{x, y, z\}$	$\{x\} \cap \{y, z\} = \varnothing$
$\mathsf{c}\{y\} = \{x, z\}$	$\{y\} \cup \{x, z\} = \{x, y, z\}$	$\{y\} \cap \{x, z\} = \varnothing$
$\mathsf{c}\{x, y\} = \{z\}$	$\{x, y\} \cup \{z\} = \{x, y, z\}$	$\{x, y\} \cap \{z\} = \varnothing$
$\mathsf{c}\{z\} = \{x, y\}$	$\{z\} \cup \{x, y\} = \{x, y, z\}$	$\{z\} \cap \{x, y\} = \varnothing$
$\mathsf{c}\{x, z\} = \{y\}$	$\{x, z\} \cup \{y\} = \{x, y, z\}$	$\{x, z\} \cap \{y\} = \varnothing$
$\mathsf{c}\{y, z\} = \{x\}$	$\{y, z\} \cup \{x\} = \{x, y, z\}$	$\{y, z\} \cap \{x\} = \varnothing$
$\mathsf{c}\{x, y, z\} = \varnothing$	$\{x, y, z\} \cup \varnothing = \{x, y, z\}$	$\{x, y, z\} \cap \varnothing = \varnothing$

Example H.2 (factors of 12). [4] The lattice $\boldsymbol{L} \triangleq (\{1, 2, 3, 4, 6, 12\}, \text{lcm}, \text{gcd}\,;\, |)$ (illustrated to the right) is *non-complemented*. In particular, the elements 2 and 6 have no complements in $\boldsymbol{L}$:

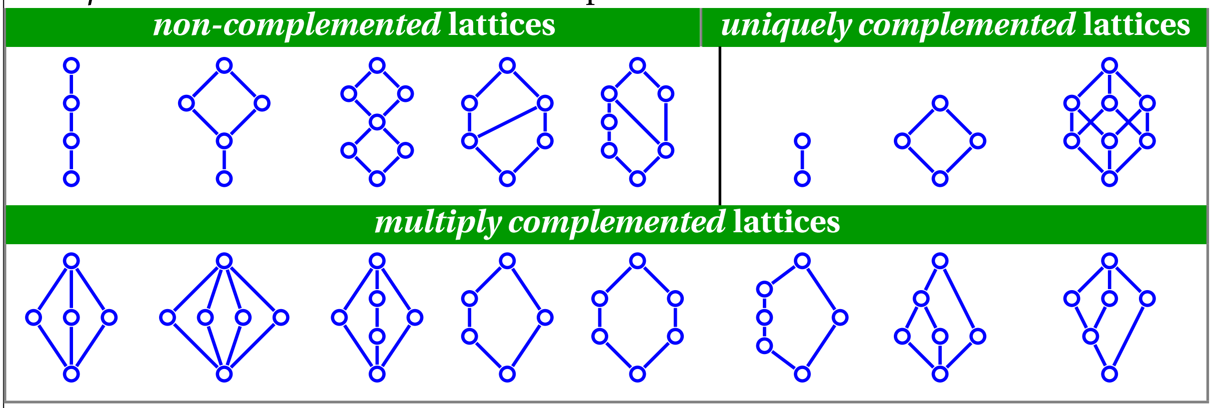
$$\begin{array}{lclclclclcl}
\text{lcm}(2,3) & = & 6 & \ne & 12 & & \gcd(2,3) & = & 1 & & \\
\text{lcm}(2,4) & = & 4 & \ne & 12 & & \gcd(2,4) & = & 2 & \ne & 1 \\
\text{lcm}(2,6) & = & 6 & \ne & 12 & & \gcd(2,2) & = & 2 & \ne & 1 \\
\text{lcm}(6,3) & = & 6 & \ne & 12 & & \gcd(6,3) & = & 3 & \ne & 1 \\
\text{lcm}(6,4) & = & 12 & & & & \gcd(6,4) & = & 2 & \ne & 1
\end{array}$$



Example H.3. [5] The lattice illustrated in the figure to the right is *complemented*. In this complemented lattice, complements are *not unique*. For example, the complement of x is both y and z, the complement of y is both x and z, and the complement of z is both x and y.



Example H.4. Here are some more examples:



Example H.5.

EX

Of the 53 unlabeled lattices on a 7 element set (Example D.13 page 124),
0 are complemented with unique complements,
17 are complemented with multiple complements, and
36 are non-complemented.

H.3 Properties

Theorem H.1 (next) is a landmark theorem in mathematics.

Theorem H.1. [6]

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[4] Durbin (2000) page 271, Saliĭ (1988) pages 26–27
[5] Durbin (2000) page 271
[6] Dilworth (1945) page 123, Saliĭ (1988) page 51, Grätzer (2003) page 378 ⟨Corollary 3.8⟩

THM *For every lattice $\boldsymbol{L}$, there exists a lattice $\boldsymbol{U}$ such that*
1. $\boldsymbol{L} \subseteq \boldsymbol{U}$ ($\boldsymbol{L}$ *is a sublattice of* $\boldsymbol{U}$) *and*
2. $\boldsymbol{U}$ *is* UNIQUELY COMPLEMENTED.

"I therefore propose the following problem…". With these words, Edward Huntington in a 1904 paper introduced one of the most famous problems in mathematical history;[7] a question that took some 40 years to answer, and that in the end had a very surprising solution. Huntington's problem was essentially this: *Are all uniquely complemented lattices also distributive?*[8] This question is significant because if a lattice is both complemented and distributive, then it is *uniquely complemented* (Corollary H.1—next) and, more importantly, is a *Boolean algebra* (Definition I.1 page 171). Being a Boolean algebra is very significant in that it implies the lattice has several powerful properties including that it satisfies *de Morgan's laws* (Theorem D.3 page 118) and that it is isomorphic to an *algebra of sets* (Theorem A.4 page 50).

A uniquely complemented lattice that satifies any one of a number of other conditions is distributive (Theorem H.2 page 167, Literature item 3 page 168). So there was ample evidence that the answer to Huntington's question is "yes". But the final answer to Huntington's problem is actually "no"—an answer that took the mathematical community 40 years to find. The resulting effort had a profound impact on lattice theory in general. In fact, George Grätzer, in a 2007 paper, identified uniquely complemented lattices as one of the "two problems that shaped a century of lattice theory".[9]

This final solution to Huntington's problem was found by Robert Dilworth and published in a 1945 paper.[10] And the answer is this: *Every lattice is a sublattice of a uniquely complemented lattice* (Theorem H.1 page 166). To understand why this answers the question, consider either the *M3 lattice* (Definition G.3 page 149) or the *N5 lattice* (Definition F.4 page 138). Neither of these lattices are *distributive* (Theorem G.2 page 150), but yet either of them can be a sublattice in a uniquely complemented lattice (by *Dilworth's theorem*). That is, it is therefore possible to have a lattice that is both *uniquely complemented* and *non-distributive*.

Corollary H.1. [11] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

COR
$$\left\{\begin{array}{ll} 1. & \boldsymbol{L} \text{ is DISTRIBUTIVE} \quad \text{and} \\ 2. & \boldsymbol{L} \text{ is COMPLEMENTED} \end{array}\right\} \implies \{\boldsymbol{L} \text{ is UNIQUELY COMPLEMENTED}\}$$

✎PROOF:

$\boldsymbol{L}$ is complemented

$\iff \forall x \in L \exists a, b$ such that a, b are complements of x in $\boldsymbol{L}$	by definition of complement page 165
$\iff x \vee a = 1,\ x \vee b = 1,\ x \wedge a = 0,\ x \wedge b = 0$	by definition of complement page 165
$\implies a = b$	by Theorem G.3 page 153
$\implies \boldsymbol{L}$ is uniquely complemented	

✏

Theorem H.2 (Huntington properties). [12] *Let* $\boldsymbol{L}$ *be a lattice.*

[7] For more discussion, see Literature item 7 page 169
[8] Huntington (1904) page 305
[9] Grätzer (2007) page 696
[10] Dilworth (1945) page 123
[11] MacLane and Birkhoff (1999) page 488, Saliĭ (1988) page 30 ⟨Theorem 10⟩
[12] Roman (2008) page 103, Adams (1990) page 79, Saliĭ (1988) page 40, Dilworth (1945) page 123, Grätzer (2007) page 698

THM $\left\{\begin{array}{l} \mathbf{L} \text{ is} \\ \text{UNIQUELY} \\ \text{COMPLEMENTED} \end{array}\right\}$ *and* $\underbrace{\left\{\begin{array}{ll} \mathbf{L} \text{ is MODULAR} & or \\ \mathbf{L} \text{ is ATOMIC} & or \\ \mathbf{L} \text{ is ORTHO-COMPLEMENTED} & or \\ \mathbf{L} \text{ has FINITE WIDTH} & or \\ \mathbf{L} \text{ has DE MORGAN properties} & \end{array}\right\}}_{\text{HUNTINGTON PROPERTIES}} \implies \left\{\begin{array}{l} \mathbf{L} \text{ is} \\ \text{DISTRIBUTIVE} \end{array}\right\}$

Theorem H.3 (Peirce's Theorem). [13] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a bounded lattice. Let* $\complement y \triangleq \{y' \in X \,|\, y' \text{ is a complement of } y\}$.

THM $\{\forall y' \in \complement y,\ x \not\le y' \implies x \wedge y \ne 0\} \implies \left\{\begin{array}{lll} 1. & \mathbf{L} \text{ is UNIQUELY COMPLEMENTED} & and \\ 2. & \mathbf{L} \text{ is DISTRIBUTIVE} & \end{array}\right\}$

H.4 Literature

Literature survey:

1. General treatment of lattice varieties:
   Jipsen and Rose (1992)
2. Distributive lattices:
   Grätzer (1971)
   Balbes and Dwinger (1975)
   Dilworth (1984)
3. Uniquely complemented lattices:
   Dilworth (1945) ⟨"Every lattice is a sublattice of a lattice with unique complements."⟩
   Saliĭ (1988) ⟨ISBN:0821845225⟩
   Adams (1990) pages 79–84
   Grätzer (2007)
   Roman (2008) page 103
   Bergman (1929) ⟨uniquely complemented + *modular* = distributive⟩
   Birkhoff (1940) ⟨uniquely complemented + *ortho-complemented* = distributive⟩
   Birkhoff and Ward (1939a) ⟨uniquely complemented + *atomic* = distr.⟩
   Birkhoff and Ward (1939b) ⟨uniquely complemented + *atomic* = distributive⟩
4. Projective distributive lattices:
   Balbes (1967)
   Balbes and Horn (1970)
5. Median property:
   Birkhoff and Kiss (1947a)
   Birkhoff and Kiss (1947b)
   Grau (1947)
   Evans (1977)
   Isbell (1980)
   Bandelt and Hedlíková (1983)
   Birkhoff and Ward (1987) pages 1–8
   Artamonov (2000) page 554 ⟨median algebras⟩
   Grätzer (2008) page 356
6. Properties of lattices
   (a) The fact that lattices are not in general *distributive* was not always universally accepted. In a famous 1880 paper, Charles S. Peirce(Peirce, 1880b)33 presents distributivity as a property of all lattices but says that "the proof is too tedius to give".

[13] Saliĭ (1988) pages 38–39 ⟨"Peirce's Theorem"⟩, Peirce ⟨1902 January 31 entry⟩, Peirce (1903) ⟨letter to Huntington⟩, Peirce (1904) ⟨letter to Huntington⟩, Huntington (1904)

7. Note about *Huntington's problem* concerning uniquely complemented lattices:

   (a) Salii[14] suggests that Huntington's problem is actually motivated by and a simple extension of *Peirce's Theorem* (Theorem H.3 page 168). That is, Huntington's problem is equivalent to asking if the uniquely complemented property is equivalent to the left hypothesis in Peirce's Theorem.

   (b) George Grätzer in a 2007 paper seems to indicate that Huntington's 1904 paper[15] is *not* the original source of "Huntington's problem". In particular, Grätzer says "...Neither gives any references as to the origin of the problem. G. Birkhoff and M. Ward, 1933, reference E. V. Huntington, 1904, for the lattice axioms, which Huntington stated as being due to E. Schröder, but not for the problem. If the reader is surprised, I suggest he try to read the original paper of E. V. Huntington, and there he may find the clue. In my earlier papers on the subject, I reference only R. P. Dilworth, 1945, but in my lattice books (e.g., [7]) I give the correct reference. But I have no recollection of reading E. V. Huntington, 1904, until the preparation for this article." (Grätzer (2007) page 699) The reference [7] is Grätzer (2003). In this reference, Dilworth's 1945 theorem is presented on page 378, and its historical background is discussed on page 392. However, this discussion does not seem to give credit for Huntington's problem to anyone other than Huntington (1904). Perhaps it is Peirce that Grätzer has in mind with these comments—but so far the person referred to by Grätzer is unclear (to me). See also `http://groups.google.com/group/sci.math/browse_thread/thread/b7790be1efe8946e#`

8. General treatment of lattice varieties:
   Jipsen and Rose (1992)

9. Atomic lattices:
   Birkhoff (1938) page 800 ⟨see footnote ‡⟩

[14] Saliǐ (1988) pages 38–39 ⟨"Peirce's Theorem"⟩
[15] Huntington (1904) page 305

APPENDIX I

BOOLEAN LATTICES

❝*That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved.... by Mr. Boole. The unity of the forms of thought in all the applications of reason, however remotely separated, will one day be matter of notoriety and common wonder: and Boole's name will be remembered in connection with one of the most important steps towards the attainment of knowledge.*❞
Augustus de Morgan (1806–1871), British mathematician and logician, [1]

I.1 Definition and properties

A *Boolean algebra* (next definition) is a *bounded* (Definition E.1 page 133), *distributive* (Definition G.2 page 146), and *complemented* (Definition H.1 page 165), *lattice* (Definition D.3 page 117).

Definition I.1. [2]

DEF

The BOUNDED LATTICE *(Definition E.1 page 133)* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1\,;\, \le)$ *is* ***Boolean*** *if*

1. $\boldsymbol{L}$ *is* COMPLEMENTED *(Definition H.1 page 165) and*
2. $\boldsymbol{L}$ *is* DISTRIBUTIVE *(Definition G.2 page 146).*

A BOUNDED LATTICE $\boldsymbol{L}$ *that is* BOOLEAN *is a* ***Boolean algebra*** *or a* ***Boolean lattice****.*
A BOOLEAN LATTICE *with* 2^N *elements is denoted* $\boldsymbol{L}_2^N$*.*

Several examples of *Boolean lattices* are illustrated in Example J.2 (page 196).

Proposition I.1.

[1] quote: DeMorgan (1872) page 80
image: http://en.wikipedia.org/wiki/Augustus_De_Morgan
[2] MacLane and Birkhoff (1999) page 488, Jevons (1864)

PRP: *The algebraic structure* $\mathbf{A} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *is a* ***Boolean algebra*** *(Definition I.1 page 171) if*

$$\begin{array}{llclll}
1. & (X, \vee, \wedge, 0, 1; \leq) \text{ is a BOUNDED LATTICE (Definition E.1 page 133)} & & & & \text{and}\\
2. & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X \text{ (DISTRIBUTIVE)} & \text{and}\\
3. & x \wedge x' & = & 0 & \forall x \in X \text{ (NON-CONTRADICTION)} & \text{and}\\
4. & x \vee x' & = & 1 & \forall x \in X \text{ (EXCLUDED MIDDLE)}. &
\end{array}.$$

✎PROOF: This follows directly from Definition I.1 (page 171). ✏

Boolean algebras support the *principle of duality* (next theorem).

Theorem I.1 (Principle of duality). [3] *Let* $\mathbf{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a Boolean algebra.*

THM:
$$\left\{\begin{array}{l}\phi \text{ is an identity on } \mathbf{B} \text{ in terms}\\ \text{of the operations}\\ \vee, \wedge, ', 0, \text{ and } 1\end{array}\right\} \implies \mathbf{T}\phi \text{ is also an identity on } \mathbf{B}$$

where the operator $\mathbf{T}$ *performs the following mapping on the operations in* X^X:

$$0 \to 1, \qquad 1 \to 0, \qquad \vee \to \wedge, \qquad \wedge \to \vee$$

✎PROOF: For each of the identities in the definition of Boolean algebras (Proposition I.5 page 188), the operator $\mathbf{T}$ produces another identity that is also in the definition:

$$\begin{array}{lclclclclclcl}
\mathbf{T}(1a) & = & \mathbf{T}[x \vee y & = & y \vee x] & = & [x \wedge y & = & y \wedge x] & = & (1b)\\
\mathbf{T}(1b) & = & \mathbf{T}[x \wedge y & = & y \wedge x] & = & [x \vee y & = & y \vee x] & = & (1a)\\
\mathbf{T}(2a) & = & \mathbf{T}[x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z)] & = & [x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z)] & = & (2b)\\
\mathbf{T}(2b) & = & \mathbf{T}[x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z)] & = & [x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z)] & = & (2a)\\
\mathbf{T}(3a) & = & \mathbf{T}[x \vee 0 & = & x] & = & [x \wedge 1 & = & x] & = & (3b)\\
\mathbf{T}(3b) & = & \mathbf{T}[x \wedge 1 & = & x] & = & [x \vee 0 & = & x] & = & (3a)\\
\mathbf{T}(4a) & = & \mathbf{T}[x \vee x' & = & 1] & = & [x \wedge x' & = & 0] & = & (4b)\\
\mathbf{T}(4b) & = & \mathbf{T}[x \wedge x' & = & 0] & = & [x \vee x' & = & 1] & = & (4a)
\end{array}$$

Therefore, if the statement ϕ is consistent with regards to the Boolean algebra $\mathbf{B}$, then $\mathbf{T}\phi$ is also consistent with regards to the Boolean algebra $\mathbf{B}$. ✏

I.2 Order properties

The definition of Boolean algebras given by Definition I.1 is a set of postulates known as *Huntington's* FIRST SET. Lemma I.1 (next) gives a link between *Huntington's* FIRST SET of Boolean algebra postulates and the *classic 10* set of Boolean algebra postulates (Theorem I.2 page 176).

Lemma I.1. [4] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a bounded lattice.*

[3] Givant and Halmos (2009) pages 20–22 ⟨Chapter 4⟩, Sikorski (1969) page 8
[4] Huntington (1904) pages 292–296 ⟨“1st set”⟩, Joshi (1989) pages 224–227

LEM

If $\forall x, y, z \in X$

$$\left\{\begin{array}{l|lcl|lcl|ll}
① & x \vee y & = & y \vee x & x \wedge y & = & y \wedge x & \text{(commutative)} & \text{and} \\
② & x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z) & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) & \text{(distributive)} & \text{and} \\
③ & x \vee 0 & = & x & x \wedge 1 & = & x & \text{(identity)} & \text{and} \\
④ & x \vee x' & = & 1 & x \wedge x' & = & 0 & \text{(complemented)} &
\end{array}\right\}$$

then $\forall x, y, z \in X$

$$\left\{\begin{array}{l|lcl|lcl|ll}
1. & x \vee x & = & x & x \wedge x & = & x & \text{(idempotent)} & \text{and} \\
2. & x \vee (y \vee z) & = & (x \vee y) \vee z & x \wedge (y \wedge z) & = & (x \wedge y) \wedge z & \text{(associative)}^{5} & \text{and} \\
3. & x \vee (x \wedge y) & = & x & x \wedge (x \vee y) & = & x & \text{(absorptive)} & \text{and} \\
4. & x \vee 1 & = & 1 & x \wedge 0 & = & 0 & \text{(bounded)} & \text{and} \\
5. & (x \vee y)' & = & x' \wedge y' & (x \wedge y)' & = & x' \vee y' & \text{(de Morgan's laws)}. &
\end{array}\right\}$$

✎PROOF: For each pair of properties, it is only necessary to prove one of them, as the other follows by the *principle of duality* (Theorem I.1 page 172). Let the *join* $\vee$ be represented by $+$, the operation *meet* $\wedge$ represented by $\cdot$ or juxtaposition, and let $\wedge$ have algebraic precedence over $\vee$.

1. Proof that $x + x = x$ and $xx = x$ (*idempotent* properties):

$$\begin{aligned}
x + x &= (x + x) \cdot 1 && \text{by } \textit{identity} \text{ property,} && ③\text{b} \\
&= (x + x)(x + x') && \text{by } \textit{complemented} \text{ property,} && ④\text{a} \\
&= x + (xx') && \text{by } \textit{distributive} \text{ property,} && ②\text{a} \\
&= x + 0 && \text{by } \textit{complemented} \text{ property,} && ④\text{b} \\
&= x && \text{by } \textit{identity} \text{ property,} && ③\text{a}
\end{aligned}$$

2. Proof that $x + 1 = 1$ and $x \cdot 0 = 0$ (*bounded* properties):

$$\begin{aligned}
x + 1 &= (x + 1) \cdot 1 && \text{by } \textit{identity} \text{ property,} && ③\text{b} \\
&= 1 \cdot (x + 1) && \text{by } \textit{commutative} \text{ property,} && ①\text{b} \\
&= (x + x')(x + 1) && \text{by } \textit{complemented} \text{ property,} && ④\text{a} \\
&= x + (x' \cdot 1) && \text{by } \textit{distributive} \text{ property,} && ②\text{a} \\
&= x + x' && \text{by } \textit{identity} \text{ property,} && ③\text{b} \\
&= 1 && \text{by } \textit{complemented} \text{ property,} && ④\text{a}
\end{aligned}$$

3. Proof that $x + (xy) = x$ and $x(x + y) = x$: (*absorptive* properties)

$$\begin{aligned}
x + (x \cdot y) &= (x \cdot 1) + (xy) && \text{by } \textit{identity} \text{ property,} && ③\text{b} \\
&= x \cdot (1 + y) && \text{by } \textit{distributive} \text{ property,} && ②\text{b} \\
&= x \cdot (y + 1) && \text{by } \textit{commutative} \text{ property,} && ①\text{a} \\
&= x \cdot 1 && \text{by item (2)} && \\
&= x && \text{by } \textit{identity} \text{ property,} && ③\text{b}
\end{aligned}$$

4. Proof that $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ (*associative* properties):
   Let $a \triangleq x(yz)$ and $b \triangleq (xy)z$.

