A Book Concerning Statistical Signal Processing

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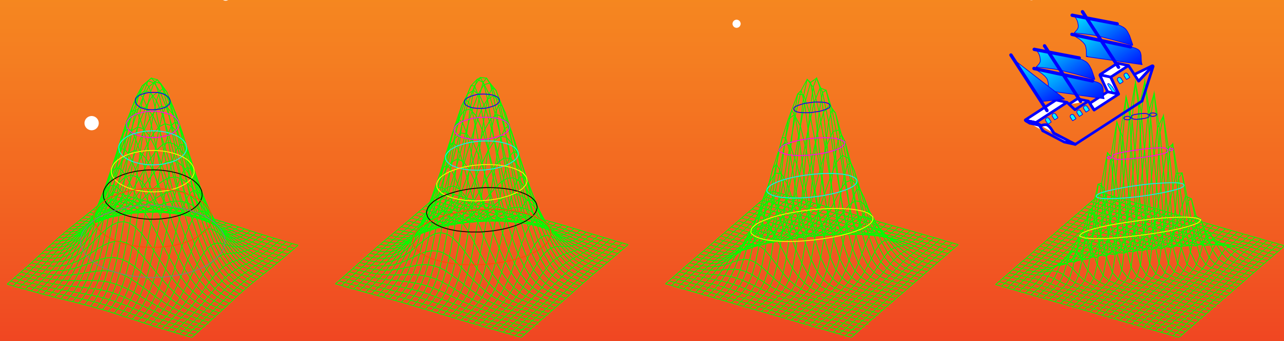
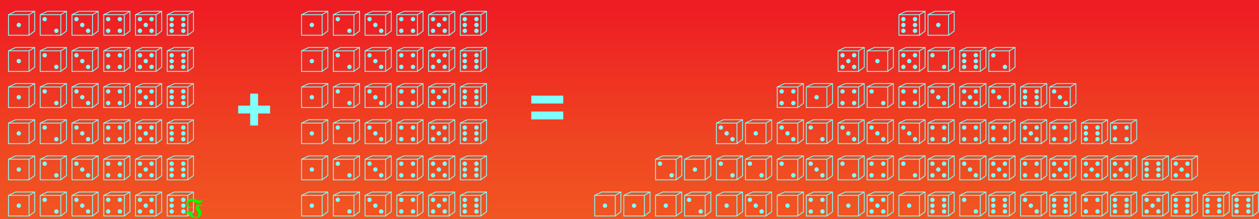
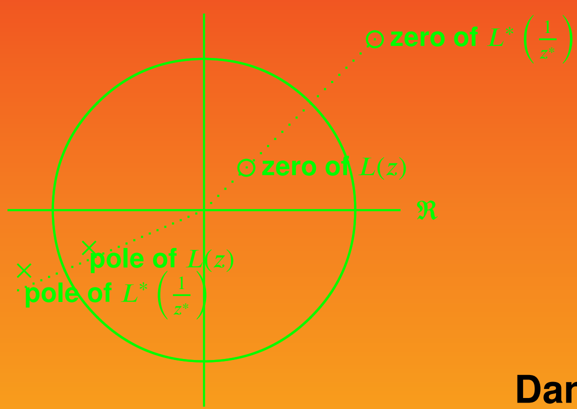
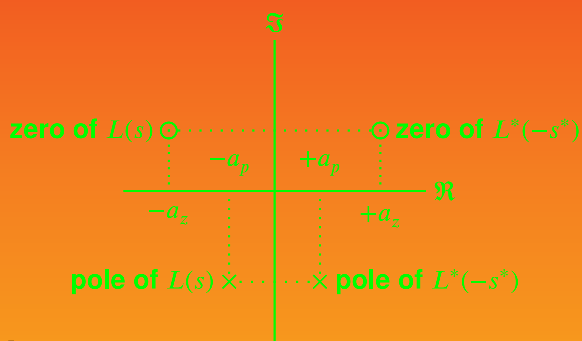
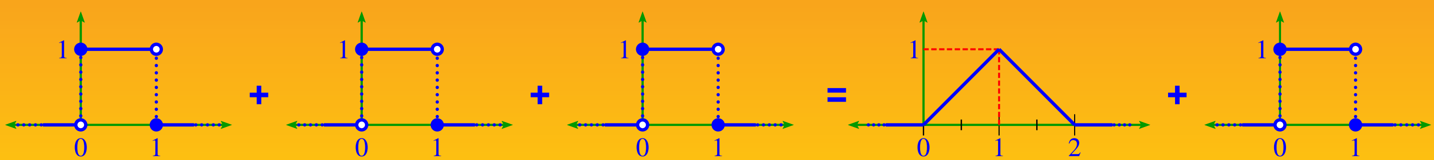








Daniel J. Greenhoe



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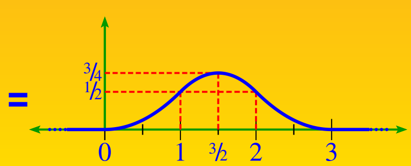

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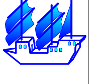
The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]



[1] Paine (2000) page 63 ⟨Golden Hind⟩

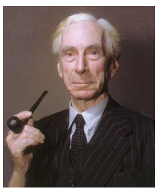
“Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?”

“Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.”

Alfred Edward Housman, English poet (1859–1936) [2]

“The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.”

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



“As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.”

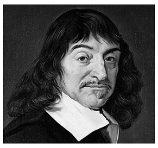
Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

[2] quote: Housman (1936), page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961), page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967), page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

❝*rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.*❞



❝*Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.*❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝*In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.*❞

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \ldots$
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$

…continued on next page…

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{Z}^-$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_o$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_e$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^+$	positive real numbers	$(0, \infty)$
$\mathbb{R}^-$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^*$	extended real numbers	$\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
Ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \triangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\not\subseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \setminus B$	set difference	
A^c	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	
0	"false" condition	
$\neg$	logical NOT operation	

...continued on next page...

symbol	description	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\leq$	reflexive ordering relation	"less than or equal to"
$\geq$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lVert\lvert\cdot\rvert\rVert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_{\mathsf{j}}(X, Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_{\mathsf{j}}(X, Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_{\mathsf{j}}(X, Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition N.2 page 309)	
$\hat{\mathbf{F}}$	*Fourier Series* operator	

...continued on next page...

symbol	description
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition O.1 page 319)
$\mathbf{Z}$	*Z-Transform* operator (Definition P.4 page 330)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\breve{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

Part I

Statistical Analysis

CHAPTER 1

EXPECTATION OPERATOR

1.1 Definitions

In a *probability space* $(\Omega, \mathbb{E}, \mathsf{P})$ (Definition A.2 page 147), all probability information is contained in the *measure* P. Often times this information is overwhelming and a simpler statistic, which does not offer so much information, is sufficient. Some of the most common statistics can be conveniently expressed in terms of the *expectation operator* E.

Definition 1.1. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE *(Definition A.2 page 147) and* X *a* RANDOM VARIABLE *(Definition ?? page ??) on* $(\Omega, \mathbb{E}, \mathsf{P})$ *with* PROBABILITY DENSITY FUNCTION p_x.

DEF

The ***expectation operator*** E_x *on* X *is defined as*

$$\mathsf{E}_\mathsf{x}\mathsf{X} \triangleq \int_{x\in\mathbb{F}} x\mathsf{p}_\mathsf{x}(x)\,\mathsf{d}x.$$

We already said that a *random variable* X is neither random nor a variable, but is rather a function of an underlying process that does appear to be random. However, because it is a function of a process that does appear random, the *random variable* X also appears to be random. That is, if we don't know the outcome of of the underlying experimental process, then we also don't know for sure what X is, and so X does indeed appear to be random. However, eventhough X appears to be random, the expected value $\mathsf{E}_\mathsf{x}\mathsf{X}$ of X is **not random**. Rather it is a fixed value (like 0 or 7.9 or -2.6).

On the other hand, eventhough EX is **not random**, note that $\mathsf{E}(\mathsf{X}|\mathsf{Y})$ **is random**. This is because $\mathsf{E}(\mathsf{X}|\mathsf{Y})$ is a function of Y. That is, once we know that Y equals some fixed value y (like 0 or 2.7 or -5.1) then $\mathsf{E}(\mathsf{X}|\mathsf{Y}=y)$ is also fixed. However, if we don't know the value of Y, then Y is still a *random variable* and the expression $\mathsf{E}(\mathsf{X}|\mathsf{Y})$ is also random (a function of *random variable* Y).

Two common statistics that are conveniently expressed in terms of the expectation operator are the *mean* and *variance*. The mean is an indicator of the "middle" of a probability distribution and the variance is an indicator of the "spread".

Definition 1.2. *Let* X *be a* RANDOM VARIABLE *on the* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$.

DEF

(1). *The* ***mean*** μ_X *of* X *is*	μ_X	$\triangleq$	$\mathsf{E}_\mathsf{x}\mathsf{X}$
(2). *The* ***variance*** $\mathsf{Var}(\mathsf{X})$ *or* σ^2_X *of* X *is*	$\mathsf{Var}(\mathsf{X})$	$\triangleq$	$\mathsf{E}_\mathsf{x}\left[(\mathsf{X}-\mathsf{E}_\mathsf{x}\mathsf{X})^2\right]$

1.2 Expectation as a linear operator

The next theorem demonstrates that the operator E is a *linear operator* (Definition M.3 page 282)—which in turn makes E part of a distinguished club of operators along with fellow member operators differentiation $\frac{d}{dx}$, integration $\int dx$, Laplace $\mathbf{L}$, Fourier $\tilde{\mathbf{F}}$, z-transform $\mathbf{Z}$, etc. Because E is a linear operator, it immediately inherits all the properties that it's linear operator birthright grants it (Corollary 1.1 page 4).

Theorem 1.1 (Linearity of E). [1] *Let* X *be a* RANDOM VARIABLE *on a* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$.

THM
$$\mathsf{E}_x(a\mathsf{X} + b\mathsf{Y} + c) = \left(a\mathsf{E}_x\mathsf{X}\right) + \left(b\mathsf{E}_x\mathsf{Y}\right) + c \quad \forall a,b,c \in \mathbb{R} \quad \text{(LINEAR)}$$

✎PROOF:

$$\begin{aligned}
\mathsf{E}_{xy}(a\mathsf{X} + b\mathsf{Y} + c) &\triangleq \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} [ax + by + c]\mathsf{p}_{xy}(x,y)\,\mathsf{d}y\,\mathsf{d}x && \text{by definition of E (Definition 1.1 page 3)}\\
&= \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} ax\mathsf{p}_{xy}(x,y)\,\mathsf{d}y\,\mathsf{d}x + \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} by\mathsf{p}_{xy}(x,y)\,\mathsf{d}y\,\mathsf{d}x + \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} c\mathsf{p}_{xy}(x,y)\,\mathsf{d}y\,\mathsf{d}x\\
&= \int_{x\in\mathbb{R}} ax \underbrace{\int_{y\in\mathbb{R}} \mathsf{p}_{xy}(x,y)\,\mathsf{d}y}_{\mathsf{p}_x(x)}\,\mathsf{d}x + \int_{y\in\mathbb{R}} by \underbrace{\int_{x\in\mathbb{R}} \mathsf{p}_{xy}(x,y)\,\mathsf{d}x}_{\mathsf{p}_y(y)}\,\mathsf{d}y + c\underbrace{\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}} \mathsf{p}_{xy}(x,y)\,\mathsf{d}x\,\mathsf{d}y}_{1}\\
&= a\underbrace{\int_{x\in\mathbb{R}} x\mathsf{p}_x(x)\,\mathsf{d}x}_{\mathsf{E}\mathsf{X}} + b\underbrace{\int_{y\in\mathbb{R}} y\mathsf{p}_y(y)\,\mathsf{d}y}_{\mathsf{E}\mathsf{Y}} + c\\
&= \left(a\mathsf{E}_x\mathsf{X}\right) + \left(b\mathsf{E}_y\mathsf{Y}\right) + c
\end{aligned}$$

✏

Corollary 1.1. *Let* E *be the* EXPECTATION OPERATOR *over a* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$. *Let spLLF be a* VECTOR SPACE OF RANDOM VARIABLES *over* $(\Omega, \mathbb{E}, \mathsf{P})$.

COR

(1). $\mathsf{E}\mathbb{0} = \mathbb{0}$ *and*

(2). $\mathsf{E}(-\mathsf{X}) = -(\mathsf{E}\mathsf{X}) \quad \forall \mathsf{X} \in \boldsymbol{L}^2_{\mathbb{F}}$ *and*

(3). $\mathsf{E}(\mathsf{X} - \mathsf{Y}) = \mathsf{E}\mathsf{X} - \mathsf{E}\mathsf{Y} \quad \forall \mathsf{X},\mathsf{Y} \in \boldsymbol{L}^2_{\mathbb{F}}$ *and*

(4). $\mathsf{E}\left(\sum_{n=1}^{N} \alpha_n \mathsf{X}_n\right) = \sum_{n=1}^{N} \alpha_n \left(\mathsf{E}\mathsf{X}_n\right) \quad \forall \alpha_n \in \mathbb{F}, \quad \forall \mathsf{X} \in \boldsymbol{L}^2_{\mathbb{F}}$

✎PROOF: These all follow immediately from the fact that E is a *linear operator* and from Theorem M.1 (page 282). ✏

Remark 1.1. Projecting a stochastic process onto a basis often yields valuable insights into the nature of the underlying data. Typical projection operators include the Fourier operator $\tilde{\mathbf{F}}$, Laplace $\mathbf{L}$, and z-transform $\mathbf{Z}$ …not to mention wavelet operators. But note that any such projection on a random sequence simply produces another random sequence. For example, the Fourier transform $\tilde{\mathbf{F}}\boldsymbol{x}(n)$ of a random sequence $\boldsymbol{x}(n)$ is another random sequence.

One way to overcome this difficulty is to simply invoke the *sampling* operator $\mathbf{S}\boldsymbol{x}(n)$ (CHAPTER **??** page **??**), yielding a deterministic sequence, and then take the Fourier transform of the resulting deterministic sequence. The problem here is that every time you resample the sequence, you will very likely get a different Fourier transform.

[1] Wilks (1963), page 73 ⟨§3.2 "Mean value of a random variable"⟩

Arguably a better approach (and the standard one at that) is to first invoke the expectation operator $\mathsf{E}x(n)$, also yielding a deterministic sequence.

The good news here is that because E and all the above mentioned operators are *linear*, we can do all the standard arithmetic acrobatics associated with linear algebra operators (next corollary).

Corollary 1.2. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be* LINEAR OPERATORS (Definition M.3 page 282).

COR

1.	$\mathsf{E}(\mathbf{MN})$	$=$	$(\mathsf{E}\mathbf{M})\mathbf{N}$	$\forall \mathsf{E}\in\mathcal{L}(\boldsymbol{Z},\boldsymbol{W}), \mathbf{M}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(ASSOCIATIVE)
2.	$\mathsf{E}(\mathbf{M} \dotplus \mathbf{N})$	$=$	$(\mathsf{E}\mathbf{M}) \dotplus (\mathsf{E}\mathbf{N})$	$\forall \mathsf{E}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \mathbf{N}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(LEFT DISTRIBUTIVE)
3.	$(\mathsf{E} \dotplus \mathbf{M})\mathbf{N}$	$=$	$(\mathsf{E}\mathbf{N}) \dotplus (\mathbf{MN})$	$\forall \mathsf{E}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathsf{E}\mathbf{M})$	$=$	$(\alpha\mathsf{E})\mathbf{M} = \mathsf{E}(\alpha\mathbf{M})$	$\forall \mathsf{E}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF: These all follow immediately from the fact that E is a *linear operator* and from Theorem M.4 (page 285). ⇨

Corollary 1.3. *Let* X *be a* RANDOM VARIABLE *on a* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$.

COR

$$\mathsf{Var}(a\mathsf{X}+b) = a^2\mathsf{Var}(\mathsf{X}) \qquad \forall a,b\in\mathbb{R}$$
$$\mathsf{Var}(\mathsf{X}) = \mathsf{E}_x(\mathsf{X}^2) - (\mathsf{E}_x\mathsf{X})^2$$

✎PROOF:

$$\begin{aligned}
\mathsf{Var}(\mathsf{X}) &\triangleq \mathsf{E}_x\left[(\mathsf{X}-\mathsf{E}_x\mathsf{X})^2\right] && \text{by definition of Var} && \text{(Definition 1.2 page 3)}\\
&= \mathsf{E}_x\left[\mathsf{X}^2 - 2\mathsf{X}\mathsf{E}_x\mathsf{X} + (\mathsf{E}_x\mathsf{X})^2\right] && \text{by } \textit{Binomial Theorem} && \text{(Theorem ?? page ??)}\\
&= \mathsf{E}_x\mathsf{X}^2 - \mathsf{E}_x\left[2\mathsf{X}\mathsf{E}_x\mathsf{X}\right] + \mathsf{E}_x(\mathsf{E}_x\mathsf{X})^2 && \text{by } \textit{linearity} \text{ of E} && \text{(Theorem 1.1 page 4)}\\
&= \mathsf{E}_x\mathsf{X}^2 - 2(\mathsf{E}_x\mathsf{X})[\mathsf{E}_x\mathsf{X}] + (\mathsf{E}_x\mathsf{X})^2\\
&= \mathsf{E}_x(\mathsf{X}^2) - (\mathsf{E}_x\mathsf{X})^2\\
\mathsf{Var}(a\mathsf{X}+b) &= \mathsf{E}_x(a\mathsf{X}+b)^2 - [\mathsf{E}_x(a\mathsf{X}+b)]^2\\
&= \mathsf{E}_x(a^2\mathsf{X}^2 + 2ab\mathsf{X} + b^2) - [a(\mathsf{E}_x\mathsf{X}) + b]^2\\
&= a^2\mathsf{E}_x\mathsf{X}^2 + 2ab\mathsf{E}_x\mathsf{X} + b^2 - \left[a^2[\mathsf{E}_x\mathsf{X}]^2 + 2ab\mathsf{E}_x\mathsf{X} + b^2\right] && \text{by } \textit{linearity} \text{ of E} && \text{(Theorem 1.1 page 4)}\\
&= a^2\left[\mathsf{E}_x\mathsf{X}^2 - (\mathsf{E}_x\mathsf{X})^2\right]\\
&\triangleq a^2\mathsf{Var}(\mathsf{X}) && \text{by previous result}
\end{aligned}$$

⇨

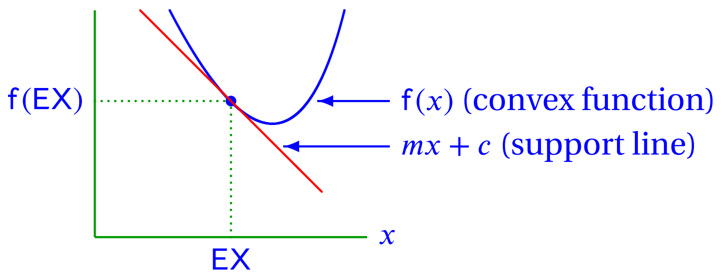


Figure 1.1: Jensen's inequality

Jensen's inequality is an extremely useful application of *convex*ity (Definition K.9 page 262) to the *expectation* operator. Jensen's inequality is stated in Corollary 1.4 (next) and illustrated in Figure 1.1 (page 5).

Corollary 1.4 (Jensen's inequality). [2] *Let* f *be a function in* $\mathbb{R}^{\mathbb{R}}$ *and* X *be a* RANDOM VARIABLE *on* $(\Omega, \mathbb{E}, \mathsf{P})$.

COR
$$\{\mathsf{f} \text{ is CONVEX}\} \implies \{\mathsf{f}(\mathsf{EX}) \le \mathsf{Ef}(\mathsf{X})\}$$

✎PROOF:

1. Proof 1: Let $mx + c$ be a "support line" under $\mathsf{f}(x)$ (Figure 1.1 page 5) such that

$$\begin{aligned} mx + c &< \mathsf{f}(x) && \text{for } x \neq \mathsf{EX} \\ mx + c &= \mathsf{f}(x) && \text{for } x = \mathsf{EX}. \end{aligned}$$

Then

$$\begin{aligned} \mathsf{f}(\mathsf{EX}) &= m[\mathsf{EX}] + c \\ &= \mathsf{E}[mX + c] \\ &\le \mathsf{Ef}(\mathsf{X}) \end{aligned}$$

2. Proof 2 (alternate proof):

$$\begin{aligned} \mathsf{f}(\mathsf{EX}) &\triangleq \mathsf{f}\left(\sum_{x \in \mathbb{E}} x\mathsf{P}(x)\right) \\ &\le \sum_{x \in \mathbb{E}} \mathsf{f}(x)\mathsf{P}(x) && \text{by } \textit{Jensen's inequality} \text{ for convex sets} && \text{(Theorem K.1 page 262)} \end{aligned}$$

⇰

Example 1.1. [3] Some examples of *Jensen's Inequality* (Corollary ?? page ??) applied to the *expectation operator* are the following:

EX
$$\begin{aligned} (\mathsf{EX})^{-1} &< \mathsf{E}\left(\mathsf{X}^{-1}\right) \\ \mathsf{E}\left(\log \mathsf{X}\right) &< \log\left(\mathsf{EX}\right) \\ e^{-\mathsf{EX}} &\le \mathsf{E}\left[e^{-\mathsf{X}}\right] \end{aligned}$$

Theorem 1.2 (Law of the Unconscious Statistician). [4]

THM
$$\mathsf{E}\left[\mathsf{g}(\mathsf{X})\right] = \int_{x \in \mathbb{R}} \mathsf{g}(x)\mathsf{p}_{\mathbf{x}}(x)\, \mathsf{d}x$$

1.3 Expectation inequalities

Theorem 1.3 (Markov's inequality). [5] *Let* $\mathsf{X} : \Omega \to [0, \infty)$ *be a non-negative valued* RANDOM VARIABLE *and* $a \in (0, \infty)$. *Then*

THM
$$\mathsf{P}\{\mathsf{X} \ge a\} \le \frac{1}{a}\mathsf{EX}$$

[2] Shao (2003) page 31 ⟨"1.3 Distributions and Their Characteristics"⟩, Cover and Thomas (1991), page 25, Jensen (1906), pages 179–180

[3] Shao (2003) pages 31–32 ⟨"Example 1.18"⟩, Dekking et al. (2006) page 110 ⟨"8.5 Solutions to the quick exercises"⟩

[4] Suhov et al. (2005) page 145 ⟨(2.69)⟩, Allen (2018) page 490 ⟨18.3.4 The Law of the Unconscious Statistician⟩, Papoulis (1990) page 124 ⟨Fundamental Theorem⟩

[5] Ross (1998), page 395

✎PROOF:

$$\begin{aligned}
I &\triangleq \begin{cases} 1 & \text{for} \quad \mathsf{X} \geq a \\ 0 & \text{for} \quad \mathsf{X} < a \end{cases} \\
aI &\leq \mathsf{X} \\
I &\leq \frac{1}{a}\mathsf{X} \\
\mathsf{E}I &\leq \mathsf{E}\left(\frac{1}{a}\mathsf{X}\right)
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}\{\mathsf{X} \geq a\} &= 1 \cdot \mathsf{P}\{\mathsf{X} \geq a\} + 0 \cdot \mathsf{P}\{\mathsf{X} < a\} \\
&= \mathsf{E}I \\
&\leq \mathsf{E}\left(\frac{1}{a}\mathsf{X}\right) \\
&= \frac{1}{a}\mathsf{E}\mathsf{X}
\end{aligned}$$

✏

Theorem 1.4 (Chebyshev's inequality). [6] *Let* X *be a* RANDOM VARIABLE *with mean* μ *and variance* σ^2.

THM
$$\mathsf{P}\{|\mathsf{X}-\mu| \geq a\} \leq \frac{\sigma^2}{a^2}$$

✎PROOF:

$$\begin{aligned}
\mathsf{P}\{|\mathsf{X}-\mu| \geq a\} &= \mathsf{P}\left\{(\mathsf{X}-\mu)^2 \geq a^2\right\} \\
&\leq \frac{1}{a^2}\mathsf{E}(\mathsf{X}-\mu)^2 && \text{by } \textit{Markov's inequality} \quad \text{(Theorem 1.3 page 6)} \\
&= \frac{\sigma^2}{a^2}
\end{aligned}$$

✏

Theorem 1.5 (Kolmogorov's inequality). [7] *Let* X *be a* RANDOM VARIABLE *with mean* μ *and variance* σ^2.

THM
$$\left\{\begin{array}{lll} \text{(A).} & (\!(\boldsymbol{x}_n)\!) \text{ are INDEPENDENT} & \text{and} \\ \text{(B).} & \text{Each } \boldsymbol{x}_n \text{ has ZERO-MEAN} & \text{and} \\ \text{(C).} & \text{Each } \boldsymbol{x}_n \text{ has variance } \sigma^2 & \end{array}\right\} \implies \mathsf{P}\left[\left|\sum_{n=1}^{N} \boldsymbol{x}_n\right| < \lambda \sum_{n=1}^{N} \boldsymbol{x}_n^2\right] \geq 1 - \frac{1}{\lambda^2}$$

1.4 Joint and conditional probability spaces

Sometimes the problem of finding the expected value of a *random variable* X can be simplified by "conditioning X on Y".

Theorem 1.6. *Let* X *and* Y *be* RANDOM VARIABLES*. Then*

THM
$$\mathsf{E}_{\mathsf{x}}\mathsf{X} = \mathsf{E}_{\mathsf{y}}\mathsf{E}_{\mathsf{x}|\mathsf{y}}(\mathsf{X}|\mathsf{Y})$$

[6] Ross (1998), page 396
[7] Wilks (1963), page 107 ⟨§4.5 "Kolmogorov's inequality"⟩

✎PROOF:

$$
\begin{aligned}
\mathsf{E}_{\mathsf{y}}\mathsf{E}_{\mathsf{x}|\mathsf{y}}(\mathsf{X}|\mathsf{Y}) &\triangleq \mathsf{E}_{\mathsf{y}}\left[\int_{x\in\mathbb{R}} x\mathsf{p}(\mathsf{X}=x|\mathsf{Y})\,\mathsf{d}x\right] \\
&\triangleq \int_{y\in\mathbb{R}}\left[\int_{x\in\mathbb{R}} x\mathsf{p}(x|\mathsf{Y}=y)\,\mathsf{d}x\right]\mathsf{p}(y)\,\mathsf{d}y \\
&= \int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}} x\mathsf{p}(x|y)\mathsf{p}(y)\,\mathsf{d}x\,\mathsf{d}y \\
&= \int_{x\in\mathbb{R}} x\int_{y\in\mathbb{R}} \mathsf{p}(x,y)\,\mathsf{d}y\,\mathsf{d}x \\
&= \int_{x\in\mathbb{R}} x\mathsf{p}(x)\,\mathsf{d}x \\
&\triangleq \mathsf{E}_{\mathsf{x}}\mathsf{X}
\end{aligned}
$$

✏

1.5 Expectation inner product space

When possible, we like to generalize any given mathematical structure to a more general mathematical structure and then take advantage of the properties of that more general structure. Such a generalization can be done with *random variables*. Random variables can be viewed as vectors in a vector space. Furthermore, the expectation of the product of two *random variables* (e.g. $\mathsf{E}(\mathsf{XY})$) can be viewed as an *inner product* in an *inner product space*. Since we have an *inner product* space, we can then immediately use all the properties of *inner product spaces*, *norm*ed spaces, vector spaces, metric spaces, and topological spaces.

Theorem 1.7. [8] *Let R be a ring, $(\Omega,\mathbb{E},\mathsf{P})$ be a* PROBABILITY SPACE, E *the expectation operator, and $\boldsymbol{V}=\{\mathsf{X}|\mathsf{X}:\Omega\to R\}$ be the set of all random vectors in* PROBABILITY SPACE $(\Omega,\mathbb{E},\mathsf{P})$.

THM		
(1).	$\boldsymbol{V}\triangleq\{\mathsf{X}|\mathsf{X}:\Omega\to R\}$	*is a* VECTOR SPACE.
(2).	$\langle\mathsf{X}\,|\,\mathsf{Y}\rangle\triangleq\mathsf{E}(\mathsf{XY}^*)$	*is an* INNER PRODUCT.
(3).	$\|\mathsf{X}\|\triangleq\sqrt{\mathsf{E}(\mathsf{XX}^*)}$	*is a* NORM.
(4).	$(\boldsymbol{V},\langle\triangle\,|\,\triangledown\rangle)$	*is an* INNER PRODUCT SPACE.

✎PROOF:

1. Proof that $\boldsymbol{V}$ is a vector space:

1)	$\forall\mathsf{X},\mathsf{Y},\mathsf{Z}\in\boldsymbol{V}$	$(\mathsf{X}+\mathsf{Y})+\mathsf{Z}=\mathsf{X}+(\mathsf{Y}+\mathsf{Z})$	(+ is associative)
2)	$\forall\mathsf{X},\mathsf{Y}\in\boldsymbol{V}$	$\mathsf{X}+\mathsf{Y}=\mathsf{Y}+\mathsf{X}$	(+ is commutative)
3)	$\exists 0\in\boldsymbol{V}$ such that $\forall\mathsf{X}\in\boldsymbol{V}$	$\mathsf{X}+0=\mathsf{X}$	(+ identity)
4)	$\forall\mathsf{X}\in\boldsymbol{V}\exists\mathsf{Y}\in\boldsymbol{V}$ such that	$\mathsf{X}+\mathsf{Y}=0$	(+ inverse)
5)	$\forall\alpha\in S$ and $\mathsf{X},\mathsf{Y}\in\boldsymbol{V}$	$\alpha\cdot(\mathsf{X}+\mathsf{Y})=(\alpha\cdot\mathsf{X})+(\alpha\cdot\mathsf{Y})$	($\cdot$ distributes over +)
6)	$\forall\alpha,\beta\in S$ and $\mathsf{X}\in\boldsymbol{V}$	$(\alpha+\beta)\cdot\mathsf{X}=(\alpha\cdot\mathsf{X})+(\beta\cdot\mathsf{X})$	($\cdot$ pseudo-distributes over +)
7)	$\forall\alpha,\beta\in S$ and $\mathsf{X}\in\boldsymbol{V}$	$\alpha(\beta\cdot\mathsf{X})=(\alpha\cdot\beta)\cdot\mathsf{X}$	($\cdot$ associates with $\cdot$)
8)	$\forall\mathsf{X}\in\boldsymbol{V}$	$1\cdot\mathsf{X}=\mathsf{X}$	($\cdot$ identity)

[8] Lindquist and Picci (2015) pages 25–26 ⟨2.1 Hilbert Space of Second-Order Random Variables. $\langle\xi\,|\,\eta\rangle=\mathsf{E}\{\xi\bar{\eta}\}$⟩, Caines (1988) page 21 ⟨$Exy=\int_\Omega x(\omega)y(\omega)dP(\omega)$⟩, Caines (2018) page 21 ⟨$Exy=\int_\Omega x(\omega)y(\omega)dP(\omega)$⟩, Moon and Stirling (2000), pages 105–106

2. Proof that $\langle \mathsf{X} \mid \mathsf{Y} \rangle \triangleq \mathsf{E}(\mathsf{X}\mathsf{Y}^*)$ is an *inner product*.

$$\begin{array}{llllll}
1) & \mathsf{E}(\mathsf{X}\mathsf{X}^*) & \geq 0 & \forall \mathsf{X} \in \boldsymbol{V} & \text{(non-negative)}\\
2) & \mathsf{E}(\mathsf{X}\mathsf{X}^*) & = 0 \iff \mathsf{X} = 0 & \forall \mathsf{X} \in \boldsymbol{V} & \text{(non-degenerate)}\\
3) & \mathsf{E}(\alpha\mathsf{X}\mathsf{Y}^*) & = \alpha\mathsf{E}(\mathsf{X}\mathsf{Y}^*) & \forall \mathsf{X},\mathsf{Y} \in \boldsymbol{V},\ \forall\alpha\in\mathbb{C} & \text{(homogeneous)}\\
4) & \mathsf{E}[(\mathsf{X}+\mathsf{Y})\mathsf{Z}^*] & = \mathsf{E}(\mathsf{X}\mathsf{Z}^*) + \mathsf{E}(Y\mathsf{Z}^*) & \forall \mathsf{X},\mathsf{Y},\mathsf{Z} \in \boldsymbol{V} & \text{(additive)}\\
5) & \mathsf{E}(\mathsf{X}\mathsf{Y}^*) & = \mathsf{E}(\mathsf{Y}\mathsf{X}^*) & \forall \mathsf{X},\mathsf{Y} \in \boldsymbol{V} & \text{(conjugate symmetric).}
\end{array}$$

3. Proof that $\|\mathsf{X}\| \triangleq \sqrt{\mathsf{E}(\mathsf{X}\mathsf{X}^*)}$ is a *norm*: This *norm* is simply induced by the above *inner product*.

4. Proof that $(\boldsymbol{V}, \langle \triangle \mid \triangledown \rangle)$ is an *inner product space*: Because $\boldsymbol{V}$ is a vector space and $\langle \triangle \mid \triangledown \rangle$ is an *inner product*, $(\boldsymbol{V}, \langle \triangle \mid \triangledown \rangle)$ is an *inner product space*.

⇨

The next theorem gives some results that follow directly from vector space properties:

Theorem 1.8. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE *with expectation functional* E.

THM

$$\begin{array}{llcll}
1. & \sqrt{\mathsf{E}\left(\sum_{n=1}^{N} \mathsf{X}_n\right)} & \leq & \sum_{n=1}^{N} \mathsf{E}(\mathsf{X}_n X_n^*) & \text{(Generalized triangle inequality)}\\
2. & \left|\mathsf{E}(\mathsf{X}\mathsf{Y}^*)\right|^2 & \leq & \mathsf{E}(\mathsf{X}\mathsf{X}^*)\,\mathsf{E}(YY^*) & \text{(Cauchy-Schwartz inequality)}\\
3. & 2\mathsf{E}(\mathsf{X}\mathsf{X}^*) + 2\mathsf{E}(YY^*) & = & \mathsf{E}[(\mathsf{X}+\mathsf{Y})(\mathsf{X}+\mathsf{Y})^*] + \mathsf{E}[(\mathsf{X}-Y)(\mathsf{X}-Y)^*] & \text{(Parallelogram Law)}
\end{array}$$

✎PROOF:

1. $\left(\mathbb{R}^{\Omega}, \mathsf{E}(\boldsymbol{x}, \boldsymbol{y})\right)$ is an *inner product space*. Proof: Theorem 1.7 (page 8).

2. Because it is an *inner product space*, the other properties follow:

1.	Generalized triangle inequality:	Theorem J.1	page 249
2.	Cauchy-Schwartz inequality:	Theorem I.2	page 234
3.	Parallelogram Law:	Theorem I.7	page 241

⇨

CHAPTER 2

RANDOM SEQUENCES

❝*A likely impossibility is always preferable to an unconvincing possibility.*❞
Aristotle (384 BC – 322 BC) [1]

❝*We are quite in danger of sending highly trained and highly intelligent young men out into the world with tables of erroneous numbers under their arms, and with a dense fog in the place where their brains ought to be. In this century, of course, they will be working on guided missiles and advising the medical profession on the control of disease, and there is no limit to the extent to which they could impede every sort of national effort.*❞
Ronald A. Fisher, (1890–1962), Statistician, at a lecture in 1958 at Michigan State University [2]

2.1 Definitions

Definition 2.1.

DEF *A* ***random sequence*** *is a* SEQUENCE *over a* PROBABILITY SPACE *(Definition A.2 page 147).*

Definition 2.2. [3] *Let* $\mathbf{x}(n)$ *and* $\mathbf{y}(n)$ *be* RANDOM SEQUENCES.

[1] quote: http://en.wikiquote.org/wiki/Aristotle
image: http://en.wikipedia.org/wiki/Aristotle

[2] quote: Yates and Mather (1963) page 107. image: http://www.genetics.org/content/154/4/1419

[3] Papoulis (1984) page 263 ⟨$R_{xy}(m) = E\{\mathbf{x}(m)\mathbf{y}^*(0)\}$⟩, Wilks (1963), page 77 ⟨§3.4 "Moments of two-dimensional random variables"⟩, Cadzow (1987) page 341 ⟨$r_{xy}(m) = E[\mathbf{x}(m)\mathbf{y}^*(0)]$⟩, MatLab (2018b) ⟨$R_{xy}(m) = E\{x_{n+m}y^*_n\}$⟩, MatLab (2018a) ⟨$R_{xy}(m) = E\{x_{n+m}y^*_n\}$⟩

DEF

$$\begin{array}{lllll}
\textit{The } \textbf{\textit{mean}} & \mu_{\mathsf{x}}(n) & \textit{of } \boldsymbol{x}(n) \textit{ is} & \mu_{\mathsf{x}}(n) & \triangleq \mathsf{E}[\boldsymbol{x}(n)] \\
\textit{The } \textbf{\textit{variance}} & \sigma^2_{\mathsf{x}}(n) & \textit{of } \boldsymbol{x}(n) \textit{ is} & \sigma^2_{\mathsf{x}}(n) & \triangleq \mathsf{E}\left(\left[\boldsymbol{x}(n) - \mu_{\mathsf{x}}(n)\right]^2\right) \\
\textit{The } \textbf{\textit{cross-correlation}} & \mathsf{R}_{\mathsf{xy}}(n,m) & \textit{of } \boldsymbol{x}(n) \textit{ and } \boldsymbol{y}(n) \textit{ is} & \mathsf{R}_{\mathsf{xy}}(n,m) & \triangleq \mathsf{E}\left[\boldsymbol{x}(n+m)\boldsymbol{y}^*(n)\right] \\
\textit{The } \textbf{\textit{auto-correlation}} & \mathsf{R}_{\mathsf{xx}}(n,m) & \textit{of } \boldsymbol{x}(n) \textit{ is} & \mathsf{R}_{\mathsf{xx}}(n,m) & \triangleq \mathsf{R}_{\mathsf{xy}}(n,m)\big|_{\boldsymbol{y}=\boldsymbol{x}}
\end{array}$$

2.2 Properties

Theorem 2.1.

THM

$$\begin{aligned}
\mathsf{R}_{\mathsf{xx}}(n,m) &= \mathsf{R}^*_{\mathsf{xx}}(n+m,-m) \\
\mathsf{R}_{\mathsf{xy}}(n,m) &= \mathsf{R}^*_{\mathsf{yx}}(n+m,-m)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{\mathsf{xy}}(n,m) &\triangleq \mathsf{E}\left[\boldsymbol{x}(n+m)\boldsymbol{y}^*(n)\right] && \text{by definition of } \mathsf{R}_{\mathsf{xy}}(n,m) && \text{(Definition 2.2 page 11)} \\
&= \mathsf{E}\left[\boldsymbol{y}^*(n)\boldsymbol{x}(n+m)\right] && \text{by \textit{commutative} property of } (\mathbb{C},+,\cdot,0,1) \\
&= \left(\mathsf{E}\left[\boldsymbol{y}(n)\boldsymbol{x}^*(n+m)\right]\right)^* && \text{by \textit{distributive} property of } *\text{-\textbf{algebras}} && \text{(Definition H.3 page 228)} \\
&= \left(\mathsf{E}\left[\boldsymbol{y}(n+m-m)\boldsymbol{x}^*(n+m)\right]\right)^* && \text{by \textit{additive identity} property of } (\mathbb{R},+,\cdot,0,1) \\
&\triangleq \mathsf{R}^*_{\mathsf{yx}}(n+m,-m) && \text{by definition of } \mathsf{R}_{\mathsf{xy}}(n,m) && \text{(Definition 2.2 page 11)}
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{\mathsf{xx}}(n,m) &= \mathsf{R}_{\mathsf{xy}}(n,m)\big|_{\boldsymbol{y}=\boldsymbol{x}} && \text{by } \boldsymbol{y}=\boldsymbol{x} \text{ constraint} \\
&= \mathsf{R}^*_{\mathsf{xy}}(n+m,-m)\big|_{\boldsymbol{y}=\boldsymbol{x}} && \text{by previous result} \\
&= \mathsf{R}^*_{\mathsf{xx}}(n+m,-m) && \text{by } \boldsymbol{y}=\boldsymbol{x} \text{ constraint}
\end{aligned}$$

⇨

2.3 Wide Sense Stationary processes

Definition 2.3. *Let* $\boldsymbol{x}(n)$ *be a* RANDOM SEQUENCE *with* MEAN $\mu_{\mathsf{x}}(n)$ *and* VARIANCE $\sigma^2(n)$ *(Definition 2.2 page 11).*

DEF

$\boldsymbol{x}(n)$ *is* ***wide sense stationary*** *(WSS) if*

1. $\mu_{\mathsf{x}}(n)$ *is* CONSTANT *with respect to* n (STATIONARY IN THE 1ST MOMENT) *and*
2. $\sigma^2_{\mathsf{x}}(n)$ *is* CONSTANT *with respect to* n (STATIONARY IN THE 2ND MOMENT)

Definition 2.4. [4] *Let* $\boldsymbol{x}(n)$ *be a* RANDOM SEQUENCE *with statistics* $\mu_{\mathsf{x}}(n)$, $\sigma^2_{\mathsf{x}}(n)$, $\mathsf{R}_{\mathsf{xx}}(n,m)$, *and* $\mathsf{R}_{\mathsf{xy}}(n,m)$ *(Definition 2.2 page 11).*

DEF

$$\left\{ \boldsymbol{x} \textit{ and } \boldsymbol{y} \textit{ are } \text{WIDE SENSE STATIONARY} \right\} \implies
\left\{\begin{array}{llllll}
(1). & \textit{The } \textbf{\textit{mean}} & \mu_{\mathsf{x}} & \textit{of } \boldsymbol{x}(n) \textit{ is} & \mu_{\mathsf{x}} & \triangleq \mathsf{E}[\boldsymbol{x}(0)] \\
(2). & \textit{The } \textbf{\textit{variance}} & \sigma^2_{\mathsf{x}} & \textit{of } \boldsymbol{x}(n) \textit{ is} & \sigma^2_{\mathsf{x}} & \triangleq \mathsf{E}\left(\left[\boldsymbol{x}(0) - \mu_{\mathsf{x}}\right]^2\right) \\
(4). & \textit{The } \textbf{\textit{cross-correlation}} & \mathsf{R}_{\mathsf{xy}}(m) & \textit{of } \boldsymbol{x}(n) \textit{ and } \boldsymbol{y}(n) \textit{ is} & \mathsf{R}_{\mathsf{xy}}(m) & \triangleq \mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{y}^*(0)\right] \\
(3). & \textit{The } \textbf{\textit{auto-correlation}} & \mathsf{R}_{\mathsf{xx}}(m) & \textit{of } \boldsymbol{x}(n) \textit{ is} & \mathsf{R}_{\mathsf{xx}}(m) & \triangleq \mathsf{R}_{\mathsf{xy}}(m)\big|_{\boldsymbol{y}=\boldsymbol{x}}
\end{array}\right\}$$

[4] Papoulis (1984) page 263 ⟨"$R_{xy}(\tau) = E\{\boldsymbol{x}(t+\tau)\boldsymbol{y}^*(t)\}$"⟩, Cadzow (1987) page 341 ⟨$r_{xy}(n) = E[\boldsymbol{x}(k+n)\boldsymbol{y}^*(k)]$ (10.41)⟩

Remark 2.1. The $\mathsf{R}_{xy}(n,m)$ of Definition 2.2 (page 11) and the $\mathsf{R}_{xy}(m)$ of Definition 2.4 (page 12) (etc.) are examples of *function overload*—that is, functions that use the same mnemonic but are distinguished by different domains. Perhaps a more common example of function overload is the "+" mnemonic. Traditionally it is used with domain of the natural numbers $\mathbb{N}$ as in $3+2$. Later it was extended for domain real numbers $\mathbb{R}$ as in $\sqrt{3}+\sqrt{2}$, or even complex numbers $\mathbb{C}$ as in $\left(\sqrt{3}+i\sqrt{2}\right)+(e+i\pi)$. And it was even more dramatically extended for use with domain $\mathbb{R}^N\times\mathbb{R}^M$ in "linear algebra" as in

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}+\begin{bmatrix}5&6\\7&8\end{bmatrix}=\begin{bmatrix}6&8\\10&12\end{bmatrix}$$

Proposition 2.1. *Let* $\mathbf{y}(n)$ *be a* RANDOM SEQUENCE, $\mathbf{x}(n)$ *a* RANDOM SEQUENCE *with* AUTO-CORRELATION $\mathsf{R}_{xx}(n,m)$, *and* R_{xy} *the* CROSS-CORRELATION *of* $\mathbf{x}$ *and* $\mathbf{y}$.

PRP
$$\left\{\begin{array}{l}\mathbf{x}\text{ and }\mathbf{y}\text{ are}\\ \text{WIDE SENSE STATIONARY}\\ \text{(WSS) (Definition 6.1 page 43)}\end{array}\right\}\implies\left\{\begin{array}{rcll}\mathsf{R}_{xx}(n,m)&=&\mathsf{R}_{xx}(m)&\forall n\in\mathbb{Z}\\ \mathsf{R}_{xy}(n,m)&=&\mathsf{R}_{xy}(m)&\forall n\in\mathbb{Z}\\ \text{(Definition 2.2 page 11)}&&\text{(Definition 2.4 page 12)}&\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{xy}(n,m)&\triangleq \mathsf{E}\left[\mathbf{x}[n+m]\mathbf{y}^*[n]\right] &&\text{by definition of } \mathsf{R}_{xy}(n,m) &&\text{(Definition 2.2 page 11)}\\
&=\mathsf{E}\left[\mathbf{x}[n-n+m]\mathbf{y}^*[n-n]\right] &&\text{by \textit{wide sense stationary} hypothesis}\\
&=\mathsf{E}\left[\mathbf{x}[m]\mathbf{y}^*[0]\right]\\
&\triangleq \mathsf{R}_{xy}(m) &&\text{by definition of } \mathsf{R}_{xy}(m) &&\text{(Definition 2.4 page 12)}\\
\mathsf{R}_{xx}(n,m)&=\left.\mathsf{R}_{xy}(n,m)\right|_{\mathbf{y}=\mathbf{x}}\\
&=\left.\mathsf{R}_{xy}(m)\right|_{\mathbf{y}=\mathbf{x}} &&\text{by previous result}\\
&=\mathsf{R}_{xx}(m)
\end{aligned}$$

✎

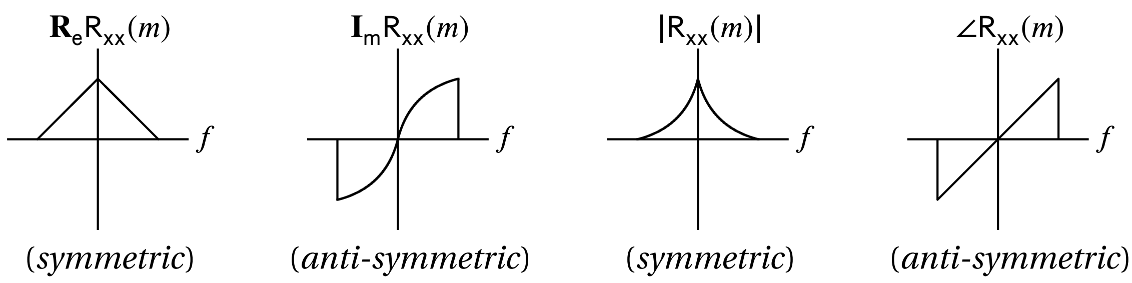


Figure 2.1: *auto-correlation* $\mathsf{R}_{xx}(m)$

Corollary 2.1. *Let* $\mathbf{x}(n)$ *be a* RANDOM SEQUENCE *with* AUTO-CORRELATION $\mathsf{R}_{xx}(n,m)$, $\mathbf{y}(n)$ *a* RANDOM SEQUENCE *with* AUTO-CORRELATION $\mathsf{R}_{yy}(n,m)$, *and* $\mathsf{R}_{xy}(n,m)$ *the* CROSS-CORRELATION *of* $\mathbf{x}$ *and* $\mathbf{y}$. *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

COR
$$\left\{\begin{array}{lll}\text{(A).}&\mathbf{x}\text{ is WSS}&\text{and}\\ \text{(B).}&\mathbf{y}\text{ is WSS}&\text{and}\\ \text{(C).}&\mathbf{S}\text{ is LTI}&\end{array}\right\}\implies\left\{\begin{array}{lrcll}\text{(1).}&\mathsf{R}_{xy}(m)&=&\mathsf{R}^*_{yx}(-m)&&\text{and}\\ \text{(2).}&\mathsf{R}_{xx}(m)&=&\mathsf{R}^*_{xx}(-m)&\text{(CONJUGATE SYMMETRIC)}&\text{and}\\ \text{(3).}&\mathbf{R_e}\mathsf{R}_{xx}(m)&=&\mathbf{R_e}\mathsf{R}_{xx}(-m)&\text{(SYMMETRIC)}&\text{and}\\ \text{(4).}&\mathbf{I_m}\mathsf{R}_{xx}(m)&=&-\mathbf{I_m}\mathsf{R}_{xx}(-m)&\text{(ANTI-SYMMETRIC)}&\text{and}\\ \text{(5).}&\left|\mathsf{R}_{xx}(m)\right|&=&\left|\mathsf{R}_{xx}(-m)\right|&\text{(SYMMETRIC)}&\text{and}\\ \text{(6).}&\angle\mathsf{R}_{xx}(m)&=&-\angle\mathsf{R}_{xx}(-m)&\text{(ANTI-SYMMETRIC)}&\end{array}\right\}$$

✎PROOF:

$$
\begin{aligned}
\mathsf{R}_{xy}(m) &= \mathsf{R}_{xy}(n,m) && \text{by Proposition 2.1 page 13} && \text{and hypotheses (A),(B)}\\
&= \mathsf{R}^*_{yx}(n+m,-m) && \text{by Theorem 2.1 page 12} && \text{and hypothesis (B)}\\
&= \mathsf{R}^*_{yx}(-m) && \text{by Proposition 2.1 page 13} && \text{and hypothesis (A)}\\
\mathsf{R}_{xx}(m) &= \mathsf{R}_{xx}(n,m) && \text{by Proposition 2.1 page 13} && \text{and hypothesis (A)}\\
&= \mathsf{R}^*_{xx}(n+m,-m) && \text{by Theorem 2.1 page 12} && \text{and hypothesis (B)}\\
&= \mathsf{R}^*_{xx}(-m) && \text{by Proposition 2.1 page 13} && \text{and hypothesis (A)}
\end{aligned}
$$

✎

2.4 Spectral density

Definition 2.5. *Let* $\mathbf{x}(n)$ *and* $\mathbf{y}(n)$ *be* WIDE SENSE STATIONARY RANDOM SEQUENCES *with auto-correlation* $\mathsf{R}_{xx}(m)$ *and cross-correlation* $\mathsf{R}_{xy}(m)$. *Let* $\mathbf{Z}$ *be the* Z-TRANSFORM OPERATOR *(Definition P.4 page 330).*

DEF

*The **z-domain cross spectral density** (**CSD**)* $\check{S}_{xy}(z)$ *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is*

$$\check{S}_{xy}(z) \triangleq \mathbf{Z}\mathsf{R}_{xy}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{xy}(m)z^{-m}$$

*The **z-domain power spectral density** (**PSD**)* $\check{S}_{xx}(z)$ *of* $\mathbf{x}$ *is* $\quad \check{S}_{xx}(z) \triangleq \check{S}_{xy}(z)\big|_{\mathbf{y}(n)=\mathbf{x}(n)}$

Definition 2.6. *Let* $\mathbf{x}(n)$ *and* $\mathbf{y}(n)$ *be* WIDE SENSE STATIONARY RANDOM SEQUENCES *with auto-correlation* $\mathsf{R}_{xx}(m)$ *and cross-correlation* $\mathsf{R}_{xy}(m)$. *Let* $\check{\mathbf{F}}$ *be the* DISCRETE TIME FOURIER TRANSFORM (DTFT) *operator (Definition O.1 page 319).*

DEF

*The **auto-spectral density*** $\check{S}_{xx}(z)$ *of* $\mathbf{x}$ *is* $\quad \tilde{S}_{xx}(z) \triangleq \check{\mathbf{F}}\mathsf{R}_{xx}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{xx}(m)e^{-i\omega m}$

*The **cross spectral density** (**CSD**)* $\check{S}_{xy}(z)$ *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is* $\quad \tilde{S}_{xy}(z) \triangleq \check{\mathbf{F}}\mathsf{R}_{xy}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{xy}(m)e^{-i\omega m}$

*The **auto-spectral density** is also called **power spectral density** (**PSD**).*

Theorem 2.2. *Let* $\mathbf{S}$ *be a system with* IMPULSE RESPONSE $\mathsf{h}(n)$, INPUT $\mathbf{x}(n)$, *and* OUTPUT $\mathbf{y}(n)$.

THM

$$\left\{ \mathbf{x} \text{ and } \mathbf{y} \text{ are WIDE SENSE STATIONARY} \right\} \implies \left\{ \begin{array}{llcll} (1). & \check{S}_{xx}(z) & = & \check{S}^*_{xx}\left(\frac{1}{z^*}\right) & \text{and}\\ (2). & \check{S}_{yx}(z) & = & \check{S}^*_{xy}\left(\frac{1}{z^*}\right) & \end{array} \right\}$$

✎PROOF:

$$
\begin{aligned}
\check{S}_{yx}(z) &\triangleq \mathbf{Z}\mathsf{R}_{yx}(m) && \text{by definition of } \check{S}_{xy}(z) && \text{(Definition 2.6 page 14)}\\
&\triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{yx}(m)z^{-m} && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)}\\
&\triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}^*_{xy}(-m)z^{-m} && \text{by Corollary 2.1 page 13} &&\\
&= \left[\sum_{m\in\mathbb{Z}} \mathsf{R}_{xy}(-m)(z^*)^{-m}\right]^* && \text{by } \textit{antiautomorphic} \text{ property of } {}^*\text{-algebras} && \text{(Definition H.3 page 228)}
\end{aligned}
$$

$$= \left[\sum_{-p\in\mathbb{Z}} \mathsf{R}_{xy}(p)(z^*)^p\right]^* \qquad \text{where } p \triangleq -m \qquad \Longrightarrow m = -p$$

$$= \left[\sum_{p\in\mathbb{Z}} \mathsf{R}_{xy}(p)(z^*)^p\right]^* \qquad \text{by } \textit{absolutely summable} \text{ property} \qquad \text{(Definition P.2 page 329)}$$

$$= \left[\sum_{p\in\mathbb{Z}} \mathsf{R}_{xy}(p)\left(\frac{1}{z^*}\right)^{-p}\right]^*$$

$$= \check{\mathsf{S}}^*_{xy}\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition P.4 page 330)}$$

$$\check{\mathsf{S}}_{xx}(z) = \check{\mathsf{S}}_{xy}(z)\big|_{y=x}$$

$$= \check{\mathsf{S}}^*_{yx}(z)\big|_{y=x}$$

$$= \check{\mathsf{S}}^*_{xy}\left(\frac{1}{z^*}\right)\bigg|_{y=x} \qquad \text{by (2)—previous result}$$

$$= \check{\mathsf{S}}^*_{xx}\left(\frac{1}{z^*}\right)$$

Corollary 2.2. *Let* **S** *be a system with* IMPULSE RESPONSE $\mathsf{h}(n)$, INPUT $x(n)$, *and* OUTPUT $y(n)$.

COR

$$\left\{\begin{array}{ll} \text{(A).} & \mathsf{h} \textit{ is } \text{LTI} \quad \textit{and} \\ \text{(B).} & x \textit{ and } y \textit{ are } \text{WSS} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lllll} \text{(1).} & \tilde{\mathsf{S}}^*_{xy}(\omega) & = & \tilde{\mathsf{S}}_{yx}(\omega) & (\text{conjugate-symmetric}) \quad \textit{and} \\ \text{(2).} & \tilde{\mathsf{S}}^*_{xx}(\omega) & = & \tilde{\mathsf{S}}_{xx}(\omega) & (\text{conjugate symmetric}) \quad \textit{and} \\ \text{(3).} & \tilde{\mathsf{S}}_{xx}(\omega) & \in & \mathbb{R} & (\text{real-valued}) \end{array}\right\}$$

✎PROOF:

$$\tilde{\mathsf{S}}^*_{xy}(\omega) = \check{\mathsf{S}}^*_{xy}(z)\big|_{z=e^{i\omega}} \qquad \text{by definition of } \textit{DTFT} \qquad \text{(Definition O.1 page 319)}$$

$$= \check{\mathsf{S}}^{**}_{yx}\left(\frac{1}{z^*}\right)\bigg|_{z=e^{i\omega}} \qquad \text{by Theorem 2.2 page 14}$$

$$= \check{\mathsf{S}}_{yx}\left(\frac{1}{z^*}\right)\bigg|_{z=e^{i\omega}} \qquad \text{by } \textit{involutory} \text{ property of } *\text{-}\mathbf{algebras} \qquad \text{(Definition H.3 page 228)}$$

$$= \check{\mathsf{S}}_{yx}\left(\frac{1}{e^{i\omega *}}\right)$$

$$= \check{\mathsf{S}}_{yx}\left(e^{i\omega}\right)$$

$$= \tilde{\mathsf{S}}_{yx}(\omega) \qquad \text{by definition of } \textit{DTFT} \qquad \text{(Definition O.1 page 319)}$$

$$\tilde{\mathsf{S}}^*_{xx}(\omega) = \check{\mathsf{S}}^*_{xx}(z)\big|_{z=e^{i\omega}} \qquad \text{by definition of } \textit{DTFT} \qquad \text{(Definition O.1 page 319)}$$

$$= \check{\mathsf{S}}^{**}_{xx}\left(\frac{1}{z^*}\right)\bigg|_{z=e^{i\omega}} \qquad \text{by Theorem 2.2 page 14}$$

$$= \check{\mathsf{S}}_{xx}\left(\frac{1}{z^*}\right)\bigg|_{z=e^{i\omega}} \qquad \text{by } \textit{involutory} \text{ property of } *\text{-}\mathbf{algebras} \qquad \text{(Definition H.3 page 228)}$$

$$= \check{\mathsf{S}}_{xx}\left(\frac{1}{e^{i\omega *}}\right)$$

$$= \check{\mathsf{S}}_{xx}\left(e^{i\omega}\right)$$

$$= \tilde{\mathsf{S}}_{xx}(\omega) \qquad \text{by definition of } \textit{DTFT} \qquad \text{(Definition O.1 page 319)}$$

$$\Longrightarrow \tilde{\mathsf{S}}_{xx}(\omega) \text{ is } \textit{real-valued}$$

$$\tilde{\mathsf{S}}^*_{xx}(\omega) = \tilde{\mathsf{S}}^*_{xy}(\omega)\big|_{y=x}$$

$$= \tilde{\mathsf{S}}_{yx}(\omega)\big|_{y=x} \qquad \text{by previous result}$$

$$= \tilde{\mathsf{S}}_{xx}(\omega)$$

2.5 Spectral Power

The term "*spectral power*" is a bit of an oxymoron because "spectral" deals with leaving the time-domain for the frequency-domain, howbeit the concept of power is solidly founded on the concept of time in that power = energy per time.

However, *Parseval's Theorem* (Proposition G.2 page 208) demonstrates that power in time can also be calculated in frequency. So, it makes some sense to speak of the term "spectral power". Moreover, one way to estimate this power is to average the Fourier Transforms of the product $|\boldsymbol{x}(n)|^2 = \boldsymbol{x}(n)\boldsymbol{x}^*(n)$...that is, to use an estimate of the auto-spectral density $\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)$. Thus, an alternate name for *auto-spectral density* is **power spectral density** (PSD).

CHAPTER 3

CONTINUOUS RANDOM PROCESSES

❝*A likely impossibility is always preferable to an unconvincing possibility.*❞
Aristotle (384 BC – 322 BC) [1]

3.1 Definitions

Definition 3.1. [2] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE.

DEF
The function $\boldsymbol{x} : \Omega \to \mathbb{R}$ *is a* ***random variable****.*
The function $\boldsymbol{y} : \mathbb{R} \times \Omega \to \mathbb{R}$ *is a* ***random process****.*

The random process $\boldsymbol{x}(t, \omega)$, where t commonly represents time and $\omega \in \Omega$ is an outcome of an experiment, can take on more specialized forms depending on whether t and ω are fixed or allowed to vary. These forms are illustrated in Figure 3.1 page 17[3] and Figure 3.2 page 18.

$\boldsymbol{x}(t, \omega)$	fixed t	variable t
fixed ω	number	time function
variable ω	random variable	random process

Figure 3.1: Specialized forms of a random process $\boldsymbol{x}(t, \omega)$

[1] quote: http://en.wikiquote.org/wiki/Aristotle
image: http://en.wikipedia.org/wiki/Aristotle
[2] Papoulis (1991), page 63, Papoulis (1991), page 285
[3] Papoulis (1991), pages 285–286

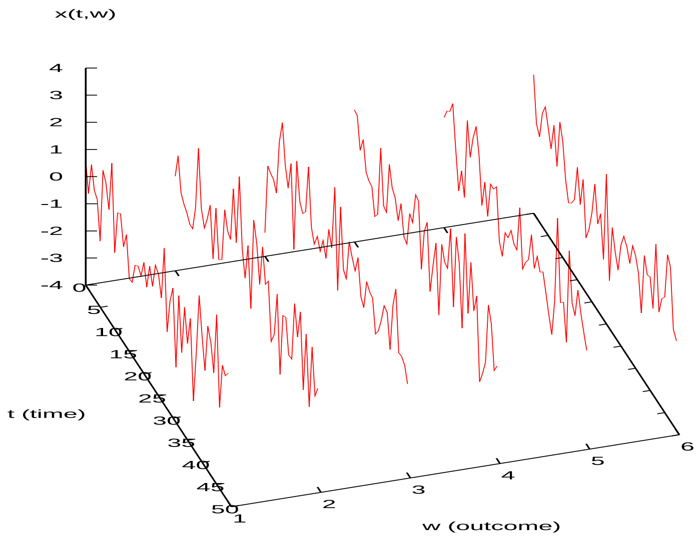


Figure 3.2: Example of a random process $\mathbf{x}(t,\omega)$

Definition 3.2. [4] *Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be random processes.*

The ***mean***	$\mu_{\mathrm{x}}(t)$	*of $\mathbf{x}(t)$ is*	$\mu_{\mathrm{x}}(t)$	$\triangleq$	$\mathsf{E}[\mathbf{x}(t)]$
The ***cross-correlation***	$\mathsf{R}_{\mathrm{xy}}(t)$	*of $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is*	$\mathsf{R}_{\mathrm{xy}}(t,u)$	$\triangleq$	$\mathsf{E}\left[\mathbf{x}(t)\mathbf{y}^*(u)\right]$
The ***autocorrelation function***	$\mathsf{R}_{\mathrm{xx}}(t)$	*of $\mathbf{x}(t)$ is*	$\mathsf{R}_{\mathrm{xx}}(t,u)$	$\triangleq$	$\mathsf{E}\left[\mathbf{x}(t)\mathbf{x}^*(u)\right]$
The ***autocorrelation operator***	$\mathbf{R}\mathsf{f}$	*of $\mathsf{f}(t)$ is*	$\mathbf{R}f$	$\triangleq$	$\displaystyle\int_{u\in\mathbb{R}} \mathsf{R}_{\mathrm{xx}}(t,u)\mathsf{f}(u)\,\mathrm{d}u$

Remark 3.1. [5] The equation $\int_{u\in\mathbb{R}} \mathsf{R}_{\mathrm{xx}}(t,u)\mathsf{f}(u)\,\mathrm{d}u$ is a *Fredholm integral equation of the first kind* and $\mathsf{R}_{\mathrm{xx}}(t,u)$ is the *kernel* of the equation.

3.2 Properties

Theorem 3.1. *Let $\mathrm{x}(t)$ and $\mathrm{y}(t)$ be random processes with cross-correlation $\mathsf{R}_{\mathrm{xy}}(t,u)$ and let $\mathsf{R}_{\mathrm{xx}}(t,u)$ be the auto-correlation of $\mathrm{x}(t)$.*

$$\begin{aligned} \mathsf{R}_{\mathrm{xx}}(t,u) &= \mathsf{R}^*_{\mathrm{xx}}(u,t) \quad \text{(CONJUGATE SYMMETRIC)} \\ \mathsf{R}_{\mathrm{xy}}(t,u) &= \mathsf{R}^*_{\mathrm{yx}}(u,t) \end{aligned}$$

✎PROOF:

$$\begin{aligned} \mathsf{R}_{\mathrm{xx}}(t,u) \triangleq \mathsf{E}\left[\mathbf{x}(t)\mathbf{x}^*(u)\right] &= \mathsf{E}\left[\mathbf{x}^*(u)\mathbf{x}(t)\right] = \left(\mathsf{E}\left[\mathbf{x}(u)\mathbf{x}^*(t)\right]\right)^* &\triangleq \mathsf{R}^*_{\mathrm{xx}}(u,t) \\ \mathsf{R}_{\mathrm{xy}}(t,u) \triangleq \mathsf{E}\left[\mathbf{x}(t)\mathbf{y}^*(u)\right] &= \mathsf{E}\left[\mathbf{y}^*(u)\mathbf{x}(t)\right] = \left(\mathsf{E}\left[\mathbf{y}(u)\mathbf{x}^*(t)\right]\right)^* &\triangleq \mathsf{R}^*_{\mathrm{yx}}(u,t) \end{aligned}$$

⇨

[4] Papoulis (1984) page 216 ⟨$R_{xy}(t_1,t_2) = E\left\{\mathbf{x}(t_1)\mathbf{y}^*(t_2)\right\}$ (9-35)⟩,

[5] Fredholm (1900), Fredholm (1903), page 365, Michel and Herget (1993), page 97, Keener (1988), page 101

Theorem 3.2. *Let* $\mathbf{R} : \boldsymbol{X} \to \boldsymbol{X}$ *be an auto-correlation operator.*

THM

$$\langle \mathbf{R}\mathsf{x} \mid \mathsf{x}\rangle \ge 0 \qquad \forall \mathsf{x} \in \boldsymbol{X} \qquad (\text{NON-NEGATIVE})$$
$$\langle \mathbf{R}\mathsf{x} \mid \mathsf{y}\rangle = \langle \mathsf{x} \mid \mathbf{R}\mathsf{y}\rangle \qquad \forall \mathsf{x},\mathsf{y} \in \boldsymbol{X} \qquad (\text{SELF-ADJOINT})$$

✎PROOF:

1. Proof that $\mathbf{R}$ is non-negative:

$$\begin{aligned}
\langle \mathbf{R}\mathsf{y} \mid \mathsf{y}\rangle &= \left\langle \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{y}(u)\,\mathsf{d}u \mid \mathsf{y}(t)\right\rangle && \text{by definition of } \mathbf{R} && \text{(Definition 3.2 page 18)}\\
&= \left\langle \int_{u\in\mathbb{R}} \mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right]\mathsf{y}(u)\,\mathsf{d}u \mid \mathsf{y}(t)\right\rangle && \text{by definition of } \mathsf{R}_{\mathsf{xx}}(t,u) && \text{(Definition 3.2 page 18)}\\
&= \mathsf{E}\left[\left\langle \int_{u\in\mathbb{R}} \mathsf{x}(t)\mathsf{x}^*(u)\mathsf{y}(u)\,\mathsf{d}u \mid \mathsf{y}(t)\right\rangle\right] && \text{by \textit{linearity} of } \langle \triangle \mid \nabla\rangle \text{ and } \int && \text{(Definition I.1 page 233)}\\
&= \mathsf{E}\left[\int_{u\in\mathbb{R}} \mathsf{x}^*(u)\mathsf{y}(u)\,\mathsf{d}u\,\langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle\right] && \text{by \textit{additivity} property of } \langle \triangle \mid \nabla\rangle && \text{(Definition I.1 page 233)}\\
&= \mathsf{E}\left[\langle \mathsf{y}(u) \mid \mathsf{x}(u)\rangle\,\langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle\right] && \text{by local definition of } \langle \triangle \mid \nabla\rangle && \text{(Definition I.1 page 233)}\\
&= \mathsf{E}\left[\langle \mathsf{x}(u) \mid \mathsf{y}(u)\rangle^*\,\langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle\right] && \text{by \textit{conjugate symmetry} prop.} && \text{(Definition I.1 page 233)}\\
&= \mathsf{E}\left|\langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle\right|^2 && \text{by definition of } |\cdot| && \text{(Definition ?? page ??)}\\
&\ge 0
\end{aligned}$$

2. Proof that $\mathbf{R}$ is self-adjoint:

$$\begin{aligned}
\langle [\mathbf{R}\mathsf{x}](t) \mid \mathsf{y}\rangle &= \left\langle \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{x}(u)\,\mathsf{d}u \mid \mathsf{y}(t)\right\rangle && \text{by definition of } \mathbf{R} && \text{(Definition 3.2 page 18)}\\
&= \int_{u\in\mathbb{R}} \mathsf{x}(u)\,\langle \mathsf{R}_{\mathsf{xx}}(t,u) \mid \mathsf{y}(t)\rangle\;\mathsf{d}u && \text{by \textit{additive} property of } \langle \triangle \mid \nabla\rangle && \text{(Definition I.1 page 233)}\\
&= \int_{u\in\mathbb{R}} \mathsf{x}(u)\,\langle \mathsf{y}(t) \mid \mathsf{R}_{\mathsf{xx}}(t,u)\rangle^*\;\mathsf{d}u && \text{by \textit{conjugate symmetry} prop.} && \text{(Definition I.1 page 233)}\\
&= \langle \mathsf{x}(u) \mid \langle \mathsf{y}(t) \mid \mathsf{R}_{\mathsf{xx}}(t,u)\rangle\rangle && \text{by local definition of } \langle \triangle \mid \nabla\rangle && \text{(Definition I.1 page 233)}\\
&= \left\langle \mathsf{x}(u) \mid \int_{t\in\mathbb{R}} \mathsf{y}(t)\mathsf{R}^*_{\mathsf{xx}}(t,u)\,\mathsf{d}t\right\rangle && \text{by local definition of } \langle \triangle \mid \nabla\rangle && \text{(Definition I.1 page 233)}\\
&= \left\langle \mathsf{x}(u) \mid \int_{t\in\mathbb{R}} \mathsf{y}(t)\mathsf{R}_{\mathsf{xx}}(u,t)\,\mathsf{d}t\right\rangle && \text{by property of } \mathsf{R}_{\mathsf{xx}} && \text{(Theorem 3.1 page 18)}\\
&= \left\langle \mathsf{x}(u) \mid \underbrace{\mathbf{R}}_{\mathbf{R}^*}\mathsf{y}\right\rangle && \text{by definition of } \mathbf{R} && \text{(Definition 3.2 page 18)}\\
\implies\quad & \mathbf{R} = \mathbf{R}^* \qquad \implies \mathbf{R} \text{ is \textit{self adjoint}}
\end{aligned}$$

⇒

Theorem 3.3. [6] *Let* $(\lambda_n)_{n\in\mathbb{Z}}$ *be the eigenvalues and* $(\psi_n)_{n\in\mathbb{Z}}$ *be the eigenfunctions of operator* $\mathbf{R}$ *such that* $\mathbf{R}\psi_n = \lambda_n\psi_n$.

THM

1. $\lambda_n \in \mathbb{R}$ — *(eigenvalues of* $\mathbf{R}$ *are* REAL*)*
2. $\lambda_n \ne \lambda_m \implies \langle \psi_n \mid \psi_m\rangle = 0$ — *(eigenfunctions associated with distinct eigenvalues are* ORTHOGONAL*)*
3. $\left\|\psi_n(t)\right\|^2 > 0 \implies \lambda_n \ge 0$ — *(eigenvalues are* NON-NEGATIVE*)*
4. $\left\|\psi_n(t)\right\|^2 > 0, \langle \mathbf{R}f \mid f\rangle > 0 \implies \lambda_n > 0$ — *(if* $\mathbf{R}$ *is* POSITIVE DEFINITE*, then eigenvalues are* POSITIVE*)*

[6] Keener (1988), pages 114–119

✎PROOF:

1. Proof that eigenvalues are *real-valued*: Because $\mathbf{R}$ is self-adjoint, its eigenvalues are real (Theorem M.17 page 298).

2. eigenfunctions associated with distinct eigenvalues are orthogonal: Because $\mathbf{R}$ is self-adjoint, this property follows (Theorem M.17 page 298).

3. Proof that eigenvalues are *non-negative*:

$$\begin{aligned}
0 &\geq \langle \mathbf{R}\psi_n \,|\, \psi_n \rangle && \text{by definition of non-negative definite}\\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by hypothesis}\\
&= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of inner-products}\\
&= \lambda_n \left\|\psi_n\right\|^2 && \text{by definition of norm induced by inner-product}
\end{aligned}$$

4. Eigenvalues are *positive* if $\mathbf{R}$ is *positive definite*:

$$\begin{aligned}
0 &> \langle \mathbf{R}\psi_n \,|\, \psi_n \rangle && \text{by definition of \textit{positive definite}}\\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by hypothesis}\\
&= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of inner-products}\\
&= \lambda_n \left\|\psi_n\right\|^2 && \text{by definition of norm induced by inner-product}
\end{aligned}$$

⇰

3.3 Basis for random processes

If a random process $\boldsymbol{x}(t)$ is white [7] and $\Psi = \{\psi_1(t), \psi_2(t), \dots, \psi_N(t)\}$ is **any** set of orthonormal basis functions, then the innerproducts $\langle n(t) \,|\, \psi_n(t)\rangle$ and $\langle n(t) \,|\, \psi_m(t)\rangle$ are *uncorrelated* for $m \neq n$. However, if $\boldsymbol{x}(t)$ is colored (not white), then the innerproducts are not in general uncorrelated. But if the elements of Ψ are chosen to be the eigenfunctions of $\mathbf{R}$ such that

$$\mathbf{R}\psi_n = \lambda_n \psi_n,$$

then by Theorem 3.1 page 18, $\{\psi_n(t)\}$ are orthogonal and the innerproducts **are** uncorrelated eventhough $\boldsymbol{x}(t)$ is not white. This criterion is called the Karhunen-Loève criterion for $\boldsymbol{x}(t)$.

Theorem 3.4 (Karhunen-Loève Expansion). [8] *Let* $\mathbf{R}$ *be the* AUTOCORRELATION OPERATOR *of a* RANDOM PROCESS $\boldsymbol{x}(t)$.

THM

$$\left\{\begin{array}{lll}
\text{(A).} & (\lambda_n)_{n\in\mathbb{Z}} \text{ are the eigenvalues of } \mathbf{R} & \text{and}\\
\text{(B).} & (\psi_n)_{n\in\mathbb{Z}} \text{ are the eigenfunctions of } \mathbf{R} & \text{such that}\\
\text{(C).} & \mathbf{R}\psi_n = \lambda_n\psi_n &
\end{array}\right\}
\implies
\left\{\begin{array}{l}
\mathsf{E}\left\{\left| x(t) - \sum_{n\in\mathbb{Z}} \langle x(t) \,|\, \psi_n(t)\rangle\, \psi_n(t)\right|^2\right\} = 0\\
(\{\psi_n(t)\} \text{ is a BASIS for } \boldsymbol{x}(t))
\end{array}\right\}$$

[7] *white noise process*: random process $\boldsymbol{x}(t)$ with autocorrelation $\mathsf{R}_{xx}(\tau) = \delta(\tau)$

[8] Keener (1988), pages 114–119

✎PROOF:

1. $\{\psi_n(t)\}$ is a basis for $\boldsymbol{x}(t)$

$$\mathsf{E}\left\{\left|x(t)-\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right|^2\right\}=0 \qquad \text{where } \dot{x}_n \triangleq \langle x(t)\,|\,\psi_n(t)\rangle$$

$$\begin{aligned}
\mathsf{E}\left[x(t)\left(\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right)^*\right] &= \mathsf{E}\left[x(t)\left(\sum_{n\in\mathbb{Z}}\int_{u\in\mathbb{R}} x(u)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\right)^*\right]\\
&= \sum_{n\in\mathbb{Z}}\left(\int_{u\in\mathbb{R}}\mathsf{E}\left[x(t)x^*(u)\right]\psi_n(u)\,\mathsf{d}u\right)\psi_n^*(t)\\
&= \sum_{n\in\mathbb{Z}}\left(\int_{u\in\mathbb{R}}\mathsf{R}_{xx}(t,u)\psi_n(u)\,\mathsf{d}u\right)\psi_n^*(t)\\
&= \sum_{n\in\mathbb{Z}}\lambda_n\psi_n(t)\psi_n^*(t)\\
&= \sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}\left[\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\left(\sum_{m\in\mathbb{Z}}\dot{x}_m\psi_m(t)\right)^*\right] &= \mathsf{E}\left[\sum_{n\in\mathbb{Z}}\int_{u\in\mathbb{R}}x(u)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\left(\sum_{m\in\mathbb{Z}}\int_v x(v)\psi_m^*(v)\,\mathsf{d}v\psi_m(t)\right)^*\right]\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\int_v\mathsf{E}\left[x(u)x^*(v)\right]\psi_m(v)\,\mathsf{d}v\right)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\psi_m^*(t)\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\int_v\mathsf{R}_{xx}(u,v)\psi_m(v)\,\mathsf{d}v\right)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\psi_m^*(t)\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\big(\lambda_m\psi_m(u)\big)\psi_n^*(u)\,\mathsf{d}u\psi_n(t)\psi_m^*(t)\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\left(\int_{u\in\mathbb{R}}\psi_m(u)\psi_n^*(u)\,\mathsf{d}u\right)\psi_n(t)\psi_m^*(t)\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\bar{\delta}_{mn}\psi_n(t)\psi_m^*(t)\\
&= \sum_{n\in\mathbb{Z}}\lambda_n\psi_n(t)\psi_n^*(t)\\
&= \sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2
\end{aligned}$$

2. Using the previous two results, we can prove the following:

$$\begin{aligned}
&\mathsf{E}\left\{\left|x(t)-\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right|^2\right\}\\
&= \mathsf{E}\left[\left[x(t)-\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right]\left[x(t)-\sum_{m\in\mathbb{Z}}\dot{x}_m\psi_m(t)\right]^*\right]\\
&= \mathsf{E}\left[x(t)x^*(t)-x(t)\left(\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right)^*-x^*(t)\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)+\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\left(\sum_{m\in\mathbb{Z}}\dot{x}_m\psi_m(t)\right)^*\right]
\end{aligned}$$

$$= \mathsf{E}\left[x(t)x^*(t)\right] - \mathsf{E}\left[x(t)\left(\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\right)^*\right] - \mathsf{E}\left[x^*(t)\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\right] + \mathsf{E}\left[\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\left(\sum_{m\in\mathbb{Z}} \dot{x}_m\psi_m(t)\right)^*\right]$$

by *Mercer's Theorem*(Theorem D.5 page 172)

$$= \sum_{n\in\mathbb{Z}} \lambda_n|\psi_n(t)|^2 - \sum_{n\in\mathbb{Z}} \lambda_n\left|\psi_n(t)\right|^2 - \left(\sum_{n\in\mathbb{Z}} \lambda_n\left|\psi_n(t)\right|^2\right)^* + \sum_{n\in\mathbb{Z}} \lambda_n\left|\psi_n(t)\right|^2$$

$$= \sum_{n\in\mathbb{Z}} \lambda_n|\psi_n(t)|^2 - \sum_{n\in\mathbb{Z}} \lambda_n\left|\psi_n(t)\right|^2 - \sum_{n\in\mathbb{Z}} \lambda_n\left|\psi_n(t)\right|^2 + \sum_{n\in\mathbb{Z}} \lambda_n\left|\psi_n(t)\right|^2$$

$$= 0$$

Part II

Statistical Processing

CHAPTER 4

OPERATIONS ON RANDOM VARIABLES

4.1 Functions of one random variable

Proposition 4.1. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a probability space, and* X *a* RANDOM VARIABLE *with* CUMULATIVE DISTRIBUTION FUNCTION $\mathsf{c_x}(x)$.

PRP
$$\left\{\begin{array}{l}\mathsf{X} \textit{ is } \text{UNIFORMLY DISTRIBUTED} \\ \text{(Definition C.1 page 161)}\end{array}\right\} \iff \mathsf{c_x}(x) = \left\{\begin{array}{ll} 0 & \textit{for } x < 0 \\ x & \textit{for } 0 < x \le 1 \\ 1 & \textit{otherwise}\end{array}\right\}$$

Theorem 4.1 (Probability integral transform). [1] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a probability space. Let* X *be a* RANDOM VARIABLE *with* PROBABILITY DENSITY FUNCTION $\mathsf{p_x}(x)$ *and* CUMULATIVE DISTRIBUTION FUNCTION $\mathsf{c_x}(x)$. *Let* Y *be a* RANDOM VARIABLE CUMULATIVE DISTRIBUTION FUNCTION $\mathsf{c_Y}(y)$.

THM
$$\left\{\begin{array}{lll} (1). & \mathsf{Y} = \mathsf{c_x}(\mathsf{X}) & \textit{and} \\ (2). & \mathsf{p_x}(x) \textit{ is } \text{CONTINUOUS} & \end{array}\right\} \implies \left\{\begin{array}{l}\mathsf{Y} \textit{ is } \text{UNIFORMLY DISTRIBUTED} \\ \text{(Definition C.1 page 161)}\end{array}\right\}$$

✎PROOF:

$$\begin{array}{lll}
\mathsf{c_Y}(y) \triangleq \mathsf{P}\{\mathsf{Y} \le y\} & \text{by definition of } \textit{cdf} & \text{(Definition B.2 page 156)} \\
= \mathsf{P}\left\{\mathsf{c_x}(\mathsf{X}) \le y\right\} & \text{by hypothesis (1)} & \\
= \mathsf{P}\left\{\mathsf{X} \le \mathsf{c_x}^{-1}(y)\right\} & \text{by hypothesis (2) and} & \text{Proposition A.2 page 150} \\
\triangleq \mathsf{c_x}\left[\mathsf{c_x}^{-1}(y)\right] & \text{by definition of } \textit{cdf} & \text{(Definition B.2 page 156)} \\
= y & & \\
\implies \mathsf{Y} \text{ is } \textit{uniformly distributed} & \text{by} & \text{Proposition 4.1 page 25}
\end{array}$$

✏

Theorem 4.2 (Inverse probability integral transform). [2] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a probability space. Let* X *be a* RANDOM VARIABLE *with* PROBABILITY DENSITY FUNCTION $\mathsf{p_x}(x)$ *and* CUMULATIVE DISTRIBUTION FUNCTION $\mathsf{c_x}(x)$. *Let* Y *be a* RANDOM VARIABLE CUMULATIVE DISTRIBUTION FUNCTION $\mathsf{c_Y}(y)$.

THM
$$\left\{\begin{array}{lll} (1). & \mathsf{Y} = \mathsf{c_z}^{-1}(\mathsf{X}) & \textit{and} \\ (2). & \mathsf{Y} \textit{ is } \text{UNIFORMLY DISTRIBUTED} & \textit{and} \\ (3). & \mathsf{p_z}(z) \textit{ is } \text{CONTINUOUS} & \end{array}\right\} \implies \left\{\begin{array}{l}\mathsf{p_Y}(y) = \mathsf{p_z}(y) \\ (\mathsf{Y} \textit{ has distribution } \mathsf{p_z}(y))\end{array}\right\}$$

[1] Angus (1994), Roussas (2014) page 232 ⟨Theorem 10⟩, Devroye (1986) page 28 ⟨Theorem 2.1⟩
[2] Devroye (1986) page 28 ⟨Theorem 2.1⟩, Balakrishnan and Lai (2009) page 624 ⟨14.2.1 Introduction⟩

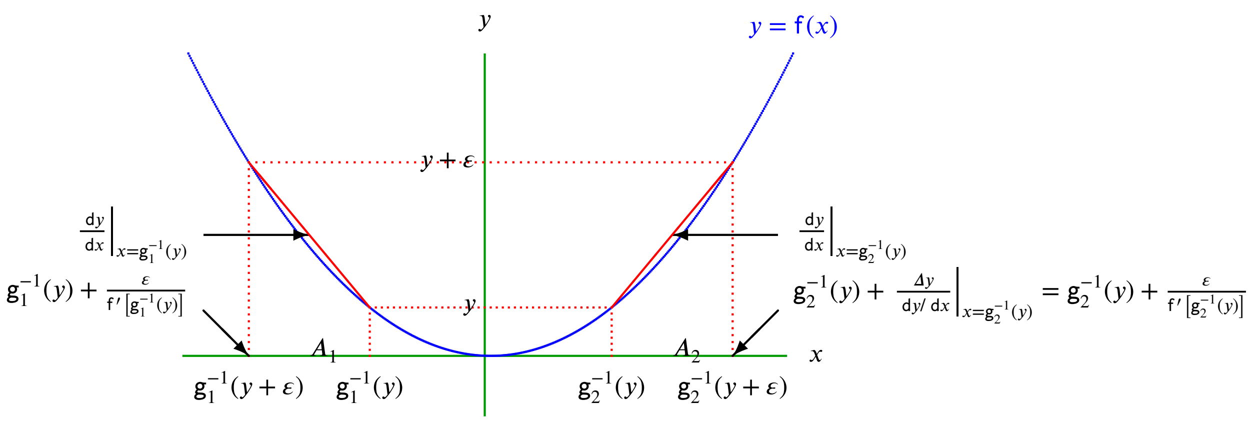


Figure 4.1: $\mathsf{Y} = \mathsf{f}(\mathsf{X})$

✎PROOF:

$\mathsf{c}_\mathsf{Y}(y) \triangleq \mathsf{P}\{\mathsf{Y} \le y\}$	by definition of c_Y	(Definition B.2 page 156)
$= \mathsf{P}\left\{\mathsf{c}_\mathsf{Z}^{-1}(\mathsf{X}) \le y\right\}$	by hypothesis (1)	
$= \mathsf{P}\left\{\mathsf{X} \le \mathsf{c}_\mathsf{Z}(y)\right\}$	by hypothesis (3) and	Proposition A.2 page 150
$\triangleq \mathsf{c}_\mathsf{X}\left[\mathsf{c}_\mathsf{Z}(y)\right]$	by definition of c_X	(Definition B.2 page 156)
$= \mathsf{c}_\mathsf{Z}(y)$	because $0 \le \mathsf{c}_\mathsf{Z}(y) \le 1$ and by	Proposition 4.1 page 25
$\implies \mathsf{p}_\mathsf{Y}(y) = \mathsf{p}_\mathsf{Z}(y)$	(Y has the distribution of Z)	

✏

Definition 4.1. [3] *Let $\mathsf{f}(x)$ be a* DIFFERENTIABLE FUNCTION *in* $\mathbb{R}^\mathbb{R}$.

DEF: *A point $p \in \mathbb{R}$ is a* ***critical point*** *of $\mathsf{f}(x)$ if* $\mathsf{f}'(p) = 0$.

Theorem 4.3. [4] *Let X and Y be* RANDOM VARIABLES *in* $\mathbb{R}^\mathbb{R}$. *Let f be a* DIFFERENTIABLE FUNCTION *in* $\mathbb{R}^\mathbb{R}$ *with N* CRITICAL POINTS *(Definition 4.1 page 26). Let the range of X be partitioned into $N+1$ partitions $\left\{A_n | n = 1, 2, \ldots, N+1\right\}$ with partition boundaries set at the N* CRITICAL POINTS *of $\mathsf{f}(x)$—as illustrated in Figure 4.1 (page 26). Let $\mathsf{g}_n(x) \triangleq \mathsf{f}(x)$ but with domain restricted to $x \in A_n$.*

THM:
$$\left\{\begin{array}{lll} (1). & \mathsf{Y} = \mathsf{f}(\mathsf{X}) & \text{and} \\ (2). & \mathsf{f} \text{ is DIFFERENTIABLE} & \end{array}\right\} \implies \left\{\mathsf{p}_\mathsf{Y}(y) = \sum_{n=1}^{N+1} \frac{\mathsf{p}_\mathsf{X}\left(\mathsf{g}_n^{-1}(y)\right)}{\left|\mathsf{f}'\left(\mathsf{g}_n^{-1}(y)\right)\right|}\right\}$$

✎PROOF:

1. The problem with a function $\mathsf{f}(x)$ with at least $N = 1$ critical point is that $\mathsf{f}^{-1}(y)$ is *not invertible*. That is, $\mathsf{f}^{-1}(y)$ has more than one solution (and thus the *relation* $\mathsf{f}^{-1}(y)$ is not a *function*). However, note that in each partition A_n, $\mathsf{f}(x)$ is *invertible* and thus $\mathsf{f}^{-1}(y)$ in that partition has a *unique* solution. Thus, each $\mathsf{g}_n(x)$ *is* invertible in it's domain (and each $\mathsf{g}_n^{-1}(y)$ exists as a function).

[3] Callahan (2010) page 189 ⟨Definition 6.1⟩
[4] Papoulis (1984) pages 95–96 ⟨"Fundamental Theorem"⟩, Papoulis (1990) page 157 ⟨"Fundamental Theorem"⟩ Papoulis (1991), page 93, Proakis (2001), page 30

2. Using item (1), the remainder of the proof follows …

$$\begin{aligned}
\mathsf{p}_\mathsf{Y}(y) &\triangleq \frac{\mathrm{d}}{\mathrm{d}y}\mathsf{P}\{\mathsf{Y}\le y\} && \text{by definition of } \mathsf{p}_\mathsf{Y} \text{ (Definition B.2 page 156)}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\mathsf{P}\{\mathsf{f}(\mathsf{X})\le y\} && \text{by hypothesis (1)}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{f}(\mathsf{X})\le y | \mathsf{X}\in A_n\right\} && \text{by \textit{sum of products} (Theorem A.3 page 149)}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{f}(\mathsf{X})\le y \mid \mathsf{X}\in A_n\right\}\mathsf{P}\left\{\mathsf{X}\in A_n\right\} && \text{by definition of } \mathsf{P}\{X|Y\} \text{ (Definition A.4 page 148)}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{g}_n(\mathsf{X})\le y \;\middle|\; \mathsf{X}\in A_n\right\}\mathsf{P}\left\{\mathsf{X}\in A_n\right\} && \text{by definition of } \mathsf{g}_n(x)\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\left\{\begin{array}{ll}\sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y) \;\middle|\; \mathsf{X}\in A_n\right\}\mathsf{P}\left\{\mathsf{X}\in A_n\right\} & \text{for } \mathsf{f}'(x)\ge 0\\ \sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{X}\ge \mathsf{g}_n^{-1}(y) \;\middle|\; \mathsf{X}\in A_n\right\}\mathsf{P}\left\{\mathsf{X}\in A_n\right\} & \text{otherwise}\end{array}\right\} && \text{by item (1)}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\left\{\begin{array}{ll}\sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y) \;\middle|\; \mathsf{X}\in A_n\right\}\mathsf{P}\left\{\mathsf{X}\in A_n\right\} & \text{for } \mathsf{f}'(x)\ge 0\\ \sum_{n=1}^{N+1}\left[1-\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y) \;\middle|\; \mathsf{X}\in A_n\right\}\mathsf{P}\left\{\mathsf{X}\in A_n\right\}\right] & \text{otherwise}\end{array}\right\}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\left\{\begin{array}{ll}\sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y) | \mathsf{X}\in A_n\right\} & \text{for } \mathsf{f}'(x)\ge 0\\ \sum_{n=1}^{N+1}\left[1-\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y) | \mathsf{X}\in A_n\right\}\right] & \text{otherwise}\end{array}\right\} && \text{by definition of } \mathsf{P}\{X|Y\}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\left\{\begin{array}{ll}\sum_{n=1}^{N+1}\left(\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y)\right\}-\mathsf{P}\left\{\mathsf{X}<\min A_{n-1}\right\}\right) & \text{for } \mathsf{f}'(x)\ge 0\\ \sum_{n=1}^{N+1}\left[1-\left(\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y)\right\}-\mathsf{P}\left\{\mathsf{X}<\min A_{n-1}\right\}\right)\right] & \text{otherwise}\end{array}\right\}\\
&= \frac{\mathrm{d}}{\mathrm{d}y}\left\{\begin{array}{ll}\sum_{n=1}^{N+1}\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y)\right\} & \text{for } \mathsf{f}'(x)\ge 0\\ \sum_{n=1}^{N+1}\left[1-\mathsf{P}\left\{\mathsf{X}\le \mathsf{g}_n^{-1}(y)\right\}\right] & \text{otherwise}\end{array}\right\} && \text{because } \frac{\mathrm{d}}{\mathrm{d}y}\mathsf{P}\{\mathsf{X}<\text{a constant}\}=0\\
&= \left\{\begin{array}{ll}\sum_{n=1}^{N+1}\frac{\mathrm{d}}{\mathrm{d}y}\mathsf{c}_\mathsf{X}\left[\mathsf{g}_n^{-1}(y)\right] & \text{for } \mathsf{f}'(x)\ge 0\\ \sum_{n=1}^{N+1}\frac{\mathrm{d}}{\mathrm{d}y}\left[1-\mathsf{c}_\mathsf{X}\left(\mathsf{g}_n^{-1}(y)\right)\right] & \text{otherwise}\end{array}\right\} && \text{by \textit{linearity} of } \frac{\mathrm{d}}{\mathrm{d}y} \text{ operator}\\
&= \left\{\begin{array}{ll}\sum_{n=1}^{N+1}\mathsf{p}_\mathsf{X}\left[\mathsf{g}_n^{-1}(y)\right]\frac{\mathrm{d}}{\mathrm{d}y}\left[\mathsf{g}_n^{-1}(y)\right] & \text{for } \mathsf{f}'(x)\ge 0\\ \sum_{n=1}^{N+1}\left[-\mathsf{p}_\mathsf{X}\left[\mathsf{g}_n^{-1}(y)\right]\frac{\mathrm{d}}{\mathrm{d}y}\left[\mathsf{g}_n^{-1}(y)\right]\right] & \text{otherwise}\end{array}\right\} && \text{by definition of } \mathsf{p}_\mathsf{X} \text{ (Definition B.2 page 156) and the \textit{chain rule}}
\end{aligned}$$

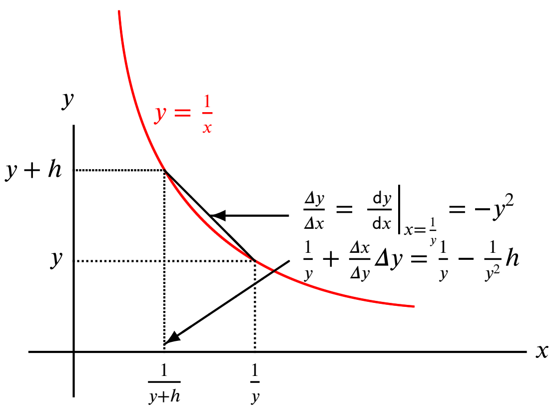


Figure 4.2: $\mathsf{Y} = \frac{1}{\mathsf{X}}$

$$= \sum_{n=1}^{N+1} \mathsf{p}_{\mathsf{X}}\left(\mathsf{g}_n^{-1}(y)\right)\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[\mathsf{g}_n^{-1}(y)\right]\right|$$

$$= \sum_{n=1}^{N+1} \frac{\mathsf{p}_{\mathsf{X}}\left(\mathsf{g}_n^{-1}(y)\right)}{\left|\mathsf{f}'\left(\mathsf{g}_n^{-1}(y)\right)\right|} \qquad \text{by Lemma ?? page ??}$$

✏

Corollary 4.1. [5] *Let* X *and* Y *be* RANDOM VARIABLES *in* $\mathbb{R}^{\mathbb{R}}$. *Let* $a, b \in \mathbb{R}$.

COR
$$\left\{\begin{array}{lll} (1). & \mathsf{Y} = a\mathsf{X} + b & \text{and} \\ (2). & a \neq 0 & \end{array}\right\} \implies \left\{\mathsf{p}_{\mathsf{Y}}(y) = \frac{1}{|a|}\mathsf{p}_{\mathsf{X}}\left(\frac{y-b}{a}\right)\right\}$$

✎PROOF:

1. Note that $\mathsf{f}(x) = ax + b$ is a *differentiable function* with $N = 0$ *critical points* and $\mathsf{f}'(x) = a$.
2. The inverse of $\mathsf{f}(x)$ is $\mathsf{g}_1(y) = \mathsf{f}^{-1}(y) = \frac{y-b}{a}$.
3. It follows that

$$\begin{aligned}
\mathsf{p}_{\mathsf{Y}}(y) &= \sum_{n=1}^{N+1} \frac{\mathsf{p}_{\mathsf{X}}\left(\mathsf{g}_n^{-1}(y)\right)}{\left|\mathsf{f}'\left(\mathsf{g}_n^{-1}(y)\right)\right|} && \text{by Theorem 4.3 (page 26)} \\
&= \frac{\mathsf{p}_{\mathsf{X}}\left(\mathsf{f}^{-1}(y)\right)}{\left|\mathsf{f}'\left(\mathsf{f}^{-1}(y)\right)\right|} && \text{because } N = 0 \\
&= \frac{\mathsf{p}_{\mathsf{X}}\left(\mathsf{f}^{-1}(y)\right)}{|a|} && \text{by item (1)} \\
&= \frac{1}{|a|}\mathsf{p}_{\mathsf{X}}\left(\frac{y-b}{a}\right) && \text{by item (2)}
\end{aligned}$$

✏

[5] Papoulis (1984) page 96 ⟨"Illustrations" 1⟩, Papoulis (1991), page 95, Proakis (2001), page 29

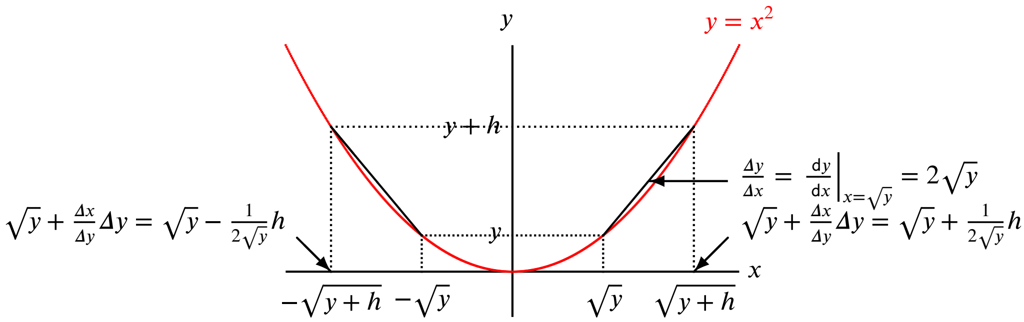


Figure 4.3: $\mathsf{Y} = rvX^2$

Corollary 4.2. [6]

$$\left\{ \mathsf{Y} = \frac{1}{\mathsf{X}} \right\} \implies \left\{ \mathsf{p}_{\mathsf{Y}}(y) = \frac{1}{y^2}\mathsf{p}_{\mathsf{X}}\left(\frac{1}{y}\right) \quad \textit{for } y > 0 \right\}$$

✎PROOF:

1. Note that $\mathsf{f}(x) = 1/x$ is a *differentiable function* in $x > 0$ with $N = 0$ *critical points* and $\mathsf{f}'(x) = -1/x^2$.
2. The inverse of $\mathsf{f}(x)$ is $\mathsf{g}_1(y) = \mathsf{f}^{-1}(y) = \frac{1}{y}$.
3. It follows that

$$\begin{aligned}
\mathsf{p}_{\mathsf{Y}}(y) &= \sum_{n=1}^{N+1} \frac{\mathsf{p}_{\mathsf{X}}\left(\mathsf{g}_n^{-1}(y)\right)}{\left|\mathsf{f}'\left(\mathsf{g}_n^{-1}(y)\right)\right|} && \text{by Theorem 4.3 (page 26)}\\
&= \frac{\mathsf{p}_{\mathsf{X}}\left(\mathsf{f}^{-1}(y)\right)}{\left|\mathsf{f}'\left(\mathsf{f}^{-1}(y)\right)\right|} && \text{because } N = 0\\
&= \frac{1}{\left|-1/(1/y)^2\right|}\mathsf{p}_{\mathsf{X}}\left(\frac{1}{y}\right)\\
&= \frac{1}{y^2}\mathsf{p}_{\mathsf{X}}\left(\frac{1}{y}\right)
\end{aligned}$$

✏

Corollary 4.3. [7] *Let* X *and* Y *be* RANDOM VARIABLES.

$$\left\{ \mathsf{Y} = \mathsf{X}^2 \right\} \implies \left\{ \mathsf{p}_{\mathsf{Y}}(y) = \frac{1}{2\sqrt{y}}\left[\mathsf{p}_{\mathsf{X}}(-\sqrt{y}) + \mathsf{p}_{\mathsf{X}}(\sqrt{y})\right] \right\}$$

✎PROOF:

1. The roots of $y = x^2$ are $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$.
2. The derivative of $\mathsf{f}(x) \triangleq y = x^2$ is $\mathsf{f}'(x) = 2x$.

[6] Papoulis (1984) page 97 ⟨Example 5-10⟩, Papoulis (1991), page 94
[7] Papoulis (1984) page 95 ⟨Example 5-9⟩, Devroye (1986) page 27 ⟨Example 4.4⟩, Papoulis (1991), page 95, Proakis (2001), page 29

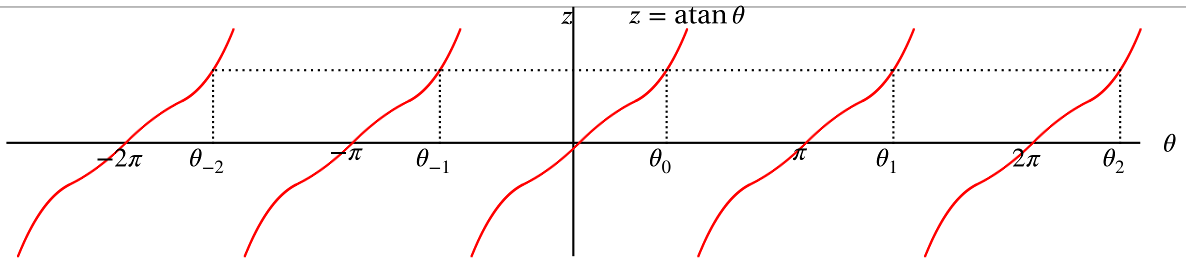


Figure 4.4: $\mathsf{Z} = \tan\Theta$

3. And so it follows that …

$$\begin{aligned}
\mathsf{p}_{\mathsf{Y}}(y) &= \sum_{n=1}^{N} \frac{\mathsf{p}_{\mathsf{X}}(x_n)}{|\mathsf{f}'(x_n)|} && \text{by Theorem 4.3 page 26}\\
&= \frac{\mathsf{p}_{\mathsf{X}}(x_1)}{|\mathsf{f}'(x_1)|} + \frac{\mathsf{p}_{\mathsf{X}}(x_2)}{|\mathsf{f}'(x_2)|} && \text{by definition of } \textstyle\sum\\
&= \frac{\mathsf{p}_{\mathsf{X}}(-\sqrt{y})}{|\mathsf{f}'(-\sqrt{y})|} + \frac{\mathsf{p}_{\mathsf{X}}(\sqrt{y})}{|\mathsf{f}'(\sqrt{y})|} && \text{by item (1)}\\
&= \frac{\mathsf{p}_{\mathsf{X}}(-\sqrt{y})}{2\sqrt{y}} + \frac{\mathsf{p}_{\mathsf{X}}(\sqrt{y})}{2\sqrt{y}} && \text{by item (2)}\\
&= \frac{1}{2\sqrt{y}}\left[\mathsf{p}_{\mathsf{X}}(-\sqrt{y}) + \mathsf{p}_{\mathsf{X}}(\sqrt{y})\right] && \text{by } \textit{linearity} \text{ of + operation}
\end{aligned}$$

✏

Corollary 4.4. [8] *Let* $\mathsf{Z} = \tan\Theta$. *Then*

COR

$$\{\mathsf{Z} = \tan\Theta\} \implies \left\{ \mathsf{p}_{\mathsf{Z}}(z) = \frac{1}{1+z^2}\sum_{n\in\mathbb{Z}} \mathsf{p}_{\theta}(\operatorname{atan}(z) + n\pi) \right\}$$

✎PROOF:

1. The roots of $z = \tan\theta$ are $\{\theta_n = \operatorname{atan} z + n\pi | n \in \mathbb{Z}\}$.

2. The derivative of $z = \tan\theta$ is $\mathsf{f}'(\theta) = \sec^2\theta$.

3. It follows that

$$\begin{aligned}
\mathsf{p}_{\mathsf{Z}}(z) &= \sum_{n=1}^{N} \frac{\mathsf{p}_{\theta}(\theta_n)}{|\mathsf{f}'(\theta_n)|}\\
&= \sum_{n} \frac{\mathsf{p}_{\theta}(\operatorname{atan} z + n\pi)}{|\mathsf{f}'(\operatorname{atan} z + n\pi)|}\\
&= \sum_{n} \frac{\mathsf{p}_{\theta}(\operatorname{atan} z + n\pi)}{|\sec^2(\operatorname{atan} z + n\pi)|}\\
&= \sum_{n} \cos^2(\operatorname{atan} z + n\pi)\mathsf{p}_{\theta}(\operatorname{atan} z + n\pi)
\end{aligned}$$

[8] Papoulis (1991), pages 99–100

$$= \cos^2(\operatorname{atan} z) \sum_n \mathsf{p}_\theta(\operatorname{atan} z + n\pi)$$
$$= \frac{1}{1+z^2} \sum_n \mathsf{p}_\theta(\operatorname{atan} z + n\pi)$$

⇨

4.2 Functions of two random variables

Theorem 4.4. [9] *Let* X, Y, *and* Z *be* RANDOM VARIABLES. *Let* $\star$ *be the* CONVOLUTION *operator* (Definition N.3 page 312).

THM
$$\left\{\begin{array}{lll} (1). & \mathsf{Z} \triangleq \mathsf{X} + \mathsf{Y} & \text{and} \\ (2). & \mathsf{X} \text{ and } \mathsf{Y} \text{ are INDEPENDENT} & \text{(Definition A.3 page 147)} \end{array}\right\} \implies \left\{\mathsf{p}_\mathsf{Z}(z) = \mathsf{p}_\mathsf{X}(z) \star \mathsf{p}_\mathsf{Y}(z)\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{p}_\mathsf{Z}(z) &\triangleq \frac{\mathsf{d}}{\mathsf{d}z}\mathsf{c}_\mathsf{Z}(z) && \text{by definition of } \mathsf{p}_\mathsf{Z} && \text{(Definition B.2 page 156)}\\
&\triangleq \frac{\mathsf{d}}{\mathsf{d}z}\mathsf{P}\{\mathsf{Z} \le z\} && \text{by definition of } \mathsf{c}_\mathsf{Z} && \text{(Definition B.2 page 156)}\\
&= \frac{\mathsf{d}}{\mathsf{d}z}\mathsf{P}\{\mathsf{X}+\mathsf{Y} \le z\} && \text{by hypothesis (1)}\\
&= \frac{\mathsf{d}}{\mathsf{d}z}\lim_{\varepsilon\to 0}\sum_{n\in\mathbb{Z}} \mathsf{P}\left\{\mathsf{X}+\mathsf{Y} \le z | y + n\varepsilon < \mathsf{Y} \le y + (n+1)\varepsilon\right\} && \text{by \textit{sum of products}} && \text{(Theorem A.3 page 149)}\\
&= \frac{\mathsf{d}}{\mathsf{d}z}\int_{y\in\mathbb{R}} \mathsf{P}\left\{\mathsf{X}+\mathsf{Y} \le z \mid \mathsf{Y} = y\right\}\mathsf{p}_\mathsf{Y}(y)\,\mathsf{d}y && \text{by definiton of } \mathsf{P}\{\mathsf{X} \mid \mathsf{Y}\} && \text{(Definition A.4 page 148)}\\
&= \frac{\mathsf{d}}{\mathsf{d}z}\int_{y\in\mathbb{R}} \mathsf{P}\left\{\mathsf{X} \le z - y \mid \mathsf{Y} = y\right\}\mathsf{p}_\mathsf{Y}(y)\,\mathsf{d}y\\
&= \frac{\mathsf{d}}{\mathsf{d}z}\int_{y\in\mathbb{R}} \mathsf{P}\{\mathsf{X} \le z - y\}\mathsf{p}_\mathsf{Y}(y)\,\mathsf{d}y && \text{by hypothesis (2)}\\
&\triangleq \frac{\mathsf{d}}{\mathsf{d}z}\int_{y\in\mathbb{R}} \mathsf{c}_\mathsf{X}(z-y)\mathsf{p}_\mathsf{Y}(y)\,\mathsf{d}y && \text{by definition of } \mathsf{c}_\mathsf{X} && \text{(Definition B.2 page 156)}\\
&= \int_{y\in\mathbb{R}} \frac{\mathsf{d}}{\mathsf{d}z}\left[\mathsf{c}_\mathsf{X}(z-y)\mathsf{p}_\mathsf{Y}(y)\right]\mathsf{d}y && \text{by \textit{linearity} of } \frac{\mathsf{d}}{\mathsf{d}z}\\
&= \int_{y\in\mathbb{R}} \left[\frac{\mathsf{d}}{\mathsf{d}z}\mathsf{c}_\mathsf{X}(z-y)\right]\mathsf{p}_\mathsf{Y}(y)\,\mathsf{d}y && \text{because } y \text{ is fixed inside the integral}\\
&\triangleq \int_y \mathsf{p}_\mathsf{X}(z-y)\mathsf{p}_\mathsf{Y}(y)\,\mathsf{d}y && \text{by definition of } \mathsf{p}_\mathsf{X} && \text{(Definition B.2 page 156)}\\
&= \mathsf{p}_\mathsf{X}(z) \star \mathsf{p}_\mathsf{Y}(z) && \text{by definition of } \star && \text{(Definition N.3 page 312)}
\end{aligned}$$

⇨

Theorem 4.5. *Let*

- X_1 *and* X_2 *be random variables with joint distribution* $\mathsf{p}_{\mathsf{X}_1,\mathsf{X}_2}(x_1, x_2)$
- $\mathsf{Y}_1 = \mathsf{f}_1(x_1, x_2)$ *and* $\mathsf{Y}_2 = \mathsf{f}_2(x_1, x_2)$

[9] Papoulis (1990) page 160 ⟨Example 5.16⟩

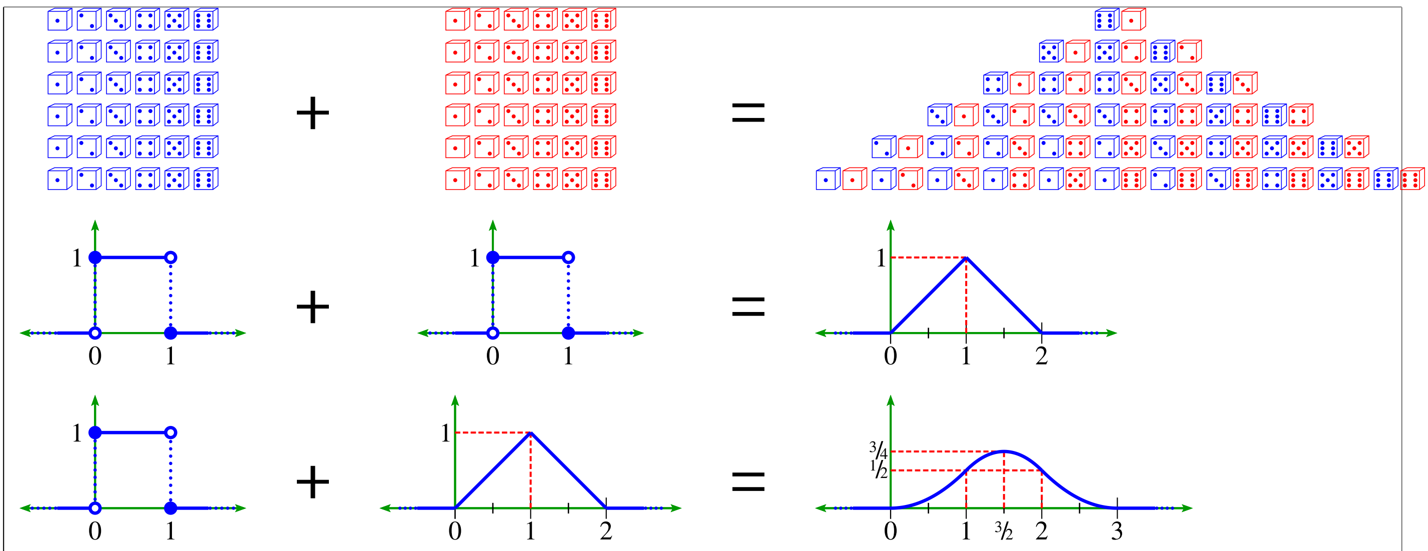


Figure 4.5: **Sum** of random variables yields **convolution** of pdfs (Theorem 4.4 page 31)

Then the joint distribution of Y_1 *and* Y_2 *is*

THM

$$\mathsf{p}_{\mathsf{Y}_1,\mathsf{Y}_2}(y_1,y_2) = \frac{\mathsf{p}_{\mathsf{X}_1,\mathsf{X}_2}(x_1,x_2)}{|J(x_1,x_2)|} = \frac{\mathsf{p}_{\mathsf{X}_1,\mathsf{X}_2}(x_1,x_2)}{\begin{vmatrix} \frac{\partial \mathsf{f}_1}{\partial x_1} & \frac{\partial \mathsf{f}_1}{\partial x_2} \\ \frac{\partial \mathsf{f}_2}{\partial x_1} & \frac{\partial \mathsf{f}_2}{\partial x_2} \end{vmatrix}} = \frac{\mathsf{p}_{\mathsf{X}_1,\mathsf{X}_2}(x_1,x_2)}{\frac{\partial \mathsf{f}_1}{\partial x_1}\frac{\partial \mathsf{f}_2}{\partial x_2} - \frac{\partial \mathsf{f}_1}{\partial x_2}\frac{\partial \mathsf{f}_2}{\partial x_1}}$$

Proposition 4.2. *Let* X *and* Y *be random variables with joint distribution* $\mathsf{p}_{\mathsf{XY}}(x,y)$ *and*

$$R^2 \triangleq X^2 + Y^2 \qquad\qquad \Theta \triangleq \operatorname{atan}\frac{\mathsf{Y}}{\mathsf{X}}.$$

Then

PRP

$$\mathsf{p}_{\mathsf{R},\Theta}(r,\theta) = r\ \mathsf{p}_{\mathsf{XY}}(r\cos\theta, r\sin\theta)$$

✎PROOF:

$$\begin{aligned}
\mathsf{p}_{\mathsf{R},\Theta}(r,\theta) &= \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{|J(x,y)|} = \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{\begin{vmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}} = \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{\begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}} \\
&= \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{\frac{x}{\sqrt{x^2+y^2}}\frac{x}{x^2+y^2} - \frac{y}{\sqrt{x^2+y^2}}\frac{-y}{x^2+y^2}} \\
&= \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{\frac{x^2+y^2}{(x^2+y^2)^{3/2}}} \\
&= \mathsf{p}_{\mathsf{XY}}(x,y)\frac{(x^2+y^2)^{3/2}}{x^2+y^2} \\
&= \mathsf{p}_{\mathsf{XY}}(r\cos\theta, r\sin\theta)\frac{r^3}{r^2} \\
&= r\ \mathsf{p}_{\mathsf{XY}}(r\cos\theta, r\sin\theta)
\end{aligned}$$

⇨

Proposition 4.3. *Let* $\mathsf{X} \sim \mathsf{N}\left(0,\sigma^2\right)$ *and* $\mathsf{Y} \sim \mathsf{N}\left(0,\sigma^2\right)$ *be independent random variables and*

$$R^2 \triangleq X^2 + Y^2 \qquad\qquad \Theta \triangleq \operatorname{atan}\frac{\mathsf{Y}}{\mathsf{X}}.$$

Then

PRP

1. *R and Θ are independent with joint distribution* $\mathsf{p}_{R,\Theta}(r,\theta) = \mathsf{p}_R(r)\mathsf{p}_\theta(\theta)$
2. *R has Rayleigh distribution* $\mathsf{p}_R(r) = \frac{r}{\sigma^2}\exp\frac{r^2}{-2\sigma^2}$
3. *Θ has uniform distribution* $\mathsf{p}_\theta(\theta) = \frac{1}{2\pi}$

✎PROOF:

$$
\begin{aligned}
\mathsf{p}_{R,\Theta}(r,\theta) &= r\ \mathsf{p}_{XY}(r\cos\theta, r\sin\theta) && \text{by Proposition 4.2 (page 32)}\\
&= r\ \mathsf{p}_X(r\cos\theta)\ \mathsf{p}_Y(r\sin\theta) && \text{by independence hypothesis}\\
&= r\ \frac{1}{\sqrt{2\pi\sigma^2}}\exp\frac{(r\cos\theta-0)^2}{-2\sigma^2}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\frac{(r\sin\theta-0)^2}{-2\sigma^2}\\
&= \frac{1}{2\pi\sigma^2}\ r\ \exp\frac{r^2(\cos^2\theta+\sin^2\theta)}{-2\sigma^2}\\
&= \frac{1}{2\pi\sigma^2}\ r\ \exp\frac{r^2}{-2\sigma^2}\\
&= \left[\frac{1}{2\pi}\right]\left[\frac{r}{\sigma^2}\exp\frac{r^2}{-2\sigma^2}\right]
\end{aligned}
$$

⇨

Proposition 4.4. *Let* X *and* Y *be* RANDOM VARIABLES *with covariance* σ_{xy} *on a* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$.

PRP

$$
\left\{\begin{array}{lll}
\text{(A).} & \mathsf{X}\text{ is GAUSSIAN with }\mathsf{N}\left(\mu_X, \sigma_X^2\right) & \text{and}\\
\text{(B).} & \mathsf{X}\text{ is GAUSSIAN with }\mathsf{N}\left(\mu_Y, \sigma_Y^2\right) & \text{and}\\
\text{(C).} & \sigma_{xy} = \operatorname{cov}[\mathsf{X},\mathsf{Y}] &
\end{array}\right\}
\implies
\left\{\mathsf{P}\{\mathsf{X} > \mathsf{Y}\} = \mathsf{Q}\left(\frac{-\mu_X+\mu_Y}{\sigma_X^2+\sigma_Y^2-2\sigma_{xy}}\right)\right\}
$$

✎PROOF: Because X and Y are jointly Gaussian, their linear combination $Z = rvX - Y$ is also Gaussian. A Gaussian distribution is completely defined by its mean and variance. So, to determine the distribution of Z, we just have to determine the mean and variance of Z.

$$
\begin{aligned}
\mathsf{E}Z &= \mathsf{E}X - \mathsf{E}Y\\
&= \mu_X - \mu_Y
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{var} Z &= \mathsf{E}Z^2 - (\mathsf{E}Z)^2\\
&= \mathsf{E}(X-Y)^2 - (\mathsf{E}X-\mathsf{E}Y)^2\\
&= \mathsf{E}(X^2 - 2XY + Y^2) - [(\mathsf{E}X)^2 - 2\mathsf{E}X\mathsf{E}Y + (\mathsf{E}Y)^2]\\
&= [\mathsf{E}X^2 - (\mathsf{E}X)^2] + [Y^2 - (\mathsf{E}Y)^2] - 2[\mathsf{E}XY - \mathsf{E}X\mathsf{E}Y]\\
&= \operatorname{var} X + \operatorname{var} Y - 2\operatorname{cov}[X,Y]\\
&\triangleq \sigma_X^2 + \sigma_Y^2 - 2\sigma_{xy}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{P}\{X > Y\} &= \mathsf{P}\{X - Y > 0\}\\
&= \mathsf{P}\{Z > 0\}\\
&= \mathsf{Q}\left(\frac{z-\mathsf{E}Z}{\operatorname{var} Z}\right)\Big|_{z=0}\\
&= \mathsf{Q}\left(\frac{0-\mu_X+\mu_Y}{\sigma_X^2+\sigma_Y^2-2\sigma_{xy}}\right)
\end{aligned}
$$

⇨

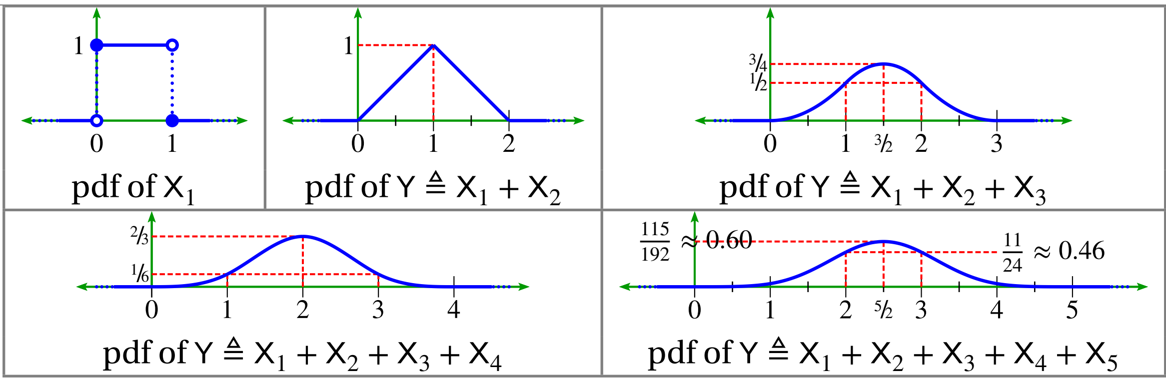


Figure 4.6: The distributions of sums of independent uniformly distributed random variables (Example 4.1 page 34)

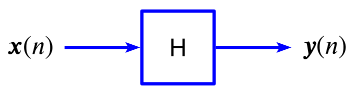
Example 4.1. Let $(\!(X_1, X_2, X_3, \ldots)\!)$ be a *sequence* of *independent* (Definition A.3 page 147) *uniformly distributed* random variables. Let $p_N(x)$ be the *probability density function* of $Y \triangleq \sum_{n=1}^{N} X_n$. Some of these distributions are illustrated in Figure 4.6 (page 34). Note that the distributions of the sequence $(\!(p_1, p_2, p_3, \ldots)\!)$ are all *B-splines* (Definition Q.2 page 343) and all form a *partition of unity*.

CHAPTER 5

OPERATORS ON DISCRETE RANDOM SEQUENCES

5.1 LTI operators on random sequences

Theorem 5.1. [1] *Let* $\boldsymbol{x}(n)$ *be a* RANDOM SEQUENCE *with* MEAN μ_{X} *and* $\boldsymbol{y}(n)$ *a* RANDOM SEQUENCE *with* MEAN μ_{Y}. *Let* $\mathbf{S}$ *be a system with* IMPULSE RESPONSE $\mathsf{h}(n)$, INPUT $\boldsymbol{x}(n)$, *and* OUTPUT $\boldsymbol{y}(n)$.



T H M

$$\left\{\ \mathbf{S} \text{ is } (\textit{LTI})\ \right\} \implies \left\{\begin{array}{llll} (1). & \mu_{\mathsf{Y}}(n) = \sum_{k\in\mathbb{Z}} \mathsf{h}(k)\mu_{\mathsf{X}}(n-k) & \triangleq \mathsf{h}(n) \star \mu_{\mathsf{X}}(n) & \text{and} \\ (2). & \mathsf{R}_{\mathsf{xy}}(n,m) = \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{\mathsf{xx}}(n-k, m+k) & & \text{and} \\ (3). & \mathsf{R}_{\mathsf{yy}}(n,m) = \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{\mathsf{yx}}(n-k, m+k) & & \end{array}\right\}$$

✎PROOF:

$\mu_{\mathsf{Y}}(n) \triangleq \mathsf{E}\left[\boldsymbol{y}(n)\right]$	by definition of μ_{Y}	(Definition 2.2 page 11)
$= \mathsf{E}\left[\sum_{k\in\mathbb{Z}} \mathsf{h}(k)\boldsymbol{x}(n-k)\right]$	by *LTI* hypothesis	
$= \sum_{k\in\mathbb{Z}} \mathsf{h}(k)\mathsf{E}[\boldsymbol{x}(n-k)]$	by *linear* property	
$= \sum_{k\in\mathbb{Z}} \mathsf{h}(k)\mu_{\mathsf{X}}(n-k)$	by definition of μ_{X}	(Definition 2.2 page 11)
$\triangleq \mathsf{h}(n) \star \mu_{\mathsf{X}}(n)$	by definition of *convolution*	(Definition P.3 page 329)

$\mathsf{R}_{\mathsf{xy}}(n,m) \triangleq \mathsf{E}\left[\boldsymbol{x}(n+m)\boldsymbol{y}^*(n)\right]$	by definition of $\mathsf{R}_{\mathsf{xy}}(n,m)$	(Definition 2.2 page 11)
$= \mathsf{E}\left[\boldsymbol{x}(n+m)(\mathsf{h}(n) \star \boldsymbol{x}(n))^*\right]$	by *LTI* hypothesis	
$\triangleq \mathsf{E}\left[\boldsymbol{x}(n+m)\left(\sum_{k\in\mathbb{Z}} h(k)\boldsymbol{x}(n-k)\right)^*\right]$	by definition of *convolution* $\star$	(Definition P.3 page 329)
$= \mathsf{E}\left[\boldsymbol{x}(n+m)\sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\boldsymbol{x}^*(n-k)\right]$	by *distributive* property of $*$-**algebras**	(Definition H.3 page 228)

[1] Papoulis (1991), page 310

$$= \mathsf{E}\left[\sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\boldsymbol{x}(n+m)\boldsymbol{x}^*(n-k)\right]$$ by *distributive* property of $(\mathbb{C}, +, \cdot, 0, 1)$

$$= \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{E}\left[\boldsymbol{x}(n-k+k+m)\boldsymbol{x}^*(n-k)\right]$$ by *linear* property of E (Theorem 1.1 page 4)

$$\triangleq \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{xx}(n-k, m+k)$$ by definition of $\mathsf{R}_{xx}(n,m)$ (Definition 2.2 page 11)

$$\mathsf{R}_{yy}(n,m) \triangleq \mathsf{E}\left[\boldsymbol{y}(n+m)\boldsymbol{y}^*(n)\right]$$ by definition of $\mathsf{R}_{xy}(n,m)$ (Definition 2.2 page 11)

$$= \mathsf{E}\left[\boldsymbol{y}(n+m)(\mathsf{h}(n) \star \boldsymbol{x}(n))^*\right]$$ by *LTI* hypothesis

$$\triangleq \mathsf{E}\left[\boldsymbol{y}(n+m)\left(\sum_{k\in\mathbb{Z}} \mathsf{h}(k)\boldsymbol{x}(n-k)\right)^*\right]$$ by definition of *convolution* (Definition P.3 page 329)

$$= \mathsf{E}\left[\boldsymbol{y}(n+m)\sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\boldsymbol{x}^*(n-k)\right]$$ by *distributive* property of **∗-algebras** (Definition H.3 page 228)

$$= \mathsf{E}\left[\sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\boldsymbol{y}(n+m)\boldsymbol{x}^*(n-k)\right]$$ by *distributive* property of $(\mathbb{C}, +, \cdot, 0, 1)$

$$= \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{E}\left[\boldsymbol{y}(n-k+k+m)\boldsymbol{x}^*(n-k)\right]$$ by *linear* property of E (Theorem 1.1 page 4)

$$\triangleq \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{yx}(n-k, m+k)$$ by definition of $\mathsf{R}_{xy}(n,m)$ (Definition 2.2 page 11)

⇒

5.2 LTI operators on WSS random sequences

Corollary 5.1. *Let* **S** *be the system defined in Theorem 5.1 (page 35).*

COR

(A). **S** *is* LTI *and*
(B). $\boldsymbol{x}(n)$ *is* WSS

$$\Longrightarrow \begin{cases} (1). & \mu_{\mathsf{Y}} = \mu_{\mathsf{X}} \sum_{n\in\mathbb{Z}} \mathsf{h}(k) & \textit{and} \\ (2). & \mathsf{R}_{xy}(m) = \mathsf{R}_{xx}(m) \star \mathsf{h}^*(-m) \triangleq \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{xx}(m+k) & \textit{and} \\ (3). & \mathsf{R}_{yy}(m) = \mathsf{R}_{yx}(m) \star \mathsf{h}^*(-m) \triangleq \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{xy}(m+k) & \textit{and} \\ (4). & \mathsf{R}_{yy}(m) = \mathsf{R}^*_{xx}(m) \star \mathsf{h}(-m) \star \mathsf{h}^*(-m) \end{cases}$$

✎PROOF:

$$\mu_{\mathsf{Y}} = \mu_{\mathsf{Y}}(n)$$ by Proposition 2.1 page 13 and hypothesis (A)

$$= \sum_{n\in\mathbb{Z}} \mathsf{h}(k)\mu_{\mathsf{X}}(n-k)$$ by Theorem 2.1 page 12 and hypothesis (B)

$$= \sum_{n\in\mathbb{Z}} \mathsf{h}(k)\mu_{\mathsf{X}}(0)$$ by Definition 6.1 page 43 and hypothesis (B)

$$= \mu_{\mathsf{X}}(0)\sum_{n\in\mathbb{Z}} \mathsf{h}(k)$$ by *linear* property of $\sum$

$$= \mu_{\mathsf{X}}\sum_{n\in\mathbb{Z}} \mathsf{h}(k)$$ by Proposition 2.1 page 13

[1] Papoulis (1991), page 323

$$
\begin{aligned}
\mathsf{R}_{xy}(m) &\triangleq \mathsf{R}_{xy}(0,m) && \text{by Proposition 2.1 page 13} && \text{and hypothesis (A)}\\
&= \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{xx}(0-k, m+k) && \text{by Theorem 5.1 page 35} && \text{and hypothesis (B)}\\
&= \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{xx}(m+k) && \text{by Proposition 2.1 page 13} && \text{and hypothesis (A)}\\
&= \mathsf{h}^*(-m)\star\mathsf{R}_{xx}(m) && \text{by Proposition P.2 page 330}\\
\mathsf{R}_{yy}(m) &\triangleq \mathsf{R}_{yy}(0,m) && \text{by Proposition 2.1 page 13} && \text{and hypothesis (A)}\\
&= \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{yx}(n-k, m+k) && \text{by Theorem 5.1 page 35} && \text{and hypothesis (B)}\\
&= \sum_{k\in\mathbb{Z}} \mathsf{h}^*(k)\mathsf{R}_{yx}(m+k) && \text{by Proposition 2.1 page 13} && \text{and hypothesis (A)}\\
&= \mathsf{h}^*(-m)\star\mathsf{R}_{yx}(m) && \text{by Proposition P.2 page 330}\\
\mathsf{R}_{yy}(m) &= \mathsf{h}^*(-m)\star\mathsf{R}_{yx}(m) && \text{by result (2)}\\
&= \mathsf{h}^*(-m)\star\mathsf{R}^*_{xy}(m) && \text{by Corollary 2.1 page 13}\\
&= \mathsf{h}^*(-m)\star\left[\mathsf{h}^*(-m)\star\mathsf{R}_{xx}(m)\right]^* && \text{by result (1)}\\
&= \mathsf{h}^*(-m)\star\mathsf{h}(-m)\star\mathsf{R}^*_{xx}(m) && \text{by \textit{distributive} property of $*$-\textbf{algebras}} && \text{(Definition H.3 page 228)}
\end{aligned}
$$

✏

Corollary 5.2. [2] *Let* **S** *be a system with* IMPULSE RESPONSE $\mathsf{h}(n)$, INPUT $\boldsymbol{x}(n)$, *and* OUTPUT $\boldsymbol{y}(n)$.

COR

$$
\left\{\begin{array}{lll}
\text{(A).} & \mathsf{h} \textit{ is } \text{LINEAR TIME INVARIANT} & \textit{and}\\
\text{(B).} & \boldsymbol{x} \textit{ and } \boldsymbol{y} \textit{ are } \text{WIDE SENSE STATIONARY} &
\end{array}\right\}
\implies
\left\{\begin{array}{lrcll}
\text{(1).} & \check{\mathsf{S}}_{xy}(z) &=& \check{\mathsf{S}}_{xx}(z)\check{\mathsf{H}}^*\left(\frac{1}{z^*}\right) & \textit{and}\\
\text{(2).} & \check{\mathsf{S}}_{yy}(z) &=& \check{\mathsf{S}}_{yx}(z)\check{\mathsf{H}}^*\left(\frac{1}{z^*}\right) & \textit{and}\\
\text{(3).} & \check{\mathsf{S}}_{yy}(z) &=& \check{\mathsf{S}}_{xx}(z)\check{\mathsf{H}}(z)\check{\mathsf{H}}^*\left(\frac{1}{z^*}\right) &
\end{array}\right\}
$$

✎PROOF: The proof is given in Proposition **??** (page **??**) (1). ✏

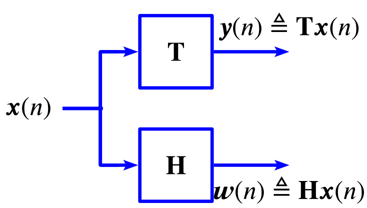
Corollary 5.3. *Let* **S** *be a system with* IMPULSE RESPONSE $\mathsf{h}(n)$, INPUT $\boldsymbol{x}(n)$, *and* OUTPUT $\boldsymbol{y}(n)$.

COR

$$
\left\{\begin{array}{llll}
\text{(A).} & \mathsf{h} \textit{ is} & \text{LTI} & \textit{and}\\
\text{(B).} & \boldsymbol{x} \textit{ and } \boldsymbol{y} \textit{ are} & \text{WSS} &
\end{array}\right\}
\implies
\left\{\begin{array}{lrcll}
\text{(1).} & \tilde{\mathsf{S}}_{xy}(\omega) &=& \tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{H}}^*(\omega) & \textit{and}\\
\text{(2).} & \tilde{\mathsf{S}}_{yy}(\omega) &=& \tilde{\mathsf{S}}_{xy}(\omega)\tilde{\mathsf{H}}(\omega) & \textit{and}\\
\text{(3).} & \tilde{\mathsf{S}}_{yy}(\omega) &=& \tilde{\mathsf{S}}_{xx}(\omega)\left|\tilde{\mathsf{H}}(\omega)\right|^2 &
\end{array}\right\}
$$

✎PROOF: The proof is given in Proposition **??** (page **??**) (1). ✏

5.3 Parallel operators on WSS random sequences



Theorem 5.2. *Let* **S** *be the* SYSTEM *illustrated to the right, where* **T** *is* NOT NECESSARILY LINEAR. *Let*

$$(\!(\mathsf{h}(n))\!) \triangleq \mathbf{H}\bar{\delta}(n) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\bar{\delta}(n-m)$$

be the IMPULSE RESPONSE *of* **H**.

[2] Papoulis (1991), page 323

THM
$$\left\{\begin{array}{lll}\text{(A).} & \boldsymbol{x}(n) \text{ is} & \textit{WSS} \text{ and} \\ \text{(B).} & \mathbf{H} \text{ is} & \textit{LTI}\end{array}\right\} \implies \left\{\begin{array}{llll}\text{(1).} & \mathsf{R}_{wy}(m) & = \sum_{n\in\mathbb{Z}} \mathsf{h}(n)\mathsf{R}_{xy}(m-n) & \text{(convolution)} \\ & & \triangleq \mathsf{h}(m) \star \mathsf{R}_{xy}(m) & \text{and} \\ \text{(2).} & \check{\mathsf{S}}_{wy}(z) & = \check{\mathsf{H}}(z)\check{\mathsf{S}}_{xy}(z) & \text{and} \\ \text{(3).} & \tilde{\mathsf{S}}_{wy}(\omega) & = \tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{xy}(\omega) & \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{wy}(m) &\triangleq \mathsf{E}\left[\boldsymbol{w}(m)\boldsymbol{y}^*(0)\right] && \text{by (A) and definition of } \mathsf{R}_{wy} && \text{(Definition 2.4 page 12)} \\
&\triangleq \mathsf{E}\left([\mathbf{H}\boldsymbol{x}](m)\boldsymbol{y}^*(0)\right) && \text{by definition of } \mathbf{S} \\
&= \mathbf{H}\mathsf{E}\left(\boldsymbol{x}(m)\boldsymbol{y}^*(0)\right) && \text{by } \textit{LTI} \text{ hypothesis} && \text{(B)} \\
&\triangleq \mathbf{H}\mathsf{R}_{xy}(m) && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}(n)\mathsf{R}_{xy}(m-n) && \text{by definition of } \mathbf{H} \textit{ impulse response } (\mathsf{h}(n)) \\
&= \left[\mathsf{h}(m) \star \mathsf{R}_{xy}(m)\right] && \text{by definition of } \textit{convolution} && \text{(Definition P.3 page 329)} \\
\check{\mathsf{S}}_{wy}(z) &\triangleq \mathbf{Z}\mathsf{R}_{wy}(m) && \text{by definition of } \check{\mathsf{S}}_{wy} && \text{(Definition 2.5 page 14)} \\
&= \left[\mathsf{h}(m) \star \mathsf{R}_{xy}(m)\right] && \text{by previous result} \\
&= \check{\mathsf{H}}(z)\check{\mathsf{S}}_{xy}(z) && \text{by } \textit{Convolution Theorem} && \text{(Theorem P.2 page 332)} \\
\tilde{\mathsf{S}}_{wy}(\omega) &\triangleq \check{\mathbf{F}}\mathsf{R}_{wy}(m) && \text{by definition of } \tilde{\mathsf{S}}_{wy} && \text{(Definition 6.3 page 44)} \\
&= \left[\mathsf{h}(m) \star \mathsf{R}_{xy}(m)\right] && \text{by previous result} \\
&= \tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{xy}(\omega) && \text{by } \textit{Convolution Theorem} && \text{(Theorem P.2 page 332)}
\end{aligned}$$

✏

5.4 Whitening discrete random sequences

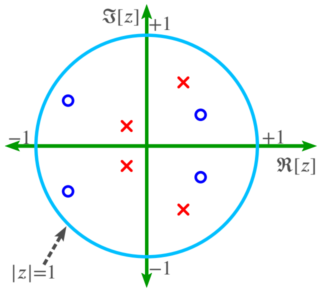


Figure 5.1: Poles (×) and zeros (o) of a *minimum phase* filter

Definition 5.1. *Let $\check{\mathsf{H}}(z)$ be the z-transform of the impulse response of a filter. If $\check{\mathsf{H}}(z)$ can be expressed as a rational expression with poles and zeros $r_n e^{i\theta_n}$, then the filter is* ***minimum phase*** *if each $r_n < 1$ (all roots lie inside the unit circle in the complex z-plane).*

See Figure 5.1 page 38.

Note that if $\mathsf{L}(z)$ has a root at $z = re^{i\theta}$, then $\mathsf{L}^*(1/z^*)$ has a root at

$$\frac{1}{z^*} = \frac{1}{\left(re^{i\theta}\right)^*} = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta}.$$

That is, if $\mathsf{L}(z)$ has a root inside the unit circle, then $\mathsf{L}^*(1/z^*)$ has a root directly opposite across the unit circle boundary (see Figure 5.2 page 39). A causal stable filter $\check{\mathsf{H}}(z)$ must have all of its poles inside the unit circle. A minimum phase filter is a filter with both its poles and zeros inside the unit circle. One advantage of a minimum phase filter is that its reciprocol (zeros become poles and poles become zeros) is also causal and stable.

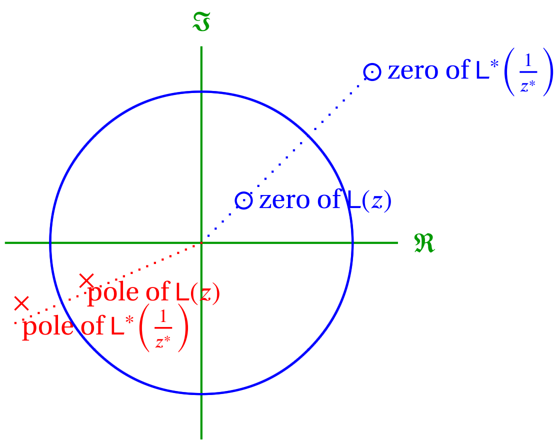


Figure 5.2: Mirrored roots in complex-z plane

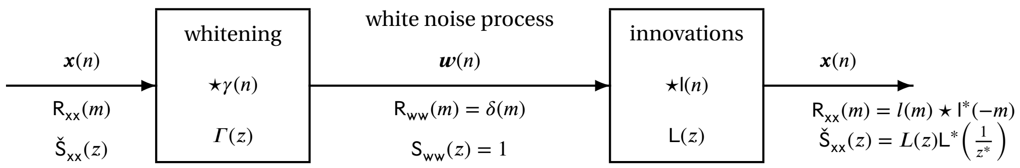


Figure 5.3: Innovations and whitening filters

The next theorem demonstrates a method for "whitening" a *random sequence* $\boldsymbol{x}(n)$ with a filter constructed from a decomposition of $\mathsf{R}_{xx}(m)$. The technique is stated precisely in Theorem 5.3 page 39 and illustrated in Figure 5.3 page 39. Both imply two filters with impulse responses $l(n)$ and $\gamma(n)$. Filter $l(n)$ is referred to as the **innovations filter** (because it generates or "innovates" $\boldsymbol{x}(n)$ from a white noise process $\mathsf{w}(n)$) and $\gamma(n)$ is referred to as the **whitening filter** because it produces a white noise sequence when the input sequence is $\boldsymbol{x}(n)$.[3]

Theorem 5.3. *Let $\boldsymbol{x}(n)$ be a WSS* RANDOM SEQUENCE *with auto-correlation $\mathsf{R}_{xx}(m)$ and spectral density $\check{\mathsf{S}}_{xx}(z)$.* ***If*** *$\check{\mathsf{S}}_{xx}(z)$ has a* ***rational expression****, then the following are true:*

1. *There exists a rational expression $\mathsf{L}(z)$ with minimum phase such that*
$$\check{\mathsf{S}}_{xx}(z) = \mathsf{L}(z)\mathsf{L}^*\left(\frac{1}{z^*}\right).$$
2. *An LTI filter for which the Laplace transform of the impulse response $\gamma(n)$ is*
$$\Gamma(z) = \frac{1}{\mathsf{L}(z)}$$
is both causal and stable.
3. *If $\boldsymbol{x}(n)$ is the input to the filter $\gamma(n)$, the output $\mathsf{y}(n)$ is a* ***white noise sequence*** *such that*
$$\mathsf{S}_{yy}(z) = 1 \qquad \mathsf{R}_{yy}(m) = \bar{\delta}(m).$$

[3] Papoulis (1991), pages 401–402

✎PROOF:

$$\begin{aligned}
\mathsf{S}_{ww}(z) &= \Gamma(z)\Gamma^*\left(\frac{1}{z^*}\right)\check{\mathsf{S}}_{xx}(z) \\
&= \frac{1}{\mathsf{L}(z)}\frac{1}{\mathsf{L}^*\left(\frac{1}{z^*}\right)}\check{\mathsf{S}}_{xx}(z) \\
&= \frac{1}{\mathsf{L}(z)}\frac{1}{\mathsf{L}^*\left(\frac{1}{z^*}\right)}\mathsf{L}(z)\mathsf{L}^*\left(\frac{1}{z^*}\right) \\
&= 1
\end{aligned}$$

⇒

CHAPTER 6

OPERATORS ON CONTINUOUS RANDOM SEQUENCES

6.1 LTI Operations on non-stationary random processes

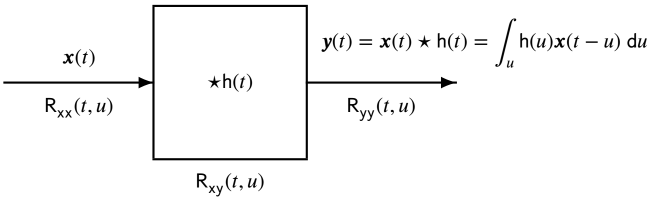


Figure 6.1: Linear system with random process input and output

Theorem 6.1. [1] *Let* $\mathsf{h} : \mathbb{R} \to \mathbb{C}$ *be the impulse response of a linear time-invariant system and Let* $\boldsymbol{y}(t) = h(t) \star \boldsymbol{x}(t) \triangleq \int_{u\in\mathbb{R}} h(u)\boldsymbol{x}(t-u)\,\mathrm{d}u$ *as illustrated in Figure 6.1* *page 41*. *Then*

THM

Correlation functions

$$\begin{aligned}
\mathsf{R}_{xy}(t,u) &= \mathsf{R}_{xx}(t,u) \star h^*(u) &&\triangleq \int_v \mathsf{h}^*(v)\mathsf{R}_{xx}(t,u-v)\,\mathrm{d}v\\
\mathsf{R}_{yy}(t,u) &= \mathsf{R}_{xy}(t,u) \star \mathsf{h}(t) &&\triangleq \int_v \mathsf{h}(v)\mathsf{R}_{xy}(t-v,u)\,\mathrm{d}v\\
\mathsf{R}_{yy}(t,u) &= \mathsf{R}_{xx}(t,u) \star \mathsf{h}(t) \star \mathsf{h}^*(u) &&\triangleq \int_w \mathsf{h}^*(w)\int_v \mathsf{h}(v)\mathsf{R}_{xx}(t-v,u-w)\,\mathrm{d}v\,\mathrm{d}\omega
\end{aligned}$$

Laplace power spectral density functions

$$\begin{aligned}
\check{S}_{xy}(s,r) &= \check{S}_{xx}(s,r)\check{\mathsf{h}}^*(r^*)\\
\check{S}_{yy}(s,r) &= \check{S}_{xy}(s,r)\check{\mathsf{h}}(s)\\
\check{S}_{yy}(s,r) &= \check{S}_{xx}(s,r)\check{\mathsf{h}}(s)\check{\mathsf{h}}^*(r^*)
\end{aligned}$$

Power spectral density functions

$$\begin{aligned}
\mathsf{S}_{xy}(f,g) &= \mathsf{S}_{xx}(f,g)\tilde{\mathsf{h}}^*(-g)\\
\mathsf{S}_{yy}(f,g) &= \mathsf{S}_{xy}(f,g)\tilde{\mathsf{h}}(\omega)\\
\mathsf{S}_{yy}(f,g) &= \mathsf{S}_{xx}(f,g)\tilde{\mathsf{h}}(\omega)\tilde{\mathsf{h}}^*(-g)
\end{aligned}$$

[1] Papoulis (1991), page 312

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{\mathsf{xy}}(t,u) &\triangleq \mathsf{E}\left[\boldsymbol{x}(t)\boldsymbol{y}^*(u)\right] \\
&= \mathsf{E}\left[\boldsymbol{x}(t)\left(\int_v \mathsf{h}(v)\boldsymbol{x}(u-v)\,\mathsf{d}v\right)^*\right] \\
&= \mathsf{E}\left[\boldsymbol{x}(t)\int_v \mathsf{h}^*(v)\boldsymbol{x}^*(u-v)\,\mathsf{d}v\right] \\
&= \int_v \mathsf{h}^*(v)\mathsf{E}\left[\boldsymbol{x}(t)\boldsymbol{x}^*(u-v)\right]\,\mathsf{d}v \\
&= \int_v \mathsf{h}^*(v)\mathsf{R}_{\mathsf{xx}}(t,u-v)\,\mathsf{d}v \\
&\triangleq \mathsf{R}_{\mathsf{xx}}(t,u) \star h^*(u)
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{\mathsf{yy}}(t,u) &\triangleq \mathsf{E}\left[\boldsymbol{y}(t)\boldsymbol{y}^*(u)\right] \\
&= \mathsf{E}\left[\left(\int_v \mathsf{h}(v)\mathsf{x}(t-v)\,\mathsf{d}v\right)\boldsymbol{y}^*(u)\right] \\
&= \int_v \mathsf{h}(v)\mathsf{E}\left[\mathsf{x}(t-v)\boldsymbol{y}^*(u)\right]\,\mathsf{d}v \\
&= \int_v \mathsf{h}(v)\mathsf{R}_{\mathsf{xy}}(t-v,u)\,\mathsf{d}v \\
&\triangleq \mathsf{R}_{\mathsf{xy}}(t,u) \star \mathsf{h}(t)
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{\mathsf{yy}}(t,u) &\triangleq \mathsf{E}\left[\boldsymbol{y}(t)\boldsymbol{y}^*(u)\right] \\
&= \mathsf{E}\left[\left(\int_v \mathsf{h}(v)\mathsf{x}(t-v)\,\mathsf{d}v\right)\left(\int_w \mathsf{h}(w)\mathsf{x}(u-w)\,\mathsf{d}\omega\right)^*\right] \\
&= \int_w \mathsf{h}^*(w)\int_v \mathsf{h}(v)\mathsf{E}\left[\mathsf{x}(t-v)\mathsf{x}^*(u-w)\right]\,\mathsf{d}v\,\mathsf{d}\omega \\
&= \int_w \mathsf{h}^*(w)\int_v \mathsf{h}(v)\mathsf{R}_{\mathsf{xx}}(t-v,u-w)\,\mathsf{d}v\,\mathsf{d}\omega \\
&= \int_w \mathsf{h}^*(w)\left[\mathsf{R}_{\mathsf{xx}}(t,u-w) \star \mathsf{h}(t)\right]\,\mathsf{d}\omega \\
&\triangleq \mathsf{R}_{\mathsf{xx}}(t,u) \star \mathsf{h}(t) \star \mathsf{h}^*(u)
\end{aligned}$$

$$\begin{aligned}
\check{S}_{\mathsf{xy}}(s,r) &\triangleq \mathbf{L}\mathsf{R}_{\mathsf{xy}}(t,u) \\
&= \mathbf{L}[\mathsf{R}_{\mathsf{xx}}(t,u) \star \mathsf{h}^*(u)] \\
&= \mathbf{L}[\mathsf{R}_{\mathsf{xx}}(t,u)]\mathbf{L}[\mathsf{h}^*(u)] \\
&= \check{S}_{\mathsf{xx}}(s,r)\int_{u\in\mathbb{R}} \mathsf{h}^*(u)e^{-ru}\,\mathsf{d}u \\
&= \check{S}_{\mathsf{xx}}(s,r)\left[\int_{u\in\mathbb{R}} \mathsf{h}(u)e^{-r^*u}\,\mathsf{d}u\right]^* \\
&= \check{S}_{\mathsf{xx}}(s,r)\check{\mathsf{h}}^*(r^*)
\end{aligned}$$

$$\begin{aligned}
\check{S}_{\mathsf{yy}}(s,r) &\triangleq \mathbf{L}\mathsf{R}_{\mathsf{yy}}(t,u) \\
&= \mathbf{L}[\mathsf{R}_{\mathsf{xy}}(t,u) \star \mathsf{h}(t)] \\
&= \mathbf{L}[\mathsf{R}_{\mathsf{xy}}(t,u)]\mathbf{L}[\mathsf{h}(t)] \\
&= \check{S}_{\mathsf{xy}}(s,r)\check{\mathsf{h}}(s)
\end{aligned}$$

$$\begin{aligned}
&= \check{S}_{xy}(s,r)\check{h}(s)\\
&= \check{S}_{xx}(s,r)\check{h}^*(r^*)\check{h}(s)\\
&= \check{S}_{xx}(s,r)\check{h}(s)\check{h}^*(r^*)
\end{aligned}$$

$$\begin{aligned}
S_{xy}(f,g) &\triangleq \tilde{\mathbf{F}}R_{xy}(t,u)\\
&= \tilde{\mathbf{F}}[R_{xx}(t,u) \star h^*(u)]\\
&= \tilde{\mathbf{F}}[R_{xx}(t,u)]\tilde{\mathbf{F}}[h^*(u)]\\
&= S_{xx}(f,g)\int_{u\in\mathbb{R}} h^*(u)e^{-i2\pi gu}\,du\\
&= S_{xx}(f,g)\left[\int_{u\in\mathbb{R}} h(u)e^{i2\pi gu}\,du\right]^*\\
&= S_{xx}(f,g)\left[\int_{u\in\mathbb{R}} h(u)e^{-i2\pi(-g)u}\,du\right]^*\\
&= S_{xx}(f,g)\tilde{h}^*(-g)
\end{aligned}$$

$$\begin{aligned}
S_{yy}(f,g) &\triangleq \tilde{\mathbf{F}}R_{yy}(t,u)\\
&= \tilde{\mathbf{F}}[R_{xy}(t,u) \star h(t)]\\
&= \tilde{\mathbf{F}}[R_{xy}(t,u)]\tilde{\mathbf{F}}[h(t)]\\
&= S_{xy}(f,g)\tilde{h}(\omega)\\
&\\
&= S_{xy}(f,g)\tilde{h}(\omega)\\
&= S_{xx}(f,g)\tilde{h}^*(-g)\tilde{h}(\omega)
\end{aligned}$$

6.2 LTI Operations on WSS random processes

Definition 6.1.

