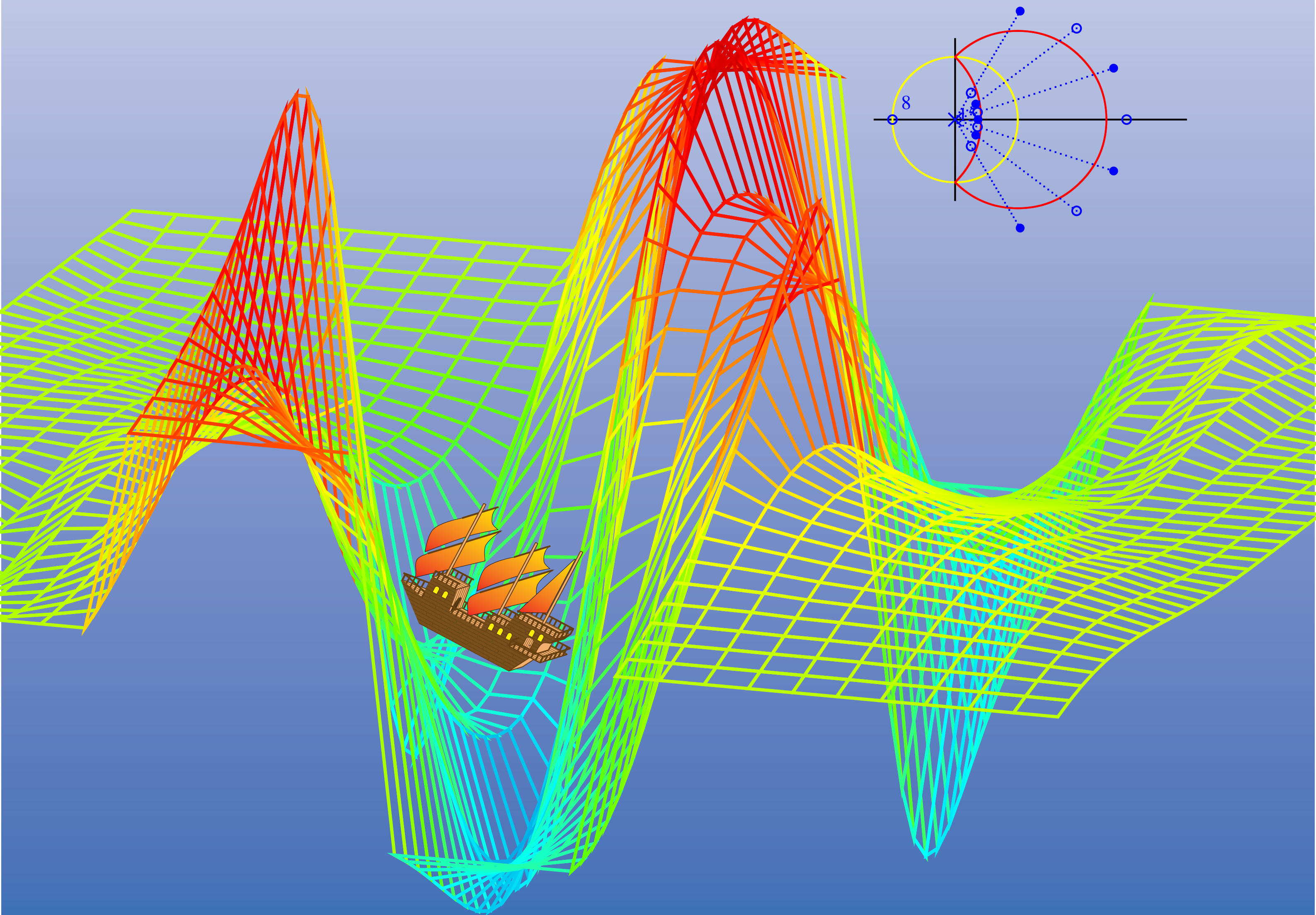
wavelet
Structure and Design



Daniel J. Greenhoe

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3

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The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]

[1] Paine (2000) page 63 ⟨Golden Hind⟩

This book is dedicated to Jesus,
who brings us over the waves
and through the storms of life,
if we but put our trust in Him;
and to my wife "Apple", for the
joy she has brought and
continues to bring to my life
and to the lives of our children,
Jonathan and Katie.

— Daniel

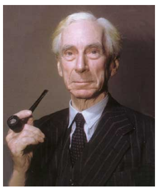
❝Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?❞

❝Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.❞

Alfred Edward Housman, English poet (1859–1936) [2]

❝The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.❞

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



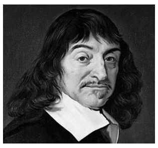
❝As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.❞

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

[2] quote: Housman (1936), page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg
[3] quote: Ewen (1961), page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg
[4] quote: Heijenoort (1967), page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.❞



❝Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.❞

Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \ldots$
$\mathbb{N}$	natural numbers	$1, 2, 3, \ldots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\ldots, -3, -2, -1, 0$

...continued on next page...

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), http://en.wikipedia.org/wiki/Descartes, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

symbol	description	
$\mathbb{Z}^-$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_o$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_e$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^+$	positive real numbers	$(0, \infty)$
$\mathbb{R}^-$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^*$	extended real numbers	$\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \triangledown)$	ordered pair	
$\lvert z \rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}(®)$	domain of a relation ®	
$\mathcal{I}(®)$	image of a relation ®	
$\mathcal{R}(®)$	range of a relation ®	
$\mathcal{N}(®)$	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\nsubseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^c	set complement	
$\lvert \cdot \rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	"true" condition	
0	"false" condition	
$\neg$	logical NOT operation	

...continued on next page...

symbol	description	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\leq$	reflexive ordering relation	"less than or equal to"
$\geq$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\varnothing$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_{\mathsf{j}}(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_{\mathsf{j}}(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_{\mathsf{j}}(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		
$\tilde{\mathbf{F}}$	*Fourier Transform* operator (Definition K.2 page 257)	
$\hat{\mathbf{F}}$	*Fourier Series* operator (Definition J.1 page 253)	

...continued on next page...

symbol	description
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator (Definition P.1 page 355)
$\mathbf{Z}$	*Z-Transform* operator (Definition O.4 page 342)
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $\left(\!\left(x_n \in \mathbb{C}\right)\!\right)_{n\in\mathbb{Z}}$
$\breve{\mathsf{x}}(z)$	*Z-Transform* of a sequence $\left(\!\left(x_n \in \mathbb{C}\right)\!\right)_{n\in\mathbb{Z}}$

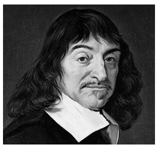
SYMBOL INDEX

CONTENTS

CHAPTER 1

TRANSVERSAL OPERATORS

❝Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:❞



❝I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them.❞

René Descartes, philosopher and mathematician (1596–1650) [1]

1.1 Families of Functions

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition 1.1 page 1). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition 1.1. *Let X and Y be sets.*

DEF: *The space Y^X represents the set of all functions with* DOMAIN *X and* RANGE *Y such that*
$$Y^X \triangleq \{\mathsf{f}(x)|\mathsf{f}(x) : X \to Y\}$$

Definition 1.2. [2] *Let X be a set.*

[1] quote: Descartes (1637b)
translation: Descartes (1637c) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

[2] Aliprantis and Burkinshaw (1998), page 126, Hausdorff (1937), page 22, de la Vallée-Poussin (1915) page 440

DEF The **indicator function** $\mathbb{1} \in \{0,1\}^{2^X}$ *is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \qquad \forall x \in X,\ A \in 2^X \\ 0 & \text{for } x \notin A \qquad \forall x \in X,\ A \in 2^X \end{cases}$$

The indicator function $\mathbb{1}$ *is also called the* ***characteristic function***.

1.2 Definitions and algebraic properties

Much of the wavelet theory developed in this text is constructed using the **translation operator T** and the **dilation operator D** (next).

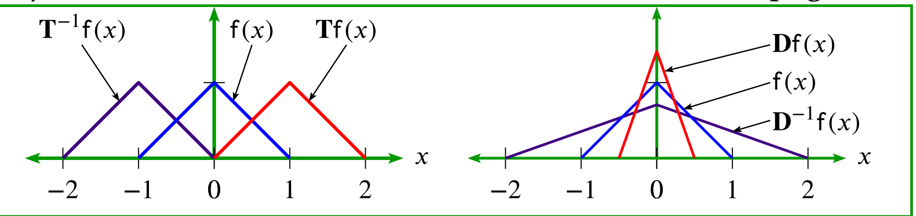
Definition 1.3. [3]

DEF
$\mathbf{T}_\tau$ *is a* ***translation operator*** *on* $\mathbb{C}^\mathbb{C}$ *if* $\quad \mathbf{T}_\tau \mathsf{f}(x) \triangleq \mathsf{f}(x-\tau) \quad \forall \mathsf{f} \in \mathbb{C}^\mathbb{C}$.

$\mathbf{D}_\alpha$ *is a* ***dilation operator*** *on* $\mathbb{C}^\mathbb{C}$ *if* $\quad \mathbf{D}_\alpha \mathsf{f}(x) \triangleq \mathsf{f}(\alpha x) \quad \forall \mathsf{f} \in \mathbb{C}^\mathbb{C}$.

Moreover, $\mathbf{T} \triangleq \mathbf{T}_1$ *and* $\mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2$.

Example 1.1. Let **T** and **D** be defined as in Definition 1.3 (page 2).

EX


Proposition 1.1. *Let* $\mathbf{T}_\tau$ *be a* TRANSLATION OPERATOR *(Definition 1.3 page 2)*.

PRP
$$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) \qquad \forall \mathsf{f} \in \mathbb{R}^\mathbb{R} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x) \text{ is PERIODIC with period } \tau \right)$$

✎PROOF:

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n \mathsf{f}(x+\tau) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n\tau+\tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 1.3 page 2)} \\ &= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m\tau) && \text{where } m \triangleq n-1 && \implies n = m+1 \\ &= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m \mathsf{f}(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition 1.3 page 2)} \end{aligned}$$

⇰

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators **T** and **D** both exist as operators, as demonstrated next.

Proposition 1.2 (transversal operator inverses). *Let* **T** *and* **D** *be as defined in Definition 1.3 page 2.*

PRP
T *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^\mathbb{C}$ *expressed by the relation*

$$\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1) \qquad \forall \mathsf{f} \in \mathbb{C}^\mathbb{C} \qquad \textit{(translation operator inverse)}.$$

D *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^\mathbb{C}$ *expressed by the relation*

$$\mathbf{D}^{-1}\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right) \qquad \forall \mathsf{f} \in \mathbb{C}^\mathbb{C} \qquad \textit{(dilation operator inverse)}.$$

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2000) page 473, Packer (2004) page 260, Benedetto and Zayed (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a), page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996), page 81, Dai and Larson (1998) page 2

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\implies \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} && \text{(Definition 1.3 page 2)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} && \text{(Definition 1.3 page 2)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\implies \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

✎

Proposition 1.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 1.3 page 2.*
Let $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP
$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^jx-n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

1.3 Linear space properties

Proposition 1.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 2.*

PRP
$$\mathbf{D}^j\mathbf{T}^n[\mathsf{f}\mathsf{g}] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

✎PROOF:

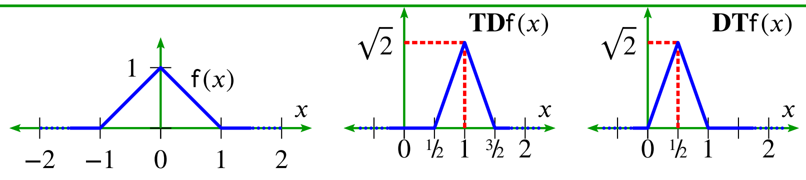
$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n[\mathsf{f}(x)\mathsf{g}(x)] &= 2^{j/2}\mathsf{f}\left(2^jx-n\right)\mathsf{g}\left(2^jx-n\right) && \text{by Proposition 1.3 page 3}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^jx-n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^jx-n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition 1.3 page 3}
\end{aligned}$$

✎

In general the operators $\mathbf{T}$ and $\mathbf{D}$ are *noncommutative* ($\mathbf{TD} \neq \mathbf{DT}$), as demonstrated by Counterexample 1.1 (next) and Proposition 1.5 (page 4).

Counterexample 1.1.

CNT As illustrated to the right, it is **not** always true that $\mathbf{TD} = \mathbf{DT}$:



Proposition 1.5 (commutator relation). [4] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 2.*

PRP

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j && \forall j,n\in\mathbb{Z} \\ \mathbf{T}^n\mathbf{D}^j &= \mathbf{D}^j\mathbf{T}^{2^j n} && \forall n,j\in\mathbb{Z} \end{aligned}$$

✎PROOF:

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^{2^j n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - 2^j n) && \text{by Proposition 1.4 page 3} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 130)} \\ &= \mathbf{T}^n 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)} \\ &= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)} \end{aligned}$$

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - n) && \text{by Proposition 1.4 page 3} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j\left[x-2^{-j/2}n\right]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition A.6 page 130)} \\ &= \mathbf{T}^{2^{-j/2}n}2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)} \\ &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)} \end{aligned}$$

✏

1.4 Inner product space properties

In an inner product space, every operator has an *adjoint* (Proposition D.3 page 169) and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator $\mathbf{U}$ coincide, then $\mathbf{U}$ is said to be *unitary* (Definition D.14 page 179). And in this case, $\mathbf{U}$ has several nice properties (see Proposition 1.9 and Theorem 1.1 page 7). Proposition 1.6 (next) gives the adjoints of $\mathbf{D}$ and $\mathbf{T}$, and Proposition 1.7 (page 5) demonstrates that both $\mathbf{D}$ and $\mathbf{T}$ are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ (Corollary K.1 page 259) and the *rotation matrix operator* (Example D.10 page 183).

Proposition 1.6. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 1.3 page 2) with* ADJOINT $\mathbf{T}^*$ *and* $\mathbf{D}$ *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$ *(Definition D.8 page 165).*

PRP

$$\begin{aligned} \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(TRANSLATION OPERATOR ADJOINT)} \\ \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(DILATION OPERATOR ADJOINT)} \end{aligned}$$

✎PROOF:

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a), page 641, Goodman et al. (1993b), page 110

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^* && \text{(Definition D.8 page 165)} \\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)} \\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1 \\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^* && \text{(Definition D.8 page 165)} \\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)} \\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\,\mathrm{d}u && \text{by definition of } \langle \triangle \,|\, \triangledown\rangle \\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^* \mathrm{d}x && \text{where } x = 2u \\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \triangledown\rangle \\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}$$

✏

Proposition 1.7. [5] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 (page 2). Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 1.2 (page 2).*

PRP: $\mathbf{T}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $\quad(\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $L^2_{\mathbb{R}})$.
$\mathbf{D}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $\quad(\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $L^2_{\mathbb{R}})$.

✎PROOF:

$$\begin{aligned}
&\mathbf{T}^{-1} = \mathbf{T}^* && \text{by Proposition 1.2 page 2 and Proposition 1.6 page 4} \\
&\implies \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition D.14 page 179)}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{D}^{-1} = \mathbf{D}^* && \text{by Proposition 1.2 page 2 and Proposition 1.6 page 4} \\
&\implies \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators (Definition D.14 page 179)}
\end{aligned}$$

✏

1.5 Normed linear space properties

Proposition 1.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR *(Definition 1.3 page 2).*

PRP: $\left\{\begin{array}{ll} (1). & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) \quad \textit{and} \\ (2). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f}\in L^2_{\mathbb{R}}$

✎PROOF:

---

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$\begin{aligned}\mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)}\\ &= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis}\end{aligned}$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$\begin{aligned}\|\mathsf{f}(x)-\mathsf{f}(x+h)\| &= \|\mathsf{f}(x)-\mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis}\\ &= \|0\|\\ &= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|\\ &\le \varepsilon\\ &\implies \forall h>0,\ \exists\varepsilon \quad \text{such that} \quad \|\mathsf{f}(x)-\mathsf{f}(x+h)\|<\varepsilon\\ &\overset{\text{def}}{\iff} \mathsf{f}(x) \text{ is } \textit{continuous}\end{aligned}$$

3. Proof that (1,2) $\implies$ *constant* property:
   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \neq \mathsf{f}(y)$.
   (b) Let $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{N}}$ be a sequence with limit x and $\left(\!\left(y_n\right)\!\right)_{n\in\mathbb{N}}$ a sequence with limit y
   (c) Then

$$\begin{aligned}0 &< \|\mathsf{f}(x)-\mathsf{f}(y)\| && \text{by assumption in item (3a) page 6}\\ &= \lim_{n\to\infty}\left\|\mathsf{f}\left(x_n\right)-\mathsf{f}\left(y_n\right)\right\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 6}\\ &= \lim_{n\to\infty}\left\|\mathsf{f}\left(2^m x_n\right)-\mathsf{f}\left(2^\ell y_n\right)\right\| \quad \forall m,\ell\in\mathbb{Z} && \text{by (1)}\\ &= 0\end{aligned}$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

⇰

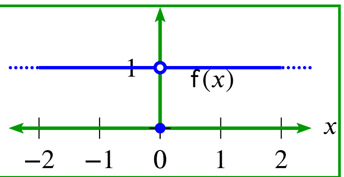
Remark 1.1.

REM In Proposition 1.8 page 5, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample 1.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x=0\\ 1 & \text{otherwise.}\end{cases}$

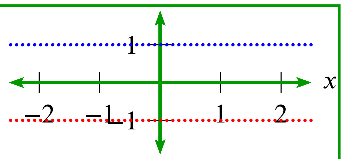
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Counterexample 1.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\backslash\mathbb{Q}$ the set of *irrational numbers*.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x\in\mathbb{Q}\\ -1 & \text{for } x\in\mathbb{R}\backslash\mathbb{Q}.\end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Proposition 1.9 (Operator norm). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 2. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 1.2 page 2. Let* $\mathbf{T}^{*}$ *and* $\mathbf{D}^{*}$ *be as in Proposition 1.6 page 4. Let* $\|\cdot\|$ *and* $\langle\triangle\,|\,\nabla\rangle$ *be as in Definition E.1 page 185. Let* $|\!|\!|\cdot|\!|\!|$ *be the operator norm* (Definition D.6 page 161) *induced by* $\|\cdot\|$.

PRP $|\!|\!|\mathbf{T}|\!|\!| = |\!|\!|\mathbf{D}|\!|\!| = |\!|\!|\mathbf{T}^{*}|\!|\!| = |\!|\!|\mathbf{D}^{*}|\!|\!| = |\!|\!|\mathbf{T}^{-1}|\!|\!| = |\!|\!|\mathbf{D}^{-1}|\!|\!| = 1$

✎PROOF: These results follow directly from the fact that **T** and **D** are *unitary* (Proposition 1.7 page 5) and from Theorem D.25 page 180 and Theorem D.26 page 180. ✏

Theorem 1.1. *Let* **T** *and* **D** *be as in Definition 1.3 page 2.*
Let $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition 1.2 page 2. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition E.1 page 185.*

THM

1.	$\|\mathbf{T}f\|$	$= \|\mathbf{D}f\|$	$= \|f\|$	$\forall f \in \mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}f - \mathbf{T}g\|$	$= \|\mathbf{D}f - \mathbf{D}g\|$	$= \|f - g\|$	$\forall f, g \in \mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
3.	$\left\|\mathbf{T}^{-1}f - \mathbf{T}^{-1}g\right\|$	$= \left\|\mathbf{D}^{-1}f - \mathbf{D}^{-1}g\right\|$	$= \|f - g\|$	$\forall f, g \in \mathcal{L}^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
4.	$\langle \mathbf{T}f \mid \mathbf{T}g \rangle$	$= \langle \mathbf{D}f \mid \mathbf{D}g \rangle$	$= \langle f \mid g \rangle$	$\forall f, g \in \mathcal{L}^2_{\mathbb{R}}$	(SURJECTIVE)
5.	$\left\langle \mathbf{T}^{-1}f \mid \mathbf{T}^{-1}g \right\rangle$	$= \left\langle \mathbf{D}^{-1}f \mid \mathbf{D}^{-1}g \right\rangle$	$= \langle f \mid g \rangle$	$\forall f, g \in \mathcal{L}^2_{\mathbb{R}}$	(SURJECTIVE)

✎PROOF: These results follow directly from the fact that **T** and **D** are *unitary* (Proposition 1.7 page 5) and from Theorem D.25 page 180 and Theorem D.26 page 180. ✏

Proposition 1.10. *Let* **T** *be as in Definition 1.3 page 2. Let* $\mathbf{A}^*$ *be the* ADJOINT *(Definition D.8 page 165) of an operator* **A**. *Let the property "*SELF ADJOINT*" be defined as in Definition D.11 (page 173).*

PRP

$$\left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* \qquad \left(\text{The operator } \left[\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right] \text{ is SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) f(x) \,\middle|\, g(x) \right\rangle &= \left\langle \sum_{n\in\mathbb{Z}} f(x-n) \,\middle|\, g(x) \right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
&= \left\langle \sum_{n\in\mathbb{Z}} f(x+n) \,\middle|\, g(x) \right\rangle && \text{by } \textit{commutative} \text{ property} && \text{(Definition A.5 page 130)}\\
&= \sum_{n\in\mathbb{Z}} \langle f(x+n) \mid g(x) \rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \sum_{n\in\mathbb{Z}} \langle f(u) \mid g(u-n) \rangle && \text{where } u \triangleq x+n\\
&= \left\langle f(u) \,\middle|\, \sum_{n\in\mathbb{Z}} g(u-n) \right\rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \left\langle f(x) \,\middle|\, \sum_{n\in\mathbb{Z}} g(x-n) \right\rangle && \text{by change of variable: } u \to x\\
&= \left\langle f(x) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathbf{T}^n g(x) \right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) &= \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* && \text{by definition of } \textit{adjoint} && \text{(Proposition D.3 page 169)}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) &\text{ is } \textit{self-adjoint} && \text{by definition of } \textit{self-adjoint} && \text{(Definition D.11 page 173)}
\end{aligned}$$

✏

1.6 Fourier transform properties

Proposition 1.11. *Let* **T** *and* **D** *be as in Definition 1.3 page 2.*
Let **B** *be the* TWO-SIDED LAPLACE TRANSFORM *defined as* $[\mathbf{B}f](s) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-sx}\, dx$.

PRP

1. $\mathbf{B}\mathbf{T}^n = e^{-sn}\mathbf{B} \qquad \forall n\in\mathbb{Z}$
2. $\mathbf{B}\mathbf{D}^j = \mathbf{D}^{-j}\mathbf{B} \qquad \forall j\in\mathbb{Z}$
3. $\mathbf{D}\mathbf{B} = \mathbf{B}\mathbf{D}^{-1} \qquad \forall n\in\mathbb{Z}$
4. $\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} = \mathbf{D} \qquad \forall n\in\mathbb{Z}$ ($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5. $\mathbf{D}\mathbf{B}\mathbf{D} = \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1} = \mathbf{B} \qquad \forall n\in\mathbb{Z}$

✎PROOF:

$$\begin{aligned}
\mathbf{B}\mathbf{T}^n \mathsf{f}(x) &= \mathbf{B}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathrm{d}u && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right]\\
&= e^{-sn}\,\mathbf{B}\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}\mathbf{D}^j \mathsf{f}(x) &= \mathbf{B}\left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}\left(2^j x\right)\right]e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathrm{d}u && \text{let } u \triangleq 2^j x \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\,\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathrm{d}u\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right] && \text{by Proposition 1.6 page 4 and} && \text{Proposition 1.7 page 5}\\
&= \mathbf{D}^{-j}\,\mathbf{B}\,\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{D}\mathbf{B}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x\right] && \text{by definition of } \mathbf{B}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathrm{d}x && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{u}{2}\right)\right]e^{-su}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\,e^{-su}\,\mathrm{d}u && \text{by Proposition 1.6 page 4 and} && \text{Proposition 1.7 page 5}\\
&= \mathbf{B}\mathbf{D}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{B}\mathbf{D} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition D.3 page 156)}\\
\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{-1} &= \mathbf{D}\mathbf{B}\mathbf{B}^{-1} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse} && \text{(Definition D.3 page 156)}\\
\mathbf{D}\mathbf{B}\mathbf{D} &= \mathbf{D}\mathbf{D}^{-1}\mathbf{B} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse} && \text{(Definition D.3 page 156)}\\
\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1} &= \mathbf{D}^{-1}\mathbf{D}\mathbf{B} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse} && \text{(Definition D.3 page 156)}
\end{aligned}$$

Corollary 1.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 2. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition K.2 page 257) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition E.1 page 185).*

COR

$$\begin{array}{llcll}
1. & \tilde{\mathbf{F}}\mathbf{T}^n & = & e^{-i\omega n}\tilde{\mathbf{F}} & \\
2. & \tilde{\mathbf{F}}\mathbf{D}^j & = & \mathbf{D}^{-j}\tilde{\mathbf{F}} & \\
3. & \mathbf{D}\tilde{\mathbf{F}} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1} & \\
4. & \mathbf{D} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} & = \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}} \\
5. & \tilde{\mathbf{F}} & = & \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} & = \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}
\end{array}$$

✎PROOF: These results follow directly from Proposition 1.11 page 7 with $\tilde{\mathbf{F}} = \mathbf{B}|_{s=i\omega}$. ✏

Proposition 1.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition 1.3 page 2. Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition K.2 page 257) of some function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition E.1 page 185).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = \frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{\mathsf{f}}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n\mathsf{f}(x) && \text{by Corollary 1.1 page 9 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}\mathsf{f}(x) && \text{by Corollary 1.1 page 9 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathsf{f}}(\omega)\\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{\mathsf{f}}\left(2^{-j}\omega\right) && \text{by Proposition 1.2 page 2}
\end{aligned}$$

✏

Proposition 1.13. *Let* $\mathbf{T}$ *be the translation operator (Definition 1.3 page 2). Let* $\tilde{\mathsf{f}}(\omega) \triangleq \tilde{\mathbf{F}}\mathsf{f}(x)$ *be the* FOURIER TRANSFORM *(Definition K.2 page 257) of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$*. Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT *(Definition P.1 page 355) of a sequence* $(a_n)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition O.2 page 341).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) \qquad \forall (a_n)\in\boldsymbol{\ell}^2_{\mathbb{R}}, \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}^n\phi(x)\\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\tilde{\mathbf{F}}\phi(x) && \text{by Corollary 1.1 page 9}\\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega)\\
&= \breve{\mathsf{a}}(\omega)\tilde{\phi}(\omega) && \text{by definition of } \textit{DTFT} \text{ (Definition P.1 page 355)}
\end{aligned}$$

✏

Definition 1.4. *Let* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition E.1 page 185). Let* $\boldsymbol{\ell}^2_{\mathbb{R}}$ *be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition E.1 page 185).*

DEF

$\mathbf{S}$ *is the* ***sampling operator*** *in* $\boldsymbol{\ell}^{2\,\boldsymbol{L}^2_{\mathbb{R}}}_{\mathbb{R}}$ *if* $\quad [\mathbf{S}\mathsf{f}(x)](n) \triangleq \mathsf{f}\left(\dfrac{2\pi}{\tau}n\right) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \tau\in\mathbb{R}^+$

Theorem 1.2 (Poisson Summation Formula—PSF). [6] *Let $\tilde{f}(\omega)$ be the* FOURIER TRANSFORM (Definition K.2 page 257) *of a function* $f(x) \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\mathbf{S}$ *be the* SAMPLING OPERATOR (Definition 1.4 page 9).

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n f(x) = \sum_{n\in\mathbb{Z}} f(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[f(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}} \tilde{f}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

✎PROOF:

1. lemma: If $h(x) \triangleq \sum_{n\in\mathbb{Z}} f(x+n\tau)$ then $h \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}h$. Proof:

   Note that $h(x)$ is *periodic* with period τ (Lemma F.3 page 200). Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $h \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}h$.

2. Proof of PSF (this theorem—Theorem 1.2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} f(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} f(x+n\tau) && \text{by (1) lemma page 10}\\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \left(\sum_{n\in\mathbb{Z}} f(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,dx\right]}_{\hat{\mathbf{F}}[\sum_{n\in\mathbb{Z}} f(x+n\tau)]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition J.1 page 253)}\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau f(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,dx\right]\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau} f(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,du\right] && \text{where } u \triangleq x+n\tau \implies x = u-n\tau\\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi kn}}\int_{u=n\tau}^{u=(n+1)\tau} f(u)e^{-i\frac{2\pi}{\tau}ku}\,du\right]\\
&= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} f(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,du\right]}_{[\tilde{\mathbf{F}}f]\left(\frac{2\pi}{\tau}k\right)} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem J.1 page 254)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}f(x)]\left(\frac{2\pi}{\tau}k\right)\right] && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition K.2 page 257)}\\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}f && \text{by definition of } \mathbf{S} \quad \text{(Definition 1.4 page 9)}\\
&= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{f}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem J.1 page 254)}
\end{aligned}$$

✏

Theorem 1.3 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{f}(\omega)$ *be the* FOURIER TRANSFORM (Definition K.2 page 257) *of a function* $f(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

[6] Andrews et al. (2001), page 624, Knapp (2005b) page 389, Lasser (1996), page 254, Rudin (1987), pages 194–195, Folland (1992), page 337
[7] Gauss (1900), page 88

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)}_{\text{summation in "frequency"}} = \underbrace{\frac{\tau}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\text{summation in "time"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(\omega) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)$, then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:
   Note that $\mathsf{h}(\omega)$ is periodic with period $2\pi/T$:
   $$\mathsf{h}\left(\omega + \frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau} + \frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + (n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \triangleq \mathsf{h}(\omega)$$
   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of IPSF (this theorem—Theorem 1.3):

$$\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)$$

$$= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) \qquad \text{by (1) lemma page 11}$$

$$= \hat{\mathbf{F}}^{-1}\underbrace{\left[\sqrt{\frac{\tau}{2\pi}}\int_0^{\frac{2\pi}{\tau}} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\,\mathrm{d}\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)\right]} \qquad \text{by definition of } \hat{\mathbf{F}} \qquad \text{(Definition J.1 page 253)}$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{\tau}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right)e^{-i\omega Tk}\,\mathrm{d}\omega\right]$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-i\left(u-\frac{2\pi}{\tau}n\right)Tk}\,\mathrm{d}u\right] \qquad \text{where } u \triangleq \omega + \frac{2\pi}{\tau}n \implies \omega = u - \frac{2\pi}{\tau}n$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi nk}}\int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{d}u\right]$$

$$= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathrm{d}u\right]$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(u)e^{iu(-\tau k)}\,\mathrm{d}u\right]}_{\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](-k\tau)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\left[\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](-k\tau)\right] \qquad \text{by value of } \tilde{\mathbf{F}}^{-1} \qquad \text{(Theorem K.1 page 258)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\,\tilde{\mathsf{f}} \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition 1.4 page 9)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\mathsf{f}(x) \qquad \text{by definition of } \tilde{\mathbf{F}} \qquad \text{(Definition K.2 page 257)}$$

$$= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathsf{f}(-k\tau) \qquad \text{by definition of } \mathbf{S} \qquad \text{(Definition 1.4 page 9)}$$

$$= \sqrt{\tau}\,\frac{1}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}}\mathsf{f}(-k\tau)e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem J.1 page 254)}$$

$$= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}}\mathsf{f}(-k\tau)e^{ik\tau\omega} \qquad \text{by definition of } \hat{\mathbf{F}}^{-1} \qquad \text{(Theorem J.1 page 254)}$$

$$= \frac{\tau}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\mathsf{f}(m\tau)e^{-i\omega m\tau} \qquad \text{let } m \triangleq -k$$

Remark 1.2. The left hand side of the *Poisson Summation Formula* (Theorem 1.2 page 10) is very similar to the *Zak Transform* $\mathbf{Z}$: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark 1.3. A generalization of the *Poisson Summation Formula* (Theorem 1.2 page 10) is the **Selberg Trace Formula**.[9]

1.7 Examples

Example 1.2 (linear functions). [10] Let $\mathbf{T}$ be the *translation operator* (Definition 1.3 page 2). Let $\mathcal{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX
1. $\{x,\ \mathbf{T}x\}$ is a *basis* for $\mathcal{L}(\mathbb{C},\mathbb{C})$ and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x \qquad \forall \mathsf{f}\in\mathcal{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition 1.3 page 2}\\
&= (ax+b)|_{x=1}\, x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of f}\\
&= (a+b)x - b(x-1)\\
&= ax + bx - bx + b\\
&= ax+b\\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of f}
\end{aligned}$$

⇨

Example 1.3 (Cardinal Series). Let $\mathbf{T}$ be the *translation operator* (Definition 1.3 page 2). The *Paley-Wiener* class of functions $\boldsymbol{PW}^2_\sigma$ (Definition N.3 page 336) are those functions which are "*bandlimited*" with respect to their Fourier transform (Definition K.2 page 257). The cardinal series forms an orthogonal basis for such a space (Theorem N.3 page 337). The *Fourier coefficients* (Definition L.11 page 278) for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of $\mathsf{f}(x)$ taken at regular intervals (Theorem N.4 page 337). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x)\,|\,\mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\,\mathsf{d}x \triangleq \mathsf{f}(n)$$

EX
1. $\left\{\mathbf{T}^n\frac{\sin(\pi x)}{\pi x}\,\middle|\,n\in\mathbb{N}\right\}$ is a *basis* for $\boldsymbol{PW}^2_\sigma$ and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty}\mathsf{f}(n)\mathbf{T}^n\frac{\sin(\pi x)}{\pi x}}_{\text{Cardinal series}} \qquad \forall \mathsf{f}\in\boldsymbol{PW}^2_\sigma,\ \sigma\le\frac{1}{2}$

✎PROOF: See Theorem N.3 page 337. ⇨

[8] Janssen (1988), page 24, Zayed (1996), page 482
[9] Lax (2002), page 349, Selberg (1956), Terras (1999)
[10] Higgins (1996) page 2

Example 1.4 (Fourier Series).