[5] K.D. Joshi comments that having the *associative* property as a result of an axiom rather than as an axiom, is a very unusual and "remarkable property" in the world of algebras. Joshi (1989) pages 225–226

(a) Proof that $a + x = b + x$:

$a + x = x(yz) + x$	by definition of a	
$= x(yz) + x1$	by *identity* property,	③b
$= x(yz + 1)$	by *distributive* property,	②a
$= x(1)$	by *bounded* property,	item (2)
$= x$	by *identity* property,	③b
$= x(x + z)$	by *absorptive* property,	item (3)
$= (x + xy)(x + z)$	by *absorptive* property,	item (3)
$= x + (xy)z$	by *distributive* property,	②b
$= (xy)z + x$	by *commutative* property,	①a,b
$= b + x$	by definition of b	

(b) Proof that $a + x' = b + x'$:

$a + x' = x(yz) + x'$	by definition of a	
$= x' + x(yz)$	by *commutative* property,	①a,b
$= (x' + x)(x' + yz)$	by *distributive* property,	②b
$= 1 \cdot (x' + yz)$	by *complemented* property,	④a
$= x' + yz$	by *identity* property,	③b
$= (x' + y)(x' + z)$	by *distributive* property,	②b
$= \left[(x' + y) \cdot 1\right](x' + z)$	by *identity* property,	③b
$= \left[1 \cdot (x' + y)\right](x' + z)$	by *commutative* property,	①b
$= \left[(x + x')(x' + y)\right](x' + z)$	by *complemented* property,	④a
$= (x' + xy)(x' + z)$	by *distributive* property,	②b
$= x' + (xy)z$	by *distributive* property,	②b
$= (xy)z + x'$	by *commutative* property,	①a
$= b + x'$	by definition of b	

(c) Proof that $x(yz) = (xy)z$:

$x(yz) \triangleq a$	by definition of a	
$= a + a$	by *idempotent* property,	item (1)
$= a + a1 + 0$	by *identity* property,	③a,b
$= a + a(x + x') + xx'$	by *complemented* property,	④a,b
$= a + ax + ax' + xx'$	by *distributive* property,	②a
$= a + ax' + xa + xx'$	by *commutative* property,	①a,b
$= aa + ax' + xa + xx'$	by *idempotent* property,	item (1)
$= a(a + x') + x(a + x')$	by *distributive* property,	②a
$= (a + x)(a + x')$	by *distributive* property,	②a
$= (b + x)(a + x')$	by item (4a)	
$= (b + x)(b + x')$	by item (4b)	
$= (b + x)b + (b + x)x'$	by *distributive* property,	②a
$= b(b + x) + x'(b + x)$	by *commutative* property,	①b
$= bb + bx + x'b + x'x$	by *distributive* property,	②a
$= b + bx + x'b + x'x$	by *idempotent* property,	item (1)
$= b + bx + bx' + x'x$	by *commutative* property,	①b

$$
\begin{aligned}
&= b + b(x+x') + x'x && \text{by } \textit{distributive} \text{ property,} && \text{②a}\\
&= b + b\cdot 1 + 0 && \text{by } \textit{complemented} \text{ property,} && \text{④a,b}\\
&= b + b && \text{by } \textit{identity} \text{ property,} && \text{③a,b}\\
&= b && \text{by } \textit{idempotent} \text{ property,} && \text{item (1)}\\
&\triangleq (xy)z && \text{by definition of } b
\end{aligned}
$$

5. Proof that $(x+y)' = x'y'$ and $(xy)' = x' + y'$: (*de Morgan* properties)

   (a) Proof that $(x+y)+(x'y') = 1$:

$$
\begin{aligned}
&(x+y)+(x'y')\\
&= \left[(x+y)+x'\right]\left[(x+y)+y'\right] && \text{by } \textit{distributive} \text{ property,} && \text{②a}\\
&= \left[x'+(x+y)\right]\left[y'+(x+y)\right] && \text{by } \textit{commutative} \text{ property,} && \text{①a}\\
&= \left[\left(x'+(x+y)\right)1\right]\left[\left(y'+(x+y)\right)1\right] && \text{by } \textit{identity} \text{ property,} && \text{③b}\\
&= \left[1\left(x'+(x+y)\right)\right]\left[1\left(y'+(y+x)\right)\right] && \text{by } \textit{distributive} \text{ property,} && \text{②b}\\
&= \left[\left(x'+x\right)\left(x'+(x+y)\right)\right]\left[\left(y'+y\right)\left(y'+(y+x)\right)\right] && \text{by } \textit{complemented} \text{ property,} && \text{④a}\\
&= \left[x'+(x(x+y))\right]\left[y'+(y(y+x))\right] && \text{by } \textit{distributive} \text{ property,} && \text{②a}\\
&= \left[x'+x\right]\left[y'+y\right] && \text{by } \textit{absorptive} \text{ property,} && \text{item (3)}\\
&= [1][1] && \text{by } \textit{complemented} \text{ property,} && \text{④a}\\
&= 1 && \text{by } \textit{bounded} \text{ property,} && \text{item (2)}
\end{aligned}
$$

   (b) Proof that $(x+y)(x'y') = 0$:

$$
\begin{aligned}
(x+y)(x'y') &= \left[x(x'y')\right]+\left[y(x'y')\right] && \text{by } \textit{distributive} \text{ property,} && \text{②b}\\
&= \left[0+x(x'y')\right]+\left[0+y(x'y')\right] && \text{by } \textit{identity} \text{ property,} && \text{③a}\\
&= \left[(xx')+x(x'y')\right]+\left[(yy')+y(x'y')\right] && \text{by } \textit{complemented} \text{ property,} && \text{④b}\\
&= \left[x\left(x'+x'y'\right)\right]+\left[y\left(y'+x'y'\right)\right] && \text{by } \textit{distributive} \text{ property,} && \text{②b}\\
&= \left[xx'\right]+\left[yy'\right] && \text{by } \textit{absorptive} \text{ property,} && \text{item (3)}\\
&= [0]+[0] && \text{by } \textit{complemented} \text{ property,} && \text{④b}\\
&= 0 && \text{by } \textit{bounded} \text{ property,} && \text{item (2)}
\end{aligned}
$$

   (c) Proof that $(x+y)' = x'y'$:
   The quanities $(x+y)$ and $x'y'$ are *complements* of each other as demonstrated by item (5a) ($(x+y)+(x'y') = 1$) and item (5b) ($(x+y)(x'y') = 0$). Therefore, $(x+y)' = x'y'$.

✏

Proposition I.2. [6] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ *be a Boolean algebra.*

PRP *The pair* $(X, \leq)$ *is an* ORDERED SET. *In particular,*

1.	$x \leq x$	$\forall x \in X$	(REFLEXIVE)	*and*
2.	$x \leq y$ *and* $y \leq z \implies x \leq z$	$\forall x,y,z \in X$	(TRANSITIVE)	*and*
3.	$x \leq y$ *and* $y \leq x \implies x = y$	$\forall x,y \in X$	(ANTI-SYMMETRIC).	

✎PROOF:

1. Proof that $\leq$ is *reflexive* in $(X, \leq)$:

$$
\begin{aligned}
x \leq x &\iff x \vee x = x && \text{by definition of } \leq \text{ (Definition I.1 page 171)}\\
&\iff \text{true} && \text{by Lemma I.1 page 172}
\end{aligned}
$$

[6] Sikorski (1969) page 7

2. Proof that $\le$ is *transitive* in $(X, \le)$:

$$\begin{aligned}
\{(x \le y) \text{ and } (y \le z)\} &\iff \{(x \vee y = y) \text{ and } (y \vee z = z)\} && \text{by definition of } \le \text{ (Definition I.1 page 171)}\\
&\implies (x \vee z)\\
&= x \vee (y \vee z)\\
&= (x \vee y) \vee z && \text{by } \textit{associative} \text{ property of Lemma I.1 page 172}\\
&= y \vee z\\
&= z
\end{aligned}$$

3. Proof that $\le$ is *anti-symmetric* in $(X, \le)$:

$$\begin{aligned}
\{(x \le y) \text{ and } (y \le x)\} &\iff \{(x \vee y = y) \text{ and } (y \vee x = x)\} && \text{by definition of } \le \text{ (Definition I.1 page 171)}\\
&\iff \{(x \vee y = y) \text{ and } (x \vee y = x)\} && \text{by } \textit{commutative} \text{ property of Definition I.1 page 171}\\
&\iff x = x \vee y = y\\
&\implies x = y
\end{aligned}$$

Proposition I.3. *Let $(X, \vee, \wedge, 0, 1; \le)$ be a Boolean algebra.*

PRP

$x \vee y$	*is the* LEAST UPPER BOUND	*of x and y in $(X, \le)$.*
$x \wedge y$	*is the* GREATEST LOWER BOUND	*of x and y in $(X, \le)$.*

Theorem I.2 (classic 10 Boolean properties). [7]

THM

$\mathbf{A} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *is a* ***Boolean algebra*** $\iff$ $\forall x, y, z \in X$

property with emphasis on $\vee$			dual property with emphasis on $\wedge$			property name	
$x \vee x$	$=$	x	$x \wedge x$	$=$	x	(IDEMPOTENT)	*and*
$x \vee y$	$=$	$y \vee x$	$x \wedge y$	$=$	$y \wedge x$	(COMMUTATIVE)	*and*
$x \vee (y \vee z)$	$=$	$(x \vee y) \vee z$	$x \wedge (y \wedge z)$	$=$	$(x \wedge y) \wedge z$	(ASSOCIATIVE)	*and*
$x \vee (x \wedge y)$	$=$	x	$x \wedge (x \vee y)$	$=$	x	(ABSORPTIVE)	*and*
$x \vee (y \wedge z)$	$=$	$(x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z)$	$=$	$(x \wedge y) \vee (x \wedge z)$	(DISTRIBUTIVE)	*and*
$x \vee 0$	$=$	x	$x \wedge 1$	$=$	x	(IDENTITY)	*and*
$x \vee 1$	$=$	1	$x \wedge 0$	$=$	0	(BOUNDED)	*and*
$x \vee x'$	$=$	1	$x \wedge x'$	$=$	0	(COMPLEMENTED)	*and*
$(x \vee y)'$	$=$	$x' \wedge y'$	$(x \wedge y)'$	$=$	$x' \vee y'$	(DE MORGAN)	*and*
$(x')'$	$=$	x				(INVOLUTORY).	

PROOF:

1. Proof that Proposition I.5 (page 188) $\implies$ Theorem I.2 (page 176):

1.	Proof that $\mathbf{A}$ is *idempotent*:	by 1	of Lemma I.1	page 172
2.	Proof that $\mathbf{A}$ is *commutative*:	by 1	of Proposition I.5	page 188
3.	Proof that $\mathbf{A}$ is *associative*:	by 2	of Lemma I.1	page 172
4.	Proof that $\mathbf{A}$ is *absorptive*:	by 3	of Lemma I.1	page 172
5.	Proof that $\mathbf{A}$ is *distributive*:	by 2	of Proposition I.5	page 188
6.	Proof that $\mathbf{A}$ is *identity*:	by 3	of Proposition I.5	page 188

[7] Huntington (1904) pages 292–293 ⟨"1st set"⟩, Huntington (1933) page 280 ⟨"4th set"⟩, MacLane and Birkhoff (1999) page 488, Givant and Halmos (2009) page 10, Müller (1909) pages 20–21, Schröder (1890), Whitehead (1898) pages 35–37

7. Proof that $\mathbf{A}$ is *bounded*: by 4 of Lemma I.1 page 172
8. Proof that $\mathbf{A}$ is *complemented*: by 4 of Proposition I.5 page 188
9. Proof that $\mathbf{A}$ is *involutory*: by Corollary H.1 page 167
10. Proof that $\mathbf{A}$ is *de Morgan*: by 5 of Lemma I.1 page 172

2. Proof that Proposition I.5 (page 188) $\Longleftarrow$ Theorem I.2 (page 176):
   1. Proof that $\mathbf{A}$ is *commutative*: by 2 of Theorem I.2 page 176
   2. Proof that $\mathbf{A}$ is *distributive*: by 5 of Theorem I.2 page 176
   3. Proof that $\mathbf{A}$ is *identity*: by 6 of Theorem I.2 page 176
   4. Proof that $\mathbf{A}$ is *complemented*: by 8 of Theorem I.2 page 176

⇨

Lemma I.2.

LEM

$$\left.\begin{array}{l}(X, \vee, \wedge, 0, 1\,;\, \leq)\\ \text{is a \textsc{Boolean algebra}}\end{array}\right\} \implies \left\{\begin{array}{llcll} 1. & x' \vee (x \wedge y) & = & x' \vee y & \forall x,y \in X \quad \text{(\textsc{Sasaki hook})} \quad \text{and}\\ 2. & x \vee (x' \wedge y) & = & x \vee y & \forall x,y \in X\end{array}\right.$$

✎PROOF:

$$\begin{aligned}
x' \vee (x \wedge y) &= \underbrace{x' \vee (x' \wedge y)}_{x'} \vee (x \wedge y) && \text{by \textit{absorption} property (Theorem I.2 page 176)}\\
&= x' \vee \left[(x' \vee x) \wedge y\right] && \text{by \textit{associative} and \textit{distributive} properties (Theorem I.2 page 176)}\\
&= x' \vee [1 \wedge y] && \text{by \textit{excluded middle} property (Theorem I.2 page 176)}\\
&= x' \vee y && \text{by definition of 1 (Definition C.21 page 114)}\\
x \vee (x' \wedge y) &= \underbrace{x \vee (x \wedge y)}_{x} \vee (x \wedge y) && \text{by \textit{absorption} property (Theorem I.2 page 176)}\\
&= x \vee \left[(x \vee x') \wedge y\right] && \text{by \textit{associative} and \textit{distributive} properties (Theorem I.2 page 176)}\\
&= x \vee [1 \wedge y] && \text{by \textit{excluded middle} property (Theorem I.2 page 176)}\\
&= x \vee y && \text{by definition of 1 (Definition C.21 page 114)}
\end{aligned}$$

⇨

Theorem I.3. [8] *Let* $|X|$ *be the number of elements in a finite set* X.

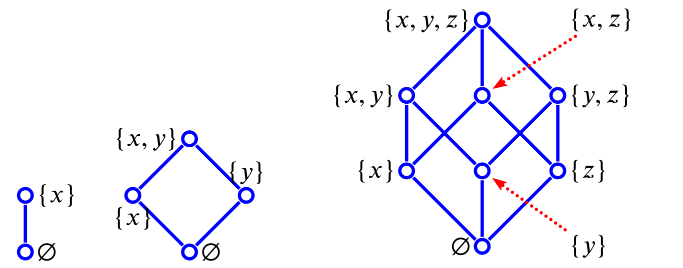
THM

$$\mathbf{A} \text{ is a \textsc{Boolean algebra}} \implies |\mathbf{A}| = 2^n \text{ for some } n \in \mathbb{N}.$$

Example I.1. Here are some lattices that are Boolean algebras.

EX

Boolean algebras as algebras of sets



[8] Joshi (1989) page 237

Theorem I.4.

THM

If $(X, \vee, \wedge, 0, 1; \leq)$ *is a* BOOLEAN ALGEBRA *then*

$$\left\{\begin{array}{llllll}
\multicolumn{3}{c}{\textit{tautology}} & \multicolumn{3}{c}{\textit{dual}} \\
\hline
\neg\left(\bigwedge_{n=1}^{N} x_n\right) & = & \bigvee_{n=1}^{N}(\neg x_n) & \neg\left(\bigvee_{n=1}^{N} x_n\right) & = & \bigwedge_{n=1}^{N}(\neg x_n) \\
\left(\bigwedge_{n=1}^{N} x_n\right) \vee y & = & \bigwedge_{n=1}^{N}(x_n \vee y) & \left(\bigvee_{n=1}^{N} x_n\right) \wedge y & = & \bigvee_{n=1}^{N}(x_n \wedge y)
\end{array}\right\} \quad \forall x_n \in X, N \in \mathbb{N}$$

✎PROOF:

1. Proof that $\neg\left(\bigwedge_{n=1}^{N} x_n\right) = \bigvee_{n=1}^{N}(\neg x_n)$ (by induction):

Proof for $N = 1$ case:

$$\begin{aligned}
\neg\left(\bigwedge_{n=1}^{N=1} x_n\right) &= \neg x_n && \text{by definition of } \wedge \\
&= \bigvee_{n=1}^{N=1}(\neg x_n) && \text{by definition of } \vee
\end{aligned}$$

Proof for $N = 2$ case:

$$\begin{aligned}
\neg\left(\bigwedge_{n=1}^{N=2} x_n\right) &= (\neg x_1) \vee (\neg x_2) && \text{by Theorem I.2 page 176} \\
&= \bigvee_{n=1}^{N=2}(\neg x_n) && \text{by definition of } \vee
\end{aligned}$$

Proof that (N case) $\implies$ ($N + 1$ case):

$$\begin{aligned}
\neg\left(\bigwedge_{n=1}^{N+1} x_n\right) &= \neg\left[\left(\bigwedge_{n=1}^{N} x_n\right) \wedge x_N\right] && \text{by definition of } \wedge \\
&= \left(\neg\bigwedge_{n=1}^{N} x_n\right) \vee (\neg x_{N+1}) && \text{by Theorem I.2 page 176} \\
&= \left[\bigvee_{n=1}^{N}(\neg x_n)\right] \vee (\neg x_{N+1}) && \text{by left hypothesis} \\
&= \bigvee_{n=1}^{N+1}(\neg x_n) && \text{by definition of } \vee
\end{aligned}$$

2. Proof that $\neg\left(\bigvee_{n=1}^{N} x_n\right) = \bigwedge_{n=1}^{N}(\neg x_n)$:

$$\begin{aligned}
\neg\left(\bigvee_{n=1}^{N} x_n\right) &= \neg\left(\bigvee_{n=1}^{N}(\neg\neg x_n)\right) && \text{by Theorem I.2 page 176} \\
&= \neg\neg\left(\bigwedge_{n=1}^{N}(\neg x_n)\right) && \text{by previous result 1.} \\
&= \bigwedge_{n=1}^{N}(\neg x_n) && \text{by Theorem I.2 page 176}
\end{aligned}$$

3. Proof that $\left(\bigwedge_{n=1}^{N} x_n\right) \vee y = \bigvee_{n=1}^{N}(x_n \vee y)$ (by induction):

Proof for $N = 1$ case:

$$\begin{aligned}
\left(\bigwedge_{n=1}^{N=1} x_n\right) \vee y &= x_1 \vee y && \text{by definition of } \wedge \\
&= \bigwedge_{n=1}^{N=1}(x_n \vee y) && \text{by definition of } \wedge
\end{aligned}$$

Proof for $N = 2$ case:

$$\begin{aligned}
\left(\bigwedge_{n=1}^{N=2} x_n\right) \vee y &= (x_1 \vee y) \wedge (x_2 \vee y) && \text{by Theorem I.2 page 176} \\
&= \bigwedge_{n=1}^{N=2}(x_n \vee y) && \text{by definition of } \wedge
\end{aligned}$$

Proof that (N case) $\Longrightarrow$ ($N + 1$ case):

$$\begin{aligned}
\left(\bigwedge_{n=1}^{N+1} x_n\right) \vee y &= \left[\left(\bigwedge_{n=1}^{N} x_n\right) \wedge x_{N+1}\right] \vee y && \text{by definition of } \wedge \\
&= \left[\left(\bigwedge_{n=1}^{N} x_n\right) \vee y\right] \wedge (x_{N+1} \vee y) && \text{by Theorem I.2 page 176} \\
&= \left[\bigwedge_{n=1}^{N}(x_n \vee y)\right] \wedge (x_{N+1} \vee y) && \text{by left hypothesis} \\
&= \bigwedge_{n=1}^{N+1}(x_n \vee y) && \text{by definition of } \wedge
\end{aligned}$$

4. Proof that $\left(\bigvee_{n=1}^{N} x_n\right) \wedge y = \bigwedge_{n=1}^{N}(x_n \wedge y)$:

$$\begin{aligned}
\left(\bigvee_{n=1}^{N} x_n\right) \wedge y &= \neg\neg\left[\left(\bigvee_{n=1}^{N} x_n\right) \wedge y\right] && \text{by Theorem I.2 page 176} \\
&= \neg\left[\neg\left(\bigvee_{n=1}^{N} x_n\right) \vee (\neg y)\right] && \text{by Theorem I.2 page 176} \\
&= \neg\left[\left(\bigwedge_{n=1}^{N}(\neg x_n)\right) \vee (\neg y)\right] && \text{by previous result 2.} \\
&= \neg\left(\bigwedge_{n=1}^{N}[(\neg x_n) \vee (\neg y)]\right) && \text{by previous result 3.} \\
&= \left(\bigvee_{n=1}^{N} \neg[(\neg x_n) \vee (\neg y)]\right) && \text{by previous result 1.} \\
&= \bigvee_{n=1}^{N}(x_n \wedge y) && \text{by Theorem I.2 page 176}
\end{aligned}$$

I.3 Additional operations

Propositional logic has a total of $2^4 = 16$ operations in the class of functions $\{0,1\}^{\{0,1\}^2}$ (see page 35). The 16 logic operations of propositional logic can all be represented using the logic operations of *disjunction* $\vee$, *conjunction* $\wedge$, and *negation* $\neg$. Using these representations, all 16 operations can be generalized to *Boolean algebras* using the equivalent Boolean algebra/lattice operations of *join*, *meet*, and *complement*.[9] Several of these additional operations for Boolean algebras are defined in Definition I.2 (next).

Definition I.2 (additional Boolean algebra operations). [10] *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be a Boolean algebra. The following table defines additional operations in* $X^{X\times X}$ *in terms of* $\vee$, $\wedge$, *and* $'$. *Let* $x' \triangleq {}'x$ *and* $y' \triangleq {}'y$.

name	symbol	definition			
rejection	$\downarrow$	$x \downarrow y$	$\triangleq$	$x' \wedge y'$	$\forall x,y\in X$
exception	$-$	$x - y$	$\triangleq$	$x \wedge y'$	$\forall x,y\in X$
adjunction	$\div$	$x \div y$	$\triangleq$	$x \vee y'$	$\forall x,y\in X$
Sheffer stroke	$\vert$	$x\vert y$	$\triangleq$	$x' \vee y'$	$\forall x,y\in X$
Boolean addition	$\triangle$	$x \triangle y$	$\triangleq$	$(x' \wedge y) \vee (x \wedge y')$	$\forall x,y\in X$
inhibit x	$\ominus$	$x \ominus y$	$\triangleq$	$x' \wedge y$	$\forall x,y\in X$
implication	$\Rightarrow$	$x \Rightarrow y$	$\triangleq$	$x' \vee y$	$\forall x,y\in X$
biconditional	$\Leftrightarrow$	$x \Leftrightarrow y$	$\triangleq$	$(x \wedge y) \vee (x' \wedge y')$	$\forall x,y\in X$

Theorem I.5. [11]

THM

$\vee$	*(join)*	*is the dual of*	$\downarrow$	*(rejection)*
$\wedge$	*(meet)*	*is the dual of*	$\vert$	*(Sheffer stroke)*
$\triangle$	*(Boolean addition)*	*is the dual of*	$\Leftrightarrow$	*(biconditional)*
$-$	*(exception)*	*is the dual of*	$\Rightarrow$	*(implication)*
$\div$	*(adjunction)*	*is the dual of*	$\ominus$	*(inhibit x)*

✎PROOF:

$$
\begin{array}{rrcll}
(\textit{join}) & (x \vee y)' &=& x' \wedge y' & \text{by } \textit{de Morgan's law} \text{ property (Theorem I.2 page 176)}\\
& &=& x \downarrow y \quad (\textit{rejection}) & \text{by definition of } \textit{rejection} \downarrow \text{ (Definition I.2 page 180)}\\
(\textit{meet}) & (x \wedge y)' &=& x' \vee y' & \text{by } \textit{de Morgan's law} \text{ property (Theorem I.2 page 176)}\\
& &=& x|y \quad (\textit{Sheffer stroke}) & \text{by definition of } \textit{Sheffer stroke} \mid \text{ (Definition I.2 page 180)}\\
(\textit{Boolean addition}) & (x \triangle y)' &=& (x'y \vee xy')' & \text{by def. of } \textit{Boolean addition} \triangle \text{ (Definition I.2 page 180)}\\
& &=& (x \vee y')(x' \vee y) & \text{by } \textit{de Morgan's law} \text{ property (Theorem I.2 page 176)}\\
& &=& xx' \vee xy \vee y'x' \vee y'y & \text{by } \textit{distributive} \text{ property (Theorem I.2 page 176)}\\
& &=& xy \vee x'y' & \\
& &=& x \Leftrightarrow y \quad (\textit{biconditional}) & \text{by def. of } \textit{biconditional} \Leftrightarrow \text{ (Definition I.2 page 180)}\\
(\textit{exception}) & (x - y)' &=& (xy')' & \text{by definition of } \textit{exception} - \text{ (Definition I.2 page 180)}\\
& &=& x' \vee y & \text{by } \textit{de Morgan's law} \text{ property (Theorem I.2 page 176)}\\
& &=& x \Rightarrow y \quad (\textit{implication}) & \text{by definition of } \textit{implication} \Rightarrow \text{ (Definition I.2 page 180)}
\end{array}
$$

[9] Givant and Halmos (2009), page 32

[10] Givant and Halmos (2009) pages 32–33, Bernstein (1934) page 876 ⟨implication ⊃⟩, Huntington (1933) page 276, Taylor (1920) page 243, Bernstein (1914) page 93, Sheffer (1913) pages 487–488, Peirce (1902) page 216, Peirce (1880a) pages 218–221

[11] Givant and Halmos (2009) page 33

$$\begin{aligned}
(adjunction)\quad (x \div y)' &= \left(x \vee y'\right)' && \text{by definition of } adjunction\ \div \text{ (Definition I.2 page 180)}\\
&= x' y && \text{by } de\ Morgan's\ law \text{ property (Theorem I.2 page 176)}\\
&= x \ominus y \quad (inhibit\, x) && \text{by definition of } inhibit\ x \ominus \text{ (Definition I.2 page 180)}\\
(complement\, x)\quad (x \oplus y)' &= \left(x'\right)' && \text{by definition of } complement\ x \oplus\\
&= x && \text{by } involutory \text{ property (Theorem I.2 page 176)}\\
&= x \dashv\!| y \quad (transfer\, x) && \text{by definition of } transfer\ x \dashv\!|\\
(complement\, y)\quad (x \oplus y)' &= \left(y'\right)' && \text{by definition of } complement\ y \oplus\\
&= y && \text{by } involutory \text{ property (Theorem I.2 page 176)}\\
&= x \,|\!\vDash y \quad (transfer\, y) && \text{by definition of } transfer\ y\ |\!\vDash
\end{aligned}$$

⇒

Theorem I.6. [12] *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM

$$\begin{array}{lclclll}
x \leq y & \iff & y' & \leq & x' & & \forall x,y \in X\\
x \leq y & \iff & x & - & y = 0 & & \forall x,y \in X\\
x \leq y & \iff & x & \Rightarrow & y = 1 & & \forall x,y \in X
\end{array}$$

✎PROOF:

1. Proof that $x \leq y \iff y' \leq x'$:

$$\begin{aligned}
x \leq y &\iff x \wedge y = x && \text{by definition of } meet\ \wedge, && \text{Definition C.22 page 114}\\
&\iff (x \wedge y)' = x' && \text{by } de\ Morgan's\ law \text{ property,} && \text{Theorem I.2 page 176}\\
&\iff x' \vee y' = x' && \text{by } de\ Morgan's\ law \text{ property,} && \text{Theorem I.2 page 176}\\
&\iff y' \leq x' && \text{by definition of } join\ \vee, && \text{Definition C.21 page 114}
\end{aligned}$$

2. Proof that $x \leq y \implies x - y = 0$:

$$\begin{aligned}
x - y &= x \wedge y' && \text{by definition of } exception\ -, && \text{Definition I.2 page 180}\\
&\leq y \wedge y' && \text{by left hypothesis}\\
&= 0 && \text{by definition of } complement, && \text{Definition H.1 page 165}
\end{aligned}$$

3. Proof that $x \leq y \impliedby x - y = 0$:

$$\begin{aligned}
x - y = 0 &\iff x \wedge y' = 0 && \text{by definition of } exception\ -, && \text{Definition I.2 page 180}\\
&\iff
\end{aligned}$$

⇒

I.4 Representation

A Boolean algebra $(X, \vee, \wedge, 0, 1; \leq)$ can be represented in terms of five operators (see Theorem I.2 page 176):

- the binary operators join $\vee$ and meet $\wedge$,

[12] Givant and Halmos (2009) page 39

- the unary operator complement $'$, and
- the nullary opeartors 0 and 1.