DEF *A random process $\mathbf{x}(t)$ is* ***wide sense stationary*** *(**WSS**) if*

(1). $\mu_X(t)$ *is* CONSTANT *with respect to* t (STATIONARY IN THE MEAN) *and*

(2). $R_{xx}(t+\tau,t)$ *is* CONSTANT *with respect to* t (STATIONARY IN CORRELATION)

If a process $\mathbf{x}(t)$ is *wide sense stationary*, mean and correlation are often written μ_X and $R_{xx}(\tau)$, respectively. If a pair of processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are *WSS*, then their cross-correlation is commonly written $R_{xy}(\tau)$.

Definition 6.2. *Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be WSS random processes.*

DEF

$$\begin{aligned}
\check{S}_{xx}(s) &\triangleq \mathbf{L}R_{xx}(\tau) \triangleq \int_{\tau\in\mathbb{R}} R_{xx}(\tau)e^{-s\tau}\,d\tau\\
\check{S}_{yy}(s) &\triangleq \mathbf{L}R_{yy}(\tau) \triangleq \int_{\tau\in\mathbb{R}} R_{yy}(\tau)e^{-s\tau}\,d\tau\\
\check{S}_{xy}(s) &\triangleq \mathbf{L}R_{xy}(\tau) \triangleq \int_{\tau\in\mathbb{R}} R_{xy}(\tau)e^{-s\tau}\,d\tau\\
\check{S}_{yx}(s) &\triangleq \mathbf{L}R_{yx}(\tau) \triangleq \int_{\tau\in\mathbb{R}} R_{yx}(\tau)e^{-s\tau}\,d\tau
\end{aligned}$$

Definition 6.3. *Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be WSS random processes.*

DEF

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) &\triangleq [\tilde{\mathbf{F}}\mathsf{R}_{\mathsf{xx}}(\tau)](\omega) \triangleq \int_{\tau\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(\tau)e^{-i\omega\tau}\,\mathsf{d}\tau\\
\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega) &\triangleq [\tilde{\mathbf{F}}\mathsf{R}_{\mathsf{yy}}(\tau)](\omega) \triangleq \int_{\tau\in\mathbb{R}} \mathsf{R}_{\mathsf{yy}}(\tau)e^{-i\omega\tau}\,\mathsf{d}\tau\\
\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) &\triangleq [\tilde{\mathbf{F}}\mathsf{R}_{\mathsf{xy}}(\tau)](\omega) \triangleq \int_{\tau\in\mathbb{R}} \mathsf{R}_{\mathsf{xy}}(\tau)e^{-i\omega\tau}\,\mathsf{d}\tau\\
\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega) &\triangleq [\tilde{\mathbf{F}}\mathsf{R}_{\mathsf{yx}}(\tau)](\omega) \triangleq \int_{\tau\in\mathbb{R}} \mathsf{R}_{\mathsf{yx}}(\tau)e^{-i\omega\tau}\,\mathsf{d}\tau
\end{aligned}$$

Definition 6.4. [2] *Let $\mathbf{x}(t)$ be a random variable that is* STATIONARY IN THE MEAN *such that* $\mathsf{E}[\mathbf{x}(t)]$ *is constant with respect to t.*

DEF

$\mathbf{x}(t)$ *is* ***ergodic in the mean*** *if*

$$\underbrace{\mathsf{E}[\mathbf{x}(t)]}_{\text{ENSEMBLE AVERAGE}} = \underbrace{\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{+\tau}\mathbf{x}(t)\,\mathsf{d}t}_{\text{TIME AVERAGE}}$$

Proposition 6.1.

PRP

$$\left\{\ \mathbf{x}(t) \text{ is NON-STATIONARY}\ \right\} \implies \left\{\ \mathbf{x}(t) \text{ is NOT ERGODIC IN THE MEAN}\ \right\}$$

✎PROOF: If $\mathbf{x}(t)$ is *non-stationary*, then $\mathsf{E}[\mathbf{x}(t)]$ is not constant with time. But $\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{+\tau}\mathbf{x}(t)\,\mathsf{d}t$ must be a constant (if it is *convergent*). ✏

Definition 6.5. [3] *Let $\mathbf{x}(t)$ be a* WIDE SENSE STATIONARY *random process.*

DEF

(1). *The* ***average power*** $\mathsf{P}_{\mathsf{avg}}$ *of $\mathbf{x}(t)$ is* $\quad \mathsf{P}_{\mathsf{avg}}\mathbf{x}(t) \triangleq \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{t\in\mathbb{R}}|\mathbf{x}(t)|^2\,\mathsf{d}t$

(2). *The* ***energy spectral density*** $|\tilde{\mathsf{x}}(\omega)|^2$ *of $\mathbf{x}(t)$ is* $\quad |\tilde{\mathsf{x}}(\omega)|^2 \triangleq \left|\tilde{\mathbf{F}}\mathbf{x}(t)\right|^2$

Remark 6.1 (**spectral power**). Why does $\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)$ deserve the name *power spectral density*? This is answered by Theorem 6.2 (next). But to elaborate further, note that $\tilde{\mathsf{S}}_{\mathsf{xx}}$ is the spectral representation of the statistical relationship (the *variance*) between samples of $\mathbf{x}(t)$. For example, if there is no relationship, then $\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) = 1$. But in the case that $\mathbf{x}(t)$ is *ergodic in the mean*, then $\tilde{\mathsf{S}}_{\mathsf{xx}}$ takes on an additional meaning—it describes the "spectral power" present in $\mathbf{x}(t)$. This is demonstrated by the next theorem.

Theorem 6.2. *Let $\mathbf{x}(t)$ be a* RANDOM PROCESS.

THM

$$\left\{\begin{array}{ll}\text{(A).} & \mathbf{x}(t) \text{ is ERGODIC IN THE MEAN} \quad \textit{and}\\ \text{(B).} & \tilde{\mathsf{x}}(\omega) \text{ EXISTS}\end{array}\right\} \implies \left\{\begin{array}{llll}\text{(1).} & \tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) &= \lim_{\tau\to\infty}\frac{1}{2\tau}\underbrace{|\tilde{\mathsf{x}}(\omega)|^2}_{\text{(ESD)}} & \textit{and}\\ \text{(2).} & \mathsf{P}_{\mathsf{avg}}[\mathbf{x}(t)] &= \int_{\omega\in\mathbb{R}}\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\,\mathsf{d}\omega & \end{array}\right\}$$

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[2] Papoulis (1984) page 246 ⟨Mean-Ergodic processes⟩, Papoulis (2002) page 523 ⟨12-1 ERGODICITY⟩, Kay (1988) page 58 ⟨3.6 ERGODICITY OF THE AUTOCORRELATION FUNCTION⟩, Manolakis et al. (2005) page 106 ⟨ERGODIC RANDOM PROCESSES⟩, Koopmans (1995) pages 53–61, Cadzow (1987) page 378 ⟨11.13 ERGODIC TIME SERIES⟩, Helstrom (1991) page 336

[3] **?** page 177

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{S}}_{xx}(\omega) &\triangleq \int_{\tau\in\mathbb{R}} \mathsf{R}_{xx}(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau && \text{by definition of } \tilde{\mathsf{S}}_{xx}(\omega) && \text{(Definition 6.3 page 44)}\\
&= \int_{\tau\in\mathbb{R}} \mathsf{E}\left[\boldsymbol{x}(t+\tau)\boldsymbol{x}^*(t)\right]e^{-i\omega\tau}\,\mathrm{d}\tau && \text{by definition of } \mathsf{R}_{xx}(t) && \text{(Definition 3.2 page 18)}\\
&= \mathsf{E}\left[\boldsymbol{x}^*(t)\int_{\tau\in\mathbb{R}} \boldsymbol{x}(t+\tau)e^{-i\omega\tau}\,\mathrm{d}\tau\right] && \text{by \textit{linearity} of E operator}\\
&= \mathsf{E}\left[\boldsymbol{x}^*(t)\int_{u\in\mathbb{R}} \boldsymbol{x}(u)e^{-i\omega(u-t)}\,\mathrm{d}u\right] && \text{where } u \triangleq t+\tau \implies \tau = u-t\\
&= \mathsf{E}\left[\boldsymbol{x}^*(t)e^{i\omega t}\int_{u\in\mathbb{R}} \boldsymbol{x}(u)e^{-i\omega u}\,\mathrm{d}u\right]\\
&= \mathsf{E}\left[\boldsymbol{x}^*(t)e^{i\omega t}\tilde{\mathsf{x}}(\omega)\right] && \text{by definition of \textit{Fourier Transform}} && \text{(Definition N.2 page 309)}\\
&= \mathsf{E}\left[\boldsymbol{x}^*(t)e^{i\omega t}\right]\tilde{\mathsf{x}}(\omega) && \text{by hypothesis (B)}\\
&= \left[\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{+\tau} \boldsymbol{x}^*(t)e^{i\omega t}\,\mathrm{d}t\right]\tilde{\mathsf{x}}(\omega) && \text{by \textit{ergodic in the mean} hypothesis} && \text{(Definition 6.4 page 44)}\\
&= \lim_{\tau\to\infty}\frac{1}{2\tau}\left[\int_{t\in\mathbb{R}} \boldsymbol{x}(t)e^{-i\omega t}\,\mathrm{d}t\right]^*\tilde{\mathsf{x}}(\omega)\\
&= \lim_{\tau\to\infty}\frac{1}{2\tau}\tilde{\mathsf{x}}^*(\omega)\tilde{\mathsf{x}}(\omega) && \text{by hypothesis (B)}\\
&= \lim_{\tau\to\infty}\frac{1}{2\tau}|\tilde{\mathsf{x}}(\omega)|^2
\end{aligned}$$

$$\begin{aligned}
\int_{\omega\in\mathbb{R}} \tilde{\mathsf{S}}_{xx}(\omega)\,\mathrm{d}\omega &= \int_{\omega\in\mathbb{R}} \lim_{\tau\to\infty}\frac{1}{2\tau}|\tilde{\mathsf{x}}(\omega)|^2\,\mathrm{d}\omega\\
&= \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{\omega\in\mathbb{R}} |\tilde{\mathsf{x}}(\omega)|^2\,\mathrm{d}\omega\\
&= \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{t\in\mathbb{R}} |\boldsymbol{x}(t)|^2\,\mathrm{d}t && \text{by \textit{Plancheral's formula}} && \text{(Theorem N.3 page 311, Theorem G.9 page 210)}\\
&= \mathsf{P}_{\text{avg}} && \text{by definition of } \mathsf{P}_{\text{avg}} && \text{(Definition 6.5 page 44)}
\end{aligned}$$

Thus, $\tilde{\mathsf{S}}_{xx}(\omega)$ is the power density of $\boldsymbol{x}(t)$ in the frequency domain. ✏

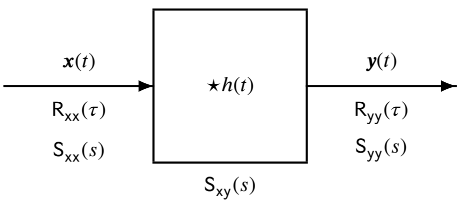


Figure 6.2: Linear system with WSS random process input and output

Theorem 6.3. [4] *Let* $\mathsf{h} : \mathbb{R} \to \mathbb{C}$ *be the impulse response of a linear time-invariant system and let* $\boldsymbol{y}(t) = h(t) \star \boldsymbol{x}(t) \triangleq \int_{u\in\mathbb{R}} h(u)\boldsymbol{x}(t-u)\,\mathrm{d}u$ *as illustrated in Figure 6.1 page 41. Then*

[4] Papoulis (1991), pages 323–324

THM

$$\begin{array}{rcllll}
\mathsf{R}_{xy}(\tau) &=& \mathsf{R}_{xx}(\tau)\star h^*(-\tau) &\triangleq& \int_{u\in\mathbb{R}} h^*(-u)\mathsf{R}_{xx}(\tau-u)\,\mathsf{d}u\\
\mathsf{R}_{yy}(\tau) &=& \mathsf{R}_{xy}(\tau)\star h(\tau) &\triangleq& \int_{u\in\mathbb{R}} h(u)\mathsf{R}_{xy}(\tau-u)\,\mathsf{d}u\\
\mathsf{R}_{yy}(\tau) &=& \mathsf{R}_{xx}(\tau)\star h(\tau)\star h^*(-\tau) &\triangleq& \int_v\int_{u\in\mathbb{R}} h(u-v)h^*(-v)\mathsf{R}_{xx}(\tau-u-v)\,\mathsf{d}u\,\mathsf{d}v\\
\\
\mathsf{S}_{xy}(s) &=& \mathsf{S}_{xx}(s)\hat{h}^*(-s^*)\\
\mathsf{S}_{yy}(s) &=& \mathsf{S}_{xy}(s)\hat{h}(s)\\
\mathsf{S}_{yy}(s) &=& \mathsf{S}_{xx}(s)\hat{h}(s)\hat{h}^*(-s^*)\\
\\
\tilde{\mathsf{S}}_{xy}(\omega) &=& \tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{h}}^*(\omega)\\
\tilde{\mathsf{S}}_{yy}(\omega) &=& \tilde{\mathsf{S}}_{xy}(\omega)\tilde{\mathsf{h}}(\omega)\\
\tilde{\mathsf{S}}_{yy}(\omega) &=& \tilde{\mathsf{S}}_{xx}(\omega)|\tilde{\mathsf{h}}(\omega)|^2
\end{array}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{xx}(\tau)\star h^*(-\tau) &\triangleq \int_{u\in\mathbb{R}} h^*(-u)\mathsf{R}_{xx}(\tau-u)\,\mathsf{d}u\\
&= \int_{u\in\mathbb{R}} h^*(-u)\mathsf{E}\left[\boldsymbol{x}(t)\boldsymbol{x}^*(t-\tau+u)\right]\,\mathsf{d}u\\
&= \mathsf{E}\left[\boldsymbol{x}(t)\int_{u\in\mathbb{R}} h^*(-u)\boldsymbol{x}^*(t-\tau+u)\,\mathsf{d}u\right]\\
&= \mathsf{E}\left[\boldsymbol{x}(t)\int_{u\in\mathbb{R}} h^*(u')\boldsymbol{x}^*(t-\tau-u')\,\mathsf{d}u'\right]\\
&= \mathsf{E}\left[\boldsymbol{x}(t)\boldsymbol{y}^*(t-\tau)\right]\\
&\triangleq \mathsf{R}_{xy}(\tau)
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{xy}(\tau)\star h(\tau) &\triangleq \int_{u\in\mathbb{R}} h(u)\mathsf{R}_{xy}(\tau-u)\,\mathsf{d}u\\
&= \int_{u\in\mathbb{R}} h(u)\mathsf{E}\left[\boldsymbol{x}(t+\tau-u)\boldsymbol{y}^*(t)\right]\,\mathsf{d}u\\
&= \mathsf{E}\left[\boldsymbol{y}^*(t)\int_{u\in\mathbb{R}} h(u)\boldsymbol{x}(t+\tau-u)\,\mathsf{d}u\right]\\
&= \mathsf{E}\left[\boldsymbol{y}^*(t)\boldsymbol{y}(t+\tau)\right]\\
&= \mathsf{E}\left[\boldsymbol{y}(t+\tau)\boldsymbol{y}^*(t)\right]\\
&\triangleq \mathsf{R}_{yy}(\tau)
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{yy}(\tau) &= \mathsf{R}_{xy}(\tau)\star h(\tau)\\
&= \mathsf{R}_{xx}(\tau)\star h^*(-\tau)\star h(\tau)\\
&= \mathsf{R}_{xx}(\tau)\star h(\tau)\star h^*(-\tau)
\end{aligned}$$

$$\begin{aligned}
\mathsf{S}_{xy}(s) &\triangleq \mathbf{L}\mathsf{R}_{xy}(\tau)\\
&\triangleq \int_{\tau\in\mathbb{R}} \mathsf{R}_{xy}(\tau)e^{-s\tau}\,d\tau\\
&= \int_{\tau\in\mathbb{R}} \left[\mathsf{R}_{xx}(\tau)\star h^*(-\tau)\right]e^{-s\tau}\,d\tau\\
&= \int_{\tau\in\mathbb{R}} \left[\int_{u\in\mathbb{R}} h^*(-u)\mathsf{R}_{xx}(\tau-u)\,\mathsf{d}u\right]e^{-s\tau}\,d\tau\\
&= \int_{u\in\mathbb{R}} h^*(-u)\int_{\tau\in\mathbb{R}} \mathsf{R}_{xx}(\tau-u)e^{-s\tau}\,d\tau\,\mathsf{d}u
\end{aligned}$$

$$
\begin{aligned}
&= \int_{u\in\mathbb{R}} h^*(-u) \int_v \mathsf{R}_{xx}(v) e^{-s(v+u)} \,\mathsf{d}v \,\mathsf{d}u && \text{where } v = \tau - u \iff \tau = v + u \\
&= \int_{u\in\mathbb{R}} h^*(-u) e^{-su} \,\mathsf{d}u \int_v \mathsf{R}_{xx}(v) e^{-sv} \,\mathsf{d}v \\
&= \int_{u\in\mathbb{R}} h^*(u) e^{-s(-u)} \,\mathsf{d}u \int_v \mathsf{R}_{xx}(v) e^{-sv} \,\mathsf{d}v \\
&= \left(\int_{u\in\mathbb{R}} h(u) e^{-(-s^*)u} \,\mathsf{d}u \right)^* \int_v \mathsf{R}_{xx}(v) e^{-sv} \,\mathsf{d}v \\
&\triangleq \hat{h}^*(-s^*) \mathsf{S}_{xx}(s)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{S}_{yy}(s) &\triangleq \mathbf{L}\mathsf{R}_{yy}(\tau) \\
&\triangleq \int_{\tau\in\mathbb{R}} \mathsf{R}_{yy}(\tau) e^{-s\tau} \, d\tau \\
&= \int_{\tau\in\mathbb{R}} \left[\mathsf{R}_{xy}(\tau) \star h(\tau) \right] e^{-s\tau} \, d\tau \\
&= \int_{\tau\in\mathbb{R}} \left[\int_{u\in\mathbb{R}} h(u) \mathsf{R}_{xy}(\tau - u) \,\mathsf{d}u \right] e^{-s\tau} \, d\tau \\
&= \int_{u\in\mathbb{R}} h(u) \int_{\tau\in\mathbb{R}} \mathsf{R}_{xy}(\tau - u) e^{-s\tau} \, d\tau \,\mathsf{d}u \\
&= \int_{u\in\mathbb{R}} h(u) \int_{\tau\in\mathbb{R}} \mathsf{R}_{xy}(v) e^{-s(v+u)} \, d\tau \,\mathsf{d}u && \text{where } v = \tau - u \iff \tau = v + u \\
&= \int_{u\in\mathbb{R}} h(u) e^{-su} \, du \int_{\tau\in\mathbb{R}} \mathsf{R}_{xy}(v) e^{-sv} \, d\tau \\
&\triangleq \hat{h}(s) \mathsf{S}_{xy}(s)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{S}_{yy}(s) &= \hat{h}(s) \mathsf{S}_{xy}(s) \\
&= \hat{h}(s) \hat{h}^*(-s^*) \mathsf{S}_{xx}(s)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathsf{S}}_{xy}(\omega) &= \left. \mathsf{S}_{xy}(s) \right|_{s=j\omega} \\
&= \left. \hat{h}^*(-s^*) \mathsf{S}_{xx}(s) \right|_{s=j\omega} \\
&= \left. \left(\int_{u\in\mathbb{R}} h(u) e^{-(-s^*)u} \,\mathsf{d}u \right)^* \int_v \mathsf{R}_{xx}(v) e^{-sv} \,\mathsf{d}v \right|_{s=j\omega} \\
&= \left(\int_{u\in\mathbb{R}} h(u) e^{-(-j\omega)^* u} \,\mathsf{d}u \right)^* \int_v \mathsf{R}_{xx}(v) e^{-j\omega v} \,\mathsf{d}v \\
&= \left(\int_{u\in\mathbb{R}} h(u) e^{-j\omega u} \,\mathsf{d}u \right)^* \int_v \mathsf{R}_{xx}(v) e^{-j\omega v} \,\mathsf{d}v \\
&\triangleq \tilde{\mathsf{h}}^*(\omega) \tilde{\mathsf{S}}_{xx}(\omega)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathsf{S}}_{yy}(\omega) &= \left. \mathsf{S}_{yy}(s) \right|_{s=j\omega} \\
&= \left. \hat{h}(s) \mathsf{S}_{xy}(s) \right|_{s=j\omega} \\
&= \left. \int_{u\in\mathbb{R}} h(u) e^{-su} \, du \int_{\tau\in\mathbb{R}} \mathsf{R}_{xy}(v) e^{-sv} \, d\tau \right|_{s=j\omega} \\
&= \int_{u\in\mathbb{R}} h(u) e^{-j\omega u} \, du \int_{\tau\in\mathbb{R}} \mathsf{R}_{xy}(v) e^{-j\omega v} \, d\tau \\
&= \tilde{\mathsf{h}}(\omega) \tilde{\mathsf{S}}_{xy}(\omega)
\end{aligned}
$$

$$\begin{aligned}\tilde{S}_{yy}(\omega) &= \tilde{h}(\omega)\tilde{S}_{xy}(\omega)\\ &= \tilde{h}(\omega)\tilde{h}^*(\omega)\tilde{S}_{xx}(\omega)\\ &= |\tilde{h}(\omega)|^2\tilde{S}_{xx}(\omega)\end{aligned}$$

⇰

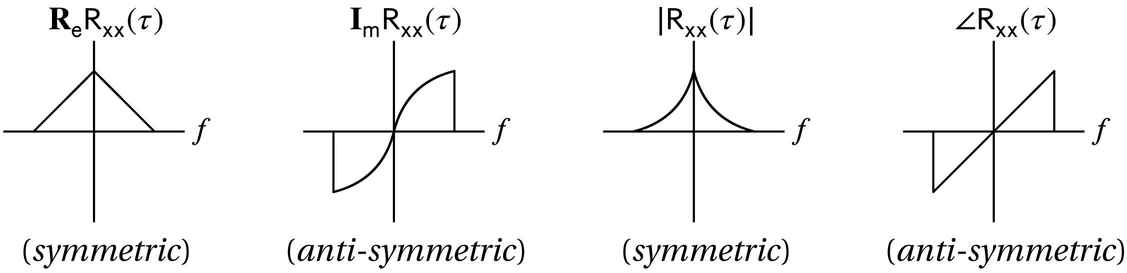


Figure 6.3: *auto-correlation* $R_{xx}(\tau)$

Theorem 6.4. *Let* $x : \mathbb{R} \to \mathbb{C}$ *be a WSS random process with auto-correlation* $R_{xx}(\tau)$*. Then* $R_{xx}(\tau)$ *is* ***conjugate symmetric*** *such that (see Figure 6.3 page 48)*

THM

$$\begin{array}{rcll} R_{xx}(\tau) &=& R^*_{xx}(-\tau) & \text{(CONJUGATE SYMMETRIC)}\\ \mathbf{R_e}\left[R_{xx}(\tau)\right] &=& \mathbf{R_e}\left[R_{xx}(-\tau)\right] & \text{(SYMMETRIC)}\\ \mathbf{I_m}\left[R_{xx}(\tau)\right] &=& -\mathbf{I_m}\left[R_{xx}(-\tau)\right] & \text{(ANTI-SYMMETRIC)}\\ |R_{xx}(\tau)| &=& |R_{xx}(-\tau)| & \text{(SYMMETRIC)}\\ \angle R_{xx}(\tau) &=& \angle R_{xx}(-\tau) & \text{(ANTI-SYMMETRIC).} \end{array}$$

✎PROOF:

$$\begin{aligned} R^*_{xx}(\tau) &\triangleq \left(\mathsf{E}\left[x(t-\tau)x^*(t)\right]\right)^* \\ &= \mathsf{E}\left[x^*(t-\tau)x(t)\right] \\ &= \mathsf{E}\left[x(t)x^*(t-\tau)\right] \\ &= \mathsf{E}\left[x(u+\tau)x^*(u)\right] && \text{where } u \triangleq t-\tau \iff t = u+\tau \\ &\triangleq R_{xx}(\tau) \end{aligned}$$

$$\begin{aligned} \mathbf{R_e}\left[R_{xx}(\tau)\right] &= \mathbf{R_e}\left[R^*_{xx}(-\tau)\right] &&= \mathbf{R_e}\left[R_{xx}(-\tau)\right]\\ \mathbf{I_m}\left[R_{xx}(\tau)\right] &= \mathbf{I_m}\left[R^*_{xx}(-\tau)\right] &&= -\mathbf{I_m}\left[R_{xx}(-\tau)\right]\\ abs R_{xx}(\tau) &= \left|R^*_{xx}(-\tau)\right| &&= \left|R_{xx}(-\tau)\right|\\ \angle R_{xx}(\tau) &= \angle R^*_{xx}(-\tau) &&= -\angle R_{xx}(-\tau) \end{aligned}$$

⇰

6.3 Whitening continuous random sequences

Simple algebraic operations on white noise processes (processes with autocorrelation $R_{xx}(\tau) = \delta(\tau)$) often produce *colored* noise (processes with autocorrelation $R_{xx}(\tau) \neq \delta(\tau)$). Sometimes we would like to process a colored noise process to produce a white noise process. This operation is known as *whitening*. Reasons for why we may want to whiten a noise process include

1. Samples from a white noise process are uncorrelated. If the noise process is Gaussian, then these samples are also independent which often greatly simplifies analysis.

2. Any orthonormal basis can be used to decompose a white noise process. This is not true of a colored noise process. Karhunen–Loève expansion can be used to decompose colored noise. [5]

Definition 6.6. *A **rational expression** $\mathsf{p}(s)$ is a polynomial divided by a polynomial such that*

DEF
$$\mathsf{p}(s) = \frac{\sum_{n=0}^{N} b_n s^n}{\sum_{n=0}^{M} a_n s^n}.$$

*The **zeros** of a rational expression are the roots of its numerator polynomial. The **poles** of a rational expression are the roots of its denominator polynomial.*

Definition 6.7. *Let $\check{\mathsf{h}}(s)$ be the Laplace transform of the impulse response of a filter. If $\check{\mathsf{h}}(s)$ can be expressed as a rational expression with poles and zeros at $a_n + ib_n$, then the filter is **minimum phase** if each $a_n < 0$ (all roots lie in the left hand side of the complex s-plane).*

Note that if $L(s)$ has a root at $s = -a + ib$, then $L^*(-s^*)$ has a root at

$$-s^* = -(-a + ib)^* = -(-a - ib) = a + ib.$$

That is, if $L(s)$ has a root in the left hand plane, then $L^*(-s^*)$ has a root directly opposite across the imaginary axis in the right hand plane (see Figure 6.4 page 49). A causal stable filter $\hat{h}(s)$ must have all of its poles in the left hand plane. A minimum phase filter is a filter with both its poles and zeros in the left hand plane. One advantage of a minimum phase filter is that its recipricol (zeros become poles and poles become zeros) is also causal and stable.

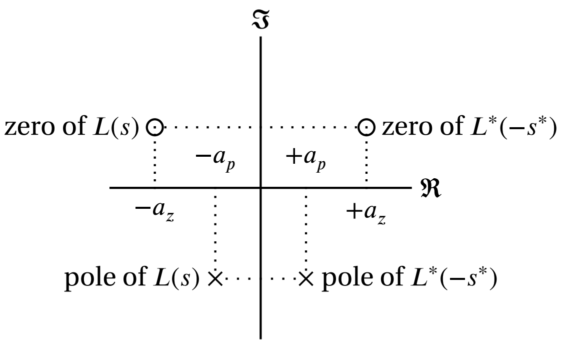


Figure 6.4: Mirrored roots in complex-s plane

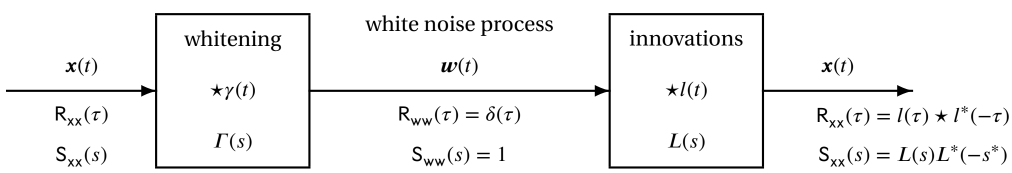


Figure 6.5: Innovations and whitening filters

The next theorem demonstrates a method for "whitening" a random process $\mathsf{x}(t)$ with a filter constructed from a decomposition of $\mathsf{R_{xx}}(\tau)$. The technique is stated precisely in Theorem 6.5 page 50

[5] *Karhunen–Loève expansion*: Section 3.3 page 20

and illustrated in Figure 6.5 page 49. Both imply two filters with impulse responses $l(t)$ and $\gamma(t)$. Filter $l(t)$ is referred to as the **innovations filter** (because it generates or "innovates" $\mathsf{x}(t)$ from a white noise process $\mathsf{w}(t)$) and $\gamma(t)$ is referred to as the **whitening filter** because it produces a white noise sequence when the input sequence is $\mathsf{x}(t)$. [6]

Theorem 6.5. *Let $\mathsf{x}(t)$ be a WSS random process with autocorrelation $\mathsf{R}_{\mathsf{xx}}(\tau)$ and spectral density $\mathsf{S}_{\mathsf{xx}}(s)$. **If** $\mathsf{S}_{\mathsf{xx}}(s)$ has a **rational expression**, then the following are true:*

1. *There exists a rational expression $L(s)$ with minimum phase such that*

$$\mathsf{S}_{\mathsf{xx}}(s) = L(s)L^*(-s^*).$$

2. *An LTI filter for which the Laplace transform of the impulse response $\gamma(t)$ is*

$$\Gamma(s) = \frac{1}{L(s)}$$

   is both causal and stable.

3. *If $\mathsf{x}(t)$ is the input to the filter $\gamma(t)$, the output $\mathsf{y}(t)$ is a **white noise sequence** such that*

$$\mathsf{S}_{\mathsf{yy}}(s) = 1 \qquad\qquad \mathsf{R}_{\mathsf{yy}}(\tau) = \delta(\tau).$$

✎PROOF:

$$\begin{aligned}
\mathsf{S}_{\mathsf{ww}}(s) &= \Gamma(s)\Gamma^*(-s^*)\mathsf{S}_{\mathsf{xx}}(s) \\
&= \frac{1}{L(s)}\frac{1}{L^*(-s^*)}\mathsf{S}_{\mathsf{xx}}(s) \\
&= \frac{1}{L(s)}\frac{1}{L^*(-s^*)}L(s)L^*(-s^*) \\
&= 1
\end{aligned}$$

⇒

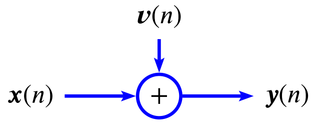
[6] Papoulis (1991), pages 401–402

CHAPTER 7

ADDITIVE NOISE ON RANDOM SEQUENCES

7.1 Additive noise and correlation

Theorem 7.1. *Let* S *be the* SYSTEM *illustrated to the right, where* **T** *is* NOT NECESSARILY LINEAR.



THM

$$\left.\begin{array}{llll}
\text{(A).} & x(n) \text{ is} & \textit{WSS} & \text{and} \\
\text{(B).} & x(n) \text{ and } v(n) \text{ are} & \textit{uncorrelated} & \text{and} \\
\text{(C).} & v(n) \text{ is} & \textit{zero-mean} &
\end{array}\right\} \implies \left\{\begin{array}{llll}
\text{(1).} & \mathsf{R}_{yv}(m) & = & \mathsf{R}_{vv}(m) \quad \text{and} \\
\text{(2).} & \mathsf{R}_{xy}(m) & = & \mathsf{R}_{xx}(m) \quad \text{and} \\
\text{(3).} & \mathsf{R}_{yy}(m) & = & \mathsf{R}_{xx}(m) + \mathsf{R}_{vv}(m) \quad \text{and} \\
\text{(4).} & \mathsf{R}_{xx}(m) & = & \mathsf{R}_{yy}(m) + \mathsf{R}_{vv}(m) - 2\mathbf{R_e}\mathsf{R}_{yv}(m)
\end{array}\right.$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{yv}(m) &\triangleq \mathsf{E}\left[y(m)v^*(0)\right] && \text{by (A) and definition of } \mathsf{R}_{yv} && \text{(Definition 2.4 page 12)} \\
&\triangleq \mathsf{E}\left[(x(m)+v(m))v^*(0)\right] && \text{by definition of } y \\
&= \mathsf{E}\left[x(m)v^*(0)\right] + \mathsf{E}\left[v(m)v^*(0)\right] && \text{by } \textit{linearity} \text{ of } \mathsf{E} && \text{(Theorem 1.1 page 4)} \\
&= \mathsf{E}x(m)\mathsf{E}v^*(0) + \mathsf{E}\left[v(m)v^*(0)\right] && \text{by } \textit{uncorrelated} \text{ hypothesis} && \text{(B)} \\
&= \mathsf{E}x(m)\cancelto{0}{\mathsf{E}v^*(0)} + \mathsf{E}\left[v(m)v^*(0)\right] && \text{by } \textit{zero-mean} \text{ hypothesis} && \text{(C)} \\
&= \mathsf{R}_{vv}(m) && \text{by definition of } \mathsf{R}_{vv} && \text{(Definition 2.4 page 12)} \\
\mathsf{R}_{xy}(m) &\triangleq \mathsf{E}\left[x(m)y^*(0)\right] && \text{by (A) and definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)} \\
&\triangleq \mathsf{E}\left(x(m)[x(0)+v(0)]^*\right) && \text{by definition of } y \\
&= \mathsf{E}\left[x(m)x^*(0)\right] + \mathsf{E}\left[x(m)v^*(0)\right] && \text{by } \textit{linearity} \text{ of } \mathsf{E} && \text{(Theorem 1.1 page 4)} \\
&= \mathsf{E}\left[x(m)x^*(0)\right] + \mathsf{E}[x(m)]\mathsf{E}\left[v^*(0)\right] && \text{by } \textit{uncorrelated} \text{ hypothesis} && \text{(B)} \\
&= \mathsf{E}\left[x(m)x^*(0)\right] + \mathsf{E}[x(m)]\cancelto{0}{\mathsf{E}\left[v^*(0)\right]} && \text{by } \textit{zero-mean} \text{ hypothesis} && \text{(C)} \\
&= \mathsf{R}_{xx}(m) && \text{by definition of } \mathsf{R}_{xx} && \text{(Definition 2.4 page 12)}
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{yy}(m) &\triangleq \mathsf{E}\left[y(m)y^*(0)\right] && \text{by (A) and definition of } \mathsf{R}_{yy} \\
&\triangleq \mathsf{E}\left[(x(m)+v(m))(x(0)+v(0))^*\right] && \text{by definition of } y \\
&= \mathsf{E}\left[x(m)x^*(0)\right] + \mathsf{E}\left[x(m)v^*(0)\right] + \mathsf{E}\left[v(m)x^*(0)\right] + \mathsf{E}\left[v(m)v^*(0)\right] \\
&= \mathsf{E}\left[x(m)x^*(0)\right] + \mathsf{E}x(m)\mathsf{E}v^*(0) + \mathsf{E}v(m)\mathsf{E}x^*(0) + \mathsf{E}\left[v(m)v^*(0)\right] && \text{by } \textit{uncorrelated} \text{ hypothesis (B)} \\
&= \mathsf{E}\left[x(m)x^*(0)\right] + \mathsf{E}x(m)\cancelto{0}{\mathsf{E}v^*(0)} + \cancelto{0}{\mathsf{E}v(m)}\mathsf{E}x^*(0) + \mathsf{E}\left[v(m)v^*(0)\right] && \text{by } \textit{zero-mean} \text{ hypothesis (C)}
\end{aligned}$$

$$\begin{aligned}
&= \mathsf{R}_{xx}(m) + \mathsf{R}_{vv}(m) && \text{by definition of } \mathsf{R}_{xx}\\
\mathsf{R}_{xx}(m) &\triangleq \mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{x}^*(0)\right]\\
&\triangleq \mathsf{E}\left(\left[\boldsymbol{y}(m)-\boldsymbol{v}(m)\right]\left[\boldsymbol{y}(0)-\boldsymbol{v}(0)\right]^*\right)\\
&= \mathsf{E}\left[\boldsymbol{y}(m)\boldsymbol{y}^*(0)\right] - \mathsf{E}\left[\boldsymbol{y}(m)\boldsymbol{v}^*(0)\right] - \mathsf{E}\left[\boldsymbol{v}(m)\boldsymbol{y}^*(0)\right] + \mathsf{E}\left[\boldsymbol{v}(m)\boldsymbol{v}^*(0)\right]\\
&\triangleq \mathsf{R}_{yy}(m) - \mathsf{R}_{yv}(m) - \mathsf{R}_{vy}(m) + \mathsf{R}_{vv}(m)\\
&= \mathsf{R}_{yy}(m) + \mathsf{R}_{vv}(m) - 2\mathbf{R}_{e}\mathsf{R}_{yv}(m)
\end{aligned}$$

⇨

Remark 7.1. Because in Theorem 7.1 $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{v}$ and $\mathsf{R}_{yy} = \mathsf{R}_{xx} + \mathsf{R}_{vv}$, one might assume that R is a kind of *linear operator* (Definition M.3 page 282) and further assume that because $\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{v}$ and $\mathsf{R}_{(-v)(-v)} = \mathsf{R}_{vv}$, that $\mathsf{R}_{xx} = \mathsf{R}_{yy} + \mathsf{R}_{vv}$. As Theorem 7.1 demonstrates, this is simply **not the case**. The problem here is that $\boldsymbol{y}$ and $\boldsymbol{v}$ are very much *correlated*—in fact $\boldsymbol{y}$ is obviously a *function* of $\boldsymbol{v}$.

Corollary 7.1. *Let* **S** *be the* SYSTEM *illustrated in Theorem 7.1 (page 51).*

COR

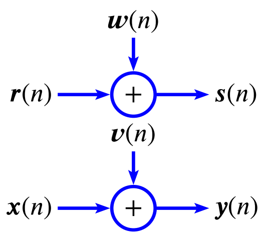
$$\left.\begin{array}{l}\textit{hypotheses of}\\ \textit{Theorem 7.1 (page 51)}\end{array}\right\} \implies \left\{\begin{array}{llcll}
(1). & \check{\mathsf{S}}_{yy}(z) &=& \check{\mathsf{S}}_{xx}(z) + \check{\mathsf{S}}_{vv}(z) & \textit{and}\\
(2). & \check{\mathsf{S}}_{yv}(z) &=& \check{\mathsf{S}}_{vv}(z) & \textit{and}\\
(3). & \check{\mathsf{S}}_{yy}(z) &=& \check{\mathsf{S}}_{yy}(z) + \check{\mathsf{S}}_{vv}(z) + \check{\mathsf{S}}_{yv}(z) + \check{\mathsf{S}}^*_{yv}(z^*) & \textit{and}\\
(4). & \tilde{\mathsf{S}}_{yy}(\omega) &=& \tilde{\mathsf{S}}_{xx}(\omega) + \tilde{\mathsf{S}}_{vv}(\omega) & \textit{and}\\
(5). & \tilde{\mathsf{S}}_{yv}(\omega) &=& \tilde{\mathsf{S}}_{vv}(\omega) & \textit{and}\\
(6). & \tilde{\mathsf{S}}_{yv}(\omega) &=& \tilde{\mathsf{S}}_{yy}(\omega) + \tilde{\mathsf{S}}_{vv}(\omega) + \tilde{\mathsf{S}}_{yv}(\omega) + \tilde{\mathsf{S}}^*_{yv}(-\omega) &
\end{array}\right.$$

✎PROOF:

$$\begin{aligned}
\check{\mathsf{S}}_{yy}(z) &\triangleq \mathbf{Z}\mathsf{R}_{yy}(m) && \text{by definition of } \check{\mathsf{S}}_{xx} && \text{(Definition 2.5 page 14)}\\
&= \mathbf{Z}\mathsf{R}_{qq}(m) + \mathbf{Z}\mathsf{R}_{vv}(m) && \text{by previous result} && (1)\\
&= \check{\mathsf{S}}_{qq}(z) + \check{\mathsf{S}}_{vv}(z) && \text{by definition of } \check{\mathsf{S}}_{yy} && \text{(Definition 6.3 page 44)}\\
\tilde{\mathsf{S}}_{yy}(\omega) &\triangleq \check{\mathbf{F}}\mathsf{R}_{yy}(m) && \text{by definition of } \tilde{\mathsf{S}}_{yy} && \text{(Definition 6.3 page 44)}\\
&= \check{\mathbf{F}}\mathsf{R}_{qq}(m) + \check{\mathbf{F}}\mathsf{R}_{vv}(m) && \text{by previous result} && (1)\\
&= \tilde{\mathsf{S}}_{qq}(\omega) + \tilde{\mathsf{S}}_{vv}(\omega) && \text{by definition of } \tilde{\mathsf{S}}_{yy} && \text{(Definition 6.3 page 44)}
\end{aligned}$$

⇨

Theorem 7.2. *Let* **S** *be the* SYSTEM *illustrated to the right:*



THM

$$\left\{\begin{array}{lrclll}
(A). & \multicolumn{3}{l}{\boldsymbol{x}(n) \text{ and } \boldsymbol{r}(n) \text{ are } \textit{wide sense stationary}} & & \text{and}\\
(B). & \mathsf{E}[\boldsymbol{x}(n)\boldsymbol{w}(n)] &=& \mathsf{E}\boldsymbol{x}(n)\mathsf{E}\boldsymbol{w}(n) & (\textit{uncorrelated}) & \text{and}\\
(C). & \mathsf{E}[\boldsymbol{r}(n)\boldsymbol{v}(n)] &=& \mathsf{E}\boldsymbol{r}(n)\mathsf{E}\boldsymbol{v}(n) & (\textit{uncorrelated}) & \text{and}\\
(D). & \mathsf{E}[\boldsymbol{w}(n)\boldsymbol{v}(n)] &=& \mathsf{E}\boldsymbol{w}(n)\mathsf{E}\boldsymbol{v}(n) & (\textit{uncorrelated}) & \text{and}\\
(E). & \mathsf{E}\boldsymbol{v}(n) = \mathsf{E}\boldsymbol{w}(n) &=& 0 & (\textit{zero-mean}) &
\end{array}\right\} \implies \left\{\begin{array}{rcl}
\mathsf{R}_{sy}(m) &=& \mathsf{R}_{sx}(m)\\
&=& \mathsf{R}_{ry}(m)\\
&=& \mathsf{R}_{rx}(m)
\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{sy}(m) &\triangleq \mathsf{E}\left[\boldsymbol{s}(m)\boldsymbol{y}^*(0)\right] && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)}\\
&\triangleq \mathsf{E}\left([\boldsymbol{r}(m)+\boldsymbol{w}(m)][\boldsymbol{x}(0)+\boldsymbol{v}(0)]^*\right) && \text{by definition of } \mathbf{S}\\
&= \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{x}^*(0)\right]+\mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{v}^*(0)\right]+\mathsf{E}\left[\boldsymbol{w}(m)\boldsymbol{x}^*(0)\right]+\mathsf{E}\left[\boldsymbol{w}(m)\boldsymbol{v}^*(0)\right]\\
&= \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{x}^*(0)\right]+\mathsf{E}\boldsymbol{r}(m)\mathsf{E}\boldsymbol{v}^*(0)\\
&\quad +\mathsf{E}\boldsymbol{w}(m)\mathsf{E}\boldsymbol{x}^*(0)+\mathsf{E}\boldsymbol{w}(m)\mathsf{E}\boldsymbol{v}^*(0) && \text{by \textit{uncorrelated} hypotheses} && \text{(B), (C), and (D)}\\
&= \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{x}^*(0)\right]+\mathsf{E}\boldsymbol{r}(m)\cancelto{0}{\mathsf{E}\boldsymbol{v}^*(0)}\\
&\quad +\cancelto{0}{\mathsf{E}\boldsymbol{w}(m)}\mathsf{E}\boldsymbol{x}^*(0)+\cancelto{0}{\mathsf{E}\boldsymbol{w}(m)}\mathsf{E}\boldsymbol{v}^*(0) && \text{by \textit{zero-mean} hypothesis} && \text{(E)}\\
&= \mathsf{R}_{rx}(m) && \text{by definition of } \mathsf{R}_{rx} && \text{(Definition 2.4 page 12)}\\
\mathsf{R}_{sx}(m) &\triangleq \mathsf{E}\left[\boldsymbol{s}(m)\boldsymbol{x}^*(0)\right] && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)}\\
&\triangleq \mathsf{E}\left([\boldsymbol{r}(m)+\boldsymbol{w}(m)]\boldsymbol{x}^*(0)\right) && \text{by definition of } \mathbf{S}\\
&= \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{x}^*(0)\right]+\mathsf{E}\boldsymbol{w}(m)\mathsf{E}\boldsymbol{x}^*(0) && \text{by \textit{uncorrelated} hypothesis} && \text{(B)}\\
&= \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{x}^*(0)\right]+\cancelto{0}{\mathsf{E}\boldsymbol{w}(m)}\mathsf{E}\boldsymbol{x}^*(0) && \text{by \textit{zero-mean} hypothesis} && \text{(E)}\\
&= \mathsf{R}_{rx}(m) && \text{by definition of } \mathsf{R}_{rx} && \text{(Definition 2.4 page 12)}\\
\mathsf{R}_{ry}(m) &\triangleq \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{y}^*(0)\right] && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)}\\
&\triangleq \mathsf{E}\left(\boldsymbol{r}(m)[\boldsymbol{x}(0)+\boldsymbol{v}(0)]^*\right) && \text{by definition of } \mathbf{S}\\
&= \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{x}^*(0)\right]+\mathsf{E}\boldsymbol{r}(m)\mathsf{E}\boldsymbol{v}^*(0) && \text{by \textit{uncorrelated} hypothesis} && \text{(C)}\\
&= \mathsf{E}\left[\boldsymbol{r}(m)\boldsymbol{x}^*(0)\right]+\mathsf{E}\boldsymbol{r}(m)\cancelto{0}{\mathsf{E}\boldsymbol{v}^*(0)} && \text{by \textit{zero-mean} hypothesis} && \text{(E)}\\
&= \mathsf{R}_{rx}(m) && \text{by definition of } \mathsf{R}_{rx} && \text{(Definition 2.4 page 12)}
\end{aligned}$$

✏

Corollary 7.2. *Let* **S** *be the* SYSTEM *illustrated in Theorem 7.2 (page 52).*

COR
$$\left\{\begin{array}{l}\textit{hypotheses of}\\ \textit{Theorem 7.2 (page 52)}\end{array}\right\} \Longrightarrow \left\{\begin{array}{llllllll}(1). & \check{\mathsf{S}}_{sy}(z) &=& \check{\mathsf{S}}_{sx}(z) &=& \check{\mathsf{S}}_{ry}(z) &=& \check{\mathsf{S}}_{rx}(z) \quad \textit{and}\\ (2). & \tilde{\mathsf{S}}_{sy}(\omega) &=& \tilde{\mathsf{S}}_{sx}(\omega) &=& \tilde{\mathsf{S}}_{ry}(\omega) &=& \tilde{\mathsf{S}}_{rx}(\omega)\end{array}\right\}$$

✎PROOF:

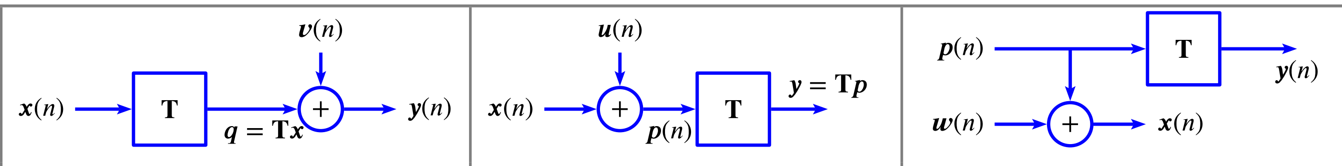
$$\begin{aligned}
\check{\mathsf{S}}_{sy}(\omega) &\triangleq \mathbf{Z}\mathsf{R}_{sy}(m) && \text{by definition of } \check{\mathsf{S}}_{xy} && \text{(Definition 2.5 page 14)}\\
&= \mathbf{Z}\mathsf{R}_{rx}(m) && \text{by previous result} && (1)\\
&= \check{\mathsf{S}}_{rx}(z) && \text{by definition of } \check{\mathsf{S}}_{xy} && \text{(Definition 2.5 page 14)}\\
\tilde{\mathsf{S}}_{sy}(\omega) &\triangleq \check{\mathbf{F}}\mathsf{R}_{sy}(m) && \text{by definition of } \tilde{\mathsf{S}}_{xy} && \text{(Definition 6.3 page 44)}\\
&= \check{\mathbf{F}}\mathsf{R}_{rx}(m) && \text{by previous result} && (1)\\
&= \tilde{\mathsf{S}}_{rx}(\omega) && \text{by definition of } \tilde{\mathsf{S}}_{xy} && \text{(Definition 6.3 page 44)}
\end{aligned}$$

✏

7.2 Additive noise and operators

Lemma 7.1. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.2 (page 55) (A).*

LEM
$$\left\{\begin{array}{lrcll}(A). & \mathsf{R}_{xx}(n_1,m) &=& \mathsf{R}_{xx}(n_2,m) & \text{(WSS)} \quad \textit{and}\\ (B). & \mathsf{E}[\boldsymbol{x}(n)\boldsymbol{v}(n)] &=& \mathsf{E}\boldsymbol{x}(n)\mathsf{E}\boldsymbol{v}(n) & \text{(UNCORRELATED)} \quad \textit{and}\\ (E). & \mathsf{E}\boldsymbol{v}(n) &=& 0 & \text{(ZERO-MEAN)}\end{array}\right\} \Longrightarrow \left\{\begin{array}{lrcll}(1). & \mathsf{R}_{xy}(m) &=& \mathsf{R}_{xq}(m) & \textit{and}\\ (2). & \check{\mathsf{S}}_{xy}(z) &=& \check{\mathsf{S}}_{xq}(z) & \textit{and}\\ (3). & \tilde{\mathsf{S}}_{xy}(\omega) &=& \tilde{\mathsf{S}}_{xq}(\omega) & \end{array}\right\}$$



(A) $\mathsf{R}_{xy}(m) = \mathsf{R}_{xq}(m)$ (Lemma 7.1 page 53)

(B) $\mathsf{R}_{py}(m) = \mathsf{R}_{xy}(m) + \mathsf{R}_{uy}(m)$ (Lemma 7.2 page 54)

(C) $\mathsf{R}_{xy}(m) = \mathsf{R}_{py}(m)$ (Lemma 7.3 page 54)

Figure 7.1: Additive noise with *linear*/**non-linear** operator **T**

✎PROOF:

$$
\begin{aligned}
\mathsf{R}_{xy}(m) &\triangleq \mathsf{E}\left[x(m)y^*(0)\right] && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)}\\
&\triangleq \mathsf{E}\left[x(m)(q(0)+v(0))^*\right] && \text{by definition of } \mathbf{S} && \text{(Figure 7.2 page 55) (A)}\\
&= \mathsf{E}\left[x(m)q^*(0) + p(m)v^*(0)\right] \\
&= \mathsf{E}\left[x(m)q^*(0)\right] + \mathsf{E}\left[x(m)v^*(0)\right] \\
&= \mathsf{E}\left[x(m)q^*(0)\right] + \left[\mathsf{E}x(m)\right]\left[\mathsf{E}v^*(0)\right] && \text{by \textit{uncorrelated} hypothesis} && \text{(B)}\\
&= \mathsf{E}\left[x(m)q^*(0)\right] + \left[\mathsf{E}p(m)\right]\cancelto{0}{\left[\mathsf{E}v^*(0)\right]} && \text{by \textit{zero-mean} hypothesis} && \text{(E)}\\
&= \mathsf{R}_{xq}(m) && \text{by definition of } \mathsf{R}_{xq} && \text{(Definition 2.4 page 12)}\\
\check{\mathsf{S}}_{xy}(z) &\triangleq \mathbf{Z}\mathsf{R}_{xy}(m) && \text{by definition of } \check{\mathsf{S}}_{xy} && \text{(Definition 2.5 page 14)}\\
&= \mathbf{Z}\mathsf{R}_{xq}(m) && \text{by previous result} && \text{(1)}\\
&= \check{\mathsf{S}}_{xq}(z) && \text{by definition of } \check{\mathsf{S}}_{xq} && \text{(Definition 2.5 page 14)}\\
\tilde{\mathsf{S}}_{xy}(\omega) &\triangleq \breve{\mathbf{F}}\mathsf{R}_{xy}(m) && \text{by definition of } \tilde{\mathsf{S}}_{xy} && \text{(Definition 6.3 page 44)}\\
&= \breve{\mathbf{F}}\mathsf{R}_{xq}(m) && \text{by previous result} && \text{(1)}\\
&= \tilde{\mathsf{S}}_{xq}(\omega) && \text{by definition of } \tilde{\mathsf{S}}_{xq} && \text{(Definition 6.3 page 44)}
\end{aligned}
$$

✏

Lemma 7.2. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.2 (page 55) (B).*

LEM

$$
\left\{\begin{array}{lll}
\text{(A).} & x(n) \text{ is} & \text{(WSS)} \quad \text{and}\\
\text{(B).} & u(n) \text{ is} & \text{(ZERO-MEAN)} \quad \text{and}\\
\text{(C).} & x(n) \text{ and } u(n) \text{ are} & \text{(UNCORRELATED)}
\end{array}\right\}
\implies
\left\{\begin{array}{llcl}
\text{(1).} & \mathsf{R}_{pq}(m) &=& \mathsf{R}_{xy}(m) + \mathsf{R}_{uy}(m) \quad \text{and}\\
\text{(2).} & \check{\mathsf{S}}_{pq}(z) &=& \check{\mathsf{S}}_{xy}(z) + \check{\mathsf{S}}_{uy}(z) \quad \text{and}\\
\text{(3).} & \tilde{\mathsf{S}}_{pq}(\omega) &=& \tilde{\mathsf{S}}_{xy}(\omega) + \tilde{\mathsf{S}}_{uy}(\omega)
\end{array}\right\}
$$

✎PROOF:

$$
\begin{aligned}
\mathsf{R}_{xy}(m) &\triangleq \mathsf{E}\left[x(m)y^*(0)\right] && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)}\\
&\triangleq \mathsf{E}\left(\left[p(m)-u(m)\right]y^*(0)\right) && \text{by definition of } \mathbf{S}\\
&= \mathsf{E}\left[p(m)y^*(0) - u(m)y^*(0)\right] \\
&= \mathsf{E}\left[p(m)y^*(0)\right] - \mathsf{E}\left[u(m)y^*(0)\right] && \text{because } \mathsf{E} \text{ is a \textit{linear operator}} && \text{(Theorem 1.1 page 4)}\\
&\triangleq \mathsf{R}_{py}(m) - \mathsf{R}_{uy}(m) && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)}
\end{aligned}
$$

✏

Lemma 7.3. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.2 (page 55) (C).*

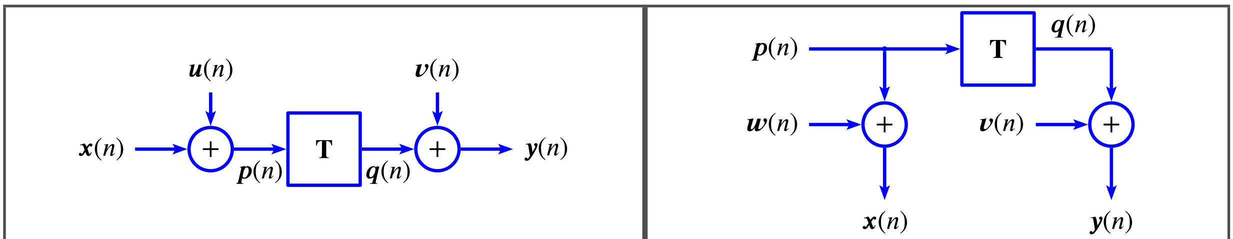
LEM

$$
\left\{\begin{array}{lll}
\text{(A).} & x(n) \text{ is} & \text{(WSS)} \quad \text{and}\\
\text{(B).} & u(n) \text{ is} & \text{(ZERO-MEAN)} \quad \text{and}\\
\text{(C).} & x(n) \text{ and } u(n) \text{ are} & \text{(UNCORRELATED)}
\end{array}\right\}
\implies
\left\{\begin{array}{llcl}
\text{(1).} & \mathsf{R}_{xy}(m) &=& \mathsf{R}_{py}(m) \quad \text{and}\\
\text{(2).} & \check{\mathsf{S}}_{xy}(z) &=& \check{\mathsf{S}}_{py}(z) \quad \text{and}\\
\text{(3).} & \tilde{\mathsf{S}}_{xy}(\omega) &=& \tilde{\mathsf{S}}_{py}(\omega)
\end{array}\right\}
$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{xy}(m) &\triangleq \mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{y}^*(0)\right] && \text{by definition of } \mathsf{R}_{py} && \text{(Definition 2.4 page 12)}\\
&\triangleq \mathsf{E}\left(\left[\boldsymbol{p}(m)+\boldsymbol{u}(m)\right]\boldsymbol{y}^*(0)\right) && \text{by definition of } \mathbf{S} && \\
&= \mathsf{E}\left[\boldsymbol{p}(m)\boldsymbol{y}^*(0)+\boldsymbol{u}(m)\boldsymbol{y}^*(0)\right] && \text{by field properties of } (\mathbb{R},+,\cdot,0,1) && \\
&= \mathsf{E}\left[\boldsymbol{p}(m)\boldsymbol{y}^*(0)\right]+\mathsf{E}\left[\boldsymbol{u}(m)\boldsymbol{y}^*(0)\right] && \text{because E is a } \textit{linear operator} && \text{(Theorem 1.1 page 4)}\\
&= \mathsf{E}\left[\boldsymbol{p}(m)\boldsymbol{y}^*(0)\right]+\mathsf{E}[\boldsymbol{u}(m)]\mathsf{E}\left[\boldsymbol{y}^*(0)\right] && \text{by } \textit{uncorrelated} \text{ hypothesis} && \text{(C)}\\
&= \mathsf{E}\left[\boldsymbol{p}(m)\boldsymbol{y}^*(0)\right]+\cancelto{0}{\mathsf{E}[\boldsymbol{u}(m)]\mathsf{E}\left[\boldsymbol{y}^*(0)\right]} && \text{by } \textit{zero-mean} \text{ hypothesis} && \text{(B)}\\
&\triangleq \mathsf{R}_{py}(m) && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)}
\end{aligned}$$

⇨



(A) *Communications additive noise system*
$\tilde{\mathsf{S}}_{pq}(\omega)=\tilde{\mathsf{S}}_{py}(\omega)=\tilde{\mathsf{S}}_{xy}(\omega)+\tilde{\mathsf{S}}_{uy}(\omega)$
(Theorem 7.3 page 55)

(B) *Measurement additive noise system*
$\tilde{\mathsf{S}}_{pq}(\omega)=\tilde{\mathsf{S}}_{py}(\omega)=\tilde{\mathsf{S}}_{xq}(\omega)=\tilde{\mathsf{S}}_{xy}(\omega)$
(Theorem 7.4 page 55)

Figure 7.2: *linear* / **non-linear** additive noise systems

Theorem 7.3 (*communications additive noise cross-correlation*).
Let **S** *be the* SYSTEM *illustrated in Figure 7.2 page 55 (A).*

THM

$$\left.\begin{array}{lll}
\text{(A).} & \boldsymbol{x}(n) \textit{ is} & \text{WSS} \quad \textit{and}\\
\text{(B).} & \boldsymbol{u}(n) \textit{ is} & \text{ZERO-MEAN} \quad \textit{and}\\
\text{(C).} & \boldsymbol{v}(n) \textit{ is} & \text{ZERO-MEAN} \quad \textit{and}\\
\text{(D).} & \boldsymbol{x}(n), \boldsymbol{u}(n), \boldsymbol{v}(n) \textit{ are} & \text{UNCORRELATED}
\end{array}\right\} \implies \left\{\begin{array}{ll}
\text{(1).} & \mathsf{R}_{pq}(m)=\mathsf{R}_{py}(m)=\mathsf{R}_{xy}(m)+\mathsf{R}_{uy}(m) \quad \textit{and}\\
\text{(2).} & \check{\mathsf{S}}_{pq}(z)=\check{\mathsf{S}}_{py}(z)=\check{\mathsf{S}}_{xy}(z)+\check{\mathsf{S}}_{uy}(z) \quad \textit{and}\\
\text{(3).} & \tilde{\mathsf{S}}_{pq}(\omega)=\tilde{\mathsf{S}}_{py}(\omega)=\tilde{\mathsf{S}}_{xy}(\omega)+\tilde{\mathsf{S}}_{uy}(\omega)
\end{array}\right.$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{pq}(m) &= \mathsf{R}_{py}(m) && \text{by Lemma 7.1 page 53} &&\\
\mathsf{R}_{pq}(m) &= \mathsf{R}_{xq}(m)+\mathsf{R}_{uq}(m) && \text{by Lemma 7.2 page 54} &&\\
\mathsf{R}_{py}(m) &\triangleq \mathsf{E}\left[\boldsymbol{p}(m)\boldsymbol{y}^*(0)\right] && \text{by definition } \mathsf{R}_{py} && \text{(Definition 2.4 page 12)}\\
&\triangleq \mathsf{E}\left([\boldsymbol{x}(m)+\boldsymbol{u}(m)]\boldsymbol{y}^*(0)\right) && \text{by definition } \mathbf{S} && \text{(Figure 7.2 page 55) (A)}\\
&= \mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{y}^*(0)+\boldsymbol{u}(m)\boldsymbol{y}^*(0)\right] && &&\\
&= \mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{y}^*(0)\right]+\mathsf{E}\left[\boldsymbol{u}(m)\boldsymbol{y}^*(0)\right] && \text{by } \textit{linearity} \text{ of E} && \text{(Theorem 1.1 page 4)}\\
&= \mathsf{R}_{xy}(m)+\mathsf{R}_{uy}(m) && \text{by definitions } \mathsf{R}_{xy} \text{ and } \mathsf{R}_{uy} && \text{(Definition 2.4 page 12)}
\end{aligned}$$

⇨

Theorem 7.4 (*measurement additive noise cross-correlation*).
Let **S** *be the* SYSTEM *illustrated in Figure 7.2 page 55 (B).*

THM

$$\left.\begin{array}{lll}
\text{(A).} & \boldsymbol{x}(n) \textit{ is} & \text{WSS} \quad \textit{and}\\
\text{(B).} & \boldsymbol{u}(n) \textit{ is} & \text{ZERO-MEAN} \quad \textit{and}\\
\text{(C).} & \boldsymbol{v}(n) \textit{ is} & \text{ZERO-MEAN} \quad \textit{and}\\
\text{(D).} & \boldsymbol{x}(n), \boldsymbol{u}(n), \boldsymbol{v}(n) \textit{ are} & \text{UNCORRELATED}
\end{array}\right\} \implies \left\{\begin{array}{ll}
\text{(1).} & \mathsf{R}_{pq}(m)=\mathsf{R}_{py}(m)=\mathsf{R}_{xq}(m)=\mathsf{R}_{xy}(m) \quad \textit{and}\\
\text{(2).} & \check{\mathsf{S}}_{pq}(z)=\check{\mathsf{S}}_{py}(z)=\check{\mathsf{S}}_{xq}(z)=\check{\mathsf{S}}_{xy}(z) \quad \textit{and}\\
\text{(3).} & \tilde{\mathsf{S}}_{pq}(\omega)=\tilde{\mathsf{S}}_{py}(\omega)=\tilde{\mathsf{S}}_{xq}(\omega)=\tilde{\mathsf{S}}_{xy}(\omega)
\end{array}\right.$$

✎PROOF:

$$
\begin{aligned}
\mathsf{R}_{pq}(m) &= \mathsf{R}_{py}(m) && \text{by Lemma 7.1 page 53} \\
\mathsf{R}_{pq}(m) &= \mathsf{R}_{xq}(m) && \text{by Lemma 7.3 page 54} \\
\mathsf{R}_{xy}(m) &\triangleq \mathsf{E}\left[x(m)y^*(0)\right] && \text{by definition } \mathsf{R}_{xy} && \text{(Definition 2.4 page 12)} \\
&\triangleq \mathsf{E}\left(\left[p(m)+u(m)\right]y^*(0)\right) && \text{by definition } \mathbf{S} && \text{(Figure 7.2 page 55) (B)} \\
&= \mathsf{E}\left[p(m)y^*(0)+u(m)y^*(0)\right] \\
&= \mathsf{E}\left[p(m)y^*(0)\right]+\mathsf{E}\left[u(m)y^*(0)\right] && \text{by } \textit{linearity} \text{ of } \mathsf{E} && \text{(Theorem 1.1 page 4)} \\
&= \mathsf{E}\left[p(m)y^*(0)\right]+\cancelto{0}{\mathsf{E}\left[u(m)y^*(0)\right]} && \text{by } \textit{uncorrelated} \text{ hypothesis} && \text{(D)} \\
&= \mathsf{R}_{py}(m) && \text{by definition of } \mathsf{R}_{py} && \text{(Definition 2.4 page 12)}
\end{aligned}
$$

✏

7.3 Additive noise and LTI operators

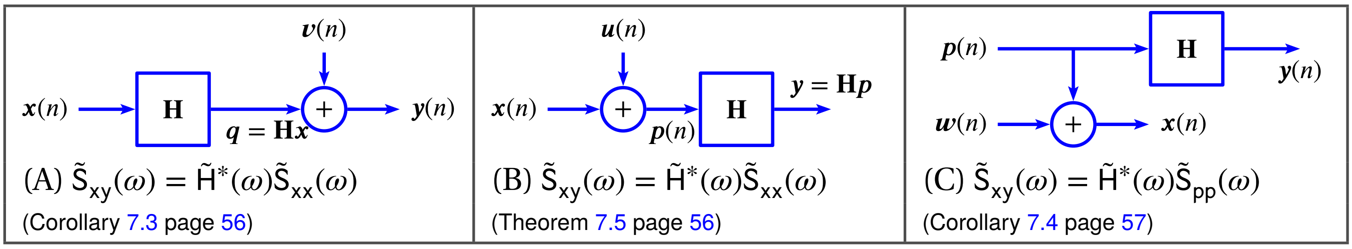


Figure 7.3: Additive noise with **LTI** operator **H**

Corollary 7.3. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.3 (page 56) (A).*

COR

$$
\left\{\begin{array}{llll}
\text{(A).} & x(n) \text{ is} & (\text{WSS}) & \text{and} \\
\text{(B).} & u(n) \text{ is} & (\text{ZERO-MEAN}) & \text{and} \\
\text{(C).} & x(n) \text{ and } u(n) \text{ are} & (\text{UNCORRELATED}) & \text{and} \\
\text{(D).} & \mathbf{H} \text{ is} & (\text{LTI}) &
\end{array}\right\}
\implies \left\{ \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega) \right\}
$$

✎PROOF:

$$
\begin{aligned}
\tilde{\mathsf{S}}_{xy}(\omega) &= \tilde{\mathsf{S}}_{xq}(\omega) && \text{by Lemma 7.1 page 53} \\
&= \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega) && \text{by Corollary 5.3 page 37}
\end{aligned}
$$

✏

Theorem 7.5. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.3 (page 56) (B).*

THM

$$
\left\{\begin{array}{llll}
\text{(A).} & x(n) \text{ is} & (\text{WSS}) & \text{and} \\
\text{(B).} & u(n) \text{ is} & (\text{ZERO-MEAN}) & \text{and} \\
\text{(C).} & x(n) \text{ and } u(n) \text{ are} & (\text{UNCORRELATED}) & \text{and} \\
\text{(D).} & \mathbf{H} \text{ is} & (\text{LTI}) &
\end{array}\right\}
\implies \left\{\begin{array}{lrcll}
\text{(1).} & \mathsf{R}_{yx}(m) & = & \mathsf{h}(m) \star \mathsf{R}_{xx}(m) & \text{and} \\
\text{(2).} & \check{\mathsf{S}}_{yx}(z) & = & \check{\mathsf{H}}(z)\check{\mathsf{S}}_{xx}(z) & \text{and} \\
\text{(3).} & \tilde{\mathsf{S}}_{xy}(\omega) & = & \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega) &
\end{array}\right\}
$$

✎PROOF:

1. definition: Let $(\mathsf{h}(n))$ be the *impulse response* of operator **H** such that
$$\mathbf{H}\bar{\delta}(n) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\bar{\delta}(n-m)$$

2. lemma: $\mathbf{H}\boldsymbol{x}(n) = \sum_{m\in\mathbb{Z}} \mathsf{h}(n)\boldsymbol{x}(m-n) = \mathsf{h}(n) \star \mathsf{R}_{\mathsf{xx}}(n)$.

   Proof: by the *linear time-invariant* hypotheses (D) and definition of *convolution* operator $\star$ (Definition N.3 page 312)

3. Proof that $\mathsf{R}_{\mathsf{yx}}(m) = \mathsf{h}(n) \star \mathsf{R}_{\mathsf{xx}}(m)$:

$$
\begin{aligned}
\mathsf{R}_{\mathsf{yx}}(m) &\triangleq \mathsf{E}\left[\boldsymbol{y}(m)\boldsymbol{x}^*(0)\right] && \text{by definition of } \mathsf{R}_{\mathsf{py}} && \text{(Definition 2.4 page 12)}\\
&= \mathsf{E}\big([\mathbf{H}\boldsymbol{x}(m) + \mathbf{H}\boldsymbol{u}(m)]\boldsymbol{x}^*(0)\big) && \text{by } \textit{linear} \text{ hypothesis} && \text{(D)}\\
&= \mathsf{E}\big(\left[\mathbf{H}\boldsymbol{x}^*(m)\right]\boldsymbol{x}^*(0) + [\mathbf{H}\boldsymbol{u}(0)]\boldsymbol{x}^*(0)\big) && &&\\
&= \mathsf{E}\big(\left[\mathbf{H}\boldsymbol{x}^*(m)\right]\boldsymbol{x}^*(0)\big) + \mathsf{E}\big([\mathbf{H}\boldsymbol{u}(0)]\boldsymbol{x}^*(0)\big) && \text{by } \textit{linearity} \text{ of } \mathbf{E} && \text{(Theorem ?? page ??)}\\
&= \mathbf{H}\mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{x}^*(0)\right] + \mathbf{H}\mathsf{E}\left[\boldsymbol{u}(m)\boldsymbol{x}^*(0)\right] && \text{by } \textit{LTI} \text{ hypotheses} && \text{(D)}\\
&= \mathbf{H}\mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{x}^*(0)\right] + \mathbf{H}\mathsf{E}\boldsymbol{u}(m)\mathsf{E}\boldsymbol{x}^*(0) && \text{by } \textit{uncorrelated} \text{ hypothesis} && \text{(C)}\\
&= \mathbf{H}\mathsf{E}\left[\boldsymbol{x}(m)\boldsymbol{x}^*(0)\right] + \cancelto{0}{\mathbf{H}\mathsf{E}\boldsymbol{u}(m)\mathsf{E}\boldsymbol{x}^*(0)} && \text{by } \textit{zero-mean} \text{ hypothesis} && \text{(B)}\\
&= \mathbf{H}\mathsf{R}_{\mathsf{xx}}(m) && \text{by definition of } \mathsf{R}_{\mathsf{xx}} && \text{(Definition 2.4 page 12)}\\
&= \mathsf{h}(m) \star \mathsf{R}_{\mathsf{xx}}(m) && \text{by (2) lemma} &&
\end{aligned}
$$

When **H** is *LTI*, what effect does the additive uncorrelated noise sources have on the cross-statistical properties of $\boldsymbol{x}$ and $\boldsymbol{y}$? Corollary 7.5 (next) demonstrates that, amazingly, under very general conditions, the noise sources have **no effect**.

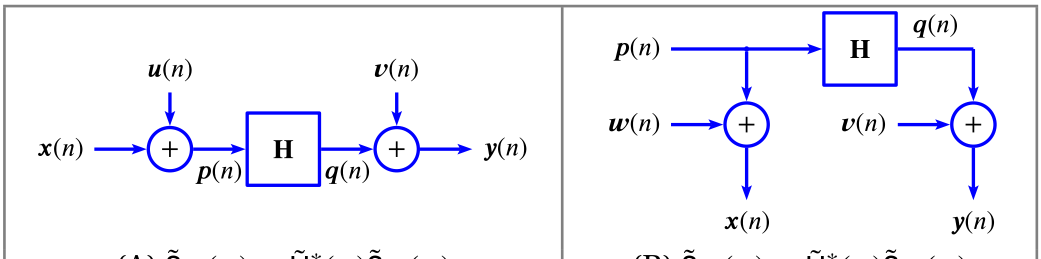
Corollary 7.4. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.3 (page 56) (C).*

COR

$$
\left\{\begin{array}{llll}
\text{(A).} & \boldsymbol{x}(n) \text{ is} & \text{(WSS)} & \text{and}\\
\text{(B).} & \boldsymbol{u}(n) \text{ is} & \text{(ZERO-MEAN)} & \text{and}\\
\text{(C).} & \boldsymbol{x}(n) \text{ and } \boldsymbol{w}(n) \text{ are} & \text{(UNCORRELATED)} & \text{and}\\
\text{(D).} & \mathbf{H} \text{ is} & \text{(LTI)} &
\end{array}\right\} \implies \left\{ \tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) = \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega) \right\}
$$

✎PROOF:

$$
\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) &= \tilde{\mathsf{S}}_{\mathsf{py}}(\omega) && \text{by Lemma 7.3 page 54}\\
&= \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega) && \text{by Corollary 5.3 page 37}
\end{aligned}
$$



(A) $\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) = \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)$ (Corollary 7.5 page 57)

(B) $\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) = \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)$ (Corollary 7.6 page 58)

Figure 7.4:

Corollary 7.5. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.4 page 57 (A).*

COR

$$
\left\{\begin{array}{llll}
\text{(A).} & \boldsymbol{x}(n) \text{ is} & \text{(WSS)} & \text{and}\\
\text{(B).} & \boldsymbol{u}(n) \text{ is} & \text{(ZERO-MEAN)} & \text{and}\\
\text{(C).} & \boldsymbol{v}(n) \text{ is} & \text{(ZERO-MEAN)} & \text{and}\\
\text{(D).} & \boldsymbol{x}(n), \boldsymbol{u}(n), \boldsymbol{v}(n) \text{ are} & \text{(UNCORRELATED)} & \text{and}\\
\text{(E).} & \mathbf{H} \text{ is} & \text{(LTI)} &
\end{array}\right\} \implies \left\{ \tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) = \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) \right\}
$$

✎PROOF:

$$\begin{aligned}
\tilde{S}_{yx}(\omega) &= \tilde{S}_{qx}(\omega) && \text{by Lemma 7.1 page 53}\\
&= \tilde{H}(\omega)\tilde{S}_{xx}(\omega) && \text{by Corollary 5.3 page 37}
\end{aligned}$$

✏

Corollary 7.6. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.4 page 57 (B).*

COR

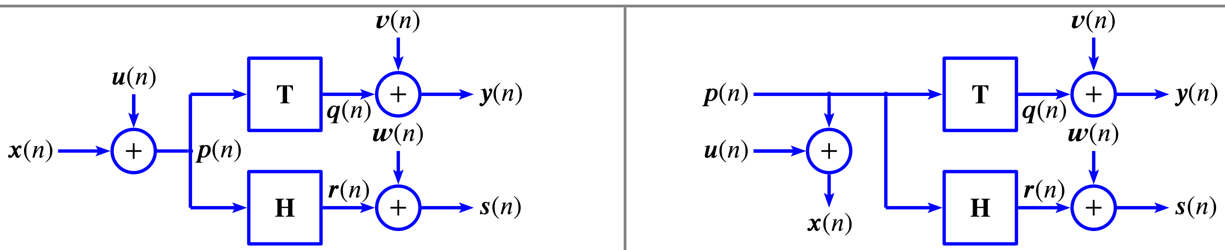
$$\left\{\begin{array}{llll}
\text{(A).} & x(n) \text{ is} & \text{WSS} & \text{and}\\
\text{(B).} & w(n) \text{ is} & \text{ZERO-MEAN} & \text{and}\\
\text{(C).} & v(n) \text{ is} & \text{ZERO-MEAN} & \text{and}\\
\text{(D).} & x(n), w(n), v(n) \text{ are} & \text{UNCORRELATED} & \text{and}\\
\text{(E).} & \mathbf{H} \text{ is} & \text{LTI} &
\end{array}\right\} \implies \left\{ \tilde{S}_{xy}(\omega) = \tilde{H}^*(\omega)\tilde{S}_{pp}(\omega) \right\}$$

✎PROOF:

$$\begin{aligned}
\tilde{S}_{yx}(\omega) &= \tilde{S}_{qx}(\omega) && \text{by Lemma 7.1 page 53}\\
&= \tilde{S}_{qp}(\omega) && \text{by Lemma 7.1 page 53}\\
&= \tilde{H}(\omega)\tilde{S}_{pp}(\omega) && \text{by Corollary 5.3 page 37}
\end{aligned}$$

✏

7.4 Additive noise and dual operators



(A) dual communications additive noise system (Corollary 7.7 page 58)

(B) dual measurement additive noise system (Corollary 7.8 page 59)

Figure 7.5: Dual Additive Noise Systems

Corollary 7.7. *Let* **S** *be the* SYSTEM *illustrated in Figure 7.5 (page 58) (A).*

COR

$$\left\{\begin{array}{llll}
\text{(A).} & \mathbf{H} \text{ is} & \text{LTI} & \text{and}\\
\text{(B).} & x(n) \text{ is} & \text{WSS} & \text{and}\\
\text{(C).} & u \text{ and } v \text{ are} & \text{ZERO-MEAN} & \text{and}\\
\text{(D).} & x, u, v \text{ are} & \text{UNCORRELATED} &
\end{array}\right\} \implies \left\{\begin{array}{llll}
\text{(1).} & \check{S}_{sy}(z) & = & \check{H}(z)\left[\check{S}_{xy}(z) + \check{S}_{uy}(z)\right] \quad \text{and}\\
\text{(2).} & \tilde{S}_{sy}(\omega) & = & \tilde{H}(\omega)\left[\tilde{S}_{xy}(\omega) + \tilde{S}_{uy}(\omega)\right]
\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\check{S}_{sy}(z) &= \check{S}_{rq}(z) && \text{by Corollary 7.2 page 53} && \text{and (B), (C) and (D)}\\
&= \check{H}(z)\check{S}_{pq}(z) && \text{by Theorem 5.2 page 37} && \text{and (A)}\\
&= \check{H}(z)\left[\check{S}_{xq}(z) + \check{S}_{uq}(z)\right] && \text{by Lemma 7.2 page 54}\\
&= \check{H}(z)\left[\check{S}_{xy}(z) + \check{S}_{uy}(z)\right] && \text{by Lemma 7.1 page 53}\\
\tilde{S}_{sy}(\omega) &= \check{S}_{sy}(z)\Big|_{z=e^{i\omega}}\\
&= \check{H}(z)\left[\check{S}_{xy}(z) + \check{S}_{uy}(z)\right]\Big|_{z=e^{i\omega}} && \text{by previous result} && (1)\\
&= \tilde{H}(\omega)\left[\tilde{S}_{xy}(\omega) + \tilde{S}_{uy}(\omega)\right]
\end{aligned}$$

Corollary 7.8. *Let* $\mathbf{S}$ *be the* SYSTEM *illustrated in Figure 7.5 (page 58) (B).*

COR

$$\left\{\begin{array}{llll} \text{(A).} & \mathbf{H} \textit{ is} & \text{LTI} & \textit{and} \\ \text{(B).} & x(n) \textit{ is} & \text{WSS} & \textit{and} \\ \text{(C).} & u \textit{ and } v \textit{ are} & \text{ZERO-MEAN} & \textit{and} \\ \text{(D).} & p, u, v \textit{ are} & \text{UNCORRELATED} & \end{array}\right\} \implies \left\{\begin{array}{llll} \text{(1).} & \check{\mathsf{S}}_{sy}(z) & = & \check{\mathsf{H}}(z)\check{\mathsf{S}}_{xy}(z) \quad \textit{and} \\ \text{(2).} & \tilde{\mathsf{S}}_{sy}(\omega) & = & \tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{xy}(\omega) \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\check{\mathsf{S}}_{sy}(z) &= \check{\mathsf{S}}_{rq}(z) && \text{by Corollary 7.2 page 53} && \text{and (B), (C) and (D)} \\
&= \check{\mathsf{H}}(z)\check{\mathsf{S}}_{pq}(z) && \text{by Theorem 5.2 page 37} && \text{and (A)} \\
&= \check{\mathsf{H}}(z)\check{\mathsf{S}}_{xq}(z) && \text{by Lemma 7.3 page 54} && \\
&= \check{\mathsf{H}}(z)\check{\mathsf{S}}_{xy}(z) && \text{by Lemma 7.1 page 53} && \\
\tilde{\mathsf{S}}_{sy}(\omega) &= \check{\mathsf{S}}_{sy}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
&= \check{\mathsf{H}}(z)\check{\mathsf{S}}_{xy}(z)\Big|_{z=e^{i\omega}} && \text{by previous result} && (1) \\
&= \tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{xy}(\omega)
\end{aligned}$$

Part III

Statistical Estimation

CHAPTER 8

ESTIMATION THEORY

Estimation types. Let $\boldsymbol{x}(t;\theta)$ be a waveform with parameter θ. There are three basic types of estimation on s:

1. *detection*:
   - The waveform $\boldsymbol{x}(t;\theta_n)$ is known except for the value of parameter θ_n.
   - The parameter θ_n is one of a finite set of values.
   - Estimate θ_n and thereby also estimate $\boldsymbol{x}(t;\theta)$.
2. *parametric* estimation:
   - The waveform $\boldsymbol{x}(t;\theta)$ is known except for the value of parameter θ.
   - The parameter θ is one of an infinite set of values.
   - Estimate θ and thereby also estimate $\boldsymbol{x}(t;\theta)$.
3. *nonparametric* estimation:
   - The waveform $\boldsymbol{x}(t)$ is unknown.
   - Estimate $\boldsymbol{x}(t)$.

Estimation criterion. Optimization requires a criterion against which the quality of an estimate is measured.[1] The most demanding and general criterion is the *Bayesian* criterion. The Bayesian criterion requires knowledge of the probability distribution functions and the definition of a *cost function*. Other criterion are special cases of the Bayesian criterion such that the cost function is defined in a special way, no cost function is defined, and/or the distribution is not known (Figure 8.2 page 66).

Estimation techniques. Estimation techniques can be classified into five groups (Figure 8.2 page 66):[2]

1. sequential decoding
2. norm minimization

[1] Mandyam D. Srinath (1996) ⟨013125295X⟩.

[2] Nelles (2001) page 26 ⟨"Fig 2.2 Overview of linear and nonlinear optimization techniques"⟩, Nelles (2001) page 33 ⟨"Fig 2.5 The Bayes method is the most general approach but…"⟩, Nelles (2001) page 63 ⟨"Table 3.3 Relationship between linear recursive and nonlinear optimization techniques"⟩, Nelles (2001) page 66

3. gradient search
4. inner product analysis
5. direct search

Sequential decoding is a non-linear estimation family. Perhaps the most famous of these is the Veterbi algorithm which uses a trellis to calculate the estimate. The Verterbi algorithm has been shown to yield an optimal estimate in the maximal likelihood (ML) sense. Norm minimization and gradient search algorithms are all linear algorithms. While this restriction to linear operations often simplifies calculations, it often yields an estimate that is not optimal in the ML sense.

8.1 Estimation criterion

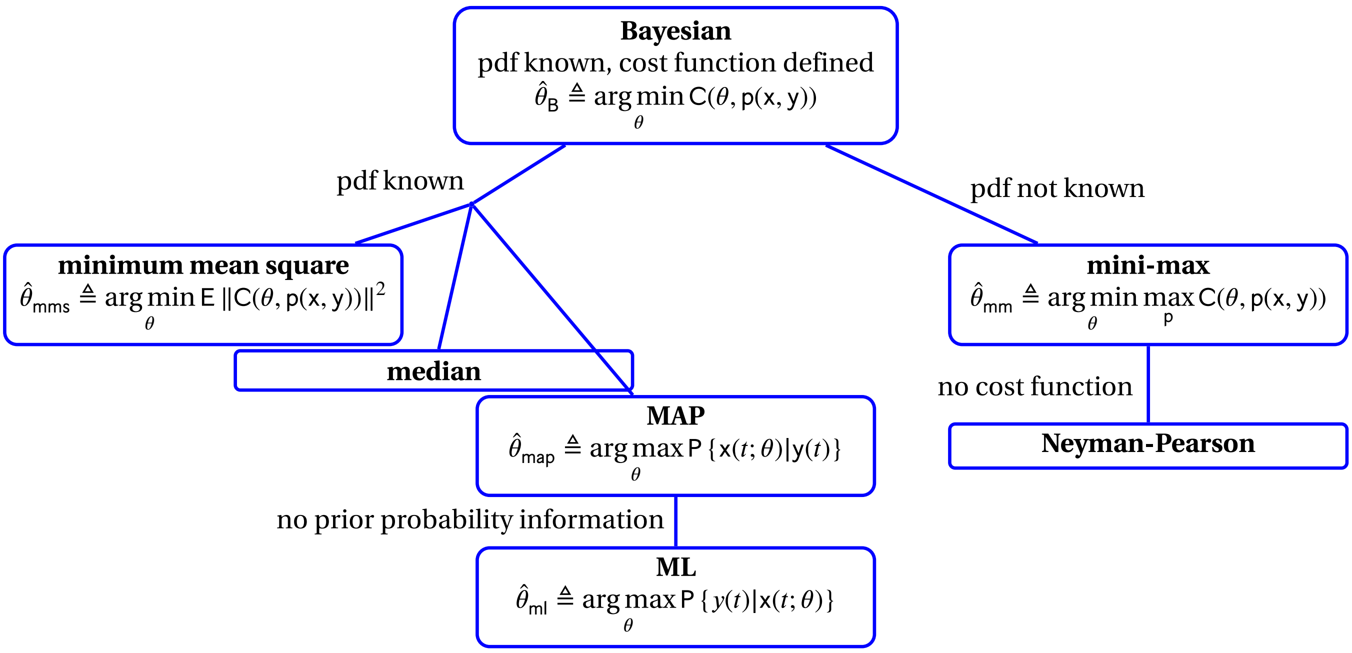


Figure 8.1: Estimation criterion

Definition 8.1. *Let*

(A). $x(t;\theta)$ *be a function with unknown parameter* θ
(B). $y(t)$ *a known function which is statistically dependent on* $x(t;\theta)$
(C). $C(\theta, \mathsf{p}(x, y))$ *be a cost function.*

Then the following ***estimates*** *are defined as follows:*

DEF

(1). ***Bayesian estimate*** $\hat{\theta}_{\mathsf{B}} \triangleq \arg\min_{\theta} C(\theta, \mathsf{p}(x, y))$

(2). ***Mean square estimate*** *(**"MS estimate"**)* $\hat{\theta}_{\mathsf{mms}} \triangleq \arg\min_{\theta} \mathsf{E}\,\|C(\theta, \mathsf{p}(x, y))\|^2$

(3). ***mini-max estimate*** *(**"MM estimate"**)* $\hat{\theta}_{\mathsf{mm}} \triangleq \arg\min_{\theta} \max_{\mathsf{p}} C(\theta, \mathsf{p}(x, y))$

(4). ***maximum a-posteriori probability estimate*** *(**"MAP estimate"**)* $\hat{\theta}_{\mathsf{map}} \triangleq \arg\max_{\theta} \mathsf{P}\{x(t;\theta)|y(t)\}$

(5). ***maximum likelihood estimate*** *(**"ML estimate"**)* $\hat{\theta}_{\mathsf{ml}} \triangleq \arg\max_{\theta} \mathsf{P}\{y(t)|x(t;\theta)\}$

Theorem 8.1. *Let* $\mathbf{x}(t;\theta)$ *be a function with unknown parameter* θ.

THM

$$\{\mathsf{P}\{\theta\} = \text{CONSTANT}\} \implies \{\hat{\theta}_{\mathsf{map}} = \hat{\theta}_{\mathsf{ml}}\}$$

✎PROOF:

$$\begin{aligned}
\hat{\theta}_{\text{map}} &\triangleq \arg\max_{\theta} \mathsf{P}\{s(t;\theta)|r(t)\} && \text{by definition of } \hat{\theta}_{\text{map}} && \text{(Definition 8.1 page 64)}\\
&= \arg\max_{\theta} \frac{\mathsf{P}\{s(t;\theta) \wedge \boldsymbol{y}(t)\}}{\mathsf{P}\{r(t)\}}\\
&= \arg\max_{\theta} \frac{\mathsf{P}\{r(t)|\boldsymbol{x}(t;\theta)\}\mathsf{P}\{s(t;\theta)\}}{\mathsf{P}\{r(t)\}}\\
&= \arg\max_{\theta} \mathsf{P}\{r(t)|\boldsymbol{x}(t;\theta)\}\mathsf{P}\{s(t;\theta)\}\\
&= \arg\max_{\theta} \mathsf{P}\{r(t)|\boldsymbol{x}(t;\theta)\}\\
&\triangleq \hat{\theta}_{\text{ml}} && \text{by definition of } \hat{\theta}_{\text{ml}} && \text{(Definition 8.1 page 64)}
\end{aligned}$$

⇨

Definition 8.2.

DEF

*The **mean square error** $\mathsf{mse}(\hat{\theta})$ of an estimate $\hat{\theta}$ of a parameter θ is defined as*

$$\mathsf{mse}(\hat{\theta}) \triangleq \mathsf{E}\left[\left(\hat{\theta}-\theta\right)^2\right]$$

The *mean square error* of $\hat{\theta}$ can be expressed as the sum of two components: the variance of $\hat{\theta}$ and the bias of $\hat{\theta}$ squared (next Theorem).

Theorem 8.2. [3]

THM

$$\mathsf{mse}(\hat{\theta}) = \underbrace{\mathsf{E}\left[\left(\hat{\theta}-\mathsf{E}\hat{\theta}\right)^2\right]}_{\text{variance of } \hat{\theta}} + \underbrace{\left[\mathsf{E}\hat{\theta}-\theta\right]^2}_{\text{bias of } \hat{\theta} \text{ squared}}$$

✎PROOF:

$$\begin{aligned}
\mathsf{mse}(\hat{\theta}) &\triangleq \mathsf{E}\left[\left(\hat{\theta}-\theta\right)^2\right] && \text{by definition of } \mathsf{mse} && \text{(Definition 8.2 page 65)}\\
&= \mathsf{E}\left[\left(\hat{\theta}-\mathsf{E}\hat{\theta}+\mathsf{E}\hat{\theta}-\theta\right)^2\right]\\
&= \mathsf{E}\left[\left(\hat{\theta}-\mathsf{E}\hat{\theta}\right)^2 + \underbrace{\left(\mathsf{E}\hat{\theta}-\theta\right)^2}_{\text{constant}} - 2\left(\hat{\theta}-\mathsf{E}\hat{\theta}\right)\left(\mathsf{E}\hat{\theta}-\theta\right)\right]\\
&= \mathsf{E}\left(\hat{\theta}-\mathsf{E}\hat{\theta}\right)^2 + \left(\mathsf{E}\hat{\theta}-\theta\right)^2 - 2\mathsf{E}\left[\hat{\theta}\mathsf{E}\hat{\theta} - \hat{\theta}\theta - \mathsf{E}\hat{\theta}\hat{\theta} + \mathsf{E}\hat{\theta}\theta\right]\\
&= \mathsf{E}\left(\hat{\theta}-\mathsf{E}\hat{\theta}\right)^2 + \left(\mathsf{E}\hat{\theta}-\theta\right)^2 - 2\left[\mathsf{E}\hat{\theta}\mathsf{E}\hat{\theta} - \mathsf{E}\hat{\theta}\mathsf{E}\theta - \mathsf{E}\hat{\theta}\mathsf{E}\hat{\theta} + \mathsf{E}\hat{\theta}\mathsf{E}\theta\right]\\
&= \mathsf{E}\left(\hat{\theta}-\mathsf{E}\hat{\theta}\right)^2 + \left(\mathsf{E}\hat{\theta}-\theta\right)^2 - 0
\end{aligned}$$

⇨

8.2 Estimation techniques

8.2.1 Sequential decoding

It has been shown that the Viterbi algorithm (trellis) produces an optimal estimate in the maximal likelihood (ML) sense. A Verterbi trellis is shown in Figure 8.3 (page 66).

[3] Kay (1988) page 45 ⟨§"3.3 ESTIMATION THEORY"⟩

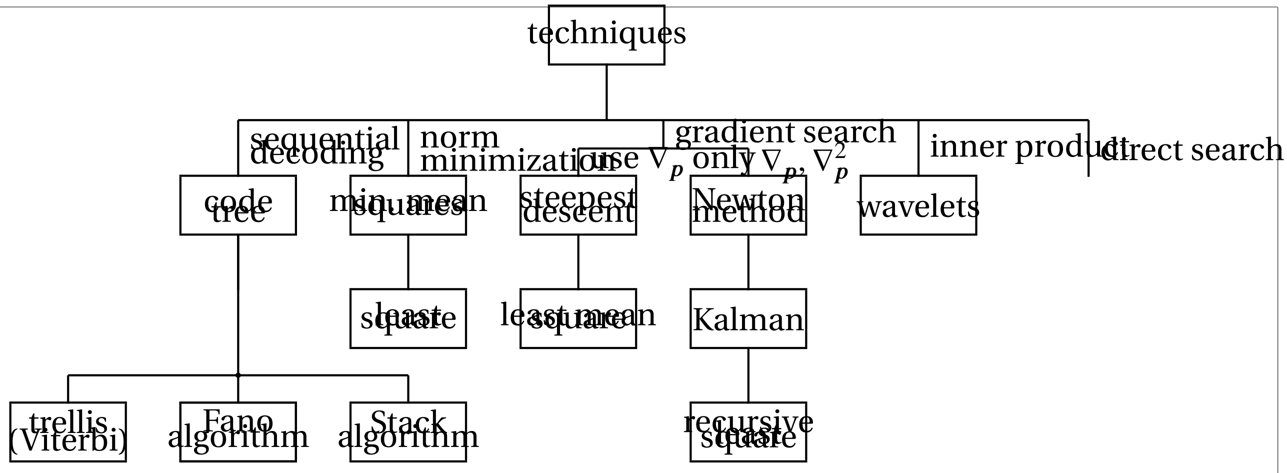


Figure 8.2: Estimation techniques

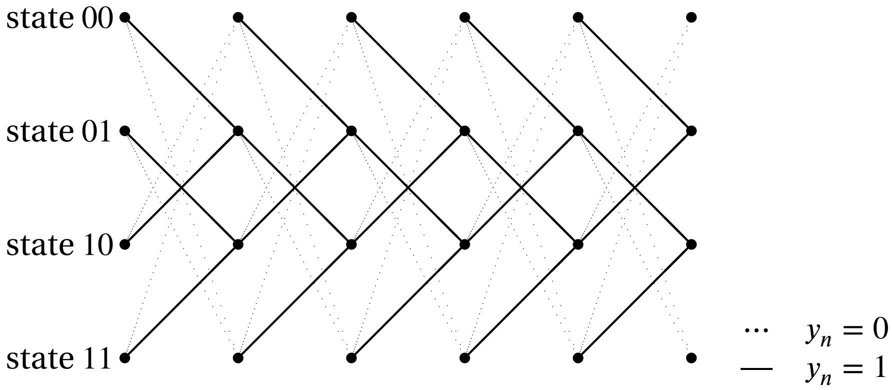


Figure 8.3: Viterbi algorithm trellis

8.2.2 Norm minimization

Norm minimization techniques are very powerful in that an optimum solution can be computed in one step without iteration or recursion. In this section we present two types of norm minimization: [4]

1. minimum mean square estimation (MMSE):
   The MMS estimate is a *stochastic* estimate. To compute the MMS estimate, the we do not need to know the actual data values, but we must know certain system statistics which are the input data autocorrelation and input/output crosscorrelation. The cost function is the expected value of the norm squared error.
2. least square estimation (LSE):
   The LS estimate is a *deterministic* estimate. To compute the LS estimate, we must know the actual data values (although these may be "noisy" measurements). The cost function is the norm squared error.

Solutions to both are given in terms of two matrices:

Y: Autocorrelation matrix
W: Crosscorrelation matrix.

[4] The Least Squares algorithm is nothing new to mathematics. It was first developed in 1795 by Gauss who was also the first to discover the FFT.

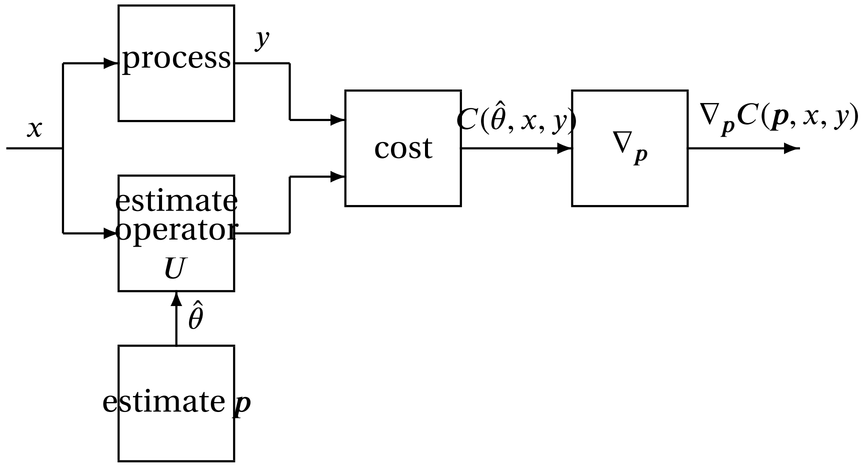


Figure 8.4: Estimation using gradient of cost function

Minimum mean square estimation

Definition 8.3. *Let the following vectors, matrices, and functions be defined as follows:*

DEF

$\boldsymbol{x}$	$\in$	$\mathbb{C}^m$	*data vector*
$\boldsymbol{y}$	$\in$	$\mathbb{C}^n$	*processed data vector*
$\hat{\boldsymbol{y}}$	$\in$	$\mathbb{C}^n$	*processed data estimate vector*
$\boldsymbol{e}$	$\in$	$\mathbb{C}^n$	*error vector*
$\boldsymbol{p}$	$\in$	$\mathbb{R}^m$	*parameter vector*
U	$\in$	C_{mn}	*regression matrix*
R	$\in$	C_{mm}	*autocorrelation matrix*
W	$\in$	$\mathbb{C}^m$	*cross-correlation vector*
C	$:$	$\mathbb{R}^m \to \mathbb{R}^{\vdash}$	*cost function*

Theorem 8.3 (Minimum mean square estimation)**.** *Let*

$$\begin{aligned}
\hat{\boldsymbol{y}}(\boldsymbol{p}) &\triangleq U^H \boldsymbol{p} \\
\boldsymbol{e}(\boldsymbol{p}) &\triangleq \hat{\boldsymbol{y}} - \boldsymbol{y} \\
\mathsf{C}(\boldsymbol{p}) &\triangleq \mathsf{E}\,\|\boldsymbol{e}\|^2 \triangleq \mathsf{E}\left[\boldsymbol{e}^H \boldsymbol{e}\right] \\
\hat{\theta}_{\mathsf{mms}} &\triangleq \arg\min_{\boldsymbol{p}} \mathsf{C}(\boldsymbol{p}) \\
R &\triangleq \mathsf{E}\left[U U^H\right] \\
W &\triangleq \mathsf{E}\left[U \boldsymbol{y}\right].
\end{aligned}$$

Then

THM

$$\begin{aligned}
\hat{\theta}_{\mathsf{mms}} &= (\mathbf{R}_{\mathsf{e}} Y)^{-1} (\mathbf{R}_{\mathsf{e}} W) \\
\mathsf{C}(\boldsymbol{p}) &= \boldsymbol{p}^H R \boldsymbol{p} - (W^H \boldsymbol{p})^* - W^H \boldsymbol{p} + \mathsf{E} \boldsymbol{y}^H \boldsymbol{y} \\
\nabla_p \mathsf{C}(\boldsymbol{p}) &= 2\mathbf{R}_{\mathsf{e}}\,[Y]\boldsymbol{p} - 2\mathbf{R}_{\mathsf{e}} W \\
\mathsf{C}(\hat{\theta}_{\mathsf{mms}}) &= (\mathbf{R}_{\mathsf{e}} W^H)(\mathbf{R}_{\mathsf{e}} Y)^{-1} R (\mathbf{R}_{\mathsf{e}} Y)^{-1} (\mathbf{R}_{\mathsf{e}} W) - 2(\mathbf{R}_{\mathsf{e}} W^H)(\mathbf{R}_{\mathsf{e}} Y)^{-1} (\mathbf{R}_{\mathsf{e}} W) + \mathsf{E} \boldsymbol{y}^H \boldsymbol{y} \\
\mathsf{C}(\hat{\theta}_{\mathsf{mms}})|_{R\ real} &= \mathsf{E} \boldsymbol{y}^H \boldsymbol{y} - (\mathbf{R}_{\mathsf{e}} W^H) R^{-1} (\mathbf{R}_{\mathsf{e}} W).
\end{aligned}$$

✎PROOF: See APPENDIX E (page 173) for a Matrix Calculus reference.

$$\begin{aligned}
\mathsf{C}(\boldsymbol{p}) &\triangleq \mathsf{E}\,\|\boldsymbol{e}\|^2 \\
&= \mathsf{E}\left[\boldsymbol{e}^H\boldsymbol{e}\right] \\
&= \mathsf{E}\left[(\hat{\boldsymbol{y}}-\boldsymbol{y})^H(\hat{\boldsymbol{y}}-\boldsymbol{y})\right] \\
&= \mathsf{E}\left[\left(U^H\boldsymbol{p}-\boldsymbol{y}\right)^H\left(U^H\boldsymbol{p}-\boldsymbol{y}\right)\right] \\
&= \mathsf{E}\left[\left(\boldsymbol{p}^H U-\boldsymbol{y}^H\right)\left(U^H\boldsymbol{p}-\boldsymbol{y}\right)\right] \\
&= \mathsf{E}\left[\boldsymbol{p}^H UU^H\boldsymbol{p}-\boldsymbol{p}^H U\boldsymbol{y}-\boldsymbol{y}^H U^H\boldsymbol{p}+\boldsymbol{y}^H\boldsymbol{y}\right] \\
&= \boldsymbol{p}^H\mathsf{E}\left[UU^H\right]\boldsymbol{p}-\boldsymbol{p}^H\mathsf{E}\left[U\boldsymbol{y}\right]-\mathsf{E}\left[\boldsymbol{y}^H U^H\right]\boldsymbol{p}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H\mathsf{E}\left[UU^H\right]\boldsymbol{p}-(\mathsf{E}\left[U\boldsymbol{y}\right]^H\boldsymbol{p})^H-\mathsf{E}\left[U\boldsymbol{y}\right]^H\boldsymbol{p}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p}-(W^H\boldsymbol{p})^H-W^H\boldsymbol{p}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p}-(W^H\boldsymbol{p})^*-W^H\boldsymbol{p}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p}-(W^H)^*\boldsymbol{p}-W^H\boldsymbol{p}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p}-2\mathbf{R}_{\mathsf{e}}\left[W^H\right]\boldsymbol{p}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\boldsymbol{p}}\mathsf{C}(\boldsymbol{p}) &= \nabla_{\boldsymbol{p}}\left[\boldsymbol{p}^H R\boldsymbol{p}-(W^H)^*\boldsymbol{p}-W^H\boldsymbol{p}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y}\right] \\
&= R\boldsymbol{p}+R^T\boldsymbol{p}-[(W^H)^*]^T-[W^H]^T+0 \\
&= R\boldsymbol{p}+(R^H)^*\boldsymbol{p}-W-W^* \\
&= R\boldsymbol{p}+R^*\boldsymbol{p}-W-W^* \\
&= (R+R^*)\boldsymbol{p}-(W+W^*) \\
&= 2(\mathbf{R}_{\mathsf{e}}Y)\boldsymbol{p}-2\mathbf{R}_{\mathsf{e}}W
\end{aligned}$$

$$\boldsymbol{p}_{\mathsf{opt}} = (\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)$$

$$\begin{aligned}
\mathsf{C}(\boldsymbol{p}_{\mathsf{opt}}) &= \boldsymbol{p}_{\mathsf{opt}}^H R\boldsymbol{p}_{\mathsf{opt}}-2\mathbf{R}_{\mathsf{e}}\left[W^H\right]\boldsymbol{p}_{\mathsf{opt}}+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= [(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)]^H R[(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)]-2\mathbf{R}_{\mathsf{e}}\left[W^H\right][(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)]+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R}_{\mathsf{e}}W^H)(\mathbf{R}_{\mathsf{e}}Y)^{-H}R(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)-2\mathbf{R}_{\mathsf{e}}\left[W^H\right](\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R}_{\mathsf{e}}W^H)(\mathbf{R}_{\mathsf{e}}R^H)^{-1}R(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)-2\mathbf{R}_{\mathsf{e}}\left[W^H\right](\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R}_{\mathsf{e}}W^H)(\mathbf{R}_{\mathsf{e}}Y)^{-1}R(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)-2(\mathbf{R}_{\mathsf{e}}W^H)(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y}
\end{aligned}$$

$$\begin{aligned}
\mathsf{C}(\boldsymbol{p}_{\mathsf{opt}})|_{R\text{ real}} &= (\mathbf{R}_{\mathsf{e}}W^H)(\mathbf{R}_{\mathsf{e}}Y)^{-1}R(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)-2(\mathbf{R}_{\mathsf{e}}W^H)(\mathbf{R}_{\mathsf{e}}Y)^{-1}(\mathbf{R}_{\mathsf{e}}W)+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R}_{\mathsf{e}}W^H)R^{-1}RR^{-1}(\mathbf{R}_{\mathsf{e}}W)-2(\mathbf{R}_{\mathsf{e}}W^H)R^{-1}(\mathbf{R}_{\mathsf{e}}W)+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R}_{\mathsf{e}}W^H)R^{-1}(\mathbf{R}_{\mathsf{e}}W)-2(\mathbf{R}_{\mathsf{e}}W^H)R^{-1}(\mathbf{R}_{\mathsf{e}}W)+\mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
&= \mathsf{E}\boldsymbol{y}^H\boldsymbol{y}-(\mathbf{R}_{\mathsf{e}}W^H)R^{-1}(\mathbf{R}_{\mathsf{e}}W)
\end{aligned}$$

⇰

In many adaptive filter and equalization applications, the autocorrelation matrix U is simply the m-element random data vector $\boldsymbol{x}(k)$ at time k, as in the *Wiener-Hopf equations* (next).

Corollary 8.1 (Wiener-Hopf equations). [5]

[5] Ifeachor and Jervis (1993) pages 547–549 ⟨§"9.3 Basic Wiener filter theory"⟩, Ifeachor and Jervis (2002) pages 651–654 ⟨§"10.3 Basic Wiener filter theory"⟩

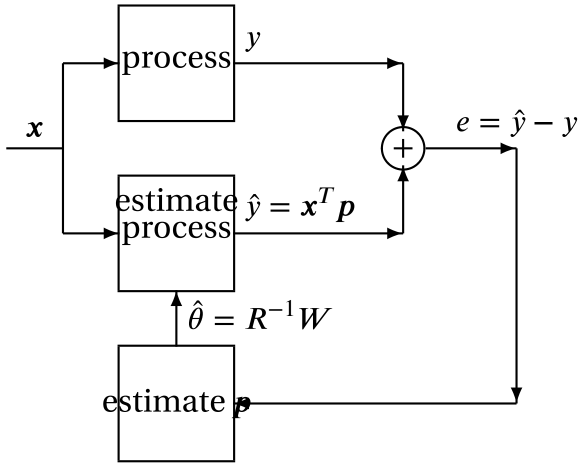


Figure 8.5: Adative filter example

COR

$$\left\{ U \triangleq \boldsymbol{x}(k) \triangleq \begin{bmatrix} x(k) \\ x(k-1) \\ x(k-2) \\ \vdots \\ x(k-m+1) \end{bmatrix} \right\} \implies \left\{ \begin{array}{rcl} \hat{\theta}_{\text{mms}} &=& R^{-1}W \\ \mathsf{C}(\hat{\theta}_{\text{mms}}) &=& W^T R^{-1} R R^{-1} W - 2W^T R^{-1} W + \mathsf{E}\boldsymbol{y}^T\boldsymbol{y} \end{array} \right\}$$

✎PROOF: This is a special case of the more general case discussed in Theorem 8.3 (page 67). Here, the dimension of U is $m \times 1$ (n=1). As a result, $\boldsymbol{y}$, $\hat{\boldsymbol{y}}$, and $\boldsymbol{e}$ are simply scalar quantities (not vectors). In this special case, we have the following results (Figure 8.5 page 69):

$$\begin{aligned}
\hat{y}(\boldsymbol{p}) &\triangleq \boldsymbol{x}^T\boldsymbol{p} \\
e(\boldsymbol{p}) &\triangleq \hat{y} - y \\
\mathsf{C}(\boldsymbol{p}) &\triangleq \mathsf{E}\,\|e\|^2 \triangleq \mathsf{E}\left[e^2\right] \\
\hat{\theta}_{\text{mms}} &\triangleq \arg\min_{\boldsymbol{p}} \mathsf{C}(\boldsymbol{p}) \\
R &\triangleq \mathsf{E}\left[\boldsymbol{x}\boldsymbol{x}^T\right] \\
W &\triangleq \mathsf{E}\left[\boldsymbol{x}y\right] \\
\mathsf{C}(\boldsymbol{p}) &= \boldsymbol{p}^T R\boldsymbol{p} - 2W^T\boldsymbol{p} + \mathsf{E}\left[\boldsymbol{y}^T\boldsymbol{y}\right] \\
\nabla_{\boldsymbol{p}}\mathsf{C}(\boldsymbol{p}) &= 2R\boldsymbol{p} - 2W \\
\mathsf{C}(\hat{\theta}_{\text{mms}})|_{R\text{ real}} &= \mathsf{E}\boldsymbol{y}^T\boldsymbol{y} - W^T R^{-1} W.
\end{aligned}$$

⇰

Least squares

Theorem 8.4 (Least squares)**.** *Let*

$$\begin{aligned}
\hat{\boldsymbol{y}}(\boldsymbol{p}) &\triangleq U^H\boldsymbol{p} \\
\boldsymbol{e}(\boldsymbol{p}) &\triangleq \hat{\boldsymbol{y}} - \boldsymbol{y} \\
\mathsf{C}(\boldsymbol{p}) &\triangleq \|\boldsymbol{e}\|^2 \triangleq \boldsymbol{e}^H\boldsymbol{e}
\end{aligned}$$

$$\hat{\theta}_{\mathsf{ls}} \triangleq \arg\min_{p} \mathsf{C}(\boldsymbol{p})$$
$$R \triangleq UU^H$$
$$W \triangleq U\boldsymbol{y}.$$

Then

THM

$$\begin{aligned}
\hat{\theta}_{\mathsf{ls}} &= (\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) \\
\mathsf{C}(\boldsymbol{p}) &= \boldsymbol{p}^H R\boldsymbol{p} - (W^H\boldsymbol{p})^* - W^H\boldsymbol{p} + \mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
\nabla_{\boldsymbol{p}}\mathsf{C}(\boldsymbol{p}) &= 2\mathbf{R_e}[Y]\boldsymbol{p} - 2\mathbf{R_e}W \\
\mathsf{C}(\hat{\theta}_{\mathsf{ls}}) &= (\mathbf{R_e}W^H)(\mathbf{R_e}Y)^{-1}R(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) - 2(\mathbf{R_e}W^H)(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) + \mathsf{E}\boldsymbol{y}^H\boldsymbol{y} \\
\mathsf{C}(\hat{\theta}_{\mathsf{ls}})|_{R \text{ real}} &= \mathsf{E}\boldsymbol{y}^H\boldsymbol{y} - (\mathbf{R_e}W^H)R^{-1}(\mathbf{R_e}W).
\end{aligned}$$

✎PROOF: See APPENDIX E (page 173) for a Matrix Calculus reference.

$$\begin{aligned}
\mathsf{C}(\boldsymbol{p}) &\triangleq \|\boldsymbol{e}\|^2 \\
&= e^H\boldsymbol{e} \\
&= (\hat{\boldsymbol{y}} - \boldsymbol{y})^H(\hat{\boldsymbol{y}} - \boldsymbol{y}) \\
&= \left(U^H\boldsymbol{p} - \boldsymbol{y}\right)^H\left(U^H\boldsymbol{p} - \boldsymbol{y}\right) \\
&= \left(\boldsymbol{p}^H U - \boldsymbol{y}^H\right)\left(U^H\boldsymbol{p} - \boldsymbol{y}\right) \\
&= \boldsymbol{p}^H UU^H\boldsymbol{p} - \boldsymbol{p}^H U\boldsymbol{y} - \boldsymbol{y}^H U^H\boldsymbol{p} + \boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p} - (W^H\boldsymbol{p})^H - W^H\boldsymbol{p} + \boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p} - (W^H\boldsymbol{p})^* - W^H\boldsymbol{p} + \boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p} - (W^H)^*\boldsymbol{p} - W^H\boldsymbol{p} + \boldsymbol{y}^H\boldsymbol{y} \\
&= \boldsymbol{p}^H R\boldsymbol{p} - 2\mathbf{R_e}\left[W^H\right]\boldsymbol{p} + \boldsymbol{y}^H\boldsymbol{y}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\boldsymbol{p}}\mathsf{C}(\boldsymbol{p}) &= \nabla_{\boldsymbol{p}}\left[\boldsymbol{p}^H R\boldsymbol{p} - (W^H)^*\boldsymbol{p} - W^H\boldsymbol{p} + \boldsymbol{y}^H\boldsymbol{y}\right] \\
&= R\boldsymbol{p} + R^T\boldsymbol{p} - [(W^H)^*]^T - [W^H]^T + 0 \\
&= R\boldsymbol{p} + (R^H)^*\boldsymbol{p} - W - W^* \\
&= R\boldsymbol{p} + R^*\boldsymbol{p} - W - W^* \\
&= (R + R^*)\boldsymbol{p} - (W + W^*) \\
&= 2(\mathbf{R_e}Y)\boldsymbol{p} - 2\mathbf{R_e}W
\end{aligned}$$

$$\boldsymbol{p}_{\mathsf{opt}} = (\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W)$$

$$\begin{aligned}
\mathsf{C}(\boldsymbol{p}_{\mathsf{opt}}) &= \boldsymbol{p}_{\mathsf{opt}}^H R\boldsymbol{p}_{\mathsf{opt}} - 2\mathbf{R_e}\left[W^H\right]\boldsymbol{p}_{\mathsf{opt}} + \boldsymbol{y}^H\boldsymbol{y} \\
&= [(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W)]^H R[(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W)] - 2\mathbf{R_e}\left[W^H\right][(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W)] + \boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R_e}W^H)(\mathbf{R_e}Y)^{-H}R(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) - 2\mathbf{R_e}\left[W^H\right](\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) + \boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R_e}W^H)(\mathbf{R_e}R^H)^{-1}R(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) - 2\mathbf{R_e}\left[W^H\right](\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) + \boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R_e}W^H)(\mathbf{R_e}Y)^{-1}R(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) - 2(\mathbf{R_e}W^H)(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) + \boldsymbol{y}^H\boldsymbol{y}
\end{aligned}$$

$$\begin{aligned}
\mathsf{C}(\boldsymbol{p}_{\mathsf{opt}})|_{R \text{ real}} &= (\mathbf{R_e}W^H)(\mathbf{R_e}Y)^{-1}R(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) - 2(\mathbf{R_e}W^H)(\mathbf{R_e}Y)^{-1}(\mathbf{R_e}W) + \boldsymbol{y}^H\boldsymbol{y} \\
&= (\mathbf{R_e}W^H)R^{-1}RR^{-1}(\mathbf{R_e}W) - 2(\mathbf{R_e}W^H)R^{-1}(\mathbf{R_e}W) + \boldsymbol{y}^H\boldsymbol{y}
\end{aligned}$$

$$= (\mathbf{R}_e W^H) R^{-1} (\mathbf{R}_e W) - 2(\mathbf{R}_e W^H) R^{-1} (\mathbf{R}_e W) + \boldsymbol{y}^H \boldsymbol{y}$$
$$= \boldsymbol{y}^H \boldsymbol{y} - (\mathbf{R}_e W^H) R^{-1} (\mathbf{R}_e W)$$

✏

Example 8.1 (Polynomial approximation).
Suppose we **know** the locations $\{(x_n, y_n) | n = 1, 2, 3, 4, 5\}$ of 5 data points. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the locations of these points such that

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \boldsymbol{y} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

Suppose we want to find a second order polynomial

$$cx^2 + bx + a$$

that best approximates these 5 points in the least squares sense. We define the matrix U (known) and vector $\hat{\theta}$ (to be computed) as follows:

$$U^H \triangleq \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix}}_{\text{Vandermonde matrix}^{6}} {}^{6} \qquad \hat{\theta} \triangleq \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Then, using Theorem 8.4 (page 69), the best coefficients $\hat{\theta}$ for the polynomial are

$$\hat{\theta} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
$$= R^{-1} W$$
$$= (U U^H)^{-1} (U \boldsymbol{y})$$
$$= \left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix}^H \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix}^H \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \right)$$

8.2.3 Gradient search techniques

One of the biggest advantages of using a gradient search technique is that they can be implemented *recursively* as shown in the next equation. The general form of the gradient search parameter estimation techniques is[7]

THM $\boxed{\boldsymbol{p}_n = \boldsymbol{p}_{n-1} - \eta_{n-1} R \left[\nabla_p \mathsf{C}(\boldsymbol{p}_n)\right]}$ where at time n

[6] Horn and Johnson (1990)29
[7] Nelles (2001), page 90

$\boldsymbol{p}_n$	is the *state*	(vector)
η_n	is the *step size*	(scalar)
Y	is the *direction*	(matrix)
$\nabla_p \mathsf{C}(\boldsymbol{p}_n)$	is the *gradient* of the cost function $\mathsf{C}(\boldsymbol{p}_n)$	(vector)

Two major categories of gradient search techniques are

- steepest descent (includes LMS)
- Newton's method (includes RLS and Kalman filters).

The key difference between the two is that ***steepest descent* uses only first derivative information**, while ***Newton's method* uses both first and second derivative information** making it converge much faster but with significantly higher complexity.

First derivative techniques

Steepest descent. In this algorithm, $R = I$ (identity matrix). First derivative information is contained in $\nabla\mathsf{C}$. Second derivative information, if present, is contained in Y. Thus, steepest descent algorithms do not use second derivative information.

THM $\boldsymbol{p}_n = \boldsymbol{p}_{n-1} - \eta_{n-1}\left[\nabla_p \mathsf{C}(\boldsymbol{p}_n)\right]$

Least Mean Squares (LMS). [8] This is a special case of *steepest descent*. In minimum mean square estimation (Section 8.2.2 page 67), the cost function $\mathsf{C}(\boldsymbol{p})$ is defined as a *statistical average* of the error vector such that $\mathsf{C}(\boldsymbol{p}) = \mathsf{E}\left[\boldsymbol{e}^H \boldsymbol{e}\right]$. In this case the gradient $\nabla\mathsf{C}$ is difficult to compute. However, the LMS algorithm greatly simplifies the problem by instead defining the cost function as a function of the *instantaneous error* such that

$$\begin{aligned}
\boldsymbol{y} &= y(n)\\
\hat{\boldsymbol{y}} &= \hat{y}(n)\\
\mathsf{C}(\boldsymbol{p}) &= \|e(n)\|^2\\
&= e^2(n)\\
&= (\hat{y}(n) - y(n))^2
\end{aligned}$$

Computing the gradient of this cost function is then just a special case of *least squares estimation* (Section 8.2.2 page 69). Using LS, we let $U = \boldsymbol{x}^T$ and hence

$$\begin{aligned}
\nabla_p \mathsf{C}(\boldsymbol{p}) &= 2U^T U\boldsymbol{p} - 2U^T\boldsymbol{y} && \text{by Theorem 8.4 page 69}\\
&= 2\boldsymbol{x}\boldsymbol{x}^T\boldsymbol{p} - 2\boldsymbol{x}y && \text{by above definitions}\\
&= 2\boldsymbol{x}\hat{y} - 2\boldsymbol{x}y\\
&= 2\boldsymbol{x}(\hat{y} - y)\\
&= 2\boldsymbol{x}e(n)
\end{aligned}$$

The LMS algorithm uses this instantaneous gradient for $\nabla\mathsf{C}$, lets $R = I$, and uses a constant step

[8] Manolakis et al. (2000), page 526

size η to give

THM $\boxed{p_n = p_{n-1} - 2\eta x_n e(n)}$

Second derivative techniques

Newton's Method. This algorithm uses the *Hessian* matrix H, which is the second derivative of the cost function $\mathsf{C}(p)$, and lets $R = H^{-1}$.

$$\begin{aligned} H_n &\triangleq \nabla_p \nabla_p \mathsf{C}(p_n) \\ p_n &= p_{n-1} - \eta_{n-1} H_n^{-1} \left[\nabla_p \mathsf{C}(p_n)\right] \end{aligned}$$

Kalman filtering [9]

$$\begin{aligned} \gamma(k) &= \frac{1}{x^T(k)P(k-1)x(k)+1} P(k-1)x(k) \\ P(k) &= (I - \gamma(k)x^T(k))P(k-1) + V \\ e(k) &= y(k) - x^T(k)\hat{p}(k-1) \\ \hat{p}(k) &= \hat{p}(k-1) + \gamma(k)e(k) \end{aligned}$$

Recursive Least Squares (RLS) [10] This algorithm is a special case of either the RLS with forgetting or the Kalman filter.

$$\begin{aligned} \gamma(k) &= \frac{1}{x^T(k)P(k-1)x(k)+1} P(k-1)x(k) \\ P(k) &= (I - \gamma(k)x^T(k))P(k-1) \\ e(k) &= y(k) - x^T(k)\hat{p}(k-1) \\ \hat{p}(k) &= \hat{p}(k-1) + \gamma(k)e(k) \end{aligned}$$

8.2.4 Direct search

A direct search algorithm may be used in cases where the cost function over p has several local minimums, making convergence difficult. Furthermore, direct search algorithms can be very computationally demanding.

[9] Nelles (2001), page 66
[10] Nelles (2001), page 66

CHAPTER 9

KL-EXPANSION APPLICATION

9.1 Sufficient statistics

Theorem 9.1 (page 75) (next) shows that the finite set $Y \triangleq \{\dot{y}_n | n = 1, 2, \ldots, N\}$ provides just as much information as having the entire $\mathbf{y}(t)$ waveform (an uncountably infinite number of values) with respect to the following cases:

1. the conditional probability of $\mathbf{x}(t; \hat{\theta})$ given $\mathbf{y}(t)$
2. the *MAP estimate* of the information sequence
3. the *ML estimate* of the information sequence.

That is, even with a drastic reduction in the amount of information from uncountably infinite to finite N, no information is lost with respect to the quantities listed above.

This amazing result is very useful in practical system implementation and also for proving other theoretical results (notably estimation and detection theorems which come later in this chapter).

Theorem 9.1 (Sufficient statistic theorem). [1] *Let* **S** *be an additive White Gaussian noise system and* Ψ *an orthonormal basis for* $\mathbf{x}(t; \hat{\theta})$ *such that*

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{x}(t; \hat{\theta}) + \mathbf{v}(t) \\ \Psi &= \{\psi_n | n = 1, 2, \ldots, N\} \end{aligned}$$

Then $Y \triangleq \{\dot{y}_n | n = 1, 2, \ldots, N\}$ *is a* ***sufficient statistic*** *for* $\mathrm{y}(t)$ *such that...*

T H M

$$\left\{ \mathbf{v}(t) \text{ is AWGN } \right\} \implies \left\{ \begin{array}{lllll} \text{(1).} & \mathsf{P}\{\mathbf{x}(t;\hat{\theta})|\mathbf{y}(t)\} & & = & \mathsf{P}\{\mathbf{x}(t;\hat{\theta})|Y\} \\ \text{(2).} & \hat{\theta}_{\mathsf{map}} & \triangleq \arg\max\limits_{\hat{\theta}} \mathsf{P}\{\mathbf{x}(t;\hat{\theta})|\mathbf{y}(t)\} & = & \arg\max\limits_{\hat{\theta}} \mathsf{P}\{\mathbf{x}(t;\hat{\theta})|Y\} \\ \text{(3).} & \hat{\theta}_{\mathsf{ml}} & \triangleq \arg\max\limits_{\hat{\theta}} \mathsf{P}\{\mathbf{y}(t)|\mathbf{x}(t;\hat{\theta})\} & = & \arg\max\limits_{\hat{\theta}} \mathsf{P}\{Y|\mathbf{x}(t;\hat{\theta})\} \end{array} \right\}$$

✎PROOF: Let $\mathbf{v}'(t) \triangleq \mathbf{v}(t) - \sum_{n=1}^{N} \dot{v}_n \psi_n(t)$.

[1] Fisher (1922) page 316 ⟨"Criterion of Sufficiency"⟩

1. The relationship between Y and $\boldsymbol{v}'(t)$ is given by

$$
\begin{aligned}
\boldsymbol{y}(t) &= \sum_{n=1}^{N} \langle \boldsymbol{y}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) + \left[\boldsymbol{y}(t) - \sum_{n=1}^{N} \langle \boldsymbol{y}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) \right] \\
&= \sum_{n=1}^{N} \langle \boldsymbol{y}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) + \left[\boldsymbol{y}(t) - \sum_{n=1}^{N} \langle \boldsymbol{x}(t) + \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) \right] \\
&= \sum_{n=1}^{N} \dot{y}_n \psi_n(t) + \left[\boldsymbol{x}(t) + \boldsymbol{v}(t) - \sum_{n=1}^{N} \langle \boldsymbol{x}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) - \sum_{n=1}^{N} \langle \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) \right] \\
&= \sum_{n=1}^{N} \dot{y}_n \psi_n(t) + \boldsymbol{x}(t) + \boldsymbol{v}(t) - \boldsymbol{x}(t) - \left[\boldsymbol{v}(t) - \boldsymbol{v}'(t) \right] \\
&= \sum_{n=1}^{N} \dot{y}_n \psi_n(t) + \boldsymbol{v}'(t).
\end{aligned}
$$

2. Proof that the set of statistics Y and the random process $\boldsymbol{v}'(t)$ are *uncorrelated*:

$$
\begin{aligned}
\mathsf{E}\left[\dot{y}_n \boldsymbol{v}'(t) \right] &= \mathsf{E}\left[\langle \boldsymbol{y}(t) \,|\, \psi_n(t) \rangle \left(\boldsymbol{v}(t) - \sum_{n=1}^{N} \langle \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) \right) \right] \\
&= \mathsf{E}\left[\langle \boldsymbol{x}(t) + \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \left(\boldsymbol{v}(t) - \sum_{n=1}^{N} \langle \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) \right) \right] \\
&= \mathsf{E}\left[\left(\langle \boldsymbol{x}(t) \,|\, \psi_n(t) \rangle + \langle \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \right) \left(\boldsymbol{v}(t) - \sum_{n=1}^{N} \langle \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) \right) \right] \\
&= \mathsf{E}\left[\left(\dot{x}_n + \dot{v}_n \right) \left(\boldsymbol{v}(t) - \sum_{n=1}^{N} \dot{v}_n \psi_n(t) \right) \right] \\
&= \mathsf{E}\left[\dot{x}_n \boldsymbol{v}(t) - \dot{x}_n \sum_{n=1}^{N} \dot{v}_n \psi_n(t) + \dot{v}_n \boldsymbol{v}(t) - \dot{v}_n \sum_{n=1}^{N} \dot{v}_n \psi_n(t) \right] \\
&= \mathsf{E}\left[\dot{x}_n \boldsymbol{v}(t) \right] - \mathsf{E}\left[\dot{x}_n \sum_{n=1}^{N} \dot{v}_n \psi_n(t) \right] + \mathsf{E}\left[\langle \boldsymbol{v}(t) \,|\, \psi_n(t) \rangle \, \boldsymbol{v}(t) \right] - \mathsf{E}\left[\sum_{m=1}^{N} \dot{v}_n \dot{v}_m \psi_m(t) \right] \\
&= \dot{x}_n \cancelto{0}{\mathsf{E}\boldsymbol{v}(t)} - \dot{x}_n \sum_{n=1}^{N} \cancelto{0}{\mathsf{E}\left[\dot{v}_n \right]} \psi_n(t) + \mathsf{E}\left[\left\langle \boldsymbol{v}(t) \boldsymbol{v}(\hat{\theta}) \,|\, \psi_n(\hat{\theta}) \right\rangle \right] - \sum_{m=1}^{N} \mathsf{E}\left[\dot{v}_n \dot{v}_m \right] \psi_m(t) \\
&= 0 - 0 + \left\langle \mathsf{E}\left[\boldsymbol{v}(t) n(\hat{\theta}) \right] \,|\, \psi_n(\hat{\theta}) \right\rangle - \sum_{m=1}^{N} N_o \bar{\delta}_{mn} \psi_m(t) \qquad \text{(because } \dot{v}_n \text{ is white)} \\
&= \left\langle N_o \delta(t - \hat{\theta}) \,|\, \psi_n(\hat{\theta}) \right\rangle - N_o \psi_n(t) \\
&= N_o \psi_n(t) - N_o \psi_n(t) \\
&= 0
\end{aligned}
$$

3. This implies $\dot{y}_n$ and $\boldsymbol{v}'(t)$ are uncorrelated. Since they are Gaussian processes (due to channel operator hypothesis), they are also independent.

4. Proof that $P\left\{ \boldsymbol{x}(t;\hat{\theta}) | \boldsymbol{y}(t) \right\} = P\left\{ \boldsymbol{x}(t;\hat{\theta}) | \dot{y}_1,\ \dot{y}_2, \ldots, \dot{y}_N \right\}$:

$$
\mathsf{P}\left\{ \boldsymbol{x}(t;\hat{\theta}) | \boldsymbol{y}(t) \right\} = \mathsf{P}\left\{ \boldsymbol{x}(t;\hat{\theta}) | \sum_{n=1}^{N} \dot{y}_n \psi_n(t) + \boldsymbol{v}'(t) \right\}
$$

$$
\begin{aligned}
&= \mathsf{P}\left\{\boldsymbol{x}(t;\hat{\theta})|R,\boldsymbol{v}'(t)\right\} && \text{because } Y \text{ and } \boldsymbol{v}'(t) \text{ can be extracted by } \langle\cdots|\psi_n(t)\rangle\\
&= \frac{\mathsf{P}\left\{R,\boldsymbol{v}'(t)|\boldsymbol{x}(t;\hat{\theta})\right\}P\left\{\boldsymbol{x}(t;\hat{\theta})\right\}}{\mathsf{P}\{R,\boldsymbol{v}'(t)\}}\\
&= \frac{\mathsf{P}\{R|\boldsymbol{x}(t;\hat{\theta})\}\mathsf{P}\{\boldsymbol{v}'(t)|\boldsymbol{x}(t;\hat{\theta})\}\mathsf{P}\{\boldsymbol{x}(t;\hat{\theta})\}}{\mathsf{P}\{Y\}\mathsf{P}\{\boldsymbol{v}'(t)\}} && \text{by \textit{independence} of } Y \text{ and } \boldsymbol{v}'(t)\\
&= \frac{\mathsf{P}\{R|\boldsymbol{x}(t;\hat{\theta})\}\mathsf{P}\{\boldsymbol{v}'(t)\}\mathsf{P}\{\boldsymbol{x}(t;\hat{\theta})\}}{\mathsf{P}\{Y\}\mathsf{P}\{\boldsymbol{v}'(t)\}}\\
&= \frac{\mathsf{P}\{R|\boldsymbol{x}(t;\hat{\theta})\}\mathsf{P}\{\boldsymbol{x}(t;\hat{\theta})\}}{\mathsf{P}\{Y\}}\\
&= \frac{\mathsf{P}\{R,\boldsymbol{x}(t;\hat{\theta})\}}{\mathsf{P}\{Y\}}\\
&= \mathsf{P}\left\{\boldsymbol{x}(t;\hat{\theta})|Y\right\}
\end{aligned}
$$

5. Proof that Y is a sufficient statistic for the *MAP estimate*:

$$
\begin{aligned}
\hat{\theta}_{\mathsf{map}} &\triangleq \arg\max_{\hat{\theta}} \mathsf{P}\{\boldsymbol{x}(t;\hat{\theta})|\boldsymbol{y}(t)\} && \text{by definition of \textit{MAP estimate}}\\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{\boldsymbol{x}(t;\hat{\theta})|R\} && \text{by result 4.}
\end{aligned}
$$

6. Proof that Y is a sufficient statistic for the *ML estimate*:

$$
\begin{aligned}
\hat{\theta}_{\mathsf{ml}} &\triangleq \arg\max_{\hat{\theta}} \mathsf{P}\{\boldsymbol{y}(t)|\boldsymbol{x}(t;\hat{\theta})\} && \text{by definition of \textit{ML estimate}}\\
&= \arg\max_{\hat{\theta}} \mathsf{P}\left\{\sum_{n=1}^{N}\dot{y}_n\psi_n(t)+\boldsymbol{v}'(t)|\boldsymbol{x}(t;\hat{\theta})\right\}\\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{R,\boldsymbol{v}'(t)|\boldsymbol{x}(t;\hat{\theta})\} && \text{because } Y \text{ and } \boldsymbol{v}'(t) \text{ can be extracted by } \langle\cdots|\psi_n(t)\rangle\\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{R|\boldsymbol{x}(t;\hat{\theta})\}\mathsf{P}\{\boldsymbol{v}'(t)|\boldsymbol{x}(t;\hat{\theta})\} && \text{by \textit{independence} of } Y \text{ and } \boldsymbol{v}'(t)\\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{R|\boldsymbol{x}(t;\hat{\theta})\}\mathsf{P}\{\boldsymbol{v}'(t)\} && \text{by \textit{independence} of } \boldsymbol{x}(t) \text{ and } \boldsymbol{v}'(t)\\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{R|\boldsymbol{x}(t;\hat{\theta})\} && \text{by \textit{independence} of } \boldsymbol{v}'(t) \text{ and } \hat{\theta}
\end{aligned}
$$

⇨

Theorem 9.2. *Let* $\mathbf{C} = \mathbf{C}_{\mathsf{a}}$ *be an additive noise channel.*

THM

$$
\underbrace{\mathbf{C} = \mathbf{C}_{\mathsf{a}}}_{\textit{additive noise channel}} \implies \left\{ \mathsf{E}(\dot{y}_n|\theta) = \dot{x}_n(\theta) + \mathsf{E}\dot{v}_n \right.
$$

✎PROOF:

$$
\begin{aligned}
\mathsf{E}(\dot{y}_n|\theta) &\triangleq \left(\langle\boldsymbol{y}(t)\,|\,\psi_n(t)\rangle\,|\theta\right)\\
&= \langle\boldsymbol{x}(t;\theta)+\mathsf{n}(t)\,|\,\psi_n(t)\rangle\\
&= \langle\boldsymbol{x}(t;\theta)\,|\,\psi_n(t)\rangle + \langle\mathsf{n}(t)\,|\,\psi_n(t)\rangle\\
&= \left\langle\sum_{k=1}^{N}\dot{x}_k(\theta)\psi_k(t)\,|\,\psi_n(t)\right\rangle + \dot{v}_n
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{k=1}^{N} \dot{x}_k(\theta) \langle \psi_k(t) \,|\, \psi_n(t) \rangle + \dot{v}_n \\
&= \dot{x}_n(\theta) + \dot{v}_n
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}(\dot{y}_n|\theta) &= \mathsf{E}\left[\dot{x}_n(\theta) + \dot{v}_n\right] \\
&= \mathsf{E}\dot{x}_n(\theta) + \mathsf{E}\dot{v}_n \\
&= \dot{x}_n(\theta)
\end{aligned}$$

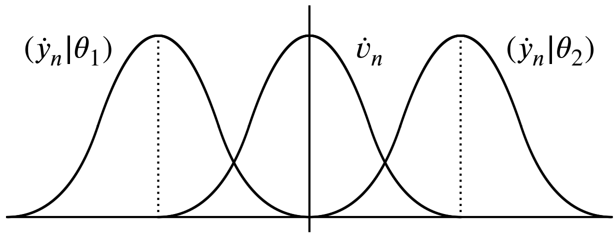


Figure 9.1: Additive Gaussian noise channel Statistics

Theorem 9.3. *Let* $\mathbf{C} = \mathbf{C}_{\mathsf{agn}}$ *be an additive gaussian noise channel with distribution* $\mathsf{n}(t) \sim \mathsf{N}\left(0, \sigma^2\right)$ *for all* t.

THM

$$\underbrace{\mathbf{C} = \mathbf{C}_{\mathsf{agn}}}_{\text{additive Gaussian channel}} \implies \left\{\begin{array}{lcll}
\mathsf{E}\dot{v}_n &=& 0 & \\
\mathsf{E}(\dot{y}_n|\theta) &=& \dot{x}_n(\theta) & \\
\dot{v}_n &\sim& \mathsf{N}\left(0, \sigma^2\right) & \text{(noise projections are Gaussian)} \\
\dot{y}_n|\theta &\sim& \mathsf{N}\left(\dot{x}_n(\theta), \sigma^2\right) & \text{(receiver projections are Gaussian)}
\end{array}\right.$$

✎PROOF:

$$\begin{aligned}
\mathsf{E}\dot{v}_n &= \mathsf{E}\,\langle \mathsf{n}(t) \,|\, \psi_n(t) \rangle \\
&= \langle \mathsf{E}\mathsf{n}(t) \,|\, \psi_n(t) \rangle \\
&= \langle 0 \,|\, \psi_n(t) \rangle \\
&= 0
\end{aligned}$$

$$\begin{aligned}
(\dot{y}_n|\theta) &\triangleq \langle \boldsymbol{y}(t) \,|\, \psi_n(t) \rangle | \theta \\
&= \langle \boldsymbol{x}(t;\theta) + \mathsf{n}(t) \,|\, \psi_n(t) \rangle \\
&= \langle \boldsymbol{x}(t;\theta) \,|\, \psi_n(t) \rangle + \langle \mathsf{n}(t) \,|\, \psi_n(t) \rangle \\
&= \left\langle \sum_{k=1}^{N} \dot{x}_k(\theta) \psi_k(t) \,\middle|\, \psi_n(t) \right\rangle + \dot{v}_n \\
&= \sum_{k=1}^{N} \dot{x}_k(\theta) \langle \psi_k(t) \,|\, \psi_n(t) \rangle + \dot{v}_n \\
&= \dot{x}_n(\theta) + \dot{v}_n
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}(\dot{y}_n|\theta) &= \mathsf{E}\left[\dot{x}_n(\theta) + \dot{v}_n\right] \\
&= \mathsf{E}\dot{x}_n(\theta) + \mathsf{E}\dot{v}_n \\
&= \dot{x}_n(\theta)
\end{aligned}$$

The distributions follow because they are linear operations on Gaussian processes.

Theorem 9.4. *Let* $\mathbf{C} = \mathbf{C}_{\mathsf{awn}}$ *be an additive white noise channel.*

THM

$$\underbrace{\mathbf{C} = \mathbf{C}_{\mathsf{awn}}}_{\text{additive white channel}} \implies \left\{ \begin{array}{lcll} \mathsf{E}\dot{v}_n &=& 0 & \text{(noise projection is zero-mean)} \\ \mathsf{E}(\dot{y}_n|\theta) &=& \dot{x}_n(\theta) & \text{(expected receiver projection = transmitted projection)} \\ \mathsf{cov}\left[\dot{v}_n, \dot{v}_m\right] &=& \sigma^2\bar{\delta}_{nm} & \text{(noise projections are uncorrelated)} \\ \mathsf{cov}\left[\dot{y}_n|\theta, \dot{y}_m|\theta\right] &=& \sigma^2\bar{\delta}_{nm} & \text{(receiver projections are uncorrelated)} \end{array} \right.$$

✎PROOF: Because the noise is additive (see Theorem 9.2 (page 77))

$$\begin{aligned} \mathsf{E}\dot{v}_n &= 0 \\ (\dot{y}_n|\theta) &= \dot{x}_n(\theta) + \dot{v}_n \\ \mathsf{E}(\dot{y}_n|\theta) &= \dot{x}_n(\theta). \end{aligned}$$

Because the noise is also white,

$$\begin{aligned} \mathsf{cov}\left[\dot{v}_m, \dot{v}_n\right] &= \mathsf{cov}\left[\langle \mathsf{n}(t) \,|\, \psi_m(t)\rangle, \langle \mathsf{n}(t) \,|\, \psi_n(t)\rangle\right] \\ &= \mathsf{E}\left[\langle \mathsf{n}(t) \,|\, \psi_m(t)\rangle \langle \mathsf{n}(t) \,|\, \psi_n(t)\rangle\right] \\ &= \mathsf{E}\left[\langle \mathsf{n}(t) \,|\, \psi_m(t)\rangle \left\langle n(\hat{\theta}) \,|\, \psi_n(\hat{\theta})\right\rangle\right] \\ &= \mathsf{E}\left[\left\langle n(\hat{\theta}) \langle \mathsf{n}(t) \,|\, \psi_m(t)\rangle \,|\, \psi_n(\hat{\theta})\right\rangle\right] \\ &= \mathsf{E}\left[\left\langle \left\langle n(\hat{\theta})\mathsf{n}(t) \,|\, \psi_m(t)\right\rangle \,|\, \psi_n(\hat{\theta})\right\rangle\right] \\ &= \left\langle \left\langle \mathsf{E}\left[n(\hat{\theta})\mathsf{n}(t)\right] \,|\, \psi_m(t)\right\rangle \,|\, \psi_n(\hat{\theta})\right\rangle \\ &= \left\langle \left\langle \sigma^2\delta(t-\hat{\theta}) \,|\, \psi_m(t)\right\rangle \,|\, \psi_n(\hat{\theta})\right\rangle \\ &= \sigma^2 \langle \psi_n(t) \,|\, \psi_m(t)\rangle \\ &= \left\{ \begin{array}{ll} \sigma^2 & \text{for } n = m \\ 0 & \text{for } n \neq m. \end{array} \right. \end{aligned}$$

$$\begin{aligned} \mathsf{cov}\left[\dot{y}_n|\theta, \dot{y}_m|\theta\right] &= \mathsf{E}\left[\dot{y}_n\dot{y}_m|\theta\right] - [\mathsf{E}\dot{y}_n|\theta][\mathsf{E}\dot{y}_m|\theta] \\ &= \mathsf{E}\left[(\dot{x}_n(\theta) + \dot{v}_n)(\dot{x}_m(\theta) + \dot{v}_m)\right] - \dot{x}_n(\theta)\dot{x}_m(\theta) \\ &= \mathsf{E}\left[(\dot{x}_n(\theta) + \dot{v}_n)(\dot{x}_m(\theta) + \dot{v}_m)\right] - \dot{x}_n(\theta)\dot{x}_m(\theta) \\ &= \mathsf{E}\left[\dot{x}_n(\theta)\dot{x}_m(\theta) + \dot{x}_n(\theta)\dot{v}_m + \dot{v}_n\dot{x}_m(\theta) + \dot{v}_n\dot{v}_m\right] - \dot{x}_n(\theta)\dot{x}_m(\theta) \\ &= \dot{x}_n(\theta)\dot{x}_m(\theta) + \dot{x}_n(\theta)\mathsf{E}\left[\dot{v}_m\right] + \mathsf{E}\left[\dot{v}_n\right]\dot{x}_m(\theta) + \mathsf{E}\left[\dot{v}_n\dot{v}_m\right] - \dot{x}_n(\theta)\dot{x}_m(\theta) \\ &= 0 + \dot{x}_n(\theta)\cdot 0 + 0\cdot\dot{x}_m(\theta) + \mathsf{cov}\left[\dot{v}_n, \dot{v}_m\right] + [\mathsf{E}\dot{v}_n][\mathsf{E}\dot{v}_m] \\ &= \sigma^2\bar{\delta}_{nm} + 0\cdot 0 \\ &= \left\{ \begin{array}{ll} \sigma^2 & \text{for } n = m \\ 0 & \text{for } n \neq m. \end{array} \right. \end{aligned}$$

⇰

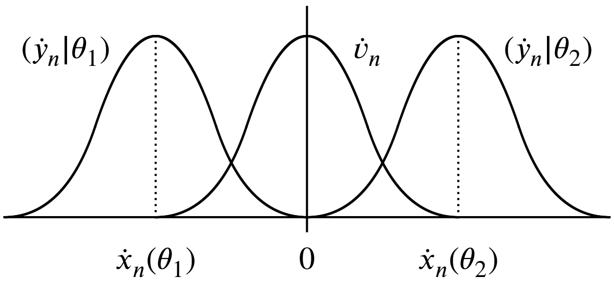


Figure 9.2: Additive white Gaussian noise channel statistics

Theorem 9.5. *Let* $\mathbf{C} = \mathbf{C}_{\mathsf{awgn}}$ *be an additive gaussian noise channel with distribution* $\mathsf{n}(t) \sim \mathsf{N}\left(0, \sigma^2\right)$ *for all* t.

THM

$$\underbrace{\mathbf{C} = \mathbf{C}_{\text{awgn}}}_{\text{AWGN}} \implies \left\{\begin{array}{rcll} \dot{v}_n & \sim & \mathsf{N}\left(0, \sigma^2\right) & \textit{(noise projections are Gaussian)} \\ \dot{y}_n|\theta & \sim & \mathsf{N}\left(\dot{x}_n(\theta), \sigma^2\right) & \textit{(receiver projections are Gaussian)} \\ \operatorname{cov}\left[\dot{v}_n, \dot{v}_m\right] & = & \sigma^2\bar{\delta}_{nm} & \textit{(noise projections are uncorrelated)} \\ \operatorname{cov}\left[\dot{y}_n, \dot{y}_m\right] & = & \sigma^2\bar{\delta}_{nm} & \textit{(receiver projections are uncorrelated)} \\ \mathsf{P}\{\dot{v}_n = a \wedge \dot{v}_m = b\} & = & \mathsf{P}\{\dot{v}_n = a\}\mathsf{P}\{\dot{v}_m = b\} & \textit{(noise projections are independent)} \\ \mathsf{P}\{\dot{y}_n = a \wedge \dot{y}_m = b\} & = & \mathsf{P}\{\dot{y}_n = a\}\mathsf{P}\{\dot{y}_m = b\} & \textit{(receiver projections are independent)} \end{array}\right.$$

✎PROOF: The distributions follow because they are linear operations on Gaussian processes.