EX

1. $\left\{ \mathbf{D}_n e^{ix} \middle| n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix} \quad \forall x\in(0:2\pi), \mathsf{f}\in\boldsymbol{L}(0:2\pi)$ where
3. $\alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x) \mathbf{D}_n e^{-ix}\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}(0:2\pi)$

✎PROOF: See Theorem J.1 page 254. ✏

Example 1.5 (Fourier Transform). [11]

EX

1. $\left\{ \mathbf{D}_\omega e^{ix} | \omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) \mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\tilde{\mathsf{f}}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \mathbf{D}_\omega e^{-ix}\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example 1.6 (Gabor Transform). [12]

EX

1. $\left\{ \left(\mathbf{T}_\tau e^{-\pi x^2} \right) \left(\mathbf{D}_\omega e^{ix} \right) \middle| \tau,\omega\in\mathbb{R} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \int_{\mathbb{R}} \mathsf{G}(\tau,\omega)\, \mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\mathsf{G}(\tau,\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x) \left(\mathbf{T}_\tau e^{-\pi x^2} \right) \left(\mathbf{D}_\omega e^{-ix} \right) \mathrm{d}x \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example 1.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX

1. $\left\{ \mathbf{D}^k \mathbf{T}^n \psi(x) \middle| k,n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha_{k,n} \mathbf{D}^k \mathbf{T}^n \psi(x) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\alpha_n \triangleq \int_{\mathbb{R}} \mathsf{f}(x) \mathbf{D}^k \mathbf{T}^n \psi^*(x)\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

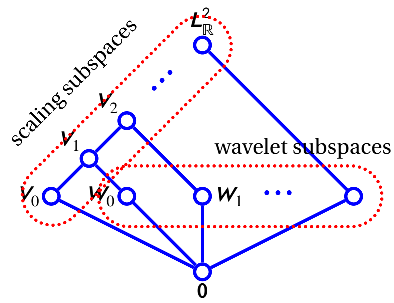
[11] cross reference: Definition K.2 page 257

[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

CHAPTER 2

MRA STRUCTURES

2.1 Introduction



In 1989, Stéphane G. Mallat introduced the *Multiresolution Analysis* (MRA, Definition 2.1 page 16) method for wavelet construction. The MRA has become the dominate wavelet construction method. This text uses the MRA method extensively, and combines the MRA "scaling subspaces" (Definition 2.1 page 16) with "wavelet subspaces" (Definition 3.1 page 43) to form a subspace structure as represented by the *Hasse diagram* to the right. The *Fast Wavelet Transform* combines both sets of subspaces as well, providing the results of projections onto both wavelet and MRA subspaces. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA.[1]

The MRA is an **analysis** of the linear space $\boldsymbol{L}^2_{\mathbb{R}}$. An analysis of a linear space $\boldsymbol{X}$ is any sequence $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ of linear subspaces of $\boldsymbol{X}$. The partial or complete reconstruction of $\boldsymbol{X}$ from $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **synthesis**.[2] An analysis is completely *characterized* by a *transform*. For example, a Fourier analysis is a sequence of subspaces with sinusoidal bases. Examples of subspaces in a Fourier analysis include $\boldsymbol{V}_1 = \operatorname{span}\left\{e^{ix}\right\}$, $\boldsymbol{V}_{2.3} = \operatorname{span}\left\{e^{i2.3x}\right\}$, $\boldsymbol{V}_{\sqrt{2}} = \operatorname{span}\left\{e^{i\sqrt{2}x}\right\}$, etc. A **transform** is loosely defined as a function that maps a family of functions into an analysis. A very useful transform (a "*Fourier transform*") for Fourier Analysis is (Definition K.2 page 257)

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x$$

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[1] Lemarié (1990), Mallat (1999) page 240

[2] The word *analysis* comes from the Greek word ἀνάλυσις, meaning "dissolution" (Perschbacher (1990), page 23 ⟨entry 359⟩), which in turn means "the resolution or separation into component parts" (Black et al. (2009), `http://dictionary.reference.com/browse/dissolution`)

2.2 Definition

A multiresolution analysis provides "coarse" approximations of a function in a linear space $\boldsymbol{L}^2_{\mathbb{R}}$ at multiple "scales" or "resolutions". Key to this process is a sequence of *scaling functions*. Most traditional transforms feature a single *scaling function* $\phi(x)$ set equal to one ($\phi(x) = 1$). This allows for convenient representation of the most basic functions, such as constants.[3] A multiresolution system, on the other hand, uses a generalized form of the scaling concept:

1. Instead of the scaling function simply being set *equal to unity* ($\phi(x) = 1$), a multiresolution system (Definition 2.3 page 25) is often constructed in such a way that the scaling function $\phi(x)$ forms a *partition of unity* (Definition Q.1 page 366) such that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi(x) = 1$.
2. Instead of there being *just one* scaling function, there is an entire sequence of scaling functions $\left(\!\left(\mathbf{D}^j\phi(x)\right)\!\right)_{j\in\mathbb{Z}}$, each corresponding to a different "*resolution*".

Definition 2.1. [4] *Let $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ be a sequence of subspaces on $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185). Let A^- be the* CLOSURE *of a set A.*

DEF

*The sequence $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **multiresolution analysis** on $\boldsymbol{L}^2_{\mathbb{R}}$ if*

1.	$\boldsymbol{V}_j = \boldsymbol{V}_j^-$	$\forall j\in\mathbb{Z}$	(CLOSED)	and
2.	$\boldsymbol{V}_j \subset \boldsymbol{V}_{j+1}$	$\forall j\in\mathbb{Z}$	(LINEARLY ORDERED)	and
3.	$\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}$		(DENSE *in* $\boldsymbol{L}^2_{\mathbb{R}}$)	and
4.	$\mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{D}\mathsf{f} \in \boldsymbol{V}_{j+1}$	$\forall j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(SELF-SIMILAR)	and
5.	$\exists\phi$ such that $\left\{\mathbf{T}^n\phi \mid n\in\mathbb{Z}\right\}$ *is a* RIESZ BASIS *for* $\boldsymbol{V}_0$.			

A MULTIRESOLUTION ANALYSIS *is also called an* ***MRA***.
*An element $\boldsymbol{V}_j$ of $\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ is a **scaling subspace** of the space $\boldsymbol{L}^2_{\mathbb{R}}$.*
*The pair $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right)\right)$ is a **multiresolution analysis space**, or **MRA space**.*
*The function ϕ is the **scaling function** of the* MRA SPACE.

The traditional definition of the *MRA* also includes the following:

1.	$\mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{T}^n\mathsf{f} \in \boldsymbol{V}_j$	$\forall n,j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(*translation invariant*)
2.	$\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \{\mathbb{0}\}$		(*greatest lower bound* is $\mathbb{0}$)

However, Proposition 2.1 (next) demonstrates that both of these follow from the *MRA* as defined in Definition 2.1.

Proposition 2.1. [5]

PRP

$$\left.\begin{array}{c}\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}} \text{ is an MRA} \\ \text{(Definition 2.1 page 16)}\end{array}\right\} \implies \left\{\begin{array}{lll} 1. & \mathsf{f} \in \boldsymbol{V}_j \iff \mathbf{T}^n\mathsf{f} \in \boldsymbol{V}_j \quad \forall n,j\in\mathbb{Z}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} & \text{(TRANSLATION INVARIANT) and} \\ 2. & \bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \{\mathbb{0}\} & \text{(GREATEST LOWER BOUND is } \mathbb{0}\text{)} \end{array}\right.$$

[3] Jawerth and Sweldens (1994) page 8
[4] Hernández and Weiss (1996) page 44, Mallat (1999) page 221 ⟨Definition 7.1⟩, Mallat (1989) page 70, Meyer (1992) page 21 ⟨Definition 2.2.1⟩, Christensen (2003) page 284 ⟨Definition 13.1.1⟩, Bachman et al. (2000) pages 451–452 ⟨Definition 7.7.6⟩, Walnut (2002) pages 300–301 ⟨Definition 10.16⟩, Daubechies (1992) pages 129–140 ⟨Riesz basis: page 139⟩
[5] Hernández and Weiss (1996) page 45 ⟨Theorem 1.6⟩, Wojtaszczyk (1997) pages 19–28 ⟨Proposition 2.14⟩, Pinsky (2002) pages 313–314 ⟨Lemma 6.4.28⟩

✎PROOF: Proof for (1):

$$\begin{aligned}
&\mathbf{T}^n \mathsf{f} \in \boldsymbol{V}_j \\
&\iff \mathbf{T}^n \mathsf{f} \in \operatorname{span}\left\{\mathbf{D}^j \mathbf{T}^m \phi | m\in\mathbb{Z}\right\} && \text{by definition of } \{\phi\} && \text{(Definition 2.1 page 16)}\\
&\iff \exists (\alpha_n)_{n\in\mathbb{Z}} \text{ such that } \mathbf{T}^n \mathsf{f}(x) = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{D}^j \mathbf{T}^k \phi(x) && \text{by definition of } \{\phi\} && \text{(Definition 2.1 page 16)}\\
&\iff \exists (\alpha_n)_{n\in\mathbb{Z}} \text{ such that } \mathsf{f}(x) = \mathbf{T}^{-n} \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{D}^j \mathbf{T}^k \phi(x) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
&\qquad = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{T}^{-n} \mathbf{D}^j \mathbf{T}^k \phi(x) \\
&\qquad = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{D}^j \mathbf{T}^{k-2n} \phi(x) && \text{by \textit{commutator relation}} && \text{(Proposition 1.5 page 4)}\\
&\qquad = \sum_{\ell\in\mathbb{Z}} \alpha_{\ell+2n} \mathbf{D}^j \mathbf{T}^{\ell} \phi(x) && \text{where } \ell \triangleq k-2n \implies && k = \ell + 2n\\
&\qquad = \sum_{\ell\in\mathbb{Z}} \beta_\ell \mathbf{D}^j \mathbf{T}^{\ell} \phi(x) && \text{where } \beta_\ell \triangleq \alpha_{\ell+2n}\\
&\qquad \iff \mathsf{f} \in \boldsymbol{V}_j && \text{by def. of } \{\mathbf{T}^n\phi\} && \text{(Definition 2.1 page 16)}
\end{aligned}$$

Proof for (2):

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace $\boldsymbol{V}_j$ such that
   $\mathbf{V}_j = \left\{\mathbf{P}_j \mathsf{f} | \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}\right\}$
2. lemma: Functions with *compact support* are *dense* in $\boldsymbol{L}^2_{\mathbb{R}}$. Therefore, we only need to prove that the proposition is true for functions with support in $[-R : R]$, for all $R > 0$.
3. For some function $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$, let $(\mathsf{f}_n)_{n\in\mathbb{Z}}$ be a sequence of functions in $\boldsymbol{L}^2_{\mathbb{R}}$ with *compact support* such that
   $\operatorname{supp}\mathsf{f}_n \subseteq [-R : R]$ for some $R > 0$ and $\mathsf{f}(x) = \lim_{n\to\infty} (\mathsf{f}_n(x))$.
4. lemma: $\bigcap \boldsymbol{V}_j = \{\mathbb{0}\} \iff \lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\| = 0 \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$. Proof:

$$\begin{aligned}
\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j &= \bigcap_{j\in\mathbb{Z}} \left\{\mathbf{P}_j \mathsf{f} | \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}\right\} && \text{by definition of } \boldsymbol{V}_j && \text{(definition 1 page 17)}\\
&= \lim_{j\to-\infty} \left\{\mathbf{P}_j \mathsf{f} | \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}\right\} && \text{by definition of } \cap\\
&= \mathbb{0} \iff \lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\| = 0 && \text{by \textit{nondegenerate} property of } \|\cdot\| && \text{(Definition D.5 page 160)}
\end{aligned}$$

5. lemma: $\lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\| = 0 \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$. Proof:
   Let $\mathbb{1}_{A(x)}$ be the *set indicator function* (Definition 1.2 page 1)

$$\begin{aligned}
&\lim_{j\to-\infty} \left\|\mathbf{P}_j \mathsf{f}\right\|^2 \\
&= \lim_{j\to-\infty} \left\|\mathbf{P}_j \lim_{n\to\infty} (\mathsf{f}_n)\right\|^2 && \text{by definition 3 page 17}\\
&\le \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbf{P}_j \lim_{n\to\infty} (\mathsf{f}_n) \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by \textit{frame property}} && \text{(Proposition L.5 page 288)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \lim_{n\to\infty} (\mathsf{f}_n) \,|\, \mathbf{D}^j \mathbf{T}^n \phi \right\rangle\right|^2 && \text{by definition of } \mathbf{P}_j && \text{(definition 1 page 17)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbb{1}_{[-R:R]}(x) \lim_{n\to\infty} (\mathsf{f}_n) \,|\, \mathbf{D}^j \mathbf{T}^n \phi(x) \right\rangle\right|^2 && \text{by definition of } (\mathsf{f}_n) && \text{(definition 3 page 17)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left|\left\langle \lim_{n\to\infty} (\!(\mathsf{f}_n)\!) \,|\, \mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x)\right\rangle\right|^2 && \text{prop. of } \langle\triangle\,|\,\nabla\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}\\
&\le \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \left\|\lim_{n\to\infty} (\!(\mathsf{f}_n)\!)\right\|^2 \left\|\mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x)\right\|^2 && \text{by \textit{CS Inequality}}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbb{1}_{[-R:R]}(x)\mathbf{D}^j\mathbf{T}^n\phi(x)\right\|^2 && \text{by definition of } (\!(\mathsf{f}_n)\!) && \text{(definition 3 page 17)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\Big[\underbrace{\mathbf{D}^j\mathbf{D}^{-j}}_{\mathbf{I}}\mathbb{1}_{[-R:R]}(x)\Big]\left[\mathbf{D}^j\mathbf{T}^n\phi(x)\right]\right\|^2 && \text{by property of } \mathbf{D} && \text{(Proposition 1.2 page 2)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|2^{j/2}\mathbf{D}^j\left\{\left[\mathbf{D}^{-j}\mathbb{1}_{[-R:R]}(x)\right]\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 && \text{by Proposition 1.4 page 3}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\left\{2^{j/2}2^{-j/2}\mathbb{1}_{[-R:R]}(2^{-j}x)\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 && \text{by property of } \mathbf{D} && \text{(Proposition 1.2 page 2)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\left\{\Big[\underbrace{\mathbf{T}^n\mathbf{T}^{-n}}_{\mathbf{I}}\mathbb{1}_{[-R:R]}(2^{-j}x)\Big]\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 && \text{by property of } \mathbf{T} && \text{(Proposition 1.2 page 2)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\left\{\left[\mathbf{T}^n\mathbb{1}_{[-R:R]}(2^{-j}x+n)\right]\left[\mathbf{T}^n\phi(x)\right]\right\}\right\|^2 && \text{by property of } \mathbf{T} && \text{(Proposition 1.2 page 2)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbf{D}^j\mathbf{T}^n\left\{\mathbb{1}_{[-R:R]}(2^{-j}x+n)\phi(x)\right\}\right\|^2 && \text{by property of } \mathbf{D} && \text{(Proposition 1.2 page 2)}\\
&= \lim_{j\to-\infty} B \sum_{n\in\mathbb{Z}} \|\mathsf{f}\|^2 \left\|\mathbb{1}_{[-R:R]}(2^{-j}x+n)\phi(x)\right\|^2 && \text{by \textit{unitary} prop.} && \text{(Theorem 1.1 page 7)}\\
&= B\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \lim_{j\to-\infty} \left\|\mathbb{1}_{[-2^jR+n:2^jR+n]}(u)\phi(2^{-j}(u-n))\right\|^2 && u \triangleq 2^jx+n \implies && x = 2^{-j}(u-n)\\
&= B\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \lim_{j\to-\infty} \int_{-2^jR+n}^{2^jR+n} \left|\phi(2^{-j}(u-n))\right|^2 \,\mathrm{d}u\\
&= B\|\mathsf{f}\|^2 \sum_{n\in\mathbb{Z}} \int_n^n |\phi(0)|^2 \,\mathrm{d}u\\
&= 0
\end{aligned}$$

6. Final step in proof that $\bigcap \boldsymbol{V}_j = \{\mathbb{0}\}$: by (4) lemma page 17 and (5) lemma page 17

✏

Proposition 2.2. [6]

PRP

$$\left\{\begin{array}{lll} (1). & (\!(\mathbf{T}^n\phi)\!) \textit{ is a } \textsc{Riesz sequence} & \textit{and}\\ (2). & \tilde{\phi}(\omega) \textit{ is } \textsc{continuous } \textit{at } 0 & \textit{and}\\ (3). & \tilde{\phi}(0) \neq 0 & \end{array}\right\} \implies \left\{\overline{\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)} = \boldsymbol{L}^2_{\mathbb{R}} \quad (\textsc{dense } \textit{in } \boldsymbol{L}^2_{\mathbb{R}})\right\}$$

✎PROOF:

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace $\boldsymbol{V}_j$ such that
   $\boldsymbol{V}_j = \left\{\mathbf{P}_j\mathsf{f} \,|\, \mathsf{f} \in \boldsymbol{H}\right\}$

2. definition: Choose $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ such that $\mathsf{f} \perp \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j$. Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier Transform* (Definition K.2 page 257) of $\mathsf{f}(x)$.

[6] Wojtaszczyk (1997) pages 28–31 ⟨Proposition 2.15⟩

3. lemma: The function f (definition 2 page 18) *exists* because the set of functions that can be chosen to be f at least contains 0 (it is not the emptyset). Proof:

$$\begin{aligned}
\mathsf{f}(x) = 0 \implies & \left\langle \mathsf{f} \,\middle|\, \left\{ \mathsf{h} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{h} \in \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right\} \right\rangle \\
&= \left\langle 0 \,\middle|\, \left\{ \mathsf{h} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{h} \in \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \right\} \right\rangle \\
&= 0 \\
\implies & \quad \mathsf{f} \perp \bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j \\
\implies & \quad \mathsf{f} \text{ exists}
\end{aligned}$$

4. lemma: $\left\|\mathbf{P}_j \mathsf{f}\right\| = 0 \quad \forall j\in\mathbb{Z}$. Proof:

$$\begin{aligned}
\left\|\mathbf{P}_j \mathsf{f}\right\| &= \|0\| && \text{by definition of } \mathsf{f} && \text{(definition 2 page 18)} \\
&= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|
\end{aligned}$$

5. definition: Choose some function $\mathsf{g} \in \boldsymbol{L}^2_{\mathbb{R}}$ such that $\tilde{\mathsf{g}}(\omega) = \tilde{\mathsf{f}}(\omega)\mathbb{1}_{[-R:R]}$ (Definition 1.2 page 1) for some $R > 0$ and such that $\|\mathsf{f} - \mathsf{g}\| < \varepsilon$. Let $\tilde{\mathsf{g}}(\omega)$ be the *Fourier Transform* (Definition K.2 page 257) of $\mathsf{g}(x)$.

6. lemma: The function g (definition 5 page 19) *exists.* Proof: For some (possibly very large) R,

$$\begin{aligned}
\varepsilon &> \left\|\tilde{\mathsf{f}}(\omega) - \tilde{\mathsf{g}}(\omega)\right\| && \text{by definition of } \mathsf{g} && \text{(definition 5 page 19)} \\
&= \left\|\tilde{\mathbf{F}}\mathsf{f}(x) - \tilde{\mathbf{F}}\mathsf{g}(x)\right\| && \text{by definition of } \tilde{\mathsf{f}} \text{ and } \tilde{\mathsf{g}} && \text{(definition 2 page 18), (definition 5 page 19)} \\
&= \left\|\tilde{\mathbf{F}}\left[\mathsf{f}(x) - \mathsf{g}(x)\right]\right\| && \text{by } \textit{linearity} \text{ of } \tilde{\mathbf{F}} && \text{(Definition D.4 page 157)} \\
&= \|\mathsf{f}(x) - \mathsf{g}(x)\| && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem K.2 page 258)} \\
\implies & \quad \mathsf{g} \text{ exists} && \text{because it's possible to satisfy definition 5 page 19}
\end{aligned}$$

7. lemma: $\left\|\mathbf{P}_j \mathsf{g}\right\| < \varepsilon \quad \forall j\in\mathbb{Z}$ for sufficiently large R. Proof:

$$\begin{aligned}
\varepsilon &> \|\mathsf{f} - \mathsf{g}\| && \text{by definition of } \mathsf{g} && \text{(definition 5 page 19)} \\
&\geq \left\|\mathbf{P}_j\left[\mathsf{f} - \mathsf{g}\right]\right\| && \text{by property of } \textit{projection operators} && \text{(Definition D.10 page 171)} \\
&= \left\|\mathbf{P}_j\mathsf{f} - \mathbf{P}_j\mathsf{g}\right\| && \text{by } \textit{additive} \text{ property of } \mathbf{P}_j && \text{(Definition D.4 page 157)} \\
&\geq \left|\left\|\mathbf{P}_j\mathsf{f}\right\| - \left\|\mathbf{P}_j\mathsf{g}\right\|\right| && \text{by } \textit{Reverse Triangle Inequality} \\
&= \left|0 - \left\|\mathbf{P}_j\mathsf{g}\right\|\right| && \text{by ((4) lemma page 19)} \\
&= \left\|\mathbf{P}_j\mathsf{g}\right\| && \text{by } \textit{strictly positive} \text{ property of } \|\cdot\| && \text{(Definition D.5 page 160)}
\end{aligned}$$

8. lemma: $\mathsf{g} = 0$. Proof:

$$\begin{aligned}
0 &= \lim_{j\to\infty} \left\|\mathbf{P}_j\mathsf{g}\right\|^2 && \text{by (7) lemma page 19} \\
&\geq \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \mathbf{P}_j\mathsf{g} \,|\, \mathbf{D}^j\mathbf{T}^n\phi \right\rangle\right|^2 && \text{by } \textit{frame property} && \text{(Proposition L.5 page 288)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \mathsf{g} \,|\, \mathbf{D}^j\mathbf{T}^n\phi \right\rangle\right|^2 && \text{by definition of } \mathbf{P}_j && \text{(item (1) page 18)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathbf{F}}\mathsf{g} \,|\, \tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\phi \right\rangle\right|^2 && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem K.2 page 258)} \\
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{\mathsf{g}}(\omega) \,|\, 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{\phi}(2^{-j}\omega) \right\rangle\right|^2 && \text{by Proposition 1.12 page 9}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{j\to\infty} A \sum_{n\in\mathbb{Z}} \left|\left\langle \tilde{g}(\omega)\tilde{\phi}^*(2^{-j}\omega) \,|\, 2^{-j/2}e^{-i2^{-j}\omega n}\right\rangle\right|^2 && \text{by property of } \langle\triangle\,|\,\nabla\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \\
&= \lim_{j\to\infty} A \left\|\tilde{g}(\omega)\tilde{\phi}^*(2^{-j}\omega)\right\|^2 && \text{by \textit{Parseval's Identity}} \quad \text{(Theorem L.9 page 280)} \\
&= A \left\|\tilde{g}(\omega)\tilde{\phi}^*(0)\right\|^2 && \text{by left hypothesis (2)} \\
&= A\left|\tilde{\phi}^*(0)\right|^2 \;\|\tilde{g}(\omega)\|^2 && \text{by \textit{homogeneous} property of } \|\cdot\| \\
&= A\left|\tilde{\phi}(0)\right|^2 \;\|g\|^2 && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} \quad \text{(Theorem K.2 page 258)} \\
&\implies \|g\| = 0 && \text{by left hypothesis (3)} \\
&\iff g = 0 && \text{by \textit{nondegenerate} property of } \|\cdot\|
\end{aligned}$$

9. Final step in proof that $\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}$:

$$\begin{aligned}
& g = 0 && \text{by (8) lemma page 19} \\
&\quad\implies f = 0 && \text{by definition of } g \quad \text{(definition 5 page 19)} \\
&\quad\implies \left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^- = \boldsymbol{L}^2_{\mathbb{R}}
\end{aligned}$$

⇨

Definition 2.1 defines an MRA on the space $\boldsymbol{L}^2_{\mathbb{R}}$, which is a special case of a *separable Hilbert space*. A Hilbert space is a *linear space* that is equipped with an *inner product*, is *complete* with respect to the *metric* induced by the inner product, and contains a subset that is *dense* in $\boldsymbol{L}^2_{\mathbb{R}}$.

An *inner product* on a linear space endows the linear space with a *topology*. The sum such as $\sum_{n=1}^{N} \alpha_n f_n$ is finite and thus suitable for a finite linear space only. An infinite space requires an infinite sum $\sum_{n=1}^{\infty} \alpha_n \phi_n$, and an infinite sum is defined in terms of a limit (Definition G.1 page 204). The limit, in turn, is defined in terms of a *topology*. The *inner product* induces a *norm* (Definition D.5 page 160) which induces a *metric* which induces a topology.

Definition 2.1 defines each subspace $\boldsymbol{V}_j$ to be *closed* ($\boldsymbol{V}_j = \boldsymbol{V}_j^-$) in $\boldsymbol{L}^2_{\mathbb{R}}$. As one might imagine, the properties of *completeness* and *closure* are closely related. Moreover, Every *complete* sequence is also *bounded*, and so each subspace $\boldsymbol{V}_j$ is *bounded* as well.

Proposition 2.3. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!)\right)$ *be an* MRA SPACE.

PRP

Each subspace $\boldsymbol{V}_j$ *is* COMPLETE.

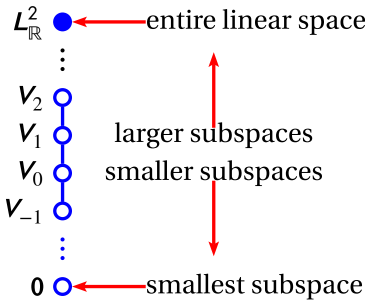
✎PROOF:

1. By definition Definition 2.1, $\boldsymbol{L}^2_{\mathbb{R}}$ is *complete*.
2. In any metric space, (which includes all inner product spaces such as $\boldsymbol{L}^2_{\mathbb{R}}$), a *closed* subspace of a *complete* metric space is itself also *complete*.
3. In any *complete* metric space $\boldsymbol{X}$ (which includes all Hilbert spaces such as $\boldsymbol{L}^2_{\mathbb{R}}$), the two properties coincide—that is, a subspace is complete *if and only if* it is closed in the space $\boldsymbol{X}$.
4. So because $\boldsymbol{L}^2_{\mathbb{R}}$ is *complete* and each $\boldsymbol{V}_j$ is *closed*, then each $\boldsymbol{V}_j$ is also *complete*.

⇨

2.3 Order structure

A *multiresolution analysis* (Definition 2.1 page 16) together with the set inclusion relation $\subseteq$ forms the *linearly ordered set* $\left(\left(\!\left(\boldsymbol{V}_j\right)\!\right), \subseteq\right)$, illustrated to the right by a *Hasse diagram*. Subspaces $\boldsymbol{V}_j$ increase in "size" with increasing j. That is, they contain more and more vectors (functions) for larger and larger j—with the upper limit of this sequence being $\boldsymbol{L}^2_{\mathbb{R}}$. Alternatively, we can say that approximation within a subspace $\boldsymbol{V}_j$ yields greater "*resolution*" for increasing j.



The *least upper bound* (*l.u.b.*) of the linearly ordered set $\left(\left(\!\left(\boldsymbol{V}_j\right)\!\right), \subseteq\right)$ is $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition 2.1 page 16):

$$\overline{\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)} = \boldsymbol{L}^2_{\mathbb{R}}.$$

The *greatest lower bound* (*g.l.b.*) of the linearly ordered set $\left(\left(\!\left(\boldsymbol{V}_j\right)\!\right), \subseteq\right)$ is $\mathbf{0}$ (Proposition 2.1 page 16):

$$\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \mathbf{0}.$$

All linear subspaces contain the zero vector. So the intersection of any two subspaces must at least contain $\mathbb{0}$. If the intersection of any two linear subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ is exactly $\{\mathbb{0}\}$, then for any vector in the sum of those subspaces ($\boldsymbol{u} \in \boldsymbol{X} \hat{+} \boldsymbol{Y}$) there are **unique** vectors $\mathsf{f} \in \boldsymbol{X}$ and $\mathsf{g} \in \boldsymbol{Y}$ such that $\boldsymbol{u} = \mathsf{f} + \mathsf{g}$. This is *not* necessarily true if the intersection contains more than just $\{\mathbb{0}\}$.

2.4 Dilation equation

Several functions in mathematics exhibit a kind of *self-similar* or *recursive* property:

- If a function $\mathsf{f}(x)$ is *linear*, then (Example 1.2 page 12)
  $$\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x.$$
- If a function $\mathsf{f}(x)$ is sufficiently *bandlimited*, then the *Cardinal series* (Example 1.3 page 12) demonstrates
  $$\mathsf{f}(x) = \sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin[\pi(x)]}{\pi(x)}.$$
- *B-splines* (Theorem M.3 page 306) are another example:
  $$\mathsf{N}_n(x) = \frac{1}{n}x\mathsf{N}_{n-1}(x) - \frac{1}{n}x\mathbf{T}\mathsf{N}_{n-1}(x) + \frac{n+1}{n}\mathbf{T}\mathsf{N}_{n-1}(x) \qquad \forall n\in\mathbb{W}\backslash\{1\}, \forall x\in\mathbb{R}.$$

The scaling function $\phi(x)$ (Definition 2.1 page 16) also exhibits a kind of *self-similar* property. By Definition 2.1 page 16, the dilation $\mathbf{D}\mathsf{f}$ of each vector f in $\boldsymbol{V}_0$ is in $\boldsymbol{V}_1$. If $\left\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\right\}$ is a basis for $\boldsymbol{V}_0$, then $\left\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\right\}$ is a basis for $\boldsymbol{V}_1$, $\left\{\mathbf{D}^2\mathbf{T}^n\phi|\, n\in\mathbb{Z}\right\}$ is a basis for $\boldsymbol{V}_2$, …; and in general $\left\{\mathbf{D}^j\mathbf{T}^m\phi|j\in\mathbb{Z}\right\}$ is a basis for $\boldsymbol{V}_j$. Also, if ϕ is in $\boldsymbol{V}_0$, then it is also in $\boldsymbol{V}_1$ (because $\boldsymbol{V}_0 \subset \boldsymbol{V}_1$). And because ϕ is in $\boldsymbol{V}_1$ and because $\left\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\right\}$ is a basis for $\boldsymbol{V}_1$, ϕ is a linear combination of the elements in $\left\{\mathbf{D}\mathbf{T}^n\phi|\, n\in\mathbb{Z}\right\}$. That is, ϕ can be represented as a linear combination of translated and dilated versions of itself. The resulting equation is called the *dilation equation* (Definition 2.2, next).[7]

[7] The property of *translation invariance* is of particular significance in the theory of *normed linear spaces* (a Hilbert

Definition 2.2. [8] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!)\right)$ *be a* MULTIRESOLUTION ANALYSIS SPACE *with scaling function* ϕ (Definition 2.1 page 16). *Let* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ *be a* SEQUENCE (Definition O.1 page 341) *in* $\ell^2_{\mathbb{R}}$ (Definition O.2 page 341).