However, it is also possible to represent a Boolean algebra with fewer operators— in fact, as few as one operator. When a set of operators can completely represent all the operators of a Boolean algebra, then that set is called *functionally complete* (next definition).

Definition I.3. [13] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a Boolean algebra.*

DEF: *A set of operators Φ is **functionally complete** in $\boldsymbol{B}$ if*
$\vee, \wedge, ', 0,$ *and* 1
can all be expressed in terms of Φ.

Here are some examples of functionally complete sets:

$\{\downarrow\}$	(*rejection*)	Theorem I.9	page 183
$\{\mid\}$	(*Sheffer stroke*)	Theorem I.10	page 183
$\{\div, 0\}$	(*adjunction* and 0)	Theorem I.12	page 184
$\{-, 1\}$	(*exception* and 1)	Theorem I.13	page 185
$\{\vee, '\}$	(*join* and *complement*)	Theorem I.7	page 182
$\{\wedge, '\}$	(*meet* and *complement*)	Theorem I.8	page 182
$\{\triangle, \wedge, 1\}$	(*Boolean addition*, *meet*, and 1)	Theorem I.14	page 185
$\{\triangle, \vee, 1\}$	(*Boolean addition*, *join*, and 1)	Theorem I.15	page 186
$\{\triangle, -, '\}$	(*Boolean addition*, *exception*, and *complement*)	Theorem I.16	page 186

Theorem I.7. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM: *The set* $\{\vee, '\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned} x \wedge y &= (x' \vee y)' && \forall x,y \in X \\ 0 &= (x \vee x')' && \forall x \in X \\ 1 &= x \vee x' && \forall x \in X \end{aligned}$$

PROOF:

$$\begin{aligned} x \wedge y &= (x \wedge y)'' && \text{by } \textit{involutory} \text{ property Theorem I.2 page 176} \\ &= (x' \vee y')' && \text{by de Morgan's Law property Theorem I.2 page 176} \\ 1 &= x \vee x' && \text{by complement property Theorem I.2 page 176} \\ 0 &= 1' \\ &= (x \vee x')' && \text{by complement property Theorem I.2 page 176} \end{aligned}$$

Theorem I.8. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM: *The set* $\{\wedge, '\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned} x \vee y &= (x' \wedge y)' && \forall x,y \in X \\ 0 &= x \wedge x' && \forall x \in X \\ 1 &= (x \wedge x')' && \forall x \in X \end{aligned}$$

13 Whitesitt (1995) page 69

✎PROOF:

$$
\begin{aligned}
x \vee y &= (x \vee y)'' && \text{by } \textit{involutory} \text{ property Theorem I.2 page 176}\\
&= \left(x' \wedge y'\right)' && \text{by de Morgan's Law property Theorem I.2 page 176}\\
0 &= x \wedge x' && \text{by complement property Theorem I.2 page 176}\\
1 &= 0' \\
&= \left(x \wedge x'\right)' && \text{by complement property Theorem I.2 page 176}
\end{aligned}
$$

✏

Theorem I.9. [14] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA. *Let* $\downarrow$ *represent the* REJECTION *operator (Definition I.2 page 180).*

THM *The set* $\{\downarrow\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$
\begin{array}{rcll}
x \vee y &=& (x \downarrow y) \downarrow (x \downarrow y) & \forall x,y \in X\\
x \wedge y &=& (x \downarrow x) \downarrow (y \downarrow y) & \forall x,y \in X\\
x' &=& x \downarrow x & \forall x \in X\\
0 &=& x \downarrow (x \downarrow x) & \forall x \in X\\
1 &=& [x \downarrow (x \downarrow x)] \downarrow [x \downarrow (x \downarrow x)] & \forall x \in X
\end{array}
$$

✎PROOF:

$$
\begin{aligned}
x' &= (x \vee x)' && \text{by Theorem I.2 page 176}\\
&= x \downarrow x && \text{by definition of } \downarrow \text{ page 180}\\
x \vee y &= (x \vee y)'' && \text{by Theorem I.2 page 176}\\
&= (x \downarrow y)' && \text{by definition of } \downarrow \text{ page 180}\\
&= (x \downarrow y) \downarrow (x \downarrow y) && \text{by previous result}\\
x \wedge y &= (x \wedge y)'' && \text{by Theorem I.2 page 176}\\
&= \left(x' \vee y'\right)' && \text{by de Morgan's Law page 176}\\
&= x' \downarrow y' && \text{by definition of } \downarrow \text{ page 180}\\
&= (x \downarrow x) \downarrow (y \downarrow y) && \text{by previous result}\\
0 &= 1' \\
&= \left(x \vee x'\right)' && \text{by Theorem I.2 page 176}\\
&= x \downarrow \left(x'\right) && \text{by definition of } \downarrow \text{ page 180}\\
&= x \downarrow (x \downarrow x) \\
1 &= \left(x \vee x'\right) && \text{by Theorem I.2 page 176}\\
&= \left(x \vee x'\right)'' && \text{by Theorem I.2 page 176}\\
&= \left(x \vee x'\right)' \downarrow \left(x \vee x'\right)' && \text{by definition of } \downarrow \text{ page 180}\\
&= \left[x \downarrow \left(x'\right)\right] \downarrow \left[x \downarrow \left(x'\right)\right] \\
&= [x \downarrow (x \downarrow x)] \downarrow [x \downarrow (x \downarrow x)]
\end{aligned}
$$

✏

Theorem I.10. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA. *Let* $|$ *represent the* SHEFFER STROKE *operator (Definition I.2 page 180).*

[14] Givant and Halmos (2009) page 33

THM *The set* $\{|\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{array}{rcll}
x \vee y &=& (x|x)|(y|y) & \forall x,y \in X\\
x \wedge y &=& (x|y)|(x|y) & \forall x,y \in X\\
x' &=& x|x & \forall x \in X\\
0 &=& [x|(x|x)]|[x|(x|x)] & \forall x \in X\\
1 &=& x|(x|x) & \forall x \in X
\end{array}$$

✎PROOF:

$$\begin{array}{rll}
x' &= (x \wedge x)' & \text{by Theorem I.2 page 176}\\
&= x|x & \text{by definition of } | \text{ page 180}\\
x \vee y &= (x \vee y)'' & \text{by Theorem I.2 page 176}\\
&= \left(x' \wedge y'\right)' & \text{by de Morgan's Law page 176}\\
&= x'|y' & \text{by definition of } | \text{ page 180}\\
&= (x|x)|(y|y) & \text{by first result}\\
x \wedge y &= (x \wedge y)'' & \text{by Theorem I.2 page 176}\\
&= (x|y)' & \text{by definition of } | \text{ page 180}\\
&= (x|y)|(x|y) & \text{by first result}\\
1 &= 0' & \\
&= \left(x \wedge x'\right)' & \text{by Theorem I.2 page 176}\\
&= x|\left(x'\right) & \text{by definition of } | \text{ page 180}\\
&= x|(x|x) & \\
0 &= \left(x \wedge x'\right) & \text{by Theorem I.2 page 176}\\
&= \left(x \wedge x'\right)'' & \text{by Theorem I.2 page 176}\\
&= \left(x \wedge x'\right)'|\left(x \wedge x'\right)' & \text{by definition of } | \text{ page 180}\\
&= \left[x|\left(x'\right)\right]|\left[x|\left(x'\right)\right] & \\
&= [x|(x|x)]|[x|(x|x)] &
\end{array}$$

✏

Besides the *rejection* singleton $\{\downarrow\}$ and the Sheffer stroke singleton $\{|\}$, there are no single opertor sets that are *functionally complete* (next theorem).

Theorem I.11. [15] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ *be a Boolean algebra. Let* $\downarrow$ *be the* REJECTION *operator and* $|$ *be the* SHEFFER STROKE *operator.*

THM $\{+\}$ *is* FUNCTIONALLY COMPLETE *in* $\boldsymbol{B}$ $\implies$ $+ = \downarrow$ *or* $+ = |$

Theorem I.12. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1\,;\,\leq)$ *be a* BOOLEAN ALGEBRA. *Let* $\div$ *represent the* ADJUNCTION *operator (Definition I.2 page 180).*

THM *The set* $\{\div, 0\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{array}{rcll}
x \vee y &=& x \div (0 \div y) & \forall x,y \in X\\
x \wedge y &=& 0 \div [(0 \div x) \div y] & \forall x,y \in X\\
x' &=& 0 \div x & \forall x \in X\\
1 &=& x \div x & \forall x \in X
\end{array}$$

15 Quine (1979) page 49, Żyliński (1925) page 208 $\langle \downarrow = \phi_{15}, | = \phi_2 \rangle$

✎PROOF:

$$\begin{aligned}
x' &= 0 \vee x' && \text{by Theorem I.2 page 176}\\
&= 0 \div x && \text{by definition of } \div \text{ (Definition I.2 page 180)}\\
x \vee y &= x \vee y'' && \text{by Theorem I.2 page 176}\\
&= x \div \left(y'\right) && \text{by definition of } \div \text{ (Definition I.2 page 180)}\\
&= x \div (0 \div y) && \text{by previous result}\\
x \wedge y &= \left(x' \vee y'\right)' && \text{by } \textit{de Morgan's law} \text{ property Theorem I.2 page 176}\\
&= \left(x' \div y\right)' && \text{by definition of } \div \text{ (Definition I.2 page 180)}\\
&= [(0 \div x) \div y]' && \text{by previous result}\\
&= 0 \div [(0 \div x) \div y] && \text{by previous result}\\
1 &= x \vee x' && \text{by } \textit{complement} \text{ property Theorem I.2 page 176}\\
&= x \div x && \text{by definition of } \div \text{ (Definition I.2 page 180)}
\end{aligned}$$

⇨

Theorem I.13. [16] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA. *Let* $-$ *represent the* EXCEPTION *operator (Definition I.2 page 180).*

THM *The set* $\{-, 1\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{array}{rcll}
x \vee y &=& 1 - [(1-x) - y] & \forall x,y \in X\\
x \wedge y &=& x - (1 - y) & \forall x,y \in X\\
x' &=& 1 - x & \forall x \in X\\
0 &=& x - x & \forall x \in X
\end{array}$$

✎PROOF:

$$\begin{aligned}
x' &= 1 \wedge x' && \text{by Theorem I.2 page 176}\\
&= 1 - x && \text{by definition of } - \text{ (Definition I.2 page 180)}\\
x \wedge y &= x \wedge y'' && \text{by Theorem I.2 page 176}\\
&= x - \left(y'\right) && \text{by definition of } - \text{ (Definition I.2 page 180)}\\
&= x - (1 - y) && \text{by previous result}\\
x \vee y &= \left(x' \wedge y'\right)' && \text{by } \textit{de Morgan's law} \text{ property Theorem I.2 page 176}\\
&= \left(x' - y\right)' && \text{by definition of } - \text{ (Definition I.2 page 180)}\\
&= [(1 - x) - y]' && \text{by previous result}\\
&= 1 - [(1 - x) - y] && \text{by previous result}\\
0 &= x \wedge x' && \text{by } \textit{complement} \text{ property Theorem I.2 page 176}\\
&= x - x && \text{by definition of } - \text{ (Definition I.2 page 180)}
\end{aligned}$$

⇨

Theorem I.14. [17] *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM *The set* $\{\triangle, \wedge, 1\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{array}{rcll}
x \vee y &=& xy \triangle x \triangle y & \forall x,y \in X\\
x' &=& x \triangle 1 & \forall x \in X\\
0 &=& x \triangle x & \forall x \in X
\end{array}$$

[16] Bernstein (1914) pages 89–91
[17] Roth (2006) page 42

✎PROOF:

$$\begin{aligned}
x' &= x' \vee 0 && \text{by Theorem I.2 page 176}\\
&= \left(x' \wedge 1\right) \vee (x \wedge 0) && \text{by Theorem I.2 page 176}\\
&= \left(x' \wedge 1\right) \vee \left(x \wedge 1'\right)\\
&= x \triangle 1 && \text{by definition of } \triangle \text{ (Definition I.2 page 180)}\\
0 &= 0 \vee 0 && \text{by Theorem I.2 page 176}\\
&= \left(x' \wedge x\right) \vee \left(x \wedge x'\right) && \text{by Theorem I.2 page 176}\\
&= x \triangle x && \text{by definition of } \triangle \text{ (Definition I.2 page 180)}\\
xy \oplus x \oplus y &= (xy) \triangle (x \triangle y) && \text{by \textit{associative} property Theorem I.2 page 176}\\
&= (xy) \oplus \left(x'y \vee xy'\right) && \text{by definition of } \triangle \text{ (Definition I.2 page 180)}\\
&= (xy)'\left(x'y \vee xy'\right) \vee (xy)\left(x'y \vee xy'\right)' && \text{by definition of } \triangle \text{ (Definition I.2 page 180)}\\
&= \left(x' \vee y'\right)\left(x'y \vee xy'\right) \vee (xy)\left[\left(x'y\right)'\left(xy'\right)'\right] && \text{by \textit{de Morgan's law} Theorem I.2 page 176}\\
&= \left(x' \vee y'\right)\left(x'y \vee xy'\right) \vee (xy)\left[\left(x'' \vee y'\right)\left(x' \vee y''\right)\right] && \text{by \textit{de Morgan's law} Theorem I.2 page 176}\\
&= \left(x' \vee y'\right)\left(x'y \vee xy'\right) \vee (xy)\left[\left(x \vee y'\right)\left(x' \vee y\right)\right]\\
&= \left(x'y \vee xy'\right) \vee (xy)\left[xy \vee x'y'\right]\\
&= \left(x'y \vee xy'\right) \vee xy\\
&= \left(x'y \vee xy'\right) \vee (xy \vee xy) && \text{by \textit{idempotent} property Theorem I.2}\\
&= \left(xy \vee x'y\right) \vee \left(xy \vee xy'\right) && \text{by Theorem I.2 page 176}\\
&= \left(x \vee x'\right)y \vee x\left(y \vee y'\right) && \text{by \textit{distributive} property Theorem I.2}\\
&= (1)y \vee x(1)\\
&= x \vee y
\end{aligned}$$

✏

Theorem I.15. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM *The set* $\{\triangle, \vee, 1\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned}
x \wedge y &= \left[\left(x \triangle 1\right) \vee \left(y \triangle 1\right)\right] \triangle 1 && \forall x,y \in X\\
x' &= x \triangle 1 && \forall x \in X\\
0 &= x \triangle x && \forall x \in X
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
0 &= x \triangle x\\
x' &= x \triangle 1\\
x \wedge y &= \left(x' \vee y'\right)'\\
&= \left[\left(x \triangle 1\right) \vee \left(y \triangle 1\right)\right] \triangle 1
\end{aligned}$$

✏

Theorem I.16. *Let* $\boldsymbol{B} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOOLEAN ALGEBRA.

THM *The set* $\{\triangle, -, '\}$ *is* FUNCTIONALLY COMPLETE *with respect to* $\boldsymbol{B}$. *In particular,*

$$\begin{aligned}
x \vee y &= (x - y) \triangle y && \forall x,y \in X\\
x \wedge y &= x - (x - y) && \forall x,y \in X\\
0 &= x \triangle x && \forall x \in X
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
x \vee y &= x(y \vee y') \vee y \\
&= xy \vee xy' \vee y && \text{by } \textit{distributive} \text{ property (Theorem I.2 page 176)} \\
&= (y \vee xy) \vee xy' && \text{by } \textit{associative} \text{ property (Theorem I.2 page 176)} \\
&= y \vee xy' && \text{by } \textit{absorptive} \text{ property (Theorem I.2 page 176)} \\
&= (y \vee x'y) \vee xy' && \text{by } \textit{absorptive} \text{ property (Theorem I.2 page 176)} \\
&= (y \vee x')y \vee (xy')y' && \text{by } \textit{distributive} \text{ and } \textit{idempotent} \text{ properties (Theorem I.2 page 176)} \\
&= (xy')'y \vee (xy')y' && \text{by } \textit{de Morgan's law} \text{ property (Theorem I.2 page 176)} \\
&= (xy') \bigtriangleup y && \text{by definition of } \bigtriangleup \text{ (Definition I.2 page 180)} \\
&= (x - y) \bigtriangleup y && \text{by definition of } - \text{ (Definition I.2 page 180)} \\
x \wedge y &= xx' \vee xy \\
&= x(x' \vee y) && \text{by } \textit{distributive} \text{ and } \textit{idempotent} \text{ properties (Theorem I.2 page 176)} \\
&= x(x''y')' && \text{by } \textit{de Morgan's law} \text{ property (Theorem I.2 page 176)} \\
&= x(xy')' && \text{by } \textit{involutory} \text{ property (Theorem I.2 page 176)} \\
&= x(x - y)' && \text{by definition of } - \text{ (Definition I.2 page 180)} \\
&= x - (x - y) && \text{by definition of } - \text{ (Definition I.2 page 180)} \\
0 &= xx' \\
&= x - (x - x') && \text{by previous result}
\end{aligned}$$

✏

I.5 Characterizations

❝The algebra of symbolic logic...has recently assumed some importance as an independent calculus; it may therefore be not without interest to consider it from a purely mathematical or abstract point of view...❞
Edward V. Huntington (1874–1952), American mathematician[18]

Order characterizations

An order characterization of Boolean algebras has already been given by Definition I.1 (page 171): A lattice is a Boolean algebra if and only if it is *distributive* and *complemented*.

Proposition I.4. [19] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* BOUNDED *and* COMPLEMENTED LATTICE.

PRP
$$\left\{ \begin{array}{l} \boldsymbol{A} \text{ is a} \\ \textbf{Boolean algebra} \end{array} \right\} \iff \left\{ \begin{array}{lll} 1. & 1' = 0 & \text{and} \\ 2. & (x \wedge y')' = y \vee (x' \wedge y') \quad \forall x,y \in X & \text{(ELKAN'S LAW)} \end{array} \right\}$$

[18] quote: Huntington (1904) page 288
image: http://en.wikipedia.org/wiki/Edward_V._Huntington
[19] Kondo and Dudek (2008) page 1035, Elkan et al. (1994) page 3 ⟨*Elkan's law*⟩

Algebraic characterizations

This section presents several algebraic characterizations. One such characterization has already been provided by Theorem I.2 (page 176)— the standard properties of Boolean algebras characterized by 19 identities. If a system satisfies these 19 identities, then that system *is* a Boolean algebra. However, the set of 19 identities is very much an *over*-specification. It is possible to characterize Boolean algebras using much fewer relationships, from which all of the 19 identities of Theorem I.2 can be derived. Here are some of these reduced characterizations:

- *Huntington's first set*: (1904) 8 relationships, Proposition I.5 page 188
- *Huntington's fourth set*: (1933) 4 relationships, Proposition I.6 page 189
- *Huntington's fifth set*: (1933) 3 relationships, Proposition I.7 page 190
- *Stone*: (1935) 7 relationships, Proposition I.8 page 190
- *Byrne's Formulation A*: (1946) 3 relationships, Proposition I.9 page 190
- *Byrne's Formulation B*: (1946) 2 relationships, Proposition I.10 page 192

All of these characterizations use 3 variables. It might be reasonable to ask if there exists a characterization that uses only two variables. The answer is "No", as demonstrated by the next theorem.

Theorem I.17. [20]

THM: *There does* NOT *exist a characterization of Boolean algebras consisting of only 2 variables.*

Proposition I.5 (Huntington's first set). [21] *Let X be a set, $\le$ a relation in 2^{XX}, $\vee$ and $\wedge$ binary operations in $X^{X\times X}$, $'$ an unary operation in X^X, and 0 and 1 nullary operations on X.*

PRP: $(X, \vee, \wedge, 0, 1; \le)$ *is a* ***Boolean algebra*** *if for all* $x, y, z \in X$

1.	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	(COMMUTATIVE)
2.	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(DISTRIBUTIVE)
3.	$x \vee 0 = x$	$x \wedge 1 = x$	(IDENTITY)
4.	$x \vee x' = 1$	$x \wedge x' = 0$	(COMPLEMENTED)

and where the relation $\le$ is defined as $x \le y \iff x \vee y = y \quad \forall x,y \in X$.