By Theorem 9.4 (page 79) (for AWN channel)

$$\begin{aligned} \mathsf{E}\dot{v}_n &= 0 \\ \operatorname{cov}\left[\dot{v}_m, \dot{v}_n\right] &= \sigma^2\bar{\delta}_{mn} \\ \dot{y}_n &= \dot{x}_n + \dot{v}_n \\ \mathsf{E}\dot{y}_n &= \dot{x}_n \\ \operatorname{cov}\left[\dot{y}_n, \dot{y}_m\right] &= \sigma^2\bar{\delta}_{mn} \end{aligned}$$

Because the processes are Gaussian, uncorrelatedness implies *independence*. ✏

9.2 Optimal symbol estimation

The AWGN projection statistics provided by Theorem 9.5 (page 79) help generate the optimal ML-estimates for a number of communication systems. These ML-estimates can be expressed in either of two standard forms:

- **Spectral decompostion**: The optimal estimate is expressed in terms of *projections* of signals onto orthonormal basis functions.
- **Matched signal**: The optimal estimate is expressed in terms of the (noisy) received signal correlated with ("matched" with) the (noiseless) transmitted signal.

Theorem 9.6 (page 80) (next) expresses the general optimal *ML estimate* in both of these forms.

Parameter detection is a special case of parameter estimation. In parameter detection, the estimate is a member of an finite set. In parameter estimation, the estimate is a member of an infinite set (Section 9.2 page 80).

Theorem 9.6 (General ML estimation). *Let Ψ be an orthonormal set spanning $\boldsymbol{x}(t)$ such that*

$$\begin{aligned} \Psi &\triangleq \left(\!\left(\psi_1(t), \psi_2(t), \ldots, \psi_n(t)\right)\!\right) \\ \dot{y}_n &\triangleq \langle \boldsymbol{y}(t) \,|\, \psi_n(t) \rangle \\ \dot{x}_n &\triangleq \langle \boldsymbol{x}(t) \,|\, \psi_n(t) \rangle \\ \boldsymbol{y}(t) &= \boldsymbol{x}(t; \hat{\theta}) + \mathsf{n}(t). \end{aligned}$$

Then the optimal ML-estimate $\hat{\theta}_{\text{ml}}$ of parameter θ is

THM

$$\begin{aligned} \hat{\theta}_{\text{ml}} &= \arg\min_{\hat{\theta}} \left[\sum_{n=1}^{N} [\dot{y}_n - \dot{x}_n(\hat{\theta})]^2\right] && \textit{(spectral decomposition)} \\ &= \arg\max_{\hat{\theta}} \left[2\left\langle \boldsymbol{y}(t) \,|\, \boldsymbol{x}(t; \hat{\theta})\right\rangle - \left\|\boldsymbol{x}(t; \hat{\theta})\right\|^2\right] && \textit{(matched signal)} \end{aligned}$$

✎PROOF:

$$\begin{aligned}
\hat{\theta}_{\mathrm{ml}} &= \arg\max_{\hat{\theta}} \mathsf{P}\left\{\boldsymbol{y}(t)|\boldsymbol{x}(t;\hat{\theta})\right\} \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\left\{\dot{y}_1, \dot{y}_2, \dots, \dot{y}_n|\boldsymbol{x}(t;\hat{\theta})\right\} && \text{by Theorem 9.1 (page 75)} \\
&= \arg\max_{\hat{\theta}} \prod_{n=1}^{N} \mathsf{P}\left\{\dot{y}_n|\boldsymbol{x}(t;\hat{\theta})\right\} \\
&= \arg\max_{\hat{\theta}} \prod_{n=1}^{N} \mathsf{p}\left[\dot{\mathsf{y}}_n|\boldsymbol{x}(\mathrm{t};\hat{\theta})\right] \\
&= \arg\max_{\hat{\theta}} \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\frac{[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2}{-2\sigma^2} && \text{by Theorem 9.5 (page 79)} \\
&= \arg\max_{\hat{\theta}} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2 \\
&= \arg\max_{\hat{\theta}} \left[-\sum_{n=1}^{N}[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2\right] \\
&= \arg\max_{\hat{\theta}} \left[-\lim_{N\to\infty}\sum_{n=1}^{N}[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2\right] && \text{by Theorem 9.1 (page 75)} \\
&= \arg\max_{\hat{\theta}} \left[-\left\|\boldsymbol{y}(t) - \boldsymbol{x}(t;\hat{\theta})\right\|^2\right] && \text{by } \textit{Plancheral's formula} \quad \text{(Theorem G.9 page 210)} \\
&= \arg\max_{\hat{\theta}} \left[-\|\boldsymbol{y}(t)\|^2 + 2\mathbf{R}_{\mathrm{e}}\left\langle \boldsymbol{y}(t)\,|\,\boldsymbol{x}(t;\hat{\theta})\right\rangle - \left\|\boldsymbol{x}(t;\hat{\theta})\right\|^2\right] \\
&= \arg\max_{\hat{\theta}} \left[2\left\langle \boldsymbol{y}(t)\,|\,\boldsymbol{x}(t;\hat{\theta})\right\rangle - \left\|\boldsymbol{x}(t;\hat{\theta})\right\|^2\right] && \text{because } \boldsymbol{y}(t) \textit{ independent of } \hat{\theta}
\end{aligned}$$

✏

Theorem 9.7 (ML amplitude estimation). [2] *Let* **S** *be an additive white gaussian noise system such that*

$$\begin{aligned}
\boldsymbol{y}(t) &= [\mathbf{C}_{\mathrm{awgn}}s](t) = \boldsymbol{x}(t;a) + \mathrm{n}(t) \\
\boldsymbol{x}(t;a) &\triangleq a\lambda(t).
\end{aligned}$$

Then

THM

$$\begin{aligned}
\hat{a}_{\mathrm{ml}} &= \frac{1}{\|\lambda(t)\|^2}\langle \boldsymbol{y}(t)\,|\,\lambda(t)\rangle && \textit{(optimal ML-estimate of } a\textit{)} \\
&= \frac{1}{\|\lambda(t)\|^2}\sum_{n=1}^{N}\dot{y}_n\dot{\lambda}_n \\
\mathsf{E}\hat{a}_{\mathrm{ml}} &= a && (\hat{a}_{\mathrm{ml}} \textit{ is } \textbf{\textit{unbiased}}) \\
\operatorname{var}\hat{a}_{\mathrm{ml}} &= \frac{\sigma^2}{\|\lambda(t)\|^2} && \textit{(variance of estimate } \hat{a}_{\mathrm{ml}}\textit{)} \\
\operatorname{var}\hat{a}_{\mathrm{ml}} &= \textit{CR lower bound} && (\hat{a}_{\mathrm{ml}} \textit{ is an } \textbf{\textit{efficient estimate}})
\end{aligned}$$

✎PROOF:

[2] Mandyam D. Srinath (1996) pages 158–159

1. *ML estimate* in "matched signal" form:

$$\begin{aligned}
\hat{a}_{\text{ml}} &= \arg\max_a \left[2\left\langle \boldsymbol{y}(t) \,|\, \boldsymbol{x}(t;\hat{\theta})\right\rangle - \|\boldsymbol{x}(t;\phi)\|^2\right] && \text{by Theorem 9.6 (page 80)}\\
&= \arg\max_a \left[2\left\langle \boldsymbol{y}(t) \,|\, a\lambda(t)\right\rangle - \|a\lambda(t)\|^2\right] && \text{by hypothesis}\\
&= \arg_a \left[\frac{\partial}{\partial a}2a\left\langle \boldsymbol{y}(t) \,|\, \lambda(t)\right\rangle - \frac{\partial}{\partial a}a^2\,\|\lambda(t)\|^2 = 0\right]\\
&= \arg_a \left[2\left\langle \boldsymbol{y}(t) \,|\, \lambda(t)\right\rangle - 2a\,\|\lambda(t)\|^2 = 0\right]\\
&= \arg_a \left[\left\langle \boldsymbol{y}(t) \,|\, \lambda(t)\right\rangle = a\,\|\lambda(t)\|^2\right]\\
&= \frac{1}{\|\lambda(t)\|^2}\left\langle \boldsymbol{y}(t) \,|\, \lambda(t)\right\rangle
\end{aligned}$$

2. *ML estimate* in "spectral decomposition" form:

$$\begin{aligned}
\hat{a}_{\text{ml}} &= \arg\min_a \left(\sum_{n=1}^{N}\left[\dot{y}_n - \dot{x}_n(a)\right]^2\right) && \text{by Theorem 9.6 (page 80)}\\
&= \arg_a \left(\frac{\partial}{\partial a}\sum_{n=1}^{N}\left[\dot{y}_n - \dot{x}_n(a)\right]^2 = 0\right)\\
&= \arg_a \left(2\sum_{n=1}^{N}\left[\dot{y}_n - \dot{x}_n(a)\right]\frac{\partial}{\partial a}\dot{x}_n(a) = 0\right)\\
&= \arg_a \left(\sum_{n=1}^{N}\left[\dot{y}_n - \left\langle a\lambda(t) \,|\, \psi_n(t)\right\rangle\right]\frac{\partial}{\partial a}\left\langle a\lambda(t) \,|\, \psi_n(t)\right\rangle = 0\right)\\
&= \arg_a \left(\sum_{n=1}^{N}\left[\dot{y}_n - a\left\langle \lambda(t) \,|\, \psi_n(t)\right\rangle\right]\frac{\partial}{\partial a}(a\left\langle \lambda(t) \,|\, \psi_n(t)\right\rangle) = 0\right)\\
&= \arg_a \left(\sum_{n=1}^{N}\left[\dot{y}_n - a\dot{\lambda}_n\right]\left\langle \lambda(t) \,|\, \psi_n(t)\right\rangle = 0\right)\\
&= \arg_a \left(\sum_{n=1}^{N}\left[\dot{y}_n - a\dot{\lambda}_n\right]\dot{\lambda}_n = 0\right)\\
&= \arg_a \left(\sum_{n=1}^{N}\dot{y}_n\dot{\lambda}_n = \sum_{n=1}^{N}a\dot{\lambda}_n^2\right)\\
&= \left(\frac{1}{\sum_{n=1}^{N}\dot{\lambda}_n^2}\right)\sum_{n=1}^{N}\dot{y}_n\dot{\lambda}_n\\
&= \frac{1}{\|\lambda(t)\|^2}\sum_{n=1}^{N}\dot{y}_n\dot{\lambda}_n
\end{aligned}$$

3. Prove that the estimate $\hat{a}_{\text{ml}}$ is **unbiased**:

$$\begin{aligned}
\mathsf{E}\hat{a}_{\text{ml}} &= \mathsf{E}\frac{1}{\|\lambda(t)\|^2}\int_{t\in\mathbb{R}}\boldsymbol{y}(t)\lambda(t)\;\mathrm{d}t && \text{by previous result}\\
&= \mathsf{E}\frac{1}{\|\lambda(t)\|^2}\int_{t\in\mathbb{R}}[a\lambda(t)+\mathsf{n}(t)]\lambda(t)\;\mathrm{d}t && \text{by hypothesis}\\
&= \frac{1}{\|\lambda(t)\|^2}\int_{t\in\mathbb{R}}\mathsf{E}[a\lambda(t)+\mathsf{n}(t)]\lambda(t)\;\mathrm{d}t && \text{by linearity of } \int\cdot\,\mathrm{d}t \text{ and } \mathsf{E}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\|\lambda(t)\|^2} a \int_{t\in\mathbb{R}} \lambda^2(t)\, \mathrm{d}t && \text{by E operation}\\
&= \frac{1}{\|\lambda(t)\|^2} a \|\lambda(t)\|^2 && \text{by definition of } \|\cdot\|^2\\
&= a
\end{aligned}
$$

4. Compute the variance of $\hat{a}_{\mathsf{ml}}$:

$$
\begin{aligned}
\mathsf{E}\hat{a}^2_{\mathsf{ml}} &= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^2}\int_{t\in\mathbb{R}} \boldsymbol{y}(t)\lambda(t)\,\mathrm{d}t\right]^2\\
&= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}} \boldsymbol{y}(t)\lambda(t)\,\mathrm{d}t \int_v \boldsymbol{y}(v)\lambda(v)\,\mathrm{d}v\right]\\
&= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_v [a\lambda(t)+\mathsf{n}(t)][a\lambda(v)+\mathsf{n}(v)]\lambda(t)\lambda(v)\,\mathrm{d}v\,\mathrm{d}t\right]\\
&= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_v [a^2\lambda(t)\lambda(v)+a\lambda(t)\mathsf{n}(v)+a\lambda(v)\mathsf{n}(t)+\mathsf{n}(t)\mathsf{n}(v)]\lambda(t)\lambda(v)\,\mathrm{d}v\,\mathrm{d}t\right]\\
&= \left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_v [a^2\lambda(t)\lambda(v)+0+0+\sigma^2\delta(t-v)]\lambda(t)\lambda(v)\,\mathrm{d}v\,\mathrm{d}t\right]\\
&= \frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_v a^2\lambda^2(t)\lambda^2(v)\,\mathrm{d}v\,\mathrm{d}t + \frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_v \sigma^2\delta(t-v)\lambda(t)\lambda(v)\,\mathrm{d}v\,\mathrm{d}t\\
&= \frac{1}{\|\lambda(t)\|^4} a^2 \int_{t\in\mathbb{R}}\lambda^2(t)\,\mathrm{d}t\int_v \lambda^2(v)\,\mathrm{d}v + \frac{1}{\|\lambda(t)\|^4}\sigma^2\int_{t\in\mathbb{R}}\lambda^2(t)\,\mathrm{d}t\\
&= a^2\frac{1}{\|\lambda(t)\|^4}\|\lambda(t)\|^2\|\lambda(v)\|^2 + \frac{1}{\|\lambda(t)\|^4}\sigma^2\|\lambda(t)\|^2\\
&= a^2 + \frac{\sigma^2}{\|\lambda(t)\|^2}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{var}\hat{a}_{\mathsf{ml}} &= \mathsf{E}\hat{a}^2_{\mathsf{ml}} - (\mathsf{E}\hat{a}_{\mathsf{ml}})^2\\
&= \left(a^2 + \frac{\sigma^2}{\|\lambda(t)\|^2}\right) - \left(a^2\right)\\
&= \frac{\sigma^2}{\|\lambda(t)\|^2}
\end{aligned}
$$

5. Compute the Cramér-Rao Bound:

$$
\begin{aligned}
\mathsf{p}\left[\boldsymbol{y}(\mathsf{t})|\mathsf{s}(\mathsf{t};\mathsf{a})\right] &= \mathsf{p}\left[\dot{\mathsf{y}}_1,\dot{\mathsf{y}}_2,\ldots,\dot{\mathsf{y}}_N|\mathsf{s}(\mathsf{t};\mathsf{a})\right]\\
&= \prod_{n=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\frac{(\dot{y}_n - a\dot{\lambda}_n)^2}{-2\sigma^2}\\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n - a\dot{\lambda}_n)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial a}\ln\mathsf{p}\left[\boldsymbol{y}(\mathsf{t})|\mathsf{s}(\mathsf{t};\mathsf{a})\right] &= \frac{\partial}{\partial a}\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n - a\dot{\lambda}_n)^2\\
&= \frac{\partial}{\partial a}\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N + \frac{\partial}{\partial a}\ln\exp\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n - a\dot{\lambda}_n)^2\\
&= \frac{\partial}{\partial a}\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n - a\dot{\lambda}_n)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{-2\sigma^2}\sum_{n=1}^{N} 2(\dot{y}_n - a\dot{\lambda}_n)(-\dot{\lambda}_n) \\
&= \frac{1}{\sigma^2}\sum_{n=1}^{N} \dot{\lambda}_n(\dot{y}_n - a\dot{\lambda}_n)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\boldsymbol{y}(t)|s(t;a)\right] &= \frac{\partial}{\partial a}\frac{\partial}{\partial a}\ln \mathsf{p}\left[\boldsymbol{y}(t)|s(t;a)\right] \\
&= \frac{\partial}{\partial a}\frac{1}{\sigma^2}\sum_{n=1}^{N} \dot{\lambda}_n(\dot{y}_n - a\dot{\lambda}_n) \\
&= \frac{1}{\sigma^2}\sum_{n=1}^{N} \dot{\lambda}_n(-\dot{\lambda}_n) \\
&= \frac{-1}{\sigma^2}\sum_{n=1}^{N} \dot{\lambda}_n^2 \\
&= \frac{-\|\lambda(t)\|^2}{\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{var}\hat{a}_{\mathsf{ml}} &\triangleq \mathsf{E}\left[\hat{a}_{\mathsf{ml}} - \mathsf{E}\hat{a}_{\mathsf{ml}}\right]^2 \\
&= \mathsf{E}\left[\hat{a}_{\mathsf{ml}} - a\right]^2 \\
&\geq \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\boldsymbol{y}(t)|s(t;a)\right]\right)} \\
&= \frac{-1}{\mathsf{E}\left(\frac{-\|\lambda(t)\|^2}{\sigma^2}\right)} \\
&= \frac{\sigma^2}{\|\lambda(t)\|^2} \qquad \text{(Cramér-Rao lower bound of the variance)}
\end{aligned}
$$

6. Prove that $\hat{a}_{\mathsf{ml}}$ is an ***efficient* estimate**:

   A estimate is *efficient* if $\operatorname{var}\hat{a}_{\mathsf{ml}}$ = CR lower bound. We have already proven this, so $\hat{a}_{\mathsf{ml}}$ is an *efficient* estimate.

   Also, even without explicitly computing the variance of $\hat{a}_{\mathsf{ml}}$, the variance equals the *Cramér-Rao lower bound* (and hence $\hat{a}_{\mathsf{ml}}$ is an *efficient* estimate) if and only if

$$
\hat{a}_{\mathsf{ml}} - a = \left(\frac{-1}{\mathsf{E}\left[\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\boldsymbol{y}(t)|s(t;a)\right]\right]}\right)\left(\frac{\partial}{\partial a}\ln \mathsf{p}\left[\boldsymbol{y}(t)|s(t;a)\right]\right)
$$

$$
\begin{aligned}
\left(\frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\boldsymbol{y}(t)|s(t;a)\right]\right)}\right)\left(\frac{\partial}{\partial a}\ln \mathsf{p}\left[\boldsymbol{y}(t)|s(t;a)\right]\right) &= \left(\frac{\sigma^2}{\|\lambda(t)\|^2}\right)\left(\frac{1}{\sigma^2}\sum_{n=1}^{N}\dot{\lambda}(\dot{y} - a\dot{\lambda})\right) \\
&= \frac{1}{\|\lambda(t)\|^2}\sum_{n=1}^{N}\dot{\lambda}\dot{y} - \frac{1}{\|\lambda(t)\|^2}\sum_{n=1}^{N}\dot{\lambda}^2 \\
&= \hat{a}_{\mathsf{ml}} - a
\end{aligned}
$$

✐

Theorem 9.8 (ML phase estimation). [3] *Let* **S** *be an additive white gaussian noise system such that*

[3] Mandyam D. Srinath (1996) pages 159–160

$$\begin{aligned} \boldsymbol{y}(t) &= [\mathbf{C}_{\text{awgn}} s](t) = \boldsymbol{x}(t;\phi) + \mathsf{n}(t) \\ \boldsymbol{x}(t;\phi) &= A\cos(2\pi f_c t + \phi). \end{aligned}$$

Then the optimal ML-estimate of parameter ϕ *is*

THM $$\hat{\phi}_{\text{ml}} = -\operatorname{atan}\left(\frac{\langle \boldsymbol{y}(t) \,|\, \sin(2\pi f_c t)\rangle}{\langle \boldsymbol{y}(t) \,|\, \cos(2\pi f_c t)\rangle}\right)$$

✎PROOF:

$$\begin{aligned}
\hat{\phi}_{\text{ml}} &= \arg\max_{\phi} \left[2\left\langle \boldsymbol{y}(t) \,|\, \boldsymbol{x}(t;\hat{\theta})\right\rangle - \|\boldsymbol{x}(t;\phi)\|^2\right] && \text{by Theorem 9.6 (page 80)} \\
&= \arg\max_{\phi} \left[2\left\langle \boldsymbol{y}(t) \,|\, \boldsymbol{x}(t;\phi)\right\rangle\right] && \text{because } \|\boldsymbol{x}(t;\phi)\| \text{ does not depend on } \phi \\
&= \arg_{\phi}\left[\frac{\partial}{\partial\phi}\left\langle \boldsymbol{y}(t) \,|\, \boldsymbol{x}(t;\phi)\right\rangle = 0\right] \\
&= \arg_{\phi}\left[\left\langle \boldsymbol{y}(t) \,|\, \frac{\partial}{\partial\phi}\boldsymbol{x}(t;\phi)\right\rangle = 0\right] && \text{because } \langle\cdot\,|\,\cdot\rangle \text{ is a linear operator} \\
&= \arg_{\phi}\left[\left\langle \boldsymbol{y}(t) \,|\, \frac{\partial}{\partial\phi}A\cos(2\pi f_c t + \phi)\right\rangle = 0\right] \\
&= \arg_{\phi}\left[\left\langle \boldsymbol{y}(t) \,|\, -A\sin(2\pi f_c t + \phi)\right\rangle = 0\right] \\
&= \arg_{\phi}\left[-A\left\langle \boldsymbol{y}(t) \,|\, \cos(2\pi f_c t)\sin\phi + \sin(2\pi f_c t)\cos\phi\right\rangle = 0\right] \\
&= \arg_{\phi}\left[\sin\phi\left\langle \boldsymbol{y}(t) \,|\, \cos(2\pi f_c t)\right\rangle = -\cos\phi\left\langle \boldsymbol{y}(t) \,|\, \sin(2\pi f_c t)\right\rangle\right] \\
&= \arg_{\phi}\left[\frac{\sin\phi}{\cos\phi} = -\frac{\langle \boldsymbol{y}(t) \,|\, \sin(2\pi f_c t)\rangle}{\langle \boldsymbol{y}(t) \,|\, \cos(2\pi f_c t)\rangle}\right] \\
&= \arg_{\phi}\left[\tan\phi = -\frac{\langle \boldsymbol{y}(t) \,|\, \sin(2\pi f_c t)\rangle}{\langle \boldsymbol{y}(t) \,|\, \cos(2\pi f_c t)\rangle}\right] \\
&= -\operatorname{atan}\left(\frac{\langle \boldsymbol{y}(t) \,|\, \sin(2\pi f_c t)\rangle}{\langle \boldsymbol{y}(t) \,|\, \cos(2\pi f_c t)\rangle}\right)
\end{aligned}$$

✏

Theorem 9.9 (ML estimation of a function of a parameter). [4] *Let* **S** *be an additive white gaussian noise system such that*

$$\begin{aligned} \boldsymbol{y}(t) &= [\mathbf{C}_{\text{awgn}} s](t) = \boldsymbol{x}(t;\hat{\theta}) + \mathsf{n}(t) \\ \boldsymbol{x}(t;\hat{\theta}) &= \mathsf{g}(\hat{\theta}) \end{aligned}$$

where g *is one-to-one and onto (invertible).*

THM

Then the optimal ML-estimate of parameter u *is* $$\hat{\theta}_{\text{ml}} = \mathsf{g}^{-1}\left(\frac{1}{N}\sum_{n=1}^{N}\dot{y}_n\right).$$

If an ML ESTIMATE $\hat{\theta}_{\text{ml}}$ *is unbiased* ($\mathsf{E}\hat{\theta}_{\text{ml}} = \theta$) *then* $$\operatorname{var}\hat{\theta}_{\text{ml}} \geq \frac{\sigma^2}{N}\frac{1}{\left[\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right]^2}.$$

If $\mathsf{g}(\theta) = \theta$ *then* $\hat{\theta}_{\text{ml}}$ *is an* ***efficient*** *estimate such that* $$\operatorname{var}\hat{\theta}_{\text{ml}} = \frac{\sigma^2}{N}.$$

✎PROOF:

$$\hat{\theta}_{\text{ml}} = \arg\min_{\hat{\theta}}\left[\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\hat{\theta})]^2\right]$$

[4] Mandyam D. Srinath (1996) pages 142–143

$$
\begin{aligned}
&= \arg_{\hat\theta}\left[\frac{\partial}{\partial\hat\theta}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\hat\theta)]^2 = 0\right]\\
&= \arg_{\hat\theta}\left[2\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\hat\theta)]\frac{\partial}{\partial\hat\theta}\mathrm{g}(\hat\theta) = 0\right]\\
&= \arg_{\hat\theta}\left[2\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\hat\theta)] = 0\right]\\
&= \arg_{\hat\theta}\left[\sum_{n=1}^{N}\dot y_n = N\mathrm{g}(\hat\theta)\right]\\
&= \arg_{\hat\theta}\left[\mathrm{g}(\hat\theta) = \frac{1}{N}\sum_{n=1}^{N}\dot y_n\right]\\
&= \arg_{\hat\theta}\left[u = \mathrm{g}^{-1}\left(\frac{1}{N}\sum_{n=1}^{N}\dot y_n\right)\right]\\
&= \mathrm{g}^{-1}\left(\frac{1}{N}\sum_{n=1}^{N}\dot y_n\right)
\end{aligned}
$$

If $\hat\theta_{\mathsf{ml}}$ is unbiased ($\mathsf{E}\hat\theta_{\mathsf{ml}} = \theta$), we can use the *Cramér-Rao bound* to find a lower bound on the variance:

$$
\begin{aligned}
\operatorname{var}\hat\theta_{\mathsf{ml}} &\triangleq \mathsf{E}\left[\hat\theta_{\mathsf{ml}} - \mathsf{E}\hat\theta_{\mathsf{ml}}\right]^2\\
&= \mathsf{E}\left[\hat\theta_{\mathsf{ml}} - \theta\right]^2\\
&\ge \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\boldsymbol{y}(t)|\mathsf{s}(t;\theta)\right]\right)}\\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\dot{\mathsf{y}}_1, \dot{\mathsf{y}}_2, \ldots, \dot{\mathsf{y}}_N|\mathsf{s}(t;\theta)\right]\right)}\\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln\left[\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\theta)]^2\right]\right)}\\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln\left[\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N\right] + \frac{\partial^2}{\partial\theta^2}\ln\left[\exp\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\theta)]^2\right]\right)}\\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\left(\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\theta)]^2\right)\right)}\\
&= \frac{2\sigma^2}{\mathsf{E}\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\theta)]^2\right)}\\
&= \frac{2\sigma^2}{\mathsf{E}\left(-2\frac{\partial}{\partial\theta}\frac{\partial\mathrm{g}(\theta)}{\partial\theta}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\theta)]\right)}\\
&= \frac{-\sigma^2}{\mathsf{E}\left(\frac{\partial\mathrm{g}^2(\theta)}{\partial\theta^2}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\theta)] + \frac{\partial\mathrm{g}(\theta)}{\partial\theta}\frac{\partial}{\partial\theta}\sum_{n=1}^{N}[\dot y_n - \mathrm{g}(\theta)]\right)}
\end{aligned}
$$

$$= \frac{-\sigma^2}{\mathsf{E}\left(\frac{\partial \mathsf{g}^2(\theta)}{\partial\theta^2}\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)] - N\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right)}$$
$$= \frac{-\sigma^2}{\frac{\partial \mathsf{g}^2(\theta)}{\partial\theta^2}\sum_{n=1}^{N}\mathsf{E}[\dot{y}_n - \mathsf{g}(\theta)] - N\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}$$
$$= \frac{-\sigma^2}{-N\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}$$
$$= \frac{\sigma^2}{N}\frac{1}{\left[\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right]^2}$$

The inequality becomes equality (an *efficient* estimate) if and only if

$$\hat{\theta}_{\mathsf{ml}} - \theta = \left(\frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\boldsymbol{y}(\mathsf{t})|\mathsf{s}(\mathsf{t};\theta)\right]\right)}\right)\left(\frac{\partial}{\partial\theta}\ln \mathsf{p}\left[\boldsymbol{y}(\mathsf{t})|\mathsf{s}(\mathsf{t};\theta)\right]\right).$$

$$\left(\frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\boldsymbol{y}(\mathsf{t})|\mathsf{s}(\mathsf{t};\theta)\right]\right)}\right)\left(\frac{\partial}{\partial\theta}\ln \mathsf{p}\left[\boldsymbol{y}(\mathsf{t})|\mathsf{s}(\mathsf{t};\theta)\right]\right) = \left(\frac{\sigma^2}{N}\frac{1}{\left[\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right]^2}\right)\left(\frac{-1}{2\sigma^2}(2)\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)]\right)$$
$$= -\frac{1}{N}\frac{1}{\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}\left(\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)]\right)$$
$$= -\frac{1}{\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}\left(\frac{1}{N}\sum_{n=1}^{N}\dot{y}_n - \mathsf{g}(\theta)\right)$$
$$= -\frac{1}{\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}\left(\hat{\theta}_{\mathsf{ml}} - \mathsf{g}(\theta)\right)$$
$$= -(\hat{\theta}_{\mathsf{ml}} - \theta)$$

9.3 Colored noise

This chapter presented several theorems whose results depended on the noise being white. However if the noise is **colored**, then these results are invalid. But there is still hope for colored noise. Processing colored signals can be accomplished using two techniques:

1. Karhunen-Loève basis functions [5]
2. whitening filter [6]

Karhunen-Loève. If the noise is white, the set $\{\langle \boldsymbol{y}(t) \,|\, \psi_n(t)\rangle\}$ is a sufficient statistic regardless of which set $\{\psi_n(t)\}$ of orthonormal basis functions are used. If the noise is colored, and if $\{\psi_n(t)\}$

[5] **Karhunen-Loève**: Section 3.3 (page 20)

[6] *Continuous data whitening*: Section 6.3 page 48
Discrete data whitening: Section 5.4 page 38

satisfy the Karhunen-Loève criterion

$$\int_{t_2} \mathsf{R}_{xx}(t_1, t_2)\psi_n(t_2)\, \mathrm{d}t_2 = \lambda_n \psi_n(t_1)$$

then $\{\langle \boldsymbol{y}(t) \,|\, \psi_n(t)\rangle\}$ is still a sufficient statistic.

Whitening filter. The whitening filter makes the received signal $\boldsymbol{y}(t)$ statistically white (uncorrelated in time). In this case, any orthonormal basis set can be used to generate sufficient statistics.

9.4 Signal matching

Detection methods. There are basically two types of detection methods:

1. signal matching
2. orthonormal decomposition.

Let S be the set of transmitted waveforms and Y be a set of orthonormal basis functions that span S. *Signal matching* computes the innerproducts of a received signal $\boldsymbol{y}(t)$ with each signal from S. *Orthonormal decomposition* computes the innerproducts of $\boldsymbol{y}(t)$ with each signal from the set Y.

In the case where $|S|$ is large, often $|R| << |S|$ making orthonormal decomposition much easier to implement. For example, in a QAM-64 modulation system, signal matching requires $|S| = 64$ innerproduct calculations, while orthonormal decomposition only requires $|R| = 2$ innerproduct calculations because all 64 signals in S can be spanned by just 2 orthonormal basis functions.

Maximizing SNR. Theorem 9.1 (page 75) shows that the innerproducts of $\boldsymbol{y}(t)$ with basis functions of Y is sufficient for optimal detection. Theorem 9.10 (page 88) (next) shows that a receiver can maximize the SNR of a received signal when signal matching is used.

Theorem 9.10. *Let $\boldsymbol{x}(t)$ be a transmitted signal, $\mathsf{n}(t)$ noise, and $\boldsymbol{y}(t)$ the received signal in an AWGN channel. Let the* SIGNAL TO NOISE RATIO *SNR be defined as*

$$\mathrm{SNR}[\boldsymbol{y}(t)] \triangleq \frac{|\langle \boldsymbol{x}(t) \,|\, \boldsymbol{x}(t)\rangle|^2}{\mathsf{E}\left[|\langle \mathsf{n}(t) \,|\, \boldsymbol{x}(t)\rangle|^2\right]}.$$

THM

$$\mathrm{SNR}[\boldsymbol{y}(t)] \le \frac{2\,\|\boldsymbol{x}(t)\|^2}{N_o} \qquad \textit{and is maximized (equality) when } \boldsymbol{x}(t) = a\boldsymbol{x}(t), \textit{ where } a \in \mathbb{R}.$$

✎PROOF:

$$\begin{aligned}
\mathrm{SNR}[\boldsymbol{y}(t)] &\triangleq \frac{|\langle \boldsymbol{x}(t) \,|\, \boldsymbol{x}(t)\rangle|^2}{\mathsf{E}\left[|\langle \mathsf{n}(t) \,|\, \boldsymbol{x}(t)\rangle|^2\right]} \\
&= \frac{|\langle \boldsymbol{x}(t) \,|\, f(t)\rangle|^2}{\mathsf{E}\left[\left[\int_{t\in\mathbb{R}} \mathsf{n}(t)\boldsymbol{x}^*(t)\, dt\right]\left[\int_{\hat{\theta}} n(\hat{\theta}) f^*(\hat{\theta})\, du\right]^*\right]} \\
&= \frac{|\langle \boldsymbol{x}(t) \,|\, \boldsymbol{x}(t)\rangle|^2}{\mathsf{E}\left[\int_{t\in\mathbb{R}}\int_{\hat{\theta}} \mathsf{n}(t) n^*(\hat{\theta})\boldsymbol{x}^*(t)\boldsymbol{x}(\hat{\theta})\, dt du\right]}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|\langle \boldsymbol{x}(t) \,|\, f(t)\rangle|^2}{\int_{t\in\mathbb{R}}\int_{\hat\theta} \mathsf{E}\left[\mathsf{n}(t)n^*(\hat\theta)\right]\boldsymbol{x}^*(t)\boldsymbol{x}(\hat\theta)\; dtdu}\\
&= \frac{|\langle \boldsymbol{x}(t) \,|\, \boldsymbol{x}(t)\rangle|^2}{\int_{t\in\mathbb{R}}\int_{\hat\theta} \frac{1}{2}N_o\delta(t-\hat\theta)\boldsymbol{x}^*(t)\boldsymbol{x}(\hat\theta)\; dtdu}\\
&= \frac{|\langle \boldsymbol{x}(t) \,|\, \boldsymbol{x}(t)\rangle|^2}{\frac{1}{2}N_o\int_{t\in\mathbb{R}} \boldsymbol{x}^*(t)\boldsymbol{x}(t)\; \mathrm{d}t}\\
&= \frac{|\langle \boldsymbol{x}(t) \,|\, \boldsymbol{x}(t)\rangle|^2}{\frac{1}{2}N_o\,\|\boldsymbol{x}(t)\|^2}\\
&\le \frac{|\,\|\boldsymbol{x}(t)\|\;\;\|\boldsymbol{x}(t)\|\,|^2}{\frac{1}{2}N_o\,\|\boldsymbol{x}(t)\|^2} && \text{by \textit{Cauchy-Schwarz Inequality}} \quad \text{(Theorem I.2 page 234)}\\
&= \frac{2\,\|\boldsymbol{x}(t)\|^2}{N_o}
\end{aligned}
$$

The Cauchy-Schwarz Inequality becomes an equality (SNR is maximized) when $\boldsymbol{x}(t) = a\boldsymbol{x}(t)$. ✎

Implementation. The innerproduct operations can be implemented using either

1. a correlator or
2. a matched filter.

A correlator is simply an integrator of the form $\langle \boldsymbol{y}(t) \,|\, f(t)\rangle = \displaystyle\int_0^T \boldsymbol{y}(t)f(t)\;\mathrm{d}t$.

A matched filter introduces a function $\mathsf{h}(t)$ such that $\mathsf{h}(t) = \boldsymbol{x}(T-t)$ (which implies $\boldsymbol{x}(t) = h(T-t)$) giving

$$
\underbrace{\langle \boldsymbol{y}(t) \,|\, \boldsymbol{x}(t)\rangle = \int_0^T \boldsymbol{y}(t)\boldsymbol{x}(t)\;\mathrm{d}t}_{\text{correlator}} = \underbrace{\left.\int_0^\infty \boldsymbol{x}(\tau)h(t-\tau)\;\mathrm{d}\tau\right|_{t=T} = \boldsymbol{x}(t)\star\mathsf{h}(t)|_{t=T}}_{\text{matched filter}}.
$$

This shows that $\mathsf{h}(t)$ is the impulse response of a filter operation sampled at time T. By Theorem 9.10 (page 88), the optimal impulse response is $\mathsf{h}(T-t) = f(t) = \boldsymbol{x}(t)$. That is, the optimal $\mathsf{h}(t)$ is just a “flipped” and shifted version of $\boldsymbol{x}(t)$.

CHAPTER 10

MOMENT ESTIMATION

Theorem 10.1. *Let* $\hat{\mu} \triangleq \sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n$ *with* $\sum_{n=1}^{N} \lambda_n = 1$ *be the* ARITHMETIC MEAN *(Definition L.4 page 273)*.

THM

$$\left\{\begin{array}{lll}
\text{(A).} & (\boldsymbol{x}_n) \text{ is WIDE SENSE STATIONARY} & \text{and} \\
\text{(B).} & \mu \triangleq \mathsf{E}\boldsymbol{x}_n & \text{and} \\
\text{(C).} & (\boldsymbol{x}_n) \text{ is UNCORRELATED} & \text{and} \\
\text{(D).} & \hat{\mu} \triangleq \sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n \text{ (ARITHMETIC MEAN)} &
\end{array}\right\} \implies \left\{\begin{array}{lrcll}
\text{(1).} & \mathsf{E}\hat{\mu} & = & \mu \quad \text{(UNBIASED)} & \text{and} \\
\text{(2).} & \operatorname{var}(\hat{\mu}) & = & \sigma^2 \sum_{n=1}^{N} \lambda_n^2 & \text{and} \\
\text{(3).} & \operatorname{mse}(\hat{\mu}) & = & \sigma^2 \sum_{n=1}^{N} \lambda_n^2 &
\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{E}\hat{\mu} &\triangleq \mathsf{E}\sum_{n\in\mathbb{Z}} \lambda_n \boldsymbol{x}_n && \text{by definition of } \textit{arithmetic mean} && \text{(Definition L.4 page 273)} \\
&= \sum_{n\in\mathbb{Z}} \lambda_n \mathsf{E}\boldsymbol{x}_n && \text{by } \textit{linearity} \text{ of } \mathsf{E} && \text{(Theorem 1.1 page 4)} \\
&= \mu \sum_{n\in\mathbb{Z}} \lambda_n && \text{by } \textit{WSS} \text{ hypothesis} && \text{(A)} \\
&= \mu && \text{by } \textstyle\sum \lambda_n = 1 \text{ hypothesis} && \text{(Definition L.4 page 273)} \\
\operatorname{var}(\hat{\mu}) &\triangleq \mathsf{E}(\hat{\mu} - \mathsf{E}\hat{\mu})^2 && \text{by definition of } \textit{variance} \\
&= \mathsf{E}(\hat{\mu} - \mu)^2 && \text{by previous result} \\
&= \mathsf{E}\left(\sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n - \mu\right)^2 && \text{by definition of } \hat{\mu} \\
&= \mathsf{E}\left[\sum_{n=1}^{N} \boldsymbol{x}_n - \mu \underbrace{\sum_{n=1}^{N} \lambda_n}_{1}\right]^2 \\
&= \mathsf{E}\left[\sum_{n=1}^{N} \lambda_n (\boldsymbol{x}_n - \mu)\right]^2
\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{E}\left[\sum_{n=1}^{N}\lambda_n(\boldsymbol{x}_n-\mu)\sum_{m=1}^{N}\lambda_m(\boldsymbol{x}_m-\mu)\right]\\
&= \sum_{n=1}^{N}\sum_{m=1}^{N}\lambda_n\lambda_m\left(\mathsf{E}\left[(\boldsymbol{x}_n-\mu)(\boldsymbol{x}_m-\mu)\right]\right)\\
&= \sum_{n=1}^{N}\sum_{m=1}^{N}\lambda_n\lambda_m\left(\mathsf{E}\left[\boldsymbol{x}_n\boldsymbol{x}_m\right]-\mu\mathsf{E}\left[\boldsymbol{x}_n\right]-\mu\mathsf{E}\left[\boldsymbol{x}_m\right]+\mu^2\right)\\
&= \sum_{n=1}^{N}\sum_{m=1}^{N}\lambda_n\lambda_m\left(\mathsf{E}\left[\boldsymbol{x}_n\boldsymbol{x}_m\right]-\mu^2-\mu^2+\mu^2\right)\\
&= \sum_{n=1}^{N}\sum_{m=1}^{N}\lambda_n\lambda_m\left(\mathsf{E}\left[\boldsymbol{x}_n\boldsymbol{x}_m\right]-\mu^2\right)\\
&= \sum_{n=1}^{N}\lambda_n^2\mathsf{E}\left[\boldsymbol{x}_n^2\right]-\mu^2+\sum_{n=1}^{N}\sum_{m\neq n}\lambda_n\lambda_m\left(\mathsf{E}\left[\boldsymbol{x}_n\boldsymbol{x}_m\right]-\mu^2\right)\\
&= \sum_{n=1}^{N}\lambda_n^2\left(\mathsf{E}\left[\boldsymbol{x}_n^2\right]-\mu^2\right)+\sum_{n=1}^{N}\sum_{m\neq n}\lambda_n\lambda_m\left(\mathsf{E}\boldsymbol{x}_n\mathsf{E}\boldsymbol{x}_m\right)-\mu^2\\
&= \sum_{n=1}^{N}\lambda_n^2\sigma^2+\cancelto{0}{\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m\neq n}\lambda_n\lambda_m\left(\mu\mu-\mu^2\right)}\\
&= \sigma^2\sum_{n=1}^{N}\lambda_n^2
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{mse}(\hat{\mu}) &= \mathsf{E}(\hat{\mu}-\mathsf{E}\hat{\mu})^2+(\mathsf{E}\hat{\mu}-\mu)^2 && \text{by Theorem 8.2 page 65}\\
&= \sigma^2\sum_{n=1}^{N}\lambda_n^2+(\mu-\mu)^2 && \text{by previous results}\\
&= \sigma^2\sum_{n=1}^{N}\lambda_n^2
\end{aligned}
$$

Corollary 10.1. [1]

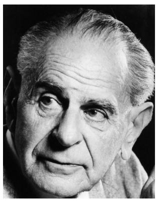
COR

$$
\left\{\begin{array}{lll}
\text{(A).} & (\!(\boldsymbol{x}_n)\!) \text{ is WIDE SENSE STATIONARY} & \text{and}\\
\text{(B).} & \mu \triangleq \mathsf{E}\boldsymbol{x}_n & \text{and}\\
\text{(C).} & (\!(\boldsymbol{x}_n)\!) \text{ is UNCORRELATED} & \text{and}\\
\text{(D).} & \hat{\mu} \triangleq \dfrac{1}{N}\displaystyle\sum_{n=1}^{N}\boldsymbol{x}_n \quad \text{(AVERAGE)} &
\end{array}\right\}
\implies
\left\{\begin{array}{lrcll}
\text{(1).} & \mathsf{E}\hat{\mu} & = & \mu & \text{(UNBIASED)} \quad \text{and}\\
\text{(2).} & \mathsf{var}(\hat{\mu}) & = & \dfrac{\sigma^2}{N} & \quad \text{and}\\
\text{(3).} & \mathsf{mse}(\hat{\mu}) & = & \dfrac{\sigma^2}{N} & \text{(CONSISTENT)}
\end{array}\right\}
$$

[1] Kay (1988) page 45 ⟨§"3.3 Estimation Theory"⟩

CHAPTER 11

SYSTEM IDENTIFICATION



❝*I can therefore gladly admit that falsificationists like myself much prefer an attempt to solve an interesting problem by a bold conjecture, even (and especially) if it so turns out to be false, to any recital of a sequence of irrelevant truisms. We prefer this because we believe that this is the way in which we can learn from our mistakes and that in finding that our conjecture was false we shall have learned much about the truth, and shall have got nearer to the truth.*❞

Karl R. Popper (1902–1994) [1]

11.1 Estimation techniques

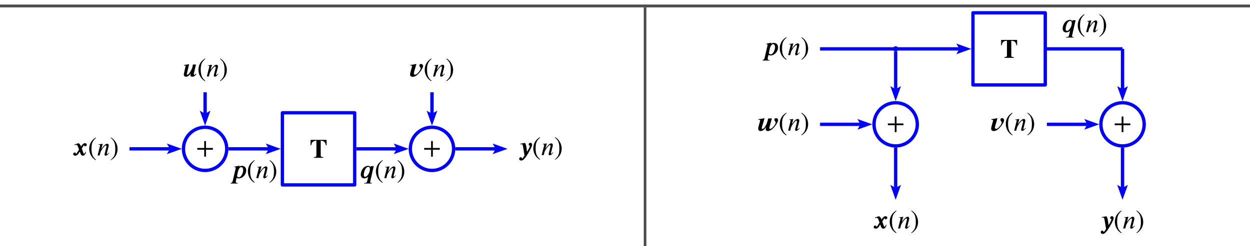
Let **S** be a *system* with *impulse response* $\mathsf{h}(n)$ with with *DTFT* $\tilde{\mathsf{H}}(\omega)$, input $\boldsymbol{x}(n)$, and output $\boldsymbol{y}(n)$. Often in the field of "digital signal processing" (DSP), **S** is a "filter" with known $\mathsf{h}(n)$ and $\tilde{\mathsf{H}}(\omega)$ because the filter **S** was designed by a designer who had direct control over $\mathsf{h}(n)$.

However in many other practical situations, **S** is some other system for which $\mathsf{h}(n)$ and $\tilde{\mathsf{H}}(\omega)$ are *not* known...but which we may want to *estimate*. Examples of such **S** is a device on an industrial shaker table, a communication channel, or the entire earth.

Determining $\mathsf{h}(n)$ and/or $\tilde{\mathsf{H}}(\omega)$ is part of an operation called "*system identication*". Determining $\tilde{\mathsf{H}}(\omega)$ in particular is referred to as "*Frequency Response Identification*"[2] or as "*Frequency Response Function*" ("*FRF*") estimation.[3] *FRF* estimation is a challenging problem and one that many have devoted much effort to. This chapter describes some of that effort.

In the early days, people used a rather obvious technique for determining $\tilde{\mathsf{H}}(\omega)$—the humble *sine sweep*. That is, they drove the input with a sine wave with slowly increasing (or decreasing) frequency while measuring the resulting output. This technique, although effective, was "very slow".[4]

[1] quote: Popper (1962), page 231, Popper (1963) page 313
image: https://en.wikipedia.org/wiki/File:Karl_Popper.jpg, "no known copyright restrictions"
[2] Shin and Hammond (2008) page 292
[3] Cobb (1988) page 1 ⟨FRF "measurement"⟩
[4] Leuridan et al. (1986)911"Stepped Sine Testing", Cobb (1988) page 1 ⟨Chapter 1—Introduction⟩, Ewins



(A) **communications additive noise model**
The "input signal" is $\boldsymbol{x}(n)$.

(B) **measurement additive noise model**
The "input signal" is $\boldsymbol{p}(n)$.

In each model, $\boldsymbol{x}(n)$ and $\boldsymbol{y}(n)$ are "known", and $\boldsymbol{u}(n)$, $\boldsymbol{v}(n)$, and $\boldsymbol{w}(n)$ are *not*.
In definition, the two models are **equivalent** under the relation $\boldsymbol{u}(n) = -\boldsymbol{w}(n)$.
In practice, they are **different**:

- in (A), $\boldsymbol{x}$ and $\boldsymbol{u}$ would be typically *uncorrelated*;
- in (B), $\boldsymbol{x}$ and $\boldsymbol{w} = -\boldsymbol{u}$ are very much *correlated* ($\boldsymbol{x}$ is a function of $\boldsymbol{u}$).

Figure 11.1: Additive noise systems with *linear*/**non-linear** operator **T**

And there is another problem here—we don't always have control over the input signal. Examples of this include earthquake and volcanic activity analysis.

An alternative to the sine-sweep input is *random sequence* input. All the techniques that follow in this chapter are of this type. A problem with using random sequences directly for estimating $\tilde{\mathsf{H}}(\omega)$ is that the estimate $\hat{\mathsf{H}}(\omega)$ is itself also random. This is not what we want. We want an estimate that we can actually write down on paper or at least plot on paper.

A solution to this is to not use the random sequences directly to estimate $\tilde{\mathsf{H}}(\omega)$, but instead to first use the *expectation* operator E (Definition 1.1 page 3). The expectation operator takes a quantity X that is inherently "random" (with some probability distribution $\mathsf{p}(x)$) and turns it into a deterministic "constant" $\mathsf{E}\mathsf{X}$.

The operator E is also used by the spectral density functions $\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)$ and $\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)$ (Definition 6.3 page 44). And $\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)$ and $\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)$ are what are typically used to calculate an estimate $\hat{\mathsf{H}}(\omega)$.

11.2 Additive noise system models

Consider the additive noise systems illustrated in Figure 12.1 (page 119).

- The illustration on the left is suitable for modeling a communications system where $\boldsymbol{x}$ is the transmitted signal, $\boldsymbol{y}$ is the received signal, $\boldsymbol{u}$ and $\boldsymbol{v}$ are thermal noise, and the "transfer function" $\mathbf{H}$ is the communications channel (air, water, wires, etc.) that one wishes to estimate.
- The illustration on the right is suitable for modeling a testing system where $\boldsymbol{p}$ is an input test signal (from an industrial shaker or from a naturally occurring signal originating from geophysical activity), $\boldsymbol{w}$ is measurement noise, $\boldsymbol{x}$ is the measured input contaminated by noise, and $\mathbf{H}$ is the device under test (a piece of equipment, a building, or the entire earth).

Note that the two models are an equivalent system $\mathbf{S}$ under the relation $\boldsymbol{u} = -\boldsymbol{w}$. But although one might expect such a sign difference to wreak mathematical havoc in resulting equations, this

(1986) pages 125–140 ⟨3.7 Use of different excitation types⟩

is simply not the case here because

$$\tilde{\mathsf{S}}_{\mathsf{ww}} = \tilde{\mathsf{F}}\mathsf{E}[\boldsymbol{w}(m)\boldsymbol{w} * (0)] = \tilde{\mathsf{F}}\mathsf{E}\big[(-\boldsymbol{u}(m))\big(-\boldsymbol{u}^*(0)\big)\big] = \tilde{\mathsf{F}}\mathsf{E}\big[(\boldsymbol{u}(m))\big(\boldsymbol{u}^*(0)\big)\big] = \tilde{\mathsf{S}}_{\mathsf{uu}}$$

So the sign difference is not that big of a difference after all. But there are some key differences in practice:

- In the communications model (on the left), the "input signal" is $\boldsymbol{x}(n)$ and the frequency-domain input *signal-to-noise ratio* (*SNR*) is $\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)/\tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)$. In the measurement model (on the right), the "input signal" is $\boldsymbol{p}(n)$ and the frequency-domain input *signal-to-noise ratio* (*SNR*) is $\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)/\tilde{\mathsf{S}}_{\mathsf{ww}}(\omega) = \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)/\tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)$.
- On the left, $\boldsymbol{x}$ and $\boldsymbol{u}$ would be typically *uncorrelated*; on the right, $\boldsymbol{x}$ and $\boldsymbol{w} = -\boldsymbol{u}$ are very much *correlated* ($\boldsymbol{x}$ is a function of $\boldsymbol{u}$).

11.3 Transfer function estimate definitions and interpretation

As a first attempt at estimating the transfer function **H** of **S**, or at least the magnitude squared of **H**, we might assume **H** to be *LTI*, take a cue from the relation $\tilde{\mathsf{S}}_{\mathsf{yy}} = \tilde{\mathsf{S}}_{\mathsf{xx}}\big|\tilde{\mathsf{H}}\big|^2$ of Corollary 5.3 (page 37), and arrive at a function called "*transmissibility*" (next definition).

Definition 11.1. [5] *Let* **S** *be a* SYSTEM *with input* $\boldsymbol{x}(n)$ *and output* $\boldsymbol{y}(n)$.

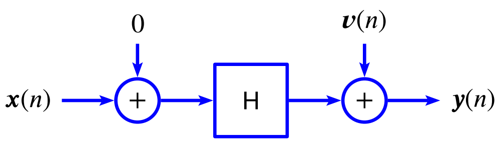
DEF: ***transmissibility*** $\tilde{\mathsf{T}}_{\mathsf{xy}}(\omega)$ *is defined as* $\quad \tilde{\mathsf{T}}_{\mathsf{xy}}(\omega) \triangleq \sqrt{\dfrac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}}$

Transmissibility is in essence the ratio of "*spectral power*" (Remark 6.1 page 44) output to *spectral power* input. Note that it is a real-valued function (because $\tilde{\mathsf{S}}_{\mathsf{xx}}$ and $\tilde{\mathsf{S}}_{\mathsf{yy}}$ are real-valued). We might suspect that we could attain better estimates of **H** by allowing the estimates to be complex-valued. And in fact, all the remaining estimates in this section are in general complex-valued.

And so to start (again), and in the very special (a.k.a unrealistic) case of **S** having *zero measurement noise* (*zero measurement error*) ($\boldsymbol{v} = \boldsymbol{u} = \boldsymbol{w} = 0$), $\mathsf{h}(n)$ being *linear time invariant* (*LTI*), and input $\boldsymbol{x}(n)$ being *wide sense stationary*...then we can determine (a.k.a "identify") $\mathsf{h}(n)$ or $\tilde{\mathsf{H}}(\omega)$ exactly by $\tilde{\mathsf{H}}(\omega) = \tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)/\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)$ (Corollary 5.3 page 37).

However, in practical situations, there is measurement noise/error. Examples may include "road noise" from a test being performed in a moving vehicle or *quantization noise* from an *analog-to-digital converter* (*ADC*).

If the measurement error is at the output only (and under the assumptions of *LTI* and *WSS*) then $\hat{\mathsf{H}}_1$ (next definition) is the ideal estimator in the sense that $\hat{\mathsf{H}}_1 = \tilde{\mathsf{H}}$ (Corollary 11.4 page 113).



Definition 11.2. [6] *Let* **S** *be a* SYSTEM *with input* $\boldsymbol{x}(n)$ *and output* $\boldsymbol{y}(n)$.

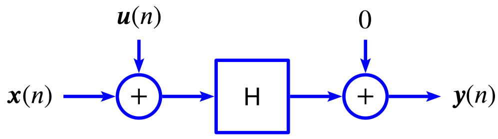
[5] Bendat and Piersol (2010) page 469 ⟨$|H(f)| = \big[G_{yy}(f)/G_{xx}(f)\big]^{1/2}$⟩, Yan and Ren (2012) page 204 ⟨(1) $\big[G_{YY}(s)\big] = [H(s)]\big[G_{FF}(s)\big]\big[H^*(s)\big]^T$⟩, Goldman (1999) page 179 ⟨Transmissibility ... $H'_{ab} = G_{bb}/G_{aa}$ (note: differs by $\sqrt{\cdot}$ from Bendat and Piersol⟩, Zhang et al. (2016), Zhou and Wahab (2018) page 824, https://link.springer.com/chapter/10.1007/978-3-319-54109-9_4

[6] Bendat and Piersol (1993) pages 106–109 ⟨5.1.1 Optimality of Calculations⟩, Bendat and Piersol (2010) page 185 ⟨$H_1(f) = G_{xy}(f)/G_{xx}(f)$ (6.37)⟩, Shin and Hammond (2008) page 293 ⟨ $H_1(f) = \tilde{\mathsf{S}}_{xy}(f)/\tilde{\mathsf{S}}_{xx}(f)$ (9.63); which dif-

DEF *The **Least Squares transfer function estimate** $\hat{\mathsf{H}}_1(\omega)$ of $\mathbf{S}$ is defined as* $\hat{\mathsf{H}}_1(\omega) \triangleq \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}$

The estimator $\hat{\mathsf{H}}_1$ is a good start. However in the early 1980s, L. D. Mitchell pointed out that in the presence of input noise, $\hat{\mathsf{H}}_1$ is far from ideal in that it is *biased* with respect to $\tilde{\mathsf{H}}$; in fact, $\hat{\mathsf{H}}_1$ *under estimates* $\tilde{\mathsf{H}}$ (Corollary 11.4 page 113). Mitchell proposed a new estimator $\hat{\mathsf{H}}_2$ (next definition).

This estimator has the special property that when there is input noise but no output noise (and under *LTI*, *WSS*, and *uncorrelated* assumptions), then it is ideal in the sense that $\hat{\mathsf{H}}_2(\omega) = \tilde{\mathsf{H}}(\omega)$ (Corollary 11.4 page 113).



Note also that in the case of both no input and no output noise, then $\hat{\mathsf{H}}_1 = \hat{\mathsf{H}}_2$ (Corollary 5.3 page 37).

Definition 11.3. [7] *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

DEF *The **Inverse Method transfer function estimate** $\hat{\mathsf{H}}_2(\omega)$ of $\mathbf{S}$ is defined as* $\hat{\mathsf{H}}_2(\omega) \triangleq \dfrac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}$

Mitchell's $\hat{\mathsf{H}}_2$ contribution "generated a flurry of activity"[8] and soon more $\tilde{\mathsf{H}}$ estimators appeared. So far we have

- $\hat{\mathsf{H}}_1$ which is ideal when there is no input noise but *under estimate*s $\tilde{\mathsf{H}}$ when there is (Corollary 11.4 page 113)
- $\hat{\mathsf{H}}_2$ which is ideal when there is no output noise but *over estimate*s $\tilde{\mathsf{H}}$ when there is (Corollary 11.4 page 113).

But what about estimators for when there is noise on both input and output? Armed with two estimators that between them account for both input and output noise, an "ad hoc" solution might be to somehow take mean values of $\hat{\mathsf{H}}_1$ and $\hat{\mathsf{H}}_2$ to induce new estimators—this approach summarizes the next three definitions. An arguably more mature approach is to find estimators that are optimal with respect to least squares measures—and this approach summarizes Definition 11.9 – Definition 11.7 (page 99).

Definition 11.4. *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

DEF *The **Arithmetic Mean transfer function estimate** $\hat{\mathsf{H}}_{am}(\omega)$ of $\mathbf{S}$ is defined as*

$$\hat{\mathsf{H}}_{am}(\omega) \triangleq \frac{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|^2 + \tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{yy}(\omega)}{2\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{xy}(\omega)}$$

fers from Definition 11.2, but see APPENDIX ?? page ?? ⟩, Bendat (1978)cited by Cobb(1988)—variance estimate for $\hat{\mathsf{H}}_1$, Allemang et al. (1979) ⟨cited by Shin(2008)⟩, Leuridan et al. (1986) page 910 ⟨*Least Squares Technique*; (8) $[G_{xx}](H) = [G_{xy}]$⟩, Abom (1986)cited by Cobb(1988)—variance estimate for $\hat{\mathsf{H}}_1$, Allemang et al. (1987) pages 54–55 ⟨5.3.1 H_1 Technique; $[H] = [G_{XF}][G_{FF}]^{-1}$ (11)⟩, Cobb (1988) page 2 ⟨$^1\hat{H}(f) = \hat{G}_{yx}(f)/\hat{G}_{xx}(f)$ (1)⟩, Goyder (1984) page 438 ⟨$H(i\omega) = S_{qp}/S_{pp}$ (3)⟩, Pintelon and Schoukens (2012) page 233 ⟨$\hat{G}(\Omega_k) = S_{yu}(j\omega_k)S_{uu}^{-1}(j\omega_k)$ (7-30)⟩, White et al. (2006) page 678 ⟨$H_1(f) = \hat{S}_{x_m y_m}(f)/\hat{S}_{x_m x_m}(f)$ (1) which differs by conjugate, references Bendat and Piersol⟩,

[7] Shin and Hammond (2008) page 293 ⟨$H_2(f) = \tilde{\mathsf{S}}_{yy}(f)/\tilde{\mathsf{S}}_{yx}(f)$ (9.65); which differs from Definition 11.3, but see APPENDIX ?? page ?? ⟩, Bendat and Piersol (2010) page 186 ⟨$H_2(f) = G_{yy}(f)/G_{yx}(f)$ (6.42)⟩, Mitchell (1980) ⟨cited by Cobb(1988)⟩, Mitchell (1982) page 278 ⟨"Define what will be called an inverse method for calculation of a FRF as…"; $H_2(f) = G_{yy}/G_{yx}$ (6); Note this differs with Definition 11.3 by a conjugate, but note that Mitchell seems to follow Bendat (see his [3] and [4]), which would explain this difference (APPENDIX ?? page ??)⟩, Cobb (1988) page 3 ⟨$^2\hat{H}(f) = \hat{G}_{yy}(f)/\hat{G}_{xy}(f)$ (1)⟩, White et al. (2006) page 678 ⟨$H_2(f) = \hat{S}_{y_m y_m}(f)/\hat{S}_{y_m x_m}(f)$ (2) which differs by conjugate, references Bendat and Piersol⟩

[8] Cobb (1988) page 3

Proposition 11.1. [9] *Let* **S** *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

PRP
$$\hat{\mathsf{H}}_{am}(\omega) = \frac{\hat{\mathsf{H}}_1(\omega) + \hat{\mathsf{H}}_2(\omega)}{2} \qquad \text{(\textbf{arithmetic mean} of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2\text{)}$$

✎PROOF:

$$\begin{aligned}
\hat{\mathsf{H}}_{am}(\omega) &\triangleq \frac{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|^2 + \tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{yy}(\omega)}{2\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{xy}(\omega)} && \text{by definition of } \hat{\mathsf{H}}_{am} && \text{(Definition 11.4 page 96)}\\
&= \frac{\tilde{\mathsf{S}}_{xy}(\omega)\tilde{\mathsf{S}}^*_{xy}(\omega)}{2\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{xy}(\omega)} + \frac{\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{yy}(\omega)}{2\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{xy}(\omega)} = \frac{\frac{\tilde{\mathsf{S}}^*_{xy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} + \frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}}{2}\\
&= \frac{\hat{\mathsf{H}}_1(\omega) + \hat{\mathsf{H}}_2(\omega)}{2} && \text{by definitions of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2 && \text{(Definition 11.2 page 95, Definition 11.3 page 96)}
\end{aligned}$$

⇨

Definition 11.5. *Let* **S** *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

DEF
The ***Geometric mean transfer function estimate*** $\hat{\mathsf{H}}_{gm}(\omega)$ *of* **S** *is defined as*
$$\hat{\mathsf{H}}_{gm}(\omega) \triangleq \frac{\tilde{\mathsf{S}}^*_{xy}(\omega)}{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|}\sqrt{\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}}$$

Proposition 11.2. [10] *Let* **S** *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

PRP
$$\pm\hat{\mathsf{H}}_{gm}(\omega) = \sqrt{\hat{\mathsf{H}}_1(\omega)\hat{\mathsf{H}}_2(\omega)} \qquad \text{(\textbf{geometric mean} of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2\text{)}$$

✎PROOF:

$$\begin{aligned}
\pm\hat{\mathsf{H}}_{gm}(\omega) &\triangleq \pm\frac{\tilde{\mathsf{S}}^*_{xy}(\omega)}{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|}\sqrt{\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}} && \text{by definition of } \hat{\mathsf{H}}_{gm} && \text{(Definition 11.5 page 97)}\\
&= \sqrt{\frac{\left[\tilde{\mathsf{S}}^*_{xy}(\omega)\right]^2}{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|^2}\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}} = \sqrt{\frac{\tilde{\mathsf{S}}^*_{xy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}} = \sqrt{\frac{\tilde{\mathsf{S}}^*_{xy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}}\\
&= \sqrt{\hat{\mathsf{H}}_1(\omega)\hat{\mathsf{H}}_2(\omega)} && \text{by definitions of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2 && \text{(Definition 11.2 page 95, Definition 11.3 page 96)}\\
&= \textbf{Geometric mean} \text{ of } \hat{\mathsf{H}}_1(\omega) \text{ and } \hat{\mathsf{H}}_2(\omega)
\end{aligned}$$

Note that for a complex number $z \triangleq |z|e^{i\phi}$, $\sqrt{z}$ has two solutions:[11]
$$\sqrt{z} = \sqrt{|z|e^{i\phi}} = \{z_1, z_2\} = \left\{\sqrt{|z|}e^{i(\phi/2)},\ \sqrt{|z|}e^{i(\phi/2+\pi)}\right\} = \pm\sqrt{|z|}e^{i(\phi/2)}$$
because $z_1^2 = z$ and $z_2^2 = z$. ⇨

Note that the *geometric mean estimator* (Definition 11.5 page 97) and *transmissibility* (Definition 11.1 page 95) are closely related (next).

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[9] Mitchell (1982) page 279 ⟨"Frequency Response Calculation: The Average Method"⟩, Zheng et al. (2002) page 918 ⟨"1.3 Arithmetic Mean Estimator H_3"⟩

[10] Zheng et al. (2002) page 918 ⟨"1.4 Geometric Mean Estimator H_4"⟩

[11] Many many thanks to Ben Cleveland for his help with this!!!

Proposition 11.3. *Let $\phi(\omega)$ be the* PHASE *of* $\tilde{\mathsf{S}}_{xy}(\omega)$ *such that* $\tilde{\mathsf{S}}_{xy}(\omega) \triangleq \left|\tilde{\mathsf{S}}_{xy}(\omega)\right| e^{-i\phi(\omega)}$

PRP

$$\hat{\mathsf{H}}_{gm}(\omega) = \tilde{\mathsf{T}}_{xy}(\omega)\, e^{-i\phi(\omega)} \quad \left(\begin{array}{rcll} \left|\hat{\mathsf{H}}_{gm}(\omega)\right| &=& \tilde{\mathsf{T}}_{xy}(\omega) & \textit{is the MAGNITUDE of } \hat{\mathsf{H}}_{gm}(\omega) \quad \textit{and} \\ \angle\hat{\mathsf{H}}_{gm}(\omega) &=& -\angle\tilde{\mathsf{S}}_{xy}(\omega) & \textit{is the PHASE of } \hat{\mathsf{H}}_{gm}(\omega) \end{array}\right)$$

✎PROOF: Let $\phi(\omega)$ be the *phase* of

$$\begin{aligned}
\hat{\mathsf{H}}_{gm}(\omega) &\triangleq \sqrt{\hat{\mathsf{H}}_1(\omega)\hat{\mathsf{H}}_2(\omega)} && \text{by definition of } \hat{\mathsf{H}}_{gm} && \text{(Definition 11.5 page 97)} \\
&\triangleq \sqrt{\frac{\tilde{\mathsf{S}}^*_{xy}(\omega)\,\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)\,\tilde{\mathsf{S}}_{xy}(\omega)}} && \text{by definitions of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2 && \text{(Definition 11.2 page 95, Definition 11.3 page 96)} \\
&= \sqrt{\frac{\tilde{\mathsf{S}}_{yy}(\omega)\,\tilde{\mathsf{S}}^*_{xy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)\,\tilde{\mathsf{S}}_{xy}(\omega)}} \\
&= \tilde{\mathsf{T}}_{xy}(\omega)\sqrt{\frac{\tilde{\mathsf{S}}^*_{xy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}} && \text{by definition of } \tilde{\mathsf{T}}_{xy} && \text{(Definition 11.1 page 95)} \\
&= \tilde{\mathsf{T}}_{xy}(\omega)\sqrt{\frac{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right| e^{-i\phi(\omega)}}{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right| e^{i\phi(\omega)}}} && \text{where } \tilde{\mathsf{S}}_{xy}(\omega) \triangleq \left|\tilde{\mathsf{S}}_{xy}(\omega)\right| e^{-i\phi(\omega)} \\
&= \tilde{\mathsf{T}}_{xy}(\omega)\sqrt{e^{-i2\phi(\omega)}} \\
&= \tilde{\mathsf{T}}_{xy}(\omega)\, e^{-i\phi(\omega)}
\end{aligned}$$

✏

Remark 11.1. *Transmissibility* $\tilde{\mathsf{T}}_{xy}$ is a kind of "black sheep" of the system identification function family. All the other members of this family ($\hat{\mathsf{H}}_1, \hat{\mathsf{H}}_2, \hat{\mathsf{H}}_v, \hat{\mathsf{H}}_s$) are *complex-valued*, but $\tilde{\mathsf{T}}_{xy}$ is only *real-valued*—a seemingly ordinary Joe born into a super-hero family. But Proposition 11.3 suggests that $\tilde{\mathsf{T}}_{xy}$ is not simply a "black sheep", but rather a "dark horse" with abilities that can easily be unleashed by slight redefinition. In particular, Proposition 11.3 demonstrates that $\tilde{\mathsf{T}}_{xy}$ is the *magnitude* of the geometric mean of $\hat{\mathsf{H}}_1$ and $\hat{\mathsf{H}}_2$. We can thus justifiably define a **complex transmissibility** function as $\hat{\mathsf{H}}_{gm}$...and the magnitude of this *complex transmissibility* function is the *ordinary transmissibility* function of Definition 11.1 (page 95).

REM

complex transmissibility $\tilde{\mathsf{T}}'_{xy}(\omega) \triangleq \hat{\mathsf{H}}_{gm}(\omega)$

Definition 11.6. *Let* **S** *be a* SYSTEM *with input* $\boldsymbol{x}(n)$ *and output* $\boldsymbol{y}(n)$.

DEF

*The **Harmonic mean transfer function estimate*** $\hat{\mathsf{H}}_{hm}(\omega)$ *of* **S** *is defined as*

$$\hat{\mathsf{H}}_{hm}(\omega) \triangleq \frac{2\tilde{\mathsf{S}}_{yy}(\omega)\tilde{\mathsf{S}}^*_{xy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{yy}(\omega) + \left|\tilde{\mathsf{S}}_{xy}(\omega)\right|^2}$$

Proposition 11.4. [12] *Let* **S** *be a* SYSTEM *with input* $\boldsymbol{x}(n)$ *and output* $\boldsymbol{y}(n)$.

PRP

$$\hat{\mathsf{H}}_{hm}(\omega) = \frac{2}{\frac{1}{\hat{\mathsf{H}}_1(\omega)} + \frac{1}{\hat{\mathsf{H}}_2(\omega)}} \quad \textit{(**Harmonic mean** of } \hat{\mathsf{H}}_1 \textit{ and } \hat{\mathsf{H}}_2\textit{)}$$

[12] Carne and Dohrmann (2006) $\langle H_C = [H_A^{-1} + H_B^{-1}]^{-1} \rangle$

✎PROOF:

$$\begin{aligned}
\hat{H}_{hm}(\omega) &\triangleq \frac{2\tilde{S}_{yy}(\omega)\tilde{S}^*_{xy}(\omega)}{\tilde{S}_{xx}(\omega)\tilde{S}_{yy}(\omega)+\left|\tilde{S}_{xy}(\omega)\right|^2} && \text{by definition of } \hat{H}_{hm} && \text{(Definition 11.6 page 98)}\\
&= \frac{2}{\dfrac{\tilde{S}_{xx}(\omega)\tilde{S}_{yy}(\omega)+\left|\tilde{S}_{xy}(\omega)\right|^2}{\tilde{S}_{yy}(\omega)\tilde{S}^*_{xy}(\omega)}} = \frac{2}{\dfrac{\tilde{S}_{xx}(\omega)}{\tilde{S}^*_{xy}(\omega)}+\dfrac{\tilde{S}_{xy}(\omega)}{\tilde{S}_{yy}(\omega)}}\\
&= \frac{2}{\dfrac{1}{\hat{H}_1(\omega)}+\dfrac{1}{\hat{H}_2(\omega)}} && \text{by definitions of } \hat{H}_1 \text{ and } \hat{H}_2 && \text{(Definition 11.2 page 95, Definition 11.3 page 96)}\\
&= \textbf{Harmonic mean} \text{ of } \hat{H}_1(\omega) \text{ and } \hat{H}_2(\omega)
\end{aligned}$$

⇰

A bit of review reveals $\hat{H}_1$ at the low end of the estimation problem, $\hat{H}_2$ at the high end, and $\hat{H}_{hm}$, $\hat{H}_{gm}$, and $\hat{H}_{am}$ somewhere between. But these three "between" estimates are not shown to be optimal in any sense—they are just conceptually interesting. What we might really like is a family of estimators that

- include $\hat{H}_1$ and $\hat{H}_2$ as limiting cases
- include the between cases
- are optimal in some sense

The estimator $\hat{H}_\kappa(\omega;\kappa)$ is one such estimator (next definition) that

- has $\hat{H}_1$ and $\hat{H}_2$ as limiting cases (Theorem 11.1 page 101),
- is optimal in the least squares sense (Theorem 11.6 page 114), and
- allows for a system designer to specify an output-input spectral noise ratio $\kappa(\omega)$ that can vary with frequency ω.

Moreover, $\hat{H}_\kappa(\omega)$ includes some special cases:

- In the case of constant κ, $\hat{H}_\kappa$ simplifies to the *Scaling transfer function estimate* $\hat{H}_s$ (Definition 11.8 page 99).
- In the case of $\kappa = 1$, $\hat{H}_\kappa$ and $\hat{H}_s$ simplify to the *Total least squares transfer function estimate* $\hat{H}_v$ (Definition 11.9 page 100).

Definition 11.7. [13] *Let* **S** *be a* SYSTEM *with input* $\boldsymbol{x}(n)$ *and output* $\boldsymbol{y}(n)$.

DEF *The* ***transfer function estimate*** $\hat{H}_\kappa(\omega;\kappa)$ *with* ***scaling function*** $\kappa(\omega)$ *is defined as*

$$\hat{H}_\kappa(\omega;\kappa) \triangleq \frac{\tilde{S}_{yy}(\omega)-\kappa(\omega)\tilde{S}_{xx}(\omega)+\sqrt{\left[\tilde{S}_{yy}(\omega)-\kappa(\omega)\tilde{S}_{xx}(\omega)\right]^2+4\kappa(\omega)\left|\tilde{S}_{xy}(\omega)\right|^2}}{2\tilde{S}_{xy}(\omega)}$$

Definition 11.8. [14] *Let* **S** *be a* SYSTEM *with input* $\boldsymbol{x}(n)$ *and output* $\boldsymbol{y}(n)$.

DEF *The* ***Scaling transfer function estimate*** $\hat{H}_s(\omega;s)$ *of* **S** *with* ***scaling parameter*** $s\in[0:\infty)$ *is defined as* $\hat{H}_s(\omega;s) \triangleq \hat{H}_\kappa(\omega;\kappa)$ *with* $\kappa(\omega)\triangleq s^2$

[13] White et al. (2006) page 679 ⟨(6)⟩, Shin and Hammond (2008) page 293 ⟨(9.67)⟩

[14] Shin and Hammond (2008) page 293 ⟨(9.67) with $\kappa(\omega)=s^2$⟩, White et al. (2006) page 679 ⟨(6) with $\kappa(\omega)=s^2$⟩, Leclere et al. (2014) ⟨(10) $\kappa(f)=1/s^2$ and x and y swapped⟩, Wicks and Vold (1986) page 898 ⟨has additional s in denominator⟩, Zheng et al. (2002) page 918 ⟨(10), seems to differ⟩

Definition 11.9. [15] *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

DEF *The* ***Total Least Squares transfer function estimate*** $\hat{\mathsf{H}}_{\mathsf{v}}(\omega)$ *of* $\mathbf{S}$ *is defined as*

$$\hat{\mathsf{H}}_{\mathsf{v}}(\omega) \triangleq \hat{\mathsf{H}}_{\kappa}(\omega;\kappa) \qquad \textit{with } \kappa(\omega) = 1$$

The previous estimators all assumed two signals: an input $\mathbf{x}(n)$ and an output $\mathbf{y}(n)$. However, in many practical systems, there is a third signal that is "driving" the system. In 1984 Goyder proposed an estimator (next definition) that is based on three signals.

Definition 11.10 (Three channel estimate)**.** [16] *Let* $\mathbf{S}$ *be a system with input* $\mathbf{x}(n)$*, output* $\mathbf{y}(n)$*, and a driver* $\mathbf{p}(n)$.

DEF *The* ***transfer function estimate*** $\hat{\mathsf{H}}_{\mathsf{c}}(\omega)$ *is defined as*

$$\hat{\mathsf{H}}_{\mathsf{c}}(\omega) \triangleq \frac{\tilde{\mathsf{S}}_{\mathsf{py}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{px}}(\omega)}$$

11.4 Estimator relationships

Lemma 11.1.

LEM

$$\frac{\mathsf{d}}{\mathsf{d}p}\left[\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] = \frac{\tilde{\mathsf{S}}_{\mathsf{xx}}\left(p\tilde{\mathsf{S}}_{\mathsf{xx}}-\tilde{\mathsf{S}}_{\mathsf{yy}}\right)+2\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2-\tilde{\mathsf{S}}_{\mathsf{xx}}\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{xx}}-\tilde{\mathsf{S}}_{\mathsf{yy}}\right)^2+4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{xx}}-\tilde{\mathsf{S}}_{\mathsf{yy}}\right)^2+4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}$$

$$\frac{\mathsf{d}}{\mathsf{d}p}\left[p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] = \frac{\tilde{\mathsf{S}}_{\mathsf{yy}}\left(p\tilde{\mathsf{S}}_{\mathsf{yy}}-\tilde{\mathsf{S}}_{\mathsf{xx}}\right)+2\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2+\tilde{\mathsf{S}}_{\mathsf{yy}}\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}}-\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2+4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}}-\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2+4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}$$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}p}\left[\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] &= -\tilde{\mathsf{S}}_{\mathsf{xx}} + \frac{-2\tilde{\mathsf{S}}_{\mathsf{xx}}\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right) + 4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}{2\sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}\\
&= \frac{4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2 - 2\tilde{\mathsf{S}}_{\mathsf{xx}}\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right) - 2\tilde{\mathsf{S}}_{\mathsf{xx}}\sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{2\sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}\\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{xx}}\left(p\tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{yy}}\right) + 2\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2 - \tilde{\mathsf{S}}_{\mathsf{xx}}\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{yy}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{yy}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}p}\left[p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] &= \tilde{\mathsf{S}}_{\mathsf{yy}} + \frac{2\tilde{\mathsf{S}}_{\mathsf{yy}}\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right) + 4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}{2\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}\\
&= \frac{4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2 + 2\tilde{\mathsf{S}}_{\mathsf{yy}}\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right) + 2\tilde{\mathsf{S}}_{\mathsf{yy}}\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{2\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}
\end{aligned}$$

[15] White et al. (2006) page 679 ⟨(6)⟩, Shin and Hammond (2008) page 294 ⟨(9.69)⟩

[16] Shin and Hammond (2008) page 297 ⟨$H_3(f) = S_{ry}(f)/S_{rx}(f)$ (9.78)⟩, Cobb (1988) page 4 ⟨${}^c\hat{H}(f) = \hat{G}_{ys}(f)/\hat{G}_{xs}(f)$ (1.4)⟩, Goyder (1984) page 440 ⟨$H(i\omega) = S_{qz}/S_{pz}$ (5)⟩, Cobb and Mitchell (1990) page 450 ⟨(1)⟩, Pintelon and Schoukens (2012) page 241 ⟨$\hat{G}(\Omega_k) = \hat{G}_{ry}(\Omega_k)\hat{G}_{ru}^{-1}(\Omega_k)$ (7-49)⟩

$$= \frac{\tilde{S}_{yy}(p\tilde{S}_{yy} - \tilde{S}_{xx}) + 2|\tilde{S}_{xy}|^2 + \tilde{S}_{yy}\sqrt{(p\tilde{S}_{yy} - \tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}{\sqrt{(p\tilde{S}_{yy} - \tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}$$

✏

Lemma 11.2.

LEM

$$\begin{aligned}
\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{(\tilde{S}_{yy} - p\tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2} &\geq 0\\
p\tilde{S}_{yy} - \tilde{S}_{xx} + \sqrt{(p\tilde{S}_{yy} - \tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2} &\geq 0
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
&\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{(\tilde{S}_{yy} - p\tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2} && \geq 0\\
\iff &\sqrt{(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2} && \geq p\tilde{S}_{xx} - \tilde{S}_{yy}\\
\iff &(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2 && \geq (p\tilde{S}_{xx} - \tilde{S}_{yy})^2\\
\iff &4p|\tilde{S}_{xy}|^2 && \geq 0\\
\iff &|\tilde{S}_{xy}| && \geq 0
\end{aligned}$$

$$\begin{aligned}
&p\tilde{S}_{yy} - \tilde{S}_{xx} + \sqrt{(p\tilde{S}_{yy} - \tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2} && \geq 0\\
\iff &\sqrt{(\tilde{S}_{xx} - p\tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2} && \geq \tilde{S}_{xx} - p\tilde{S}_{yy}\\
\iff &(\tilde{S}_{xx} - p\tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2 && \geq (\tilde{S}_{xx} - p\tilde{S}_{yy})^2\\
\iff &4p|\tilde{S}_{xy}|^2 && \geq 0\\
\iff &|\tilde{S}_{xy}| && \geq 0
\end{aligned}$$

✏

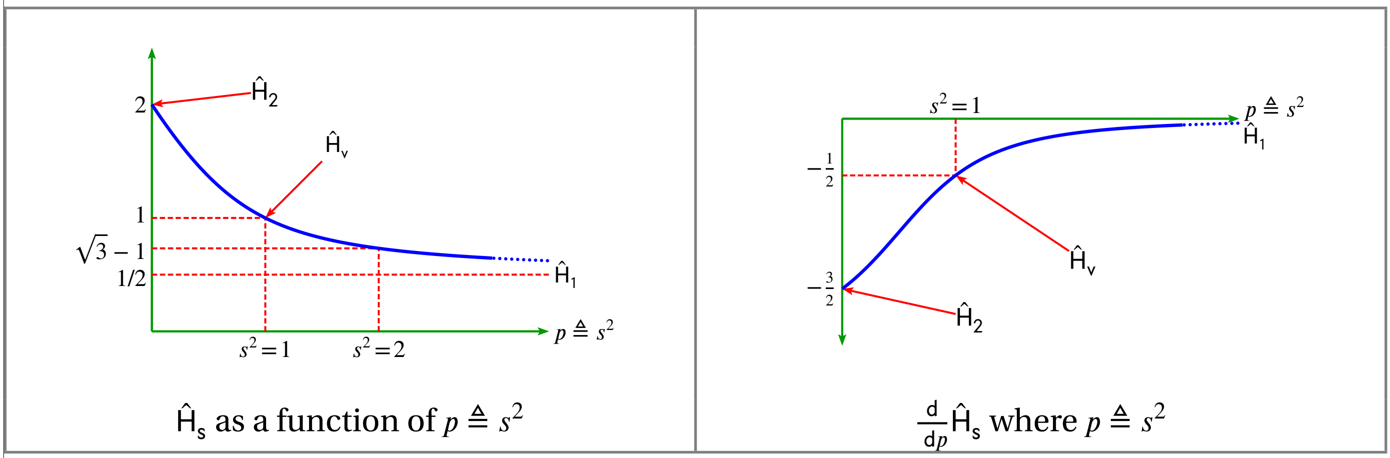


Figure 11.2: $\hat{H}_s$ with $\tilde{S}_{xx} = \tilde{S}_{yy} = 1$ and $\tilde{S}_{xy} = \frac{1}{2}$

Theorem 11.1. *Let $\hat{H}_s$ be defined as in Definition 11.8 (page 99).*

THM

$$\begin{aligned}
\{s_1 < s_2\} &\implies \left|\hat{H}_s(\omega; s_2)\right| \leq \hat{H}_s(\omega; s_1) && (\hat{H}_s \textit{ is } \textbf{\textit{monotonically decreasing}} \textit{ in } s)\\
\left|\hat{H}_1(\omega)\right| &\leq \left|\hat{H}_s(\omega; s)\right| \leq \left|\hat{H}_2(\omega)\right|\\
\hat{H}_s(\omega; s)\Big|_{s=0} &= \hat{H}_2(\omega)\\
\hat{H}_s(\omega; s)\Big|_{s=1} &= \hat{H}_v(\omega)\\
\lim_{s\to\infty} \hat{H}_s(\omega; s) &= \hat{H}_1(\omega)
\end{aligned}$$

✎PROOF: I. Proofs for equalities:

$$\hat{H}_s(\omega;s)\Big|_{s=0} \triangleq \left.\frac{\tilde{S}_{yy} - s^2\tilde{S}_{xx} + \sqrt{[\tilde{S}_{yy} - s^2\tilde{S}_{xx}]^2 + 4s^2|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}}\right|_{s=0} \qquad \text{by def. of } \hat{H}_s \qquad \text{(Definition 11.8 page 99)}$$

$$= \frac{\tilde{S}_{yy} - 0 + \sqrt{[\tilde{S}_{yy} - 0]^2 + 0}}{2\tilde{S}_{xy}} = \frac{\tilde{S}_{yy}}{\tilde{S}_{xy}}$$

$$\triangleq \hat{H}_2 \qquad \text{by def. of } \hat{H}_2 \qquad \text{(Definition 11.3 page 96)}$$

$$\hat{H}_s(\omega;s)\Big|_{s=1} \triangleq \left.\frac{\tilde{S}_{yy} - s^2\tilde{S}_{xx} + \sqrt{[\tilde{S}_{yy} - s^2\tilde{S}_{xx}]^2 + 4s^2|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}}\right|_{s=1} \qquad \text{by def. of } \hat{H}_s \qquad \text{(Definition 11.8 page 99)}$$

$$= \frac{\tilde{S}_{yy} - \tilde{S}_{xx} + \sqrt{[\tilde{S}_{yy} - \tilde{S}_{xx}]^2 + 4|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}}$$

$$\triangleq \hat{H}_v \qquad \text{by def. of } \hat{H}_v \qquad \text{(Definition 11.9 page 100)}$$

$$\lim_{s\to\infty}\hat{H}_s(\omega;s) \triangleq \lim_{s\to\infty}\frac{\tilde{S}_{yy} - s^2\tilde{S}_{xx} + \sqrt{[\tilde{S}_{yy} - s^2\tilde{S}_{xx}]^2 + 4s^2|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}} \qquad \text{by def. of } \hat{H}_s \qquad \text{(Definition 11.8 page 99)}$$

$$\triangleq \lim_{p\to 0}\frac{\tilde{S}_{yy} - {}^1\!/\!_p\tilde{S}_{xx} + \sqrt{[\tilde{S}_{yy} - {}^1\!/\!_p\tilde{S}_{xx}]^2 + 4{}^1\!/\!_p|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}} \qquad \text{where } p \triangleq \frac{1}{s^2}$$

$$= \lim_{p\to 0}\frac{p\tilde{S}_{yy} - \tilde{S}_{xx} + \sqrt{[p\tilde{S}_{yy} - \tilde{S}_{xx}]^2 + 4p|\tilde{S}_{xy}|^2}}{2p\tilde{S}_{xy}} \qquad \text{by mult. by } 1 = \frac{p}{p}$$

$$= \lim_{p\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}p}\left[p\tilde{S}_{yy} - \tilde{S}_{xx} + \sqrt{[p\tilde{S}_{yy} - \tilde{S}_{xx}]^2 + 4p|\tilde{S}_{xy}|^2}\right]}{\frac{\mathrm{d}}{\mathrm{d}p}[2p\tilde{S}_{xy}]} \qquad \text{by } \textit{l'Hôpital's rule}$$

$$= \lim_{p\to 0}\frac{\dfrac{\tilde{S}_{yy}(p\tilde{S}_{yy} - \tilde{S}_{xx}) + 2|\tilde{S}_{xy}|^2 + \tilde{S}_{yy}\sqrt{(p\tilde{S}_{yy} - \tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}{\sqrt{(p\tilde{S}_{yy} - \tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}}{2\tilde{S}_{xy}} \qquad \text{by Lemma 11.1 page 100}$$

$$= \frac{\tilde{S}_{yy}(-\tilde{S}_{xx}) + 2|\tilde{S}_{xy}|^2 + \tilde{S}_{yy}\sqrt{(-\tilde{S}_{xx})^2}}{2\tilde{S}_{xy}\sqrt{(-\tilde{S}_{xx})^2}}$$

$$= \frac{2|\tilde{S}_{xy}|^2}{2\tilde{S}_{xx}\tilde{S}_{xy}} = \frac{\tilde{S}^*_{xy}}{\tilde{S}_{xx}} \triangleq \hat{H}_1 \qquad \text{by def. of } \hat{H}_1 \qquad \text{(Definition 11.2 page 95)}$$

II. Proof for monoticity:

1. Let $p \triangleq s^2$
2. lemma:

$$\boxed{\left[2|\tilde{S}_{xy}|^2 + \tilde{S}_{xx}(p\tilde{S}_{xx} - \tilde{S}_{yy})\right]^2}$$

$$= 4|\tilde{S}_{xy}|^4 + 4\tilde{S}_{xx}|\tilde{S}_{xy}|^2(p\tilde{S}_{xx} - \tilde{S}_{yy}) + \tilde{S}^2_{xx}(p\tilde{S}_{xx} - \tilde{S}_{yy})^2$$

$$\boxed{\le} 4\left|\tilde{S}_{xy}\right|^2\tilde{S}_{xx}\tilde{S}_{yy} + 4p\tilde{S}_{xx}\left|\tilde{S}_{xy}\right|^2\tilde{S}_{xx} - 4\tilde{S}_{xx}\left|\tilde{S}_{xy}\right|^2\tilde{S}_{yy} + \tilde{S}_{xx}^2\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 \quad \left(\text{by } \textit{Cauchy Schwartz inequality} \text{ (Theorem I.2 page 234)}\right)$$

$$= \cancelto{0}{4\tilde{S}_{xx}\tilde{S}_{yy}\left|\tilde{S}_{xy}\right|^2} + 4p\tilde{S}_{xx}\left|\tilde{S}_{xy}\right|^2\tilde{S}_{xx} - \cancelto{0}{4\tilde{S}_{xx}\tilde{S}_{yy}\left|\tilde{S}_{xy}\right|^2} + \tilde{S}_{xx}^2\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2$$

$$= \tilde{S}_{xx}^2\left[\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2\right]$$

$$= \boxed{\left[\tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}\right]^2}$$

3. lemma: $2\left|\tilde{S}_{xy}\right|^2 + \tilde{S}_{xx}\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right) \le \tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}$. Proof:

$$2\left|\tilde{S}_{xy}\right|^2 + \tilde{S}_{xx}\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right) \le \tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}$$

$$\iff \left[2\left|\tilde{S}_{xy}\right|^2 + \tilde{S}_{xx}\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)\right]^2 \le \left[\tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}\right]^2 \quad \left(\text{because } \mathsf{f}(x)\triangleq x^2 \text{ is } \textit{strictly monotonic increasing}\right)$$

The previous inequality is true by (2) lemma, so (3) lemma also true.

4. Proof that $\frac{\mathsf{d}}{dp}\left|\hat{H}_s\right| \le 0$:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}p}\left|\hat{H}_s\right| &\triangleq \frac{\mathsf{d}}{\mathsf{d}p}\left|\frac{\tilde{S}_{yy} - s^2\tilde{S}_{xx} + \sqrt{\left(\tilde{S}_{yy}-s^2\tilde{S}_{xx}\right)^2 + 4s^2\left|\tilde{S}_{xy}\right|^2}}{2\tilde{S}_{xy}}\right| && \text{by def. of } \hat{H}_s \text{ (Definition 11.8 page 99)}\\
&\triangleq \frac{\mathsf{d}}{\mathsf{d}p}\left[\frac{\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{\left(\tilde{S}_{yy}-p\tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}{2\tilde{S}_{xy}}\right] && \text{by definition of } p \text{ (item (1) page 102)}\\
&= \frac{\mathsf{d}}{\mathsf{d}p}\left(\frac{\left|\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{\left(\tilde{S}_{yy}-p\tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}\right|}{\left|2\tilde{S}_{xy}\right|}\right)\\
&= \frac{\mathsf{d}}{\mathsf{d}p}\left(\frac{\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{\left(\tilde{S}_{yy}-p\tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}{2\left|\tilde{S}_{xy}\right|}\right) && \text{by Lemma 11.2 page 101}\\
&= \frac{\dfrac{\tilde{S}_{xx}\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right) + 2\left|\tilde{S}_{xy}\right|^2 - \tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}{\sqrt{\left(\tilde{S}_{yy}-p\tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}}{2\left|\tilde{S}_{xy}\right|} && \text{by Lemma 11.1 page 100}\\
&= \frac{\tilde{S}_{xx}\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right) + 2\left|\tilde{S}_{xy}\right|^2 - \tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}{2\left|\tilde{S}_{xy}\right|\sqrt{\left(\tilde{S}_{yy}-p\tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}\\
&\le 0 && \text{by (3) lemma}
\end{aligned}$$

⇨

Theorem 11.2. *Let* **S** *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

THM
$$\left|\hat{H}_1(\omega)\right| \le \left|\hat{H}_{hm}(\omega)\right| \le \left|\hat{H}_{gm}(\omega)\right| \le \left|\hat{H}_{am}(\omega)\right| \le \left|\hat{H}_2(\omega)\right|$$

✎PROOF:

1. lemma: $\hat{\mathsf{H}}_1(\omega) \le \hat{\mathsf{H}}_2(\omega)$. Proof:

$$\begin{aligned}
\left|\hat{\mathsf{H}}_1\right| &\triangleq \left|\frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}\right| && \text{by definition of } \hat{\mathsf{H}}_1 && \text{(Definition 11.2 page 95)}\\
&= \left|\frac{\langle \boldsymbol{y} \,|\, \boldsymbol{x}\rangle}{\|\boldsymbol{x}\|^2}\right| = \frac{|\langle \boldsymbol{y} \,|\, \boldsymbol{x}\rangle|}{\|\boldsymbol{x}\|^2}\\
&\le \frac{|\langle \boldsymbol{y} \,|\, \boldsymbol{x}\rangle|}{\|\boldsymbol{x}\|^2}\left|\frac{\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|}{\langle \boldsymbol{y} \,|\, \boldsymbol{x}\rangle}\right|^2 && \text{by \textit{Cauchy Schwartz inequality}} && \text{Theorem I.2 page 234}\\
&= \frac{\|\boldsymbol{y}\|^2}{|\langle \boldsymbol{y} \,|\, \boldsymbol{x}\rangle|} = \left|\frac{\|\boldsymbol{y}\|^2}{\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle}\right| = \left|\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}\right|\\
&= \left|\hat{\mathsf{H}}_2\right| && \text{by definition of } \hat{\mathsf{H}}_2 && \text{(Definition 11.3 page 96)}
\end{aligned}$$

2. remainder of the proof:

$$\begin{aligned}
\left|\hat{\mathsf{H}}_1(\omega)\right| &= \min\left\{\hat{\mathsf{H}}_1(\omega),\, \hat{\mathsf{H}}_2(\omega)\right\} && \text{by (1) lemma}\\
&\le \left|\hat{\mathsf{H}}_{\mathsf{hm}}(\omega)\right| && \text{by Corollary L.1 page 274} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&\le \left|\hat{\mathsf{H}}_{\mathsf{gm}}(\omega)\right| && \text{by Corollary L.1 page 274} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&\le \left|\hat{\mathsf{H}}_{\mathsf{am}}(\omega)\right| && \text{by Corollary L.1 page 274} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&\le \max\left\{\hat{\mathsf{H}}_1(\omega),\, \hat{\mathsf{H}}_2(\omega)\right\} && \text{by Corollary L.1 page 274} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&= \left|\hat{\mathsf{H}}_2(\omega)\right| && \text{by (1) lemma}
\end{aligned}$$

✏

Theorem 11.2 (page 103) compared the magnitudes of several transfer function estimates and demonstrated a simple *linear* relationship. What about phase? The phase of those estimates is even simpler than the magnitude, as demonstrated next.

Proposition 11.5 (Estimator phase). *Let $z \triangleq |z|e^{i\phi}$ be a* COMPLEX *number in the set of complex numbers $\mathbb{C}$. Let $\angle z \triangleq \phi$ be the* PHASE *of z.*

PRP

$$\begin{aligned}
\angle\hat{\mathsf{H}}_1(\omega) &= \angle\hat{\mathsf{H}}_{\mathsf{hm}}(\omega) = \angle\hat{\mathsf{H}}_{\mathsf{gm}}(\omega) = \angle\hat{\mathsf{H}}_{\mathsf{am}}(\omega) = \angle\hat{\mathsf{H}}_2(\omega) = \angle\hat{\mathsf{H}}_{\mathsf{s}}(\omega) = \angle\hat{\mathsf{H}}_{\mathsf{v}}(\omega) = \angle\hat{\mathsf{H}}_{\kappa}(\omega)\\
&= \angle\mathsf{C}_{xy}(\omega)\\
&= -\angle\tilde{\mathsf{S}}_{xy}(\omega)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\angle\hat{\mathsf{H}}_1 &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yx}}{\tilde{\mathsf{S}}_{xx}} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.2 page 95)}\\
\angle\hat{\mathsf{H}}_{hm} &\triangleq \angle\frac{2\tilde{\mathsf{S}}_{yy}\tilde{\mathsf{S}}^*_{xy}}{\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{yy}+|\tilde{\mathsf{S}}_{xy}|^2} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.6 page 98)}\\
\angle\hat{\mathsf{H}}_{gm} &\triangleq \angle\frac{\tilde{\mathsf{S}}^*_{xy}}{|\tilde{\mathsf{S}}_{xy}|}\sqrt{\frac{\tilde{\mathsf{S}}_{yy}}{\tilde{\mathsf{S}}_{xx}}} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.5 page 97)}\\
\angle\hat{\mathsf{H}}_{am} &\triangleq \angle\frac{|\tilde{\mathsf{S}}_{xy}|^2+\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{yy}}{2\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.4 page 96)}\\
\angle\hat{\mathsf{H}}_2 &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}}{\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.3 page 96)}\\
\angle\hat{\mathsf{H}}_s &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}-s^2\tilde{\mathsf{S}}_{xx}+\sqrt{[\tilde{\mathsf{S}}_{yy}-s^2\tilde{\mathsf{S}}_{xx}]^2+4s^2|\tilde{\mathsf{S}}_{xy}|^2}}{2\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.8 page 99)}\\
\angle\hat{\mathsf{H}}_v &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}-\tilde{\mathsf{S}}_{xx}+\sqrt{[\tilde{\mathsf{S}}_{yy}-\tilde{\mathsf{S}}_{xx}]^2+4|\tilde{\mathsf{S}}_{xy}|^2}}{2\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.9 page 100)}\\
\angle\hat{\mathsf{H}}_\kappa &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}-\kappa\tilde{\mathsf{S}}_{xx}+\sqrt{[\tilde{\mathsf{S}}_{yy}-\kappa\tilde{\mathsf{S}}_{xx}]^2+4\kappa|\tilde{\mathsf{S}}_{xy}|^2}}{2\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.7 page 99)}\\
\angle\mathsf{C}_{xy} &\triangleq \angle\frac{\tilde{\mathsf{S}}^*_{xy}}{\sqrt{\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{yy}}} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.12 page 116)}
\end{aligned}$$

✏

11.5 Alternate forms

Any standard kit of algebraic tricks should arguably always include the ability to swap the location of a square root between numerator and denominator. If you are of this persuasion, after traveling from the definition of $\hat{\mathsf{H}}_s$ on page 99, you won't be dissappointed when arriving at the next proposition (Proposition 11.6 page 105). But it has more use than just allowing you to entertain friends at social occasions. It also makes it very easy to see (using only algebra) what previously employed *l'Hôpital's rule* (using calculus) in the proof of Theorem 11.1—that $\lim_{s\to\infty}\hat{\mathsf{H}}_s = \hat{\mathsf{H}}_1$.

Proposition 11.6. [17] *Let* $\hat{\mathsf{H}}_\kappa(\omega;\kappa)$ *be defined as in Definition 11.7 (page 99).*

[17] Shin and Hammond (2008) page 293 ⟨(9.67)⟩, Leclere et al. (2014) ⟨(10) $\kappa(f) = 1/s^2$ and x and y swapped⟩

PRP

$$\begin{aligned}
\hat{H}_\kappa(\omega;s) &= \frac{2\kappa(\omega)\tilde{S}_{yx}(\omega)}{\kappa(\omega)\tilde{S}_{xx}(\omega) - \tilde{S}_{yy}(\omega) + \sqrt{\left[\kappa(\omega)\tilde{S}_{xx}(\omega) - \tilde{S}_{yy}(\omega)\right]^2 + 4\kappa(\omega)\left|\tilde{S}_{xy}(\omega)\right|^2}}\\
&= \frac{2\tilde{S}^*_{xy}}{\tilde{S}_{xx} - \frac{1}{\kappa(\omega)}\tilde{S}_{yy} + \sqrt{\left[\tilde{S}_{xx} - \frac{1}{\kappa(\omega)}\tilde{S}_{yy}\right]^2 + \frac{4}{\kappa(\omega)}\left|\tilde{S}_{xy}\right|^2}}
\end{aligned}$$

✎PROOF: We can transform $\hat{H}_\kappa$ from that found in Definition 11.8 (page 99) to the forms in this proposition by the technique of "*rationalizing the denominator*"[18]

$$\begin{aligned}
\hat{H}_\kappa &\triangleq \frac{\tilde{S}_{yy} - \kappa\tilde{S}_{xx} + \sqrt{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}}{2\tilde{S}_{xy}} && \text{by definition of } \hat{H}_\kappa \text{ (Definition 11.8 page 99)}\\
&= \frac{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx} + \sqrt{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}\right]\overbrace{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx} - \sqrt{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}\right]}^{\text{"rationalizing factor"}}}{2\tilde{S}_{xy}\underbrace{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx} - \sqrt{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}\right]}_{\text{"rationalizing factor"}}}\\
&= \frac{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 - \left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 - 4\kappa\left|\tilde{S}_{xy}\right|^2}{2\tilde{S}_{xy}\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx} - \sqrt{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}\right]}
= \frac{-4\kappa\left|\tilde{S}_{xy}\right|^2}{2\tilde{S}_{xy}\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx} - \sqrt{\left[\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right]^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}\right]}\\
&= \boxed{\frac{2\kappa\tilde{S}^*_{xy}}{\kappa\tilde{S}_{xx} - \tilde{S}_{yy} + \sqrt{\left[\kappa\tilde{S}_{xx} - \tilde{S}_{yy}\right]^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}}}
= \frac{2\frac{\kappa}{\kappa}\tilde{S}^*_{xy}}{\frac{\kappa}{\kappa}\tilde{S}_{xx} - \frac{1}{\kappa}\tilde{S}_{yy} + \sqrt{\frac{1}{s^4}\left[\kappa\tilde{S}_{xx} - \tilde{S}_{yy}\right]^2 + \frac{4\kappa}{s^4}\left|\tilde{S}_{xy}\right|^2}}\\
&= \frac{2\tilde{S}^*_{xy}}{\tilde{S}_{xx} - \frac{1}{\kappa}\tilde{S}_{yy} + \sqrt{\left[\frac{\kappa}{\kappa}\tilde{S}_{xx} - \frac{1}{\kappa}\tilde{S}_{yy}\right]^2 + \frac{4}{\kappa}\left|\tilde{S}_{xy}\right|^2}}
= \boxed{\frac{2\tilde{S}^*_{xy}}{\tilde{S}_{xx} - \frac{1}{\kappa}\tilde{S}_{yy} + \sqrt{\left[\tilde{S}_{xx} - \frac{1}{\kappa}\tilde{S}_{yy}\right]^2 + \frac{4}{\kappa}\left|\tilde{S}_{xy}\right|^2}}}
\end{aligned}$$

Integrity check for $s = 0$ and $s \to \infty$ cases: Let $p \triangleq \kappa$.

$$\begin{aligned}
\lim_{p\to\infty}\hat{H}_\kappa &= \lim_{p\to\infty}\frac{2\tilde{S}_{yx}}{\tilde{S}_{xx} - \frac{1}{p}\tilde{S}_{yy} + \sqrt{\left[\tilde{S}_{xx} - \frac{1}{p}\tilde{S}_{yy}\right]^2 + \frac{4}{p}\left|\tilde{S}_{xy}\right|^2}} = \frac{2\tilde{S}_{yx}}{\tilde{S}_{xx} + \sqrt{\left[\tilde{S}_{xx}\right]^2}}\\
&= \frac{\tilde{S}_{yx}}{\tilde{S}_{xx}} \triangleq \hat{H}_1 && \text{by def. of } \hat{H}_1 \text{ (Definition 11.2 page 95)}
\end{aligned}$$

$$\begin{aligned}
\lim_{p\to 0}\hat{H}_\kappa &= \lim_{p\to 0}\frac{2p\tilde{S}_{yx}}{p\tilde{S}_{xx} - \tilde{S}_{yy} + \sqrt{\left[p\tilde{S}_{xx} - \tilde{S}_{yy}\right]^2 + 4p\left|\tilde{S}_{xy}\right|^2}}\\
&= \lim_{p\to 0}\frac{\frac{d}{dp}\left(2p\tilde{S}_{yx}\right)}{\frac{d}{dp}\left(p\tilde{S}_{xx} - \tilde{S}_{yy} + \sqrt{\left[p\tilde{S}_{xx} - \tilde{S}_{yy}\right]^2 + 4p\left|\tilde{S}_{xy}\right|^2}\right)} && \text{by } \textit{l'Hôpital's rule}
\end{aligned}$$

[18] Slaught and Lennes (1915), page 274 ⟨"197. Rationalizing the Denominator."⟩ `https://archive.org/details/elementaryalgebr00slaurich/page/274` Note that the operation in the proof of Proposition 11.6 is being performed essentially in reverse…or rather "rationalizing the numerator".

$$= \lim_{p\to 0} \frac{2\tilde{S}_{yx}}{\tilde{S}_{xx}\left(p\tilde{S}_{xx} - \tilde{S}_{yy}\right) + 2|\tilde{S}_{xy}|^2 + \tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx} - \tilde{S}_{yy}\right)^2 + 4p|\tilde{S}_{xy}|^2}} \quad \text{by Lemma 11.1 page 100}$$

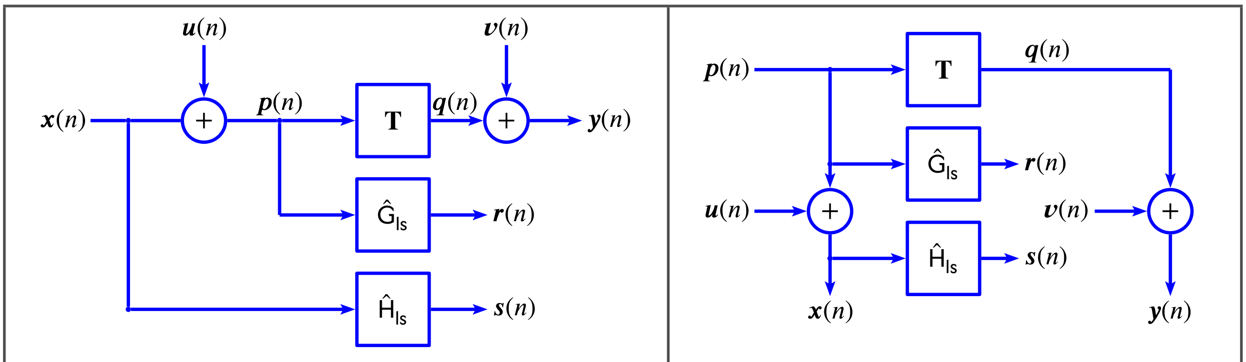
$$\sqrt{\left(p\tilde{S}_{xx} - \tilde{S}_{yy}\right)^2 + 4p|\tilde{S}_{xy}|^2}$$

$$= \frac{2\tilde{S}_{yy}\tilde{S}_{yx}}{-\tilde{S}_{xx}\tilde{S}_{yy} + 2|\tilde{S}_{xy}|^2 + \tilde{S}_{xx}\tilde{S}_{yy}} = \frac{2\tilde{S}_{yy}\tilde{S}_{yx}}{2|\tilde{S}_{xy}|^2} = \frac{\tilde{S}_{yy}}{\tilde{S}_{xy}}$$

$$\triangleq \hat{H}_2 \quad \text{by def. of } \hat{H}_2 \text{ (Definition 11.3 page 96)}$$

11.6 Least squares estimates of non-linear systems

> *"The legendary Hungarian mathematician John von Neumann once referred to the the theory of nonequilibrium systems as the "theory of non-elephants," ...Nevertheless, such a theory of non-elephants will be attempted here."*
>
> Per Bak, in *"how nature works..."* [19]

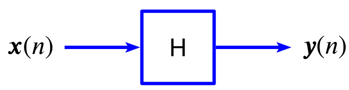


(A) *Least squares estimation* for *communications additive noise model* (Corollary 11.1 page 110)

(B) *Least squares estimations* for *measurement additive noise model* (Corollary 11.2 page 111)

Figure 11.3: Least Square estimation (Theorem 11.3 page 108)

Let **S** be the *system* illustrated to the right. **If** there is no measurement noise on the input and output and **if** **H** is *linear time invariant*, then $\tilde{H} = \tilde{S}_{xy}/\tilde{S}_{xx}$ (Corollary 5.1 page 36). But what if there is output measurement noise? And what if **H** is not *LTI*? What is the best least-squares estimate of $\tilde{H}$? The answer depends on how you define "the best".



The defintion of "best" or "optimal" is given by a *cost function* $C(\hat{H})$. There are several possible cost functions. Definition 11.11 provides some. Theorem 11.3 then demonstrate optimal solutions with respect to these definitions.

Definition 11.11. *Let* **S** *be a system defined as in Figure 11.3 (page 107) (A) or (B). Define the following* COST FUNCTIONS *for spectral* LEAST-SQUARES *estimates:*

DEF

$$C_{rq}(\hat{G}) \triangleq \tilde{F}\,\|r(n) - q(n)\|^2 \triangleq \tilde{F}\,\langle r(n) - q(n) \,|\, r(0) - q(0)\rangle \triangleq \tilde{F}E\left(\left[r(n) - q(n)\right]\left[r(0) - q(0)\right]^*\right)$$

$$C_{sy}(\hat{H}) \triangleq \tilde{F}\,\|s(n) - y(n)\|^2 \triangleq \tilde{F}\,\langle s(n) - y(n) \,|\, s(0) - y(0)\rangle \triangleq \tilde{F}E\left(\left[s(n) - y(n)\right]\left[s(0) - y(0)\right]^*\right)$$

[19] Bak (2013) page 29 ⟨§ Systems in Balance Are Not Complex⟩

Lemma 11.3. *Let* $\mathsf{C}_{rq}(\hat{\mathsf{G}})$ *and* $\mathsf{C}_{sy}(\hat{\mathsf{H}})$ *be defined as in Definition 11.11 (page 107).*

LEM

$$\begin{aligned}
\mathsf{C}_{rq}(\hat{\mathsf{G}}) &= \tilde{\mathsf{S}}_{pp}(\omega)\left|\hat{\mathsf{G}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{py}(\omega)\hat{\mathsf{G}}(\omega) - \tilde{\mathsf{S}}^*_{py}(\omega)\hat{\mathsf{G}}^*(\omega) + \tilde{\mathsf{S}}_{qq}(\omega)\\
\mathsf{C}_{sy}(\hat{\mathsf{H}}) &= \tilde{\mathsf{S}}_{xx}(\omega)\left|\hat{\mathsf{H}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{xy}(\omega)\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}^*_{xy}(\omega)\hat{\mathsf{H}}^*(\omega) + \tilde{\mathsf{S}}_{yy}(\omega)
\end{aligned}$$

✎PROOF:

$\boxed{\mathsf{C}_{rq}(\hat{\mathsf{G}})}$

$\triangleq \check{\mathbf{F}}\mathsf{E}\left(\left[\boldsymbol{r}(n)-\boldsymbol{q}(n)\right]\left[\boldsymbol{r}(0)-\boldsymbol{q}(0)\right]^*\right)$	by definition of C_{rq}	(Definition 11.11 page 107)
$= \tilde{\mathbf{F}}\left[\mathsf{E}\left[\boldsymbol{r}(n)\boldsymbol{r}^*(0)\right] - \mathsf{E}\left[\boldsymbol{r}(n)\boldsymbol{q}^*(0)\right] - \mathsf{E}\left[\boldsymbol{q}(n)\boldsymbol{r}^*(0)\right] + \mathsf{E}\left[\boldsymbol{q}(n)\boldsymbol{q}^*(0)\right]\right]$	by *linearity* of E	(Theorem 1.1 page 4)
$\triangleq \tilde{\mathbf{F}}\left[\mathsf{R}_{rr}(m) - \mathsf{R}_{rq}(m) - \mathsf{R}_{qr}(m) + \mathsf{R}_{qq}(m)\right]$	by definition of R_{xy}	(Definition 2.4 page 12)
$\triangleq \left[\tilde{\mathsf{S}}_{rr}(\omega) - \tilde{\mathsf{S}}_{rq}(\omega) - \tilde{\mathsf{S}}_{qr}(\omega) + \tilde{\mathsf{S}}_{qq}(\omega)\right]$	by definition of $\tilde{\mathsf{S}}_{xy}$	(Definition 6.3 page 44)
$= \boxed{\tilde{\mathsf{S}}_{pp}(\omega)\left|\hat{\mathsf{G}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{py}(\omega)\hat{\mathsf{G}}(\omega) - \tilde{\mathsf{S}}^*_{py}(\omega)\hat{\mathsf{G}}^*(\omega) + \tilde{\mathsf{S}}_{qq}(\omega)}$	by (A)–(D) and	Corollary 7.8 page 59

$\boxed{\mathsf{C}_{sy}(\hat{\mathsf{H}})}$

$\triangleq \check{\mathbf{F}}\mathsf{E}\left(\left[\boldsymbol{s}(n)-\boldsymbol{y}(n)\right]\left[\boldsymbol{s}(0)-\boldsymbol{y}(0)\right]^*\right)$	by definition of C_{sy}	(Definition 11.11 page 107)
$= \tilde{\mathbf{F}}\left[\mathsf{E}\left[\boldsymbol{s}(n)\boldsymbol{s}^*(0)\right] - \mathsf{E}\left[\boldsymbol{s}(n)\boldsymbol{y}^*(0)\right] - \mathsf{E}\left[\boldsymbol{y}(n)\boldsymbol{s}^*(0)\right] + \mathsf{E}\left[\boldsymbol{y}(n)\boldsymbol{y}^*(0)\right]\right]$	by *linearity* of E	(Theorem 1.1 page 4)
$\triangleq \tilde{\mathbf{F}}\left[\mathsf{R}_{ss}(m) - \mathsf{R}_{sy}(m) - \mathsf{R}_{ys}(m) + \mathsf{R}_{yy}(m)\right]$	by definition of R_{xy}	(Definition 2.4 page 12)
$\triangleq \left[\tilde{\mathsf{S}}_{ss}(\omega) - \tilde{\mathsf{S}}_{sy}(\omega) - \tilde{\mathsf{S}}_{ys}(\omega) + \tilde{\mathsf{S}}_{yy}(\omega)\right]$	by definition of $\tilde{\mathsf{S}}_{xy}$	(Definition 6.3 page 44)
$= \boxed{\tilde{\mathsf{S}}_{xx}(\omega)\left|\hat{\mathsf{H}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{xy}(\omega)\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}^*_{xy}(\omega)\hat{\mathsf{H}}^*(\omega) + \tilde{\mathsf{S}}_{yy}(\omega)}$	by (A)–(D) and	Corollary 7.8 (page 59)

✏

Theorem 11.3. *Let* $\mathbf{S}$ *be the* SYSTEM *illustrated in Figure 11.3 page 107 (A) or (B).*

THM

$$\left.\begin{array}{lll}
\text{(A).} & \boldsymbol{x}, \boldsymbol{u}, \textit{ and } \boldsymbol{v} \textit{ are } \text{WSS} & \textit{and}\\
\text{(B).} & \boldsymbol{x}, \boldsymbol{u}, \textit{ and } \boldsymbol{v} \textit{ are } \text{UNCORRELATED} & \textit{and}\\
\text{(C).} & \mathsf{E}\boldsymbol{u} = \mathsf{E}\boldsymbol{v} = 0 \quad (\text{ZERO-MEAN}) & \textit{and}\\
\text{(D).} & \hat{\mathsf{G}}_{ls} \textit{ and } \hat{\mathsf{H}}_{ls} \textit{ are } \text{LTI} &
\end{array}\right\} \implies \left\{\begin{array}{lll}
\text{(1).} & \arg\min_{\hat{\mathsf{G}}} \mathsf{C}_{rq}(\hat{\mathsf{G}}) = \dfrac{\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)}\\
\text{(2).} & \arg\min_{\hat{\mathsf{H}}} \mathsf{C}_{sy}(\hat{\mathsf{H}}) = \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}
\end{array}\right.$$

✎PROOF:

1. Define shorthand function names $\hat{\mathsf{G}} \triangleq \hat{\mathsf{G}}_{ls}$ and $\hat{\mathsf{H}} \triangleq \hat{\mathsf{H}}_{ls}$.
2. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{G}}_R}\mathsf{C}_{rq}(\hat{\mathsf{G}})\\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_R}\left(\tilde{\mathsf{S}}_{pp}|\hat{\mathsf{G}}|^2 - \hat{\mathsf{G}}\tilde{\mathsf{S}}_{py} - \hat{\mathsf{G}}^*\tilde{\mathsf{S}}^*_{py} + \tilde{\mathsf{S}}_{qq}\right) && \text{by Lemma 11.3 page 108}\\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_R}\left(\tilde{\mathsf{S}}_{pp}\left[\hat{\mathsf{G}}_R^2 + \hat{\mathsf{G}}_I^2\right] - (\hat{\mathsf{G}}_R + i\hat{\mathsf{G}}_I)\tilde{\mathsf{S}}_{py} - (\hat{\mathsf{G}}_R + i\hat{\mathsf{G}}_I)^*\tilde{\mathsf{S}}^*_{py} + \tilde{\mathsf{S}}_{qq}\right)\\
&= 2\hat{\mathsf{G}}_R\tilde{\mathsf{S}}_{pp} - \tilde{\mathsf{S}}_{py} - \tilde{\mathsf{S}}^*_{py} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{G}}_R}\tilde{\mathsf{S}}_{qq}} && \text{because } \boldsymbol{q} \text{ does not vary with } \hat{\mathsf{G}}\\
&= 2\hat{\mathsf{G}}_R\tilde{\mathsf{S}}_{pp} - 2\mathbf{R_e}\tilde{\mathsf{S}}_{py}\\
&= 2\hat{\mathsf{G}}_R\tilde{\mathsf{S}}_{pp} - 2\mathbf{R_e}\tilde{\mathsf{S}}_{yp} && \text{by Corollary 2.2 page 15}\\
\implies & \boxed{\hat{\mathsf{G}}_R(\omega) = \frac{\mathbf{R_e}\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)}}
\end{aligned}$$

3. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{G}}_I} \mathsf{C}_{\mathsf{rq}}(\hat{\mathsf{G}}) \\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_I}\left(\tilde{\mathsf{S}}_{\mathsf{pp}}|\hat{\mathsf{G}}|^2 - \hat{\mathsf{G}}\tilde{\mathsf{S}}_{\mathsf{py}} - \hat{\mathsf{G}}^*\tilde{\mathsf{S}}^*_{\mathsf{py}} + \tilde{\mathsf{S}}_{\mathsf{qq}}\right) && \text{by Lemma 11.3 page 108} \\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_I}\left[\tilde{\mathsf{S}}_{\mathsf{pp}}[\hat{\mathsf{G}}_R^2 + \hat{\mathsf{G}}_I^2] - (\hat{\mathsf{G}}_R + i\hat{\mathsf{G}}_I)\tilde{\mathsf{S}}_{\mathsf{py}} - (\hat{\mathsf{G}}_R - i\hat{\mathsf{G}}_I)\tilde{\mathsf{S}}^*_{\mathsf{py}} + \tilde{\mathsf{S}}_{\mathsf{qq}}\right] \\
&= 2\hat{\mathsf{G}}_I\tilde{\mathsf{S}}_{\mathsf{pp}} - i\tilde{\mathsf{S}}_{\mathsf{py}} + i\tilde{\mathsf{S}}^*_{\mathsf{py}} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{G}}_I}\tilde{\mathsf{S}}_{\mathsf{qq}}} && \text{because } \boldsymbol{q} \text{ does not vary with } \hat{\mathsf{G}} \\
&= 2\hat{\mathsf{G}}_I\tilde{\mathsf{S}}_{\mathsf{pp}} - 2i(i\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{py}}) \\
&= 2\hat{\mathsf{G}}_I\tilde{\mathsf{S}}_{\mathsf{pp}} + 2i(i\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{yp}}) && \text{by Corollary 2.2 page 15} \\
\implies & \boxed{\hat{\mathsf{G}}_I(\omega) = \frac{\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}}
\end{aligned}$$

4. Proof for $\hat{\mathsf{G}} \triangleq \hat{\mathsf{G}}_{\mathsf{ls}}$ expression:

$$\begin{aligned}
\boxed{\hat{\mathsf{G}}(\omega)} &= \hat{\mathsf{G}}_R(\omega) + i\hat{\mathsf{G}}_I(\omega) \\
&= \frac{\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)} + \frac{i\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)} && \text{by (2) lemma and (3) lemma} \\
&= \boxed{\frac{\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}}
\end{aligned}$$

5. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{H}}_R} \mathsf{C}_{\mathsf{sy}}(\hat{\mathsf{H}}) \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_R}\left(\tilde{\mathsf{S}}_{\mathsf{xx}}|\hat{\mathsf{H}}|^2 - \hat{\mathsf{H}}\tilde{\mathsf{S}}_{\mathsf{xy}} - \hat{\mathsf{H}}^*\tilde{\mathsf{S}}^*_{\mathsf{xy}} + \tilde{\mathsf{S}}_{\mathsf{yy}}\right) && \text{by Lemma 11.3 page 108} \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_R}(\tilde{\mathsf{S}}_{\mathsf{xx}}[\hat{\mathsf{H}}_R^2 + \hat{\mathsf{H}}_I^2] - (\hat{\mathsf{H}}_R + i\hat{\mathsf{H}}_I)\tilde{\mathsf{S}}_{\mathsf{xy}} - (\hat{\mathsf{H}}_R + i\hat{\mathsf{H}}_I)^*\tilde{\mathsf{S}}^*_{\mathsf{xy}} + \tilde{\mathsf{S}}_{\mathsf{yy}}) \\
&= 2\hat{\mathsf{H}}_R\tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{xy}} - \tilde{\mathsf{S}}^*_{\mathsf{xy}} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{H}}_R}\tilde{\mathsf{S}}_{\mathsf{yy}}} && \text{because } \boldsymbol{y} \text{ does not vary with } \hat{\mathsf{H}} \\
&= 2\hat{\mathsf{H}}_R\tilde{\mathsf{S}}_{\mathsf{xx}} - 2\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{xy}} \\
&= 2\hat{\mathsf{H}}_R\tilde{\mathsf{S}}_{\mathsf{xx}} - 2\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{yx}} && \text{by Corollary 2.2 page 15} \\
\implies & \boxed{\hat{\mathsf{H}}_R(\omega) = \frac{\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}}
\end{aligned}$$

6. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{H}}_I} \mathsf{C}_{\mathsf{sy}}(\hat{\mathsf{H}}) \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_I}\left(\tilde{\mathsf{S}}_{\mathsf{xx}}|\hat{\mathsf{H}}|^2 - \tilde{\mathsf{S}}_{\mathsf{xy}}\hat{\mathsf{H}} - \tilde{\mathsf{S}}^*_{\mathsf{xy}}\hat{\mathsf{H}}^* + \tilde{\mathsf{S}}_{\mathsf{yy}}\right) && \text{by Lemma 11.3 page 108} \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_I}\left[\tilde{\mathsf{S}}_{\mathsf{xx}}[\hat{\mathsf{H}}_R^2 + \hat{\mathsf{H}}_I^2] - \tilde{\mathsf{S}}_{\mathsf{xy}}(\hat{\mathsf{H}}_R + i\hat{\mathsf{H}}_I) - \tilde{\mathsf{S}}^*_{\mathsf{xy}}(\hat{\mathsf{H}}_R - i\hat{\mathsf{H}}_I) + \tilde{\mathsf{S}}_{\mathsf{yy}}\right] \\
&= 2\hat{\mathsf{H}}_I\tilde{\mathsf{S}}_{\mathsf{xx}} - i\tilde{\mathsf{S}}_{\mathsf{xy}} + i\tilde{\mathsf{S}}^*_{\mathsf{xy}} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{H}}_R}\tilde{\mathsf{S}}_{\mathsf{yy}}} && \text{because } \boldsymbol{q} \text{ does not vary with } \hat{\mathsf{H}}
\end{aligned}$$

$$
\begin{aligned}
&= 2\hat{\mathsf{H}}_I\tilde{\mathsf{S}}_{xx} - 2i(i\mathbf{I}_m\tilde{\mathsf{S}}_{xy}) \\
&= 2\hat{\mathsf{H}}_I\tilde{\mathsf{S}}_{xx} + 2i(i\mathbf{I}_m\tilde{\mathsf{S}}_{yx}) && \text{by Corollary 2.2 page 15} \\
&= 2\hat{\mathsf{H}}_I\tilde{\mathsf{S}}_{xx} - 2\mathbf{I}_m\tilde{\mathsf{S}}_{yx} \\
\implies & \boxed{\hat{\mathsf{H}}_I(\omega) = \frac{\mathbf{I}_m\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}}
\end{aligned}
$$

7. Proof for $\hat{\mathsf{H}} \triangleq \hat{\mathsf{H}}_{ls}$ expression:

$$
\begin{aligned}
\boxed{\hat{\mathsf{H}}(\omega)} &= \hat{\mathsf{H}}_R(\omega) + i\hat{\mathsf{H}}_I(\omega) \\
&= \frac{\mathbf{R}_e\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} + \frac{i\mathbf{I}_m\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by (5) lemma and (6) lemma} \\
&= \frac{\mathbf{R}_e\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} + \frac{i\mathbf{I}_m\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Theorem 7.4 page 55} \\
&= \boxed{\frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}}
\end{aligned}
$$

⇨

Using Theorem 11.3 (previous) we can see that the optimal **least-squares** operators $\hat{\mathsf{G}}_{ls}$ and $\hat{\mathsf{H}}_{ls}$ for the **non-linear** operator $\mathbf{T}$ in Figure 11.3 (page 107) (A) and (B) are (next two corollaries)

$$
\boxed{
\begin{array}{llcll}
(1). & \hat{\mathsf{G}}_{ls}(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega) + \tilde{\mathsf{S}}_{uu}(\omega)} & \text{for (A)—}\textit{communication system} \\
(2). & \hat{\mathsf{G}}_{ls}(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega) - \tilde{\mathsf{S}}_{uu}(\omega)} & \text{for (B)—}\textit{measurement system} \\
(3). & \hat{\mathsf{H}}_{ls}(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} & \text{for either (A) or (B)}
\end{array}
}
$$

Corollary 11.1. *Let* **S** *be the* SYSTEM *illustrated in Figure 11.3 page 107 (A).*

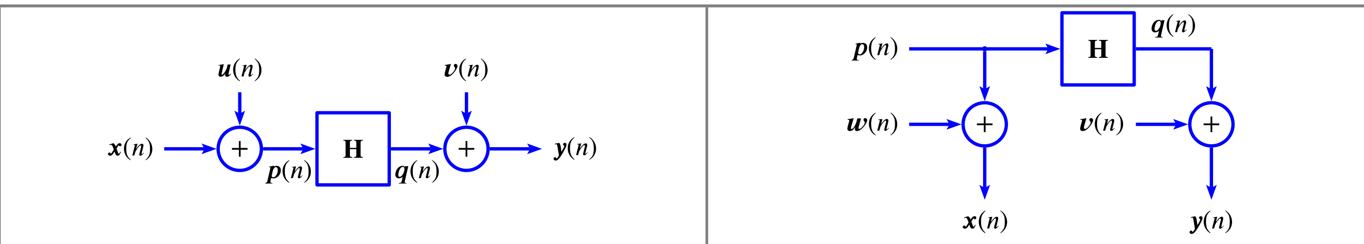
THM

$$
\left\{ \textit{hypotheses of Theorem 11.3 page 108} \right\} \implies \left\{
\begin{array}{llcl}
(1). & \arg\min\limits_{\hat{\mathsf{G}}_{ls}} \mathsf{C}_{rq}(\hat{\mathsf{G}}_{ls}) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega) + \tilde{\mathsf{S}}_{uu}(\omega)} \\
(2). & \arg\min\limits_{\hat{\mathsf{H}}_{ls}} \mathsf{C}_{sy}(\hat{\mathsf{H}}_{ls}) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}
\end{array}
\right\}
$$

✎PROOF:

$$
\begin{aligned}
\hat{\mathsf{G}}_{ls} &= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)} && \text{by Theorem 11.3 page 108} \\
&= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega) + \tilde{\mathsf{S}}_{uu}(\omega)} && \text{by Theorem 7.1 page 51} \\
\hat{\mathsf{H}}_{ls} &= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Theorem 11.3 page 108}
\end{aligned}
$$

⇨



(A) **communications LTI additive noise model** (B) **measurement LTI additive noise model**

Figure 11.4: Additive noise systems with **LTI** operator **H**

Corollary 11.2. *Let* **S** *be the* SYSTEM *illustrated in Figure 11.3 page 107 (B).*

THM

$$\left\{ \textit{hypotheses of Theorem 11.3 page 108} \right\} \implies \left\{ \begin{array}{llcl} (1). & \underset{\hat{\mathsf{G}}_{\mathsf{ls}}}{\arg\min}\, \mathsf{C}_{\mathsf{rq}}(\hat{\mathsf{G}}_{\mathsf{ls}}) & = & \dfrac{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)} \\ (2). & \underset{\hat{\mathsf{H}}_{\mathsf{ls}}}{\arg\min}\, \mathsf{C}_{\mathsf{sy}}(\hat{\mathsf{H}}_{\mathsf{ls}}) & = & \dfrac{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} \end{array} \right\}$$

✎PROOF:

$$\begin{aligned} \hat{\mathsf{G}} &= \frac{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)} && \text{by Theorem 11.3 page 108} \\ &= \frac{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)} && \text{by Theorem 7.1 page 51} \\ \hat{\mathsf{H}} &= \frac{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} && \text{by Theorem 11.3 page 108} \end{aligned}$$

⇨

It follows immediately from Corollary 11.1 (page 110) and Corollary 11.2 (page 111) that, in the special case of no input noise ($\boldsymbol{u}(n) = 0$), $\hat{\mathsf{H}}_1$ is the optimal least-squares estimate of $\tilde{\mathsf{H}}$ (next corollary).

Corollary 11.3. [20] *Let* **S** *be the* SYSTEM *illustrated in Figure 11.3 page 107 (A) or (B).*

COR

$$\left\{ \begin{array}{ll} (1). & \textit{hypotheses of Theorem 11.3 and} \\ (2). & \boldsymbol{u}(n) = 0 \end{array} \right\} \implies \left\{ \hat{\mathsf{G}}_{\mathsf{ls}}(\omega) = \hat{\mathsf{H}}_{\mathsf{ls}}(\omega) = \hat{\mathsf{H}}_1(\omega) \right\}$$

11.7 Least squares estimates of linear systems

The previous section did assume the estimates $\hat{\mathsf{H}}_1$ and $\hat{\mathsf{H}}_2$ to be *linear time invariant* (*LTI*), but it did *not* assume that the system transfer function **T** itself to be *LTI*. But making the LTI assumption on **H** yields some interesting and insightful results, such as those in this section.

Theorem 11.4 (Estimating H in communication additive noise system)**.** *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 page 111 (A).*

[20] Bendat and Piersol (1980) pages 98–100 ⟨5.1.1 Optimal Character of Calculations; note: proof minimizing $\tilde{\mathsf{S}}_{\mathsf{vv}}$ but yields same result⟩, Bendat and Piersol (1993) pages 106–109 ⟨5.1.1 Optimality of Calculations⟩, Bendat and Piersol (2010) pages 187–190 ⟨6.1.4 Optimum Frequency Response Functions⟩

THM

$$\left\{\begin{array}{llll}\text{(A).} & \mathbf{H} \textit{ is} & \text{LINEAR TIME INVARIANT} & (\text{LTI}) \textit{ and} \\ \text{(B).} & x(n) \textit{ is} & \text{WIDE-SENSE STATIONARY} & (\text{WSS}) \textit{ and} \\ \text{(C).} & x(n), u(n), \textit{and } v(n) \textit{ are} & \text{UNCORRELATED} & \end{array}\right\} \implies \left\{\begin{array}{llll}\text{(1).} & \hat{\mathsf{H}}_1(\omega) & = & \tilde{\mathsf{H}}(\omega) \quad \textit{and} \\ \text{(2).} & \hat{\mathsf{H}}_2(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{vv}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega)} + \tilde{\mathsf{H}}(\omega)\left[1 + \dfrac{\tilde{\mathsf{S}}_{uu}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}\right]\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\hat{\mathsf{H}}_1(\omega) &\triangleq \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by definition of } \hat{\mathsf{H}}_1 && \text{(Definition 11.2 page 95)}\\
&= \frac{\tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{xx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Corollary 7.5 page 57}\\
&= \tilde{\mathsf{H}}(\omega)\\
\hat{\mathsf{H}}_2(\omega) &\triangleq \frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)} && \text{by definition of } \hat{\mathsf{H}}_2 && \text{(Definition 11.3 page 96)}\\
&= \frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Corollary 7.5 page 57}\\
&= \frac{\tilde{\mathsf{S}}_{vv}(\omega) + \tilde{\mathsf{S}}_{qq}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Theorem 7.1 page 51}\\
&= \frac{\tilde{\mathsf{S}}_{vv}(\omega) + \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{pp}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Corollary 5.3 page 37}\\
&= \frac{\tilde{\mathsf{S}}_{vv}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega)} + \frac{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{H}}(\omega)\left[\tilde{\mathsf{S}}_{xx}(\omega) + \tilde{\mathsf{S}}_{uu}(\omega)\right]}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega)}\\
&= \frac{\tilde{\mathsf{S}}_{vv}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{xx}(\omega)} + \tilde{\mathsf{H}}(\omega)\left[1 + \frac{\tilde{\mathsf{S}}_{uu}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}\right]
\end{aligned}$$

✏

Theorem 11.5 (Estimating H in measurement additive noise system). [21] *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 page 111 (B).*

THM

$$\left\{\begin{array}{lll}\text{(A).} & \mathbf{H} \textit{ is} & \text{LINEAR TIME INVARIANT} \textit{ and} \\ \text{(B).} & x(n) \textit{ is} & \text{WIDE-SENSE STATIONARY} \textit{ and} \\ \text{(C).} & x(n), u(n), \textit{and } v(n) \textit{ are} & \text{UNCORRELATED} \end{array}\right\} \implies \left\{\begin{array}{lllll}\text{(1).} & \hat{\mathsf{H}}_1(\omega) & = & \tilde{\mathsf{H}}(\omega)\left[\dfrac{1}{1 + \dfrac{\tilde{\mathsf{S}}_{ww}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)}}\right] & (\text{UNDER-ESTIMATED}) \textit{ and} \\ \text{(2).} & \hat{\mathsf{H}}_2(\omega) & = & \tilde{\mathsf{H}}(\omega)\left[1 + \dfrac{\tilde{\mathsf{S}}_{vv}(\omega)}{\tilde{\mathsf{S}}_{qq}(\omega)}\right] & (\text{OVER-ESTIMATED})\end{array}\right\}$$

[21] Shin and Hammond (2008) page 294 ⟨$H_1(f) = H(f)$ (9.70); $H_2(f) = H(f)\left(1 + S_{n_yn_y}(f)/S_{yy}(f)\right)$ (9.71)⟩, Shin and Hammond (2008) page 294 ⟨$H_1(f) = H(f)/\left(1 + S_{n_xn_x}/S_{xx}(f)\right)$ (9.72); $H_2(f) = H(f)$ (9.73)⟩, Mitchell (1982) page 277 ⟨$H_1(f) = H_0(f)/(1 + G_{nn}/G_{uu})$⟩ Mitchell (1982) page 278 ⟨$H_2(f) = H_0(f)(1 + G_{mm}/G_{vv})$⟩

✎PROOF:

$$\begin{aligned}
\hat{H}_1(\omega) &\triangleq \frac{\tilde{S}_{yx}(\omega)}{\tilde{S}_{xx}(\omega)} && \text{by definition of } \hat{H}_1 && \text{(Definition 11.2 page 95)} \\
&= \frac{\tilde{S}_{pp}(\omega)\tilde{H}(\omega)}{\tilde{S}_{xx}(\omega)} && \text{by Corollary 7.4 page 57} \\
&= \frac{\tilde{S}_{pp}(\omega)\tilde{H}(\omega)}{\tilde{S}_{pp}(\omega)+\tilde{S}_{ww}(\omega)} && \text{by hypothesis (A)} && \text{and Corollary 5.3 page 37} \\
&= \tilde{H}(\omega)\left[\frac{1}{1+\frac{\tilde{S}_{ww}(\omega)}{\tilde{S}_{pp}(\omega)}}\right] \\
\hat{H}_2(\omega) &\triangleq \frac{\tilde{S}_{yy}(\omega)}{\tilde{S}_{xy}(\omega)} && \text{by definition of } \hat{H}_2 && \text{(Definition 11.3 page 96)} \\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{xy}(\omega)} && \text{by hypothesis (C)} && \text{and Corollary 7.1 page 52} \\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{xq}(\omega)} && \text{by hypothesis (C)} && \text{and Theorem 7.4 page 55} \\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{pq}(\omega)} && \text{by hypothesis (C)} && \text{and Lemma 7.3 page 54} \\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{qq}(\omega)/\tilde{H}(\omega)} && \text{by \textit{LTI} hypothesis (A)} && \text{and Corollary 5.3 page 37} \\
&= \tilde{H}(\omega)\left[1+\frac{\tilde{S}_{vv}(\omega)}{\tilde{S}_{qq}(\omega)}\right] && \text{by hypotheses (A) and (B)} && \text{and Corollary 5.3 page 37}
\end{aligned}$$

✏

Corollary 11.4. *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 (page 111).*

COR

$$\left\{\begin{array}{ll} \text{(A).} & \textit{hypotheses of Theorem 11.5 and} \\ \text{(B).} & u(n) = u(n) = 0 \quad (\text{NO INPUT NOISE}) \end{array}\right\} \implies \left\{ \hat{H}_1(\omega) = \tilde{H}(\omega) \quad (\text{UNBIASED}) \right\}$$

$$\left\{\begin{array}{ll} \text{(A).} & \textit{hypotheses of Theorem 11.5 and} \\ \text{(B).} & v(n) = 0 \quad (\text{NO OUTPUT NOISE}) \end{array}\right\} \implies \left\{ \hat{H}_2(\omega) = \tilde{H}(\omega) \quad (\text{UNBIASED}) \right\}$$

Lemma 11.4. *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 (page 111).*

LEM

$$\left\{ \text{There exists } \kappa(\omega) \text{ such that } \tilde{S}_{vv}(\omega) = \kappa(\omega)\tilde{S}_{uu}(\omega) \right\}$$
$$\implies \left\{ \tilde{S}_{uu}(\omega) = \frac{\left|\hat{H}(\omega)\right|^2\tilde{S}_{xx}(\omega) - \hat{H}(\omega)\tilde{S}_{xy}(\omega) - \hat{H}^*(\omega)\tilde{S}^*_{xy}(\omega) + \tilde{S}_{yy}(\omega)}{\kappa(\omega) + \left|\hat{H}(\omega)\right|^2} \right\}$$

✎PROOF:

1. Development based on results of previous chapters:

$$\begin{aligned}
\tilde{S}_{vv} &= \tilde{S}_{yy} - \tilde{S}_{qq} && \text{by Corollary 7.1 page 52} \\
&= \tilde{S}_{yy} - \tilde{S}_{pq}\hat{H} && \text{by Corollary 5.3 page 37} \\
&= \tilde{S}_{yy} - \tilde{S}_{xy}\hat{H} && \text{by Theorem 7.4 page 55} \\
\tilde{S}_{uu} &= \tilde{S}_{xx} - \tilde{S}_{pp} && \text{by Corollary 7.1 page 52}
\end{aligned}$$

$$= \tilde{S}_{xx} - \frac{\tilde{S}_{qp}}{\hat{H}}$$ by Corollary 5.3 page 37

$$= \tilde{S}_{xx} - \frac{\tilde{S}_{yx}}{\hat{H}}$$ by Theorem 7.4 page 55

$$\tilde{S}_{uu}\left[\left|\hat{H}\right|^2 + \kappa\right] = \left|\hat{H}\right|^2\tilde{S}_{uu} + \kappa\tilde{S}_{uu}$$

$$\triangleq \tilde{S}_{uu}\left|\hat{H}\right|^2 + \tilde{S}_{vv}$$ by definition of $\kappa(\omega)$

$$= \left|\hat{H}\right|^2\left[\tilde{S}_{xx} - \frac{\tilde{S}_{yx}}{\hat{H}}\right] + \left[\tilde{S}_{yy} - \tilde{S}_{xy}\hat{H}\right]$$

$$= \left|\hat{H}\right|^2\tilde{S}_{xx} - \hat{H}^*\tilde{S}_{yx} - \tilde{S}_{xy}\hat{H} + \tilde{S}_{yy}$$

$$\implies \boxed{\tilde{S}_{uu}(\omega) = \frac{\left|\hat{H}(\omega)\right|^2\tilde{S}_{xx}(\omega) - \hat{H}(\omega)\tilde{S}_{xy}(\omega) - \hat{H}^*(\omega)\tilde{S}^*_{xy}(\omega) + \tilde{S}_{yy}(\omega)}{\kappa(\omega) + \left|\hat{H}(\omega)\right|^2}}$$

2. Development of Wicks and Vold (Wicks and Vold (1986)):

$$\tilde{Y} - \tilde{V} = \tilde{Q} = \hat{H}\tilde{P} = \hat{H}(\tilde{X} - \tilde{U})$$ by definition of $\hat{H}$

$$\hat{H}\tilde{U} - \tilde{V} = \hat{H}\tilde{X} - \tilde{Y}$$ by *left distributive* prop. (Theorem M.4 page 285)

$$E\left(\left[\hat{H}\tilde{U} - \tilde{V}\right]\left[\hat{H}\tilde{U} - \tilde{V}\right]^*\right) = E\left(\left[\hat{H}\tilde{X} - \tilde{Y}\right]\left[\hat{H}\tilde{X} - \tilde{Y}\right]^*\right)$$

$$\left|\hat{H}\right|^2\tilde{S}_{uu} - \hat{H}\cancelto{0}{\tilde{S}_{uv}} - \hat{H}^*\cancelto{0}{\tilde{S}_{vu}} + \tilde{S}_{vv} = \left|\hat{H}\right|^2\tilde{S}_{xx} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy}$$ because $\boldsymbol{u}$ and $\boldsymbol{v}$ are *uncorrelated*

$$\left|\hat{H}\right|^2\tilde{S}_{uu} + \kappa\tilde{S}_{uu} = \left|\hat{H}\right|^2\tilde{S}_{xx} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy}$$ by hypothesis

✏

Theorem 11.6. [22] *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 (page 111). Let* $\hat{H}_\kappa(\omega)$ *be the transfer function estimate defined in Definition 11.7 (page 99).*

THM

$$\left\{\begin{array}{ll}(1). & \text{There exists } \kappa(\omega) \text{ such that} \\ & \tilde{S}_{vv}(\omega) = \kappa(\omega)\tilde{S}_{uu}(\omega) \quad \text{and} \\ (2). & C(\hat{H}_{ls}) = \tilde{S}_{uu}(\omega)\end{array}\right\} \implies \left\{\begin{array}{l}\arg\min_{\hat{H}} C(\hat{H}) = \hat{H}_\kappa(\omega) \\ (\hat{H}_\kappa \text{ is the “optimal” estimator for minimizing system noise})\end{array}\right\}$$

✎PROOF:

1. Let $F \triangleq \left|\hat{H}(\omega)\right|^2\tilde{S}_{xx}(\omega) - \hat{H}(\omega)\tilde{S}_{xy}(\omega) - \hat{H}^*(\omega)\tilde{S}^*_{xy}(\omega) + \tilde{S}_{yy}(\omega)$ (numerator in Lemma 11.4) and
$G \triangleq \kappa(\omega) + \left|\hat{H}(\omega)\right|^2$ (denominator in Lemma 11.4)

2. lemma $\left(\frac{\partial}{\partial \hat{H}_R}\tilde{S}_{uu}\right)$:

$$\boxed{0} = \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_R}\tilde{S}_{uu}$$ set $\frac{\partial}{\partial\hat{H}_R}\tilde{S}_{uu} = 0$ to find optimum $\hat{H}_R$

$$= \frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_R}\frac{F}{G}$$ by Lemma 11.4 page 113

$$= \frac{1}{2}G^2\frac{(F'G - G'F)}{G^2}$$ by *Quotient Rule*

$$= \frac{1}{2}(F'G - G'F)$$

[22] Wicks and Vold (1986) page 898 ⟨has additional s in denominator⟩, Shin and Hammond (2008) page 293 ⟨(9.67)⟩, White et al. (2006) page 679 ⟨(6)⟩

$$= \frac{1}{2}\left[2\hat{H}_R\tilde{S}_{xx} - \tilde{S}_{xy} - \tilde{S}^*_{xy}\right]G - \frac{1}{2}2\hat{H}_R F \qquad \text{by definition of } F, G \qquad \text{(item (1) page 114)}$$

$$= \boxed{\hat{H}_R\tilde{S}_{xx}G - G\mathbf{R_e}\tilde{S}_{xy} - \hat{H}_R F}$$

3. lemma $\left(\frac{\partial}{\partial \hat{H}_I}\tilde{S}_{uu}\right)$:

$$\boxed{0} = \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial \hat{H}_I}\tilde{S}_{uu} \qquad \text{set } \frac{\partial}{\partial \hat{H}_I}\tilde{S}_{uu} = 0 \text{ to find optimum } \hat{H}_I$$

$$= \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial \hat{H}_I}\frac{F}{G} \qquad \text{by Lemma 11.4 page 113}$$

$$= \frac{1}{2}G^2\frac{(F'G - G'F)}{G^2} \qquad \text{by \textit{Quotient Rule}}$$

$$= \frac{1}{2}(F'G - G'F)$$

$$= \frac{1}{2}\left[2\hat{H}_I\tilde{S}_{xx} - i\tilde{S}_{xy} + i\tilde{S}^*_{xy}\right]G - \frac{1}{2}2\hat{H}_I F \qquad \text{by definition of } F, G \qquad \text{(item (1) page 114)}$$

$$= \boxed{\hat{H}_I\tilde{S}_{xx}G + G\mathbf{I_m}\tilde{S}_{xy} - \hat{H}_I F}$$

4. Solve for $\hat{H}$ ...

$$0 = 0 + i0 = \frac{1}{2}G^2 0 + \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial \hat{H}_R}\tilde{S}_{uu} + i\frac{1}{2}G^2\frac{\partial}{\partial \hat{H}_I}\tilde{S}_{uu}$$

$$= \left[\hat{H}_R\tilde{S}_{xx}G - G\mathbf{R_e}\tilde{S}_{xy} - \hat{H}_R F\right] + i\left[\hat{H}_I\tilde{S}_{xx}G + G\mathbf{I_m}\tilde{S}_{xy} - \hat{H}_I F\right] \qquad \text{by (2) lemma and (3) lemma}$$

$$= \hat{H}\tilde{S}_{xx}G - \tilde{S}^*_{xy}G - \hat{H}F \qquad \text{because } \mathbf{R_e}(z) + i\mathbf{I_m}(z) = z \text{ and } \mathbf{R_e}(z) - i\mathbf{I_m}(z) = z^*$$

$$= \hat{H}\tilde{S}_{xx}G - \tilde{S}_{yx}G - \hat{H}F \qquad \text{by Corollary 2.2 page 15}$$

$$= \hat{H}\tilde{S}_{xx}\left(\kappa + |\hat{H}|^2\right) - \tilde{S}_{yx}\left(\kappa + |\hat{H}|^2\right) - \hat{H}\left(|\hat{H}|^2\tilde{S}_{xx} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy}\right) \qquad \text{by } F, G \text{ defs.}$$

$$= \hat{H}\tilde{S}_{xx}\left(\kappa + \cancel{|\hat{H}|^2}\right) - \tilde{S}_{yx}\left(\kappa + |\hat{H}|^2\right) - \hat{H}\left(\cancel{|\hat{H}|^2\tilde{S}_{xx}} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy}\right)$$

$$= \kappa\hat{H}\tilde{S}_{xx} - \tilde{S}_{yx}\left(\kappa + |\hat{H}|^2\right) + \left(\hat{H}^2\tilde{S}_{xy} + |\hat{H}|^2\tilde{S}^*_{xy} - \hat{H}\tilde{S}_{yy}\right)$$

$$= \kappa\hat{H}\tilde{S}_{xx} - \kappa\tilde{S}_{yx} - \cancel{\tilde{S}_{yx}|\hat{H}|^2} + \left(\hat{H}^2\tilde{S}_{xy} + \cancel{|\hat{H}|^2\tilde{S}^*_{xy}} - \hat{H}\tilde{S}_{yy}\right)$$

$$= \hat{H}^2\tilde{S}_{xy} + \hat{H}\left[\kappa\tilde{S}_{xx} - \tilde{S}_{yy}\right] - \kappa\tilde{S}^*_{xy}$$

$$\Longrightarrow \boxed{\hat{H} = \frac{(\tilde{S}_{yy} - \kappa\tilde{S}_{xx}) \pm \sqrt{(\tilde{S}_{yy} - \kappa\tilde{S}_{xx})^2 + 4\kappa|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}}} \qquad \text{by \textit{Quadratic Equation}}$$

⇨

11.8 Coherence

11.8.1 Application

Coherence has two basic purposes:

1. The *coherence* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is a measure of how closely $\boldsymbol{x}$ and $\boldsymbol{y}$ are statistically related. That is, it is an indication of how much $\boldsymbol{x}$ and $\boldsymbol{y}$ "cohere" or "stick" together

2. The *coherence* of $\mathbf{x}$ and $\mathbf{y}$ is a measure of how reliable are the estimates $\hat{\mathsf{H}}_1$ and $\hat{\mathsf{H}}_2$ (Definition 11.2 page 95, Definition 11.3 page 96). If the coherence is 0.70 or above, then we can have high confidence that the estimates $\hat{\mathsf{H}}_1$ and $\hat{\mathsf{H}}_2$ are "good" estimates.[23]

11.8.2 Definitions

Definition 11.12. [24] *Let* **S** *be a* SYSTEM *with input* $\mathbf{x}(n)$ *and output* $\mathbf{y}(n)$.