DEF *The* EQUATION $\left\{\phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi(x) \qquad \forall x\in\mathbb{R}\right\}$ *is called the* ***dilation equation****.* *It is also called the* ***refinement equation****,* ***two-scale difference equation****, and* ***two-scale relation****.*

Remark 2.1.

REM The *dilation equation* under the defintions of $\mathbf{T}$ and $\mathbf{D}$ evaluates to
$$\phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\phi(2x-n).$$

✎PROOF:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\phi(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n\phi(2x-n) && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)}
\end{aligned}$$

✏

Theorem 2.1 (dilation equation). *Let an* MRA SPACE *and* SCALING FUNCTION *be as defined in Definition 2.1 page 16.*

THM
$$\left\{\begin{array}{l}\left(L^2_{\mathbb{R}}, (\!(V_j)\!)\right) \text{ is an MRA SPACE} \\ \text{with SCALING FUNCTION } \phi\end{array}\right\} \implies \underbrace{\left\{\begin{array}{l}\exists (\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}} \text{ such that} \\ \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) \qquad \forall x\in\mathbb{R}\end{array}\right\}}_{\text{DILATION EQUATION IN "TIME"}}$$

✎PROOF:

$$\begin{aligned}
\phi &\in V_0 && \text{by definition of MRA} && \text{(Definition 2.1 page 16)} \\
&\subseteq V_1 && \text{by definition of MRA} && \text{(Definition 2.1 page 16)} \\
&\triangleq \operatorname{span}\{\mathbf{D}\mathbf{T}^n\phi(x) | n\in\mathbb{Z}\} && \text{by definition of } V_j && \text{(Definition 2.1 page 16)} \\
&\implies \exists (\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}} \text{ such that } \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) && \text{by definition of span} && \text{(Definition L.2 page 267)}
\end{aligned}$$

✏

Lemma 2.1. [9] *Let* $\phi(x)$ *be a function in* $L^2_{\mathbb{R}}$ (Definition E.1 page 185)*. Let* $\tilde{\phi}(\omega)$ *be the* FOURIER TRANSFORM (Definition K.2 page 257) *of* $\phi(x)$*. Let* $\breve{\mathsf{h}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM (Definition P.1 page 355) *of a sequence* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$*.*

LEM
$$\begin{aligned}
&(A)\quad \phi(x) = \sum_{n\in\mathbb{Z}} \mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) \quad \forall x\in\mathbb{R} &&\iff \tilde{\phi}(\omega) = \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \forall\omega\in\mathbb{R} && (1) \\
& &&\iff \tilde{\phi}(\omega) = \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) && \forall n\in\mathbb{N}, \omega\in\mathbb{R} && (2)
\end{aligned}$$

space is a complete normed linear space equipped with an inner product).

[8] Jawerth and Sweldens (1994), page 7

[9] Mallat (1999) page 228

✎PROOF:

1. Proof that (A) $\implies$ (1):

$$\begin{aligned}
\tilde{\phi}(\omega) &\triangleq \tilde{\mathbf{F}}\phi \\
&= \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) && \text{by (A)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \underbrace{\frac{\sqrt{2}}{2} e^{-i\frac{\omega}{2}n} \phi\left(\frac{\omega}{2}\right)}_{\tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n\phi(x)} && \text{by Proposition 1.12 page 9} \\
&= \underbrace{\frac{\sqrt{2}}{2}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n e^{-i\frac{\omega}{2}n}\right]}_{\breve{\mathsf{h}}(\omega/2)} \tilde{\phi}\left(\frac{\omega}{2}\right) \\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \textit{DTFT} \text{ (Definition P.1 page 355)}
\end{aligned}$$

2. Proof that (A) $\impliedby$ (1):

$$\begin{aligned}
\phi(x) &= \tilde{\mathbf{F}}^{-1}\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega) \\
&= \tilde{\mathbf{F}}^{-1}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by (1)} \\
&= \tilde{\mathbf{F}}^{-1}\frac{\sqrt{2}}{2}\sum_{n\in\mathbb{Z}} h_n e^{-i\frac{\omega}{2}n}\,\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \textit{DTFT} \text{ (Definition P.1 page 355)} \\
&= \frac{\sqrt{2}}{2}\sum_{n\in\mathbb{Z}} h_n \tilde{\mathbf{F}}^{-1} e^{-i\frac{\omega}{2}n}\,\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by property of linear operators} \\
&= \frac{\sqrt{2}}{2}\sum_{n\in\mathbb{Z}} h_n \tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n\phi && \text{by Proposition 1.12 page 9} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi(x) && \text{by definition of } \textit{operator inverse}
\end{aligned}$$

3. Proof that (1) $\implies$ (2):

(a) Proof for $N = 1$ case:

$$\begin{aligned}
\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\bigg|_{N=1} &= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \\
&= \tilde{\phi}(\omega) && \text{by (1)}
\end{aligned}$$

(b) Proof that [N case] $\implies$ [$N+1$ case]:

$$\begin{aligned}
\tilde{\phi}\left(\frac{\omega}{2^{N+1}}\right)\prod_{n=1}^{N+1}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) &= \left[\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\right]\underbrace{\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^{N+1}}\right)\tilde{\phi}\left(\frac{\omega}{2^{N+1}}\right)}_{\tilde{\phi}(\omega/2^N)} \\
&= \tilde{\phi}(\omega/2^N)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) \\
&= \tilde{\phi}(\omega) && \text{by [} N \text{ case] hypothesis}
\end{aligned}$$

4. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\tilde{\phi}(\omega) &= \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right)\Bigg|_{N=1} && \text{by (2)}\\
&= \tilde{\phi}\left(\frac{\omega}{2}\right)\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)
\end{aligned}$$

✎

Lemma 2.2. *Let $\phi(x)$ be a function in $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185). Let $\tilde{\phi}(\omega)$ be the* Fourier transform *(Definition K.2 page 257) of $\phi(x)$. Let $\breve{\mathsf{h}}(\omega)$ be the* Discrete time Fourier transform *(Definition P.1 page 355) of $(\!(\mathsf{h}_n)\!)$. Let $\prod_{n=1}^{\infty} x_n \triangleq \lim_{N\to\infty}\prod_{n=1}^{N} x_n$, with respect to the standard norm in $\boldsymbol{L}^2_{\mathbb{R}}$.*

LEM

$$\left\{\begin{array}{ll}\tilde{\phi}(\omega) = C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \\ \forall C>0,\omega\in\mathbb{R} & \text{(A)}\end{array}\right\} \implies \begin{array}{rcll}\phi(x) &=& \sum_{n\in\mathbb{Z}}\mathsf{h}_n\mathbf{D}\mathbf{T}^n\phi(x) & \forall x\in\mathbb{R} \quad (1)\\ \iff \tilde{\phi}(\omega) &=& \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) & \forall\omega\in\mathbb{R} \quad (2)\\ \iff \tilde{\phi}(\omega) &=& \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \forall n\in\mathbb{N},\omega\in\mathbb{R} \quad (3)\end{array}$$

✎Proof:

1. Proof that (1) $\iff$ (2) $\iff$ (3): by Lemma 2.1 page 22
2. Proof that (A) $\implies$ (2):

$$\begin{aligned}
\tilde{\phi}(\omega) &= C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) && \text{by left hypothesis}\\
&= C\,\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^{n+1}}\right)\\
&= C\,\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega/2}{2^n}\right)\\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\left[C\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega/2}{2^n}\right)\right]\\
&= \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by left hypothesis}
\end{aligned}$$

✎

Proposition 2.4. *Let $\phi(x)$ be a function in $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185). Let $\tilde{\phi}(\omega)$ be the* Fourier transform *(Definition K.2 page 257) of $\phi(x)$. Let $\breve{\mathsf{h}}(\omega)$ be the* Discrete time Fourier transform *(Definition P.1 page 355) of $(\!(\mathsf{h}_n)\!)$. Let $\prod_{n=1}^{\infty} x_n \triangleq \lim_{N\to\infty}\prod_{n=1}^{N} x_n$, with respect to the standard norm in $\boldsymbol{L}^2_{\mathbb{R}}$.*

PRP

$$\left\{\begin{array}{l}\tilde{\phi}(\omega)\text{ is}\\ \text{CONTINUOUS}\\ \text{at }\omega=0\end{array}\right\} \implies \left\{\begin{array}{lrcll}
 & \phi(x) & = & \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) & \forall x\in\mathbb{R} \quad (1)\\
\iff & \tilde{\phi}(\omega) & = & \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) & \forall \omega\in\mathbb{R} \quad (2)\\
\iff & \tilde{\phi}(\omega) & = & \tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \forall n\in\mathbb{N},\omega\in\mathbb{R} \quad (3)\\
\iff & \tilde{\phi}(\omega) & = & \tilde{\phi}(0)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) & \omega\in\mathbb{R} \quad (4)
\end{array}\right\}$$

✎PROOF:

1. Proof that (1) $\iff$ (2) $\iff$ (3): by Lemma 2.1 page 22
2. Proof that (3) $\implies$ (4):

$$\begin{aligned}
\tilde{\phi}(0)\prod_{n=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) &= \lim_{N\to\infty}\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{n=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^n}\right) && \text{by } continuity \text{ and definition of } \prod_{n=1}^{\infty} x_n\\
&= \tilde{\phi}(\omega) && \text{by (3) and Lemma 2.1 page 22}
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4): by Lemma 2.2 page 24

✏

Definition 2.3 (next) formally defines the coefficients that appear in Theorem 2.1 (page 22).

Definition 2.3. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!)\right)$ *be a multiresolution analysis space with scaling function* ϕ. *Let* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ *be a sequence of coefficients such that* $\phi = \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi$.

DEF *A* ***multiresolution system*** *is the tuple* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$. *The sequence* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$ *is the* ***scaling coefficient sequence***. *A multiresolution system is also called an* ***MRA system***. *An* MRA SYSTEM *is an* ***orthonormal MRA system*** *if* $\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}$ *is* ORTHONORMAL.

Theorem 2.2. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM (Definition 2.3 page 25). *Let* span$A$ *be the* LINEAR SPAN (Definition L.2 page 267) *of a set* A.

THM

$$\underbrace{\operatorname{span}\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\} = \boldsymbol{V}_0}_{\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}\text{ is a BASIS for }\boldsymbol{V}_0} \implies \underbrace{\operatorname{span}\{\mathbf{D}^j\mathbf{T}^n\phi|\, n\in\mathbb{Z}\} = \boldsymbol{V}_j \quad \forall j\in\mathbb{W}}_{\{\mathbf{D}^j\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}\text{ is a BASIS for }\boldsymbol{V}_j}$$

✎PROOF: Proof is by induction:[10]

1. induction basis (proof for $j=0$ case):

$$\begin{aligned}
\operatorname{span}\{\mathbf{D}^j\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}\big|_{j=0} &= \operatorname{span}\{\mathbf{T}^n\phi|\, n\in\mathbb{Z}\}\\
&= \boldsymbol{V}_0 && \text{by left hypothesis}
\end{aligned}$$

[10] Smith (2011) page 4

2. induction step (proof that j case $\implies$ $j+1$ case):

$$
\begin{aligned}
&\operatorname{span}\left\{\mathbf{D}^{j+1}\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} \\
&= \left\{ \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\!(\alpha_n)\!) \quad \text{such that} \quad \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j+1}\mathbf{T}^n\phi \right\} && \text{by definition of span} && \text{(Definition L.2 page 267)} \\
&= \left\{ \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\!(\alpha_n)\!) \quad \text{such that} \quad \mathsf{f}(x) = \mathbf{D}\sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi \right\} \\
&= \left\{ \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\!(\alpha_n)\!) \quad \text{such that} \quad \mathbf{D}^{-1}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi \right\} \\
&= \left\{ [\mathbf{D}\mathsf{f}] \in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\!(\alpha_n)\!) \quad \text{such that} \quad \mathbf{D}^{-1}[\mathbf{D}\mathsf{f}(x)] = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi \right\} \\
&= \mathbf{D}\left\{ \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \exists (\!(\alpha_n)\!) \quad \text{such that} \quad \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}^{j}\mathbf{T}^n\phi \right\} \\
&= \mathbf{D}\operatorname{span}\left\{\mathbf{D}^{j}\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\} && \text{by definition of span} && \text{(Definition L.2 page 267)} \\
&= \mathbf{D}\boldsymbol{V}_j && \text{by induction hypothesis} \\
&= \boldsymbol{V}_{j+1} && \text{by } \textit{self-similar} \text{ property} && \text{(Definition 2.1 page 16)}
\end{aligned}
$$

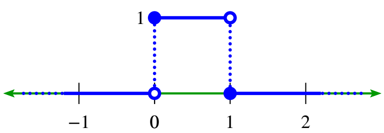
✎

Example 2.1.

EX

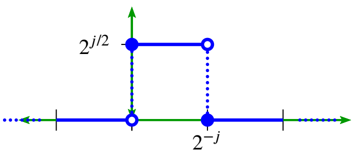
In the *Haar* MRA, the scaling function $\phi(x)$ is the *pulse function*

$$\phi(x) = \begin{cases} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise.} \end{cases}$$



In the subspace $\boldsymbol{V}_j$ ($j\in\mathbb{Z}$) the scaling functions are

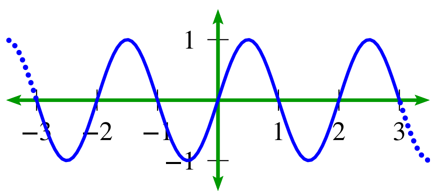
$$\mathbf{D}^j\phi(x) = \begin{cases} (2)^{j/2} & \text{for } x \in \left[0 : \left(2^{-j}\right)\right) \\ 0 & \text{otherwise.} \end{cases}$$



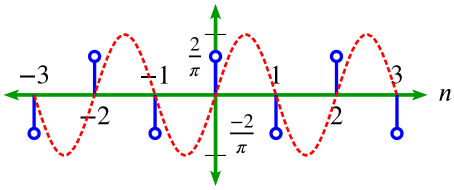
The scaling subspace $\boldsymbol{V}_0$ is the span $\boldsymbol{V}_0 \triangleq \operatorname{span}\left\{\mathbf{T}^n\phi \middle| n\in\mathbb{Z}\right\}$. The scaling subspace $\boldsymbol{V}_j$ is the span $\boldsymbol{V}_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi | n \in \mathbb{Z}\right\}$. Note that $\left\|\mathbf{D}^j\mathbf{T}^n\phi\right\|$ for each resolution j and shift n is unity:

$$
\begin{aligned}
\left\|\mathbf{D}^j\mathbf{T}^n\phi\right\|^2 &= \|\phi\|^2 && \text{by } \textit{unitary} \text{ properties of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Theorem 1.1 page 7)} \\
&= \int_0^1 |1|^2 \,\mathrm{d}x && \text{by definition of } \|\cdot\| \text{ on } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)} \\
&= 1
\end{aligned}
$$

Let $\mathsf{f}(x) = \sin(\pi x)$. Suppose we want to project $\mathsf{f}(x)$ onto the subspaces $\boldsymbol{V}_0, \boldsymbol{V}_1, \boldsymbol{V}_2, \ldots$.



The values of the transform coefficients for the subspace $\boldsymbol{V}_j$ are given by



$$
\begin{aligned}
\left[\mathbf{R}_j \mathsf{f}(x)\right](n) &= \frac{1}{\|\mathbf{D}^j\mathbf{T}^n\phi\|^2}\left\langle \mathsf{f}(x) \,|\, \mathbf{D}^j\mathbf{T}^n\phi \right\rangle \\
&= \frac{1}{\cancelto{1}{\|\phi\|^2}}\left\langle \mathsf{f}(x) \,|\, 2^{j/2}\phi\left(2^j x - n\right)\right\rangle && \text{by Proposition 1.3 page 3} \\
&= 2^{j/2}\left\langle \mathsf{f}(x) \,|\, \phi\left(2^j x - n\right)\right\rangle \\
&= 2^{j/2}\int_{2^{-j}n}^{2^{-j}(n+1)} \mathsf{f}(x)\, \mathsf{d}x \\
&= 2^{j/2}\int_{2^{-j}n}^{2^{-j}(n+1)} \sin(\pi x)\, \mathsf{d}x \\
&= 2^{j/2}\left(-\frac{1}{\pi}\right)\cos(\pi x)\Big|_{2^{-j}n}^{2^{-j}(n+1)} \\
&= \frac{2^{j/2}}{\pi}\left[\cos\left(2^{-j}n\pi\right) - \cos\left(2^{-j}(n+1)\pi\right)\right]
\end{aligned}
$$

And the projection $\mathbf{A}_n\mathsf{f}(x)$ of the function $\mathsf{f}(x)$ onto the subspace $\boldsymbol{V}_j$ is

$$
\begin{aligned}
\mathbf{A}_j \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \left\langle \mathsf{f}(x) \,|\, \mathbf{D}^j\mathbf{T}^n\phi \right\rangle \mathbf{D}^j\mathbf{T}^n\phi \\
&= \frac{2^{j/2}}{\pi}\sum_{n\in\mathbb{Z}}\left[\cos\left(2^{-j}n\pi\right) - \cos\left(2^{-j}(n+1)\pi\right)\right]2^{j/2}\phi\left(2^j x - n\right) \\
&= \frac{2^{j}}{\pi}\sum_{n\in\mathbb{Z}}\left[\cos\left(2^{-j}n\pi\right) - \cos\left(2^{-j}(n+1)\pi\right)\right]\phi\left(2^j x - n\right)
\end{aligned}
$$

The transforms of $\sin(\pi x)$ into the subspaces $\boldsymbol{V}_0$, $\boldsymbol{V}_1$, and $\boldsymbol{V}_2$, as well as the approximations in those subspaces are as illustrated in Figure 2.1 (page 28).

2.5 Necessary Conditions

Theorem 2.3 (admissibility condition). *Let* $\check{\mathsf{h}}(z)$ *be the* Z-TRANSFORM (Definition O.4 page 342) *and* $\breve{\mathsf{h}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM (Definition P.1 page 355) *of a sequence* $(\!(\mathsf{h}_n)\!)_{n\in\mathbb{Z}}$.

THM

$\left\{\left(L^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)\right.$ *is an* MRA SYSTEM (Definition 2.3 page 25)$\left.\right\}$

$$
\begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array}
\underbrace{\left\{\sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2}\right\}}_{\text{(1) ADMISSIBILITY in "time"}}
\iff
\underbrace{\left\{\check{\mathsf{h}}(z)\Big|_{z=1} = \sqrt{2}\right\}}_{\text{(2) ADMISSIBILITY in "z domain"}}
\iff
\underbrace{\left\{\breve{\mathsf{h}}(\omega)\Big|_{\omega=0} = \sqrt{2}\right\}}_{\text{(3) ADMISSIBILITY in "frequency"}}
$$

✎PROOF:

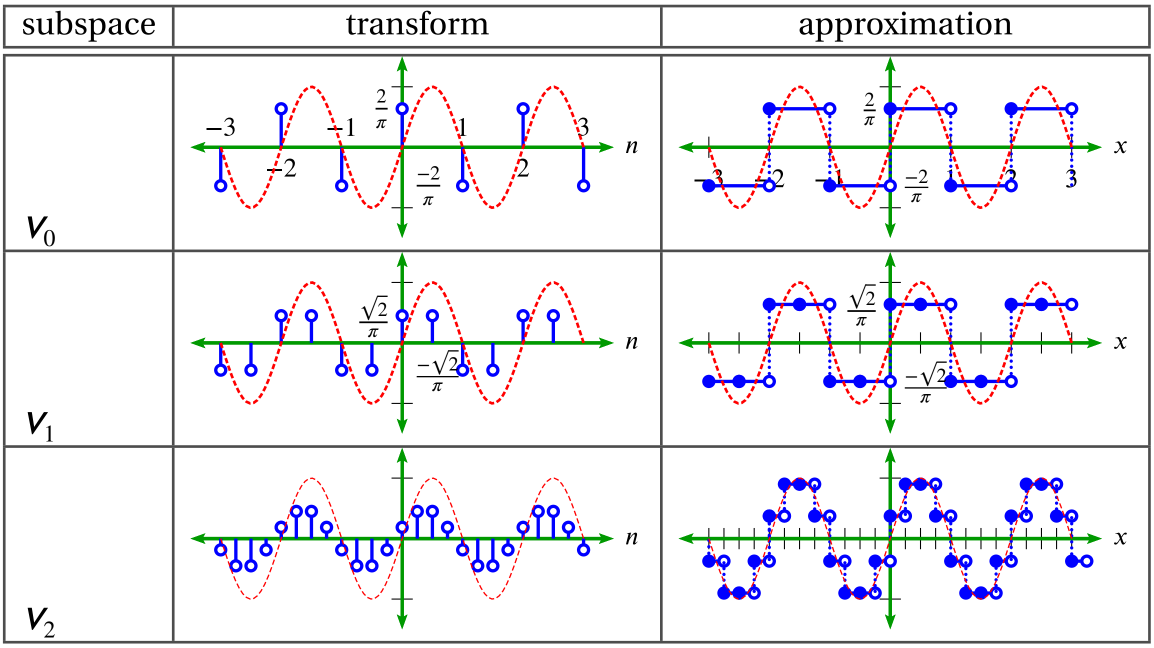


Figure 2.1: Projections of $\sin(\pi x)$ on Haar subspaces (Example 2.1 page 26)

1. Proof that MRA system $\implies$ (1):

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{h}_n &= \frac{\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x}{\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \\
&= \frac{1}{\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(x)\, \mathrm{d}x \\
&= \frac{1}{\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \frac{\sqrt{2}}{\sqrt{2}} \phi(2y-n) 2\, \mathrm{d}y && \text{let } y \triangleq \frac{x+n}{2} \implies x = 2y-n \implies \mathrm{d}x = 2\,\mathrm{d}y \\
&= \frac{2}{\sqrt{2}} \frac{1}{\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(y)\, \mathrm{d}y && \text{by definitions of } \mathbf{T} \text{ and } \mathbf{D} \text{ (Definition 1.3 page 2)} \\
&= \sqrt{2} \frac{1}{\int_{\mathbb{R}} \phi(x)\, \mathrm{d}x} \int_{\mathbb{R}} \phi(y)\, \mathrm{d}y && \text{by \textit{dilation equation} (Theorem 2.1 page 22)} \\
&= \sqrt{2}
\end{aligned}
$$

2. Alternate proof that MRA system $\implies$ (1):
Let $\mathsf{f}(x) \triangleq 1 \quad \forall x \in \mathbb{R}$.

$$
\begin{aligned}
\langle \phi \,|\, \mathsf{f} \rangle &= \left\langle \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi \,\middle|\, \mathsf{f} \right\rangle && \text{by \textit{dilation equation}} && \text{(Theorem 2.1 page 22)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \mathbf{D}\mathbf{T}^n \phi \,|\, \mathsf{f} \rangle && \text{by linearity of } \langle \triangle \,|\, \triangledown \rangle \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \phi \,|\, (\mathbf{D}\mathbf{T}^n)^* \mathsf{f} \rangle && \text{by definition of operator adjoint} && \text{(Theorem D.13 page 170)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \langle \phi \,|\, (\mathbf{T}^*)^n \mathbf{D}^* \mathsf{f} \rangle && \text{by property of operator adjoint} && \text{(Theorem D.13 page 170)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,\middle|\, (\mathbf{T}^{-1})^n \mathbf{D}^{-1} \mathsf{f} \right\rangle && \text{by unitary property of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Proposition 1.7 page 5)} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,\middle|\, (\mathbf{T}^{-1})^n \frac{\sqrt{2}}{2} \mathsf{f} \right\rangle && \text{because } \mathsf{f} \text{ is a constant hypothesis} && \text{and by Proposition 1.2 page 2}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \left\langle \phi \,|\, \tfrac{\sqrt{2}}{2}\mathsf{f} \right\rangle && \text{by } \mathsf{f}(x) = 1 \text{ definition}\\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \tfrac{\sqrt{2}}{2} \langle \phi \,|\, \mathsf{f} \rangle && \text{by property of } \langle \triangle \,|\, \nabla \rangle\\
&= \tfrac{\sqrt{2}}{2} \langle \phi \,|\, \mathsf{f} \rangle \sum_{n\in\mathbb{Z}} \mathsf{h}_n\\
&\implies \sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2}
\end{aligned}
$$

3. Proof that (1) $\iff$ (2) $\iff$ (3): by Proposition P.2 page 357.

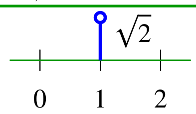
4. Proof for $\not\Leftarrow$ part: by Counterexample 2.1 page 29.

✏

Counterexample 2.1. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ be an *MRA system* (Definition 2.3 page 25).

CNT

$$\left\{ \left(\!\left(\mathsf{h}_n\right)\!\right) \triangleq \sqrt{2}\bar{\delta}_{n-1} \triangleq \begin{cases} \sqrt{2} & \text{for } n = 1\\ 0 & \text{otherwise.}\end{cases} \right\} \implies \{\phi(x) = 0\}$$



which means

$$\left\{ \sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2} \right\} \quad \not\Longrightarrow \quad \left\{ \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right) \text{ is an MRA system for } \boldsymbol{L}^2_{\mathbb{R}}.\right\}$$

✎PROOF:

$$
\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 22)}\\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x - n) && \text{by definitions of } \mathbf{D} \text{ and } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\sqrt{2}\bar{\delta}_{n-1}}_{\left(\!\left(\mathsf{h}_n\right)\!\right)} \phi(2x - n) && \text{by definitions of } \left(\!\left(\mathsf{h}_n\right)\!\right)\\
&= \sqrt{2}\phi(2x - 1) && \text{by definition of } \phi(x)\\
\implies \phi(x) &= 0
\end{aligned}
$$

This implies $\phi(x) = 0$, which implies that $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ is *not* an *MRA system* for $\boldsymbol{L}^2_{\mathbb{R}}$ because

$$\left(\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j\right)^{-} = \left(\bigcup_{j\in\mathbb{Z}} \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi|_{n\in\mathbb{Z}}\right\}\right)^{-} \neq \boldsymbol{L}^2_{\mathbb{R}}$$

(the *least upper bound* is *not* $\boldsymbol{L}^2_{\mathbb{R}}$).

✏

Theorem 2.4 (Quadrature condition in "time"). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 25).*

THM

$$\sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}^*_k \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \langle \phi \,|\, \mathbf{T}^n \phi \rangle \qquad \forall n\in\mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\langle \phi \,|\, \mathbf{T}^n \phi \rangle &= \left\langle \sum_{m\in\mathbb{Z}} \mathsf{h}_m \mathbf{D}\mathbf{T}^m \phi \,\middle|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} \mathsf{h}_k \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by dilation equation (Theorem 2.1 page 22)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \mathbf{D}\mathbf{T}^m \phi \,|\, \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by properties of } \langle \triangle \,|\, \nabla \rangle\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by definition of operator adjoint (Proposition D.3 page 169)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \mathbf{D}\mathbf{T}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by Proposition 1.5 page 4}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, \mathbf{T}^{*m} \mathbf{D}^* \mathbf{D}\mathbf{T}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by operator star-algebra properties (Theorem D.13 page 170)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, \mathbf{T}^{-m} \mathbf{D}^{-1} \mathbf{D}\mathbf{T}^{2n} \mathbf{T}^k \phi \right\rangle && \text{by Proposition 1.7 page 5}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{k\in\mathbb{Z}} \mathsf{h}_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k} \phi \right\rangle
\end{aligned}$$

✏

Theorem 2.5 (next) presents the *quadrature necessary conditions* of a *wavelet system*. These relations simplify dramatically in the special case of an *orthonormal wavelet system* (Theorem P.4 page 361).

Theorem 2.5 (Quadrature condition in "frequency"). [11] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 25). Let* $\tilde{x}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition P.1 page 355) for a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\tilde{\mathsf{S}}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM *(Definition R.3 page 373) of* ϕ.

THM
$$\left|\breve{\mathsf{h}}(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) \;=\; 2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)$$

Note: $\tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1$ *for* ORTHONORMAL *MRA* *(Lemma 5.1 page 71)*

✎PROOF:

$$\begin{aligned}
&2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)\\
&= 2(2\pi) \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(2\omega+2\pi n)\right|^2 && \text{by Theorem R.1 page 373}\\
&= 2(2\pi) \sum_{n\in\mathbb{Z}} \left| \frac{\sqrt{2}}{2} \breve{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right) \tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 && \text{by Lemma 2.1 page 22}\\
&= 2\pi \sum_{n\in\mathbb{Z}_e} \left|\breve{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 + 2\pi \sum_{n\in\mathbb{Z}_o} \left|\breve{\mathsf{h}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\\
&= 2\pi \sum_{n\in\mathbb{Z}} \left|\breve{\mathsf{h}}(\omega+2\pi n)\right|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi \sum_{n\in\mathbb{Z}} \left|\breve{\mathsf{h}}(\omega+2\pi n+\pi)\right|^2 \left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2\\
&= 2\pi \sum_{n\in\mathbb{Z}} \left|\breve{\mathsf{h}}(\omega)\right|^2 \left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi \sum_{n\in\mathbb{Z}} \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2 \left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 && \text{by Proposition P.1 page 355}\\
&= \left|\breve{\mathsf{h}}(\omega)\right|^2 \left(2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega+2\pi n)\right|^2\right) + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2 \left(2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right)\\
&= \left|\breve{\mathsf{h}}(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) && \text{by Theorem R.1 page 373}
\end{aligned}$$

✏

[11] Chui (1992), page 135, Goswami and Chan (1999), page 110

2.6 Sufficient conditions

Theorem 2.6 (next) gives a set of *sufficient* conditions on the *scaling function* (Definition 2.1 page 16) ϕ to generate an *MRA*. Theorem 5.2 (page 74) provides a set of sufficient conditions on the *scaling coefficients* (Definition 2.3 page 25) $(\!(h_n)\!)_{n\in\mathbb{Z}}$ to generate an *MRA*; howbeit, this set results in the more restrictive *orthonormal* MRA.

Theorem 2.6. [12] *Let* $V_j \triangleq \operatorname{span}\{\mathbf{T}\phi(x) \mid n\in\mathbb{Z}\}$ *(Definition L.2 page 267).*

THM

$$\left\{\begin{array}{lll}
(1). & (\!(\mathbf{T}^n\phi)\!) \text{ is a } \textsc{Riesz sequence} \text{ (Definition L.14 page 285)} & \text{and}\\
(2). & \exists (\!(h_n)\!) \quad \text{such that} \quad \phi(x) = \sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n\phi(x) & \text{and}\\
(3). & \tilde{\phi}(\omega) \text{ is } \textsc{continuous} \text{ at } 0 & \text{and}\\
(4). & \tilde{\phi}(0) \neq 0 &
\end{array}\right\}
\implies
\left\{\begin{array}{c}(\!(V_j)\!)_{j\in\mathbb{Z}} \text{ is an MRA}\\ \text{(Definition 2.1 page 16)}\end{array}\right\}$$

✎PROOF: For this to be true, each of the conditions in the definition of an *MRA* (Definition 2.1 page 16) must be satisfied:

1. Proof that each V_j is *closed*: by definition of span
2. Proof that $(\!(V_j)\!)$ is *linearly ordered*:
$$V_j \subseteq V_{j+1} \iff \operatorname{span}\{\mathbf{D}^j\mathbf{T}^n\phi\} \subseteq \operatorname{span}\{\mathbf{D}^{j+1}\mathbf{T}^n\phi\} \qquad \Longleftarrow (2)$$
3. Proof that $\bigcup_{j\in\mathbb{Z}} V_j$ is *dense* in $L^2_{\mathbb{R}}$: by Proposition 2.2 page 18
4. Proof of *self-similar* property:
$$\{\mathsf{f} \in V_j \iff \mathbf{D}\mathsf{f} \in V_{j+1}\} \iff \mathsf{f} \in \operatorname{span}\{\mathbf{T}^n\phi\} \iff \mathbf{D}\mathsf{f} \in \operatorname{span}\{\mathbf{D}\mathbf{T}^n\phi\} \qquad \Longleftarrow (2)$$
5. Proof for *Riesz basis*: by (1) and Proposition 2.2 page 18.

✏

2.7 Support size

The *support* of a function is what it's non-zero part "sits" on. If the support of the scaling coefficients $(\!(h_n)\!)$ goes from say $[0,3]$ in $\mathbb{Z}$, what is the support of the scaling function $\phi(x)$? The answer is $[0,3]$ in $\mathbb{R}$—essentially the same as the support of $(\!(h_n)\!)$ except that the two functions have different domains ($\mathbb{Z}$ versus $\mathbb{R}$). This concept is defined in Definition 2.4 (next definition), and proven in Theorem 2.7 (next theorem).