The property $x \vee x' = 1$ is referred to as "the law of the EXCLUDED MIDDLE*". The property $x \wedge x' = 0$ is referred to as "the law of* NON-CONTRADICTION*".*

✎PROOF:

1. Proof that $\boldsymbol{A}$ is a Boolean algebra $\implies$ $\boldsymbol{A}$ is a *distributive complemented lattice*:
   (a) Proof that $\boldsymbol{A}$ is *distributive*: by Definition I.1 page 171
   (b) Proof that $\boldsymbol{A}$ is *complemented*: by Definition I.1 page 171
   (c) Proof that $\boldsymbol{A}$ is *bounded*: by Lemma I.1 page 172
   (d) Proof that $\boldsymbol{A}$ is a *lattice*:
      i. Proof that $\boldsymbol{A}$ is *idempotent*: by Lemma I.1 page 172
      ii. Proof that $\boldsymbol{A}$ is *commutative*: by Definition I.1 page 171
      iii. Proof that $\boldsymbol{A}$ is *associative*: by Lemma I.1 page 172
      iv. Proof that $\boldsymbol{A}$ is *absorptive*: by Lemma I.1 page 172

[20] Sikorski (1969) page 3, Diamond and McKinsey (1947) page 961, Gerrish (1978) page 36

[21] Gerrish (1978) page 35, Saliĭ (1988) page 33 ⟨"Huntington's Theorem"⟩, Joshi (1989) page 222 ⟨(B1)–(B4)⟩, Huntington (1904) pages 292–293 ⟨"1st set"⟩, Huntington (1933) page 277 ⟨"1st set"⟩, Givant and Halmos (2009) page 10

v. Therefore, by Theorem D.3 (page 118), **A** is a *lattice*

2. Proof that **A** is a Boolean algebra $\Longleftarrow$ **A** is a *distributive complemented lattice*:

   (a) Proof that **A** is *commutative*: by property of lattices, Theorem D.3 page 118

   (b) Proof that **A** is *distributive*: by right hypothesis

   (c) Proof that **A** has *identity*:

$$\begin{aligned} x \vee 0 &= x \vee (x \wedge x') && \text{by } \textit{complemented} \text{ property in right hypothesis} \\ &= x && \text{by } \textit{absorptive} \text{ property of lattices Theorem D.3 page 118} \\ x \wedge 1 &= x \wedge (x \vee x') && \text{by } \textit{complemented} \text{ property in right hypothesis} \\ &= x && \text{by } \textit{absorptive} \text{ property of lattices Theorem D.3 page 118} \end{aligned}$$

   (d) Proof that **A** is *complemented*: by right hypothesis

✏

Huntington's fourth set (next) characterizes Boolean algebras in terms of the standard properties of *idempotent*, *commutative*, and *associative* (see Theorem I.2 page 176), and also in terms of an additional property called *Huntington's axiom*,[22] or (in terms of x and y), x *commutes* y. Huntington's axiom is significant in the context of *orthomodular* lattices in that an orthomodular lattice that satisfies Huntington's axiom is a Boolean algebra.[23]

Proposition I.6 (Huntington's fourth set). [24] *Let* **A** $\triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

A** is a **Boolean algebra $\iff$

$$\left\{\begin{array}{lllllll} 1. & x \vee x & = & x & \forall x \in X & (\text{IDEMPOTENT}) & \text{and} \\ 2. & x \vee y & = & y \vee x & \forall x,y \in X & (\text{COMMUTATIVE}) & \text{and} \\ 3. & (x \vee y) \vee z & = & x \vee (y \vee z) & \forall x,y,z \in X & (\text{ASSOCIATIVE}) & \text{and} \\ 4. & (x' \vee y')' \vee (x' \vee y)' & = & x & \forall x,y \in X. & (\text{HUNTINGTON'S AXIOM}) & \end{array}\right\}$$

✎PROOF:

1. Proof that [**A** is a *Boolean algebra*] $\Longrightarrow$ [**A** satisfies the 4 pairs of properties]:

   (a) Proof that $x \vee x = x$ (*idempotent* property with respect to $\vee$): by 1a of Lemma I.1 (page 172).

   (b) Proof that $x \vee y = y \vee x$ (*commutative* property with respect to $\vee$): by 1a of this proposition.

   (c) Proof that $(x \vee y) \vee z = x \vee (y \vee z)$ (*associative* property with respect to $\vee$): by 2a of Lemma I.1 (page 172).

   (d) Proof that $(x \wedge y) \vee (x \wedge y') = x$ (*Huntington's axiom*):

$$\begin{aligned} (x \wedge y) \vee (x \wedge y') &= x \wedge (y \vee y') && \text{by 2a} && (\textit{distributive} \text{ property wrt } \vee) \\ &= x \wedge 1 && \text{by 3a} && (\textit{complemented} \text{ property wrt } \vee) \\ &= x && \text{by 4b} && (\textit{identity} \text{ property wrt } \wedge) \end{aligned}$$

2. Proof that [**A** is a *Boolean algebra*] $\Longleftarrow$ [**A** satisfies the 4 pairs of properties]:

[22] Givant and Halmos (2009) page 13 ⟨problem 7⟩
[23] Renedo et al. (2003) page 72 ⟨Definition 3⟩, Beran (1985) page 52, Beran (1982)
[24] Huntington (1933) page 280 ⟨"4th set"⟩

(a) Proof that $x \vee y = y \vee x$: by 2 of Definition I.1 page 171.

(b) Proof that $x \wedge y = y \wedge x$:

(c) Proof that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$:

(d) Proof that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$:

(e) Proof that $x \vee x' = 1$:

(f) Proof that $x \wedge x' = 0$:

(g) Proof that $x \vee 0 = x$:

(h) Proof that $x \wedge 1 = x$:

✏

Proposition I.7 (Huntington's fifth set). [25] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\boldsymbol{A} \text{ is a Boolean algebra} \iff \left\{\begin{array}{llllll} 1. & x'' & = & x & \forall x,y,z \in X & \text{and} \\ 2. & x \vee \left(y \vee y'\right)' & = & x & \forall x,y \in X & \text{and} \\ 3. & x \vee (y \vee z)' & = & \left[\left(y' \vee x\right)' \vee \left(z' \vee x\right)'\right]' & \forall x,y,z \in X. & \end{array}\right\}$$

Proposition I.8 (Stone). [26] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\boldsymbol{A} \text{ is a Boolean algebra} \iff \left\{\begin{array}{lllllll} 1. & x \vee y & = & y \vee x & \forall x,y \in X & (\text{JOIN COMMUTATIVE}) & \text{and} \\ 2. & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) & \forall x,y,z \in X & (\text{LEFT DISTRIBUTIVE}) & \text{and} \\ 3. & (x \vee y) \wedge z & = & (x \wedge z) \vee (y \wedge z) & \forall x,y,z \in X & (\text{RIGHT DISTRIBUTIVE}) & \text{and} \\ 4. & x \vee 0 & = & x & \forall x \in X & (\text{JOIN IDENTITY}) & \text{and} \\ 5. & \exists x' \text{ such that } x \vee x' & = & 1 \text{ and } x \wedge x' = 0 & \forall x \in X & (\text{COMPLEMENTED}) & \text{and} \\ 6. & x \vee x & = & x & \forall x \in X & (\text{IDEMPOTENT}) & \text{and} \\ 7. & x \wedge x & = & x & \forall x \in X & & \end{array}\right\}$$

Proposition I.9 (Byrne's FORMULATION A). [27] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$$\boldsymbol{A} \text{ is a Boolean algebra} \iff \left\{\begin{array}{llllll} 1. & x \vee y & = & y \vee x & \forall x,y \in X & (\text{COMMUTATIVE}) \quad \text{and} \\ 2. & (x \vee y) \vee z & = & x \vee (y \vee z) & \forall x,y,z \in X & (\text{ASSOCIATIVE}) \quad \text{and} \\ 3. & x \vee y' = z \vee z' & \iff & x \vee y = x & \forall x,y,z \in X. & \end{array}\right\}$$

✎PROOF:

1. Proof that $\boldsymbol{A}$ is a Boolean algebra $\implies$ 3 identities:

   (a) *commutative* property: By Theorem I.2 (page 176), all Boolean algebras are *commutative*.

   (b) *associative* property: By Theorem I.2 (page 176), all Boolean algebras are *associative*.

   (c) Proof that $x \vee y' = y \vee y' \implies x \vee y = x$:

$$\begin{aligned} x \vee y &= y \vee x && \text{by Boolean hypothesis and Theorem I.2 page 176} \\ &= y \vee \left(x'\right)' && \text{by Boolean hypothesis and Theorem I.2 page 176} \\ &= y \vee \left(x'\right)' && \text{by Boolean hypothesis and Theorem I.2 page 176} \\ &= x' \vee \left(x'\right)' && \text{by } x \vee y' = y \vee y' \text{ hypothesis} \\ &= x' \vee x && \text{by Boolean hypothesis and Theorem I.2 page 176} \\ &= x && \text{by Boolean hypothesis and Theorem I.2 page 176} \end{aligned}$$

[25] Givant and Halmos (2009) page 13, Huntington (1933) page 286 ⟨"5th set"⟩

[26] Stone (1935) page 705

[27] Givant and Halmos (2009) page 13, Byrne (1946) page 270 ⟨"FORMULATION A"⟩

(d) Proof that $x \vee y' = y \vee y' \iff x \vee y = x$:

$$\begin{aligned}
x \vee y' &= (x \vee y) \vee y' && \text{by } x \vee y = x \text{ hypothesis} \\
&= x \vee \left(y \vee y'\right) && \text{by Boolean hypothesis and Theorem I.2 page 176} \\
&= x \vee 1 && \text{by Boolean hypothesis and Theorem I.2 page 176} \\
&= x && \text{by Boolean hypothesis and Theorem I.2 page 176}
\end{aligned}$$

2. Proof that $\boldsymbol{A}$ is a Boolean algebra $\Longleftarrow$ 3 identities:

(a) Proof that $x \vee x = x$ (*idempotent* property): because $x \vee x' = x \vee x'$ and by identity 3

(b) Proof that $x \vee x' = y \vee y'$: by item (2a) and identity 3

(c) Proof that $x \vee y = x$ and $y \vee z = y \implies x \vee z = x$:

$$\begin{aligned}
x \vee z &= (x \vee y) \vee z && \text{by } x \vee y = x \text{ hypothesis} \\
&= x \vee (y \vee z) && \text{by identity 2 (\textit{associative} property)} \\
&= x \vee y && \text{by } y \vee z = y \text{ hypothesis} \\
&= x && \text{by } x \vee y = x \text{ hypothesis}
\end{aligned}$$

(d) Proof that $x'' = x$ (*involutory* property):

$$\begin{aligned}
x'' \vee x' &= x' \vee x'' && \text{by identity 1 (\textit{commutative} property)} && \text{(I.1)} \\
&= z \vee z' && \text{by item (2b)} \\
x'' \vee x &= x'' && \text{by equation (I.1) and identity 3} && \text{(I.2)} \\
x''' \vee x' &= x''' && \text{by equation (I.2)} && \text{(I.3)} \\
x'''' \vee x'' &= x'''' && \text{by equation (I.2)} && \text{(I.4)} \\
x'''' \vee x &= x'''' && \text{by equation (I.4), equation (I.5), and item (2c)} && \text{(I.5)} \\
x'''' \vee x' &= z \vee z' && \text{by equation (I.5) and identity 3} && \text{(I.6)} \\
x' \vee x''' &= x' && \text{by equation (I.6) and identity 3} && \text{(I.7)} \\
x''' &= x''' \vee x' && \text{by equation (I.3)} && \text{(I.8)} \\
&= x' \vee x''' && \text{by identity 1 (\textit{commutative} property)} \\
&= x' && \text{by equation (I.7)} \\
x \vee x''' &= x \vee x' && \text{by equation (I.8)} && \text{(I.9)} \\
&= z \vee z' && \text{by item (2b)} \\
x \vee x'' &= x && \text{by equation (I.9) and identity 3} && \text{(I.10)} \\
x'' &= x'' \vee x && \text{by equation (I.2)} \\
&= x \vee x'' && \text{by identity 1 (\textit{commutative} property)} \\
&= x && \text{by equation (I.10)}
\end{aligned}$$

(e) Proof that $x \vee (x' \vee y)'' = z \vee z'$:

$$\begin{aligned}
x \vee (x' \vee y)'' &= x \vee (x' \vee y) && \text{by item (2d) (\textit{involutory} property)} \\
&= (x \vee x') \vee y && \text{by identity 2 (\textit{associative} property)} \\
&= y \vee (x \vee x') && \text{by identity 1 (\textit{commutative} property)} \\
&= y \vee (y \vee y') && \text{by item (2b)} \\
&= (y \vee y) \vee y' && \text{by identity 2 (\textit{associative} property)} \\
&= y \vee y' && \text{by item (2a)} \\
&= z \vee z' && \text{by item (2b)}
\end{aligned}$$

(f) Proof that $x \vee (x' \vee y)' = x$: by item (2e) and identity 3

(g) Proof that $x \vee y'' \vee (x \vee y)' = z \vee z'$:

$$\begin{aligned} x \vee y'' \vee (x \vee y)' &= x \vee y \vee (x \vee y)' && \text{by item (2d)} \\ &= z \vee z' && \text{by item (2b)} \end{aligned}$$

(h) Proof that $x \vee (x \vee y)' = x \vee y'$:

$$\begin{aligned} x \vee (x \vee y)' &= x \vee (x \vee y)' \vee y' && \text{by item (2g) and identity 3} \\ &= x \vee y' \vee (x \vee y)' && \text{by identity 1 (\textit{commutative} property)} \\ &= x \vee y' \vee \left[(x \vee y')' z\right] && \text{by item (2f)} \\ &= x \vee y' && \text{by item (2f)} \end{aligned}$$

(i) Proof that $\left[\left(x' \vee y'\right)' \vee \left(x' \vee y\right)'\right] \vee x' = z \vee z'$:

$$\begin{aligned} \left[\left(x' \vee y'\right)' \vee \left(x' \vee y\right)'\right] \vee x' &= x' \vee \left[\left(x' \vee y'\right)' \vee \left(x' \vee y\right)'\right] && \text{by identity 1 (\textit{commutative} property)} \\ &= \left[x' \vee \left(x' \vee y'\right)'\right] \vee \left(x' \vee y\right)' && \text{by identity 2 (\textit{associative} property)} \\ &= \left(x' \vee y''\right) \vee \left(x' \vee y\right)' && \text{by item (2h)} \\ &= \left(x' \vee y\right) \vee \left(x' \vee y\right)' && \text{by item (2d) (\textit{involutory})} \\ &= z \vee z' && \text{by item (2b)} \end{aligned}$$

(j) Proof that $\left(x' \vee y'\right)' \vee \left(x' \vee y\right)' = x$ (*Huntington's axiom*):

$$\begin{aligned} \underbrace{\left(x' \vee y'\right)' \vee \left(x' \vee y\right)'}_{\text{“}x\text{” in identity 3}} &= \underbrace{(x' \vee y')' \vee (x' \vee y)'}_{\text{“}x\text{” in identity 3}} \vee \underbrace{x}_{\text{“}y\text{”}} && \text{by item (2i) and identity 3} \\ &= \underbrace{x \vee (x' \vee y)'}_{x \text{ by item (2f)}} \vee (x' \vee y')' && \text{by identity 1 (\textit{commutative} property)} \\ &= \underbrace{x \vee (x' \vee y')'}_{x \text{ by item (2f)}} && \text{by item (2f)} \\ &= x && \text{by item (2f)} \end{aligned}$$

(k) The three identities therefore imply that $\boldsymbol{A}$

i. is *idempotent* (item (2a)),
ii. is *commutative* (identity 1),
iii. is *associative* (identity 2), and
iv. satisfies *Huntington's axiom* (item (2j)).

Therefore, by Proposition I.6 page 189 (*Huntington's Fourth Set*), $\boldsymbol{A}$ is a *Boolean algebra*.

✏

Proposition I.10 (Byrne's Formulation B). [28] *Let* $\boldsymbol{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

PRP

$\boldsymbol{A}$ ***is a Boolean algebra*** $\iff$

$$\left\{\begin{array}{llllll} 1. & x \vee y' = z \vee z' & \iff & x \vee y = x & \forall x,y,z \in X & \text{and} \\ 2. & (x \vee y) \vee z & = & (y \vee z) \vee x & \forall x,y,z \in X. & \end{array}\right\}$$

[28] Byrne (1946) page 271 ⟨“Formulation B”⟩

Theorem I.18. [29] *Let* $\mathbf{A} \triangleq (X, \vee, \wedge; \leq)$ *be an* ALGEBRAIC STRUCTURE.

THM

$$\mathbf{A} \text{ is a \textbf{Boolean algebra}} \iff \left\{\begin{array}{llcll} 1. & x \wedge (x \vee y) & = & x & \forall x,y \in X \quad \text{and} \\ 2. & x \wedge (y \vee z) & = & (z \wedge x) \vee (y \wedge x) & \forall x,y,z \in X \quad \text{and} \\ 3. & \exists y' \text{ such that } x \wedge (y \vee y') & = & x \vee (y \wedge y') & \forall x,y \in X. \end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{A}$ is a *distributive lattice*: by 1 and 2 and by Theorem G.4 (page 156).
2. Define $0 \triangleq x \wedge x'$ and $1 \triangleq x \vee x'$.
3. Proof that 0 is the *join-identity* element and that 1 is the *meet-identity* element:

$$\begin{aligned}
x \vee 0 &= x \vee (y \wedge y') && \text{by definition of 0 (item (2) page 193)} \\
&= (x \vee x) \vee (y \wedge y') && \text{by \textit{idempotent} property of lattices (Theorem D.3 page 118)} \\
&= x \vee [x \vee (y \wedge y')] && \text{by \textit{associative} property of lattices (Theorem D.3 page 118)} \\
&= x \vee [x \wedge (y \vee y')] && \text{by 3} \\
&= x && \text{by \textit{absorptive} property of lattices (Theorem D.3 page 118)}
\end{aligned}$$

$$\begin{aligned}
x \wedge 1 &= x \wedge (y \vee y') && \text{by definition of 1 (item (2) page 193)} \\
&= (x \wedge x) \wedge (y \vee y') && \text{by \textit{idempotent} property of lattices (Theorem D.3 page 118)} \\
&= x \wedge [x \wedge (y \vee y')] && \text{by \textit{associative} property of lattices (Theorem D.3 page 118)} \\
&= x \wedge [x \vee (y \wedge y')] && \text{by 3} \\
&= x && \text{by \textit{absorptive} property of lattices (Theorem D.3 page 118)}
\end{aligned}$$

4. Proof that $\mathbf{A}$ is *bounded* with 0 being the *greatest lower bound* and 1 being the *least upper bound*:

$$\begin{aligned}
x \wedge 0 &= (x \vee 0) \wedge 0 && \text{by \textit{identity} property (item (3) page 193)} \\
&= 0 \wedge (0 \vee x) && \text{by \textit{commutative} property of lattices (Theorem D.3 page 118)} \\
&= 0 && \text{by \textit{absorptive} property of lattices (Theorem D.3 page 118)}
\end{aligned}$$

$$\begin{aligned}
x \vee 1 &= (x \wedge 1) \vee 1 && \text{by \textit{identity} property (item (3) page 193)} \\
&= 1 \vee (1 \wedge x) && \text{by \textit{commutative} property of lattices (Theorem D.3 page 118)} \\
&= 1 && \text{by \textit{absorptive} property of lattices (Theorem D.3 page 118)}
\end{aligned}$$

5. Proof that $\mathbf{A}$ is *complemented*: Because $\mathbf{A}$ is *bounded* with greatest lower bound 0 and least upper bound 1 (item (4)) and because $x \wedge x' = 0$ and $x \vee x' = 1$ (definition of 0 and 1 (item (2) page 193)).
6. Proof that $\mathbf{A}$ is a *Boolean algebra*: Because $\mathbf{A}$ is *distributive* (item (1)) and *complemented* (item (5)), and by Definition I.1 (page 171).

✏

I.6 Literature

Literature survey:

[29] Sholander (1951) pages 28–29, $P1$, $P2$, $P3^*$

1. General information about Boolean algebras:
   Sikorski (1969)
   Dwinger (1971)
      Dwinger (1961)
   Halmos (1972)
   Monk (1989)
   Givant and Halmos (2009)

2. Characterizations:

   (a) Survey of characterizations:
   Padmanabhan and Rudeanu (2008)

   (b) Characterizations in terms of traditional *binary* operations *join* $\vee$, *meet* $\wedge$, and *complement* $'$:
   Huntington (1904) ⟨
   ⟩ Huntington (1933) ⟨
   ⟩ Diamond (1933)
   Diamond (1934)
   Stone (1935)
   Hoberman and McKinsey (1937)
   Frink (1941) ⟨4 identities involving $\vee$, $\wedge$, $'$⟩
   Newman (1941)
   Braithwaite (1942)
   Byrne (1946) ⟨Form. A and B⟩
   Gerrish (1978) ⟨independence of Huntington's characterizations⟩

   (c) Characterizations in terms of non-traditional *binary* operations:
   Sheffer (1913) ⟨rejection $\downarrow$⟩
   Bernstein (1914) ⟨exception $-$⟩
   Bernstein (1916) ⟨rejection $\downarrow$⟩
   Bernstein (1933) ⟨rejection $\downarrow$⟩
   Bernstein (1934) ⟨implication $\Rightarrow$⟩
   Bernstein (1936) ⟨complete disjuction $\triangle$⟩
   Byrne (1948) ⟨inclusion⟩
   Byrne (1951) ⟨ring operations⟩
   Miller (1952) ⟨ring operations⟩

   (d) Characterizations in terms of *ternary* operations:
   Whiteman (1937) *ternary rejection*

   (e) Characterizations involving *Elkan's law*:
   Kondo and Dudek (2008) ⟨for bounded lattices⟩
   Renedo et al. (2003) ⟨for orthomodular lattices⟩
   Trillas et al. (2004) ⟨for orthocomplemented lattices⟩

3. Analytic properties:
   Vladimirov (2002)

4. Miscellaneous:
   Montague and Tarski (1954)
   Rudeanu (1961) ⟨referenced by Sikorski (1969)⟩

5. Actually, "Boolean algebras" are not really "algebras". Rather, they are "a commutative ring with unit, without nilpotents, and having idempotents which stood for classes"
   Hailperin (1981) page 184

6. Pioneering works related to Boolean algebras:
   Boole (1847)
   Boole (1854)
   Jevons (1864) ⟨join and meet operations⟩
   Peirce (1870a) ⟨order concepts⟩
   Huntington (1904) ⟨axiomization⟩

7. History of development of Boolean algebra:
   Burris (2000)

APPENDIX J

ORTHOCOMPLEMENTED LATTICES

Orthocomplemented lattices (Definition J.1 page 196) are a kind of generalization of *Boolean algebras*. The relationship between lattices of several types, including orthocomplemented and Boolean lattices, is stated in Theorem J.7 (page 207) and illustrated in Figure J.1 (page 195).

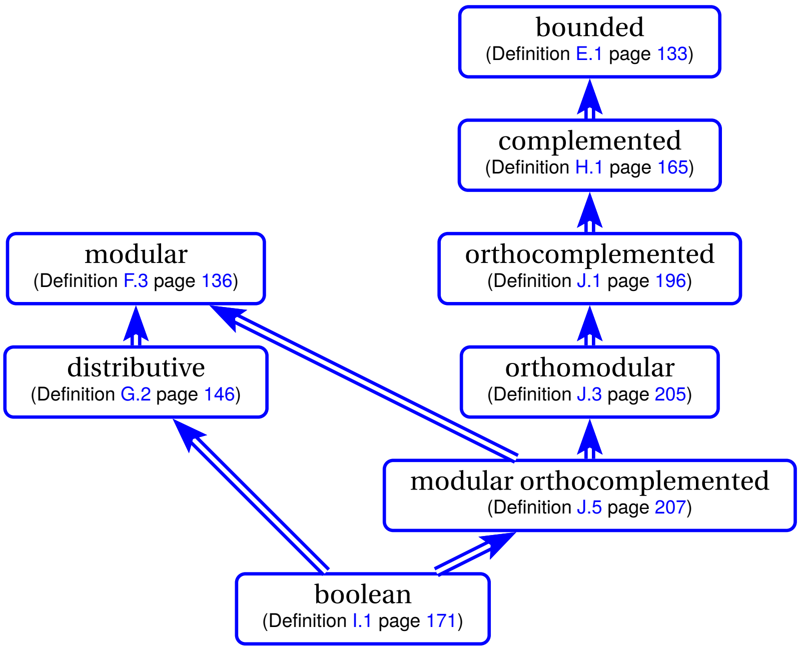


Figure J.1: lattice of orthocomplemented lattices

J.1 Orthocomplemented Lattices

J.1.1 Definition

Definition J.1. [1] *Let $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ be a* BOUNDED LATTICE *(Definition E.1 page 133).*

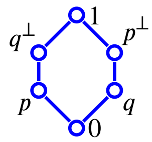
DEF

*An element $x^\perp \in X$ is an **orthocomplement** of an element $x \in X$ if*

1. $x^{\perp\perp} = x$ (INVOLUTORY) *and*
2. $x \wedge x^\perp = 0$ (NON-CONTRADICTION) *and*
3. $x \leq y \implies y^\perp \leq x^\perp \quad \forall y \in X$ (ANTITONE).