DEF

*The **complex coherence** function is defined as* $$\mathsf{C}_{xy}(\omega) \triangleq \frac{\tilde{\mathsf{S}}^*_{xy}(\omega)}{\sqrt{\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{yy}(\omega)}}$$

*The **ordinary coherence** function is defined as* $$\gamma^2_{xy}(\omega) \triangleq \frac{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|^2}{\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{yy}(\omega)}$$

Proposition 11.7.

PRP

$$\gamma^2_{xy}(\omega) = \frac{\hat{\mathsf{H}}_1(\omega)}{\hat{\mathsf{H}}_2(\omega)}$$

✎PROOF:

$$\boxed{\gamma^2_{xy}(\omega)} \triangleq \frac{\left|\tilde{\mathsf{S}}_{xy}\right|^2}{\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{yy}} \qquad \text{by definition of } \gamma^2_{xy} \qquad \text{(Definition 11.12 page 116)}$$

$$= \frac{\tilde{\mathsf{S}}^*_{xy}/\tilde{\mathsf{S}}_{xx}}{\tilde{\mathsf{S}}_{yy}/\tilde{\mathsf{S}}_{xy}} \triangleq \boxed{\frac{\hat{\mathsf{H}}_1}{\hat{\mathsf{H}}_2}} \qquad \text{by definitions of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2 \qquad \text{(Definition 11.2 page 95, Definition 11.3 page 96)}$$

⇨

Remark 11.2. Note that the *complex transmissibility* $\tilde{\mathsf{T}}'_{xy}$ of Remark 11.1 provides a nice mathematical symmetry (always a good sign of good direction) with *coherence* in the system identification family tree. In particular, note that the following:

REM

$$\mathsf{C}_{xy} \triangleq \sqrt{\frac{\hat{\mathsf{H}}^*_1}{\hat{\mathsf{H}}_2}} \text{ whereas } \tilde{\mathsf{T}}'_{xy} \triangleq \sqrt{\hat{\mathsf{H}}_1\hat{\mathsf{H}}_2}$$

✎PROOF:

$$\sqrt{\frac{\hat{\mathsf{H}}^*_1(\omega)}{\hat{\mathsf{H}}_2(\omega)}} \qquad \text{by definition of } \hat{\mathsf{H}}_{gm} \qquad \text{(Definition 11.5 page 97)}$$

⇨

11.8.3 A warning

Estimators yield, as the name implies, estimates. These estimates in general contain some error.

[23] Liang and Lee (2015) pages 363–365 ⟨7.4.2COHERECE FUNCTION⟩

[24] Chen et al. (2012) page 4699(1), (2), Liang and Lee (2015) pages 363–365 ⟨7.4.2 Coherence function⟩, Ewins (1986) page 131 ⟨$\gamma^2 = H_1(\omega)/H_2(\omega)$ (3.8)⟩

Example 11.1 (The K=1 Welch estimate of coherence). Suppose we have two *uncorrelated* stationary sequences $\boldsymbol{x}(n)$ and $\boldsymbol{y}(n)$. Then, there CSD $\mathsf{S}_{\mathsf{xy}}(\omega)$ should be 0 because

$$\begin{aligned}
\mathsf{S}_{\mathsf{xy}}(\omega) &\triangleq \check{\mathbf{F}}\mathsf{E}\mathsf{R}_{\mathsf{xy}}(m) \\
&= \check{\mathbf{F}}\mathsf{E}[x(n)y[n+m]] \\
&= \check{\mathbf{F}}\left[\mathsf{E}_{\mathsf{x}}(n)\right]\left[\mathsf{E}_{\mathsf{y}}[n+m]\right] \\
&= \check{\mathbf{F}}[0][0] \\
&= 0
\end{aligned}$$

This will give a coherence of 0 also:

$$C(\omega) = \frac{\mathsf{S}_{\mathsf{xy}}}{\sqrt{\mathsf{S}_{\mathsf{xx}}\mathsf{S}_{\mathsf{yy}}}} = 0$$

However, the Welch estimate with $K = 1$ will yield

$$\begin{aligned}
|C(\omega)| &= \left|\frac{\mathsf{S}_{\mathsf{xy}}}{\sqrt{\mathsf{S}_{\mathsf{xx}}\mathsf{S}_{\mathsf{yy}}}}\right| \\
&= \left|\frac{(\tilde{\mathbf{F}}x)(\tilde{\mathbf{F}}y)^*}{\sqrt{\left|\tilde{\mathbf{F}}x\right|^2\left|\tilde{\mathbf{F}}y\right|^2}}\right| \\
&= 1
\end{aligned}$$

CHAPTER 12

ESTIMATING NOISE

Estimating noise in a system is difficult and many estimation methods are possible.

- Thong et al. (2001)
- Zheng et al. (2002)
- Kim and Kamel (2004)
- Kamel and Sim (2004)

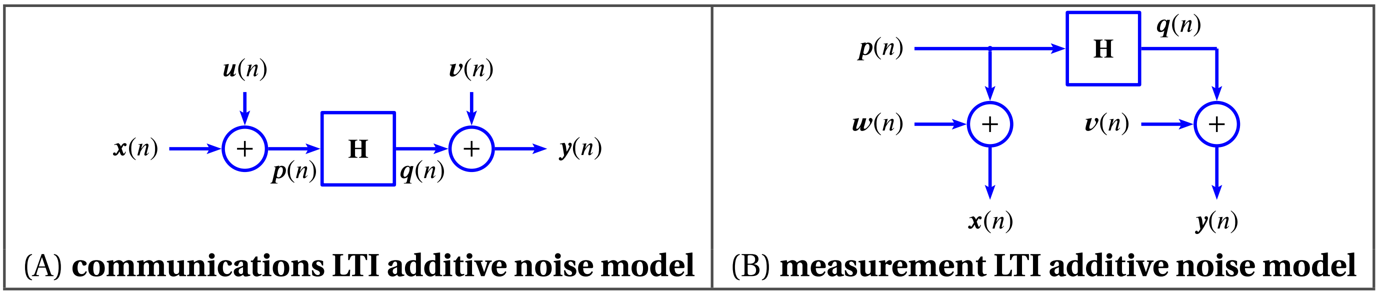


Figure 12.1: Additive noise systems with **LTI** operator **H**

CHAPTER 13

COMMUNICATION CHANNELS

13.1 System model

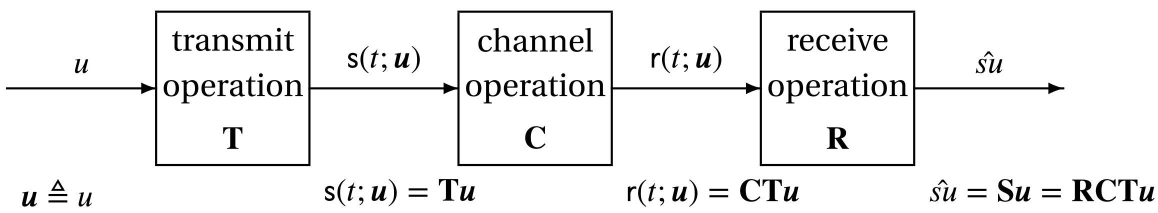


Figure 13.1: Communication system model

A communication system is an operator. $\mathbf{S}$ over an information sequence u that generates an estimated information sequence $\hat{su}$. The system operator factors into a receive operator $\mathbf{R}$, a channel operator $\mathbf{C}$, and a transmit operator $\mathbf{T}$ such that

$$\mathbf{S} = \mathbf{RCT}.$$

The transmit operator operates on an information sequence u to generate a channel signal $\mathsf{s}(t;\boldsymbol{u})$. The channel operator operates on the transmitted signal $\mathsf{s}(t;\boldsymbol{u})$ to generate the received signal $\mathsf{r}(t;\boldsymbol{u})$. The receive operator operates on the received signal $\mathsf{r}(t;\boldsymbol{u})$ to generate the estimate $\hat{su}$ (see Figure 13.1 (page 121)).

Definition 13.1. *Let U be the set of all sequences u and let*

DEF

- $\mathbf{S} : U \to U$ *(system operator)*
- $\mathbf{T} : U \to \mathbb{R}^\infty$ *(transmit operator)*
- $\mathbf{C} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ *(channel operator)*
- $\mathbf{R} : \mathbb{R}^\infty \to U$ *(receive operator)*

be operators. A ***digital communication system*** *is the operation $\mathbf{S}$ on the set of information sequences U such that $\mathbf{S} \triangleq \mathbf{RCT}$.*

Communication systems can be continuous or discrete valued in time and/or amplitude:

$s(t) = a(t)\psi(t)$	continuous time t	discrete time t
continuous amplitude $a(t)$	analog communications	discrete-time communications
discrete amplitude $a(t)$	—	digital communications

In this document, we normally take the approach that

1. **C** is stochastic
2. There is no structural constraint on **R**.
3. **R** is optimum with respect to the ML-criterion.

These characteristics are explained more fully below.

13.1.1 Channel operator

Real-world physical channels perform a number of operations on a signal. Often these operations are closely modeled by a channel operator **C**. Properties that characterize a particular channel operator associated with some physical channel include

- linear or non-linear
- time-invariant or time-variant
- memoryless or non-memoryless
- deterministic or stochastic.

Examples of physical channels include free space, air, water, soil, copper wire, and fiber optic cable. Information is carried through a channel using some physical process. These processes include:

Process	Example
electromagnetic waves	free space, air
acoustic waves	water, soil
electric field potential (voltage)	wire
light	fiber optic cable
quantum mechanics	

13.1.2 Receive operator

Let **I** be the *identity operator*.[1] Ideally, **R** is selected such that $\mathbf{RCT} = \mathbf{I}$. In this case we say that **R** is the *left inverse*[2] of **CT** and denote this left inverse by **C**. One example of a system where this inverse exists is the noiseless ISI system. While this is quite useful for mathematical analysis and system design, **C** does not actually exist for any real-world system.

When **C** does not exist, the "ideal" **R** is one that is optimum

1. with respect to some *criterion* (or cost function)
2. and sometimes under some structural *constraint*.

[1] **I** is the *identity operator* if for any operator **X**, $\mathbf{XI} = \mathbf{IX} = \mathbf{X}$.

[2]

$\mathbf{X}^{-1}X$ is the	*left inverse*	of **X** if	$\mathbf{X}^{-1}X\mathbf{X} = \mathbf{I}$.
$\mathbf{X}^{-1}X$ is the	*right inverse*	of **X** if	$\mathbf{X}\mathbf{X}^{-1}X = \mathbf{I}$.
$\mathbf{X}^{-1}X$ is the	*inverse*	of **X** if	$\mathbf{X}^{-1}X\mathbf{X} = \mathbf{X}\mathbf{X}^{-1}X = \mathbf{I}$.

When a structural constraint is imposed on $\mathbf{R}$, the solution is called *structured*; otherwise, it is called *non-structured*.[3] A common example of a structured approach is the use of a transversal filter (FIR filter in DSP) in which optimal coefficients are found for the filter. A structured $\mathbf{R}$ is only optimal with respect to the imposed constraint. Even though $\mathbf{R}$ may be optimal with respect to this structure, $\mathbf{R}$ may not be optimal in general; that is, there may be another structure that would lead to a "better" solution. In a non-structured approach, $\mathbf{R}$ is free to take any form whatsoever (practical or impractical) and therefore leads to the best of the best solutions.

The nature of $\mathbf{R}$ depends heavily on the nature of $\mathbf{C}$. If $\mathbf{C}$ does not exist, then the ideal $\mathbf{R}$ is one that is optimal with respect to some criterion. If $\mathbf{C}$ is deterministic, then appropriate optimization criterion may include

- least square error (LSE) criterion
- minimum absolute error criterion
- minimum peak distortion criterion.

If $\mathbf{C}$ is stochastic then appropriate optimization criterion may include

Bayes:	pdf known and cost function defined
Maximum aposteriori probability (MAP):	pdf known and uniform cost function
Maximum likelihood (ML):	pdf known and no prior probability information
mini-max:	pdf not known but a cost function is defined
Neyman-Pearson:	pdf not known and no cost function defined.

Making $\mathbf{R}$ optimum with respect to one of these criterion leads to an *estimate* $\hat{s}u = \mathbf{RCT}\boldsymbol{u}$ that is also optimum with respect to the same criterion (Definition 8.1 page 64).

13.2 Optimization in the case of additional operations

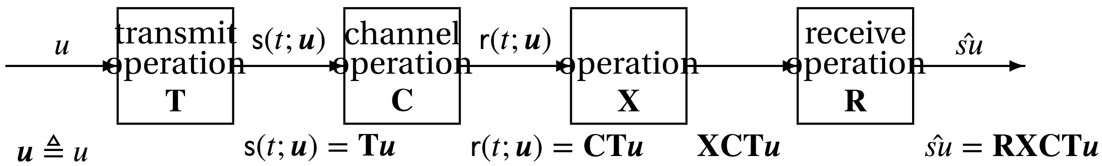


Figure 13.2: Theorem of reversibility

Often in communication systems, an additional operator $\mathbf{X}$ is inserted such that (see Figure 13.2 (page 123))

$$\mathbf{S} = \mathbf{RXCT}.$$

An example of such an operator $\mathbf{X}$ is a receive filter. Is it still possible to find an $\mathbf{R}$ that will perform as well as the case where $\mathbf{X}$ is not inserted? In general, the answer is "no". For example, if $\mathbf{X}r = 0$, then all received information is lost and obviously there is no $\mathbf{R}$ that can recover from this event. However, in the case where the right inverse $\mathbf{X}^{-1}X$ of $\mathbf{X}$ exists, then the answer to the question is "yes" and an optimum $\mathbf{R}$ still exists. That is, it doesn't matter if an $\mathbf{X}$ is inserted into system as long as $\mathbf{X}$ is invertible. This is stated formally in the next theorem.

Theorem 13.1 (Theorem of Reversibility). [4] *Let*

[3] Trees (2001) page 12
[4] Trees (2001) pages 289–290

- $\hat{\theta}\_\mathbf{RCT}\boldsymbol{u}$ *be the optimum estimate of* $\boldsymbol{u}$
- $\mathbf{X}$ *be an operator with right inverse* $\mathbf{X}^{-1}X$.

Then there exists some $\mathbf{R}'$ *such that*

THM $\hat{\theta}\_\mathbf{R}'\mathbf{XCT}\boldsymbol{u}$.

✎Proof: Let $\mathbf{R}' = \mathbf{RX}^{-1}X$. Then

$$\mathbf{R}'\mathbf{XCT}\boldsymbol{u} = \mathbf{RX}^{-1}X\mathbf{CT}\boldsymbol{u} = \mathbf{RCT}\boldsymbol{u} = \hat{\theta}.$$

⇰

13.3 Channel Statistics

The receiver needs to make a decision as to what sequence $(\!(u)\!)$ the transmitter has sent. This decision should be optimal in some sense. Very often the optimization criterion is chosen to be the *maximal likelihood (ML)* criterion. The information that the receiver can use to make an optimal decision is the received signal $\mathsf{r}(t)$.

If the symbols in $\mathsf{r}(t)$ are statistically *independent*, then the optimal estimate of the current symbol depends only on the current symbol period of $\mathsf{r}(t)$. Using other symbol periods of $\mathsf{r}(t)$ has absolutely no additional benefit. Note that the AWGN channel is *memoryless*; that is, the way the channel treats the current symbol has nothing to do with the way it has treated any other symbol. Therefore, if the symbols sent by the transmitter into the channel are independent, the symbols coming out of the channel are also independent.

However, also note that the symbols sent by the transmitter are often very intentionally not independent; but rather a strong relationship between symbols is intentionally introduced. This relationship is called *channel coding*. With proper channel coding, it is theoretically possible to reduce the probability of communication error to any arbitrarily small value as long as the channel is operating below its *channel capacity*.

This chapter assumes that the received symbols are statistically independent; and therefore optimal decisions at the receiver for the current symbol are made only from the current symbol period of $\mathsf{r}(t)$.

The received signal $\mathsf{r}(t)$ over a single symbol period contains an uncountably infinite number of points. That is a lot. It would be nice if the receiver did not have to look at all those uncountably infinite number of points when making an optimal decision. And in fact the receiver does indeed not have to. As it turns out, a single finite set of *statistics* $\{\dot{r}_1,\ \dot{r}_2, \ldots, \dot{r}_N\}$ is sufficient for the receiver to make an optimal decision as to which value the transmitter sent.

Definition 13.2. *Let* $\mathbf{C}$ *be an additive noise channel*

CHAPTER 14

OPTIMAL SYMBOL DETECTION

14.1 ML Estimation

Theorem 14.1. *In an AWGN channel with received signal* $\mathsf{r}(t) = s(t;\phi) + n(t)$ *Let*

- $\mathsf{r}(t) = s(t;\phi) + n(t)$ *be the received signal in an AWGN channel*
- $n(t)$ *a Gaussian white noise process*
- $\mathsf{s}(t;\phi)$ *the transmitted signal such that*

$$s(t;\phi) = \sum_{n\in\mathbb{Z}} a_n \lambda(t-nT)\cos(2\pi f_c t + \theta_n + \phi).$$

Then the optimal ML estimate of ϕ *is either of the two equivalent expressions*

THM

$$\begin{aligned}\hat{\phi}_{\mathsf{ml}} &= -\operatorname{atan}\left[\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathsf{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathsf{d}t}\right] \\ &= \arg_\phi\left(\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\lambda(t-nT)\sin(2\pi f_c t+\theta_n+\phi)\right]\,\mathsf{d}t = 0\right).\end{aligned}$$

✎PROOF:

$$\begin{aligned}\hat{\phi}_{\mathsf{ml}} &= \arg_\phi\left(2\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathsf{d}t = \frac{\partial}{\partial\phi}\int_{t\in\mathbb{R}} s^2(t;\phi)\,\mathsf{d}t\right) && \text{by Theorem 9.6 page 80} \\ &= \arg_\phi\left(2\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathsf{d}t = \frac{\partial}{\partial\phi}\|s(t;\phi)\|^2\,\mathsf{d}t\right) \\ &= \arg_\phi\left(2\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathsf{d}t = 0\right) \\ &= \arg_\phi\left(\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}\sum_{n\in\mathbb{Z}} a_n\lambda(t-nT)\cos(2\pi f_c t+\theta_n+\phi)\right]\,\mathsf{d}t = 0\right)\end{aligned}$$

$$
\begin{aligned}
&= \arg_\phi \left(-\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t) \left[\lambda(t-nT)\sin(2\pi f_c t + \theta_n + \phi)\right] \,\mathrm{d}t = 0 \right) \\
&= \arg_\phi \left(\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT) \left[\sin(2\pi f_c t + \theta_n)\cos(\phi) + \sin(\phi)\cos(2\pi f_c t + \theta_n)\right] \,\mathrm{d}t = 0 \right) \\
&= \arg_\phi \left(\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\sin(\phi)\cos(2\pi f_c t + \theta_n) \,\mathrm{d}t = -\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n)\cos(\phi) \,\mathrm{d}t \right) \\
&= \arg_\phi \left(\sin(\phi) \sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\cos(2\pi f_c t + \theta_n) \,\mathrm{d}t = -\cos(\phi) \sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n) \,\mathrm{d}t \right) \\
&= \arg_\phi \left(\frac{\sin(\phi)}{\cos(\phi)} = -\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n) \,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\cos(2\pi f_c t + \theta_n) \,\mathrm{d}t} \right) \\
&= \arg_\phi \left(\tan(\phi) = -\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n) \,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\cos(2\pi f_c t + \theta_n) \,\mathrm{d}t} \right) \\
&= \arg_\phi \left(\phi = -\operatorname{atan}\left(\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n) \,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\cos(2\pi f_c t + \theta_n) \,\mathrm{d}t} \right) \right) \\
&= -\operatorname{atan}\left(\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n) \,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathrm{r}(t)\lambda(t-nT)\cos(2\pi f_c t + \theta_n) \,\mathrm{d}t} \right)
\end{aligned}
$$

14.2 Generalized coherent modulation

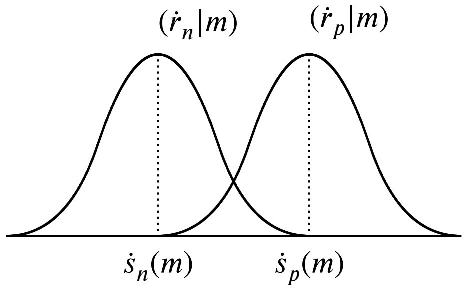


Figure 14.1: Distributions of orthonormal components

Theorem 14.2. *Let*

- $(V, \langle\cdot\,|\,\cdot\rangle, S)$ *be a modulation space*
- $\Psi \triangleq \{\psi_n(t) : n = 1, 2, \ldots, N\}$ *be a set of orthonormal functions that span* S
- $\dot{r}_n \triangleq \langle \mathrm{r}(t)\,|\,\psi_n(t)\rangle$
- $R \triangleq \{\dot{r}_n : n = 1, 2, \ldots, N\}$
- $\dot{s}_n(m) \triangleq \langle s(t;m)\,|\,\psi_n(t)\rangle$

and let V *be partitioned into* ***decision regions***

$$\{D_m : m = 1, 2, \ldots, |S|\}$$

such that

$$\mathrm{r}(t) \in D_{\hat{m}} \iff \hat{m} = \arg\max_m \mathrm{P}\{s(t;m)|\mathrm{r}(t)\}.$$

*Then the **probability of detection error** is*

THM
$$\mathsf{P}\{error\} = 1 - \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \sum_m \mathsf{P}\{m\ sent\} \int_{\boldsymbol{r}\in D_m} \exp\frac{-1}{2\sigma^2}\sum_{n=1}^N [\dot{r}_n - \dot{s}_n(m)]^2\, d\boldsymbol{r}.$$

✎PROOF:

$$\begin{aligned}
\mathsf{P}\{\text{error}\} &= 1 - \mathsf{P}\{\text{no error}\}\\
&= 1 - \sum_m \mathsf{P}\left\{(m\text{ sent}) \wedge (\hat{m} = m\text{ detected})\right\}\\
&= 1 - \sum_m \mathsf{P}\left\{(\hat{m} = m\text{ detected})|(m\text{ sent})\right\}\mathsf{P}\{m\text{ sent}\}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\,\mathsf{P}\{\boldsymbol{r}|(m\text{ sent})\}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \mathsf{p}\left[\boldsymbol{r}|(\mathsf{m}\text{ sent})\right] d\boldsymbol{r}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \prod_n \mathsf{p}\left[\dot{\mathsf{r}}_n|\mathsf{m}\right]\, d\boldsymbol{r}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \prod_{n=1}^N \frac{1}{\sqrt{2\pi\sigma^2}}\exp\frac{-[\dot{r}_n - \mathsf{E}\dot{r}_n]^2}{2\sigma^2}\, d\boldsymbol{r}\\
&= 1 - \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \exp\frac{-1}{2\sigma^2}\sum_{n=1}^N[\dot{r}_n - \dot{s}_n(m)]^2\, d\boldsymbol{r}
\end{aligned}$$

✎

14.3 Frequency Shift Keying (FSK)

Theorem 14.3. *In an FSK modulation space, the optimal ML estimator of m is*

THM
$$\hat{m} = \arg\max_m \dot{r}_m.$$

✎PROOF:

$$\begin{aligned}
\hat{m} &= \arg\max_m \mathsf{P}\{\mathsf{r}(t)|s(t;m)\}\\
&= \arg\min_m \sum_{n=1}^N [\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 9.6 (page 80)}\\
&= \arg\min_m \sum_{n=1}^N [\dot{r}_n^2 - 2\dot{r}_n\dot{s}_n(m) + \dot{s}_n^2(m)]\\
&= \arg\min_m \sum_{n=1}^N [-2\dot{r}_n\dot{s}_n(m) + \dot{s}_n^2(m)] && \dot{r}_n^2 \text{ is independent of } m\\
&= \arg\min_m \sum_{n=1}^N [-2\dot{r}_n a\bar{\delta}_{mn} + a^2\bar{\delta}_{mn}]\\
&= \arg\min_m [-2a\dot{r}_m + a^2]
\end{aligned}$$

$$= \arg\min_m [-\dot{r}_m] \qquad a \text{ and } 2 \text{ independent of } m$$
$$= \arg\max_m [\dot{r}_m]$$

✏

Theorem 14.4. *If an FSK modulation space let*

$$\begin{array}{rcl|lcl|l} z_2 & \triangleq & \dot{r}_1(1) - \dot{r}_2(1) & z_2 > 0 & \implies & \dot{r}_1 > \dot{r}_2 & m = 1 \\ z_3 & \triangleq & \dot{r}_1(1) - \dot{r}_3(1) & z_3 > 0 & \implies & \dot{r}_1 > \dot{r}_3 & m = 1 \\ & \vdots & & & & & \\ z_M & \triangleq & \dot{r}_1(1) - \dot{r}_M(1) & z_M > 0 & \implies & \dot{r}_1 > \dot{r}_M & m = 1 \end{array}$$

Then the ***probability of detection error*** *is*

THM
$$\boxed{\mathsf{P}\{error\} = 1 - \frac{M-1}{M} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \mathsf{p}(z_2, z_3, \ldots, z_M)\, dz_2 dz_3 \cdots dz_M}$$ *where*

$$\mathsf{p}(z_2, z_3, \ldots, z_M) = \frac{1}{(2\pi)^{\frac{M-1}{2}} \sqrt{\det R}} \exp -\frac{1}{2} \begin{bmatrix} z_2 - \dot{s} \\ z_3 - \dot{s} \\ \vdots \\ z_M - \dot{s} \end{bmatrix}^T R^{-1} \begin{bmatrix} z_2 - \dot{s} \\ z_3 - \dot{s} \\ \vdots \\ z_M - \dot{s} \end{bmatrix}$$

and

$$R = \begin{bmatrix} \operatorname{cov}[z_2, z_2] & \operatorname{cov}[z_2, z_3] & \cdots & \operatorname{cov}[z_2, z_M] \\ \operatorname{cov}[z_3, z_2] & \operatorname{cov}[z_3, z_3] & \cdots & \operatorname{cov}[z_3, z_M] \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cov}[z_M, z_2] & \operatorname{cov}[z_M, z_3] & \cdots & \operatorname{cov}[z_M, z_M] \end{bmatrix} = N_o \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$$

The inverse matrix R^{-1} is equivalent to (????)

$$R^{-1} \stackrel{?}{=} \frac{1}{MN_o} \begin{bmatrix} M-1 & -1 & \cdots & -1 \\ -1 & M-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & M-1 \end{bmatrix}$$

✎PROOF:

$$\begin{aligned} \mathsf{E} z_k &= \mathsf{E}\left[\dot{r}_{11} - \dot{r}_{1k}\right] \\ &= \mathsf{E}\dot{r}_{11} - \mathsf{E}\dot{r}_{1k} \\ &= \dot{s} - 0 \\ &= \dot{s} \end{aligned}$$

$$\begin{aligned}
\operatorname{cov}\left[z_m, z_n\right] &= \mathsf{E}\left[z_m z_n\right] - [\mathsf{E} z_m][\mathsf{E} z_n] \\
&= \mathsf{E}\left[(\dot{r}_{11} - \dot{r}_{1m})(\dot{r}_{11} - \dot{r}_{1n})\right] - \dot{s}^2 \\
&= \mathsf{E}\left[\dot{r}_{11}^2 - \dot{r}_{11}\dot{r}_{1n} - \dot{r}_{1m}\dot{r}_{11}\dot{r}_{1m}\dot{r}_{1n}\right] - \dot{s}^2 \\
&= [\operatorname{var}\dot{r}_{11} + (\mathsf{E}\dot{r}_{11})^2] - \mathsf{E}\left[\dot{r}_{11}\right]\mathsf{E}\left[\dot{r}_{1n}\right] - \mathsf{E}\left[\dot{r}_{1m}\right]\mathsf{E}\left[\dot{r}_{11}\right] + [\operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] + (\mathsf{E}\dot{r}_{1m})(\mathsf{E}\dot{r}_{1n})] - \dot{s}^2 \\
&= [\operatorname{var}\dot{r}_{11} + \dot{s}^2] - a\cdot 0 - 0\cdot a + [\operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] + 0\cdot 0] - \dot{s}^2 \\
&= \operatorname{var}\dot{r}_{11} + \operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] \\
&= N_o + \operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] \\
&= \begin{cases} 2N_o & \text{for } m = n \\ N_o & \text{for } m \neq n. \end{cases}
\end{aligned}$$

$$\begin{aligned}
P\{\text{error}\} &= 1 - P\{\text{no error}\} \\
&= 1 - \sum_{m=1}^{M} P\{\text{m transmitted}) \wedge (\forall k \neq m, \dot{r}_m > \dot{r}_k\} \\
&= 1 - (M-1)P\{1 \text{ transmitted}) \wedge (\dot{r}_{11} > \dot{r}_{12}) \wedge (\dot{r}_{11} > \dot{r}_{13}) \wedge \cdots \wedge (\dot{r}_{11} > \dot{r}_{1M})\} \\
&= 1 - (M-1)P\{(\dot{r}_{11} - \dot{r}_{12} > 0) \wedge (\dot{r}_{11} - \dot{r}_{13} > 0) \wedge \cdots \wedge (\dot{r}_{11} - \dot{r}_{1M} > 0)|1 \text{ transmitted})\}P\{1 \text{ transmitted})\} \\
&= 1 - \frac{M-1}{M} P\{(z_2 > 0) \wedge (z_3 > 0) \wedge \cdots \wedge (z_M > 0)|1 \text{ transmitted})\} \\
&= 1 - \frac{M-1}{M} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \mathsf{p}(z_2, z_3, \ldots, z_M)\, dz_2 dz_3 \cdots dz_M.
\end{aligned}$$

14.4 Quadrature Amplitude Modulation (QAM)

14.4.1 Receiver statistics

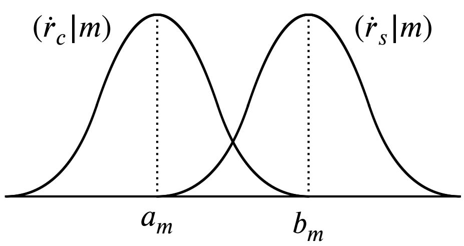


Figure 14.2: Distributions of QAM components

Theorem 14.5. *Let $(V, \langle \cdot \mid \cdot \rangle$ be a QAM modulation space such that*

$$\begin{aligned}
\mathsf{r}(t) &= \mathsf{s}(t; m) + \mathsf{n}(t) \\
\dot{r}_c &\triangleq \langle \mathsf{r}(t) \mid \psi_c(t) \rangle \\
\dot{r}_s &\triangleq \langle \mathsf{r}(t) \mid \psi_s(t) \rangle.
\end{aligned}$$

Then $(\dot{r}_c|m)$ *and* $(\dot{r}_s|m)$ *are* ***independent*** *and have* ***marginal distributions***

$$\begin{aligned}(\dot{r}_c|m) &\sim \mathsf{N}\left(a_m, \sigma^2\right) = \mathsf{N}\left(r_m \cos\theta_m, \sigma^2\right)\\ (\dot{r}_s|m) &\sim \mathsf{N}\left(b_m, \sigma^2\right) = \mathsf{N}\left(r_m \sin\theta_m, \sigma^2\right).\end{aligned}$$

✎PROOF: See Theorem 9.5 (page 79) page 79. ✏

14.4.2 Detection

Theorem 14.6. *Let* $(V, \langle\cdot\,|\,\cdot\rangle, S)$ *be a QAM modulation space with*

$$\begin{aligned}\mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t)\\ \dot{r}_c &\triangleq \langle \mathsf{r}(t)\,|\,\psi_c(t)\rangle\\ \dot{r}_s &\triangleq \langle \mathsf{r}(t)\,|\,\psi_s(t)\rangle\,.\end{aligned}$$

Then $\{\dot{r}_c, \dot{r}_s\}$ *are sufficient statistics for optimal ML detection and the optimal ML estimate of* m *is*

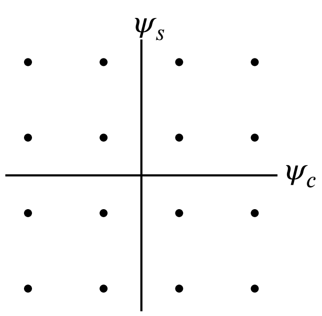
$$\hat{u}_{\mathsf{ml}}[m] = \arg\min_m \left[(\dot{r}_c - a_m)^2 + (\dot{r}_s - b_m)^2\right].$$

✎PROOF:

$$\begin{aligned}\hat{u}_{\mathsf{ml}}[m] &= \arg\max_m \mathsf{P}\{\mathsf{r}(t)|s(t;m)\} && \text{by Definition 8.1 (page 64)}\\ &= \arg\min_m \sum_{n=1}^{N}[\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 9.6 (page 80)}\\ &= \arg\min_m \left[(\dot{r}_c - a_m)^2 + (\dot{r}_s - b_m)^2\right]\end{aligned}$$

✏

14.4.3 Probability of error



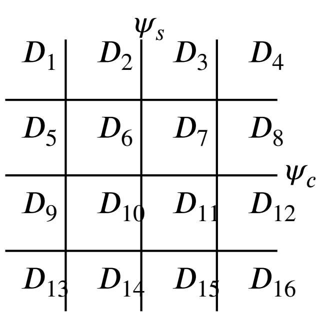


Figure 14.3: QAM-16 costellation and decision regions

Theorem 14.7. *In a QAM-16 constellation as shown in Figure 14.3 (page 130), the probability of error is*

$$\mathsf{P}\{error\} = \frac{9}{4}Q^2\left(\frac{\dot{s}_{21} - \dot{s}_{11}}{2N_o}\right).$$

✎PROOF: Let

$$d \triangleq \dot{s}_{21} - \dot{s}_{11}.$$

Then

$$\begin{aligned}
\mathsf{P}\{\text{error}\} &= \sum_{m=1}^{M} \mathsf{P}\left\{[s(t;m)\text{ transmitted }] \land [(\dot{r}_1,\dot{r}_2) \notin D_m]\right\} \\
&= \sum_{m=1}^{M} \mathsf{P}\left\{[(\dot{r}_1,\dot{r}_2) \notin D_m] | [s(t;m)\text{ transmitted }]\right\} \mathsf{P}\left\{[s(t;m)\text{ transmitted }]\right\} \\
&= \frac{1}{M}\sum_{m=1}^{M} \mathsf{P}\left\{[(\dot{r}_1,\dot{r}_2) \notin D_m] | [s(t;m)\text{ transmitted }]\right\} \\
&= \frac{1}{M}\left[4\mathsf{P}\left\{(\dot{r}_1,\dot{r}_2) \notin D_1 | s_1(t)\right\} + 8\mathsf{P}\left\{(\dot{r}_1,\dot{r}_2) \notin D_2 | s_2(t)\right\} + 4\mathsf{P}\left\{(\dot{r}_1,\dot{r}_2) \notin D_6 | s_6(t)\right\}\right] \\
&= \frac{1}{M}\left[4\int\int_{(x,y)\notin D_1} \mathsf{p}_{xy|1}(x,y)\,\mathsf{d}x\,\mathsf{d}y + 8\int\int_{(x,y)\notin D_2} \mathsf{p}_{xy|2}(x,y)\,\mathsf{d}x\,\mathsf{d}y + \right. \\
&\left. 4\int\int_{(x,y)\notin D_6} \mathsf{p}_{xy|6}(x,y)\,\mathsf{d}x\,\mathsf{d}y\right] \\
&= \frac{1}{M}\left[4\int\int_{(x,y)\notin D_1} \mathsf{p}_{x|1}(x)\mathsf{p}_{y|1}(y)\,\mathsf{d}x\,\mathsf{d}y + 8\int\int_{(x,y)\notin D_2} \mathsf{p}_{x|2}(x)\mathsf{p}_{y|2}(y)\,\mathsf{d}x\,\mathsf{d}y + \right. \\
&\left. 4\int\int_{(x,y)\notin D_6} \mathsf{p}_{x|6}(x)\mathsf{p}_{y|6}(y)\,\mathsf{d}x\,\mathsf{d}y\right] \\
&= \frac{1}{M}\left[4Q\left(\frac{d}{2N_o}\right)Q\left(\frac{d}{2N_o}\right) + 8Q\left(\frac{d}{2N_o}\right)2Q\left(\frac{d}{2N_o}\right) + 4\cdot 2Q\left(\frac{d}{2N_o}\right)2Q\left(\frac{d}{2N_o}\right)\right] \\
&= \frac{9}{4}Q^2\left(\frac{d}{2N_o}\right)
\end{aligned}$$

⇨

14.5 Phase Shift Keying (PSK)

14.5.1 Receiver statistics

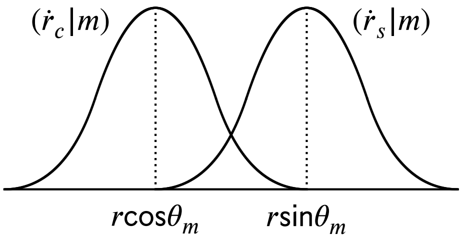


Figure 14.4: Distributions of PSK components

Theorem 14.8. *Let*

$$\begin{aligned}
\dot{r}_c &\triangleq \langle \mathsf{r}(t) \,|\, \psi_c(t) \rangle \\
\dot{r}_s &\triangleq \langle \mathsf{r}(t) \,|\, \psi_s(t) \rangle \\
\theta_m &\triangleq \operatorname{atan}\left[\frac{\dot{r}_s(m)}{\dot{r}_c(m)}\right].
\end{aligned}$$

The statistics $(\dot{r}_c|m)$ *and* $(\dot{r}_s|m)$ *are* ***independent*** *with marginal distributions*

$$\begin{aligned}
(\dot{r}_c|m) &\sim \mathsf{N}\left(r\cos\theta_m, \sigma^2\right) \\
(\dot{r}_s|m) &\sim \mathsf{N}\left(r\sin\theta_m, \sigma^2\right) \\
\mathsf{p}_{\theta_m}(\theta|m) &= \int_0^\infty x \mathsf{p}_{\dot{r}_c}(x|m)\mathsf{p}_{\dot{r}_s}(x\tan\theta|m)dx.
\end{aligned}$$

✎PROOF:

Indepence and marginal distributions of $\dot{r}_1(m)$ and $\dot{r}_2(m)$ follow directly from Theorem 9.5 (page 79) (page 79).

Let $X \triangleq \dot{r}_1(m)$, $Y \triangleq \dot{r}_2(m)$ and $\Theta \triangleq \theta_m$. Then[1]

$$\begin{aligned}
\mathsf{p}_\theta(\theta)d\theta &\triangleq \mathsf{P}\{\theta < \Theta \le \theta + d\theta\} \\
&= \mathsf{P}\left\{\theta < \operatorname{atan}\frac{Y}{X} \le \theta + d\theta\right\} \\
&= \mathsf{P}\left\{\tan(\theta) < \frac{Y}{X} \le \tan(\theta + d\theta)\right\} \\
&= \mathsf{P}\left\{\tan(\theta) < \frac{Y}{X} \le \tan\theta + (1+\tan^2\theta)\,\mathsf{d}\theta\right\} \\
&= \int_0^\infty \mathsf{P}\left\{\left[\tan\theta < \frac{Y}{X} \le \tan\theta + (1+\tan^2\theta)\,\mathsf{d}\theta\right] \wedge \left[(x < X \le x + \,\mathsf{d}x)\right]\right\} \\
&= \int_0^\infty \mathsf{P}\left\{\tan\theta < \frac{Y}{x} \le \tan\theta + (1+\tan^2\theta)\,\mathsf{d}\theta \,\middle|\, x < X \le x + dx\right\} \mathsf{P}\{x < X \le x + \,\mathsf{d}x\} \\
&= \int_0^\infty \mathsf{P}\left\{x\tan\theta < Y \le x\tan\theta + x(1+\tan^2\theta)\,\mathsf{d}\theta) \,\middle|\, X = x\right\} \mathsf{p}_x(x)\,\mathsf{d}x \\
&= \int_0^\infty [\mathsf{p}_\mathsf{Y}(x\tan\theta)x(1+\tan^2\theta)]\mathsf{p}_\mathsf{X}(x)\,\mathsf{d}x\,\mathsf{d}\theta \\
&= (1+\tan^2\theta)\int_0^\infty x\mathsf{p}_\mathsf{Y}(x\tan\theta)\mathsf{p}_\mathsf{X}(x)\,\mathsf{d}x\,\mathsf{d}\theta
\end{aligned}$$

$$\Longrightarrow$$

$$\mathsf{p}_\theta(\theta)d\theta = (1+\tan^2\theta)\int_0^\infty x\mathsf{p}_\mathsf{Y}(x\tan\theta)\mathsf{p}_\mathsf{X}(x)\,\mathsf{d}x$$

⇰

[1] A similar example is in Papoulis (1991), page 138

14.5.2 Detection

Theorem 14.9. *Let $(V, \langle \cdot \mid \cdot \rangle, S)$ be a PSK modulation space with*

$$\begin{aligned}
\mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t) \\
\dot{r}_c &\triangleq \langle \mathsf{r}(t) \mid \psi_c(t) \rangle \\
\dot{r}_s &\triangleq \langle \mathsf{r}(t) \mid \psi_s(t) \rangle .
\end{aligned}$$

Then $\{\dot{r}_c, \dot{r}_s\}$ are sufficient statistics for optimal ML detection and the optimal ML estimate of m is

$$\hat{u}_{\mathsf{ml}}[m] = \arg\min_m \left[(\dot{r}_1 - r\cos\theta_m)^2 + (\dot{r}_2 - r\sin\theta_m)^2 \right].$$

✎PROOF:

$$\begin{aligned}
\hat{u}_{\mathsf{ml}}[m] &= \arg\max_m \mathsf{P}\{\mathsf{r}(t) | s(t;m)\} && \text{by Definition 8.1 (page 64)} \\
&= \arg\min_m \sum_{n=1}^{N} [\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 9.6 (page 80)} \\
&= \arg\min_m \left[(\dot{r}_1 - r\cos\theta_m)^2 + (\dot{r}_2 - r\sin\theta_m)^2 \right].
\end{aligned}$$

✏

14.5.3 Probability of error

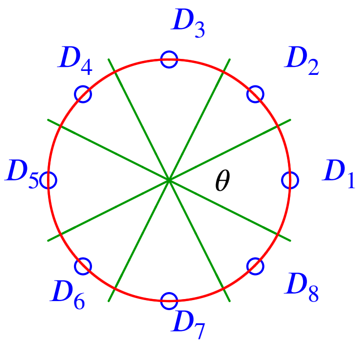


Figure 14.5: PSK-8 Decision regions

Theorem 14.10. *The probability of error using PSK modulation is*

$$\mathsf{P}\{error\} = M \left[1 - \int_{\frac{2\pi}{M}\left(m-\frac{3}{2}\right)}^{\frac{2\pi}{M}\left(m-\frac{1}{2}\right)} \mathrm{p}_{\theta_1}(\theta)\, d\theta \right].$$

✎PROOF: See Figure 14.5 (page 133).

$$\begin{aligned}
\mathsf{P}\{\text{error}\} &= \sum_{m=1}^{M} \mathsf{P}\left\{\text{error}|s(t;m) \text{ was transmitted}\right\} \\
&= M\,\mathsf{P}\left\{\text{error}|s_1(t) \text{ was transmitted}\right\} \\
&= M\left[1 - \int_{\frac{2\pi}{M}\left(m-\frac{3}{2}\right)}^{\frac{2\pi}{M}\left(m-\frac{1}{2}\right)} \mathrm{p}_{\theta_1}(\theta)\, d\theta\right].
\end{aligned}$$

14.6 Pulse Amplitude Modulation (PAM)

14.6.1 Receiver statistics

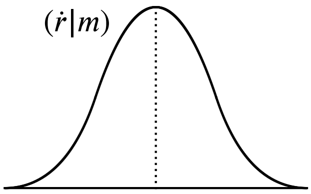


Figure 14.6: Distribution of PAM component

Theorem 14.11. *Let* $(V, \langle\cdot\,|\,\cdot\rangle$ *be a PAM modulation space such that*

$$\begin{aligned}
\mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t) \\
\dot{r}_c &\triangleq \langle \mathsf{r}(t)\,|\,\psi_c(t)\rangle \\
\dot{r}_s &\triangleq \langle \mathsf{r}(t)\,|\,\psi_s(t)\rangle\,.
\end{aligned}$$

Then $(\dot{r}|m)$ *has* ***distribution***

$$\dot{r}(m) \sim \mathsf{N}\left(a_m, \sigma^2\right).$$

✎PROOF: This follows directly from Theorem 9.5 (page 79) (page 79).

14.6.2 Detection

Theorem 14.12. *Let* $(V, \langle\cdot\,|\,\cdot\rangle, S)$ *be a PAM modulation space with*

$$\begin{aligned}
\mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t) \\
\dot{r} &\triangleq \langle \mathsf{r}(t)\,|\,\psi(t)\rangle\,.
\end{aligned}$$

Then $\dot{r}$ *is a sufficient statistic for the optimal ML detection of* m *and the optimal ML estimate of* m *is*

$$\hat{u}_{\mathsf{ml}}[m] = \arg\min_m |\dot{r} - a_m|.$$

✎PROOF:

$$\begin{aligned}
\hat{u}_{\mathrm{ml}}[m] &= \arg\max_m \mathsf{P}\left\{\mathsf{r}(t)|a_m\right\} && \text{by Definition 8.1 (page 64)}\\
&= \arg\min_m \sum_{n=1}^{N} [\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 9.6 (page 80)}\\
&= \arg\min_m [\dot{r} - \dot{s}(m)]^2\\
&= \arg\min_m |\dot{r} - \dot{s}(m)|
\end{aligned}$$

✏

14.6.3 Probability of error

Theorem 14.13. *The probability of detection error in a PAM modulation space is*

$$\mathsf{P}\{error\} = 2\frac{M-1}{M}\mathsf{Q}\left[\frac{a_2 - a_1}{2\sqrt{N_o}}\right].$$

✎PROOF: Let $d \triangleq a_2 - a_1$ and $\sigma \triangleq \sqrt{\operatorname{var}\dot{r}} = \sqrt{N_o}$. Also, let the decision regions D_m be as illustrated in Figure 14.7 (page 135). Then

$$\begin{aligned}
\mathsf{P}\{error\} &= \sum_{m=1}^{M} \mathsf{P}\left\{s(t;m) \text{ sent } \wedge r \notin D_m\right\}\\
&= \sum_{m=1}^{M} \mathsf{P}\left\{\dot{r} \notin D_m | \mathsf{s}(t;m) \text{ sent }\right\} \mathsf{P}\left\{s(t;m) \text{ sent }\right\}\\
&= \sum_{m=1}^{M} \mathsf{P}\left\{\dot{r}_m \notin D_m\right\} \frac{1}{M}\\
&= \frac{1}{M}\left(\mathsf{Q}\left[\frac{d}{2\sigma}\right] + 2\mathsf{Q}\left[\frac{d}{2\sigma}\right] + \ldots 2\mathsf{Q}\left[\frac{d}{2\sigma}\right] + \mathsf{Q}\left[\frac{d}{2\sigma}\right]\right)\\
&= 2\frac{M-1}{M}\mathsf{Q}\left[\frac{d}{2\sigma}\right]\\
&= 2\frac{M-1}{M}\mathsf{Q}\left[\frac{\dot{s}_2 - \dot{s}_1}{2\sqrt{N_o}}\right]
\end{aligned}$$

✏

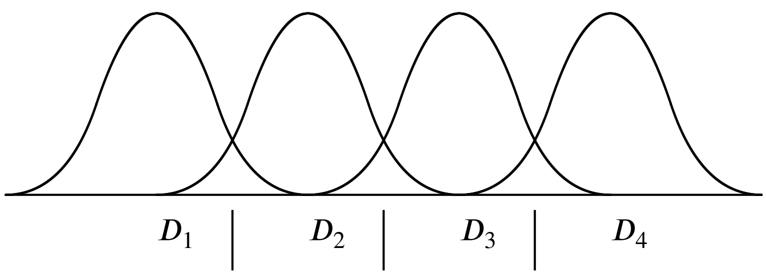


Figure 14.7: 4-ary PAM in AWGN channel

CHAPTER 15

NETWORK DETECTION

15.1 Detection

For detection, we need

1. Cost function: for hard decisions, its range must be linearly ordered. For soft decisions, it can be a lattice.
2. system joint and marginal probabilities (for Bayesian detection)

15.2 Bayesian Estimation

Definition 15.1.

DEF

H	$\triangleq$	$\{h_1, h_2, h_3, \ldots\}$	*set of hypotheses*
$\boldsymbol{D}$	$\triangleq$	$\{D_1, D_2, D_3, \ldots\}$	*partition—decision regions*
X	$\triangleq$	$\{X_1, X_2, X_3, \ldots\}$	*set of sensor inputs*

$$
\begin{aligned}
cost(h; P) &= \min_{\boldsymbol{D}} \sum_i \mathsf{P}\left\{ \left[X \in D_i \right] \wedge \left[H \neq h_i \right] \right\} \\
&= \min_{\boldsymbol{D}} \sum_i \mathsf{P}\left\{ X \in D_i \;\middle|\; H \neq h_i \right\} \mathsf{P}\left\{ H \neq h_i \right\} \\
&= \min_{\boldsymbol{D}} \sum_i \sum_{j \neq i} \left[1 - \mathsf{P}\left\{ X \in D_i \;\middle|\; H = h_i \right\} \right] \sum_{j \neq i} \left[1 - \mathsf{P}\left\{ H = h_i \right\} \right]
\end{aligned}
$$

$$\hat{h} = \arg_h cost(h; P)$$

15.3 Joint Gaussian Model

Assume convexity …

$$\begin{aligned}
\boldsymbol{D} &= \arg_{\boldsymbol{D}} \min_{\boldsymbol{D}} cost(h;\boldsymbol{P}) \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) \underbrace{\mathsf{p}(H \neq h_i)}_{c} d\boldsymbol{x} = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} c \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) d\boldsymbol{x} = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} \sum_i \left[1 - \sum_{j\neq i} \int_{D_i} \mathsf{p}(\boldsymbol{x}|H = h_i) d\boldsymbol{x} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} \sum_i \left[1 - \sum_{j\neq i} \int_{D_i} \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{M}|}} \exp -\frac{1}{2} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x})^T \boldsymbol{M}^{-1} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x}) d\boldsymbol{z} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \sum_i \left[1 - \sum_{j\neq i} \frac{\partial}{\partial \boldsymbol{D}} \int_{D_i} \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{M}|}} \exp -\frac{1}{2} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x})^T \boldsymbol{M}^{-1} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x}) d\boldsymbol{z} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \sum_i \left[1 - \sum_{j\neq i} \begin{bmatrix} \frac{\partial}{\partial D_1} \\ \frac{\partial}{\partial D_2} \\ \vdots \; \frac{\partial}{\partial D_n} \end{bmatrix} \int_{D_i} \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{M}|}} \exp -\frac{1}{2} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x})^T \boldsymbol{M}^{-1} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x}) d\boldsymbol{z} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \sum_i \left[1 - \sum_{j\neq i} \underbrace{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}}_{\text{Jacobian matrix}} \right] \right\}
\end{aligned}$$

For two variable Gaussian …

$$\begin{aligned}
cost &= \min_{\boldsymbol{D}} \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) \underbrace{\mathsf{p}(H \neq h_i)}_{c} d\boldsymbol{x} \\
&= \min_{\boldsymbol{D}} c \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) d\boldsymbol{x} \\
&= \min_{\boldsymbol{D}} c \sum_i \left[1 - \sum_{j\neq i} \int_{D_i} \mathsf{p}(\boldsymbol{x}|H = h_i) d\boldsymbol{x} \right]
\end{aligned}$$

$$= \min_D c \sum_i \left[1 - \sum_{j \neq i} \int_{D_i} \frac{1}{2\pi\sqrt{|M|}} \exp\left(\frac{z_1^2 \mathsf{E}[z_2 z_2] - 2 z_1 z_2 \mathsf{E}[z_1 z_2] + z_2^2 \mathsf{E}[z_1 z_1]}{-2|M|} \right) dz \right]$$

15.4 2 hypothesis, 2 sensor detection

Theorem 15.1 (centralized case). *Let $(\Omega, \mathbb{E}, \mathsf{P})$ be a probability space. Let $D \subsetneq \mathbb{E}$ be the* DECISION REGION *indicating hypothesis $H = h_1$. Let $\pi_0 \triangleq \mathsf{P}\{H = h_0\}$ and $\pi_1 \triangleq \mathsf{P}\{H = h_1\}$.*

THM

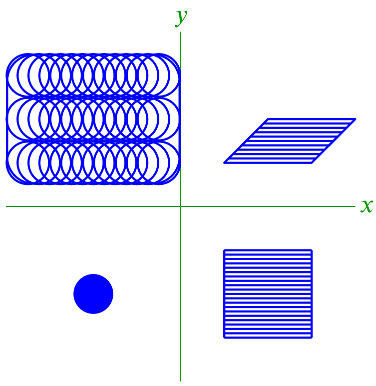
$$\begin{aligned} D &= \arg\min_D \left[\underbrace{\mathsf{P}\left\{(x,y) \in D | H = h_0\right\} \pi_0}_{\text{error for } H = h_0} + \underbrace{\mathsf{P}\left\{(x,y) \in D^c | H = h_1\right\} \pi_1}_{\text{error for } H = h_1} \right] \\ &= \arg\min_D \left[\underbrace{\pi_0 \int_D \mathrm{p}_0(x,y)\, \mathrm{d}x\, \mathrm{d}y}_{\text{error for } H = h_0} + \underbrace{\pi_1 \int_D \mathrm{p}_1(x,y)\, \mathrm{d}x\, \mathrm{d}y}_{\text{error for } H = h_1} \right] \end{aligned}$$

✎PROOF:

$$\begin{aligned} D &= \arg\min_D [\mathsf{P}\{\text{error}\}] && \text{by definition of decision region } D \\ &= \arg\min_D [\mathsf{P}\{\text{error} \wedge H = h_0\} + \mathsf{P}\{\text{error} \wedge H = h_1\}] \\ &= \arg\min_D [\mathsf{P}\{\text{error} | H = h_0\} \pi_0 + \mathsf{P}\{\text{error} | H = h_1\} \pi_1] \\ &= \arg\min_D [\mathsf{P}\{(x,y) \in D | H = h_0\} \pi_0 + \mathsf{P}\{(x,y) \in D^c | H = h_1\} \pi_1] \\ &= \arg\min_D \left[\pi_0 \int_D \mathrm{p}_0(x,y)\, \mathrm{d}x\, \mathrm{d}y + \pi_1 \int_D \mathrm{p}_1(x,y)\, \mathrm{d}x\, \mathrm{d}y \right] \end{aligned}$$

⇨

Example 15.1. In the centralized case, the decision regions D in the xy-plane can be any arbitrary shape, as illustrated to the right.



Definition 15.2.

DEF

Let $\mathbf{P}_x$ and $\mathbf{P}_y$ be ***set projection operators*** *such that*

$$\begin{aligned} D_x &\triangleq \mathbf{P}_x D \\ D_y &\triangleq \mathbf{P}_y D \end{aligned}$$

Proposition 15.1. *Let* $+$ *represent* MINKOWSKI ADDITION

PRP

$$\mathbf{D} = \mathbf{D}_x + \mathbf{D}_y$$

Theorem 15.2 (distributed AND case). *Let $(\Omega, \mathbb{E}, \mathsf{P})$ be a probability space. Let $D \subsetneq \mathbb{E}$ be the* DECISION REGION *indicating hypothesis $H = h_1$. Let $\pi_0 \triangleq \mathsf{P}\{H = h_0\}$ and $\pi_1 \triangleq \mathsf{P}\{H = h_1\}$. Let $E \triangleq D^c$.*

THM
$$D = \arg\min_D \begin{pmatrix} \mathsf{P}\{ & x \in E, & y \in E & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in E, & y \in D & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D, & y \in E & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D, & y \in D & \}\{H = h_0\}\pi_0 & \end{pmatrix}$$

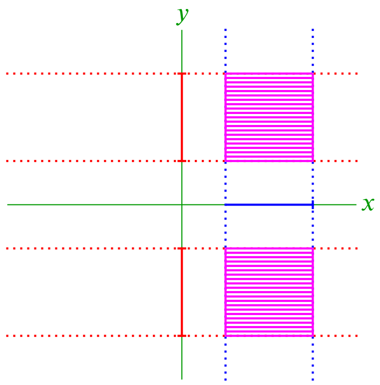
✎PROOF:

x	y	H	$x \wedge y$	
0	0	0	0	
0	1	0	0	
1	0	0	0	
1	1	0	1	error
0	0	1	0	error
0	1	1	0	error
1	0	1	0	error
1	1	1	1	

$$\begin{aligned} D &= \arg\min_D [\mathsf{P}\{\text{error}\}] && \text{by definition of decision region } D \\ &= \arg\min_D \left[\mathsf{P}\{\text{error} \wedge H = h_0\} + \mathsf{P}\{\text{error} \wedge H = h_1\}\right] \\ &= \arg\min_D \left[\mathsf{P}\{\text{error} | H = h_0\}\pi_0 + \mathsf{P}\{\text{error} | H = h_1\}\pi_1\right] \\ &= \arg\min_D \begin{pmatrix} \mathsf{P}\{ & x \in E_x, & y \in E_y & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D_x, & y \in E_y & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in E_x, & y \in D_y & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D_x, & y \in D_y & \}\{H = h_0\}\pi_0 & \end{pmatrix} \end{aligned}$$

✏

Example 15.2. In the distributed AND case, the decision regions D in the xy-plane are only simple rectangular shapes, as illustrated to the right.



Proposition 15.2.

PRP
In general, distributed AND *detection is suboptimal.*

✎PROOF: Because only rectangular decision regions are possible, detection is suboptimal. ✏

Theorem 15.3. [1]

THM
For the distributed AND *detection*
$$D_x = \left\{ x | \pi_0 \int_{D_y} \mathrm{p}_0(x, y)\, \mathrm{d}x\, \mathrm{d}y \le \pi_1 \int_{D_y} \mathrm{p}_1(x, y)\, \mathrm{d}x\, \mathrm{d}y \right\}$$

[1] Willett et al. (2000), page 3268

✎PROOF:

$$\begin{aligned} D_x = \left\{x|y \in D_y \right. &\implies \mathsf{P}\left\{(x,y)\,|H=h_0\right\}\pi_0 \le \mathsf{P}\left\{(x,y)\,|H=h_1\right\}\pi_1\Big\} \\ &= \left\{x|\pi_0 \int_{D_y} \mathrm{p}_0\,(x,y)\;\mathrm{d}x\;\mathrm{d}y \le \pi_1 \int_{D_y} \mathrm{p}_1\,(x,y)\;\mathrm{d}x\;\mathrm{d}y\right\} \end{aligned}$$

⇰

Part IV

Appendices

APPENDIX A

PROBABILITY SPACE

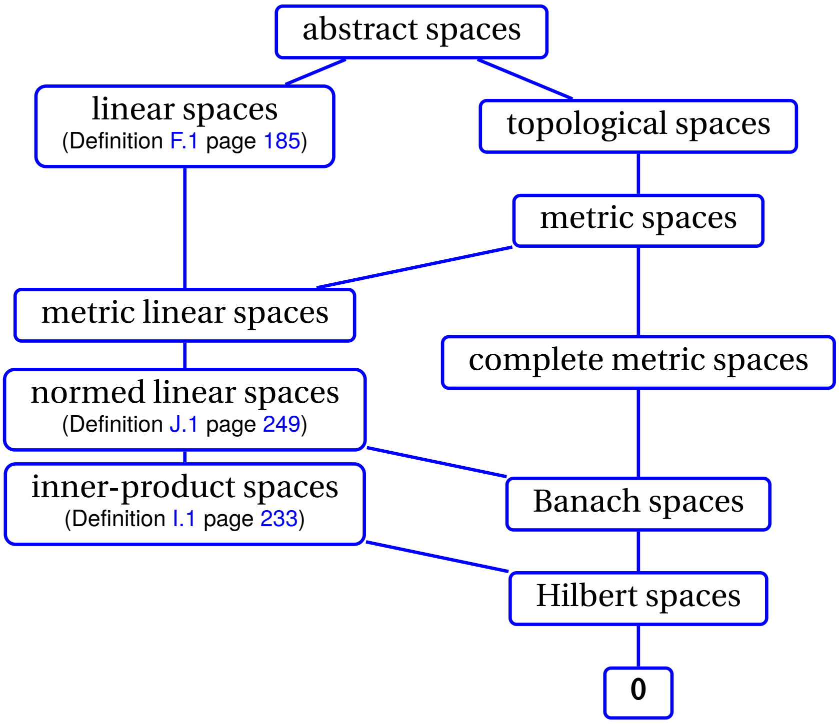


Figure A.1: Lattice of mathematical spaces

❝*It is not certain that everything is certain.*❞
Blaise Pascal (1623–1662), mathematician [1]

[1] quote: http://en.wikiquote.org/wiki/Blaise_Pascal
image: http://en.wikipedia.org/wiki/Image:Blaise_pascal.jpg

A.1 Probability functions

Definition A.1. [2] *Let* $(X, \vee, \wedge, \neg, 0, 1; \leq)$ *be a* LATTICE WITH NEGATION.

DEF

The function P *is a* ***probability function*** *if*

(1).	$\mathsf{P}(1) = 1$		(NORMALIZED)	and
(2).	$\mathsf{P}(x) \geq 0$	$\forall x \in X$	(NONNEGATIVE)	and
(3).	$\bigwedge_{n=1}^{\infty} x_n = 0 \implies \mathsf{P}\left(\bigvee_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} \mathsf{P}(x_n)$	$\forall x_n \in X$	(σ-ADDITIVE)	.

Remark A.1. The advantage of this definition is that P is a *measure*, and hence all the power of measure theory is subsequently at one's disposal in using P. However, it has often been argued that the requirement of *σ-additivity* is unnecessary for a probability function. Even as early as 1930, de Finetti argued against it, in what became a kind of polite running debate with Fréchet.[3] In fact, Kolmogorov himself provided some argument against *σ-additivity* when referring to the closely related *Axiom of Continuity* saying, “Since the new axiom is essential for infinite fields of probability only, it is almost impossible to elucidate its empirical meaning…For, in describing any observable random process we can obtain only finite fields of probability.…” But in its support he added, “This limitation has been found expedient in researches of the most diverse sort.”[4]

There are several other definitions of probability that only require *additivity* rather than *σ-additivity*. On a *Boolean lattice*, the **traditional probability** function is defined as[5]

(1).	$\mathsf{P}(1) = 1$		(*normalized*)	and
(2).	$\mathsf{P}(x) \geq 0$	$\forall x \in X$	(*nonnegative*)	and
(3).	$x \wedge y = 0 \implies \mathsf{P}(x \vee y) = \mathsf{P}(x) + \mathsf{P}(y)$	$\forall x,y \in X$	(*additive*)	.

This definition implies (on a *Boolean lattice*) that

(a).	$\mathsf{P}(0) = 0$		(*nondegenerate*)	and
(b).	$\mathsf{P}(x) \leq 1$	$\forall x \in X$	(*upper bounded*)	and
(c).	$\mathsf{P}(x) = 1 - \mathsf{P}(x^{\perp})$	$\forall x \in X$		and
(d).	$\mathsf{P}(x \vee y) \leq \mathsf{P}(x) + \mathsf{P}(y)$	$\forall x,y \in X$	(*subadditive*)	and
(e).	$\mathsf{P}(x \vee y) = \mathsf{P}(x) + \mathsf{P}(y) - \mathsf{P}(x \wedge y)$	$\forall x,y \in X$		and
(f).	$x \leq y \implies \mathsf{P}(x) \leq \mathsf{P}(y)$	$\forall x,y \in X$	(*monotone*)	.

On a *distributive pseudocomplemented lattice*, the **generalized probability** function has been defined as[6]

(1).	$\mathsf{P}(0) = 0$		(*nondegenerate*)	and
(2).	$\mathsf{P}(1) = 1$		(*normalized*)	and
(3).	$0 \leq \mathsf{P}(1) \leq 1$			and
(4).	$\mathsf{P}(x \vee y) = \mathsf{P}(x) + \mathsf{P}(y) - \mathsf{P}(x \wedge y)$	$\forall x,y \in X$		.

On an *orthomodular lattice*, or a *finite modular lattice*, the **quantum probability** function is defined as[7]

(1).	$\mathsf{P}(0) = 0$		(*nondegenerate*)	and
(2).	$\mathsf{P}(1) = 1$		(*normalized*)	and
(3).	$x \perp y \implies \mathsf{P}(x \vee y) = \mathsf{P}(x) + \mathsf{P}(y)$	$\forall x,y \in X$	(*additive*)	.

However, for lattices that are not *distributive*, *modular*, or *orthomodular*, none of these definitions

[2] Billingsley (1995) pages 22–23 ⟨Probability Measures⟩, Kolmogorov (1933a), Kolmogorov (1933b), page 16 ⟨*field of probability*⟩, Pap (1995) pages 8–9 ⟨Definition 2.3(13)⟩, Kalmbach (1986) page 27

[3] de Finetti (1930a), Fréchet (1930a), de Finetti (1930b), Fréchet (1930b), de Finetti (1930c), Cifarelli and Regazzini (1996) pages 258–260

[4] Kolmogorov (1933b), page 15

[5] Papoulis (1991) pages 21–22, Kolmogorov (1933b), page 2 ⟨§1. Axioms I–V⟩

[6] Narens (2014) page 118, Narens (2007)

[7] Greechie (1971) page 126 ⟨DEFINITIONS⟩, Narens (2014) page 118

work out so well. Take for example the O_6 *lattice* with the "very reasonable" probability function given in Example ?? (page ??). This probability space $(\mathsf{O}_6, \mathsf{P})$ fails to be any of the 4 probability functions defined in this Remark. It fails to be a *measure-theoretic* or *traditional probability* function because

$$a \wedge b = 0 \qquad \text{but} \qquad \mathsf{P}(a \vee b) = \mathsf{P}(1) = 1 \neq \tfrac{1}{3} + \tfrac{1}{2} = \mathsf{P}(a) + \mathsf{P}(b)\,.$$

It fails to be a *generalized probability* function because

$$\mathsf{P}(a \vee b) = \mathsf{P}(1) = 1 \neq \tfrac{1}{3} + \tfrac{1}{2} - 0 = \mathsf{P}(a) + \mathsf{P}(b) - \mathsf{P}(0) = \mathsf{P}(a) + \mathsf{P}(b) - \mathsf{P}(a \wedge b)\,.$$

It fails to be an *quantum probability* function because

$$a \perp b = 0 \qquad \text{but} \qquad \mathsf{P}(a \vee b) = \mathsf{P}(1) = 1 \neq \tfrac{1}{3} + \tfrac{1}{2} = \mathsf{P}(a) + \mathsf{P}(b)\,.$$

In each of these cases, the function P fails to be *additive*. The solution of Definition A.1 (page 146) is simply to "switch off" *additivity* when the lattice is not *distributive*. This method is a little "crude", but at least it allows us to define probability on a very wide class of lattices, while retaining compatibility with the *Boolean* case.

A.2 Probability Space

In mathematics, a *space* is simply a set and in the most general definition, nothing else. However, normally for a space to actually be useful, some additional structure is added. One of the most general additional structures is a *topology*; and a space together with a topology is called a *topological space*. A topological space imposes additional structure on a space in the form of subsets and guarantees that these subsets are closed under such fundamental operations as set *union* and set *intersection*. With the additional structure available in a topological space, we are able to analyze such basic concepts as continuity, convergence, and connectivity.

However for a great number of mathematical applications, we need to *measure* mathematical objects—the most general measurement being measures on subsets of some set. Examples of measurement in mathematics include integration and probability. Before measurement can be effectively performed on a set, the set must be equipped with a subset structure. In analysis, arguably the most fundamental subset structure is the humble *topology* (Definition ?? page ??). However, a simple topology does not provide sufficient structure for effective measurement. For example, often we would not only like to measure some subset A, but also its complement A^c. A topology is not closed under the complement operation. So instead of a topology only, we equip the space with a more powerful (and thus less general) structure called a *σ-algebra* (*sigma-algebra*) (Definition ?? page ??). A *σ-algebra* is a subset structure that is closed under set complement. A set together with a *σ-algebra* is called a *measurable space*. And a set together with a *σ-algebra* and a *measure* on that *σ-algebra* is called a *measure space* (Definition ?? page ??).

The next definition presents a very important measure space—the *probability space*.

Definition A.2.
DEF
The triple $(\Omega, \mathbb{E}, \mathsf{P})$ *is a* ***probability space*** *if*

(1). Ω *is a* SET *and*
(2). $\mathbb{E}$ *is a* σ-ALGEBRA *on* Ω (Definition ?? page ??) *and*
(3). $\mathsf{P} : \mathbb{E} \to [0, 1]$ *is a* MEASURE *on* $\mathbb{E}$ (Definition ?? page ??) .

If $\boldsymbol{S} \triangleq (\Omega, \mathbb{E}, \mathsf{P})$ *is a* PROBABILITY SPACE *then* x *is an* ***outcome*** *in* $\boldsymbol{S}$ *if* $x \in \Omega$*,* A *is an* ***event*** *in* $\boldsymbol{S}$ *if* $A \in \mathbb{E}$*, and* $\mathsf{P}A$ *is the* ***probability*** *of* A *in* $\boldsymbol{S}$ *if* A *is an* EVENT *in* $\boldsymbol{S}$.

Definition A.3. [8] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE (Definition A.2 page 147).

[8] Papoulis (1990) page 52 ⟨Independent Events⟩

DEF: *Two* EVENTS A *and* B *in* $\mathbb{E}$ *are* ***independent*** *if*

$$\mathsf{P}(A \cap B) = \mathsf{P}(A)\mathsf{P}(B)$$

Definition A.4. [9] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE (Definition A.2 page 147). *Let* x *and* y *be* EVENTS *in* L.

DEF: *The* ***conditional probability of*** x ***given*** y *is defined as*

$$\mathsf{P}(A|B) \triangleq \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)}$$

A.3 Properties

Proposition A.1.

PRP:

$$\underbrace{(\Omega, \mathbb{E}, \mathsf{P}) \text{ is a PROBABILITY SPACE} \implies (\Omega, \mathbb{E}, \mathsf{P}) \text{ is a MEASURE SPACE}}_{\text{(every probability space is a measure space)}}$$

Theorem A.1. [10] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE (Definition A.2 page 147).

THM:

(1).	0	$\le$	$\mathsf{P}(x) \le 1$	$\forall x \in X$	(BOUNDED)	and
(2).	$\mathsf{P}(x)$	$=$	$1 - \mathsf{P}(x^\perp)$	$\forall x \in X$	(PARTITION OF UNITY)	and
(3).	$x \le y$	$\implies$	$\mathsf{P}(y^\perp) \le \mathsf{P}(x^\perp)$	$\forall x, y \in X$	(ANTITONE)	

✎PROOF:

1. Proof for $0 \le \mathsf{P}(x) \le 1$:

$$\begin{aligned}
\boxed{0} &= \mathsf{P}(0) && \text{by by } \textit{nondegenerate} \text{ property of } \mathsf{P} \text{ (Definition A.2 page 147)}\\
&\le \boxed{\mathsf{P}(x)} && \text{because } 0 \le x \text{ and } \textit{monotone} \text{ property of } \mathsf{P}\\
&\le \mathsf{P}(1) && \text{because } x \le 1 \text{ and } \textit{monotone} \text{ property of } \mathsf{P}\\
&= \boxed{1} && \text{by } \textit{normalized} \text{ property of } \mathsf{P}
\end{aligned}$$

2. Proof for $\mathsf{P}(x) = 1 - \mathsf{P}(x^\perp)$:
   (a) Proof that P is *additive* (Definition A.2 page 147) over $\{0, x, x^\perp\} \subseteq X$:
      i. $\{0, x, x^\perp\}$ is *distributive*.
      ii. $x \wedge x^\perp = 0$ for all $x \in X$ by the *non-contradiction* property of *orthocomplemented lattices*.
      iii. Therefore, by Definition A.2, P is *additive* over $\{0, x, x^\perp\}$.
   (b) Then ...

$$\begin{aligned}
1 - \mathsf{P}(x^\perp) &= \mathsf{P}(1) - \mathsf{P}(x^\perp) && \text{by } \textit{normalized} \text{ property of } \mathsf{P} && \text{(Definition A.2 page 147)}\\
&= \mathsf{P}(x \vee x^\perp) - \mathsf{P}(x^\perp) && \text{by } \textit{excluded middle} \text{ property of ortho. lat.}\\
&= \mathsf{P}(x) + \mathsf{P}(x^\perp) - \mathsf{P}(x^\perp) && \text{by } \textit{additive} \text{ property of } (\Omega, \mathbb{E}, \mathsf{P}) && \text{(item (2a) page 148)}\\
&= \mathsf{P}(x) && \text{by field property of } (\mathbb{R}, +, \cdot, 0, 1)
\end{aligned}$$

3. Proof for $x \le y \implies \mathsf{P}(y^\perp) \le \mathsf{P}(x^\perp)$:

$$\begin{aligned}
x \le y &\implies y^\perp \le x^\perp && \text{by } \textit{antitone} \text{ property of } \textit{orthocomplemented lattices}\\
&\implies \mathsf{P}(y^\perp) \le \mathsf{P}(x^\perp) && \text{by } \textit{monotone} \text{ property of } \mathsf{P} && \text{(Definition A.2 page 147)}
\end{aligned}$$

[9] Papoulis (1990) page 45 ⟨2-3 Conditional Probability and Indenpendence⟩
[10] property (1): Papoulis (1991) page 21 ⟨(2-11)⟩

Theorem A.2. [11] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE *(Definition A.2 page 147)*.

THM: $\left.\begin{array}{l} \boldsymbol{L} \text{ is BOOLEAN} \\ \text{(Definition ?? page ??)} \end{array}\right\} \implies \left\{\begin{array}{llll} 1. & \mathsf{P}(x \vee y) = \mathsf{P}(x) + \mathsf{P}(y) - \mathsf{P}(x \wedge y) & \forall x,y \in X & \text{and} \\ 2. & \mathsf{P}(x \vee y) \le \mathsf{P}(x) + \mathsf{P}(y) & \forall x,y \in X & \text{(BOOLE'S INEQUALITY)} \end{array}\right.$

PROOF:

1. lemma: Proof that $\mathsf{P}((\neg x) \wedge y) = \mathsf{P}(y) - \mathsf{P}(x \wedge y)$:

$$\begin{aligned} \mathsf{P}(y) - \mathsf{P}(xy) &= \mathsf{P}(1 \wedge y) - \mathsf{P}(xy) && \text{by definition of 1 and } \wedge \\ &= \mathsf{P}\left[(x \vee x^{\perp})y\right] - \mathsf{P}(xy) && \text{by \textit{excluded middle} property of \textit{Boolean lattices}} \\ &= \mathsf{P}(xy \vee x^{\perp}y) - \mathsf{P}(xy) && \text{by \textit{distributive} property of \textit{Boolean lattices}} \\ &= \mathsf{P}(xy) + \mathsf{P}(x^{\perp}y) - \mathsf{P}(xy) && \text{because } (xy)(x^{\perp}y) = 0 \text{ and by \textit{additive} property} \\ &= \mathsf{P}(x^{\perp}y) \end{aligned}$$

2. Proof that $\mathsf{P}(x \vee y) = \mathsf{P}(x) + \mathsf{P}(y) - \mathsf{P}(x \wedge y)$:

$$\begin{aligned} \mathsf{P}(x \vee y) &= \mathsf{P}(x \vee x^{\perp}y) && \text{by property of \textit{Boolean lattices}} \\ &= \mathsf{P}(x) + \mathsf{P}(x^{\perp}y) && \text{because } (x)(x^{\perp}y) = 0 \text{ and by \textit{additive} property} \\ &= \mathsf{P}(x) + \mathsf{P}(y) - \mathsf{P}(x \wedge y) && \text{by item (1) (page 149)} \end{aligned}$$

Theorem A.3 (sum of products). *Let* $(X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE, $(\Omega, \mathbb{E}, \mathsf{P})$ *a* PROBABILITY SPACE *(Definition A.2 page 147)*, *and* $\{y, x_1, x_2, x_3, \dots\}$ *a subset of* X.

THM: $\left\{\begin{array}{lll} 1. & \boldsymbol{L} \text{ is DISTRIBUTIVE} & \text{and} \\ 2. & \{x_1, x_2, \dots\} \text{ is a PARTITION of } y & \end{array}\right\} \implies \left\{\begin{array}{llll} 1. & \mathsf{P}(y) = \sum_n \mathsf{P}(x_n) & \text{and} \\ 2. & \mathsf{P}(y) = \sum_n \mathsf{P}(y \wedge x_n) & \text{and} \\ 3. & \mathsf{P}(z \wedge y) = \sum_n \mathsf{P}(z \wedge x_n) & \end{array}\right\}$

PROOF:

1. Proof that P is *additive* (Definition A.2 page 147) on $(\Omega, \mathbb{E}, \mathsf{P})$:
   (a) Proof that $(yx_n) \wedge (yx_m) = 0$ for $n \ne m$:

$$\begin{aligned} (yx_n) \wedge (yx_m) &= y(x_n x_m) && \text{by definition of } \wedge \\ &= y \wedge 0 && \text{by \textit{mutually exclusive} property of \textit{partitions}} \\ &= 0 && \text{by \textit{lower bounded} property of \textit{bounded lattices}} \end{aligned}$$

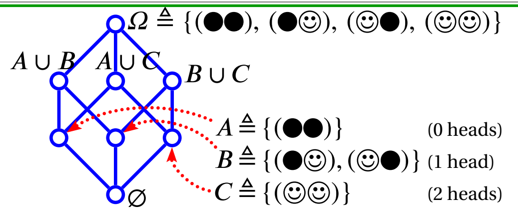
   (b) Proof that $\boldsymbol{L}$ is *distributive*: by *distributive* hypothesis

2. Proof that $\mathsf{P}(y) = \sum_n \mathsf{P}(x_n)$

$$\begin{aligned} \mathsf{P}(y) &= \mathsf{P}(yx_1 \vee yx_2 \vee \cdots \vee yx_n) \\ &= \sum_n \mathsf{P}(yx_n) && \text{by item (1) and \textit{additive} property} && \text{(Definition A.2 page 147)} \\ &= \sum_n \mathsf{P}(y|x_n)\mathsf{P}(x_n) && \text{by \textit{conditional probability}} && \text{(Definition A.4 page 148)} \end{aligned}$$

[11] Papoulis (1991) page 21 ⟨(2-13)⟩, Feller (1970) pages 22–23 ⟨(7.4),(7.6)⟩

EX



$$\begin{aligned} \mathsf{P}(\varnothing) &= 0 \\ \mathsf{P}(A) &= (1-p)^2 \\ \mathsf{P}(B) &= 2p(1-p) \\ \mathsf{P}(C) &= p^2 \end{aligned} \qquad \begin{aligned} \mathsf{P}(\Omega) &= 1 \\ \mathsf{P}(A \cup B) &= (1-p)^2 + 2p(1-p) \\ \mathsf{P}(A \cup C) &= 2p^2 - 2p + 1 \\ \mathsf{P}(B \cup C) &= -p^2 + 2p \end{aligned}$$

Figure A.2: *Double coin toss* (Example A.2 page 150)

As described in Definition A.2 (page 147), every *probability space* $(\Omega, \mathbb{E}, \mathsf{P})$ contains a probability *measure* $\mathsf{P} : \mathbb{E} \to [0, 1]$. This probability *measure* has some basic properties as described in Theorem A.4 (next).

Theorem A.4. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE. *Let* B *be a set and* $\{B_n | n = 1, 2, \ldots, N\}$ *a set of sets.*

THM

$$\left\{ \begin{array}{l} \{B_n | n = 1, 2, \ldots, N\} \text{ is a} \\ \text{PARTITION of } B. \end{array} \right\} \implies \left\{ \begin{array}{llll} (1). & \mathsf{P}(B) = \sum_{n=1}^{N} \mathsf{P}(B_n) & \forall B \in \mathbb{E} & \text{and} \\ (2). & \mathsf{P}(AB) = \sum_{n=1}^{N} \mathsf{P}(AB_n) & \forall A, B \in \mathbb{E} & \end{array} \right\}$$

✎PROOF: P is a *measure* and by Definition ?? (page ??). ⇒

Proposition A.2. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a probability space, and* X *a* RANDOM VARIABLE *with* PROBABILITY DENSITY FUNCTION $\mathsf{p}_\mathsf{x}(x)$ *and* CUMULATIVE DISTRIBUTION FUNCTION $\mathsf{c}_\mathsf{x}(x)$.

PRP

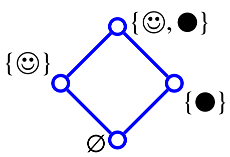
(1). $\mathsf{c}_\mathsf{x}(x)$ *is* MONOTONE *and*
(2). $\mathsf{p}_\mathsf{x}(x)$ *is* CONTINUOUS $\implies$ $\mathsf{c}_\mathsf{x}(x)$ *is* STRICTLY MONOTONE *and*
(3). $\mathsf{p}_\mathsf{x}(x)$ *is* CONTINUOUS $\implies$ $\mathsf{c}_\mathsf{x}(x)$ *is* INVERTIBLE

A.4 Examples

Example A.1 (single coin toss). Let ☺ represent "heads" and ● represent "tails" in a coin toss. Let $0 < p < 1$ be the probability of a head. A *probability space* $(\Omega, \mathbb{E}, \mathsf{P})$ for a single coin toss is as follows:

EX

$$\begin{aligned} \Omega &= \{☺, ●\} \\ \mathbb{E} &= \left\{ \underbrace{\varnothing}_{\text{neither}}, \underbrace{\{☺\}}_{\text{heads}}, \underbrace{\{●\}}_{\text{tails}}, \underbrace{\{☺, ●\}}_{\text{heads or tails}} \right\} \\ \mathsf{P}(X) &= \left\{ \begin{array}{llll} 0 & \text{for } X = \varnothing & & \text{(neither heads nor tails)} \\ p & \text{for } X = \{☺\} & & \text{(heads)} \\ 1-p & \text{for } X = \{●\} & & \text{(tails)} \\ 1 & \text{for } X = \{☺, ●\} & & \text{(either heads or tails)} \end{array} \right\} \end{aligned}$$



$\mathsf{P}(\{☺, ●\}) = 1$
$\mathsf{P}(\{☺\}) = p$
$\mathsf{P}(\{●\}) = 1 - p$
$\mathsf{P}(\varnothing) = 0$

Example A.2 (Double coin toss). Let ☺ represent "heads" and ● represent "tails" in a double coin toss in which each toss is *independent* (Definition A.3 page 147) of the other. Let $0 < p < 1$ be the probability of a head. The *probability space* $(\Omega, \mathbb{E}, \mathsf{P})$ is illustrated in Figure A.2 (page 150).

✎PROOF:

$$\begin{aligned}
\mathsf{P}(\Omega) &= 1 && \text{by \textit{normalized} property of } \mathsf{P} && \text{(Definition A.1 page 146)}\\
\mathsf{P}(C) &= \mathsf{P}\{☺☺\} && \text{by definition of } C\\
&= \mathsf{P}(☺)\mathsf{P}(☺) && \text{by definition of \textit{independence}} && \text{(Definition A.3 page 147)}\\
&= p^2 && \text{by definition of } p\\
\mathsf{P}(A) &= \mathsf{P}\{●●\} && \text{by definition of } A\\
&= \mathsf{P}(●)\mathsf{P}(●) && \text{by definition of \textit{independence}} && \text{(Definition A.3 page 147)}\\
&= \{1-\mathsf{P}(☺)\}\{1-\mathsf{P}(☺)\} && \text{by \textit{antitone} property of } \mathsf{P} && \text{(Theorem A.1 page 148)}\\
&= (1-p)^2 && \text{by definition of } p\\
\mathsf{P}(B) &= \mathsf{P}\{(●☺),(☺●)\} && \text{by definition of } B\\
&= \mathsf{P}\{●☺\}+\mathsf{P}\{☺●\} && \text{by \textit{additive} property of } \mathsf{P} && \text{(Definition A.1 page 146)}\\
&= \mathsf{P}\{●\}\mathsf{P}\{☺\}+\mathsf{P}\{☺\}\mathsf{P}\{●\} && \text{by definition of \textit{independence}} && \text{(Definition A.3 page 147)}\\
&= (1-p)p+p(1-p) && \text{by \textit{antitone} property of } \mathsf{P} && \text{(Theorem A.1 page 148) and definition of } p\\
&= -2p^2+p+1\\
\mathsf{P}(A\cup B) &= \mathsf{P}(A)+\mathsf{P}(B)-\mathsf{P}(A\cap B) && \text{by Theorem A.2}\\
&= \mathsf{P}(A)+\mathsf{P}(B)-\mathsf{P}(\varnothing)\\
&= (1-p)^2+(-2p^2+p+1)+0 && \text{by previous results}\\
&= -p^2-p+1\\
\mathsf{P}(\varnothing) &= 0 && \text{by \textit{nondegenerate} property of } \mathsf{P} && \text{(Definition A.1 page 146)}
\end{aligned}$$

✎

EX

$$\begin{aligned}
\Omega &= \{⚀,⚁,⚂,⚃,⚄,⚅\}\\
\mathbb{E} &= \left\{\underbrace{\{\;\}}_{\varnothing},\ \underbrace{\{⚀,⚂,⚄\}}_{\text{odd}},\ \underbrace{\{⚁,⚃,⚅\}}_{\text{even}},\ \underbrace{\{⚀,⚁,⚂,⚃,⚄,⚅\}}_{\Omega}\right\}\\
\mathsf{P}(X) &= \begin{cases}
0 & \text{for } e=\{\;\} & (\varnothing)\\
1 & \text{for } e=\{⚀,⚁,⚂,⚃,⚄,⚅\} & (\Omega)\\
p & \text{for } e=\{⚀,⚂,⚄\} & (\text{odd})\\
1-p & \text{for } e=\{⚁,⚃,⚅\} & (\text{even})
\end{cases}
\end{aligned}$$

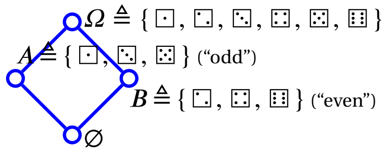


Figure A.3: *even/odd die probability space* (Example A.3 page 151)

Example A.3 (even/odd die toss). The *probability space* for an **even/odd die toss**, with $0 < p < 1$ being the probability of the die toss being odd, is illustrated in Figure A.3 (page 151).

✎PROOF:

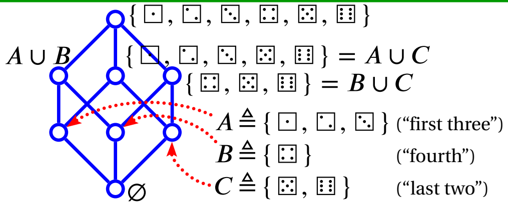
$$\begin{aligned}
\mathsf{P}(\Omega) &= 1 && \text{by \textit{normalized} property of } \mathsf{P} && \text{(Definition A.1 page 146)}\\
\mathsf{P}(C) &= \mathsf{P}\{☺☺\} && \text{by definition of } C\\
&= \mathsf{P}(☺)\mathsf{P}(☺) && \text{by definition of \textit{independence}} && \text{(Definition A.3 page 147)}\\
&= p^2 && \text{by definition of } p\\
\mathsf{P}(A) &= \mathsf{P}\{⚀,⚂,⚄\} && \text{by definition of } A\\
&= p && \text{by definition of } p\\
\mathsf{P}(B) &= \mathsf{P}\{⚁,⚃,⚅\} && \text{by definition of } B\\
&= \mathsf{P}\{⚀,⚂,⚄\}^{\mathsf{c}} && \text{by definition of set complement } \mathsf{c}\\
&= \mathsf{P}A^{\mathsf{c}} && \text{by definition of } A
\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{P}(\neg A) && \text{by definition of } \neg\\
&= 1 - \mathsf{P}(A) && \text{by Theorem A.1 page 148}\\
&= 1 - p && \text{by definition of } p\\
\mathsf{P}(\varnothing) &= 0 && \text{by \textit{nondegenerate} property of } \mathsf{P} \qquad \text{(Definition A.1 page 146)}
\end{aligned}
$$

⇨

The two previous *even/odd die* example (Example A.5 page 152) is in essence the same as the *single coin toss* (Example A.1 page 150). The next offers a little more complexity.

EX



$$
\begin{array}{rclrcl}
\mathsf{P}(\varnothing) &=& 0 & \mathsf{P}(\Omega) &=& 1\\
\mathsf{P}(A) &=& (1-p)^2 & \mathsf{P}(A\cup B) &=& (1-p)^2 + 2p(1-p)\\
\mathsf{P}(B) &=& 2p(1-p) & \mathsf{P}(A\cup C) &=& 2p^2 - 2p + 1\\
\mathsf{P}(C) &=& p^2 & \mathsf{P}(B\cup C) &=& -p^2 + 2p
\end{array}
$$

Figure A.4: 3-4-2 die example (Example A.4 page 152)

Example A.4. Suppose we have a “fair” die and we are primarily interested in the events of the first three {⚀, ⚁, ⚂}, the next two, {⚃, ⚄} and the final one {⚅}. The resulting *probability space* is illustrated in Figure A.4 (page 152).

The two previous examples (Example A.5 page 152, Example A.4 page 152) illustrate a *probabilty spaces* in which the events are *mutually exclusive*. The (next) illustrates one where events are *not*.

Example A.5. Suppose we have a “fair” die and we are primarily interested in the events of the first four ({⚀,⚁,⚂,⚃}) (that is, whether one roll of the die will produce a value in the set {⚀, ⚁, ⚂, ⚃}) and the last three ({⚃,⚄,⚅}) However, these events do not by themselves form a σ-*algebra*. Rather under the ∩ and ∪ operations, these two events generate a total of eight possible events that together form a σ-*algebra*. The resulting *probability space* is illustrated in Figure A.5 (page 153).

But why go through all the trouble of requiring a σ-algebra? Having a σ-algebra in place ensures that anything we might possibly want to measure *can* be measured. It makes sure all possible combinations are taken into account. And why go through the additional trouble of requiring a measure space? With a measure space available, expressing the measure over a complex set is often greatly simplified because the measure space provides nice algebraic properties (namely the σ-*additive* property). Example A.6 (next) illustrates how a rather complex σ-algebra (64 elements) can be compactly represented in a measure space.

Example A.6. Suppose we have a “fair” dice and we are interested in measuring over the power set of events (largest possible algebra—$2^6 = 64$ events). This leads to the probability space $(\Omega, \mathbb{E}, \mathsf{P})$ where

EX

$$
\begin{array}{lll}
\Omega &= \{⚀,⚁,⚂,⚃,⚄,⚅\} & \\
\mathbb{E} &= \mathcal{P}(\Omega) & \text{(the power-set of } \Omega)\\
\mathsf{P}(e) &= \frac{1}{6}|e| & (\tfrac{1}{6} \text{ times the number of possible outcomes in event } e)
\end{array}
$$

Example A.7 (Gaussian distribution on $\mathbb{R}$). Let $\boldsymbol{B}$ be the *Borel algebra on* $\mathbb{R}$. Let $\boldsymbol{L} \triangleq (\boldsymbol{B}, \subseteq)$ be the lattice formed by the elements of $\boldsymbol{B}$—this lattice is a *Boolean algebra*. Let

$$\mathsf{P}(A) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{\frac{x^2}{2\sigma^2}}\, \mathrm{d}x \text{ for } A \subseteq \boldsymbol{B}$$

and where $\int$ is the *Lebesgue integral* (Definition ?? page ??). Then $(\boldsymbol{L}, \mathsf{P})$ is a **probability space.**

EX

$$
\begin{aligned}
\Omega &= \{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}\\
\mathbb{E} &= \left\{ \underbrace{\{\ \}}_{\varnothing}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\}}_{\Omega}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\}}_{\text{first four}}, \underbrace{\{\text{⚃},\text{⚄},\text{⚅}\}}_{\text{last three}}, \right.\\
&\qquad \left. \underbrace{\{\text{⚃}\}}_{\{1234\}\cap\{456\}}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂},\text{⚄},\text{⚅}\}}_{\{4\}^c}, \underbrace{\{\text{⚄},\text{⚅}\}}_{\{4\}^c\cap\{456\}}, \underbrace{\{\text{⚀},\text{⚁},\text{⚂}\}}_{\{1234\}\cap\{4\}^c}, \right\}\\
\mathsf{P}(e) &= \begin{cases}
0 & \text{for } e = \{\ \} & (\varnothing)\\
1 & \text{for } e = \{\text{⚀},\text{⚁},\text{⚂},\text{⚃},\text{⚄},\text{⚅}\} & (\Omega)\\
\frac{2}{3} & \text{for } e = \{\text{⚀},\text{⚁},\text{⚂},\text{⚃}\} & (\text{first four})\\
\frac{1}{2} & \text{for } e = \{\text{⚃},\text{⚄},\text{⚅}\} & (\text{last three})\\
\frac{1}{6} & \text{for } e = \{\text{⚃}\} & (\{1234\}\cap\{456\})\\
\frac{5}{6} & \text{for } e = \{\text{⚀},\text{⚁},\text{⚂},\text{⚄},\text{⚅}\} & (\{4\}^c)\\
\frac{1}{3} & \text{for } e = \{\text{⚄},\text{⚅}\} & (\{4\}^c\cap\{456\})\\
\frac{1}{2} & \text{for } e = \{\text{⚀},\text{⚁},\text{⚂}\} & (\{1234\}\cap\{4\}^c)
\end{cases}
\end{aligned}
$$

Figure A.5: First 4 / last 3 die example (Example A.5 page 152)

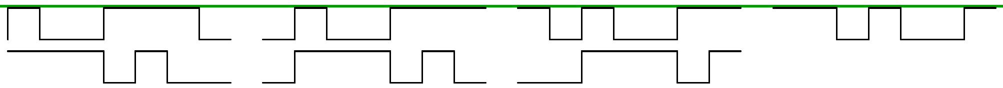
Example A.8 (Gaussian noise). Let $X \sim \mathsf{N}\left(0,\sigma^2\right)$ be a random variable with Gaussian distribution. We can construct the following probability space $(\Omega, \mathbb{E}, \mathsf{P})$:

EX

$$
\begin{aligned}
\Omega &= \mathbb{R}\\
\mathbb{E} &= \{\varnothing, \Omega\} \cup \{(a,b) | a,b \in \mathbb{R}, a < b\}\\
\mathsf{P}x &= \begin{cases}
0 & \text{for } x = \varnothing\\
1 & \text{for } x = \Omega\\
\frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{\frac{x^2}{2\sigma^2}}\, \mathrm{d}x & \text{otherwise}
\end{cases}
\end{aligned}
$$

Example A.9. The set of outcomes Ω can also be a set of waveforms:

EX

$\Omega = \Bigg\{$  $\Bigg\}$

$\mathbb{E} = \mathcal{P}(\Omega)$

$\mathsf{P}(e) = \frac{1}{7}|e|$

A.5 Probability subspaces

Example A.10. Suppose a random process is capable of producing three values $\Omega \triangleq \{x, y, z\}$. There are five *algebras of sets* on Ω and therefore five probability spaces $(\Omega, \mathbb{E}_n, \mathsf{P})$ on Ω with the five values of $\mathbb{E}_n$ listed below: [12]

[12] *algebra of sets*: Definition ?? page ??

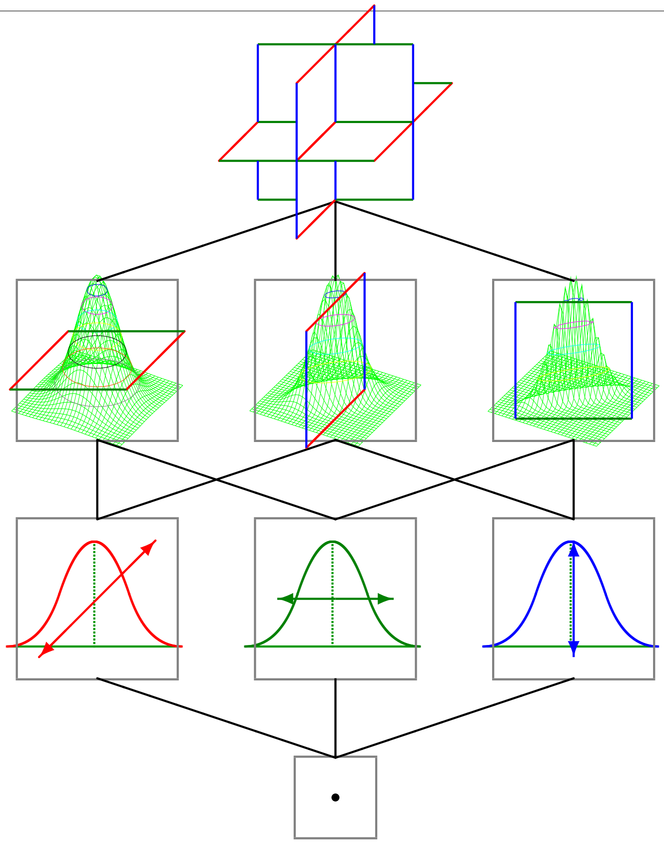
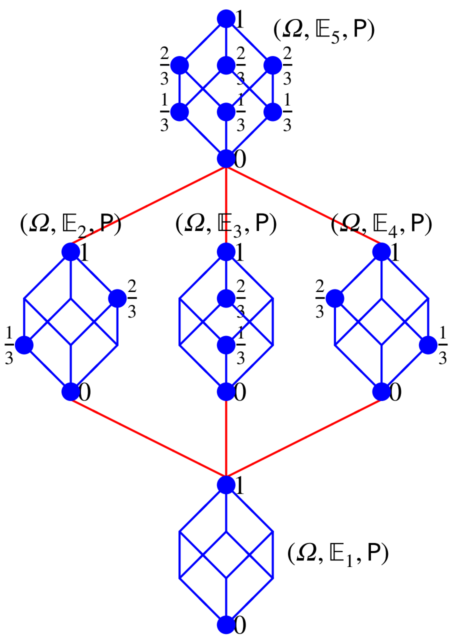


Figure A.6: Euclidean 3-dimensional space partitioned as a power lattice



$$
\begin{array}{llllllllll}
\mathbb{E}_1 = \{ & \varnothing, & & & & & & & X & \} \\
\mathbb{E}_2 = \{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X & \} \\
\mathbb{E}_3 = \{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X & \} \\
\mathbb{E}_4 = \{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X & \} \\
\mathbb{E}_5 = \{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X & \}
\end{array}
$$

Suppose the samples of Ω are generated by a physical process such that they are all equally likely to occur. Then by varying the *σ-algebra* $\mathbb{E}_n$ effectively varies the probability distribution of the probability space $(\Omega, \mathbb{E}_n, \mathsf{P})$. This is illustrated by the figure to the left.

APPENDIX B

PROBABILITY DENSITY FUNCTIONS

❝*While writing my book I had an argument with Feller. He asserted that everyone said "random variable" and I asserted that everyone said "chance variable." We obviously had to use the same name in our books, so we decided the issue by a stochastic procedure. That is, we tossed for it and he won.*❞

Joseph Leonard Doob (1910–2004), pioneer of and key contributor to mathematical probability[1]

B.1 Random variables

The concept of the *random variable* is widely used in probability and random processes. Before discussing what a *random variable is*, note two things that a *random variable* is *not* (next remark).

Remark B.1. [2] As pointed out by others, the term "random variable" is a "misnomer":

REM
- A *random variable* is **not random**.
- A *random variable* is **not a variable**.

What is it then? It is a *function* (next definition). In particular, it is a function that maps from an underlying stochastic process into $\mathbb{R}$. Any "*randomness*" (whatever that means) it may *appear* to have comes from the stochastic process it is mapping *from*. But the function itself (the *random*

[1] quote: Snell (1997), page 307, Snell (2005), page 251
image: http://www.dartmouth.edu/~chance/Doob/conversation.html

[2] Miller (2006) page 130, Feldman and Valdez-Flores (2010) page 4 ⟨"The name "random variable" is actually a misnomer, since it is not random and not a variable....the *random variable* simply maps each point (outcome) in the sample space to a number on the real line...Technically, the space into which the *random variable* maps the sample space may be more general than the real line..."⟩, Curry and Feldman (2010) page 4, Trivedi (2016) page 2.1 ⟨"The term "random variable" is actually a misnomer, since a *random variable* X is really a function whose domain is the sample space S, and whose range is the set of all real numbers, $\mathbb{R}$."⟩

variable itself) is very deterministic and well-defined. What gives it the appearance of being random is that the outcome ω of the experiment appears to be random to the observer. So the *random variable* $\mathsf{X}(\omega)$ is simply a function of an underlying mechanism that appears to be random.

Definition B.1. [3] *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE *(Definition A.2 page 147)*.

DEF: *A* ***random variable*** X *is any function in* $\mathbb{R}^{\Omega}$.

B.2 Probability distributions

The probability information about σ-algebra $\mathbb{E}$ in a *probability space* (Definition A.2 page 147) is completely specified by *measure* P. However, sometimes it is more convenient to express this same *measure* information in terms of the *probability density function* or the *cummulative distribution function* of the *probability space*.

Definition B.2. [4] *Let* X *be a* RANDOM VARIABLE *(Definition B.1 page 156) on a* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$.

DEF: X *has* ***cummulative distribution function*** ***(cdf)*** $\mathsf{c}_{\mathsf{x}}(x)$ *if*

$$\mathsf{c}_{\mathsf{x}}(x) \triangleq \mathsf{P}\{x \in \mathbb{E} | \mathsf{X} < x\}$$

X ***probability density function*** ***(pdf)*** $\mathsf{p}_{\mathsf{x}}(x)$ *if*

$$\mathsf{p}_{\mathsf{x}}(x) \triangleq \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{x}}(x) \triangleq \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{P}\{x \in \mathbb{E} | \mathsf{X} < x\}$$

Remark B.2. Suppose X be a *random variable* on a *probability space* $(\Omega, \mathbb{E}, \mathsf{P})$. Note that

- Both X and $\mathbb{E}$ are *functions*.
- But X is a function that maps from Ω to $\mathbb{R}$,
- whereas P is a function that maps from $\mathbb{E}$ to $\mathbb{R}$.

Definition B.3. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE *(Definition A.2 page 147) and* X *and* $\mathsf{Y} : \Omega \to \mathbb{R}$ *random variables. Then a* ***joint probability density function*** $\mathsf{p}_{\mathsf{x}} : \mathbb{E} \times \Omega \to [0:1]$ *and a* ***joint cumulative distribution function*** $\mathsf{c}_{\mathsf{x}} : \mathbb{E} \times \Omega \to [0:1]$ *are defined as*

DEF:

$$\begin{aligned}
\mathsf{c}_{\mathsf{XY}}(x,y) &\triangleq \mathsf{P}\{\mathsf{X} \le x | \mathsf{Y} \le y\} && \text{(JOINT CUMULATIVE DISTRIBUTION FUNCTION)}\\
\mathsf{p}_{\mathsf{XY}}(x,y) &\triangleq \frac{\mathsf{d}}{\mathsf{d}y}\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{XY}}(x,y) && \text{(JOINT PROBABILITY DENSITY FUNCTION)}
\end{aligned}$$

Definition B.4. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* PROBABILITY SPACE *(Definition A.2 page 147) and* X *a random variable. Then a* ***conditional probability density function*** $\mathsf{p}_{\mathsf{x}} : \mathbb{E} \times \Omega \to [0:1]$ *and a* ***conditional cumulative distribution function*** $\mathsf{c}_{\mathsf{x}} : \mathbb{E} \times \Omega \to [0:1]$ *are defined as*

DEF:

$$\begin{aligned}
\mathsf{c}_{\mathsf{x}}(x|y) &\triangleq \mathsf{P}\{\mathsf{X} \le x | \mathsf{Y} = y\} && \text{(CONDITIONAL CUMULATIVE DISTRIBUTION FUNCTION—CDF)}\\
\mathsf{p}_{\mathsf{x}}(x|y) &\triangleq \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{x}}(x|y) && \text{(CONDITIONAL PROBABILITY DENSITY FUNCTION—PDF)}
\end{aligned}$$

B.3 Properties

Definition B.2 (page 156) defines the pdf and cdf of a *probability space* $(\Omega, \mathbb{E}, \mathsf{P})$ in terms of *measure* P. Conversely, the probability *measure* $\mathsf{P}\{a \le \mathsf{X} < b\}$ of an event $\{a \le \mathsf{X} < b\}$ can be expressed in terms of either the pdf or cdf.

[3] Papoulis (1991), page 63
[4] von der Linden et al. (2014) page 93 ⟨Definitions 7.1, 7.2⟩

Proposition B.1. *Let* X *a* RANDOM VARIABLE *with* PDF p_X *and* CDF c_X (Definition B.2 page 156) *on the* PROBABILITY SPACE $(\Omega, \mathbb{E}, P)$ (Definition A.2 page 147).

PRP

$$\left\{\begin{array}{ll}(1). & c_X(x) \text{ and } c_Y(y) \text{ are CONTINUOUS} \quad \text{OR} \\ (2). & p_X(x) \text{ and } p_Y(y) \text{ are CONTINUOUS}\end{array}\right\} \implies \left\{\begin{array}{rcl} p_X(x) &=& \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} P\{x \le X < x+\varepsilon\} \\ p_{XY}(x,y) &=& \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} P\{x \le X < x+\varepsilon \wedge y \le Y < y+\varepsilon\}\end{array}\right\}$$

✎PROOF:

$$\begin{aligned} p_X(x) &\triangleq \frac{d}{dx} c_X(x) && \text{by definition of } p_X && \text{(Definition B.2 page 156)} \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} P\{x \in \mathbb{R} | x \le X < x+\varepsilon\} && \text{by definition of } \frac{d}{dx} && \text{(Definition ?? page ??)} \end{aligned}$$

⇨

Theorem B.1. *Let* $(\Omega, \mathbb{E}, P)$ *be a probability space,* X *be a random variable, and* (a, b) *a real interval.*

THM

$$\left\{\begin{array}{ll}(1). & c_X(x) \text{ is CONTINUOUS} \quad \text{OR} \\ (2). & p_X(x) \text{ is CONTINUOUS}\end{array}\right\} \implies \left\{ P\{a < X \le b\} = c_X(b) - c_X(a) = \int_a^b p_X(x)\, dx \right\}$$

✎PROOF:

$$\begin{aligned} P\{a < X \le b\} &= P\{X \le b\} - P\{X < a\} && \text{by \textit{sum of products}} && \text{(Theorem A.3 page 149)} \\ &= P\{X \le b\} - P\{X \le a\} && \text{by \textit{continuity} hypothesis} \\ &\triangleq c_X(b) - c_X(a) && \text{by definition of } c_X && \text{(Definition B.2 page 156)} \end{aligned}$$

$$\begin{aligned} \int_a^b p_X(x)\, dx &\triangleq \int_a^b \left[\frac{d}{dx} c_X(x)\right] dx && \text{by definition of } p_X && \text{(Definition B.2 page 156)} \\ &= c_X(x)\big|_{x=b} - c_X(x)\big|_{x=a} && \text{by \textit{Fundamental theorem of calculus}} \\ &= c_X(b) - c_X(a) \end{aligned}$$

⇨

Theorem B.2. *Let* $(\Omega, \mathbb{E}, P)$ *be a* PROBABILITY SPACE, X *be a* RANDOM VARIABLE, *and* $(a : b)$ *a* REAL INTERVAL.

THM

$$P\{a \le X < b\} = \int_a^b p_X(x)\, dx = \int_{-\infty}^b c_X(x)\, dx - \int_{-\infty}^a c_X(x)\, dx$$

The properties of the pdf follow closely the properties of *measure* P.

Theorem B.3. [5]

THM

$$\left\{\begin{array}{ll}(A). & c_X(x) \text{ is CONTINUOUS} \quad \text{OR} \\ (B). & p_X(x) \text{ is CONTINUOUS}\end{array}\right\} \implies \left\{\begin{array}{llcl}(1). & p_{X|Y}(x|y) &=& \dfrac{p_{XY}(x,y)}{p_Y(y)} \quad \text{and} \\ (2). & p_X(x) &=& \int_{y\in\mathbb{R}} p_{XY}(x,y)\, dy\end{array}\right\}$$

[5] Papoulis (1990) page 158 ⟨Auxiliary Variable⟩

✎PROOF:

$$\begin{aligned}
\mathsf{p}_{\mathsf{X}|\mathsf{Y}}(x|y) &\triangleq \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{X}|\mathsf{Y}}(x|y) && \text{by definition of } \mathsf{c}_{\mathsf{X}} && \text{(Definition A.4 page 148)}\\
&\triangleq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathsf{P}\{x \le \mathsf{X} < x+\varepsilon | \mathsf{Y} = y\} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition ?? page ??)}\\
&\triangleq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{\mathsf{P}\{(x \le \mathsf{X} < x+\varepsilon)\wedge(\mathsf{Y} = y)\}}{\mathsf{P}\{\mathsf{Y} = y\}} && \text{by definition of } \mathsf{P}\{A|B\} && \text{(Definition A.4 page 148)}\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{\mathsf{P}\{(x \le \mathsf{X} < x+\varepsilon)\wedge(y \le \mathsf{Y} < y+\varepsilon)\}}{\mathsf{P}\{y \le \mathsf{Y} < y+\varepsilon\}} && \text{by } \textit{continuity} \text{ hypothesis}\\
&= \frac{\lim\limits_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathsf{P}\{(x \le \mathsf{X} < x+\varepsilon)\wedge(y \le \mathsf{Y} < y+\varepsilon)\}}{\lim\limits_{\varepsilon\to 0}\mathsf{P}\{y \le \mathsf{Y} < y+\varepsilon\}} && \text{by property of } \lim_{\varepsilon\to 0}\\
&= \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)} && \text{by Proposition B.1 page 157}
\end{aligned}$$

$$\begin{aligned}
\int_{y\in\mathbb{R}} \mathsf{p}_{\mathsf{XY}}(x,y)\,\mathsf{d}y &\triangleq \int_{y\in\mathbb{R}}\left[\frac{\mathsf{d}}{\mathsf{d}y}\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{XY}}(x,y)\right]\mathsf{d}y && \text{by definition of } \mathsf{p}_{\mathsf{X}} && \text{(Definition B.2 page 156)}\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{XY}}(x,y)\\
&\triangleq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{y\in\mathbb{R}}\mathsf{P}\{x \le \mathsf{X} < x+\varepsilon, y \le \mathsf{Y} < y+\varepsilon\}\,\mathsf{d}y && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition ?? page ??)}\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathsf{P}\{x \le \mathsf{X} < x+\varepsilon\}\\
&= \mathsf{p}_{\mathsf{X}}(x) && \text{by Proposition B.1 page 157}
\end{aligned}$$

✏

Theorem B.4.

THM
$$\begin{aligned}
\mathsf{c}_{\mathsf{X}}(\sup\mathbb{R}) &= 1\\
\mathsf{c}_{\mathsf{X}}(\inf\mathbb{R}) &= 0
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{c}_{\mathsf{X}}(\sup\mathbb{R}) &\triangleq \mathsf{P}\left\{\mathsf{X} \le \sup\mathbb{R}\right\} && \text{by definition of } \mathsf{c}_{\mathsf{X}} && \text{(Definition B.2 page 156)}\\
&= 1\\
\mathsf{c}_{\mathsf{X}}(\inf\mathbb{R}) &\triangleq \mathsf{P}\left\{\mathsf{X} \le \inf\mathbb{R}\right\} && \text{by definition of } \mathsf{c}_{\mathsf{X}} && \text{(Definition B.2 page 156)}\\
&= 0
\end{aligned}$$

✏

The properties of the pdf follow closely the properties of measure P.

Theorem B.5.

THM
$$\mathsf{c}_{\mathsf{X}|\mathsf{Y}}(x|y) = \frac{\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{c}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)} \qquad \mathsf{p}_{\mathsf{X}|\mathsf{Y}}(x|y) = \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)}$$

✎PROOF:

$$
\begin{aligned}
\mathsf{c}_{\mathsf{X}|\mathsf{Y}}(x|y) &\triangleq \mathsf{P}\{\mathsf{X} \le x \mid \mathsf{Y} = y\} && \text{by definition of } \mathsf{c}_{\mathsf{X}|\mathsf{Y}} && \text{(Definition B.4 page 156)}\\
&\triangleq \frac{\mathsf{P}\{\mathsf{X} \le x|\mathsf{Y} = y\}}{\mathsf{P}\{\mathsf{Y} = y\}} && \text{by definition of } \mathsf{P}\{\mathsf{X}|\mathsf{Y}\} && \text{(Definition A.4 page 148)}\\
&= \lim_{\varepsilon\to 0} \frac{\mathsf{P}\{\mathsf{X} \le x|y < \mathsf{Y} \le y+\varepsilon\}}{\mathsf{P}\{y < \mathsf{Y} \le y+\varepsilon\}}\\
&= \lim_{\varepsilon\to 0} \frac{[\mathsf{P}\{\mathsf{X} \le x|\mathsf{Y} \le y+\varepsilon\} - \mathsf{P}\{\mathsf{X} \le x|\mathsf{Y} \le y\}]/\varepsilon}{[\mathsf{P}\{\mathsf{Y} \le y+\varepsilon\} - \mathsf{P}\{\mathsf{Y} \le y\}]/\varepsilon}\\
&\triangleq \lim_{\varepsilon\to 0} \frac{\left[\mathsf{c}_{\mathsf{XY}}(x, y+\varepsilon) - \mathsf{c}_{\mathsf{XY}}(x,y)\right]/\varepsilon}{\left[\mathsf{c}_{\mathsf{Y}}(y+\varepsilon) - \mathsf{c}_{\mathsf{Y}}(y)\right]/\varepsilon} && \text{by definition of } \mathsf{c}_{\mathsf{XY}} && \text{(Definition B.3 page 156)}\\
&\triangleq \frac{\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{c}_{\mathsf{XY}}(x,y)}{\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{c}_{\mathsf{Y}}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}(y)\\
&\triangleq \frac{\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{c}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)} && \text{by definition of } \mathsf{p}_{\mathsf{Y}} && \text{(Definition B.2 page 156)}\\
&= \frac{\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{c}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)} && \text{because } y \text{ is fixed}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{p}_{\mathsf{X}|\mathsf{Y}}(x|y) &\triangleq \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{X}|\mathsf{Y}}(x|y) && \text{by definition of } \mathsf{p}_{\mathsf{X}|\mathsf{Y}} && \text{(Definition B.4 page 156)}\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\frac{\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{c}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)} && \text{by previous result}\\
&= \frac{\frac{\mathsf{d}}{\mathsf{d}x}\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{c}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)} && \text{because } \mathsf{p}_{\mathsf{Y}}(y) \text{ is not a function of } x\\
&\triangleq \frac{\mathsf{p}_{\mathsf{XY}}(x,y)}{\mathsf{p}_{\mathsf{Y}}(y)} && \text{by definition of } \mathsf{p}_{\mathsf{XY}}(x,y) && \text{(Definition B.3 page 156)}
\end{aligned}
$$

✏

Theorem B.6. *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a probability space.*

THM

$$
\begin{aligned}
\int_{x\in\mathbb{R}} \mathsf{p}_{\mathsf{X}}(x)\,\mathsf{d}x &= 1\\
\int_{y\in\mathbb{R}} \mathsf{p}_{\mathsf{XY}}(x,y)\,\mathsf{d}y &= \mathsf{p}_{\mathsf{X}}(x) \quad \forall x\in\Omega
\end{aligned}
\qquad
\begin{aligned}
\int_{x\in\mathbb{R}} \mathsf{p}_{\mathsf{X}|\mathsf{Y}}(x|y)\,\mathsf{d}x &= 1\\
\int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} \mathsf{p}_{\mathsf{XY}}(x,y)\,\mathsf{d}y\,\mathsf{d}x &= 1
\end{aligned}
$$

✎PROOF:

$$
\begin{aligned}
\int_{\mathbb{R}} \mathsf{p}_{\mathsf{X}}(x)\,\mathsf{d}x &= \mathsf{c}_{\mathsf{X}}(\sup\mathbb{R}) - \mathsf{c}_{\mathsf{X}}(\inf\mathbb{R}) && \text{by Theorem B.1 page 157}\\
&= 1 - 0\\
&= 1 && \text{because 0 is the additive identity element in } (\mathbb{R}, +, \cdot, 0, 1)\\
\int_{x\in\mathbb{R}} \mathsf{p}_{\mathsf{X}|\mathsf{Y}}(x|y)\,\mathsf{d}x &\triangleq \int_{x\in\mathbb{R}} \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{c}_{\mathsf{X}|\mathsf{Y}}(x|y)\,\mathsf{d}x && \text{by definition of } \mathsf{p}_{\mathsf{X}|\mathsf{Y}}(x|y) \text{ (Definition B.4 page 156)}\\
&= \mathsf{c}_{\mathsf{X}|\mathsf{Y}}(\sup\mathbb{R}|y) - \mathsf{c}_{\mathsf{X}|\mathsf{Y}}(\inf\mathbb{R}|y) && \text{by \textit{Fundamental theorem of calculus}}
\end{aligned}
$$

$$\begin{aligned}
&= 1 - 0 \\
&= 1 && \text{because 0 is the additive identity element in } (\mathbb{R}, +, \cdot, 0, 1) \\
\int_{y\in\mathbb{R}} \mathsf{p}_{XY}(x,y)\,\mathrm{d}y &= \int_{y\in\mathbb{R}} \mathsf{p}_{YX}(y,x)\,\mathrm{d}y \\
&= \int_{y\in\mathbb{R}} \mathsf{p}_{Y|X}(y|x)\mathsf{p}_X(x)\,\mathrm{d}y && \text{by Theorem B.5 page 158} \\
&= \mathsf{p}_X(x)\int_{y\in\mathbb{R}} \mathsf{p}_{Y|X}(y|x)\,\mathrm{d}y && \text{because } \mathsf{p}_X(x) \text{ is not a function of } y \\
&= \mathsf{p}_X(x)\cdot 1 && \text{by previous result} \\
&= \mathsf{p}_X(x) && \text{because 1 is the multiplicative identity element in } (\mathbb{R}, +, \cdot, 0, 1)
\end{aligned}$$

$$\begin{aligned}
\int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} \mathsf{p}_{XY}(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_{x\in\mathbb{R}} \mathsf{p}_X(x)\,\mathrm{d}x && \text{by previous result} \\
&= 1 && \text{by previous result}
\end{aligned}$$

✎

APPENDIX C

SOME PROBABILITY DENSITY FUNCTIONS

C.1 Discrete distributions

Example C.1. [1] Suppose we throw two "fair" dice and want to know the probabilities of their sum. Let X represent the sum of the face values of the two dice. The resulting probability distribution is illustrated in Figure C.1 (page 162) and has probability space as follows:

EX

$$\begin{aligned} \Omega &= \{\text{⚀⚀},\ \text{⚀⚁},\ \text{⚀⚂},\ \ldots,\ \text{⚅⚅}\} \\ \mathbb{E} &= \left\{2^{\{X=n \mid n=2,3,\ldots,10,11,\text{ or }12\}}\right\} \\ \mathsf{P}(e) &= \frac{1}{36}|e| \end{aligned}$$

C.2 Continuous distributions

C.2.1 Uniform distribution

Definition C.1. *The* ***uniform distribution*** $\mathsf{p}_{\mathsf{x}}(x)$ *is defined as*

DEF

$$\mathsf{p}_{\mathsf{x}}(x) \triangleq \left\{ \begin{array}{ll} 1 & \text{for } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{array} \right\}$$

Note that although "simple" in form, in light of *Wold's Theorem*, the value of the *uniform distribution* should *not* be taken lightly.

[1] Osgood (2002)

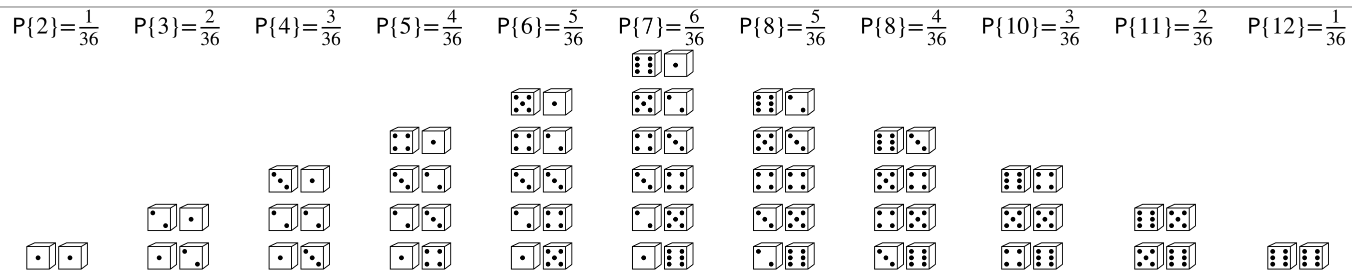
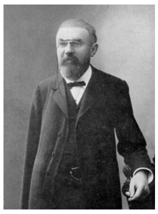


Figure C.1: Probability distribution for two dice (see Example C.1 page 161)

C.2.2 Gaussian distribution

❝*Tout le monde y croit cependant, me disait un jour M. Lippmann, car les expérimentateurs s'imaginent que c'est un théorème de mathématiques, et les mathématiciens que c'est un fait expérimental.*❞



❝*Everyone believes in it [(the normal distribution)] however, said to me one day Mr. Lippmann, because the experimenters imagine that it is a theorem of mathematics, and mathematicians that it is an experimental fact.*❞

Bernard A. Lippmann as told by Henri Poincaré [2]

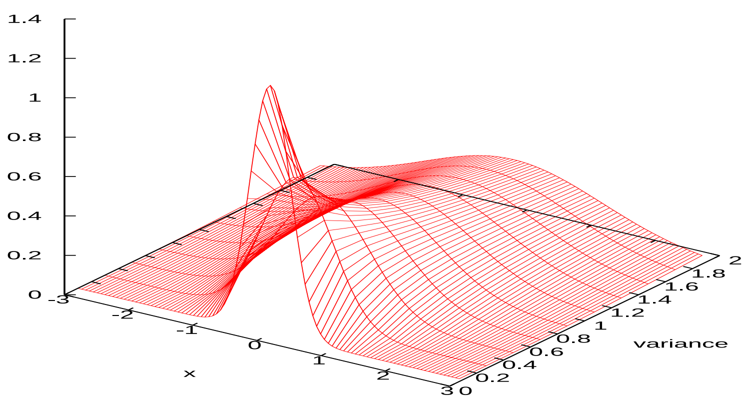


Figure C.2: Gaussian pdf with $\mu = 0$ and $\sigma \in [0.1, 2]$.

Definition C.2. *The **Gaussian distribution** (or **normal distribution** has pdf*

DEF $$\mathsf{p}_{\mathsf{x}}(x) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$ *A random variable* X *with this distribution is denoted*

DEF $$X \sim \mathsf{N}\left(\mu, \sigma^2\right)$$ *The function $Q(x)$ is defined as the area under a Gaussian PDF with zero mean*

[2] quote: Poincaré (1912), page 171
translation: assisted by Google Translate
image:

and variance equal to one from x to infinity such that

DEF
$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-u^2}{2}}\, \mathrm{d}u$$

C.2.3 Gamma distribution

Definition C.3. [3] *Let $b \in \mathbb{R}$. The **gamma function** $\Gamma(b)$ is*

DEF
$$\Gamma(b) \triangleq \int_0^\infty x^{b-1} e^{-x}\, \mathrm{d}x$$

Proposition C.1. [4] *Let $b \in \mathbb{R}$ and $n \in \mathbb{N}$.*

PRP
$$\begin{aligned} \Gamma(b) &= (b-1)\Gamma(b-1) \\ \Gamma(n) &= (n-1)! \end{aligned}$$

✎PROOF: Let

$$\begin{aligned} u &= x^{b-1} & \mathrm{d}u &= (b-1)x^{b-1}\, \mathrm{d}x \\ \mathrm{d}v &= e^{-x}\, \mathrm{d}x & v &= -e^{-x} \end{aligned}$$

$$\begin{aligned} \Gamma(b) &\triangleq \int_0^\infty x^{b-1} e^{-x}\, \mathrm{d}x \\ &= \int_{x=0}^\infty u\, \mathrm{d}v \\ &= uv|_{x=0}^\infty - \int_{x=0}^\infty v\, \mathrm{d}u \\ &= -x^{b-1}e^{-x}|_{x=0}^\infty + (b-1)\int_{x=0}^\infty e^{-x}x^{b-1}\, \mathrm{d}x \\ &= (-0+0) + (b-1)\Gamma(b-1) \end{aligned}$$

Note that

$$\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}\, \mathrm{d}x = \int_0^\infty e^{-x}\, \mathrm{d}x = -e^{-x}|_0^\infty = -0+1 = 1$$

$$\begin{aligned} \Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) \\ &= (n-1)(n-2)(n-3)\Gamma(n-3) \\ &\vdots \\ &= (n-1)(n-2)(n-3)\cdots(1)\Gamma(1) \\ &= (n-1)(n-2)(n-3)\cdots(1) \\ &\triangleq (n-1)! \end{aligned}$$

⇨

Definition C.4. *A **Gamma distribution** (b, λ) has pdf*

DEF
$$\mathsf{p}_\mathsf{x}(x) \triangleq \frac{\lambda}{\Gamma(b)} e^{-\lambda x}(\lambda x)^{b-1}$$

[3] Papoulis (1991), page 79, Ross (1998), page 222
[4] Ross (1998), page 223

Theorem C.1. [5] *Let* X *and* Y *be* RANDOM VARIABLES *on a* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$.

THM

$$\left\{\begin{array}{lll}\text{(A).} & \mathsf{X} \text{ and } \mathsf{Y} \text{ are INDEPENDENT} & \text{and}\\ \text{(B).} & \mathsf{X} \text{ has GAMMA DISTRIBUTION } (a,\lambda) & \text{and}\\ \text{(C).} & \mathsf{Y} \text{ has GAMMA DISTRIBUTION } (b,\lambda) & \text{and}\\ \text{(D).} & \mathsf{Z} \triangleq \mathsf{X}+\mathsf{Y} & \end{array}\right\} \implies \left\{\begin{array}{l}\mathsf{Z} \text{ has Gamma distribution}\\ (a+b,\lambda).\end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{p_z}(z) &= \mathsf{p_x}(z) \star \mathsf{p_y}(z)\\
&= \int_{u\in\mathbb{R}} \mathsf{p_x}(u)\mathsf{p_y}(z-u)\,\mathrm{d}u && \text{by definition of } \textit{convolution} \quad \text{(Definition N.3 page 312)}\\
&= \int_0^z \frac{1}{\Gamma(a)}\lambda e^{-\lambda u}(\lambda u)^{a-1}\frac{1}{\Gamma(b)}\lambda e^{-\lambda(z-u)}(\lambda(z-u))^{b-1}\,\mathrm{d}u\\
&= \frac{1}{\Gamma(a)}\frac{1}{\Gamma(b)}e^{-\lambda z}\lambda^{1+1+a-1+b-1}\int_0^z u^{a-1}(z-u)^{b-1}\,\mathrm{d}u\\
&= \frac{1}{\Gamma(a)}\frac{1}{\Gamma(b)}\lambda e^{-\lambda z}\lambda^{a+b-1}\int_0^1 (vz)^{a-1}(z-vz)^{b-1}z\,\mathrm{d}v\\
&= \frac{1}{\Gamma(a)}\frac{1}{\Gamma(b)}\lambda e^{-\lambda z}\lambda^{a+b-1}z^{a-1+b-1+1}\int_0^1 v^{a-1}(1-v)^{b-1}\,\mathrm{d}v\\
&= \left[\frac{1}{\Gamma(a)}\frac{1}{\Gamma(b)}\int_0^1 v^{a-1}(1-v)^{b-1}\,\mathrm{d}v\right]\lambda e^{-\lambda z}(\lambda z)^{a+b-1}\\
&= C\lambda e^{-\lambda z}(\lambda z)^{a+b-1} && \text{where } C \text{ is some constant}\\
&= \frac{\lambda}{\Gamma(a+b)}e^{-\lambda z}(\lambda z)^{a+b-1} && C \text{ must be the value that makes } \int_z \mathsf{p_z}(z) = 1
\end{aligned}$$

$\implies \mathsf{p_z}(z)$ is a $(a+b,\lambda)$ Gamma distribution

✎

C.2.4 Chi-squared distributions

Definition C.5. [6] *Let* $\mathsf{p}(x)$ *be a* PROBABILITY DENSITY FUNCTION *on a* PROBABILITY SPACE $(\Omega, \mathbb{E}, \mathsf{P})$.

DEF

$\mathsf{p}(x)$ *is a* ***chi-square distribution*** *if*

$$\mathsf{p}(x) \triangleq \left\{\begin{array}{ll} 0 & \text{if } x<0\\ \dfrac{1}{\sqrt{2\pi\sigma^2 x}}\exp\left[-\dfrac{x}{2\sigma^2}\right] & \text{if } x \ge 0\end{array}\right\} \qquad \text{for } \sigma > 0$$

Theorem C.2. [7]

THM

The following distributions are equivalent:

(1). *chi-squared distribution* *and*
(2). *distribution of* X^2 *where* $\mathsf{X} \sim \mathsf{N}\left(0,\sigma^2\right)$ *and*
(3). *Gamma distribution* $\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$

✎PROOF:

[5] Ross (1998), page 266
[6] Proakis (2001), page 41, Papoulis (1990) page 219 ⟨7-4 Special Distributions of Statistics, (7-78)⟩
[7] Ross (1998), page 267

1. Proof that X^2 has chi-squared distribution:

$$\begin{aligned}
\mathsf{p}_\mathsf{Y}(y) &= \frac{1}{2\sqrt{y}}\left[\mathsf{p}_\mathsf{X}(-\sqrt{y})+\mathsf{p}_\mathsf{X}(\sqrt{y})\right] && \text{by Corollary 4.3 page 29}\\
&= \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi\sigma^2}}\exp-\frac{(-\sqrt{y}-0)^2}{2\sigma^2}+\frac{1}{\sqrt{2\pi\sigma^2}}\exp-\frac{(+\sqrt{y}-0)^2}{2\sigma^2}\right]\\
&= \frac{1}{2\sqrt{y}}\left[2\frac{1}{\sqrt{2\pi\sigma^2}}\exp-\frac{y}{2\sigma^2}+\right]\\
&= \frac{1}{\sqrt{2\pi\sigma^2 y}}\exp-\frac{y}{2\sigma^2}
\end{aligned}$$

2. Proof that chi-distribution is a Gamma distribution (b,λ):

$$\begin{aligned}
b &\triangleq \frac{1}{2}\\
\lambda &\triangleq \frac{1}{2\sigma^2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma^2 y}}\exp-\frac{y}{2\sigma^2} &= \frac{1}{\sqrt{\pi}}\lambda^{1/2}\lambda^{1/2}(\lambda y)^{-1/2}e^{-\lambda y}\\
&= \frac{\lambda}{\sqrt{\pi}}(\lambda y)^{b-1}e^{-\lambda y}
\end{aligned}$$

✏

Definition C.6. [8] *The **Chi-squared distribution with** n **degrees of freedom** has pdf*

DEF
$$\mathsf{p}_\mathsf{Y}(y) \triangleq \begin{cases} 0 & : y<0\\ \dfrac{1}{2\sigma^2\Gamma(n/2)}\left(\dfrac{y}{2\sigma^2}\right)^{\frac{n}{2}-1}\exp-\dfrac{y}{2\sigma^2} & : y\ge 0\end{cases}$$

Theorem C.3. [9] *The following distributions are equivalent:*

1. *chi-squared distribution with n degrees of freedom*
2. *the distribution of $\sum_{k=1}^{n} X_k^2$ where $\left\{X_k | X_k \sim \mathsf{N}\left(0,\sigma^2\right), k=1,2,\ldots,n\right\}$ are independent random variables.*
3. *Gamma distribution $\left(\frac{n}{2},\frac{1}{2\sigma^2}\right)$.*

✎Proof:

1. Prove chi-squared distribution with n degrees of freedom is the Gamma distribution $\left(\frac{n}{2},\frac{1}{2\sigma^2}\right)$:

$$\begin{aligned}
\lambda &\triangleq \frac{1}{2\sigma^2}\\
b &\triangleq \frac{1}{2}\\
\frac{1}{2\sigma^2\Gamma(n/2)}\left(\frac{y}{2\sigma^2}\right)^{\frac{n}{2}-1}\exp-\frac{y}{2\sigma^2} &= \frac{\lambda}{\Gamma(nb)}(\lambda y)^{nb-1}\exp-\lambda y
\end{aligned}$$

[8] Proakis (2001), page 41
[9] Ross (1998), page 267

2. Prove $\sum_{k=1}^{n} X^2$ is Gamma $\left(\frac{n}{2}, \frac{1}{2\sigma^2}\right)$:

   (a) By Theorem C.2, X_k has Gamma distribution $\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$.

   (b) By Theorem C.1, $\sum_{k=1}^{n} X_k^2$ has distribution $\left(\frac{n}{2}, \frac{1}{2\sigma^2}\right)$.

✏

Definition C.7. [10] *A **noncentral chi-square distribution** (μ, σ^2) has pdf*

DEF
$$\mathsf{p}_{\mathsf{Y}}(y) = \frac{1}{\sqrt{2\pi\sigma^2 y}} \exp\frac{y+\mu^2}{-2\sigma^2} \cosh\frac{\mu\sqrt{y}}{\sigma^2}$$

Theorem C.4.

THM
The following distributions are equivalent:

(1). NON-CENTRAL CHI-SQUARED DISTRIBUTION (μ, σ^2)

(2). *distribution of* X^2 *where* $\mathsf{X} \sim \mathsf{N}\left(\mu, \sigma^2\right)$

✎PROOF:

1. Proof that $\mathsf{Y} = X^2$ has a non-central chi-squared distribution:

$$\begin{aligned}
\mathsf{p}_{\mathsf{Y}}(y) &= \frac{1}{2\sqrt{y}}\left[\mathsf{p}_{\mathsf{X}}(-\sqrt{y}) + \mathsf{p}_{\mathsf{X}}(\sqrt{y})\right] && \text{by Corollary 4.3 page 29}\\
&= \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi\sigma^2}}\exp -\frac{(-\sqrt{y}-\mu)^2}{2\sigma^2} + \frac{1}{\sqrt{2\pi\sigma^2}}\exp -\frac{(+\sqrt{y}-\mu)^2}{2\sigma^2}\right]\\
&= \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi\sigma^2}}\exp -\frac{y+\mu^2}{2\sigma^2}\exp\frac{-2\mu\sqrt{y}}{2\sigma^2} + \frac{1}{\sqrt{2\pi\sigma^2}}\exp -\frac{y+\mu^2}{2\sigma^2}\exp\frac{2\mu\sqrt{y}}{2\sigma^2}\right]\\
&= \frac{1}{\sqrt{2\pi\sigma^2 y}}\exp -\frac{y+\mu^2}{2\sigma^2}\frac{1}{2}\left[\exp\frac{2\mu\sqrt{y}}{2\sigma^2} + \exp\frac{-2\mu\sqrt{y}}{2\sigma^2}\right]\\
&= \frac{1}{\sqrt{2\pi\sigma^2 y}}\exp\frac{y+\mu^2}{-2\sigma^2}\cosh\frac{\mu\sqrt{y}}{\sigma^2}
\end{aligned}$$

✏

Definition C.8. [11] *The α**th-order modified Bessel function of the first kind** $I_\alpha(x)$ is*

DEF
$$I_\alpha(x) = \sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(\alpha+k+1)}\left(\frac{x}{2}\right)^{\alpha+2k}$$

Definition C.9. [12] *The **noncentral chi-square with** n**-degrees of freedom** distribution has pdf*

DEF
$$\mathsf{p}_{\mathsf{Y}}(y) = \frac{1}{2\sigma^2}\left(\frac{y}{s^2}\right)^{\frac{n-2}{4}}\exp\frac{y+s^2}{-2\sigma^2} I_{n/2-1}\left(\sqrt{y}\frac{s}{\sigma^2}\right) \qquad \textit{where } s^2 \triangleq \sum_{k=1}^{n}\mu_k^2$$

[10] Proakis (2001), page 42
[11] Proakis (2001), page 43
[12] Proakis (2001), page 43

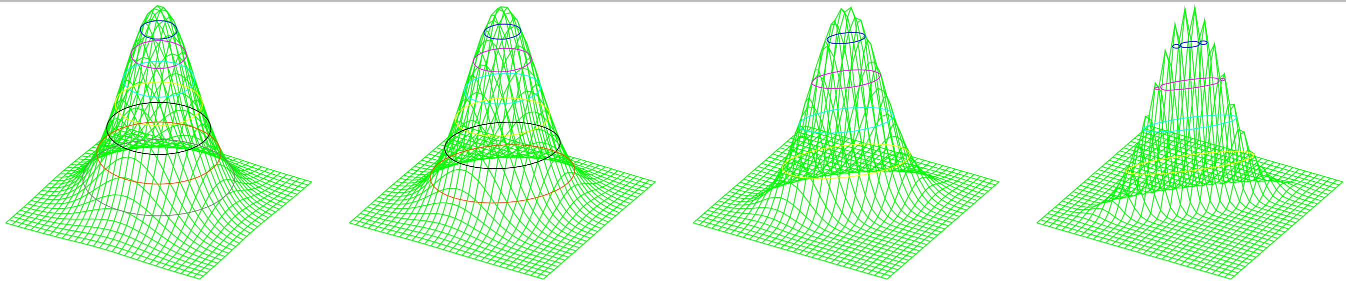


Figure C.3: *Joint Gaussian distributions* $\mathsf{p}_{\mathsf{XY}}(x,y)$ with varying correlations

C.2.5 Radial distributions

Definition C.10. [13] *The **Rayleigh distribution** is the pdf*

$$\mathsf{p}_{\mathsf{R}}(r) = \begin{cases} 0 & \textit{for} \quad r < 0 \\ \frac{r}{\sigma^2}\exp -\frac{r^2}{2\sigma^2} & \textit{for} \quad r \geq 0 \end{cases}$$

Note that by Proposition 4.3, this distribution is equivalent to the distribution of $R = \sqrt{X^2+Y^2}$ where X and Y are independent random variables each with distribution $\mathsf{N}\left(0,\sigma^2\right)$.

Definition C.11. [14] *The **Rice distribution** is the pdf*

$$\mathsf{p}_{\mathsf{R}}(r) = \begin{cases} 0 & \textit{for} \quad r < 0 \\ \frac{r}{\sigma^2}\exp \frac{r^2+s^2}{-2\sigma^2} I_o\left(\frac{rs}{\sigma^2}\right) & \textit{for} \quad r \geq 0 \end{cases}$$

C.3 Joint Gaussian distributions

Definition C.12 (Joint Gaussian pdf). [15]

$$\begin{aligned}
\mathsf{p}(x_1,x_2,\ldots,x_n) &\triangleq \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{M}|}}\exp -\frac{1}{2}(\boldsymbol{x}-\mathsf{E}\boldsymbol{x})^T\boldsymbol{M}^{-1}(\boldsymbol{x}-\mathsf{E}\boldsymbol{x}) && \text{(Gaussian joint pdf)} \\
\boldsymbol{x} &\triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
\mathsf{Z}_k &\triangleq X_k - \mathsf{E}X_k && \text{(zero mean random variables)} \\
\boldsymbol{M} &\triangleq \begin{bmatrix} \mathsf{E}[\mathsf{Z}_1\mathsf{Z}_1] & \mathsf{E}[\mathsf{Z}_1\mathsf{Z}_2] & \cdots & \mathsf{E}[\mathsf{Z}_1\mathsf{Z}_n] \\ \mathsf{E}[\mathsf{Z}_2\mathsf{Z}_1] & \mathsf{E}[\mathsf{Z}_2\mathsf{Z}_2] & & \mathsf{E}[\mathsf{Z}_2\mathsf{Z}_n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{E}[\mathsf{Z}_n\mathsf{Z}_1] & \mathsf{E}[\mathsf{Z}_n\mathsf{Z}_2] & \cdots & \mathsf{E}[\mathsf{Z}_n\mathsf{Z}_n] \end{bmatrix} && \text{(correlation matrix)}
\end{aligned}$$

Example C.2 (1 variable joint Gaussian pdf). The **Gaussian distribution** (or **normal distribution**) has pdf

[13] Proakis (2001), page 44
[14] Proakis (2001), page 46
[15] Proakis (2001), page 49, Moon and Stirling (2000), page 34

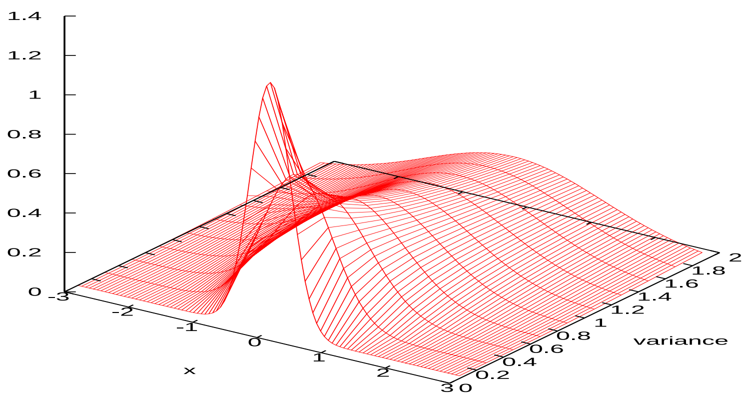


Figure C.4: Gaussian pdf with $\mu = 0$ and $\sigma \in [0.1, 2]$.

EX $\mathsf{p}_{\mathsf{x}}(x) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$$
\begin{aligned}
t &= \arg_t \min_t \left[\frac{1}{2}\int_t^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} + \frac{1}{2}\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\eta)^2}{2\sigma^2}} \right] \\
&= \arg_t \left\{ \frac{\partial}{\partial t}\left[\frac{1}{2}\int_t^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} + \frac{1}{2}\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\eta)^2}{2\sigma^2}} \right] = 0 \right\} \\
&= \arg_t \left\{ \frac{1}{2\sqrt{2\pi\sigma^2}} \left[\frac{\partial}{\partial t}\int_t^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}} + \frac{\partial}{\partial t}\int_{-\infty}^{t} e^{\frac{-(x-\eta)^2}{2\sigma^2}} \right] = 0 \right\} \\
&= \arg_t \left\{ \left[\left(e^{\frac{-(\infty-\mu)^2}{2\sigma^2}} 0 - e^{\frac{-(t-\mu)^2}{2\sigma^2}} 1 \right) + \left(e^{\frac{-(t-\mu)^2}{2\sigma^2}} 1 - e^{\frac{-(\infty-\mu)^2}{2\sigma^2}} 0 \right) \right] = 0 \right\} \\
&= \arg_t \left\{ \left[e^{\frac{-(t-\eta)^2}{2\sigma^2}} - e^{\frac{-(t-\mu)^2}{2\sigma^2}} \right] = 0 \right\} \\
&= \arg_t \left\{ (t-\eta)^2 = (t-\mu)^2 \right\} \\
&= \frac{\mu+\eta}{2}
\end{aligned}
$$

Example C.3 (2 variable joint Gaussian pdf).

EX

$$
\begin{aligned}
z_1 &\triangleq x_1 - \mathsf{E}x_1 \\
z_2 &\triangleq x_2 - \mathsf{E}x_2 \\
|M| &\triangleq \left| \mathsf{E}[z_1 z_1]\mathsf{E}[z_2 z_2] - \mathsf{E}[z_1 z_2]\mathsf{E}[z_1 z_2] \right| \\
\mathsf{p}(x_1, x_2) &\triangleq \frac{1}{2\pi\sqrt{|M|}} \exp\left(\frac{z_1^2 \mathsf{E}[z_2 z_2] - 2 z_1 z_2 \mathsf{E}[z_1 z_2] + z_2^2 \mathsf{E}[z_1 z_1]}{-2|M|} \right)
\end{aligned}
$$

APPENDIX D

SPECTRAL THEORY

D.1 Operator Spectrum

Definition D.1. [1] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator over the linear spaces* $\boldsymbol{X} = (X, F, \oplus, \otimes)$ *and* $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$. *Let* $\mathcal{N}(\mathbf{A})$ *be the* NULL SPACE *of* $\mathbf{A}$.

DEF
An ***eigenvalue*** *of* $\mathbf{A}$ *is any value* λ *such that there exists* $\mathbf{x}$ *such that* $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.
The ***eigenspace*** $\boldsymbol{H}_\lambda$ *of* $\mathbf{A}$ *at eigenvalue* λ *is* $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$.
An ***eigenvector*** *of* $\mathbf{A}$ *associated with eigenvalue* λ *is any element of* $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$.

Example D.1. [2] Let $\mathbf{D}$ be the differntial operator.

EX
The set $\left\{e^{\lambda x} | \lambda \in \mathbb{C}\right\}$ are the eigenvectors of $\mathbf{D}$.

$$\begin{aligned}
\rho(\mathbf{D}) &= \varnothing && \text{($\mathbf{D}$ has no non-spectral points whatsoever)}\\
\sigma_{\mathsf{p}}(\mathbf{D}) &= \sigma(\mathbf{D}) && \text{(the spectrum of $\mathbf{D}$ is all eigenvalues)}\\
\sigma_{\mathsf{c}}(\mathbf{D}) &= \varnothing && \text{($\mathbf{D}$ has no continuous spectrum)}\\
\sigma_{\mathsf{r}}(\mathbf{D}) &= \varnothing && \text{($\mathbf{D}$ has no resolvent spectrum)}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
(\mathbf{D} - \lambda\mathbf{I})e^{\lambda x} &= \mathbf{D}e^{\lambda x} - \lambda\mathbf{I}e^{\lambda x}\\
&= \lambda e^{\lambda x} - \lambda e^{\lambda x}\\
&= 0 && \forall \lambda \in \mathbb{C}
\end{aligned}$$

This theorem and proof needs more work and investigation to prove/disprove its claims. ✎

Definition D.2. [3] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator over the linear spaces* $\boldsymbol{X} = (X, F, \oplus, \otimes)$ *and* $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$.

[1] Bollobás (1999), page 168, Descartes (1637), Descartes (1954), Cayley (1858), Hilbert (1904), page 67, Hilbert (1912),
[2] Pedersen (2000), page 79
[3] Michel and Herget (1993), page 439

quantity		$\mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\}$ ($\boldsymbol{x} = \mathbb{0}$ is the only solution)	$\overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X}$ (dense)	$(\mathbf{A}-\lambda\mathbf{I})^{-1} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ (continuous/bounded)
$\rho(\mathbf{A})$	(resolvent set)	1	1	1
$\sigma_{\mathsf{p}}(\mathbf{A})$	(point spectrum)	0		
$\sigma_{\mathsf{r}}(\mathbf{A})$	(residual spectrum)	1	0	
$\sigma_{\mathsf{c}}(\mathbf{A})$	(continuous spectrum)	1	1	0

Table D.1: Spectrum of an operator $\mathbf{A}$

DEF

*The **resolvent set** $\rho(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\rho(\mathbf{A}) \triangleq \left\{ \lambda \in F \middle| \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \text{(no non-zero eigenvectors)} \quad \text{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X} & \text{(the range is dense in } \boldsymbol{X}\text{).} \quad \text{and} \\ 3. & (\mathbf{A}-\lambda\mathbf{I})^{-1} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) & \text{(inverse is continuous/bounded).} \end{array} \right\}$$

*The **spectrum** $\sigma(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma(\mathbf{A}) \triangleq F \backslash \rho(\mathbf{A}).$$

Definition D.3. [4] *Let $\mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ be an operator over the linear spaces $\boldsymbol{X} = (X, F, \oplus, \otimes)$ and $\boldsymbol{Y} \triangleq (Y, F, \oplus, \otimes)$.*

DEF

*The **point spectrum** $\sigma_{\mathsf{p}}(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

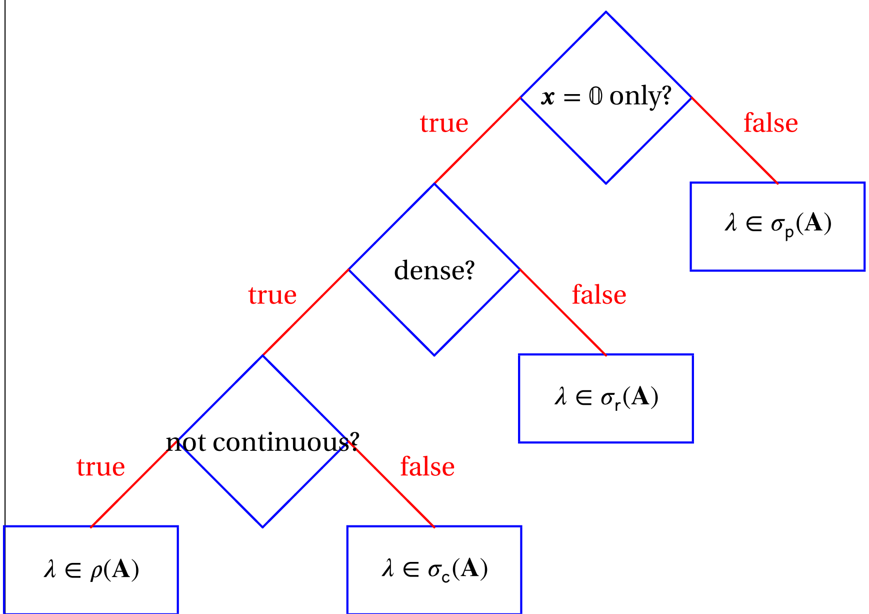
$$\sigma_{\mathsf{p}}(\mathbf{A}) \triangleq \left\{ \lambda \in F \middle| \; 1. \; \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) \supsetneq \{\mathbb{0}\} \quad \text{(has non-zero eigenvector)} \right\}$$

*The **residual spectrum** $\sigma_{\mathsf{r}}(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma_{\mathsf{r}}(\mathbf{A}) \triangleq \left\{ \lambda \in F \middle| \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \text{(no non-zero eigenvectors)} \quad \text{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} \neq \boldsymbol{X} & \text{(not dense in } \boldsymbol{X}\text{—has gaps).} \end{array} \right\}$$

*The **continuous spectrum** $\sigma_{\mathsf{c}}(\mathbf{A})$ of operator $\mathbf{A}$ is defined as*

$$\sigma_{\mathsf{c}}(\mathbf{A}) \triangleq \left\{ \lambda \in F \middle| \begin{array}{lll} 1. & \mathcal{N}(\mathbf{A}-\lambda\mathbf{I}) = \{\mathbb{0}\} & \text{(no non-zero eigenvectors)} \quad \text{and} \\ 2. & \overline{\mathcal{R}(\mathbf{A}-\lambda\mathbf{I})} = \boldsymbol{X} & \text{(dense in } \boldsymbol{X}\text{).} \quad \text{and} \\ 3. & (\mathbf{A}-\lambda\mathbf{I})^{-1} \notin \mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) & \text{(not continuous / not bounded)} \end{array} \right\}$$



The spectral components' definitions are illustrated in the figure to the left and summarized in Table D.1 (page 170). Let a family of operators $\mathbf{B}(\lambda)$ be defined with respect to an operator $\mathbf{A}$ such that $\mathbf{B}(\lambda) \triangleq (\mathbf{A} - \lambda\mathbf{I})$. Normally, we might expect a "normal" or "regular" or even "mundane" operator $\mathbf{B}(\lambda)$ to have the properties

1. $\mathbf{B}(\lambda)\boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \mathbb{0}$
2. $\mathbf{B}(\lambda)\boldsymbol{x}$ spans virtually all of $\boldsymbol{X}$ as we vary $\boldsymbol{x}$
3. $\mathbf{B}^{-1}(\lambda)$ is continuous.