Definition 2.4. *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 25). Let* X^- *represent the* CLOSURE *of a set* X *in* $L^2_{\mathbb{R}}$, $\vee X$ *the* LEAST UPPER BOUND *of an* ORDERED SET $(X, \leq)$, $\wedge X$ *the* GREATEST LOWER BOUND *of* $(X, \leq)$, *and*

[12] Wojtaszczyk (1997) page 28 ⟨Theorem 2.13⟩, Pinsky (2002) page 313 ⟨Theorem 6.4.27⟩

$$\lfloor x \rfloor \triangleq \bigvee \{n \in \mathbb{Z} | n \le x\} \quad \forall x \in \mathbb{R} \quad (\text{FLOOR of } x)$$
$$\lceil x \rceil \triangleq \bigwedge \{n \in \mathbb{Z} | n \ge x\} \quad \forall x \in \mathbb{R} \quad (\text{CEILING of } x).$$

DEF *The set* $\operatorname{supp} f$ *of a function* $f \in Y^X$ *is the* ***support*** *of* f *if*

$$\operatorname{supp} f \triangleq \begin{cases} \{x \in \mathbb{R} | f(x) \ne 0\}^- & \text{for } X = \mathbb{R} \quad (\text{domain of } f \text{ is } \mathbb{R}) \quad \text{and} \\ \{x \in \mathbb{R} | f(\lfloor x \rfloor) \ne 0 \text{ and } f(\lceil x \rceil) \ne 0\}^- & \text{for } X = \mathbb{Z} \quad (\text{domain of } f \text{ is } \mathbb{Z}) \end{cases}.$$

Theorem 2.7 (support size). [13] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM (Definition 2.3 page 25). *Let* $\operatorname{supp} f$ *be the support of a function* f (Definition 2.4 page 31).

THM $\operatorname{supp}\phi = \operatorname{supp} h$

✎PROOF:

1. Definitions: $\operatorname{supp}\phi \triangleq [a, b]$
   $\operatorname{supp} h \triangleq [k, m]$.

2. lemma: $\operatorname{supp}\phi(x) = [a, b] \iff \operatorname{supp}\phi(2x) = \left[\frac{a}{2}, \frac{b}{2}\right]$

3. lemma: $\operatorname{supp}[\lambda\phi(x)] = \operatorname{supp}[\phi(x)] \quad \forall \lambda \in \mathbb{R}\backslash 0$

4. Proof that $k = a$:

$$\begin{aligned}
a &= \bigwedge \operatorname{supp}\phi(x) && \text{by definition of } a \quad \text{(item (1) page 32)} \\
&\triangleq \bigwedge \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} h_n \mathbf{D}\mathbf{T}^n \phi(x)\right] && \text{by } \textit{dilation equation} \quad \text{(Theorem 2.1 page 22)} \\
&= \bigwedge \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} h_n \phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} \quad \text{(Definition 1.3 page 2)} \\
&= \bigwedge \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} h_n \phi(2x-n)\right] && \text{by (3) lemma} \\
&= \bigwedge \operatorname{supp}[h_k \phi(2x-k)] && \text{because } n = k \text{ is the } \textit{least value} \text{ of } n \text{ for which } h_n \ne 0 \\
&= \bigwedge \operatorname{supp}[\phi(2x-k)] && \text{by (3) lemma} \\
&= \bigwedge \operatorname{supp}\left[\phi\left(2\left[x - \frac{k}{2}\right]\right)\right] \\
&= \bigwedge \left\{t | \phi\left(2\left[x - \frac{k}{2}\right]\right) \ne 0\right\} && \text{by definition of supp} \quad \text{(Definition 2.4 page 31)} \\
&= x \quad \text{such that} \quad x - \frac{k}{2} = \frac{a}{2} && \text{by (2) lemma} \\
&= \frac{k}{2} + \frac{a}{2}
\end{aligned}$$

$$\implies \quad \frac{k}{2} = a - \frac{a}{2}$$
$$\iff \quad k = a$$

[13] Mallat (1999) pages 243–244

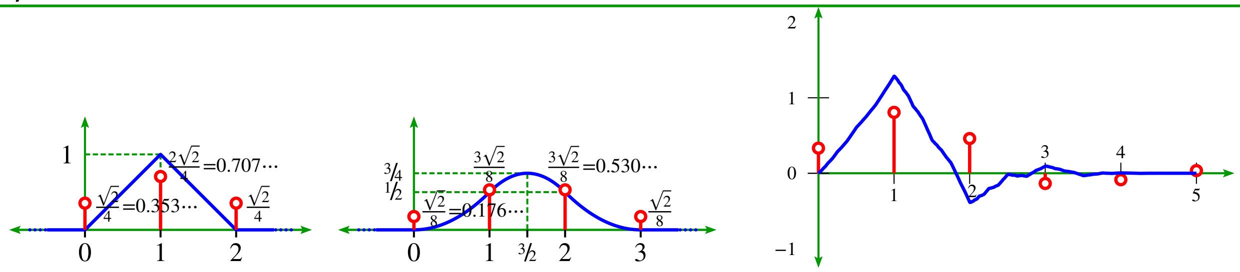
5. Proof that $m = b$:

$$
\begin{aligned}
b &= \bigvee \operatorname{supp}\phi(x) && \text{by definition of } b && \text{(item (1) page 32)}\\
&\triangleq \bigvee \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x)\right] && \text{by } \textit{dilation equation} && \text{(Theorem 2.1 page 22)}\\
&= \bigvee \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 2)}\\
&= \bigvee \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}_n \phi(2x-n)\right] && \text{by (3) lemma}\\
&= \bigvee \operatorname{supp}\left[\mathsf{h}_m \phi(2x-m)\right] && \text{because } n=m \text{ is the } \textit{greatest value} \text{ of } n \text{ for which } \mathsf{h}_n \neq 0\\
&= \bigvee \operatorname{supp}[\phi(2x-m)] && \text{by (3) lemma}\\
&= \bigvee \operatorname{supp}\left[\phi\left(2\left[x-\frac{m}{2}\right]\right)\right]\\
&= \bigvee \left\{t \,|\, \phi\left(2\left[x-\frac{m}{2}\right]\right) \neq 0\right\} && \text{by definition of } \operatorname{supp} && \text{(Definition 2.4 page 31)}\\
&= x \quad \text{such that} \quad x-\frac{m}{2} = \frac{b}{2} && \text{by (2) lemma}\\
&= \frac{m}{2}+\frac{b}{2}\\
&\Longrightarrow \qquad \frac{m}{2} = b-\frac{b}{2}\\
&\Longleftrightarrow \qquad m = b
\end{aligned}
$$

✏

Example 2.2.

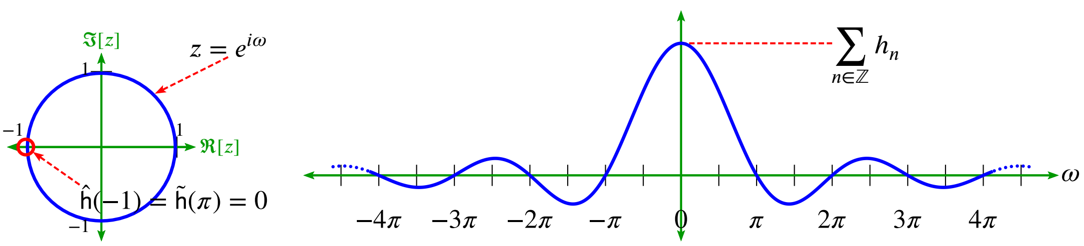
EX



B-spline $\mathsf{N}_1(x)$ (Example M.2 page 300) | *B-spline* $\mathsf{N}_2(x)$ (Example M.3 page 300) | *Daubechies-3 scaling function* (Example 6.5 page 111)

2.8 Scaling functions with partition of unity

The Z transform (Definition O.4 page 342) of a sequence $\langle h_n \rangle$ with sum $\sum_{n\in\mathbb{Z}}(-1)^n \mathsf{h}_n = 0$ has a zero at $z=-1$. Somewhat surprisingly, the *partition of unity* and *zero at* $z=-1$ properties are actually equivalent (next theorem).



Theorem 2.8. [14] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 2.3 page 25). Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* FOURIER TRANSFORM *(Definition K.2 page 257) of a function* $\mathsf{f} \in L^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION *(Definition L.12 page 278). Let* c *be some contant in* $\mathbb{R}\backslash 0$.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi = c}_{\text{(1) PARTITION OF UNITY}} \iff \underbrace{\sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n = 0}_{\text{(2) ZERO AT } z=-1} \iff \underbrace{\sum_{n\in\mathbb{Z}} \mathsf{h}_{2n} = \sum_{n\in\mathbb{Z}} \mathsf{h}_{2n+1} = \frac{\sqrt{2}}{2}}_{\text{(3) sum of even = sum of odd = } \frac{\sqrt{2}}{2}}$$

✎PROOF: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

1. Proof that (1) $\Longleftarrow$ (2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\phi &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n \left[\sum_{m\in\mathbb{Z}} \mathsf{h}_m \mathbf{D}\mathbf{T}^m\phi\right] && \text{by } \textit{dilation equation} \quad \text{(Theorem 2.1 page 22)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathbf{D}\mathbf{T}^m\phi \\
&= \sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^m\phi && \text{by } \textit{commutator relation} \quad \text{(Proposition 1.5 page 4)}\\
&= \mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \sum_{n\in\mathbb{Z}} \mathbf{T}^{2n}\mathbf{T}^m\phi \\
&= \mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \left[\sqrt{\frac{2\pi}{2}}\hat{\mathbf{F}}^{-1}\mathbf{S}_2\tilde{\mathbf{F}}(\mathbf{T}^m\phi)\right] && \text{by } \textit{PSF} \quad \text{(Theorem 1.2 page 10)}\\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1}\mathbf{S}_2 e^{-i\omega m}\tilde{\mathbf{F}}\phi && \text{by Corollary 1.1 page 9}\\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1} e^{-i\frac{2\pi}{2}km}\mathbf{S}_2\tilde{\mathbf{F}}\phi && \text{by definition of } \mathbf{S} \quad \text{(Definition 1.4 page 9)}\\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \hat{\mathbf{F}}^{-1}(-1)^{km}\mathbf{S}_2\tilde{\mathbf{F}}\phi \\
&= \sqrt{\pi}\mathbf{D}\sum_{m\in\mathbb{Z}} \mathsf{h}_m \left[\frac{\sqrt{2}}{2}\sum_{k\in\mathbb{Z}}(-1)^{km}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\frac{2\pi}{2}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem J.1 page 254)}\\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}\mathsf{h}_m \\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}\mathsf{h}_m + \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\sum_{m\in\mathbb{Z}}(-1)^{km}\mathsf{h}_m \\
&= \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_e}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}}\mathsf{h}_m}_{\sqrt{2}} + \frac{\sqrt{2\pi}}{2}\mathbf{D}\sum_{k\in\mathbb{Z}_o}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx}\underbrace{\sum_{m\in\mathbb{Z}}(-1)^{m}\mathsf{h}_m}_{0} \\
&= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e}(\mathbf{S}_2\tilde{\mathbf{F}}\phi)e^{i\pi kx} && \text{by Theorem 2.3 (page 27) and right hypothesis}\\
&= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}_e}\tilde{\phi}\left(\frac{2\pi}{2}k\right)e^{i\pi kx} && \text{by definitions of } \tilde{\mathbf{F}} \text{ and } \mathbf{S}_2\\
&= \sqrt{\pi}\mathbf{D}\sum_{k\in\mathbb{Z}}\tilde{\phi}(2\pi k)e^{i2\pi kx} && \text{by definition of } \mathbb{Z}_e
\end{aligned}$$

[14] Jawerth and Sweldens (1994) page 8, Chui (1992), page 123

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\mathbf{D}\left\{\sqrt{2\pi}\sum_{k\in\mathbb{Z}}\tilde{\phi}(2\pi k)e^{i2\pi kx}\right\} \\
&= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n\in\mathbb{Z}}\phi(x+n) && \text{by } \textit{PSF} && \text{(Theorem 1.2 page 10))} \\
&= \frac{1}{\sqrt{2}}\mathbf{D}\sum_{n}\mathbf{T}^n\phi && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)}
\end{aligned}
$$

The above equation sequence demonstrates that

$$\mathbf{D}\sum_{n}\mathbf{T}^n\phi = \sqrt{2}\sum_{n}\mathbf{T}^n\phi$$

(essentially that $\sum_n \mathbf{T}^n\phi$ is equal to it's own dilation). This implies that $\sum_n \mathbf{T}^n\phi$ is a constant (Proposition 1.8 page 5).

2. Proof that (1) $\Longrightarrow$ (2):

$$
\begin{aligned}
c &= \sum_{n\in\mathbb{Z}}\mathbf{T}^n\phi && \text{by left hypothesis} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\phi && \text{by } \textit{PSF} && \text{(Theorem 1.2 page 10)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\underbrace{\sqrt{2}\left(\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}}\mathsf{h}_n e^{-i\omega n}\right)\left(\mathbf{D}^{-1}\tilde{\mathbf{F}}\phi\right)}_{\tilde{\mathbf{F}}\phi} && \text{by Lemma 2.1 page 22} \\
&= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\mathbf{D}^{-1}\sum_{n\in\mathbb{Z}}\mathsf{h}_n e^{-i\omega n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) && \text{by Corollary 1.1 page 9} \\
&= 2\sqrt{\pi}\,\hat{\mathbf{F}}^{-1}\left(\mathbf{S}\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}\mathsf{h}_n e^{-i\frac{\omega}{2}n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) && \text{by evaluation of } \mathbf{D}^{-1} && \text{(Proposition 1.2 page 2)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}\mathsf{h}_n e^{-i\frac{2\pi k}{2}n}\right)\left(\mathbf{S}\tilde{\mathbf{F}}\mathbf{D}\phi\right) && \text{by definition of } \mathbf{S} && \text{(Definition 1.4 page 9)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^{kn}\right)\left(\mathbf{S}\mathbf{D}^{-1}\mathbf{F}\phi\right) \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^{kn}\right)\left(\mathbf{S}\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{\omega}{2}\right)\right) && \text{by definition of } \mathbf{S} && \text{(Definition 1.4 page 9)} \\
&= \sqrt{2\pi}\,\hat{\mathbf{F}}^{-1}\left(\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^{kn}\right)\left(\frac{1}{\sqrt{2}}\tilde{\phi}\left(\frac{2\pi k}{2}\right)\right) \\
&= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} && \text{by definition of } \hat{\mathbf{F}}^{-1} && \text{(Theorem J.1 page 254)} \\
&= \sqrt{\pi}\sum_{k\text{ even}}\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} + \sqrt{\pi}\sum_{k\text{ odd}}\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^{kn}\,\tilde{\phi}(\pi k)e^{i2\pi kx} \\
&= \sqrt{\pi}\sum_{k\text{ even}}\left(\cancelto{\sqrt{2}}{\sum_{n\in\mathbb{Z}}\mathsf{h}_n}\right)\tilde{\phi}(\pi k)e^{i2\pi kx} + \sqrt{\pi}\sum_{k\text{ odd}}\left(\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^n\right)\tilde{\phi}(\pi k)e^{i2\pi kx} \\
&= \sqrt{\pi}\sum_{k\in\mathbb{Z}}\sqrt{2}\,\tilde{\phi}(\pi 2k)e^{i2\pi 2kx} + \sqrt{\pi}\sum_{k\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^n\right)\tilde{\phi}(\pi[2k+1])e^{i2\pi[2k+1]x} && \text{by Theorem 2.3 page 27} \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\tilde{\phi}(0) + \sqrt{\pi}e^{i2\pi x}\sum_{n\in\mathbb{Z}}\mathsf{h}_n(-1)^n\sum_{k\in\mathbb{Z}}\tilde{\phi}(\pi[2k+1])e^{i4\pi kx} && \text{by left hypothesis and Theorem Q.1 page 366}
\end{aligned}
$$

$$\implies \left(\sum_{n\in\mathbb{Z}} \mathsf{h}_n(-1)^n\right) = 0 \qquad \text{because the right side must equal } c$$

3. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}_e} \mathsf{h}_n = \sum_{n\in\mathbb{Z}_o} \mathsf{h}_n &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n && \text{by (2) and Proposition P.4 page 358}\\
&= \frac{\sqrt{2}}{2} && \text{by } \textit{admissibility condition} \text{ (Theorem 2.3 page 27)}
\end{aligned}$$

4. Proof that (2) $\impliedby$ (3):

$$\begin{aligned}
\frac{\sqrt{2}}{2} &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n\mathsf{h}_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n\mathsf{h}_n}_{\text{odd terms}} && \text{by (3)}\\
&\implies \sum_{n\in\mathbb{Z}}(-1)^n\mathsf{h}_n = 0 && \text{by Proposition P.4 page 358}
\end{aligned}$$

✐

Not every function that forms a *partition of unity* is a *basis* for an *MRA*, as formerly stated next and demonstrated by Counterexample 2.2 (page 36) and Counterexample 2.3 (page 38).

Proposition 2.5.

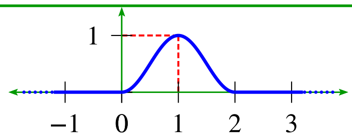
PRP: $\phi(x)$ *generates a* PARTITION OF UNITY $\;\;\not\Longrightarrow\;\;$ $\phi(x)$ *generates an* MRA *system.*

✎PROOF: By Counterexample 2.2 (page 36) and Counterexample 2.3 (page 38). ✐

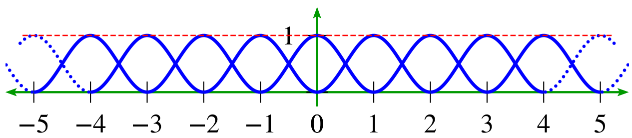
Counterexample 2.2. Let a function ϕ be defined in terms of the sine function (Definition H.3 page 215) as follows:

CNT:

$$\phi(x) \triangleq \begin{cases} \sin^2\left(\frac{\pi}{2}x\right) & \text{for } x \in [0:2]\\ 0 & \text{otherwise}\end{cases}$$



Then $\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x = 1$ and ϕ induces a *partition of unity*



but $\left\{\mathbf{T}^n\phi|_{n\in\mathbb{Z}}\right\}$ does **not** generate an *MRA*.

✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 1) on a set A.

1. Proof that $\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x = 1$: by Example Q.3 (page 368)

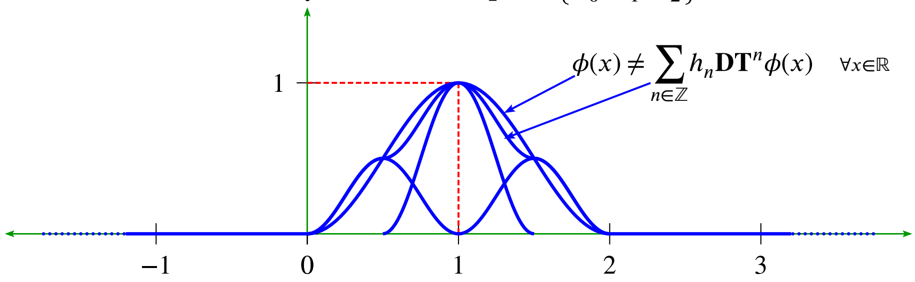
2. Proof that $\phi(x)$ forms a *partition of unity*: by Example Q.3 (page 368)

3. Proof that $\phi(x) \notin \operatorname{span}\{\mathbf{DT}^n\phi(x)|_{n\in\mathbb{Z}}\}$ (and so does not generate an *MRA*):

   (a) Note that the *support* (Definition 2.4 page 31) of ϕ is $\operatorname{supp}\phi = [0:2]$.

   (b) Therefore, the *support* of $(\!(h_n)\!)$ is $\operatorname{supp}(\!(h_n)\!) = \{0,1,2\}$ (Theorem 2.7 page 32).

(c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\{h_0, h_1, h_2\}$ (the rest would be 0).



Here would be the values of $\{h_1, h_2, h_3\}$:

$$\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n) \\
&= \sum_{n=0}^{2} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2x-n)\right)\mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 2.7 page 32}
\end{aligned}$$

(d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$ (see the diagram in item (3c) page 37):

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot\frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(1-n)\right)\mathbb{1}_{[0:2]}(1-n) \\
&= \mathsf{h}_0 \sin^2\left(\frac{\pi}{2}(1-0)\right)\mathbb{1}_{[0:2]}(1-0) + \mathsf{h}_1 \sin^2\left(\frac{\pi}{2}(1-1)\right)\mathbb{1}_{[0:2]}(1-1) \\
&\qquad + \mathsf{h}_2 \sin^2\left(\frac{\pi}{2}(1-2)\right)\mathbb{1}_{[0:2]}(1-2) \\
&= \mathsf{h}_0\cdot 1\cdot 1 + \mathsf{h}_1\cdot 0\cdot 1 + \mathsf{h}_2(-1)\cdot 0 \\
&= \mathsf{h}_0
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}}{2}\cdot 1 &= \frac{\sqrt{2}}{2}(1)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{\pi}{2}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi(1) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(2-n)\right)\mathbb{1}_{[0:2]}(2-n) \\
&= \mathsf{h}_0 \sin^2\left(\frac{\pi}{2}(2-0)\right)\mathbb{1}_{[0:2]}(2-0) + \mathsf{h}_1 \sin^2\left(\frac{\pi}{2}(2-1)\right)\mathbb{1}_{[0:2]}(2-1) \\
&\qquad + \mathsf{h}_2 \sin^2\left(\frac{\pi}{2}(2-2)\right)\mathbb{1}_{[0:2]}(2-2) \\
&= \mathsf{h}_0\cdot 0\cdot 1 + \mathsf{h}_1\cdot 1\cdot 1 + \mathsf{h}_2\cdot 0\cdot 1
\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{h}_1 \\
\frac{\sqrt{2}}{2}\cdot\frac{1}{2} &= \frac{\sqrt{2}}{2}\left(\frac{1}{-\sqrt{2}}\right)^2 \\
&= \frac{\sqrt{2}}{2}\sin^2\left(\frac{3\pi}{4}\right) \\
&\triangleq \frac{\sqrt{2}}{2}\phi\left(\frac{3}{2}\right) \\
&= \frac{\sqrt{2}}{2}\sqrt{2}\sum_{n\in\mathbb{Z}} \mathsf{h}_n \sin^2\left(\frac{\pi}{2}(3-n)\right)\mathbb{1}_{[0:2]}(3-n) \\
&= \mathsf{h}_0\sin^2\left(\frac{\pi}{2}(3-0)\right)\mathbb{1}_{[0:2]}(3-0) + \mathsf{h}_1\sin^2\left(\frac{\pi}{2}(3-1)\right)\mathbb{1}_{[0:2]}(3-1) \\
&\quad + \mathsf{h}_2\sin^2\left(\frac{\pi}{2}(3-2)\right)\mathbb{1}_{[0:2]}(3-2) \\
&= \mathsf{h}_0\cdot(-1)\cdot 0 + \mathsf{h}_1\cdot 0\cdot 1 + \mathsf{h}_2 1\cdot 1 \\
&= \mathsf{h}_2
\end{aligned}
$$

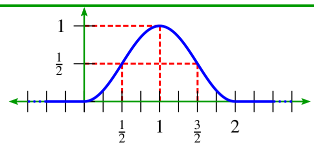
(e) These values for (h_0, h_1, h_2) are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 2.1 page 22). In particular,
$$\phi(x) \neq \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n\phi(x)$$
(see the diagram in item (3c) page 37)

✑

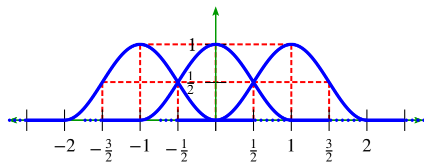
Counterexample 2.3 (raised sine). [15] Let a function f be defined in terms of a shifted cosine function (Definition H.2 page 215) as follows:

CNT

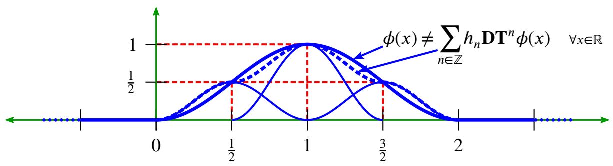
$$\phi(x) \triangleq \begin{cases} \frac{1}{2}\left\{1+\cos[\pi(|x-1|)]\right\} & \text{for } 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$



Then ϕ forms a *partition of unity*:



but $\left\{\mathbf{T}^n\phi \mid n\in\mathbb{Z}\right\}$ does **not** generate an *MRA*.



✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 1) on a set A.

1. Proof that $\phi(x)$ forms a *partition of unity*:

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}}\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}}\mathbf{T}^n\phi(x+1) && \text{by Proposition 1.1 page 2} \\
&= \sum_{n\in\mathbb{Z}}\phi(x+1-n) && \text{by Definition 1.3 page 2} \\
&= \sum_{n\in\mathbb{Z}}\frac{1}{2}\{1+\cos[\pi(|x-1+1-n|)]\}\mathbb{1}_{[0:2)}(x+1-n) && \text{by definition of } \phi(x)
\end{aligned}
$$

[15] Proakis (2001) pages 560–561

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \frac{1}{2}\{1+\cos[\pi(|x-n|)]\}\mathbb{1}_{[-1:1)}(x-n) && \text{by Definition 1.2 page 1}\\
&= \underbrace{\sum_{n\in\mathbb{Z}} \frac{1}{2}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n|-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_{[-1:1)}(x-n)\bigg|_{\beta=1}}_{\textit{raised cosine} \text{ (Example Q.4 page 369) with } \beta=1}\\
&= 1 && \text{by Example Q.4 page 369}
\end{aligned}
$$

2. Proof that $\phi(x) \notin \operatorname{span}\{\mathbf{DT}^n\phi(x) | n\in\mathbb{Z}\}$ (and so does not generate an *MRA*):
   (a) Note that the *support* (Definition 2.4 page 31) of ϕ is $\operatorname{supp}\phi = [0:2]$.
   (b) Therefore, the *support* of $(\!(h_n)\!)$ is $\operatorname{supp}(\!(h_n)\!) = \{0,1,2\}$ (Theorem 2.7 page 32).
   (c) So if $\phi(x)$ *is* an *MRA*, we only need to compute $\{h_0, h_1, h_2\}$ (the rest would be 0). Here would be the values of $\{h_1, h_2, h_3\}$:

$$
\begin{aligned}
\phi(x) &= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n\phi(x)\\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{DT}^n \frac{1}{2}\left\{1+\cos[\pi(|x-1|)]\right\}\mathbb{1}_{[0:2]}(x) && \text{by definition of } \phi(x)\\
&= \sum_{n\in\mathbb{Z}} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{1+\cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n) && \text{by Definition 1.3 page 2}\\
&= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{1+\cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n) && \text{by Theorem 2.7 page 32}
\end{aligned}
$$

   (d) The values of $(\!(h_0, h_1, h_2)\!)$ can be conveniently calculated at the knot locations $x=\frac{1}{2}$, $x=1$, and $x=\frac{3}{2}$ (see the diagram in item (3c) page 37):

$$
\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{1+\cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n)\bigg|_{x=\frac{1}{2}}\\
&= \mathsf{h}_0 \frac{\sqrt{2}}{2}\left\{1+\cos[1-1-0]\right\}\\
&= \mathsf{h}_0\sqrt{2}\\
&\implies \mathsf{h}_0 = \frac{\sqrt{2}}{4}
\end{aligned}
$$

$$
\begin{aligned}
1 &= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{1+\cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n)\bigg|_{x=1}\\
&= \mathsf{h}_1 \frac{\sqrt{2}}{2}\left\{1+\cos[2-1-1]\right\}\\
&= \mathsf{h}_1\sqrt{2}\\
&\implies \mathsf{h}_1 = \frac{\sqrt{2}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} &= \sum_{n=0}^{2} \mathsf{h}_n \frac{\sqrt{2}}{2}\left\{1+\cos[\pi(|2x-1-n|)]\right\}\mathbb{1}_{[0:2]}(2x-n)\bigg|_{x=\frac{3}{2}}\\
&= \mathsf{h}_2 \frac{\sqrt{2}}{2}\left\{1+\cos[1-1-0]\right\}
\end{aligned}
$$

$$= \mathsf{h}_2\sqrt{2}$$
$$\implies \mathsf{h}_2 = \frac{\sqrt{2}}{4}$$

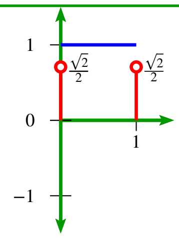
(e) These values for $(\!(h_0, h_1, h_2)\!)$ are valid for the knot locations $x = \frac{1}{2}$, $x = 1$, and $x = \frac{3}{2}$, **but** they don't satisfy the *dilation equation* (Theorem 2.1 page 22). In particular (see diagram),
$$\phi(x) \neq \sum_{n\in\mathbb{Z}} \mathsf{h}_n \mathbf{D}\mathbf{T}^n \phi(x)\,.$$

✏

Example 2.3 (2 coefficient case/Haar wavelet system/order 0 B-spline wavelet system). [16]
Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be an *wavelet system*.

EX

$$\left\{\begin{array}{lll} 1. & \operatorname{supp}\phi(x) = [0:1] & \text{(Theorem 2.7 page 32) and} \\ 2. & \textit{admissibility condition} & \text{(Theorem 2.3 page 27) and} \\ 3. & \textit{partition of unity} & \text{(Theorem 2.8 page 34)} \end{array}\right\} \implies \left\{\begin{array}{r|c} n & \mathsf{h}_n \\ \hline 0 & \frac{\sqrt{2}}{2} \\ 1 & \frac{\sqrt{2}}{2} \\ \text{other} & 0 \end{array}\right\}$$



✎PROOF:

1. Proof that (1) $\implies$ that only h_0 and h_1 are non-zero: by Theorem 2.7 page 32.

2. Proof for values of h_0 and h_1:

   (a) Method 1: Under the constraint of two non-zero scaling coefficients, a scaling function design is fully constrained using the *admissibility equation* (Theorem 2.3 page 27) and the *partition of unity* constraint (Definition Q.1 page 366). The partition of unity formed by $\phi(x)$ is illustrated in Example M.13 (page 315).

   Here are the equations:

$\mathsf{h}_0 + \mathsf{h}_1 = \sqrt{2}$	(admissibility equation	Theorem 2.3	page 27)
$\mathsf{h}_0 - \mathsf{h}_1 = 0$	(partition of unity/zero at -1	Theorem 2.8	page 34)

   Here are the calculations for the coefficients:
   $$\begin{aligned} (h_0 + h_1) + (h_0 - h_1) = 2h_0 &= \sqrt{2} && \text{(add two equations together)} \\ (h_0 + h_1) - (h_0 - h_1) = 2h_1 &= \sqrt{2} && \text{(subtract second from first)} \end{aligned}$$

   (b) Method 2: By Theorem M.11 page 324.