The LATTICE $\mathbf{L}$ *is **orthocomplemented** ($\mathbf{L}$ is an **orthocomplemented lattice**) if every element x in X has an* ORTHOCOMPLEMENT $x^\perp$ *in X.*

Definition J.2. [2]



DEF

*The **O_6 lattice** is the ordered set $\left(\{0, p, q, p^\perp, q^\perp, 1\}, \leq\right)$ with cover relation $<\!\!= \left\{(0,p), (0,q), (p,q^\perp), (q,p^\perp), (p^\perp,1), (q^\perp,1)\right\}$.*

The O_6 lattice is illustrated by the Hasse diagram to the right.

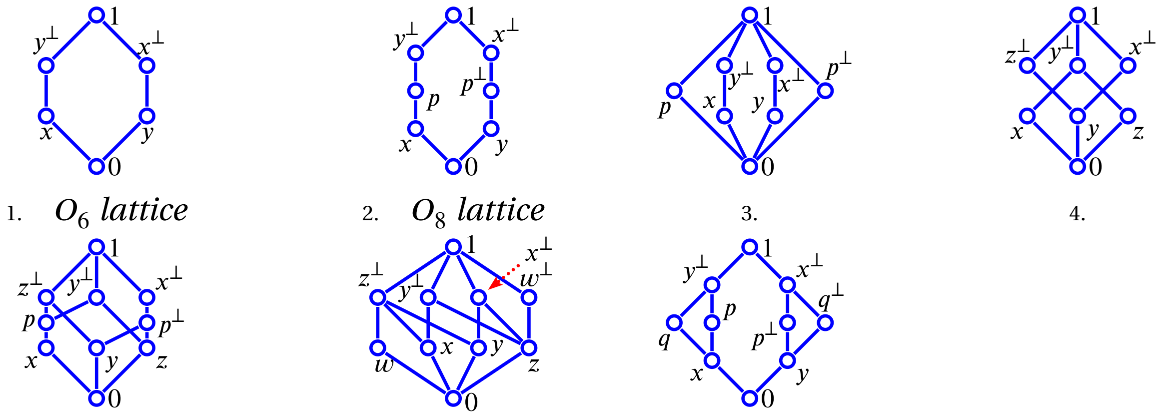
Example J.1. [3]

EX

The O_6 **lattice** (Definition J.2 page 196) is an **orthocomplemented lattice** (Definition J.1 page 196).

Example J.2. [4]There are a total of 10 **orthocomplemented lattices** with 8 elements or less. These 10, along with 3 other orthocomplemented lattices with 10 elements, are illustrated next:

Lattices that are **orthocomplemented** but *non-orthomodular* and hence also *not modular orthocomplemented* and *non-Boolean*:



1. O_6 *lattice* 2. O_8 *lattice* 3. 4. 5. 6. 7.

Lattices that are **orthocomplemented** and **orthomodular** but *not modular orthocomplemented* and hence also *non-Boolean*:

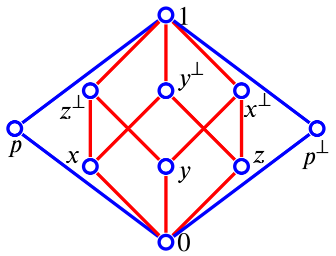
---

[1] Stern (1999) page 11, Beran (1985) page 28, Kalmbach (1983) page 16, Gudder (1988) page 76, Loomis (1955) page 3, Birkhoff and Neumann (1936) page 830 ⟨L71–L73⟩

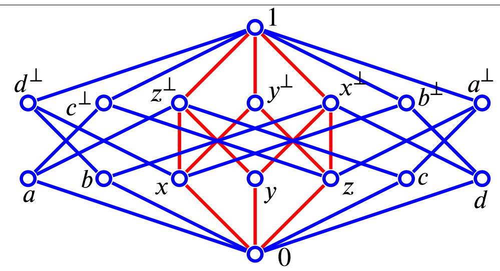
[2] Kalmbach (1983) page 22, Holland (1970) page 50, Beran (1985) page 33, Stern (1999) page 12, The O_6 *lattice* is also called the **Benzene ring** or the **hexagon**.

[3] Holland (1963) page 50

[4] Beran (1985) pages 33–42, Maeda (1966) page 250, Kalmbach (1983) page 24 ⟨Figure 3.2⟩, Stern (1999) page 12, Holland (1970) page 50

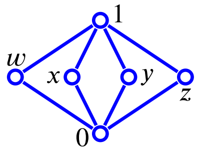


8.

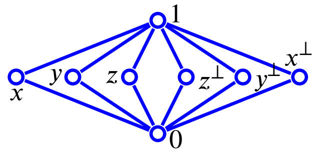


9.

Lattices that are **orthocomplemented**, **orthomodular**, and **modular orthocomplemented** but *non-Boolean*:



10. M_4 *lattice*

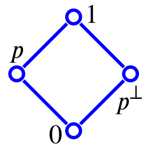


11. M_6 *lattice*

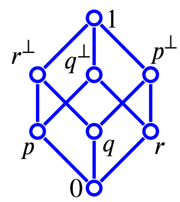
Lattices that are **orthocomplemented**, **orthomodular**, **modular orthocomplemented** and **Boolean**:

12. L_1 *lattice*

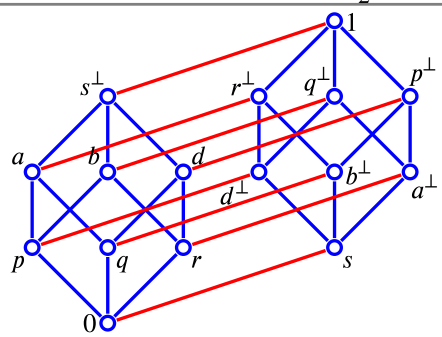
13. L_2 *lattice*



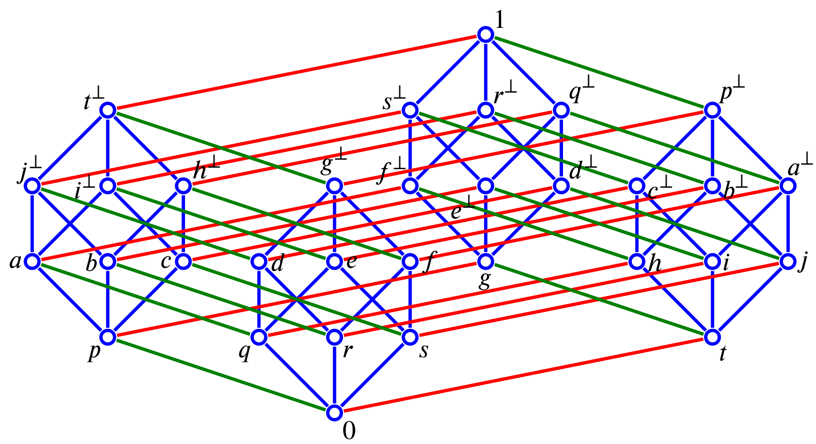
14. L_2^2 *lattice*



15. L_2^3 *lattice*



16. L_2^4 *lattice*



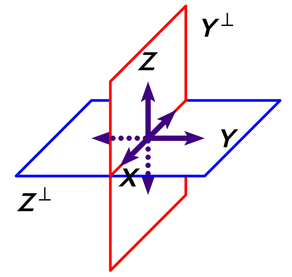
17. L_2^5 *lattice*

Example J.3.

EX

The structure $\left(2^{\mathbb{R}^N}, +, \cap, \varnothing, \boldsymbol{H} ; \subseteq\right)$ is an **orthocomplemented lattice** where

- $\mathbb{R}^N$ is an **Euclidean space** with dimension N
- $2^{\mathbb{R}^N}$ is the set of all subspaces of $\mathbb{R}^N$
- $\boldsymbol{V} + \boldsymbol{W}$ is the *Minkowski sum* of subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$
- $\boldsymbol{V} \cap \boldsymbol{W}$ is the *intersection* of subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$



Example J.4.

EX

The structure $\left(2^{\boldsymbol{H}}, \oplus, \cap, \varnothing, \boldsymbol{H} ; \subseteq\right)$ is an **orthocomplemented lattice** where

- $\boldsymbol{H}$ is a **Hilbert space**
- $2^{\boldsymbol{H}}$ is the set of all closed subspaces of $\boldsymbol{H}$
- $\boldsymbol{X} + \boldsymbol{Y}$ is the *Minkowski sum* of subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$
- $\boldsymbol{X} \oplus \boldsymbol{Y} \triangleq (\boldsymbol{X} + \boldsymbol{Y})^-$ is the *closure* of $\boldsymbol{X} + \boldsymbol{Y}$
- $\boldsymbol{X} \cap \boldsymbol{Y}$ is the *intersection* of subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$

J.1.2 Properties

Theorem J.1. [5] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *be a* BOUNDED LATTICE.

THM

$$\left\{\begin{array}{l}\boldsymbol{L} \textit{ is}\\ \text{ORTHOCOMPLEMENTED}\\ \text{(Definition J.1 page 196)}\end{array}\right\} \implies \left\{\begin{array}{lrcll}(1). & 0^\perp &=& 1 & & \text{(BOUNDARY CONDITION)} \quad \textit{and}\\ (2). & 1^\perp &=& 0 & & \text{(BOUNDARY CONDITION)} \quad \textit{and}\\ (3). & (x \vee y)^\perp &=& x^\perp \wedge y^\perp & \forall x,y \in X & \text{(DISJUNCTIVE DE MORGAN)} \quad \textit{and}\\ (4). & (x \wedge y)^\perp &=& x^\perp \vee y^\perp & \forall x,y \in X & \text{(CONJUNCTIVE DE MORGAN)} \quad \textit{and}\\ (5). & x \vee x^\perp &=& 1 & \forall x \in X & \text{(EXCLUDED MIDDLE).}\end{array}\right.$$

✎PROOF: Let $x^\perp \triangleq \neg x$, where $\neg$ is an *ortho negation* function (Definition 1.3 page 2). Then, this theorem follows directly from Theorem 1.5 (page 6). ⇨

Corollary J.1. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133)*.

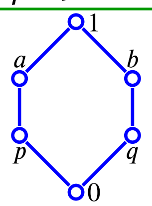
COR

$$\left\{\begin{array}{l}\boldsymbol{L} \textbf{\textit{ is orthocomplemented}}\\ \text{(Definition J.1 page 196)}\end{array}\right\} \implies \left\{\begin{array}{l}\boldsymbol{L} \textbf{\textit{ is complemented}}\\ \text{(Definition H.1 page 165)}\end{array}\right\}$$

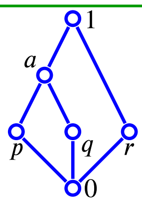
✎PROOF: This follows directly from the definition of *orthocomplemented lattices* (Definition J.1 page 196) and *complemented lattices* (Definition H.1 page 165). ⇨

Example J.5.

EX



The O_6 *lattice* (Definition J.2 page 196) illustrated to the left is both **orthocomplemented** (Definition J.1 page 196) and **multiply complemented** (Definition H.1 page 165). The lattice illustrated to the right is **multiply complemented**, but is **non-orthocomplemented**.



✎PROOF:

1. Proof that O_6 *lattice* is multiply complemented: b and q are both *complements* of p.

[5] Beran (1985) pages 30–31, Birkhoff and Neumann (1936) page 830 ⟨L74⟩, Cohen (1989) page 37 ⟨3B.13. Theorem⟩

2. Proof that the right side lattice is multiply complemented: a, p, and q are all *complements* of r.

⇒

Lemma J.1 (next) is useful in proving that *de Morgan*'s laws (Theorem A.8 page 58) hold in orthocomplemented lattices (Theorem J.1 page 198) and in proving the characterization of Theorem J.2 (page 200).

Lemma J.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

LEM

$$\underbrace{x \leq y \implies y^{\perp} \leq x^{\perp}}_{\text{ANTITONE}} \iff \underbrace{\left\{\begin{array}{rclll} (x \vee y)^{\perp} & = & x^{\perp} \wedge y^{\perp} & {\scriptstyle x,y \in X} & \text{and} \\ (x \wedge y)^{\perp} & = & x^{\perp} \vee y^{\perp} & {\scriptstyle x,y \in X} & \end{array}\right.}_{\text{DE MORGAN}}$$

✎PROOF: This follows directly from Lemma 1.2 (page 4). ⇒

Lemma J.2. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

LEM

The set $\{0, x, x^{\perp}\}$ *is* DISTRIBUTIVE *(Definition G.1 page 145) for all* $x \in X$.

✎PROOF:

$0 \wedge (x \vee x^{\perp}) = 0$	by *lower bounded* property	(Proposition E.2 page 133)
$= 0 \vee 0$	by *join identity*	(Proposition E.2 page 133)
$= (0 \wedge x) \vee (0 \wedge x^{\perp})$	by *lower bounded* property	(Proposition E.2 page 133)
$0 \wedge (x^{\perp} \vee x) = 0$	by *lower bounded* property	(Proposition E.2 page 133)
$= 0 \vee 0$	by *join identity*	(Proposition E.2 page 133)
$= (0 \wedge x^{\perp}) \vee (0 \wedge x)$	by *lower bounded* property	(Proposition E.2 page 133)
$x \wedge (x^{\perp} \vee 0) = x \wedge x^{\perp}$	by *join identity*	(Proposition E.2 page 133)
$= 0$	by *non-contradiction* property	(Definition J.1 page 196)
$= 0 \vee 0$	by *join identity*	(Proposition E.2 page 133)
$= (x \wedge x^{\perp}) \vee 0$	by *non-contradiction* property	(Definition J.1 page 196)
$= (x \wedge x^{\perp}) \vee (x \wedge 0)$	by *lower bounded* property	(Proposition E.2 page 133)
$x \wedge (0 \vee x^{\perp}) = x \wedge (x^{\perp} \vee 0)$	by *commutative* property of *lattices*	(Theorem D.3 page 118)
$= (x \wedge x^{\perp}) \vee (x \wedge 0)$	by previous result	
$= (x \wedge 0) \vee (x \wedge x^{\perp})$	by *commutative* property of *lattices*	(Theorem D.3 page 118)
$x^{\perp} \wedge (x \vee 0) = (x^{\perp} \wedge x) \vee (x^{\perp} \wedge 0)$	by $x \wedge (x^{\perp} \vee 0)$ result	
$x^{\perp} \wedge (0 \vee x) = (x^{\perp} \wedge 0) \vee (x^{\perp} \wedge x)$	by $x \wedge (0 \vee x^{\perp})$ result	

⇒

[6] Beran (1985) pages 30–31, Fáy (1967) ⟨cf Beran 1985 page 30⟩, Nakano and Romberger (1971) ⟨cf Beran 1985⟩

J.1.3 Characterization

Theorem J.2. [7] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an algebraic structure.*

THM *$\boldsymbol{L}$ is an orthocomplemented lattice* $\iff$
$$\left\{\begin{array}{lllll} 1. & (z^\perp \wedge y^\perp)^\perp \vee x & = & (x \vee y) \vee z & \forall x,y,z \in X \quad \text{and} \\ 2. & x \wedge (x \vee y) & = & x & \forall x,y \in X \quad \text{and} \\ 3. & x \vee (y \wedge y^\perp) & = & x & \forall x,y \in X. \end{array}\right.$$

✎PROOF:

1. Proof that orthocomplemented lattice $\implies$ 3 properties:

$$\begin{aligned}
(z^\perp \wedge y^\perp)^\perp \vee x &= \left[(z^\perp)^\perp \vee (y^\perp)^\perp\right] \vee x && \text{by } \textit{de Morgan} \text{ property (Theorem J.1 page 198)} \\
&= (z \vee y) \vee x && \text{by } \textit{involutory} \text{ property (Definition J.1 page 196)} \\
&= x \vee (z \vee y) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)} \\
&= x \vee (y \vee z) && \text{by } \textit{commutative} \text{ property (Theorem D.3 page 118)} \\
&= (x \vee y) \vee z && \text{by } \textit{associative} \text{ property (Theorem D.3 page 118)}
\end{aligned}$$

$$x \wedge (x \vee y) = x \qquad \text{by } \textit{absorptive} \text{ property (Theorem D.3 page 118)}$$

$$\begin{aligned}
x \vee (y \wedge y^\perp) &= x \vee 0 && \text{by } \textit{complemented} \text{ property (Definition J.1 page 196)} \\
&= x
\end{aligned}$$

2. Proof that orthocomplemented lattice $\impliedby$ 3 properties:
   (a) Proof that $\boldsymbol{L}$ is *meet-idempotent*:

$$\begin{aligned}
x \wedge x &= x \wedge \left[x \vee \left(y \wedge y^\perp\right)\right] && \text{by (3)} \\
&= x \wedge \left[x \vee \left(y \wedge y^\perp\right)\right] && \text{by (3)} \\
&= x && \text{by (2)}
\end{aligned}$$

   (b) Define $0 \triangleq xx^\perp$ for some $x \in X$. Proof that 0 is the *greatest lower bound* of $\boldsymbol{L}$: The element 0 is the greatest lower bound if and only if $xx^\perp = yy^\perp \quad \forall x, y \in X \ldots$
      i. Proof that $(xx^\perp)^{\perp\perp} = (xx^\perp) \quad \forall x \in X$:

$$\begin{aligned}
(xx^\perp)^{\perp\perp} &= (xx^\perp)^{\perp\perp} + (xx^\perp) && \text{by (3)} \\
&= \left[(xx^\perp)^\perp (xx^\perp)^\perp\right]^\perp + (xx^\perp) && \text{by item (2a)} \\
&= \left[(xx^\perp) + (xx^\perp)\right] + (xx^\perp) && \text{by (1)} \\
&= \left[(xx^\perp)\right] + (xx^\perp) && \text{by (3)} \\
&= (xx^\perp) && \text{by (3)}
\end{aligned}$$

      ii. Proof that $a = (xx^\perp) + a \quad \forall a, x \in X$:

$$\begin{aligned}
a &= a + (xx^\perp) && \text{by (3)} \\
&= \left[a + (xx^\perp)\right] + (xx^\perp) && \text{by (3)} \\
&= \left[(xx^\perp)^\perp (xx^\perp)^\perp\right]^\perp + a && \text{by (1)} \\
&= \left[(xx^\perp)^\perp\right]^\perp + a && \text{by item (2a)} \\
&= (xx^\perp) + a && \text{by item (2(b)i)}
\end{aligned}$$

[7] Beran (1985) pages 31–33, Beran (1976) pages 251–252

iii. Proof that $(xx^\perp) = (yy^\perp) \quad \forall x, y \in X$:

$$\begin{aligned}
(xx^\perp) &= (xx^\perp) + (yy^\perp) && \text{by (3)}\\
&= (yy^\perp) && \text{by item (2(b)ii)}
\end{aligned}$$

(c) Proof that $x + 0 = 0 + x = x \quad \forall x \in X$ (*join identity*):

$$\begin{aligned}
x + 0 &= x + (yy^\perp) && \text{by item (2(b)iii)}\\
&= x && \text{by (3)}\\
0 + x &= (uu^\perp) + x && \text{by item (2(b)iii)}\\
&= x && \text{by item (2(b)ii)}
\end{aligned}$$

(d) Proof that $x + y = (y^\perp x^\perp)^\perp \quad \forall x, y \in X$:

$$\begin{aligned}
(y^\perp x^\perp)^\perp &= (y^\perp x^\perp)^\perp + 0 && \text{by item (2c)}\\
&= (0 + x) + y && \text{by (1)}\\
&= x + y && \text{by item (2c)}
\end{aligned}$$

(e) Proof that $x + x = x^{\perp\perp} \quad \forall x \in X$:

$$\begin{aligned}
x + x &= (x^\perp x^\perp)^\perp && \text{by item (2d)}\\
&= (x^\perp)^\perp && \text{by item (2a)}
\end{aligned}$$

(f) Proof that $x + y = y + x \quad \forall x, y \in X$ (*join-commutative*):

$$\begin{aligned}
x + y &= (x + 0) + y && \text{by item (2c)}\\
&= (0^\perp x^\perp)^\perp + y && \text{by item (2d)}\\
&= (y + x) + 0 && \text{by (1)}\\
&= y + x && \text{by item (2c)}
\end{aligned}$$

(g) Proof that $(x + y) + z = x + (y + z) \quad \forall x, y, z \in X$ (*join-associative*):

$$\begin{aligned}
(x + y) + z &= (z^\perp y^\perp)^\perp + x && \text{by (1)}\\
&= (y + z) + x && \text{by item (2d)}\\
&= x + (y + z) && \text{by item (2f)}
\end{aligned}$$

(h) Proof that $x^{\perp\perp} = x \quad \forall x \in X$ (*involutory*):

$$\begin{aligned}
x^{\perp\perp} &= \left(x^\perp\right) \perp && \text{by definition of } x^{\perp\perp}\\
&= \left[x^\perp\left(x^\perp + x\right)\right] \perp && \text{by (2)}\\
&= \left[x^\perp\left(x^\perp x^{\perp\perp}\right)^\perp\right] \perp && \text{by item (2d)}\\
&= \left(x^\perp x^{\perp\perp}\right) + x && \text{by item (2d)}\\
&= (0) + x && \text{by item (2b)}\\
&= x && \text{by item (2c)}
\end{aligned}$$

(i) Proof of *de Morgan*'s laws:

$$\begin{aligned}
(x + y)^\perp &= (y + x)^\perp && \text{by item (2g)}\\
&= \left[\left(x^\perp y^\perp\right)^\perp\right]^\perp && \text{by item (2d)}\\
&= x^\perp y^\perp && \text{by item (2h)}
\end{aligned}$$

$$\begin{aligned}
(xy)^\perp &= \left(x^{\perp\perp} y^{\perp\perp}\right)^\perp && \text{by item (2h)}\\
&= y^\perp + x^\perp && \text{by item (2d)}\\
&= x^\perp + y^\perp && \text{by item (2g)}
\end{aligned}$$

(j) Proof that $(xy)z = x(yz) \quad \forall x, y, z \in X \quad$ (*meet-commutative*):

$$\begin{aligned}
xy &= (xy)^{\perp\perp} && \text{by item (2h)}\\
&= \left(x^{\perp} + y^{\perp}\right)^{\perp} && \text{by item (2i)}\\
&= \left(y^{\perp} + x^{\perp}\right)^{\perp} && \text{by item (2g)}\\
&= y^{\perp\perp}x^{\perp\perp} && \text{by item (2i)}\\
&= yx && \text{by item (2i)}
\end{aligned}$$

(k) Proof that $(xy)z = x(yz) \quad \forall x, y, z \in X \quad$ (*meet-associative*):

$$\begin{aligned}
(xy)z &= [(xy)z]^{\perp\ \perp} && \text{by item (2h)}\\
&= \left[(xy)^{\perp} + z^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[\left(x^{\perp} + y^{\perp}\right) + z^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp} + \left(y^{\perp} + z^{\perp}\right)\right]^{\perp} && \text{by item (2g)}\\
&= x^{\perp\perp}\left(y^{\perp} + z^{\perp}\right)^{\perp} && \text{by item (2i)}\\
&= x^{\perp\perp}\left(y^{\perp\perp}z^{\perp\perp}\right) && \text{by item (2i)}\\
&= x(yz) && \text{by item (2h)}
\end{aligned}$$

(l) Proof that $x + (xz) = x$ (*join-meet-absorptive*):

$$\begin{aligned}
x \vee (xz) &= [x + (xz)]^{\perp\perp} && \text{by item (2h)}\\
&= \left[x^{\perp}(xz)^{\perp}\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp}\left(x^{\perp} + z^{\perp}\right)\right]^{\perp} && \text{by item (2i)}\\
&= \left[x^{\perp}\right]^{\perp} && \text{by (2)}\\
&= x && \text{by item (2h)}
\end{aligned}$$

(m) Because $\boldsymbol{L}$ is *commutative* (item (2f) and item (2j)), *associative* (item (2g) and item (2k)), and *absorptive* ((2) and item (2l)), and by Theorem D.8 (page 126), $\boldsymbol{L}$ is a *lattice*.