After all, these are the properties that we would have if $\mathbf{B}(\lambda)$ were simply an affine operator in the

[4] Bollobás (1999), page 168, Hilbert (1906) pages 169–172

field of real numbers— such as $[\mathbf{B}(\lambda)](x) \triangleq [\lambda](x) = \lambda x$ which is 0 if and only if $x = 0$, has range $\boldsymbol{R}(\lambda) = \mathbb{R}$, and its inverse $\lambda^{-1}x$ is continuous.

If for some λ the operator $\mathbf{B}(\lambda)$ does have all these "regular" properties, then that λ part of the *resolvent set* of $\mathbf{A}$ and λ is called *regular*. However if for some λ the operator $\mathbf{B}(\lambda)$ fails any of these conditions, then that λ part of the *spectrum* of $\mathbf{A}$. And which conditions it fails determines which component of the spectrum it is in.

Theorem D.1. [5] *Let* $\mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator.*

THM

$$\sigma(\mathbf{A}) = \sigma_{\mathrm{p}}(\mathbf{A}) \cup \sigma_{\mathrm{c}}(\mathbf{A}) \cup \sigma_{\mathrm{r}}(\mathbf{A})$$

Theorem D.2 (Spectral Theorem). [6] [7] *Let* $\mathbf{N} \in Y^X$ *be an operator.*

THM

$$\left.\begin{array}{ll} (1). & \underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \textit{ is NORMAL}} \\ (2). & \mathbf{N} \textit{ is COMPACT} \end{array}\right\} \implies \left\{\begin{array}{lrcl} (1). & \mathbf{N} &=& \sum_n \lambda_n \mathbf{P}_n \\ (2). & \sum_n \mathbf{P}_n &=& \mathbf{I} \\ (3). & \mathbf{P}_n\mathbf{P}_m &=& \bar{\delta}_{n-m}\mathbf{P}_n \\ (4). & \dim(\boldsymbol{H}_n) &<& \infty \\ (5). & \left|\{\lambda_n | \lambda_n \neq 0\}\right| & \multicolumn{2}{l}{\textit{is COUNTABLY INFINITE}} \end{array}\right.$$

where

$$\begin{array}{rcll} (\lambda_n)_{n\in\mathbb{Z}} &\triangleq& \sigma_{\mathrm{p}}(\mathbf{N}) & \textit{(eigenvalues of } \mathbf{N}\textit{)} \\ \boldsymbol{H}_n &\triangleq& \mathcal{N}(\mathbf{N} - \lambda_n\mathbf{I}) & \textit{(}\lambda_n \textit{ is the eigenspace of } \mathbf{N} \textit{ at } \lambda_n \textit{ in } \boldsymbol{Y}\textit{)} \\ \boldsymbol{H}_n &=& \mathbf{P}_n\boldsymbol{Y} & \textit{(}\mathbf{P}_n \textit{ is the projection operator that generates } \boldsymbol{H}_n\textit{)} \end{array}$$

D.2 Fredholm kernels

Definition D.4. [8]

DEF

A ***Fredholm operator*** $\mathbf{K}$ *is defined as*

$$\underbrace{[\mathbf{K}\mathrm{f}](t) \triangleq \int_a^b \underbrace{\kappa(t,s)}_{\textit{kernel}} \mathrm{f}(s)\,\mathrm{d}s}_{\textit{Fredholm integral equation of the first kind}^{9}} \qquad \forall \mathrm{f} \in \boldsymbol{L}_2([a,b])$$

Example D.2. Examples of Fredholm operators include

1.	Fourier Transform	$[\tilde{\mathbf{F}}\mathrm{x}](f)$	$=$	$\int_t \mathrm{x}(t)e^{-i2\pi ft}\,\mathrm{d}t$	$\kappa(t,f)$	$=$	$e^{-i2\pi ft}$
2.	Inverse Fourier Transform	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathrm{x}}](t)$	$=$	$\int_f \tilde{\mathrm{x}}(f)e^{i2\pi ft}\,\mathrm{d}f$	$\kappa(f,t)$	$=$	$e^{i2\pi ft}$
3.	Laplace operator	$[\mathbf{L}\mathrm{x}](s)$	$=$	$\int_t \mathrm{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t,s)$	$=$	e^{-st}
4.	autocorrelation operator	$[\mathbf{R}\mathrm{x}](t)$	$=$	$\int_s R(t,s)\mathrm{x}(s)\,\mathrm{d}s$	$\kappa(t,s)$	$=$	$R(t,s)$

Theorem D.3. *Let* $\mathbf{K}$ *be a Fredholm operator with kernel* $\kappa(t,s)$ *and adjoint* $\mathbf{K}^*$.

THM

$$[\mathbf{K}\mathrm{f}](t) = \int_A \kappa(t,s)\mathrm{f}(s)\,\mathrm{d}s \iff [\mathbf{K}^*\mathrm{f}](t) = \int_A \kappa^*(s,t)\mathrm{f}(s)\,\mathrm{d}s$$

[5] Michel and Herget (1993), page 440
[6] Michel and Herget (1993), page 457, Bollobás (1999), page 200
[7] Hilbert (1906), Hilbert (1912), von Neumann (1929), de Witt (1659)
[8] Michel and Herget (1993), page 425
[9] The equation $\int_u \kappa(t,s)\mathrm{f}(s)\,\mathrm{d}s$ is a **Fredholm integral equation of the first kind** and $\kappa(t,u)$ is the **kernel** of the equation. References: Fredholm (1900), Fredholm (1903), page 365, Michel and Herget (1993), page 97, Keener (1988), page 101

✎PROOF:

$$
\begin{aligned}
& [\mathbf{K}\mathsf{f}](t) = \int_A \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s \\
\iff & \langle [\mathbf{K}\mathsf{f}](t) \,|\, \mathsf{g}(t)\rangle = \left\langle \int_s \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s \,\middle|\, \mathsf{g}(t)\right\rangle && \text{by left hypothesis} \\
& = \int_s \mathsf{f}(s)\,\langle \kappa(t,s) \,|\, \mathsf{g}(t)\rangle\,\mathrm{d}s && \text{by additivity property of } \langle \triangle \,|\, \triangledown\rangle \text{ (Definition I.1 page 233)} \\
& = \int_s \mathsf{f}(s)\,\langle \mathsf{g}(t) \,|\, \kappa(t,s)\rangle^*\,\mathrm{d}s && \text{by conjugate symmetry property of } \langle \triangle \,|\, \triangledown\rangle \text{ (Definition I.1 page 233)} \\
& = \langle \mathsf{f}(s) \,|\, \langle \mathsf{g}(t) \,|\, \kappa(t,s)\rangle\rangle && \text{by local definition of } \langle \triangle \,|\, \triangledown\rangle \\
& = \left\langle \mathsf{f}(s) \,\middle|\, \underbrace{\int_t \kappa^*(t,s)\mathsf{g}(t)\,\mathrm{d}t}_{[\mathbf{K}^*\mathsf{g}](s)} \right\rangle && \text{by local definition of } \langle \triangle \,|\, \triangledown\rangle \\
\iff & \left[\mathbf{K}^*\mathsf{g}\right](s) = \int_A \kappa^*(t,s)\mathsf{g}(t)\,\mathrm{d}t && \text{by right hypothesis} \\
\iff & \left[\mathbf{K}^*\mathsf{g}\right](\sigma) = \int_A \kappa^*(\tau,\sigma)\mathsf{g}(\tau)\,\mathrm{d}\tau && \text{by change of variable: } \tau = t,\ \sigma = s \\
\iff & \left[\mathbf{K}^*\mathsf{f}\right](t) = \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathrm{d}s && \text{by change of variable: } t = \sigma,\ s = \tau,\ \mathsf{f} = \mathsf{g}
\end{aligned}
$$

⇨

Theorem D.4. [10] *Let* $\mathbf{K}$ *be an Fredholm operator with kernel* $\kappa(t,s)$ *and adjoint* $\mathbf{K}^*$.

THM

$$\underbrace{\mathbf{K} = \mathbf{K}^*}_{\mathbf{K} \text{ is self-adjoint}} \iff \underbrace{\kappa(t,s) = \kappa^*(s,t)}_{\text{kernel is conjugate symmetric}}$$

✎PROOF:

$$
\begin{aligned}
\mathbf{K} = \mathbf{K}^* &\iff \int_A \kappa(t,s)\mathsf{f}(s)\,\mathrm{d}s = \int_A \kappa^*(s,t)\mathsf{f}(s)\,\mathrm{d}s && \text{by Theorem D.3 page 171} \\
&\iff \kappa(t,s) = \kappa^*(s,t)
\end{aligned}
$$

⇨

Theorem D.5 (Mercer's Theorem). [11] *Let* $\mathbf{K}$ *be an Fredholm operator with kernel* $\kappa(t,s)$ *and eigensystem* $(\!(\lambda_n, \phi_n(t))\!)_{n\in\mathbb{Z}}$.

THM

$$\left.\begin{array}{l} 1.\ \underbrace{\displaystyle\int_a^b\int_a^b \kappa(t,s)\mathsf{f}(t)\mathsf{f}^*(s)\,\mathrm{d}t \ge 0}_{\text{positive}} \quad \textit{and} \\ 2.\ \kappa(t,s) \textit{ is continuous on } [a,b]\times[a,b] \end{array}\right\} \implies \left\{\begin{array}{l} 1.\ \kappa(t,s) = \sum_n \lambda_n\phi_n(t)\phi_n^*(s) \quad \textit{and} \\ 2.\ \kappa(t,s) \textit{ converges absolutely and} \\ \quad \textit{uniformly on } [a,b]\times[a,b] \end{array}\right.$$

✎PROOF:

⇨

[10] Michel and Herget (1993), page 430

[11] Gohberg et al. (2003), page 198, Courant and Hilbert (1930), pages 138–140, Mercer (1909), page 439

APPENDIX E

MATRIX CALCULUS

Optimization problems often require finding the value of some parameter which results in some measure reaching a minimum or maximum value. Often this optimal parameter value can be found by solving the single equation generated by the partial derivative of the measure with respect to the parameter. When there are several parameters, optimization often requires several simultaneous equations generated by the partial derivatives of the measure with respect to each parameter. The need for several partial derivatives and several simultaneous equations leads to a natual union of two branches of mathematics— partial differential equations and linear algebra. In general, we would like to not only be able to take the partial derivative of a scalar with respect to another scalar, but to be able to take the partial derivative of a vector with respect to another vector. This generalization is the problem addressed in this section. Other references are also available.[1]

E.1 First derivative of a vector with respect to a vector

Definition E.1.

DEF

$\boldsymbol{x}$ is a vector with the following properties:

1. $$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ *(n element column vector)*

2. $$\frac{\partial}{\partial x_k} x_j = \bar{\delta}_{kj}$$ *($(x_1, x_2, \ldots, x_n)$ are mutually independent)*

Definition E.2 (Jacobian matrix). [2] *The **gradient of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$**, as well as the **gradient of $\boldsymbol{y}^T$ with respect to $\boldsymbol{x}$**, is defined as*

---

[1] Graham (1981) ⟨Chapter 4⟩, Haykin (2001) ⟨Appendix B⟩, Moon and Stirling (2000) ⟨Appendix E⟩, Scharf (1991), pages 274–276, Trees (2002) ⟨Section A.7⟩, Felippa (1999)

[2] Graham (1981), page 52, Scharf (1991), page 274, Trees (2002), page 1398

DEF

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \triangleq \frac{\partial \boldsymbol{y}^T}{\partial \boldsymbol{x}} \triangleq \underbrace{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}}_{n \times m \text{ matrix}} \qquad \forall \boldsymbol{y} \in \mathbb{C}^m$$

Remark E.1. Depending on whether $\boldsymbol{x}$ and $\boldsymbol{y}$ are scalars or vectors, $\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}$ takes on the following forms:[3]

	y scalar	y vector
x scalar	$\frac{\partial y}{\partial x}$	$\frac{\partial \boldsymbol{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} & \cdots & \frac{\partial y_m}{\partial x} \end{bmatrix}$
x vector	$\frac{\partial y}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}$	$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \triangleq \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$

Lemma E.1. *Let $\boldsymbol{x} \in \mathbb{R}^n$ be a vector. Then*

LEM

$$\frac{\partial}{\partial x_k} x_i x_j = \bar{\delta}_{ik} x_j + \bar{\delta}_{jk} x_i = \begin{cases} 2x_k & \text{for } i = j = k \\ x_j & \text{for } i = k \text{ and } j \neq k \\ x_i & \text{for } i \neq k \text{ and } j = k \\ 0 & \text{otherwise} \end{cases}$$

Lemma E.2.

LEM

$$\left(\boldsymbol{x}^H \mathbf{A} \boldsymbol{x}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i^* x_j \qquad \forall \underbrace{\mathbf{A} \in (\mathbb{C}^n \times \mathbb{C}^n)}_{n \times n \text{ array}} \text{ and } \underbrace{\boldsymbol{x} \in \mathbb{C}^n}_{n \text{ element column vector}}$$

✎PROOF:

$$\begin{aligned}
\boldsymbol{x}^H \mathbf{A} \boldsymbol{x} &\triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^* \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} && \text{by definitions of } \mathbf{A} \text{ and } \boldsymbol{x} \\
&= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^* \sum_{i=1}^{n} x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \\
&= \sum_{i=1}^{n} x_i \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^* \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ji} x_j^*
\end{aligned}$$

[3] For the generalization of the partial derivative of a matrix with respect to a matrix, see Graham (1981) ⟨chapter 6⟩. Graham uses *kronecker products* to handle the additional dimensions(?)

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i^* x_j$$

✏

Lemma E.3.

LEM

$$\frac{\partial}{\partial \boldsymbol{x}}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] = \mathsf{a}(\boldsymbol{x})\left[\frac{\partial}{\partial \boldsymbol{x}}\mathsf{b}(\boldsymbol{x})\right] + \left[\frac{\partial}{\partial \boldsymbol{x}}\mathsf{a}(\boldsymbol{x})\right]\mathsf{b}(\boldsymbol{x}) \qquad \underbrace{\forall \mathsf{a},\mathsf{b} : \mathbb{R}^n \to \mathbb{R}}_{\mathsf{a}(\boldsymbol{x}),\,\mathsf{b}(\boldsymbol{x}) \text{ are functions from a vector } \boldsymbol{x} \text{ to a scalar in } \mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] &= \begin{bmatrix} \frac{\partial}{\partial x_1}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] \\ \frac{\partial}{\partial x_2}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] \\ \vdots \\ \frac{\partial}{\partial x_n}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_1} + \mathsf{b}(\boldsymbol{x})\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_1} \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_2} + \mathsf{b}(\boldsymbol{x})\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_2} \\ \vdots \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_n} + \mathsf{b}(\boldsymbol{x})\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_1} \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_2} \\ \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_n} \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_1}\mathsf{b}(\boldsymbol{x}) \\ \frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_2}\mathsf{b}(\boldsymbol{x}) \\ \vdots \\ \frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_n}\mathsf{b}(\boldsymbol{x}) \end{bmatrix} \\
&= \mathsf{a}(\boldsymbol{x})\left[\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right] + \left[\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right]\mathsf{b}(\boldsymbol{x})
\end{aligned}$$

✏

Theorem E.1. [4]

LEM

$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x} = \mathbf{I} \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x} &= \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial x_n} & \frac{\partial x_2}{\partial x_n} & \cdots & \frac{\partial x_n}{\partial x_n} \end{bmatrix} && \text{by Definition E.2 page 173} \\
&= \begin{bmatrix} \bar{\delta}_{11} & \bar{\delta}_{21} & \cdots & \bar{\delta}_{n1} \\ \bar{\delta}_{12} & \bar{\delta}_{22} & \cdots & \bar{\delta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\delta}_{1n} & \bar{\delta}_{2n} & \cdots & \bar{\delta}_{nn} \end{bmatrix} && \text{by Definition E.1 page 173 (mutual independence property)}
\end{aligned}$$

[4] Scharf (1991), page 274, Trees (2002), page 1398

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \text{by definition of kronecker delta function } \bar{\delta}$$

$$= \mathbf{I} \qquad \text{by definition of identity operator } \mathbf{I}$$

Theorem E.2.

THM

$$\frac{\partial}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x}) = \mathbf{A}^T + \sum_{i=1}^{n} \left(\frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi} \end{bmatrix} \right) x_i \qquad \forall \boldsymbol{x} \in \mathbb{R}^n,\ \mathbf{A} \in \mathbb{C}^m \times \mathbb{C}^n$$

PROOF: Let

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{A} \triangleq \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\frac{\partial}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{x}} \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) \qquad \text{by definition of } A \text{ and } \boldsymbol{x}$$

$$= \frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} \sum_{i=1}^{n} a_{1i} x_i \\ \sum_{i=1}^{n} a_{2i} x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} x_i \end{bmatrix} \qquad \text{by matrix multiplication}$$

$$= \frac{\partial}{\partial \boldsymbol{x}} \sum_{i=1}^{n} \begin{bmatrix} a_{1i} x_i \\ a_{2i} x_i \\ \vdots \\ a_{mi} x_i \end{bmatrix}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} x_i \\ a_{2i} x_i \\ \vdots \\ a_{mi} x_i \end{bmatrix}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial a_{1i} x_i}{\partial x_1} & \frac{\partial a_{2i} x_i}{\partial x_1} & \cdots & \frac{\partial a_{mi} x_i}{\partial x_1} \\ \frac{\partial a_{1i} x_i}{\partial x_2} & \frac{\partial a_{2i} x_i}{\partial x_2} & \cdots & \frac{\partial a_{mi} x_i}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_{1i} x_i}{\partial x_n} & \frac{\partial a_{2i} x_i}{\partial x_n} & \cdots & \frac{\partial a_{mi} x_i}{\partial x_n} \end{bmatrix} \qquad \text{by Definition E.2 page 173}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} a_{1i} \frac{\partial x_i}{\partial x_1} + \frac{\partial a_{1i}}{\partial x_1} x_i & a_{2i} \frac{\partial x_i}{\partial x_1} + \frac{\partial a_{2i}}{\partial x_1} x_i & \cdots & a_{mi} \frac{\partial x_i}{\partial x_1} + \frac{\partial a_{mi}}{\partial x_1} x_i \\ a_{1i} \frac{\partial x_i}{\partial x_2} + \frac{\partial a_{1i}}{\partial x_2} x_i & a_{2i} \frac{\partial x_i}{\partial x_2} + \frac{\partial a_{2i}}{\partial x_2} x_i & \cdots & a_{mi} \frac{\partial x_i}{\partial x_2} + \frac{\partial a_{mi}}{\partial x_2} x_i \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i} \frac{\partial x_i}{\partial x_n} + \frac{\partial a_{1i}}{\partial x_n} x_i & a_{2i} \frac{\partial x_i}{\partial x_n} + \frac{\partial a_{2i}}{\partial x_n} x_i & \cdots & a_{mi} \frac{\partial x_i}{\partial x_n} + \frac{\partial a_{mi}}{\partial x_n} x_i \end{bmatrix} \qquad \text{by Lemma E.3 page 175}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} a_{1i}\frac{\partial x_i}{\partial x_1} & a_{2i}\frac{\partial x_i}{\partial x_1} & \cdots & a_{mi}\frac{\partial x_i}{\partial x_1} \\ a_{1i}\frac{\partial x_i}{\partial x_2} & a_{2i}\frac{\partial x_i}{\partial x_2} & \cdots & a_{mi}\frac{\partial x_i}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i}\frac{\partial x_i}{\partial x_n} & a_{2i}\frac{\partial x_i}{\partial x_n} & \cdots & a_{mi}\frac{\partial x_i}{\partial x_n} \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial a_{1i}}{\partial x_1}x_i & \frac{\partial a_{2i}}{\partial x_1}x_i & \cdots & \frac{\partial a_{mi}}{\partial x_1}x_i \\ \frac{\partial a_{1i}}{\partial x_2}x_i & \frac{\partial a_{2i}}{\partial x_2}x_i & \cdots & \frac{\partial a_{mi}}{\partial x_2}x_i \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_{1i}}{\partial x_n}x_i & \frac{\partial a_{2i}}{\partial x_n}x_i & \cdots & \frac{\partial a_{mi}}{\partial x_n}x_i \end{bmatrix}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} a_{1i}\bar{\delta}_{i1} & a_{2i}\bar{\delta}_{i1} & \cdots & a_{mi}\bar{\delta}_{i1} \\ a_{1i}\bar{\delta}_{i2} & a_{2i}\bar{\delta}_{i2} & \cdots & a_{mi}\bar{\delta}_{i2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i}\bar{\delta}_{in} & a_{2i}\bar{\delta}_{in} & \cdots & a_{mi}\bar{\delta}_{in} \end{bmatrix} + \sum_{i=1}^{n}\left(\frac{\partial}{\partial \mathbf{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi}\end{bmatrix}\right)x_i \qquad \text{by Lemma E.1}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} + \sum_{i=1}^{n}\left(\frac{\partial}{\partial \mathbf{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi}\end{bmatrix}\right)x_i \qquad \text{by definition of } \bar{\delta}$$

$$= \mathbf{A}^T + \sum_{i=1}^{n}\left(\frac{\partial}{\partial \mathbf{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi}\end{bmatrix}\right)x_i$$

⇨

Theorem E.3 (Affine equations). [5]

THM

$\mathbf{A}$ *and* $\mathbf{B}$ *are independent of* $\mathbf{x}$ $\implies \begin{cases} \frac{\partial}{\partial \mathbf{x}}(\mathbf{A}\mathbf{x}) = \mathbf{A}^T & \forall \mathbf{x}\in\mathbb{R}^n,\ \mathbf{A}\in\mathbb{C}^m\times\mathbb{C}^n \\ \frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^T\mathbf{B}\right) = \mathbf{B} & \forall \mathbf{x}\in\mathbb{R}^n,\ \mathbf{B}\in\mathbb{C}^n\times\mathbb{C}^m \end{cases}$

✎PROOF: Let $\mathbf{B} \triangleq \mathbf{A}^T$.

1. Proof that $\frac{\partial}{\partial x}(A\mathbf{x}) = \mathbf{A}^T$:

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{A}\mathbf{x}) = \mathbf{A}^T + \sum_{i=1}^{n}\left(\frac{\partial}{\partial \mathbf{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi}\end{bmatrix}\right)x_i \qquad \text{by Theorem E.2 page 176}$$

$$= \mathbf{A}^T + \sum_{i=1}^{n}\begin{bmatrix} \frac{\partial}{\partial \mathbf{x}}a_{1i} & \frac{\partial}{\partial \mathbf{x}}a_{2i} & \cdots & \frac{\partial}{\partial \mathbf{x}}a_{mi}\end{bmatrix}x_i$$

$$= \mathbf{A}^T + \sum_{i=1}^{n}\begin{bmatrix} 0 & 0 & \cdots & 0\end{bmatrix}x_i \xrightarrow{} 0 \qquad \text{by left hypothesis}$$

$$= \mathbf{A}^T$$

2. Proof that $\frac{\partial}{\partial x}\left(\mathbf{x}^T\mathbf{B}\right) = \mathbf{B}$:

$$\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^T\mathbf{B}\right) = \frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^T\mathbf{A}^T\right) \qquad \text{by definition of } \mathbf{B}$$

$$= \frac{\partial}{\partial \mathbf{x}}\left[(\mathbf{A}\mathbf{x})^T\right]$$

$$= \frac{\partial}{\partial \mathbf{x}}(\mathbf{A}\mathbf{x}) \qquad \text{by Definition E.2 page 173}$$

$$= \mathbf{A}^T \qquad \text{by Theorem E.3 page 177}$$

$$= \mathbf{B} \qquad \text{by definition of } \mathbf{B}$$

⇨

[5] Graham (1981), page 54

Theorem E.4 (Product rule). [6] *Let $\boldsymbol{y}$ and $\boldsymbol{z}$ be functions of $\boldsymbol{x}$ and*

THM
$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{z}^T\boldsymbol{y} = \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}}\boldsymbol{y} + \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\boldsymbol{z} \qquad \forall \boldsymbol{x}\in\mathbb{R}^n,\ \boldsymbol{y}\in\mathbb{R}^m,\ \boldsymbol{z}\in\mathbb{R}^m$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{z}^T\boldsymbol{y} &= \frac{\partial}{\partial \boldsymbol{x}}\sum_{k=1}^{m} z_k y_k \\
&= \sum_{k=1}^{m}\frac{\partial}{\partial \boldsymbol{x}} z_k y_k \\
&= \sum_{k=1}^{m}\frac{\partial z_k}{\partial \boldsymbol{x}} y_k + \sum_{k=1}^{m}\frac{\partial y_k}{\partial \boldsymbol{x}} z_k && \text{by Lemma E.3 page 175} \\
&= \begin{bmatrix}
\frac{\partial z_1}{\partial x_1}y_1 & + & \frac{\partial z_2}{\partial x_1}y_2 & + & \cdots & + & \frac{\partial z_n}{\partial x_1}y_n \\
\frac{\partial z_1}{\partial x_1}y_1 & + & \frac{\partial z_2}{\partial x_1}y_2 & + & \cdots & + & \frac{\partial z_n}{\partial x_1}y_n \\
\vdots & & & & \ddots & & \vdots \\
\frac{\partial z_1}{\partial x_1}y_1 & + & \frac{\partial z_2}{\partial x_1}y_2 & + & \cdots & + & \frac{\partial z_n}{\partial x_1}y_n
\end{bmatrix} + \begin{bmatrix}
\frac{\partial y_1}{\partial x_1}z_1 & + & \frac{\partial y_2}{\partial x_1}z_2 & + & \cdots & + & \frac{\partial y_n}{\partial x_1}z_n \\
\frac{\partial y_1}{\partial x_1}z_1 & + & \frac{\partial y_2}{\partial x_1}z_2 & + & \cdots & + & \frac{\partial y_n}{\partial x_1}z_n \\
\vdots & & & & \ddots & & \vdots \\
\frac{\partial y_1}{\partial x_1}z_1 & + & \frac{\partial y_2}{\partial x_1}z_2 & + & \cdots & + & \frac{\partial y_n}{\partial x_1}z_n
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_1} \\
\frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_1} \\
\vdots & & \ddots & \vdots \\
\frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_1}
\end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\
\vdots & & \ddots & \vdots \\
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1}
\end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}}\boldsymbol{y} + \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\boldsymbol{z}
\end{aligned}$$

✏

Theorem E.5.

THM
$$\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T A\boldsymbol{x}\right) = \mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x} + \left[\sum_{i=1}^{n}\left(\frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni}\end{bmatrix}\right)x_i\right]\boldsymbol{x} \qquad \forall \boldsymbol{x}\in\mathbb{R}^n, \mathbf{A}\in\mathbb{R}^n\times\mathbb{R}^n$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T\mathbf{A}\boldsymbol{x}\right) &= \left[\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}\right]\mathbf{A}\boldsymbol{x} + \left[\frac{\partial}{\partial \boldsymbol{x}}\mathbf{A}\boldsymbol{x}\right]\boldsymbol{x} && \text{by Theorem E.4 page 178} \\
&= \mathbf{I}\mathbf{A}\boldsymbol{x} + \left[\mathbf{A}^T + \sum_{i=1}^{n}\left(\frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni}\end{bmatrix}\right)x_i\right]\boldsymbol{x} && \text{by Theorem E.1 and Theorem E.2} \\
&= \mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x} + \left[\sum_{i=1}^{n}\left(\frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni}\end{bmatrix}\right)x_i\right]\boldsymbol{x} && \text{by definition of identity operator } \mathbf{I}
\end{aligned}$$

✏

Theorem E.6 (Quadratic form). [7]

THM
$$\mathbf{A} \text{ is independent of } \boldsymbol{x} \implies \frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T A\boldsymbol{x}\right) = A\boldsymbol{x} + A^T\boldsymbol{x} \qquad \forall \boldsymbol{x}\in\mathbb{R}^n, \mathbf{A}\in\mathbb{R}^n\times\mathbb{R}^n$$

[6] Scharf (1991), page 274, Trees (2002), page 1398
[7] Graham (1981), page 54

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T \mathbf{A}\boldsymbol{x}\right) &= \left[\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}\right]\mathbf{A}\boldsymbol{x} + \left[\frac{\partial}{\partial \boldsymbol{x}}\mathbf{A}\boldsymbol{x}\right]\boldsymbol{x} && \text{by Theorem E.4 page 178}\\
&= \mathbf{I}\mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x} && \text{by Theorem E.1 page 175 and Theorem E.3 page 177}
\end{aligned}$$

⇨

Corollary E.1. [8]

COR
$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T\boldsymbol{x}) = 2\boldsymbol{x} \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T\boldsymbol{x}) &= \frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T\mathbf{I}\boldsymbol{x}) && \text{by property of identity operator } I\\
&= \mathbf{I}\boldsymbol{x} + \mathbf{I}^T\boldsymbol{x} && \text{by previous result 3.}\\
&= \boldsymbol{x} + \boldsymbol{x} && \text{by property of identity operator } I\\
&= 2\boldsymbol{x}
\end{aligned}$$

⇨

Theorem E.7 (Chain rule). [9] *Let* $\boldsymbol{z}$ *be a function of* $\boldsymbol{y}$ *and* $\boldsymbol{y}$ *a function of* $\boldsymbol{x}$ *and*

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \boldsymbol{y} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \qquad \boldsymbol{z} \triangleq \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}$$

THM
$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{z} = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{y}}$$

✎PROOF:

$$\begin{aligned}
\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}} &= \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_k}{\partial x_1} \\ \frac{\partial z_1}{\partial x_2} & \frac{\partial z_2}{\partial x_2} & \cdots & \frac{\partial z_k}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_1}{\partial x_n} & \frac{\partial z_2}{\partial x_n} & \cdots & \frac{\partial z_k}{\partial x_n} \end{bmatrix}\\
&= \begin{bmatrix} \sum_{j=0}^m \frac{\partial z_1}{\partial y_j}\frac{\partial y_j}{\partial x_1} & \sum_{j=0}^m \frac{\partial z_2}{\partial y_j}\frac{\partial y_j}{\partial x_1} & \cdots & \sum_{j=0}^m \frac{\partial z_k}{\partial y_j}\frac{\partial y_j}{\partial x_1} \\ \sum_{j=0}^m \frac{\partial z_1}{\partial y_j}\frac{\partial y_j}{\partial x_2} & \sum_{j=0}^m \frac{\partial z_2}{\partial y_j}\frac{\partial y_j}{\partial x_2} & \cdots & \sum_{j=0}^m \frac{\partial z_k}{\partial y_j}\frac{\partial y_j}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^m \frac{\partial z_1}{\partial y_j}\frac{\partial y_j}{\partial x_n} & \sum_{j=0}^m \frac{\partial z_2}{\partial y_j}\frac{\partial y_j}{\partial x_n} & \cdots & \sum_{j=0}^m \frac{\partial z_k}{\partial y_j}\frac{\partial y_j}{\partial x_n} \end{bmatrix}\\
&= \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_2}{\partial y_1} & \cdots & \frac{\partial z_k}{\partial y_1} \\ \frac{\partial z_1}{\partial y_2} & \frac{\partial z_2}{\partial y_2} & \cdots & \frac{\partial z_k}{\partial y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_1}{\partial y_m} & \frac{\partial z_2}{\partial y_m} & \cdots & \frac{\partial z_k}{\partial y_m} \end{bmatrix}\\
&= \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{y}}
\end{aligned}$$

[8] Graham (1981), page 54
[9] Graham (1981), pages 54–55

E.2 First derivative of a matrix with respect to a scalar

Definition E.3. *Let* $x \in \mathbb{R}$, $\{y_{jk} \in \mathbb{C} | j = 1,2,\ldots,m;\ k = 1,2,\ldots,n\}$ *and*

$$Y = \underbrace{\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}}_{m \times n \text{ matrix}}$$

*The **derivative of** Y **with respect to** x is*

DEF

$$\frac{\mathrm{d}Y}{\mathrm{d}x} \triangleq \underbrace{\begin{bmatrix} \frac{\mathrm{d}y_{11}}{\mathrm{d}x} & \frac{\mathrm{d}y_{12}}{\mathrm{d}x} & \cdots & \frac{\mathrm{d}y_{1n}}{\mathrm{d}x} \\ \frac{\mathrm{d}y_{21}}{\mathrm{d}x} & \frac{\mathrm{d}y_{22}}{\mathrm{d}x} & \cdots & \frac{\mathrm{d}y_{2n}}{\mathrm{d}x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}y_{m1}}{\mathrm{d}x} & \frac{\mathrm{d}y_{m2}}{\mathrm{d}x} & \cdots & \frac{\mathrm{d}y_{mn}}{\mathrm{d}x} \end{bmatrix}}_{m \times n \text{ matrix}}$$

Theorem E.8. [10] *Let* $x \in \mathbb{R}$, $\{y_{jp} \in \mathbb{C} | j = 1,2,\ldots,m;\ p = 1,2,\ldots,n\}$, $\{w_{jp} \in \mathbb{C} | j = 1,2,\ldots,n;\ p = 1,2,\ldots,k\}$, *and*

$$Y = \underbrace{\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}}_{m \times n \text{ matrix}} \qquad W = \underbrace{\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{p1} & w_{p2} & \cdots & w_{pk} \end{bmatrix}}_{p \times k \text{ matrix}}$$

THM

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y + W\right) &= \frac{\mathrm{d}}{\mathrm{d}x}Y + \frac{\mathrm{d}}{\mathrm{d}x}W && \text{(for } p = m,\ k = n\text{)} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(YW\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}Y\right)W + Y\left(\frac{\mathrm{d}}{\mathrm{d}x}W\right) && \text{(for } p = n\text{)} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^T\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}Y\right)^T \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^{-1}\right) &= -Y^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x}Y\right)Y^{-1} && \text{(for } m = n \text{ and } Y \text{ invertible)}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y + W\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{bmatrix}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} y_{11} + w_{11} & y_{12} + w_{12} & \cdots & y_{1n} + w_{1n} \\ y_{21} + w_{21} & y_{22} + w_{22} & \cdots & y_{2n} + w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} + w_{m1} & y_{m2} + w_{m2} & \cdots & y_{mn} + w_{mn} \end{bmatrix}
\end{aligned}$$

[10] Gradshteyn and Ryzhik (1980), pages 1106–1107

$$
\begin{aligned}
&= \begin{bmatrix} (y_{11}+w_{11})' & (y_{12}+w_{12})' & \cdots & (y_{1n}+w_{1n})' \\ (y_{21}+w_{21})' & (y_{22}+w_{22})' & \cdots & (y_{2n}+w_{2n})' \\ \vdots & \vdots & \ddots & \vdots \\ (y_{m1}+w_{m1})' & (y_{m2}+w_{m2})' & \cdots & (y_{mn}+w_{mn})' \end{bmatrix} \\
&= \begin{bmatrix} y'_{11}+w'_{11} & y'_{12}+w'_{12} & \cdots & y'_{1n}+w'_{1n} \\ y'_{21}+w'_{21} & y'_{22}+w'_{22} & \cdots & y'_{2n}+w'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{m1}+w'_{m1} & y'_{m2}+w'_{m2} & \cdots & y'_{mn}+w'_{mn} \end{bmatrix} \\
&= \begin{bmatrix} y'_{11} & y'_{12} & \cdots & y'_{1n} \\ y'_{21} & y'_{22} & \cdots & y'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{m1} & y'_{m2} & \cdots & y'_{mn} \end{bmatrix} + \begin{bmatrix} w'_{11} & w'_{12} & \cdots & w'_{1n} \\ w'_{21} & w'_{22} & \cdots & w'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w'_{m1} & w'_{m2} & \cdots & w'_{mn} \end{bmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} + \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{bmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}Y + \frac{\mathrm{d}}{\mathrm{d}x}W
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(YW\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nk} \end{bmatrix}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} \sum_{j=1}^{n} y_{1j}w_{j1} & \sum_{j=1}^{n} y_{1j}w_{j2} & \cdots & \sum_{j=1}^{n} y_{1j}w_{jk} \\ \sum_{j=1}^{n} y_{2j}w_{j1} & \sum_{j=1}^{n} y_{2j}w_{j2} & \cdots & \sum_{j=1}^{n} y_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} y_{mj}w_{j1} & \sum_{j=1}^{n} y_{mj}w_{j2} & \cdots & \sum_{j=1}^{n} y_{mj}w_{jk} \end{bmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=1}^{n}\begin{bmatrix} y_{1j}w_{j1} & y_{1j}w_{j2} & \cdots & y_{1j}w_{jk} \\ y_{2j}w_{j1} & y_{2j}w_{j2} & \cdots & y_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y_{mj}w_{j1} & y_{mj}w_{j2} & \cdots & y_{mj}w_{jk} \end{bmatrix} \\
&= \sum_{j=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} y_{1j}w_{j1} & y_{1j}w_{j2} & \cdots & y_{1j}w_{jk} \\ y_{2j}w_{j1} & y_{2j}w_{j2} & \cdots & y_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y_{mj}w_{j1} & y_{mj}w_{j2} & \cdots & y_{mj}w_{jk} \end{bmatrix} \\
&= \sum_{j=1}^{n}\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x}(y_{1j}w_{j1}) & \frac{\mathrm{d}}{\mathrm{d}x}(y_{1j}w_{j2}) & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}(y_{1j}w_{jk}) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y_{2j}w_{j1}) & \frac{\mathrm{d}}{\mathrm{d}x}(y_{2j}w_{j2}) & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}(y_{2j}w_{jk}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}x}(y_{mj}w_{j1}) & \frac{\mathrm{d}}{\mathrm{d}x}(y_{mj}w_{j2}) & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}(y_{mj}w_{jk}) \end{bmatrix} \\
&= \sum_{j=1}^{n}\begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ y'_{1j}w_{j1}+y_{1j}w'_{j1} & y'_{1j}w_{j2}+y_{1j}w'_{j2} & \cdots & y'_{1j}w_{jk}+y_{1j}w'_{jk} \\ y'_{2j}w_{j1}+y_{2j}w'_{j1} & y'_{2j}w_{j2}+y_{2j}w'_{j2} & \cdots & y'_{2j}w_{jk}+y_{2j}w'_{jk} \\ y'_{mj}w_{j1}+y_{mj}w'_{j1} & y'_{mj}w_{j2}+y_{mj}w'_{j2} & \cdots & y'_{mj}w_{jk}+y_{mj}w'_{jk} \end{bmatrix}
\end{aligned}
$$

$$= \sum_{j=1}^{n} \left(\begin{bmatrix} y'_{1j}w_{j1} & y'_{1j}w_{j2} & \cdots & y'_{1j}w_{jk} \\ y'_{2j}w_{j1} & y'_{2j}w_{j2} & \cdots & y'_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{mj}w_{j1} & y'_{mj}w_{j2} & \cdots & y'_{mj}w_{jk} \end{bmatrix} + \begin{bmatrix} y_{1j}w'_{j1} & y_{1j}w'_{j2} & \cdots & y_{1j}w'_{jk} \\ y_{2j}w'_{j1} & y_{2j}w'_{j2} & \cdots & y_{2j}w'_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y_{mj}w'_{j1} & y_{mj}w'_{j2} & \cdots & y_{mj}w'_{jk} \end{bmatrix} \right)$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x} Y\right) W + Y \left(\frac{\mathrm{d}}{\mathrm{d}x} W\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^T\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}^T \right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ y_{12} & y_{22} & \cdots & y_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} y'_{11} & y'_{21} & \cdots & y'_{n1} \\ y'_{12} & y'_{22} & \cdots & y'_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{1n} & y'_{2n} & \cdots & y'_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} y'_{11} & y'_{12} & \cdots & y'_{1n} \\ y'_{21} & y'_{22} & \cdots & y'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{m1} & y'_{m2} & \cdots & y'_{mn} \end{bmatrix}^T$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} \right)^T$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^{-1}\right) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\operatorname{adj} Y}{|Y|}$$

$\vdots$

no proof at this time

$\vdots$

$$= -Y^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x} Y\right) Y^{-1}$$

E.3 Second derivative of a scalar with respect to a vector

Definition E.4. [11] *Let*

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

[11] Lieb and Loss (2001), page 240, Horn and Johnson (1990), page 167

The ***Hessian matrix*** *of a scalar y with respect to the vector $\boldsymbol{x}$ is*

DEF

$$\frac{\partial^2 y}{\partial \boldsymbol{x}^2} = \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial y}{\partial \boldsymbol{x}}\right) = \frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix}\frac{\partial y}{\partial x_1}\\ \frac{\partial y}{\partial x_2}\\ \vdots\\ \frac{\partial y}{\partial x_n}\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\partial}{\partial x_1}\frac{\partial y}{\partial x_1} & \frac{\partial}{\partial x_1}\frac{\partial y}{\partial x_2} & \cdots & \frac{\partial}{\partial x_1}\frac{\partial y}{\partial x_n}\\ \frac{\partial}{\partial x_2}\frac{\partial y}{\partial x_1} & \frac{\partial}{\partial x_2}\frac{\partial y}{\partial x_2} & \cdots & \frac{\partial}{\partial x_2}\frac{\partial y}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial}{\partial x_n}\frac{\partial y}{\partial x_1} & \frac{\partial}{\partial x_n}\frac{\partial y}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n}\frac{\partial y}{\partial x_n}\end{bmatrix}}_{n\times n \text{ matrix}}$$

E.4 Multiple derivatives of a vector with respect to a scalar

Definition E.5. *Let*

$$\boldsymbol{y} \triangleq \begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_m\end{bmatrix}$$

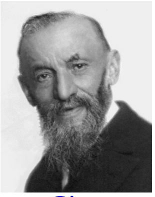
The derivative of a vector $\boldsymbol{y}$ with respect to the scalar x is

DEF

$$\begin{bmatrix}\boldsymbol{y}\\ \frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{y}\\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{y}\\ \vdots\\ \frac{\mathrm{d}^n}{\mathrm{d}x^n}\boldsymbol{y}\end{bmatrix} = \underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_m\\ \frac{\mathrm{d}}{\mathrm{d}x}y_1 & \frac{\mathrm{d}}{\mathrm{d}x}y_2 & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}y_m\\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}y_1 & \frac{\mathrm{d}^2}{\mathrm{d}x^2}y_2 & \cdots & \frac{\mathrm{d}^2}{\mathrm{d}x^2}y_m\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\mathrm{d}^n}{\mathrm{d}x^n}y_1 & \frac{\mathrm{d}^n}{\mathrm{d}x^n}y_2 & \cdots & \frac{\mathrm{d}^n}{\mathrm{d}x^n}y_m\end{bmatrix}}_{(n+1)\times m \text{ matrix}}$$

APPENDIX F

LINEAR SPACES



❝*The geometric calculus, in general, consists in a system of operations on geometric entities, and their consequences, analogous to those that algebra has on the numbers. It permits the expression in formulas of the results of geometric constructions, the representation with equations of propositions of geometry, and the substitution of a transformation of equations for a verbal argument.*❞

Giuseppe Peano (1858–1932), Italian mathematician, credited with being one of the first to introduce the concept of the *linear space* (*vector space*).[1]

F.1 Definition and basic results

A *metric space* is a *set* together with nothing else save a *metric* that gives the space a *topology* (Definition ?? page ??). A *linear space* (next definition) in general has no topology but does have some additional *algebraic* structure that is useful in generalizing a number of mathematical concepts. If one wishes to have both algebraic structure and a topology, then this can be accomplished by appending a *topology* to a *linear space* giving a *topological linear space* (Definition ?? page ??), a *metric* giving a *metric linear space*, an *inner product* giving an *inner product space* (Definition I.1 page 233), or a *norm* giving a *normed linear space* (Definition J.1 page 249).

Definition F.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD. *Let* X *be a set, let* $+$ *be an* OPERATOR *(Definition M.1 page 281) in* X^{X^2}, *and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

---

[1] quote: Peano (1888b), page ix
image http://en.wikipedia.org/wiki/File:Giuseppe_Peano.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991), page 41, Halmos (1948), pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b), pages 119–120, Banach (1922) pages 134–135

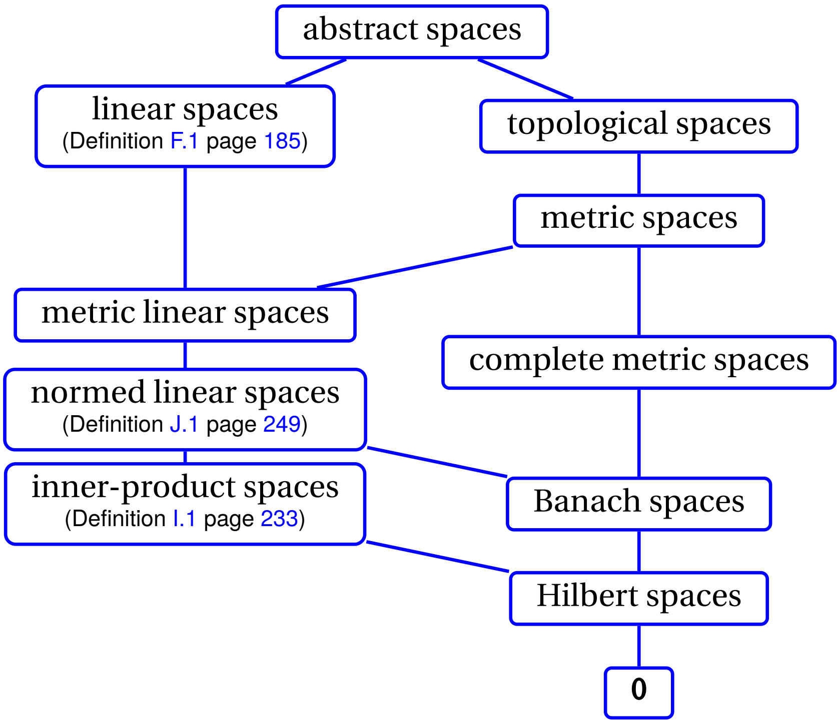


Figure F.1: Lattice of mathematical spaces

DEF

The structure $\boldsymbol{\Omega} \triangleq \left(X,\, +,\, \cdot,\, (\mathbb{F},\, \dot{+},\, \dot{\times})\right)$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1.	$\exists \mathbb{0} \in X$ such that	$\boldsymbol{x} + \mathbb{0} = \boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	(+ IDENTITY)	*
2.	$\exists \boldsymbol{y} \in X$ such that	$\boldsymbol{x} + \boldsymbol{y} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	(+ INVERSE)	
3.		$(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$	(+ is ASSOCIATIVE)	
4.		$\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	(+ is COMMUTATIVE)	
5.		$1 \cdot \boldsymbol{x} = \boldsymbol{x}$	$\forall \boldsymbol{x} \in X$	($\cdot$ IDENTITY)	
6.		$\alpha \cdot (\beta \cdot \boldsymbol{x}) = (\alpha \cdot \beta) \cdot \boldsymbol{x}$	$\forall \alpha,\beta \in S$ and $\boldsymbol{x} \in X$	($\cdot$ ASSOCIATES with $\cdot$)	
7.		$\alpha \cdot (\boldsymbol{x} + \boldsymbol{y}) = (\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y})$	$\forall \alpha \in S$ and $\boldsymbol{x},\boldsymbol{y} \in X$	($\cdot$ DISTRIBUTES over +)	
8.		$(\alpha + \beta) \cdot \boldsymbol{x} = (\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x})$	$\forall \alpha,\beta \in S$ and $\boldsymbol{x} \in X$	($\cdot$ PSEUDO-DISTRIBUTES over +)	

The set X *is called the* ***underlying set****. The elements of* X *are called* ***vectors****. The elements of* $\mathbb{F}$ *are called* ***scalars****. A* LINEAR SPACE *is also called a* ***vector space****. If* $\mathbb{F} \triangleq \mathbb{R}$*, then* $\boldsymbol{\Omega}$ *is a* ***real linear space****. If* $\mathbb{F} \triangleq \mathbb{C}$*, then* $\boldsymbol{\Omega}$ *is a* ***complex linear space****.*

Definition F.2. *Let* $\boldsymbol{L}_1 \triangleq \left(X_1,\, +,\, \cdot,\, (\mathbb{F}_1,\, \dot{+},\, \dot{\times})\right)$ *and* $\boldsymbol{L}_2 \triangleq \left(X_2,\, +,\, \cdot,\, (\mathbb{F}_2,\, \dot{+},\, \dot{\times})\right)$.

DEF

$\boldsymbol{\Omega}_2$ *is a* ***linear subspace*** *of* $\boldsymbol{\Omega}_1$ *if*

1. $\boldsymbol{L}_1$ *is a* LINEAR SPACE (Definition F.1 page 185) *and*
2. $\boldsymbol{L}_2$ *is a* LINEAR SPACE (Definition F.1 page 185) *and*
3. $\mathbb{F}_2 \subseteq \mathbb{F}_1$ *and*
4. $X_2 \subseteq X_1$ *and*

Remark F.1. [3] By the first four conditions ($*$]) listed in Definition F.1, $(X, +)$ is a **commutative group** (or **abelian group**).

[3] Akhiezer and Glazman (1993), page 1, Haaser and Sullivan (1991), page 41

Often when discussing a linear space, the operator $\cdot$ is simply expressed with juxtaposition (e.g. $\alpha\mathbf{x}$ is equivalent to $\alpha\cdot\mathbf{x}$). In doing this, there is no risk of ambiguity between scalar-vector multiplication and scalar-scalar multiplication because the operands uniquely identify the precise operator.[4]

Example F.1 (tuples in $\mathbb{F}^N$). [5] Let $(\!|x_n|\!)_1^N$ be an N-*tuple* (Definition P.1 page 329) over a *field* $(\mathbb{F},+,\cdot,0,1)$.

EX
$$\begin{array}{rrcll} \text{Let} & X & \triangleq & \left\{(\!|x_n|\!)_1^N \,|\, x_n\in\mathbb{F}\right\} & & \text{and}\\ & (\!|x_n|\!)_1^N+(\!|y_n|\!)_1^N & \triangleq & (\!|x_n\dot{+}y_n|\!)_1^N & \forall (\!|x_n|\!)\in X & \text{and}\\ & \alpha\cdot(\!|x_n|\!)_1^N & \triangleq & (\!|\alpha\dot{\times}x_n|\!)_1^N & \forall (\!|x_n|\!)\in X,\alpha\in\mathbb{F}. & \end{array}$$
Then the structure $\left(X,\,+,\,\cdot,\,(\mathbb{F},\,\dot{+},\,\dot{\times})\right)$ is a *linear space*.

Example F.2 (real numbers). [6] Let $(\mathbb{R},+,\cdot,0,1)$ be the field of real numbers.

EX
The structure $(\mathbb{R},\,+,\,\cdot,\,(\mathbb{R},\,+,\,\cdot))$ is a *linear space*.
That is, the field of real numbers forms a linear space over itself.

Example F.3 (functions). [7] Let $(\mathbb{F},+,\cdot,0,1)$ be a field. Let Y^X be the set of all functions with domain X and range Y.

EX
$$\begin{array}{rrcllr} \text{Let} & [\mathsf{f}+\mathsf{g}](x) & \triangleq & \mathsf{f}(x)+\mathsf{g}(x) & \forall \mathsf{f},\mathsf{g}\in Y^X & (\textit{pointwise addition}) \quad \text{and}\\ & [\alpha\cdot\mathsf{f}](x) & \triangleq & \alpha\cdot[\mathsf{f}(x)] & \forall \mathsf{f}\in Y^X,\alpha\in\mathbb{F}. & \end{array}$$
Then the structure $\left(Y^X,\,+,\,\cdot,\,(\mathbb{F},\,\dot{+},\,\dot{\times})\right)$ is a *linear space*.

Example F.4 (functions onto $\mathbb{F}$). [8] Let $(\mathbb{F},+,\cdot,0,1)$ be a field. Let $\mathbb{F}^{\boldsymbol{X}}$ be the set of all functions with domain X and range $\mathbb{F}$.

EX
$$\begin{array}{rrcllr} \text{Let} & [\mathsf{f}+\mathsf{g}](x) & \triangleq & \mathsf{f}(x)+\mathsf{g}(x) & \forall \mathsf{f},\mathsf{g}\in \mathbb{F}^{\boldsymbol{X}} & (\textit{pointwise addition}) \quad \text{and}\\ & [\alpha\cdot\mathsf{f}](x) & \triangleq & \alpha\cdot[\mathsf{f}(x)] & \forall \mathsf{f}\in \mathbb{F}^{\boldsymbol{X}},\alpha\in\mathbb{F}. & \end{array}$$
Then the structure $\left(\mathbb{F}^{\boldsymbol{X}},\,+,\,\cdot,\,(\mathbb{F},\,+,\,\cdot)\right)$ is a *linear space*.

Theorem F.1 (Additive identity properties). [9] *Let* $\left(X,\,+,\,\cdot,\,(\mathbb{F},\,\dot{+},\,\dot{\times})\right)$ *be a linear space,* 0 *the* ADDITIVE IDENTITY ELEMENT *(Definition* **??** *page* **??***) with respect to* $\dot{+}$*, and* $\mathbb{0}$ *the* ADDITIVE IDENTITY ELEMENT *with respect to* $+$.

THM
1. $0\mathbf{x} = \mathbb{0}$ $\quad\forall \mathbf{x}\in X$
2. $\alpha\mathbb{0} = \mathbb{0}$ $\quad\forall \alpha\in\mathbb{F}$
3. $\alpha\mathbf{x} = \mathbb{0} \implies \alpha = 0$ *or* $\mathbf{x} = \mathbb{0}$
4. $\mathbf{x}+\mathbf{x} = \mathbf{x} \implies \mathbf{x} = \mathbb{0}$
5. $\alpha\neq 0$ *and* $\mathbf{x}\neq\mathbb{0} \implies \alpha\mathbf{x}\neq\mathbb{0}$

✎PROOF:

[4] *Operator overload* is a technique in which two fundamentally different operators or functions share the same symbol or label. It is inherent in the programming language C++ and is therein called *operator overload*. In C++, you can define two (or more) operators or functions that share the same symbol or name, but yet are completely different. Two such operators (or functions) are distinguished from each other by the type of their operands. So for example, in C++, you can define an $m\times n$ matrix *type* and use operator overload to define a $+$ operator that operates on this new matrix type. So if variables x and y are of floating point type and A and B are of the matrix type, you can then add either type using the same syntax style:

x+y (add two floating point numbers)
A+B (add two matrices)

Even though both of these operations "look" the same, they are of course fundamentally different.

[5] Kubrusly (2001) page 41 ⟨Example 2D⟩

[6] Kubrusly (2001) page 41 ⟨Example 2D⟩, Hamel (1905)

[7] Kubrusly (2001) page 42 ⟨Example 2F⟩

[8] Kubrusly (2001) page 41 ⟨Example 2E⟩

[9] Berberian (1961) page 6 ⟨Theorem 1⟩, Michel and Herget (1993) page 77

1. Proof that $0\mathbf{x} = \mathbb{0}$:

$$\begin{aligned}
0\mathbf{x} &= 0\mathbf{x} + 0\mathbb{0} && \text{by definition of + additive identity element}\\
&= 0\mathbf{x} + 0\mathbf{x} + (-0\mathbf{x}) && \text{by definition of + additive inverse}\\
&= (0+0)\mathbf{x} + (-0\cdot\mathbf{x}) && \text{by definition of + additive identity element}\\
&= 0\mathbf{x} + (-0\mathbf{x}) && \text{by Definition F.1 property 4}\\
&= \mathbb{0} && \text{by definition of + additive identity element}
\end{aligned}$$

2. Proof that $\alpha\mathbb{0} = \mathbb{0}$:

$$\begin{aligned}
\alpha\mathbb{0} &= \alpha(0\mathbf{x}) && \text{by item 1}\\
&= (\alpha 0)\mathbf{x} && \text{by Definition F.1 property 6}\\
&= 0\mathbf{x}\\
&= \mathbb{0} && \text{by item 1}
\end{aligned}$$

3. Proof that $\alpha \neq 0$ and $\mathbf{x} \neq \mathbb{0} \implies \alpha\mathbf{x} \neq \mathbb{0}$: Suppose $\alpha\mathbf{x} = \mathbb{0}$. Then

$$\begin{aligned}
\mathbf{x} &= \left(\frac{1}{\alpha}\alpha\right)\mathbf{x}\\
&= \frac{1}{\alpha}(\alpha\mathbf{x})\\
&= \frac{1}{\alpha}\mathbb{0}\\
&= \mathbb{0} && \text{by item 2}\\
\implies \mathbf{x} &= \mathbb{0}
\end{aligned}$$

This is a *contradiction* and so $\alpha\mathbf{x} \neq \mathbb{0}$.

4. Proof that $\alpha\mathbf{x} = \mathbb{0} \implies \alpha = 0$ or $\mathbf{x} = \mathbb{0}$: contrapositive argument of item 3

5. Proof that $\mathbf{x} + \mathbf{x} = \mathbf{x} \implies \mathbf{x} = \mathbb{0}$:

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} + \mathbb{0} && \text{by *additive identity* property (Definition F.1 page 185)}\\
&= \mathbf{x} + [\mathbf{x} + (-\mathbf{x})] && \text{by *additive inverse* property (Definition F.1 page 185)}\\
&= [\mathbf{x} + \mathbf{x}] + (-\mathbf{x}) && \text{by *associative* property (Definition F.1 page 185)}\\
&= \mathbf{x} + (-\mathbf{x}) && \text{by left hypothesis}\\
&= \mathbb{0} && \text{by *additive inverse* property (Definition F.1 page 185)}
\end{aligned}$$

Definition F.3. [10] *Let $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a linear space with vectors $\mathbf{x}, \mathbf{y} \in X$. Let $-\mathbf{y}$ be the additive inverse of $\mathbf{y}$ such that $\mathbf{y} + (-\mathbf{y}) = \mathbb{0}$.*

DEF: *The **difference** of $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x} + (-\mathbf{y})$ and is denoted $\mathbf{x} - \mathbf{y}$.*

Theorem F.2 (Additive inverse properties). [11] *Let $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a linear space, $\mathbb{0}$ the* ADDITIVE IDENTITY ELEMENT *with respect to +, and $-\mathbf{x}$ the* ADDITIVE INVERSE *(Definition ?? page ??) of $\mathbf{x}$ with respect to +.*

THM:

1.	$\mathbf{x} + \mathbf{y}$	$=$	$\mathbb{0} \implies \mathbf{x} = -\mathbf{y}$	$\forall \mathbf{x},\mathbf{y} \in X$	(additive inverse is UNIQUE)
2.	$(-\alpha)\mathbf{x}$	$=$	$-(\alpha\mathbf{x}) = \alpha(-\mathbf{x})$	$\forall \mathbf{x} \in X, \alpha \in \mathbb{F}$	
3.	$\alpha(\mathbf{x} - \mathbf{y})$	$=$	$\alpha\mathbf{x} - \alpha\mathbf{y}$	$\forall \mathbf{x},\mathbf{y} \in X, \alpha \in \mathbb{F}$	(DISTRIBUTIVE)
4.	$(\alpha - \beta)\mathbf{x}$	$=$	$\alpha\mathbf{x} - \beta\mathbf{x}$	$\forall \mathbf{x} \in X, \alpha,\beta \in \mathbb{F}$	(DISTRIBUTIVE)

[10] Berberian (1961) page 7 ⟨Definition 1⟩

[11] Berberian (1961) page 7 ⟨Corollary 1⟩, Michel and Herget (1993) page 77, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

✎PROOF:

1. Proof that $x + y = 0 \implies x = -y$:

$$\begin{aligned}
x &= x - \mathbb{0} \\
&= x - (x + y) && \text{by left hypothesis} \\
&= (x - x) - y \\
&= \mathbb{0} - y \\
&= -y
\end{aligned}$$

2. Proof that $(-\alpha)x = -(\alpha x)$:

$$\begin{aligned}
\mathbb{0} &= 0x && \text{by Theorem F.1 page 187} \\
&= (\alpha - \alpha)x && \text{by field property of } \mathbb{F} \\
&= [\alpha + (-\alpha)]x && \text{by field property of } \mathbb{F} \\
&= \alpha x + (-\alpha)x && \text{by Definition F.1 page 185} \\
\implies -(\alpha x) &= (-\alpha)x && \text{by item (1) page 189}
\end{aligned}$$

3. Proof that $\alpha(-x) = -(\alpha x)$:

$$\begin{aligned}
\mathbb{0} &= \alpha\mathbb{0} && \text{by Theorem F.1 page 187} \\
&= \alpha[x + (-x)] && \text{by definition of additive identity element } -x \\
&= \alpha x + \alpha(-x) && \text{by Definition F.1 page 185} \\
&= \alpha x + \alpha(-x) \\
\implies -(\alpha x) &= \alpha(-x) && \text{by item (1) page 189}
\end{aligned}$$

4. Proof that $\alpha(x - y) = \alpha x - \alpha y$:

$$\begin{aligned}
\alpha(x - y) &= \alpha\left[x + (-y)\right] && \text{by Definition F.3 page 188} \\
&= \alpha x + \alpha(-y) && \text{by Definition F.1 page 185} \\
&= \alpha x + (-\alpha y) && \text{by item (3) page 189} \\
&= \alpha x - \alpha y && \text{by Definition F.3 page 188}
\end{aligned}$$

5. Proof that $(\alpha - \beta)x = \alpha x - \beta x$:

$$\begin{aligned}
(\alpha - \beta)x &= [\alpha + (-\beta)]x && \text{by field properties of } \mathbb{F} \\
&= \alpha x + (-\beta)x && \text{by Definition F.1} \\
&= \alpha x + [-(\beta x)] && \text{by item (2) page 189} \\
&= \alpha x - (\beta x) && \text{by Definition F.3 page 188}
\end{aligned}$$

✏

Theorem F.3. [12] *Let $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a linear space, $\mathbb{0}$ the additive identity element with respect to $+$, and $-x$ additive inverse of x with respect to $+$.*

THM				
1.	$\alpha x = \alpha y$ and $\alpha \neq 0$	$\implies x = y$	$\forall x,y \in X$
2.	$\alpha x = \beta x$ and $x \neq \mathbb{0}$	$\implies \alpha = \beta$	$\forall x,y \in X,\ \alpha,\beta \in \mathbb{F}$
3.	$z + x = z + y$	$\implies x = y$	$\forall x,y,z \in X$

[12] Michel and Herget (1993) page 77, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b) pages 119–120, Banach (1922) pages 134–135

✎PROOF:

1. Proof that $\alpha x = \alpha y$ and $\alpha \neq 0 \implies x = y$:

$$\begin{aligned}
\mathbb{0} &= \frac{1}{\alpha}(\mathbb{0}) && \text{by left hypothesis } (\alpha \neq 0)\\
&= \frac{1}{\alpha}\Big(\alpha x - \alpha y\Big) && \text{by left hypothesis } (\alpha x = \alpha y)\\
&= \frac{1}{\alpha}\alpha\Big(x - y\Big) && \text{by Definition F.1 page 185}\\
&= x - y
\end{aligned}$$

2. Proof that $\alpha x = \beta x$ and $x \neq \mathbb{0} \implies \alpha = \beta$:

$$\begin{aligned}
\mathbb{0} &= \alpha x + (-\alpha x) && \text{by definition of additive inverse}\\
&= \beta x + (-\alpha x) && \text{by left hypothesis}\\
&= \beta x + (-\alpha) x && \text{by Theorem F.2 page 188}\\
&= [\beta + (-\alpha)] x && \text{by Definition F.1 page 185}\\
\implies \beta - \alpha &= 0 && \text{by Theorem F.1 page 187}\\
\implies \alpha &= \beta && \text{by field properties of } \mathbb{F}
\end{aligned}$$

3. Proof that $z + x = z + y \implies x = y$:

$$\begin{aligned}
\mathbb{0} &= (z + x) - (z + y)\\
&= (x + z) - (z + y) && \text{by Definition F.1 property 1}\\
&= (x + z) + [(-1)z + (-1)y] && \text{by Definition F.1 property 3}\\
&= (x + z) + (-z - y) && \text{by previous result 2.}\\
&= x + (z - z) - y\\
&= x - y
\end{aligned}$$

✏

F.2 Order on Linear Spaces

Definition F.4. [13] *Let $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times})\big)$ be a real linear space.*

DEF

*The pair $(\boldsymbol{\Omega}, \leq)$ is an **ordered linear space** if*

1. $x \leq y \implies x + z \leq y + z \quad \forall z \in X$ *and*
2. $x \leq y \implies \alpha x \leq \alpha y \quad \forall \alpha \in \mathbb{F}$

*A vector x is **positive** if $\mathbb{0} \leq x$.*
*The **positive cone** X^+ of $(X, \leq)$ is the set $X^+ \triangleq \{x \in X | \mathbb{0} \leq x\}$.*

Definition F.5. [14] *Let $(X, \leq)$ be an ordered linear space.*

DEF

*The tuple $L \triangleq (X, \vee, \wedge; \leq)$ is a **Riesz space** if L is a **lattice**.*
*A RIESZ SPACE is also called a **vector lattice**.*

Theorem F.4. [15] *Let $(X, \vee, \wedge; \leq)$ be a Riesz space* (Definition F.5 page 190).

THM

$x \vee y = -\big[(-x) \wedge (-y)\big]$	$x \wedge y = -\big[(-x) \vee (-y)\big]$	$\forall x, y \in X$
$x + (y \vee z) = (x + y) \vee (x + z)$	$x + (y \wedge z) = (x + y) \wedge (x + z)$	$\forall x, y, z \in X$
$\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$	$\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$	$\forall x, y \in X, \alpha \geq 0$
$x + y = (x \wedge y) + (x \vee y)$		$\forall x, y \in X, \alpha \in \mathbb{F}$

[13] Aliprantis and Burkinshaw (2006) pages 1–2
[14] Aliprantis and Burkinshaw (2006) page 2
[15] Aliprantis and Burkinshaw (2006) page 3 ⟨Theorem 1.2⟩

✎PROOF:

1. Proof that $x \vee y = -\left[(-x) \wedge (-y)\right]$:

$\begin{aligned}(-x) \wedge (-y) &\le -x\\ x &\le -\left[(-x) \wedge (-y)\right]\\ x \vee y &\le -\left[(-x) \wedge (-y)\right]\end{aligned}$	$\begin{aligned}(-x) \wedge (-y) &\le -y\\ y &\le -\left[(-x) \wedge (-y)\right]\end{aligned}$
$\begin{aligned}x &\le x \vee y\\ -(x \vee y) &\le -x\\ -(x \vee y) &\le (-x) \wedge (-y)\\ -\left[(-x) \wedge (-y)\right] &\le x \vee y\end{aligned}$	$\begin{aligned}y &\le x \vee y\\ -(x \vee y) &\le -y\end{aligned}$

2. Proof that $x \wedge y = -\left[(-x) \vee (-y)\right]$:

$x \vee y$	$=$	$-\left[(-x) \wedge (-y)\right]$	by item (1)
$(-x) \vee (-y)$	$=$	$-\left[(-(-x)) \wedge (-(-y))\right]$	replace x with $-x$ and y with y
$(-x) \vee (-y)$	$=$	$-\left[x \wedge y\right]$	$-(-x) = x$
$-\left[x \wedge y\right]$	$=$	$(-x) \vee (-y)$	by symmetry of $=$ relation
$x \wedge y$	$=$	$-\left[(-x) \vee (-y)\right]$	multipy both sides by -1

3. Proof that $x + (y \vee z) = (x + y) \vee (x + z)$:

$\begin{aligned}x + y &\le x + (y \vee z)\\ (x + y) \vee (x + z) &\le x + (y \vee z)\end{aligned}$	$x + z \le x + (y \vee z)$
$\begin{aligned}y &= -x + (x + y)\\ &\le -x + \left[(x + y) \vee (x + z)\right]\\ y \vee z &\le -x + \left[(x + y) \vee (x + z)\right]\\ x + (y \vee z) &\le (x + y) \vee (x + z)\end{aligned}$	$\begin{aligned}z &= -x + (x + z)\\ &\le -x + \left[(x + y) \vee (x + z)\right]\end{aligned}$

4. Proof that $x + (y \wedge z) = (x + y) \wedge (x + z)$:

$\begin{aligned}x + y &\ge x + (y \wedge z)\\ (x + y) \wedge (x + z) &\ge x + (y \wedge z)\end{aligned}$	$x + z \ge x + (y \wedge z)$
$\begin{aligned}y &= -x + (x + y)\\ &\ge -x + \left[(x + y) \wedge (x + z)\right]\\ y \wedge z &\ge -x + \left[(x + y) \wedge (x + z)\right]\\ x + (y \wedge z) &\ge (x + y) \wedge (x + z)\end{aligned}$	$\begin{aligned}z &= -x + (x + z)\\ &\ge -x + \left[(x + y) \wedge (x + z)\right]\end{aligned}$

5. Proof that $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$ for $\alpha \ge 0$:

$\begin{aligned}x &\le x \vee y\\ \alpha x &\le \alpha(x \vee y)\\ (\alpha x) \vee (\alpha y) &\le \alpha(x \vee y)\end{aligned}$	$\begin{aligned}y &\le x \vee y\\ \alpha y &\le \alpha(x \vee y)\end{aligned}$	by Definition F.4 page 190
$\begin{aligned}\alpha x &\le (\alpha x) \vee (\alpha y)\\ x &\le \alpha^{-1}(\alpha x) \vee (\alpha y)\\ x \vee y &\le \alpha^{-1}(\alpha x) \vee (\alpha y)\\ \alpha(x \vee y) &\le (\alpha x) \vee (\alpha y)\end{aligned}$	$\begin{aligned}\alpha y &\le (\alpha x) \vee (\alpha y)\\ y &\le \alpha^{-1}(\alpha x) \vee (\alpha y)\end{aligned}$	

6. Proof that $\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$ for $\alpha \ge 0$:

$\begin{aligned}x &\ge x \wedge y\\ \alpha x &\ge \alpha(x \wedge y)\\ (\alpha x) \wedge (\alpha y) &\ge \alpha(x \wedge y)\end{aligned}$	$\begin{aligned}y &\ge x \wedge y\\ \alpha y &\ge \alpha(x \wedge y)\end{aligned}$	by Definition F.4 page 190

$\begin{aligned} \alpha x &\geq (\alpha x) \wedge (\alpha y) \\ x &\geq \alpha^{-1}(\alpha x) \wedge (\alpha y) \\ x \wedge y &\geq \alpha^{-1}(\alpha x) \wedge (\alpha y) \\ \alpha(x \wedge y) &\geq (\alpha x) \wedge (\alpha y) \end{aligned}$	$\begin{aligned} \alpha y &\geq (\alpha x) \wedge (\alpha y) \\ y &\geq \alpha^{-1}(\alpha x) \wedge (\alpha y) \end{aligned}$	

7. Proof that $x + y = (x \wedge y) + (x \vee y)$:

$\begin{aligned} x &\leq x \vee y \\ x + y &\leq (x \vee y) + y \\ x + y - (x \vee y) &\leq y \\ x + y - (x \vee y) &\leq x \wedge y \\ x + y &\leq (x \vee y) + (x \wedge y) \end{aligned}$	$\begin{aligned} y &\leq x \vee y \\ x + vy &\leq x + (x \vee y) \\ x + vy - (x \vee y) &\leq x \end{aligned}$
$\begin{aligned} x \wedge y &\leq x \\ 0 &\leq x - (x \wedge y) \\ y &\leq y + x - (x \wedge y) \\ y &\leq x + y - (x \wedge y) \\ x \vee y &\leq x + y - (x \wedge y) \\ (x \wedge y) + (x \vee y) &\leq x + y \end{aligned}$	$\begin{aligned} x \wedge y &\leq y \\ 0 &\leq y - (x \wedge y) \\ x &\leq x + y - (x \wedge y) \\ x &\leq x + y - (x \wedge y) \end{aligned}$

⊨

Definition F.6. [16] *Let* $(X, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition F.5 page 190).

DEF

x^+	*is defined as* $x^+ \triangleq x \vee \mathbb{0}$	*and is called the* ***positive part*** *of* x.
x^-	*is defined as* $x^- \triangleq (-x) \vee \mathbb{0}$	*and is called the* ***negative part*** *of* x.
$\lvert x\rvert$	*is defined as* $\lvert x\rvert \triangleq x \vee (-x)$	*and is called the* ***absolute value*** *of* x.

Theorem F.5. [17] *Let* $(X, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition F.5 page 190).

THM

$$\left.\begin{aligned} y - z &= x \quad \text{and} \\ y \wedge z &= \mathbb{0} \end{aligned}\right\} \iff \left\{\begin{aligned} y &= x^+ \quad \text{and} \\ z &= x^- \end{aligned}\right.$$

✎PROOF:

1. Proof that left hypothesis $\implies$ right hypothesis:

$$\begin{aligned}
x^+ &= x \vee \mathbb{0} && \text{by definition of } x^+ \text{ Definition F.6 page 192} \\
&= (y - z) \vee \mathbb{0} && \text{by left hypothesis} \\
&= (y - z) \vee (z - z) \\
&= (y \vee z) - z && \text{by Theorem F.4 page 190} \\
&= \left[y + z - (y \wedge z)\right] - z && \text{by Theorem F.4 page 190} \\
&= y - (y \wedge z) \\
&= y - \mathbb{0} && \text{by left hypothesis} \\
&= y \\
x^- &= (-x) \vee \mathbb{0} && \text{by definition of } x^- \text{ Definition F.6 page 192} \\
&= (z - y) \vee \mathbb{0} && \text{by left hypothesis} \\
&= (z - y) \vee (y - y) \\
&= (z \vee y) - y && \text{by Theorem F.4 page 190}
\end{aligned}$$

[16] Aliprantis and Burkinshaw (2006) page 4, Istrăţescu (1987) page 129
[17] Aliprantis and Burkinshaw (2006) page 4 ⟨Theorem 1.3⟩

$$= [z + y - (z \wedge y)] - z \quad \text{by Theorem F.4 page 190}$$
$$= z - (z \wedge y)$$
$$= z - \mathbb{0} \quad \text{by left hypothesis}$$
$$= z$$

2. Proof that left hypothesis $\Longleftarrow$ right hypothesis:

$$\begin{aligned}
y - z &= x^+ - x^- && \text{by right hypothesis}\\
&= [x \vee \mathbb{0}] - [(-x) \vee \mathbb{0}] && \text{by Definition F.6 page 192}\\
&= (x \vee \mathbb{0}) + (x \wedge \mathbb{0}) && \text{by Theorem F.4 page 190}\\
&= x && \text{by Theorem F.4 page 190}\\
y \wedge z &= x^+ \wedge x^- && \text{by right hypothesis}\\
&= [x^- + (x^+ - x^-)] \wedge [x^- + \mathbb{0}]\\
&x^- + [(x^+ - x^-) \wedge \mathbb{0}] && \text{by Theorem F.4 page 190}\\
&x^- + [(y - z) \wedge \mathbb{0}] && \text{by right hypothesis}\\
&x^- + (x \wedge \mathbb{0}) && \text{by previous result}\\
&x^- - [-x \vee \mathbb{0}] && \text{by Theorem F.4 page 190}\\
&x^- - x^- && \text{by definition of } x^- \text{ (Definition F.6 page 192)}\\
&\mathbb{0}
\end{aligned}$$

⇨

Theorem F.6. [18] *Let* $(\boldsymbol{X}, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (*Definition F.5 page 190*). *Let* x^+ *the* POSITIVE PART *of* $x \in \boldsymbol{X}$, x^- *the* NEGATIVE PART *of* $x \in \boldsymbol{X}$, *and* $|x|$ *the* ABSOLUTE VALUE (*Definition F.6 page 192*) *of* $x \in \boldsymbol{X}$.

THM
$$\begin{aligned}
|x| &= x^+ + x^- && \forall x \in \boldsymbol{X}\\
x &= (x - y)^+ + (x \wedge y) && \forall x \in \boldsymbol{X}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
|x| &= x \vee (-x) && \text{by definition of } |x| \text{ (Definition F.6 page 192)}\\
&= (2x - x) \vee (\mathbb{0} - x)\\
&= (-x + 2x) \vee (-x + \mathbb{0}) && \text{by \textit{commutative} property (Definition F.1 page 185)}\\
&= (-x) + (2x \vee \mathbb{0}) && \text{by Theorem F.4 page 190}\\
&= (2x \vee 2\mathbb{0}) - x && \text{by the \textit{commutative} property (Definition F.1 page 185)}\\
&= 2(x \vee \mathbb{0}) - x && \text{by Theorem F.4 page 190}\\
&= 2x^+ - x && \text{by definition of } x^+ \text{ (Definition F.6 page 192)}\\
&= 2x^+ - (x^+ - x^-) && \text{by 1}\\
&= x^+ + x^-\\
x &= x + \mathbb{0} && x + y - y\\
&= (x \vee y) + (x \wedge y) - y && \text{by Theorem F.4 page 190}\\
&= [(x - y) \vee (y - y)] + (x \wedge y) && \text{by Theorem F.4 page 190}\\
&= [(x - y) \vee \mathbb{0}] + (x \wedge y)\\
&= (x - y)^+ + (x \wedge y) && \text{by definition of } x^+ \text{ (Definition F.6 page 192)}
\end{aligned}$$

⇨

[18] Aliprantis and Burkinshaw (2006) page 4

Theorem F.7. [19] *Let* $(\boldsymbol{X}, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition F.5 page 190). *Let* $\boldsymbol{x}^+$ *the* POSITIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}^-$ *the* NEGATIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, *and* $|\boldsymbol{x}|$ *the* ABSOLUTE VALUE (Definition F.6 page 192) *of* $\boldsymbol{x} \in \boldsymbol{X}$.

THM

$$\begin{array}{llcll}
1. & \boldsymbol{x} \vee \boldsymbol{y} & = & \frac{1}{2}(\boldsymbol{x}+\boldsymbol{y}+|\boldsymbol{x}-\boldsymbol{y}|) & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}\\
2. & \boldsymbol{x} \wedge \boldsymbol{y} & = & \frac{1}{2}(\boldsymbol{x}+\boldsymbol{y}-|\boldsymbol{x}-\boldsymbol{y}|) & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}\\
3. & |\boldsymbol{x}-\boldsymbol{y}| & = & (\boldsymbol{x} \vee \boldsymbol{y}) - (\boldsymbol{x} \wedge \boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}\\
4. & |\boldsymbol{x}| \vee |\boldsymbol{y}| & = & \frac{1}{2}(|\boldsymbol{x}+\boldsymbol{y}|+|\boldsymbol{x}-\boldsymbol{y}|) & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}\\
5. & |\boldsymbol{x}| \vee |\boldsymbol{y}| & = & \frac{1}{2}\big||\boldsymbol{x}+\boldsymbol{y}|-|\boldsymbol{x}-\boldsymbol{y}|\big| & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}
\end{array}$$

✎PROOF:

$$\begin{aligned}
(\boldsymbol{x}+\boldsymbol{y}+|\boldsymbol{x}-\boldsymbol{y}|) &= (\boldsymbol{x}+\boldsymbol{y}) + \big[(\boldsymbol{x}-\boldsymbol{y}) \vee (\boldsymbol{y}-\boldsymbol{x})\big] && \text{by Definition F.6 page 192}\\
&= \big[(\boldsymbol{x}+\boldsymbol{y})+(\boldsymbol{x}-\boldsymbol{y})\big] \vee \big[(\boldsymbol{x}+\boldsymbol{y})+(\boldsymbol{y}-\boldsymbol{x})\big] && \text{by Theorem F.4 page 190}\\
&= (2\boldsymbol{x}) \vee (2\boldsymbol{y})\\
&= 2(\boldsymbol{x} \vee \boldsymbol{y}) && \text{by Theorem F.4 page 190}\\
(\boldsymbol{x}+\boldsymbol{y}-|\boldsymbol{x}-\boldsymbol{y}|) &= (\boldsymbol{x}+\boldsymbol{y}) - \big[(\boldsymbol{x}-\boldsymbol{y}) \vee (\boldsymbol{y}-\boldsymbol{x})\big] && \text{by Definition F.6 page 192}\\
&= (\boldsymbol{x}+\boldsymbol{y}) - \big[(-(\boldsymbol{y}-\boldsymbol{x})) \vee (-(\boldsymbol{x}-\boldsymbol{y}))\big]\\
&= (\boldsymbol{x}+\boldsymbol{y}) + \big[(\boldsymbol{y}-\boldsymbol{x}) \wedge (\boldsymbol{x}-\boldsymbol{y})\big] && \text{by Theorem F.4 page 190}\\
&= \big[(\boldsymbol{x}+\boldsymbol{y})+(\boldsymbol{y}-\boldsymbol{x})\big] \wedge \big[(\boldsymbol{x}+\boldsymbol{y})+(\boldsymbol{x}-\boldsymbol{y})\big] && \text{by Theorem F.4 page 190}\\
&= (2\boldsymbol{y}) \wedge (2\boldsymbol{x})\\
&= 2(\boldsymbol{y} \wedge \boldsymbol{x}) && \text{by Theorem F.4 page 190}\\
&= 2(\boldsymbol{x} \wedge \boldsymbol{y})\\
|\boldsymbol{x}-\boldsymbol{y}| &= \frac{1}{2}(\boldsymbol{x}+\boldsymbol{y}+|\boldsymbol{x}-\boldsymbol{y}|) - \frac{1}{2}(\boldsymbol{x}+\boldsymbol{y}+|\boldsymbol{x}-\boldsymbol{y}|)\\
&= (\boldsymbol{x} \vee \boldsymbol{y}) - (\boldsymbol{x} \wedge \boldsymbol{y}) && \text{by 1 and 2}\\
|\boldsymbol{x}+\boldsymbol{y}| + |\boldsymbol{x}-\boldsymbol{y}| &= \frac{1}{2}(\mathbb{0} + |2\boldsymbol{x}+2\boldsymbol{y}|) + |\boldsymbol{x}-\boldsymbol{y}|\\
&= \frac{1}{2}\big[(\boldsymbol{x}+\boldsymbol{y}) + (-\boldsymbol{x}-\boldsymbol{y}) + |(\boldsymbol{x}+\boldsymbol{y}) - (-\boldsymbol{x}-\boldsymbol{y})|\big] + |\boldsymbol{x}-\boldsymbol{y}|\\
&= \big[(\boldsymbol{x}+\boldsymbol{y}) \vee (-\boldsymbol{x}-\boldsymbol{y})\big] + |\boldsymbol{x}-\boldsymbol{y}| && \text{by 1}\\
&= \big[(\boldsymbol{x}+\boldsymbol{y}) + |\boldsymbol{x}-\boldsymbol{y}|\big] \vee \big[(-\boldsymbol{x}-\boldsymbol{y}) + |\boldsymbol{x}-\boldsymbol{y}|\big] && \text{by Theorem F.4 page 190}\\
&= 2(\boldsymbol{x} \vee \boldsymbol{y}) \vee 2\big[(-\boldsymbol{y}) + (-\boldsymbol{x}) + |(-\boldsymbol{y}) - (-\boldsymbol{x})|\big] && \text{by 1}\\
&= 2(\boldsymbol{x} \vee \boldsymbol{y}) \vee 2\big[(-\boldsymbol{y}) \vee (-\boldsymbol{x})\big] && \text{by 1}\\
&= 2([\boldsymbol{x} \vee (-\boldsymbol{x})] \vee (\boldsymbol{y} \vee (-\boldsymbol{y})))\\
&= 2(|\boldsymbol{x}| \vee |\boldsymbol{y}|) && \text{by Definition F.6 page 192}\\
\big||\boldsymbol{x}+\boldsymbol{y}| - |\boldsymbol{x}-\boldsymbol{y}|\big| &= 2(|\boldsymbol{x}+\boldsymbol{y}| \vee |\boldsymbol{x}-\boldsymbol{y}|) - (|\boldsymbol{x}+\boldsymbol{y}| + |\boldsymbol{x}-\boldsymbol{y}|) && \text{by 1}\\
&= (|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{x}-\boldsymbol{y}| + |\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{y}|) - 2(|\boldsymbol{x}| \vee |\boldsymbol{y}|) && \text{by 3}\\
&= 2(|\boldsymbol{x}|+|\boldsymbol{y}|) - 2(|\boldsymbol{x}| \vee |\boldsymbol{y}|)\\
&= 2(|\boldsymbol{x}| \vee |\boldsymbol{y}|) && \text{by Theorem F.4 page 190}
\end{aligned}$$

✎

Definition F.7. [20] *Let* $(\boldsymbol{X}, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition F.5 page 190). *Let* $\boldsymbol{x}^+$ *the* POSITIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}^-$ *the* NEGATIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, *and* $|\boldsymbol{x}|$ *the* ABSOLUTE VALUE (Definition F.6 page 192) *of* $\boldsymbol{x} \in \boldsymbol{X}$.

DEF

$\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are* ***disjoint***, *denoted by* $\boldsymbol{x} \perp \boldsymbol{y}$, *if*

$$|\boldsymbol{x}| \wedge |\boldsymbol{y}| = \mathbb{0}.$$

Two subsets $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *of* $\boldsymbol{X}$ *are* ***disjoint***, *denoted by* $\boldsymbol{U} \perp \boldsymbol{V}$ *if*

$$\boldsymbol{x} \perp \boldsymbol{y} \qquad \forall \boldsymbol{x} \in \boldsymbol{U} \text{ and } \boldsymbol{y} \in \boldsymbol{V}$$

[19] Aliprantis and Burkinshaw (2006) page 5 ⟨Theorem 1.4⟩
[20] Aliprantis and Burkinshaw (2006) page 5

Definition F.8. [21] *Let* $(\boldsymbol{X}, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition F.5 page 190). *Let* $\boldsymbol{x}^+$ *the* POSITIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}^-$ *the* NEGATIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, *and* $|\boldsymbol{x}|$ *the* ABSOLUTE VALUE (Definition F.6 page 192) *of* $\boldsymbol{x} \in \boldsymbol{X}$. *Let* $\boldsymbol{Y}$ *be a subset of* $\boldsymbol{X}$.

DEF

$\boldsymbol{Y}^d$ *is the* ***disjoint complement*** *of* $\boldsymbol{Y}$ *if* $\quad \boldsymbol{Y}^d \triangleq \{\boldsymbol{x} \in \boldsymbol{X} | \boldsymbol{x} \perp \boldsymbol{y} \quad \forall \boldsymbol{y} \in \boldsymbol{Y}\}$.

The quantity $\boldsymbol{Y}^{dd}$ *is defined as* $\left(\boldsymbol{Y}^d\right)^d$.

Definition F.9. [22] *Let* $(\boldsymbol{X}, \vee, \wedge; \leq)$ *be a* RIESZ SPACE (Definition F.5 page 190). *Let* $\boldsymbol{x}^+$ *the* POSITIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}^-$ *the* NEGATIVE PART *of* $\boldsymbol{x} \in \boldsymbol{X}$, *and* $|\boldsymbol{x}|$ *the* ABSOLUTE VALUE (Definition F.6 page 192) *of* $\boldsymbol{x} \in \boldsymbol{X}$.

DEF

$$\begin{aligned}
|A| &\triangleq \{|\boldsymbol{a}| | \boldsymbol{a} \in A\} \\
A^+ &\triangleq \left\{\boldsymbol{a}^+ | \boldsymbol{a} \in A\right\} \\
A^- &\triangleq \{\boldsymbol{a}^- | \boldsymbol{a} \in A\} \\
A \vee B &\triangleq \left\{\boldsymbol{a} \vee \boldsymbol{b} | \boldsymbol{a} \in A \text{ and } \boldsymbol{b} \in B\right\} \\
A \wedge B &\triangleq \left\{\boldsymbol{a} \wedge \boldsymbol{b} | \boldsymbol{a} \in A \text{ and } \boldsymbol{b} \in B\right\} \\
\boldsymbol{x} \vee A &\triangleq \{\boldsymbol{x} \vee \boldsymbol{a} | \boldsymbol{a} \in A\} \\
\boldsymbol{x} \wedge A &\triangleq \{\boldsymbol{x} \wedge \boldsymbol{a} | \boldsymbol{a} \in A\}
\end{aligned}$$

[21] Aliprantis and Burkinshaw (2006) page 5
[22] Aliprantis and Burkinshaw (2006) page 7

APPENDIX G

LINEAR COMBINATIONS

G.1 Linear combinations in linear spaces

A *linear space* (Definition F.1 page 185) in general is not equipped with a *topology*. Without a topology, it is not possible to determine whether an *infinite sum* of vectors converges. Therefore in this section (dealing with linear spaces), all definitions related to sums of vectors will be valid for *finite* sums(Definition L.1 page 267) only (finite "N").

Definition G.1. [1] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF: *A vector* $\boldsymbol{x} \in X$ *is a* ***linear combination*** *of the vectors in* $\{\boldsymbol{x}_n\}$ *if there exists* $\{\alpha_n \in \mathbb{F} \mid n=1,2,\ldots,N\}$ *such that* $\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$.

Definition G.2. [2] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space and* Y *be a subset of* X.

DEF: *The* ***linear span*** *of* Y *is defined as* $\operatorname{span} Y \triangleq \left\{ \sum_{\gamma \in \Gamma} \alpha_\gamma \boldsymbol{y}_\gamma \,\middle|\, \alpha_\gamma \in \mathbb{F},\ \boldsymbol{y}_\gamma \in Y \right\}$.
The set Y ***spans*** *a set* A *if* $A \subseteq \operatorname{span} Y$.

Proposition G.1. [3] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

PRP:
1. $\operatorname{span}\{\boldsymbol{x}_n\}$ *is a* LINEAR SPACE (Definition F.1 page 185) *and*
2. $\operatorname{span}\{\boldsymbol{x}_n\}$ *is a* LINEAR SUBSPACE *of* $\boldsymbol{L}$ (Definition F.2 page 186).

Definition G.3. [4] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

DEF: *The set* $Y \triangleq \{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *is* ***linearly independent*** *in* $\boldsymbol{L}$ *if*

$$\left\{\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = 0\right\} \implies \{\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0\}.$$

The set Y *is* ***linearly dependent*** *in* $\boldsymbol{L}$ *if* Y *is not linearly independent in* $\boldsymbol{L}$.

[1] Berberian (1961) page 11 ⟨Definition I.4.1⟩, Kubrusly (2001) page 46
[2] Michel and Herget (1993) page 86 ⟨3.3.7 Definition⟩, Kurdila and Zabarankin (2005) page 44, Searcóid (2002) page 71 ⟨Definition 3.2.5—more general definition⟩
[3] Kubrusly (2001) page 46
[4] Bachman and Narici (1966) pages 3–4, Christensen (2003) page 2, Heil (2011) page 156 ⟨Definition 5.7⟩

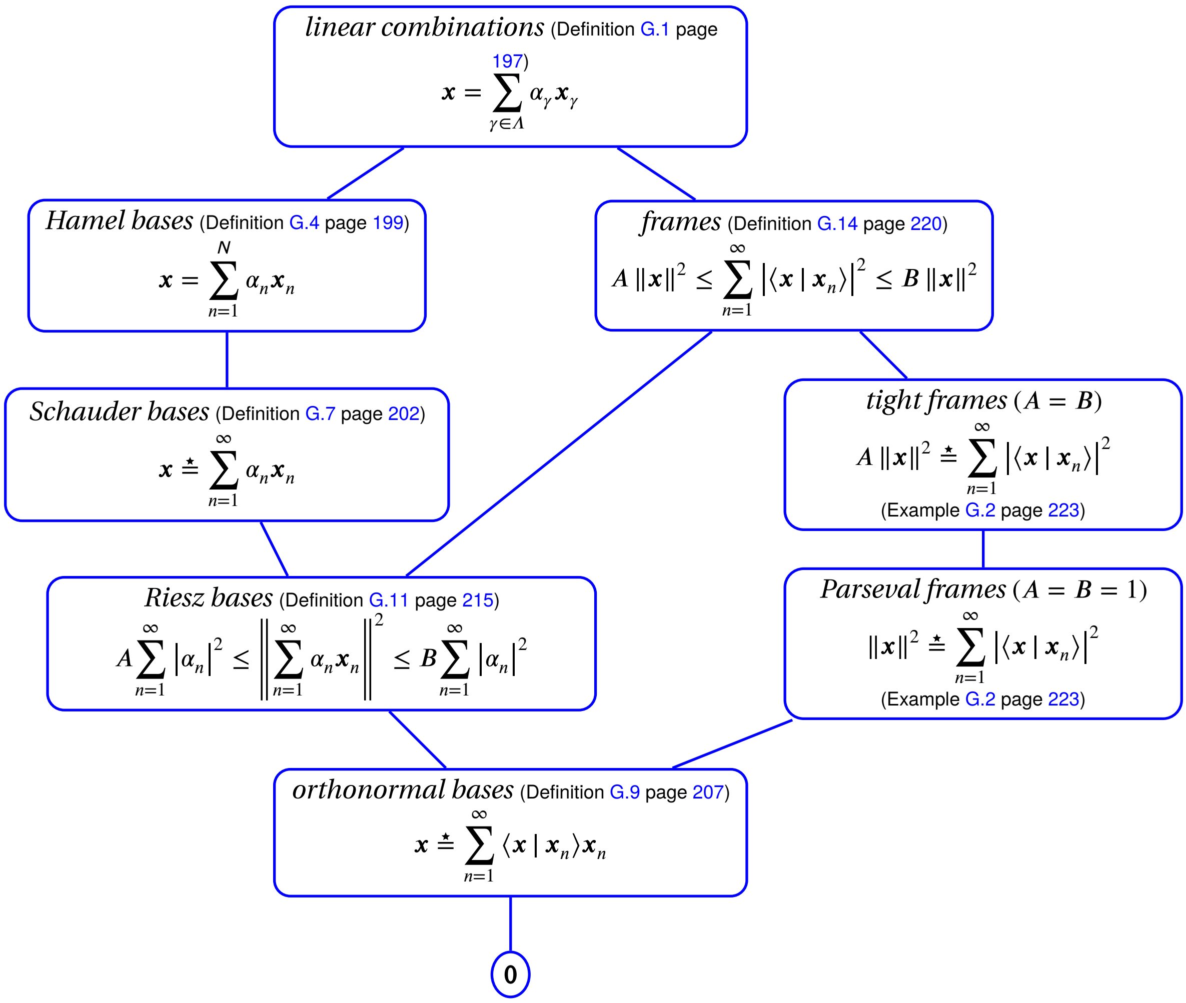


Figure G.1: Lattice of *linear combinations*

Definition G.4. [5] *Let* $\{x_n \in X \,|\, n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $\mathbf{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF

The set $\{x_n\}$ *is a* ***Hamel basis*** *for* $\mathbf{L}$ *if*

1. $\{x_n\}$ SPANS $\mathbf{L}$ (Definition G.2 page 197) *and*
2. $\{x_n\}$ *is* LINEARLY INDEPENDENT *in* $\mathbf{L}$ (Definition G.1 page 197) .

A HAMEL BASIS *is also called a* ***linear basis***.

Definition G.5. [6] *Let* $\mathbf{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* x *be a* VECTOR *in* $\mathbf{L}$ *and* $Y \triangleq \{x_n \in X \,|\, n=1,2,\ldots,N\}$ *be a set of vectors in* $\mathbf{L}$.

DEF

The expression $\sum_{n=1}^{N} \alpha_n x_n$ *is the* ***expansion*** *of* x *on* Y *in* $\mathbf{L}$ *if* $\quad x = \sum_{n=1}^{N} \alpha_n x_n$.

In this case, the sequence $(\alpha_n)_{n=1}^{N}$ *is the* ***coordinates*** *of* x *with respect to* Y *in* $\mathbf{L}$.

If $\alpha_N \neq 0$, *then* N *is the* ***dimension*** $\dim \mathbf{L}$ *of* $\mathbf{L}$.

Theorem G.1. [7] *Let* $\{x_n \,|\, n=1,2,\ldots,N\}$ *be a* HAMEL BASIS (Definition G.4 page 199) *for a* LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

THM

$$\left\{ x = \sum_{n=1}^{N} \alpha_n x_n = \sum_{n=1}^{N} \beta_n x_n \right\} \implies \underbrace{\alpha_n = \beta_n \quad \forall n=1,2,\ldots,N}_{\text{coordinates of } x \text{ are UNIQUE}} \qquad \forall x \in X$$

✎PROOF:

$$\begin{aligned}
\mathbb{0} &= x - x \\
&= \sum_{n=1}^{N} \alpha_n x_n - \sum_{n=1}^{N} \beta_n x_n \\
&= \sum_{n=1}^{N} (\alpha_n - \beta_n) x_n \\
&\implies \{x_n\} \text{ is } \textit{linearly dependent} \text{ if } (\alpha_n - \beta_n) \neq 0 \qquad \forall n = 1,2,\ldots,N \\
&\implies (\alpha_n - \beta_n) = 0 \qquad \forall n = 1,2,\ldots,N \qquad \text{(because } \{x_n\} \text{ is a } \textit{basis} \text{ and therefore must be } \textit{linearly independent}\text{)} \\
&\implies \alpha_n = \beta_n \text{ for } n = 1,2,\ldots,N
\end{aligned}$$

✏

Theorem G.2. [8] *Let* $\mathbf{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

THM

$$\left\{ \begin{array}{lll} 1. & \{x_n \in X \,|\, n=1,2,\ldots,N\} \text{ is a HAMEL BASIS for } \mathbf{L} & \text{and} \\ 2. & \{y_n \in X \,|\, n=1,2,\ldots,M\} \text{ is a set of LINEARLY INDEPENDENT vectors in } \mathbf{L} & \end{array} \right\}$$
$$\implies \left\{ \begin{array}{lll} 1. & M \leq N & \text{and} \\ 2. & M = N \implies \{y_n \,|\, n=1,2,\ldots,M\} \text{ is a BASIS for } \mathbf{L} & \text{and} \\ 3. & M \neq N \implies \{y_n \,|\, n=1,2,\ldots,M\} \text{ is NOT a basis for } \mathbf{L} & \end{array} \right\}$$

✎PROOF:

[5] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[6] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[7] Michel and Herget (1993) pages 89–90 ⟨Theorem 3.3.25⟩

[8] Michel and Herget (1993) pages 90–91 ⟨Theorem 3.3.26⟩

1. Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ is a *basis* for L:
   (a) Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ *spans* L:
      i. Because $\{x_n|_{n=1,2,\ldots,N}\}$ is a *basis* for L, there exists $\beta \in \mathbb{F}$ and $\{\alpha_n \in \mathbb{F}|_{n=1,2,\ldots,N}\}$ such that $\beta y_1 + \sum_{n=1}^{N} \alpha_n x_n = 0.$
      ii. Select an n such that $\alpha_n \neq 0$ and renumber (if necessary) the above indices such that $x_n = -\frac{\beta}{\alpha_n} y_1 + \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n} x_n.$
      iii. Then, for any $y \in X$, we can write

$$\begin{aligned}
y &= \sum_{n=1}^{N} \gamma_{n\in\mathbb{Z}} x_n \\
&= \left(\sum_{n=1}^{N-1} \gamma_{n\in\mathbb{Z}} x_n\right) + \gamma_{n\in\mathbb{Z}}\left(-\frac{\beta}{\alpha_n} y_1 - \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n} x_n\right) \\
&= -\frac{\beta\gamma_n}{\alpha_n} y_1 + \sum_{n=1}^{N-1}\left(\gamma_n - \frac{\alpha_n\gamma_n}{\alpha_n}\right) x_n \\
&= \delta y_1 + \sum_{n=1}^{N-1} \delta_{n\in\mathbb{Z}} x_n
\end{aligned}$$

      iv. This implies that $\{y_1, x_1, \ldots, x_{N-1}\}$ *spans* L:
   (b) Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly independent*:
      i. If $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly dependent*, then there exists $\{\epsilon, \epsilon_1, \ldots, \epsilon_{N-1}\}$ such that $\epsilon y_1 + \left(\sum_{n=1}^{N-1} \epsilon_{n\in\mathbb{Z}} x_n\right) + 0x_n = 0.$
      ii. item (1(b)i) implies that the coordinate of y_1 associated with x_n *is* 0. $y_1 = -\left(\sum_{n=1}^{N-1} \frac{\epsilon_n}{\epsilon} x_n\right) + 0x_n = 0.$
      iii. item (1(a)i) implies that the coordinate of y_1 associated with x_n is *not* 0. $y_1 = -\sum_{n=1}^{N} \frac{\alpha_n}{\beta} x_n.$
      iv. This implies that item (1(b)i) (that the set is linearly dependent) is *false* because item (1(b)ii) and item (1(b)iii) *contradict* each other.
      v. This implies $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly independent*.
2. Proof that $\{y_1, y_2, x_1, \ldots, x_{N-2}\}$ is a *basis*: Repeat item (1).
3. Suppose $m = n$. Proof that $\{y_1, y_2, \ldots, y_M\}$ is a *basis*: Repeat item (1) $M-1$ times.
4. Proof that $M \not> N$:
   (a) Suppose that $M = N+1$.
   (b) Then because $\{y_n|_{n=1,2,\ldots,N}\}$ is a *basis*, there exists $\{\zeta_n|_{n=1,2,\ldots,N+1}\}$ such that $\sum_{n=1}^{N+1} \zeta_{n\in\mathbb{Z}} y_{n\in\mathbb{Z}} = 0.$
   (c) This implies that $\{y_n|_{n=1,2,\ldots,N+1}\}$ is *linearly dependent*.

(d) This implies that $\{\boldsymbol{y}_n|n=1,2,\ldots,N+1\}$ is *not* a basis.

(e) This implies that $M \not> N$.

5. Proof that $M \neq N \implies \{\boldsymbol{y}_n|n=1,2,\ldots,M\}$ is *not* a basis for $\boldsymbol{L}$:

   (a) Proof that $M > N \implies \{\boldsymbol{y}_n|n=1,2,\ldots,M\}$ is *not* a basis for $\boldsymbol{L}$: same as in item (4).

   (b) Proof that $M < N \implies \{\boldsymbol{y}_n|n=1,2,\ldots,M\}$ is *not* a basis for $\boldsymbol{L}$:

      i. Suppose $m = N - 1$.

      ii. Then $\{\boldsymbol{y}_n|n=1,2,\ldots,N-1\}$ is a *basis* and there exists λ such that
      $$\sum_{n=1}^{N} \lambda_{n\in\mathbb{Z}} \boldsymbol{y}_{n\in\mathbb{Z}} = 0.$$

      iii. This implies that $\{\boldsymbol{y}_n|n=1,2,\ldots,N\}$ is *linearly dependent* and is *not* a basis.

      iv. But this contradicts item (3), therefore $M \neq N - 1$.

      v. Because $M = N$ yields a basis but $M = N - 1$ does not, $M < N - 1$ also does not yield a basis.

⇒

Corollary G.1. [9] *Let $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a linear space.*

COR

$$\underbrace{\left\{\begin{array}{ll} 1. & \{\boldsymbol{x}_n \in X|n=1,2,\ldots,N\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \quad \text{and} \\ 2. & \{\boldsymbol{y}_n \in X|n=1,2,\ldots,M\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \end{array}\right\} \implies \{N = M\}}_{\text{(all Hamel bases for } \boldsymbol{L} \text{ have the same number of vectors)}}$$

✎PROOF: This follows from Theorem G.2 (page 199). ⇒

G.2 Bases in topological linear spaces

A linear space supports the concept of the *span* of a set of vectors (Definition G.2 page 197). In a topological linear space $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$, a set A is said to be *total* in $\boldsymbol{\Omega}$ if the span of A is *dense* in $\boldsymbol{\Omega}$. In this case, A is said to be a *total set* or a *complete set*. However, this use of "complete" in a "*complete set*" is not equivalent to the use of "complete" in a "*complete metric space*". [10] In this text, except for these comments and Definition G.6, "complete" refers to the metric space definition only.

If a set is both *total* and *linearly independent* (Definition G.3 page 197) in $\boldsymbol{\Omega}$, then that set is a *Hamel basis* (Definition G.4 page 199) for $\boldsymbol{\Omega}$.

Definition G.6. [11] *Let A^- be the* CLOSURE *of a A in a* TOPOLOGICAL LINEAR SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$. *Let $\operatorname{span} A$ be the* SPAN (Definition G.2 page 197) *of a set A.*

DEF

*A set of vectors A is **total** (or **complete** or **fundamental**) in $\boldsymbol{\Omega}$ if*
$$(\operatorname{span} A)^- = \boldsymbol{\Omega} \qquad \text{(SPAN of } A \text{ is DENSE in } \boldsymbol{\Omega}).$$

[9] Kubrusly (2001) page 52 ⟨Theorem 2.7⟩, Michel and Herget (1993) page 91 ⟨Theorem 3.3.31⟩

[10] Haaser and Sullivan (1991) pages 296–297 ⟨6·Orthogonal Bases⟩, Rynne and Youngson (2008) page 78 ⟨Remark 3.50⟩, Heil (2011) page 21 ⟨Remark 1.26⟩

[11] Young (2001) page 19 ⟨Definition 1.5.1⟩, Sohrab (2003) page 362 ⟨Definition 9.2.3⟩, Gupta (1998) page 134 ⟨Definition 2.4⟩, Bachman and Narici (1966) pages 149–153 ⟨Definition 9.3, Theorems 9.9 and 9.10⟩

G.3 Schauder bases in Banach spaces

Definition G.7. [12] *Let* $\boldsymbol{B} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* BANACH SPACE. *Let* $\doteq$ *represent* STRONG CONVERGENCE *in* $\boldsymbol{B}$.

DEF

The countable set $\{\boldsymbol{x}_n \in X | n \in \mathbb{N}\}$ *is a* ***Schauder basis*** *for* $\boldsymbol{B}$ *if for each* $\boldsymbol{x} \in X$

1. $\exists (\!(\alpha_n \in \mathbb{F})\!)_{n\in\mathbb{N}}$ such that $\boldsymbol{x} \doteq \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ (STRONG CONVERGENCE *in* $\boldsymbol{B}$) *and*
2. $\left\{ \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \doteq \sum_{n=1}^{\infty} \beta_n \boldsymbol{x}_n \right\} \implies \{(\!(\alpha_n)\!) = (\!(\beta_n)\!)\}$ (COEFFICIENT FUNCTIONALS *are* UNIQUE)

In this case, $\sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ *is the* ***expansion*** *of* $\boldsymbol{x}$ *on* $\{\boldsymbol{x}_n | n\in\mathbb{N}\}$ *and the elements of* $(\!(\alpha_n)\!)$ *are the* ***coefficient functionals*** *associated with the basis* $\{\boldsymbol{x}_n\}$. *Coefficient functionals are also called* ***coordinate functionals***.

In a Banach space, the existence of a Schauder basis implies that the space is *separable* (Theorem G.3 page 202). The question of whether the converse is also true was posed by Banach himself in 1932,[13] and became know as "*The basis problem*". This remained an open question for many years. The question was finally answered some 41 years later in 1973 by Per Enflo (University of California at Berkley), with the answer being "no". Enflo constructed a counterexample in which a separable Banach space does *not* have a Schauder basis.[14] Life is simpler in Hilbert spaces where the converse *is* true: a Hilbert space has a Schauder basis *if and only if* it is separable (Theorem G.11 page 214).

Theorem G.3. [15] *Let* $\boldsymbol{B} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* BANACH SPACE. *Let* $\mathbb{Q}$ *be the field of rational numbers.*

THM

$$\left\{ \begin{array}{ll} 1. & \boldsymbol{B} \text{ has a SCHAUDER BASIS and} \\ 2. & \mathbb{Q} \text{ is DENSE in } \mathbb{F}. \end{array} \right\} \implies \{ \boldsymbol{B} \text{ is SEPARABLE} \}$$

✎PROOF:

1. lemma:

$$\begin{aligned} \left| \left\{ \boldsymbol{x} | \exists (\!(\alpha_n \mathbb{Q})\!)_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty} \left\| \boldsymbol{x} - \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \right\| = 0 \right\} \right| &= |\mathbb{Q} \times \mathbb{N}| \\ &= |\mathbb{Z} \times \mathbb{Z}| \\ &= |\mathbb{Z}| \\ &= \textit{countably infinite} \end{aligned}$$

[12] Carothers (2005) pages 24–25, Christensen (2003) pages 46–49 ⟨Definition 3.1.1 and page 49⟩, Young (2001) page 19 ⟨Section 6⟩, Singer (1970), page 17, Schauder (1927), Schauder (1928)

[13] Banach (1932a), page 111

[14] Enflo (1973), Lindenstrauss and Tzafriri (1977) pages 84–95 ⟨Section 2.d⟩

[15] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Giles (2000) page 17, Heil (2011) page 21 ⟨Theorem 1.27⟩

2. remainder of proof:

$$\begin{aligned}
&\boldsymbol{B} \text{ has a } \textit{Schauder basis } (\boldsymbol{x}_n)_{n\in\mathbb{N}} \\
\implies &\text{for every } \boldsymbol{x}\in\boldsymbol{B}, \text{ there exists } (\alpha_n\in\mathbb{F})_{n\in\mathbb{N}} \text{ such that } \boldsymbol{x} \doteq \sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n && \text{by Definition G.7 page 202}\\
\implies &\text{for every } \boldsymbol{x}\in\boldsymbol{B}, \text{ there exists } (\alpha_n\in\mathbb{F})_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0 \\
\implies &\text{for every } \boldsymbol{x}\in\boldsymbol{B}, \text{ there exists } (\alpha_n\in\mathbb{Q})_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0 && \text{because } \mathbb{Q}^- = \mathbb{F}\\
\implies &\boldsymbol{B} = \left\{\boldsymbol{x}\,|\,\exists\,(\alpha_n\mathbb{Q})_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\| = 0\right\} \\
\implies &\boldsymbol{B} = \left\{\boldsymbol{x}\,|\,\exists\,(\alpha_n\mathbb{Q})_{n\in\mathbb{N}} \text{ such that } \boldsymbol{x} = \lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\}^- \\
\implies &\boldsymbol{B} \text{ is } \textit{separable} && \text{by (1) lemma page 202}
\end{aligned}$$

⇨

Definition G.8. [16] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

DEF $\{\boldsymbol{x}_n\}$ *is* ***equivalent*** *to* $\{\boldsymbol{y}_n\}$
if there exists a BOUNDED INVERTIBLE *operator* $\mathbf{R}$ *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *such that* $\mathbf{R}\boldsymbol{x}_n = \boldsymbol{y}_n \quad \forall n\in\mathbb{Z}$

Theorem G.4. [17] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

THM
$$\left\{\{\boldsymbol{x}_n\} \text{ is EQUIVALENT to } \{\boldsymbol{y}_n\}\right\} \iff \left\{\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n \text{ is CONVERGENT} \iff \sum_{n=1}^{\infty}\alpha_n\boldsymbol{y}_n \text{ is CONVERGENT}\right\}$$

Lemma G.1. [18] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$ *be a topological linear space. Let* $\operatorname{span}A$ *be the* SPAN *of a set* A (Definition G.2 page 197). *Let* $\tilde{\mathsf{f}}(\omega)$ *and* $\tilde{\mathsf{g}}(\omega)$ *be the* FOURIER TRANSFORMS (Definition N.2 page 309) *of the functions* $\mathsf{f}(x)$ *and* $\mathsf{g}(x)$*, respectively, in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition ?? page ??). *Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT (Definition O.1 page 319) *of a sequence* $(a_n)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ (Definition P.2 page 329).

LEM
$$\left\{\begin{array}{ll}(1). & \{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega} \quad \text{and}\\ (2). & \{\mathbf{T}^n\mathsf{g}\,|\,n\in\mathbb{Z}\} \text{ is a SCHAUDER BASIS for } \boldsymbol{\Omega}\end{array}\right\} \implies \left\{\begin{array}{l}\exists\,(a_n)_{n\in\mathbb{Z}} \quad \text{such that}\\ \tilde{\mathsf{f}}(\omega) = \breve{\mathsf{a}}(\omega)\tilde{\mathsf{g}}(\omega)\end{array}\right\}$$

✎PROOF: Let V_0' be the space spanned by $\{\mathbf{T}^n\phi|\,n\in\mathbb{Z}\}$.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)}\\
&= \tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}\mathsf{g} && \text{by (2)}\\
&= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}\mathsf{g}
\end{aligned}$$

[16] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[17] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[18] Daubechies (1992), page 140

$$= \underbrace{\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}}_{\breve{a}(\omega)} \underbrace{\tilde{\mathbf{F}}\mathrm{g}}_{\tilde{\mathrm{g}}(\omega)} \qquad \text{by Corollary ?? page ??}$$

$$= \breve{a}(\omega)\tilde{\mathrm{g}}(\omega) \qquad \text{by definition of } \breve{\mathbf{F}} \text{ and } \tilde{\mathbf{F}} \quad \text{by (Definition O.1 page 319, Definition N.2 page 309)}$$

$$\begin{aligned}
V_0 &\triangleq \left\{ \mathrm{f}(x) | \mathrm{f}(x) = \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathrm{g}(x) \right\} \\
&= \left\{ \mathrm{f}(x) | \tilde{\mathbf{F}}\mathrm{f}(x) = \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathrm{g}(x) \right\} \\
&= \left\{ \mathrm{f}(x) | \tilde{\mathrm{f}}(\omega) = \tilde{\mathrm{b}}(\omega)\tilde{\mathrm{g}}(\omega) \right\} \\
&= \left\{ \mathrm{f}(x) | \tilde{\mathrm{f}}(\omega) = \tilde{\mathrm{b}}(\omega)\breve{\mathrm{a}}(\omega)\tilde{\mathrm{f}}(\omega) \right\} \\
&= \left\{ \mathrm{f}(x) | \tilde{\mathrm{f}}(\omega) = \tilde{\mathrm{c}}(\omega)\tilde{\mathrm{f}}(\omega) \right\} && \text{where } \tilde{\mathrm{c}}(\omega) \triangleq \tilde{\mathrm{b}}(\omega)\breve{\mathrm{a}}(\omega) \\
&= \left\{ \mathrm{f}(x) | \mathrm{f}(x) = \sum_{n\in\mathbb{Z}} \mathrm{c}_n \mathrm{f}(x-n) \right\} \\
&\triangleq V_0'
\end{aligned}$$

✏

G.4 Linear combinations in inner product spaces

In an *inner product space*, *orthogonality* is a special case of *linear independence*; or alternatively, linear independence is a generalization of orthogonality (next theorem).

Theorem G.5. [19] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition I.1 page 233)* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \triangledown \rangle\right)$.

THM

$$\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is ORTHOGONAL} \\ \text{(Definition I.4 page 245)} \end{array} \right\} \implies \left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is LINEARLY INDEPENDENT} \\ \text{(Definition G.1 page 197)} \end{array} \right\}$$

✎PROOF:

1. Proof using *Pythagorean theorem* (Theorem I.10 page 246):
   Let $(\!(\alpha_n)\!)_{n\in\mathbb{N}}$ be a sequence with at least one nonzero element.

$$\begin{aligned}
\left\| \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \right\|^2 &= \sum_{n=1}^{N} \left\| \alpha_n \boldsymbol{x}_n \right\|^2 && \text{by left hypoth. and } \textit{Pythagorean Theorem} \quad \text{(Theorem I.10 page 246)} \\
&= \sum_{n=1}^{N} |\alpha_n|^2 \left\| \boldsymbol{x}_n \right\|^2 && \text{by definition of } \|\cdot\| \quad \text{(Definition J.1 page 249)} \\
&> 0 \\
\implies \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n &\neq 0 \\
\implies (\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{N}} &\text{ is } \textit{linearly independent} && \text{by definition of } \textit{linear independence} \quad \text{(Definition G.3 page 197)}
\end{aligned}$$

[19] Aliprantis and Burkinshaw (1998) page 283 ⟨Corollary 32.8⟩, Kubrusly (2001) page 352 ⟨Proposition 5.34⟩

2. Alternative proof:

$$\begin{aligned}
\sum_{n=1}^{N} \alpha_n x_n = \mathbb{0} &\implies \left\langle \sum_{n=1}^{N} \alpha_n x_n \,|\, x_m \right\rangle = \langle \mathbb{0} \,|\, x_m \rangle \\
&\implies \sum_{n=1}^{N} \alpha_n \langle x_n \,|\, x_m \rangle = 0 \\
&\implies \sum_{n=1}^{N} \alpha_n \bar{\delta}(k-m) = 0 \\
&\implies \alpha_m = 0 \qquad \text{for } m = 1, 2, \ldots, N
\end{aligned}$$

⇨

Theorem G.6 (Bessel's Equality). [20] *Let* $\{ x_n \in X | _{n=1,2,\ldots,N} \}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition I.1 page 233)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ *and with* $\|x\| \triangleq \sqrt{\langle x \,|\, x \rangle}$ *(Definition I.2 page 238).*

THM

$$\left\{ \begin{array}{c} \{x_n\} \text{ is ORTHONORMAL} \\ \text{(Definition I.4 page 245)} \end{array} \right\} \implies \left\{ \underbrace{\left\| x - \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2}_{\text{approximation error}} = \|x\|^2 - \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 \quad \forall x \in X \right\}$$

✎PROOF:

$$\begin{aligned}
&\left\| x - \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2 \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2 - 2\Re \left\langle x \,\middle|\, \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\rangle && \text{by \textit{polar identity}} && \text{(Lemma I.1 page 237)} \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x \,|\, x_n \rangle x_n \right\|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle x \,|\, x_n \rangle \right)^* \langle x \,|\, x_n \rangle \right] && \text{by property of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition I.1 page 233)} \\
&= \|x\|^2 + \sum_{n=1}^{N} \| \langle x \,|\, x_n \rangle x_n \|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle x \,|\, x_n \rangle \right)^* \langle x \,|\, x_n \rangle \right] && \text{by \textit{Pythagorean Theorem}} && \text{(Theorem I.10 page 246)} \\
&= \|x\|^2 + \sum_{n=1}^{N} \| \langle x \,|\, x_n \rangle x_n \|^2 - 2\Re \left(\sum_{n=1}^{N} \langle x \,|\, x_n \rangle^* \langle x \,|\, x_n \rangle \right) \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 \underbrace{\|x_n\|^2}_{1} - 2\Re \left(\sum_{n=1}^{N} \langle x \,|\, x_n \rangle^* \langle x \,|\, x_n \rangle \right) && \text{by property of } \|\cdot\| && \text{(Definition J.1 page 249)} \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 \cdot 1 - 2\Re \left(\sum_{n=1}^{N} \langle x \,|\, x_n \rangle^* \langle x \,|\, x_n \rangle \right) && \text{by def. of \textit{orthonormality}} && \text{(Definition I.4 page 245)} \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 - 2\Re \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x \,|\, x_n \rangle|^2 && \text{because } |\cdot| \text{ is real}
\end{aligned}$$

[20] Bachman et al. (2000) page 103, Pedersen (2000) pages 38–39

$$= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2$$

⇰

Theorem G.7 (Bessel's inequality). [21] *Let* $\{ \boldsymbol{x}_n \in X | _{n=1,2,\ldots,N} \}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition I.1 page 233)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and with* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$ *(Definition I.2 page 238).*

THM

$$\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is ORTHONORMAL} \\ \text{(Definition I.4 page 245)} \end{array} \right\} \implies \left\{ \sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 \leq \|\boldsymbol{x}\|^2 \quad \forall \boldsymbol{x} \in X \right\}$$

✎PROOF:

$$\begin{aligned}
0 &\leq \left\| \boldsymbol{x} - \sum_{n=1}^{N} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by definition of } \|\cdot\| && \text{(Definition J.1 page 249)} \\
&= \|\boldsymbol{x}\|^2 - \sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 && \text{by } \textit{Bessel's Equality} && \text{(Theorem G.6 page 205)}
\end{aligned}$$

⇰

The *Best Approximation Theorem* (next) shows that

- the best sequence for representing a vector is the sequence of projections of the vector onto the sequence of basis functions
- the error of the projection is orthogonal to the projection.

Theorem G.8 (Best Approximation Theorem). [22] *Let* $\{ \boldsymbol{x}_n \in X | _{n=1,2,\ldots,N} \}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition I.1 page 233)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and with* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$ *(Definition I.2 page 238).*

THM

$$\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is} \\ \text{ORTHONORMAL} \\ \text{(Definition I.4 page 245)} \end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & \arg\min_{(\!(\alpha_n)\!)_{n=1}^{N}} \left\| \boldsymbol{x} - \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \right\| = \underbrace{(\!(\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle)\!)_{n=1}^{N}}_{\text{best } \alpha_n = \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle} & \forall \boldsymbol{x} \in X \quad \text{and} \\ 2. & \underbrace{\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right)}_{\text{approximation}} \perp \underbrace{\left(\boldsymbol{x} - \sum_{n=1}^{N} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right)}_{\text{approximation error}} & \forall \boldsymbol{x} \in X \end{array} \right\}$$

✎PROOF:

[21] Giles (2000) pages 54–55 ⟨3.13 Bessel's inequality⟩, Bollobás (1999) page 147, Aliprantis and Burkinshaw (1998) page 284

[22] Walter and Shen (2001), pages 3–4, Pedersen (2000), page 39, Edwards (1995), pages 94–100, Weyl (1940)

1. Proof that $(\langle x \mid x_n \rangle)$ is the best sequence:

$$
\begin{aligned}
&\left\| x - \sum_{n=1}^{N} \alpha_n x_n \right\|^2 \\
&= \|x\|^2 - 2\Re \left\langle x \mid \sum_{n=1}^{N} \alpha_n x_n \right\rangle + \left\| \sum_{n=1}^{N} \alpha_n x_n \right\|^2 \\
&= \|x\|^2 - 2\Re \left(\sum_{n=1}^{N} \alpha_n^* \langle x \mid x_n \rangle \right) + \sum_{n=1}^{N} \|\alpha_n x_n\|^2 && \text{by } \textit{Pythagorean Theorem} && \text{(Theorem I.10 page 246)} \\
&= \|x\|^2 - 2\Re \left(\sum_{n=1}^{N} \alpha_n^* \langle x \mid x_n \rangle \right) + \sum_{n=1}^{N} |\alpha_n|^2 + \underbrace{\left[\sum_{n=1}^{N} |\langle x \mid x_n \rangle|^2 - \sum_{n=1}^{N} |\langle x \mid x_n \rangle|^2 \right]}_{0} \\
&= \left[\|x\|^2 - \sum_{n=1}^{N} |\langle x \mid x_n \rangle|^2 \right] + \sum_{n=1}^{N} \left[|\langle x \mid x_n \rangle|^2 - 2\mathbf{R_e} \left[\alpha_n^* \langle x \mid x_n \rangle \right] + |\alpha_n|^2 \right] \\
&= \left[\|x\|^2 - \sum_{n=1}^{N} |\langle x \mid x_n \rangle|^2 \right] + \sum_{n=1}^{N} \left[|\langle x \mid x_n \rangle|^2 - \alpha_n^* \langle x \mid x_n \rangle - \alpha_n \langle x \mid x_n \rangle^* + |\alpha_n|^2 \right] \\
&= \left\| x - \sum_{n=1}^{N} \langle x \mid x_n \rangle x_n \right\|^2 + \sum_{n=1}^{N} |\langle x \mid x_n \rangle - \alpha_n|^2 && \text{by } \textit{Bessel's Equality} && \text{(Theorem G.6 page 205)} \\
&\geq \left\| x - \sum_{n=1}^{N} \langle x \mid x_n \rangle x_n \right\|^2
\end{aligned}
$$

2. Proof that the approximation and approximation error are orthogonal:

$$
\begin{aligned}
\left\langle \sum_{n=1}^{N} \langle x \mid x_n \rangle x_n \mid x - \sum_{n=1}^{N} \langle x \mid x_n \rangle x_n \right\rangle &= \left\langle \sum_{n=1}^{N} \langle x \mid x_n \rangle x_n \mid x \right\rangle - \left\langle \sum_{n=1}^{N} \langle x \mid x_n \rangle x_n \mid \sum_{n=1}^{N} \langle x \mid x_n \rangle x_n \right\rangle \\
&= \sum_{n=1}^{N} \langle x \mid x_n \rangle^* \langle x \mid x_n \rangle - \sum_{n=1}^{N} \sum_{m=1}^{N} \langle x \mid x_n \rangle \langle x \mid x_m \rangle^* \langle x_n \mid x_m \rangle \\
&= \sum_{n=1}^{N} |\langle x \mid x_n \rangle|^2 - \sum_{n=1}^{N} \sum_{m=1}^{N} \langle x \mid x_n \rangle \langle x \mid x_m \rangle^* \bar{\delta}_{nm} \\
&= \sum_{n=1}^{N} |\langle x \mid x_n \rangle|^2 - \sum_{n=1}^{N} |\langle x \mid x_n \rangle|^2 \\
&= 0
\end{aligned}
$$

✏

G.5 Orthonormal bases in Hilbert spaces

Definition G.9. *Let* $\left\{ x_n \in X \middle| n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition I.1 page 233)* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \triangledown \rangle \right)$.

DEF

The set $\{x_n\}$ *is an* ***orthogonal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{x_n\}$ *is* ORTHOGONAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.
The set $\{x_n\}$ *is an* ***orthonormal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{x_n\}$ *is* ORTHONORMAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.

Definition G.10. [23] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a Hilbert space.*

DEF

Suppose there exists a set $\left\{x_n \in X \,\middle|\, n\in\mathbb{N}\right\}$ *such that* $x \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n$.
Then the quantities $\langle x \mid x_n \rangle$ *are called the* ***Fourier coefficients*** *of* x *and the sum* $\sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n$ *is called the* ***Fourier expansion*** *of* x *or the* ***Fourier series*** *for* x.

Lemma G.2 (Perfect reconstruction). *Let* $\left\{x_n \in X \,\middle|\, n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

LEM

$$\left\{\begin{array}{lll} (1). & (\!(x_n)\!) \text{ is a BASIS for } \boldsymbol{H} & \text{and} \\ (2). & (\!(x_n)\!) \text{ is ORTHONORMAL} & \end{array}\right\} \implies x \stackrel{\star}{=} \overbrace{\sum_{n=1}^{\infty} \underbrace{\langle x \mid x_n \rangle}_{\text{Fourier coefficient}} x_n}^{\text{Fourier expansion}} \qquad \forall x\in X$$

✎PROOF:

$$\begin{aligned}
\langle x \mid x_n \rangle &= \left\langle \sum_{m\in\mathbb{Z}} \alpha_m x_m \,\middle|\, x_n \right\rangle && \text{by left hypothesis (1)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \langle x_m \mid x_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition I.1 page 233)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \bar{\delta}_{n-m} && \text{by left hypothesis (2)} && \text{(Definition I.4 page 245)} \\
&= \alpha_n
\end{aligned}$$

✏

Proposition G.2. [24] *Let* $\left\{x_n \in X \,\middle|\, n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

PRP

$$\underbrace{\|x\|^2 \stackrel{\star}{=} \sum_{n=1}^{\infty} \left|\langle x \mid x_n \rangle\right|^2}_{\text{PARSEVAL FRAME}} \iff \underbrace{x \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n}_{\text{FOURIER EXPANSION (Definition G.10 page 208)}} \qquad \forall x\in X$$

✎PROOF:

[23] Fabian et al. (2010) page 27 ⟨Theorem 1.55⟩, Young (2001) page 6, Young (1980) page 6
[24] Han et al. (2007) pages 93–94 ⟨Proposition 3.11⟩

1. Proof that *Parseval frame* $\Longleftarrow$ *Fourier expansion*

$$\begin{aligned}
\|x\|^2 &\triangleq \langle x \mid x\rangle && \text{by definition of } \|\cdot\| && \text{(Definition J.1 page 249)}\\
&= \left\langle \sum_{n=1}^{\infty} \langle x \mid x_n\rangle\, x \mid x_n \right\rangle && \text{by right hypothesis}\\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle x \mid x_n\rangle \langle x \mid x_n\rangle && \text{by property of } \langle \triangle \mid \nabla\rangle && \text{(Definition I.1 page 233)}\\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle x \mid x_n\rangle \langle x \mid x_n\rangle^* && \text{by property of } \langle \triangle \mid \nabla\rangle && \text{(Definition I.1 page 233)}\\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} |\langle x \mid x_n\rangle|^2 && \text{by property of } \mathbb{C} && \text{(Definition H.7 page 231)}
\end{aligned}$$

2. Proof that *Parseval frame* $\Longrightarrow$ *Fourier expansion*

   (a) Let $(e_n)_{n\in\mathbb{N}}$ be the *standard othornormal basis* such that the nth element of e_n is 1 and all other elements are 0.

   (b) Let $\mathbf{M}$ be an operator in $\boldsymbol{H}$ such that $\mathbf{M}x \triangleq \sum_{n=1}^{\infty} \langle x \mid x_n\rangle\, e_n$.

   (c) lemma: $\mathbf{M}$ is *isometric*. Proof:

$$\begin{aligned}
\|\mathbf{M}x\|^2 &= \left\|\sum_{n=1}^{\infty} \langle x \mid x_n\rangle\, e_n\right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2b) page 209)}\\
&= \sum_{n=1}^{\infty} \|\langle x \mid x_n\rangle\, e_n\|^2 && \text{by \textit{Pythagorean Theorem}} && \text{(Theorem I.10 page 246)}\\
&= \sum_{n=1}^{\infty} |\langle x \mid x_n\rangle|^2\, \|e_n\|^2 && \text{by \textit{homogeneous} property of } \|\cdot\| && \text{(Definition J.1 page 249)}\\
&= \sum_{n=1}^{\infty} |\langle x \mid x_n\rangle|^2 && \text{by definition of \textit{orthonormal}} && \text{(Definition I.4 page 245)}\\
&= \|x\|^2 && \text{by Parseval frame hypothesis}\\
\Longrightarrow &\quad \mathbf{M} \text{ is \textit{isometric}} && \text{by definition of \textit{isometric}} && \text{(Definition M.10 page 300)}
\end{aligned}$$

   (d) Let $(u_n)_{n\in\mathbb{N}}$ be an *orthornormal basis* for $\boldsymbol{H}$.

   (e) Proof for *Fourier expansion*:

$$\begin{aligned}
x &= \sum_{n=1}^{\infty} \langle x \mid u_n\rangle\, u_n && \text{by \textit{Fourier expansion} (Proposition G.3 page 212)}\\
&= \sum_{n=1}^{\infty} \langle \mathbf{M}x \mid \mathbf{M}u_n\rangle\, u_n && \text{by (2c) lemma page 209 and Theorem M.21 page 301}\\
&= \sum_{n=1}^{\infty} \left\langle \sum_{m=1}^{\infty} \langle x \mid x_m\rangle\, e_m \mid \sum_{k=1}^{\infty} \langle u_n \mid x_k\rangle\, e_k \right\rangle u_n && \text{by item (2b) page 209}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle x \mid x_m\rangle \sum_{k=1}^{\infty} \langle u_n \mid x_k\rangle^* \langle e_m \mid e_k\rangle\, u_n && \text{by Definition I.1 page 233}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle x \mid x_m\rangle \langle u_n \mid x_m\rangle^*\, u_n && \text{by item (2a) page 209 and Definition I.4 page 245}
\end{aligned}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m\rangle \langle \boldsymbol{x}_m \mid \boldsymbol{u}_n\rangle \boldsymbol{u}_n \qquad \text{by Definition I.1 page 233}$$

$$= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m\rangle \sum_{n=1}^{\infty} \langle \boldsymbol{x}_m \mid \boldsymbol{u}_n\rangle \boldsymbol{u}_n$$

$$= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m\rangle \boldsymbol{x}_m \qquad \text{by item (2d) page 209}$$

⇒

When is a set of orthonormal vectors in a Hilbert space $\boldsymbol{H}$ *total*? Theorem G.9 (next) offers some help.

Theorem G.9 (The Fourier Series Theorem). [25] *Let* $\{\boldsymbol{x}_n \in X \mid n \in \mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$ (Definition I.2 page 238).

THM

(A) $\{\boldsymbol{x}_n\}$ *is* ORTHONORMAL *in* $\boldsymbol{H}$ $\implies$

$$\left\{\begin{array}{llrcll}
 & (1). & (\operatorname{span}\{\boldsymbol{x}_n\})^- & = & \boldsymbol{H} & (\{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{H})\\
\iff & (2). & \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \langle \boldsymbol{y} \mid \boldsymbol{x}_n\rangle^* \quad \forall \boldsymbol{x},\boldsymbol{y} \in X & (\text{GENERALIZED PARSEVAL'S IDENTITY})\\
\iff & (3). & \|\boldsymbol{x}\|^2 & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle|^2 \quad \forall \boldsymbol{x} \in X & (\text{PARSEVAL'S IDENTITY})\\
\iff & (4). & \boldsymbol{x} & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_n \quad \forall \boldsymbol{x} \in X & (\text{FOURIER SERIES EXPANSION})
\end{array}\right\}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle &\stackrel{\star}{=} \left\langle \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_n \;\middle|\; \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m\rangle \boldsymbol{x}_m \right\rangle && \text{by (A) and (1)}\\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \left\langle \boldsymbol{x}_n \;\middle|\; \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m\rangle \boldsymbol{x}_m \right\rangle && \text{by property of } \langle \cdot \mid \cdot \rangle \quad \text{(Definition I.1 page 233)}\\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m\rangle^* \langle \boldsymbol{x}_n \mid \boldsymbol{x}_m\rangle && \text{by property of } \langle \cdot \mid \cdot \rangle \quad \text{(Definition I.1 page 233)}\\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \sum_{m=1}^{\infty} \langle \boldsymbol{y} \mid \boldsymbol{x}_m\rangle^* \bar{\delta}_{mn} && \text{by (A)}\\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \langle \boldsymbol{y} \mid \boldsymbol{x}_n\rangle^* && \text{by definition of } \bar{\delta}_n \quad \text{(Definition I.3 page 245)}
\end{aligned}$$

[25] Bachman and Narici (1966) pages 149–155 ⟨Theorem 9.12⟩, Kubrusly (2001) pages 360–363 ⟨Theorem 5.48⟩, Aliprantis and Burkinshaw (1998) pages 298–299 ⟨Theorem 34.2⟩, Christensen (2003) page 57 ⟨Theorem 3.4.2⟩, Berberian (1961) pages 52–53 ⟨Theorem II§8.3⟩, Heil (2011) pages 34–35 ⟨Theorem 1.50⟩, Bracewell (1978) page 112 ⟨Rayleigh's theorem⟩

2. Proof that (2) $\implies$ (3):

$$
\begin{aligned}
\|x\|^2 &\triangleq \langle x \,|\, x\rangle && \text{by definition of } \textit{induced norm} && \text{(Theorem I.4 page 238)}\\
&= \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \langle x \,|\, x_n\rangle^* && \text{by (2)}\\
&= \sum_{n=1}^{\infty} \left|\langle x \,|\, x_n\rangle\right|^2
\end{aligned}
$$

3. Proof that (3) $\iff$ (4) *not* using (A): by Proposition G.2 page 208

4. Proof that (3) $\implies$ (1) (proof by contradiction):

   (a) Suppose $\{x_n\}$ is *not total*.

   (b) Then there must exist a vector y in H such that the set $B \triangleq \{x_n\} \cup y$ is *orthonormal*.

   (c) Then $1 = \|y\|^2 \neq \sum_{n=1}^{\infty} \left|\langle y \,|\, x_n\rangle\right|^2 = 0$.

   (d) But this contradicts (3), and so $\{x_n\}$ must be *total* and (3) $\implies$ (1).

5. Extraneous proof that (3) $\implies$ (4) (this proof is not really necessary here):

$$
\begin{aligned}
\left\| x - \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle x_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x \,|\, x_n\rangle|^2 && \text{by } \textit{Bessel's Equality} && \text{(Theorem G.6 page 205)}\\
&= 0 && \text{by (3)}\\
\implies \quad x &\doteq \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle x_n && \text{by definition of } \doteq
\end{aligned}
$$

6. Extraneous proof that (A) $\implies$ (4) (this proof is not really necessary here)

   (a) The sequence $\sum_{n=1}^{N} \left|\langle x \,|\, x_n\rangle\right|^2$ is *monotonically increasing* in n.

   (b) By Bessel's inequality (page 206), the sequence is upper bounded by $\|x\|^2$:

$$\sum_{n=1}^{N} \left|\langle x \,|\, x_n\rangle\right|^2 \leq \|x\|^2$$

   (c) Because this sequence is both monotonically increasing and bounded in n, it must equal its bound in the limit as n approaches infinity:

$$\lim_{N\to\infty} \sum_{n=1}^{N} \left|\langle x \,|\, x_n\rangle\right|^2 = \|x\|^2 \tag{G.1}$$

   (d) If we combine this result with *Bessel's Equality* (Theorem G.6 page 205) we have

$$
\begin{aligned}
\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} \langle x \,|\, x_n\rangle x_n \right\|^2 &= \|x\|^2 - \lim_{N\to\infty} \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 && \text{by } \textit{Bessel's Equality} \text{ (Theorem G.6 page 205)}\\
&= \|x\|^2 - \|x\|^2 && \text{by equation (G.1) page 211}\\
&= 0
\end{aligned}
$$

Proposition G.3 (Fourier expansion). *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

PRP

$$\underbrace{\{x_n\} \text{ is an ORTHONORMAL BASIS for } \boldsymbol{H}}_{(A)} \implies \left\{ \underbrace{x \stackrel{\star}{=} \sum_{n=1}^{\infty} \alpha_n x_n}_{(1)} \iff \underbrace{\alpha_n = \langle x \mid x_n \rangle}_{(2)} \right\}$$

✎PROOF:

1. Proof that (1) $\implies$ (2): by Lemma G.2 page 208
2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\left\| x - \sum_{n=1}^{\infty} \alpha_n x_{n\in\mathbb{Z}} \right\|^2 &= \left\| x - \sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_{n\in\mathbb{Z}} \right\|^2 && \text{by right hypothesis} \\
&= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x \mid x_n \rangle|^2 && \text{by Bessel's equality} && \text{(Theorem G.6 page 205)} \\
&= 0 && \text{by } \textit{Parseval's Identity} && \text{(Theorem G.9 page 210)} \\
&\stackrel{\text{def}}{\iff} \quad x \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n && \text{by definition of } \textit{strong convergence}
\end{aligned}$$

⇒

Proposition G.4 (Riesz-Fischer Theorem).[26] *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

PRP

$$\underbrace{\{x_n\} \text{ is an ORTHONORMAL BASIS for } \boldsymbol{H}}_{(A)} \implies \left\{ \underbrace{\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty}_{(1)} \iff \underbrace{\exists x\in\boldsymbol{H} \text{ such that } \alpha_n = \langle x \mid x_n \rangle}_{(2)} \right\}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) If (1) is true, then let $x \triangleq \sum_{n\in\mathbb{N}} \alpha_n x_n$.
   (b) Then

$$\begin{aligned}
\langle x \mid x_n \rangle &= \left\langle \sum_{m\in\mathbb{N}} \alpha_m x_m \mid x_n \right\rangle && \text{by definition of } x \\
&= \sum_{m\in\mathbb{N}} \alpha_m \langle x_m \mid x_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition I.1 page 233)} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \bar{\delta}_{mn} && \text{by (A)} \\
&= \sum_{m\in\mathbb{N}} \alpha_n && \text{by definition of } \bar{\delta} && \text{(Definition I.3 page 245)}
\end{aligned}$$

[26] Young (2001) page 6

2. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |\alpha_n|^2 &= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle|^2 && \text{by (2)}\\
&\le \|\boldsymbol{x}\|^2 && \text{by } \textit{Bessel's Inequality} \quad \text{(Theorem G.7 page 206)}\\
&\le \infty
\end{aligned}$$

Theorem G.10. [27]

THM *All* SEPARABLE HILBERT SPACES *are* ISOMORPHIC. *That is,*

$$\left\{\begin{array}{l} \boldsymbol{X} \text{ is a separable} \\ \text{Hilbert space} \quad \text{and} \\ \boldsymbol{Y} \text{ is a separable} \\ \text{Hilbert space} \end{array}\right\} \implies \left\{\begin{array}{lllll} \multicolumn{5}{l}{\text{there is a BIJECTIVE operator } \mathbf{M} \in \boldsymbol{Y}^{\boldsymbol{X}} \text{ such that}} \\ (1). & \boldsymbol{y} & = \mathbf{M}\boldsymbol{x} & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} & \text{and} \\ (2). & \|\mathbf{M}\boldsymbol{x}\| & = \|\boldsymbol{x}\| & \forall x\in\boldsymbol{X} & \text{and} \\ (3). & \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y}\rangle & = \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} & \end{array}\right\}$$

✎PROOF:

1. Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \nabla\rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{\boldsymbol{x}_n | n\in\mathbb{N}\}$.
   Let $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \nabla\rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{\boldsymbol{y}_n | n\in\mathbb{N}\}$.

2. Proof that there exists *bijective* operator $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$:

   (a) Let $\mathbf{M}$ be defined such that $\boldsymbol{y}_n \triangleq \mathbf{M}\boldsymbol{x}_n$.

   (b) Thus $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$.

   (c) Because $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$, $\mathbf{M}$ has an inverse operator $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$ such that $\boldsymbol{x}_n = \mathbf{M}^{-1}\boldsymbol{y}_n$.

3. Proof that $\mathbf{M}$ and $\mathbf{M}^{-1}$ are *bijective* operators between $\boldsymbol{X}$ and $\boldsymbol{Y}$:

   (a) Proof that $\mathbf{M}$ maps $\boldsymbol{X}$ *into* $\boldsymbol{Y}$:

$$\begin{aligned}
\boldsymbol{x} \in \boldsymbol{X} &\iff \boldsymbol{x} \stackrel{\star}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_n && \text{by } \textit{Fourier expansion} \quad \text{(Theorem G.9 page 210)}\\
&\implies \exists \boldsymbol{y} \in \boldsymbol{Y} \quad \text{such that} \quad \langle \boldsymbol{y} \mid \boldsymbol{y}_n\rangle = \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle && \text{by } \textit{Riesz-Fischer Thm.} \quad \text{(Proposition G.4 page 212)}\\
&\implies\\
\boldsymbol{y} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n\rangle \boldsymbol{y}_n && \text{by } \textit{Fourier expansion} \quad \text{(Theorem G.9 page 210)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{y}_n && \text{by } \textit{Riesz-Fischer Thm.} \quad \text{(Proposition G.4 page 212)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \mathbf{M}\boldsymbol{x}_n && \text{by definition of } \mathbf{M} \quad \text{(item (2a) page 213)}\\
&= \mathbf{M}\sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_n && \text{by prop. of linear ops.} \quad \text{(Theorem M.1 page 282)}\\
&= \mathbf{M}\boldsymbol{x} && \text{by definition of } \boldsymbol{x}
\end{aligned}$$

[27] Young (2001) page 6

(b) Proof that $\mathbf{M}^{-1}$ maps $\boldsymbol{Y}$ *into* $\boldsymbol{X}$:

$$\begin{aligned}
\boldsymbol{y} \in \boldsymbol{Y} &\iff \boldsymbol{y} \stackrel{*}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 210)}\\
&\implies \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle = \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition G.4 page 212)}\\
&\implies \\
\boldsymbol{x} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 210)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{x}_n && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition G.4 page 212)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \mathbf{M}^{-1}\boldsymbol{y}_n && \text{by definition of } \mathbf{M}^{-1} && \text{(item (2c) page 213)}\\
&= \mathbf{M}^{-1} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \mid \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by prop. of \textit{linear ops.}} && \text{(Theorem M.1 page 282)}\\
&= \mathbf{M}^{-1}\boldsymbol{y} && \text{by definition of } \boldsymbol{y}
\end{aligned}$$

4. Proof for (2):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\|\mathbf{M}\sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n\right\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 210)}\\
&= \left\|\sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n\right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem M.1 page 282)}\\
&= \left\|\sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{y}_n\right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2a) page 213)}\\
&= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition G.4 page 212)}\\
&= \left\|\sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n\right\|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition G.4 page 212)}\\
&= \|\boldsymbol{x}\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem G.9 page 210)}
\end{aligned}$$

5. Proof for (3): by (2) and Theorem M.21 page 301

✏

Theorem G.11. [28] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has a* SCHAUDER BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

Theorem G.12. [29] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: $\boldsymbol{H}$ *has an* ORTHONORMAL BASIS $\iff$ $\boldsymbol{H}$ *is* SEPARABLE

[28] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Berberian (1961) page 53 ⟨Theorem II§8.3⟩
[29] Kubrusly (2001) page 357 ⟨Proposition 5.43⟩

G.6 Riesz bases in Hilbert spaces

Definition G.11. [30] *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

DEF: $\{x_n\}$ *is a* ***Riesz basis*** *for* $\boldsymbol{H}$ *if* $\{x_n\}$ *is* EQUIVALENT (Definition G.8 page 203) *to some* ORTHONORMAL BASIS (Definition G.9 page 207) *in* $\boldsymbol{H}$.

Definition G.12. [31] *Let* $(x_n \in X)_{n\in\mathbb{N}}$ *be a sequence of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

DEF: *The sequence* (x_n) *is a* ***Riesz sequence*** *for* $\boldsymbol{H}$ *if*

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 \qquad \forall(\alpha_n)\in\ell^2_{\mathbb{F}}.$$

Definition G.13. *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be an* INNER PRODUCT SPACE (Definition I.1 page 233).