3. Note: h_0 and h_1 can also be produced using other systems of equations including the following:

   (a) *admissibility condition* and *orthonormality* (Example 5.3 page 90)

   (b) *Daubechies-p1* wavelets computed using spectral techniques (Example 6.3 page 109)

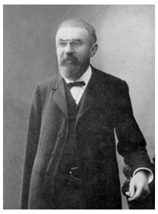
✏

[16] Haar (1910), Wojtaszczyk (1997) pages 14–15 ⟨"Sources and comments"⟩

CHAPTER 3

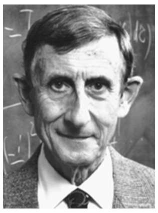
WAVELET STRUCTURES

❝. . . on fait la science avec des faits comme une maison avec des pierres ; mais une accumulation de faits n'est pas plus une science qu'un tas de pierres n'est une maison.❞



❝Science is built up of facts, as a house is built of stones; but an accumulation of facts is no more a science than a heap of stones is a house.❞

Jules Henri Poincaré (1854-1912), physicist and mathematician [1]



❝The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit. The problem is the overall design.❞

Freeman Dyson (1923–), physicist and mathematician [2]

3.1 Introduction

3.1.1 What are wavelets?

In Fourier analysis, *continuous* dilations (Definition 1.3 page 2) of the *complex exponential* (Definition H.5 page 220) form a *basis* (Definition L.7 page 272) for the *space of square integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185) such that

$$\boldsymbol{L}^2_{\mathbb{R}} = \operatorname{span}\left\{\mathbf{D}_\omega e^{ix} | \omega\in\mathbb{R}\right\}.$$

[1] quote: Poincaré (1902a) ⟨Chapter IX, paragraph 7⟩
translation: Poincaré (1902b), page 141
image: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Poincare.html

[2] quote: Albers and Dyson (1994), page 20
image: http://www.isepp.org/Media/Speaker%20Images/95-96%20Images/dyson.jpg

In Fourier series analysis (Theorem J.1 page 254), *discrete* dilations of the complex exponential form a basis for $\mathbf{L}^2_{\mathbb{R}}(0:2\pi)$ such that

$$\mathbf{L}^2_{\mathbb{R}}(0:2\pi) = \operatorname{span}\left\{ \mathbf{D}_j e^{ix} \middle| j\in\mathbb{Z} \right\}.$$

In Wavelet analysis, for some *mother wavelet* (Definition 3.1 page 43) $\psi(x)$,

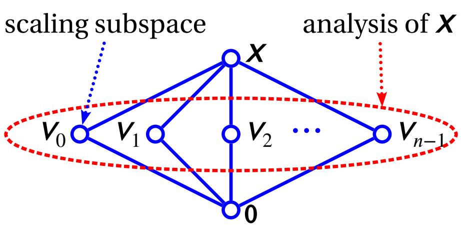
$$\mathbf{L}^2_{\mathbb{R}} = \operatorname{span}\left\{ \mathbf{D}_\omega \mathbf{T}_\tau \psi(x) | \omega,\tau\in\mathbb{R} \right\}.$$

However, the ranges of parameters ω and τ can be much reduced to the countable set $\mathbb{Z}$ resulting in a *dyadic* wavelet basis such that for some mother wavelet $\psi(x)$,

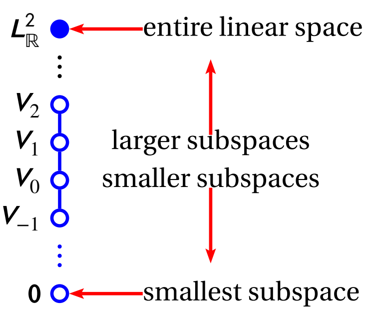
$$\mathbf{L}^2_{\mathbb{R}} = \operatorname{span}\left\{ \mathbf{D}^j \mathbf{T}^n \psi(x) | j,n\in\mathbb{Z} \right\}.$$

This text deals almost exclusively with dyadic wavelets. Wavelets that are both *dyadic* and *compactly supported* have the attractive feature that they can be easily implemented in hardware or software by use of the *Fast Wavelet Transform* (Figure T.1 page 387).

3.1.2 Analyses

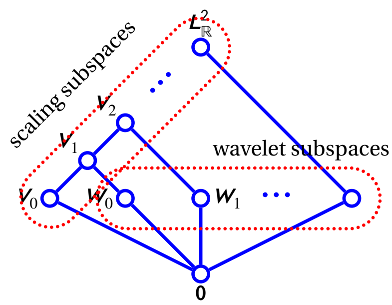


An analysis can be partially characterized by its order structure with respect to an order relation such as the set inclusion relation $\subseteq$. Most transforms have a very simple M-n order structure, as illustrated to the right. The M-n lattices for $n\geq 3$ are *modular* but not *distributive*. Analyses typically have one subspace that is a *scaling* subspace; and this subspace is often simply a family of constants (as is the case with Fourier Analysis).



A special characteristic of wavelet analysis is that there is not just one scaling subspace, but an entire sequence of scaling subspaces. These scaling subspaces are *linearly ordered* with respect to the ordering relation $\subseteq$. In wavelet theory, this structure is called a *multiresolution analysis*, or *MRA* (Definition 2.1 page 16).

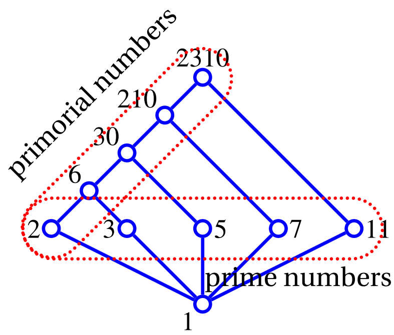
The MRA was introduced by Stéphane G. Mallat in 1989. The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan, and later as the *Gaussian Pyramid* by Burt and Adelson in the 1980s in the West.[3]



A second special characteristic of wavelet analysis is that it's order structure with respect to the $\subseteq$ relation is not a simple M-n lattice (as is with the case of Fourier and other analyses). Rather, it is a lattice of the form illustrated to the right. This lattice is *non-complemented*, *non-distributive*, *non-modular*, and *non-Boolean* (Proposition 3.1 page 45).

[3] Mallat (1989) page 70, Iijima (1959), Burt and Adelson (1983), Adelson and Burt (1981), Lindeberg (1993), Alvarez et al. (1993), Guichard et al. (2012), Weickert (1999) ⟨historical survey⟩

The wavelet subspace structure is similar in form to that of the *Primorial numbers*,[4] illustrated to the right by a *Hasse diagram*.



An analysis can be represented using three different structures:

① sequence of subspaces
② sequence of basis coefficients
③ sequence of basis vectors

These structures are isomorphic to each other, and can therefore be used interchangeably.

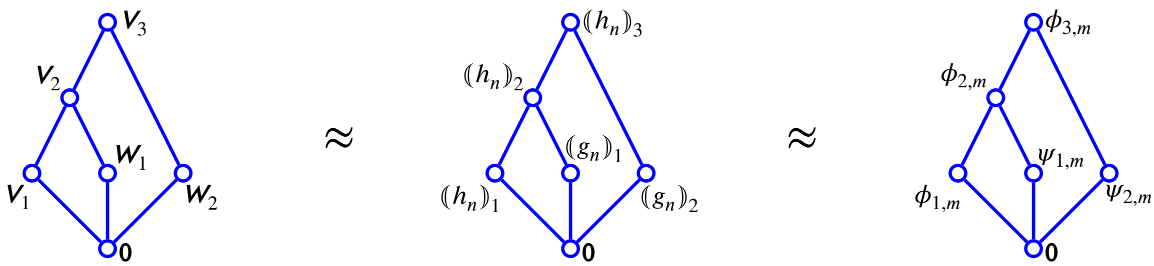


Figure 3.1 (page 53) illustrate the order structures of some analyses, including two wavelet analyses:

3.2 Definition

The term "wavelet" comes from the French word "*ondelette*", meaning "small wave". And in essence, wavelets are "small waves" (as opposed to the "long waves" of Fourier analysis) that form a basis for the Hilbert space $\boldsymbol{L}^2_{\mathbb{R}}$.[6]

Definition 3.1. [7] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition 1.3 page 2.*

DEF

A function $\psi(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ *is a **wavelet function** for* $\boldsymbol{L}^2_{\mathbb{R}}$ *if*

$$\left\{\mathbf{D}^j\mathbf{T}^n\psi\,|_{j,n\in\mathbb{Z}}\right\} \text{ is a RIESZ BASIS for } \boldsymbol{L}^2_{\mathbb{R}}.$$

In this case, ψ *is also called the **mother wavelet** of the basis* $\left\{\mathbf{D}^j\mathbf{T}^n\psi\,|_{j,n\in\mathbb{Z}}\right\}$. *The sequence of subspaces* $(\!(\boldsymbol{W}_j)\!)_{j\in\mathbb{Z}}$ *is the **wavelet analysis** induced by* ψ*, where each subspace* $\boldsymbol{W}_j$ *is defined as*

$$\boldsymbol{W}_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\psi \,\middle|\, n\in\mathbb{Z}\right\}.$$

A *wavelet analysis* $(\!(\boldsymbol{W}_j)\!)$ is often constructed from a *multiresolution anaysis* (Definition 2.1 page 16) $(\!(\boldsymbol{V}_j)\!)$ under the relationship

$$\boldsymbol{V}_{j+1} = \boldsymbol{V}_j \dotplus \boldsymbol{W}_j, \qquad \text{where } \dotplus \text{ is subspace addition (}\textit{Minkowski addition}\text{).}$$

By this relationship alone, $(\!(\boldsymbol{W}_j)\!)$ is in no way uniquely defined in terms of a multiresolution analysis $(\!(\boldsymbol{V}_j)\!)$. In general there are many possible complements of a subspace $\boldsymbol{V}_j$. To uniquely define

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4 Sloane (2014) ⟨http://oeis.org/A002110⟩

6 Strang and Nguyen (1996) page ix, Atkinson and Han (2009) page 191

7 Wojtaszczyk (1997) page 17 ⟨Definition 2.1⟩

such a wavelet subspace, one or more additional constraints are required. One of the most common additional constraints is *orthogonality*, such that $\boldsymbol{V}_j$ and $\boldsymbol{W}_j$ are orthogonal to each other (see CHAPTER 5 page 67).

3.3 Dilation equation

Suppose $(\mathbf{T}^n\psi)_{n\in\mathbb{Z}}$ is a basis for $\boldsymbol{W}_0$. By Definition 3.1 page 43, the wavelet subspace $\boldsymbol{W}_0$ is contained in the scaling subspace $\boldsymbol{V}_1$. By Definition 2.1 page 16, the sequence $(\mathbf{DT}^n\phi)_{n\in\mathbb{Z}}$ is a basis for $\boldsymbol{V}_1$. Because $\boldsymbol{W}_0$ is contained in $\boldsymbol{V}_1$, the sequence $(\mathbf{DT}^n\phi)_{n\in\mathbb{Z}}$ is also a basis for $\boldsymbol{W}_0$.

Theorem 3.1 (wavelet dilation equation). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\boldsymbol{V}_j), \phi, (h_n)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 2.3 page 25) and* $(\boldsymbol{W}_j)_{j\in\mathbb{Z}}$ *be a* WAVELET ANALYSIS *(Definition 3.1 page 43) with respect to* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\boldsymbol{V}_j), \phi, (h_n)\right)$ *and with* WAVELET FUNCTION ψ *(Definition 3.1 page 43).*

THM

$$\exists (g_n)_{n\in\mathbb{Z}} \quad \text{such that} \quad \psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{DT}^n\phi$$

✎PROOF:

$$\begin{aligned}
\psi &\in \boldsymbol{W}_0 && \text{by Definition 3.1 page 43}\\
&\subseteq \boldsymbol{V}_1 && \text{by Definition 3.1 page 43}\\
&= \operatorname{span}(\mathbf{DT}^n\phi(x))_{n\in\mathbb{Z}} && \text{by Definition 2.1 page 16 (MRA)}\\
&\implies \exists (g_n)_{n\in\mathbb{Z}} \quad \text{such that} \quad \psi = \sum_{n\in\mathbb{Z}} g_n \mathbf{DT}^n\phi
\end{aligned}$$

✏

A *wavelet system* (next definition) consists of two subspace sequences:

- A **multiresolution analysis** $(\boldsymbol{V}_j)$ (Definition 2.1 page 16) provides "coarse" approximations of a function in $\boldsymbol{L}^2_{\mathbb{R}}$ at different "scales" or resolutions.
- A **wavelet analysis** $(\boldsymbol{W}_j)$ provides the "detail" of the function missing from the approximation provided by a given scaling subspace (Definition 3.1 page 43).

Definition 3.2. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\boldsymbol{V}_j), \phi, (h_n)\right)$ *be a* MULTIRESOLUTION SYSTEM *(Definition 2.1 page 16) and* $(\boldsymbol{W}_j)_{j\in\mathbb{Z}}$ *a wavelet analysis (Definition 3.1 page 43) with respect to* $(\boldsymbol{V}_j)_{j\in\mathbb{Z}}$. *Let* $(g_n)_{n\in\mathbb{Z}}$ *be a sequence of coefficients.*

DEF

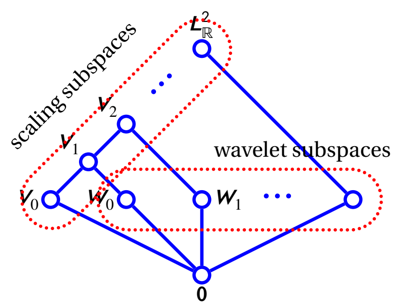
*A **wavelet system** is the tuple* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\boldsymbol{V}_j), (\boldsymbol{W}_j), \phi, \psi, (h_n), (g_n)\right)$
and the sequence $(g_n)_{n\in\mathbb{Z}}$ *that satisfies the equation* $\psi = \sum_{n\in\mathbb{Z}} g_n\mathbf{DT}^n\phi$
*is the **wavelet coefficient sequence**.*

Remark 3.1. The pair of coefficient sequences $\left((h_n), (g_n)\right)$ generates the scaling function $\phi(x)$ (Definition 2.1 page 16) and the wavelet function $\psi(x)$ (Definition 3.1 page 43). These functions in turn generate the multiresolution analysis $(\boldsymbol{V}_j)$ (Definition 2.1 page 16) and the wavelet analysis $(\boldsymbol{W}_j)$ (Definition 3.1 page 43). Therefore, the coefficient sequence pair $\left((h_n), (g_n)\right)$ totally defines a wavelet system $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\boldsymbol{V}_j), (\boldsymbol{W}_j), \phi, \psi, (h_n), (g_n)\right)$ (Definition 3.2 page 44).

Furthermore, especially in the case of orthonormal wavelets, the wavelet coefficient sequence $(\!(g_n)\!)_{n\in\mathbb{Z}}$ is often defined in terms of the scaling coefficient sequence $(\!(h_n)\!)_{n\in\mathbb{Z}}$ in a very simple and straightforward manner. Therefore, in the case of an orthonormal wavelet system, the coefficient scaling sequence $(\!(h_n)\!)_{n\in\mathbb{Z}}$ often totally defines the entire wavelet system. And in this case, designing a wavelet system is only a matter of finding a handful of scaling coefficients $(\!(h_1,\, h_2,\, \ldots,\, h_n)\!)$...because once you have these, you can generate everything else.

3.4 Order structure

The *wavelet system* $\left(\boldsymbol{L}^2_{\mathbb{R}},\, (\!(\boldsymbol{V}_j)\!)\,,\, (\!(\boldsymbol{W}_j)\!)\,,\, \phi,\, \psi,\, (\!(h_n)\!)\,,\, (\!(g_n)\!)\right)$ (Definition 3.2 page 44) together with the set inclusion relation $\subseteq$ forms an *ordered set*, illustrated to the right by a *Hasse diagram*.



Proposition 3.1. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}},\, (\!(\boldsymbol{V}_j)\!)\,,\, (\!(\boldsymbol{W}_j)\!)\,,\, \phi,\, \psi,\, (\!(h_n)\!)\,,\, (\!(g_n)\!)\right)$ *be a wavelet system with order relation* $\subseteq$. *The lattice* $\boldsymbol{L} \triangleq \left(\, (\!(\boldsymbol{V}_j)\!)\,,(\!(\boldsymbol{W}_j)\!)\,,\, \vee,\, \wedge;\, \subseteq\right)$ *has the following properties:*

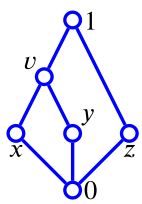
PRP

1. $\boldsymbol{L}$ *is* NONDISTRIBUTIVE.
2. $\boldsymbol{L}$ *is* NONMODULAR.
3. $\boldsymbol{L}$ *is* NONCOMPLEMENTED.
4. $\boldsymbol{L}$ *is* NONBOOLEAN.

✎PROOF:

1. Proof that $\boldsymbol{L}$ is *nondistributive*:
   (a) $\boldsymbol{L}$ contains the $N5$ lattice.
   (b) Because $\boldsymbol{L}$ contains the $N5$ lattice, $\boldsymbol{L}$ is *nondistributive*.
2. Proof that $\boldsymbol{L}$ is *nonmodular* and *nondistributive*:
   (a) $\boldsymbol{L}$ contains the $N5$ lattice.
   (b) Because $\boldsymbol{L}$ contains the $N5$ lattice, $\boldsymbol{L}$ is *nonmodular*.
3. Proof that $\boldsymbol{L}$ is *noncomplemented*:

$$\begin{aligned} x' &= y' = v' = z \\ z' &= \{x, y, v\} \\ x'' &= \left(x'\right)' \\ &= z' \\ &= \{x, y, v\} \\ &\neq x \end{aligned}$$



4. Proof that $\boldsymbol{L}$ is *nonBoolean*:
   (a) $\boldsymbol{L}$ is *nondistributive* (item (1)).
   (b) Because $\boldsymbol{L}$ is *nondistributive*, it is *nonBoolean*.

3.5 Subspace algebraic structure

Theorem 3.2. *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \left(\!\left(W_j\right)\!\right), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM (Definition 3.2 page 44). *Let* $V_1 \dotplus V_2$ *represent* MINKOWSKI ADDITION *of two subspaces* V_1 *and* V_2 *of a Hilbert space* H.

THM

$$\begin{aligned}
L^2_{\mathbb{R}} &= \lim_{j\to\infty} V_j && (L^2_{\mathbb{R}} \text{ is equivalent to one very large scaling subspace}) \\
&= V_j \dotplus W_j \dotplus W_{j+1} \dotplus W_{j+2} \dotplus \cdots && \left(\begin{array}{l} L^2_{\mathbb{R}} \text{ is equivalent to one scaling space} \\ \text{and a sequence of wavelet subspaces} \end{array}\right) \\
&= \cdots \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus \cdots && (L^2_{\mathbb{R}} \text{ is equivalent to a sequence of wavelet subspaces})
\end{aligned}$$

✎PROOF:

1. Proof for (1):

$$L^2_{\mathbb{R}} = \lim_{j\to\infty} V_j \qquad \text{by Definition 2.1 page 16}$$

2. Proof for (2):

$$\begin{aligned}
\underbrace{V_j \dotplus W_j}_{V_{j+1}} \dotplus W_{j+1} \dotplus W_{j+2} \dotplus \cdots &= \underbrace{V_{j+1} \dotplus W_{j+1}}_{V_{j+2}} \dotplus W_{j+2} \dotplus W_{j+3} \dotplus \cdots \\
&= \underbrace{V_{j+2} \dotplus W_{j+2}}_{V_{j+3}} \dotplus W_{j+3} \dotplus W_{j+4} \dotplus \cdots \\
&= \underbrace{V_{j+3} \dotplus W_{j+3}}_{V_{j+4}} \dotplus W_{j+4} \dotplus W_{j+5} \dotplus \cdots \\
&= \underbrace{V_{j+5} \dotplus W_{j+5}}_{V_{j+5}} \dotplus W_{j+6} \dotplus W_{j+6} \dotplus \cdots \\
&= \lim_{j\to\infty} V_{j+5} \dotplus W_{j+5} \dotplus W_{j+6} \dotplus W_{j+6} \dotplus \cdots \\
&= L^2_{\mathbb{R}}
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
L^2_{\mathbb{R}} &= \underbrace{V_0}_{V_{-1}\dotplus W_{-1}} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots && \text{by (2)} \\
&= \underbrace{V_{-1}}_{V_{-2}\dotplus W_{-2}} W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\
&= \underbrace{V_{-2}}_{V_{-3}\dotplus W_{-3}} W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\
&= \underbrace{V_{-3}}_{V_{-4}\dotplus W_{-4}} W_{-3} \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots \\
&\;\vdots \\
&= \cdots \dotplus W_{-3} \dotplus W_{-2} \dotplus W_{-1} \dotplus W_0 \dotplus W_1 \dotplus W_2 \dotplus W_3 \dotplus \cdots
\end{aligned}$$

Remark 3.2. In the special case that two subspaces $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are *orthogonal* to each other, then the *subspace addition* operation $\boldsymbol{W}_1 \dotplus \boldsymbol{W}_2$ is frequently expressed as $\boldsymbol{W}_1 \oplus \boldsymbol{W}_2$. In the case of an *orthonormal wavelet system* (Definition 5.1 page 67), the expressions in Theorem 3.2 (page 46) could be expressed as

$$\begin{aligned}
\boldsymbol{L}^2_{\mathbb{R}} &= \lim_{j\to\infty} \boldsymbol{V}_j \\
&= \boldsymbol{V}_j \oplus \boldsymbol{W}_j \oplus \boldsymbol{W}_{j+1} \oplus \boldsymbol{W}_{j+2} \oplus \cdots \\
&= \cdots \oplus \boldsymbol{W}_{-2} \oplus \boldsymbol{W}_{-1} \oplus \boldsymbol{W}_0 \oplus \boldsymbol{W}_1 \oplus \boldsymbol{W}_2 \oplus \cdots .
\end{aligned}$$

.

3.6 Necessary conditions

Theorem 3.3 (quadrature conditions in "time"). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system* (Definition 3.2 page 44).

THM

1. $\displaystyle\sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \phi \,|\, \mathbf{T}^n \phi \right\rangle \quad \forall n\in\mathbb{Z}$
2. $\displaystyle\sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \psi \,|\, \mathbf{T}^n \psi \right\rangle \quad \forall n\in\mathbb{Z}$
3. $\displaystyle\sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle = \left\langle \phi \,|\, \mathbf{T}^n \psi \right\rangle \quad \forall n\in\mathbb{Z}$

✎PROOF:

1. Proof for (1): by Theorem 2.4 page 29.

2. Proof for (2):

$$\begin{aligned}
\left\langle \psi \,|\, \mathbf{T}^n \psi \right\rangle &= \left\langle \sum_{m\in\mathbb{Z}} g_m \mathbf{D}\mathbf{T}^m \phi \,\middle|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} g_k \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by \textit{wavelet dilation equation} (Theorem 3.1 page 44)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \mathbf{D}\mathbf{T}^m \phi \,|\, \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by properties of } \langle \triangle \,|\, \nabla \rangle \text{ (Definition D.9 page 168)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \, \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by def. of operator adjoint (Proposition D.3 page 169)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \, \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k \phi \right\rangle && \text{by Proposition 1.5 page 4} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{*m}\mathbf{D}^* \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k \phi \right\rangle && \text{by operator star-algebra prop. (Theorem D.13 page 170)} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{-m}\mathbf{D}^{-1} \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k \phi \right\rangle && \text{by Proposition 1.7 page 5} \\
&= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k} \phi \right\rangle
\end{aligned}$$

3. Proof for (3):

$$
\begin{aligned}
&\langle \phi \,|\, \mathbf{T}^n \psi \rangle \\
&= \left\langle \sum_{m\in\mathbb{Z}} h_m \mathbf{D}\mathbf{T}^m \phi \,\middle|\, \mathbf{T}^n \sum_{k\in\mathbb{Z}} g_k \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by Theorem 2.1 page 22 and Theorem 3.1 page 44} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \mathbf{D}\mathbf{T}^m \phi \,|\, \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by properties of } \langle \triangle \,|\, \triangledown \rangle \text{ (Definition D.9 page 168)} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \, \mathbf{T}^n \mathbf{D}\mathbf{T}^k \phi \right\rangle && \text{by definition of operator adjoint (Proposition D.3 page 169)} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, (\mathbf{D}\mathbf{T}^m)^* \, \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k \phi \right\rangle && \text{by Proposition 1.5 page 4} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{*m} \mathbf{D}^* \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k \phi \right\rangle && \text{by operator star-algebra properties (Theorem D.13 page 170)} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{-m} \mathbf{D}^{-1} \mathbf{D}\mathbf{T}^{2n}\mathbf{T}^k \phi \right\rangle && \text{by Proposition 1.7 page 5} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k^* \left\langle \phi \,|\, \mathbf{T}^{2n-m+k} \phi \right\rangle
\end{aligned}
$$

✏

Proposition 3.2. *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \left(\!\left(W_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system. Let* $\tilde{\phi}(\omega)$ *and* $\tilde{\psi}(\omega)$ *be the* FOURIER TRANSFORMS *(Definition K.2 page 257) of* $\phi(x)$ *and* $\psi(x)$*, respectively. Let* $\breve{g}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *(Definition P.1 page 355) of* $\left(\!\left(g_n\right)\!\right)$*.*

PRP
$$\tilde{\psi}(\omega) = \frac{\sqrt{2}}{2}\, \breve{g}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right)$$

✎PROOF:

$$
\begin{aligned}
\tilde{\psi}(\omega) &\triangleq \tilde{\mathbf{F}}\psi \\
&= \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} g_n \mathbf{D}\mathbf{T}^n \phi && \text{by } \textit{wavelet dilation equation} \text{ (Theorem 3.1 page 44)} \\
&= \sum_{n\in\mathbb{Z}} g_n \tilde{\mathbf{F}} \mathbf{D}\mathbf{T}^n \phi \\
&= \sum_{n\in\mathbb{Z}} g_n \mathbf{D}^{-1} \tilde{\mathbf{F}} \mathbf{T}^n \phi && \text{by Corollary 1.1 page 9} \\
&= \sum_{n\in\mathbb{Z}} g_n \mathbf{D}^{-1} e^{-i\omega n} \tilde{\mathbf{F}} \phi && \text{by Corollary 1.1 page 9} \\
&= \sum_{n\in\mathbb{Z}} g_n \sqrt{2} \left(\mathbf{D}^{-1} e^{-i\omega n}\right) \left(\mathbf{D}^{-1} \tilde{\mathbf{F}} \phi\right) && \text{by Proposition 1.4 page 3} \\
&= \sqrt{2} \left(\mathbf{D}^{-1} \sum_{n\in\mathbb{Z}} g_n e^{-i\omega n}\right) \left(\mathbf{D}^{-1} \tilde{\mathbf{F}} \phi\right) \\
&= \sqrt{2} \left(\mathbf{D}^{-1} \breve{\mathbf{F}} \left(\!\left(g_n\right)\!\right)\right) \left(\mathbf{D}^{-1} \tilde{\mathbf{F}} \phi\right) && \text{by definition of } \breve{\mathbf{F}} \text{ (Definition P.1 page 355)} \\
&= \sqrt{2} \frac{\sqrt{2}}{2} \breve{g}\left(\frac{\omega}{2}\right) \frac{\sqrt{2}}{2} \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by property of } \mathbf{D} \text{ (Proposition 1.2 page 2)} \\
&= \frac{\sqrt{2}}{2} \breve{g}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right)
\end{aligned}
$$

✏

Theorem 3.4 (next) presents the *quadrature* necessary conditions of a wavelet system. These relations simplify dramatically in the special case of an *orthonormal wavelet system* (Theorem P.4 page 361).

Theorem 3.4 (Quadrature conditions in "frequency"). [8] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system. Let* $\tilde{\mathsf{x}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM (Definition P.1 page 355) *for a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\tilde{\mathsf{S}}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM (Definition R.3 page 373) *of* ϕ, $\tilde{\mathsf{S}}_{\psi\psi}(\omega)$ *be the* AUTO-POWER SPECTRUM *of* ψ, *and* $\tilde{\mathsf{S}}_{\phi\psi}(\omega)$ *be the* CROSS-POWER SPECTRUM *of* ϕ *and* ψ.

THM

1. $\left|\breve{\mathsf{h}}(\omega)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)$
2. $|\breve{\mathsf{g}}(\omega)|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + |\breve{\mathsf{g}}(\omega+\pi)|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\psi\psi}(2\omega)$
3. $\breve{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega)\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \breve{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\phi\psi}(2\omega)$

✎PROOF:

1. Proof for (1): by Theorem 2.5 page 30.
2. Proof for (2):

$$\begin{aligned}
2\tilde{\mathsf{S}}_{\psi\psi}(2\omega) &\triangleq 2(2\pi)\sum_{n\in\mathbb{Z}}|\tilde{\psi}(2\omega+2\pi n)|^2 \\
&= 2(2\pi)\sum_{n\in\mathbb{Z}}\left|\frac{\sqrt{2}}{2}\breve{\mathsf{g}}\left(\frac{2\omega+2\pi n}{2}\right)\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 && \text{by Lemma 2.1 page 22} \\
&= 2\pi\sum_{n\in\mathbb{Z}_e}\left|\breve{\mathsf{g}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 + \\
&\qquad 2\pi\sum_{n\in\mathbb{Z}_o}\left|\breve{\mathsf{g}}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2\left|\tilde{\phi}\left(\frac{2\omega+2\pi n}{2}\right)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}|\breve{\mathsf{g}}(\omega+2\pi n)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}|\breve{\mathsf{g}}(\omega+2\pi n+\pi)|^2\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}|\breve{\mathsf{g}}(\omega)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 + 2\pi\sum_{n\in\mathbb{Z}}|\breve{\mathsf{g}}(\omega+\pi)|^2\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 \\
&= |\breve{\mathsf{g}}(\omega)|^2\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2 + \right)|\breve{\mathsf{g}}(\omega+\pi)|^2\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right) \\
&= |\breve{\mathsf{g}}(\omega)|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + |\breve{\mathsf{g}}(\omega+\pi)|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) && \text{by Theorem R.1 page 373}
\end{aligned}$$

3. Proof for (3):

$$\begin{aligned}
2\tilde{\mathsf{S}}_{\phi\psi}(2\omega) &= 2(2\pi)\sum_{n\in\mathbb{Z}}\tilde{\phi}(2\omega+2\pi n)\tilde{\psi}^*(2\omega+2\pi n) \\
&= 2(2\pi)\sum_{n\in\mathbb{Z}}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}(\omega+\pi n)\,\tilde{\phi}(\omega+\pi n)\,\frac{\sqrt{2}}{2}\breve{\mathsf{g}}^*(\omega+\pi n)\,\tilde{\phi}^*(\omega+\pi n) && \text{by Lemma 2.1 page 22} \\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega+\pi n)\,\breve{\mathsf{g}}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}_o}\breve{\mathsf{h}}(\omega+\pi n)\,\breve{\mathsf{g}}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2 \\
&\quad + 2\pi\sum_{n\in\mathbb{Z}_e}\breve{\mathsf{h}}(\omega+\pi n)\,\breve{\mathsf{g}}^*(\omega+\pi n)\left|\tilde{\phi}(\omega+\pi n)\right|^2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega+2\pi n+\pi)\,\breve{\mathsf{g}}^*(\omega+2\pi n+\pi)\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 \\
&\quad + 2\pi\sum_{n\in\mathbb{Z}}\breve{\mathsf{h}}(\omega+2\pi n)\,\breve{\mathsf{g}}^*(\omega+2\pi n)\left|\tilde{\phi}(\omega+2\pi n)\right|^2
\end{aligned}$$

[8] Chui (1992), page 135, Goswami and Chan (1999), page 110

$$
\begin{aligned}
&= 2\pi \sum_{n\in\mathbb{Z}} \check{\mathsf{h}}(\omega+\pi)\,\breve{\mathsf{g}}^*(\omega+\pi)\left|\tilde{\phi}(\omega+2\pi n+\pi)\right|^2 + 2\pi \sum_{n\in\mathbb{Z}} \check{\mathsf{h}}(\omega)\,\breve{\mathsf{g}}^*(\omega)\left|\tilde{\phi}(\omega+2\pi n)\right|^2 \\
&= \check{\mathsf{h}}(\omega)\,\breve{\mathsf{g}}^*(\omega)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2\right) \\
&\quad + \check{\mathsf{h}}(\omega+\pi)\,\breve{\mathsf{g}}^*(\omega+\pi)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right) \\
&= \check{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2\right) + \check{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\left(2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+\pi+2\pi n)\right|^2\right) \\
&= \check{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega)\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \check{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi)\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) && \text{by Theorem R.1 page 373}
\end{aligned}
$$

⇨

3.7 Sufficient condition

In this text, an often used sufficient condition for designing the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 3.2 page 44) is the *conjugate quadrature filter condition* (Definition O.9 page 349). It expresses the sequence $(\!(g_n)\!)$ in terms of the *scaling coefficient sequence* (Definition 2.3 page 25) and a "shift" integer N as $g_n = \pm(-1)^n h^*_{N-n}$. The *CQF condition* has the following "nice" properties:

1. Given a *scaling coefficient sequence* $(\!(h_n)\!)$ (Definition 2.3 page 25), it is extremely simple to compute the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 3.2 page 44).
2. If $\{\mathbf{T}\phi\}$ of a *wavelet system* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ (Definition 3.2 page 44) is *orthonormal* and $\left((\!(g_n)\!), (\!(h_n)\!), N\right)$ satisfies the *CQF condition*, then $\left\{\mathbf{T}^n\psi\right\}$ is also *orthnormal* (Theorem 5.5 page 89).
3. If $\{\mathbf{T}\phi\}$ of a *wavelet system* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ (Definition 3.2 page 44) is *orthonormal* and $\left((\!(g_n)\!), (\!(h_n)\!), N\right)$ satisfies the *CQF condition*, then the *wavelet subspace* W_0 is *orthnormal* to the *scaling subspace* V_0 ($W_0 \perp V_0$) (Theorem 5.5 page 89).