(n) Define $1 \triangleq x + x^{\perp}$ for some $x \in X$. Proof that 1 is the *least upper bound* of $\boldsymbol{L}$: The element 1 is the least upper bound if and only if $x + x^{\perp} = y + y^{\perp} \quad \forall x, y \in X\ldots$

$$\begin{aligned}
1 &= \left(x + x^{\perp}\right) && \text{by definition of 1}\\
&= \left(x + x^{\perp}\right)^{\perp\perp} && \text{by item (2h)}\\
&= \left(x^{\perp}x\right)^{\perp} && \text{by item (2h)}\\
&= \left(xx^{\perp}\right)^{\perp} && \text{by item (2j)}\\
&= \left(yy^{\perp}\right)^{\perp} && \text{by item (2(b)iii)}\\
&= y^{\perp} + y^{\perp\perp} && \text{by item (2i)}\\
&= y^{\perp} + y && \text{by item (2h)}\\
&= y + y^{\perp} && \text{by item (2f)}
\end{aligned}$$

(o) Proof that $\boldsymbol{L}$ is *antitone*: by Theorem 1.4 (page 6).

(p) Proof that $\boldsymbol{L}$ is *complemented*: by item (2(b)iii) and item (2n).

(q) Because $\boldsymbol{L}$ is a *bounded* (item (2b) and item (2n)) lattice (item (2m)), and because $\boldsymbol{L}$ is *complemented* (item (2p)), is *involutory* (item (2h)), and is *antitone* (item (2o)), and by Definition J.1 (page 196), $\boldsymbol{L}$ is an *orthocomplemented lattice*.

J.1.4 Restrictions resulting in Boolean algebras

Proposition J.1. [8] *Let* $\mathbf{L} = (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* LATTICE *(Definition D.3 page 117).*

PRP $\left\{\begin{array}{lll} 1. & \mathbf{L} \text{ is } \textbf{\textit{orthocomplemented}} & \text{(Definition J.1 page 196) and} \\ 2. & \mathbf{L} \text{ is } \textbf{\textit{distributive}} & \text{(Definition G.2 page 146)} \end{array}\right\} \implies \left\{\begin{array}{l} \mathbf{L} \text{ is } \textbf{\textit{Boolean}} \\ \text{(Definition I.1 page 171)} \end{array}\right\}$

✎PROOF: To be a *Boolean algebra*, **L** must satisfy the 8 requirements of *boolean algebras* (Definition I.1 page 171):

1. Proof for *commutative* properties: These are true for *all* lattices (Definition D.3 page 117).
2. Proof for *join-distributive* property: by hypothesis (2).
3. Proof for *meet-distributive* property: by *join-distributive* property and the *Principle of duality* (Theorem D.4 page 119) for lattices.
4. Proof for *identity* properties: because **L** is a *bounded lattice* and by definitions of 1 (*least upper bound*), 0 (*greatest lower bound*), ∨, and ∧.
5. Proof for *complemented* properties: by hypothesis (1) and definition of *orthocomplemented lattices* (Definition J.1 page 196).

✏

Proposition J.2. *Let* $\mathbf{L} = (X, \vee, \wedge, 0, 1\,;\, \leq)$ *be a* LATTICE *(Definition D.3 page 117).*

PRP $\left\{\begin{array}{lll} 1. & \mathbf{L} \text{ is } \textbf{\textit{orthocomplemented}} & \text{(Definition J.1 page 196) and} \\ 2. & \text{Every } x \in \mathbf{L} \text{ is in the } \textbf{\textit{center}} \text{ of } \mathbf{L} & \text{(Definition K.4 page 214)} \end{array}\right\} \iff \left\{\begin{array}{l} \mathbf{L} \text{ is} \\ \textbf{\textit{Boolean}} \end{array}\right\}$

✎PROOF:

1. Proof that (1,2) $\implies$ *Boolean*: **L** is *Boolean* because it satisfies *Huntington's Fourth Set* (Proposition I.6 page 189), as demonstrated by the following ...
   (a) Proof that $x \vee x = x$ (*idempotent*): **L** is a *lattice* (by definition of **L**), and all lattices are *idempotent* (Definition D.3 page 117).
   (b) Proof that $x \vee y = y \vee x$ (*commutative*): **L** is a *lattice* (by definition of **L**), and all lattices are *commutative* (Definition D.3 page 117).
   (c) Proof that $(x \vee y) \vee z = x \vee (y \vee z)$ (*associative*): **L** is a *lattice* (by definition of **L**), and all lattices are *associative* (Definition D.3 page 117).
   (d) Proof that $(x^\perp \vee y^\perp)^\perp \vee (x^\perp \vee y)^\perp = x$ (*Huntington's axiom*):

$$\begin{aligned} (x^\perp \vee y^\perp)^\perp \vee (x^\perp \vee y)^\perp &= (x^\perp \perp \wedge y^\perp \perp) \vee (x^\perp \perp \wedge y^\perp) && \text{by } \textit{de Morgan} \text{ property (Theorem J.1 page 198)} \\ &= (x \wedge y) \vee (x \wedge y^\perp) && \text{by } \textit{involution} \text{ property (Definition J.1 page 196)} \\ &= x && \text{by definition of } \textit{center} \text{ (Definition K.4 page 214)} \end{aligned}$$

2. Proof that (1) $\impliedby$ *Boolean*:
   (a) Proof that $x \vee x^\perp = 1$: by definition of *Boolean algebras* (Definition I.1 page 171).
   (b) Proof that $x \wedge x^\perp = 0$: by definition of *Boolean algebras* (Definition I.1 page 171).

[8] Kalmbach (1983) page 22

(c) Proof that $x^{\perp\perp} = x$: by *involutory* property of *Boolean algebra* (Theorem I.2 page 176).

(d) Proof that $x \le y \implies y^\perp \le x^\perp$:

$$\begin{aligned}
y^\perp \le x^\perp \iff y^\perp &= y^\perp \wedge x^\perp && \text{by Lemma D.1 page 119}\\
\iff y^{\perp\perp} &= \left(y^\perp \wedge x^\perp\right)^\perp\\
\iff y^{\perp\perp} &= y^{\perp\perp} \vee x^{\perp\perp} && \text{by \textit{de Morgan} property (Theorem I.2 page 176)}\\
\iff y &= y \vee x && \text{by \textit{involutory} property (Theorem I.2 page 176)}\\
\iff y &= y && \text{by } x \le y \text{ hypothesis}
\end{aligned}$$

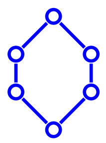
3. Proof that (2) $\Longleftarrow$ *Boolean*: for all $x, y \in \boldsymbol{L}$

$$\begin{aligned}
(x \wedge y) \vee \left(x \wedge y^\perp\right) &= [(x \wedge y) \vee x] \wedge \left[(x \wedge y) \vee y^\perp\right] && \text{by \textit{distributive} property (Theorem I.2 page 176)}\\
&= x \wedge \left[(x \wedge y) \vee y^\perp\right] && \text{by \textit{absorptive} property (Theorem I.2 page 176)}\\
&= x \wedge \left[\left(x \vee y^\perp\right) \wedge \left(y \vee y^\perp\right)\right] && \text{by \textit{distributive} property (Theorem I.2 page 176)}\\
&= x \wedge \left(x \vee y^\perp\right) \wedge 1 && \text{by \textit{complement} property (Theorem I.2 page 176)}\\
&= x && \text{by \textit{absorptive} property (Theorem I.2 page 176)}\\
&\implies x \copyright y \quad \forall x, y \in \boldsymbol{L} && \text{by Definition K.2 page 211}\\
&\implies x \text{ is in the \textit{center} of } \boldsymbol{L} \text{ for all } x \in \boldsymbol{L} && \text{by Definition K.4 page 214}
\end{aligned}$$

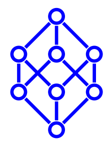
✏

Example J.6.

EX



The O_6 *lattice* (Definition J.2 page 196) illustrated to the left is **orthocomplemented** (Definition J.1 page 196) but **non-join-distributive** (Definition G.2 page 146),and hence *non-Boolean*. The lattice illustrated to the right is **orthocomplemented** *and* **distributive** and hence also **Boolean** (Proposition J.1 page 203). Alternatively, the right side lattice is **orthocomplemented** *and* every element is in the *center*, and hence also **Boolean** (Proposition J.2 page 203).



Note that of the 5 lattices on 5 element sets (Example D.11 page 124), the 15 lattices on 6 element sets (Example D.12 page 124), and 53 lattices on 7 element sets (Example D.13 page 124), **none** are **uniquely complemented**.

✎PROOF:

1. Proof that the O_6 *lattice* is *non-join-distributive*:

$$\begin{aligned}
x \vee (x^\perp \wedge z^\perp) &= x \vee 0\\
&= x\\
&\ne z^\perp\\
&= 1 \wedge z^\perp\\
&= (x \vee x^\perp) \wedge (x \vee z^\perp)
\end{aligned}$$

2. Proof that the O_6 *lattice* is also *non-meet-distributive*:

$$\begin{aligned}
z^\perp \wedge (x \vee z) &= z^\perp \wedge 1\\
&= z^\perp\\
&\ne x\\
&= x \vee 1\\
&= (z^\perp \wedge x) \vee (z^\perp \wedge z)
\end{aligned}$$

✏

J.2 Orthomodular lattices

J.2.1 Properties

Definition J.3. [9] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an algebraic structure.*

DEF

L *is an* ***orthomodular lattice*** *if*

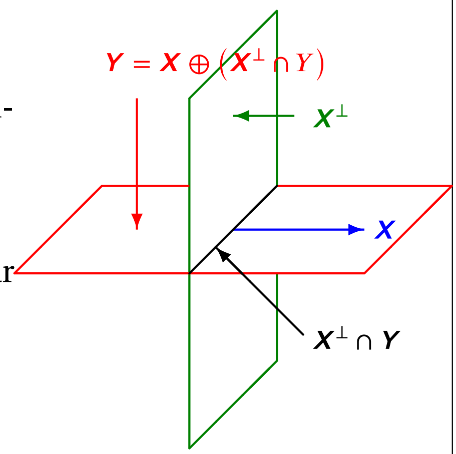
1. ***L*** *is an* ORTHOCOMPLEMENTED LATTICE *and*
2. $x \leq y \implies x \vee (x^{\perp} \wedge y) = y \quad \forall x,y \in X$ (ORTHOMODULAR IDENTITY)

Example J.7.

EX

The O_6 *lattice* (Definition J.2 page 196) is *orthocomplemented*, but *non-orthomodular* (and hence, *non-modular* and *non-Boolean*).

Example J.8. [10] Let $\boldsymbol{H}$ be a Hilbert space and $2^{\boldsymbol{H}}$ the set of closed linear subspaces of $\boldsymbol{H}$.



EX

$(2^{\boldsymbol{H}}, \oplus, \cap, \varnothing, \boldsymbol{H}; \subseteq)$ is an orthomodular lattice.

This concept is illustrated to the right where $\boldsymbol{X}, \boldsymbol{Y} \in 2^{\boldsymbol{H}}$ are linear subspaces of the linear space $\boldsymbol{H}$ and

$$\boldsymbol{X} \subseteq \boldsymbol{Y} \implies \boldsymbol{Y} = \boldsymbol{X} \oplus (\boldsymbol{X}^{\perp} \cap Y).$$

Theorem J.3. [11] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a lattice.*

THM

$$\left.\begin{array}{ll} 1. & \mathbf{L} \text{ is ORTHOMODULAR and} \\ 2. & y \copyright x \text{ and } z \copyright x \end{array}\right\} \implies (x, y, z) \in \text{Ⓓ}$$

J.2.2 Characterizations

Theorem J.4. [12] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE (Definition J.1 page 196). *Let* Ⓜ *and* Ⓜ* *be the modularity relation and dual modularity relation, respectively* (Definition F.1 page 135), $\perp$ *the orthogonality relation (Definition K.1 page 209), and* © *the commutes relation (Definition K.2 page 211).*

[9] Kalmbach (1983) page 22, Lidl and Pilz (1998) page 90, Husimi (1937)
[10] Iturrioz (1985) pages 56–57
[11] Kalmbach (1983) page 25, Holland (1963) pages 69–70 ⟨Theorem 3⟩, Foulis (1962) page 68 ⟨Theorem 5⟩
[12] Kalmbach (1983) page 22, Stern (1999) page 12, Nakamura (1957), Holland (1963), Foulis (1962), Maeda and Maeda (1970) page 132 ⟨Theorem 29.13⟩

THM *The following statements are* EQUIVALENT:

1. $\mathbf{L}$ *is* ORTHOMODULAR
$\iff$ 2. $x \le y$ *and* $y \wedge x^{\perp} = 0 \implies x = y$
$\iff$ 3. $\mathbf{L}$ *does* NOT *contain the* O_6 *lattice*
$\iff$ 4. $x \circledcirc y \iff y \circledcirc x$ (© *is* SYMMETRIC)
$\iff$ 5. $x \text{Ⓜ} x^{\perp} \qquad \forall x \in X$
$\iff$ 6. $x \text{Ⓜ}^{*} x^{\perp} \qquad \forall x \in X$
$\iff$ 7. $x \vee \left[x^{\perp} \wedge (x \vee y)\right] = x \vee y \qquad \forall x, y \in X$
$\iff$ 8. $x \le y \implies \exists p \in X$ such that $x \perp p$ *and* $x \vee p = y$

✎PROOF:

1. Proof that *orthomodular* $\iff$ *symmetric*: by Proposition K.3 (page 212).

✏

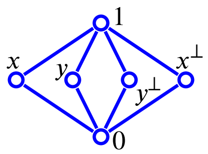
J.2.3 Restrictions resulting in Boolean algebras

Theorem J.5. [13] *Let* $\mathbf{L} = (X, \vee, \wedge, 0, 1; \le)$ *be an algebraic structure.*

THM
$$\left\{\begin{array}{ll} \mathbf{L} \text{ is an \textbf{orthomodular lattice}} & \text{and} \\ \underbrace{\left(x \wedge y^{\perp}\right)^{\perp} = y \vee \left(x^{\perp} \wedge y^{\perp}\right)}_{\text{ELKAN'S LAW}} & \forall x, y \in X \end{array}\right\} \implies \left\{\begin{array}{l} \mathbf{L} \text{ is a} \\ \textbf{Boolean algebra} \\ \text{(Definition I.1 page 171)} \end{array}\right\}$$

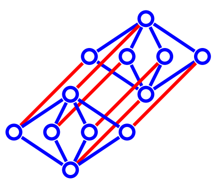
Definition J.4. [14]

DEF *The* $\mathbf{MO_2}$ ***lattice*** *is the ordered set* $\left(\left\{0, x, y, x^{\perp}, y^{\perp}, 1\right\}, \le\right)$ *with cover relation*
$$\lessdot = \left\{(0, x), (0, y), \left(0, x^{\perp}\right), \left(0, y^{\perp}\right), (x, 1), (y, 1), \left(x^{\perp}, 1\right), \left(y^{\perp}, 1\right)\right\}$$
This lattice is also called the ***Chinese lantern***.



MO_2

L_2



$\mathrm{MO}_2 \times \mathrm{L}_2$

Theorem J.6. [15] *Let* $\mathbf{M} = (X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOMODULAR *lattice.*

THM
$$\left\{\begin{array}{l} \mathbf{M} \text{ is} \\ \text{BOOLEAN} \end{array}\right\} \iff \left\{\begin{array}{ll} 1. & \mathbf{M} \text{ does not contain the } \mathrm{MO}_2 \text{ lattice (Definition J.4 page 206)} \quad \text{and} \\ 2. & \mathbf{M} \text{ does not contain the } \mathrm{MO}_2 \times \mathrm{L}_2 \text{ lattice.} \end{array}\right\}$$

[13] Renedo et al. (2003) page 72
[14] Iturrioz (1985) page 57, Davey and Priestley (2002) pages 18–19 ⟨1.25 Products⟩
[15] Iturrioz (1985) page 57, Carrega (1982) ⟨cf Iturrioz 1985 page 57⟩

J.3 Modular orthocomplemented lattices

Definition J.5. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be a* BOUNDED LATTICE *(Definition E.1 page 133).*

DEF

***L** is a **modular orthocomplemeted lattice** if*

1. ***L** is **orthocomplemented*** (Definition J.1 page 196) *and*
2. ***L** is **modular*** (Definition F.3 page 136)

J.4 Relationships between orthocomplemented lattices

Theorem J.7. [16] *Let **L** be a lattice.*

THM

$$\left\{\begin{array}{l}\mathbf{L} \textit{ is}\\ \text{BOOLEAN}\end{array}\right\} \implies \left\{\begin{array}{l}\mathbf{L} \textit{ is}\\ \text{MODULAR}\\ \text{ORTHOCOM-}\\ \text{PLEMENTED}\end{array}\right\} \implies \left\{\begin{array}{l}\mathbf{L} \textit{ is } \text{ORTHO-}\\ \text{MODULAR}\end{array}\right\} \implies \left\{\begin{array}{l}\mathbf{L} \textit{ is}\\ \text{ORTHOCOM-}\\ \text{PLEMENTED}\end{array}\right\}$$

Remark J.1. [17] Lattice number 8 in Example J.2 (page 196) was originally introduced by Dilworth as a counterexample to *Husimi's conjecture* (1937). Kalmbach(1983) points out that this lattice was the first example of a *finite orthomodular* lattice.

[16] Kalmbach (1983) page 32 ⟨20.⟩, Iturrioz (1985) page 57

[17] Dilworth (1940), Dilworth (1990), Kalmbach (1983) page 9

APPENDIX K

RELATIONS ON LATTICES WITH NEGATION

The relations in this chapter are typically defined on an *orthocomplemented lattice* (Definition J.1 page 196). Here, some relations are generalized to a *lattice with negation* (Definition 1.5 page 3). A *lattice* (Definition D.3 page 117) with an *ortho negation* negation successfully defined on it is an *orthocomplemented lattice* (Definition J.1 page 196). In many cases, these relations only work well on an *orthocomplemented lattice*, and thus many results are restricted to orthocomplemented lattices.

K.1 Orthogonality

Proposition K.1. *Let* $(X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

PRP

$$x \leq y \implies \left\{ \begin{array}{rcrcll} x^\perp & \vee & y & = & 1 & \text{and} \\ x & \wedge & y^\perp & = & 0 & \end{array} \right\} \qquad \forall x,y \in X$$

✎PROOF:

$$\begin{aligned}
x \leq y &\implies x \vee x^\perp \leq y \vee x^\perp && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition D.1 page 119)} \\
&\implies 1 \leq y \vee x^\perp && \text{by } \textit{excluded middle} \text{ property of } \textit{ortho lattices} \text{ (Definition J.1 page 196)} \\
&\implies x^\perp \vee y = 1 && \text{by } \textit{upper bounded} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)} \\
x \leq y &\implies x \wedge y^\perp \leq y \wedge y^\perp && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition D.1 page 119)} \\
&\implies x \wedge y^\perp \leq 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{ortho lattices} \text{ (Definition J.1 page 196)} \\
&\implies x \wedge y^\perp = 0 && \text{by } \textit{lower bounded} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}
\end{aligned}$$

✏

Definition K.1. [1] *Let* $(X, \vee, \wedge, \neg, 0, 1; \leq)$ *be a* LATTICE WITH NEGATION *(Definition 1.5 page 3).*

DEF

The ***orthogonality*** *relation* $\perp \in 2^{XX}$ *is defined as*

$$x \perp y \quad \overset{\text{def}}{\iff} \quad x \leq \neg y$$

If $x \perp y$*, we say that* x *is* ***orthogonal*** *to* y*.*

[1] Stern (1999) page 12, Loomis (1955) page 3

Lemma K.1. *Let* $(X, \vee, \wedge, \neg, 0, 1\,;\, \leq)$ *be a* LATTICE WITH NEGATION *(Definition 1.5 page 3).*

LEM
$$\left\{\, x \perp y \quad (\text{ORTHOGONAL Definition K.1 page 209}) \,\right\} \implies \left\{\, y \perp x \quad (\text{SYMMETRIC}) \,\right\}$$

✎PROOF:

$$\begin{aligned}
x \perp y &\implies x \leq \neg y && \text{by definition of } \perp \text{ (Definition K.1 page 209)}\\
&\implies (\neg\neg y) \leq \neg x && \text{by } \textit{antitone} \text{ property (Definition J.1 page 196)}\\
&\implies y \leq \neg x && \text{by } \textit{weak double negation} \text{ property of } \textit{negation} \text{ (Definition 1.2 page 2)}\\
&\implies y \perp x && \text{by definition of } \perp \text{ (Definition K.1 page 209)}
\end{aligned}$$

✏

Lemma K.2. [2] *Let* $(X, \vee, \wedge, 0, 1\,;\, \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

LEM
$$\underbrace{x \perp y}_{\text{ORTHOGONAL (Definition K.1 page 209)}} \implies \left\{\begin{array}{lrcll} 1. & x \wedge y & = & 0 & \text{and}\\ 2. & x^\perp \vee y^\perp & = & 1 & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
x \perp y &\implies x \leq y^\perp && \text{by definition of } \perp \text{ (Definition K.1 page 209)}\\
&\implies x \wedge y \leq y^\perp \wedge y && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition D.1 page 119)}\\
&\implies x \wedge y \leq y \wedge y^\perp && \text{by } \textit{commutative} \text{ property of lattices (Theorem D.3 page 118)}\\
&\implies x \wedge y \leq 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{ortho negation} \text{ (Definition 1.3 page 2)}\\
&\implies x \wedge y = 0 && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}
\end{aligned}$$

$$\begin{aligned}
x \perp y &\implies x \leq y^\perp && \text{by definition of } \perp \text{ (Definition K.1 page 209)}\\
&\implies x^\perp \vee x \leq x^\perp \vee y^\perp && \text{by } \textit{monotone} \text{ property of } \textit{lattices} \text{ (Proposition D.1 page 119)}\\
&\implies x \vee x^\perp \leq x^\perp \vee y^\perp && \text{by } \textit{commutative} \text{ property of } \textit{lattices} \text{ (Theorem D.3 page 118)}\\
&\implies 1 \leq x^\perp \vee y^\perp && \text{by } \textit{excluded middle} \text{ property of } \textit{ortho lattices} \text{ (Theorem 1.5 page 6)}\\
&\implies x^\perp \vee y^\perp && \text{by } \textit{upper bound} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}
\end{aligned}$$

✏

Remark K.1. In an *orthocomplemented lattice* $\boldsymbol{L}$, the *orthogonality* relation $\perp$ is in general *non-associative*. That is

$$\left\{\begin{array}{cccl} x & \perp & y & \text{and}\\ y & \perp & z & \end{array}\right\} \;\not\Longrightarrow\; x \perp z$$

✎PROOF: Consider the $\boldsymbol{L}_2^4$ *Boolean lattice* in Example J.2 (page 196).

- $a^\perp \perp p$ because $a^\perp \leq p^\perp$.
- $p \perp r$ because $p \leq r^\perp$.
- But yet $a^\perp$ is *not* orthogonal to r because $a^\perp \nleq r^\perp$.

✏

Example K.1.