DEF: *The sequences* $(x_n \in X)_{n\in\mathbb{Z}}$ *and* $(y_n \in X)_{n\in\mathbb{Z}}$ *are* ***biorthogonal*** *with respect to each other in* $\boldsymbol{X}$ *if* $\langle x_n \mid y_m \rangle = \bar{\delta}_{nm}$

Lemma G.3. [32] *Let* $\{x_n \mid n\in\mathbb{N}\}$ *be a sequence in a* HILBERT SPACE $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$. *Let* $\{y_n \mid n\in\mathbb{N}\}$ *be a sequence in a* HILBERT SPACE $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$. *Let*

LEM:

$$\left\{\begin{array}{lll}
\text{(i).} & \{x_n\} \text{ is TOTAL in } \boldsymbol{X} & \text{and}\\
\text{(ii).} & \text{There exists } A>0 \text{ such that } A\sum_{n\in C}|a_n|^2 \le \left\|\sum_{n\in C}a_n x_n\right\|^2 \quad \text{for finite } C & \text{and}\\
\text{(iii).} & \text{There exists } B>0 \text{ such that } \left\|\sum_{n=1}^{\infty}b_n y_n\right\|^2 \le B\sum_{n=1}^{\infty}|b_n|^2 \quad \forall(b_n)_{n\in\mathbb{N}}\in\ell^2_{\mathbb{F}} &
\end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lll}
\text{(1).} & \mathbf{R}^\circ \text{ is a linear bounded operator that maps from } \operatorname{span}\{x_n\} \text{ to } \operatorname{span}\{y_n\} & \\
 & \quad \text{where } \mathbf{R}^\circ \sum_{n\in C} c_n x_n \triangleq \sum_{n\in C} c_n y_n, \text{ for some sequence } (c_n) \text{ and finite set } C & \text{and}\\
\text{(2).} & \mathbf{R} \text{ has a unique extension to a bounded operator } \mathbf{R} \text{ that maps from } \boldsymbol{X} \text{ to } \boldsymbol{Y} & \text{and}\\
\text{(3).} & |||\mathbf{R}^\circ||| \le \frac{B}{A} & \text{and}\\
\text{(4).} & |||\mathbf{R}||| \le \frac{B}{A} &
\end{array}\right\}$$

Theorem G.13. [33] *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM:

$$\left\{\begin{array}{l}\{x_n\} \text{ is a RIESZ BASIS}\\ \text{for } \boldsymbol{H}\end{array}\right\} \iff \left\{\begin{array}{ll}
\text{(1).} \ \{x_n\} \text{ is TOTAL in } \boldsymbol{H} & \text{and}\\
\text{(2).} \ \exists A,B\in\mathbb{R}^+ \quad \text{such that} \quad \forall(\alpha_n)\in\ell^2_{\mathbb{F}}, & \\
\quad A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 &
\end{array}\right\}$$

✎PROOF:

[30] Young (2001) page 27 ⟨Definition 1.8.2⟩, Christensen (2003) page 63 ⟨Definition 3.6.1⟩, Heil (2011) page 196 ⟨Definition 7.9⟩

[31] Christensen (2003) pages 66–68 ⟨page 68 and (3.24) on page 66⟩, Wojtaszczyk (1997) page 20 ⟨Definition 2.6⟩

[32] Christensen (2003) pages 65–66 ⟨Lemma 3.6.5⟩

[33] Young (2001) page 27 ⟨Theorem 1.8.9⟩, Christensen (2003) page 66 ⟨Theorem 3.6.6⟩, Heil (2011) pages 197–198 ⟨Theorem 7.13⟩, Christensen (2008) pages 61–62 ⟨Theorem 3.3.7⟩

1. Proof for ($\Longrightarrow$) case:
   (a) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (1): all bases for $\boldsymbol{H}$ are *total* in $\boldsymbol{H}$.
   (b) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (2):
      i. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthonormal basis* for $\boldsymbol{H}$.
      ii. Let $\mathbf{R}$ be a *bounded bijective* operator such that $\boldsymbol{x}_n = \mathbf{R}\boldsymbol{u}_n$.
      iii. Proof for upper bound B:

$$
\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \left\|\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii))}\\
&= \left\|\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem M.1 page 282}\\
&\le \|\|\mathbf{R}\|\|^2\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by Theorem M.6 page 288}\\
&= \|\|\mathbf{R}\|\|^2\sum_{n=1}^{\infty}\left\|\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by \textit{Pythagorean Theorem}} && \text{(Theorem I.10 page 246)}\\
&= \|\|\mathbf{R}\|\|^2\sum_{n=1}^{\infty}|\alpha|^2\left\|\boldsymbol{u}_n\right\|^2 && \text{by \textit{homogeneous} property of norms} && \text{(Definition J.1 page 249)}\\
&= \underbrace{\|\|\mathbf{R}\|\|^2}_{B}\sum_{n=1}^{\infty}|\alpha|^2 && \text{by definition of \textit{orthonormality}} && \text{(Definition I.4 page 245)}
\end{aligned}
$$

      iv. Proof for lower bound A:

$$
\begin{aligned}
\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 &= \frac{\|\|\mathbf{R}^{-1}\|\|^2}{\|\|\mathbf{R}^{-1}\|\|^2}\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{because } \|\|\mathbf{R}^{-1}\|\| > 0 && \text{(Proposition M.1 page 286)}\\
&\ge \frac{1}{\|\|\mathbf{R}^{-1}\|\|^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\right\|^2 && \text{by Theorem M.6 page 288}\\
&= \frac{1}{\|\|\mathbf{R}^{-1}\|\|^2}\left\|\mathbf{R}^{-1}\sum_{n=1}^{\infty}\alpha_n\mathbf{R}\boldsymbol{u}_n\right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii) page 216)}\\
&= \frac{1}{\|\|\mathbf{R}^{-1}\|\|^2}\left\|\mathbf{R}^{-1}\mathbf{R}\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem M.1 page 282)}\\
&= \frac{1}{\|\|\mathbf{R}^{-1}\|\|^2}\left\|\sum_{n=1}^{\infty}\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by definition of inverse op.} && \text{(Definition M.2 page 281)}\\
&= \frac{1}{\|\|\mathbf{R}^{-1}\|\|^2}\sum_{n=1}^{\infty}\left\|\alpha_n\boldsymbol{u}_n\right\|^2 && \text{by \textit{Pythagorean Theorem}} && \text{(Theorem I.10 page 246)}\\
&= \frac{1}{\|\|\mathbf{R}^{-1}\|\|^2}\sum_{n=1}^{\infty}|\alpha_n|^2\left\|\boldsymbol{u}_n\right\|^2 && \text{by } \|\cdot\| \text{ \textit{homogeneous} prop.} && \text{(Definition J.1 page 249)}\\
&= \underbrace{\frac{1}{\|\|\mathbf{R}^{-1}\|\|^2}}_{A}\sum_{n=1}^{\infty}|\alpha_n|^2 && \text{by def. of \textit{orthonormality}} && \text{(Definition I.4 page 245)}
\end{aligned}
$$

2. Proof for ($\Longrightarrow$) case:

(a) Let $\{u_n|_{n\in\mathbb{N}}\}$ be an *orthonormal basis* for $\boldsymbol{H}$.

(b) Using (2) and Lemma G.3 (page 215), construct an bounded extension operator $\mathbf{R}$ such that $\mathbf{R}u_n = x_n$ for all $n \in \mathbb{N}$.

(c) Using (2) and Lemma G.3 (page 215), construct an bounded extension operator $\mathbf{S}$ such that $\mathbf{S}x_n = u_n$ for all $n \in \mathbb{N}$.

(d) Then, $\mathbf{RV}x = \mathbf{VR}x \implies \mathbf{V} = \mathbf{R}^{-1}$, and so $\mathbf{R}$ is a bounded invertible operator

(e) and $\{x_n\}$ is a *Riesz sequence*.

✏

Theorem G.14. [34] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle\triangle\,|\,\nabla\rangle)$ *be a* SEPARABLE HILBERT SPACE.

THM

$$\left\{ \begin{array}{l} (\!(x_n \in \boldsymbol{H})\!)_{n\in\mathbb{Z}} \text{ is a} \\ \text{RIESZ BASIS for } \boldsymbol{H} \end{array} \right\} \implies \left\{ \begin{array}{lll} \multicolumn{3}{l}{\text{There exists } (\!(y_n \in \boldsymbol{H})\!)_{n\in\mathbb{Z}} \text{ such that}} \\ (1). & (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ are BIORTHOGONAL} & \text{and} \\ (2). & (\!(y_n)\!) \text{ is also a RIESZ BASIS for } \boldsymbol{H} & \text{and} \\ (3). & \exists B>A>0 \text{ such that} & \\ \multicolumn{3}{l}{\displaystyle A\sum_{n=1}^{\infty}|a_n|^2 \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\|^2 = \left\|\sum_{n=1}^{\infty} a_n y_n\right\|^2 \le B\sum_{n=1}^{\infty}|a_n|^2} \\ \multicolumn{3}{l}{\forall (\!(a_n)\!)_{n\in\mathbb{N}} \in \ell^2_{\mathbb{F}}} \end{array} \right\}$$

✎PROOF:

1. Proof for (1):

   (a) Let e_n be the *unit vector* in $\boldsymbol{H}$ such that the nth element of e_n is 1 and all other elements are 0.

   (b) Let $\mathbf{M}$ be an operator on $\boldsymbol{H}$ such that $\mathbf{M}e_n = x_n$.

   (c) Note that $\mathbf{M}$ is *isometric*, and as such $\|\mathbf{M}x\| = \|x\| \quad \forall x\in\boldsymbol{H}$.

   (d) Let $y_n \triangleq (\mathbf{M}^{-1})^*$.

   (e) Then,

$$\begin{aligned}
\langle y_n \,|\, x_m\rangle &= \left\langle (\mathbf{M}^{-1})^* e_n \,|\, \mathbf{M}e_m \right\rangle \\
&= \left\langle e_n \,|\, \mathbf{M}^{-1}\mathbf{M}e_m \right\rangle \\
&= \langle e_n \,|\, e_m\rangle \\
&= \bar{\delta}_{nm} \\
&\implies \{x_n\} \text{ and } \{y_n\} \text{ are } \textit{biorthogonal} && \text{by Definition I.4 page 245}
\end{aligned}$$

[34] Wojtaszczyk (1997) page 20 ⟨Lemma 2.7(a)⟩

2. Proof for (3):

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{y}_n\right\| &= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \left(\mathbf{M}^{-1}\right)^* \boldsymbol{e}_n\right\| && \text{by definition of } \boldsymbol{y}_n && \text{(Proposition 1d page 217)}\\
&= \left\|\left(\mathbf{M}^{-1}\right)^* \sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{by property of \textit{linear ops.}}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \left(\mathbf{M}^{-1}\right)^* \text{ is \textit{isometric}} && \text{(Definition M.10 page 300)}\\
&= \left\|\mathbf{M}\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \mathbf{M} \text{ is \textit{isometric}} && \text{(Definition M.10 page 300)}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \mathbf{M}\boldsymbol{e}_n\right\| && \text{by property of \textit{linear operators}}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{x}_n\right\| && \text{by definition of } \mathbf{M}\\
&\implies \left\{\boldsymbol{y}_n\right\} \text{ is a \textit{Riesz basis}} && \text{by left hypothesis}
\end{aligned}$$

3. Proof for (2): by (3) and definition of *Riesz basis* (Definition G.11 page 215)

Proposition G.5. [35] *Let* $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$.

P R P

$$\left\{\begin{array}{l}\left\{\boldsymbol{x}_n\right\} \text{ is a RIESZ BASIS for } \boldsymbol{H} \text{ with}\\ A\displaystyle\sum_{n=1}^{\infty}\left|a_n\right|^2 \le \left\|\sum_{n=1}^{\infty} a_n\boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}\left|a_n\right|^2\\ \forall\left\{a_n\right\}\in\ell^2_{\mathbb{F}}\end{array}\right\} \implies \left\{\begin{array}{l}\left\{\boldsymbol{x}_n\right\} \text{ is a FRAME for } \boldsymbol{H} \text{ with}\\ \underbrace{\frac{1}{B}\|\boldsymbol{x}\|^2 \le \displaystyle\sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \le \frac{1}{A}\|\boldsymbol{x}\|^2}_{\text{STABILITY CONDITION}}\\ \forall \boldsymbol{x}\in\boldsymbol{H}\end{array}\right\}$$

✎PROOF:

1. Let $\left\{\boldsymbol{y}_n \middle| n\in\mathbb{N}\right\}$ be a *Riesz basis* that is *biorthogonal* to $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ (Theorem G.14 page 217).

2. Let $\boldsymbol{x} \triangleq \displaystyle\sum_{n=1}^{\infty} a_n \boldsymbol{y}_n$.

3. lemma:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 &= \sum_{n=1}^{\infty}\left|\left\langle \sum_{m=1}^{\infty} a_n \boldsymbol{y}_m \mid \boldsymbol{x}_n\right\rangle\right|^2 && \text{by definition of } \boldsymbol{x} && \text{(item (2) page 218)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \langle \boldsymbol{y}_m \mid \boldsymbol{x}_n\rangle\right|^2 && \text{by \textit{homogeneous} property of } \langle \triangle \mid \nabla \rangle && \text{(Definition I.1 page 233)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \bar{\delta}_{mn}\right|^2 && \text{by definition of \textit{biorthogonal}} && \text{(Definition G.13 page 215)}\\
&= \sum_{n=1}^{\infty}\left|a_n\right|^2 && \text{by definition of } \bar{\delta} && \text{(Definition I.3 page 245)}
\end{aligned}$$

[35] Igari (1996) page 220 ⟨Lemma 9.8⟩, Wojtaszczyk (1997) pages 20–21 ⟨Lemma 2.7(a)⟩

4. Then

$$\begin{array}{llllll}
& A\sum_{n=1}^{\infty}|a_n|^2 & \le & \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{x}_n\right\|^2 & \le & B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 218)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le & \left\|\sum_{n=1}^{\infty}a_n\boldsymbol{y}_n\right\|^2 & \le & B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} \text{ (item (1) page 218)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le & \|\boldsymbol{x}\|^2 & \le & B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \boldsymbol{x} \text{ (item (2) page 218)}\\
\implies & A\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \le & \|\boldsymbol{x}\|^2 & \le & B\sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \text{by (3) lemma}\\
\implies & \frac{1}{B}\|\boldsymbol{x}\|^2 & \le & \sum_{n=1}^{\infty}|\langle\boldsymbol{x}\,|\,\boldsymbol{x}_n\rangle|^2 & \le & \frac{1}{A}\|\boldsymbol{x}\|^2 &
\end{array}$$

⇨

Theorem G.15 (Battle-Lemarié orthogonalization). [36] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM *(Definition N.2 page 309) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM

$$\left\{\begin{array}{ll}1. & \{\mathbf{T}^n\mathsf{g}\,|\,n\in\mathbb{Z}\} \text{ is a RIESZ BASIS for } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{and}\\ 2. & \tilde{\mathsf{f}}(\omega) \triangleq \dfrac{\tilde{\mathsf{g}}(\omega)}{\sqrt{2\pi\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2}}\end{array}\right\} \implies \left\{\begin{array}{l}\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}\\ \text{is an ORTHONORMAL BASIS for } \boldsymbol{L}^2_{\mathbb{R}}\end{array}\right\}$$

✎PROOF:

1. Proof that $\{\mathbf{T}^n\mathsf{f}\,|\,n\in\mathbb{Z}\}$ is orthonormal:

$$\begin{aligned}
\tilde{S}_{\phi\phi}(\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem ?? page ??}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{2\pi\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi(n-m))|^2}}\right|^2 && \text{by left hypothesis}\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{1}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2|\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= \frac{1}{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= 1\\
\implies & \{\mathsf{f}_n\,|\,n\in\mathbb{Z}\} \text{ is orthonormal} && \text{by Theorem ?? page ??}
\end{aligned}$$

[36] Wojtaszczyk (1997) page 25 ⟨Remark 2.4⟩, Vidakovic (1999), page 71, Mallat (1989), page 72, Mallat (1999), page 225, Daubechies (1992) page 140 ⟨(5.3.3)⟩

2. Proof that $\{\mathbf{T}^n \mathsf{f} \mid n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_0$: by Lemma G.1 page 203.

✏

G.7 Frames in Hilbert spaces

Definition G.14. [37] *Let* $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

DEF

The set $\{\boldsymbol{x}_n\}$ *is a* ***frame*** *for* $\boldsymbol{H}$ *if* (STABILITY CONDITION)

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\,\|\boldsymbol{x}\|^2 \le \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle\right|^2 \le B\,\|\boldsymbol{x}\|^2 \qquad \forall x\in X.$$

The quantities A and B are ***frame bounds***.
The quantity A' *is the* ***optimal lower frame bound*** *if*
$A' = \sup\{A \in \mathbb{R}^+ \mid A \text{ is a lower frame bound}\}$.
The quantity B' *is the* ***optimal upper frame bound*** *if*
$B' = \inf\{B \in \mathbb{R}^+ \mid B \text{ is an upper frame bound}\}$.
A frame is a ***tight frame*** *if* $A = B$.
A frame is a ***normalized tight frame*** *(or a* ***Parseval frame****) if* $A = B = 1$.
A frame $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ *is an* ***exact frame*** *if for some* $m \in \mathbb{Z}$, $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\} \setminus \{\boldsymbol{x}_m\}$ *is* NOT *a frame.*

A frame is a *Parseval frame* (Definition G.14) if it satisfies *Parseval's Identity* (Theorem G.9 page 210). All orthonormal bases are Parseval frames (Theorem G.9 page 210); but not all Parseval frames are orthonormal bases.

Definition G.15. *Let* $\{\boldsymbol{x}_n\}$ *be a* ***frame*** *(Definition G.14 page 220) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$. *Let* $\mathbf{S}$ *be an* OPERATOR *on* $\boldsymbol{H}$.

DEF

$\mathbf{S}$ *is a* ***frame operator*** *for* $\{\boldsymbol{x}_n\}$ *if* $\quad \mathbf{S}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \qquad \forall \mathsf{f}\in\boldsymbol{H}.$

Theorem G.16. [38] *Let* $\mathbf{S}$ *be a* FRAME OPERATOR *(Definition G.15 page 220) of a* FRAME $\{\boldsymbol{x}_n\}$ *(Definition G.14 page 220) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM

(1). $\mathbf{S}$ *is* INVERTIBLE. *and*
(2). $\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \rangle \boldsymbol{x}_n = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n \rangle \mathbf{S}^{-1}\boldsymbol{x}_n \quad \forall \mathsf{f} \in \boldsymbol{H}$

Theorem G.17. [39] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM

$\{\boldsymbol{x}_n\}$ *is a* FRAME *for* $\operatorname{span}\{\boldsymbol{x}_n\}$.

✎PROOF:

[37] Young (2001) pages 154–155, Christensen (2003) page 88 ⟨Definitions 5.1.1, 5.1.2⟩, Heil (2011) pages 204–205 ⟨Definition 8.2⟩, Jørgensen et al. (2008) page 267 ⟨Definition 12.22⟩, Duffin and Schaeffer (1952) page 343, Daubechies et al. (1986), page 1272

[38] Christensen (2008) pages 100–102 ⟨Theorem 5.1.7⟩

[39] Christensen (2003) page 3

1. Upper bound: Proof that there exists B such that $\sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 \leq B \|\boldsymbol{x}\|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

$$\begin{aligned}
\sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 &\leq \sum_{n=1}^{N} \langle \boldsymbol{x}_n \mid \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by } \textit{Cauchy-Schwarz inequality} \text{ (Theorem I.2 page 234)}\\
&= \underbrace{\left\{ \sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2 \right\}}_{B} \|\boldsymbol{x}\|^2
\end{aligned}$$

2. Lower bound: Proof that there exists A such that $A \|\boldsymbol{x}\|^2 \leq \sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

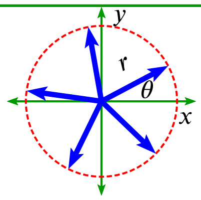
$$\begin{aligned}
\sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 &= \sum_{n=1}^{N} \left| \left\langle \boldsymbol{x}_n \mid \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right\rangle \right|^2 \|\boldsymbol{x}\|^2 \\
&\geq \underbrace{\left(\inf_{\boldsymbol{y}} \left\{ \sum_{n=1}^{N} |\langle \boldsymbol{x}_n \mid \boldsymbol{y} \rangle|^2 \mid \|\boldsymbol{y}\| = 1 \right\} \right)}_{A} \|\boldsymbol{x}\|^2
\end{aligned}$$

✏

Example G.1. Let $(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be an inner product space with $\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \mid \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1 x_2 + y_1 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition G.15 page 220) with *inverse* $\mathbf{S}^{-1}$.

EX

Let $N \in \{3, 4, 5, \ldots\}$, $\theta \in \mathbb{R}$, and $r \in \mathbb{R}^+$ $(r > 0)$.
Let $\boldsymbol{x}_n \triangleq r \begin{bmatrix} \cos(\theta + 2n\pi/N) \\ \sin(\theta + 2n\pi/N) \end{bmatrix} \quad \forall n \in \{0, 1, \ldots, N-1\}$.
Then, $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{Nr^2}{2}$.



Moreover, $\mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{S}^{-1} = \frac{2}{Nr^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\boldsymbol{v} = \frac{2}{Nr^2} \sum_{n=0}^{N-1} \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \quad \forall \boldsymbol{v} \in \mathbb{R}^2$.

✎PROOF:

1. Proof that $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})$ is a *tight frame* with *frame bound* $A = \frac{Nr^2}{2}$: Let $\boldsymbol{v} \triangleq (x, y) \in \mathbb{R}^2$.

$$\begin{aligned}
\sum_{n=0}^{N-1} |\langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle|^2 &\triangleq \sum_{n=0}^{N-1} \left| \boldsymbol{v}^{\mathbf{H}} r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \right|^2 && \text{by definitions of } \boldsymbol{v} \text{ of } \langle \boldsymbol{y} \mid \boldsymbol{x} \rangle \\
&\triangleq \sum_{n=0}^{N-1} r^2 \left| x\cos\left(\theta + \frac{2n\pi}{N}\right) + y\sin\left(\theta + \frac{2n\pi}{N}\right) \right|^2 && \text{by definition of } \boldsymbol{y}^{\mathbf{H}}\boldsymbol{x} \text{ operation} \\
&= r^2 x^2 \sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 y^2 \sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 xy \sum_{n=0}^{N-1} \cos\left(\theta + \frac{2n\pi}{N}\right) \sin\left(\theta + \frac{2n\pi}{N}\right) \\
&= r^2 x^2 \frac{N}{2} + r^2 y^2 \frac{N}{2} + r^2 xy 0 && \text{by Corollary ?? page ??} \\
&= (x^2 + y^2) \frac{Nr^2}{2} = \left(\frac{Nr^2}{2} \right) \boldsymbol{v}^{\mathbf{H}} \boldsymbol{v} \triangleq \underbrace{\left(\frac{Nr^2}{2} \right)}_{A} \|\boldsymbol{v}\|^2 && \text{by definition of } \|\boldsymbol{v}\|
\end{aligned}$$

2. Proof that $\mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

(a) Let $\boldsymbol{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) lemma: $\mathbf{S}\boldsymbol{e}_1 = \frac{Nr^2}{2}\boldsymbol{e}_1$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_1 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\cos\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \cos^2\left(\theta + \frac{2n\pi}{N}\right) \\ \cos\left(\theta + \frac{2n\pi}{N}\right)\sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} N/2 \\ 0 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_1 && \text{by \textit{Summation around unit circle} (Corollary ?? page ??)}
\end{aligned}$$

(c) lemma: $\mathbf{S}\boldsymbol{e}_2 = \frac{Nr^2}{2}\boldsymbol{e}_2$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_2 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_2 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\sin\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \sin\left(\theta + \frac{2n\pi}{N}\right)\cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin^2\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} 0 \\ N/2 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_2 && \text{by \textit{Summation around unit circle} (Corollary ?? page ??)}
\end{aligned}$$

(d) Complete the proof of item (2) using *Eigendecomposition* $\mathbf{S} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \frac{Nr^2}{2}\boldsymbol{e}_1 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_1 \text{ is an \textit{eigenvector} of } \mathbf{S} \text{ with \textit{eigenvalue} } \frac{Nr^2}{2} \\
\mathbf{S}\boldsymbol{e}_2 &= \frac{Nr^2}{2}\boldsymbol{e}_2 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_2 \text{ is an \textit{eigenvector} of } \mathbf{S} \text{ with \textit{eigenvalue} } \frac{Nr^2}{2}
\end{aligned}$$

$$\mathbf{S} = \overbrace{\underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}^{-1}}_{\mathbf{Q}^{-1}}}^{\textit{Eigendecomposition} \text{ of } \mathbf{S}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Proof that $\mathbf{S}^{-1} = \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$\begin{aligned}
\mathbf{S}\mathbf{S}^{-1} &= \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)} \\
\mathbf{S}^{-1}\mathbf{S} &= \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)}
\end{aligned}$$

4. Proof that $\boldsymbol{v} = \frac{2}{Nr^2}\sum_{n=0}^{N-1} (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n = \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,|\, \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n && \text{by item (3)} \\
&= \frac{2}{Nr^2}\sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \frac{2}{Nr^2}\sum_{n=0}^{N-1} (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n && \text{by definition of } \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle
\end{aligned}$$

Example G.2 (Peace Frame/Mercedes Frame). [40] Let $(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ be an inner product space with $\langle (x_1, y_1) \mid (x_2, y_2) \rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition G.15 page 220) with *inverse* $\mathbf{S}^{-1}$.

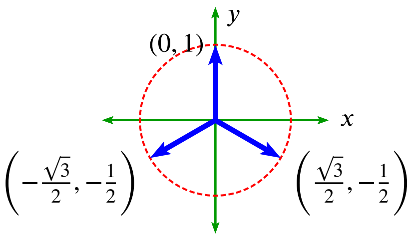
EX

Let $\boldsymbol{x}_1 \triangleq \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\boldsymbol{x}_2 \triangleq \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}$, and $\boldsymbol{x}_3 \triangleq \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$.

Then, $⦅\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3⦆$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{3}{2}$.

Moreover, $\mathbf{S} = \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \mathbf{S}^{-1} = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

and $\quad \boldsymbol{v} = \frac{2}{3}\sum_{n=1}^{3} \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \triangleq \frac{2}{3}\sum_{n=1}^{3} (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n \qquad \forall \boldsymbol{v} \in \mathbb{R}^2.$



✎PROOF:

1. This frame is simply a special case of the frame presented in Example G.1 (page 221) with $r = 1$, $N = 3$, and $\theta = \pi/2$.

2. Let's give it a try! Let $\boldsymbol{v} \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned}
\sum_{n=1}^{3} \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n &= \frac{2}{3}\sum_{n=1}^{3} (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n)\boldsymbol{x}_n && \text{by Example G.1 page 221} \\
&= (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_1)\boldsymbol{x}_1 + (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_2)\boldsymbol{x}_2 + (\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_3)\boldsymbol{x}_3 \\
&= \frac{2}{3}\left(\left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{2}{3}\cdot\frac{1}{2}\left(\left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{1}{3}\left((2)\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \left(-\sqrt{3} - 1\right)\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix} + \left(\sqrt{3} - 1\right)\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right) \\
&= \frac{1}{6}\begin{bmatrix} 2(0) & + & (-\sqrt{3} - 1)(-\sqrt{3}) & + & (\sqrt{3} - 1)(\sqrt{3}) \\ 2(2) & + & (-\sqrt{3} - 1)(-1) & + & (\sqrt{3} - 1)(-1) \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 0 & + & (3 + \sqrt{3}) & + & (3 - \sqrt{3}) \\ 4 & + & (1 + \sqrt{3}) & + & (1 - \sqrt{3}) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \triangleq \boldsymbol{v}
\end{aligned}$$

In Example G.1 (page 221) and Example G.2 (page 223), the frame operator $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ were computed. In general however, it is not always necessary or even possible to compute these, as illustrated in Example G.3 (next).

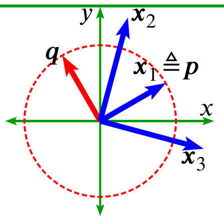
Example G.3. [41] Let $(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ be an inner product space with $\langle (x_1, y_1) \mid (x_2, y_2) \rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition G.15 page 220) with *inverse* $\mathbf{S}^{-1}$.

[40] Heil (2011) pages 204–205 ⟨$r = 1$ case⟩, Byrne (2005) page 80 ⟨$r = 1$ case⟩, Han et al. (2007) page 91 ⟨Example 3.9, $r = \sqrt{2/3}$ case⟩

[41] Christensen (2003) pages 7–8 ⟨?⟩

EX

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be *orthonormal* vectors in $\boldsymbol{X} \triangleq \operatorname{span}\{\boldsymbol{p}, \boldsymbol{q}\}$.
Let $\boldsymbol{x}_1 \triangleq \boldsymbol{p}$, $\boldsymbol{x}_2 \triangleq \boldsymbol{p} + \boldsymbol{q}$, and $\boldsymbol{x}_3 \triangleq \boldsymbol{p} - \boldsymbol{q}$.
Then, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, , \boldsymbol{x}_3\}$ is a **frame** for $\boldsymbol{X}$ with *frame bounds* $A = 0$ and $B = 5$.



Moreover,
$\mathbf{S}^{-1}\boldsymbol{x}_1 = {}^1\!/\!_3\boldsymbol{p}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_2 = {}^1\!/\!_3\boldsymbol{p} + {}^1\!/\!_2\boldsymbol{q}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_3 = {}^1\!/\!_3\boldsymbol{p} - {}^1\!/\!_2\boldsymbol{q}$.

✎PROOF:

1. Proof that $(\!(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\!)$ is a *frame* with *frame bounds* $A = 0$ and $B = 5$:

$$\begin{aligned}
\sum_{n=1}^{3} \left|\langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle\right|^2 &\triangleq |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle|^2 && \text{by definitions of } \boldsymbol{x}_1, \boldsymbol{x}_2, \text{ and } \boldsymbol{x}_3 \\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 && \text{by } \textit{additivity} \text{ of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition I.1 page 233)} \\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right) \\
&\quad + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 - \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right) \\
&= 3|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + 2|\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 \\
&\boxed{\leq} 3\,\|\boldsymbol{v}\|\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{v}\|\,\|\boldsymbol{q}\| && \text{by } \textit{CS Inequality} \text{ (Theorem I.2 page 234)} \\
&= \|\boldsymbol{v}\|\,(3\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{q}\|) \\
&= \boxed{5}\,\|\boldsymbol{v}\| && \text{by } \textit{orthonormality} \text{ of } \boldsymbol{p} \text{ and } \boldsymbol{q}
\end{aligned}$$

2. lemma: $\mathbf{S}\boldsymbol{p} = 3\boldsymbol{p}$, $\mathbf{S}\boldsymbol{q} = 2\boldsymbol{q}$, $\mathbf{S}^{-1}\boldsymbol{p} = \frac{1}{3}\boldsymbol{p}$, and $\mathbf{S}^{-1}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{p} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{p} \,|\, \boldsymbol{x}_n \rangle\, \boldsymbol{x}_n \\
&= \langle \boldsymbol{p} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{p} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\, (\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{p} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\, (\boldsymbol{p}-\boldsymbol{q}) \\
&= (1)\boldsymbol{p} + (1+0)(\boldsymbol{p}+\boldsymbol{q}) + (1-0)(\boldsymbol{p}-\boldsymbol{q}) \\
&= 3\boldsymbol{p} \\
\implies \mathbf{S}^{-1}\boldsymbol{p} &= \frac{1}{3}\boldsymbol{p} \\
\mathbf{S}\boldsymbol{q} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{q} \,|\, \boldsymbol{x}_n \rangle\, \boldsymbol{x}_n \\
&= \langle \boldsymbol{q} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{q} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\, (\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{q} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\, (\boldsymbol{p}-\boldsymbol{q}) \\
&= (0)\boldsymbol{q} + (0+1)(\boldsymbol{p}+\boldsymbol{q}) + (0-1)(\boldsymbol{p}-\boldsymbol{q}) \\
&= 2\boldsymbol{q} \\
\implies \mathbf{S}^{-1}\boldsymbol{q} &= \frac{1}{2}\boldsymbol{q}
\end{aligned}$$

3. Remark: Without knowing $\boldsymbol{p}$ and $\boldsymbol{q}$, from (2) lemma it follows that it is not possible to compute $\mathbf{S}$ or $\mathbf{S}^{-1}$ explicitly.

4. Proof that $\mathbf{S}^{-1}\boldsymbol{x}_1 = {}^1\!/\!_3\boldsymbol{p}$, $\mathbf{S}^{-1}\boldsymbol{x}_2 = {}^1\!/\!_3\boldsymbol{p} + {}^1\!/\!_2\boldsymbol{q}$ and $\mathbf{S}^{-1}\boldsymbol{x}_3 = {}^1\!/\!_3\boldsymbol{p} - {}^1\!/\!_2\boldsymbol{q}$:

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_1 &\triangleq \mathbf{S}^{-1}\boldsymbol{p} && \text{by definition of } \boldsymbol{x}_1 \\
&= \frac{1}{3}\boldsymbol{p} && \text{by (2) lemma} \\
\mathbf{S}^{-1}\boldsymbol{x}_2 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2 \\
&= \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_3 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2\\
&= \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

5. Check that $\boldsymbol{v} = \sum_n \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=1}^{3} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n\\
&= \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q})\\
&= \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q})\\
&= \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \left(\frac{1}{3}-\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{q} + \left(\frac{1}{2}-\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{p} + \left(\frac{1}{2}+\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}\\
&= \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}
\end{aligned}$$

APPENDIX H

NORMED ALGEBRAS

H.1 Algebras

All *linear spaces* (Definition F.1 page 185) are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to be "multiplied" together. But linear spaces in general have no operation that allows two vectors to be multiplied together. A linear space together with such an operator is an **algebra**.[1]

There are many many possible algebras—many more than one can shake a stick at, as indicated by Michiel Hazewinkel in his book, *Handbook of Algebras*: "Algebra, as we know it today (2005), consists of many different ideas, concepts and results. A reasonable estimate of the number of these different items would be somewhere between 50,000 and 200,000. Many of these have been named and many more could (and perhaps should) have a "name" or other convenient designation."[2]

Definition H.1. [3] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF *An algebra $\boldsymbol{A}$ is **unital** if* $\exists u \in \boldsymbol{A}$ such that $ux = xu = x \qquad \forall x \in \boldsymbol{A}$

Definition H.2. [4] *Let $\boldsymbol{A}$ be an* UNITAL ALGEBRA *(Definition H.1 page 227) with unit e.*

DEF

$$\begin{array}{llcl} \textit{The } \textbf{\textit{spectrum}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \sigma(x) & \triangleq & \{\lambda \in \mathbb{C} | \lambda e - x \textit{ is not invertible}\}. \\ \textit{The } \textbf{\textit{resolvent}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & \rho_x(\lambda) & \triangleq & (\lambda e - x)^{-1} \qquad \forall \lambda \notin \sigma(x). \\ \textit{The } \textbf{\textit{spectral radius}} \textit{ of } x \in \boldsymbol{A} \textit{ is} & r(x) & \triangleq & \sup \{|\lambda| \,|\, \lambda \in \sigma(x)\}. \end{array}$$

[1] Fuchs (1995) page 2
[2] Hazewinkel (2000) page v
[3] Folland (1995) page 1
[4] Folland (1995) pages 3–4

H.2 Star-Algebras

Definition H.3. [5] *Let $\boldsymbol{A}$ be an* ALGEBRA.

DEF

*The pair $(\boldsymbol{A}, *)$ is a $*$-**algebra**, or **star-algebra**, if*

1.	$(x+y)^* = x^* + y^*$	$\forall x,y \in \boldsymbol{A}$	(DISTRIBUTIVE)	and
2.	$(\alpha x)^* = \bar{\alpha} x^*$	$\forall x \in \boldsymbol{A}, \alpha \in \mathbb{C}$	(CONJUGATE LINEAR)	and
3.	$(xy)^* = y^* x^*$	$\forall x,y \in \boldsymbol{A}$	(ANTIAUTOMORPHIC)	and
4.	$x^{**} = x$	$\forall x \in \boldsymbol{A}$	(INVOLUTORY)	

*The operator $*$ is called an **involution** on the algebra $\boldsymbol{A}$.*

Proposition H.1. [6] *Let $(\boldsymbol{A}, *)$ be an* UNITAL $*$-ALGEBRA.

PRP

$$x \text{ is invertible} \implies \begin{cases} 1. & x^* \text{ is INVERTIBLE} \quad \forall x \in \boldsymbol{A} \quad \text{and} \\ 2. & \left(x^*\right)^{-1} = \left(x^{-1}\right)^* \quad \forall x \in \boldsymbol{A} \end{cases}$$

✎PROOF: Let e be the unit element of $(\boldsymbol{A}, *)$.

1. Proof that $e^* = e$:

$$\begin{aligned}
x\,e^* &= \left(x\,e^*\right)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \left(x^*\,e^{**}\right)^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \left(x^*\,e\right)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \left(x^*\right)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}\\
e^*\,x &= \left(e^*\,x\right)^{**} && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \left(e^{**}\,x^*\right)^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \left(e\,x^*\right)^* && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \left(x^*\right)^* && \text{by definition of } e\\
&= x && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}
\end{aligned}$$

2. Proof that $\left(x^*\right)^{-1} = \left(x^{-1}\right)^*$:

$$\begin{aligned}
\left(x^{-1}\right)^*\left(x^*\right) &= \left[x\left(x^{-1}\right)\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition H.3 page 228)}\\
&= e^*\\
&= e && \text{by item (1) page 228}\\
\left(x^*\right)\left(x^{-1}\right)^* &= \left[x^{-1}\,x\right]^* && \text{by } \textit{antiautomorphic} \text{ and } \textit{involution} \text{ properties of } * && \text{(Definition H.3 page 228)}\\
&= e^*\\
&= e && \text{by item (1) page 228}
\end{aligned}$$

⇨

Definition H.4. [7] *Let $(\boldsymbol{A}, \|\cdot\|)$ be a $*$-*ALGEBRA *(Definition H.3 page 228).*

DEF

- *An element $x \in \boldsymbol{A}$ is **hermitian** or **self-adjoint** if $x^* = x$.*
- *An element $x \in \boldsymbol{A}$ is **normal** if $xx^* = x^*x$.*
- *An element $x \in \boldsymbol{A}$ is a **projection** if $xx = x$ (INVOLUTORY) and $x^* = x$ (HERMITIAN).*

[5] Rickart (1960), page 178, Gelfand and Naimark (1964), page 241

[6] Folland (1995) page 5

[7] Rickart (1960), page 178, Gelfand and Naimark (1964), page 242

Theorem H.1. [8] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition H.3 page 228).*

THM
$$\underbrace{x = x^* \text{ and } y = y^*}_{x \text{ and } y \text{ are hermitian}} \implies \left\{\begin{array}{lll} x+y = (x+y)^* & & (x+y \text{ is self adjoint}) \\ x^* = (x^*)^* & & (x^* \text{ is self adjoint}) \\ \underbrace{xy = (xy)^*}_{(xy) \text{ is hermitian}} \iff \underbrace{xy = yx}_{\text{commutative}} \end{array}\right.$$

✎PROOF:

$$\begin{aligned}(x+y)^* &= x^* + y^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition H.3 page 228)}\\ &= x + y && \text{by left hypothesis}\end{aligned}$$

$$\left(x^*\right)^* = x \qquad \text{by } \textit{involutory} \text{ property of } * \qquad \text{(Definition H.3 page 228)}$$

Proof that $xy = (xy)^* \implies xy = yx$

$$\begin{aligned}xy &= (xy)^* && \text{by left hypothesis}\\ &= y^*x^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition H.3 page 228)}\\ &= yx && \text{by left hypothesis}\end{aligned}$$

Proof that $xy = (xy)^* \impliedby xy = yx$

$$\begin{aligned}(xy)^* &= (yx)^* && \text{by left hypothesis}\\ &= x^*y^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition H.3 page 228)}\\ &= xy && \text{by left hypothesis}\end{aligned}$$

⇨

Definition H.5 (Hermitian components). [9] *Let* $(\boldsymbol{A}, \|\cdot\|)$ *be a* $*$-ALGEBRA *(Definition H.3 page 228).*

DEF
*The **real part** of* x *is defined as* $\mathbf{R}_{\mathrm{e}}x \triangleq \frac{1}{2}\left(x + x^*\right)$

*The **imaginary part** of* x *is defined as* $\mathbf{I}_{\mathrm{m}}x \triangleq \frac{1}{2i}\left(x - x^*\right)$

Theorem H.2. [10] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition H.3 page 228).*

THM
$$\begin{array}{lllll} \Re x &=& (\Re x)^* & \forall x \in \boldsymbol{A} & (\Re x \text{ is hermitian}) \\ \Im x &=& (\Im x)^* & \forall x \in \boldsymbol{A} & (\Im x \text{ is hermitian}) \end{array}$$

✎PROOF:

$$\begin{aligned}(\Re x)^* &= \left(\frac{1}{2}\left(x + x^*\right)\right)^* && \text{by definition of } \Re && \text{(Definition H.5 page 229)}\\ &= \frac{1}{2}\left(x^* + x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition H.3 page 228)}\\ &= \frac{1}{2}\left(x^* + x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}\\ &= \Re x && \text{by definition of } \Re && \text{(Definition H.5 page 229)}\end{aligned}$$

$$(\Im x)^* = \left(\frac{1}{2i}\left(x - x^*\right)\right)^* \qquad \text{by definition of } \Im \qquad \text{(Definition H.5 page 229)}$$

[8] Michel and Herget (1993) page 429
[9] Michel and Herget (1993) page 430, Rickart (1960), page 179, Gelfand and Naimark (1964), page 242
[10] Michel and Herget (1993) page 430, Halmos (1998) page 42

$$
\begin{aligned}
&= \frac{1}{2i}\left(x^* - x^{**}\right) && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \frac{1}{2i}\left(x^* - x\right) && \text{by } \textit{involutory} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \Im x && \text{by definition of } \Im && \text{(Definition H.5 page 229)}
\end{aligned}
$$

Theorem H.3 (Hermitian representation). [11] *Let* $(\boldsymbol{A}, *)$ *be a* $*$-ALGEBRA *(Definition H.3 page 228).*

THM

$$a = x + iy \iff x = \Re a \quad \textit{and} \quad y = \Im a$$

✎PROOF:

Proof that $a = x + iy \implies x = \Re a$ and $y = \Im a$:

$$
\begin{aligned}
& & a &= x + iy && \text{by left hypothesis}\\
&\implies & a^* &= (x+iy)^* && \text{by definition of } \textit{adjoint} && \text{(Definition H.4 page 228)}\\
& & &= x^* - iy^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition H.3 page 228)}\\
& & &= x - iy && \text{by Theorem H.2 page 229}\\
&\implies & x &= a - iy && \text{by solving for } x \text{ in } a = x+iy \text{ equation}\\
& & x &= a^* + iy && \text{by solving for } x \text{ in } a^* = x - iy \text{ equation}\\
&\implies & x + x &= a + a^* && \text{by adding previous 2 equations}\\
&\implies & 2x &= a + a^* && \text{by solving for } x \text{ in previous equation}\\
&\implies & x &= \frac{1}{2}\left(a + a^*\right)\\
& & &= \Re a && \text{by definition of } \Re && \text{(Definition H.5 page 229)}
\end{aligned}
$$

$$
\begin{aligned}
& & iy &= a - x && \text{by solving for } iy \text{ in } a = x+iy \text{ equation}\\
& & iy &= -a^* + x && \text{by solving for } iy \text{ in } a = x+iy \text{ equation}\\
&\implies & iy + iy &= a - a^* && \text{by adding previous 2 equations}\\
&\implies & y &= \frac{1}{2i}\left(a - a^*\right) && \text{by solving for } iy \text{ in previous equations}\\
& & &= \Im a && \text{by definition of } \Im && \text{(Definition H.5 page 229)}
\end{aligned}
$$

Proof that $a = x + iy \impliedby x = \Re a$ and $y = \Im a$:

$$
\begin{aligned}
x + iy &= \Re a + i\,\Im a && \text{by right hypothesis}\\
&= \underbrace{\frac{1}{2}\left(a + a^*\right)}_{\Re a} + i\underbrace{\frac{1}{2i}\left(a - a^*\right)}_{\Im a} && \text{by definition of } \Re \text{ and } \Im && \text{(Definition H.5 page 229)}\\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \cancelto{0}{\left(\frac{1}{2}a^* - \frac{1}{2}a^*\right)}\\
&= a
\end{aligned}
$$

[11] Michel and Herget (1993) page 430, Rickart (1960), page 179, Gelfand and Neumark (1943b), page 7

H.3 Normed Algebras

Definition H.6. [12] *Let $\boldsymbol{A}$ be an algebra.*

DEF

*The pair $(\boldsymbol{A}, \|\cdot\|)$ is a **normed algebra** if*

$$\|xy\| \le \|x\|\,\|y\| \qquad \forall x, y \in \boldsymbol{A} \qquad \textit{(multiplicative condition)}$$

*A normed algebra $(\boldsymbol{A}, \|\cdot\|)$ is a **Banach algebra** if $(\boldsymbol{A}, \|\cdot\|)$ is also a Banach space.*

Proposition H.2.

PRP

$$(\boldsymbol{A}, \|\cdot\|) \text{ is a normed algebra} \implies \text{multiplication is \textbf{continuous} in } (\boldsymbol{A}, \|\cdot\|)$$

✎PROOF:

1. Define $\mathsf{f}(x) \triangleq zx$. That is, the function f represents multiplication of x times some arbitrary value z.
2. Let $\delta \triangleq \|x - y\|$ and $\epsilon \triangleq \|\mathsf{f}(x) - \mathsf{f}(y)\|$.
3. To prove that multiplication (f) is *continuous* with respect to the metric generated by $\|\cdot\|$, we have to show that we can always make ϵ arbitrarily small for some $\delta > 0$.
4. And here is the proof that multiplication is indeed continuous in $(\boldsymbol{A}, \|\cdot\|)$:

$$\begin{aligned}
\|\mathsf{f}(x) - \mathsf{f}(y)\| &\triangleq \|zx - zy\| && \text{by definition of } \mathsf{f} && \text{(item (1) page 231)}\\
&= \|z(x-y)\| \\
&\le \|z\|\,\|x-y\| && \text{by definition of \textit{normed algebra}} && \text{(Definition H.6 page 231)}\\
&\triangleq \|z\|\;\delta && \text{by definition of } \delta && \text{(item (2) page 231)}\\
&\le \epsilon && \text{for some value of } \delta > 0
\end{aligned}$$

✏

Theorem H.4 (Gelfand-Mazur Theorem). [13] *Let $\mathbb{C}$ be the field of complex numbers.*

THM

$$\left.\begin{array}{l} (\boldsymbol{A}, \|\cdot\|) \text{ is a Banach algebra} \\ \text{every nonzero } x \in \boldsymbol{A} \text{ is invertible} \end{array}\right\} \implies \boldsymbol{A} \equiv \mathbb{C} \quad (\boldsymbol{A} \text{ is isomorphic to } \mathbb{C})$$

H.4 C* Algebras

Definition H.7. [14]

DEF

*The triple $(\boldsymbol{A}, \|\cdot\|, *)$ is a C^* **algebra** if*

1. $(\boldsymbol{A}, \|\cdot\|)$ *is a Banach algebra* *and*
2. $(\boldsymbol{A}, *)$ *is a $*$-algebra* *and*
3. $\left\|x^* x\right\| = \|x\|^2 \quad \forall x \in \boldsymbol{A}$.

*A C^* **algebra** $(\boldsymbol{A}, \|\cdot\|, *)$ is also called a **C star algebra**.*

[12] Rickart (1960), page 2, Berberian (1961) page 103 ⟨Theorem IV.9.2⟩
[13] Folland (1995) page 4, Mazur (1938) ⟨(statement)⟩, Gelfand (1941) ⟨(proof)⟩
[14] Folland (1995) page 1, Gelfand and Naimark (1964), page 241, Gelfand and Neumark (1943a), Gelfand and Neumark (1943b)

Theorem H.5. [15] *Let $\boldsymbol{A}$ be an algebra.*

THM $(\boldsymbol{A}, \|\cdot\|, *)$ *is a C^** ***algebra*** $\implies$ $\|x^*\| = \|x\|$

✎PROOF:

$$\begin{aligned}
\|x\| &= \frac{1}{\|x\|}\ \|x\|^2 \\
&= \frac{1}{\|x\|}\ \|x^* x\| && \text{by definition of } C^*\text{-}\textit{algebra} && \text{(Definition H.7 page 231)} \\
&\le \frac{1}{\|x\|}\ \|x^*\|\ \|x\| && \text{by definition of } \textit{normed algebra} && \text{(Definition H.6 page 231)} \\
&= \|x^*\| \\
\|x^*\| &\le \|x^{**}\| && \text{by previous result} \\
&= \|x\| && \text{by } \textit{involution} \text{ property of } * && \text{(Definition H.3 page 228)}
\end{aligned}$$

✎

[15] Folland (1995) page 1, Gelfand and Neumark (1943b), page 4, Gelfand and Neumark (1943a)

APPENDIX I

INNER PRODUCT SPACES

I.1 Definition and basic results

Definition I.1. [1] *Let* $\Omega \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition F.1 page 185).*

D E F

A FUNCTIONAL $\langle \triangle \mid \triangledown \rangle \in \mathbb{F}^{X \times X}$ *is an **inner product** on* Ω *if*

1. $\langle \alpha x \mid y \rangle = \alpha \langle x \mid y \rangle \quad \forall x,y \in X,\ \forall \alpha \in \mathbb{C}$ (HOMOGENEOUS) *and*
2. $\langle x + y \mid u \rangle = \langle x \mid u \rangle + \langle y \mid u \rangle \quad \forall x,y,u \in X$ (ADDITIVE) *and*
3. $\langle x \mid y \rangle = \langle y \mid x \rangle^* \quad \forall x,y \in X$ (CONJUGATE SYMMETRIC). *and*
4. $\langle x \mid x \rangle \geq 0 \quad \forall x \in X$ (NON-NEGATIVE) *and*
5. $\langle x \mid x \rangle = 0 \iff x = \mathbb{0} \quad \forall x \in X$ (NON-ISOTROPIC)

*An inner product is also called a **scalar product**.*
The tuple $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *is called an **inner product space**.*

Theorem I.1. [2] *Let* $\Omega \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a* LINEAR SPACE *(Definition F.1 page 185).*

T H M

1. $\langle x \mid y + z \rangle = \langle x \mid y \rangle + \langle x \mid z \rangle \quad \forall x,y,z \in X$
2. $\langle x \mid \alpha y \rangle = \alpha^* \langle x \mid y \rangle \quad \forall x,y \in X,\ \alpha \in \mathbb{F}$
3. $\langle x \mid \mathbb{0} \rangle = \langle \mathbb{0} \mid x \rangle = 0 \quad \forall x \in X$
4. $\langle x - y \mid z \rangle = \langle x \mid z \rangle - \langle y \mid z \rangle \quad \forall x,y,z \in X$
5. $\langle x \mid y - z \rangle = \langle x \mid y \rangle - \langle x \mid z \rangle \quad \forall x,y,z \in X$
6. $\langle x \mid z \rangle = \langle y \mid z \rangle \quad \forall z \in X \neq \{\mathbb{0}\} \iff x = y$
7. $\langle x \mid y \rangle = 0 \quad \forall x \in X \iff y = \mathbb{0}$

✎PROOF:

$$
\begin{aligned}
\langle x \mid y + z \rangle &= \langle y + z \mid x \rangle^* && \text{by \textit{conjugate symmetric} property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition I.1 page 233)} \\
&= (\langle y \mid x \rangle + \langle z \mid x \rangle)^* && \text{by \textit{additive} property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition I.1 page 233)} \\
&= \langle y \mid x \rangle^* + \langle z \mid x \rangle^* && \text{by \textit{distributive} property of } * && \text{(Definition H.3 page 228)} \\
&= \langle x \mid y \rangle + \langle x \mid z \rangle && \text{by \textit{conjugate symmetric} property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition I.1 page 233)} \\
\langle x \mid \alpha y \rangle &= \langle \alpha y \mid x \rangle^* && \text{by \textit{conjugate symmetric} property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition I.1 page 233)}
\end{aligned}
$$

[1] Istrăţescu (1987) page 111 ⟨Definition 4.1.1⟩, Bollobás (1999) pages 130–131, Haaser and Sullivan (1991) page 277, Aliprantis and Burkinshaw (1998), page 276, Peano (1888b) page 72
[2] Berberian (1961) page 27, Haaser and Sullivan (1991) page 277

$$
\begin{aligned}
&= (\alpha \langle y \mid x\rangle)^* && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
&= \alpha^* \langle y \mid x\rangle^* && \text{by } \textit{antiautomorphic} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \alpha^* \langle x \mid y\rangle && \text{by } \textit{conjugate symmetric} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
\langle x \mid \mathbb{0}\rangle &= \langle \mathbb{0} \mid x\rangle^* && \text{by } \textit{conjugate symmetric} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
&= \langle 0\cdot y \mid x\rangle^* && && \\
&= (0\cdot\langle y \mid x\rangle)^* && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
&= 0 && && \\
\langle \mathbb{0} \mid x\rangle &= \langle 0\cdot y \mid x\rangle && && \\
&= (0\cdot\langle y \mid x\rangle) && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
&= 0 && && \\
\langle x - y \mid z\rangle &= \langle x + (-y) \mid z\rangle && \text{by definition of } + && \\
&= \langle x \mid z\rangle + \langle -y \mid z\rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
&= \langle x \mid z\rangle - \langle y \mid z\rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
\langle x \mid y - z\rangle &= \langle y - z \mid x\rangle^* && \text{by } \textit{conjugate symmetric} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}\\
&= (\langle y \mid x\rangle - \langle z \mid x\rangle)^* && \text{by 4.} && \\
&= \langle y \mid x\rangle^* - \langle z \mid x\rangle^* && \text{by } \textit{distributive} \text{ property of } * && \text{(Definition H.3 page 228)}\\
&= \langle x \mid y\rangle - \langle x \mid z\rangle && \text{by } \textit{conjugate symmetric} \text{ property of } \langle \triangle \mid \triangledown\rangle && \text{(Definition I.1 page 233)}
\end{aligned}
$$

$$
\begin{aligned}
& & \langle x \mid z\rangle = \langle y \mid z\rangle & \quad \forall z & & \\
&\iff & \langle x \mid z\rangle - \langle y \mid z\rangle = 0 & \quad \forall z & & \text{by property of complex numbers}\\
&\iff & \langle x - y \mid z\rangle = 0 & \quad \forall z & & \text{by 4.}\\
&\iff & x - y = \mathbb{0} & \quad \forall z & & \text{by } \textit{non-isotropic} \text{ property of } \langle \triangle \mid \triangledown\rangle \text{ (Definition I.1 page 233)}
\end{aligned}
$$

Proof that $\langle x \mid y\rangle = 0 \implies y = \mathbb{0}$:

1. Suppose $y \neq \mathbb{0}$;
2. Then $\langle y \mid y\rangle \neq \mathbb{0}$ by the *non-isotropic* property of $\langle \triangle \mid \triangledown\rangle$ (Definition I.1 page 233)
3. But because $y \in X$, the left hypothesis implies that $\langle y \mid y\rangle = 0$.
4. This is a *contradiction*.
5. Therefore $y \neq \mathbb{0}$ must be incorrect and $y = \mathbb{0}$ must be correct.

Proof that $\langle x \mid y\rangle = 0 \impliedby y = \mathbb{0}$:

$$
\begin{aligned}
\langle x \mid y\rangle &= \langle x \mid \mathbb{0}\rangle && \text{by right hypothesis}\\
&= 0 && \text{by Theorem I.1 page 233}
\end{aligned}
$$

⇨

One of the most useful and widely used inequalities in analysis is the *Cauchy-Schwarz Inequality* (sometimes also called the *Cauchy-Bunyakovsky-Schwarz Inequality*). In fact, we will use this inequality shortly to prove that every inner product space *has* a norm and therefore every inner product space *is* a normed linear space.

Theorem I.2 (Cauchy-Schwarz Inequality). [3] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown\rangle\right)$ *be an* INNER PRODUCT SPACE *and* $|\cdot| \in \mathbb{R}^{\mathbb{C}}$ *an* ABSOLUTE VALUE *function. Let* $\|\cdot\|$ *be a function in* $\mathbb{R}^{\mathbb{F}}$ *such that* $\|x\| \triangleq$

[3] Haaser and Sullivan (1991) page 278, Aliprantis and Burkinshaw (1998), page 278, Cauchy (1821) page 455, Bunyakovsky (1859) page 6, Schwarz (1885)

$\sqrt{\langle x \mid x\rangle}$.[4]

THM

$$\begin{array}{rcllr}
|\langle x \mid y\rangle|^2 & \le & \langle x \mid x\rangle \, \langle y \mid y\rangle & & \forall x,y\in X \\
|\langle x \mid y\rangle|^2 & = & \langle x \mid x\rangle \, \langle y \mid y\rangle & \iff \exists \alpha \in \mathbb{F} \text{ such that } y = \alpha x & \forall x,y\in X \\
|\langle x \mid y\rangle| & \le & \|x\| \, \|y\| & & \forall x,y\in X \\
|\langle x \mid y\rangle| & = & \|x\| \, \|y\| & \iff \exists \alpha \in \mathbb{F} \text{ such that } y = \alpha x & \forall x,y\in X
\end{array}$$

✎PROOF:

1. Proof that $|\langle x \mid y\rangle| \le \|x\| \, \|y\|$: [5]

   (a) $y = \mathbb{0}$ case:

$$\begin{aligned}
|\langle x \mid y\rangle|^2 &= |\langle x \mid \mathbb{0}\rangle|^2 && \text{by } y = \mathbb{0} \text{ hypothesis} \\
&= |\langle \mathbb{0} \mid x\rangle|^2 && \text{by Definition I.1 page 233} \\
&= |\langle 0\mathbb{0} \mid x\rangle|^2 && \text{by Definition F.1 page 185} \\
&= |0 \, \langle \mathbb{0} \mid x\rangle|^2 && \text{by Definition I.1 page 233} \\
&= 0 \\
&= \langle x \mid x\rangle \, \langle \mathbb{0} \mid \mathbb{0}\rangle \\
&= \langle x \mid x\rangle \, \langle y \mid y\rangle && \text{by } y = \mathbb{0} \text{ hypothesis}
\end{aligned}$$

   (b) $y \ne \mathbb{0}$ case: Let $\lambda \triangleq \frac{\langle x \mid y\rangle}{\langle y \mid y\rangle}$.

$$\begin{aligned}
0 &\le \langle x - \lambda y \mid x - \lambda y\rangle && \text{by Definition I.1} \\
&= \langle x \mid x - \lambda y\rangle + \langle -\lambda y \mid x - \lambda y\rangle && \text{by Definition I.1} \\
&= \langle x - \lambda y \mid x\rangle^* - \lambda \, \langle y \mid x - \lambda y\rangle && \text{by Definition I.1} \\
&= \langle x \mid x\rangle^* + \langle -\lambda y \mid x\rangle^* - \lambda \, \langle x - \lambda y \mid y\rangle^* && \text{by Definition I.1} \\
&= \langle x \mid x\rangle^* - \lambda^* \, \langle y \mid x\rangle^* - \lambda \, \langle x \mid y\rangle^* - \lambda \, \langle -\lambda y \mid y\rangle^* && \text{by Definition I.1} \\
&= \langle x \mid x\rangle - \lambda^* \, \langle x \mid y\rangle - \lambda \, \langle x \mid y\rangle^* + \lambda\lambda^* \, \langle y \mid y\rangle^* && \text{by Definition I.1} \\
&= \langle x \mid x\rangle + \cancelto{0}{\left[\frac{\langle x \mid y\rangle}{\langle y \mid y\rangle} \lambda^* \, \langle y \mid y\rangle - \lambda^* \, \langle x \mid y\rangle\right]} - \frac{\langle x \mid y\rangle}{\langle y \mid y\rangle} \, \langle x \mid y\rangle^* && \text{by definition of } \lambda \\
&= \langle x \mid x\rangle - \frac{1}{\langle y \mid y\rangle} \, |\langle x \mid y\rangle|^2
\end{aligned}$$

$$\implies |\langle x \mid y\rangle|^2 \le \langle x \mid x\rangle \, \langle y \mid y\rangle$$

2. Proof that $|\langle x \mid y\rangle|^2 = \langle x \mid x\rangle \, \langle y \mid y\rangle \iff y = ax$:
   Let $\frac{1}{a} \triangleq \lambda \triangleq \frac{\langle x \mid y\rangle}{\langle y \mid y\rangle}$. Then...

$$\begin{aligned}
& y = ax \\
\iff\quad & x = \lambda y \\
\iff\quad & x - \lambda y = \mathbb{0} \\
\iff\quad & 0 = \langle x - \lambda y \mid x - \lambda y\rangle && \text{by Definition I.1 page 233} \\
& \quad = \langle x \mid x - \lambda y\rangle + \langle -\lambda y \mid x - \lambda y\rangle && \text{by Definition I.1 page 233} \\
& \quad = \langle x - \lambda y \mid x\rangle^* - \lambda \, \langle y \mid x - \lambda y\rangle && \text{by Definition I.1 page 233} \\
& \quad \vdots && \text{(same steps as in 1(b))}
\end{aligned}$$

[4] The function $\|\cdot\|$ is a *norm* (Theorem I.4 page 238) and is called the *norm induced by the inner product* $\langle \triangle \mid \triangledown\rangle$ (Definition I.2 page 238).

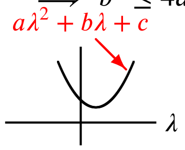
[5] Haaser and Sullivan (1991), page 278

$$= \langle \mathbf{x} \,|\, \mathbf{x} \rangle - \frac{1}{\langle \mathbf{y} \,|\, \mathbf{y} \rangle} |\langle \mathbf{x} \,|\, \mathbf{y} \rangle|^2$$

$$\iff \qquad |\langle \mathbf{x} \,|\, \mathbf{y} \rangle|^2 = \langle \mathbf{x} \,|\, \mathbf{x} \rangle \ \langle \mathbf{y} \,|\, \mathbf{y} \rangle$$

3. Alternate proof for $|\langle \mathbf{x} \,|\, \mathbf{y} \rangle| \le \|\mathbf{x}\| \, \|\mathbf{y}\|$: (Note: This is essentially the same proof as used by Schwarz). [6]

   (a) Proof that $\left(a\lambda^2 + b\lambda + c \ge 0 \quad \forall \lambda \in \mathbb{R}\right) \implies b^2 \le 4ac$ (quadratic discriminate inequality):



   Let $k \in (0, \infty)$, and $r_1, r_2 \in \mathbb{C}$ be the roots of $a\lambda^2 + b\lambda + c = 0$. Then

$$\begin{aligned}
0 &\le a\lambda^2 + b\lambda + c && \text{by left hypothesis} \\
&= k(\lambda - r_1)(\lambda - r_2) && \text{by definition of } r_1 \text{ and } r_2 \\
&= k(\lambda^2 - r_1\lambda - r_2\lambda + r_1 r_2) \\
\implies \quad & \lambda^2 - r_1\lambda - r_2\lambda + r_1 r_2 \ge 0 \\
\implies \quad & r_1 = r_2^* && \text{because } r_1 r_2 \ge 0 \text{ for } \lambda = 0
\end{aligned}$$

   The *quadratic equation* places another constraint on r_1 and r_2:

$$\begin{aligned}
\frac{b^2 + \sqrt{b^2 - 4ac}}{2a} &= r_1 && \text{by quadratic equation} \\
&= r_2^* && \text{by previous result} \\
&= \left(\frac{b^2 - \sqrt{b^2 - 4ac}}{2a}\right)^* && \text{by quadratic equation}
\end{aligned}$$

   The only way for this to be true is if $b^2 \le 4ac$ (the **discriminate** is non-positive).

   (b) Proof that $\langle \mathbf{y} \,|\, \mathbf{y} \rangle \lambda^2 + 2|\langle \mathbf{x} \,|\, \mathbf{y} \rangle|\lambda + \langle \mathbf{x} \,|\, \mathbf{x} \rangle \ge 0 \quad \forall \lambda \in \mathbb{R}$:

$$\begin{aligned}
0 &\le \langle \mathbf{x} + \alpha\mathbf{y} \,|\, \mathbf{x} + \alpha\mathbf{y} \rangle && \text{by Definition I.1 page 233} \\
&= \langle \mathbf{x} \,|\, \mathbf{x} + \alpha\mathbf{y} \rangle + \langle \alpha\mathbf{y} \,|\, \mathbf{x} + \alpha\mathbf{y} \rangle && \text{by Definition I.1 page 233} \\
&= \langle \mathbf{x} \,|\, \mathbf{x} + \alpha\mathbf{y} \rangle + \alpha \langle \mathbf{y} \,|\, \mathbf{x} + \alpha\mathbf{y} \rangle && \text{by Definition I.1 page 233} \\
&= \langle \mathbf{x} + \alpha\mathbf{y} \,|\, \mathbf{x} \rangle^* + \alpha \langle \mathbf{x} + \alpha\mathbf{y} \,|\, \mathbf{y} \rangle^* && \text{by Definition I.1 page 233} \\
&= \langle \mathbf{x} \,|\, \mathbf{x} \rangle^* + \langle \alpha\mathbf{y} \,|\, \mathbf{x} \rangle^* + \alpha \langle \mathbf{x} \,|\, \mathbf{y} \rangle^* + \alpha \langle \alpha\mathbf{y} \,|\, \mathbf{y} \rangle^* && \text{by Definition I.1 page 233} \\
&= \langle \mathbf{x} \,|\, \mathbf{x} \rangle^* + \alpha^* \langle \mathbf{y} \,|\, \mathbf{x} \rangle^* + \alpha \langle \mathbf{x} \,|\, \mathbf{y} \rangle^* + \alpha\alpha^* \langle \mathbf{y} \,|\, \mathbf{y} \rangle^* && \text{by Definition I.1 page 233} \\
&= \langle \mathbf{x} \,|\, \mathbf{x} \rangle + \alpha^* \langle \mathbf{x} \,|\, \mathbf{y} \rangle + \left(\alpha^* \langle \mathbf{x} \,|\, \mathbf{y} \rangle\right)^* + |\alpha|^2 \langle \mathbf{y} \,|\, \mathbf{y} \rangle && \text{by Definition I.1 page 233} \\
&= \langle \mathbf{x} \,|\, \mathbf{x} \rangle + 2\Re\left(\alpha^* \langle \mathbf{x} \,|\, \mathbf{y} \rangle\right) + |\alpha|^2 \langle \mathbf{y} \,|\, \mathbf{y} \rangle \\
&\le \langle \mathbf{x} \,|\, \mathbf{x} \rangle + 2\left|\alpha^* \langle \mathbf{x} \,|\, \mathbf{y} \rangle\right| + |\alpha|^2 \langle \mathbf{y} \,|\, \mathbf{y} \rangle \\
&= \langle \mathbf{x} \,|\, \mathbf{x} \rangle + 2|\langle \mathbf{x} \,|\, \mathbf{y} \rangle||\alpha| + \langle \mathbf{y} \,|\, \mathbf{y} \rangle |\alpha|^2 \\
&= \langle \mathbf{y} \,|\, \mathbf{y} \rangle |\alpha|^2 + 2\,|\langle \mathbf{x} \,|\, \mathbf{y} \rangle|\,|\alpha| + \langle \mathbf{x} \,|\, \mathbf{x} \rangle \\
&= \underbrace{\langle \mathbf{y} \,|\, \mathbf{y} \rangle}_{a}\lambda^2 + \underbrace{2|\langle \mathbf{x} \,|\, \mathbf{y} \rangle|}_{b}\lambda + \underbrace{\langle \mathbf{x} \,|\, \mathbf{x} \rangle}_{c} && \text{because } \lambda \triangleq |\alpha| \in \mathbb{R}
\end{aligned}$$

   (c) The above equation is in the quadratic form used in the lemma of part (a).

$$\begin{aligned}
&\left(\underbrace{2|\langle \mathbf{x} \,|\, \mathbf{y} \rangle|}_{b}\right)^2 \le 4 \underbrace{\langle \mathbf{y} \,|\, \mathbf{y} \rangle}_{a} \ \underbrace{\langle \mathbf{x} \,|\, \mathbf{x} \rangle}_{c} && \text{by the results of parts (a) and (b)} \\
\implies \ & |\langle \mathbf{x} \,|\, \mathbf{y} \rangle|^2 \le \langle \mathbf{x} \,|\, \mathbf{x} \rangle \ \langle \mathbf{y} \,|\, \mathbf{y} \rangle
\end{aligned}$$

[6] Aliprantis and Burkinshaw (1998), page 278, Steele (2004), page 11

4. Proof that $|\langle x \mid y\rangle| \le \|x\|\,\|y\|$:
   This follows directly from the definition $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

5. Proof that $|\langle x \mid y\rangle| = \|x\|\,\|y\| \iff \exists \alpha \in \mathbb{C}$ such that $y = \alpha x$:
   This follows directly from the definition $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

⇨

Corollary I.1. [7] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla\rangle\right)$ *be an* INNER PRODUCT SPACE.

COR: $\langle x \mid y\rangle$ *is* CONTINUOUS *in both* x *and* y.

✎PROOF: Let $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.

$$\begin{aligned}
|\langle x+\epsilon \mid y\rangle - \langle x \mid y\rangle|^2 &= |\langle x+\epsilon-x \mid y\rangle|^2 && \text{by } \textit{additivity} \text{ of } \langle\triangle\mid\nabla\rangle && \text{(Definition I.1 page 233)}\\
&= |\langle \epsilon \mid y\rangle|^2 \\
&\le \|\epsilon\|^2\,\|y\| && \text{by } \textit{Cauchy-Schwarz Inequality} && \text{(Theorem I.2 page 234)}
\end{aligned}$$

⇨

I.2 Relationship between norms and inner products

I.2.1 Norms induced by inner products

Lemma I.1 (Polar Identity). [8] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla\rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition I.1 page 233). Let* $\Re z$ *represent the real part of* $z \in \mathbb{C}$. *Let* $\|\cdot\|$ *be a function in* $\mathbb{R}^{\mathbb{F}}$ *such that* $\|x\| \triangleq \sqrt{\langle x \mid x\rangle}$.[9]

LEM: $\|x+y\|^2 = \|x\|^2 + 2\mathbf{R}_e\left[\langle x \mid y\rangle\right] + \|y\|^2 \qquad \forall x,y \in X$

✎PROOF:

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y \mid x+y\rangle && \text{by definition of } \textit{induced norm} && \text{(Theorem I.4 page 238)}\\
&= \langle x \mid x+y\rangle + \langle y \mid x+y\rangle && \text{by Definition I.1 page 233}\\
&= \langle x+y \mid x\rangle^* + \langle x+y \mid y\rangle^* && \text{by Definition I.1 page 233}\\
&= \langle x \mid x\rangle^* + \langle y \mid x\rangle^* + \langle x \mid y\rangle^* + \langle y \mid y\rangle^* && \text{by Definition I.1 page 233}\\
&= \langle x \mid x\rangle + \langle x \mid y\rangle + \langle x \mid y\rangle^* + \langle y \mid y\rangle && \text{by definition of } \textit{inner product} && \text{(Definition I.1 page 233)}\\
&= \|x\|^2 + 2\Re\,\langle x \mid y\rangle + \|y\|^2 && \text{by definition of } \textit{induced norm} && \text{(Theorem I.4 page 238)}
\end{aligned}$$

⇨

[7] Bollobás (1999) page 132, Aliprantis and Burkinshaw (1998) page 279 ⟨Lemma 32.4⟩

[8] Conway (1990) page 4, Heil (2011) page 27 ⟨Lemma 1.36(a)⟩

[9] The function $\|\cdot\|$ is a *norm* (Theorem I.4 page 238) and is called the *norm induced by the inner product* $\langle\triangle\mid\nabla\rangle$ (Definition I.2 page 238).

Theorem I.3 (Minkowski's inequality). [10] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *be an* INNER PRODUCT SPACE. *Let* $\|\cdot\|$ *be a function in* $\mathbb{R}^{\mathbb{F}}$ *such that* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.[11]

THM $$\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \qquad \forall \boldsymbol{x},\boldsymbol{y} \in X$$

✎PROOF:

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \|\boldsymbol{x}\|^2 + 2\Re\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle + \|\boldsymbol{y}\|^2 && \text{by } \textit{Polar Identity} \text{ (Lemma I.1 page 237)}\\
&\le \|\boldsymbol{x}\|^2 + 2|\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle| + \|\boldsymbol{y}\|^2 \\
&\le \|\boldsymbol{x}\|^2 + 2\sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}\sqrt{\langle \boldsymbol{y} \mid \boldsymbol{y}\rangle} + \|\boldsymbol{y}\|^2 && \text{by } \textit{Cauchy-Schwarz Inequality} \text{ (Theorem I.2 page 234)}\\
&= (\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2
\end{aligned}$$

✎

Theorem I.4 (induced norm). [12] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition I.1 page 233)*.

THM $$\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle} \quad \Longrightarrow \quad \|\cdot\| \text{ is a NORM}$$

✎PROOF: For a function to be a norm, it must satisfy the four properties listed in Definition J.1 (page 249).

1. Proof that $\|\cdot\|$ is a norm:
   (a) Proof that $\|\boldsymbol{x}\| > 0$ for $\boldsymbol{x} \neq 0$ (non-negative):
   By Definition I.1 page 233, all inner products have this property.
   (b) Proof that $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = 0$ (non-isometric):
   By Definition I.1, all inner products have this property.
   (c) Prove $\|a\boldsymbol{x}\| = |a|\,\|\boldsymbol{x}\|$ (homogeneous):
   $$\|a\boldsymbol{x}\| \triangleq \sqrt{\langle a\boldsymbol{x} \mid a\boldsymbol{x}\rangle} = \sqrt{aa^*\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle} = \sqrt{|a|^2\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle} = |a|\,\|\boldsymbol{x}\|$$
   (d) Proof that $\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (subadditive): This is true by Minkowski's inequality page 238
2. Proof that every inner product space is a normed linear space:
   Since every inner product induces a norm, so every inner product space has a norm (the norm induced by the inner product) and is therefore a normed linear space.

✎

Theorem I.4 (previous theorem) demonstrates that in any inner product space $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$, the function $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$ is a norm. That is, $\|\boldsymbol{x}\|$ is the *norm induced by the inner product*. This norm is formally defined next.

Definition I.2. *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *be an* INNER PRODUCT SPACE *(Definition I.1 page 233)*.

DEF *The **norm induced by the inner product*** $\langle \triangle \mid \nabla \rangle$ *is defined as*
$$\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$$

[10] Aliprantis and Burkinshaw (1998) pages 278–279 ⟨Theorem 32.3⟩, Maligranda (1995), Minkowski (1910) page 115

[11] The function $\|\cdot\|$ is a *norm* (Theorem I.4 page 238) and is called the *normed induced by the inner product* $\langle \triangle \mid \nabla \rangle$ (Definition I.2 page 238).

[12] Aliprantis and Burkinshaw (1998), pages 278–279, Haaser and Sullivan (1991) page 278

I.2.2 Inner products induced by norms

Theorem I.4 (page 238) demonstrates that if a *linear space* (Definition F.1 page 185) has an *inner product* (Definition I.1 page 233), then that inner product always induces a *norm* (Definition J.1 page 249), and the relationship between the two is simply $\|x\| = \sqrt{\langle x \mid x\rangle}$ (Definition I.2 page 238). But what about the converse? What if a linear space has a norm—can that norm also induce an inner product? The answer in general is "no": Not all norms can induce an inner product. But a less harsh answer is "sometimes": Some norms **can** induce inner products. This leads to some important and interesting questions:

1. How many different inner products can be induced from a single norm? The answer turns out to be **at most** one, but maybe none (Theorem I.5 page 239).
2. When a norm *can* induce an inner product, what is that (unique) inner product? The inner product expressed in terms of the norm is given by the *Polarization Identity* (Theorem I.6 page 240).
3. Which norms can induce an inner product and which ones cannot? The answer is that norms that satisfy the *parallelogram law* (Theorem I.7 page 241) **can** induce an inner product; and the ones that don't, cannot (Theorem I.7 page 241).

Theorem I.5. [13] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

THM

$$\underbrace{\left.\begin{array}{ll}\exists\ \langle \triangle \mid \triangledown\rangle \text{ and } (\cdot \mid \cdot) & \text{such that}\\ \|x\|^2 = \langle x \mid x\rangle = (x \mid x) & \forall x\in X\end{array}\right\}}_{\text{If a norm is induced by two inner products...}} \implies \underbrace{\langle x \mid y\rangle = (x \mid y) \quad \forall x,y\in X}_{\text{...then those two inner products are equivalent.}}$$

✎PROOF:

$$\begin{aligned}
2\langle x \mid y\rangle &= \big[\langle x \mid y\rangle + \langle y \mid x\rangle\big] + \big[\langle x \mid y\rangle - \langle y \mid x\rangle\big]\\
&= \big[\langle x \mid y\rangle + \langle y \mid x\rangle\big] + i\big[-i\langle x \mid y\rangle + i\langle y \mid x\rangle\big]\\
&= \big[\langle x \mid y\rangle + \langle y \mid x\rangle\big] + i\big[-\langle ix \mid y\rangle - \langle y \mid ix\rangle\big]\\
&= \left(\underbrace{\big[\langle x \mid y\rangle + \langle y \mid x\rangle + \langle x \mid x\rangle + \langle y \mid y\rangle\big]}_{\langle x+y \mid x+y\rangle} - \underbrace{\big[\langle x \mid x\rangle + \langle y \mid y\rangle\big]}_{\text{residue}}\right)\\
&\quad - i\left(\underbrace{\big[\langle ix \mid y\rangle + \langle y \mid ix\rangle + \langle ix \mid ix\rangle + \langle y \mid y\rangle\big]}_{\langle ix+y \mid ix+y\rangle} - \underbrace{\big[\langle ix \mid ix\rangle + \langle y \mid y\rangle\big]}_{\text{residue}}\right)\\
&= \Big(\big[\langle x+y \mid x+y\rangle\big] - \big[\langle x \mid x\rangle + \langle y \mid y\rangle\big]\Big) - i\Big(\big[\langle ix+y \mid ix+y\rangle\big] - \big[\langle ix \mid ix\rangle + \langle y \mid y\rangle\big]\Big)\\
&= \Big(\big[(x+y \mid x+y)\big] - \big[(x \mid x) + (y \mid y)\big]\Big) - i\Big(\big[(ix+y \mid ix+y)\big] - \big[(ix \mid ix) + (y \mid y)\big]\Big)\\
&= \left(\underbrace{\big[(x \mid y) + (y \mid x) + (x \mid x) + (y \mid y)\big]}_{(x+y \mid x+y)} - \underbrace{\big[(x \mid x) + (y \mid y)\big]}_{\text{residue}}\right)\\
&\quad - i\left(\underbrace{\big[(ix \mid y) + (y \mid ix) + (ix \mid ix) + (y \mid y)\big]}_{(ix+y \mid ix+y)} - \underbrace{\big[(ix \mid ix) + (y \mid y)\big]}_{\text{residue}}\right)
\end{aligned}$$

[13] Aliprantis and Burkinshaw (1998), page 280, Bollobás (1999), page 132, Jordan and von Neumann (1935) page 721

$$
\begin{aligned}
&= \left[(x\,|\,y) + (y\,|\,x)\right] + i\left[-(ix\,|\,y) - (y\,|\,ix)\right]\\
&= \left[(x\,|\,y) + (y\,|\,x)\right] + i\left[-i(x\,|\,y) + i(y\,|\,x)\right]\\
&= \left[(x\,|\,y) + (y\,|\,x)\right] + \left[(x\,|\,y) - (y\,|\,x)\right]\\
&= 2(x\,|\,y)
\end{aligned}
$$

✏

Theorem I.6 (Polarization Identities). [14] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space,* $\langle \triangle \,|\, \nabla \rangle \in \mathbb{F}^{X\times X}$ *a function, and* $\|x\| \triangleq \sqrt{\langle x\,|\,x\rangle}$.

THM

$\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle\right)$ *is an inner product space* $\Longrightarrow$

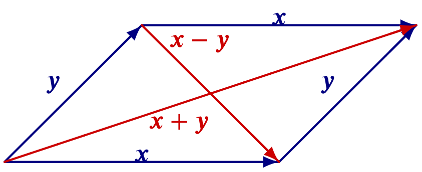
$$
\underbrace{4\langle x\,|\,y\rangle = \begin{cases} \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 & \text{for } \mathbb{F}=\mathbb{C} \quad \forall x,y\in X\\ \|x+y\|^2 - \|x-y\|^2 & \text{for } \mathbb{F}=\mathbb{R} \quad \forall x,y\in X \end{cases}}_{\text{inner product induced by norm}}
$$

✎PROOF:

1. These follow directly from properties of *bilinear functionals* (Theorem ?? page ??).
2. Alternative proof for $\mathbb{F} = \mathbb{C}$ case:

$$
\begin{aligned}
&\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\\
&= \underbrace{\|x\|^2 + \|y\|^2 + 2\Re\langle x\,|\,y\rangle}_{\langle x+y\,|\,x+y\rangle} - \underbrace{\left(\|x\|^2 + \|-y\|^2 + 2\Re\langle x\,|\,-y\rangle\right)}_{\langle x-y\,|\,x-y\rangle}\\
&\quad + \underbrace{i\left(\|x\|^2 + \|iy\|^2 + 2\Re\langle x\,|\,iy\rangle\right)}_{i\langle x+iy\,|\,x+iy\rangle} - \underbrace{i\left(\|x\|^2 + \|-iy\|^2 + 2\Re\langle x\,|\,-iy\rangle\right)}_{i\langle x-iy\,|\,x-iy\rangle} && \text{by Lemma I.1 page 237}\\
&= \underbrace{\|x\|^2 + \|y\|^2 + 2\Re\langle x\,|\,y\rangle}_{\langle x+y\,|\,x+y\rangle} - \underbrace{\left(\|x\|^2 + \|y\|^2 - 2\Re\langle x\,|\,y\rangle\right)}_{\langle x-y\,|\,x-y\rangle}\\
&\quad + \underbrace{i\left(\|x\|^2 + \|y\|^2 + 2\Re\langle x\,|\,iy\rangle\right)}_{i\langle x+iy\,|\,x+iy\rangle} - \underbrace{i\left(\|x\|^2 + \|y\|^2 - 2\Re\langle x\,|\,iy\rangle\right)}_{i\langle x-iy\,|\,x-iy\rangle} && \text{by Definition I.1 page 233}\\
&= 4\Re\langle x\,|\,y\rangle + 4i\Re\langle x\,|\,iy\rangle\\
&= \underbrace{2\left(\langle x\,|\,y\rangle + \langle x\,|\,y\rangle^*\right)}_{4\Re\langle x\,|\,y\rangle} + \underbrace{2i\left(\langle x\,|\,iy\rangle + \langle x\,|\,iy\rangle^*\right)}_{4i\Re\langle x\,|\,iy\rangle}\\
&= 2\left(\langle x\,|\,y\rangle + \langle x\,|\,y\rangle^*\right) + 2i\left(i^*\langle x\,|\,y\rangle + (i^{**})\langle x\,|\,y\rangle^*\right)\\
&= 2\left(\langle x\,|\,y\rangle + \langle x\,|\,y\rangle^*\right) + 2i\left(-i\langle x\,|\,y\rangle + i\langle x\,|\,y\rangle^*\right) && \text{by Definition I.1 page 233}\\
&= 2\langle x\,|\,y\rangle + 2\langle x\,|\,y\rangle^* + 2\langle x\,|\,y\rangle - 2\langle x\,|\,y\rangle^*\\
&= 4\langle x\,|\,y\rangle
\end{aligned}
$$

✏



In plane geometry ($\mathbb{R}^2$), the *parallelogram law* states that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of its diagonals. This is illustrated in the figure to the left.

[14] Berberian (1961) pages 29–30 ⟨Theorem II.3.3⟩, Istrăţescu (1987) page 110 ⟨Proposition 4.1.5⟩, Bollobás (1999), page 132, Jordan and von Neumann (1935) page 721

Acutally, the parallelogram law can be generalized to *any inner product space* (not just in the plane). And if the parallelogram law happens to hold true in a normed linear space, then that normed linear space is actually an *inner product space*. The parallelogram law and its relation to inner product spaces is stated in the next theorem.

Theorem I.7 (Parallelogram law). [15] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *and* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.

THM

$$\boldsymbol{\Omega} \text{ is an inner product space} \iff \underbrace{2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2 = \|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2}_{\text{PARALLELOGRAM LAW / VON NEUMANN-JORDAN CONDITION}} \qquad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{\Omega}$$

✎PROOF:

1. Proof that $[\exists \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ such that $\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle] \implies$ [parallelogram law is true]:

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 &= \left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle\right]\right] \qquad + \left[\|\boldsymbol{x}\|^2 + \|-\boldsymbol{y}\|^2 + 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid -\boldsymbol{y}\rangle\right]\right] \\
&\qquad \text{by Lemma I.1 page 237} \\
&= \left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle\right]\right] \qquad + \left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\mathbf{R_e}\left[2\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle\right]\right] \\
&= 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2
\end{aligned}$$

2. Proof that $[\exists \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ such that $\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle] \impliedby$ [parallelogram law is true]:
Note that **if** an inner product exists in the norm linear space $(\boldsymbol{\Omega}, \|\cdot\|)$, **then** that norm linear space is actually an inner product space $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$. And if it is an inner product space, then by Theorem I.6 page 240 that inner product must be given by the **Polarization Identity**
$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \|a\boldsymbol{x}+\boldsymbol{y}\|^2 - \|a\boldsymbol{x}-\boldsymbol{y}\|^2 + i\,\|a\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\,\|a\boldsymbol{x}-i\boldsymbol{y}\|^2$.
Therefore, here we must use the parallelogram law to show that the bilinear function $\mathsf{f}(\boldsymbol{x},\boldsymbol{y}) \triangleq \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ given on the left hand side of the "=" relation is indeed an inner product— that is, that it satisfies the requirements of Definition I.1 page 233.

   (a) Proof that $\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle \geq 0$ (non-negative):

$$\begin{aligned}
4\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle &\triangleq \|\boldsymbol{x}+\boldsymbol{x}\|^2 - \cancelto{0}{\|\boldsymbol{x}-\boldsymbol{x}\|^2} + i\,\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - i\,\|\boldsymbol{x}-i\boldsymbol{x}\|^2 && \text{by } \textit{Polarization Identity} \\
&= \|2\boldsymbol{x}\|^2 - \mathbb{0} + i\left(\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - \|\boldsymbol{x}-i\boldsymbol{x}\|^2\right) && \text{by Definition J.1 page 249} \\
&= |2|^2\,\|\boldsymbol{x}\|^2 + i\left(\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - |i|\,\|\boldsymbol{x}-i\boldsymbol{x}\|^2\right) \\
&= 4\|\boldsymbol{x}\|^2 + i\,\cancelto{0}{\left(\|\boldsymbol{x}+i\boldsymbol{x}\|^2 - \|i\boldsymbol{x}+\boldsymbol{x}\|^2\right)} && \text{by Definition J.1 page 249} \\
&= 4\|\boldsymbol{x}\|^2 \\
&\geq 0 && \text{by Definition J.1 page 249}
\end{aligned}$$

   (b) Proof that $\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle = 0 \iff \boldsymbol{x} = \mathbb{0}$ (non-isotropic):

$$\begin{aligned}
4\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle &= 4\|\boldsymbol{x}\|^2 && \text{by result of part (a)} \\
&= 0 \iff \boldsymbol{x} = \mathbb{0} && \text{by Definition J.1 page 249}
\end{aligned}$$

[15] Amir (1986), page 8, Istrăţescu (1987) page 110, Day (1973), page 151, Halmos (1998), page 14, Aliprantis and Burkinshaw (1998) pages 280–281 ⟨Theorem 32.6⟩, Riesz (1934) page 36?, Jordan and von Neumann (1935) pages 721–722

(c) Proof that $\langle \boldsymbol{x}+\boldsymbol{u} \mid \boldsymbol{y}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle + \langle \boldsymbol{u} \mid \boldsymbol{y}\rangle$ (additive):[16]

$$\begin{aligned}
4\langle \boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{z}\rangle &= 8\left\langle \frac{\boldsymbol{x}+\boldsymbol{y}}{2} \mid \boldsymbol{z}\right\rangle && \text{by Definition I.1 page 233}\\
&= 2\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}+\boldsymbol{z}\right\|^2 - 2\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}-\boldsymbol{z}\right\|^2\\
&\quad + 2i\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}+\boldsymbol{z}\right\|^2 - 2i\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}-i\boldsymbol{z}\right\|^2 && \text{by \textit{Polarization Identity}}\\
&= \left(2\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}+\boldsymbol{z}\right\|^2 + 2\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{2}\right\|^2\right)\\
&\quad - \left(2\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}-\boldsymbol{z}\right\|^2 + 2\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{2}\right\|^2\right)\\
&\quad + \left(2i\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}+\boldsymbol{z}\right\|^2 + 2i\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{2}\right\|^2\right)\\
&\quad - \left(2i\left\|\frac{\boldsymbol{x}+\boldsymbol{y}}{2}-i\boldsymbol{z}\right\|^2 + 2i\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{2}\right\|^2\right)\\
&= \left(\|\boldsymbol{x}+\boldsymbol{z}\|^2 + \|\boldsymbol{y}+\boldsymbol{z}\|^2\right) - \left(\|\boldsymbol{x}-\boldsymbol{z}\|^2 + \|\boldsymbol{y}-\boldsymbol{z}\|^2\right)\\
&\quad + \left(i\|\boldsymbol{x}+\boldsymbol{z}\|^2 + i\|\boldsymbol{y}+\boldsymbol{z}\|^2\right) - \left(i\|\boldsymbol{x}-i\boldsymbol{z}\|^2 + i\|\boldsymbol{y}-i\boldsymbol{z}\|^2\right) && \text{by parallelogram law}\\
&= \left(\|\boldsymbol{x}+\boldsymbol{z}\|^2 - \|\boldsymbol{x}-\boldsymbol{z}\|^2 + i\|\boldsymbol{x}+\boldsymbol{z}\|^2 - i\|\boldsymbol{x}-i\boldsymbol{z}\|^2\right)\\
&\quad + \left(\|\boldsymbol{y}+\boldsymbol{z}\|^2 - \|\boldsymbol{y}-\boldsymbol{z}\|^2 + i\|\boldsymbol{y}+\boldsymbol{z}\|^2 - i\|\boldsymbol{y}-i\boldsymbol{z}\|^2\right)\\
&= 4\langle \boldsymbol{x} \mid \boldsymbol{z}\rangle + 4\langle \boldsymbol{y} \mid \boldsymbol{z}\rangle && \text{by Polarization Identity}
\end{aligned}$$

(d) Proof that $\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle = \langle \boldsymbol{y} \mid \boldsymbol{y}\rangle^*$ (*conjugate symmetric*):

$$\begin{aligned}
4\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle &\triangleq \|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2 + i\|\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\|\boldsymbol{x}-i\boldsymbol{y}\|^2 && \text{by Polarization Identity}\\
&= \|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}-\boldsymbol{x}\|^2 + i\|i(\boldsymbol{y}-i\boldsymbol{x})\|^2 - i\|-i(\boldsymbol{y}+i\boldsymbol{x})\|^2 && \text{by Definition F.1 page 185}\\
&= \|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}-\boldsymbol{x}\|^2 + i\|\boldsymbol{y}-i\boldsymbol{x}\|^2 - i\|\boldsymbol{y}+i\boldsymbol{x}\|^2 && \text{by Definition J.1 page 249}\\
&= \left(\|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}-\boldsymbol{x}\|^2 - i\|\boldsymbol{y}-i\boldsymbol{x}\|^2 + i\|\boldsymbol{y}+i\boldsymbol{x}\|^2\right)^*\\
&= \left(\|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}-\boldsymbol{x}\|^2 + i\|\boldsymbol{y}+i\boldsymbol{x}\|^2 - i\|\boldsymbol{y}-i\boldsymbol{x}\|^2\right)^*\\
&\triangleq 4\langle \boldsymbol{y} \mid \boldsymbol{x}\rangle^* && \text{by \textit{Polarization Identity}}
\end{aligned}$$

(e) Proof that $\langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ (*homogeneous*): [17]

i. Proof that $\langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle$ is linear in α:

$$\begin{aligned}
0 &\le \left|\|\alpha\boldsymbol{x}+\boldsymbol{y}\| - \|\beta\boldsymbol{x}+\boldsymbol{y}\|\right|\\
&\le \left|\|(\alpha\boldsymbol{x}+\boldsymbol{y}) - (\beta\boldsymbol{x}+\boldsymbol{y})\|\right| && \text{by Theorem J.2 page 250}\\
&\le \left|\|(\alpha-\beta)\boldsymbol{x}\|\right|
\end{aligned}$$

This implies that as $a \to \beta$, $\|\alpha\boldsymbol{x}+\boldsymbol{y}\| \to \|\beta\boldsymbol{x}+\boldsymbol{y}\|$, which by definition implies that $\|\alpha\boldsymbol{x}+\boldsymbol{y}\|$ linear in α. And by the parallelogram law, $\langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle$ is also linear in α.

ii. Proof that $\langle n\boldsymbol{x} \mid \boldsymbol{y}\rangle = n\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ for $n \in \mathbb{Z}$ (integer case):

A. Proof for $n = \pm 1$:

$$\begin{aligned}
\langle n\boldsymbol{x} \mid \boldsymbol{y}\rangle &= \langle \pm 1\boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by } n = \pm 1 \text{ hypothesis}\\
&= \pm 1\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by definition of \textit{inner product}} \quad \text{(Definition I.1 page 233)}\\
&= n\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by } n = \pm 1 \text{ hypothesis}
\end{aligned}$$

[16] Aliprantis and Burkinshaw (1998), page 281
[17] Aliprantis and Burkinshaw (1998), page 138

B. Proof for $n = 0$:

$$\begin{aligned}
\langle n\boldsymbol{x} \mid \boldsymbol{y}\rangle &= \langle 0\boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by } n = 0 \text{ hypothesis} \\
&= \langle \boldsymbol{x} - \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle + \langle -1\boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle - 1\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle - \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= 0\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= n\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by } n = 0 \text{ hypothesis}
\end{aligned}$$

C. Proof for $n = \pm 2$:

$$\begin{aligned}
\langle n\boldsymbol{x} \mid \boldsymbol{y}\rangle &= \langle \pm 2\boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by } n = \pm 1 \text{ hypothesis} \\
&= \langle \pm(\boldsymbol{x} + \boldsymbol{x}) \mid \boldsymbol{y}\rangle \\
&= \pm\,\langle \boldsymbol{x} + \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by definition of \textit{inner product}} \quad \text{(Definition I.1 page 233)} \\
&= \pm\,(\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle + \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle) && \text{by additive property} \\
&= \pm 2\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= n\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by } n = \pm 2 \text{ hypothesis}
\end{aligned}$$

D. Proof that [n case] $\Longrightarrow$ [$n \pm 1$ case]:

$$\begin{aligned}
\langle (n \pm 1)\boldsymbol{x} \mid \boldsymbol{y}\rangle &= \langle n\boldsymbol{x} \pm 1\boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= \langle n\boldsymbol{x} \mid \boldsymbol{y}\rangle + \langle \pm 1\boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by additive property} \\
&= n\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \pm 1\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by left hypothesis} \\
&= (n \pm 1)\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle
\end{aligned}$$

iii. Proof that $\langle q\boldsymbol{x} \mid \boldsymbol{y}\rangle = q\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ for $q \in \mathbb{Q}$ (rational number case):

$$\begin{aligned}
\frac{n}{m}\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle &= \frac{n}{m}\left\langle \frac{m}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle && \text{where } n, m \in \mathbb{Z} \text{ and } m \neq 0 \\
&= \frac{nm}{m}\left\langle \frac{1}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle && \text{by previous result} \\
&= \frac{m}{m}\left\langle \frac{n}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle && \text{by previous result} \\
&= \left\langle \frac{n}{m}\boldsymbol{x} \,\middle|\, \boldsymbol{y}\right\rangle
\end{aligned}$$

iv. Proof that $\langle r\boldsymbol{x} \mid \boldsymbol{y}\rangle = r\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ for all $r \in \mathbb{R}$ (real number case):
Because $\mathbb{Q}$ is dense in $\mathbb{R}$ and because $\|\alpha\boldsymbol{x} + \boldsymbol{y}\|$ is continuous in α, so $\langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle = \alpha\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ for all $\alpha \in \mathbb{R}$.

v. Proof that $\langle c\boldsymbol{x} \mid \boldsymbol{y}\rangle = c\,\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$ for all $c \in \mathbb{C}$ (complex number case):
No proof at this time.

✏

Remark I.1. [18] The inner product has already been defined in Definition I.1 (page 233) as a bilinear function that is *non-negative*, *non-isotropic*, *homogeneous*, *additive*, and *conjugate symmetric*. However, given a normed linear space, we could alternatively define the inner product using the *parallelogram law* (Theorem I.7 page 241) together with the *Polarization Identity* (Theorem I.6 page 240). Under this new definition, an inner product *exists* if the parallelogram law is satisfied, and is *specified*, in terms of the norm, by the Polarization Identity.

[18] Loomis (1953), pages 23–24, Kubrusly (2001) page 317

Of the uncountably infinite number of $\ell^p_{\mathbb{F}}$ norms, only the norm for $p = 2$ induces an inner product (Proposition I.1, next).

Proposition I.1. [19] *Let* $\left\|(x_n)_{n\in\mathbb{Z}}\right\|_p$ *be the* $\ell^p_{\mathbb{F}}$ *norm of the sequence* (x_n) *in the space* $\ell^p_{\mathbb{F}}$.

PRP: $\left\|(x_n)\right\|_p$ *induces an inner product* $\iff$ $p = 2$

✎PROOF:

1. Proof that $\|\cdot\|_p$ induces an inner product $\Longleftarrow$ $p = 2$ (using the *Parallelogram law* page 241):

$$\begin{aligned}
&\|x+y\|_p^2 + \|x-y\|_p^2 \\
&= \|x+y\|_2^2 + \|x-y\|_2^2 && \text{by right hypothesis}\\
&= \left(\sum_{n\in\mathbb{Z}} |x_n+y_n|^2\right)^{\frac{2}{2}} + \left(\sum_{n\in\mathbb{Z}} |x_n-y_n|^2\right)^{\frac{2}{2}} && \text{by definition of } \|\cdot\|_p\\
&= \sum_{n\in\mathbb{Z}} (x_n+y_n)(x_n+y_n)^* + \sum_{n\in\mathbb{Z}} (x_n-y_n)(x_n-y_n)^*\\
&= \sum_{n\in\mathbb{Z}} \left(|x_n|^2 + |y_n|^2 + 2\Re(x_n y_n)\right) + \sum_{n\in\mathbb{Z}} \left(|x_n|^2 + |y_n|^2 - 2\Re(x_n y_n)\right)\\
&= 2\sum_{n\in\mathbb{Z}} |x_n|^2 + 2\sum_{n\in\mathbb{Z}} |y_n|^2\\
&= 2\|x\|_2^2 + 2\|y\|_2^2 && \text{by definition of } \|\cdot\|_p\\
&= 2\|x\|_p^2 + 2\|y\|_p^2 && \text{by right hypothesis}\\
&\implies \|\cdot\|_2 \text{ induces an inner product} && \text{by Theorem I.7 page 241}
\end{aligned}$$

2. Proof that $\|\cdot\|_p$ induces an inner product $\Longrightarrow$ $p = 2$:

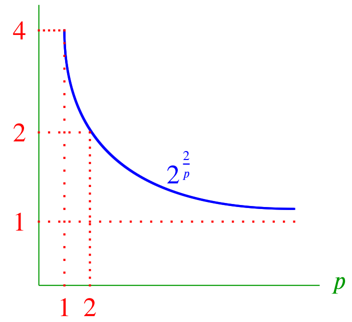
   (a) Let $x \triangleq (1, 0)$ and $y \triangleq (0, 1)$. Then [20]

$$\begin{aligned}
\|x+y\|_p^2 + \|x-y\|_p^2 &= \left(\sum_{n\in\mathbb{Z}} |x_n+y_n|^p\right)^{\frac{2}{p}} + \left(\sum_{n\in\mathbb{Z}} |x_n-y_n|^p\right)^{\frac{2}{p}} && \text{by definition of } \|\cdot\|_p\\
&= \left(|1+0|^p + |0+1|^p\right)^{\frac{2}{p}} + \left(|1-0|^p + |0-1|^p\right)^{\frac{2}{p}} && \text{by definitions of } x \text{ and } y\\
&= 2^{\frac{2}{p}} + 2^{\frac{2}{p}}\\
&= 2\cdot 2^{\frac{2}{p}}\\
2\|x\|_p^2 + 2\|y\|_p^2 &= 2\left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{2}{p}} + 2\left(\sum_{n\in\mathbb{Z}} |y_n|^p\right)^{\frac{2}{p}} && \text{by definition of } \|\cdot\|_p\\
&= 2\left(|1|^p + |0|^p\right)^{\frac{2}{p}} + 2\left(|1|^p + |0|^p\right)^{\frac{2}{p}} && \text{by definitions of } x \text{ and } y\\
&= 2+2\\
&= 4\\
2\cdot 2^{\frac{2}{p}} = 4 &= \iff 2^{\frac{2}{p}} = 2\\
&= \implies p = 2
\end{aligned}$$

[19] Kubrusly (2001) pages 318–319 ⟨Example 5B⟩
[20] http://groups.google.com/group/sci.math/msg/531b1173f08871e9

(b) Proof that $2^{2/p}$ is monotonic decreasing in p (and so $p = 2$ is the only solution):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p} 2^{\frac{2}{p}} &= \frac{\mathrm{d}}{\mathrm{d}p} e^{\ln 2^{\frac{2}{p}}} \\
&= \left(e^{\ln 2^{\frac{2}{p}}}\right) \frac{\mathrm{d}}{\mathrm{d}p} \ln 2^{\frac{2}{p}} \\
&= \left(2^{\frac{2}{p}}\right) \frac{\mathrm{d}}{\mathrm{d}p} (2 \ln 2) \frac{1}{p} \\
&= \left(2^{\frac{2}{p}}\right) 2 \ln 2 \left(-\frac{1}{p^2}\right) \\
&< 0 \qquad \forall p \in (0, \infty)
\end{aligned}$$



I.3 Orthogonality

Definition I.3.

DEF: *The **Kronecker delta function** $\bar{\delta}_n$ is defined as* $\bar{\delta}_n \triangleq \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0. \end{cases}$ *and* $\forall n \in \mathbb{Z}$

Definition I.4. *Let* $\left(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *be an* INNER PRODUCT SPACE (Definition I.1 page 233).

DEF:
Two vectors $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* X *are **orthogonal** if*

$$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \begin{cases} 0 & \text{for } \boldsymbol{x} \neq \boldsymbol{y} \\ c \in \mathbb{F} \setminus 0 & \text{for } \boldsymbol{x} = \boldsymbol{y} \end{cases}$$

The notation $\boldsymbol{x} \perp \boldsymbol{y}$ *implies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are* ORTHOGONAL.
A set $Y \in 2^X$ *is **orthogonal** if* $\boldsymbol{x} \perp \boldsymbol{y} \quad \forall \boldsymbol{x}, \boldsymbol{y} \in Y$.
A set Y *is **orthonomal** if it is* ORTHOGONAL *and* $\langle \boldsymbol{y} \mid \boldsymbol{y} \rangle = 1 \quad \forall \boldsymbol{y} \in Y$.
A sequence $(\boldsymbol{x}_n \in X)_{n \in \mathbb{Z}}$ *is **orthogonal** if* $\langle \boldsymbol{x}_n \mid \boldsymbol{x}_m \rangle = c\bar{\delta}_{nm}$ *for some* $c \in \mathbb{R} \setminus 0$.
A sequence $(\boldsymbol{x}_n \in X)_{n \in \mathbb{Z}}$ *is **orthonormal** if* $\langle \boldsymbol{x}_n \mid \boldsymbol{x}_m \rangle = \bar{\delta}_{nm}$.

The definition of the orthogonality relation $\perp$ has several immediate consequences (next theorem):

Theorem I.8. [21] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *be an* INNER PRODUCT SPACE.

THM:

1.	$\boldsymbol{x} \perp \boldsymbol{x} \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	
2.	$\boldsymbol{x} \perp \boldsymbol{y} \implies \alpha\boldsymbol{x} \perp \boldsymbol{y}$	$\forall \alpha \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in X$	(homogeneous)
3.	$\boldsymbol{x} \perp \boldsymbol{y} \iff \boldsymbol{y} \perp \boldsymbol{x}$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(symmetry)
4.	$\boldsymbol{x} \perp \boldsymbol{y}$ *and* $\boldsymbol{y} \perp \boldsymbol{z} \implies \boldsymbol{x} \perp (\boldsymbol{y} + \boldsymbol{z})$	$\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$	(additive)
5.	$\exists \beta \in \mathbb{R}$ such that $\boldsymbol{x} \perp (\beta\boldsymbol{x} + \boldsymbol{y})$	$\forall \boldsymbol{x} \in X \setminus \mathbb{0}, \boldsymbol{y} \in X$	

Theorem I.9. *Let* $\left(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ *be an* INNER PRODUCT SPACE.

THM:

$$\left.\begin{array}{lrcl} 1. & \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle & = & 0 \quad \textit{and} \\ 2. & \boldsymbol{x} + \boldsymbol{y} & = & \mathbb{0} \end{array}\right\} \iff \left\{\begin{array}{lrcl} 1. & \boldsymbol{x} & = & \mathbb{0} \quad \textit{and} \\ 2. & \boldsymbol{y} & = & \mathbb{0} \end{array}\right. \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in X$$

✎PROOF:

[21] James (1945), page 292, Drljević (1989) page 232

1. Proof that $\boldsymbol{x} = \boldsymbol{y} = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbb{0} \,|\, \mathbb{0} \rangle && \text{by } \textit{non-isotropic} \text{ property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition I.1 page 233)}\\
&= \langle \boldsymbol{x}+\boldsymbol{y} \,|\, \boldsymbol{x}+\boldsymbol{y} \rangle && \text{by left hypothesis 2}\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x}+\boldsymbol{y} \rangle + \langle \boldsymbol{y} \,|\, \boldsymbol{x}+\boldsymbol{y} \rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition I.1 page 233)}\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle + \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle + \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle^* + \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle && \text{by } \textit{conjugate symmetric} \text{ and } \textit{additive} \text{ properties of } \langle \triangle \,|\, \nabla \rangle\\
&= \underbrace{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}_{\geq 0} + 0 + 0 + \underbrace{\langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle}_{\geq 0} && \text{by left hypothesis 1}\\
&\implies \boldsymbol{x} = \mathbb{0} \text{ and } \boldsymbol{y} = \mathbb{0} && \text{by } \textit{non-negative} \text{ and } \textit{non-isotropic} \text{ properties of } \langle \triangle \,|\, \nabla \rangle
\end{aligned}$$

2. Proof that $\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle = 0$:

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle &= \langle \mathbb{0} \,|\, \mathbb{0} \rangle && \text{by right hypotheses}\\
&= 0 && \text{by } \textit{non-isotropic} \text{ property of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition I.1 page 233)}
\end{aligned}$$

3. Proof that $\boldsymbol{x} + \boldsymbol{y} = \mathbb{0}$:

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{y} &= \mathbb{0} + \mathbb{0} && \text{by right hypotheses}\\
&= \mathbb{0}
\end{aligned}$$

✏

The *triangle inequality* theorem for vectors in a *normed linear space* (Theorem J.1 page 249) demonstrates that

$$\left\| \sum_{n=1}^{N} \boldsymbol{x}_n \right\| \leq \sum_{n=1}^{N} \|\boldsymbol{x}_n\|.$$

The *Pythagorean Theorem* (next) demonstrates that this *inequality* becomes *equality* when the set $\{\boldsymbol{x}_n\}$ is orthogonal.

Theorem I.10 (Pythagorean Theorem). [22] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition I.1 page 233)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}$ *(Definition I.2 page 238).*

THM

$$\{\boldsymbol{x}_n\} \text{ is ORTHOGONAL} \iff \left\| \sum_{n=1}^{N} \boldsymbol{x}_n \right\|^2 = \sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2 \quad \forall N \in \mathbb{N}$$

✎PROOF: 1. Proof for ($\implies$) case:

$$\begin{aligned}
\left\| \sum_{n=1}^{N} \boldsymbol{x}_n \right\|^2 &= \left\langle \sum_{n=1}^{N} \boldsymbol{x}_n \,\middle|\, \sum_{m=1}^{N} \boldsymbol{x}_m \right\rangle && \text{by def. of } \|\cdot\| && \text{(Definition J.1 page 249)}\\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_m \rangle && \text{by def. of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition I.1 page 233)}\\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_m \rangle \, \bar{\delta}_{n-m} && \text{by left hypothesis}\\
&= \sum_{n=1}^{N} \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_n \rangle && \text{by def. of } \bar{\delta} && \text{(Definition I.3 page 245)}\\
&= \sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2 && \text{by def. of } \|\cdot\| && \text{(Definition J.1 page 249)}
\end{aligned}$$

[22] Aliprantis and Burkinshaw (1998) pages 282–283 ⟨Theorem 32.7⟩, Kubrusly (2001) page 324 ⟨Proposition 5.8⟩, Bollobás (1999) pages 132–133 ⟨Theorem 3⟩

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
&4\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= \|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2 + i\,\|\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\,\|\boldsymbol{x}-i\boldsymbol{y}\|^2 && \text{by \textit{polarization identity} (Theorem I.6 page 240)} \\
&= \left(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2\right) - \left(\|\boldsymbol{x}\|^2+\|-\boldsymbol{y}\|^2\right) + i\left(\|\boldsymbol{x}\|^2+\|i\boldsymbol{y}\|^2\right) - i\left(\|\boldsymbol{x}\|^2+\|-i\boldsymbol{y}\|^2\right) && \text{by right hypothesis} \\
&= \left(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2\right) - \left(\|\boldsymbol{x}\|^2+|-1|^2\,\|\boldsymbol{y}\|^2\right) + i\left(\|\boldsymbol{x}\|^2+|i|^2\,\|\boldsymbol{y}\|^2\right) - i\left(\|\boldsymbol{x}\|^2+|-i|^2\,\|\boldsymbol{y}\|^2\right) && \text{by definition of } \|\cdot\| \\
&= \left(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2\right) - \left(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2\right) + i\left(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2\right) - i\left(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2\right) && \text{by def. of } |\cdot| \\
&= 0
\end{aligned}$$

⇰

APPENDIX J

NORMED LINEAR SPACES

J.1 Definition and basic results

Definition J.1. [1] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition F.1 page 185) and* $|\cdot| \in \mathbb{R}^{\mathbb{F}}$ *the* ABSOLUTE VALUE *function.*

DEF

A functional $\|\cdot\|$ *in* $\mathbb{R}^X$ *is a* ***norm*** *if*

1. $\|x\| \geq 0$	$\forall x \in X$	(STRICTLY POSITIVE)	*and*
2. $\|x\| = 0 \iff x = \mathbb{0}$	$\forall x \in X$	(NONDEGENERATE)	*and*
3. $\|\alpha x\| = \lvert\alpha\rvert\, \|x\|$	$\forall x \in X,\ \alpha \in \mathbb{C}$	(HOMOGENEOUS)	*and*
4. $\|x + y\| \leq \|x\| + \|y\|$	$\forall x, y \in X$	(SUBADDITIVE/TRIANGLE INQUALITY).	

A ***normed linear space*** *is the tuple* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

The definition of the *norm* (Definition J.1 page 249) requires that any two vectors in a norm space be *subadditive* (they satisfy the *triangle inequality* property) such that $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$. Acutally, in **any** normed linear space, this property holds true for **any** finite number of vectors—not just two—such that $\|x_1 + x_2 + \cdots + x_N\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_N\|$ (next theorem).

Theorem J.1 (triangle inequality). [2] *Let* $(x_n \in X)_1^N$ *be an* N-TUPLE *(Definition P.1 page 329) of vectors in a* NORMED LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

THM

$$\left\|\sum_{n=1}^{N} x_n\right\| \leq \sum_{n=1}^{N} \|x_n\| \qquad \forall N \in \mathbb{N},\ x_n \in V$$

✎PROOF: Proof is by induction:

[1] Aliprantis and Burkinshaw (1998), pages 217–218, Banach (1932a), page 53, Banach (1932b), page 33, Banach (1922) page 135

[2] Michel and Herget (1993), page 344, Euclid (circa 300BC) ⟨Book I Proposition 20⟩

1. Proof for the $N = 1$ case:

$$\begin{aligned}\left\|\sum_{n=1}^{1} \boldsymbol{x}_n\right\| &= \|\boldsymbol{x}_1\| \\ &= \sum_{n=1}^{1} \|\boldsymbol{x}_1\|\end{aligned}$$

2. Proof for the $N = 2$ case:

$$\begin{aligned}\left\|\sum_{n=1}^{2} \boldsymbol{x}_n\right\| &= \left\|\sum_{n=1}^{2} \boldsymbol{x}_n\right\| \\ &= \|\boldsymbol{x}_1 + \boldsymbol{x}_2\| \\ &\le \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\| && \text{by Definition J.1 page 249 (triangle inequality)} \\ &= \sum_{n=1}^{2} \|\boldsymbol{x}_n\|\end{aligned}$$

3. Proof that [N case] $\implies$ [$N + 1$ case]:

$$\begin{aligned}\left\|\sum_{n=1}^{N+1} \boldsymbol{x}_n\right\| &= \left\|\sum_{n=1}^{N} \boldsymbol{x}_n + \boldsymbol{x}_{n+1}\right\| \\ &\le \left\|\sum_{n=1}^{N} \boldsymbol{x}_n\right\| + \|\boldsymbol{x}_{n+1}\| && \text{by Definition J.1 page 249 (triangle inequality)} \\ &\le \sum_{n=1}^{N} \|\boldsymbol{x}_n\| + \|\boldsymbol{x}_{n+1}\| && \text{by left hypothesis} \\ &= \sum_{n=1}^{N+1} \|\boldsymbol{x}_n\|\end{aligned}$$

⇨

Theorem J.2 (Reverse Triangle Inequality). [3] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

THM

$$\underbrace{\big|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|\big| \le \|\boldsymbol{x} - \boldsymbol{y}\|}_{\text{REVERSE TRIANGLE INEQUALITY}} \quad \le \quad \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in X$$

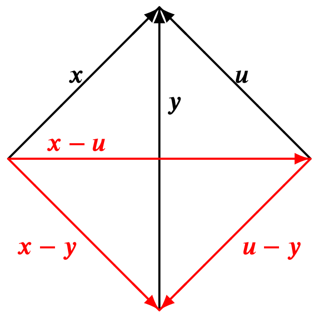
✎PROOF:

$$\begin{aligned}\big|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|\big| &= \big|\|(\boldsymbol{x} - \boldsymbol{y}) + \boldsymbol{y}\| - \|\boldsymbol{y}\|\big| \\ &\le \big|\|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y}\| - \|\boldsymbol{y}\|\big| && \text{by Definition J.1 page 249} \\ &= \big|\|\boldsymbol{x} - \boldsymbol{y}\|\big| \\ &= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by Definition J.1 page 249}\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{x} - \boldsymbol{y}\| &\le \|\boldsymbol{x} - \mathbb{0}\| + \|\mathbb{0} - \boldsymbol{y}\| && \text{by previous result with } \boldsymbol{u} = 0 \\ &= \|\boldsymbol{x}\| + |-1|\,\|\boldsymbol{y}\| && \text{by Definition J.1 page 249} \\ &= \|\boldsymbol{x}\| + \|\boldsymbol{y}\|\end{aligned}$$

⇨

[3] Aliprantis and Burkinshaw (1998), page 218, Giles (2000) page 2, Banach (1922) page 136



The shortest distance between two vectors is always the difference of the vectors. This is proven in next and illustrated to the left in the Euclidean space $\mathbb{R}^2$ (the plane)

Proposition J.1. [4] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

PRP
$$\|\boldsymbol{x}-\boldsymbol{y}\| \leq \|\boldsymbol{x}-\boldsymbol{u}\| + \|\boldsymbol{u}-\boldsymbol{y}\| \qquad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u} \in X$$

✎PROOF:

$$\begin{aligned}\|\boldsymbol{x}-\boldsymbol{y}\| &= \|(\boldsymbol{x}-\boldsymbol{u}) + (\boldsymbol{u}-\boldsymbol{y})\| \\ &\leq \|\boldsymbol{x}-\boldsymbol{u}\| + \|\boldsymbol{u}-\boldsymbol{y}\| && \text{by Definition J.1 page 249}\end{aligned}$$

⇰

Example J.1 (The usual norm). [5] Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions with domain and range the set of *real numbers* $\mathbb{R}$.

EX The **absolute value** (Definition **??** page **??**) $|\cdot| \in \mathbb{R}^{\mathbb{R}}$ is a *norm*.

Example J.2 (l_p norms). Let $(\!(x_n)\!)_{n\in\mathbb{Z}}$ be a *sequence* (Definition P.1 page 329) of real numbers. An uncountably infinite number of norms is provided by the $\ell^p_{\mathbb{F}}$ norms $\|(\!(x_n)\!)\|_p$:

EX
$$\|(\!(x_n)\!)\|_p \triangleq \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{1}{p}} \qquad \text{is a norm for } p \in [1 : \infty]$$

J.2 Relationship between metrics and norms

J.2.1 Metrics generated by norms

The concept of *length* is very closely related to the concept of *distance*. Thus it is not surprising that a *norm* (a "length" function) can be used to define a *metric* (a "distance" function) on any *metric linear space* (Definition **??** page **??**). Another way to say this is that the norm of a normed linear space *induces* a metric on this space. And so every normed linear space also has a metric. And because every normed linear space has a metric, **every normed linear space is also a metric space**. Actually this can be generalized one step further in that every metric space is also a *topological space*. And so **every normed linear space is also a topological space.** In symbols,

$$\text{normed linear space} \implies \text{metric space} \implies \text{topological space.}$$

[4] Aliprantis and Burkinshaw (1998), page 218
[5] Giles (1987), page 3

Theorem J.3. [6] *Let* $\mathsf{d} \in \mathbb{R}^{X \times X}$ *be a function on a* REAL *normed linear space* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \|\cdot\|)$. *Let* $\mathsf{B}(\boldsymbol{x}, r) \triangleq \{\boldsymbol{y} \in \boldsymbol{X} \mid \|\boldsymbol{y} - \boldsymbol{x}\| < r\}$ *be the* OPEN BALL *of radius* r *centered at a point* $\boldsymbol{x}$.

THM: $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\|$ *is a metric on* $\boldsymbol{X}$

✎PROOF: The proof follows directly from the definition of a metric (not included in this text) the definition of *norm* (Definition J.1 page 249). ✏

The previous theorem defined a metric $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y})$ induced by the norm $\|\boldsymbol{x}\|$. The next definition defines this metric formally.

Definition J.2. [7] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

DEF: *The **metric induced by the norm*** $\|\cdot\|$ *is the function* $\mathsf{d} \in \mathbb{R}^{\boldsymbol{X}}$ *such that*

$$\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\| \qquad \forall x, y \in X$$

Due to its algebraic structure, every norm is *continuous* (Corollary J.1 page 252). This makes norm spaces very useful in analysis. For a function f be to *continuous*, for every $\epsilon > 0$ there must exist a $\delta > 0$ such that $|\mathsf{f}(x + \delta) - \mathsf{f}(x)| < \epsilon$. The *Reverse Triangle Inequality* (Theorem J.2 page 250) shows this to be true when $\mathsf{f}(\cdot) \triangleq \|\cdot\|$.

Corollary J.1. [8] *Let* $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

COR: *The norm* $\|\cdot\|$ *is* CONTINUOUS *in* $\boldsymbol{\Omega}$.

✎PROOF: This follows from these concepts:

1. The fact that $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\|$ is a *metric* (Theorem J.3 page 252).
2. *Continuity* in a metric space.
3. The *Reverse Triangle Inequality* (Theorem J.2 page 250).

✏

Theorem J.4 (next) demonstrates that **all open or closed** balls in **any normed linear space** are *convex*. However, the converse is not true—that is, a metric not generated by a norm may still produce a ball that is convex.

Theorem J.4. [9] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d})$ *be a* METRIC LINEAR SPACE *(Definition **??** page **??**). Let* B *be the* OPEN BALL $\mathsf{B}(\boldsymbol{p}, r) \triangleq \{\boldsymbol{x} \in \boldsymbol{X} \mid \mathsf{d}(\boldsymbol{p}, \boldsymbol{x}) < r\}$ *(open ball with respect to metric* d *centered at point* $\boldsymbol{p}$ *and with radius* r*).*

THM:

$$\left.\begin{array}{l} \exists \|\cdot\| \in \mathbb{R}^{\boldsymbol{X}} \text{ such that} \\ \underbrace{\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{y} - \boldsymbol{x}\|}_{\mathsf{d} \text{ is generated by a norm}} \end{array}\right\} \implies \left\{\begin{array}{lll} 1. & \mathsf{B}(\boldsymbol{x}, r) = \boldsymbol{x} + B(0, r) & \\ 2. & \mathsf{B}(\mathbb{0}, r) = r\,\mathsf{B}(\mathbb{0}, 1) & \\ 3. & \mathsf{B}(\boldsymbol{x}, r) \text{ is CONVEX} & \\ 4. & \boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) \iff -\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) & \text{(SYMMETRIC)} \end{array}\right.$$

[6] Michel and Herget (1993), page 344, Banach (1932a) page 53
[7] Giles (2000) page 1 ⟨1.1 Definition⟩
[8] Giles (2000) page 2
[9] Giles (2000) page 2 ⟨1.2 Remarks⟩, Giles (1987) pages 22–26 ⟨2.4 Theorem, 2.11 Theorem⟩

✎PROOF:

1. Proof that $\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+v\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y})$ (invariant):

$$\begin{aligned}
\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+v\boldsymbol{z}) &= \|(\boldsymbol{y}+v\boldsymbol{z}) - (\boldsymbol{x}+\boldsymbol{z})\| && \text{by left hypothesis}\\
&= \|\boldsymbol{y}-\boldsymbol{x}\|\\
&= \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) && \text{by left hypothesis}
\end{aligned}$$

2. Proof that $\mathsf{B}(\boldsymbol{x}, r) = \boldsymbol{x} + B(0, r)$:

$$\begin{aligned}
\mathsf{B}(\boldsymbol{x}, r) &= \{\boldsymbol{y}\in\boldsymbol{X} | \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) < r\} && \text{by definition of open ball } B\\
&= \{\boldsymbol{y}\in\boldsymbol{X} | \mathsf{d}(\boldsymbol{y}-\boldsymbol{y}, \boldsymbol{y}-\boldsymbol{x}) < r\} && \text{by right result 1.}\\
&= \{\boldsymbol{y}\in\boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{y}-\boldsymbol{x}) < r\}\\
&= \{\boldsymbol{u}+\boldsymbol{x}\in\boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\} && \text{let } \boldsymbol{u} \triangleq \boldsymbol{y}-\boldsymbol{x}\\
&= \boldsymbol{x} + \{\boldsymbol{u}\in\boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\}\\
&= \boldsymbol{x} + B(0, r) && \text{by definition of open ball } B
\end{aligned}$$

3. Proof that $\mathsf{B}(\mathbb{0}, r) = r\,\mathsf{B}(\mathbb{0}, 1)$:

$$\begin{aligned}
\mathsf{B}(\mathbb{0}, r) &= \{\boldsymbol{y}\in\boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{y}) < r\} && \text{by definition of open ball } B\\
&= \left\{\boldsymbol{y}\in\boldsymbol{X} | \frac{1}{r}\mathsf{d}(\mathbb{0}, \boldsymbol{y}) < 1\right\}\\
&= \left\{\boldsymbol{y}\in\boldsymbol{X} | \frac{1}{r}\|\boldsymbol{y}-\mathbb{0}\| < 1\right\} && \text{by left hypothesis}\\
&= \left\{\boldsymbol{y}\in\boldsymbol{X} | \left\|\frac{1}{r}\boldsymbol{y}-\frac{1}{r}\mathbb{0}\right\| < 1\right\} && \text{by homogeneous property of } \|\cdot\| \text{ page 249}\\
&= \left\{\boldsymbol{y}\in\boldsymbol{X} | \mathsf{d}\left(\frac{1}{r}\mathbb{0},\ \frac{1}{r}\boldsymbol{y}\right) < 1\right\} && \text{by left hypothesis}\\
&= \{r\boldsymbol{u}\in\boldsymbol{X} | \mathsf{d}(\mathbb{0},\ \boldsymbol{u}) < 1\} && \text{let } \boldsymbol{u} \triangleq \frac{1}{r}\boldsymbol{y}\\
&= r\ \{\boldsymbol{u}\in\boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < 1\}\\
&= r\,B(\mathbb{0},\ 1) && \text{by definition of open ball } B
\end{aligned}$$

4. Proof that $\mathsf{B}(\boldsymbol{p}, r)$ is convex:
We must prove that for any pair of points $\boldsymbol{x}$ and $\boldsymbol{y}$ in the open ball $\mathsf{B}(\boldsymbol{p}, r)$, any point $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}$ is also in the open ball. That is, the distance from any point $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}$ to the ball's center $\boldsymbol{p}$ must be less than r.

$$\begin{aligned}
\mathsf{d}(\boldsymbol{p}, \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= \|\boldsymbol{p} - \lambda\boldsymbol{x} - (1-\lambda)\boldsymbol{y}\| && \text{by left hypothesis}\\
&= \Bigg\|\underbrace{\lambda\boldsymbol{p} + (1-\lambda)\boldsymbol{p}}_{\boldsymbol{p}} - \lambda\boldsymbol{x} - (1-\lambda)\boldsymbol{y}\Bigg\|\\
&= \|\lambda\boldsymbol{p} - \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{p} - (1-\lambda)\boldsymbol{y}\|\\
&\le \|\lambda\boldsymbol{p} - \lambda\boldsymbol{x}\| + \|(1-\lambda)\boldsymbol{p} - (1-\lambda)\boldsymbol{y}\| && \text{by subadditivity property of } \|\cdot\| \text{ page 249}\\
&= |\lambda|\,\|\boldsymbol{p}-\boldsymbol{x}\| + |1-\lambda|\,\|\boldsymbol{p}-\boldsymbol{y}\| && \text{by homogeneous property of } \|\cdot\| \text{ page 249}\\
&= \lambda\,\|\boldsymbol{p}-\boldsymbol{x}\| + (1-\lambda)\,\|\boldsymbol{p}-\boldsymbol{y}\| && \text{because } 0 \le \lambda \le 1\\
&\le \lambda r + (1-\lambda) r && \text{because } \boldsymbol{x}, \boldsymbol{y} \text{ are in the ball } \mathsf{B}(\boldsymbol{p}, r)\\
&= r
\end{aligned}$$

5. Proof that $\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) \iff -\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r)$ (symmetric):

$$\begin{aligned}
\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r) &\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{y}) < r\} && \text{by definition of open ball } B \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} | \, \|\boldsymbol{y} - \mathbb{0}\| < r\} && \text{by left hypothesis} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} | \, \|\boldsymbol{y}\| < r\} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} | \, \|(-1)(-\boldsymbol{y})\| < r\} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} | \, |-1| \, \|-\boldsymbol{y}\| < r\} && \text{by homogeneous property of } \|\cdot\| \text{ page 249} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} | \, \|-\boldsymbol{y} - \mathbb{0}\| < r\} \\
&\iff \boldsymbol{x} \in \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, -\boldsymbol{y}) < r\} && \text{by left hypothesis} \\
&\iff \boldsymbol{x} \in \{-\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\} && \text{let } \boldsymbol{u} \triangleq -\boldsymbol{y} \\
&\iff \boldsymbol{x} \in (-\{\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\}) \\
&\iff \boldsymbol{x} \in (-\mathsf{B}(\mathbb{0}, r)) \\
&\iff -\boldsymbol{x} \in \mathsf{B}(\mathbb{0}, r)
\end{aligned}$$

⇰

Theorem J.4 (page 252) demonstrates that if a metric d in a metric space $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$ is generated by a norm, then the ball $B(x, r)$ in that metric linear space is *convex*. However, the converse is not true. That is, it is possible for the balls in a metric space $(\boldsymbol{Y}, \mathsf{p})$ to be *convex*, but yet the metric p not be generated by a norm.

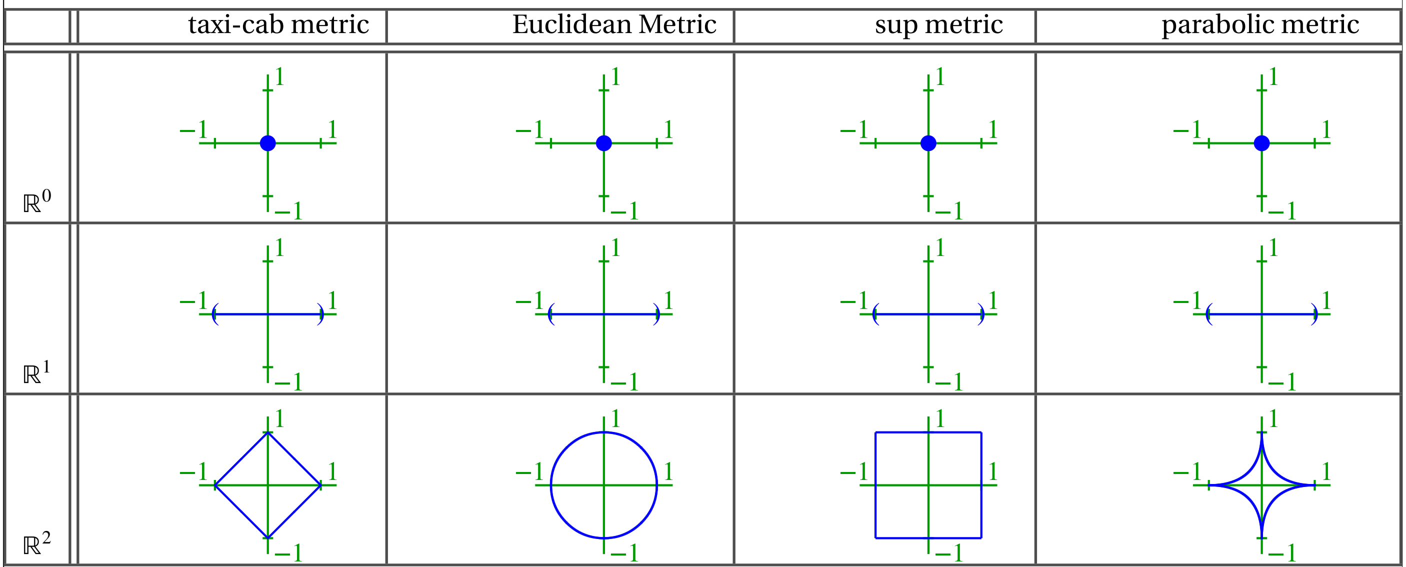


Figure J.1: Open balls in $(\mathbb{R}^0, \mathsf{d}_n)$, $(\mathbb{R}, \mathsf{d}_n)$, $(\mathbb{R}^2, \mathsf{d}_n)$, and $(\mathbb{R}^3, \mathsf{d}_n)$.

J.2.2 Norms generated by metrics

Every normed linear space is also a metric linear space (Theorem J.3 page 252). That is, a metric linear space generates a *normed linear space*. However, the converse is not true—not every metric linear space is a *normed linear space*. A characterization of metric linear spaces that *are* normed linear spaces is given by Theorem J.5 page 255.

Lemma J.1. [10] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$ *be a* METRIC LINEAR SPACE. *Let* $\|\boldsymbol{x}\| \triangleq \mathsf{d}(\boldsymbol{x}, \mathbb{0})\ \forall \boldsymbol{x} \in X$.

[10] Oikhberg and Rosenthal (2007), page 599

LEM
$$\underbrace{\mathsf{d}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in X}_{\text{TRANSLATION INVARIANT}} \implies \left\{\begin{array}{lrcll} 1. & \|\boldsymbol{x}\| &=& \|-\boldsymbol{x}\| & \forall \boldsymbol{x}\in X \quad \text{and} \\ 2. & \|\boldsymbol{x}\| &=& 0 \iff \boldsymbol{x}=0 & \forall \boldsymbol{x}\in X \quad \text{and} \\ 3. & \|\boldsymbol{x}+\boldsymbol{y}\| &\le& \|\boldsymbol{x}\|+\|\boldsymbol{y}\| & \forall \boldsymbol{x},\boldsymbol{y}\in X \end{array}\right.$$

✎PROOF:

1. Proof that $\|\boldsymbol{x}\| = \|-\boldsymbol{x}\|$:

$$\begin{aligned}
\|\boldsymbol{x}\| &= \mathsf{d}(\boldsymbol{x},\mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\boldsymbol{x}-\boldsymbol{x},\mathbb{0}-\boldsymbol{x}) && \text{by translation invariance hypothesis} \\
&= \mathsf{d}(\mathbb{0},-\boldsymbol{x}) \\
&= \|-\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

2a. Proof that $\|\boldsymbol{x}\| = 0 \implies \boldsymbol{x} = 0$:

$$\begin{aligned}
0 &= \|\boldsymbol{x}\| && \text{by left hypothesis} \\
&= \mathsf{d}(\boldsymbol{x},\mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\boldsymbol{x},\mathbb{0}) && \text{by definition of } \|\cdot\| \\
&\implies \boldsymbol{x} = \mathbb{0} && \text{by property of metrics}
\end{aligned}$$

2b. Proof that $\|\boldsymbol{x}\| = 0 \impliedby \boldsymbol{x} = 0$:

$$\begin{aligned}
\|\boldsymbol{x}\| &= \mathsf{d}(\boldsymbol{x},\mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\mathbb{0},\mathbb{0}) && \text{by right hypothesis} \\
&= 0 && \text{by property of metrics}
\end{aligned}$$

3. Proof that $\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\|+\|\boldsymbol{y}\|$:

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\| &= \mathsf{d}(\boldsymbol{x}+\boldsymbol{y},\mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{y},\mathbb{0}-\boldsymbol{y}) && \text{by translation invariance hypothesis} \\
&= \mathsf{d}(\boldsymbol{x},-\boldsymbol{y}) \\
&\le \mathsf{d}(\boldsymbol{x},\mathbb{0}) + \mathsf{d}(\mathbb{0},\boldsymbol{y}) && \text{by property of metrics} \\
&= \mathsf{d}(\boldsymbol{x},\mathbb{0}) + \mathsf{d}(\boldsymbol{y},\mathbb{0}) && \text{by property of metrics} \\
&= \|\boldsymbol{x}\| + \|\boldsymbol{y}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

⇰

Theorem J.5. [11] *Let* $(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}))$ *be a* LINEAR SPACE. *Let* $\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \triangleq \|\boldsymbol{x}-\boldsymbol{y}\|\ \forall \boldsymbol{x},\boldsymbol{y}\in X$.

THM
$$\left.\begin{array}{lll} 1. & \mathsf{d}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in X & \text{(TRANSLATION INVARIANT) and} \\ 2. & \mathsf{d}(\alpha\boldsymbol{x},\alpha\boldsymbol{y}) = |\alpha|\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y}\in X,\alpha\in\mathbb{F} & \text{(HOMOGENEOUS)} \end{array}\right\} \iff \|\cdot\| \text{ is a NORM}$$

✎PROOF:

1. Proof of $\implies$ assertion:
   (a) Proof that $\|\cdot\|$ is *strictly positive*: This follows directly from the definition of d.
   (b) Proof that $\|\cdot\|$ is *nondegenerate*: This follows directly from Lemma J.1 (page 254).
   (c) Proof that $\|\cdot\|$ is *homogeneous*: This follows from the second left hypothesis.

[11] Bollobás (1999), page 21

(d) Proof that $\|\cdot\|$ satisfies the *triangle-inequality*: This follows directly from Lemma J.1 (page 254).

2. Proof of $\Longleftarrow$ assertion:

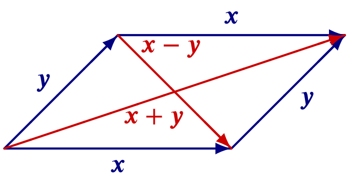
$$\begin{aligned}
\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) &= \|(\boldsymbol{x}+\boldsymbol{z}) - (\boldsymbol{y}+\boldsymbol{z})\| && \text{by definition of } \mathsf{d}\\
&= \|\boldsymbol{x}-\boldsymbol{y}\| \\
&= \mathsf{d}(\boldsymbol{x},\boldsymbol{y}) && \text{by definition of } \mathsf{d}\\
\mathsf{d}(\alpha\boldsymbol{x}, \alpha\boldsymbol{y}) &= \|(\alpha\boldsymbol{x}) - (\alpha\boldsymbol{y})\| && \text{by definition of } \mathsf{d}\\
&= \|\alpha(\boldsymbol{x}-\boldsymbol{y})\| \\
&= |\alpha|\,\|\boldsymbol{x}-\boldsymbol{y}\| && \text{by definition of } \|\cdot\| \text{ page 249}\\
&= |\alpha|\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) && \text{by definition of } \mathsf{d}
\end{aligned}$$

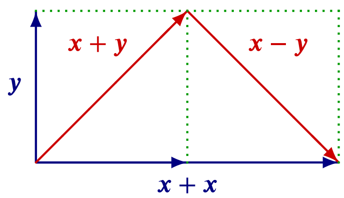
✏

J.3 Orthogonality on normed linear spaces

Traditionally, *orthogonality* (Definition I.4 page 245) is a property defined in *inner product spaces* (Definition I.1 page 233) However, the concept of orthogonality can be extended to *normed linear spaces* (Definition J.1 page 249). Here are some examples:

① *Isosceles orthogonality*: Definition J.3 page 256
② *Pythagorean orthogonality*: Definition J.4 page 258
③ *Birkhoff orthogonality*: Definition J.5 page 258





Isosceles orthogonality (Definition J.3 page 256) can be illustrated using a *parallelogram*, as illustrated in the figure to the upper left. In this case, orthogonality implies that the parallelogram is a rectangle, which in turn implies that the lengths of the two diagonals are equal ($\|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}-\boldsymbol{y}\|$). Isosceles orthogonality can also be illustrated with a triangle where the sides are of lengths $\|\boldsymbol{x}+\boldsymbol{y}\|$ and $\|\boldsymbol{x}-\boldsymbol{y}\|$ and base of length $\|\boldsymbol{x}+\boldsymbol{x}\|$. In this case if $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal, then the triangle is *isosceles*. This is illustrated in figure to the lower left. Isosceles orthogonality is formally defined next.

Definition J.3. [12] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

DEF *Two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are* ***orthogonal in the sense of James*** *if*

$$\|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}-\boldsymbol{y}\|\,.$$

This property is also called ***isosceles orthogonality*** *or* ***James orthogonality****.*

Theorem J.6. *Let* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be an* INNER-PRODUCT SPACE *(Definition I.1 page 233) with induced norm* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$, ISOSCELES ORTHOGONALITY *(Definition J.3 page 256) relation* ⓘ*, and inner-product relation* ORTHOGONALITY *(Definition I.4 page 245) relation* $\perp$.

THM

$$\underbrace{\boldsymbol{x} \text{ ⓘ } \boldsymbol{y}}_{\text{orthogonal in the sense of James}} \iff \underbrace{\boldsymbol{x} \perp \boldsymbol{y}}_{\text{orthogonal in the sense of inner-product space}}$$

[12] James (1945) page 292 ⟨DEFINITION 2.1⟩, Amir (1986) page 24, Dunford and Schwartz (1957), page 93

✎PROOF:

1. Proof that $\boldsymbol{x} \circledcirc \boldsymbol{y} \implies \boldsymbol{x} \perp \boldsymbol{y}$:

$$\begin{aligned}
&4\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle \\
&= \underbrace{\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2}_{0 \text{ by } \boldsymbol{x} \circledcirc \boldsymbol{y} \text{ hypothesis}} + i\|\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\|\boldsymbol{x}-i\boldsymbol{y}\|^2 && \text{by \textit{polarization identity} (Theorem I.6 page 240)} \\
&= 0 + i\|\boldsymbol{x}+i\boldsymbol{y}\|^2 - i\|\boldsymbol{x}-i\boldsymbol{y}\|^2 && \text{by } \boldsymbol{x} \circledcirc \boldsymbol{y} \text{ hypothesis} \\
&= i\left[\|\boldsymbol{x}\|^2 + \|i\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid i\boldsymbol{y}\rangle\right] \\
&\quad - i\left[\|\boldsymbol{x}\|^2 + \|-i\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid -i\boldsymbol{y}\rangle\right] && \text{by \textit{Polar Identity} (Lemma I.1 page 237)} \\
&= i\left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid i\boldsymbol{y}\rangle\right] \\
&\quad - i\left[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid -i\boldsymbol{y}\rangle\right] && \text{by Definition J.1 page 249 and Definition I.1 page 233} \\
&= 4i\Re\langle \boldsymbol{x} \mid i\boldsymbol{y}\rangle \\
&= 4i\Re\left[i^*\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle\right] \\
&= 0 && \text{because inner-product space is \textit{real} } (\mathbb{F} = \mathbb{R})
\end{aligned}$$

2. Proof that $\boldsymbol{x} \circledcirc \boldsymbol{y} \impliedby \boldsymbol{x} \perp \boldsymbol{y}$:

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by \textit{Polar Identity} (Lemma I.1 page 237)} \\
&= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 0 && \text{by } \boldsymbol{x} \perp \boldsymbol{y} \text{ hypothesis} \\
&= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\Re\cancel{\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle} \; 0 \text{ when } \boldsymbol{x} \perp \boldsymbol{y} && \text{by } \boldsymbol{x} \perp \boldsymbol{y} \text{ hypothesis} \\
&= \|\boldsymbol{x}\|^2 + \|-\boldsymbol{y}\|^2 + 2\Re\langle \boldsymbol{x} \mid -\boldsymbol{y}\rangle \\
&= \|\boldsymbol{x}-\boldsymbol{y}\|^2 && \text{by \textit{Polar Identity} (Lemma I.1 page 237)}
\end{aligned}$$

✏

Theorem J.7. [13] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a normed linear space and with* ISOSCELES ORTHOGONALITY *(Definition J.3 page 256) relation* $\circledcirc$.

THM
$$\boldsymbol{x} \circledcirc \boldsymbol{y} \iff \boldsymbol{y} \circledcirc \boldsymbol{x} \iff \alpha\boldsymbol{x} \circledcirc \alpha\boldsymbol{y} \quad \forall\alpha\in\mathbb{F}$$

✎PROOF:

$$\begin{aligned}
\boldsymbol{x} \circledcirc \boldsymbol{y} &\implies \|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}-\boldsymbol{y}\| && \text{by Definition J.3 page 256} \\
&\implies \|\boldsymbol{x}+\boldsymbol{y}\| = |-1|\;\|\boldsymbol{x}-\boldsymbol{y}\| \\
&\implies \|\boldsymbol{x}+\boldsymbol{y}\| = \|-(\boldsymbol{x}-\boldsymbol{y})\| && \text{by Definition J.1 page 249} \\
&\implies \|\boldsymbol{y}+\boldsymbol{x}\| = \|\boldsymbol{y}-\boldsymbol{x}\| && \text{by Definition F.1 page 185} \\
&\implies \boldsymbol{y} \circledcirc \boldsymbol{x} && \text{by Definition J.3 page 256}
\end{aligned}$$

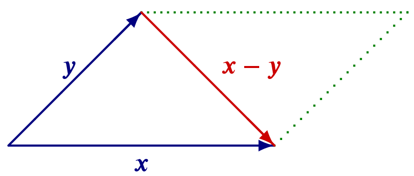
$$\begin{aligned}
\boldsymbol{y} \circledcirc \boldsymbol{x} &\implies \|\boldsymbol{y}+\boldsymbol{x}\| = \|\boldsymbol{y}-\boldsymbol{x}\| && \text{by Definition J.3 page 256} \\
&\implies |\alpha|\;\|\boldsymbol{y}+\boldsymbol{x}\| = |\alpha|\;\|\boldsymbol{y}-\boldsymbol{x}\| \\
&\implies \|\alpha(\boldsymbol{y}+\boldsymbol{x})\| = \|\alpha(\boldsymbol{y}-\boldsymbol{x})\| && \text{by Definition J.1 page 249} \\
&\implies \|\alpha\boldsymbol{y}+\alpha\boldsymbol{x}\| = \|\alpha\boldsymbol{y}-\alpha\boldsymbol{x}\| \\
&\implies \|\alpha\boldsymbol{x}+\alpha\boldsymbol{y}\| = \|-(\alpha\boldsymbol{x}-\alpha\boldsymbol{y})\| && \text{by Definition F.1 page 185}
\end{aligned}$$

[13] Amir (1986) page 24

$$\begin{aligned}
&\implies \|\alpha x + \alpha y\| &&= |-1|\ \|\alpha x - \alpha y\| && \text{by Definition J.1 page 249}\\
&\implies \|\alpha x + \alpha y\| &&= \|\alpha x - \alpha y\| &&\\
&\implies \alpha x \operp \alpha y &&&& \text{by Definition J.3 page 256}
\end{aligned}$$

$$\begin{aligned}
\alpha x \operp \alpha y &\implies \|\alpha x + \alpha y\| &&= \|\alpha x - \alpha y\| && \text{by Definition J.3 page 256}\\
&\implies \|\alpha(x + y)\| &&= \|\alpha(x - y)\| && \text{by Definition F.1 page 185}\\
&\implies |\alpha|\ \|x + y\| &&= |\alpha|\ \|x - y\| && \text{by Definition J.1 page 249}\\
&\implies \|x + y\| &&= \|x - y\| && \text{by Definition J.1 page 249}\\
&\implies x \operp y &&&& \text{by Definition J.3 page 256}
\end{aligned}$$

⇨



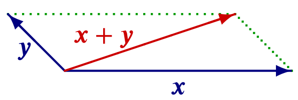
If a triangle in a plane has two perpendicular sides of lengths a and b and a hypotenuse of length c, then by the *Pythagorean Theorem* (Theorem I.10 page 246), $a^2 + b^2 = c^2$. This concept of orthogonality can be generalized to normed linear spaces. Two vectors x and y (with lengths $\|x\|$ and $\|y\|$) are orthogonal when $\|x\|^2 + \|y\|^2 = \|x - y\|^2$ ($x - y$ is a kind of "*hypotenuse*"). This kind of orthogonality is defined next and illustrated in the figure to the left.

Definition J.4. [14] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

DEF *Two vectors x and y are **orthogonal in the Pythagorean sense** if*
$$\|x - y\|^2 = \|x\|^2 + \|y\|^2.$$
*This relationship is also called **Pythagorean orthogonality**.*

Theorem J.8. [15] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be an* INNER-PRODUCT SPACE *(Definition I.1 page 233) with induced norm* $\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$, PYTHAGOREAN ORTHOGONALITY *(Definition J.4 page 258) relation* $\operp$, *and inner-product relation* ORTHOGONALITY *(Definition I.4 page 245) relation* $\perp$.

THM
$$\underbrace{x \operp y}_{\text{orthogonal in the Pythagorean sense}} \iff \underbrace{x \perp y}_{\text{orthogonal in the sense of inner-product space}}$$



Besides *isosceles orthogonality* (Definition J.3 page 256), orthogonality in normed linear spaces can be defined using *Birkhoff orthogonality*, as defined in Definition J.5 (next) and illustrated to the left.

Definition J.5. [16] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* NORMED LINEAR SPACE *(Definition J.1 page 249).*

DEF *Two vectors x and y are **orthogonal in the sense of Birkhoff** if*
$$\|x\| \le \|x + \alpha y\| \qquad \forall \alpha \in \mathbb{F}.$$
*This relationship is also called **Birkhoff orthogonality**.*

Theorem J.9. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be an* INNER-PRODUCT SPACE *(Definition I.1 page 233) with induced norm* $\|x\| \triangleq \sqrt{\langle x \mid x \rangle}$, BIRKHOFF ORTHOGONALITY *relation* $\operp$ *(Definition J.5 page 258), and inner-product relation* ORTHOGONALITY *relation* $\perp$ *(Definition I.4 page 245).*

THM
$$\underbrace{x \operp y}_{\text{orthogonal in the sense of Birkhoff}} \iff \underbrace{x \perp y}_{\text{orthogonal in the sense of inner-product space}}$$

[14] James (1945) page 292 ⟨DEFINITION 2.2⟩, Amir (1986) page 57, Drljević (1989) page 232
[15] Amir (1986) page 57
[16] Amir (1986) page 33, Dunford and Schwartz (1957), page 93, James (1947) page 265

APPENDIX K

INTERVALS AND CONVEXITY

K.1 Intervals

In the real number system, for $a \le b$, the *interval* $[a : b]$ is the set a and b and all the numbers inbetween, as in $[a : b] \triangleq \{x \in \mathbb{R} | a \le x \le b\}$. This concept can be easily generalized:

- In an **ordered set**, if two elements x and y are *comparable* and $x \le y$, then we say that x and y and all the elements inbetween, as determined by the ordering relation $\le$, are the interval $[a : b]$.
- In a **lattice**, the concept of the *interval* can be generalized even further. In an arbitrary ordered set, the interval $[x : y]$ of item (K.1) is restricted to the case in which x and y are *comparable*. This restriction can be lifted (Definition K.2 page 259) with the additional structure of upper and lower bounds provided by lattices.
- A **metric space** in general has no *order relation* $\le$. But intervals can still be defined (Definition K.4 page 260) in a metric space in terms of the *triangle inequality*.
- A **linear space** (Definition F.1 page 185) over a real or complex field in general has no *order relation* that compares *vectors* in the space, but the standard order relation $\le$ for real numbers $\mathbb{R}$ can still be used (Definition K.5 page 260) to define an interval in a linear space.

Definition K.1 (intervals on ordered sets). [1] *Let* $(X, \le)$ *be an* ORDERED SET.

D E F	*The set* $[x : y]$	$\triangleq \{z \in X	x \le z \le y\}$	*is called a* ***closed interval***	*and*
	The set $(x : y]$	$\triangleq \{z \in X	x < z \le y\}$	*is called a* ***half-open interval***	*and*
	The set $[x : y)$	$\triangleq \{z \in X	x \le z < y\}$	*is called a* ***half-open interval***	*and*
	The set $(x : y)$	$\triangleq \{z \in X	x < z < y\}$	*is called an* ***open interval***.	

Definition K.2 (intervals on lattices). [2] *Let* $(X, \vee, \wedge; \le)$ *be a* LATTICE.

D E F	*The set* $[x : y]$	$\triangleq \{z \in X | x \wedge y \le z \le x \vee y\}$	*is called a* ***closed interval***.
	The set $(x : y]$	$\triangleq \{z \in X | x \wedge y < z \le x \vee y\}$	*is called a* ***half-open interval***.
	The set $[x : y)$	$\triangleq \{z \in X | x \wedge y \le z < x \vee y\}$	*is called a* ***half-open interval***.
	The set $(x : y)$	$\triangleq \{z \in X | x \wedge y < z < x \vee y\}$	*is called an* ***open interval***.

When x and y *are* comparable and $x \le y$, then Definition K.2 (previous) simplifies to item (K.1)

[1] Apostol (1975) page 4, Ore (1935) page 409
[2] Duthie (1942) page 2, Ore (1935) page 425 ⟨*quotient structures*⟩

(page 259).

Definition K.3. [3] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *with dual* $\mathbf{L}^*$. *Let* $[x : y]$ *be a* CLOSED INTERVAL (Definition K.2 page 259) *on set* X. *The sublattices* $\mathbf{L}[x : y]$ *and* $\mathbf{L}^*[x : y]$ *are defined as follows:*

DEF
$$\begin{array}{lcll} \mathbf{L}[x : y] & \triangleq & \{z \in \mathbf{L} | z \in [x : y]\} & \forall x,y \in X \\ \mathbf{L}^*[x : y] & \triangleq & \{z \in \mathbf{L}^* | z \in [x : y]\} & \forall x,y \in X \end{array}$$

Definition K.4. [4]

DEF
In a METRIC SPACE (X, d),
the set $[a : b]$ *is the* ***closed interval*** *from* x *to* y *and is defined as*
$$[x : y] \triangleq \{z \in X | \mathsf{d}(x, z) + \mathsf{d}(z, y) = \mathsf{d}(x, y)\}.$$
An element $z \in X$ *is* ***geodesically between*** x *and* y *if* $z \in [x : y]$.