Theorem 3.5. *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 44). Let* $\breve{\mathsf{g}}(\omega)$ *be the* DTFT *(Definition P.1 page 355) and* $\breve{\mathsf{g}}(z)$ *the* Z-TRANSFORM *(Definition O.4 page 342) of* $(\!(g_n)\!)$.

THM

$$
\begin{aligned}
\underbrace{g_n = \pm(-1)^n h^*_{N-n},\ N\in\mathbb{Z}}_{\text{CONJUGATE QUADRATURE FILTER}} \iff && \breve{\mathsf{g}}(\omega) &= \pm(-1)^N e^{-i\omega N}\check{\mathsf{h}}^*(\omega+\pi)\Big|_{\omega=\pi} && (1)\\
\implies && \sum_{n\in\mathbb{Z}}(-1)^n g_n &= \sqrt{2} && (2)\\
\iff && \breve{\mathsf{g}}(z)\Big|_{z=-1} &= \sqrt{2} && (3)\\
\iff && \breve{\mathsf{g}}(\omega)\Big|_{\omega=\pi} &= \sqrt{2} && (4)
\end{aligned}
$$

✎PROOF:

1. Proof that CQF $\iff$ (1): by Theorem O.5 page 349

2. Proof that CQF $\Longrightarrow$ (4):

$$\begin{aligned}
\breve{g}(\pi) &= \breve{g}(\omega)\Big|_{\omega=\pi} \\
&= \pm(-1)^N e^{-i\omega N}\breve{h}^*(\omega+\pi)\Big|_{\omega=\pi} && \text{by } \textit{CQF theorem} && \text{(Theorem O.5 page 349)} \\
&= \pm(-1)^N e^{-i\pi N}\breve{h}^*(2\pi) \\
&= \pm(-1)^N(-1)^N\breve{h}^*(0) && \text{by } \textit{DTFT periodicity} && \text{(Proposition P.1 page 355)} \\
&= \sqrt{2} && \text{by } \textit{admissibility condition} && \text{(Theorem 2.3 page 27)}
\end{aligned}$$

3. Proof that (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4): by Proposition P.4 page 358

✏

3.8 Support size

Theorem 3.6 (support size). [9] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 44) induced by the* CQF CONDITIONS *(Theorem 3.5 page 50). Let* $\operatorname{supp} \mathsf{f}$ *be the support of a function* f *(Definition 2.4 page 31).*

THM

$$\begin{aligned}
\operatorname{supp}\phi &= \operatorname{supp}\mathsf{h} \\
\operatorname{supp}\psi &= \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right]
\end{aligned}$$

✎PROOF:

1. Proof that $\operatorname{supp}\phi = \operatorname{supp}\mathsf{h}$: by Theorem 2.7 (page 32)

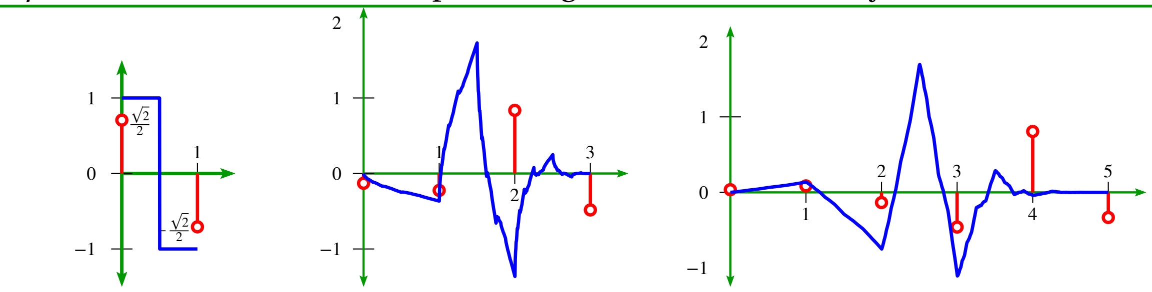
2. Proof that $\operatorname{supp}\psi = \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right]$:

$$\begin{aligned}
\operatorname{supp}\psi(x) &= \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} g_n \mathbf{D}\mathbf{T}^n\phi(x)\right] && \text{by } \textit{wavelet dilation equation} && \text{(Theorem 3.1 page 44)} \\
&= \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} g_n\phi(2x-n)\right] && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 2)} \\
&= \operatorname{supp}\left[\sqrt{2}\sum_{n\in\mathbb{Z}} \pm(-1)^N\mathsf{h}(N-n)\phi(2x-n)\right] && \text{by } \textit{CQF conditions} && \text{(Theorem 3.5 page 50)} \\
&= \operatorname{supp}\left[\sum_{n\in\mathbb{Z}} \mathsf{h}(N-n)\phi(2x-n)\right] && \text{by (3) lemma (page 32)} \\
&= \overline{\left\{x\in\mathbb{R} \,\middle|\, \sum_{n\in\mathbb{Z}} \mathsf{h}(N-n)\phi(2x-n)\neq 0\right\}} && \text{by definition of supp} && \text{(Definition 2.4 page 31)} \\
&= \left[\frac{n_1}{2}+\frac{N-n_2}{2} : \frac{n_2}{2}+\frac{N-n_1}{2}\right] \\
&= \left[\frac{N-(n_2-n_1)}{2} : \frac{N+(n_2-n_1)}{2}\right]
\end{aligned}$$

9 Mallat (1999) pages 243–244

Example 3.1. Here are some examples using *Daubechies wavelet functions*.

EX



Daubechies-1 (Example 6.3 page 109) | *Daubechies-2* (Example 6.4 page 110) | *Daubechies-3* (Example 6.5 page 111)

3.9 Examples

No further examples of wavelets are presented in this section. Examples begin in the next chapter which is about a property called the *partition of unity*. Other design constraints leading to wavelets with more "powerful" properties include *vanishing moments* (Chapter 4 page 55), *orthonormality* (Chapter 5 page 67), *compact support* (Chapter 6 page 95), and *minimum phase* (Definition **??** page **??**).

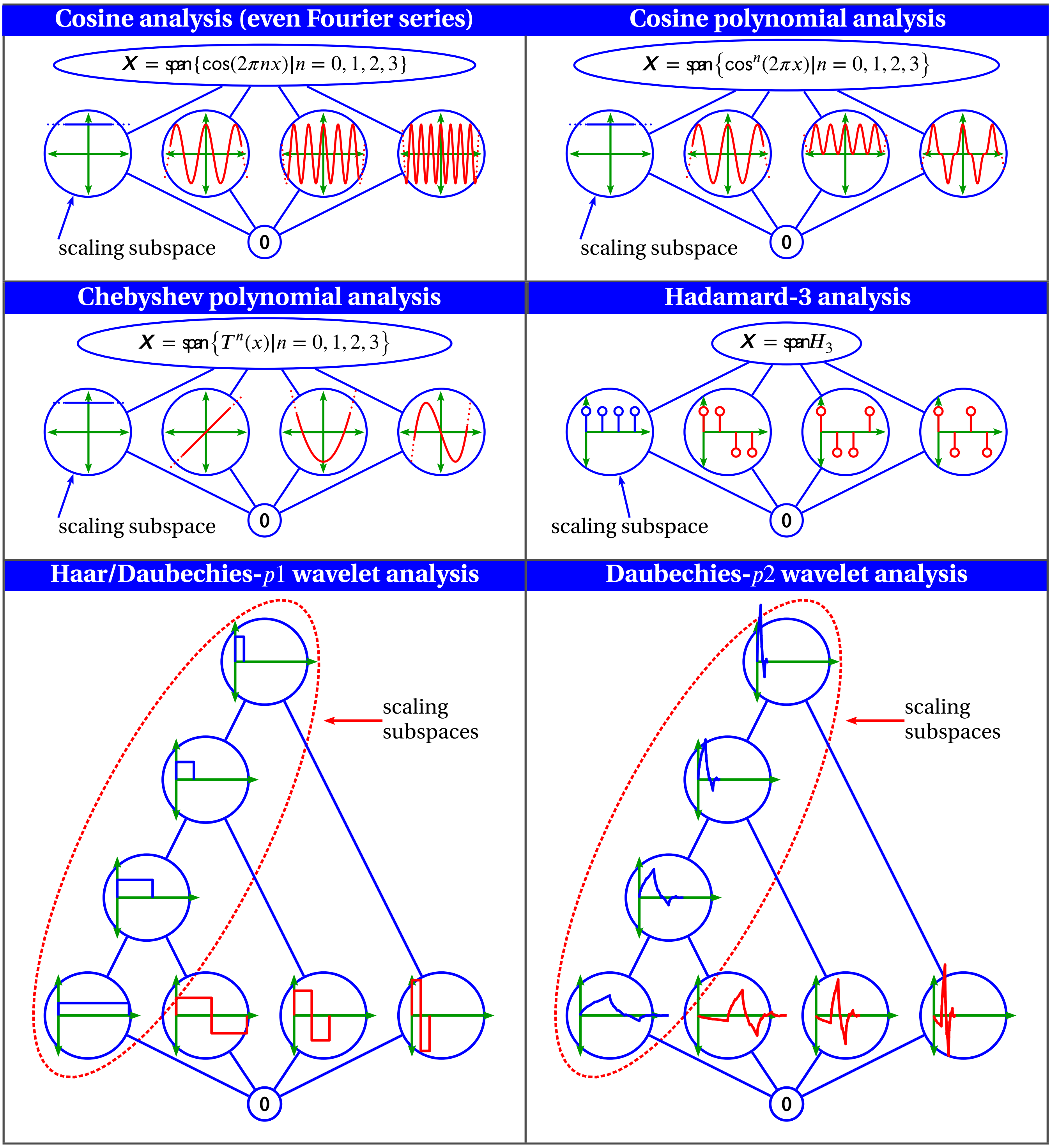


Figure 3.1: examples of the order structures of some analyses

CHAPTER 4

VANISHING MOMENTS CONSTRAINT

One of the most common wavelet design constraints is the number of *vanishing moments* p on the wavelet function such that

$\langle \psi(x) \mid x^n \rangle = 0 \qquad \text{for} \quad n = 0, 1, 2, \ldots, p-1.$

This chapter investigates wavelet design under the vanishing moment constraint.

4.1 Moments

Definition 4.1. [1]

DEF *The quantity M_n is the **nth moment** of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ if*

$$M_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\, \mathrm{d}x \qquad \textit{for } n \in \mathbb{W}.$$

Definition 4.2. [2] *Let M_n be the nTH MOMENT (Definition 4.1 page 55) of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185).*

DEF *The function $\mathsf{f}(x)$ has an **nth vanishing moment** if $M_n = 0$.*
*The function $\mathsf{f}(x)$ has p **vanishing moments** if $M_0 = M_1 = \cdots = M_{p-1} = 0$ and $M_p \neq 0$.*

Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j \right)\!\right), \left(\!\left(\boldsymbol{W}_j \right)\!\right), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a *wavelet system* (Definition 3.2 page 44). Let $\mathbf{P}$ be the *projection operator* that generates the subspace $\boldsymbol{V}_0$ and $\mathbf{Q}$ the *projection operator* that generates the subspace $\boldsymbol{W}_0$. The number of *vanishing moments* of a wavelet is important when using wavelets for the analysis and synthesis of a function $\mathsf{f}(x)$ as in

$$\mathsf{f}(x) = \underbrace{\sum_{n\in\mathbb{Z}} \underbrace{\langle \mathsf{f}(x) \mid \mathbf{T}^n\phi(x)\rangle}_{\text{Fourier coefficient}} \mathbf{T}^n\phi(x)}_{\textit{Fourier expansion} \text{ (Definition L.11 page 278) of } \mathbf{P}\mathsf{f}(x)} + \underbrace{\sum_{k=0}^{\infty}\sum_{n\in\mathbb{Z}} \underbrace{\left\langle \mathsf{f}(x) \mid \mathbf{D}^k\mathbf{T}^n\psi(x)\right\rangle}_{\text{Fourier coefficient}} \mathbf{D}^k\mathbf{T}^n\psi(x)}_{\textit{Fourier expansion} \text{ (Definition L.11 page 278) of } \mathbf{Q}\mathsf{f}(x)}$$

[1] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83

[2] Mallat (1999), page 241

All differentiable functions have a *Taylor polynomial expansion* (Theorem C.13 page 153). The *Fourier coefficients* (Definition L.11 page 278) of this polynomial decrease as the order of each term increases. The inner-product of the function and a wavelet with many vanishing moments results in a smaller value, giving a better approximation for a fixed number of Fourier coefficients.

Theorem 4.1.

THM

$$\left.\begin{array}{ll} 1. & \textit{a function } \mathsf{f}(x) \textit{ has } p \textit{ vanishing moments} \quad \textit{and} \\ 2. & \textit{a polynomial } \mathsf{q}(x) \textit{ is of order } p-1 \textit{ or less} \end{array}\right\} \implies \underbrace{\langle \mathsf{f} \,|\, \mathsf{q} \rangle = 0}_{\mathsf{f} \textit{ is ORTHOGONAL to } \mathsf{q}}$$

✎PROOF:

$$\left\langle \mathsf{f}(x) \,\middle|\, \sum_{n=0}^{p-1} a_n x^n \right\rangle = \sum_{n=0}^{p-1} a_n^* \langle \mathsf{f}(x) \,|\, x^n \rangle = \sum_{n=0}^{p-1} a_n^* \cdot 0 = 0$$

✏

4.2 Vanishing moments and the wavelet function

The number of vanishing moments a wavelet has is closely related to how the derivatives of the Fourier transforms of the wavelet and wavelet coefficients behave (next theorem).

Theorem 4.2. [3] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system. Let* $\tilde{\psi}(\omega)$ *be the* FOURIER TRANSFORM (Definition K.2 page 257) *of* $\psi(x)$. *Let* $\breve{\mathsf{h}}(\omega)$ *and* $\breve{\mathsf{g}}(\omega)$ *be the* DTFTs (Definition P.1 page 355) *of* $(\!(h_n)\!)$ *and* $(\!(g_n)\!)$, *respectively.*

THM

$$\left\{\begin{array}{llcll} \text{(A)} & \underbrace{\langle \psi(x) \,|\, x^n \rangle = 0}_{\psi(x) \textit{ has a vanishing } n\textit{th moment}} & \iff & \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\psi}(\omega)\Big|_{\omega=0} = 0 & \text{(B)} \\ & & \iff & \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{g}}(\omega)\Big|_{\omega=0} = 0 & \text{(C)} \\ & & \iff & \sum_{k\in\mathbb{Z}} k^n g_n = 0 & \text{(D)} \end{array}\right\} \forall n \in \mathbb{W}$$

✎PROOF:

1. Proof that $(A) \iff (B)$: by Lemma K.2 page 262
2. Proof that $(B) \implies (C)$:

$$\begin{aligned} 0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\psi}(2\omega)\Big|_{\omega=0} && \text{by left hypothesis} \\ &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{g}}(\omega)\tilde{\phi}(\omega)\Big|_{\omega=0} && \text{by Proposition 3.2 page 48} \\ &= \sum_{k=0}^{n} \binom{n}{k} \breve{\mathsf{g}}^{(k)}(\omega)\tilde{\phi}^{(n-k)}(\omega)\Big|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma E.2 page 187)} \\ &= \left[\breve{\mathsf{g}}^{(n)}(\omega)\tilde{\phi}(\omega) + \sum_{k=0}^{n-1} \binom{n}{k} \breve{\mathsf{g}}^{(k)}(\omega)\tilde{\phi}^{(n-k)}(\omega)\right]_{\omega=0} \\ &= \breve{\mathsf{g}}^{(n)}(0)\tilde{\phi}(0) + \sum_{k=0}^{n-1} \binom{n}{k} \breve{\mathsf{g}}^{(k)}(0)\tilde{\phi}^{(n-k)}(0) \end{aligned}$$

[3] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

$$\begin{array}{rrcl}
\implies & \breve{g}^{(0)}(0) & = & 0 \\
\implies & \breve{g}^{(1)}(0) & = & 0 \\
\implies & \breve{g}^{(2)}(0) & = & 0 \\
\implies & \breve{g}^{(3)}(0) & = & 0 \\
\implies & \breve{g}^{(4)}(0) & = & 0 \\
\vdots & \vdots & & \\
\implies & \breve{g}^{(n)}(0) & = & 0 \\
\implies & \breve{g}^{(n)}(0) & = & 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{array}$$

3. Proof that $(B) \impliedby (C)$:

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\psi}(2\omega)\bigg|_{\omega=0} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{g}(\omega)\tilde{\phi}(\omega)\bigg|_{\omega=0} && \text{by Proposition 3.2 page 48} \\
&= \sum_{k=0}^{n} \binom{n}{k} \breve{g}^{(k)}(\omega)\tilde{\phi}^{(n-k)}(\omega)\bigg|_{\omega=0} && \text{by } \textit{Leibnitz GPR} \quad \text{(Lemma E.2 page 187)} \\
&= \sum_{k=0}^{n} \binom{n}{k}\, 0\, \tilde{\phi}^{(n-k)}(\omega)\bigg|_{\omega=0} && \text{by right hypothesis} \\
&= 0
\end{aligned}$$

4. Proof that $(C) \iff (D)$: by Theorem P.5 page 363

✏

4.3 Vanishing moments and the scaling function

4.3.1 Results

Lemma 4.1. *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system and* M_n *the* nTH MOMENT *(Definition 4.1 page 55) of* $\phi(x)$.

LEM

$$\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{h}(\omega)\bigg|_{\omega=\pi} = 0 \quad \implies \quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\phi}(\omega)\bigg|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)M_n\delta_k$$

✎PROOF:

1. By Lemma K.1 page 262, $\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\phi}(\omega)\bigg|_{\omega=0} = \frac{1}{\sqrt{2\pi}}(-i)^n M_n.$

2. Then,

$$
\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\phi}(\omega)\Big|_{\omega=2\pi k} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)\Big|_{\omega=2\pi k} && \text{by Proposition 3.2 page 48}\\
&= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}\left(\frac{\omega}{2}\right)\tilde{\phi}^{(n-m)}\left(\frac{\omega}{2}\right)\Bigg|_{\omega=2\pi k} && \text{by Leibnitz GPR} \quad \text{(Lemma E.2 page 187)}\\
&= \sum_{m=0}^{n}\binom{n}{m}\underbrace{\breve{\mathsf{h}}^{(m)}(\pi k)}_{0 \text{ for } k \text{ odd}}\tilde{\phi}^{(n-m)}(\pi k)\\
&= \begin{cases}\displaystyle\sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(\pi k)\tilde{\phi}^{(n-m)}(\pi k) & \text{for } k \text{ even}\\ 0 & \text{for } k \text{ odd}\end{cases} && \text{by left hypothesis}
\end{aligned}
$$

3. And so

$$
\begin{aligned}
\tilde{\phi}^{(n)}(2\pi 0) &= (-i)^n M_n\\
\tilde{\phi}^{(n)}(2\pi 1) &= 0\\
\tilde{\phi}^{(n)}(2\pi 2) &= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(2\pi 1)\underbrace{\tilde{\phi}^{(n-m)}(2\pi 1)}_{0} = 0\\
\tilde{\phi}^{(n)}(2\pi 3) &= 0\\
\tilde{\phi}^{(n)}(2\pi 4) &= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(2\pi 2)\underbrace{\tilde{\phi}^{(n-m)}(2\pi 2)}_{0} = 0\\
\tilde{\phi}^{(n)}(2\pi 5) &= 0\\
\tilde{\phi}^{(n)}(2\pi 6) &= \sum_{m=0}^{n}\binom{n}{m}\breve{\mathsf{h}}^{(m)}(2\pi 3)\underbrace{\tilde{\phi}^{(n-m)}(2\pi 3)}_{0} = 0\\
&\vdots\;\vdots\\
\tilde{\phi}^{(n)}(2\pi k) &= \bar{\delta}_k(-i)^n M_n
\end{aligned}
$$

✏

Theorem 4.3. [4] *Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a* WAVELET SYSTEM *(Definition 3.2 page 44). Let $\breve{\mathsf{h}}(\omega)$ and $\breve{\mathsf{g}}(\omega)$ be the* DTFTs *(Definition P.1 page 355) of $(\!(h_n)\!)$ and $(\!(g_n)\!)$, respectively.*

THM *Let $g_n = \pm(-1)^n h^*_{N-n}$ (*CQF CONDITION*, Definition O.9 page 349). Then*

$$
\left\{\begin{array}{rcrcl}
\underbrace{\langle\psi(x)\,|\,x^n\rangle = 0}_{(1)\ n\text{TH VANISHING MOMENT}} & \iff & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{g}}(\omega)\Big|_{\omega=0} & = & 0 \quad (2)\\
& \iff & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{\mathsf{h}}(\omega)\Big|_{\omega=\pi} & = & 0 \quad (3)\\
& \iff & \displaystyle\sum_{k\in\mathbb{Z}}(-1)^k k^n h_k & = & 0 \quad (4)\\
& \iff & \displaystyle\sum_{k\in\mathbb{Z}} k^n g_k & = & 0 \quad (5)
\end{array}\right\} \quad \forall n\in\mathbb{W}
$$

✎PROOF:

1. Proof that (1) $\iff$ (2): by Theorem 4.2 (page 56)

[4] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

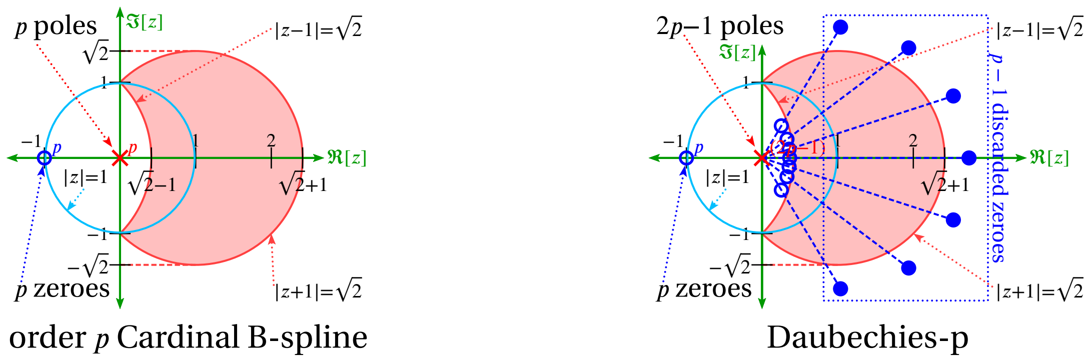


Figure 4.1: Zero locations for B-cardinal spline $N_p(x)$ and Daubechies-p scaling coefficients

2. Proof that (2) $\iff$ (3): by Theorem O.6 page 351
3. Proof that (3) $\iff$ (4): by Theorem P.5 page 363
4. Proof that (2) $\iff$ (5): by Theorem P.5 page 363; see also Theorem 4.2 page 56.
5. Proof that CQF $\not\Leftarrow$ (1): by Theorem P.5 page 363; see also Theorem 4.2 page 56.

✏

These relationships imply that a compactly supported scaling function with p vanishing moments has p zeros at $z = -1$ (due to the spline component) as illustrated in Figure 4.1.

4.3.2 Scaling function and polynomials

Linear combinations of $\left\{\mathbf{T}^n\phi(x) | n\in\mathbb{Z}\right\}$ of a wavelet system with p vanishing moments can perfectly represent any polynomial of order $p-1$ or less (Theorem 4.4 page 60). Example 4.1 page 60 illustrates this using the Strang-Fix condition (Lemma 4.2 page 59) to represent any polynomial of dgree 4 or less.

Lemma 4.2. [5] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system and* M_n *the* nTH MOMENT *(Definition 4.1 page 55) of* $\phi(x)$.

LEM
$$\left\{\begin{array}{l}\psi \text{ has } p\\ \text{vanishing}\\ \text{moments}\end{array}\right\} \implies \left\{\begin{array}{llcll} 1. & \displaystyle\sum_{k\in\mathbb{Z}}(x-k)^n\phi(x-k) & = & M_n & \text{and}\\ 2. & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\phi}(\omega)\Big|_{\omega=2\pi k} & = & \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k M_n & \end{array}\right\} \quad \text{for } n=1,2,\ldots,p-1$$

✎PROOF:

1. Proof that (1) $\iff$ (2): by Lemma K.3 page 263
2. Proof for ($\implies$) assertion:

$$\begin{aligned}
\langle\psi(x)\,|\,x^n\rangle = 0 \iff& \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\breve{\mathsf{h}}(\omega)\Big|_{\omega=\pi} = 0 && \text{by Theorem 4.3 page 58}\\
\implies& \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\phi}(\omega)\Big|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)M_n\delta_k && \text{by Lemma 4.1 page 57}\\
\iff& \sum_{k\in\mathbb{Z}}(x-k)^n\phi(x-k) = M_n && \text{by \textit{Strang-Fix condition}} \quad \text{(Lemma K.3 page 263)}
\end{aligned}$$

[5] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83, Goswami and Chan (1999), page 102, Mallat (1999), pages 241–243

Theorem 4.4. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system with* p *vanishing moments. Let* $\mathsf{q}(x)$ *be a* POLYNOMIAL. *Let*

$Q_p \triangleq \left\{\mathsf{q}(x) | \mathsf{q}(x) \text{ is a POLYNOMIAL of order } p-1 \text{ or less}\right\}.$

THM

$$\left\{\begin{array}{ll} 1. & \psi \text{ has } p \text{ vanishing moments} \quad \text{and} \\ 2. & \mathsf{q}(x) \text{ is of order } p-1 \end{array}\right\} \implies \left\{\begin{array}{ll} 1. & \text{There exists } \left(\!\left(b_n\right)\!\right)_0^{p-1} \text{ such that} \\ & \mathsf{q}(x) = \sum_{n=0}^{p-1} b_n \sum_{k\in\mathbb{Z}} k^n \phi(x-k) \quad \text{and} \\ 2. & \left\{\mathbf{T}^n\phi(x) | n\in\mathbb{Z}\right\} \text{ is a BASIS for } Q_p. \end{array}\right\}$$

✎PROOF: This follows from Lemma 4.2 page 59 (Strang-Fix condition). See Example 4.1 page 60. ⇨

Example 4.1. Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ be a *wavelet system* (Definition 3.2 page 44) in which $\psi(x)$ has 5 vanishing moments.

EX

$$\begin{array}{lclllllllllll}
\sum_{k\in\mathbb{Z}} \phi(x-k) &=& M_0 \\
\sum_{k\in\mathbb{Z}} k\phi(x-k) &=& M_0 x &-& M_1 \\
\sum_{k\in\mathbb{Z}} k^2\phi(x-k) &=& M_0 x^2 &-& 2M_1 x &+& M_2 \\
\sum_{k\in\mathbb{Z}} k^3\phi(x-k) &=& M_0 x^3 &-& 3M_1 x^2 &+& 3M_2 x &-& M_3 \\
\sum_{k\in\mathbb{Z}} k^4\phi(x-k) &=& M_0 x^4 &-& 4M_1 x^3 &+& 6M_2 x^2 &-& 4M_3 x &+& M_4 \\
\sum_{k\in\mathbb{Z}} k^5\phi(x-k) &=& M_0 x^5 &-& 5M_1 x^4 &+& 10M_2 x^3 &-& 10M_3 x^2 &+& 5M_4 x &-& M_5
\end{array}$$

These equations can be represented in matrix algebra form as

EX

$$\begin{bmatrix} \sum_{k\in\mathbb{Z}} k^4\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k^3\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k^2\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k\phi(x-k) \\ \sum_{k\in\mathbb{Z}} \phi(x-k) \end{bmatrix} = \begin{bmatrix} M_0 & -4M_1 & 6M_2 & -4M_3 & M_4 \\ 0 & M_0 & -3M_1 & 3M_2 & -M_3 \\ 0 & 0 & M_0 & -2M_1 & M_2 \\ 0 & 0 & 0 & M_0 & -M_1 \\ 0 & 0 & 0 & 0 & M_0 \end{bmatrix} \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{bmatrix}$$

The operator matrix is triagular and therefore it's inverse always exists. The monomials $1, x, x^2, x^3$ and x^4 can be computed by computing the inverse of the operator matrix.

EX

$$\begin{array}{lcl}
1 &=& \frac{1}{M_0}\sum_{k\in\mathbb{Z}} \phi(x-k) \\
x &=& \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k\phi(x-k) + \frac{M_1}{M_0} \\
x^2 &=& \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{2M_1}{M_0^2}\sum_{k\in\mathbb{Z}} k\phi(x-k) + \frac{2M_1^2-M_2M_0}{M_0^2} \\
x^3 &=& \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^3\phi(x-k) + \frac{3M_1}{M_0^2}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{6M_1^2-3M_2M_0}{M_0^3}\sum_{k\in\mathbb{Z}} k\phi(x-k) \\
&& + \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^3}
\end{array}$$

Note that each monomial x^n is simply a linear combination of $\left(\!\left(\phi(x-k)\right)\!\right)_{k\in\mathbb{Z}}$.