EX

In the O_6 *lattice* (Definition J.2 page 196), there are a total of $\binom{6}{2} = \frac{6!}{(6-2)!2!} = \frac{6\times 5}{2} = 15$ distinct unordered (the $\perp$ relation is *symmetric* by Lemma K.1 page 210 so the order doesn't matter) pairs of elements.

Of these 15 pairs, 8 are orthogonal to each other, and 0 is orthogonal to itself, making a total of 9 orthogonal pairs:

x	$\perp$	y	x	$\perp$	0	$y^\perp$	$\perp$	0
x	$\perp$	$x^\perp$	y	$\perp$	0	1	$\perp$	0
y	$\perp$	$y^\perp$	$x^\perp$	$\perp$	0	0	$\perp$	0

[2] Holland (1963) page 67

Example K.2.

EX

In lattice 5 of Example J.2 (page 196), there are a total of $\binom{10}{2} = \frac{10!}{(10-2)!2!} = \frac{10\times 9}{2} = 45$ distinct unordered pairs of elements.

Of these 45 pairs, 18 are orthogonal to each other, and 0 is orthogonal to itself, making a total of 19 orthogonal pairs:

$p \perp p^\perp$	$x \perp x^\perp$	$y \perp z$	$x^\perp \perp 0$
$p \perp x^\perp$	$x \perp y$	$y \perp 0$	$y^\perp \perp 0$
$p \perp y$	$x \perp z$	$z \perp z^\perp$	$z^\perp \perp 0$
$p \perp z$	$x \perp 0$	$z \perp 0$	$0 \perp 0$
$p \perp 0$	$y \perp y^\perp$	$p^\perp \perp 0$	

Example K.3.

EX

In the $\mathbb{R}^3$ **Euclidean space** illustrated in Example J.3 (page 197),

$$\begin{array}{llll} \boldsymbol{X} \subseteq \boldsymbol{Y}^\perp & \implies \boldsymbol{X} \perp \boldsymbol{Y} & \boldsymbol{Y} \subseteq \boldsymbol{X}^\perp & \implies \boldsymbol{Y} \perp \boldsymbol{X} \\ \boldsymbol{X} \subseteq \boldsymbol{Z}^\perp & \implies \boldsymbol{X} \perp \boldsymbol{Z} & \boldsymbol{Y} \subseteq \boldsymbol{Z}^\perp & \implies \boldsymbol{Y} \perp \boldsymbol{Z} \\ \boldsymbol{X} \wedge \boldsymbol{Y} = \boldsymbol{X} \wedge \boldsymbol{Z} = \boldsymbol{Y} \wedge \boldsymbol{Z} = \boldsymbol{0} & & & \end{array}$$

K.2 Commutativity

The *commutes* relation is defined next. Motivation for the name "commutes" is provided by Proposition K.4 (page 214) which shows that if x commutes with y in a lattice $\boldsymbol{L}$, then x and y commute in the *Sasaki projection* $\phi_x(y)$ on $\boldsymbol{L}$.

Definition K.2. [3] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1 ; \leq)$ *be a* LATTICE WITH NEGATION (Definition 1.5 page 3).

DEF

The ***commutes*** *relation* © *is defined as*

$$x © y \overset{\text{def}}{\iff} x = (x \wedge y) \vee (x \wedge \neg y) \qquad \forall x,y \in X,$$

in which case we say, "x ***commutes*** *with y in* $\boldsymbol{L}$*".*
That is, © *is a relation in* 2^{XX} *such that*

$$© \triangleq \{(x,y) \in X^2 | x = (x \wedge y) \vee (x \wedge \neg y)\}$$

Proposition K.2. [4] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1 ; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE.

PRP

$x©0$ *and* $0©x \quad \forall x \in X$	$x©y \iff x©y^\perp \quad \forall x,y \in X$
$x©1$ *and* $1©x \quad \forall x \in X$	$x \leq y \implies x©y \quad \forall x,y \in X$
$x©x \quad \forall x \in X$	$x \perp y \implies x©y \quad \forall x,y \in X$

✎PROOF:

$$\begin{aligned}
(x \wedge 0) \vee (x \wedge 0^\perp) &= 0 \vee (x \wedge 0^\perp) && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}\\
&= 0 \vee (x \wedge 1) && \text{by } \textit{boundary condition} \text{ of } \textit{ortho negation} \text{ (Theorem 1.5 page 6)}\\
&= 0 \vee (x) && \text{by } \textit{upper bound} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}\\
&= x && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}\\
&\implies x©0 && \text{by definition of © relation (Definition K.2 page 211)}\\
(0 \wedge x) \vee (0 \wedge x^\perp) &= 0 \vee (0) && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}\\
&= 0 && \text{by } \textit{lower bound} \text{ property of } \textit{bounded lattices} \text{ (Definition E.1 page 133)}\\
&\implies 0©x && \text{by definition of © relation (Definition K.2 page 211)}
\end{aligned}$$

[3] Kalmbach (1983) page 20, Holland (1970) page 79 ⟨A. Commutativity⟩, Maeda (1958) page 227 ⟨Hilfssatz (Lemma) XII.1.2⟩, Sasaki (1954) page 301 ⟨Def.5.2, cf Foulis 1962⟩, Birkhoff (1936b) page 833 ⟨"$a = (a \cap x) \cup (a \cap x')$"⟩

[4] Holland (1963) page 67

$(x \wedge 1) \vee (x \wedge 1^\perp) = x \vee (x \wedge 1^\perp)$	by *lower bound* property of *bounded lattices* (Definition E.1 page 133)
$= x \vee (x \wedge 0)$	by *boundary condition* of *ortho negation* (Theorem 1.5 page 6)
$= (x) \vee (0)$	by *lower bound* property of *bounded lattices* (Definition E.1 page 133)
$= x$	by *lower bound* property of *bounded lattices* (Definition E.1 page 133)
$\implies x © 1$	by definition of © relation (Definition K.2 page 211)
$(1 \wedge x) \vee (1 \wedge x^\perp) = (x) \vee (x^\perp)$	by *non-contradiction* prop. of *ortho negation* (Definition 1.3 page 2)
$= 1$	by *excluded middle* property of *ortho negation* (Theorem 1.5 page 6)
$\implies 1 © x$	by definition of © relation (Definition K.2 page 211)
$(x \wedge x) \vee (x \wedge x^\perp) = x \vee (x \wedge x^\perp)$	by *idempotent* property of *lattices* (Theorem D.3 page 118)
$= x \vee (0)$	by *non-contradiction* prop. of *ortho negation* (Definition 1.3 page 2)
$= x$	by *lower bound* property of *bounded lattices* (Definition E.1 page 133)
$\implies x © x$	by definition of © relation (Definition K.2 page 211)
$x © y \implies (x \wedge y^\perp) \vee (x \wedge y^{\perp\perp})$	by definition of © (Definition K.2 page 211)
$= (x \wedge y^\perp) \vee (x \wedge y)$	by *involution* property of $\perp$ (Definition J.1 page 196)
$= (x \wedge y) \vee (x \wedge y^\perp)$	by *commutative* property of *lattices* (Definition D.3 page 117)
$= x$	by $x © y$ hypothesis and Definition K.2 page 211
$\implies x © y^\perp$	by definition of © relation (Definition K.2 page 211)
$x © y^\perp \implies (x \wedge y) \vee (x \wedge y^\perp)$	by definition of © (Definition K.2 page 211)
$= (x \wedge y^{\perp\perp}) \vee (x \wedge y^\perp)$	by *involution* property of $\perp$ (Definition J.1 page 196)
$= (x \wedge y^\perp) \vee (x \wedge y^{\perp\perp})$	by *commutative* property of *lattices* (Definition D.3 page 117)
$= x$	by $x © y^\perp$ hypothesis and Definition K.2 page 211
$\implies x © y$	by definition of © relation (Definition K.2 page 211)
$x \le y \implies (x \wedge y) \vee (x \wedge y^\perp)$	by definition of © (Definition K.2 page 211)
$= x \vee (x \wedge y^\perp)$	by $x \le y$ hypothesis
$= x$	by *absorptive* property (Theorem D.3 page 118)
$\implies x © y$	by definition of © (Definition K.2 page 211)
$x \perp y \implies (x \wedge y) \vee (x \wedge y^\perp)$	by definition of © (Definition K.2 page 211)
$= 0 \vee (x \wedge y^\perp)$	by Lemma K.2 page 210
$= 0 \vee x$	by $x \perp y$ hypothesis ($x \perp y \implies x \le y^\perp$)
$= x \vee 0$	by *commutative* property (Theorem D.3 page 118)
$= x$	by *identity* property of *bounded lattices*
$\implies x © y$	by definition of © (Definition K.2 page 211)

⇨

Definition K.3. *Let © be the* COMMUTES *relation (Definition K.2 page 211) on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ *(Definition 1.5 page 3).*

DEF *$\boldsymbol{L}$ is **symmetric** if*

$$x © y \implies y © x \qquad \forall x,y \in X$$

In general, the commutes relation is not *symmetric*. But Proposition K.3 (next) describes some conditions under which it *is* symmetric.

Proposition K.3. [5] *Let $(X, \vee, \wedge, 0, 1; \le)$ be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

[5] Holland (1963) page 68, Nakamura (1957) page 158

PRP

$$\underbrace{\{x \circledC y \implies y \circledC x\}}_{\circledC \text{ is SYMMETRIC at } (x,y) \; (1)} \iff \left\{x \le y \implies y = x \vee \left(x^\perp \wedge y\right)\right\} \quad \text{(ORTHOMODULAR IDENTITY)} \quad (2)$$

$$\iff \left\{x \le y \implies x = y \wedge \left(x \vee y^\perp\right)\right\} \quad (x = \phi_y(x) \text{ (SASAKI PROJECTION)}) \quad (3)$$

$$\iff \left\{y = (x \wedge y) \vee \left[y \wedge (x \wedge y)^\perp\right]\right\} \quad (4)$$

$$\iff \left\{x = (x \vee y) \wedge \left[x \vee (x \vee y)^\perp\right]\right\} \quad (5)$$

✎PROOF:

1. Proof that (2) $\iff$ (3):

$$\begin{aligned}
x \le y \implies y^\perp &\le x^\perp && \text{by \textit{antitone} property (Definition J.1 page 196)}\\
\implies x^\perp &= y^\perp \vee \left(y^{\perp\perp} \wedge x^\perp\right) && \text{by left hypothesis}\\
\implies \left(x^\perp\right)^\perp &= \left[y^\perp \vee \left(y^{\perp\perp} \wedge x^\perp\right)\right]^\perp &&\\
\implies x &= \left[y^\perp \vee \left(y^{\perp\perp} \wedge x^\perp\right)\right]^\perp && \text{by \textit{involutory} property (Definition J.1 page 196)}\\
&= y^{\perp\perp} \wedge \left(y^{\perp\perp} \wedge x^\perp\right)^\perp && \text{by \textit{de Morgan} property (Theorem J.1 page 198)}\\
&= y \wedge \left(y \wedge x^\perp\right)^\perp && \text{by \textit{involutory} property (Definition J.1 page 196)}\\
&= y \wedge \left(y^\perp \vee x^{\perp\perp}\right) && \text{by \textit{de Morgan} property (Theorem J.1 page 198)}\\
&= y \wedge \left(y^\perp \vee x\right) && \text{by \textit{involutory} property (Definition J.1 page 196)}\\
&= y \wedge \left(x \vee y^\perp\right) && \text{by \textit{commutative} property (Theorem D.3 page 118)}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies y^\perp &\le x^\perp && \text{by \textit{antitone} property (Definition J.1 page 196)}\\
\implies y^\perp &= x^\perp \wedge \left(y^\perp \vee x^{\perp\perp}\right) && \text{by right hypothesis}\\
\implies \left(y^\perp\right)^\perp &= \left[x^\perp \wedge \left(y^\perp \vee x^{\perp\perp}\right)\right]^\perp &&\\
\implies y &= \left[x^\perp \wedge \left(y^\perp \vee x^{\perp\perp}\right)\right]^\perp && \text{by \textit{involutory} property (Definition J.1 page 196)}\\
&= x^{\perp\perp} \vee \left(y^\perp \vee x^{\perp\perp}\right)^\perp && \text{by \textit{de Morgan} property (Theorem J.1 page 198)}\\
&= x \vee \left(y^\perp \vee x\right)^\perp && \text{by \textit{involutory} property (Definition J.1 page 196)}\\
&= x \vee \left(y^{\perp\perp} \wedge x^\perp\right) && \text{by \textit{de Morgan} property (Theorem J.1 page 198)}\\
&= x \vee \left(y \wedge x^\perp\right) && \text{by \textit{involutory} property (Definition J.1 page 196)}\\
&= x \vee \left(x^\perp \wedge y\right) && \text{by \textit{commutative} property (Theorem D.3 page 118)}
\end{aligned}$$

2. Proof that (2) $\iff$ (4):

$$\begin{aligned}
(xy) \vee \left[y(xy)^\perp\right] &= u \vee \left[yu^\perp\right] && \text{where } u \triangleq xy \le y\\
&= u \vee \left[u^\perp y\right] && \text{by \textit{commutative} property of lattices (Theorem D.3 page 118)}\\
&= y && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies x \vee \left(x^\perp y\right) &= xy \vee \left[(xy)^\perp y\right] && \text{by } x \le y \text{ hypothesis}\\
&= xy \vee \left[y(xy)^\perp\right] && \text{by \textit{commutative} property of lattices (Theorem D.3 page 118)}\\
&= y && \text{by right hypothesis}
\end{aligned}$$

3. Proof that (3) $\iff$ (5):

$$\begin{aligned}
(x \vee y)\left[x \vee (x \vee y)^\perp\right] &= u\left[x \vee u^\perp\right] && \text{where } x \le u \triangleq x \vee y\\
&= x && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
x \le y \implies y\left(x \vee y^\perp\right) &= (x \vee y)\left[x \vee (x \vee y)^\perp\right] && \text{by } x \le y \text{ hypothesis}\\
&= x && \text{by right hypothesis}
\end{aligned}$$

4. Proof that (1) $\implies$ (2):

$x \le y \implies x \copyright y$	by Proposition K.2 page 211
$\implies y \copyright x$	by *symmetry* hypothesis (left hypothesis)
$\implies y = (y \wedge x) \vee (y \wedge x^\perp)$	by definition of © (Definition K.2 page 211)
$\implies y = x \vee (y \wedge x^\perp)$	by $x \le y$ hypothesis
$\implies y = x \vee (x^\perp \wedge y)$	by *commutative* property of lattices (Theorem D.3 page 118)

5. Proof that (2) $\implies$ (4):

(a) lemma: proof that $x \copyright y \implies x^\perp y = (xy)^\perp y$:

$x \copyright y \implies x^\perp y = (xy \vee xy^\perp)^\perp y$	by definition of © (Definition K.2 page 211)
$= (xy)^\perp (xy^\perp)^\perp y$	by *de Morgan*'s law (Theorem 1.4 page 6)
$= (xy)^\perp \left[\left(x^\perp \vee y^{\perp\perp}\right) y\right]$	by *de Morgan*'s law (Theorem 1.4 page 6)
$= (xy)^\perp [(x^\perp \vee y) y]$	by *involutory*'s property (Definition J.1 page 196)
$= (xy)^\perp y$	by *absorptive* property of lattices (Theorem D.3 page 118)

(b) Completion of proof for (2) $\implies$ (4):

$x \copyright y \implies xy \vee y(xy)^\perp = xy \vee (xy)^\perp y$	by *commutative* property (Theorem D.3 page 118)
$= xy \vee x^\perp y$	by $x \copyright y$ hypothesis and item (5a)
$= (yx) \vee [yx^\perp]$	by *commutative* property (Theorem D.3 page 118)
$\implies y \copyright x$	by definition of © (Definition K.2 page 211)

✏

Theorem K.1. [6] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

THM

$$\{x \copyright c \quad \forall x \in X\} \iff \{\boldsymbol{L} \text{ is ISOMORPHIC to } [0:c] \times [0:c^\perp]\}$$
with isomorphism $\theta(x) \triangleq ([0:c], [0:c^\perp])$.

Proposition K.4. [7] *Let* $(X, \vee, \wedge, 0, 1; \le)$ *be an* ORTHOMODULAR *lattice.*

PRP

$$x \copyright y \iff \phi_x(y) = \phi_y(x) = x \wedge y \qquad \forall x,y \in X$$

K.3 Center

An element in an *orthocomplemented lattice* (Definition J.1 page 196) is in the *center* of the lattice if that element *commutes* (Definition K.2 page 211) with every other element in the lattice (next definition). *All* the elements of an *orthocomplemented lattice* are in the *center* if and only if that lattice is *Boolean* (Proposition J.2 page 203).

Definition K.4. [8] *Let* © *be the* COMMUTES *relation (Definition K.2 page 211) on a* LATTICE WITH NEGATION $\boldsymbol{L} \triangleq (X, \vee, \wedge, \neg, 0, 1; \le)$ *(Definition 1.5 page 3).*

DEF

The **center** *of* $\boldsymbol{L}$ *is defined as*
$$\{x \in X \mid x \copyright y \quad \forall y \in X\}$$

[6] Kalmbach (1983) page 20, MacLaren (1964)
[7] Foulis (1962) page 66, Sasaki (1954) ⟨cf Foulis 1962⟩
[8] Holland (1970) page 80

Proposition K.5. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

PRP: 0 *and* 1 *are in the* ***center*** *of* $\boldsymbol{L}$.

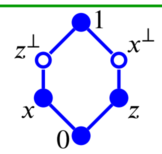
✎PROOF: This follows directly from Definition K.2 (page 211) and Proposition K.2 (page 211). ✑

Theorem K.2. [9] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

THM: *The* CENTER *of* $\boldsymbol{L}$ *is* BOOLEAN *(Definition I.1 page 171).*

Example K.4.

EX: The **center** of the O_6 **lattice** (Definition J.2 page 196) is the set $\{0, x, z, 1\}$. The elements $x^\perp$ and $z^\perp$ are **not** in the center of $\boldsymbol{L}$. The O_6 lattice is illustrated to the right, with the center elements as solid dots. Note that the center is the *Boolean* lattice $\boldsymbol{L}_2^2$ (Proposition J.2 page 203).



✎PROOF:

1. Proof that 0 and 1 are in the *center* of $\boldsymbol{L}$: by Proposition K.5 (page 215).
2. Proof that x is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(x \wedge x) \vee (x \wedge x^\perp) &= x \vee 0 &&= x &&\implies x \copyright x\\
(x \wedge z) \vee (x \wedge z^\perp) &= 0 \vee x &&= x &&\implies x \copyright z
\end{aligned}$$

   $x \copyright x$, $x \copyright x^\perp$, $x \copyright z^\perp$, $x \copyright 0$, and $x \copyright 1$ by Proposition K.2 (page 211).
3. Proof that z is in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(z \wedge z) \vee (z \wedge z^\perp) &= z \vee 0 &&= z &&\implies z \copyright z\\
(z \wedge x) \vee (z \wedge x^\perp) &= 0 \vee z &&= z &&\implies z \copyright x
\end{aligned}$$

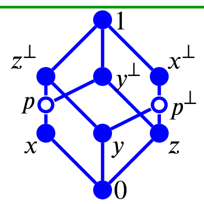
   $z \copyright z$, $z \copyright x^\perp$, $z \copyright z^\perp$, $z \copyright 0$, and $z \copyright 1$ by Proposition K.2 (page 211).
4. Proof that $x^\perp$ and $z^\perp$ are *not* in the *center* of $\boldsymbol{L}$:

$$\begin{aligned}
(x^\perp \wedge y) \vee (x^\perp \wedge y^\perp) &= y \vee 0 &&= y &&\implies x^\perp \not\copyright y\\
(z^\perp \wedge x) \vee (z^\perp \wedge x^\perp) &= x \vee 0 &&= x &&\implies z^\perp \not\copyright x
\end{aligned}$$

✑

Example K.5.

EX: The **center** the lattice illustrated to the right (Example J.2 page 196), with center elements as solid dots, is the set $\{0, 1, p, y, z, x^\perp, y^\perp, z^\perp,\}$. The elements x and $p^\perp$ are *not* in the *center* of $\boldsymbol{L}$. Note that the center is the *Boolean* lattice $\boldsymbol{L}_2^3$ (Proposition J.2 page 203).



[9] Jeffcott (1972) page 645 ⟨§5. Main theorem⟩

✎PROOF:

1. Proof that 0 and 1 are in the *center* of $\boldsymbol{L}$: by Proposition K.5 (page 215).

2. Proof that x is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{lll} (x \wedge p) \vee (x \wedge p^{\perp}) = x \vee 0 & = x & \Longrightarrow\ x \copyright p \\ (x \wedge y) \vee (x \wedge y^{\perp}) = 0 \vee x & = x & \Longrightarrow\ x \copyright y \\ (x \wedge z) \vee (x \wedge z^{\perp}) = 0 \vee x & = x & \Longrightarrow\ x \copyright z \end{array}$$

$x \copyright x$, $x \copyright x^{\perp}$, $x \copyright p^{\perp}$, $x \copyright y^{\perp}$, $x \copyright z^{\perp}$, $x \copyright 0$, and $x \copyright 1$ by Proposition K.2 (page 211).

3. Proof that y is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{lll} (y \wedge x) \vee (y \wedge x^{\perp}) = 0 \vee y & = y & \Longrightarrow\ y \copyright x \\ (y \wedge p) \vee (y \wedge p^{\perp}) = 0 \vee y & = y & \Longrightarrow\ y \copyright p \\ (y \wedge z) \vee (y \wedge z^{\perp}) = 0 \vee y & = y & \Longrightarrow\ y \copyright z \end{array}$$

$y \copyright y$, $y \copyright x^{\perp}$, $y \copyright p^{\perp}$, $y \copyright y^{\perp}$, $y \copyright z^{\perp}$, $y \copyright 0$, and $y \copyright 1$ by Proposition K.2 (page 211).

4. Proof that z is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{lll} (z \wedge x) \vee (z \wedge x^{\perp}) = 0 \vee z & = z & \Longrightarrow\ z \copyright x \\ (z \wedge p) \vee (z \wedge p^{\perp}) = 0 \vee z & = z & \Longrightarrow\ z \copyright p \\ (z \wedge y) \vee (z \wedge y^{\perp}) = 0 \vee z & = z & \Longrightarrow\ z \copyright y \end{array}$$

$z \copyright z$, $z \copyright x^{\perp}$, $z \copyright p^{\perp}$, $z \copyright y^{\perp}$, $z \copyright z^{\perp}$, $z \copyright 0$, and $z \copyright 1$ by Proposition K.2 (page 211).

5. Proof that $x^{\perp}$ is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{lll} (p^{\perp} \wedge x) \vee (p^{\perp} \wedge x^{\perp}) = 0 \vee p^{\perp} & = p^{\perp} & \Longrightarrow\ p^{\perp} \copyright x \\ (p^{\perp} \wedge y) \vee (p^{\perp} \wedge y^{\perp}) = y \vee z & = p^{\perp} & \Longrightarrow\ p^{\perp} \copyright y \\ (p^{\perp} \wedge z) \vee (p^{\perp} \wedge z^{\perp}) = z \vee y & = p^{\perp} & \Longrightarrow\ p^{\perp} \copyright z \end{array}$$

$p^{\perp} \copyright x^{\perp}$, $p^{\perp} \copyright p^{\perp}$, $p^{\perp} \copyright y^{\perp}$, $p^{\perp} \copyright z^{\perp}$, $p^{\perp} \copyright 0$, and $p^{\perp} \copyright 1$ by Proposition K.2 (page 211).

6. Proof that $y^{\perp}$ is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{lll} (y^{\perp} \wedge x) \vee (y^{\perp} \wedge x^{\perp}) = x \vee z & = y^{\perp} & \Longrightarrow\ y^{\perp} \copyright x \\ (y^{\perp} \wedge p) \vee (y^{\perp} \wedge p^{\perp}) = p \vee z & = y^{\perp} & \Longrightarrow\ y^{\perp} \not\copyright p \\ (y^{\perp} \wedge z) \vee (y^{\perp} \wedge z^{\perp}) = z \vee p & = y^{\perp} & \Longrightarrow\ y^{\perp} \copyright z \end{array}$$

$p^{\perp} \copyright x^{\perp}$, $p^{\perp} \copyright p^{\perp}$, $p^{\perp} \copyright y^{\perp}$, $p^{\perp} \copyright z^{\perp}$, $p^{\perp} \copyright 0$, and $p^{\perp} \copyright 1$ by Proposition K.2 (page 211).