Definition K.5. [5]

DEF
In a LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ (Definition F.1 page 185),
$$\begin{array}{lcll} [\mathbf{x} : \mathbf{y}] & \triangleq & \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \mathbf{z} | 0 \leq \lambda \leq 1\} & \text{is called a } \textbf{closed interval} \quad \text{and} \\ (\mathbf{x} : \mathbf{y}] & \triangleq & \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \mathbf{z} | 0 < \lambda \leq 1\} & \text{is called a } \textbf{half-open interval} \quad \text{and} \\ [\mathbf{x} : \mathbf{y}) & \triangleq & \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \mathbf{z} | 0 \leq \lambda < 1\} & \text{is called a } \textbf{half-open interval} \quad \text{and} \\ (\mathbf{x} : \mathbf{y}) & \triangleq & \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \mathbf{z} | 0 < \lambda < 1\} & \text{is called an } \textbf{open interval}. \end{array}$$

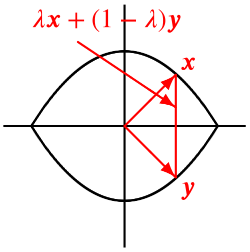
K.2 Convex sets

Using the concept of the *interval* (previous section), we can define the *convex set* (next definition).

Definition K.6. [6] *Let* X *be a* SET *in an* ORDERED SET $(X, \leq)$, *a* LATTICE $(X, \vee, \wedge; \leq)$, *a* METRIC SPACE (X, d), *or a* LINEAR SPACE $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$.

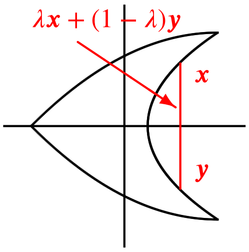
DEF
A subset $D \subseteq X$ *is a* ***convex set*** *in* X *if*
$$x, y \in D \quad \Longrightarrow \quad [x : y] \subseteq D.$$
A set that is ***not*** *convex is* ***concave***.

Example K.1. Consider the *Euclidean space* $\mathbb{R}^2$ (a special case of a *linear space*).



$\Leftarrow$ {The figure to the left is a *convex* set in $\mathbb{R}^2$.

The figure to the right is a *concave* set in $\mathbb{R}^2$.} $\Rightarrow$



Example K.2. In a *metric space*, examples of *convex sets* are convex *balls*. Examples include those balls generated by the following metrics:

- Taxi-cab metric
- Euclidean metric
- Sup metric
- Tangential metric

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[3] Maeda and Maeda (1970), page 1
[4] van de Vel (1993) page 8
[5] Barvinok (2002) page 2
[6] Barvinok (2002) page 5

Examples of metrics generating balls which are *not* convex include the following:

- Parabolic metric
- Exponential metric

K.3 Convex functions

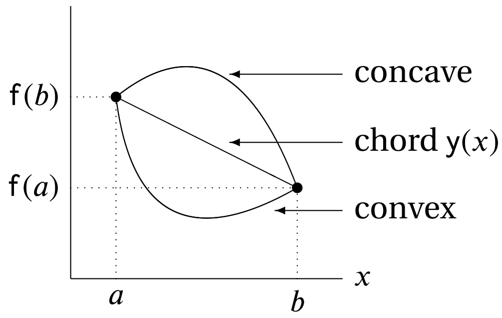


Figure K.1: Convex and concave functions

Definition K.7. [7] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition F.1 page 185) *and* D *a* CONVEX SET (Definition K.6 page 260) *in* X.

DEF

A function $f \in F^D$ *is **convex** if*

$$f(\lambda x + [1-\lambda]y) \le \lambda f(x) + (1-\lambda) f(y) \qquad \forall x, y \in D \text{ and } \forall \lambda \in (0,1)$$

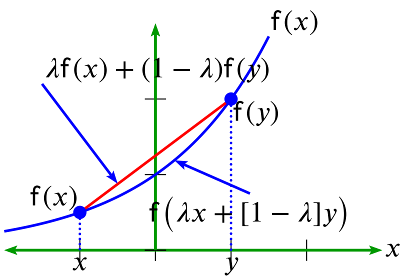
A function $g \in F^D$ *is **strictly convex** if*

$$g(\lambda x + [1-\lambda]y) = \lambda g(x) + (1-\lambda) g(y) \qquad \forall x, y \in D,\ x \neq y,\ \text{and } \forall \lambda \in (0,1)$$

A function $f \in F^D$ *is **concave** if* $-f$ *is* CONVEX.

A function $f \in F^D$ *is **affine** if* f *is* CONVEX *and* CONCAVE.

Example K.3. The function $f(x) = 2^x$ is a *convex function* (Definition K.7 page 261), as illustrated to the right.



Definition K.8. [8] *Let* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE (Definition F.1 page 185).

DEF

*The **epigraph** epi(f) and **hypograph** hyp(f) of a functional* $f \in \mathbb{R}^X$ *are defined as*

$$\text{epi}(f) \triangleq \{(x,y) \in X \times \mathbb{R} | y \ge f(x)\}$$

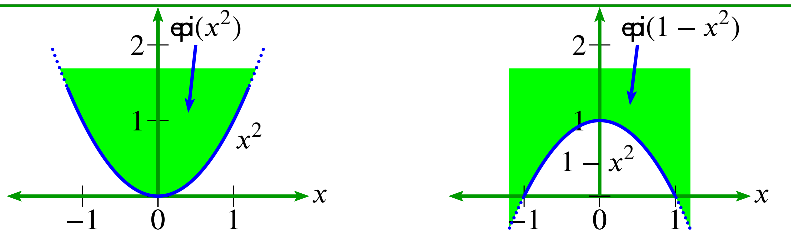
$$\text{hyp}(f) \triangleq \{(x,y) \in X \times \mathbb{R} | y \le f(x)\}$$

Example K.4.

[7] Simon (2011) page 2, Barvinok (2002) page 2, Bollobás (1999), page 3, Jensen (1906), page 176, Clarkson (1936) ⟨strictly convex⟩

[8] Beer (1993) page 13 ⟨§1.3⟩, Aubin and Frankowska (2009) page 222, Aubin (2011) page 223

EX



Proposition K.1. [9] *Let* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition F.1 page 185). Let* f *be a* FUNCTIONAL *in* $\mathbb{R}^X$.

PRP
$$\left\{\begin{array}{l}\mathsf{f} \textit{ is a}\\ \text{CONVEX FUNCTION}\end{array}\right\} \iff \left\{\begin{array}{l}\mathrm{epi}(\mathsf{f}) \textit{ is a}\\ \text{CONVEX SET}\end{array}\right\}$$

Often a function can be proven to be *convex* or *concave*. *Convex* and *concave* functions are defined in Definition K.9 (page 262) (next) and illustrated in Figure K.1 (page 261).

Definition K.9. *Let*
$$\mathsf{y}(x) = \frac{\mathsf{f}(b) - \mathsf{f}(a)}{b-a}(x-a) + \mathsf{f}(a).$$
A function $\mathsf{f} : \mathbb{R} \to \mathbb{R}$ *is*

DEF

(1).	***convex***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$\le$	$\mathsf{y}(x)$	*for* $x \in (a:b)$
(2).	***concave***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$\ge$	$\mathsf{y}(x)$	*for* $x \in (a:b)$
(3).	***strictly convex***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$<$	$\mathsf{y}(x)$	*for* $x \in (a:b)$
(4).	***strictly concave***	*in* $(a:b)$ *if*	$\mathsf{f}(x)$	$>$	$\mathsf{y}(x)$	*for* $x \in (a:b)$

Theorem K.1 (Jensen's Inequality). [10] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE *(Definition F.1 page 185), D a subset of X, and f a functional in* $\mathbb{F}^D$. *Let* $\sum$ *be the* SUMMATION OPERATOR *(Definition L.1 page 267).*

THM
$$\left\{\begin{array}{lll} 1. & D \text{ is CONVEX} & \textit{and}\\ 2. & \mathsf{f} \text{ is CONVEX} & \textit{and}\\ 3. & \displaystyle\sum_{n=1}^{N} \lambda_n = 1 & (\text{WEIGHTS}) \end{array}\right\} \implies \mathsf{f}\left(\sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n\right) \le \sum_{n=1}^{N} \lambda_n \mathsf{f}(\boldsymbol{x}_n) \qquad \forall \boldsymbol{x}_n \in D,\ N \in \mathbb{N}$$

✎PROOF: Proof is by induction:

1. Proof that statement is true for $N = 1$:
$$\begin{aligned}\mathsf{f}\left(\sum_{n=1}^{N=1} \lambda_n \boldsymbol{x}_n\right) &= \mathsf{f}(\lambda_1 \boldsymbol{x}_1)\\ &\le \mathsf{f}(\lambda_1 \boldsymbol{x}_1)\\ &= \sum_{n=1}^{N=1} \lambda_n \mathsf{f}(\boldsymbol{x}_n)\end{aligned}$$

[9] Udriste (1994) page 63, Kurdila and Zabarankin (2005) page 178 ⟨Proposition 6.1.1⟩, Rockafellar (1970) page 23 ⟨Section 4 Convex Functions⟩, Çinlar and Vanderbei (2013) page 86 ⟨5.4 Theorem⟩

[10] Mitrinović et al. (2010) page 6, Bollobás (1999) page 3, Lay (1982) page 7, Jensen (1906), pages 179–180

2. Proof that statement is true for $N = 2$:

$$\begin{aligned}
\mathsf{f}\left(\sum_{n=1}^{N=2} \lambda_n \boldsymbol{x}_n\right) &= \mathsf{f}(\lambda_1\boldsymbol{x}_1 + \lambda_2\boldsymbol{x}_2) \\
&\le \lambda_1\mathsf{f}(\boldsymbol{x}_1) + \lambda_2\mathsf{f}(\boldsymbol{x}_2) && \text{by convexity hypothesis} \\
&= \sum_{n=1}^{N=2} \lambda_n \mathsf{f}(\boldsymbol{x}_n)
\end{aligned}$$

3. Proof that if the statement is true for N, then it is also true for $N + 1$:

$$\begin{aligned}
\mathsf{f}\left(\sum_{n=1}^{N+1} \lambda_n \boldsymbol{x}_n\right) &= \mathsf{f}\left(\sum_{n=1}^{N} \lambda_n \boldsymbol{x}_n + \lambda_{N+1}\boldsymbol{x}_{N+1}\right) \\
&= \mathsf{f}\left([1-\lambda_{N+1}]\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \boldsymbol{x}_n + \lambda_{N+1}\boldsymbol{x}_{N+1}\right) \\
&\le [1-\lambda_{N+1}]\mathsf{f}\left(\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \boldsymbol{x}_n\right) + \lambda_{N+1}\mathsf{f}(\boldsymbol{x}_{N+1}) && \text{by convexity hypothesis} \\
&\le [1-\lambda_{N+1}]\sum_{n=1}^{N} \frac{\lambda_n}{1-\lambda_{N+1}} \mathsf{f}(\boldsymbol{x}_n) + \lambda_{N+1}\mathsf{f}(\boldsymbol{x}_{N+1}) && \text{by “true for } N\text{” hypothesis} \\
&= \sum_{n=1}^{N} \lambda_n\mathsf{f}(\boldsymbol{x}_n) + \lambda_{N+1}\mathsf{f}(\boldsymbol{x}_{N+1}) \\
&= \sum_{n=1}^{N+1} \lambda_n\mathsf{f}(\boldsymbol{x}_n)
\end{aligned}$$

4. Since the statement is true for $N = 1$, $N = 2$, and true for $N \implies$ true for $N + 1$, then it is true for $N = 1, 2, 3, 4, \ldots$

✏

The next theorem gives another form of convex functions that is a little less intuitive but provides powerful analytic results.

Theorem K.2. *Let* $\mathsf{f} : \mathbb{R} \to \mathbb{R}$. *For every* $x_1, x_2 \in (a, b)$ *and* $\lambda \in [0, 1]$

THM f *is convex in* (a, b) $\iff$ $\mathsf{f}(\lambda x_1 + (1-\lambda)x_2) \le \lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)$

✎PROOF:

1. prove f is convex $\implies \mathsf{f}(\lambda x_1 + (1-\lambda)x_2) \le \lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)$:

$$\begin{aligned}
\mathsf{f}(\lambda x_1 + (1-\lambda)x_2) &\le \frac{\mathsf{f}(b)-\mathsf{f}(a)}{b-a}[\lambda x_1 + (1-\lambda)x_2 - a] + \mathsf{f}(a) \\
&= \frac{\mathsf{f}(x_2)-\mathsf{f}(x_1)}{x_2-x_1}[\lambda x_1 + (1-\lambda)x_2 - x_1] + \mathsf{f}(x_1) \\
&= \frac{\mathsf{f}(x_2)-\mathsf{f}(x_1)}{x_2-x_1}[(x_2-x_1)(1-\lambda)] + \mathsf{f}(x_1) \\
&= (1-\lambda)\mathsf{f}(x_2) - (1-\lambda)\mathsf{f}(x_1) + \mathsf{f}(x_1) \\
&= \lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)
\end{aligned}$$

2. prove f is convex $\Longleftarrow$ $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda \mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2)$:

   Let $x = \lambda(b-a)+a$ Notice that as λ varies from 0 to 1, x varies from b to a. So free variable λ works as a change of variable for free variable x.

$$\begin{aligned}
\lambda &= \frac{x-a}{b-a} \\
\mathsf{f}(x) &= \mathsf{f}(\lambda(b-a)+a) \\
&\le \lambda\mathsf{f}(b) + (1-\lambda)\mathsf{f}(a) \\
&= \lambda[\mathsf{f}(b)-\mathsf{f}(a)] + \mathsf{f}(a) \\
&= \frac{\mathsf{f}(b)-\mathsf{f}(a)}{b-a}(x-a) + \mathsf{f}(a)
\end{aligned}$$

⇨

Taking the second derivative of a function provides a convenient test for whether that function is convex.

Theorem K.3. [11]

THM $\mathsf{f}''(x) > 0 \implies \mathsf{f}$ *is convex*

✎PROOF:

$$\begin{aligned}
\mathsf{f}(x) &= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x-x_0) + \mathsf{f}''(c)(x-x_0)^2 \\
&\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x-x_0) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x-\lambda x_1-(1-\lambda)x_2) \\
\\
\mathsf{f}(x_1) &\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x_1-\lambda x_1-(1-\lambda)x_2) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)(1-\lambda)(x_1-x_2) \\
&= \mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2-x_1) \\
\\
\mathsf{f}(x_2) &\ge \mathsf{f}(x_0) + \mathsf{f}'(x_0)(x_2-\lambda x_1-(1-\lambda)x_2) \\
&= \mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1) \\
\\
\lambda\mathsf{f}(x_1) + (1-\lambda)\mathsf{f}(x_2) &\ge \lambda\left[\mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2-x_1)\right] + (1-\lambda)\left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] \\
&= \lambda\left[\mathsf{f}(x_0) - \mathsf{f}'(x_0)(1-\lambda)(x_2-x_1)\right] + \left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] - \lambda\left[\mathsf{f}(x_0) + \mathsf{f}'(x_0)\lambda(x_2-x_1)\right] \\
&= \mathsf{f}(x_0) \\
&= \mathsf{f}(\lambda x_1 + (1-\lambda)x_2)
\end{aligned}$$

By Theorem K.2 (page 263), $\mathsf{f}(x)$ is convex. ⇨

K.4 Literature

Literature survey:

11 Cover and Thomas (1991), pages 24–25

1. Abstract convexity:
   Edelman and Jamison (1985)
   van de Vel (1993)
   Hörmander (1994)
2. Order convexity (lattice theory):
   Edelman (1986)
3. Metric convexity:
   Menger (1928)
   Blumenthal (1970) page 41 ⟨?⟩
   Khamsi and Kirk (2001) pages 35–38

APPENDIX L

FINITE SUMS

❝ *I think that it was Harald Bohr who remarked to me that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."* ❞

G.H. Hardy (1877–1947) in his "Presidential Address" to the London Mathematical Society on November 8, 1928, about a remark that he suggested was from Harald Bohr (1887–1951), Danish mathematician pictured to the left. [1]

L.1 Summation

Definition L.1. [2] *Let $+$ be an addition operator on a tuple $(\!|x_n|\!)_m^N$.*

DEF

*The **summation** of $(\!|x_n|\!)$ from index m to index N with respect to $+$ is*

$$\sum_{n=m}^{N} x_n \triangleq \begin{cases} 0 & \text{for } N < m \\ \left(\sum_{n=m}^{N-1} x_n\right) + x_N & \text{for } N \geq m \end{cases}$$

Theorem L.1 (Generalized associative property). [3] *Let $+$ be an addition operator on a tuple $(\!|x_n|\!)_m^N$.*

THM

$+$ *is* ASSOCIATIVE $\implies$

$$\underbrace{\sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right) = \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n \qquad \text{for } m < L < M \leq N}_{\sum_{n=m}^{N} \text{ is ASSOCIATIVE}}$$

✎PROOF:

[1] quote: Hardy (1929), page 64
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Bohr_Harald.html

[2] reference: Berberian (1961) page 8 ⟨Definition I.3.1⟩
"$\sum$" notation: Fourier (1820) page 280

[3] Berberian (1961) pages 9–10 ⟨Theorem I.3.1⟩

1. Proof for $N < m$ case: $\sum_{n=m}^{N} x_n = 0$.

2. Proof for $N = m$ case: $\sum_{n=m}^{m} x_n = \left(\sum_{n=m}^{m-1} x_n\right) + x_m = 0 + x_m = x_m$.

3. Proof for $N = m+1$ case: $\sum_{n=m}^{m+1} x_n = \left(\sum_{n=m}^{m} x_n\right) + x_{m+1} = x_m + x_{m+1}$

4. Proof for $N = m+2$ case:

$$\begin{aligned}
\sum_{n=m}^{m+2} x_n &= \left(\sum_{n=m}^{m+1} x_n\right) + x_{m+2} && \text{by Definition L.1 page 267}\\
&= \left(x_m + x_{m+1}\right) + x_{m+2} && \text{by item (3)}\\
&= x_m + \left(x_{m+1} + x_{m+2}\right) && \text{by left hypothesis}
\end{aligned}$$

5. Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\sum_{n=m}^{N+1} x_n &= \underbrace{\left(\sum_{n=m}^{N} x_n\right)}_{\text{associative}} + x_{N+1} && \text{by Definition L.1 page 267}\\
&= \left(\sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right)\right) + x_{N+1}\\
&= \left(\left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n\right) + x_{N+1}\\
&= \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N} x_n + x_{N+1}\right)\\
&= \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N+1} x_n\right)
\end{aligned}$$

L.2 Means

L.2.1 Weighted ϕ-means

Definition L.2. [4]

DEF

The $(\lambda_n)_1^N$ *weighted* ϕ**-mean** *of a tuple* $(x_n)_1^N$ *is defined as*

$$\mathsf{M}_\phi((x_n)) \triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right)$$

where ϕ *is a* CONTINUOUS *and* STRICTLY MONOTONIC *function in* $\mathbb{R}^{\mathbb{R}}$

and $(\lambda_n)_{n=1}^N$ *is a sequence of weights for which* $\sum_{n=1}^{N} \lambda_n = 1$.

Lemma L.1. [5] *Let* $\mathsf{M}_\phi((x_n))$ *be the* $(\lambda_n)_1^N$ *weighted* ϕ*-mean of a tuple* $(x_n)_1^N$*. Let the property* CONVEX *be defined as in Definition K.7 (page 261).*

LEM

$\phi\psi^{-1}$ is CONVEX	and	ϕ is INCREASING	$\Longrightarrow$	$\mathsf{M}_\phi((x_n)) \geq \mathsf{M}_\psi((x_n))$
$\phi\psi^{-1}$ is CONVEX	and	ϕ is DECREASING	$\Longrightarrow$	$\mathsf{M}_\phi((x_n)) \leq \mathsf{M}_\psi((x_n))$
$\phi\psi^{-1}$ is CONCAVE	and	ϕ is INCREASING	$\Longrightarrow$	$\mathsf{M}_\phi((x_n)) \leq \mathsf{M}_\psi((x_n))$
$\phi\psi^{-1}$ is CONCAVE	and	ϕ is DECREASING	$\Longrightarrow$	$\mathsf{M}_\phi((x_n)) \geq \mathsf{M}_\psi((x_n))$

✎PROOF:

1. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *increasing*:

$$\begin{aligned}
\mathsf{M}_\phi((x_n)) &\triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi \text{ (Definition L.2 page 269)}\\
&= \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\geq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^{N} \lambda_n \psi(x_n)\right) && \text{by \textit{Jensen's Inequality} (Theorem K.1 page 262)}\\
&= \left(\psi^{-1}\sum_{n=1}^{N} \lambda_n \psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\triangleq \mathsf{M}_\psi((x_n)) && \text{by definition of } \mathsf{M}_\psi \text{ (Definition L.2 page 269)}
\end{aligned}$$

2. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *decreasing*:

$$\begin{aligned}
\mathsf{M}_\phi((x_n)) &\triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi \text{ (Definition L.2 page 269)}\\
&= \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\leq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^{N} \lambda_n \psi(x_n)\right) && \text{by \textit{Jensen's Inequality} and because } \phi^{-1} \text{ is \textit{decreasing}}
\end{aligned}$$

[4] Bollobás (1999) page 5
[5] Pečarić et al. (1992) page 107, Bollobás (1999) page 5, Hardy et al. (1952) page 75

$$= \left(\psi^{-1}\sum_{n=1}^{N}\lambda_n\psi(x_n)\right) \qquad \text{by definition of } \psi^{-1}$$

$$\triangleq \mathsf{M}_{\psi}(\langle x_n \rangle) \qquad \text{by definition of } \mathsf{M}_{\psi} \text{ (Definition L.2 page 269)}$$

✏

One of the most well known inequalities in mathematics is *Minkowski's Inequality* (1910, Theorem L.5 page 276). In 1946, H.P. Mulholland submitted a result[6] that generalizes Minkowski's Inequality to an equal weighted ϕ-mean. And Milovanović and Milovanovć (1979) generalized this even further to a *weighted* ϕ-mean (Theorem L.2, next).

Theorem L.2. [7]

THM

$$\left\{\begin{array}{lll} 1. & \phi \textit{ is } \text{CONVEX} & \textit{and} \\ 2. & \phi \textit{ is } \text{STRICTLY MONOTONIC} & \textit{and} \\ 3. & \phi(0) = 0 & \textit{and} \\ 4. & \log \circ \phi \circ \exp \textit{ is } \text{CONVEX} & \end{array}\right\} \implies$$

$$\phi^{-1}\left(\sum_{n=1}^{N}\lambda_n\phi(x_n+y_n)\right) \le \phi^{-1}\left(\sum_{n=1}^{N}\lambda_n\phi(x_n)\right) + \phi^{-1}\left(\sum_{n=1}^{N}\lambda_n\phi(y_n)\right)$$

L.2.2 Power means

Definition L.3. [8] *Let* $\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle)$ *be the* $\langle \lambda_n \rangle_1^N$ *weighted* ϕ*-mean of a* NON-NEGATIVE *tuple* $\langle x_n \rangle_1^N$ *(Definition L.2 page 269).*

DEF

A mean $\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle)$ *is a* ***power mean*** *with parameter* r *if* $\phi(x) \triangleq x^r$. *That is,*

$$\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle) = \left(\sum_{n=1}^{N}\lambda_n(x_n)^r\right)^{\frac{1}{r}}$$

Theorem L.3. [9] *Let* $\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle)$ *be* POWER MEAN *with parameter* r *of an* N*-tuple* $\langle x_n \rangle_1^N$. *Let* $\mathbb{R}^*$ *be the set of extended real numbers* $(\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\})$.[10]

THM

$$\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle) \triangleq \left(\sum_{n=1}^{N}\lambda_n(x_n)^r\right)^{\frac{1}{r}} \quad \textit{is } \text{CONTINUOUS} \textit{ and } \text{STRICTLY INCREASING} \textit{ in } \mathbb{R}^*.$$

$$\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle) = \begin{cases} \min\limits_{n=1,2,\ldots,N}\langle x_n \rangle & \textit{for } r = -\infty \\ \prod\limits_{n=1}^{N} x_n^{\lambda_n} & \textit{for } r = 0 \\ \max\limits_{n=1,2,\ldots,N}\langle x_n \rangle & \textit{for } r = +\infty \end{cases}$$

✎PROOF:

[6] Mulholland (1950)
[7] Milovanović and Milovanović (1979), Bullen (2003) page 306 ⟨Theorem 9⟩
[8] Bullen (2003) page 175, Bollobás (1999) page 6
[9] Bullen (2003) pages 175–177 ⟨see also page 203⟩, Bollobás (1999) pages 6–8, Besso (1879), Bienaymé (1840) page 68
[10] Rana (2002) pages 385–388 ⟨Appendix A⟩

1. Proof that $M_{\phi(x;r)}$ is *strictly increasing* in r:

   (a) Let r and s be such that $-\infty < r < s < \infty$.

   (b) Let $\phi_r \triangleq x^r$ and $\phi_s \triangleq x^s$. Then $\phi_r\phi_s^{-1} = x^{\frac{r}{s}}$.

   (c) The composite function $\phi_r\phi_s^{-1}$ is *convex* or *concave* depending on the values of r and s:

	$r<0$ (ϕ_r decreasing)	$r>0$ (ϕ_r increasing)
$s<0$	*convex*	(not possible)
$s>0$	*convex*	*concave*

   (d) Therefore by Lemma L.1 (page 269),
   $$-\infty < r < s < \infty \quad\Longrightarrow\quad \mathsf{M}_{\phi(x;r)}\left(\langle x_n\rangle\right) < \mathsf{M}_{\phi(x;s)}\left(\langle x_n\rangle\right).$$

2. Proof that $\mathsf{M}_{\phi(x;r)}$ is continuous in r for $r\in\mathbb{R}\backslash 0$: The sum of continuous functions is continuous. For the cases of $r\in\{-\infty,\ 0,\ \infty\}$, see the items that follow.

3. Lemma: $\mathsf{M}_{\phi(x;-r)}\left(\langle x_n\rangle\right) = \left\{\mathsf{M}_{\phi(x;r)}\left(\langle x_n^{-1}\rangle\right)\right\}^{-1}$. Proof:

$$\begin{aligned}
\left\{\mathsf{M}_{\phi(x;r)}\left(\langle x_n^{-1}\rangle\right)\right\}^{-1} &= \left\{\left(\sum_{n=1}^{N}\lambda_n\left(x_n^{-1}\right)^r\right)^{\frac{1}{r}}\right\}^{-1} && \text{by definition of } \mathsf{M}_\phi\\
&= \left(\sum_{n=1}^{N}\lambda_n\left(x_n\right)^{-r}\right)^{\frac{1}{-r}}\\
&= \mathsf{M}_{\phi(x;-r)}\left(\langle x_n\rangle\right) && \text{by definition of } \mathsf{M}_\phi
\end{aligned}$$

4. Proof that $\lim\limits_{r\to\infty}\mathsf{M}_\phi\left(\langle x_n\rangle\right) = \max\limits_{n\in\mathbb{Z}}\langle x_n\rangle$:

   (a) Let $x_m \triangleq \max\limits_{n\in\mathbb{Z}}\langle x_n\rangle$

   (b) Note that $\lim\limits_{r\to\infty}\mathsf{M}_\phi \le \max\limits_{n\in\mathbb{Z}}\langle x_n\rangle$ because

$$\begin{aligned}
\lim_{r\to\infty}\mathsf{M}_\phi\left(\langle x_n\rangle\right) &= \lim_{r\to\infty}\left(\sum_{n=1}^{N}\lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi\\
&\le \lim_{r\to\infty}\left(\sum_{n=1}^{N}\lambda_n x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x)\triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing}\\
&= \lim_{r\to\infty}\left(x_m^r\underbrace{\sum_{n=1}^{N}\lambda_n}_{1}\right)^{\frac{1}{r}} && \text{because } x_m \text{ is a constant}\\
&= \lim_{r\to\infty}\left(x_m^r\cdot 1\right)^{\frac{1}{r}}\\
&= x_m\\
&= \max_{n\in\mathbb{Z}}\langle x_n\rangle && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}$$

(c) But also note that $\lim_{r\to\infty} \mathsf{M}_\phi \geq \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$ because

$$\begin{aligned}
\lim_{r\to\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty}\left(\sum_{n=1}^{N} \lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi\\
&\geq \lim_{r\to\infty}\left(w_m x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x) \triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing}\\
&= \lim_{r\to\infty} w_m^{\frac{1}{r}} x_m^{\frac{r}{r}}\\
&= x_m\\
&= \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}$$

(d) Combining items (b) and (c) we have $\lim_{r\to\infty} \mathsf{M}_\phi = \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$.

5. Proof that $\lim_{r\to-\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) = \min_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$:

$$\begin{aligned}
\lim_{r\to-\infty} \mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty} \mathsf{M}_{\phi(x;-r)}\left(\llparenthesis x_n \rrparenthesis\right) && \text{by change of variable } r\\
&= \lim_{r\to\infty}\left\{\mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)\right\}^{-1} && \text{by Lemma in item (3) page 271}\\
&= \lim_{r\to\infty} \frac{1}{\mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)}\\
&= \frac{\lim_{r\to\infty} 1}{\lim_{r\to\infty} \mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)} && \text{by property of lim [11]}\\
&= \frac{1}{\max_{n\in\mathbb{Z}} \llparenthesis x_n^{-1} \rrparenthesis} && \text{by item (4)}\\
&= \frac{1}{\left(\min_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis\right)^{-1}}\\
&= \min_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis
\end{aligned}$$

6. Proof that $\lim_{r\to 0} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) = \prod_{n=1}^{N} x_n^{\lambda_n}$:

$$\begin{aligned}
\lim_{r\to 0} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to 0} \exp\left\{\ln\left\{\mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right)\right\}\right\}\\
&= \lim_{r\to 0} \exp\left\{\ln\left\{\left(\sum_{n=1}^{N} \lambda_n\left(x_n^r\right)\right)^{\frac{1}{r}}\right\}\right\} && \text{by definition of } \mathsf{M}_\phi\\
&= \exp\left\{\frac{\frac{\partial}{\partial r}\ln\left(\sum_{n=1}^{N} \lambda_n\left(x_n^r\right)\right)}{\frac{\partial}{\partial r} r}\right\}_{r=0} && \text{by l'Hôpital's rule[12]}
\end{aligned}$$

[11] Rudin (1976) page 85 ⟨4.4 Theorem⟩

$$= \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial r}\left(x_n^r\right)}{\sum_{n=1}^{N}\lambda_n\left(x_n^r\right)}\right\}_{r=0}$$

$$= \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial r}\exp\left(\ln\left(x_n^r\right)\right)}{\sum_{n=1}^{N}\lambda_n}\right\}_{r=0}$$

$$= \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial r}\exp\left(r\ln\left(x_n\right)\right)}{1}\right\}_{r=0}$$

$$= \exp\left\{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial r}\exp\left(r\ln\left(x_n\right)\right)\right\}_{r=0}$$

$$= \exp\left\{\sum_{n=1}^{N}\lambda_n\exp\left\{r\ln x_n\right\}\ln\left(x_n\right)\right\}_{r=0}$$

$$= \exp\left\{\sum_{n=1}^{N}\lambda_n\ln\left(x_n\right)\right\}$$

$$= \exp\left\{\sum_{n=1}^{N}\ln\left(x_n^{\lambda_n}\right)\right\}$$

$$= \exp\left\{\ln\prod_{n=1}^{N}x_n^{\lambda_n}\right\}$$

$$= \prod_{n=1}^{N}x_n^{\lambda_n}$$

⇨

Definition L.4. *Let* $\left(x_n\right)_1^N$ *be a tuple. Let* $\left(\lambda_n\right)_1^N$ *be a tuple of weighting values such that* $\sum_{n=1}^{N}\lambda_n = 1$.

[12] Rudin (1976) page 109 ⟨5.13 Theorem⟩

DEF

*The **harmonic mean** of (x_n) is defined as* $\mu_h \triangleq \left(\sum_{n=1}^{N} \lambda_n \frac{1}{x_n}\right)^{-1}$

*The **geometric mean** of (x_n) is defined as* $\mu_g \triangleq \sum_{n=1}^{N} x_n^{\lambda_n}$

*The **arithmetic mean** of (x_n) is defined as* $\mu_a \triangleq \sum_{n=1}^{N} \lambda_n x_n$

*The **average** of (x_n) is defined as* $\mu_a \triangleq \frac{1}{N}\sum_{n=1}^{N} x_n$

Corollary L.1. [13] *Let $(x_n)_1^N$ be a tuple. Let $(\lambda_n)_1^N$ be a tuple of weighting values such that $\sum_{n=1}^{N} \lambda_n = 1$.*

COR

$$\min (x_n) \le \underbrace{\left(\sum_{n=1}^{N} \lambda_n \frac{1}{x_n}\right)^{-1}}_{\text{harmonic mean}} \le \underbrace{\prod_{n=1}^{N} x_n^{\lambda_n}}_{\text{geometric mean}} \le \underbrace{\sum_{n=1}^{N} \lambda_n x_n}_{\text{arithmetic mean}} \le \max (x_n)$$

✎PROOF:

1. These five means are all special cases of the *power mean* $\mathsf{M}_{\phi(x:r)}$ (Definition L.3 page 270):
   - $r = \infty$: $\max (x_n)$
   - $r = 1$: arithmetic mean
   - $r = 0$: geometric mean
   - $r = -1$: harmonic mean
   - $r = -\infty$: $\min (x_n)$
2. The inequalities follow directly from Theorem L.3 (page 270).
3. Generalized AM-GM inequality: If one is only concerned with the arithmetic mean and geometric mean, their relationship can be established directly using *Jensen's Inequality*:

$$\begin{aligned}
\sum_{n=1}^{N} \lambda_n x_n &= b^{\log_b\left(\sum_{n=1}^{N} \lambda_n x_n\right)} \\
&\ge b^{\left(\sum_{n=1}^{N} \lambda_n \log_b x_n\right)} && \text{by } \textit{Jensen's Inequality} \text{ (Theorem K.1 page 262)} \\
&= \prod_{n=1}^{N} b^{(\lambda_n \log_b x_n)} \\
&= \prod_{n=1}^{N} b^{(\log_b x_n)\lambda_n} \\
&= \prod_{n=1}^{N} x_n^{\lambda_n}
\end{aligned}$$

⇨

[13] Bullen (2003) page 71, Bollobás (1999) page 5, Cauchy (1821) pages 457–459 ⟨Note II, theorem 17⟩, Jensen (1906) page 183

Lemma L.2 (Young's Inequality). [14]

L E M

$$xy < \frac{x^p}{p} + \frac{y^q}{q} \quad \textit{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty, \; x, y \geq 0, \quad \textit{but } y \neq x^{p-1}$$

$$xy = \frac{x^p}{p} + \frac{y^q}{q} \quad \textit{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty, \; x, y \geq 0, \quad \textit{and } y = x^{p-1}$$

✎PROOF:

1. Proof that $\frac{1}{p-1} = q - 1$:

$$\begin{aligned}
\frac{1}{p} + \frac{1}{q} = 1 &\iff \frac{q}{q} + \frac{q}{p} = q \\
&\iff q\left(1 - \frac{1}{p}\right) = 1 \\
&\iff q = \frac{1}{1 - \frac{1}{p}} \\
&\iff q = \frac{p}{p-1} \\
&\iff q - 1 = \frac{p}{p-1} - \frac{p-1}{p-1} \\
&\iff q - 1 = \frac{p - (p-1)}{p-1} \\
&\iff q - 1 = \frac{1}{p-1}
\end{aligned}$$

2. Proof that $v = u^{p-1} \iff u = v^{q-1}$:

$$\begin{aligned}
u &= v^{\frac{1}{p-1}} && \text{by left hypothesis} \\
&= v^{q-1} && \text{by item (1)}
\end{aligned}$$

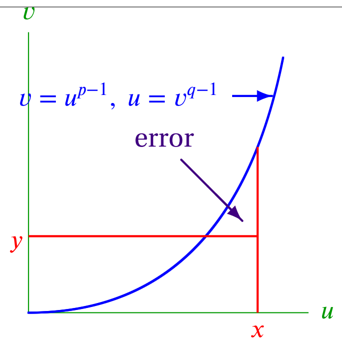
3. Proof that $v = u^{p-1}$ is monotonically increasing in u and $u = v^{q-1}$ is monotonically increasing in v:

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u} u^{p-1} \\
&= (p-1)u^{p-2} \\
&> 0 \\
\frac{\mathrm{d}u}{\mathrm{d}v} &= \frac{\mathrm{d}}{\mathrm{d}v} v^{q-1} \\
&= (q-1)v^{q-2} \\
&> 0
\end{aligned}$$

4. Proof that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$:

[14] Carothers (2000), page 43, Tolsted (1964), page 5, Maligranda (1995), page 257, Hardy et al. (1952) ⟨Theorem 24⟩, Young (1912) page 226

$$\begin{aligned}
xy &\le \int_0^x u^{p-1}\,\mathrm{d}u + \int_0^y v^{q-1}\,\mathrm{d}v \\
&= \left.\frac{u^p}{p}\right|_0^x + \left.\frac{v^q}{q}\right|_0^y \\
&= \frac{x^p}{p} + \frac{y^q}{q}
\end{aligned}$$



Theorem L.4 (Hölder's Inequality).[15] *Let* $(x_n \in \mathbb{C})_1^N$ *and* $(y_n \in \mathbb{C})_1^N$ *be complex* N*-tuples.*

THM
$$\underbrace{\sum_{n=1}^N |x_n y_n|}_{\|x\cdot y\|_1} \le \underbrace{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}}_{\|x\|_p}\underbrace{\left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}}_{\|y\|_q} \quad \text{with} \quad \frac{1}{p}+\frac{1}{q}=1 \quad \forall\, 1<p<\infty$$

✎PROOF: Let $\|x_n\|_p \triangleq \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}$.

$$\begin{aligned}
\sum_{n=1}^N |x_n y_n| &= \|(x_n)\|_p \|(y_n)\|_q \sum_{n=1}^N \frac{|x_n y_n|}{\|(x_n)\|_p \|(y_n)\|_q} \\
&= \|(x_n)\|_p \|(y_n)\|_q \sum_{n=1}^N \frac{|x_n|}{\|(x_n)\|_p} \frac{|y_n|}{\|(y_n)\|_q} \\
&\le \|(x_n)\|_p \|(y_n)\|_q \sum_{n=1}^N \left(\frac{1}{p}\frac{|x_n|^p}{\|(x_n)\|_p^p} + \frac{1}{q}\frac{|y_n|^q}{\|(y_n)\|_q^q}\right) && \text{by \textit{Young's Inequality} (Lemma L.2 page 275)} \\
&= \|(x_n)\|_p \|(y_n)\|_q \left(\frac{1}{p}\cdot\frac{\sum|x_n|^p}{\|(x_n)\|_p^p} + \frac{1}{q}\cdot\frac{\sum|y_n|^q}{\|(y_n)\|_q^q}\right) \\
&= \|(x_n)\|_p \|(y_n)\|_q \left(\frac{1}{p}\frac{\|(x_n)\|_p^p}{\|(x_n)\|_p^p} + \frac{1}{q}\frac{\|(y_n)\|_q^q}{\|(y_n)\|_q^q}\right) && \text{by definition of } \|\cdot\| \\
&= \|(x_n)\|_p \|(y_n)\|_q \underbrace{\left(\frac{1}{p}+\frac{1}{q}\right)}_{1} \\
&= \|(x_n)\|_p \|(y_n)\|_q && \text{by } \frac{1}{p}+\frac{1}{q}=1 \text{ constraint}
\end{aligned}$$

⇒

Theorem L.5 (Minkowski's Inequality for sequences).[16] *Let* $(x_n \in \mathbb{C})_1^N$ *and* $(y_n \in \mathbb{C})_1^N$ *be complex* N*-tuples.*

[15] Bullen (2003) page 178 ⟨2.1⟩, Carothers (2000), page 44, Tolsted (1964), page 6, Maligranda (1995), page 257, Hardy et al. (1952) ⟨Theorem 11⟩, Hölder (1889)

[16] Bullen (2003) page 179, Carothers (2000), page 44, Tolsted (1964), page 7, Maligranda (1995), page 258, Hardy et al. (1952) ⟨Theorem 24⟩, Minkowski (1910) page 115

THM

$$\left(\sum_{n=1}^{N}\left|x_n+y_n\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{N}\left|x_n\right|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N}\left|y_n\right|^p\right)^{\frac{1}{p}} \qquad \forall\ 1<p<\infty$$

✎PROOF:

1. Define q in terms of p such that $\frac{1}{p}+\frac{1}{q}=1$
2. Proof that $\frac{1}{q}=\frac{p-1}{p}$:

$$\begin{aligned}\frac{1}{p}+\frac{1}{q}=1 &\iff \frac{1}{q}=1-\frac{1}{p}\\ &\iff \frac{1}{q}=\frac{p-1}{p}\end{aligned}$$

3. Define $\|\cdot\|$ as follows:

$$\|\boldsymbol{x}\|_p \triangleq \left(\sum_{n=1}^{N}\left|x_n\right|^p\right)^{\frac{1}{p}}$$

4. Proof that $\left\|x_n+y_n\right\|_p \le \left\|x_n\right\|_p + \left\|y_n\right\|_p$:

$$\begin{aligned}
\left\|x_n+y_n\right\|_p^p &= \sum_{n=1}^{N}\left|x_n+y_n\right|^p && \text{by definition of } \|\cdot\|_p\\
&= \sum_{n=1}^{N}\left|x_n+y_n\right|\left|x_n+y_n\right|^{p-1} && \text{by homogeneous property of } |\cdot|\\
&\le \sum_{n=1}^{N}\left|x_n\right|\left|x_n+y_n\right|^{p-1} + \sum_{n=1}^{N}\left|y_n\right|\left|x_n+y_n\right|^{p-1} && \text{by subadditive property of } |\cdot|\\
&= \sum_{n=1}^{N}\left|x_n(x_n+y_n)^{p-1}\right| + \sum_{n=1}^{N}\left|y_n(x_n+y_n)^{p-1}\right| && \text{by homogeneous property of } |\cdot|\\
&\le \left\|x_n\right\|_p \left\|(x_n+y_n)^{p-1}\right\|_q + \left\|y_n\right\|_p \left\|(x_n+y_n)^{p-1}\right\|_q && \text{by Hölder's Inequality page 276}\\
&= \left(\left\|x_n\right\|_p+\left\|y_n\right\|_p\right)\left\|(x_n+y_n)^{p-1}\right\|_q\\
&= \left(\left\|x_n\right\|_p+\left\|y_n\right\|_p\right)\left(\sum_{n=1}^{N}\left|(x_n+y_n)^{p-1}\right|^q\right)^{\frac{1}{q}} && \text{by definition of } \|\cdot\|_p\\
&= \left(\left\|x_n\right\|_p+\left\|y_n\right\|_p\right)\left(\sum_{n=1}^{N}\left|(x_n+y_n)^{p-1}\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} && \text{by item (2)}\\
&= \left(\left\|x_n\right\|_p+\left\|y_n\right\|_p\right)\left(\sum_{n=1}^{N}\left|(x_n+y_n)\right|^p\right)^{\frac{p-1}{p}}\\
&= \left(\left\|x_n\right\|_p+\left\|y_n\right\|_p\right)\left\|x_n+y_n\right\|^{p-1}\\
&\implies \left\|x_n+y_n\right\|_p \le \left\|x_n\right\|_p+\left\|y_n\right\|_p
\end{aligned}$$

⇒

❝*Cauchy is the only one occupied with pure mathematics: Poisson, Fourier, Ampere, etc., busy themselves exclusively with magnetism and other physical subjects.*❞
in an 1826 letter written by Niels Henrik Abel [17]

Theorem L.6 (Cauchy-Schwarz Inequality). [18] *Let* $(\!(x_n \in \mathbb{C})\!)_1^N$ *and* $(\!(y_n \in \mathbb{C})\!)_1^N$ *be complex* N*-tuples.*

THM

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 \le \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \qquad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$$

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 = \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \iff \exists a\in\mathbb{C} \text{ such that } \boldsymbol{y} = a\boldsymbol{x} \qquad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$$

✎PROOF:

1. The Cauchy-Schwarz inequality for sequences is a special case of the Hölder inequality (Theorem L.4) for $p = q = 2$.
2. Alternatively, the Cauchy-Schwarz inequality for sequences is a special case of the *Cauchy-Schwarz inequality* in inner-product spaces:
   (a) $\langle x_n \mid y_n \rangle \triangleq \sum_{n=1}^{N} x_n y_n$ is an inner-product and $\left((\!(x_n)\!), \langle \triangle \mid \triangledown \rangle \right)$ is an inner-product space.
   (b) By the *Cauchy-Schwarz Inequality for inner-product spaces* (Theorem I.2 page 234),
   $$|\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle|^2 \le \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle \ \langle \boldsymbol{y} \mid \boldsymbol{y} \rangle$$
3. Not only does the Hölder inequality imply the Cauchy-Schwarz inequality, but somewhat surprisingly, the converse is also true: The Cauchy-Schwarz inequality implies the Hölder inequality.[19]

⇨

Proposition L.1. [20]

PRP

$$(x+y)^p \le 2^p(x^p + y^p) \qquad \forall x, y \ge 0,\ 1 < p < \infty$$

✎PROOF:

$$\begin{aligned}(x+y)^p &\le (2\max\{x,y\})^p \\ &= 2^p(\max\{x,y\})^p \\ &= 2^p(\max\{x^p,y^p\}) \\ &\le 2^p(x^p+y^p)\end{aligned}$$

⇨

[17] quote: Boyer and Merzbach (2011) page 462
image: http://en.wikipedia.org/wiki/File:Augustin-Louis_Cauchy_1901.jpg, public domain
[18] Aliprantis and Burkinshaw (1998), page 278, Scharz (1885), Bouniakowsky (1859), Hardy et al. (1952) page 25 ⟨Theorem 11⟩, Cauchy (1821) page 455 ⟨???⟩
[19] Bullen (2003) pages 183–185 ⟨Theorem 5⟩
[20] Carothers (2000), page 43

L.3 Power Sums

Theorem L.7 (Geometric Series). [21]

THM
$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \qquad \forall r \in \mathbb{C}\backslash 0$$

✎PROOF:

$$\begin{aligned}
\sum_{k=0}^{n-1} r^k &= \left(\frac{1}{1-r}\right)\left[(1-r)\sum_{k=0}^{n-1} r^k\right] \\
&= \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - r\sum_{k=0}^{n-1} r^k\right] \\
&= \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - \left(\sum_{k=0}^{n-1} r^k - 1 + r^n\right)\right] \\
&= \left(\frac{1}{1-r}\right)\left[1-r^n\right] \\
&= \frac{1-r^n}{1-r}
\end{aligned}$$

⇨

Lemma L.3. *Let* $\mathsf{f}(t)$ *be a function.*

LEM
$$\mathsf{S}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) = \mathsf{S}(x+\tau) \qquad (\mathsf{S}(x) \text{ is PERIODIC with period } \tau)$$

✎PROOF:

$$\begin{aligned}
\mathsf{S}(x+\tau) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+\tau+n\tau) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x+(n+1)\tau) \\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x+m\tau) && \text{where } m \triangleq n+1 \\
&\triangleq \mathsf{S}(x)
\end{aligned}$$

⇨

Proposition L.2 (Power Sums). [22]

PRP
$$\begin{aligned}
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} && \forall n\in\mathbb{N} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} && \forall n\in\mathbb{N} \\
\sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4} && \forall n\in\mathbb{N} \\
\sum_{k=1}^{n} k^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} && \forall n\in\mathbb{N}
\end{aligned}$$

[21] Hall and Knight (1894), page 39 ⟨article 55⟩
[22] Amann and Escher (2008) pages 51–57, Menini and Oystaeyen (2004) page 91 ⟨Exercises 5.36–5.39⟩

✎PROOF:

1. Proof that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$: (proof by induction)

$$\begin{aligned}
\sum_{k=1}^{n=1} k &= 1 \\
&= \frac{1(1+1)}{2} \\
&= \left.\frac{n(n+1)}{2}\right|_{n=1}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \left(\sum_{k=1}^{n} k\right) + \left(n+1\right) \\
&= \left(\frac{n(n+1)}{2}\right) + \left(n+1\right) && \text{by left hypothesis} \\
&= \left(n+1\right)\left(\frac{n}{2}+1\right) \\
&= \left(n+1\right)\left(\frac{n+2}{2}\right) \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

2. Proof that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$: (proof by induction)

$$\begin{aligned}
\sum_{k=1}^{n=1} k^2 &= 1 \\
&= \frac{1(1+1)(2+1)}{6} \\
&= \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \left(\sum_{k=1}^{n} k^2\right) + \left(n+1\right)^2 \\
&= \left(\frac{n(n+1)(2n+1)}{6}\right) + \left(n+1\right)^2 && \text{by left hypothesis} \\
&= \left(n+1\right)\left(\frac{n(2n+1)+6(n+1)}{6}\right) \\
&= \left(n+1\right)\left(\frac{2n^2+7n+6}{6}\right) \\
&= \left(n+1\right)\left(\frac{(n+2)(2n+3)}{6}\right) \\
&= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}
\end{aligned}$$

⇨

APPENDIX M

OPERATORS ON LINEAR SPACES

“And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.”

Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

M.1 Operators on linear spaces

M.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition M.1. [2]

DEF *A function $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ is an **operator** in $\boldsymbol{Y}^{\boldsymbol{X}}$ if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are both* LINEAR SPACES *(Definition F.1 page 185).*

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}\boldsymbol{x} = \mathbf{B}\boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{X}\boldsymbol{Y}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}\boldsymbol{x} = \mathbf{I}$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

Definition M.2. [3] *Let $\boldsymbol{X}^{\boldsymbol{X}}$ be the set of all operators with from a* LINEAR SPACE *$\boldsymbol{X}$ to $\boldsymbol{X}$. Let $\mathbf{I}$ be an operator in $\boldsymbol{X}^{\boldsymbol{X}}$. Let $①(\boldsymbol{X})$ be the* IDENTITY ELEMENT *in $\boldsymbol{X}^{\boldsymbol{X}}$.*

---

[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Heil (2011) page 42

[3] Michel and Herget (1993) page 411

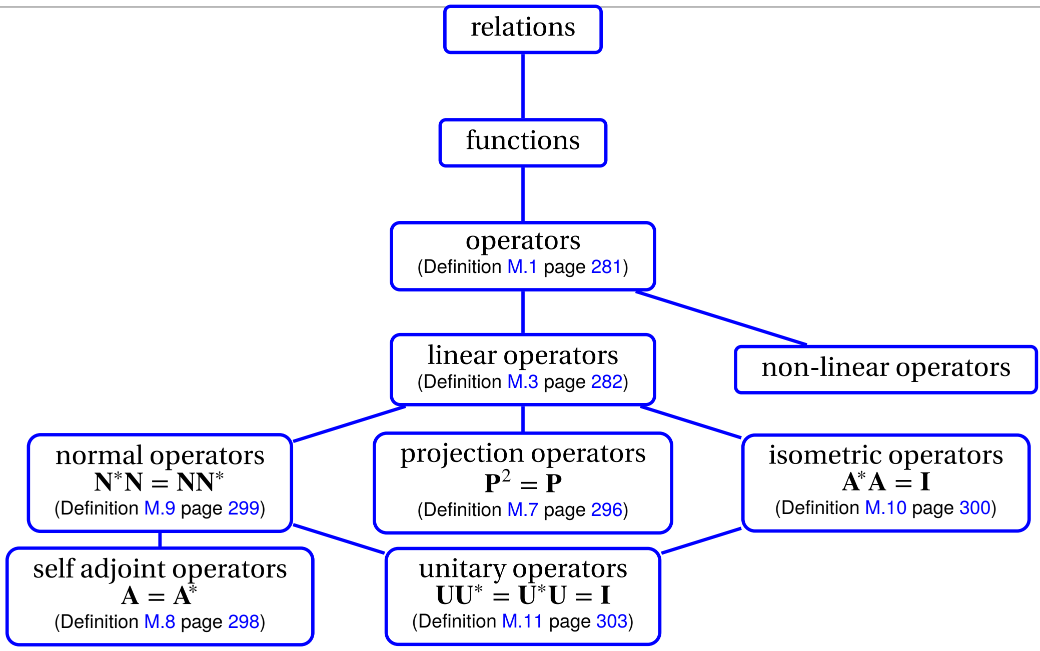


Figure M.1: Some operator types

DEF $\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\qquad \mathbf{I} = \textcircled{1}(\boldsymbol{X}).$

M.1.2 Linear operators

Definition M.3. [4] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be linear spaces.*

DEF *An operator* $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *is* ***linear*** *if*

1. $\mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$ (ADDITIVE) *and*
2. $\mathbf{L}(\alpha\boldsymbol{x}) = \alpha\mathbf{L}\boldsymbol{x} \quad \forall \boldsymbol{x} \in \boldsymbol{X}, \quad \forall \alpha \in \mathbb{F}$ (HOMOGENEOUS).

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *such that*

$$\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \{\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}} | \mathbf{L} \text{ is linear}\}.$$

Theorem M.1. [5] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$*.*

THM

$$\mathbf{L} \text{ is LINEAR} \implies \begin{cases} 1.\ \mathbf{L}\mathbb{0} = \mathbb{0} & \text{and} \\ 2.\ \mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \boldsymbol{X} & \text{and} \\ 3.\ \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X} & \text{and} \\ 4.\ \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n (\mathbf{L}\boldsymbol{x}_n) \quad \boldsymbol{x}_n \in \boldsymbol{X}, \alpha_n \in \mathbb{F} & \end{cases}$$

✎PROOF:

[4] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[5] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned}
\mathbf{L}\mathbb{0} &= \mathbf{L}(0 \cdot \mathbb{0}) && \text{by additive identity property (Theorem F.1 page 187)}\\
&= 0 \cdot (\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \text{ (Definition M.3 page 282)}\\
&= \mathbb{0} && \text{by additive identity property (Theorem F.1 page 187)}
\end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned}
\mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1 \cdot \boldsymbol{x}) && \text{by additive inverse property (Theorem F.2 page 188)}\\
&= -1 \cdot (\mathbf{L}\boldsymbol{x}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \text{ (Definition M.3 page 282)}\\
&= -(\mathbf{L}\boldsymbol{x}) && \text{by additive inverse property (Theorem F.2 page 188)}
\end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) = \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y}$:

$$\begin{aligned}
\mathbf{L}(\boldsymbol{x} - \boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x} + (-\boldsymbol{y})) && \text{by additive inverse property (Theorem F.2 page 188)}\\
&= \mathbf{L}(\boldsymbol{x}) + \mathbf{L}(-\boldsymbol{y}) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \text{ (Definition M.3 page 282)}\\
&= \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} && \text{by 2.}
\end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n \left(\mathbf{L}\boldsymbol{x}_n\right)$:

   (a) Proof for $N = 1$:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_1 \boldsymbol{x}_1\right) && \text{by } N = 1 \text{ hypothesis}\\
&= \alpha_1 \left(\mathbf{L}\boldsymbol{x}_1\right) && \text{by } \textit{homogeneous} \text{ property of Definition M.3 page 282}
\end{aligned}$$

   (b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1} \alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1} + \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right)\\
&= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1}\right) + \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of Definition M.3 page 282}\\
&= \alpha_{N+1}\mathbf{L}\left(\boldsymbol{x}_{N+1}\right) + \sum_{n=1}^{N} \mathbf{L}\left(\alpha_n \boldsymbol{x}_n\right) && \text{by left } N+1 \text{ hypothesis}\\
&= \sum_{n=1}^{N+1} \mathbf{L}\left(\alpha_n \boldsymbol{x}_n\right)
\end{aligned}$$

⇨

Theorem M.2. [6] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
$\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathcal{N}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{X}$*	$\forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathcal{I}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{Y}$*	$\forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$	

✎PROOF:

[6] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩

1. Proof that $\mathcal{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0} \in \mathcal{N}(\mathbf{L}) \implies \mathcal{N}(\mathbf{L}) \neq \varnothing$
   (b) $\mathcal{N}(\mathbf{L}) \triangleq \{\boldsymbol{x} \in \boldsymbol{X} | \mathbf{L}\boldsymbol{x} = \mathbb{0}\} \subseteq \boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{N}(\mathbf{L}) \implies \mathbb{0} = \mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) = \mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x} \in \mathcal{N}(\mathbf{L})$
   (d) $\alpha \in \mathbb{F},\ \boldsymbol{x} \in \boldsymbol{X} \implies \mathbb{0} = \mathbf{L}\boldsymbol{x} \implies \mathbb{0} = \alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0} = \mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x} \in \mathcal{N}(\mathbf{L})$
2. Proof that $\mathcal{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0} \in \mathcal{I}(\mathbf{L}) \implies \mathcal{I}(\mathbf{L}) \neq \varnothing$
   (b) $\mathcal{I}(\mathbf{L}) \triangleq \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \text{ such that } \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\} \subseteq \boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{I}(\mathbf{L}) \implies \exists \boldsymbol{v} \in \boldsymbol{X} \text{ such that } \mathbf{L}\boldsymbol{v} = \boldsymbol{x}+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x} \in \mathcal{I}(\mathbf{L})$
   (d) $\alpha \in \mathbb{F},\ \boldsymbol{x} \in \mathcal{I}(\mathbf{L}) \implies \exists \boldsymbol{x} \in \boldsymbol{X} \text{ such that } \boldsymbol{y} = \mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y} = \alpha\mathbf{L}\boldsymbol{x} = \mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x} \in \mathcal{I}(\mathbf{L})$

✎

Example M.1. [7] Let $C([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX $C([a:b],\mathbb{R})$ is a linear space.

Theorem M.3. [8] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of a linear operator $\mathbf{L} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM

$\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y} \in \mathcal{N}(\mathbf{L})$
$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}$

✎PROOF:

1. Proof that $\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \implies \boldsymbol{x}-\boldsymbol{y} \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} && \text{by Theorem M.1 page 282}\\
&= \mathbb{0} && \text{by left hypothesis}\\
&\implies \boldsymbol{x}-\boldsymbol{y} \in \mathcal{N}(\mathbf{L}) && \text{by definition of } \textit{null space}
\end{aligned}$$

2. Proof that $\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \impliedby \boldsymbol{x}-\boldsymbol{y} \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}\boldsymbol{y} &= \mathbf{L}\boldsymbol{y} + \mathbb{0} && \text{by definition of linear space (Definition F.1 page 185)}\\
&= \mathbf{L}\boldsymbol{y} + \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) && \text{by right hypothesis}\\
&= \mathbf{L}\boldsymbol{y} + (\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y}) && \text{by Theorem M.1 page 282}\\
&= (\mathbf{L}\boldsymbol{y} - \mathbf{L}\boldsymbol{y}) + \mathbf{L}\boldsymbol{x} && \text{by } \textit{associative} \text{ and } \textit{commutative} \text{ properties (Definition F.1 page 185)}\\
&= \mathbf{L}\boldsymbol{x}
\end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned}
\mathbf{L} \text{ is } \textit{injective} &\iff \{(\mathbf{L}\boldsymbol{x} = \mathbf{L}\boldsymbol{y} \iff \boldsymbol{x} = \boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y} \in X\}\\
&\iff \left\{\left[\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{y} = \mathbb{0} \iff (\boldsymbol{x}-\boldsymbol{y}) = \mathbb{0}\right] \quad \forall \boldsymbol{x},\boldsymbol{y} \in X\right\}\\
&\iff \left\{\left[\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbb{0} \iff (\boldsymbol{x}-\boldsymbol{y}) = \mathbb{0}\right] \quad \forall \boldsymbol{x},\boldsymbol{y} \in X\right\}\\
&\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}
\end{aligned}$$

[7] Eidelman et al. (2004) page 3
[8] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

Theorem M.4. [9] *Let $\boldsymbol{W}$, $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ be linear spaces over a field $\mathbb{F}$.*

THM

1.	$\mathbf{L}(\mathbf{MN})$	$=$	$(\mathbf{LM})\mathbf{N}$	$\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{Z},\boldsymbol{W}), \mathbf{M}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(ASSOCIATIVE)
2.	$\mathbf{L}\left(\mathbf{M}\dotplus\mathbf{N}\right)$	$=$	$(\mathbf{LM})\dotplus(\mathbf{LN})$	$\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \mathbf{N}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(LEFT DISTRIBUTIVE)
3.	$\left(\mathbf{L}\dotplus\mathbf{M}\right)\mathbf{N}$	$=$	$(\mathbf{LN})\dotplus(\mathbf{MN})$	$\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{N}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{LM})$	$=$	$(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{Y},\boldsymbol{Z}), \mathbf{M}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{MN}) = (\mathbf{LM})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}\left(\mathbf{M}\dotplus\mathbf{N}\right) = (\mathbf{LM})\dotplus(\mathbf{LN})$:

$$\begin{aligned}
\left[\mathbf{L}\left(\mathbf{M}\dotplus\mathbf{N}\right)\right]\boldsymbol{x} &= \mathbf{L}\left[\left(\mathbf{M}\dotplus\mathbf{N}\right)\boldsymbol{x}\right] \\
&= \mathbf{L}\left[(\mathbf{M}\boldsymbol{x})\dotplus(\mathbf{N}\boldsymbol{x})\right] \\
&= [\mathbf{L}(\mathbf{M}\boldsymbol{x})]\dotplus[\mathbf{L}(\mathbf{N}\boldsymbol{x})] && \text{by } \textit{additive} \text{ property Definition M.3 page 282} \\
&= [(\mathbf{LM})\boldsymbol{x}]\dotplus[(\mathbf{LN})\boldsymbol{x}]
\end{aligned}$$

3. Proof that $\left(\mathbf{L}\dotplus\mathbf{M}\right)\mathbf{N} = (\mathbf{LN})\dotplus(\mathbf{MN})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{LM}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{LM}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned}
[\alpha(\mathbf{LM})]\boldsymbol{x} &= \alpha[(\mathbf{LM})\boldsymbol{x}] \\
&= \mathbf{L}[\alpha(\mathbf{M}\boldsymbol{x})] && \text{by } \textit{homogeneous} \text{ property Definition M.3 page 282} \\
&= \mathbf{L}[(\alpha\mathbf{M})\boldsymbol{x}] \\
&= [\mathbf{L}(\alpha\mathbf{M})]\boldsymbol{x}
\end{aligned}$$

Theorem M.5 (Fundamental theorem of linear equations). *Michel and Herget (1993) page 99* *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator* $\mathbf{L}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *and* $\mathcal{I}(\mathbf{L})$ *the* IMAGE SET *of* $\mathbf{L}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *(Definition ?? page ??).*

THM

$$\dim \mathcal{I}(\mathbf{L}) + \dim \mathcal{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1,2,\ldots,p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for

[9] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

$\mathcal{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{ \boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x} \} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \mathbb{0} \right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ✎

M.2 Operators on Normed linear spaces

M.2.1 Operator norm

Definition M.4. [10] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [11]

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{\boldsymbol{x} \in \boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \qquad \forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition M.1 (next) shows that the functional defined in Definition M.4 (previous) is a *norm*.[12]

Proposition M.1. [13] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ and $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1.	$\vert\vert\vert\mathbf{A}\vert\vert\vert$	$\ge$	0	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NON-NEGATIVE)	and
2.	$\vert\vert\vert\mathbf{A}\vert\vert\vert$	$=$	$0 \iff \mathbf{A} \doteq \mathbb{0}$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NONDEGENERATE)	and
3.	$\vert\vert\vert\alpha\mathbf{A}\vert\vert\vert$	$=$	$\vert\alpha\vert \, \vert\vert\vert\mathbf{A}\vert\vert\vert$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), \alpha \in \mathbb{F}$	(HOMOGENEOUS)	and
4.	$\vert\vert\vert\mathbf{A} \mathring{+} \mathbf{B}\vert\vert\vert$	$\le$	$\vert\vert\vert\mathbf{A}\vert\vert\vert + \vert\vert\vert\mathbf{B}\vert\vert\vert$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

[10] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225
[11] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990), page 290
[12] *norm* $\|\cdot\|$: Definition J.1 (page 249)
[13] Rudin (1991) page 93

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \neq \mathbb{0}$:

$$\begin{aligned}|||\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\\ &> 0\end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \triangleq \mathbb{0}$:

$$\begin{aligned}|||\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\\ &= \sup_{x \in \boldsymbol{X}} \{\|\mathbb{0}x\| \mid \|x\| \le 1\}\\ &= 0\end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha|\ |||\mathbf{A}|||$:

$$\begin{aligned}|||\alpha\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{\|\alpha\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\\ &= \sup_{x \in \boldsymbol{X}} \{|\alpha|\ \|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\\ &= |\alpha| \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} && \text{by definition of sup}\\ &= |\alpha|\ \ |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\end{aligned}$$

4. Proof that $\left|\left|\left|\mathbf{A} \mathring{+} \mathbf{B}\right|\right|\right| \le |||\mathbf{A}||| + |||\mathbf{B}|||$:

$$\begin{aligned}\left|\left|\left|\mathbf{A} \mathring{+} \mathbf{B}\right|\right|\right| &\triangleq \sup_{x \in \boldsymbol{X}} \left\{\left\|(\mathbf{A} \mathring{+} \mathbf{B})x\right\| \mid \|x\| \le 1\right\} && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\\ &= \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x + \mathbf{B}x\| \mid \|x\| \le 1\}\\ &\le \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| + \|\mathbf{B}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\\ &\le \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \le 1\} + \sup_{x \in \boldsymbol{X}} \{\|\mathbf{B}x\| \mid \|x\| \le 1\}\\ &\triangleq |||\mathbf{A}||| + |||\mathbf{B}||| && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)}\end{aligned}$$

✏

Lemma M.1. *Let* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

LEM

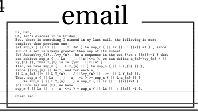
$$|||\mathbf{L}||| = \sup_{x} \{\|\mathbf{L}x\| \mid \|x\| = 1\} \qquad \forall x \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF: [14]

1. Proof that $\sup_x \{\|\mathbf{L}x\| \mid \|x\| \le 1\} \ge \sup_x \{\|\mathbf{L}x\| \mid \|x\| = 1\}$:

$$\sup_x \{\|\mathbf{L}x\| \mid \|x\| \le 1\} \ge \sup_x \{\|\mathbf{L}x\| \mid \|x\| = 1\} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

[14] email



Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ y \in X \mid \begin{array}{ll} 1. & \|\mathbf{L}y\| = \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} \text{ and} \\ 2. & 0 < \|y\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_x \{\|\mathbf{L}x\| \mid \|x\| \le 1\} \le \sup_x \{\|\mathbf{L}x\| \mid \|x\| = 1\}$:

$$\begin{aligned}
\sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} &= \|\mathbf{L}y\| && \text{by definition of set } Y \\
&= \frac{\|y\|}{\|y\|}\|\mathbf{L}y\| \\
&= \|y\| \left\| \frac{1}{\|y\|}\mathbf{L}y \right\| && \text{by homogeneous property (page 249)} \\
&= \|y\| \left\| \mathbf{L}\frac{y}{\|y\|} \right\| && \text{by homogeneous property (page 282)} \\
&\le \|y\| \sup_{y\in Y}\left\{ \left\| \mathbf{L}\frac{y}{\|y\|} \right\| \right\} && \text{by definition of supremum} \\
&= \|y\| \sup_{y\in Y}\left\{ \left\| \mathbf{L}\frac{y}{\|y\|} \right\| \;\middle|\; \left\| \frac{y}{\|y\|} \right\| = 1 \right\} && \text{because } \left\| \frac{y}{\|y\|} \right\| = 1 \text{ for all } y \in Y \\
&\le \sup_{y\in Y}\left\{ \left\| \mathbf{L}\frac{y}{\|y\|} \right\| \;\middle|\; \left\| \frac{y}{\|y\|} \right\| = 1 \right\} && \text{because } 0 < \|y\| \le 1 \\
&\le \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| = 1\} && \text{because } \frac{y}{\|y\|} \in X \quad \forall y \in Y
\end{aligned}$$

4. By (1) and (3),

$$\sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \sup_{x\in X}\{\|\mathbf{L}x\| \mid \|x\| = 1\}$$

✏

Proposition M.2. [15] *Let $\mathbf{I}$ be the identity operator in the normed space of linear operators $(\mathcal{L}(\boldsymbol{X},\boldsymbol{X}), |||\cdot|||)$.*

PRP	$|||\mathbf{I}||| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition M.4 page 286)} \\
&= \sup\{\|x\| \mid \|x\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition M.2 page 281)} \\
&= 1
\end{aligned}$$

✏

Theorem M.6. [16] *Let $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM			
$\|\mathbf{L}x\|$	$\le$	$|||\mathbf{L}|||\ \|x\|$	$\forall \mathbf{L} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{Y}),\ x \in \boldsymbol{X}$
$|||\mathbf{KL}|||$	$\le$	$|||\mathbf{K}|||\ |||\mathbf{L}|||$	$\forall \mathbf{K},\mathbf{L} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$

[15] Michel and Herget (1993) page 410
[16] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

✎PROOF:

1. Proof that $\|\mathbf{L}x\| \le \|\!|\mathbf{L}|\!\|\ \|x\|$:

$$\begin{aligned}
\|\mathbf{L}x\| &= \frac{\|x\|}{\|x\|}\|\mathbf{L}x\| \\
&= \|x\|\left\|\frac{1}{\|x\|}\mathbf{L}x\right\| && \text{by property of norms} \\
&= \|x\|\left\|\mathbf{L}\frac{x}{\|x\|}\right\| && \text{by property of linear operators} \\
&\triangleq \|x\|\ \|\mathbf{L}y\| && \text{where } y \triangleq \frac{x}{\|x\|} \\
&\le \|x\| \sup_y \|\mathbf{L}y\| && \text{by definition of supremum} \\
&= \|x\| \sup_y \{\|\mathbf{L}y\| \mid \|y\| = 1\} && \text{because } \|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1 \\
&\triangleq \|x\|\ \|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

2. Proof that $\|\!|\mathbf{KL}|\!\| \le \|\!|\mathbf{K}|\!\|\ \|\!|\mathbf{L}|\!\|$:

$$\begin{aligned}
\|\!|\mathbf{KL}|\!\| &\triangleq \sup_{x\in\boldsymbol{X}}\{\|(\mathbf{KL})x\| \mid \|x\| \le 1\} && \text{by Definition M.4 page 286 } (\|\!|\cdot|\!\|) \\
&= \sup_{x\in\boldsymbol{X}}\{\|\mathbf{K}(\mathbf{L}x)\| \mid \|x\| \le 1\} \\
&\le \sup_{x\in\boldsymbol{X}}\{\|\!|\mathbf{K}|\!\|\ \|\mathbf{L}x\| \mid \|x\| \le 1\} && \text{by 1.} \\
&\le \sup_{x\in\boldsymbol{X}}\{\|\!|\mathbf{K}|\!\|\ \|\!|\mathbf{L}|\!\|\ \|x\| \mid \|x\| \le 1\} && \text{by 1.} \\
&= \sup_{x\in\boldsymbol{X}}\{\|\!|\mathbf{K}|\!\|\ \|\!|\mathbf{L}|\!\|\ 1 \mid \|x\| \le 1\} && \text{by definition of sup} \\
&= \|\!|\mathbf{K}|\!\|\ \|\!|\mathbf{L}|\!\| && \text{by definition of sup}
\end{aligned}$$

⇰

M.2.2 Bounded linear operators

Definition M.5. [17] *Let $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \|\!|\cdot|\!\|)$ be a normed space of linear operators.*

DEF
*An operator $\mathbf{B}$ is **bounded** if $\|\!|\mathbf{B}|\!\| < \infty$.*
*The quantity $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ is the set of all **bounded linear operators** on $(\boldsymbol{X},\boldsymbol{Y})$ such that*
$$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \{\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \mid \|\!|\mathbf{L}|\!\| < \infty\}.$$

Theorem M.7. [18] *Let $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \|\!|\cdot|\!\|)$ be the set of linear operators over normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

THM
The following conditions are all EQUIVALENT:

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $x_0 \in X$ $\quad \forall\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point $x \in X$)* $\quad \forall\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
3. $\|\!|\mathbf{L}|\!\| < \infty$ *($\mathbf{L}$ is* BOUNDED*)* $\quad \forall\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}x\| \le M\,\|x\|$ $\quad \forall\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), x\in X$

[17] Rudin (1991) pages 92–93
[18] Aliprantis and Burkinshaw (1998) page 227

✎PROOF:

1. Proof that 1 $\implies$ 2:

$$\begin{aligned}
\epsilon &> \left\|\mathbf{L}x - \mathbf{L}x_0\right\| && \text{by hypothesis 1}\\
&= \left\|\mathbf{L}(x - x_0)\right\| && \text{by linearity (Definition M.3 page 282)}\\
&= \left\|\mathbf{L}(x + y - x_0 - y)\right\| && \\
&= \left\|\mathbf{L}(x + y) - \mathbf{L}(x_0 + y)\right\| && \text{by linearity (Definition M.3 page 282)}\\
&\implies \mathbf{L} \text{ is continuous at point } x + y && \\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}$$

2. Proof that 2 $\implies$ 1: obvious:

3. Proof that 4 $\implies$ 2:[19]

$$\begin{aligned}
|||\mathbf{L}x||| \le M\,\|x\| &\implies |||\mathbf{L}(x-y)||| \le M\,\|x-y\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}x - \mathbf{L}y||| \le M\,\|x-y\| && \text{by linearity of } \mathbf{L} \text{ (Definition M.3 page 282)}\\
&\implies |||\mathbf{L}x - \mathbf{L}y||| \le \epsilon \text{ whenever } M\,\|x-y\| < \epsilon && \\
&\implies |||\mathbf{L}x - \mathbf{L}y||| \le \epsilon \text{ whenever } \|x-y\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that 3 $\implies$ 4:

$$\begin{aligned}
\|\mathbf{L}x\| &\le \underbrace{|||\mathbf{L}|||}_{M}\;\|x\| && \text{by Theorem M.6 page 288}\\
&= M\;\|x\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that 1 $\implies$ 3:[20]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \infty\\
&\implies \exists\,(\!(x_n)\!) \;\; \text{such that} \;\; \left\|x_n\right\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\left\|\mathbf{L}x_n\right\| \mid \left\|x_n\right\| \le 1\right\} = \infty\\
&\implies \left\|x_n\right\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \left\|\mathbf{L}x_n\right\|\\
&\implies \left\|x_n\right\| = 1 \text{ and } \left\|\mathbf{L}x_n\right\| \ge n\\
&\implies \frac{1}{n}\left\|x_n\right\| = \frac{1}{n} \text{ and } \frac{1}{n}\left\|\mathbf{L}x_n\right\| \ge 1\\
&\implies \left\|\frac{x_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{x_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

⇰

[19] Bollobás (1999), page 29

[20] Aliprantis and Burkinshaw (1998), page 227

M.2.3 Adjoints on normed linear spaces

Definition M.6. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *of $\boldsymbol{X}$.*

DEF: $\mathbf{B}^*$ *is the* ***adjoint*** *of an operator* $\mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *if*

$$\mathsf{f}(\mathbf{B}\boldsymbol{x}) = \left[\mathbf{B}^*\mathsf{f}\right](\boldsymbol{x}) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ \boldsymbol{x} \in \boldsymbol{X}$$

Theorem M.8. [21] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM:
$$\begin{array}{rcll}
(\mathbf{A} \stackrel{\circ}{+} \mathbf{B})^* &=& \mathbf{A}^* \stackrel{\circ}{+} \mathbf{B}^* & \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) \\
(\lambda\mathbf{A})^* &=& \lambda\mathbf{A}^* & \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) \\
(\mathbf{A}\mathbf{B})^* &=& \mathbf{B}^*\mathbf{A}^* & \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})
\end{array}$$

✎PROOF:

$$\begin{aligned}
\left[\mathbf{A} \stackrel{\circ}{+} \mathbf{B}\right]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \stackrel{\circ}{+} \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint (Definition M.6 page 291)} \\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators (Definition M.3 page 282)} \\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} \\
&= \mathbf{A}^*\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} \text{ (Definition M.6 page 291)} \\
&= \left[\mathbf{A}^* + \mathbf{B}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{linear functional}
\end{aligned}$$

$$\begin{aligned}
[\lambda\mathbf{A}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} \text{ (Definition M.6 page 291)} \\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} \\
&= \left[\lambda\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} \text{ (Definition M.6 page 291)}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{A}\mathbf{B}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} \text{ (Definition M.6 page 291)} \\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of } \textit{linear operators} \text{ (Definition M.3 page 282)} \\
&= \left[\mathbf{A}^*\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{adjoint} \text{ (Definition M.6 page 291)} \\
&= \mathbf{B}^*\left[\mathbf{A}^*\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of } \textit{adjoint} \text{ (Definition M.6 page 291)} \\
&= \left[\mathbf{B}^*\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} \text{ (Definition M.6 page 291)}
\end{aligned}$$

⇨

Theorem M.9. [22] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^*$ be the adjoint of an operator $\mathbf{B}$.*

THM:
$$|||\mathbf{B}||| = \left|\left|\left|\mathbf{B}^*\right|\right|\right| \qquad \forall \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$$

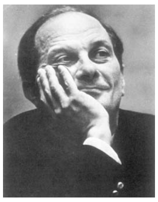
✎PROOF:

$$\begin{aligned}
|||\mathbf{B}||| &\triangleq \sup\left\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\right\} && \text{by Definition M.4 page 286} \\
&\stackrel{?}{=} \sup\left\{|\mathsf{g}(\mathbf{B}\boldsymbol{x}; \boldsymbol{y}^*)| \,|\, \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x}; \mathbf{B}^*\boldsymbol{y}^*)| \,|\, \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&\triangleq \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&= \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&\triangleq \left|\left|\left|\mathbf{B}^*\right|\right|\right| && \text{by Definition M.4 page 286}
\end{aligned}$$

⇨

[21] Bollobás (1999), page 156
[22] Rudin (1991) page 98

M.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞

Stanislaus M. Ulam (1909–1984), Polish mathematician [23]

Theorem M.10 (Mazur-Ulam theorem). [24] *Let* $\phi \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *be a function on normed linear spaces* $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ *and* $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. *Let* $\mathbf{I} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$ *be the identity operator on* $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.

THM

$$\left.\begin{array}{lll} 1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} & \text{and} \\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}}_{\text{isometric}} & \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$

   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$

2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$

3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned} \left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis} \\ &= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1} \end{aligned}$$

[23] quote: Ulam (1991), page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[24] Oikhberg and Rosenthal (2007), page 598, Väisälä (2003), page 634, Giles (2000), page 11, Dunford and Schwartz (1957), page 91, Mazur and Ulam (1932)

4. Proof that $gz = z$:

$$\begin{aligned}
2\lambda &= 2\sup\{\|gz - z\|\} && \text{by definition of } \lambda \text{ item (2)}\\
&\le 2\|gz - z\| && \text{by definition of sup}\\
&= \|2z - 2gz\|\\
&= \|\psi gz - gz\| && \text{by definition of } \psi \text{ item (1)}\\
&= \left\|g^{-1}\psi gz - g^{-1}gz\right\| && \text{by item (3)}\\
&= \left\|g^{-1}\psi gz - z\right\| && \text{by definition of } g^{-1}\\
&= \left\|\psi g^{-1}\psi gz - z\right\|\\
&= \left\|g^{*}z - z\right\|\\
&\le \lambda && \text{by definition of } \lambda \text{ item (2)}\\
&\implies 2\lambda \le \lambda\\
&\implies \lambda = 0\\
&\implies gz = z
\end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}\phi x + \frac{1}{2}\phi y$:

$$\begin{aligned}
\phi\left(\frac{1}{2}x + \frac{1}{2}y\right) &=\\
&= \frac{1}{2}\phi x + \frac{1}{2}\phi y
\end{aligned}$$

6. Proof that $\phi([1-\lambda]x + \lambda y) = [1-\lambda]\phi x + \lambda\phi y$:

$$\begin{aligned}
\phi([1-\lambda]x + \lambda y) &=\\
&= [1-\lambda]\phi x + \lambda\phi y
\end{aligned}$$

Theorem M.11 (Neumann Expansion Theorem). [25] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM
$$\left.\begin{array}{ll}
1. & \mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{X}) \quad (\mathbf{A} \text{ is bounded})\\
2. & |||\mathbf{A}||| < 1
\end{array}\right\} \implies \left\{\begin{array}{llcl}
1. & (\mathbf{I}-\mathbf{A})^{-1} & & \textit{exists}\\
2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \le & \dfrac{1}{1 - |||\mathbf{A}|||}\\
3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \displaystyle\sum_{n=0}^{\infty} \mathbf{A}^n\\
 & \multicolumn{3}{l}{\textit{with uniform convergence}}
\end{array}\right.$$

M.3 Operators on Inner product spaces

M.3.1 General Results

Theorem M.12. [26] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$ *be bounded linear operators on an inner product space* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

THM
$$\begin{array}{lllcll}
\langle \mathbf{B}x \mid x\rangle = 0 & \forall x \in X & \iff & \mathbf{B} = \mathbb{0} & \forall x \in X\\
\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle & \forall x \in X & \iff & \mathbf{A} = \mathbf{B} &
\end{array}$$

[25] Michel and Herget (1993) page 415

[26] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

✎PROOF:

1. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x+\mathbf{B}x) \mid (x+\mathbf{B}x)\rangle + i\,\langle \mathbf{B}(x+i\mathbf{B}x) \mid (x+i\mathbf{B}x)\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}x+\mathbf{B}^2x) \mid x+\mathbf{B}x\right\rangle\right\} + i\left\{\left\langle \mathbf{B}x+i\mathbf{B}^2x) \mid x+i\mathbf{B}x\right\rangle\right\} && \text{by Definition M.3 page 282}\\
&= \left\{\langle \mathbf{B}x \mid x\rangle + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + \left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\} && \text{by Definition I.1 page 233}\\
&\quad + i\left\{\langle \mathbf{B}x \mid x\rangle - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2\left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle\right\} + \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle - \left\langle \mathbf{B}^2x \mid x\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle\\
&= 2\,\|\mathbf{B}x\|^2\\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition J.1 page 249}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products (Theorem I.1 page 233).

3. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \implies \mathbf{A} \doteq \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \mid x\rangle - \langle \mathbf{B}x \mid x\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}x - \mathbf{B}x \mid x\rangle && \text{by \textit{additivity} property of } \langle \triangle \mid \triangledown\rangle \text{ (Definition I.1 page 233)}\\
&= \langle (\mathbf{A}-\mathbf{B})x \mid x\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A}-\mathbf{B})x &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \impliedby \mathbf{A} \doteq \mathbf{B}$:

$$\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \qquad \text{by } \mathbf{A} \doteq \mathbf{B} \text{ hypothesis}$$

⇨

M.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition M.3 page 294). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem M.13 page 295).
- Both support decomposition into "real" and "imaginary" parts (Theorem H.3 page 230).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *null space* of an operator (Theorem M.14 page 296).

Proposition M.3. [27] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of bounded linear operators on a Hilbert space $\boldsymbol{H}$.*[28]

PRP *An operator $\mathbf{B}^*$ is the* ADJOINT *of $\mathbf{B} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ if*
$$\langle \mathbf{B}x \mid y\rangle = \langle x \mid \mathbf{B}^*y\rangle \qquad \forall x,y \in \boldsymbol{H}.$$

[27] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000), page 182, von Neumann (1929) page 49, Stone (1932) page 41

[28] *bounded operator*: Definition M.5 (page 289); *adjoint*: Definition M.6 (page 291)

✎PROOF:

1. For fixed y, $\mathsf{f}(\boldsymbol{x}) \triangleq \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is a *functional* in $\mathbb{F}^{\boldsymbol{X}}$.
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle &\triangleq \mathsf{f}(\mathbf{B}\boldsymbol{x}) \\ &\triangleq \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by Definition M.6 (page 291)} \\ &= \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle\end{aligned}$$

✏

Example M.2 (Matrix algebra: $\mathbf{A}^* = \mathbf{A}^H$). In matrix algebra,

EX
- The inner product operation $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is represented by $\boldsymbol{y}^H\boldsymbol{x}$.
- The linear operator is represented as a matrix $\boldsymbol{A}$.
- The operation of $\boldsymbol{A}$ on vector $\boldsymbol{x}$ is represented as $\boldsymbol{A}\boldsymbol{x}$.
- The adjoint of matrix $\boldsymbol{A}$ is the Hermitian matrix $\boldsymbol{A}^H$.

✎PROOF:

$$\langle \boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H\boldsymbol{A}\boldsymbol{x} = [(\boldsymbol{A}\boldsymbol{x})^H\boldsymbol{y}]^H = [\boldsymbol{x}^H\boldsymbol{A}^H\boldsymbol{y}]^H = (\boldsymbol{A}^H\boldsymbol{y})^H\boldsymbol{x} \triangleq \left\langle \boldsymbol{x} \mid \boldsymbol{A}^H\boldsymbol{y} \right\rangle$$

✏

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*", Definition H.3 page 228). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[29]

Theorem M.13 (operator star-algebra). [30] *Let* $\boldsymbol{H}$ *be a Hilbert space with operators* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM
The pair $(\boldsymbol{H}, *)$ *is a* $*$-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \overset{\circ}{+} \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A,B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A,B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{AB})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A,B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A,B} \in \boldsymbol{H}$	(INVOLUTARY)	

✎PROOF:

$$\begin{aligned}\left\langle \boldsymbol{x} \mid (\mathbf{A} \overset{\circ}{+} \mathbf{B})^*\boldsymbol{y} \right\rangle &= \left\langle (\mathbf{A} \overset{\circ}{+} \mathbf{B})\boldsymbol{x} \mid \boldsymbol{y} \right\rangle && \text{by definition of adjoint} && \text{(Proposition M.3 page 294)} \\ &= \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition I.1 page 233)} \\ &= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of operator addition} \\ &= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} + \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition I.1 page 233)} \\ &= \langle \boldsymbol{x} \mid (\mathbf{A}^* + \mathbf{B}^*)\boldsymbol{y} \rangle && \text{by definition of operator addition}\end{aligned}$$

$$\begin{aligned}\langle \boldsymbol{x} \mid (\alpha\mathbf{A})^*\boldsymbol{y} \rangle &= \langle (\alpha\mathbf{A})\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition M.3 page 294)} \\ &= \langle \alpha(\mathbf{A}\boldsymbol{x}) \mid \boldsymbol{y} \rangle && \text{by definition of scalar multiplication} \\ &= \alpha \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition I.1 page 233)} \\ &= \alpha \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition M.3 page 294)}\end{aligned}$$

[29] Sakai (1998) page 1
[30] Halmos (1998), pages 39–40, Rudin (1991) page 311

$$= \langle \boldsymbol{x} \,|\, \alpha^* \mathbf{A}^* \boldsymbol{y} \rangle$$ by definition of inner product (Definition I.1 page 233)

$$\langle \boldsymbol{x} \,|\, (\mathbf{AB})^* \boldsymbol{y} \rangle = \langle (\mathbf{AB})\boldsymbol{x} \,|\, \boldsymbol{y} \rangle$$ by definition of adjoint (Proposition M.3 page 294)

$$= \langle \mathbf{A}(\mathbf{B}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle$$ by definition of operator multiplication

$$= \langle (\mathbf{B}\boldsymbol{x}) \,|\, \mathbf{A}^* \boldsymbol{y} \rangle$$ by definition of adjoint (Proposition M.3 page 294)

$$= \langle \boldsymbol{x} \,|\, \mathbf{B}^* \mathbf{A}^* \boldsymbol{y} \rangle$$ by definition of adjoint (Proposition M.3 page 294)

$$\langle \boldsymbol{x} \,|\, \mathbf{A}^{**} \boldsymbol{y} \rangle = \langle \mathbf{A}^* \boldsymbol{x} \,|\, \boldsymbol{y} \rangle$$ by definition of adjoint (Proposition M.3 page 294)

$$= \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle^*$$ by definition of inner product (Definition I.1 page 233)

$$= \langle \mathbf{A}\boldsymbol{y} \,|\, \boldsymbol{x} \rangle^*$$ by definition of adjoint (Proposition M.3 page 294)

$$= \langle \boldsymbol{x} \,|\, \mathbf{A}\boldsymbol{y} \rangle$$ by definition of inner product (Definition I.1 page 233)

Theorem M.14. [31] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\mathcal{N}(\mathbf{A}) = \mathcal{I}(\mathbf{A}^*)^\perp$$
$$\mathcal{N}(\mathbf{A}^*) = \mathcal{I}(\mathbf{A})^\perp$$

PROOF:

$$\mathcal{I}(\mathbf{A}^*)^\perp = \left\{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}^*) \right\}$$

$$= \left\{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\}$$

$$= \{ y \in \boldsymbol{H} | \langle \mathbf{A}\boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \}$$ by definition of $\mathbf{A}^*$ (Proposition M.3 page 294)

$$= \{ y \in \boldsymbol{H} | \mathbf{A}\boldsymbol{y} = 0 \}$$

$$= \mathcal{N}(\mathbf{A})$$ by definition of $\mathcal{N}(\mathbf{A})$

$$\mathcal{I}(\mathbf{A})^\perp = \{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}) \}$$

$$= \{ y \in \boldsymbol{H} | \langle \boldsymbol{y} \,|\, \mathbf{A}\boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \}$$ by definition of $\mathcal{I}$

$$= \left\{ y \in \boldsymbol{H} | \langle \mathbf{A}^* \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\}$$ by definition of $\mathbf{A}^*$ (Proposition M.3 page 294)

$$= \left\{ y \in \boldsymbol{H} | \mathbf{A}^* \boldsymbol{y} = 0 \right\}$$

$$= \mathcal{N}(\mathbf{A}^*)$$ by definition of $\mathcal{N}(\mathbf{A})$

M.4 Special Classes of Operators

M.4.1 Projection operators

Definition M.7. [32] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF

$\mathbf{P}$ *is a* ***projection*** *operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[31] Rudin (1991) page 312
[32] Rudin (1991) page 133 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 6, Halmos (1958) page 73 ⟨§41. Projections⟩

Theorem M.15. [33] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ with* NULL SPACE $\mathcal{N}(\mathbf{P})$ *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

$$\left.\begin{array}{lrcll}
1. & \mathbf{P}^2 & = & \mathbf{P} & (\mathbf{P} \text{ is a projection operator}) \text{ and} \\
2. & \boldsymbol{\Omega} & = & \boldsymbol{X} \dotplus \boldsymbol{Y} & (\boldsymbol{Y} \text{ compliments } \boldsymbol{X} \text{ in } \boldsymbol{\Omega}) \text{ and} \\
3. & \mathbf{P}\boldsymbol{\Omega} & = & \boldsymbol{X} & (\mathbf{P} \text{ projects onto } \boldsymbol{X})
\end{array}\right\} \implies \left\{\begin{array}{lrcll}
1. & \mathcal{I}(\mathbf{P}) & = & \boldsymbol{X} & \text{and} \\
2. & \mathcal{N}(\mathbf{P}) & = & \boldsymbol{Y} & \text{and} \\
3. & \boldsymbol{\Omega} & = & \mathcal{I}(\mathbf{P}) \dotplus \mathcal{N}(\mathbf{P}) &
\end{array}\right.$$

✎PROOF:

$$\begin{aligned}
\mathcal{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\
&= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\
&= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\
&= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\
&= \boldsymbol{\Omega}_1
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\} \\
&= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\
&= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\
&= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\
&= \boldsymbol{\Omega}_2
\end{aligned}$$

✏

Theorem M.16. [34] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

THM

$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P} \text{ is a projection operator}} \iff \underbrace{(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})}_{(\mathbf{I} - \mathbf{P}) \text{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned}
(\mathbf{I} - \mathbf{P})^2 &= (\mathbf{I} - \mathbf{P})(\mathbf{I} - \mathbf{P}) \\
&= \mathbf{I}(\mathbf{I} - \mathbf{P}) + (-\mathbf{P})(\mathbf{I} - \mathbf{P}) \\
&= \mathbf{I} - \mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2 \\
&= \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} && \text{by left hypothesis} \\
&= \mathbf{I} - \mathbf{P}
\end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned}
\mathbf{P}^2 &= \underbrace{\mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P}^2}_{(\mathbf{I} - \mathbf{P})^2} - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\
&= (\mathbf{I} - \mathbf{P})^2 - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\
&= (\mathbf{I} - \mathbf{P}) - (\mathbf{I} - \mathbf{P} - \mathbf{P}) && \text{by right hypothesis} \\
&= \mathbf{P}
\end{aligned}$$

✏

[33] Michel and Herget (1993) pages 120–121
[34] Michel and Herget (1993) page 121

M.4.2 Self Adjoint Operators

Definition M.8. [35] *Let* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a bounded operator with adjoint* $\mathbf{B}^*$ *on a Hilbert space* $\boldsymbol{H}$*.*

DEF

The operator $\mathbf{B}$ *is said to be* ***self-adjoint*** *or* ***hermitian*** *if* $\mathbf{B} \stackrel{\circ}{=} \mathbf{B}^*$.

Example M.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}.$$

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}}\mathsf{f}(u)\,\mathsf{d}u$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The autocorrelation operator $\mathbf{R}$ is *self-adjoint*)

✎PROOF:

1. First note that the *autocorrelation kernel* $\mathsf{R}_{\mathsf{xx}}(t,u)$ is *hermitian symmetric*:

$$\begin{aligned}\mathsf{R}_{\mathsf{xx}}(t,u) &\triangleq \mathsf{E}\mathsf{x}(t)\mathsf{x}^*(u) = [\mathsf{E}\mathsf{x}^*(t)\mathsf{x}(u)]^* = [\mathsf{E}\mathsf{x}(u)\mathsf{x}^*(t)]^* \\ &= \mathsf{R}^*_{\mathsf{xx}}(u,t)\end{aligned}$$

2. Proof that the *autocorrelation operator* $\mathbf{R}$ is *self-adjoint*:

$$\begin{aligned}
\langle \mathbf{R}\mathsf{f} \,|\, \mathsf{g}\rangle &= \left\langle \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{f}(u)\,\mathsf{d}u \,\Big|\, \mathsf{g}(t)\right\rangle && \text{by definition of } \mathbf{R}\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\,\langle \mathsf{R}_{\mathsf{xx}}(t,u) \,|\, \mathsf{g}(t)\rangle\;\mathsf{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\int_t \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{g}^*(t)\,\mathsf{d}t\,\mathsf{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\int_t \mathsf{R}^*_{\mathsf{xx}}(u,t)\mathsf{g}^*(t)\,\mathsf{d}t\,\mathsf{d}u && \text{by 1.}\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\int_t \mathsf{R}_{\mathsf{xx}}(u,t)\mathsf{g}(t)\,\mathsf{d}t\right]^*\mathsf{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\,[\mathbf{R}\mathsf{g}]^*\;\mathsf{d}u && \text{by definition of } \mathbf{R}\\
&= \langle \mathsf{f} \,|\, \mathbf{R}\mathsf{g}\rangle
\end{aligned}$$

✏

Theorem M.17. [36] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a Hilbert space* $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle}$*.*

THM

$$\underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S}\text{ is SELF ADJOINT}} \implies \begin{cases} 1.\ \ \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle \in \mathbb{R} & \text{(the hermitian quadratic form of } \mathbf{S} \text{ is real)}\\ 2.\ \ \lambda_n \in \mathbb{R} & \text{(eigenvalues of } \mathbf{S} \text{ are real)}\\ 3.\ \ \lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m\rangle = 0 & \text{(eigenfunctions associated with distinct eigenvalues are orthogonal)}\end{cases}$$

✎PROOF:

[35] Historical works regarding self-adjoint operators: von Neumann (1929) page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932) page 50 ⟨"self-adjoint transformations"⟩

[36] Lax (2002), pages 315–316, Keener (1988), pages 114–119

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \in \mathbb{R}$:

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition I.1 page 233}
\end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned}
\lambda_n \|\psi_n\|^2 &= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition} \\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition I.1 page 233} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_n \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \lambda_n^* \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition I.1 page 233} \\
&= \lambda_n^* \|\psi_n\|^2 && \text{by definition}
\end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition I.1 page 233} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_m \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \lambda_m^* \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition I.1 page 233} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

⇨

M.4.3 Normal Operators

Definition M.9. [37] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{N}^*$ be the adjoint of an operator $\mathbf{N} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF: $\mathbf{N}$ *is* ***normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

Theorem M.18. [38] *Let $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of bounded linear operators on a Hilbert space $\boldsymbol{H}$. Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$.*

THM: $$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \text{ is normal}} \iff \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\| \quad \forall \boldsymbol{x} \in \boldsymbol{H}$$

✎PROOF:

[37] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969), page 167, Frobenius (1878), Frobenius (1968) page 391

[38] Rudin (1991) pages 312–313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \mid \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition M.3 page 294 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \mid \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (}\mathbf{N}\text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition M.3 page 294 (definition of } \mathbf{N}^*)\\
&= \left\|\mathbf{N}^*\boldsymbol{x}\right\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition M.3 page 294 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem M.13 page 295 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \left\|\mathbf{N}^*\boldsymbol{x}\right\|^2 && \text{by right hypothesis } (\left\|\mathbf{N}^*\boldsymbol{x}\right\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Proposition M.3 page 294 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem M.19. [39] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of bounded linear operators on a Hilbert space $\boldsymbol{H}$. Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM

$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same null space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \left\{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{(definition of } \mathcal{N})\\
&= \left\{\boldsymbol{x} | \left\|\mathbf{N}^*\boldsymbol{x}\right\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition J.1 page 249)}\\
&= \{\boldsymbol{x} | \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\}\\
&= \{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} && \text{by definition of } \|\cdot\| \text{ (Definition J.1 page 249)}\\
&= \mathcal{N}(\mathbf{N}) && \text{(definition of } \mathcal{N})
\end{aligned}$$

✏

M.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition M.10. *Let $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ be* NORMED LINEAR SPACES *(Definition J.1 page 249).*

DEF

An operator $\mathbf{M} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ is **isometric** *if*

$$\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \qquad \forall x \in X.$$

Theorem M.20. [40] *Let $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ be* NORMED LINEAR SPACES. *Let $\mathbf{M}$ be a linear operator in $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM

$$\underbrace{\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall x \in X}_{\text{isometric in length}} \iff \underbrace{\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall x,y \in X}_{\text{isometric in distance}}$$

[39] Rudin (1991) pages 312–313

[40] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

✎PROOF:

1. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \implies \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| &= \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\| && \text{by definition of linear operators (Definition M.3 page 282)}\\
&= \|\mathbf{M}\boldsymbol{u}\| && \text{let } \boldsymbol{u} \triangleq \boldsymbol{x} - \boldsymbol{y}\\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by left hypothesis}
\end{aligned}$$

2. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \impliedby \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \|\mathbf{M}(\boldsymbol{x} - \mathbb{0})\|\\
&= \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition M.3 page 282)}\\
&= \|\boldsymbol{x} - \mathbb{0}\| && \text{by right hypothesis}\\
&= \|\boldsymbol{x}\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition M.10 page 300) in the more general normed linear spaces, while Theorem M.20 (page 300) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \iff \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem M.21 (next) demonstrates two additional equivalent properties.

Theorem M.21. [41] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of bounded linear operators on a normed linear space $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \|\cdot\|\right)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle}$.*

THM *The following conditions are all **equivalent**:*

1.	$\mathbf{M}^*\mathbf{M}$	$=$	$\mathbf{I}$			$\iff$
2.	$\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y}\rangle$	$=$	$\langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	($\mathbf{M}$ *is surjective*)	$\iff$
3.	$\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|$	$=$	$\|x - y\|$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	(*isometric in distance*)	$\iff$
4.	$\|\mathbf{M}\boldsymbol{x}\|$	$=$	$\|\boldsymbol{x}\|$	$\forall x \in X$	(*isometric in length*)	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y}\rangle &= \langle \boldsymbol{x} \,|\, \mathbf{M}^*\mathbf{M}\boldsymbol{y}\rangle && \text{by Proposition M.3 page 294 (definition of adjoint)}\\
&= \langle \boldsymbol{x} \,|\, \mathbf{I}\boldsymbol{y}\rangle && \text{by (1)}\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by Definition M.2 page 281 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \sqrt{\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{x}\rangle} && \text{by definition of } \|\cdot\|\\
&= \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle} && \text{by right hypothesis}\\
&= \|\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y}\rangle &= \|\mathbf{M}\boldsymbol{x} + \mathbf{M}\boldsymbol{y}\|^2 - \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|^2 + i\,\|\mathbf{M}\boldsymbol{x} + i\mathbf{M}\boldsymbol{y}\|^2 - i\,\|\mathbf{M}\boldsymbol{x} - i\mathbf{M}\boldsymbol{y}\|^2 && \text{by polarization id.}\\
&= \|\mathbf{M}(\boldsymbol{x} + \boldsymbol{y})\|^2 - \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\|^2 + i\,\|\mathbf{M}(\boldsymbol{x} + i\boldsymbol{y})\|^2 - i\,\|\mathbf{M}(\boldsymbol{x} - i\boldsymbol{y})\|^2 && \text{by Definition M.3}\\
&= \|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 + i\,\|\boldsymbol{x} + i\boldsymbol{y}\|^2 - i\,\|\boldsymbol{x} - i\boldsymbol{y}\|^2 && \text{by left hypothesis}
\end{aligned}$$

[41] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

4. Proof that (3) $\iff$ (4): by Theorem M.20 page 300

5. Proof that (4) $\implies$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition M.3 page 294 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem M.13 page 295 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y = 0\\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Definition M.2 page 281 (definition of } \mathbf{I})\\
\implies \qquad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}$$

⇰

Theorem M.22. [42] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M} \text{ is isometric}} \implies \left\{\begin{array}{lcll} |||\mathbf{M}||| &=& 1 & \text{(UNIT LENGTH)} \quad \textit{and}\\ |\lambda| &=& 1 & \forall \lambda \in \Lambda \end{array}\right.$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \implies |||\mathbf{M}||| = 1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{\boldsymbol{x}\in X} \{\|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Definition M.4 page 286}\\
&= \sup_{\boldsymbol{x}\in X} \{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Theorem M.21 page 301}\\
&= \sup_{\boldsymbol{x}\in X} \{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x}, \lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|} \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|} \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem M.21 page 301}\\
&= \frac{1}{\|\boldsymbol{x}\|} \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|} |\lambda| \; \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

⇰

Example M.4 (One sided shift operator). [43] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ $(0, 1, 2, \ldots)$ and shift operators defined as

1. $\mathbf{S}_r \left(\!\left(x_0, x_1, x_2, \ldots\right)\!\right) \triangleq \left(\!\left(0, x_0, x_1, x_2, \ldots\right)\!\right)$ (right shift operator)
2. $\mathbf{S}_l \left(\!\left(x_0, x_1, x_2, \ldots\right)\!\right) \triangleq \left(\!\left(x_1, x_2, x_3, \ldots\right)\!\right)$ (left shift operator)

[42] Michel and Herget (1993) page 432
[43] Michel and Herget (1993) page 441

EX
1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r \left(\!\left(x_0, x_1, x_2, \ldots\right)\!\right) \mid \left(\!\left(y_0, y_1, y_2, \ldots\right)\!\right)\rangle &= \langle \left(\!\left(0, x_0, x_1, x_2, \ldots\right)\!\right) \mid \left(\!\left(y_0, y_1, y_2, \ldots\right)\!\right)\rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\, y_n^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \langle \left(\!\left(x_0, x_1, x_2, \ldots\right)\!\right) \mid \left(\!\left(y_1, y_2, y_3, \ldots\right)\!\right)\rangle \\
&= \left\langle \left(\!\left(x_0, x_1, x_2, \ldots\right)\!\right) \mid \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} \left(\!\left(y_0, y_1, y_2, \ldots\right)\!\right) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✎

M.4.5 Unitary operators

Definition M.11. [44] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF *The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition M.4. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of bounded linear operators on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be bounded linear operators in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP
$$\left.\begin{array}{l}\mathbf{U} \textit{ is UNITARY} \quad \textit{and} \\ \mathbf{V} \textit{ is UNITARY}\end{array}\right\} \implies (\mathbf{UV}) \textit{ is UNITARY.}$$

[44] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\left(\mathbf{V}^*\mathbf{U}^*\right) && \text{by Theorem M.8 page 291}\\
&= \mathbf{U}\left(\mathbf{VV}^*\right)\mathbf{U}^* && \text{by associative property}\\
&= \mathbf{UIU}^* && \text{by definition of unitary operators—Definition M.11 page 303}\\
&= \mathbf{I} && \text{by definition of unitary operators—Definition M.11 page 303}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \left(\mathbf{V}^*\mathbf{U}^*\right)(\mathbf{UV}) && \text{by Theorem M.8 page 291}\\
&= \mathbf{V}^*\left(\mathbf{U}^*\mathbf{U}\right)\mathbf{V} && \text{by associative property}\\
&= \mathbf{V}^*\mathbf{IV} && \text{by definition of unitary operators—Definition M.11 page 303}\\
&= \mathbf{I} && \text{by definition of unitary operators—Definition M.11 page 303}
\end{aligned}$$

⇨

Theorem M.23. [45] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of bounded linear operators on a Hilbert space $\boldsymbol{H}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, and $\mathcal{I}(\mathbf{U})$ the* IMAGE SET *of $\mathbf{U}$.*

THM *The following conditions are* ***equivalent***:

1. $\mathbf{UU}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ *(unitary)* $\Longleftrightarrow$
2. $\langle \mathbf{U}x \,|\, \mathbf{U}y\rangle = \langle \mathbf{U}^*x \,|\, \mathbf{U}^*y\rangle = \langle x \,|\, y\rangle$ *and* $\mathcal{I}(\mathbf{U}) = X$ *(surjective)* $\Longleftrightarrow$
3. $\|\mathbf{U}x - \mathbf{U}y\| = \left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x - y\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ *(isometric in distance)* $\Longleftrightarrow$
4. $\|\mathbf{U}x\| = \|x\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ *(isometric in length)*

✎PROOF:

1. Proof that (1) $\Longrightarrow$ (2):
   (a) $\langle \mathbf{U}x \,|\, \mathbf{U}y\rangle = \langle \mathbf{U}^*x \,|\, \mathbf{U}^*y\rangle = \langle x \,|\, y\rangle$ by Theorem M.21 (page 301).
   (b) Proof that $\mathcal{I}(\mathbf{U}) = X$:

$$\begin{aligned}
X &\supseteq \mathcal{I}(\mathbf{U}) && \text{because } \mathbf{U} \in X^X\\
&\supseteq \mathcal{I}(\mathbf{UU}^*)\\
&= \mathcal{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{UU}^* = \mathbf{I})\\
&= X && \text{by Definition M.2 page 281 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4): by Theorem M.21 page 301.
3. Proof that (3) $\Longrightarrow$ (1):
   (a) Proof that $\|\mathbf{U}x - \mathbf{U}y\| = \|x - y\| \Longrightarrow \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem M.21 page 301
   (b) Proof that $\left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x - y\| \Longrightarrow \mathbf{UU}^* = \mathbf{I}$:

$$\begin{aligned}
\left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x - y\| \Longrightarrow \quad \mathbf{U}^{**}\mathbf{U}^* &= \mathbf{I} && \text{by Theorem M.21 page 301}\\
\mathbf{UU}^* &= \mathbf{I} && \text{by Theorem M.13 page 295}
\end{aligned}$$

⇨

Theorem M.24. *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of bounded linear operators on a Hilbert space $\boldsymbol{H}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathcal{N}(\mathbf{U})$ the* NULL SPACE *of $\mathbf{U}$, and $\mathcal{I}(\mathbf{U})$ the* IMAGE SET *of $\mathbf{U}$.*

THM
$$\underbrace{\mathbf{UU}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U}\text{ is unitary}} \Longrightarrow \left\{\begin{array}{lllll}
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \\
\mathcal{I}(\mathbf{U}) & = & \mathcal{I}(\mathbf{U}^*) & = & X\\
\mathcal{N}(\mathbf{U}) & = & \mathcal{N}(\mathbf{U}^*) & = & \{\mathbb{0}\}\\
|||\mathbf{U}||| & = & |||\mathbf{U}^*||| & = & 1 \quad \text{(UNIT LENGTH)}
\end{array}\right.$$

[45] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both **isometric** and **normal**:

$$\begin{array}{lclcl}
\mathbf{U}^*\mathbf{U} & & & = \mathbf{I} & \implies \mathbf{U} \text{ is isometric}\\
\mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = \mathbf{I} & \implies \mathbf{U}^* \text{ is isometric}\\
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \implies \mathbf{U}^{-1} \text{ is isometric}
\end{array}$$

$$\begin{array}{lclcl}
\mathbf{U}^*\mathbf{U} & = & \mathbf{U}\mathbf{U}^* & = & \implies \mathbf{U} \text{ is normal}\\
\mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & \implies \mathbf{U}^* \text{ is normal}\\
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \implies \mathbf{U}^{-1} \text{ is normal}
\end{array}$$

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem M.23 page 304.

3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}^{-1})$:

$$\begin{aligned}
\mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}) & & & \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both normal and by Theorem M.18 page 299}\\
&= \mathcal{I}(\mathbf{U})^\perp & & \text{by Theorem M.14 page 296}\\
&= X^\perp & & \text{by above result}\\
&= \{\mathbb{0}\} & & \text{by Proposition ?? page ??}
\end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \left\|\mathbf{U}^{-1}\right\| = \left\|\mathbf{U}^*\right\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem M.22 page 302.

✏

Example M.5. Examples of *Fredholm integral operators* include

$$\begin{array}{llrclrcl}
1. & \text{Fourier Transform} & [\tilde{\mathbf{F}}\mathsf{x}](f) & = & \int_t \mathsf{x}(t)e^{-i2\pi ft}\,\mathsf{d}t & \kappa(t,f) & = & e^{-i2\pi ft}\\
2. & \text{Inverse Fourier Transform} & [\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t) & = & \int_f \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathsf{d}f & \kappa(f,t) & = & e^{i2\pi ft}\\
3. & \text{Laplace operator} & [\mathbf{L}\mathsf{x}](s) & = & \int_t \mathsf{x}(t)e^{-st}\,\mathsf{d}t & \kappa(t,s) & = & e^{-st}
\end{array}$$

Example M.6 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathsf{f}(x) \triangleq \mathsf{f}(x-1) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

EX

$$\begin{array}{lrcll}
1. & \mathbf{T}^{-1}\mathsf{f}(x) & = & \mathsf{f}(x+1) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \text{(inverse translation operator)}\\
2. & \mathbf{T}^* & = & \mathbf{T}^{-1} & (\mathbf{T} \text{ is invertible})\\
3. & \mathbf{T}^*\mathbf{T} & = & \mathbf{T}\mathbf{T}^* = \mathbf{I} & (\mathbf{T} \text{ is unitary})
\end{array}$$

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$
$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned}
\langle \mathbf{T}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \langle \mathsf{f}(x-1) \,|\, \mathsf{g}(x)\rangle & & \text{by definition of } \mathbf{T}\\
&= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathsf{d}x\\
&= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathsf{d}x\\
&= \langle \mathsf{f}(x) \,|\, \mathsf{g}(x+1)\rangle\\
&= \left\langle \mathsf{f}(x) \,\middle|\, \underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x)\right\rangle & & \text{by 1.}
\end{aligned}$$

Example M.7 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

E X

1. $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ (inverse dilation operator)
2. $\mathbf{D}^* = \mathbf{D}^{-1}$ ($\mathbf{D}$ is invertible)
3. $\mathbf{D}^*\mathbf{D} = \mathbf{D}\mathbf{D}^* = \mathbf{I}$ ($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x) \,|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D} \\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\, \mathrm{d}x \\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u \\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^* \mathrm{d}u \\
&= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle \\
&= \left\langle \mathsf{f}(x) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

Example M.8 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

E X

The delay operator $\mathbf{D}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n-1}\right)\!\right)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n+1}\right)\!\right)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\left(\!\left(x_n\right)\!\right) \,|\, \left(\!\left(y_n\right)\!\right)\rangle &= \langle \left(\!\left(x_{n-1}\right)\!\right) \,|\, \left(\!\left(y_n\right)\!\right)\rangle && \text{by definition of } \mathbf{D} \\
&= \sum_n x_{n-1}\, y_n^* \\
&= \sum_n x_n\, y_{n+1}^* \\
&= \left\langle \left(\!\left(x_n\right)\!\right) \,|\, \left(\!\left(y_{n+1}\right)\!\right)\right\rangle \\
&= \left\langle \left(\!\left(x_n\right)\!\right) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\left(\!\left(y_n\right)\!\right)\right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary.

Example M.9 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator

$$[\tilde{\mathbf{F}}\mathsf{x}](f) \triangleq \int_t \mathsf{x}(t)\underbrace{e^{-i2\pi ft}}_{\kappa(t,f)}\,\mathrm{d}t \qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](t) \triangleq \int_f \tilde{\mathsf{x}}(f)\underbrace{e^{i2\pi ft}}_{\kappa^*(t,f)}\,\mathrm{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{x} \,|\, \tilde{\mathsf{y}} \right\rangle &= \left\langle \int_t \mathsf{x}(t)e^{-i2\pi ft}\,\mathrm{d}t \,|\, \tilde{\mathsf{y}}(f) \right\rangle \\
&= \int_t \mathsf{x}(t)\left\langle e^{-i2\pi ft} \,|\, \tilde{\mathsf{y}}(f)\right\rangle\,\mathrm{d}t \\
&= \int_t \mathsf{x}(t)\int_f e^{-i2\pi ft}\tilde{\mathsf{y}}^*(f)\,\mathrm{d}f\,\mathrm{d}t \\
&= \int_t \mathsf{x}(t)\left[\int_f e^{i2\pi ft}\tilde{\mathsf{y}}(f)\,\mathrm{d}f\right]^*\mathrm{d}t \\
&= \left\langle \mathsf{x}(t) \,|\, \int_f \tilde{\mathsf{y}}(f)e^{i2\pi ft}\,\mathrm{d}f \right\rangle \\
&= \left\langle \mathsf{x} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathsf{y}} \right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✐

Example M.10 (Rotation matrix). [46] Let the rotation matrix $\mathbf{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

EX
1. $\mathbf{R}^{-1}_\theta = \mathbf{R}_{-\theta}$
2. $\mathbf{R}^*_\theta = \mathbf{R}^{-1}_\theta$ ($\mathbf{R}$ is unitary)

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of Hermetian transpose operator } H \\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{by Theorem ?? page ??} \\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R} \\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✐

[46] Noble and Daniel (1988), page 311

M.5 Operator order

Definition M.12. [47] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF

$\mathbf{P}$ *is **positive** if* $\quad \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle \geq 0 \; \forall \boldsymbol{x} \in \boldsymbol{X}.$

This condition is denoted $\quad \mathbf{P} \geq 0.$

Theorem M.25. [48]

THM

$$\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \implies \left\{ \begin{array}{llll} (\mathbf{P}+\mathbf{Q}) & \geq 0 & & ((\mathbf{P}+\mathbf{Q}) \text{ is positive}) \\ \mathbf{A}^*\mathbf{P}\mathbf{A} & \geq 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{X}) & (\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive}) \\ \mathbf{A}^*\mathbf{A} & \geq 0 & \forall \mathbf{A} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{X}) & (\mathbf{A}^*\mathbf{A} \text{ is positive}) \end{array} \right.$$

✎PROOF:

$$\begin{aligned}
\langle (\mathbf{P}+\mathbf{Q})\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle + \langle \mathbf{Q}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by additive property of } \langle \triangle \mid \triangledown \rangle \text{ (Definition I.1 page 233)} \\
&\geq \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \mathbf{P}\mathbf{A}\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} \rangle && \text{by definition of adjoint (Proposition M.3 page 294)} \\
&= \langle \mathbf{P}\boldsymbol{y} \mid \boldsymbol{y} \rangle && \text{where } \boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition M.2 page 281)} \\
&\geq 0 && \text{by non-negative property of } \langle \triangle \mid \triangledown \rangle \text{ (Definition I.1 page 233)} \\
&\implies \mathbf{I} \text{ is positive} \\
\langle \mathbf{A}^*\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x} \rangle &= \langle \mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by definition of } \mathbf{I} \text{ (Definition M.2 page 281)} \\
&\geq 0 && \text{by two previous results}
\end{aligned}$$

✏

Definition M.13. [49] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be bounded operators.*

DEF

$$\underbrace{\mathbf{A} \geq \mathbf{B}}_{\text{"}\mathbf{A} \text{ is greater than or equal to } \mathbf{B}\text{"}} \iff \underbrace{(\mathbf{A}-\mathbf{B}) \geq 0}_{\text{"}(\mathbf{A}-\mathbf{B}) \text{ is positive"}}$$

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[47] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩
[48] Michel and Herget (1993) page 429
[49] Michel and Herget (1993) page 429

APPENDIX N

FOURIER TRANSFORM

❝*The analytical equations . . . extend to all general phenomena. There cannot be a language more universal and more simple, more free from errors and from obscurities, . . . mathematical analysis is as extensive as nature itself; . . . it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them.*❞

Joseph Fourier (1768–1830) [1]

N.1 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on ⊠, and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \int_{\mathbb{R}} |\mathsf{f}|^2 \, \mathsf{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \,|\, \triangledown \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \, \mathsf{d}\mu$ such that

$$\langle \mathsf{f} \,|\, \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, \mathsf{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \,|\, \triangledown \rangle \right)$ is a *Hilbert space*.

Definition N.1. *Let* κ *be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF *The function* κ *is the **Fourier kernel** if* $\quad \kappa(x,\omega) \triangleq e^{i\omega x} \quad \forall x,\omega \in \mathbb{R}$

Definition N.2. [2] *Let* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

[1] quote: Fourier (1878), pages 7–8 ⟨Preliminary Discourse⟩
image: http://en.wikipedia.org/wiki/File:Fourier2.jpg, public domain

[2] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965), page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

DEF *The* ***Fourier Transform*** *operator* $\tilde{\mathbf{F}}$ *is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathrm{d}x \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

This definition of the Fourier Transform is also called the ***unitary Fourier Transform***.

Remark N.1 (**Fourier transform scaling factor**). [3] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B\int_{\mathbb{R}} \mathsf{F}(\omega)e^{i\omega x}\,\mathrm{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\,e^{i2\pi f x}\,\mathrm{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{i\omega x}\,\mathrm{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int\cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem N.2 page 310)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

N.2 Operator properties

Theorem N.1 (Inverse Fourier transform). [4] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator* (Definition N.2 page 309). *The inverse* $\tilde{\mathbf{F}}^{-1}$ *of* $\tilde{\mathbf{F}}$ *is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\,\mathrm{d}\omega \qquad \forall\tilde{\mathsf{f}}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem N.2. *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with inverse* $\tilde{\mathbf{F}}^{-1}$ *and adjoint* $\tilde{\mathbf{F}}^*$.

THM
$$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{f} \mid \mathsf{g}\right\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\,e^{-i\omega x}\,\mathrm{d}x \mid \mathsf{g}(\omega)\right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 309}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\left\langle e^{-i\omega x} \mid \mathsf{g}(\omega)\right\rangle\,\mathrm{d}x && \text{by } \textit{additive property} \text{ of } \langle\triangle\mid\triangledown\rangle \text{ page 233}\\
&= \int_{\mathbb{R}} \mathsf{f}(x)\,\frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \mid e^{-i\omega x}\right\rangle^*\,\mathrm{d}x && \text{by } \textit{conjugate symmetric property} \text{ of } \langle\triangle\mid\triangledown\rangle \text{ page 233}
\end{aligned}$$

[3] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005b) pages 374–375
[4] Chorin and Hald (2009) page 13

$$= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2\pi}} \left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle \right\rangle \qquad \text{by definition of } \langle \triangle \,|\, \nabla \rangle$$

$$= \left\langle \mathsf{f} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*} \mathsf{g} \right\rangle \qquad \text{by Theorem N.1 page 310}$$

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary*[5] (Corollary N.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem N.3 page 311).

Corollary N.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

COR

$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem N.2 page 310). ⇨

Theorem N.3. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \nabla \rangle)$. *Let* $\mathcal{R}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

THM

$$\begin{array}{rclcll}
\mathcal{R}(\mathbf{F}\tau) & = & \mathcal{R}(\tilde{\mathbf{F}}^{-1}) & = & \boldsymbol{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| & = & |||\tilde{\mathbf{F}}^{-1}||| & = & 1 & (\text{UNITARY}) \\
\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g} \rangle & = & \langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g} \rangle & = & \langle \mathsf{f} \,|\, \mathsf{g} \rangle & (\text{PARSEVAL'S EQUATION}) \\
\|\tilde{\mathbf{F}}\mathsf{f}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f}\| & = & \|\mathsf{f}\| & (\text{PLANCHEREL'S FORMULA}) \\
\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\| & = & \|\mathsf{f} - \mathsf{g}\| & (\text{ISOMETRIC})
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary N.1 page 311) and from the properties of unitary operators (Theorem M.24 page 304). ⇨

Theorem N.4 (Shift relations)**.** *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator.*

THM

$$\begin{array}{rcl}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) & = & e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \\
\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) & = & \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v)
\end{array}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x-u) e^{-i\omega x} \,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{v\in\mathbb{R}} \mathsf{f}(v) e^{-i\omega(u+v)} \,\mathrm{d}v && \text{where } v \triangleq x - u \implies t = u + v \\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{v\in\mathbb{R}} \mathsf{f}(v) e^{-i\omega v} \,\mathrm{d}v \\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x} \,\mathrm{d}x && \text{by change of variable } t = v \\
&= e^{-i\omega u} \left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)} \\
\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ivx}\mathsf{g}(x) e^{-i\omega x} \,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)}
\end{aligned}$$

[5] *unitary operators*: Definition M.11 page 303

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{g}(x) e^{-i(\omega - v)x}\, \mathrm{d}x$$

$$= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v) \qquad \text{by definition of } \tilde{\mathbf{F}} \qquad \text{(Definition N.2 page 309))}$$

Theorem N.5 (Complex conjugate). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

THM

$$\begin{array}{rcll} \tilde{\mathbf{F}}\mathsf{f}^*(-x) &=& -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* & \forall \mathsf{f} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \\ \mathsf{f} \text{ is real} \implies \tilde{\mathsf{f}}(-\omega) &=& \left[\tilde{\mathsf{f}}(\omega)\right]^* & \forall \omega \in \mathbb{R} \qquad \text{REALITY CONDITION} \end{array}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}} \int \mathsf{f}^*(-x) e^{-i\omega x}\, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)} \\
&= \frac{1}{\sqrt{2\pi}} \int \mathsf{f}^*(u) e^{i\omega u}(-1)\, \mathrm{d}u && \text{where } u \triangleq -x \implies \mathrm{d}x = -\,\mathrm{d}u \\
&= -\left[\frac{1}{\sqrt{2\pi}} \int \mathsf{f}(u) e^{-i\omega u}\, \mathrm{d}u\right]^* \\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)} \\
\tilde{\mathsf{f}}(-\omega) &\triangleq \frac{1}{\sqrt{2\pi}} \int \mathsf{f}(x) e^{-i(-\omega)x}\, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)} \\
&= \left[\frac{1}{\sqrt{2\pi}} \int \mathsf{f}^*(x) e^{-i\omega x}\, \mathrm{d}x\right]^* \\
&= \left[\frac{1}{\sqrt{2\pi}} \int \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x\right]^* && \text{by } \mathsf{f} \text{ is real hypothesis} \\
&\triangleq \tilde{\mathsf{f}}^*(\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition N.2 page 309)}
\end{aligned}$$

N.3 Convolution

Definition N.3. [6]

DEF

*The **convolution operation** is defined as*

$$\left[\mathsf{f} \star \mathsf{g}\right](x) \triangleq \mathsf{f}(x) \star \mathsf{g}(x) \triangleq \int_{u \in \mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\, \mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem P.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem N.6 (convolution theorem). [7] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $\star$ the convolution operator.*

[6] Bachman (1964), page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩
[7] Bracewell (1978) page 110

THM

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega) &= \tilde{\mathbf{F}}\left[\int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\, \mathrm{d}u\right](\omega) && \text{by definition of } \star \text{ (Definition N.3 page 312)}\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\tilde{\mathbf{F}}\mathsf{g}(x-u)\right](\omega)\, \mathrm{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega)\, \mathrm{d}u && \text{by Theorem N.4 page 311}\\
&= \sqrt{2\pi}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\omega u}\, \mathrm{d}u\right)}_{\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)}\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)\\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition N.2 page 309)}\\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \tilde{\mathbf{F}}\left[\left(\tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathsf{f}(x)\right)\mathsf{g}(x)\right](\omega) && \text{by definition of operator inverse (page 281)}\\
&= \tilde{\mathbf{F}}\left[\left(\frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)e^{ivx}\, \mathrm{d}v\right)\mathsf{g}(x)\right](\omega) && \text{by Theorem N.1 page 310}\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\left(e^{ivx}\,\mathsf{g}(x)\right)\right](\omega,v)\, \mathrm{d}v\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-v)\, \mathrm{d}v && \text{by Theorem N.4 page 311}\\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \star \text{ (Definition N.3 page 312)}
\end{aligned}$$

✏

N.4 Real valued functions

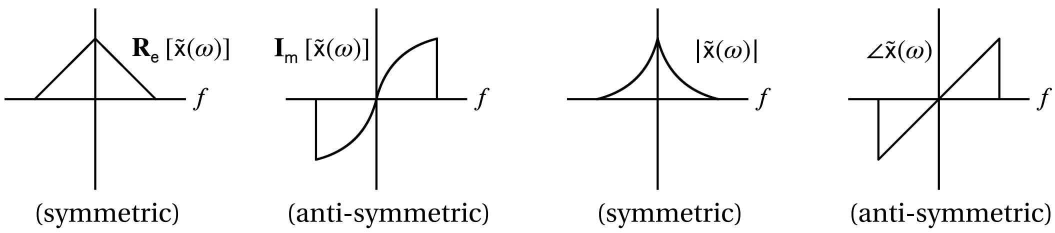


Figure N.1: Fourier transform components of real-valued signal

Theorem N.7. *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM

$$\left\{\begin{array}{l}\mathsf{f}(x) \text{ is REAL-VALUED}\\ (\mathsf{f} \in \mathbb{R}^{\mathbb{R}})\end{array}\right\} \implies \left\{\begin{array}{rcll}
\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & \text{(HERMITIAN SYMMETRIC)}\\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(SYMMETRIC)}\\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(ANTI-SYMMETRIC)}\\
\left|\tilde{\mathsf{f}}(\omega)\right| &=& \left|\tilde{\mathsf{f}}(-\omega)\right| & \text{(SYMMETRIC)}\\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & \text{(ANTI-SYMMETRIC).}
\end{array}\right\}$$

✎PROOF:

$$\begin{array}{lllllll}
\tilde{f}(\omega) & \triangleq & [\tilde{\mathbf{F}}f(x)](\omega) & \triangleq & \left\langle f(x) \,|\, e^{i\omega x} \right\rangle & = & \left\langle f(x) \,|\, e^{i(-\omega)x} \right\rangle^* \triangleq \tilde{f}^*(-\omega) \\
\mathbf{R_e}\left[\tilde{f}(\omega)\right] & = & \mathbf{R_e}\left[\tilde{f}^*(-\omega)\right] & = & \mathbf{R_e}\left[\tilde{f}(-\omega)\right] \\
\mathbf{I_m}\left[\tilde{f}(\omega)\right] & = & \mathbf{I_m}\left[\tilde{f}^*(-\omega)\right] & = & -\mathbf{I_m}\left[\tilde{f}(-\omega)\right] \\
|\tilde{f}(\omega)| & = & |\tilde{f}^*(-\omega)| & = & |\tilde{f}(-\omega)| \\
\angle\tilde{f}(\omega) & = & \angle\tilde{f}^*(-\omega) & = & -\angle\tilde{f}(-\omega)
\end{array}$$

⇨

N.5 Moment properties

Definition N.4. [8]

DEF: *The quantity* M_n *is the* n***th moment*** *of a function* $f(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n f(x)\, dx \qquad \text{for } n \in \mathbb{W}.$$

Lemma N.1. [9] *Let* M_n *be the* nTH MOMENT (Definition N.4 page 314) *and* $\tilde{f}(\omega) \triangleq \left[\tilde{\mathbf{F}}f\right](\omega)$ *the* FOURIER TRANSFORM (Definition N.2 page 309) *of a function* $f(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition ?? page ??).

LEM:

$$\begin{array}{rcll}
\mathsf{M}_n & = & \sqrt{2\pi}(i)^n \left[\frac{d}{d\omega}\right]^n \tilde{f}(\omega)\Big|_{\omega=0} & \forall n\in\mathbb{W}, f\in\boldsymbol{L}^2_{\mathbb{R}} \\
\left[\frac{d}{d\omega}\right]^n \tilde{f}(\omega)\Big|_{\omega=0} & = & \frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n & \forall n\in\mathbb{W}, f\in\boldsymbol{L}^2_{\mathbb{R}}
\end{array}$$

✎PROOF:

$$\begin{aligned}
\sqrt{2\pi}(i)^n \left[\left[\frac{d}{d\omega}\right]^n \tilde{f}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\frac{d}{d\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition N.2 page 309)} \\
&= (i)^n \int_{\mathbb{R}} f(x) \left[\left[\frac{d}{d\omega}\right]^n e^{-i\omega x}\right] dx \Big|_{\omega=0} \\
&= (i)^n \int_{\mathbb{R}} f(x) \left[(-i)^n x^n e^{-i\omega x}\right] dx \Big|_{\omega=0} \\
&= (-i^2)^n \int_{\mathbb{R}} f(x) x^n\, dx \\
&= \int_{\mathbb{R}} f(x) x^n\, dx \\
&\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \text{ (Definition N.4 page 314)}
\end{aligned}$$

⇨

Lemma N.2. [10] *Let* M_n *be the* nTH MOMENT (Definition N.4 page 314) *and* $\tilde{f}(\omega) \triangleq \left[\tilde{\mathbf{F}}f\right](\omega)$ *the* FOURIER TRANSFORM (Definition N.2 page 309) *of a function* $f(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition ?? page ??).

LEM:

$$\mathsf{M}_n = 0 \qquad \Longleftrightarrow \qquad \left[\frac{d}{d\omega}\right]^n \tilde{f}(\omega)\Big|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

[8] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83

[9] Goswami and Chan (1999), pages 38–39

[10] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
0 &= \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by left hypothesis}\\
&= \sqrt{2\pi}(-i)^{-n}\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \text{by Lemma N.1 page 314}\\
\implies \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} &= 0
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \text{by right hypothesis}\\
&= \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x\Big|_{\omega=0} && \text{by definition of } \tilde{\mathsf{f}}(\omega)\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega x}\,\mathsf{d}x\Big|_{\omega=0}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\left[(-i)^n x^n e^{-i\omega x}\right]\,\mathsf{d}x\Big|_{\omega=0}\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)x^n\,\mathsf{d}x\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by definition of } \langle\cdot\,|\,\cdot\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \text{ (Definition ?? page ??)}
\end{aligned}$$

✏

Lemma N.3 (Strang-Fix condition). [11] *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* M_n *the* nTH MOMENT *(Definition N.4 page 314) of* $\mathsf{f}(x)$. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition ?? page ??).*

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "time"}} \iff \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k\mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "frequency"}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}}\left[\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by Definition N.2 page 309}\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\int_{\mathbb{R}} x^n\mathsf{f}(x)e^{-i\omega x}\,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}(x-k)^n\mathsf{f}(x-k)\bar{\delta}_k && \text{by } \textit{PSF} \text{ (Theorem ?? page ??)}\\
&= \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k\mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

[11] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83, Mallat (1999), pages 241–243, Fix and Strang (1969)

2. Proof for ($\Longleftarrow$) case:

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[(-i)^n \bar{\delta}_k \mathsf{M}_n\right] e^{-i2\pi kx} && \text{by definition of } \bar{\delta} \quad \text{(Definition I.3 page 245)}\\
&= \sum_{k\in\mathbb{Z}} \left[\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis}\\
&= \sum_{k\in\mathbb{Z}} \left[\left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx}\\
&= \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\, \mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\, \mathsf{d}x\right]\Big|_{\omega=2\pi k} e^{-i2\pi kx}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } \textit{PSF} \quad \text{(Theorem ?? page ??)}
\end{aligned}
$$

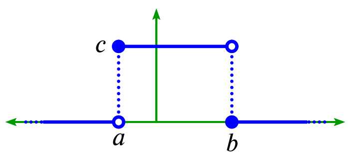
$\Rightarrow$

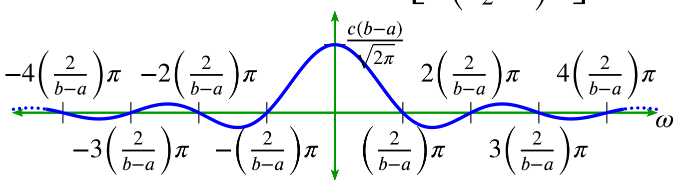
N.6 Examples

Example N.1 (rectangular pulse). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right]$$





✎PROOF:

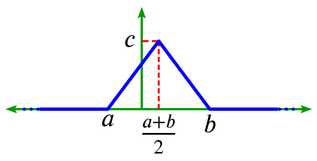
$$
\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega)\\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[\mathsf{f}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} \quad \text{(Theorem N.4 page 311)}\\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by definition of } \mathsf{f}(x)\\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} \quad \text{(Definition ?? page ??)}\\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x) e^{-i\omega x}\, \mathsf{d}x && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition N.2 page 309)}\\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} c e^{-i\omega x}\, \mathsf{d}x && \text{by definition of } \mathbb{1} \quad \text{(Definition ?? page ??)}\\
&= \frac{c}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \frac{1}{-i\omega} e^{-i\omega x}\Big|_{-\frac{b-a}{2}}^{\frac{b-a}{2}}\\
&= \frac{2c}{\sqrt{2\pi}\omega} e^{-i\left(\frac{a+b}{2}\right)\omega} \left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i}\right]
\end{aligned}
$$

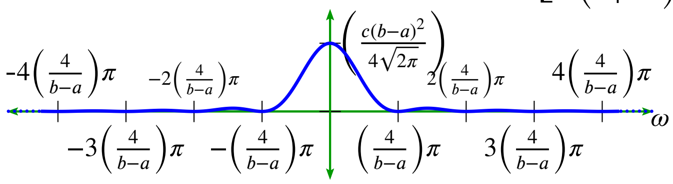
$$= \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)} \right]$$ by *Euler formulas* (Corollary ?? page ??)

Example N.2 (triangle). Let $\tilde{f}(\omega)$ be the *Fourier transform* of a function $f(x) \in L^2_{\mathbb{R}}$.

EX

$$f(x) = \begin{cases} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)} \right]^2$$





✎PROOF:

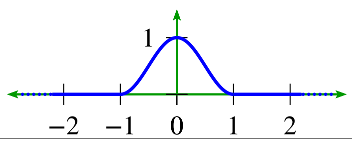
$$\begin{aligned}
\tilde{f}(\omega) &= \tilde{\mathbf{F}}[f(x)](\omega) && \text{by definition of } \tilde{f}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[f\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by \textit{shift relation}} && \text{(Theorem N.4 page 311)} \\
&= \tilde{\mathbf{F}}\left[c\left(1 - \frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) && \text{by definition of } f(x) \\
&= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x) \star \mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega) \\
&= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] && \text{by \textit{convolution theorem}} && \text{(Theorem P.2 page 332)} \\
&= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2 \\
&= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 && \text{by \textit{Rectangular pulse} ex.} && \text{Example N.1 page 316} \\
&= \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2
\end{aligned}$$

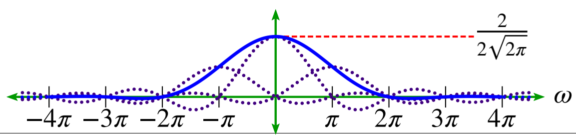
⇒

Example N.3. Let a function f be defined in terms of the cosine function (Definition ?? page ??) as follows:

EX

$$f(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\text{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\text{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\text{sinc}(\omega-\pi)}\right]$$





✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition ?? page ??) on a set A.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition N.2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathsf{f}(x)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \text{ (Definition ??)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \left[\frac{e^{i\frac{\pi}{2}x}+e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\,\mathrm{d}x && \text{by Corollary ?? page ??}\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1} \left[2+e^{i\pi x}+e^{-i\pi x}\right]e^{-i\omega x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\int_{-1}^{1} 2e^{-i\omega x}+e^{-i(\omega+\pi)x}+e^{-i(\omega-\pi)x}\,\mathrm{d}x\\
&= \frac{1}{4\sqrt{2\pi}}\left[2\frac{e^{-i\omega x}}{-i\omega}+\frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}\\
&= \frac{1}{2\sqrt{2\pi}}\left[2\frac{e^{-i\omega}-e^{+i\omega}}{-2i\omega}+\frac{e^{-i(\omega+\pi)}-e^{+i(\omega+\pi)}}{-2i(\omega+\pi)}+\frac{e^{-i(\omega-\pi)}-e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}\\
&= \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)}+\underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)}+\underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]
\end{aligned}$$

✏

APPENDIX O

DISCRETE TIME FOURIER TRANSFORM

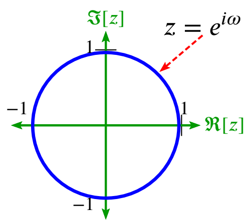
O.1 Definition

Definition O.1.

DEF *The **discrete-time Fourier transform** $\breve{\mathbf{F}}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$\left[\breve{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition O.1 page 319) to the definition of the Z-transform (Definition P.4 page 330), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.



O.2 Properties

Proposition O.1 (DTFT periodicity). *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition O.1 page 319) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP

$$\underbrace{\breve{x}(\omega) = \breve{x}(\omega + 2\pi n)}_{\text{PERIODIC with period } 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\breve{x}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} & &= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi nm}} \\
&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} & &= \breve{x}(\omega)
\end{aligned}$$

⇒

Theorem O.1. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition O.1 page 319) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM

$$\left\{ \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x[n])\!) \right\} \implies \left\{ \begin{array}{llcll} (1). & \breve{\mathbf{F}}(\!(x[-n])\!) & = & \tilde{x}(-\omega) & \text{and} \\ (2). & \breve{\mathbf{F}}(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition O.1 page 319)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}\\
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x^*[n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition O.1 page 319)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\mathbf{algebras} && \text{(Definition H.3 page 228)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{-i(-\omega)n}\right)^*\\
&\triangleq \tilde{x}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}(\!(x^*[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition O.1 page 319)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[-n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\mathbf{algebras} && \text{(Definition H.3 page 228)}\\
&= \left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n = -m\\
&\triangleq \tilde{x}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

✏

Theorem O.2. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x[n])\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition O.1 page 319) of a sequence* $(\!(x[n])\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM

$$\left\{ \begin{array}{lll} (1). & \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}(\!(x[n])\!) & \text{and} \\ (2). & (\!(x[n])\!) \text{ is REAL-VALUED} & \end{array} \right\} \implies \left\{ \begin{array}{llcllll} (1). & \breve{\mathbf{F}}(\!(x[-n])\!) & = & \tilde{x}(-\omega) & & & \text{and} \\ (2). & \breve{\mathbf{F}}(\!(x^*[n])\!) & = & \tilde{x}^*(-\omega) & = & \tilde{x}(\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}(\!(x^*[-n])\!) & = & \tilde{x}^*(\omega) & = & \tilde{x}(-\omega) & \end{array} \right\}$$

✎PROOF:

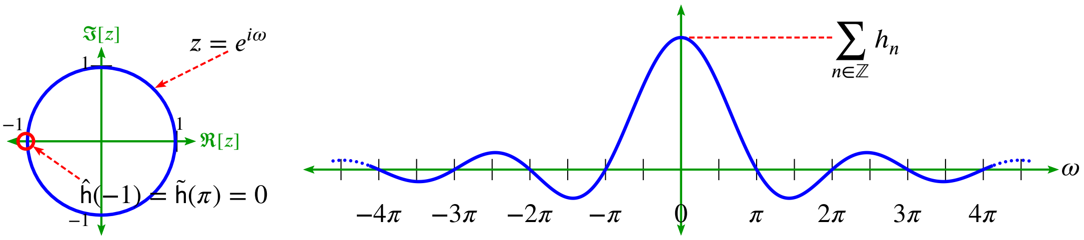
$$\begin{aligned}
\breve{\mathbf{F}}(\!(x[-n])\!) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition O.1 page 319)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}
\end{aligned}$$

$$\triangleq \breve{x}(-\omega) \qquad \text{by left hypothesis}$$

$$\begin{aligned}
\boxed{\tilde{x}^*(-\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(x^*[n])\!)} && \text{by Theorem O.1 page 320}\\
&= \breve{\mathbf{F}}\,(\!(x[n])\!) && \text{by \textit{real-valued} hypothesis}\\
&= \boxed{\breve{x}(\omega)} && \text{by definition of } \breve{x}(\omega) \quad \text{(Definition O.1 page 319)}
\end{aligned}$$

$$\begin{aligned}
\boxed{\tilde{x}^*(\omega)} &= \boxed{\breve{\mathbf{F}}\,(\!(x^*[-n])\!)} && \text{by Theorem O.1 page 320}\\
&= \breve{\mathbf{F}}\,(\!(x[-n])\!) && \text{by \textit{real-valued} hypothesis}\\
&= \boxed{\breve{x}(-\omega)} && \text{by result (1)}
\end{aligned}$$

⇒



Proposition O.2. *Let* $\breve{x}(z)$ *be the* Z-TRANSFORM *(Definition P.4 page 330) and* $\breve{x}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition O.1 page 319) of* $(\!(x_n)\!)$.

PRP

$$\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\breve{x}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{x}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}},\ c\in\mathbb{R}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{x}(z)\Big|_{z=1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \text{by definition of } \breve{x}(z) \text{ (Definition P.4 page 330)}\\
&= \sum_{n\in\mathbb{Z}} x_n && \text{because } z^n = 1 \text{ for all } n\in\mathbb{Z}\\
&= c && \text{by hypothesis (1)}
\end{aligned}$$

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\breve{x}(\omega)\Big|_{\omega=0} &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} && \text{by definition of } \breve{x}(\omega) \quad \text{(Definition O.1 page 319)}\\
&= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} \\
&= \breve{x}(z)\Big|_{z=1} && \text{by definition of } \breve{x}(z) \quad \text{(Definition P.4 page 330)}\\
&= c && \text{by hypothesis (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=0} \\
&= \breve{\mathrm{x}}(\omega) && \text{by definition of } \breve{\mathrm{x}}(\omega) && \text{(Definition O.1 page 319)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

Proposition O.3. *If the coefficients are* ***real****, then the magnitude response (MR) is* ***symmetric****.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathrm{h}}(-\omega)\right| &\triangleq \left|\breve{\mathrm{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\sum_{m\in\mathbb{Z}} \mathrm{x}[m] z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} \mathrm{x}[m] e^{i\omega m}\right| && = \left|\left(\sum_{m\in\mathbb{Z}} \mathrm{x}^*[m] e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} \mathrm{x}[m] e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } real} && = \left|\sum_{m\in\mathbb{Z}} \mathrm{x}[m] e^{-i\omega m}\right| \\
&\triangleq \left|\breve{\mathrm{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\tilde{\mathrm{h}}(\omega)\right|
\end{aligned}$$

Proposition O.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}} (-1)^n x_n = c}_{\text{(1) in "time"}} \iff \underbrace{\breve{\mathrm{x}}(z)\big|_{z=-1} = c}_{\text{(2) in "z domain"}} \iff \underbrace{\breve{\mathrm{x}}(\omega)\big|_{\omega=\pi} = c}_{\text{(3) in "frequency"}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{\mathrm{x}}(z)\big|_{z=-1} &= \left.\sum_{n\in\mathbb{Z}} x_n z^{-n}\right|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \sum_{n\in\mathbb{Z}} z^{-n} x_n\Big|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n\Big|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\implies$ (4):

   (a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

   (b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

   (d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

(e) Solutions to these equations give

$$\sum_{n\in\mathbb{Z}} x_{2n} \triangleq A = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)$$

$$\sum_{n\in\mathbb{Z}} x_{2n+1} \triangleq B = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1} \\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1} \\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)} \\
&= c
\end{aligned}$$

Lemma O.1. *Let* $\tilde{f}(\omega)$ *be the* DTFT (Definition O.1 page 319) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

LEM
$$\underbrace{(x_n \in \mathbb{R})_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{x}(\omega)|^2 = |\breve{x}(-\omega)|^2}_{\text{EVEN}} \qquad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
|\breve{x}(\omega)|^2 &= |\breve{x}(z)|^2\Big|_{z=e^{i\omega}} \\
&= \breve{x}(z)\breve{x}^*(z)\Big|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}} \\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^* (z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}} \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right] \\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right]$$
$$= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2 \sum_{n\in\mathbb{Z}} \sum_{m>n} x_n x_m \cos[\omega(m-n)]$$

Since cos is real and even, then $|\breve{x}(\omega)|^2$ must also be real and even. ✏

Theorem O.3 (inverse DTFT). [2] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM (Definition O.1 page 319) *of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\breve{x}^{-1}$ be the inverse of $\breve{x}$.*

THM

$$\underbrace{\left\{ \breve{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\breve{x}(\omega)\triangleq\breve{\mathbf{F}}(x_n)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega) e^{i\omega n}\, d\omega \quad \forall\alpha\in\mathbb{R} \right\}}_{(x_n)=\breve{\mathbf{F}}^{-1}\breve{\mathbf{F}}(x_n)} \quad \forall (x_n)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \breve{x}(\omega)e^{i\omega n}\, d\omega &= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m}\right]}_{\breve{x}(\omega)} e^{i\omega n}\, d\omega && \text{by definition of } \breve{x}(\omega) \\
&= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\, d\omega \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\, d\omega \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \left[2\pi\bar{\delta}_{m-n}\right] \\
&= x_n
\end{aligned}$$

✏

Theorem O.4 (orthonormal quadrature conditions). [3] *Let $\breve{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM (Definition O.1 page 319) *of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\bar{\delta}_n$ be the* KRONECKER DELTA FUNCTION *at n* (Definition I.3 page 245).

THM

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} &= 0 &\iff& \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) &= 0 && \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}} \\
\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} &= \bar{\delta}_n &\iff& |\breve{x}(\omega)|^2 + |\breve{x}(\omega+\pi)|^2 &= 2 && \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}}
\end{aligned}$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩
[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

1. Proof that $2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)$:

$$\begin{aligned}
&2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n}\left(1+e^{i\pi n}\right)\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n\\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_m z^{m} + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_m e^{+i(\omega+\pi)m}\\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^*\\
&\triangleq \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)
\end{aligned}$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 0 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n}\\
&= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 2 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

⇰

O.3 Derivatives

Theorem O.5. [4] *Let* $\breve{x}(\omega)$ *be the* DTFT (Definition O.1 page 319) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$.

THM

$$\text{(A)}\quad \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} = 0 \iff \sum_{k\in\mathbb{Z}} k^n x_k = 0 \quad \text{(B)} \quad \forall n\in\mathbb{W}$$

$$\text{(C)}\quad \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=\pi} = 0 \iff \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 \quad \text{(D)} \quad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} && \text{by hypothesis (A)}\\
&= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \breve{x}(\omega) \text{ (Definition O.1 page 319)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=0} &= \left[\frac{d}{d\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \tilde{g}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{d}{d\omega}\right]^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k\\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

[4] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

3. Proof that $(C) \implies (D)$:

$$
\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathrm{x}}(\omega)\bigg|_{\omega=\pi} && \text{by hypothesis (C)} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Bigg|_{\omega=\pi} && \text{by definition of } \breve{\mathrm{x}} \text{ (Definition O.1 page 319)} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Bigg|_{\omega=\pi} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Bigg|_{\omega=\pi} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right] \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k
\end{aligned}
$$

4. Proof that $(C) \impliedby (D)$:

$$
\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathrm{x}}(\omega)\bigg|_{\omega=\pi} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Bigg|_{\omega=\pi} && \text{by definition of } \breve{\mathrm{x}} \text{ (Definition O.1 page 319)} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Bigg|_{\omega=\pi} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Bigg|_{\omega=\pi} \\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right] \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k \\
&= 0 && \text{by hypothesis (D)}
\end{aligned}
$$

APPENDIX P

OPERATIONS ON SEQUENCES

P.1 Convolution operator

Definition P.1. [1] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF

*A function f in X^Y is a **sequence** over X if $Y = \mathbb{Z}$.*
A sequence may be denoted in the form $\left(x_n\right)_{n\in\mathbb{Z}}$ or simply as $\left(x_n\right)$.

Definition P.2. [2] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a* FIELD *(Definition **??** page **??**).*

DEF

*The **space of all absolutely square summable sequences** $\ell^2_{\mathbb{F}}$ over $\mathbb{F}$ is defined as*

$$\ell^2_{\mathbb{F}} \triangleq \left\{ \left(x_n\right)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} \left|x_n\right|^2 < \infty \right\}$$

The space $\ell^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\ell^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\ell^2_{\mathbb{R}}$ is isomorphic to $L^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition P.3.