✎PROOF:

$$\begin{aligned} M_0 &= \sum_{k\in\mathbb{Z}} (x-k)^0\phi(x-k) \\ &= \sum_{k\in\mathbb{Z}} \phi(x-k) \end{aligned}$$

$$M_1 = \sum_{k\in\mathbb{Z}} (x-k)^1\phi(x-k)$$

$$
\begin{aligned}
&= t\sum_{k\in\mathbb{Z}}\phi(x-k) - \sum_{k\in\mathbb{Z}} k\phi(x-k)\\
&= M_0 t - \sum_{k\in\mathbb{Z}} k\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_2 &= \sum_{k\in\mathbb{Z}}(x-k)^2\phi(x-k)\\
&= x^2\sum_{k\in\mathbb{Z}}\phi(x-k) - 2t\sum_{k\in\mathbb{Z}} k\phi(x-k) + \sum_{k\in\mathbb{Z}} k^2\phi(x-k)\\
&= x^2 M_0 - 2t(M_0 t - M_1) + \sum_{k\in\mathbb{Z}} k^2\phi(x-k)\\
&= -M_0 x^2 + 2M_1 x + \sum_{k\in\mathbb{Z}} k^2\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= \sum_{k\in\mathbb{Z}}(x-k)^3\phi(x-k)\\
&= x^3\sum_{k\in\mathbb{Z}}\phi(x-k) - 3x^2\sum_{k\in\mathbb{Z}} k\phi(x-k) + 3x\sum_{k\in\mathbb{Z}} k^2\phi(x-k) - \sum_{k\in\mathbb{Z}} k^3\phi(x-k)\\
&= x^3 M_0 - 3x^2\left[M_0 t - M_1\right] + 3x\left[M_0 x^2 - 2M_1 x + M_2\right] - \sum_{k\in\mathbb{Z}} k^3\phi(x-k)\\
&= (M_0 - 3M_0 + 3M_0)x^3 + (3M_1 - 6M_1)x^2 + 3M_2 x - \sum_{k\in\mathbb{Z}} k^3\phi(x-k)\\
&= M_0 x^3 - 3M_1 x^2 + 3M_2 t - \sum_{k\in\mathbb{Z}} k^3\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_4 &= \sum_{k\in\mathbb{Z}}(x-k)^4\phi(x-k)\\
&= x^4\sum_{k\in\mathbb{Z}}\phi(x-k) - 4x^3\sum_{k\in\mathbb{Z}} k\phi(x-k) + 6x^2\sum_{k\in\mathbb{Z}} k^2\phi(x-k) - 4x\sum_{k\in\mathbb{Z}} k^3\phi(x-k) + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= x^4 M_0 - 4x^3\left[M_0 t - M_1\right] + 6x^2\left[M_0 x^2 - 2M_1 x + M_2\right] - 4x\left[M_0 x^3 - 3M_1 x^2 + 3M_2 t - M_3\right]\\
&\quad + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= x^4 M_0 + \left[-4M_0 x^4 + 4M_1 x^3\right] + \left[6M_0 x^4 - 12M_1 x^3 + 6M_2 x^2\right]\\
&\quad + \left[-4M_0 x^4 + 12M_1 x^3 - 12M_2 x^2 + 4M_3 x\right] + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= (M_0 - 4M_0 + 6M_0 - 4M_0)x^4 + (4M_1 - 12M_1 + 12M_1)x^3 + (6M_2 - 12M_2)x^2 + 4M_3 x\\
&\quad + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)\\
&= -M_0 x^4 + 4M_1 x^3 - 6M_2 x^2 + 4M_3 x + \sum_{k\in\mathbb{Z}} k^4\phi(x-k)
\end{aligned}
$$

$$
\begin{aligned}
M_5 &= \sum_{k\in\mathbb{Z}}(x-k)^5\phi(x-k)\\
&= x^5\sum_{k\in\mathbb{Z}}\phi(x-k) - 5x^4\sum_{k\in\mathbb{Z}} k\phi(x-k) + 10x^3\sum_{k\in\mathbb{Z}} k^2\phi(x-k) - 10x^2\sum_{k\in\mathbb{Z}} k^3\phi(x-k)\\
&\quad + 5t\sum_{k\in\mathbb{Z}} k^4\phi(x-k) - \sum_{k\in\mathbb{Z}} k^5\phi(x-k)\\
&= x^5 M_0 - 5x^4\left[M_0 x - M_1\right] + 10x^3\left[M_0 x^2 - 2M_1 x + M_2\right] - 10x^2\left[M_0 x^3 - 3M_1 x^2 + 3M_2 x - M_3\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ 5t\left[M_0x^4 - 4M_1x^3 + 6M_2x^2 - 4M_3x + M_4\right] - \sum_{k\in\mathbb{Z}} k^5\phi(x-k) \\
&= x^5M_0\left[-5M_0x^5 + 5M_1x^4\right] + \left[10M_0x^5 - 20M_1x^4 + 10M_2x^3\right] \\
&\quad + \left[-10M_0x^5 + 30M_1x^4 - 30M_2x^3 + 10M_3x^2\right] \\
&\quad + \left[5M_0x^5 - 20M_1x^4 + 30M_2x^3 - 20M_3x^2 + 5M_4x\right] - \sum_{k\in\mathbb{Z}} k^5\phi(x-k) \\
&= (M_0 - 5M_0 + 10M_0 + 5M_0 - 10M_0)x^5 + (5M_1 - 20M_1 + 30M_1 - 20M_1)x^4 \\
&\quad + (10M_2 - 30M_2 + 30M_2)x^3 + (10M_3 - 20M_3)x^2 + 5M_4x - \sum_{k\in\mathbb{Z}} k^5\phi(x-k) \\
&= M_0x^5 - 5M_1x^4 + 10M_2x^3 - 10M_3x^2 + 5M_4x - \sum_{k\in\mathbb{Z}} k^5\phi(x-k)
\end{aligned}
$$

$$
\left|\begin{array}{cccccccc}
M_0 & -3M_1 & 3M_2 & -M_3 & 1 & 0 & 0 & 0 \\
0 & M_0 & -2M_1 & M_2 & 0 & 1 & 0 & 0 \\
0 & 0 & M_0 & -M_1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{array}\right| \rightarrow
$$

$$
\left|\begin{array}{cccccccc}
M_0 & -3M_1 & 3M_2 & 0 & 1 & 0 & 0 & \frac{M_3}{M_0} \\
0 & M_0 & -2M_1 & 0 & 0 & 1 & 0 & -\frac{M_2}{M_0} \\
0 & 0 & M_0 & 0 & 0 & 0 & 1 & \frac{M_1}{M_0} \\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{array}\right| \rightarrow
$$

$$
\left|\begin{array}{cccccccc}
M_0 & -3M_1 & 0 & 0 & 1 & 0 & -\frac{3M_2}{M_0} & -\frac{3M_2M_1}{M_0^2} + \frac{M_3}{M_0} \\
0 & M_0 & 0 & 0 & 0 & 1 & \frac{2M_1}{M_0} & \frac{2M_1^2}{M_0^2} - \frac{M_2}{M_0} \\
0 & 0 & M_0 & 0 & 0 & 0 & 1 & \frac{M_1}{M_0} \\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{array}\right| \rightarrow
$$

$$
\left|\begin{array}{cccccccc}
M_0 & 0 & 0 & 0 & 1 & \frac{3M_1}{M_0} & \frac{6M_1^2}{M_0^2} - \frac{3M_2}{M_0} & \frac{6M_1^3}{M_0^3} - \frac{3M_1M_2}{M_0^2} - \frac{3M_2M_1}{M_0^2} + \frac{M_3}{M_0} \\
0 & M_0 & 0 & 0 & 0 & 1 & \frac{2M_1}{M_0} & \frac{2M_1^2}{M_0^2} - \frac{M_2}{M_0} \\
0 & 0 & M_0 & 0 & 0 & 0 & 1 & \frac{M_1}{M_0} \\
0 & 0 & 0 & M_0 & 0 & 0 & 0 & 1
\end{array}\right| \rightarrow
$$

$$
\left|\begin{array}{cccccccc}
1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{3M_1}{M_0^2} & \frac{6M_1^2}{M_0^3} - \frac{3M_2}{M_0^2} & \frac{6M_1^3}{M_0^4} - \frac{3M_1M_2}{M_0^3} - \frac{3M_2M_1}{M_0^3} + \frac{M_3}{M_0^2} \\
0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{2M_1}{M_0^2} & \frac{2M_1^2}{M_0^3} - \frac{M_2}{M_0^2} \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{M_1}{M_0^2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0}
\end{array}\right| \rightarrow
$$

$$
\left|\begin{array}{cccccccc}
1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{3M_1}{M_0^2} & \frac{6M_1^2-3M_2M_0}{M_0^3} & \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^4} \\
0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{2M_1}{M_0^2} & \frac{2M_1^2-M_2M_0}{M_0^3} \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0} & \frac{M_1}{M_0^2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{M_0}
\end{array}\right| \rightarrow
$$

$$\begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{M_0} & \frac{3M_1}{M_0^2} & \frac{6M_1^2-3M_2M_0}{M_0^3} & \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^4} \\ 0 & \frac{1}{M_0} & \frac{2M_1}{M_0^2} & \frac{2M_1^2-M_2M_0}{M_0^3} \\ 0 & 0 & \frac{1}{M_0} & \frac{M_1}{M_0^2} \\ 0 & 0 & 0 & \frac{1}{M_0} \end{bmatrix} \begin{bmatrix} \sum_{k\in\mathbb{Z}} k^3\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k^2\phi(x-k) \\ \sum_{k\in\mathbb{Z}} k\phi(x-k) \\ \sum_{k\in\mathbb{Z}} \phi(x-k) \end{bmatrix}$$

$$\begin{aligned}
x^3 &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^3\phi(x-k) + \frac{3M_1}{M_0^2}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{6M_1^2-3M_2M_0}{M_0^3}\sum_{k\in\mathbb{Z}} k\phi(x-k) \\
&\quad + \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^4}\underbrace{\sum_{k\in\mathbb{Z}} \phi(x-k)}_{M_0} \\
&= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^3\phi(x-k) + \frac{3M_1}{M_0^2}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{6M_1^2-3M_2M_0}{M_0^3}\sum_{k\in\mathbb{Z}} k\phi(x-k) \\
&\qquad + \frac{6M_1^3-6M_2M_1M_0+M_3M_0^2}{M_0^3} \\
x^2 &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k^2\phi(x-k) + \frac{2M_1}{M_0^2}\sum_{k\in\mathbb{Z}} \phi(x-k) + \frac{2M_1^2-M_2M_0}{M_0^2} \\
x &= \frac{1}{M_0}\sum_{k\in\mathbb{Z}} k\phi(x-k) + \frac{M_1}{M_0}
\end{aligned}$$

Conjecture 4.1. [6] Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a wavelet system with p vanishing moments and let M_n be the nth moment (Definition 4.1 page 55) of $\phi(x)$.

CNJ
$$\boxed{\sum_{k\in\mathbb{Z}} k^n\phi(x-k) = \sum_{m=0}^{n}\binom{n}{m}(-1)^m M_m x^{n-m}}$$

✎PROOF: This conjecture is supported by the results of Example 4.1 (page 60). However, I have no proof at this time.

$$\begin{aligned}
(-1)^n M_n &= (-1)^n \sum_{k\in\mathbb{Z}} (x-k)^n\phi(x-k) \\
&= (-1)^n \sum_{k\in\mathbb{Z}}\sum_{m=0}^{n}\binom{n}{m}x^{n-m}(-k)^m\phi(x-k) \\
&= (-1)^n \sum_{m=0}^{n}\binom{n}{m}(-1)^m x^{n-m}\sum_{k\in\mathbb{Z}} k^m\phi(x-k) \\
&= \left[(-1)^{2n}\sum_{k\in\mathbb{Z}} k^n\phi(x-k)\right] + (-1)^n\sum_{m=0}^{n-1}\binom{n}{m}(-1)^m x^{n-m}\sum_{k\in\mathbb{Z}} k^m\phi(x-k) \\
&= \left[\sum_{k\in\mathbb{Z}} k^n\phi(x-k)\right] + (-1)^n\sum_{m=0}^{n-1}\binom{n}{m}(-1)^m x^{n-m}\sum_{k\in\mathbb{Z}} k^m\phi(x-k)
\end{aligned}$$

[6] Zumkeller (2005) http://oeis.org/A110555 $\langle T(n,k) \triangleq \sum_{k=0,1,\dots,n}\binom{n}{k}(-1)^k\rangle$

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} k^n\phi(x-k) &= -(-1)^n\sum_{m=0}^{n-1}\binom{n}{m}(-1)^m x^{n-m}\sum_{k\in\mathbb{Z}} k^m\phi(x-k) + (-1)^n M_n \\
&= \sum_{m=0}^{n-1}\binom{n}{m}(-1)^m \underbrace{\left[(-1)^{n-1}\sum_{k\in\mathbb{Z}} k^m\phi(x-k)\right]}_{M_m?} x^{n-m} + (-1)^n M_n
\end{aligned}
$$

$$
\begin{aligned}
M_m &\stackrel{?}{=} (-1)^{n-1}\sum_{k\in\mathbb{Z}} k^m\phi(x-k) \\
M_m &= \sum_{k\in\mathbb{Z}}(x-k)^m\phi(x-k) \\
&= \sum_{k\in\mathbb{Z}}\sum_{p=0}^{m}\binom{m}{p}x^{m-p}(-k)^p\phi(x-k) \\
&= (-1)^m\sum_{k\in\mathbb{Z}} k^m\phi(x-k) + \underbrace{\sum_{k\in\mathbb{Z}}\sum_{p=0}^{m-1}\binom{m}{p}x^{m-p}(-k)^p\phi(x-k)}_{0?}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} k^n\phi(x-k) &= -(-1)^n\sum_{m=0}^{n-1}\binom{n}{m}(-1)^m x^{n-m}\sum_{k\in\mathbb{Z}} k^m\phi(x-k) + (-1)^n M_n \\
&= \sum_{m=0}^{n-1}\binom{n}{m}(-1)^m(-1)^{n-1}\left[\sum_{k\in\mathbb{Z}} k^m\phi(x-k)\right] x^{n-m} + (-1)^n M_n \\
&= \sum_{m=0}^{n-1}\binom{n}{m}(-1)^m(-1)^{n-1}\left[\sum_{p=0}^{m-1}\binom{m}{p}(-1)^p(-1)^{m-1}\left[\sum_{k\in\mathbb{Z}} k^p\phi(x-k)\right] x^{m-p} + (-1)^m M_m\right] x^{n-m} \\
&\quad + (-1)^n M_n
\end{aligned}
$$

✏

4.3.3 Scaling function and continuity

Theorem 4.5. *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM (Definition 3.2 page 44). *Let* $\tilde{\phi}(\omega)$ *be the* FOURIER TRANSFORM (Definition K.2 page 257) *of* $\phi(x)$.

THM
$$
\left\{\begin{array}{lll}(1). & \psi \text{ has } p \geq 2 \text{ VANISHING MOMENTS} & \text{and} \\ (2). & g_n = (-1)^n h^*_{N-n} & \text{[CQF CONDITION (Definition O.9 page 349)]}\end{array}\right\} \implies \left\{\begin{array}{l}\tilde{\phi}(\omega) \text{ is CONTINUOUS} \\ \text{at } \omega = 0\end{array}\right\}
$$

✎PROOF:

$$
\begin{aligned}
\text{(1) and (2)} &\implies \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n\tilde{\phi}(\omega)\bigg|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k M_n && \text{by Lemma 4.2 page 59} \\
&\implies \frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{\phi}(\omega)\bigg|_{\omega=0} = \frac{1}{\sqrt{2\pi}}(-i)^1 M_1 \\
&\implies \frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{\phi}(\omega)\bigg|_{\omega=0} = 0 && \text{by (1) and def. of } \textit{vanishing moments} \text{ (Definition 4.2 page 55)} \\
&\implies \frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{\phi}(\omega)\bigg|_{\omega=0} < \infty \\
&\implies \tilde{\phi}(\omega) \text{ is } \textit{continuous} \text{ at } \omega = 0
\end{aligned}
$$

4.4 Sufficient conditions

How can we design a *wavelet system* (Definition 3.2 page 44) to have p *vanishing moments*? One way is to design the *scaling coefficient sequence* (Definition 2.3 page 25) $(\!(h_n)\!)$ in the *z-domain* (Definition O.4 page 342) such that $\check{\mathsf{h}}(z)$ includes the factor $\left(\frac{1+z^{-1}}{2}\right)^p$ (next theorem). This factor is included in the $\check{\mathsf{h}}(z)$ of the *Daubechies-p wavelet system* (Definition 6.1 page 105). A similar factor (Lemma M.5 page 312) also appears in the *Fourier transform* (Definition K.2 page 257) of *B-splines* (Definition M.2 page 297).

Lemma 4.3. [7] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 44). Let* $(\!(q_n)\!)_{n\in\mathbb{Z}}$ *be a sequence with* Z-TRANSFORM *(Definition O.4 page 342)* $\check{\mathsf{q}}(z)$.

$$\text{LEM}\quad \left\{\begin{array}{llll} (1). & \check{\mathsf{h}}(z) &= \left(\dfrac{1+z^{-1}}{2}\right)^p \check{\mathsf{q}}(z) & \text{and} \\ (2). & g_n &= (-1)^n h^*_{N-n} & \text{[CQF CONDITION (Definition O.9 page 349)]} \end{array}\right\} \implies \left\{\begin{array}{l} \psi \text{ has } p \\ \text{vanishing moments} \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
& \check{\mathsf{h}}(z) = \left(\frac{1+z^{-1}}{2}\right)^p \check{\mathsf{q}}(z) && \text{by (1)}\\
\implies\quad & \check{\mathsf{h}}(\omega) = \left(\frac{1+e^{i\omega}}{2}\right)^p \tilde{\mathsf{q}}(\omega) && \text{by Definition P.1 page 355}\\
\implies\quad & \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \check{\mathsf{h}}(\omega)\Big|_{\omega=\pi} = \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \left(\frac{1+e^{i\omega}}{2}\right)^p \tilde{\mathsf{q}}(\omega)\Big|_{\omega=\pi} \\
& = \sum_{k=0}^{n}\binom{n}{k}\left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^k \left(\frac{1+e^{i\omega}}{2}\right)^p\right]\left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-k}\tilde{\mathsf{q}}(\omega)\right]\Bigg|_{\omega=\pi} && \text{by Lemma E.2 page 187}\\
& = \sum_{k=0}^{n}\binom{n}{k}\left[\frac{p!}{2^p(p-k)!}\prod_{k=1}^{p}\left[ie^{i\omega}\left(1+e^{i\omega}\right)^k\right]\right]\left[\frac{\mathsf{d}}{\mathsf{d}\omega}^{n-k}\tilde{\mathsf{q}}(\omega)\right]\Bigg|_{\omega=\pi}\\
& = 0\\
\iff\quad & \psi \text{ has } p \text{ vanishing moments} && \text{by Theorem 4.3 page 58}
\end{aligned}$$

⇒

[7] Daubechies (1992), page 155, Vidakovic (1999), pages 80–82

CHAPTER 5

ORTHONORMALITY CONSTRAINT

5.1 Definition

In any Hilbert space $\boldsymbol{L}^2_{\mathbb{R}}$ with a subspace $\boldsymbol{Y}$, $\boldsymbol{Y} \dotplus \boldsymbol{Y}^{\perp} = \boldsymbol{L}^2_{\mathbb{R}}$ (*Projection Theorem*). The subspace $\boldsymbol{Y}^{\perp}$ is the *orthogonal complement* of $\boldsymbol{Y}$ in $\boldsymbol{L}^2_{\mathbb{R}}$. In wavelet theory, the orthogonal complement of a scaling subspace $\boldsymbol{V}_n$ in the subspace $\boldsymbol{V}_{n+1}$ is the *wavelet subspace* $\boldsymbol{W}_n$ (next definition). The wavelet subspace $\boldsymbol{W}_n$ is *unique* with respect to $\boldsymbol{V}_n$ because in a Hilbert space, $\boldsymbol{W}_n^{\perp} = \boldsymbol{V}_n^{\perp\perp} = \boldsymbol{V}_n$.

Definition 3.2 (page 44) defines a *general* wavelet system. An *orthonormal wavelet system* is simply a wavelet system with additional constraints of orthonormality (Definition 5.1 page 67).

Definition 5.1. *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 3.2 page 44) and* $\boldsymbol{\Gamma} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, (\!(h_n)\!)\right)$ *its component* MRA SYSTEM *(Definition 2.3 page 25). Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION *(Definition L.12 page 278).*

DEF

1. *The* MRA SYSTEM $\boldsymbol{\Gamma}$ *is **orthonormal** if* $\langle \phi \,|\, \mathbf{T}^n \phi \rangle = \bar{\delta}_n \quad \forall n \in \mathbb{Z}$.
2. *The* WAVELET SYSTEM $\boldsymbol{\Omega}$ *is **orthonormal** if* $\langle \psi \,|\, \mathbf{T}^n \psi \rangle = \bar{\delta}_n \quad \forall n \in \mathbb{Z}$ *and* $\langle \phi \,|\, \mathbf{T}^n \psi \rangle = 0 \quad \forall n \in \mathbb{Z}$.

In the case of (1), $\boldsymbol{\Gamma}$ *is an **orthonormal MRA system**.*
In the case of (2), $\boldsymbol{\Omega}$ *is an **orthonormal wavelet system**.*

5.2 Orthonormal MRA systems

5.2.1 Scaling coefficients from scaling function

The *dilation equation* (Theorem 2.1 page 22) demonstrates that if we know the *scaling coefficients* $(\!(\mathsf{h}_n)\!)$ (Definition 2.3 page 25), we can (recursively) compute the scaling function ϕ (Definition 2.1 page 16). The converse is in general not so easy to compute. However, in an *orthonormal MRA system*, we can compute the scaling coefficients $(\!(\mathsf{h}_n)\!)$ from the scaling function ϕ (next).

Proposition 5.1. [1] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 25).*

[1] Mallat (1999) page 228 ⟨(7.29)⟩, Wojtaszczyk (1997) page 38

PRP

$$\underbrace{\langle \phi \,|\, \mathbf{T}^n \phi \rangle = \bar{\delta}_n}_{\{\mathbf{T}^n\phi\} \text{ is ORTHONORMAL}} \implies \mathsf{h}_n = \langle \phi \,|\, \mathbf{D}\mathbf{T}^n \phi \rangle = \frac{\sqrt{2}}{2} \int_{\mathbb{R}} \phi(x)\phi^*(2x-n)\,\mathrm{d}t \qquad \forall n \in \mathbb{Z}$$

✎Proof:

$$\begin{aligned}
\langle \phi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle &= \left\langle \sum_{m\in\mathbb{Z}} h_m \mathbf{D}\mathbf{T}^m \phi \,\middle|\, \mathbf{D}\mathbf{T}^n\phi \right\rangle && \text{by the \textit{dilation equation}} && \text{(Theorem 2.1 page 22)}\\
&= \sum_{m\in\mathbb{Z}} h_m \langle \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle && \text{by \textit{additive} property of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition D.9 page 168)}\\
&= \sum_{m\in\mathbb{Z}} h_m \left\langle \sqrt{2}\phi(2x-m) \,|\, \sqrt{2}\phi(2x-n) \right\rangle && \text{by definition of } \mathbf{T} \text{ and } \mathbf{D} && \text{(Definition 1.3 page 2)}\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{x\in\mathbb{R}} \sqrt{2}\phi(2x-m)\sqrt{2}\phi^*(2x-n)\,\mathrm{d}x && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{u\in\mathbb{R}} \sqrt{2}\phi(u-m)\sqrt{2}\phi^*(u-n)\tfrac{1}{2}\,\mathrm{d}u && \text{where } u \triangleq 2x \implies && \mathrm{d}u = 2\,\mathrm{d}x\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n)\,\mathrm{d}u\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n)\,\mathrm{d}u\\
&= \sum_{m\in\mathbb{Z}} h_m \int_{v\in\mathbb{R}} \phi(v)\phi^*(v-(n-m))\,\mathrm{d}v && \text{where } v \triangleq u-m \implies && u = v+m\\
&= \sum_{m\in\mathbb{Z}} h_m \langle \phi \,|\, \mathbf{T}^{n-m}\phi \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}\\
&= \sum_{m\in\mathbb{Z}} h_m \bar{\delta}_{m-n} && \text{by left hypothesis}\\
&= \mathsf{h}_n && \text{by definition of } \bar{\delta} && \text{(Definition L.12 page 278)}
\end{aligned}$$

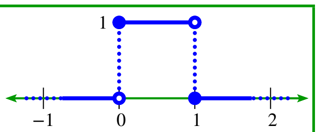
$$\begin{aligned}
\langle \phi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle &= \langle \phi \,|\, \mathbf{D}\phi(x-n) \rangle && \text{by definition of } \mathbf{T} && \text{(Definition 1.3 page 2)}\\
&= \left\langle \phi \,\middle|\, \tfrac{\sqrt{2}}{2}\phi(2x-n) \right\rangle && \text{by definition of } \mathbf{D} && \text{(Definition 1.3 page 2)}\\
&= \frac{\sqrt{2}}{2} \int_{\mathbb{R}} \phi(x)\phi^*(2x-n)\,\mathrm{d}t && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}
\end{aligned}$$

✏

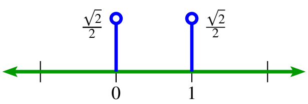
Example 5.1 (Haar scaling function).

EX

Let $\phi(x) \triangleq \begin{cases} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise.} \end{cases}$



Then, $\mathsf{h}_n = \begin{cases} \frac{\sqrt{2}}{2} & \text{for } n \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$



✎Proof:

$\mathsf{h}_n = \langle \phi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle$ by Proposition 5.1 page 67

$= \left\langle \mathbb{1}_{[0,1)}(x) \,\middle|\, \mathbf{D}\mathbf{T}^{-n}\mathbb{1}_{[0,1)}(x) \right\rangle$	by definition of ϕ	
$= \left\langle \mathbb{1}_{[0,1)}(x) \,\middle|\, \sqrt{2}\mathbb{1}_{[0,1)}(2x-n) \right\rangle$	by definition of $\mathbf{T}$ and $\mathbf{D}$	(Definition 1.3 page 2)
$= \sqrt{2}\left\langle \mathbb{1}_{[0,1)}(x) \,\middle|\, \mathbb{1}_{[0,1)}(2x-n) \right\rangle$	by properties of $\langle \triangle \mid \nabla \rangle$	(Definition D.9 page 168)
$= \sqrt{2}\int_{\mathbb{R}} \mathbb{1}_{[0,1)}(x)\,\mathbb{1}_{[0,1)}(2x-n)\,\mathrm{d}x$	by definition of $\langle \triangle \mid \nabla \rangle$ in $\boldsymbol{L}^2_{\mathbb{R}}$	(Definition E.1 page 185)
$= \sqrt{2}\int_0^1 \mathbb{1}_{[0,1)}(2x-n)\,\mathrm{d}x$	by definition of $\mathbb{1}$	(Definition 1.2 page 1)
$= \begin{cases} \sqrt{2}\int_0^{\frac{1}{2}} 1\,\mathrm{d}x = \frac{\sqrt{2}}{2} & \text{for } n=0 \\ \sqrt{2}\int_{\frac{1}{2}}^{1} 1\,\mathrm{d}x = \frac{\sqrt{2}}{2} & \text{for } n=1 \\ 0 = 0 & \text{otherwise} \end{cases}$	by definition of $\mathbb{1}$	(Definition 1.2 page 1)

✏

5.2.2 Necessary conditions

Proposition 5.2. [2] *Let $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ be an* MRA SYSTEM *(Definition 2.3 page 25). Let $\tilde{\phi}(\omega)$ be the* FOURIER TRANSFORM *(Definition K.2 page 257) of $\phi(x)$.*

PRP

$$\left\{\begin{array}{lll} \text{(A).} & (\!(\mathbf{T}^n\phi)\!) \text{ is ORTHONORMAL} & \text{and} \\ \text{(B).} & \tilde{\phi}(\omega) \text{ is CONTINUOUS at } 0 & \end{array}\right\} \implies \left\{\begin{array}{llll} \text{(1).} & \left|\tilde{\phi}(0)\right| & = \frac{1}{\sqrt{2\pi}} & \text{and} \\ \text{(2).} & \left|\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x\right| & = 1 & \end{array}\right\}$$

✎PROOF:

1. Let $\mathbf{P}_j$ be the *projection operator* that generates the scaling subspace $\boldsymbol{V}_j$ such that
$\mathbf{V}_j = \{\mathbf{P}_j\mathsf{f} | \mathsf{f} \in \boldsymbol{H}\}$

2. Let $\mathsf{g}(x)$ a function in $\boldsymbol{L}^2_{\mathbb{R}}$ such that its *Fourier transform* (Definition K.2 page 257) $\tilde{\mathsf{g}}(\omega)$ has support $\operatorname{supp}\tilde{\mathsf{g}} \subseteq [-1:1]$

3. lemma: The set $\left\{ \frac{1}{2^{j/2}\sqrt{2\pi}} e^{-i\frac{\omega}{2^j}n} \,\middle|\, n\in\mathbb{Z} \right\}$ is an *orthonormal basis* for functions in $\boldsymbol{L}^2_{\mathbb{R}}$ that have support in $[-2^j\pi : 2^j\pi]$. Proof: This follows from Theorem J.3 page 256.

$$\begin{aligned}
\int_{-2^j\pi}^{2^j\pi} \left[\frac{1}{2^{j/2}\sqrt{2\pi}} e^{-i\frac{\omega}{2^j}n}\right]\left[\frac{1}{2^{j/2}\sqrt{2\pi}} e^{-i\frac{\omega}{2^j}m}\right]^* \mathrm{d}\omega &= \int_{-2^j\pi}^{2^j\pi} \frac{1}{2^j(2\pi)} e^{-i\frac{\omega}{2^j}(n-m)}\,\mathrm{d}\omega \\
&= \frac{1}{2^j(2\pi)}\left(2^j\pi + 2^j\pi\right)\bar{\delta}_{nm} \\
&= \bar{\delta}_{nm}
\end{aligned}$$

[2] Wojtaszczyk (1997) page 31 ⟨Proposition 2.16⟩, Pinsky (2002) pages 315–316 ⟨6.4.3.1 Additional remarks⟩

4. Proof for (1): Let $A \triangleq [-1:1]$.

$$\begin{aligned}
\|\mathsf{g}\|^2 &= \left\|\lim_{j\to\infty}\mathbf{P}_j\mathsf{g}\right\|^2 \\
&= \lim_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\left\langle \mathbf{P}_j\mathsf{g}\,|\,\mathbf{D}^j\mathbf{T}^n\phi\right\rangle\right|^2 && \text{by (A) and \textit{Parseval's Identity}} && \text{(Theorem L.9 page 280)}\\
&= \lim_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\left\langle \mathsf{g}\,|\,\mathbf{D}^j\mathbf{T}^n\phi\right\rangle\right|^2 && \text{by definition of } \mathbf{P}_j && \text{(item (1) page 69)}\\
&= \lim_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\left\langle \tilde{\mathbf{F}}\mathsf{g}\,|\,\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n\phi\right\rangle\right|^2 && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} && \text{(Theorem K.2 page 258)}\\
&= \lim_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\left\langle \tilde{\mathsf{g}}(\omega)\,\middle|\,\frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{\phi}\left(\frac{\omega}{2^j}\right)\right\rangle\right|^2 && \text{by Proposition 1.12 page 9}\\
&= \lim_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\int_{\mathbb{R}}\tilde{\mathsf{g}}(\omega)\tilde{\phi}^*\left(\frac{\omega}{2^j}\right)\frac{1}{2^{j/2}}e^{i\frac{\omega}{2^j}n}\right|^2\,\mathsf{d}\omega && \text{by definition of } \langle\triangle\,|\,\nabla\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}\\
&= \lim_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\left\langle \sqrt{2\pi}\tilde{\mathsf{g}}(\omega)\tilde{\phi}^*\left(\frac{\omega}{2^j}\right)\,|\,\frac{1}{2^{j/2}\sqrt{2\pi}}e^{-i\frac{\omega}{2^j}n}\right\rangle\right|^2 && \text{by definition of } \langle\triangle\,|\,\nabla\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}\\
&= \lim_{j\to\infty}\left\|\sqrt{2\pi}\tilde{\mathsf{g}}(\omega)\tilde{\phi}^*\left(\frac{\omega}{2^j}\right)\right\|^2 && \text{by (3) lemma and \textit{Parseval's Identity}} && \text{(Theorem L.9 page 280)}\\
&= \left\|\sqrt{2\pi}\tilde{\mathsf{g}}(\omega)\tilde{\phi}^*(0)\right\|^2 \\
&= 2\pi\left|\tilde{\phi}^*(0)\right|^2\ \|\tilde{\mathsf{g}}(\omega)\|^2 && \text{by \textit{homogeneous} property of } \|\cdot\| && \text{(Definition D.5 page 160)}\\
&= 2\pi\left|\tilde{\phi}(0)\right|^2\ \|\mathsf{g}\|^2 && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} && \text{(Theorem K.2 page 258)}\\
&\implies \left|\tilde{\phi}(0)\right| = \frac{1}{\sqrt{2\pi}}
\end{aligned}$$

5. Proof for (2):

$$\begin{aligned}
\left|\int_{\mathbb{R}}\phi(x)\,\mathsf{d}x\right| &= \left|\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\phi(x)e^{-i\omega x}\,\mathsf{d}x\right|_{\omega=0} \\
&= \left|\sqrt{2\pi}\tilde{\phi}(\omega)\right|_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)}\\
&= \left|\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\right| && \text{by (1)} && \text{(item (4) page 70)}\\
&= 1
\end{aligned}$$

✎

Every *MRA system* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$, regardless of whether it is *orthonormal* or not, has the following necessary conditions on $(\!(\mathsf{h}_n)\!)$:

1. $\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}$ — *admissibility condition* (Theorem 2.3 page 27)
2. $\langle\phi\,|\,\mathbf{T}^n\phi\rangle = \sum_{m\in\mathbb{Z}}\mathsf{h}_m\sum_{k\in\mathbb{Z}}\mathsf{h}_k^*\left\langle\phi\,|\,\mathbf{T}^{2n-m+k}\phi\right\rangle$ — *quadrature condition* in "time" (Theorem 2.4 page 29)
3. $\tilde{\phi}(\omega) = \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$ — *frequency dilation equation* (Lemma 2.1 page 22)
4. $\left|\breve{\mathsf{h}}(\omega)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega) + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2\tilde{\mathsf{S}}_{\phi\phi}(\omega+\pi) = 2\tilde{\mathsf{S}}_{\phi\phi}(2\omega)$ — *quadrature condition* in "frequency" (Theorem 2.5 page 30)

In an *orthonormal MRA system*, the *orthonormal* property is equivalent to $\tilde{\mathsf{S}}_{\phi\phi}(\omega) = 1$ (where $\tilde{\mathsf{S}}_{\phi\phi}$ is the *auto-power spectrum*), and so the quadrature conditions simplify considerably (next theorem).

The *orthonormal quadrature conditions* in “time” and “frequency” are also equivalent to each other and are implied by the $\tilde{S}_{\phi\phi}(\omega) = 1$ condition. However, the *orthonormal quadrature conditions* are *not* equivalent to the $\tilde{S}_{\phi\phi}(\omega) = 1$ condition (Counterexample 5.1 page 71).