7. Proof that $z^{\perp}$ is in the *center* of $\boldsymbol{L}$:

$$\begin{array}{lll} (z^{\perp} \wedge x) \vee (z^{\perp} \wedge x^{\perp}) = x \vee y & = z^{\perp} & \Longrightarrow\ z^{\perp} \copyright x \\ (z^{\perp} \wedge p) \vee (z^{\perp} \wedge p^{\perp}) = p \vee y & = z^{\perp} & \Longrightarrow\ y^{\perp} \not\copyright p \\ (z^{\perp} \wedge y) \vee (z^{\perp} \wedge y^{\perp}) = z \vee p & = z^{\perp} & \Longrightarrow\ y^{\perp} \copyright z \end{array}$$

$z^{\perp} \copyright x^{\perp}$, $z^{\perp} \copyright p^{\perp}$, $z^{\perp} \copyright y^{\perp}$, $z^{\perp} \copyright z^{\perp}$, $z^{\perp} \copyright 0$, and $z^{\perp} \copyright 1$ by Proposition K.2 (page 211).

8. Proof that p and $x^\perp$ are *not* in the *center* of $\boldsymbol{L}$:

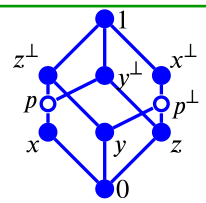
$$\begin{aligned}
(p \wedge x) \vee (p \wedge x^\perp) &= x \vee 0 &&= x &&\implies p \not{\mathbb{C}} x\\
(x^\perp \wedge p) \vee (x^\perp \wedge p^\perp) &= 0 \vee p^\perp &&= p^\perp &&\implies x^\perp \not{\mathbb{C}} p
\end{aligned}$$

✏

Example K.6.

EX

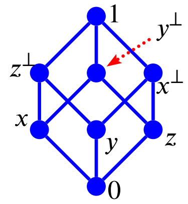
The **center** of the lattice illustrated to the right is illustrated with solid dots. Note that the center is the *Boolean* lattice $\boldsymbol{L}_2^2$ (Proposition J.2 page 203).



Example K.7.

EX

In a *Boolean* lattice, such as the one illustrated to the right, every element is in the center (Proposition J.2 page 203).



APPENDIX L

VALUATIONS ON LATTICES

Definition L.1. [1] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition D.3 page 117).*

DEF

A function $\mathsf{v} \in \mathbb{R}^X$ *is a **valuation** on* $\boldsymbol{L}$ *if*

$$\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) = \mathsf{v}(x) + \mathsf{v}(y) \qquad \forall x,y \in X$$

Proposition L.1. *Let* $\mathsf{v} \in \mathbb{R}^X$ *be a* FUNCTION *on a* LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *(Definition D.3 page 117).*

PRP

$$\left\{ \boldsymbol{L} \text{ is LINEAR (Definition D.3 page 117)} \right\} \implies \left\{ \mathsf{v} \text{ is a VALUATION (Definition L.1 page 219)} \right\}$$

✎PROOF: Let $x, y \in X$ such that $x \leq y$ or $y \lesssim x$.

$$\mathsf{v}(x \vee y) + \mathsf{v}(x \wedge y) = \mathsf{v}(x) + \mathsf{v}(y) \qquad \text{because } \boldsymbol{L} \text{ is } linear$$

✏

Example L.1. [2] Consider the *real valued lattice* $\boldsymbol{L} \triangleq (\mathbb{R}, \vee, \wedge; \leq)$. The *absolute value* function $|\cdot|$ is a *valuation* on $\boldsymbol{L}$.

✎PROOF: $\boldsymbol{L}$ is *linear* (Definition D.3 page 117), so v is a *valuation* by Proposition L.1 (page 219). ✏

Definition L.2. [3] *Let* X *be a set and* $\mathbb{R}^{\vdash}$ *the set of non-negative real numbers.*

DEF

A function $\mathsf{d} \in \mathbb{R}^{\vdash X \times X}$ *is a **metric** on* X *if*

1.	$\mathsf{d}(x, y) \geq 0$	$\forall x,y \in X$	(NON-NEGATIVE)	*and*
2.	$\mathsf{d}(x, y) = 0 \iff x = y$	$\forall x,y \in X$	(NONDEGENERATE)	*and*
3.	$\mathsf{d}(x, y) = \mathsf{d}(y, x)$	$\forall x,y \in X$	(SYMMETRIC)	*and*
4.	$\mathsf{d}(x, y) \leq \mathsf{d}(x, z) + \mathsf{d}(z, y)$	$\forall x,y,z \in X$	(SUBADDITIVE/TRIANGLE INEQUALITY).[4]	

*A **metric space** is the pair* (X, d). *A metric is also called a **distance function**.*

---

[1] Istrăţescu (1987) page 127, Birkhoff (1967) page 230 ⟨Definition X.1(V1)⟩, Blyth (2005) page 58 ⟨Exercise 4.25⟩, Deza and Laurent (1997) page 105 ⟨(8.1.1)⟩, Deza and Deza (2006) page 143 ⟨§10.3⟩, Deza and Deza (2009) page 193 ⟨§10.3⟩

[2] Khamsi and Kirk (2001) page 119 ⟨§5.7⟩

[3] Dieudonné (1969) page 28, Copson (1968) page 21, Hausdorff (1937) page 109, Fréchet (1928), Fréchet (1906) page 30

[4] Euclid (circa 300BC) ⟨Book I Proposition 20⟩

Actually, it is possible to significantly simplify the definition of a metric to an equivalent statement requiring only half as many conditions. These equivalent conditions (a "*characterization*") are stated in Theorem L.1 (next).

Theorem L.1 (metric characterization). [5] *Let* d *be a function in* $(\mathbb{R}^{\vdash})^{X\times X}$.

THM
$$\mathsf{d}(x,y) \text{ is a metric} \iff \left\{\begin{array}{lllll} 1. & \mathsf{d}(x,y) = 0 & \iff x=y & \forall x,y\in X & \text{and}\\ 2. & \mathsf{d}(x,y) \le \mathsf{d}(z,x)+\mathsf{d}(z,y) & & \forall x,y,z\in X & \end{array}\right.$$

Definition L.3 (next) defines the *open ball*. In a *metric space* (Definition L.2 page 219), sets are often specified in terms of an *open ball*; and an open ball is specified in terms of a metric.

Definition L.3. [6] *Let* (X,d) *be a* METRIC SPACE *(Definition L.2 page 219).*

DEF
An ***open ball*** *centered at* x *with radius* r *is the set* $\mathsf{B}(x,r) \triangleq \{y\in X \mid \mathsf{d}(x,y) < r\}$.
A ***closed ball*** *centered at* x *with radius* r *is the set* $\overline{\mathsf{B}}(x,r) \triangleq \{y\in X \mid \mathsf{d}(x,y) \le r\}$.
A ***unit ball*** *centered at* x *is the set* $\mathsf{B}(x,1)$.
A ***closed unit ball*** *centered at* x *is the set* $\overline{\mathsf{B}}(x,1)$.

Theorem L.2. [7] *Let* $\mathsf{v}\in\mathbb{R}^{\boldsymbol{X}}$ *be a function on a* LATTICE $\boldsymbol{L} \triangleq (X,\vee,\wedge;\le)$ *(Definition D.3 page 117).*

THM
$$\left.\begin{array}{llll} 1. & \mathsf{v}(x\vee y)+\mathsf{v}(x\wedge y) = \mathsf{v}(x)+\mathsf{v}(y) & \forall x,y\in X & \text{(VALUATION) and}\\ 2. & x\le y \implies \mathsf{v}(x)\le\mathsf{v}(y) & \forall x,y\in X & \text{(ISOTONE)}\end{array}\right\} \implies \left\{\begin{array}{l}\mathsf{d}(x,y)\triangleq\\ \mathsf{v}(x\vee y)-\mathsf{v}(x\wedge y)\\ \text{is a METRIC on } \boldsymbol{L}\end{array}\right.$$

Definition L.4. [8] *Let* v *be a* VALUATION *(Definition L.1 page 219) on a* LATTICE $\boldsymbol{L}\triangleq(X,\vee,\wedge;\le)$ *(Definition D.3 page 117). Let* $\mathsf{d}(x,y)$ *be the* METRIC *defined in Theorem L.2 (page 220).*

DEF
The pair $(\boldsymbol{L},\mathsf{d})$ *is called a* METRIC LATTICE.

For *finite modular* lattices, the *height* function $\mathsf{h}(x)$ (Definition E.3 page 134) can serve as the isotone valuation that induces a metric (next proposition). Such a height function actually satisfies the stronger condition of being *positive* (rather than just being *isotone*)—all *positive* functions are also *isotone*.

Proposition L.2. [9] *Let* $\mathsf{h}(x)$ *be the* HEIGHT *(Definition E.3 page 134) of a point* x *in a* BOUNDED LATTICE *(Definition E.1 page 133)* $\boldsymbol{L}\triangleq(X,\vee,\wedge,0,1;\le)$.

PRP
$$\left\{\begin{array}{ll} 1. & \boldsymbol{L} \text{ is MODULAR and}\\ 2. & \boldsymbol{L} \text{ is FINITE}\end{array}\right\}$$
$$\implies \left\{\begin{array}{llll} 1. & \mathsf{h}(x\vee y)+\mathsf{h}(x\wedge y)=\mathsf{h}(x)+\mathsf{h}(y) & \forall x,y\in X & \text{(VALUATION) and}\\ 2. & x\lesssim y \implies \mathsf{h}(x)\lesssim\mathsf{h}(y) & \forall x,y\in X & \text{(POSITIVE)}\end{array}\right\}$$
$$\implies \left\{\begin{array}{llll} 1. & \mathsf{h}(x\vee y)+\mathsf{h}(x\wedge y)=\mathsf{h}(x)+\mathsf{h}(y) & \forall x,y\in X & \text{(VALUATION) and}\\ 2. & x\le y \implies \mathsf{h}(x)\le\mathsf{h}(y) & \forall x,y\in X & \text{(ISOTONE)}\end{array}\right\}$$

Theorem L.3. [10] *Let* v *be a* VALUATION *(Definition L.1 page 219) on a* LATTICE $\boldsymbol{L}\triangleq(X,\vee,\wedge;\le)$ *(Definition D.3 page 117). Let* $\mathsf{d}(x,y)$ *be the* METRIC *defined in Theorem L.2 (page 220).*

THM
$$\left\{\begin{array}{c}(\boldsymbol{L},\mathsf{d}) \text{ is a METRIC LATTICE}\\ \text{(Definition L.4 page 220)}\end{array}\right\} \implies \left\{\begin{array}{c}\boldsymbol{L} \text{ is MODULAR}\\ \text{(Definition F.3 page 136)}\end{array}\right\}$$

[5] Michel and Herget (1993) page 264, Giles (1987) page 18
[6] Aliprantis and Burkinshaw (1998) page 35
[7] Deza and Laurent (1997) page 105 ⟨(8.1.2)⟩, Birkhoff (1967) pages 230–231
[8] Deza and Laurent (1997) page 105, Birkhoff (1967) page 231 ⟨§X.2⟩
[9] Birkhoff (1967) page 230
[10] Birkhoff (1967) page 232Theorem X.2, Deza and Laurent (1997) pages 105–106, Blyth (2005) page 58 ⟨Exercise 4.25⟩

Example L.2. The function h on the *Boolean* (and thus also *modular*) lattice $\boldsymbol{L}_2^3$ illustrated to the right is a *valuation* (Definition L.1 page 219) that is *positive* (and thus also *isotone*, Proposition L.2 page 220). Therefore

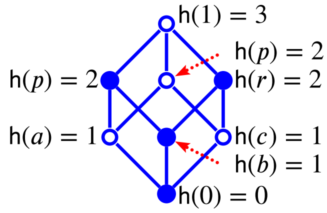
$$\mathsf{d}(x, y) \triangleq \mathsf{h}(x \vee y) - \mathsf{h}(x \wedge y) \qquad \forall x,y \in X$$

is a *metric* (Definition L.4 page 220) on $\boldsymbol{L}_2^3$. For example,

$$\mathsf{d}(b, q) \triangleq \mathsf{h}(b \vee q) - \mathsf{h}(b \wedge q) = \mathsf{h}(1) - \mathsf{h}(0) = 3 - 0 = 3\,.$$

The *closed unit ball* centered at b (Definition L.3 page 220) and illustrated with solid dots to the right is

$$\mathsf{B}(b, 1) \triangleq \{x \in X | \mathsf{d}(b, x) \leq 1\} = \{b, p, r, 0\}$$

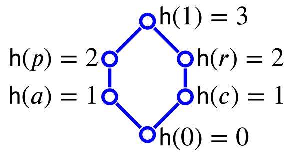


Example L.3. The *height* function h (Definition E.3 page 134) on the *orthocomplemented* but *non-modular* lattice O_6 illustrated to the right is *not* a *valuation* because for example

$$\mathsf{h}(a \vee c) + \mathsf{h}(a \wedge c) = \mathsf{h}(1) + \mathsf{h}(0) = 3 + 0 = 3 \neq 2 = 1 + 1 = \mathsf{h}(a) + \mathsf{h}(b).$$

Moreover, we might expect the "distance" from a to c to be 2. However, if we attempt to use $\mathsf{h}(x)$ to define a metric on O_6, then we get

$$\mathsf{d}(a, c) \triangleq \mathsf{h}(a \vee c) - \mathsf{h}(a \wedge c) = \mathsf{h}(1) - \mathsf{h}(0) = 3 - 0 = 3 \neq 2.$$



L.1 Projections

Definition L.5. [11] *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ *be an* ORTHOCOMPLEMENTED LATTICE *(Definition J.1 page 196).*

DEF

A function $\phi_x \in X^X$ *is a* ***Sasaki projection*** *on* $x \in X$ *if*

$$\phi_x(y) \triangleq \left(y \vee x^{\perp}\right) \wedge x.$$

The SASAKI PROJECTIONS ϕ_x *and* ϕ_y *are* ***permutable*** *if*

$$\phi_x \circ \phi_y(u) = \phi_y \circ \phi_x(u) \quad \forall u \in X.$$

Proposition L.3. *Let* $\phi_x(y)$ *be the* SASAKI PROJECTION OF y ONTO x *(Definition L.5 page 222) in an* ORTHOCOMPLEMENTED LATTICE $\boldsymbol{L} \triangleq (X, \vee, \wedge, 0, 1; \leq)$.

PRP

(1).	$x \leq y$	$\Longrightarrow$	$\phi_x(y) = x$		$\forall x,y \in X$
(2).	$y \leq x$	$\Longrightarrow$	$y \leq \phi_x(y) \leq x$		$\forall x,y \in X$
(3).	$y \leq x$ *and* $\boldsymbol{L}$ *is* BOOLEAN	$\Longrightarrow$	$\phi_x(y) = y$		$\forall x,y \in X$

✎PROOF:

$$\begin{aligned}
x \leq y \implies \phi_x(y) &\triangleq \left(y \vee x^{\perp}\right) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition L.5 page 222)}\\
&= 1 \wedge x && \text{by } x \leq y \text{ hypothesis and Proposition K.1 page 209}\\
&= x && \text{by property of bounded lattices (Proposition E.2 page 133)}\\
y \leq x \implies \boxed{y} &= y \wedge x && \text{by } y \leq x \text{ hypothesis}\\
&\leq (y \vee x^{\perp}) \wedge x && \text{by definition of } \vee \text{ (Definition C.21 page 114)}\\
&= \boxed{\phi_x(y)} && \text{by definition of } \textit{Sasaki projection} \text{ (Definition L.5 page 222)}\\
&\leq (y \vee x^{\perp}) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition L.5 page 222)}\\
&\leq \boxed{x} && \text{by definition of } \wedge \text{ (Definition C.22 page 114)}\\
y \leq x \text{ and } \textit{Boolean} \implies \phi_x(y) &= (y \vee x^{\perp}) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition L.5 page 222)}\\
&= (y \wedge x) \vee (x^{\perp} \wedge x) && \text{by } \textit{distributive} \text{ prop. of } \textit{Boolean lattices} \text{ (Theorem I.2 page 176)}\\
&= (y \wedge x) \vee 0 && \text{by } \textit{non-contradiction} \text{ of Boolean lat. (Theorem I.2 page 176)}\\
&= (y \wedge x) && \text{by } \textit{boundary} \text{ prop. of } \textit{bounded lattices} \text{ (Proposition E.2 page 133)}\\
&= y && \text{by } y \leq x \text{ hypothesis and definition of } \wedge \text{ (Definition C.22 page 114)}
\end{aligned}$$

✏

Proposition L.4. *Let* $\phi_x(y)$ *be the* SASAKI PROJECTION OF y ONTO x *(Definition L.5 page 222) in an* ORTHOCOMPLEMENTED LATTICE $(X, \vee, \wedge, 0, 1; \leq)$.

PRP

(1).	$\phi_0(y)$	$=$	0	$\forall y \in X$
(2).	$\phi_x(0)$	$=$	0	$\forall x \in X$
(3).	$\phi_1(y)$	$=$	1	$\forall y \in X$
(4).	$\phi_x(1)$	$=$	x	$\forall x \in X$
(5).	$\phi_x\left(x^{\perp}\right)$	$=$	0	$\forall x \in X$

[11] Nakamura (1957) pages 158–159 ⟨equation (S)⟩
Sasaki (1954) page 300 ⟨Def.5.1, cf Foulis 1962⟩
Kalmbach (1983) page 117

✎PROOF:

$$\begin{aligned}
\phi_0(y) &= 0 && \text{because } 0 \le y \text{ and by Proposition L.3 page 222}\\
\phi_x(0) &\triangleq \left(0 \vee x^\perp\right) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition L.5 page 222)}\\
&= x^\perp \wedge x && \text{by property of bounded lattices (Proposition E.2 page 133)}\\
&= 0 && \text{by definition of } \textit{orthocomplemented} \text{ (Definition J.1 page 196)}\\
\phi_1(y) &\triangleq \left(y \vee 1^\perp\right) \wedge 1 && \text{by definition of } \textit{Sasaki projection} \text{ (Definition L.5 page 222)}\\
&= (y \vee 0) \wedge 1 && \text{by } \textit{boundary condition} \text{ (Theorem 1.5 page 6)}\\
&= y \wedge 1 && \text{by property of bounded lattices (Proposition E.2 page 133)}\\
&= 1 && \text{by property of bounded lattices (Proposition E.2 page 133)}\\
\phi_x(1) &= x && \text{because } x \le 1 \text{ and by Proposition L.3 page 222}\\
\phi_x\left(x^\perp\right) &\triangleq \left(x^\perp \vee x^\perp\right) \wedge x && \text{by definition of } \textit{Sasaki projection} \text{ (Definition L.5 page 222)}\\
&= x^\perp \wedge x && \text{by } \textit{idempotency} \text{ of lattices (Theorem D.3 page 118)}\\
&= 0 && \text{by } \textit{non-contradiction} \text{ property of } \textit{orthocomplemented lattice} \text{ (Definition J.1 page 196)}
\end{aligned}$$

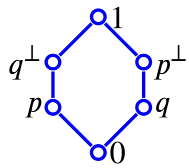
✏

Example L.4.

EX

Here are some examples of projections in the O_6 *lattice* onto the element x:

$$\begin{array}{llllllllllll}
\phi_p(q) & \triangleq & (q & \vee & p^\perp) & \wedge & p & = & p^\perp & \wedge & p & = 0 & \text{(because } p \perp q)\\
\phi_p\left(p^\perp\right) & \triangleq & (p^\perp & \vee & p^\perp) & \wedge & p & = & p^\perp & \wedge & p & = 0 & \text{(because } p \perp p^\perp)\\
\phi_p\left(q^\perp\right) & \triangleq & (q^\perp & \vee & p^\perp) & \wedge & p & = & 1 & \wedge & p & = p & \text{(because } p \le q^\perp)\\
\phi_{q^\perp}(p) & \triangleq & (p & \vee & q) & \wedge & q^\perp & = & 1 & \wedge & q^\perp & = q^\perp & \text{(because } q^\perp \le 1)\\
\phi_p(1) & \triangleq & (1 & \vee & p^\perp) & \wedge & p & = & 1 & \wedge & p & = p & \text{(because } p \le 1)\\
\phi_p(0) & \triangleq & (0 & \vee & p^\perp) & \wedge & p & = & p^\perp & \wedge & p & = 0 & \text{(because } p \perp 0)
\end{array}$$



Example L.5.

EX

Here are some examples of projections in lattice 5 of Example J.2 (page 196):

$$\begin{array}{lllllllllll}
\phi_x(p) & \triangleq & (p & \vee & x^\perp) & \wedge & x & = 1 & \wedge & x & = x & \\
\phi_x(y) & \triangleq & (y & \vee & x^\perp) & \wedge & x & = x^\perp & \wedge & x & = 0 & \text{(because } x \perp y)\\
\phi_x(z) & \triangleq & (z & \vee & x^\perp) & \wedge & x & = x^\perp & \wedge & x & = 0 & \text{(because } x \perp z)\\
\phi_x\left(p^\perp\right) & \triangleq & (p^\perp & \vee & x^\perp) & \wedge & x & = p^\perp & \wedge & x & = 0 & \\
\phi_x\left(x^\perp\right) & \triangleq & (x^\perp & \vee & x^\perp) & \wedge & x & = x^\perp & \wedge & x & = 0 & \text{(because } x \perp x^\perp)\\
\phi_x\left(y^\perp\right) & \triangleq & (y^\perp & \vee & x^\perp) & \wedge & x & = 1 & \wedge & x & = x & \text{(because } x \le y^\perp)\\
\phi_x\left(z^\perp\right) & \triangleq & (z^\perp & \vee & x^\perp) & \wedge & x & = 1 & \wedge & x & = x & \text{(because } x \le z^\perp)\\
\phi_x(1) & \triangleq & (1 & \vee & x^\perp) & \wedge & x & = 1 & \wedge & x & = x & \text{(because } x \le 1)\\
\phi_x(0) & \triangleq & (0 & \vee & x^\perp) & \wedge & x & = x^\perp & \wedge & x & = 0 & \text{(because } x \perp 0)
\end{array}$$

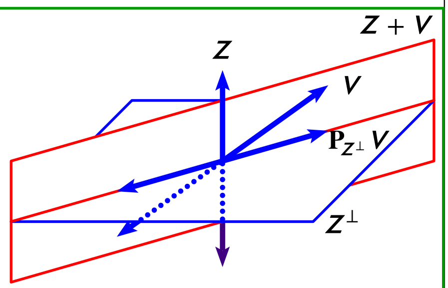
Example L.6.

EX

Let $\mathbb{R}^3$ be the *3-dimensional Euclidean space* (Example J.3 page 197) with subspaces $\boldsymbol{Z}$ and $\boldsymbol{V}$. Then the projection operator $\mathbf{P}_{\boldsymbol{Z}^\perp}$ onto $\boldsymbol{Z}^\perp$ is a *sasaki projection* $\phi_{\boldsymbol{Z}^\perp}$. In particular

$$\begin{aligned}
\mathbf{P}_{\boldsymbol{Z}^\perp}\boldsymbol{V} &\triangleq \phi_{\boldsymbol{Z}^\perp}(\boldsymbol{V})\\
&\triangleq \left(\boldsymbol{V} + \boldsymbol{Z}^{\perp\perp}\right) \cap \boldsymbol{Z}^\perp\\
&= (\boldsymbol{V} + \boldsymbol{Z}) \cap \boldsymbol{Z}^\perp
\end{aligned}$$

as illustrated to the right.



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