DEF

*The **convolution** operation $\star$ is defined as*

$$\left(x_n\right) \star \left(y_n\right) \triangleq \left(\sum_{m\in\mathbb{Z}} x_m y_{n-m} \right)_{n\in\mathbb{Z}} \qquad \forall \left(x_n\right)_{n\in\mathbb{Z}}, \left(y_n\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

Proposition P.1. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition P.3 page 329).*

PRP

$$\left(x_n\right) \star \left(y_n\right) = \left(y_n\right) \star \left(x_n\right) \qquad \forall \left(x_n\right)_{n\in\mathbb{Z}}, \left(y_n\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

[1] Bromwich (1908), page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[2] Kubrusly (2011) page 347 ⟨Example 5.K⟩

✎PROOF:

$$
\begin{aligned}
[x \star y](n) &\triangleq \sum_{m\in\mathbb{Z}} x_m y_{n-m} && \text{by Definition P.3 page 329}\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{where } k = n-m \iff m = n-k\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{by change commutivity of addition}\\
&= \sum_{m\in\mathbb{Z}} x_{n-m} y_m && \text{by change of variables}\\
&= \sum_{m\in\mathbb{Z}} y_m x_{n-m} && \text{by commutative property of the field over } \mathbb{C}\\
&\triangleq (y \star x)_n && \text{by Definition P.3 page 329}
\end{aligned}
$$

⇨

Proposition P.2. *Let* $\star$ *be the* CONVOLUTION OPERATOR *(Definition P.3 page 329). Let* $\ell^2_{\mathbb{R}}$ *be the set of* ABSOLUTELY SUMMABLE *sequences (Definition P.2 page 329).*

PRP

$$
\left\{\begin{array}{lllll} \text{(A).} & \mathsf{x}(n) & \in & \ell^2_{\mathbb{R}} & \textit{and} \\ \text{(B).} & \mathsf{y}(n) & \in & \ell^2_{\mathbb{R}} & \end{array}\right\} \implies \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] = \mathsf{x}[-n] \star \mathsf{y}(n) \right\}
$$

✎PROOF:

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \implies k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by \textit{absolutely summable} hypothesis} && \text{(Definition P.2 page 329)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \implies \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of \textit{convolution} } \star && \text{(Definition P.3 page 329)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n] &&
\end{aligned}
$$

⇨

P.2 Z-transform

Definition P.4. [3]

DEF

The ***z-transform*** $\mathbf{Z}$ *of* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *is defined as*

$$
[\mathbf{Z}\,(\!(x_n)\!)](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\textit{Laurent series}} \qquad \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}}
$$

Theorem P.1. *Let* $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ *be the* Z-TRANSFORM *of* $\mathsf{x}[n]$*.*

THM

$$
\left\{ \check{\mathsf{x}}(z) \triangleq \mathbf{Z}\,(\!(\mathsf{x}[n])\!) \right\} \implies \left\{\begin{array}{llllll}
(1). & \mathbf{Z}\,(\!(\alpha\mathsf{x}[n])\!) & = & \alpha\check{\mathsf{x}}(z) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} & \textit{and}\\
(2). & \mathbf{Z}\,(\!(\mathsf{x}[n-k])\!) & = & z^{-k}\check{\mathsf{x}}(z) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} & \textit{and}\\
(3). & \mathbf{Z}\,(\!(\mathsf{x}[-n])\!) & = & \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} & \textit{and}\\
(4). & \mathbf{Z}\,(\!(\mathsf{x}^*[n])\!) & = & \check{\mathsf{x}}^*\left(z^*\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} & \textit{and}\\
(5). & \mathbf{Z}\,(\!(\mathsf{x}^*[-n])\!) & = & \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (\!(x_n)\!)\in\ell^2_{\mathbb{R}} &
\end{array}\right\}
$$

[3] *Laurent series*: Abramovich and Aliprantis (2002) page 49

✎PROOF:

$$
\begin{aligned}
\alpha \mathbb{Z}\check{x}(z) &\triangleq \alpha \mathbf{Z}(\mathsf{x}[n]) && \text{by definition of } \check{x}(z) \\
&\triangleq \alpha \sum_{n\in\mathbb{Z}} \mathsf{x}[n] z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator} \\
&\triangleq \sum_{n\in\mathbb{Z}} (\alpha \mathsf{x}[n]) z^{-n} && \text{by \textit{distributive} property} \\
&\triangleq \mathbf{Z}(\alpha \mathsf{x}[n]) && \text{by definition of } \mathbf{Z} \text{ operator} \\
z^{-k}\check{x}(z) &= z^{-k}\mathbf{Z}(\mathsf{x}[n]) && \text{by definition of } \check{x}(z) && \text{(left hypothesis)} \\
&\triangleq z^{-k} \sum_{n=-\infty}^{n=+\infty} \mathsf{x}[n] z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
&= \sum_{n=-\infty}^{n=+\infty} \mathsf{x}[n] z^{-n-k} \\
&= \sum_{m-k=-\infty}^{m-k=+\infty} \mathsf{x}[m-k] z^{-m} && \text{where } m \triangleq n+k && \implies n = m-k \\
&= \sum_{m=-\infty}^{m=+\infty} \mathsf{x}[m-k] z^{-m} \\
&= \sum_{n=-\infty}^{n=+\infty} \mathsf{x}[n-k] z^{-n} && \text{where } n \triangleq m \\
&\triangleq \mathbf{Z}(\mathsf{x}[n-k]) && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
\mathbf{Z}(\mathsf{x}^*[n]) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}^*[n] z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
&\triangleq \left(\sum_{n\in\mathbb{Z}} \mathsf{x}[n] (z^*)^{-n} \right)^* && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
&\triangleq \check{x}^*(z^*) && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
\mathbf{Z}(\mathsf{x}[-n]) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}[-n] z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
&= \sum_{-m\in\mathbb{Z}} \mathsf{x}[m] z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m] z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition P.2 page 329)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}[m] \left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition P.2 page 329)} \\
&\triangleq \check{x}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
\mathbf{Z}(\mathsf{x}^*[-n]) &\triangleq \sum_{n\in\mathbb{Z}} \mathsf{x}^*[-n] z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition P.4 page 330)} \\
&= \sum_{-m\in\mathbb{Z}} \mathsf{x}^*[m] z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}^*[m] z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition P.2 page 329)} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{x}^*[m] \left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition P.2 page 329)} \\
&= \left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m] \left(\frac{1}{z^*}\right)^{-m} \right)^* && \text{by \textit{absolutely summable} property} && \text{(Definition P.2 page 329)}
\end{aligned}
$$

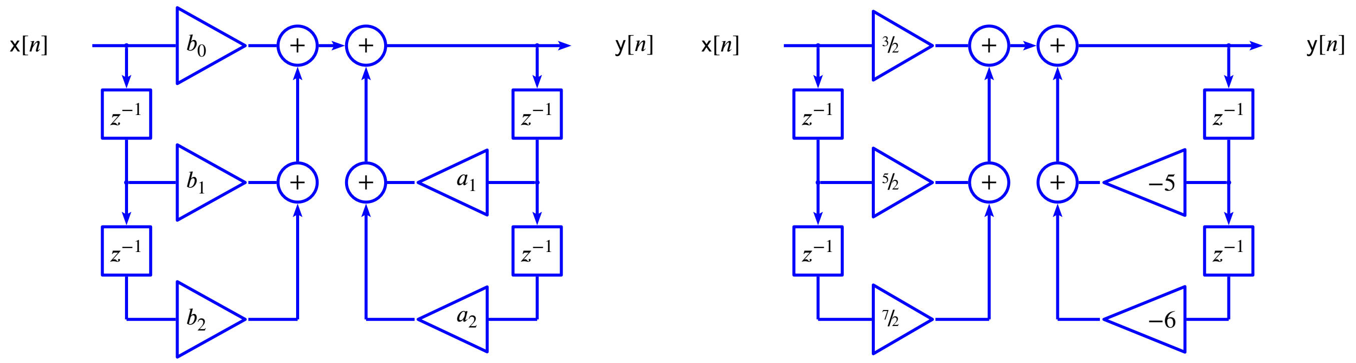


Figure P.1: Direct form 1 order 2 IIR filters

$$\triangleq \breve{x}^*\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition P.4 page 330)}$$

Theorem P.2 (convolution theorem). *Let $\star$ be the convolution operator (Definition P.3 page 329).*

THM

$$\underbrace{\mathbf{Z}\left(\left(\!\left(x_n\right)\!\right) \star \left(\!\left(y_n\right)\!\right)\right)}_{\text{sequence convolution}} = \underbrace{\left(\mathbf{Z}\left(\!\left(x_n\right)\!\right)\right)\left(\mathbf{Z}\left(\!\left(y_n\right)\!\right)\right)}_{\text{series multiplication}} \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}, \left(\!\left(y_n\right)\!\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x \star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by Definition P.3 page 329}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by Definition P.4 page 330}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k = n-m \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right]\\
&\triangleq \left(\mathbf{Z}\left(\!\left(x_n\right)\!\right)\right)\left(\mathbf{Z}\left(\!\left(y_n\right)\!\right)\right) && \text{by Definition P.4 page 330}
\end{aligned}$$

P.3 From z-domain back to time-domain

$$\breve{y}(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) - a_1z^{-1}Y(z) + a_2z^{-2}Y(z)$$

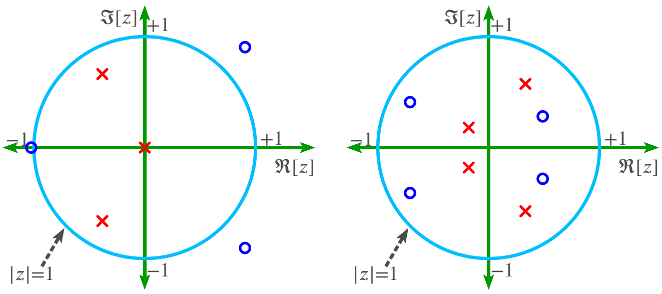
$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] - a_1y[n-1] - a_2y[n-2]$$

Example P.1. See Figure P.1 (page 332)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2(z^2+5z+6)} = \frac{\left(3/2z^2+5/2z+7/2\right)}{z^2+5z+6} = \frac{\left(3/2+5/2z^{-1}+7/2z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

P.4 Zero locations

The system property of *minimum phase* is defined in Definition P.5 (next) and illustrated in Figure P.2 (page 333).



not minimum phase   *minimum phase*

Figure P.2: Minimum Phase filter

Definition P.5. [4] *Let* $\breve{x}(z) \triangleq \mathbf{Z}\left(\!\left(x_n\right)\!\right)$ *be the* Z TRANSFORM (Definition P.4 page 330) *of a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\left(\!\left(z_n\right)\!\right)_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\breve{x}(z)$.

DEF

The sequence $\left(\!\left(x_n\right)\!\right)$ *is* ***minimum phase*** *if*

$$\underbrace{\left|z_n\right| < 1 \qquad \forall n \in \mathbb{Z}}_{\breve{x}(z)\text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem P.3 (Robinson's Energy Delay Theorem). [5] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM

$$\left\{\begin{array}{ll} \mathsf{p} & \text{is MINIMUM PHASE} \quad \text{and} \\ \mathsf{q} & \text{is NOT } \textit{minimum phase} \end{array}\right\} \implies \underbrace{\sum_{n=0}^{m-1}\left|a_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{p}(z)} \geq \underbrace{\sum_{n=0}^{m-1}\left|b_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{q}(z)} \qquad \forall 0 \leq m \leq N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle.

Example P.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a

[4] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36

[5] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976), pages 52–53

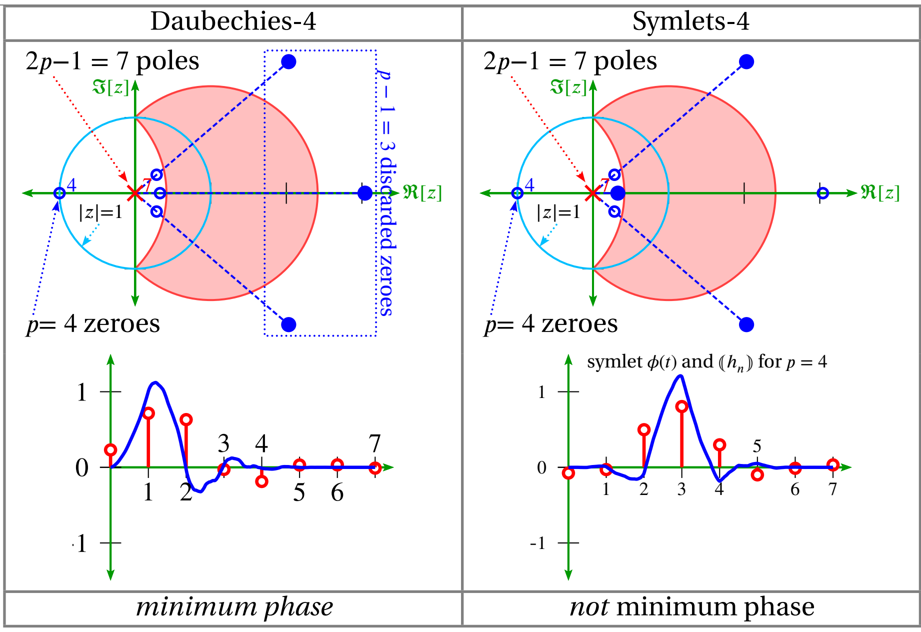


Figure P.3: Daubechies-4 and Symlet-4 scaling functions pole-zero plots

scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure P.3 (page 334).

P.5 Pole locations

Definition P.6.

> **DEF** *A filter (or system or operator)* **H** *is* ***causal***
> *if its current output does not depend on future inputs.*

Definition P.7.

> **DEF** *A filter (or system or operator)* **H** *is* ***time-invariant***
> *if the mapping it performs does not change with time.*

Definition P.8.

> **DEF** *An operation* **H** *is* ***linear*** *if any output y_n can be described as a linear combination of inputs x_n as in*
> $$y_n = \sum_{m \in \mathbb{Z}} \mathsf{h}(m)\mathsf{x}(n-m).$$

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem P.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Example P.3. Stable/unstable filters are illustrated in Figure P.4 (page 335).

True or False? This filter has no poles:

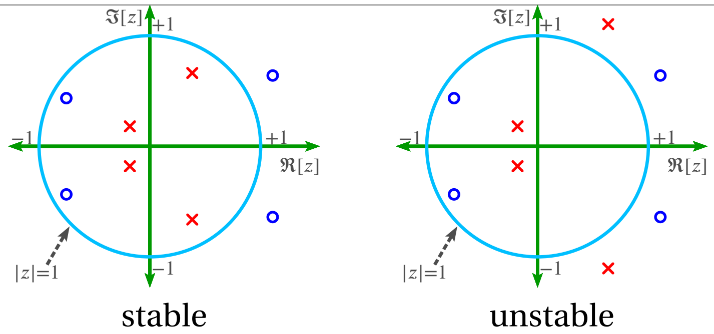
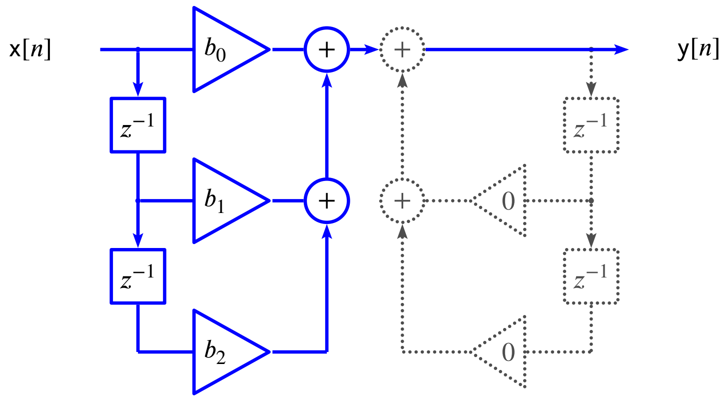
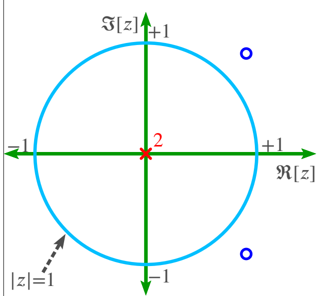


Figure P.4: Pole-zero plot stable/unstable causal LTI filters (Example P.3 page 334)

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$



$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$



P.6 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

Proposition P.3.

PRP $\left\{ \begin{array}{l} \text{ZEROS } \textit{and} \text{ POLES} \\ \textit{occur in} \text{ CONJUGATE PAIRS} \end{array} \right\} \implies \left\{ \begin{array}{l} \text{COEFFICIENTS} \\ \textit{are} \text{ REAL.} \end{array} \right\}$

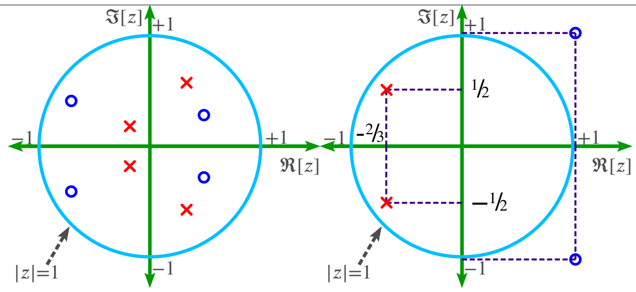


Figure P.5: Conjugate pair structure yielding real coefficients

✎PROOF:

$$\begin{aligned}
(z-p_1)(z-p_1^*) &= [z-(a+ib)][z-(a-ib)] \\
&= z^2 + [-a+ib-ib-a]z - [ib]^2 \\
&= z^2 - 2az + b^2
\end{aligned}$$

⇒

Example P.4. See Figure P.5 (page 336).

$$\begin{aligned}
H(z) &= G\frac{[z-z_1][z-z_2]}{[z-p_1][z-p_2]} = G\frac{[z-(1+i)][z-(1-i)]}{[z-(-\tfrac{2}{3}+i\tfrac{1}{2})][z-(-\tfrac{2}{3}-i\tfrac{1}{2})]} \\
&= G\frac{z^2 - z[(1-i)+(1+i)] + (1-i)(1+i)}{z^2 - z[(-\tfrac{2}{3}+i\tfrac{1}{2}) + (-\tfrac{2}{3}+i\tfrac{1}{2})] + (-\tfrac{2}{3}+i\tfrac{1}{2})(-\tfrac{2}{3}+i\tfrac{1}{2})} \\
&= G\frac{z^2-2z+2}{z^2-\tfrac{4}{3}z+(\tfrac{4}{9}+\tfrac{1}{4})} = G\frac{z^2-2z+2}{z^2-\tfrac{4}{3}z+\tfrac{19}{12}}
\end{aligned}$$

P.7 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma P.1. *A causal LTI operator* $\mathbf{H}$ *can be expressed as a rational expression* $\check{\mathsf{h}}(z)$.

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \frac{\sum_{n=0}^{N} b_n z^{-n}}{1 + \sum_{n=1}^{N} a_n z^{-n}}
\end{aligned}$$

A filter operation $\check{\mathsf{h}}(z)$ can be expressed as a product of its roots (poles and zeros).

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \alpha\frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}
\end{aligned}$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with $\times$ and zeros with $\bigcirc$. An example is shown in Figure P.6 page 337. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

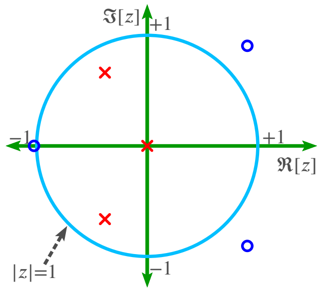


Figure P.6: Pole-zero plot for rational expression with real coefficients

P.8 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{x}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{y}(z)$ is a *high-pass filter*.

Definition P.9. [6] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES (Definition P.1 page 329) *in* $\ell^2_{\mathbb{R}}$ (Definition P.2 page 329).

DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter***.

Any triple $\left((\!(x_n)\!), (\!(y_n)\!), N\right)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition***.

Theorem P.5 (CQF theorem). [7] *Let* $\check{y}(\omega)$ *and* $\check{x}(\omega)$ *be the* DTFT*s* (Definition O.1 page 319) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$, *respectively, in* $\ell^2_{\mathbb{R}}$ (Definition P.2 page 329).

THM

$$\begin{array}{rrcll}
\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{\text{(1) CQF in "time"}} & \iff \check{y}(z) & = & \pm(-1)^N z^{-N}\check{x}^*\left(\frac{-1}{z^*}\right) & \text{(2) CQF in "z-domain"} \\
& \iff \check{y}(\omega) & = & \pm(-1)^N e^{-i\omega N}\check{x}^*(\omega+\pi) & \text{(3) CQF in "frequency"} \\
& \iff x_n & = & \pm(-1)^N(-1)^n y^*_{N-n} & \text{(4) "reversed" CQF in "time"} \\
& \iff \check{x}(z) & = & \pm z^{-N}\check{y}^*\left(\frac{-1}{z^*}\right) & \text{(5) "reversed" CQF in "z-domain"} \\
& \iff \check{x}(\omega) & = & \pm e^{-i\omega N}\check{y}^*(\omega+\pi) & \text{(6) "reversed" CQF in "frequency"}
\end{array}$$

$\forall N\in\mathbb{Z}$

✎PROOF:

[6] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

[7] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{y}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-transform} && \text{(Definition P.4 page 330)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm \sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies && n = N-m\\
&= \pm(-1)^N z^{-N} \sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N} \sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*\\
&= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by definition of } z\text{-transform} && \text{(Definition P.4 page 330)}
\end{aligned}$$

2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\check{y}(z) &= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by (2)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-transform} && \text{(Definition P.4 page 330)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x^*_m \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-transform} && \text{(Definition P.4 page 330)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x^*_m z^{-(N-m)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^n x^*_{N-n} z^{-n} && \text{where } n = N-m \implies && m \triangleq N-n\\
&\implies x_n = \pm(-1)^n x^*_{N-n}
\end{aligned}$$

3. Proof that (1) $\implies$ (3):

$$\begin{aligned}
\check{y}(\omega) &\triangleq \check{x}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition O.1 page 319)}\\
&= \left[\pm(-1)^N z^{-N} \check{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)}\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i\pi} e^{i\omega}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i(\omega+\pi)}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition O.1 page 319)}
\end{aligned}$$

4. Proof that (1) $\implies$ (6):

$$\begin{aligned}
\check{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition O.1 page 319)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x^*_{N-n}}_{CQF} e^{-i\omega n} && \text{by (1)}\\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x^*_m e^{-i\omega(N-m)} && \text{where } m \triangleq N-n \implies && n = N-m\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} (-1)^m x^*_m e^{i\omega m}
\end{aligned}$$

$$
\begin{aligned}
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} e^{i\pi m} x_m^* e^{i\omega m} \\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} x_m^* e^{i(\omega+\pi)m} \\
&= \pm(-1)^N e^{-i\omega N} \left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^* \\
&= \pm(-1)^N e^{-i\omega N} \breve{x}^*(\omega+\pi) && \text{by definition of } \textit{DTFT} \quad \text{(Definition O.1 page 319)}
\end{aligned}
$$

5. Proof that (1) $\impliedby$ (3):

$$
\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{y}(\omega)e^{i\omega n}\,d\omega && \text{by } \textit{inverse DTFT} \quad \text{(Theorem O.3 page 325)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\breve{x}^*(\omega+\pi)}_{\text{right hypothesis}} e^{i\omega n}\,d\omega && \text{by right hypothesis} \\
&= \pm(-1)^N \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{x}^*(\omega+\pi)e^{i\omega(n-N)}\,d\omega && \text{by right hypothesis} \\
&= \pm(-1)^N \frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{i(v-\pi)(n-N)}\,dv && \text{where } v \triangleq \omega+\pi \implies \omega = v-\pi \\
&= \pm(-1)^N e^{-i\pi(n-N)} \frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{iv(n-N)}\,dv \\
&= \pm(-1)^N \underbrace{(-1)^N}_{e^{i\pi N}} \underbrace{(-1)^n}_{e^{-i\pi n}} \left[\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}(v)e^{iv(N-n)}\,dv\right]^* \\
&= \pm(-1)^n x_{N-n}^* && \text{by } \textit{inverse DTFT} \quad \text{(Theorem O.3 page 325)}
\end{aligned}
$$

6. Proof that (1) $\iff$ (4):

$$
\begin{aligned}
y_n = \pm(-1)^n x_{N-n}^* &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x_{N-n}^* \\
&\iff \pm(-1)^n y_n = x_{N-n}^* \\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x_{N-n}^*\right)^* \\
&\iff \pm(-1)^n y_n^* = x_{N-n} \\
&\iff x_{N-n} = \pm(-1)^n y_n^* \\
&\iff x_m = \pm(-1)^{N-m} y_{N-m}^* && \text{where } m \triangleq N-n \implies n = N-m \\
&\iff x_m = \pm(-1)^{N-m} y_{N-m}^* \\
&\iff x_m = \pm(-1)^N(-1)^m y_{N-m}^* \\
&\iff x_n = \pm(-1)^N(-1)^n y_{N-n}^* && \text{by change of free variables}
\end{aligned}
$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

⇒

Theorem P.6. [8] *Let $\breve{y}(\omega)$ and $\breve{x}(\omega)$ be the* DTFT*s* (Definition O.1 page 319) *of the sequences $(y_n)_{n\in\mathbb{Z}}$ and $(x_n)_{n\in\mathbb{Z}}$, respectively, in $\ell^2_{\mathbb{R}}$* (Definition P.2 page 329).

THM

Let $y_n = \pm(-1)^n x_{N-n}^*$ (CQF CONDITION, *Definition P.9 page 337*). *Then*

$$
\left\{
\begin{array}{llcll}
\text{(A)} & \left[\frac{d}{d\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} = 0 & \iff & \left[\frac{d}{d\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} = 0 & \text{(B)} \\
& & \iff & \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 & \text{(C)} \\
& & \iff & \sum_{k\in\mathbb{Z}} k^n y_k = 0 & \text{(D)}
\end{array}
\right\} \quad \forall n\in\mathbb{W}
$$

[8] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

✎PROOF:

1. Proof that (A) $\implies$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{y}(\omega)\bigg|_{\omega=0} && \text{by (A)}\\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi)\bigg|_{\omega=0} && \text{by \textit{CQF theorem}} \quad \text{(Theorem P.5 page 337)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{\ell} \left[e^{-i\omega N}\right] \cdot \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\bigg|_{\omega=0} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma ?? page ??)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} e^{-i\omega N} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\bigg|_{\omega=0}\\
&= (\pm)(-1)^N e^{-i0N} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\bigg|_{\omega=0}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{x}^{(0)}(\pi) &= 0\\
&\implies & \breve{x}^{(1)}(\pi) &= 0\\
&\implies & \breve{x}^{(2)}(\pi) &= 0\\
&\implies & \breve{x}^{(3)}(\pi) &= 0\\
&\implies & \breve{x}^{(4)}(\pi) &= 0\\
&\vdots & &\vdots\\
&\implies & \breve{x}^{(n)}(\pi) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

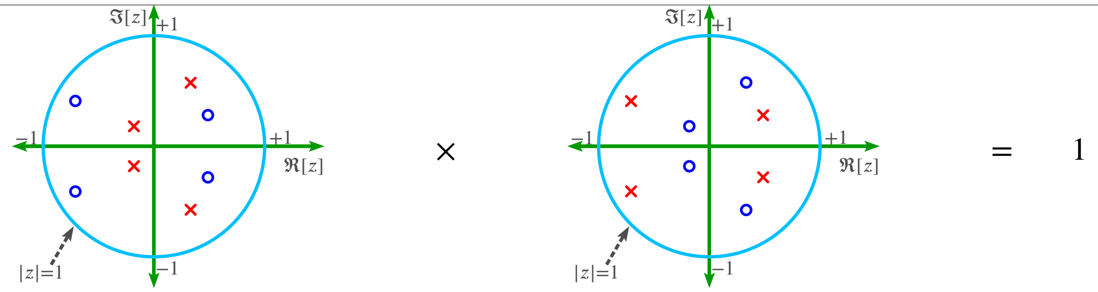
2. Proof that (A) $\impliedby$ (B):

$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{x}(\omega)\bigg|_{\omega=\pi} && \text{by (B)}\\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n (\pm)e^{-i\omega N}\breve{y}^*(\omega+\pi)\bigg|_{\omega=\pi} && \text{by \textit{CQF theorem}} \quad \text{(Theorem P.5 page 337)}\\
&= (\pm) \sum_{\ell=0}^{n} \binom{n}{\ell} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{\ell} \left[e^{-i\omega N}\right] \cdot \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma ?? page ??)}\\
&= (\pm) \sum_{\ell=0}^{n} \binom{n}{\ell} (-iN)^{\ell} e^{-i\omega N} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}\\
&= (\pm)e^{-i\pi N} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\bigg|_{\omega=\pi}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{y}^{(0)}(0) &= 0\\
&\implies & \breve{y}^{(1)}(0) &= 0\\
&\implies & \breve{y}^{(2)}(0) &= 0\\
&\implies & \breve{y}^{(3)}(0) &= 0\\
&\implies & \breve{y}^{(4)}(0) &= 0\\
&\vdots & &\vdots\\
&\implies & \breve{y}^{(n)}(0) &= 0\\
&\implies & \breve{y}^{(n)}(0) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

3. Proof that (B) $\iff$ (C): by Theorem O.5 page 327

4. Proof that (A) $\iff$ (D): by Theorem O.5 page 327



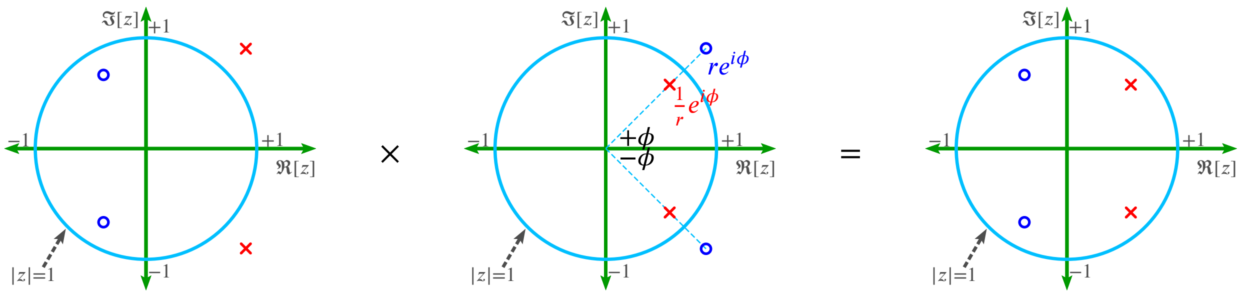
$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

5. Proof that (CQF) $\not\Leftarrow$ (A): Here is a counterexample: $\breve{y}(\omega) = 0$.

✏

P.9 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.



unstable all-pass stable!

$$\begin{aligned}
\left|A(z)\right|_{z=e^{i\omega}} &= \frac{1}{r}\left|\frac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|e^{i\phi}\left(\frac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|-z\left(\frac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\frac{1}{e^{-iv}}\left(\frac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right| \\
&= 1
\end{aligned}$$

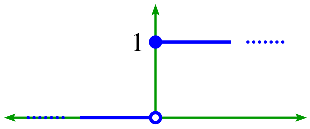
$$\begin{aligned}
&= \left|\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
&= \left|z\left(\frac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
&= \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\frac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
&= \left|\frac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right|
\end{aligned}$$

APPENDIX Q

B-SPLINES

Q.1 Definitions

Definition Q.1. *Let X be a set.*



DEF *The **step function** $\sigma \in \mathbb{R}^{\mathbb{R}}$ is defined as*

$$\sigma(x) \triangleq \mathbb{1}_{[0:\infty)}(x) \qquad \forall x \in \mathbb{R}.$$

Lemma Q.1. *Let $\sigma(x)$ be the* STEP FUNCTION *(Definition Q.1 page 343).*

LEM $$\{g(x) > 0\} \implies \left\{\sigma\left[g(x)f(x)\right] = \sigma[f(x)]\right\} \qquad \forall f, g \in \mathbb{R}^{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\sigma\left[g(x)f(x)\right] &\triangleq \mathbb{1}_{[0:\infty)}\left[g(x)f(x)\right] && \text{by definition of } \sigma(x) && \text{(Definition Q.1 page 343)}\\
&\triangleq \begin{cases} 1 & \text{for } g(x)f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition ?? page ??)}\\
&= \begin{cases} 1 & \text{for } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} && \text{by } g(x) > 0 \text{ hypothesis}\\
&\triangleq \mathbb{1}_{[0:\infty)}[f(x)] && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition ?? page ??)}\\
&\triangleq \sigma[f(x)] && \text{by definition of } \sigma(x) && \text{(Definition Q.1 page 343)}
\end{aligned}$$

✏

Definition Q.2. [1] *Let $\mathbb{1}$ be the* SET INDICATOR *function (Definition ?? page ??). Let $f(x) \star g(x)$ represent the* CONVOLUTION *operation (Definition N.3 page 312).*

DEF *The n**th order cardinal B-spline** $N_n(x)$ for $n \in \mathbb{W}$ is defined as*

$$N_n(x) \triangleq \begin{cases} \mathbb{1}_{[0:1)}(x) & \text{for } n = 0 \\ N_{n-1}(x) \star N_0(x) & \text{for } n \in \mathbb{W}\backslash 0 \end{cases} \qquad \forall x \in \mathbb{R}$$

Lemma Q.2. [2]

[1] 🕮 Chui (1992) page 85 ⟨(4.2.1)⟩, 🕮 Christensen (2008) page 140, 🕮 Chui (1988) page 1

[2] 🕮 Christensen (2008) page 140, 🕮 Chui (1992) page 85 ⟨(4.2.1)⟩, 🕮 Chui (1988) page 1, 🕮 Prasad and Iyengar (1997) page 145

LEM
$$\mathsf{N}_n(x) = \int_{\tau=0}^{\tau=1} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau \qquad \forall n \in \{1,2,3,\ldots\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{N}_n(x) &\triangleq \mathsf{N}_{n-1}(x) \star \mathsf{N}_0(x) && \text{by definition of } \mathsf{N}_n(x) && \text{(Definition Q.2 page 343)}\\
&\triangleq \int_{\mathbb{R}} \mathsf{N}_{n-1}(x-\tau)\mathsf{N}_0(\tau)\,\mathrm{d}\tau && \text{by definition of convolution operation } \star && \text{(Definition N.3 page 312)}\\
&\triangleq \int_{\mathbb{R}} \mathsf{N}_{n-1}(x-\tau)\mathbb{1}_{[0:1)}(\tau)\,\mathrm{d}\tau && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition Q.2 page 343)}\\
&= \int_{[0:1)} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition ?? page ??)}\\
&= \int_{[0:1]} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau\\
&\triangleq \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau
\end{aligned}$$

⇰

Lemma Q.3. *Let* $\mathsf{f}(x)$ *be a* FUNCTION *in* $\mathbb{R}^{\mathbb{R}}$. *Let* $\mathsf{F}(x)$ *be the* ANTI-DERIVATIVE *of* $\mathsf{f}(x)$. *Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition Q.1 page 343)*.

LEM
$$\begin{aligned}
&\int_{y=a}^{y=b} \mathsf{f}(x-y)\sigma(x-y)\,\mathrm{d}y\\
&= \left\{\begin{array}{ll} -\int_{y=x-a}^{y=x-b} \mathsf{f}(y)\,\mathrm{d}y & \text{for } x \ge b\\ -\int_{y=x-a}^{y=0} \mathsf{f}(y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} = \left\{\begin{array}{ll} \mathsf{F}(x-a)-\mathsf{F}(x-b) & \text{for } x \ge b\\ \mathsf{F}(x-a)-\mathsf{F}(0) & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\}\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(0)-\mathsf{F}(x-b)]\sigma(x-b)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\int_{y=a}^{y=b} \mathsf{f}(x-y)\sigma(x-y)\,\mathrm{d}y &= \left\{\begin{array}{ll} \int_{y=a}^{y=b} \mathsf{f}(x-y)\,\mathrm{d}y & \text{for } x \ge b\\ \int_{y=a}^{y=x} \mathsf{f}(x-y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by definition of } \sigma \text{ (Definition Q.1 page 343)}\\
&= \left\{\begin{array}{ll} -\int_{u=x-a}^{u=x-b} \mathsf{f}(u)\,\mathrm{d}u & \text{for } x \ge b\\ -\int_{u=x-a}^{u=0} \mathsf{f}(u)\,\mathrm{d}u & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{where } u \triangleq x-y \implies y = x-u\\
&= \left\{\begin{array}{ll} -\int_{y=x-a}^{y=x-b} \mathsf{f}(y)\,\mathrm{d}y & \text{for } x \ge b\\ -\int_{y=x-a}^{y=0} \mathsf{f}(y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by change of dummy variable } (u \to y)\\
&= \left\{\begin{array}{ll} \mathsf{F}(x-a)-\mathsf{F}(x-b) & \text{for } x \ge b\\ \mathsf{F}(x-a)-\mathsf{F}(0) & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by \textit{Fundamental Theorem of Calculus}}\\
&= [\mathsf{F}(x-a)-\mathsf{F}(x-b)]\sigma(x-b) + [\mathsf{F}(x-a)-\mathsf{F}(0)][\sigma(x-a)-\sigma(x-b)]\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(x-a)-\mathsf{F}(x-b)-\mathsf{F}(x-a)+\mathsf{F}(0)]\sigma(x-b)\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(0)-\mathsf{F}(x-b)]\sigma(x-b)
\end{aligned}$$

⇰

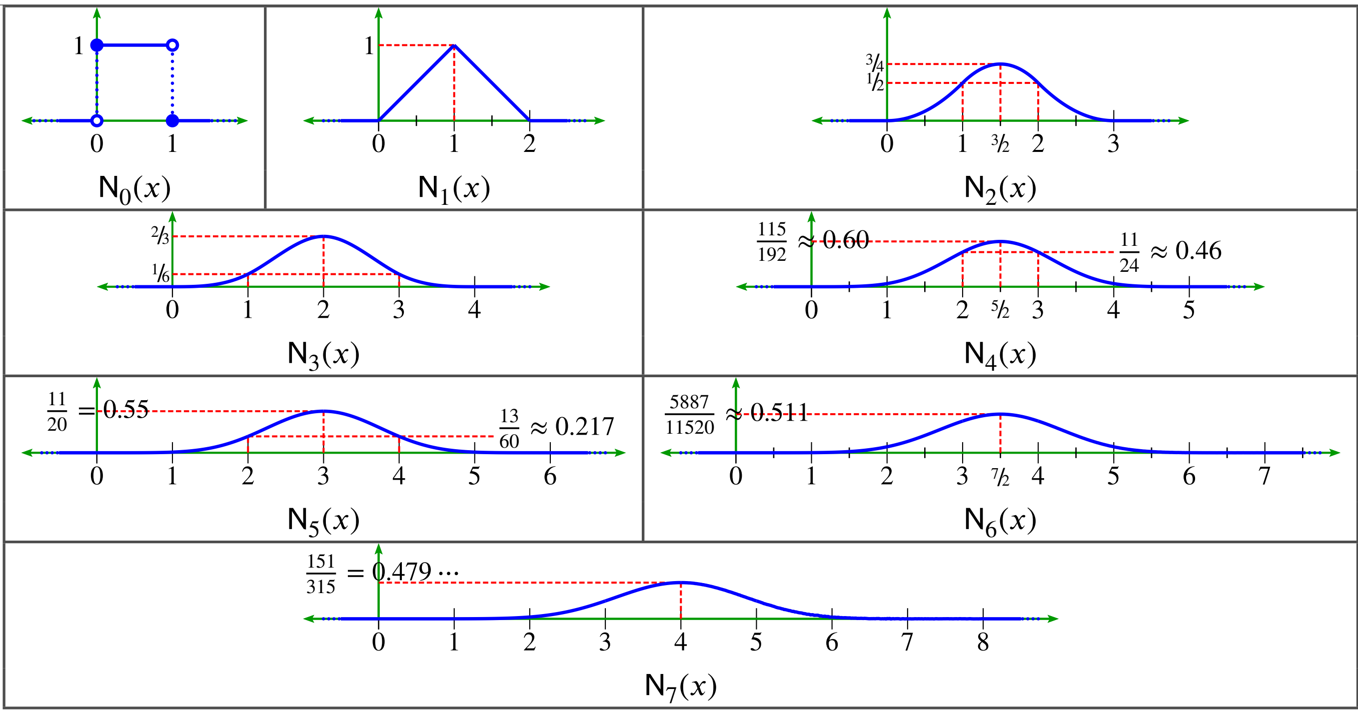


Figure Q.1: some low order B-splines (Example Q.1 page 345)

Lemma Q.4. *Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition Q.1 page 343).*

LEM

$$\int_{\tau=0}^{\tau=1} (x-\tau-k)^n \sigma(x-\tau-k)\,\mathrm{d}\tau = \frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k) - (x-k-1)^{n+1}\sigma(x-k-1)\right]$$

✎PROOF:

$$\begin{aligned}
&\int_{\tau=0}^{\tau=1} (x-\tau-k)^n \sigma(x-\tau-k)\,\mathrm{d}\tau \\
&= \int_{y=k}^{y=k+1} (x-y)^n \sigma(x-y)\,\mathrm{d}y && \text{where } y \triangleq \tau + k \implies \tau = y-k \\
&= [\mathsf{F}(x-k) - \mathsf{F}(0)]\sigma(x-k) + [\mathsf{F}(0) - \mathsf{F}(x-k-1)]\sigma(x-k-1) && \text{by Lemma Q.3 (page 344), where } \mathsf{f}(x) \triangleq x^n \\
&= \frac{\left[(x-k)^{n+1} - 0\right]\sigma(x-k) + \left[0 - (x-k-1)^{n+1}\right]\sigma(x-k-1)}{n+1} && \text{because } \mathsf{F}(x) \triangleq \int \mathsf{f}(x)\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c \\
&= \frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k) - (x-k-1)^{n+1}\sigma(x-k-1)\right]
\end{aligned}$$

⇒

Example Q.1. [3] Let $\sigma(x)$ be the *step function* (Definition Q.1 page 343). Let $\binom{n}{k}$ be the *binomial coefficient* (Definition ?? page ??). The 0*th order B-spline* (Definition Q.2 page 343) $\mathsf{N}_0(x)$ can be expressed as follows:

EX

$$\mathsf{N}_0(x) = \left\{\begin{array}{ll} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise} \end{array}\right\} = \left\{\sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-k)^0\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_0(x)$ is illustrated in Figure Q.1 (page 345).

[3] Schumaker (2007) page 136 ⟨Table 1⟩

✎PROOF:

$$\begin{aligned}
\mathsf{N}_0(x) &= \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition Q.2 page 343)}\\
&= \sigma(x) - \sigma(x-1) && \text{by definition of } \sigma(x) && \text{(Definition Q.1 page 343)}\\
&= \left[\binom{1}{0}\sigma(x) - \binom{1}{1}\sigma(x-1)\right] && \text{by definition of \textit{binomial coefficient} } \binom{n}{k} && \text{(Definition ?? page ??)}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-k)^0\sigma(x-k) && \text{by definition of } \textstyle\sum \text{ operator}
\end{aligned}$$

⇨

Example Q.2. [4] Let $\sigma(x)$ be the *step function*. Let $\binom{n}{k}$ be the *binomial coefficient*. The $1st$ *order B-spline* $\mathsf{N}_1(x)$ can be expressed as follows:

EX
$$\mathsf{N}_1(x) = \left\{\begin{array}{rl} x & \text{for } x\in[0:1] \\ -x+2 & \text{for } x\in[1:2] \\ 0 & \text{otherwise}\end{array}\right\} = \left\{\sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_1(x)$ is illustrated in Figure Q.1 (page 345).

✎PROOF:

$$\begin{aligned}
\mathsf{N}_1(x) &= \int_{\tau=0}^{\tau=1}\mathsf{N}_0(x-\tau)\,\mathrm{d}\tau && \text{by Lemma Q.2 page 343}\\
&= \int_{\tau=0}^{\tau=1}\sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-\tau-k)^0\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by Example Q.1 page 345}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}\int_{\tau=0}^{\tau=1}(x-\tau-k)^0\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by \textit{linearity} of } \int \mathrm{d}\tau \text{ operator}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}\frac{1}{0+1}\left[(x-k)^{0+1}\sigma(x-k)-(x-k-1)^{0+1}\sigma(x-k-1)\right] && \text{by Lemma Q.4 page 345}\\
&= \begin{pmatrix} 1\{(x-0)\sigma(x-0)-(x-1)\sigma(x-1)\} \\ -1\{(x-1)\sigma(x-1)-(x-2)\sigma(x-2)\}\end{pmatrix}\\
&= x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k) && \text{by def. of } \binom{n}{k} \text{ (Definition ?? page ??)}\\
&= \left\{\begin{array}{ll} x & \text{for } x\in[0:1) \\ -x+2 & \text{for } x\in[1:2] \\ 0 & \text{otherwise}\end{array}\right\} && \text{by def. of } \sigma(x) \text{ (Definition Q.1 page 343)}
\end{aligned}$$

⇨

Example Q.3. [5] Let $\sigma(x)$ be the *step function*. Let $\binom{n}{k}$ be the *binomial coefficient*. The $2nd$ *order B-spline* $\mathsf{N}_2(x)$ can be expressed as follows:

EX
$$\mathsf{N}_2(x) = \frac{1}{2}\left\{\begin{array}{rl} x^2 & \text{for } x\in[0:1) \\ -2x^2+6x-3 & \text{for } x\in[1:2] \\ x^2-6x+9 & \text{for } x\in[2:3] \\ 0 & \text{otherwise}\end{array}\right\} = \left\{\frac{1}{2}\sum_{k=0}^{3}(-1)^k\binom{3}{k}(x-k)^2\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_2(x)$ is illustrated in Figure Q.1 (page 345).

[4] Christensen (2008) page 148 ⟨Exercise 6.2⟩, Christensen (2010) page 212 ⟨Exercise 10.2⟩, Heil (2011) pages 142–143 ⟨Definition 4.22 (The Schauder System)⟩, Schumaker (2007) page 136 ⟨Table 1⟩, Stoer and Bulirsch (2002) page 124

[5] Christensen (2008) page 148 ⟨Exercise 6.2⟩, Christensen (2010) page 212 ⟨Exercise 10.2⟩, Schumaker (2007) page 136 ⟨Table 1⟩, Stoer and Bulirsch (2002) page 124

✎PROOF:

$$
\begin{aligned}
\mathsf{N}_2(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_1(x-\tau)\,\mathrm{d}\tau && \text{by Lemma Q.2 page 343}\\
&= \int_{\tau=0}^{\tau=1} \sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-\tau-k)\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by Example Q.2 page 346}\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\int_{\tau=0}^{\tau=1}(x-\tau-k)\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by \textit{linearity} of } \int \mathrm{d}\tau \text{ operator}\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\frac{1}{1+1}\left[(x-k)^{1+1}\sigma(x-k)-(x-k-1)^{1+1}\sigma(x-k-1)\right] && \text{by Lemma Q.4 page 345}\\
&= \frac{1}{2}\left(\begin{array}{rl} 1 & \left\{(x-0)^2\sigma(x-0)-(x-1)^2\sigma(x-1)\right\}\\ -2 & \left\{(x-1)^2\sigma(x-1)-(x-2)^2\sigma(x-2)\right\}\\ +1 & \left\{(x-2)^2\sigma(x-2)-(x-3)^2\sigma(x-3)\right\}\end{array}\right)\\
&= \frac{1}{2}\left[x^2\sigma(x)-3(x-1)^2\sigma(x-1)+3(x-2)^2\sigma(x-2)-(x-3)^2\sigma(x-3)\right]\\
&= \frac{1}{2}\sum_{k=0}^{3}(-1)^k\binom{3}{k}(x-k)^2\sigma(x-k) && \text{by def. of } \binom{n}{k} \text{ (Definition ?? page ??)}\\
&= \frac{1}{2}\left\{\begin{array}{rl} x^2 & \text{for } x\in[0:1)\\ -2x^2+6x-3 & \text{for } x\in[1:2]\\ x^2-6x+9 & \text{for } x\in[2:3]\\ 0 & \text{otherwise}\end{array}\right\} && \text{by def. of } \sigma(x) \text{ (Definition Q.1 page 343)}
\end{aligned}
$$

The final steps of this proof can be calculated "by hand" or by using the free and open source software package *Maxima* along with the script file listed in Section **??** (page **??**). ✏

Q.2 Algebraic properties

Theorem Q.1 (next) presents a closed form expression for an *nth order B-spline* $\mathsf{N}_n(x)$ based on the definition of $\mathsf{N}_n(x)$ given in Definition Q.2 (page 343). Alternatively, Theorem Q.1 could serve as the definition and Definition Q.2 as a property.

Theorem Q.1. [6] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition Q.2 page 343).*
Let $\sigma(x)$ *be the* STEP FUNCTION *(Definition Q.1 page 343).*

THM

$$\mathsf{N}_n(x) = \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n\sigma(x-k) \qquad \forall n\in\{0,1,2,\ldots\}=\mathbb{W}$$

✎PROOF: Proof follows by induction:

1. base case (choose one):
   - Proof for $n=0$ case: by Example Q.1 (page 345).
   - Proof for $n=1$ case: by Example Q.2 (page 346).
   - Proof for $n=2$ case: by Example Q.3 (page 346).

[6] Christensen (2008) page 142 ⟨Theorem 6.1.3⟩, Chui (1992) page 84 ⟨(4.1.12)⟩

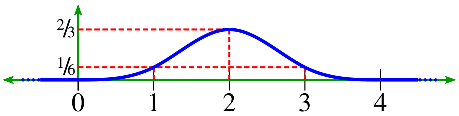
2. inductive step—proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\mathsf{N}_{n+1}(x)\\
&= \int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma Q.2 page 343}\\
&= \int_0^1 \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-\tau-k)^n\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by induction hypothesis}\\
&= \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\int_0^1(x-\tau-k)^n\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by \textit{linearity} of } \int \mathrm{d}\tau \text{ operator}\\
&= \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k)-(x-k-1)^{n+1}\sigma(x-k-1)\right] && \text{by Lemma Q.4 page 345}\\
&= \frac{1}{(n+1)!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\left[(x-k)^{n+1}\sigma(x-k)-(x-k-1)^{n+1}\sigma(x-k-1)\right]\\
&= \frac{1}{(n+1)!}\left[\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)-\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k-1)^{n+1}\sigma(x-k-1)\right]\\
&= \frac{1}{(n+1)!}\left[\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)-\sum_{m=1}^{m=n+2}(-1)^{m-1}\binom{n+1}{m-1}(x-m)^{n+1}\sigma(x-m)\right]\\
&\qquad \text{where } m \triangleq k+1 \implies k=m-1\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l}\displaystyle\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)\\ \displaystyle -\sum_{m=1}^{m=n+2}(-1)^{m-1}\left[\binom{n+2}{m}-\binom{n+1}{m}\right](x-m)^{n+1}\sigma(x-m)\end{array}\right) && \text{by \textit{Pascal's identity / Stifel formula}} \\
& && \text{(Theorem ?? page ??)}\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l}\displaystyle\sum_{m=1}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m)\\ \displaystyle -\sum_{m=1}^{m=n+2}(-1)^m\binom{n+1}{m}(x-m)^{n+1}\sigma(x-m)\\ \displaystyle +\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)\end{array}\right) && \text{note } (-1)^{m-1}=-(-1)^m\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l|ll}\displaystyle\sum_{m=0}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m) & \text{(A)} & \text{desired } n+1 \text{ case}\\ \displaystyle -\quad(-1)^0\binom{n+2}{0}(x-0)^{n+1}\sigma(x-0) & \text{(B)} & \text{cancelled by (F)}\\ \displaystyle -\sum_{m=1}^{m=n+1}(-1)^m\binom{n+1}{m}(x-m)^{n+1}\sigma(x-m) & \text{(C)} & \text{cancelled by (E)}\\ \displaystyle -\quad(-1)^{n+2}\binom{n+1}{n+2}(x-n-2)^{n+1}\sigma(x-n-2) & \text{(D)} & \binom{n+1}{n+2}=0 \text{ by Proposition ?? page ??}\\ \displaystyle +\sum_{k=1}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k) & \text{(E)} & \text{cancelled by (C)}\\ \displaystyle +\quad(-1)^0\binom{n+1}{0}(x-0)^{n+1}\sigma(x-0) & \text{(F)} & \binom{n+2}{0}=\binom{n+1}{0}=1, \text{ so (F) is cancelled by (B)}\end{array}\right)\\
&= \frac{1}{(n+1)!}\sum_{m=0}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m) && (n+1 \text{ case})
\end{aligned}$$

Example Q.4. [7] Let $\mathrm{N}_3(x)$ be the 3*rd order B-spline* (Definition Q.2 page 343).[8]

E X

$$\mathrm{N}_3(x) = \frac{1}{6}\begin{cases} x^3 & \text{for } 0 \le x \le 1 \\ -3x^3 + 12x^2 - 12x + 4 & \text{for } 1 \le x \le 2 \\ 3x^3 - 24x^2 + 60x - 44 & \text{for } 2 \le x \le 3 \\ -x^3 + 12x^2 - 48x + 64 & \text{for } 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$



✎PROOF: This expression can be calculated "by hand" using Theorem Q.1 (page 347) or by using the free and open source software package *Maxima* along with the script file listed in Section **??** (page **??**). ⇒

Example Q.5. Let $\mathrm{N}_4(x)$ be the 4*th order B-spline* (Definition Q.2 page 343).

E X

$$\mathrm{N}_4(x) = \frac{1}{24}\begin{cases} x^4 & \text{for } 0 \le x <= 1 \\ -4x^4 + 20x^3 - 30x^2 + 20x - 5 & \text{for } 1 \le x <= 2 \\ 6x^4 - 60x^3 + 210x^2 - 300x + 155 & \text{for } 2 \le x <= 3 \\ -4x^4 + 60x^3 - 330x^2 + 780x - 655 & \text{for } 3 \le x <= 4 \\ x^4 - 20x^3 + 150x^2 - 500x + 625 & \text{for } 4 \le x <= 5 \\ 0 & \text{otherwise} \end{cases}$$

✎PROOF: This expression can be calculated "by hand" using Theorem Q.1 (page 347) or by using the free and open source software package *Maxima* along with the script file listed in Section **??** (page **??**). ⇒

Example Q.6. Let $\mathrm{N}_5(x)$ be the 5*th order B-spline* (Definition Q.2 page 343).

E X

$$\mathrm{N}_5(x) = \frac{1}{120}\begin{cases} x^5 & \text{for } 0 \le x <= 1 \\ -5x^5 + 30x^4 - 60x^3 + 60x^2 - 30x + 6 & \text{for } 1 \le x <= 2 \\ 10x^5 - 120x^4 + 540x^3 - 1140x^2 + 1170x - 474 & \text{for } 2 \le x <= 3 \\ -10x^5 + 180x^4 - 1260x^3 + 4260x^2 - 6930x + 4386 & \text{for } 3 \le x <= 4 \\ 5x^5 - 120x^4 + 1140x^3 - 5340x^2 + 12270x - 10974 & \text{for } 4 \le x <= 5 \\ -x^5 + 30x^4 - 360x^3 + 2160x^2 - 6480x + 7776 & \text{for } 5 \le x <= 6 \\ 0 & \text{otherwise} \end{cases}$$

The 5*th order B-spline* $\mathrm{N}_5(x)$ is illustrated in Figure Q.1 (page 345).

✎PROOF: This expression can be calculated "by hand" using Theorem Q.1 (page 347) or by using the free and open source software package *Maxima* along with the script file listed in Section **??** (page **??**). ⇒

Example Q.7. Let $\mathrm{N}_6(x)$ be the 6*th order B-spline* (Definition Q.2 page 343).

E X

$$\mathrm{N}_6(x) = \frac{1}{720}\begin{cases} x^6 & \text{for } 0 \le x \le 1 \\ -6x^6 + 42x^5 - 105x^4 + 140x^3 - 105x^2 + 42x - 7 & \text{for } 1 \le x \le 2 \\ 15x^6 - 210x^5 + 1155x^4 - 3220x^3 + 4935x^2 - 3990x + 1337 & \text{for } 2 \le x \le 3 \\ -20x^6 + 420x^5 - 3570x^4 + 15680x^3 - 37590x^2 + 47040x - 24178 & \text{for } 3 \le x \le 4 \\ 15x^6 - 420x^5 + 4830x^4 - 29120x^3 + 96810x^2 - 168000x + 119182 & \text{for } 4 \le x \le 5 \\ -6x^6 + 210x^5 - 3045x^4 + 23380x^3 - 100065x^2 + 225750x - 208943 & \text{for } 5 \le x \le 6 \\ x^6 - 42x^5 + 735x^4 - 6860x^3 + 36015x^2 - 100842x + 117649 & \text{for } 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

The 6*th order B-spline* $\mathrm{N}_6(x)$ is illustrated in Figure Q.1 (page 345).

✎PROOF: This expression can be calculated "by hand" using Theorem Q.1 (page 347) or by using the free and open source software package *Maxima* along with the script file listed in Section **??** (page **??**). ⇒

[7] Schumaker (2007) page 136 ⟨Table 1⟩, Shizgal (2015) page 92 ⟨(2.199)⟩, Szabó and Horváth (2004) page 146 ⟨(4)⟩, Wei and Billings (2006) page 578 ⟨Table 1⟩, Maleknejad et al. (2013) ⟨(9)⟩

[8] For help with plotting *B-splines*, see APPENDIX R (page 375).

Example Q.8. Let $\mathsf{N}_7(x)$ be the 7*th order B-spline* (Definition Q.2 page 343).

EX

$$7!\mathsf{N}_7(x) = 5040\mathsf{N}_7(x) =$$

$$\begin{cases}
x^7 & \text{for } 0 \le x \le 1\\
-7x^7 + 56x^6 - 168x^5 + 280x^4 - 280x^3 + 168x^2 - 56x + 8 & \text{for } 1 \le x \le 2\\
21x^7 - 336x^6 + 2184x^5 - 7560x^4 + 15400x^3 - 18648x^2 + 12488x - 3576 & \text{for } 2 \le x \le 3\\
-35x^7 + 840x^6 - 8400x^5 + 45360x^4 - 143360x^3 + 267120x^2 - 273280x + 118896 & \text{for } 3 \le x \le 4\\
35x^7 - 1120x^6 + 15120x^5 - 111440x^4 + 483840x^3 - 1238160x^2 + 1733760x - 1027984 & \text{for } 4 \le x \le 5\\
-21x^7 + 840x^6 - 14280x^5 + 133560x^4 - 741160x^3 + 2436840x^2 - 4391240x + 3347016 & \text{for } 5 \le x \le 6\\
7x^7 - 336x^6 + 6888x^5 - 78120x^4 + 528920x^3 - 2135448x^2 + 4753336x - 4491192 & \text{for } 6 \le x \le 7\\
-x^7 + 56x^6 - 1344x^5 + 17920x^4 - 143360x^3 + 688128x^2 - 1835008x + 2097152 & \text{for } 7 \le x \le 8\\
0 & \text{otherwise}
\end{cases}$$

The 7*th order B-spline* $\mathsf{N}_7(x)$ is illustrated in Figure Q.1 (page 345).

✎PROOF: This expression can be calculated "by hand" using Theorem Q.1 (page 347) or by using the free and open source software package *Maxima* along with the script file listed in Section **??** (page **??**). ✏

Example Q.9. [9] The $(n+1)^2$ coefficients of the order $n, n-1, \ldots, 0$ monomials of each *B-spline* $\mathsf{N}_n(x)$ multiplied by $n!$ induce an *integer sequence*

$\mathbb{x} \triangleq (\!(1, 1, 0, -1, 2, 1, 0, 0, -2, 6, -3, 1, -6, 9, 1, 0, 0, 0, -3, 12, -12, 4, 3, -24, 60, -44, -1, 12, -48, 64, \ldots)\!)$

as more fully listed in Table Q.1 (page 374). In this sequence $\mathbb{x} \triangleq (\!(x_0, x_1, x_2, \ldots)\!)$, the coefficients for the *order* n *B-spline* $\mathsf{N}_n(x)$ begin at the sequence index value

$$p \triangleq \sum_{k=0}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \text{and end at index value } p + (n+1)^2 - 1.$$

For example, the coefficients for $\mathsf{N}_3(x)$ begin at index value $p \triangleq 0 + 1 + 4 + 9 = 14$ and end at index value $p + 4^2 - 1 = 29$. Using these coefficients gives the following expression for $\mathsf{N}_3(x)$:

$$\mathsf{N}_3(x) = \left[\begin{array}{rrrr|l} 1 & 0 & 0 & 0 & \text{for } 0 \le x < 1\\ -3 & 12 & -12 & 4 & \text{for } 1 \le x < 2\\ 3 & -24 & 60 & -44 & \text{for } 2 \le x < 3\\ -1 & 12 & -48 & 64 & \text{for } 3 \le x < 4 \end{array}\right] \begin{bmatrix} x^3\\ x^2\\ x\\ 1\end{bmatrix} = \begin{cases} x^3 & \text{for } 0 \le x < 1\\ -3x^3 + 12x^2 - 12x + 4 & \text{for } 1 \le x < 2\\ 3x^3 - 24x^2 + 60x - 44 & \text{for } 2 \le x < 3\\ -x^3 + 12x^2 - 48x + 64 & \text{for } 3 \le x < 4\\ 0 & \text{otherwise}\end{cases}$$

...which agrees with the result presented in Example Q.4 (page 349).

✎PROOF:

1. The coefficients for the sequence $\mathbb{x}$ may be computed with assistance from *Maxima* together with the script file listed in Section **??** (page **??**).

2. Proof that $\sum_{k=0}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$: The summation is a *power sum*. The relation may be proved using *induction*.[10]

   (a) Base case: n=0 case ...

$$\begin{aligned}\sum_{k=0}^{k=0} k^2 &= 0\\ &= \frac{0(0+1)(2\cdot 0+1)}{6}\\ &= \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=0}\end{aligned}$$

[9] Greenhoe (2017b)

[10] Greenhoe (2017a), pages 186–187 ⟨Proposition 11.2 (Power Sums)⟩

(b) Base case: n=1 case ...

$$\begin{aligned}
\sum_{k=0}^{k=1} k^2 &= 0+1 \\
&= \frac{1(1+1)(2\cdot 1+1)}{6} \\
&= \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1}
\end{aligned}$$

(c) inductive step—proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
\sum_{k=0}^{n+1} k^2 &= \left(\sum_{k=0}^{n} k^2\right) + \left(n+1\right)^2 \\
&= \left(\frac{n(n+1)(2n+1)}{6}\right) + \left(n+1\right)^2 && \text{by } n \text{ case hypothesis} \\
&= \left(n+1\right)\left(\frac{n(2n+1)+6(n+1)}{6}\right) \\
&= \left(n+1\right)\left(\frac{2n^2+7n+6}{6}\right) \\
&= \left(n+1\right)\left(\frac{(n+2)(2n+3)}{6}\right) \\
&= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}
\end{aligned}$$

✏

Theorem Q.2. [11]

THM $\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_n(x) = \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1) \qquad \forall n\in\mathbb{N},\ \forall x\in\mathbb{R}$

✎PROOF:

1. Proof using Lemma Q.2 (page 343) and the *Fundamental Theorem of Calculus*:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{dx}}\mathsf{N}_n(x) &= \frac{\mathsf{d}}{\mathsf{dx}}\int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathsf{d}\tau && \text{by Lemma Q.2 page 343} \\
&= \frac{\mathsf{d}}{\mathsf{dx}}\int_{x-u=0}^{x-u=1} \mathsf{N}_{n-1}(u)(-1)\,\mathsf{d}u && \text{where } u \triangleq x-\tau \implies \tau = x-u \\
&= \frac{\mathsf{d}}{\mathsf{dx}}\int_{u=x-1}^{u=x} \mathsf{N}_{n-1}(u)\,\mathsf{d}u \\
&= \left\{\frac{\mathsf{d}}{\mathsf{dx}}\left.\int \mathsf{N}_{n-1}(u)\,\mathsf{d}u\right|_{u=x}\right\} - \left\{\frac{\mathsf{d}}{\mathsf{dx}}\left.\int \mathsf{N}_{n-1}(u)\,\mathsf{d}u\right|_{u=x-1}\right\} && \text{by \textit{Fundamental Theorem of Calculus}}^{12} \\
&= \left\{\mathsf{N}_{n-1}(x)\frac{\mathsf{d}}{\mathsf{dx}}(x)\right\} - \left\{\mathsf{N}_{n-1}(x-1)\frac{\mathsf{d}}{\mathsf{dx}}(x-1)\right\} && \text{by \textit{Chain Rule}}^{13} \\
&= \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)
\end{aligned}$$

[11] Höllig (2003) page 25 ⟨3.2⟩, Schumaker (2007) page 121 ⟨Theorem 4.16⟩
[12] Hijab (2011) page 163 ⟨Theorem 4.4.3⟩
[13] Hijab (2011) pages 73–74 ⟨Theorem 3.1.2⟩

2. Proof using Lemma Q.2 (page 343) and *induction*:

(a) Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\mathsf{N}_0(x) - \mathsf{N}_0(x-1) &= \underbrace{\sigma(x) - \sigma(x-1)}_{\mathsf{N}_0(x)} - \underbrace{[\sigma(x-1) - \sigma(x-2)]}_{\mathsf{N}_0(x-1)} && \text{by Example Q.1 page 345}\\
&= \sigma(x) - 2\sigma(x-1) + \sigma(x-2)\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\sigma(x-k)\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k)\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_1(x) && \text{by Example Q.2 page 346}
\end{aligned}$$

(b) Base case ...proof for $n = 2$ case:

$$\begin{aligned}
&\mathsf{N}_1(x) - \mathsf{N}_1(x-1)\\
&= \underbrace{x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)}_{\mathsf{N}_1(x)}\\
&\quad - \underbrace{[(x-1)\sigma(x-1) - 2(x-2)\sigma(x-2) + (x-3)\sigma(x-3)]}_{\mathsf{N}_1(x-1)} && \text{by Example Q.2 page 346}\\
&= x\sigma(x) + [-2x + 2 - x + 1]\sigma(x-1) + [x - 2 + 2x - 4]\sigma(x-2) + [-x+3]\sigma(x-3)\\
&= x\sigma(x) + [-3x+3]\sigma(x-1) + [3x-6]\sigma(x-2) + [-x+3]\sigma(x-3)\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\left\{\begin{array}{l}\frac{1}{2}x^2\sigma(x) + \left[-\frac{3}{2}x^2 + 3x - \frac{1}{2}\right]\sigma(x-1) + \left[\frac{3}{2}x^2 - 6x + 3\right]\sigma(x-2)\\ \qquad + \left[-\frac{1}{2}x^2 + 3x - \frac{5}{2}\right]\sigma(x-3)\end{array}\right\}\\
&= \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_2(x) && \text{by Example Q.3 page 346}
\end{aligned}$$

(c) Proof that n case $\implies n+1$ case:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_{n+1}(x) &= \frac{\mathsf{d}}{\mathsf{d}x}\int_0^1 \mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by Lemma Q.2 page 343}\\
&= \int_0^1 \frac{\partial}{\partial\tau}\mathsf{N}_n(x-\tau)\,\mathsf{d}\tau && \text{by \textit{Leibniz Integration Rule} (Theorem ?? page ??)}\\
&= \int_0^1 \left[\mathsf{N}_{n-1}(x-\tau) - \mathsf{N}_{n-1}(x-1-\tau)\right]\mathsf{d}\tau && \text{by left hypothesis}\\
&= \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathsf{d}\tau - \int_0^1 \mathsf{N}_{n-1}(x-1-\tau)\,\mathsf{d}\tau\\
&= \mathsf{N}_n(x) - \mathsf{N}_n(x-1) && \text{by Lemma Q.2 page 343}
\end{aligned}$$

✏

Theorem Q.3 (*B-spline recursion*). [14] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition Q.2 page 343).*

THM
$$\mathsf{N}_n(x) \;=\; \frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1) \qquad \forall n \in \{1,2,3,\ldots\},\ \forall x \in \mathbb{R}$$

[14] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972)

✎PROOF:

1. Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\frac{x}{1}\mathsf{N}_0(x) + \frac{1+1-x}{1}\mathsf{N}_0(x-1) &= \frac{x}{1}\underbrace{[\sigma(x)-\sigma(x-1)]}_{\mathsf{N}_0(x)} + \frac{1+1-x}{1}\underbrace{[\sigma(x-1)-\sigma(x-2)]}_{\mathsf{N}_0(x-1)} \\
&= x\sigma(x) + [-x-x+2]\sigma(x-1) + [x-2]\sigma(x-2) \\
&= \mathsf{N}_1(x) \qquad \text{by Example Q.2 page 346}
\end{aligned}$$

2. Induction step ...proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1) + c_1 \\
&= \int \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1)\right\}\,\mathrm{d}x \\
&= \int \underbrace{\frac{1}{n+1}\mathsf{N}_n(x) + \frac{x}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{N}_n(x)}_{\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{n+1}\mathsf{N}_n(x)} + \underbrace{\frac{-1}{n+1}\mathsf{N}_n(x-1) + \frac{n+2-x}{n}\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{N}_n(x-1)}_{\frac{\mathrm{d}}{\mathrm{d}x}\frac{n+2-x}{n+1}\mathsf{N}_n(x-1)}\,\mathrm{d}x \\
&\qquad \text{by } \textit{product rule} \\
&= \int \frac{1}{n+1}\underbrace{\left[\frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1)\right]}_{\text{by } n \text{ hypothesis}} + \frac{x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)\right]}_{\text{by Theorem Q.2 page 351}} \\
&\qquad - \underbrace{\left[\frac{x-1}{n^2+n}\mathsf{N}_{n-1}(x-1) + \frac{n-x+2}{n(n+1)}\mathsf{N}_{n-1}(x-2)\right]}_{\text{by induction hypothesis}} \\
&\qquad + \frac{n+2-x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x-1) - \mathsf{N}_{n-1}(x-2)\right]}_{\text{by Theorem Q.2 page 351}}\,\mathrm{d}x \\
&= \int \left[\frac{x}{n(n+1)} + \frac{x}{n+1}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n-x+1}{n(n+1)} - \frac{x-1}{n(n+1)} + \frac{n+2-2x}{n+1}\right]\mathsf{N}_{n-1}(x-1) \\
&\qquad + \left[\frac{-n-2+x}{n(n+1)} + \frac{-n-2+x}{n+1}\right]\mathsf{N}_{n-1}(x-2)\,\mathrm{d}x \\
&= \int \left[\frac{x+nx}{n(n+1)}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x+n(n+2-2x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-1) \\
&\qquad + \left[\frac{-n-2+x+n(-n-2+x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-2)\,\mathrm{d}x \\
&= \int \left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x}{n}\right]\mathsf{N}_{n-1}(x-1) + \left[\frac{-n-2+x}{n}\right]\mathsf{N}_{n-1}(x-2)\,\mathrm{d}x \\
&= \int \underbrace{\left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+1-x}{n}\right]\mathsf{N}_{n-1}(x-1)}_{\mathsf{N}_n(x)} \\
&\qquad \underbrace{-\left[\frac{x-1}{n}\right]\mathsf{N}_{n-1}(x-1) - \left[\frac{n+2-x}{n}\right]\mathsf{N}_{n-1}(x-2)}_{\mathsf{N}_{n-1}(x-1)}\,\mathrm{d}x \\
&= \int \mathsf{N}_n(x) - \mathsf{N}_n(x-1)\,\mathrm{d}x \qquad \text{by } n \text{ hypothesis} \\
&= \int \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{N}_{n+1}(x)\,\mathrm{d}x \qquad \text{by Theorem Q.2 page 351} \\
&= \mathsf{N}_{n+1}(x) + c_2
\end{aligned}$$

Proof that $c_1 = c_2$: By item (2) (page 354), $\mathsf{N}_n(x) = 0$ for $x < 0$. Therefore, $c_1 = c_2$.

✏

Theorem Q.4 (*B-spline general form*). [15] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition Q.2 page 343). Let* $\operatorname{supp} \mathsf{f}$ *be the* SUPPORT *of a function* $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$.

THM

1.	$\mathsf{N}_n(x) \geq 0$	$\forall n \in \mathbb{W}, \quad \forall x \in \mathbb{R}$	(NON-NEGATIVE)
2.	$\operatorname{supp} \mathsf{N}_n(x) = [0 : n+1]$	$\forall n \in \mathbb{W}$	(CLOSED SUPPORT)
3.	$\int_{\mathbb{R}} \mathsf{N}_n(x) \, \mathrm{d}x = 1$	$\forall n \in \mathbb{W}$	(UNIT AREA)
4.	$\mathsf{N}_n\left(\frac{n+1}{2} - x\right) = \mathsf{N}_n\left(\frac{n+1}{2} + x\right)$	$\forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}$	(SYMMETRIC *about* $x = \frac{n+1}{2}$)

✎PROOF:

1. Proof that $\mathsf{N}_n(x) \geq 0$ (proof by induction):

   (a) base case...proof that $\mathsf{N}_0(x) \geq 0$:

$$\begin{aligned}
\mathsf{N}_0(x) &\triangleq \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition Q.2 page 343)}\\
&\geq 0 && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition ?? page ??)}
\end{aligned}$$

   (b) inductive step—proof that $\{\mathsf{N}_n(x) \geq 0\} \implies \{\mathsf{N}_{n+1}(x) \geq 0\}$:

$$\begin{aligned}
\mathsf{N}_{n+1}(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau) \, \mathrm{d}\tau && \text{by Lemma Q.2 page 343}\\
&\geq 0 && \text{by induction hypothesis } (\mathsf{N}_n(x) \geq 0)
\end{aligned}$$

2. Proof that $\operatorname{supp} \mathsf{N}_n(x) = [0 : n+1]$ (proof by induction):

   (a) Base case ...proof that $\operatorname{supp} \mathsf{N}_0 = [0 : 1]$:

$$\begin{aligned}
\operatorname{supp} \mathsf{N}_0 &\triangleq \operatorname{supp} \mathbb{1}_{[0:1)} && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition Q.2 page 343)}\\
&= \{[0:1)\}^- && \text{by definition of \textit{support} operator}\\
&= [0:1] && \text{by definition of \textit{closure} operator}
\end{aligned}$$

   (b) Induction step ...proof that $\{\operatorname{supp} \mathsf{N}_n = [0 : n+1]\} \implies \{\operatorname{supp} \mathsf{N}_{n+1} = [0 : n+2]\}$:

$$\begin{aligned}
\operatorname{supp} \mathsf{N}_{n+1}(x) &= \operatorname{supp} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau) \, \mathrm{d}\tau && \text{by Lemma Q.2 page 343}\\
&= \operatorname{supp} \int_{[0:1]} \mathsf{N}_n(x-\tau) \, \mathrm{d}\tau && \text{by def. of \textit{Lebesgue integration}}\\
&= \left\{x \in \mathbb{R} | (x-\tau) \in [0 : n+1] \text{ for some } \tau \in [0:1]\right\}^- && \text{by induction hypothesis}\\
&= [0 : n+1] \cup [0+1 : n+1+1]^-\\
&= [0 : n+2]^-\\
&= [0 : n+2] && \text{by property of \textit{closure} operator}
\end{aligned}$$

3. Proof that $\int_{\mathbb{R}} \mathsf{N}_n(x) \, \mathrm{d}x = 1$ (proof by induction):

[15] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972), Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2⟩

(a) Base case ...proof that $\int_{\mathbb{R}} \mathsf{N}_0(x) = 1$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{N}_0(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \mathbb{1}_{[0:1)}\,\mathrm{d}x && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition Q.2 page 343)}\\
&= \int_{[0:1)} 1\,\mathrm{d}x && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition ?? page ??)}\\
&= \int_{[0:1]} 1\,\mathrm{d}x && \text{by property of \textit{Lebesgue integration}}\\
&= 1
\end{aligned}$$

(b) Induction step ...proof that $\left\{\int_{\mathbb{R}} \mathsf{N}_n(x) = 1\right\} \implies \left\{\int_{\mathbb{R}} \mathsf{N}_{n+1} = 1\right\}$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{N}_{n+1}(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau\,\mathrm{d}x && \text{by Lemma Q.2 page 343}\\
&= \int_0^1\int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&= \int_0^1\int_{\mathbb{R}} \mathsf{N}_n(u)\,\mathrm{d}u\,\mathrm{d}\tau && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \int_0^1 1\,\mathrm{d}\tau && \text{by induction hypothesis}\\
&= 1
\end{aligned}$$

4. Proof that $\mathsf{N}_n(x)$ is *symmetric* for $n \in \{1,2,3,\ldots\}$:

(a) Note that $\mathsf{N}_0(x)$ $(n=0)$ is *not symmetric* (in particular it fails at $x = 1/2$) because
$\mathsf{N}_0\left(\frac{0+1}{2} - \frac{1}{2}\right) = \mathsf{N}_0(0) = 1 \neq 0 = \mathsf{N}_1(1) = \mathsf{N}_0\left(\frac{0+1}{2} + \frac{1}{2}\right)$

(b) Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\mathsf{N}_1\left(\frac{1+1}{2} - x\right) &= \mathsf{N}_1(1-x)\\
&= \left\{\begin{array}{rl} (1-x) & \text{for } 1-x \in [0:1]\\ -(1-x)+2 & \text{for } 1-x \in [1:2]\\ 0 & \text{otherwise}\end{array}\right\} && \text{by Example Q.2 page 346}\\
&= \left\{\begin{array}{rl} -x+1 & \text{for } -x \in [-1:0]\\ x+1 & \text{for } -x \in [0:1]\\ 0 & \text{otherwise}\end{array}\right\}\\
&= \left\{\begin{array}{rl} x+1 & \text{for } x \in [-1:0]\\ -x+1 & \text{for } x \in [0:1]\\ 0 & \text{otherwise}\end{array}\right\}\\
&= \left\{\begin{array}{rl} (1+x) & \text{for } 1+x \in [0:1]\\ -(1+x)+2 & \text{for } 1+x \in [1:2]\\ 0 & \text{otherwise}\end{array}\right\}\\
&= \mathsf{N}_1(1+x) && \text{by Example Q.2 page 346}\\
&= \mathsf{N}_1\left(\frac{1+1}{2} + x\right)
\end{aligned}$$

(c) Induction step ...proof that $n-1$ case $\implies$ n case:

$$\begin{aligned}
&\mathsf{N}_n\left(\frac{n+1}{2}+x\right)\\
&=\frac{\frac{n+1}{2}+x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}+x\right)+\frac{n+1-\left(\frac{n+1}{2}+x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}+x-1\right) && \text{by Theorem Q.3 page 352}\\
&=\frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x+\frac{1}{2}\right]\right)+\frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x-\frac{1}{2}\right]\right)\\
&=\frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x+\frac{1}{2}\right]\right)+\frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x-\frac{1}{2}\right]\right) && \text{by induction hypothesis}\\
&=\frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\left[\frac{n+1}{2}-x\right]-1\right)+\frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}-x\right)\\
&=\mathsf{N}_n\left(\frac{n+1}{2}-x\right) && \text{by Theorem Q.3 page 352}
\end{aligned}$$

Q.3 Projection properties

In the case where $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is to be used as a basis in some subspace of $\boldsymbol{L}^2_{\mathbb{R}}$, one may want to *project* a function $\mathsf{f}(x)$ onto a basis function $\mathsf{N}_n(x-k)$. This is especially true when $(\!(\mathsf{N}_n(x-k))\!)$ is *orthogonal*; but in the case of *B-spline*s this is only true when $n=0$ (Theorem Q.8 page 366). Nevertheless, projection of a function onto $\mathsf{N}_n(x-k)$, or the projection of $\mathsf{N}_n(x)$ onto another basis fucntion (such as the complex exponential in the case of *Fourier analysis* as in Lemma Q.5 page 358), is still useful. Projection in an *inner product space* is typically performed using the *inner product* $\langle \mathsf{f}(x)\,|\,\mathsf{N}_n(x-k)\rangle$; and in the space $\boldsymbol{L}^2_{\mathbb{R}}$, this inner product is typically defined as an *integral* such that

$\langle \mathsf{f}(x)\,|\,\mathsf{N}_n(x-k)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_n(x-k)\,\mathrm{d}x.$

As it turns out, there is a way to compute this inner product that only involves the function $\mathsf{f}(x)$ and the order parameter n (next theorem).

Theorem Q.5. [16] *Let* f *be a* CONTINUOUS *function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* nTH DERIVATIVE *of* $\mathsf{f}(x)$.

THM

(1). $$\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_n(x)\,\mathrm{d}x = \int_{[0:1)^{n+1}} \mathsf{f}(x_1+x_2+\cdots+x_{n+1})\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n+1}$$

(2). $$\int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathrm{d}x = \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)$$

✎PROOF:

1. Proof for (1) (proof by induction):

(a) Base case ...proof for $n=0$ case:

$$\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_0(x)\,\mathrm{d}x = \int_{[0:1)} \mathsf{f}(x)\,\mathrm{d}x \qquad \text{by definition of } \mathsf{N}_0(x) \qquad \text{(Definition Q.2 page 343)}$$

[16] Chui (1992) page 85 ⟨(4.2.2), (4.2.3)⟩, Christensen (2008) page 140 ⟨Theorem 6.1.1⟩

(b) Inductive step—proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_{n+1}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left[\int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau\right]\mathsf{f}(x)\,\mathrm{d}x && \text{by Lemma Q.2 page 343} \\
&= \int_{[0:1)}\int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\mathsf{f}(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_{[0:1)}\int_{\mathbb{R}} \mathsf{N}_n(u)\mathsf{f}(u+\tau)\,\mathrm{d}u\,\mathrm{d}\tau && \text{where } u \triangleq x-\tau \implies x = u+\tau \\
&= \int_{[0:1)}\int_{[0:1)^{n+1}} \mathsf{f}(u_1+u_2+\cdots+u_{n+1}+\tau)\,\mathrm{d}u_1\ \mathrm{d}u_2\ \cdots\ \mathrm{d}u_{n+1}\ \mathrm{d}\tau && \text{by induction hypothesis} \\
&= \int_{[0:1)^{n+2}} \mathsf{f}(u_1+u_2+\cdots+u_{n+1}+u_{n+2})\,\mathrm{d}u_1\ \mathrm{d}u_2\ \cdots\ \mathrm{d}u_{n+2}\ \mathrm{d}\tau \\
&= \int_{[0:1)^{n+2}} \mathsf{f}(x_1+x_2+\cdots+x_{n+1}+x_{n+2})\,\mathrm{d}x_1\ \mathrm{d}x_2\ \cdots\ \mathrm{d}x_{n+2} && \text{by change of variables } u_k \to x_k
\end{aligned}$$

2. Proof for (2):

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathrm{d}x &= \int_{[0:1)^{n+1}} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n+1} x_k\right)\mathrm{d}x_1\ \mathrm{d}x_2\cdots\ \mathrm{d}x_{n+1} && \text{by (1)} \\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by Theorem ?? page ??}
\end{aligned}$$

✏

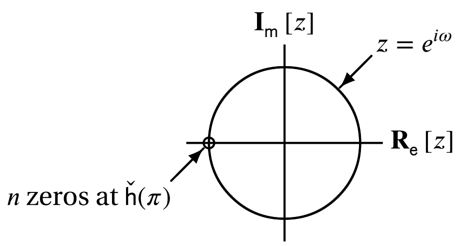


Figure Q.2: Zero locations for B-cardinal spline $N_n(x)$ scaling coefficients

Q.4 Fourier analysis

Simply put, no matter what new and fancy basis sequences are discovered, the *Fourier transform* never goes out of style. This is largely because the *kernel* of the Fourier transform—the *complex exponential* function—has two properties that makes it extremely special:

- The complex exponential is an eigenvalue of any *linear time invariant* (*LTI*) operator (Theorem ?? page ??).
- The complex exponential generates a *continuous point spectrum* for the *differential operator*.

Thus, we might expect the projection of the *B-spline* function $\mathsf{N}_n(x)$ onto the complex exponential (essentially the *Fourier transform* of $\mathsf{N}_n(x)$,...next lemma) to be useful. Such a hunch would be confirmed because it is useful for proving that

- the sequence $(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$ is a *Riesz basis* (Lemma Q.6 page 361, Theorem Q.8 page 366) and
- the sequence $(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$ is a *multiresolution analysis* (Theorem Q.10 page 369).

Lemma Q.5. [17] *Let* $\tilde{\mathbf{F}}$ *be the* FOURIER TRANSFORM *operator (Definition N.2 page 309).*

LEM
$$\tilde{\mathbf{F}}\mathsf{N}_n(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1} = \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1} \triangleq \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1}$$

✎PROOF:

1. Proof using Theorem Q.5 page 356:

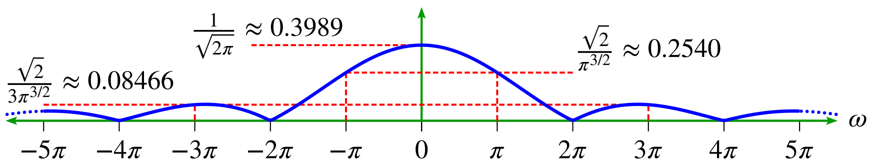
$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathsf{N}_n(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition N.2 page 309)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{[0:1)^{n+1}} e^{-i\omega(x_1+x_2+\cdots+x_{n+1})}\,\mathrm{d}x_1\ \mathrm{d}x_2\ \cdots,\ \mathrm{d}x_{n+1} && \text{by Theorem Q.5}\\
&= \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{n+1}\left(\int_{[0:1)} e^{-i\omega x_k}\,\mathrm{d}x_k\right) && \text{because } e^{x+y}=e^x e^y\\
&= \frac{1}{\sqrt{2\pi}}\left(\int_0^1 e^{-i\omega x}\,\mathrm{d}x\right)^{n+1} = \frac{1}{\sqrt{2\pi}}\left(\left.\frac{e^{-i\omega x}}{-i\omega}\right|_0^1\right)^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1} = \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}}{i\omega}\right)\right]^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{2i\sin\left(\frac{\omega}{2}\right)}{\frac{2i\omega}{2}}\right)\right]^{n+1} && \text{by \textit{Euler formulas}} \quad \text{(Corollary ?? page ??)}\\
&= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1}
\end{aligned}$$

2. Proof using *rectangular pulse* example (Example N.1 page 316) and *Convolution Theorem* (Theorem P.2 page 332):

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \left[\sqrt{2\pi}\right]^n\left[\tilde{\mathbf{F}}\mathsf{N}_0\right]^{n+1} && \text{by \textit{Convolution Theorem}} \quad \text{(Theorem P.2 page 332)}\\
&= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\frac{c(b-a)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left(\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right)\right]^{n+1} && \text{by \textit{rectangular pulse} example with } a=0,\ b=c=1 \quad \text{(Example N.1 page 316)}\\
&= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{\omega}{2}\right)}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{(\omega/2)}\right)\right]^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{(n+1)\omega}{2}\right)}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{(\omega/2)}\right)^{n+1}
\end{aligned}$$

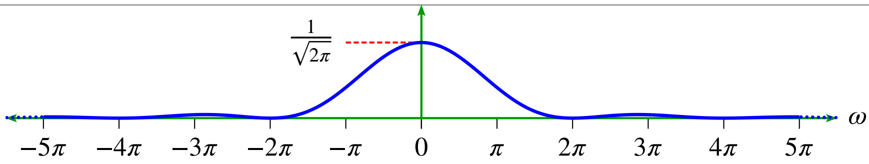
✏

Example Q.10. *The Fourier transform magnitude* $\left|\left[\tilde{\mathbf{F}}\mathsf{N}_0\right](\omega)\right|$ *of the 0 order B-spline* $\mathsf{N}_0(x)$ *is illustrated to the right.*

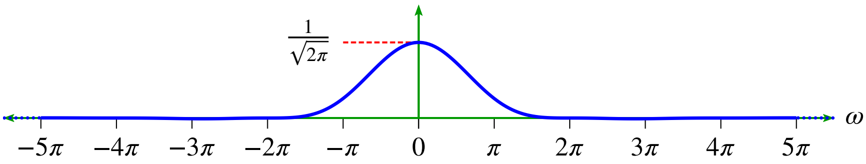


[17] Christensen (2008) page 142 ⟨Corollary 6.1.2⟩

Example Q.11. The *Fourier transform magnitude* $\left|\left[\tilde{\mathbf{F}}\mathsf{N}_1\right](\omega)\right|$ of the *1st order B-spline* $\mathsf{N}_1(x)$ is illustrated to the right.



Example Q.12. The *Fourier transform magnitude* $\left|\left[\tilde{\mathbf{F}}\mathsf{N}_2\right](\omega)\right|$ of the *2nd order B-spline* $\mathsf{N}_2(x)$ is illustrated to the right.



Q.5 Basis properties

Q.5.1 Uniqueness properties

Coefficients of a *basis sequence* are not always *unique*. Take for example a very trivial sequence (α_1, α_2) in which the coefficients are summed. If $\mathsf{f}(x) \triangleq \alpha_1 + \alpha_2$ and $\mathsf{g}(x) \triangleq \beta_1 + \beta_2$,

$$\begin{array}{lll} \text{then} & \left\{(\alpha_1,\alpha_2) = (\beta_1,\beta_2)\right\} & \implies \mathsf{f}(x) = \mathsf{g}(x) \\ \text{but} & \mathsf{f}(x) = \mathsf{g}(x) & \centernot\implies \left\{(\alpha_1,\alpha_2) = (\beta_1,\beta_2)\right\}, \end{array}$$

because for example if $(\alpha_1, \alpha_2) = (1, 2)$ and $(\beta_1, \beta_2) = (-6, 9)$, then $\mathsf{f}(x) = \mathsf{g}(x)$, but $(\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)$. This example demonstrates that the "if and only if" condition $\iff$ does not hold and coefficients are not unique in all *basis sequence*s. But arguably a minimal requirement for any practical basis sequence is that the coefficients are *unique* (the "if and only if" condition $\iff$ holds). And indeed, in a *B-spline* basis sequence $\left(\!\left(\mathsf{N}_n(x-k)\right)\!\right)_{k\in\mathbb{Z}}$, the coefficients $\left(\!\left(\alpha_k\right)\!\right)_{k\in\mathbb{Z}}$ are *unique*, as demonstrated by Theorem Q.6 (next).

Theorem Q.6. [18] *Let* $\mathsf{N}_n(x)$ *be the* nTH-ORDER B-SPLINE (Definition Q.2 page 343). *Let*

$$\mathsf{f}(x) \triangleq \sum_{k\in\mathbb{Z}} \alpha_k \mathsf{N}_n(x-k) \quad \textit{and} \quad \mathsf{g}(x) \triangleq \sum_{k\in\mathbb{Z}} \beta_k \mathsf{N}_n(x-k).$$

THM

$$\underbrace{\left\{\ \mathsf{f}(x) = \mathsf{g}(x) \quad \forall x \in \mathbb{R}\ \right\} \iff \left\{\left(\!\left(\alpha_k\right)\!\right)_{k\in\mathbb{Z}} = \left(\!\left(\beta_k\right)\!\right)_{k\in\mathbb{Z}}\right\}}_{\textit{coefficients are } \text{UNIQUE}}$$

✎PROOF:

1. Proof that $\impliedby$ condition holds:

$$\begin{aligned} \mathsf{f}(x) &\triangleq \sum_{k\in\mathbb{Z}} \alpha_k \mathsf{N}_n(x-k) && \text{by definition of } \mathsf{f}(x) \\ &= \sum_{k\in\mathbb{Z}} \beta_k \mathsf{N}_n(x-k) && \text{by right hypothesis} \\ &\triangleq \mathsf{g}(x) && \text{by definition of } \mathsf{g}(x) \end{aligned}$$

2. Proof that $\implies$ condition holds (proof by contradiction):

   (a) Suppose it does *not* hold.

[18] Wojtaszczyk (1997) page 55 ⟨Theorem 3.11⟩

(b) Then there exists sequences $\langle\!\langle \alpha_k \rangle\!\rangle_{k\in\mathbb{Z}}$ and $\langle\!\langle \beta_k \rangle\!\rangle_{k\in\mathbb{Z}}$ such that
$\langle\!\langle \alpha_k \rangle\!\rangle - \langle\!\langle \beta_k \rangle\!\rangle \triangleq \langle\!\langle alpha_k - \beta_k \rangle\!\rangle \neq \langle\!\langle 0, 0, 0, \cdots \rangle\!\rangle$
but also such that $\mathsf{f}(x) - \mathsf{g}(x) = 0\ \forall x \in \mathbb{R}$.

(c) If this were possible, then

$$\begin{aligned}
0 &= \mathsf{f}(x) - \mathsf{g}(x) \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \mathsf{N}_n(x-m) - \sum_{m\in\mathbb{Z}} \beta_m \mathsf{N}_n(x-m) \\
&= \sum_{m\in\mathbb{Z}} (\alpha_m - \beta_m) \mathsf{N}_n(x-m) \\
&= \sum_{m=0}^{m=n} (\alpha_m - \beta_m) \frac{1}{n!}\left[\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x-k)^n \sigma(x-k)\right] && \text{by Theorem Q.1 page 347}
\end{aligned}$$

(d) But this is *impossible* because $\mathsf{N}(x)$ is *non-negative* (Theorem Q.4 page 354).

(e) Therefore, there is a contradiction, and the $\Longrightarrow$ condition *does* hold.

✏

Q.5.2 Partition of unity properties

In the case in which a sequence of *B-splines* $\langle\!\langle \mathsf{N}_n(x-k) \rangle\!\rangle_{k\in\mathbb{Z}}$ is to be used as a *basis* for some subspace of $\boldsymbol{L}^2_{\mathbb{R}}$, arguably one of the most important properties for the sequence to have is the *partition of unity* property such that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1$. This allows for convenient representation of the most basic functions, such as constants.[19] As it turns out, B-splines *do* have this property (next theorem).

Theorem Q.7 (*B-spline partition of unity*). [20] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition Q.2 page 343).*

THM	
$\displaystyle\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1 \quad \forall n \in \mathbb{W}$	(PARTITION OF UNITY)

✎PROOF:

1. lemma: $\sum_{k\in\mathbb{Z}} \mathsf{N}_0(x-k) = 1$. Proof:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{N}_0(x-k) &= \sum_{k\in\mathbb{Z}} \mathbb{1}_{[0:1)}(x-k) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition Q.2 page 343)} \\
&= 1 && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition ?? page ??)}
\end{aligned}$$

2. Proof for this theorem follows from the $n = 0$ case ((1) lemma page 360), the definition of $\mathsf{N}_n(x)$ (Definition Q.2 page 343), and Corollary **??** (page **??**).

3. Alternatively, this theorem can be proved by *induction*:

   (a) Base case ($n = 0$ case): by (1) lemma.

[19] Jawerth and Sweldens (1994) page 8

[20] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972)

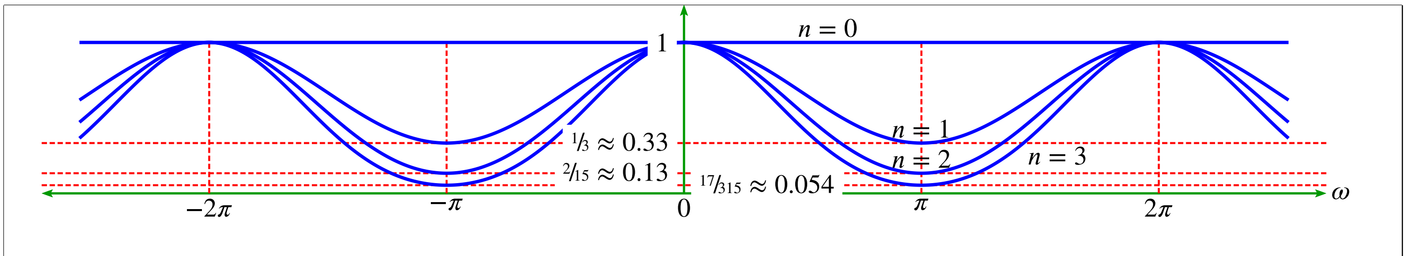
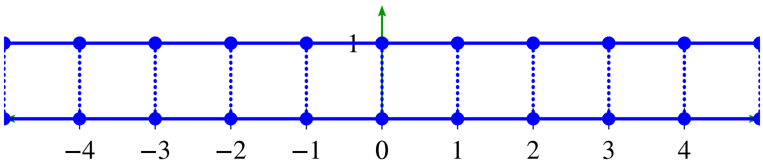


Figure Q.3: *auto-power spectrum* $\tilde{S}_n(\omega)$ plots of *B-splines* $\mathsf{N}_n(x)$ (Lemma Q.6 page 361) For C and LaTeX source code to generate such a plot, see Section ?? (page ??).

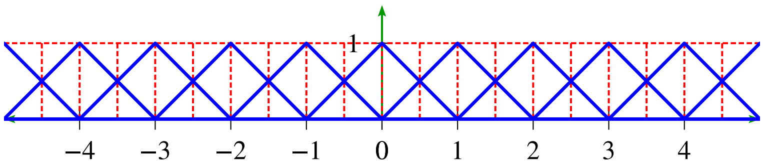
(b) Inductive step—proof that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1 \implies \sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) = 1$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) &= \sum_{k\in\mathbb{Z}} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-k-\tau)\,\mathsf{d}\tau && \text{by Lemma Q.2 page 343}\\
&= \sum_{k\in\mathbb{Z}} \int_{x-u=0}^{x-u=1} \mathsf{N}_n(u-k)(-1)\,\mathsf{d}u && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \sum_{k\in\mathbb{Z}} \int_{u=x-1}^{u=x} \mathsf{N}_n(u-k)\,\mathsf{d}u\\
&= \int_{u=x-1}^{u=x} \left(\sum_{k\in\mathbb{Z}} \mathsf{N}_n(u-k)\right) \mathsf{d}u\\
&= \int_{u=x-1}^{u=x} 1\,\mathsf{d}u && \text{by induction hypothesis}\\
&= 1
\end{aligned}$$

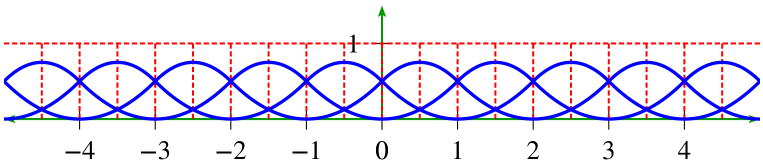
Example Q.13. The *partition of unity* property for the 0 *order B-spline* $\mathsf{N}_0(x)$ (Example Q.1 page 345) is illustrated to the right.



Example Q.14. The *partition of unity* property for the 1*st order B-spline* $\mathsf{N}_1(x)$ (Example Q.2 page 346) is illustrated to the right.



Example Q.15. The *partition of unity* property for the 2*nd order B-spline* $\mathsf{N}_2(x)$ (Example Q.3 page 346) is illustrated to the right.



Q.5.3 Riesz basis properties

Lemma Q.6. *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition Q.2 page 343).*
Let $\tilde{S}_n(\omega) \triangleq 2\pi \sum_{k\in\mathbb{Z}} \left|\tilde{\mathbf{F}}\mathsf{N}_n(\omega - 2\pi k)\right|^2$ *be the* AUTO-POWER SPECTRUM *(Definition ?? page ??) of* $\mathsf{N}_n(x)$.

LEM

(1). $0 < \tilde{S}_n(\omega) \le 1 \quad \forall \omega \in \mathbb{R} \quad , \quad \forall n \in \mathbb{W}$
(2). $\tilde{S}_n(\omega) = 1 \quad \forall \omega \in \mathbb{R} \quad , \quad \text{for } n = 0$
(3). $\tilde{S}_n(0) = 1 \quad \forall n \in \mathbb{W}$
(4). $\tilde{S}_n(\pi) \le \frac{1}{3} \quad \forall n \in \mathbb{W} \backslash \{0\}$

(Note: see illustration in Figure Q.3 page 361.)

✎PROOF:

1. lemma: $\tilde{S}_n(\omega) = \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}$. Proof:

$$\begin{aligned}
\tilde{S}_n(\omega) &\triangleq 2\pi \sum_{k\in\mathbb{Z}} \left|\tilde{\mathbf{F}}\mathsf{N}_n(\omega - 2\pi k)\right|^2 && \text{by Definition ?? page ??}\\
&= 2\pi \sum_{k\in\mathbb{Z}} \left|\frac{1}{\sqrt{2\pi}} e^{-i\frac{(n+1)(\omega-2\pi k)}{2}} \left(\frac{\sin\left(\frac{\omega-2\pi k}{2}\right)}{\frac{\omega-2\pi k}{2}}\right)^{n+1}\right|^2 && \text{by Lemma Q.5 page 358}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega-2\pi k}{2}\right)}{\frac{\omega-2\pi k}{2}}\right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2} - k\pi\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{(-1)^k \sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}
\end{aligned}$$

2. lemma (one sided series form):

$$\begin{aligned}
\tilde{S}_n(\omega) &= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)} && \text{by (1) lemma}\\
&= \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)} \left(\sum_{k=1}^{\infty} \left[\frac{1}{2k - \frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty} \left[\frac{1}{2k + \frac{\omega}{\pi}}\right]^{2(n+1)}\right)
\end{aligned}$$

3. lemma: $\tilde{S}_n(\omega)$ is *continuous* for all $\omega \in \mathbb{R}$.
   Proof: $\sin(\omega/2)$ and $\omega/2$ are *continuous*, so $\tilde{S}_n(\omega)$ is *continuous* as well.

4. lemma: $\tilde{S}_n(\omega)$ is *periodic* with period 2π (and so we only need to examine $\tilde{S}_n(\omega)$ for $\omega \in [0 : 2\pi)$). Proof of *periodicity*: This follows directly from Proposition ?? (page ??).

5. lemma: $\tilde{S}_n(-\omega) = \tilde{S}_n(\omega)$ (*symmetric* about 0) and $\tilde{S}_n(\pi - \omega) = \tilde{S}_n(\pi + \omega)$ (*symmetric* about π). Proof: This follows directly from Proposition ?? (page ??).

6. Proof that $\tilde{S}_n(0) = 1$:

$$\begin{aligned}
\tilde{S}_n(0) &= \lim_{\omega\to 0} \tilde{S}_n(\omega) && \text{by (3) lemma}\\
&= \lim_{\omega\to 0}\left[\left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right)\right] && \text{by (2) lemma}\\
&= \lim_{\omega\to 0}\left[\frac{\cos\left(\frac{\omega}{2}\right)}{-\frac{1}{2}}\right]^{2(n+1)} + 0 \qquad \text{by \textit{l'Hôpital's rule}}\\
&= (-1)^{2(n+1)} = 1
\end{aligned}$$

7. Proof that $\tilde{S}_n(\pi)$ converges to some value > 0:

(a) Proof that $\tilde{S}_n(\pi) > 0$:

$$\begin{aligned}
\tilde{S}_n(\pi) &= \left[\frac{\sin(\pi/2)}{(\pi/2)}\right]^{2(n+1)} + \left[\frac{\sin(\pi/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\pi}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\pi}{\pi}}\right]^{2(n+1)}\right) && \text{by (2) lemma}\\
&= \left(\frac{2}{\pi}\right)^{2(n+1)}\left[1 + \left(\frac{1}{1}\right)^{2(n+1)} + \left(\frac{1}{3}\right)^{2(n+1)} + \left(\frac{1}{3}\right)^{2(n+1)} + \left(\frac{1}{5}\right)^{2(n+1)} + \left(\frac{1}{5}\right)^{2(n+1)} + \cdots\right]\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \underbrace{\sum_{k=1}^{\infty}\left[\frac{1}{2k-1}\right]^{2(n+1)}}_{\text{Dirichlet Lambda function } \lambda(2n+2)}\\
&> 0 \qquad \text{because } x^2 > 0 \text{ for all } x \in \mathbb{R}\backslash\{0\}
\end{aligned}$$

(b) Proof that $\tilde{S}_n(\pi)$ *converges*:

$$\begin{aligned}
\tilde{S}_n(\pi) &= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^{2(n+1)} && \text{by item (7a)}\\
&\le 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{2(n+1)}\\
&\le 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{2}\\
&\implies \textit{convergence} && \text{by \textit{comparison test}}
\end{aligned}$$

(c) Tighter bounds for $\tilde{S}_n(\pi)$ for certain values of $n \in \{0, 1, 2, 3, 4\}$:

$$\begin{aligned}
\tilde{S}_n(\pi) &= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^{2(n+1)} && \text{by item (7a)}\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)} U_{2(n+1)} && \text{by Jolley (1961), pages 56–57 } \langle(307)\rangle\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \left[\frac{\pi^{2(n+1)}\alpha_{n+1}}{(4)[(2n+2)!]}\right] && \text{by Jolley (1961), pages 56–57 } \langle(307)\rangle\\
&= \frac{2^{2n+1}\alpha_{n+1}}{(2n+2)!}
\end{aligned}$$

$$= \begin{Bmatrix} \frac{2^1(1)}{2!} & \text{for } n=0 & (\alpha_1 = 1) \\ \frac{2^3(1)}{4!} & \text{for } n=1 & (\alpha_2 = 1) \\ \frac{2^5(3)}{6!} & \text{for } n=2 & (\alpha_3 = 3) \\ \frac{2^7(17)}{8!} & \text{for } n=3 & (\alpha_4 = 17) \\ \frac{2^9(155)}{10!} & \text{for } n=4 & (\alpha_5 = 155) \end{Bmatrix} \qquad \text{by Jolley (1961), page 234 } \langle(1130)\rangle$$

$$= \begin{Bmatrix} 1 & \text{for } n=0 \\ \frac{1}{3} & \text{for } n=1 \\ \frac{2}{15} & \text{for } n=2 \\ \frac{17}{315} & \text{for } n=3 \\ \frac{62}{2835} & \text{for } n=4 \end{Bmatrix} = \begin{Bmatrix} 1 & \text{for } n=0 \\ 0.3333333333333\cdots & \text{for } n=1 \\ 0.1333333333333\cdots & \text{for } n=2 \\ 0.0539682539682\cdots & \text{for } n=3 \\ 0.0218694885361\cdots & \text{for } n=4 \end{Bmatrix}$$

(d) Being important for the $n=0$ case, note that[21]

$$\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^2}_{\text{Dirichlet Lambda function } \lambda(2)} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

(e) Proof that $\tilde{S}_n(\pi) \le \frac{1}{3}$: because $\tilde{S}_n(\pi) = \frac{1}{3}$ for $n=1$ (item (7c) page 363) and because $\tilde{S}_n(\pi)$ is decreasing for increasing n.

8. lemma: $\tilde{S}_n(\omega)$ *converges* to some value $>0\ \forall\omega\in\mathbb{R}$. Proof:

(a) For $\omega = 0$, $\tilde{S}_n(\omega) = 1$ by item (6).

(b) Proof that $\tilde{S}_n(\omega) > 0$ for $\omega \in (0 : 2\pi)$:

$$\tilde{S}_n(\omega) = \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)} \left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right) \qquad \text{by (2) lemma}$$
$$> 0$$

(c) Proof that $\tilde{S}_n(\omega)$ *converges*:

i. lemma: $\sum_{k=1}^{\infty}\left[\frac{1}{2k\pm\frac{\omega}{\pi}}\right]^{2(n+1)}$ *converges*. Proof:

$$\begin{aligned}
\lim_{b\to\infty}\int_1^b \left[\frac{1}{2y\pm\frac{\omega}{\pi}}\right]^{2(n+1)} \mathrm{d}y &= \lim_{b\to\infty}\int_1^b \left[2y\pm\frac{\omega}{\pi}\right]^{-2n-2} \mathrm{d}y \\
&= \lim_{b\to\infty} \left.\frac{\left[2y\pm\frac{\omega}{\pi}\right]^{-2n-1}}{2(-2n-1)}\right|_1^b \\
&= \lim_{b\to\infty}\left(\frac{-1}{2(2n+1)}\right)\left[\frac{1}{\left[2b\pm\frac{\omega}{\pi}\right]^{2n+1}} - \frac{1}{\left[2\pm\frac{\omega}{\pi}\right]^{2n+1}}\right] \\
&= 0 + \frac{1}{2(2n+1)\left[2\pm\frac{\omega}{\pi}\right]^{2n+1}} \\
&< \infty \qquad \forall\omega\in[0:2\pi) \\
&\implies \sum_{k=1}^{\infty}\left[\frac{1}{2k\pm\frac{\omega}{\pi}}\right]^{2(n+1)} \textit{ converges} \qquad \text{by \textit{integral test}}
\end{aligned}$$

[21] Nahin (2011) page 153, Bailey et al. (2013) page 334 ⟨Catalan's Constant⟩, Bailey et al. (2011) ⟨(15)⟩, Wells (1987) page 36 ⟨Dictionary entry for π: pages 31–37⟩, Heinbockel (2010) page 94 ⟨(2.27) *Dirichlet Lambda function*⟩

ii. completion of proof using (8(c)i) lemma ...

$$\begin{aligned}
&\tilde{S}_n(\omega)\\
&= \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right) && \text{by (2) lemma}\\
&\implies \tilde{S}_n(\omega) \textit{ converges } \forall\omega\in(0:2\pi) && \text{by (8(c)i) lemma}
\end{aligned}$$

9. lemma (an expression for $\tilde{S}_n'(\omega)$):

$$\begin{aligned}
\tilde{S}_n'(\omega) &\triangleq \frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{S}_n(\omega)\\
&= \frac{\mathrm{d}}{\mathrm{d}\omega}\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2(n+1)} && \text{by (1) lemma page 362}\\
&= \sum_{k\in\mathbb{Z}}\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2(n+1)} && \text{by \textit{linearity} of } \frac{\mathrm{d}}{\mathrm{d}\omega} \text{ operator}\\
&= \sum_{k\in\mathbb{Z}} 2(n+1)\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right] && \text{by \textit{power rule}}\\
&= 2(n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\frac{1}{2}\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)-\sin\left(\frac{\omega}{2}\right)\left(-\frac{1}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by \textit{quotient rule}}\\
&= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right]
\end{aligned}$$

10. lemma: $\tilde{S}_n'(0) = \tilde{S}_n'(\pi) = 0$. Proof: This follows from Proposition **??** (page **??**). Here is alternate proof:

$$\begin{aligned}
\tilde{S}_n'(0) &= \lim_{\omega\to 0}\tilde{S}_n'(\omega)\\
&= \lim_{\omega\to 0}(n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by (9) lemma}\\
&= \lim_{\omega\to 0}(n+1)\left[\frac{\sin\left(\frac{\omega}{2}\right)}{-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(-\frac{\omega}{2}\right)^2}\right]\\
&= (n+1)\lim_{\omega\to 0}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{-\frac{\omega}{2}}\right]^{2n+1}\lim_{\omega\to 0}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(-\frac{\omega}{2}\right)^2}\right]\\
&= (n+1)[-1]^{2n+1}\lim_{\omega\to 0}\left[\frac{-\frac{1}{2}\sin\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\cos\left(\frac{\omega}{2}\right)\left(-\frac{1}{2}\right)+\cos\left(\frac{\omega}{2}\right)\left(\frac{1}{2}\right)}{-\frac{2}{2}\left(-\frac{\omega}{2}\right)}\right] && \text{by \textit{l'Hôpital's rule}}\\
&= (1)(0)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\tilde{S}'_n(\pi) &= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\pi}{2}\right)}{k\pi-\frac{\pi}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\pi}{2}\right)\left(k\pi-\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)}{\left(k\pi-\frac{\pi}{2}\right)^2}\right] && \text{by (9) lemma}\\
&= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{1}{k\pi-\frac{\pi}{2}}\right]^{2n+1}\left[\frac{0\left(k\pi-\frac{\pi}{2}\right)+1}{\left(k\pi-\frac{\pi}{2}\right)^2}\right]\\
&= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\sum_{k\in\mathbb{Z}}\left[\frac{1}{2k-1}\right]^{2n+3}\\
&= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\left[\left(\frac{1}{1}\right)^{2n+3}+\left(\frac{1}{-1}\right)^{2n+3}+\left(\frac{1}{3}\right)^{2n+3}+\left(\frac{1}{-3}\right)^{2n+3}+\ldots\right]\\
&= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\sum_{k=1}^{\infty}(-1)^{k+1}\alpha_k \qquad \text{where } \alpha_k \triangleq \left\{\begin{array}{ll}\left(\frac{1}{k}\right)^{2n+3} & \text{for } k \text{ odd}\\ \left(\frac{1}{k-1}\right)^{2n+3} & \text{for } k \text{ even}\end{array}\right\}\\
&= 0 \qquad \text{because } \lim_{k\to\infty}\alpha_k = 0 \text{ and by } \textit{Alternating Series Test}
\end{aligned}$$

11. lemma: $\tilde{S}_n(\omega)$ is *decreasing* with respect to $\omega \in [0:\pi]$. Proof:

$$\begin{aligned}
\tilde{S}'_n(\omega) &= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by (9) lemma page 365}\\
&= \underbrace{(n+1)\left(\sin\frac{\omega}{2}\right)^{2n+1}}_{\geq 0 \text{ for } \omega\in[0:2\pi)}\underbrace{\sum_{k\in\mathbb{Z}}\left[\frac{1}{k\pi-\frac{\omega}{2}}\right]^{2n+2}}_{>0}\left[\underbrace{\left(\cos\frac{\omega}{2}\right)}_{\text{sign change at } \omega=\pi}+\frac{\overbrace{\sin\frac{\omega}{2}}^{>0 \text{ for } \omega\in(0:2\pi)}}{\underbrace{k\pi-\frac{\omega}{2}}_{\textit{decreasing} \text{ w.r.t. } \omega\in\mathbb{R}}}\right]
\end{aligned}$$

12. lemma: $\tilde{S}_n(\omega)$ is *increasing* with respect to $\omega \in [\pi:2\pi)$. Proof: This is true because $\tilde{S}_n(\omega)$ is *decreasing* in $[0:\pi]$ ((11) lemma) and because $\tilde{S}_n(\omega)$ is *symmetric* about $\omega=\pi$ ((5) lemma).

13. Proof that $0 < \tilde{S}_n(\omega) \leq 1$:

    (a) $\tilde{S}_n(\omega) > 0$ by (8) lemma and
    (b) $\tilde{S}_n(0) = 1$ by item (6) and
    (c) $\tilde{S}_n(\omega)$ is *decreasing* from $\omega=0$ to $\omega=\pi$ by (11) lemma and
    (d) $\tilde{S}_n(\omega)$ is *increasing* from $\omega=\pi$ to $\omega=2\pi$ by (12) lemma and
    (e) $\tilde{S}_n(2\pi)=1$ because $\tilde{S}_n(2\pi)=\tilde{S}_n(0)$ by (4) lemma.

⇰

Theorem Q.8. [22]

T H M					
1.	$(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ *is a*	RIESZ BASIS	*for* span$(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$		$\forall n\in\mathbb{W}$
2.	$(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ *is an*	ORTHONORMAL BASIS	*for* span$(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$	$\iff$	$n=0$

✎PROOF:

[22] Wojtaszczyk (1997) page 56 ⟨Proposition 3.12⟩, Prasad and Iyengar (1997) page 148 ⟨Theorem 6.3⟩, Forster and Massopust (2009) page 66 ⟨Theorem 2.25⟩

1. Proof that $(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$ is a *Riesz basis* for $\operatorname{span}(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$:

$$\begin{aligned} & 0 < \tilde{S}_n(\omega) \le 1 && \text{by Lemma Q.6 page 361 (1)} \\ \implies & (\mathsf{N}_n(x-k))_{k\in\mathbb{Z}} \text{ is a } \textit{Riesz basis} \text{ for } \operatorname{span}(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}} && \text{by Theorem ?? page ??} \end{aligned}$$

2. Proof that $\{n=0\} \iff (\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$ is an *orthonormal basis* for $\operatorname{span}(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$:

$$\begin{aligned} n=0 \iff & \tilde{S}_n(\omega) = 1 && \text{by Lemma Q.6 page 361 (2), (4)} \\ \iff & (\mathsf{N}_n(x-k))_{k\in\mathbb{Z}} \text{ is an } \textit{orthonormal basis} \text{ for } \operatorname{span}(\mathsf{N}_n(x-k)) && \text{by Theorem ?? page ??} \end{aligned}$$

⇒

Q.6 Mutiresolution properties

Q.6.1 Introduction

In 1989, Stéphane G. Mallat introduced the *Multiresolution Analysis* (MRA) structure (Definition ?? page ??) An MRA is very powerful because it can be used to approximate functions at incrementally increasing "scales" of resolution, and furthermore induces a *wavelet*. In fact, the MRA has become the dominate wavelet construction method. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA. [23]

Q.6.2 B-spline dyadic decomposition

One key feature of an MRA is *dyadic decomposition* such that $\mathsf{N}_n(x) = \sum_k \alpha_n \mathsf{N}_n(2x-n)$ for some sequence (α_n). As it turns out, *B-spline*s also have this property (next theorem).

Theorem Q.9 (*B-spline dyadic decomposition*). [24] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE.

THM
$$\mathsf{N}_n(x) = \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-k) \qquad \forall n\in\mathbb{W}, \forall x\in\mathbb{R}$$

✎PROOF:

1. Base case ...proof for $n=0$ case:

$$\begin{aligned} \mathsf{N}_0(x) &= \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathbb{1}_A(x) \quad \text{(Definition ?? page ??)} \\ &= \mathbb{1}_{[0:\frac{1}{2})}(x) + \mathbb{1}_{[\frac{1}{2}:1)}(x) \\ &= \mathbb{1}_{[2\times 0:2\times\frac{1}{2})}(2x) + \mathbb{1}_{[2\times\frac{1}{2}-1:2\times 1-1)}(2x-1) \\ &= \mathbb{1}_{[0:1)}(2x) + \mathbb{1}_{[0:1)}(2x-1) \\ &= \frac{1}{2^0}\sum_{k=0}^{0+1}\binom{0+1}{k}\mathsf{N}_n(2x-k) \end{aligned}$$

[23] Mallat (1999) page 240, Definition ?? (page ??)
[24] Prasad and Iyengar (1997) pages 151–152 ⟨proof using Fourier transform⟩

2. Induction step…proof that n case $\implies$ $n+1$ case:

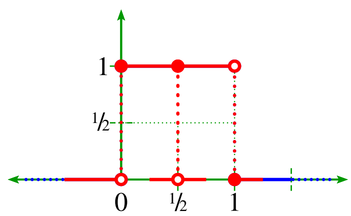
$$\begin{aligned}
\mathsf{N}_{n+1}(x) &= \int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma Q.2 page 343}\\
&= \int_0^1 \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-2\tau-k)\,\mathrm{d}\tau && \text{by induction hypothesis}\\
&= \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\int_{\tau=0}^{\tau=1}\mathsf{N}_n(2x-2\tau-k)\,\mathrm{d}\tau && \text{by } \textit{linearity} \text{ of } \textstyle\sum \text{ operator}\\
&= \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\int_{u=0}^{u=2}\mathsf{N}_n(2x-u-k)\frac{1}{2}\,\mathrm{d}u && \text{where } u \triangleq 2\tau \implies \tau = \frac{1}{2}u\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\int_{u=0}^{u=1}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u + \int_{u=1}^{u=2}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u\right]\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\int_{u=0}^{u=1}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u + \int_{v=0}^{v=1}\mathsf{N}_n(2x-k-v-1)\,\mathrm{d}v\right] && \text{where } v \triangleq u-1 \implies u = v+1\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\mathsf{N}_n(2x-k)+\mathsf{N}_n(2x-k-1)\right] && \text{by Lemma Q.2 page 343}\\
&= \frac{1}{2^{n+1}}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-k) + \sum_{m=1}^{n+2}\binom{n+1}{m-1}\mathsf{N}_n(2x-m)\right] && \text{where } m \triangleq k+1 \implies k = m-1\\
&= \frac{1}{2^{n+1}}\left[\underbrace{\sum_{k=1}^{n+1}\left[\binom{n+1}{k}+\binom{n+1}{k-1}\right]\mathsf{N}_n(2x-k)}_{\text{common indices of above two summations}} + \underbrace{\binom{n+1}{0}\mathsf{N}_n(2x-0)}_{k=0 \text{ term}} + \underbrace{\binom{n+2}{n+2}\mathsf{N}_n(2x-n-2)}_{m=n+2 \text{ term}}\right]\\
&= \frac{1}{2^{n+1}}\left[\underbrace{\sum_{k=1}^{n+1}\binom{n+2}{k}\mathsf{N}_n(2x-k)}_{\text{by } \textit{Stifel formual} \text{ (Theorem ?? page ??)}} + \underbrace{\binom{n+2}{0}\mathsf{N}_n(2x-0)}_{\text{because } \binom{n+1}{0} = 1 = \binom{n+2}{0}} + \binom{n+2}{n+2}\mathsf{N}_n(2x-n-2)\right]\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+2}\binom{n+2}{k}\mathsf{N}_n(2x-k)
\end{aligned}$$

✏

Example Q.16. [25]The 0 *order B-spline dyadic decomposition*

$$\mathsf{N}_0(x) = \frac{1}{1}\sum_{k=0}^{k=1}\binom{1}{k}\mathsf{N}_0(2x-k)$$

is illustrated to the right.

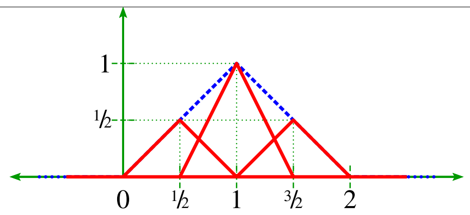


[25] Strang (1989) page 615 ⟨Box function⟩, Strang and Nguyen (1996) page 441 ⟨Box function⟩

Example Q.17. [26]The 1*st order B-spline dyadic decomposition*

$$\mathsf{N}_1(x) = \frac{1}{2}\sum_{k=0}^{k=2}\binom{2}{k}\mathsf{N}_1(2x-k)$$

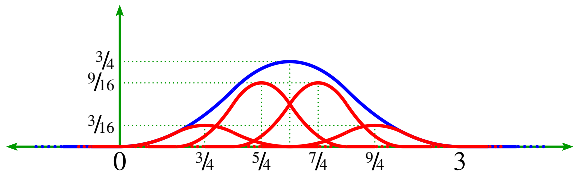
is illustrated to the right.



Example Q.18. The 2*nd order B-spline dyadic decomposition*

$$\mathsf{N}_2(x) = \frac{1}{4}\sum_{k=0}^{k=3}\binom{3}{k}\mathsf{N}_2(2x-k)$$

is illustrated to the right.



Q.6.3 B-spline MRA scaling functions

Theorem Q.10. *Let f $N_n(x)$ be the n*TH ORDER B-SPLINE (Definition Q.2 page 343).
Let $\boldsymbol{V}_j \triangleq \operatorname{span}\left(\!\left(\mathsf{N}_n\left(2^j x - k\right)\right)\!\right)_{k\in\mathbb{Z}}$.

THM: $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ *is a* MULTIRESOLUTION ANALYSIS *on* $\boldsymbol{L}^2_{\mathbb{R}}$ *with* SCALING FUNCTION $\phi(x) \triangleq \mathsf{N}_n(x)$

✎PROOF:

1. lemma: $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is a *Riesz sequence*. Proof: by Theorem Q.8 (page 366).
2. lemma: $\exists\, (\!(h_k)\!)$ such that $\mathsf{N}_n(x) = \sum_{k\in\mathbb{Z}} h_k \mathsf{N}_n(2x-k)$. Proof: by Theorem Q.9 (page 367). In fact, note that $h_k = \frac{1}{2^n\sqrt{2}}\binom{n+1}{k}$
3. lemma: $\tilde{\mathbf{F}}\mathsf{N}_n(\omega)$ is *continuous* at 0. Proof:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1} && \text{by Lemma Q.5 page 358}\\
&\implies \textit{continuous} \text{ at } 0 && \text{by known property of sinc function}
\end{aligned}$$

4. lemma: $\tilde{\phi}(0) \neq 0$. Proof:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(0) &= \left.\frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1}\right|_{\omega=0} && \text{by Lemma Q.5 page 358}\\
&= 1\cdot\frac{1}{1/2} = 2 && \text{by } \textit{l'Hôpital's rule}\\
&\neq 0
\end{aligned}$$

5. The completion of this proof follows directly from (1) lemma, (2) lemma, (3) lemma, (4) lemma, and Theorem ?? (page ??).

⇨

[26] Strang (1989) page 615 ⟨Hat function⟩, Strang and Nguyen (1996) page 442 ⟨Hat function⟩, Heil (2011) page 380 ⟨Fig. 12.10⟩

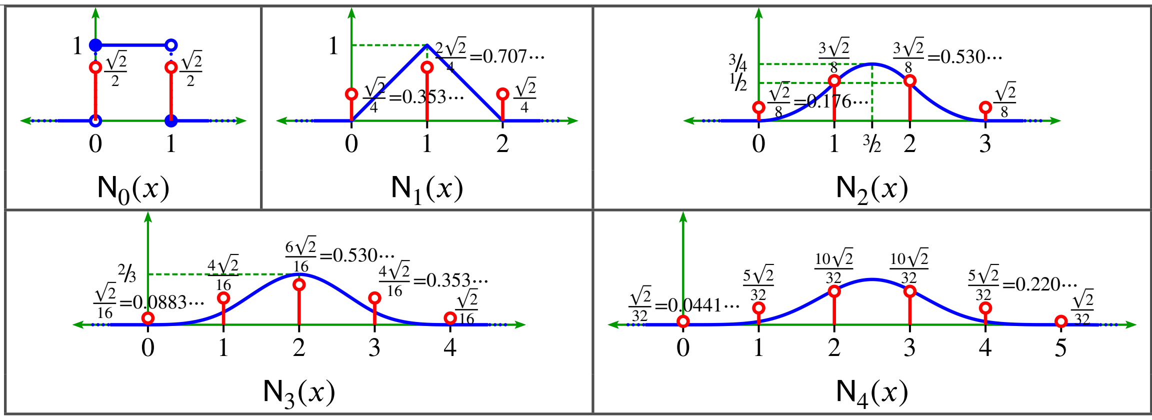


Figure Q.4: *dilation equation* demonstrations for selected B-splines (Example Q.19 page 370)

Q.6.4 B-spline MRA coefficient sequences

Because each *B-spline* $\mathsf{N}_n(x)$ is the *scaling function* for an *MRA* (Theorem Q.10 page 369), each *B-spline* also satisfies the *dilation equation* (Theorem ?? page ??) such that

$$\mathsf{N}_n(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \mathsf{N}(2x-k) \quad \text{where} \quad h_k = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2^{n+1}}\binom{n+1}{k} & \text{for } n = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{array}\right\}$$

The resulting *sequence* $(\!(h_k)\!)_{k\in\mathbb{Z}}$ is the *order n B-spline MRA coefficient sequence* induced by the *order n B-spline MRA scaling sequence* $\phi(x) \triangleq \mathsf{N}_n(x)$.[27]

Example Q.19. See Figure Q.4 (page 370) for some *dilation equation* demonstrations of selected B-splines.

Theorem Q.11 (B-spline scaling coefficients). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition ?? page ??). Let* $\mathsf{N}_n(x)$ *be a n*TH ORDER B-SPLINE *(Definition Q.2 page 343).*

THM

$$\underbrace{\phi(x) \triangleq \mathsf{N}_n(x)}_{\text{(1) B-spline scaling function}} \implies (\!(h_k)\!) = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2^{n+1}}\binom{n}{k} & \text{for } k = 0,1,\ldots,n \\ 0 & \text{otherwise}\end{array}\right. \quad \text{(2) scaling sequence in "time"}$$

$$\iff \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} = \frac{\sqrt{2}}{2^n}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} \quad \text{(3) scaling sequence in "z domain"}$$

$$\iff \tilde{\mathsf{h}}(\omega) = 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} \quad \text{(4) scaling sequence in "frequency"}$$

✎PROOF:

1. Proof that (1) $\implies$ (3): By Theorem Q.10 page 369 we know that $\mathsf{N}_n(x)$ is a *scaling function* (Definition ?? page ??). So then we know that we can use Lemma ?? page ??.

$$\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \sqrt{2}\frac{\tilde{\phi}(2\omega)}{\tilde{\phi}(\omega)} && \text{by Lemma ?? page ??} \\
&= \sqrt{2}\frac{\tilde{\mathsf{N}}_n(2\omega)}{\tilde{\mathsf{N}}_n(\omega)} && \text{by (1)} \\
&= \sqrt{2}\frac{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i2\omega}}{2i\omega}\right)^{n+1}}{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1}} && \text{by Lemma Q.5 page 358}
\end{aligned}$$

[27] For *Octave*/*MatLab* code useful for plotting a function given a sequence of coefficients $(\!(h_k)\!)$, see Section ?? (page ??).

$$= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{1-z^{-2}}{1-z^{-1}}\right)^{n+1}\Bigg|_{z=e^{i\omega}}$$

$$= \frac{\sqrt{2}}{2^{n+1}}\left[\left(\frac{1-z^{-2}}{1-z^{-1}}\right)\left(\frac{1+z^{-1}}{1+z^{-1}}\right)\right]^{n+1}\Bigg|_{z=e^{i\omega}}$$

$$= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{(1-z^{-2})(1+z^{-1})}{1-z^{-2}}\right)^{n+1}\Bigg|_{z=e^{i\omega}}$$

$$= \frac{\sqrt{2}}{2^{n}}(1+z^{-1})^{n+1}\Big|_{z\triangleq e^{i\omega}}$$

2. Proof that (3) $\iff$ (2):

$$\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \frac{\sqrt{2}}{2^{n}}(1+z^{-1})^{n+1}\Big|_{z\triangleq e^{i\omega}} && \text{by (3)}\\
&= \frac{\sqrt{2}}{2^{n}}\left(\sum_{k=0}^{n+1}\binom{n}{k}z^{-k}\right)\Bigg|_{z\triangleq e^{i\omega}} && \text{by \textit{binomial theorem}}\\
\iff \qquad h_k &= \frac{\sqrt{2}}{2^{n+1}}\binom{n}{k} && \text{by definition of \textit{Z transform} (Definition P.4 page 330)}
\end{aligned}$$

3. Proof that (3) $\implies$ (4):

$$\begin{aligned}
\tilde{\mathsf{h}}(\omega) = \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} & && \text{by definition of \textit{DTFT} (Definition O.1 page 319)}\\
&= \frac{\sqrt{2}}{2^{n}}(1+z^{-1})^{n+1}\Big|_{z\triangleq e^{i\omega}} && \text{by (3)}\\
&= \frac{\sqrt{2}}{2^{n}}(1+e^{-i\omega})^{n+1} && \text{by definition of } z\\
&= \frac{\sqrt{2}}{2^{n}}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1}\\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1}
\end{aligned}$$

4. Proof that (3) $\impliedby$ (4):

$$\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \check{\mathsf{h}}(e^{i\omega})\\
&= \tilde{\mathsf{h}}(\omega)\\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} && \text{by (4)}\\
&= \frac{\sqrt{2}}{2^{n}}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}(1+e^{-i\omega})^{n+1}\\
&= \frac{\sqrt{2}}{2^{n}}(1+z^{-1})^{n+1}\Big|_{z\triangleq e^{i\omega}}
\end{aligned}$$

Example Q.20 (2 coefficient case). [28] Let $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ be an *MRA system* (Definition ?? page ??).

EX
$$\underbrace{\left\{\begin{array}{ll} 1. & \operatorname{supp}\phi(x) = [0:1] \quad \text{and} \\ 2. & \left(\!\left(\phi(x-k)\right)\!\right) \text{ forms a} \\ & \textit{partition of unity} \end{array}\right\}}_{\text{(A)}} \iff \underbrace{h_n = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2} & \text{for } n = 0 \\ \frac{\sqrt{2}}{2} & \text{for } n = 1 \\ 0 & \text{otherwise} \end{array}\right\}}_{\text{(B)}} \iff \underbrace{\left\{\phi(x) = \mathrm{N}_0(x)\right\}}_{\text{(C)}}$$

✎PROOF:

1. Proof that (A) $\implies$ (B):

   (a) lemma: Only h_0 and h_1 are *non-zero*; All other coefficients h_k are 0. Proof: This follows from $\operatorname{supp}\phi(x) = [0:1]$ (Definition ?? page ??) and by Theorem ?? page ??.

   (b) lemma (equations for $\left(\!\left(h_k\right)\!\right)$): Because $\left(\!\left(h_k\right)\!\right)$ is a *scaling coefficient sequence* (Definition ?? page ??), it must satisfy the *admissibility equation* (Theorem ?? page ??). And because $\left(\!\left(\phi(x-k)\right)\!\right)$ forms a *partition of unity*, it must satisfy the equations given by Theorem ?? (page ??). (1a) lemma and these two constraints yield two simultaneous equations and two unknowns:
$$\begin{array}{rcll} h_0 + h_1 & = & \sqrt{2} & (\textit{admissibility condition}) \\ h_0 - h_1 & = & 0 & (\textit{partition of unity / zero at } -1 \textit{ / vanishing 0th moment}) \end{array}$$

   (c) lemma: The equations provided by (1b) lemma can be expressed in matrix algebra form as follows…
$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

   (d) lemma: The *inverse* $\mathbf{A}^{-1}$ of $\mathbf{A}$ can be expressed as demonstrated below…
$$\left[\begin{array}{rr|rr} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rr|rr} 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rr|rr} 2 & 0 & 1 & 1 \\ 0 & -1 & -1/2 & 1/2 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right]$$
$$\implies \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (e) Proof for the values of $\left(\!\left(h_k\right)\!\right)$ (B):
$$\begin{aligned} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} &= \mathbf{A}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} && \text{by (1c) lemma} \\ &= \mathbf{A}^{-1}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} && \text{by (1c) lemma} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} && \text{by (1d) lemma} \\ &= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

[28] Haar (1910), Wojtaszczyk (1997) pages 14–15 ⟨"Sources and comments"⟩

2. Proof that (B) $\implies$ (C):

$$\begin{aligned}
\text{(B)} \implies \phi(x) &= \sum_{k\in\mathbb{Z}} h_k \sqrt{2}\phi(2x-k) && \textit{dilation equation} && \text{(Theorem ?? page ??)}\\
&= \sum_{k=0}^{k=1} \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\phi(2x-k) && \text{by item (1e) page 372}\\
&= \sum_{k=0}^{k=1} \phi(2x-k)\\
&= \sum_{k=0}^{k=1} \binom{1}{k}\phi(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition ?? page ??)}\\
&\implies \text{(D)} && \text{by } \textit{B-spline dyadic decomposition} && \text{(Theorem Q.9 page 367)}
\end{aligned}$$

3. Proof that (B) $\impliedby$ (C):

$$\begin{aligned}
\text{(C)} \implies \mathrm{N}_0(x) &= \sum_{k=0}^{k=1} \binom{1}{k}\mathrm{N}_0(2x-k) && \text{by } \textit{B-spline dyadic decomposition} && \text{(Theorem Q.9 page 367)}\\
&= \sum_{k=0}^{k=1} \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathrm{N}_0(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition ?? page ??)}\\
&= \sum_{k\in\mathbb{Z}} h_k\sqrt{2}\mathrm{N}_0(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition ?? page ??)}\\
&\implies \text{(B)}
\end{aligned}$$

4. Proof that (A) $\impliedby$ (C):
   1. Proof that (C) $\implies \operatorname{supp}\phi(x) = [0:1]$: by Theorem Q.4 (page 354)
   2. Proof that (C) $\implies \langle\!\langle \phi(x-k) \rangle\!\rangle$ forms a *partition of unity*: by Theorem Q.7 (page 360)

$\Rightarrow$

EX

n=0,	(÷0!)	1;								
n=1,	(÷1!)	1,	0;							
		−1,	2;							
n=2,	(÷2!)	1,	0,	0;						
		−2,	6,	−3;						
		1,	−6,	9;						
n=3,	(÷3!)	1,	0,	0,	0;					
		−3,	12,	−12,	4;					
		3,	−24,	60,	−44;					
		−1,	12,	−48,	64;					
n=4,	(÷4!)	1,	0,	0,	0,	0;				
		−4,	20,	−30,	20,	−5;				
		6,	−60,	210,	−300,	155;				
		−4,	60,	−330,	780,	−655;				
		1,	−20,	150,	−500,	625;				
n=5,	(÷5!)	1,	0,	0,	0,	0,	0;			
		−5,	30,	−60,	60,	−30,	6;			
		10,	−120,	540,	−1140,	1170,	−474;			
		−10,	180,	−1260,	4260,	−6930,	4386;			
		5,	−120,	1140,	−5340,	12270,	−10974;			
		−1,	30,	−360,	2160,	−6480,	7776;			
n=6,	(÷6!)	1,	0,	0,	0,	0,	0,	0;		
		−6,	42,	−105,	140,	−105,	42,	−7;		
		15,	−210,	1155,	−3220,	4935,	−3990,	1337;		
		−20,	420,	−3570,	15680,	−37590,	47040,	−24178;		
		15,	−420,	4830,	−29120,	96810,	−168000,	119182;		
		−6,	210,	−3045,	23380,	−100065,	225750,	−208943;		
		1,	−42,	735,	−6860,	36015,	−100842,	117649;		
n=7,	(÷7!)	1,	0,	0,	0,	0,	0,	0,	0;	
		−7,	56,	−168,	280,	−280,	168,	−56,	8;	
		21,	−336,	2184,	−7560,	15400,	−18648,	12488,	−3576;	
		−35,	840,	−8400,	45360,	−143360,	267120,	−273280,	118896;	
		35,	−1120,	15120,	−111440,	483840,	−1238160,	1733760,	−1027984;	
		−21,	840,	−14280,	133560,	−741160,	2436840,	−4391240,	3347016;	
		7,	−336,	6888,	−78120,	528920,	−2135448,	4753336,	−4491192;	
		−1,	56,	−1344,	17920,	−143360,	688128,	−1835008,	2097152;	
n=8,	(÷8!)	1,	0,	0,	0,	0,	0,	0,	0,	0;
		−8,	72,	−252,	504,	−630,	504,	−252,	72,	−9
		28,	−504,	3780,	−15624,	39690,	−64008,	64260,	−36792,	9207
		−56,	1512,	−17388,	111384,	−436590,	1079064,	−1650348,	1432872,	−541917
		70,	−2520,	39060,	−340200,	1821330,	−6146280,	12800340,	−15082200,	7715619
		−56,	2520,	−49140,	541800,	−3691170,	15903720,	−42324660,	63667800,	−41503131
		28,	−1512,	35532,	−474264,	3929310,	−20674584,	67410252,	−124449192,	99584613
		−8,	504,	−13860,	217224,	−2121210,	13208328,	−51179940,	112731192,	−107948223
		1,	−72,	2268,	−40824,	459270,	−3306744,	14880348,	−38263752,	43046721

Table Q.1: Coefficients of the *B-spline*s $\mathsf{N}_n(x)$ multiplied by $n!$ (Example Q.9 page 350)

APPENDIX R

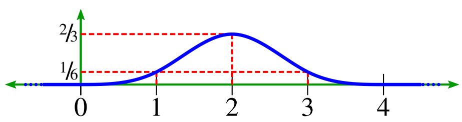
SOURCE CODE

The free and open source software package Maxima has been used to compute some of the algebraic expressions for *B-splines* used in APPENDIX Q (page 343):

```
/*========================================================================*
 * Daniel J. Greenhoe                                                     *
 * Maxima script file                                                     *
 * To execute this script, start Maxima in a command window               *
 * in the subdirectory containing this file (e.g. c:\math\maxima\)        *
 * and then after the (%i...) prompt enter                                *
 *    batchload("bspline.max")$                                           *
 * Data produced will be written to the file "bsplineout.txt".            *
 * reference: http://maxima.sourceforge.net/documentation.html            *
 *========================================================================*/
/*--------------------------------------
 * initialize script
 *--------------------------------------*/
reset();
kill(all);
load(orthopoly);
display2d:false; /* 2-dimensional display */
writefile("bsplineout.txt");
/*--------------------------------------
 * n = B-spline order parameter
 * may be set to any value in {1,2,3,...}
 *--------------------------------------*/
n:2;
print("=======================================================================");
print("Daniel J. Greenhoe");
print("Output file for nth order B-spline Nn(x) calculation, n=",n," .");
print("Output produced using Maxima running the script file bspline.max");
print("=======================================================================");
Nnx:(1/n!)*sum((-1)^k*binomial(n+1,k)*(x-k)^n*unit_step(x-k),k,0,n+1);
print("-----------------------------------------------------------");
print("           n+1      k  (n+1)       n                   ");
print(" n! Nn(x) = SUM (-1)  (    )  (x-k)   step(x-k)  ,n=",n);
print("           k=0        ( k )                            ");
print("           ",n+1,"       k (",n+1,")        ",n);
print(n,"! Nn(x) = SUM (-1)  (    )  (x-k)   step(x-k)");
print("           k=0        ( k )");
print("         = ",expand(n!*Nnx));
print("-----------------------------------------------------------");
assume(x<=0);   print(n!,"N(x)= ",expand(n!*Nnx)," for x<=0");  forget(x<=0);
for i:0 thru n step 1 do(
  assume(x>i,x<(i+1)),
  print(n!,"N(x)= ",expand(n!*Nnx)," for ",i,"<x<",i+1),
  tex(expand(n!*Nnx),"djg.tex"),/*write output in TeX format to file "djg.tex"*/
  forget(x>i,x<(i+1))
  );
assume(x>(n+1)); print(n!,"N(x)= ",expand(n!*Nnx)," for x>",n+1); forget(x>(n+1));
```

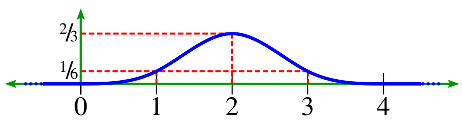
```
print("-------------------------------------------------------");
print(" values at some specific points x:                      ");
print("-------------------------------------------------------");
y:Nnx,x=(n+1)/2;print("N(",(n+1)/2,")= ",y," (center value)");
y:Nnx,x=(n+2)/2;print("N(",(n+2)/2,")= ",y);
y:Nnx,x=(n+3)/2;print("N(",(n+3)/2,")= ",y);
/*---------------------------------------
 * close output file
 *---------------------------------------*/
closefile();
```

Once the polynomial expressions for a *B-spline* have been calculated, they can be plotted within a LaTeX environment using the pst-plot package along with a LaTeX source file such as the following:[1]




```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% N_3(x) B-spline
% nominal unit = 10mm
%==========================================================================
\begin{pspicture}(-1,-0.5)(5,1)
  %---------------------------------------
  % parameters
  %---------------------------------------
  \psset{plotpoints=64,labelsep=1pt}%
  %---------------------------------------
  % axes
  %---------------------------------------
  \psaxes[linewidth=0.75pt,linecolor=axis,yAxis=false,ticks=x,labels=x]{<->}(0,0)(-1,0)(5,1)% x axis
  \psaxes[linewidth=0.75pt,linecolor=axis,xAxis=false,ticks=x,labels=x]{->}(0,0)(-1,0)(5,1)%  y axis
  %---------------------------------------
  % annotation
  %---------------------------------------
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](2,0)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.667)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](1,0)(1,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](3,0)(3,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.1667)(3,0.1667)%
  \uput[180](0,0.667){$\sfrac{2}{3}$}%
  \uput[180](0,0.1667){$\sfrac{1}{6}$}%
  %---------------------------------------
  % function plot
  %---------------------------------------
  \psplot{0}{1}{+1 x 3 exp mul                                          6 div}% for 0<=x<=1
  \psplot{1}{2}{-3 x 3 exp mul +12 x 2 exp mul add -12 x mul add  +4 add 6 div}% for 1<=x<=2
  \psplot{2}{3}{+3 x 3 exp mul -24 x 2 exp mul add +60 x mul add -44 add 6 div}% for 2<=x<=3
  \psplot{3}{4}{-1 x 3 exp mul +12 x 2 exp mul add -48 x mul add +64 add 6 div}% for 3<=x<=4
  \psline(0,0)(-0.5,0)\psline[linestyle=dotted](-0.5,0)(-0.75,0)%           % for x<=0
  \psline(4,0)(4.5,0)\psline[linestyle=dotted](4.5,0)(4.75,0)%              % for x>=4
\end{pspicture}%
```

Alternatively, one can plot $\mathsf{N}_3(x)$ more or less directly from the equation given in Theorem Q.1 (page 347) without first calculating the polynomial expressions:



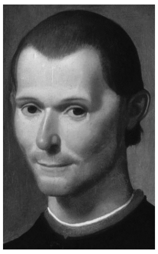

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% N_3(x) B-spline
% nominal unit = 10mm
%==========================================================================
\begin{pspicture}(-1,-0.5)(5,1)
  %---------------------------------------
  % parameters
  %---------------------------------------
  \psset{plotpoints=64,labelsep=1pt}%
```

[1] For help with PostScript® math operators, see Adobe (1999), pages 508–509 ⟨Arithmetic and Math Operators⟩.

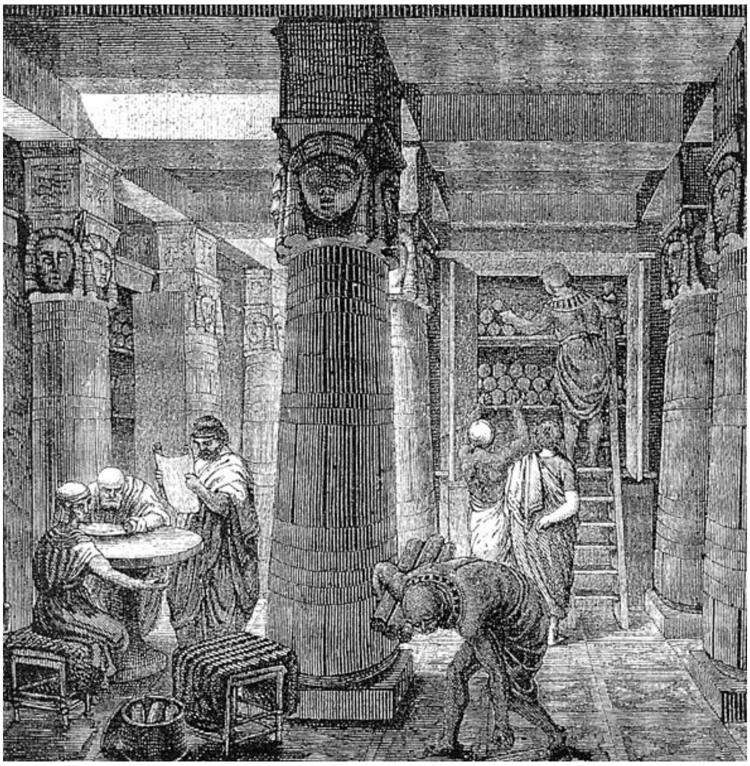
```
 %-------------------------------------
 % axes
 %-------------------------------------
  \psaxes[linewidth=0.75pt,linecolor=axis,yAxis=false,ticks=x,labels=x]{<->}(0,0)(-1,0)(5,1)% x axis
  \psaxes[linewidth=0.75pt,linecolor=axis,xAxis=false,ticks=x,labels=x]{->}(0,0)(-1,0)(5,1)%  y axis
 %-------------------------------------
 % annotation
 %-------------------------------------
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](2,0)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.667)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](1,0)(1,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](3,0)(3,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.1667)(3,0.1667)%
  \uput[180](0,0.667){$\sfrac{2}{3}$}%
  \uput[180](0,0.1667){$\sfrac{1}{6}$}%
 %--------------------------------------------------------------------
 % for n=3
 %         1   n+1         (n+1)                      1    4            ( 4 )
 % N_n(x)--- SUM (-1)^k (     ) (x-k)^n s(x-k) = --- SUM  (-1)^k (     ) (x-k)^3 s(x-k)
 %         n!  k=1         ( k )                      3!  k=1          ( k )
 %
 % where s(x) = 0 for x<0 and 1 for x>=0  (step function)
 %--------------------------------------------------------------------
  \psplot{0}{1}  {1 x 0 sub 3 exp mul   6 div}%
  \psplot{1}{2}  {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 div}%
  \psplot{2}{3}  {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   6 div}%
  \psplot{3}{4}  {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   4 x 3 sub 3 exp mul sub   6 div}%
  \psplot{4}{4.5}{1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   4 x 3 sub 3 exp mul sub   1 x 4 sub 3 exp mul add 6 div}%
 %    N_3(x) = [(4choose0)(x-0)^3    - (4choose1)(x-1)^3      + (4choose2)(x-2)^3      - (4choose3)(x-3)^3      + (4choose4)(x-4)^3      ]/3!
 %           = [         1 (x-0)^3    -          4 (x-1)^3      +          6 (x-2)^3      -          4 (x-3)^3      +          1 (x-4)^3      ]/6
  \psline(0,0)(-0.5,0)%
  \psline[linestyle=dotted](-0.5,0)(-0.75,0)%
  \psline[linestyle=dotted](4.5,0)(4.75,0)%
\end{pspicture}%
```

Back Matter

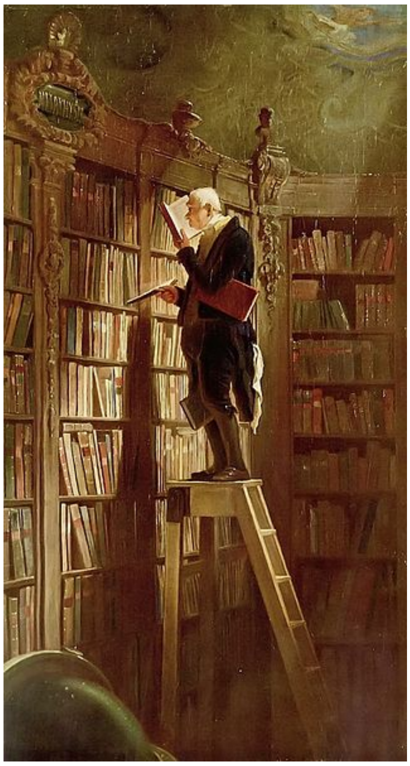
❝It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.❞
Niels Henrik Abel (1802–1829), Norwegian mathematician [2]



❝When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.❞
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [3]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850 [4]

❝To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.❞
Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [5]

[2] quote: Simmons (2007), page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

[3] quote: Machiavelli (1961), page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

[4] http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

[5] quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

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