Lemma 5.1 (orthonormal quadrature conditions). [3] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 25). Let* $\breve{h}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORM *of* $(\!(h_n)\!)$ *(Definition P.1 page 355). Let* $\tilde{S}_{\phi\phi}(\omega)$ *be the* AUTO-POWER SPECTRUM *of* $\phi(x)$ *(Definition R.3 page 373).*

LEM

$$\underbrace{\langle \phi \,|\, \mathbf{T}^n\phi \rangle = \bar{\delta}_n \quad \forall n\in\mathbb{Z}}_{\text{(1) } \{\mathbf{T}^n\phi\} \text{ is ORTHONORMAL}} \iff \underbrace{\tilde{S}_{\phi\phi}(\omega) = 1}_{\text{(2) auto-power spectrum is 1}}$$

$$\begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} \underbrace{\left\{ \sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n \quad \forall n\in\mathbb{Z} \right\}}_{\text{(3) ORTHOGONAL QUADRATURE CONDITION in “time”}} \iff \underbrace{\left\{ \left|\breve{h}(\omega)\right|^2 + \left|\breve{h}(\omega+\pi)\right|^2 = 2 \right\}}_{\text{(4) ORTHOGONAL QUADRATURE CONDITION in “frequency”}}$$

✎PROOF:

1. Proof that (1) $\iff$ (2): by Theorem R.3 page 379.

2. Proof that [(1)/(2)] $\Longrightarrow$ (3):

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} &= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h^*_k \bar{\delta}_{k-m+2n} && \text{by definition of } \bar{\delta} && \text{(Definition L.12 page 278)}\\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} h^*_k \left\langle \phi \,|\, \mathbf{T}^{2n-m+k}\phi \right\rangle && \text{by (1)}\\
&= \langle \phi \,|\, \mathbf{T}^n \phi \rangle && \text{by \textit{quadrature condition in “time”}} && \text{(Theorem 2.4 page 29)}\\
&= \bar{\delta}_n && \text{by (1)}
\end{aligned}$$

3. Proof that (3) $\not\Longrightarrow$ [(1)/(2)]: by Counterexample 5.1 page 71.

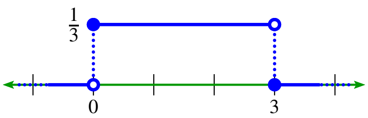
4. Proof that (3) $\iff$ (4): by Lemma 5.2 page 81.

✏

Counterexample 5.1. [4] Let $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ be an *MRA system* (Definition 2.3 page 25).

CNT

Let $\phi(x) \triangleq \begin{cases} \frac{1}{3} & \text{for } x \in [0:3) \\ 0 & \text{otherwise.} \end{cases}$



$$\text{Then, } \left\{ \begin{array}{rcll} \breve{h}(\omega) + \breve{h}(\omega+\pi) &=& 2 & \text{but} \\ \tilde{S}_{\phi\phi}(\omega) &=& \frac{1}{9}[3 + 4\cos(\omega) + 2\cos(2\omega)] & \text{and so} \\ \breve{h}(\omega) + \breve{h}(\omega+\pi) = 2 &\not\Longrightarrow& \tilde{S}_{\phi\phi}(\omega) = 1. & \end{array} \right\}$$

[3] Chui (1992), page 135, Mallat (1999) pages 229–238, Goswami and Chan (1999), page 110, Vaidyanathan (1990), page 65

[4] Vidakovic (1999) page 57 ⟨Remark 3.3.3⟩

✎PROOF:

$$
\begin{aligned}
\tilde{\phi}(\omega) &\triangleq \tilde{\mathbf{F}}\phi(x) \\
&\triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) e^{-i\omega x}\, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{1}_{[0:3)} e^{-i\omega x}\, \mathrm{d}x && \text{by definition of } \phi(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_0^3 e^{-i\omega x}\, \mathrm{d}x && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 1)} \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega x}}{-i\omega}\right)\Big|_0^3 \\
&= \left(\frac{3i}{\sqrt{2\pi}}\right)\left(\frac{e^{-i3\omega}-1}{3\omega}\right)
\end{aligned}
$$

$$
\begin{aligned}
\breve{\mathsf{h}}(\omega) &= \sqrt{2}\frac{\tilde{\phi}(2\omega)}{\tilde{\phi}(\omega)} && \text{by Lemma 2.1 page 22} \\
&= \sqrt{2}\frac{\left(\frac{3i}{\sqrt{2\pi}}\right)\left(\frac{e^{-i4\omega}-1}{6\omega}\right)}{\left(\frac{3i}{\sqrt{2\pi}}\right)\left(\frac{e^{-i2\omega}-1}{3\omega}\right)} && \text{by Lemma M.5 page 312} \\
&= \frac{\sqrt{2}}{2}\left(\frac{e^{-i6\omega}-1}{e^{-i3\omega}-1}\right) \\
&= \frac{\sqrt{2}}{2}\left(\frac{e^{-i6\omega}-1}{e^{-i3\omega}-1}\right)\left(\frac{e^{-i3\omega}+1}{e^{-i3\omega}+1}\right) \\
&= \frac{\sqrt{2}}{2}\left(\frac{\left(e^{-i6\omega}-1\right)\left(e^{-i3\omega}+1\right)}{e^{-i6\omega}-1}\right) \\
&= \frac{\sqrt{2}}{2}\left(1+e^{-i3\omega}\right) \\
&= \left(\frac{\sqrt{2}}{2}\right)e^{-i\frac{3}{2}\omega}\left(e^{i\frac{3}{2}\omega}+e^{-i\frac{3}{2}\omega}\right) \\
&= \left(\frac{\sqrt{2}}{2}\right)e^{-i\frac{3}{2}\omega}\left[2\cos\left(\frac{3}{2}\omega\right)\right] \\
&= \sqrt{2}e^{-i\frac{3}{2}\omega}\cos\left(\frac{3}{2}\omega\right)
\end{aligned}
$$

$$
\begin{aligned}
&\breve{\mathsf{h}}(\omega+\pi) \\
&= \frac{\sqrt{2}}{2}\left(1+e^{-i3(\omega+\pi)}\right) && \text{by previous } \breve{\mathsf{h}}(\omega) \text{ result} \\
&= \frac{\sqrt{2}}{2}\left(1+e^{-i3\omega}e^{-i3\pi}\right) \\
&= \frac{\sqrt{2}}{2}\left(1-e^{-i3\omega}\right) \\
&= \frac{\sqrt{2}}{2}e^{-i\frac{3}{2}\omega}\left(e^{i\frac{3}{2}\omega}-e^{-i\frac{3}{2}\omega}\right) \\
&= \frac{\sqrt{2}}{2}e^{-i\frac{3}{2}\omega}\left[2i\sin\left(\frac{3}{2}\omega\right)\right] \\
&= i\sqrt{2}e^{-i\frac{3}{2}\omega}\sin\left(\frac{3}{2}\omega\right)
\end{aligned}
$$

$$\left|\breve{\mathsf{h}}(\omega)\right|^2 + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2$$

$$\begin{aligned}
&= \left|\sqrt{2}e^{-i\frac{3}{2}\omega}\cos\left(\frac{3}{2}\omega\right)\right|^2 + \left|i\sqrt{2}e^{-i\frac{3}{2}\omega}\sin\left(\frac{3}{2}\omega\right)\right|^2 \\
&= 2\cos^2\left(\frac{3}{2}\omega\right) + 2\sin^2\left(\frac{3}{2}\omega\right) \\
&= 2\left[\cos^2\left(\frac{3}{2}\omega\right) + \sin^2\left(\frac{3}{2}\omega\right)\right] \\
&= 2 && \text{by \textit{half-angle formulas}} && \text{(Theorem H.11 page 228)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{S}}_{\phi\phi}(\omega) &\triangleq \check{\mathsf{S}}_{\phi\phi}(z)\Big|_{z=e^{in\omega}} && \text{by definition of } \tilde{\mathsf{S}}_{\phi\phi} && \text{(Definition R.3 page 373)} \\
&\triangleq \sum_{n\in\mathbb{Z}} \langle \phi \,|\, \mathbf{T}^n\phi \rangle\, z^{-n}\Big|_{z=e^{i\omega}} && \text{by definition of } \tilde{\mathsf{S}}_{\phi\phi} && \text{(Definition R.2 page 373)} \\
&\triangleq \sum_{n\in\mathbb{Z}} \left[\int_{\mathbb{R}} \frac{1}{3}\mathbb{1}_{[0:3)}(x)\frac{1}{3}\mathbb{1}_{[-n:3-n)}(x)\,\mathrm{d}x\right] z^{-n}\Big|_{z=e^{i\omega}} && \text{by definition of } \phi \\
&= \frac{1}{9}\sum_{n\in\mathbb{Z}} \left[\int_0^3 \mathbb{1}_{[-n:3-n)}(x)\,\mathrm{d}x\right] z^{-n}\Big|_{z=e^{i\omega}} && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 1)} \\
&= \frac{1}{9}\left[z^{-2} + 2z^{-1} + 3 + 2z + z^2\right]\Big|_{z=e^{i\omega}} \\
&= \frac{1}{9}\left[e^{-i2\omega} + 2e^{-i\omega} + 3 + 2e^{-i\omega} + e^{-i2\omega}\right] \\
&= \frac{1}{9}[3 + 4\cos(\omega) + 2\cos(2\omega)]
\end{aligned}$$

✏

Theorem 5.1. *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ *be a wavelet system* (Definition 3.2 page 44). *Let* $\check{\mathsf{h}}(z)$ *be the* Z-TRANSFORM (Definition O.4 page 342) *of* h.

THM

$$\underbrace{\langle \phi \,|\, \mathbf{T}^n\phi \rangle = \bar{\delta}_n \qquad \forall n\in\mathbb{Z}}_{\{\mathbf{T}^n\phi\}\text{ is ORTHONORMAL}} \implies \left\{\begin{array}{llcll}
1. & \displaystyle\sum_{m\in\mathbb{Z}} (-1)^m \mathsf{h}_m &=& 0 & \text{(LOW-PASS FILTER)} \quad \textit{and} \\
2. & \displaystyle\sum_{m\in\mathbb{Z}} \mathbf{T}^m\phi &=& 1 & \text{(PARTITION OF UNITY)} \quad \textit{and} \\
3. & \check{\mathsf{h}}(z)\Big|_{z=-1} &=& 0 & (\text{ZERO AT } z=-1)
\end{array}\right\}$$

✎PROOF:

1. Proof for (1):

$$\begin{aligned}
0 &= \left[-2 + \left|\check{\mathsf{h}}(\omega)\right|^2 + \left|\check{\mathsf{h}}(\omega+\pi)\right|^2\right]_{\omega=0} && \text{by \textit{orthonormal quadrature condition}} && \text{(Lemma 5.2 page 81)} \\
&= -2 + \left|\check{\mathsf{h}}(0)\right|^2 + \left|\check{\mathsf{h}}(\pi)\right|^2 \\
&= -2 + \left|\sqrt{2}\right|^2 + \left|\check{\mathsf{h}}(\pi)\right|^2 && \text{by \textit{admissibility condition}} && \text{(Theorem 2.3 page 27)} \\
&= 0 + \left|\check{\mathsf{h}}(\pi)\right|^2 \\
&= \left|\sum_{n\in\mathbb{Z}} h_n e^{-i\omega n}\right|^2_{\omega=\pi} && \text{by definition of } \check{\mathsf{h}} && \text{(Definition P.1 page 355)} \\
&= \left|\sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n\right|^2 && \text{because } e^{-i\pi n} = \left(e^{-i\pi}\right)^n = (-1)^n \\
\implies & \sum_{n\in\mathbb{Z}} (-1)^n \mathsf{h}_n = \pm 0 = 0
\end{aligned}$$

2. Proof for (2): by (1) and Theorem 2.8 page 34.

3. Proof for (3): by (1) and Proposition P.4 page 358.

⇒

5.2.3 Sufficient conditions

The *admissibility condition* (Theorem 2.3 page 27) and *quadrature condition* (Lemma 5.2 page 81) give **necessary** conditions for $(\!(\mathsf{h}_n)\!)$ to generate an orthogonal scaling function. Theorem 5.2 (next) shows that these two conditions are not only **necessary**, but also **sufficient** conditions.

Theorem 5.2. [5] *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *and* $\mathbf{D}$ *the* DILATION OPERATOR *(Definition 1.3 page 2). Let* $\{\mathsf{h}_n | n\in\mathbb{Z}\}$ *be a sequence in* $\ell^2_{\mathbb{F}}$ *over some field* $\mathbb{F}$ *and with* DTFT *(Definition P.1 page 355)* $\breve{\mathsf{h}}(\omega)$. *Let* $\phi(x)$ *be a function in* $L^2_{\mathbb{R}}$ *with* FOURIER TRANSFORM *(Definition K.2 page 257)* $\tilde{\phi}(\omega)$ *and defined in terms of* $(\!(\mathsf{h}_n)\!)$ *as*

$$\tilde{\phi}(\omega) \triangleq \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right) \qquad \text{and} \qquad \boldsymbol{V}_j \triangleq \operatorname{span}\left\{\mathbf{D}^j\mathbf{T}^n\phi(x) \middle| n\in\mathbb{Z}\right\}\ \forall j\in\mathbb{Z}.$$

Let an MRA *be defined as in Definition 2.1 page 16.*

THM

$$\left\{\begin{array}{lll}
\text{(A).} & \displaystyle\sum_{n\in\mathbb{Z}} \mathsf{h}_n = \sqrt{2} & \text{and}\\
\text{(B).} & \displaystyle\sum_{m\in\mathbb{Z}} \mathsf{h}_m h^*_{m-2n} = \bar{\delta}_n & \text{and}\\
\text{(C).} & \tilde{\phi}(\omega) \text{ is CONTINUOUS at } 0 & \text{and}\\
\text{(D).} & \displaystyle\inf_{\omega\in[-\pi/2:\pi/2]} \left|\breve{\mathsf{h}}(\omega)\right| > 0 &
\end{array}\right\} \implies \left\{\begin{array}{l}
\left(L^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right) \text{ is an}\\
\text{ORTHONORMAL MRA SYSTEM}\\
\text{(Definition 2.3 page 25)}
\end{array}\right\}$$

✎PROOF:

1. Proof that (A) $\Longleftarrow$ (right hypothesis): by Theorem 2.3 page 27.

2. Proof that (B) $\Longleftarrow$ (right hypothesis): by Lemma 5.2 page 81.

3. Proof that $(\!(\mathbf{T}^n\phi)\!)$ is *orthonormal*:

   (a) lemma: $(\!(\mathbf{T}^n\phi)\!)$ is *orthonormal* $\iff \int_{\mathbb{R}}\left|\tilde{\phi}(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega = \bar{\delta}_n$. Proof:

$$\begin{aligned}
\bar{\delta}_n &= \langle \phi \,|\, \mathbf{T}^n\phi\rangle && \text{by left hypothesis} && (\text{for } \Longrightarrow \text{ case})\\
&= \left\langle \tilde{\mathbf{F}}\phi \,|\, \tilde{\mathbf{F}}\mathbf{T}^n\phi\right\rangle && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem K.2 page 258)}\\
&= \left\langle \tilde{\phi}(\omega) \,|\, e^{-i\omega n}\tilde{\phi}(\omega)\right\rangle && \text{by Corollary 1.1 page 9} &&\\
&= \int_{\mathbb{R}} \tilde{\phi}(\omega)\left[\tilde{\phi}(\omega)e^{-i\omega n}\right]^*\,\mathrm{d}\omega && \text{by definition of } \langle\triangle\,|\,\nabla\rangle \text{ in } L^2_{\mathbb{R}} && \text{(Definition D.9 page 168)}\\
&= \int_{\mathbb{R}} \left|\tilde{\phi}(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega && &&\\
&= \bar{\delta}_n && \text{by right hypothesis} && (\text{for } \Longleftarrow \text{ case})
\end{aligned}$$

[5] Mallat (1999) pages 229–234, Mallat (1989), page 76, Mallat (2009) pages 271–276 ⟨Theorem 7.2⟩, Meyer (1992)

(b) definition: $\tilde{\phi}_N(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \prod_{k=1}^{N} \frac{\sqrt{2}}{2} \breve{h}\left(\frac{\omega}{2^k}\right) \mathbb{1}_A(\omega)$ and where $A \triangleq \left[-2^N\pi : 2^N\pi\right]$.

(c) definition: Let $\mathrm{I}_N(n) \triangleq \int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N-1} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega$

(d) lemma: $\mathrm{I}_N(n) = 2\pi\bar{\delta}_n \quad \forall N\in\mathbb{N}$. Proof:

i. $N = 1$ case:

$$\begin{aligned}
\mathrm{I}_1(n) &\triangleq \left.\int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N-1} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega\right|_{N=1} && \text{by definition of } \mathrm{I}_N \quad \text{(definition 3c page 75)}\\
&= \int_0^{2\pi} e^{in\omega} \prod_{k=1}^{0} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega\\
&= \int_0^{2\pi} e^{in\omega} \,\mathrm{d}\omega\\
&= 2\pi\bar{\delta}_n
\end{aligned}$$

ii. Proof that $\mathrm{I}_{N+1}(n) = \mathrm{I}_N(n)$:

$$\begin{aligned}
&\mathrm{I}_{N+1}(n)\\
&= \int_0^{2^{N+1}\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega && \text{by definition of } \mathrm{I}_N \text{ (definition 3c page 75)}\\
&= \int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega + \int_{\omega=2^N\pi}^{\omega=2^{N+1}\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega\\
&= \int_0^{2^N\pi} e^{in\omega} \prod_{k=1}^{N} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega + \int_{v=0}^{v=2^N\pi} e^{i(v-2^N\pi)n} \prod_{k=1}^{N} \frac{1}{2} \left|\breve{h}\left(\frac{v+2^N\pi}{2^k}\right)\right|^2 \,\mathrm{d}v\\
&\quad \text{where } v \triangleq \omega - 2^N\pi \implies \omega = v + 2^N\pi\\
&= \int_0^{2^N\pi} e^{in\omega} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^N}\right)\right|^2 \prod_{k=1}^{N-1} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega\\
&+ \int_{v=0}^{v=2^N\pi} e^{in(v} \frac{1}{2} \left|\breve{h}\left(\frac{v}{2^N}+\pi\right)\right| \prod_{k=1}^{N-1} \frac{1}{2} \left|\breve{h}\left(\frac{v}{2^k}+2^{N-k}\pi\right)\right|^2 \,\mathrm{d}v\\
&= \int_0^{2^N\pi} e^{in\omega} \frac{1}{2} \left[\left|\breve{h}\left(\frac{\omega}{2^N}\right)\right|^2 + \left|\breve{h}\left(\frac{\omega}{2^N}+\pi\right)\right|^2\right] \prod_{k=1}^{N-1} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega && \text{by Proposition P.1 page 355}\\
&= \int_0^{2^N\pi} e^{in\omega} \frac{1}{2} [2] \prod_{k=1}^{N-1} \frac{1}{2} \left|\breve{h}\left(\frac{\omega}{2^k}\right)\right|^2 \,\mathrm{d}\omega && \text{by (B) and \textit{orthonormal quad. cond.} (Lemma 5.1 page 71)}\\
&= \mathrm{I}_N(n) && \text{by definition of } \mathrm{I}_N(n) \text{ (definition 3c page 75)}
\end{aligned}$$

iii. Proof that $\mathrm{I}_N(n) = 2\pi\bar{\delta}_n$:

$$\begin{aligned}
\mathrm{I}_N(n) &= \mathrm{I}_1(n) && \text{by induction and (3(d)ii) lemma page 75}\\
&= 2\pi\bar{\delta}_n && \text{by (3(d)i) lemma page 75}
\end{aligned}$$

(e) lemma: $\int_{\mathbb{R}} \left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega = \bar{\delta}_n \quad \forall N\in\mathbb{N}$. Proof:

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\,\mathrm{d}\omega \\
&= \int_{\mathbb{R}} e^{in\omega}\left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\mathbb{1}_{[-2^N\pi:2^N\pi]}\right|^2\mathrm{d}\omega && \text{by definition of } \tilde{\phi}_N(\omega) \text{ (definition 3b page 75)} \\
&= \frac{1}{2\pi}\int_{-2^N\pi}^{2^N\pi} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-2^N\pi}^{0} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{v=0}^{v=2^N\pi} e^{i(v-2^N\pi)n}\prod_{k=1}^{N}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{v}{2^k}-2^{N-k}\pi\right)\right|^2\mathrm{d}v \\
&\quad + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\prod_{k=1}^{N}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega && \text{where } v \triangleq \omega+2^N\pi \implies \omega = v-2^N\pi \\
&= \frac{1}{2\pi}\int_{v=0}^{v=2^N\pi} e^{ivn}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{v}{2^k}-\pi\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{v}{2^k}\right)\right|^2\mathrm{d}v \\
&\quad + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^N}\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega && \text{by Proposition P.1 page 355} \\
&= \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^N}-\pi+2\pi\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega \\
&\quad + \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^N}\right)\right|^2\prod_{k=1}^{N-1}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega && \text{by Proposition P.1 page 355} \\
&= \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}\left[\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^N}\right)\right|^2+\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^N}+\pi\right)\right|^2\right]\prod_{k=1}^{N-1}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{0}^{2^N\pi} e^{in\omega}\frac{1}{2}[2]\prod_{k=1}^{N-1}\frac{1}{2}\left|\breve{\mathrm{h}}\left(\frac{\omega}{2^k}\right)\right|^2\mathrm{d}\omega && \text{by (B) and Lemma 5.2 page 81} \\
&= \frac{1}{2\pi}\mathtt{I}_N(n) && \text{by definition of } \mathtt{I}_N(n) \text{ (definition 3c page 75)} \\
&= \bar{\delta}_n && \text{by (3d) lemma page 75}
\end{aligned}$$

(f) lemma: $\tilde{\phi}(\omega)$ is in $\boldsymbol{L}^2_{\mathbb{R}}$ (alternatively, proof that $\left|\tilde{\phi}(\omega)\right|^2$ is *integrable*). Proof:

$$\begin{aligned}
\int_{\mathbb{R}}\left|\tilde{\phi}(\omega)\right|^2\mathrm{d}\omega &= \int_{\mathbb{R}}\lim_{N\to\infty}\left|\tilde{\phi}_N(\omega)\right|^2\mathrm{d}\omega && \text{by definition of } \tilde{\phi}_N(\omega) && \text{(definition 3b page 75)} \\
&\le \lim_{N\to\infty}\int_{\mathbb{R}}\left|\tilde{\phi}_N(\omega)\right|^2\mathrm{d}\omega && \text{by \textit{Fatou's Lemma}} \\
&= 1 && \text{by (3e) lemma page 76} \\
&\implies \tilde{\phi}(\omega) \text{ is in } \boldsymbol{L}^2_{\mathbb{R}} && \text{by definition of } \langle\triangle\,|\,\triangledown\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}
\end{aligned}$$

(g) lemma (*upper-bound condition*): $\exists C \ge 1$ such that $\left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right| \le C\left|\tilde{\phi}(\omega)\right|^2$. Proof:

i. Proof for $|\omega| > 2^N\pi$:

$$\begin{aligned}
\left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right| &= \left|\tilde{\phi}_N(\omega)\right|^2 && \text{by definition of } |\cdot| \\
&= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\mathbb{1}_{[-2^N\pi:2^N\pi]}(\omega)\right|^2 && \text{by def. of } \tilde{\phi}_N(\omega) && \text{(definition 3b page 75)} \\
&= 0 \qquad \forall|\omega| > 2^N\pi && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 1)} \\
&\le \left|\tilde{\phi}(\omega)\right|^2 && \text{by definition of } |\cdot|
\end{aligned}$$

ii. lemma: $\left|\tilde{\phi}(\omega)\right|^2 \ge \frac{1}{2\pi C}\ \forall|\omega|\le\pi \quad\Longrightarrow\quad \left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right| \le C\left|\tilde{\phi}(\omega)\right|^2\ \forall|\omega|\le 2^N\pi$

$$\begin{aligned}
\left|\tilde{\phi}(\omega)\right|^2 &= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by definition of } \tilde{\phi}(\omega) \\
&= \left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by Lemma 2.2 page 24} \\
&= 2\pi\left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\right|^2\left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \\
&= 2\pi\left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\right|^2\left|\tilde{\phi}_N(\omega)\right|^2 && \text{by def. of } \tilde{\phi}_N(\omega) && \text{(definition 3b page 75)} \\
&\ge (2\pi)\frac{1}{2\pi C}\left|\tilde{\phi}_N(\omega)\right|^2 && \text{by left hypothesis} \\
&= \frac{1}{C}\left|\left|\tilde{\phi}_N(\omega)\right|^2 e^{in\omega}\right|
\end{aligned}$$

$\Longrightarrow$ right hypothesis

iii. Proof for $0 \le |\omega| < \varepsilon$:

A. lemma: There exists $\varepsilon > 0$ such that $\left|\breve{\mathsf{h}}(\omega)\right|^2 \le 2 = \left|\breve{\mathsf{h}}(0)\right|^2$. Proof:

$$\begin{aligned}
\left|\breve{\mathsf{h}}(\omega)\right|^2 &\le \left|\breve{\mathsf{h}}(\omega)\right|^2 + \left|\breve{\mathsf{h}}(\omega+\pi)\right|^2 \\
&= 2 && \text{by (B) and (Lemma 5.1 page 71)} \\
&= \left|\breve{\mathsf{h}}(0)\right|^2 && \text{by (A) and Proposition P.2 page 357}
\end{aligned}$$

B. It follows from (3(g)iiiA) lemma that there exists $\varepsilon > 0$ such that $\forall|\omega| \le \varepsilon$

$$\begin{aligned}
0 &\ge \log_e\left(\tfrac{1}{2}\left|\breve{\mathsf{h}}(\omega)\right|^2\right) && \text{by (3(g)iiiA) lemma} \\
-|\omega| &\le \log_e\left(\tfrac{1}{2}\left|\breve{\mathsf{h}}(\omega)\right|^2\right) \le 0
\end{aligned}$$

C. last step in proof for $0 < \varepsilon < |\omega| \le 2^N\pi$:

$$\begin{aligned}
\left|\tilde{\phi}(\omega\right|^2 &= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by definition of } \tilde{\phi}(\omega) \\
&= \tfrac{1}{2\pi}\prod_{k=1}^{\infty}\tfrac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 \\
&= \tfrac{1}{2\pi}\exp\log_e\left[\prod_{k=1}^{\infty}\tfrac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\right] \\
&= \tfrac{1}{2\pi}\exp\left\{\log_e\left[\prod_{k=1}^{\infty}\tfrac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2\pi}\exp\left\{\sum_{k=1}^{\infty}\log_e\left[\tfrac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\right]\right\}\\
&\ge \tfrac{1}{2\pi}\exp\left\{\log_e\left[\tfrac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^0}\right)\right|^2\right]\right\}\\
&\ge \tfrac{1}{2\pi}e^{-|\omega|}\\
&\ge \tfrac{1}{2\pi}e^{-\varepsilon}\\
&\implies \text{upper bound condition} && \text{by (3(g)ii) lemma page 77}
\end{aligned}
$$

iv. Proof for $\varepsilon \le |\omega| \le \pi$:

A. Choose N such that $2^{-N}\pi < \varepsilon$

B. Let $K = \inf_{\omega\in[-\omega/2:\omega/2]}\left|\breve{\mathsf{h}}(\omega)\right| > 0$

C. Then

$$
\begin{aligned}
\left|\tilde{\phi}(\omega)\right|^2 &= \left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by definition of } \tilde{\phi}(\omega)\\
&= \left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by Lemma 2.2 page 24}\\
&= 2\pi\left|\tilde{\phi}\left(\frac{\omega}{2^N}\right)\right|^2\left|\frac{1}{\sqrt{2\pi}}\prod_{k=1}^{N}\frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2\\
&\ge 2\pi\frac{1}{2\pi}e^{-\varepsilon}\frac{1}{2\pi}\prod_{k=1}^{N}\frac{1}{2}\left|\breve{\mathsf{h}}\left(\frac{\omega}{2^k}\right)\right|^2 && \text{by item (3(g)ivA) and (3(g)iiiC) lemma page 77}\\
&\ge \frac{1}{2\pi}e^{-\varepsilon}\prod_{k=1}^{N}\frac{1}{2}|K|^2\\
&= \frac{1}{2\pi}e^{-\varepsilon}\frac{K^{2N}}{2^N}
\end{aligned}
$$

(h) Final step in proof for (2) (proof that $(\!(\mathbf{T}^n\phi)\!)$ is *orthonormal*):

$$
\begin{aligned}
\int_{\mathbb{R}}\left|\tilde{\phi}(\omega)e^{in\omega}\right|^2 d\omega &= \int_{\mathbb{R}}\lim_{N\to\infty}\left|\tilde{\phi}_N(\omega)e^{in\omega}\right|^2 d\omega\\
&= \lim_{N\to\infty}\int_{\mathbb{R}}\left|\tilde{\phi}_N(\omega)e^{in\omega}\right|^2 d\omega && \text{by (3g) lemma and \textit{Dominated Convergence Theorem}}\\
&= \bar{\delta}_n && \text{by (3e) lemma page 76}\\
\implies (2) & && \text{by (3a) lemma page 74}
\end{aligned}
$$

4. Proof that $(\!(V_j)\!)$ is an *MRA* (Definition 2.1 page 16):

(a) $V_j \triangleq \operatorname{span}(\!(\mathbf{T}^n\mathbf{D}^j\phi)\!)$, so we just need to prove that ϕ is a *scaling function* (Definition 2.1 page 16), and then the claim that $(\!(V_j)\!)$ is an *MRA* follows from Theorem 2.6 page 31.

(b) Proof that $\{\mathbf{T}^n\phi\}$ is an *MRA*:

i. Proof that $(\!(\mathbf{T}^n\phi)\!)$ is a *Riesz basis*: by (1) (*orthonormal* property, item (3) page 74) and by Theorem L.13 page 286.

ii. Proof that $\exists\,(\!(\mathsf{h}_n)\!)$ such that $\phi(x) = \sum_{n\in\mathbb{Z}} h_n\mathbf{D}\mathbf{T}^n\phi(x)$: by definition of $\tilde{\phi}(\omega)$, hypothesis (C), and Lemma 2.2 page 24.

iii. Proof that $\tilde{\phi}(\omega)$ is *continuous* at 0: by hypothesis (C).

iv. Proof that $\tilde{\phi}(0) \ne 0$: by hypothesis (D).

5. Proof that $(\!(\mathsf{h}_n)\!)$ is a *scaling coefficient sequence* (Definition 2.3 page 25):

$$\begin{aligned}
\tilde{\phi}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\frac{\sqrt{2}}{2}\check{\mathsf{h}}\left(\frac{\omega}{2^k}\right) && \text{by definition of } \tilde{\phi}(\omega)\\
\implies \phi(x) &= \sum_{n\in\mathbb{Z}}\mathsf{h}_n\mathbf{DT}^n\phi(x) && \text{by Lemma 2.2 page 24}\\
\implies &\quad (\!(\mathsf{h}_n)\!) \text{ is a } \textit{scaling coefficient sequence} && \text{by (Definition 2.3 page 25)}
\end{aligned}$$

✏

5.2.4 Orthonormal basis from Riesz basis

The definition of a general wavelet system (Definition 3.2 page 44) requires an MRA. The last condition of the definition of an MRA (Definition 2.1 page 16) only requires that a *Riesz basis* exist for the scaling subspace $\boldsymbol{V}_0$. The MRA definition does *not* require this basis to be *orthonormal*. However, the definition of an *orthonormal wavelet system* (Definition 5.1 page 67) does require an orthonormal basis for $\boldsymbol{V}_0$. The good news is that if you have a Riesz basis for $\boldsymbol{V}_0$ (which of course you do since you have an MRA), then from this Riesz basis you can generate an orthonormal basis. One way of doing this is given by Theorem 5.3 (next). That is, Theorem 5.3 gives a sufficient condition for generating an orthonormal basis $\{\mathbf{T}^n\phi\}$ from a Riesz basis $\{\mathbf{T}^n\theta\}$.

Theorem 5.3 (Battle-Lemarié orthogonalization). [6] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 25). Let* $\tilde{\phi}(\omega)$ *be the Fourier Transform (Definition K.2 page 257) of an arbitrary function* $\phi \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM

$$\left\{\begin{array}{ll}
1. & \{\mathbf{T}^n\theta \,|\, n\in\mathbb{Z}\} \text{ is a } \textsc{Riesz basis} \text{ for } \boldsymbol{V}_0 \quad \text{and}\\
2. & \tilde{\phi}(\omega) \triangleq \dfrac{\tilde{\theta}(\omega)}{\sqrt{2\pi\sum_n\left|\tilde{\theta}(\omega+2\pi n)\right|^2}}
\end{array}\right\} \implies \left\{\begin{array}{l}\{\mathbf{D}^m\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\}\\ \text{is an } \textsc{orthonormal basis} \text{ for } \boldsymbol{V}_m\end{array}\right\}$$

✎PROOF:

1. Proof that $\{\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\}$ is orthonormal: by Theorem L.15 page 289
2. Proof that $\{\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_0$: by Lemma L.1 page 273.
3. Proof that $\{\mathbf{D}^m\mathbf{T}^n\phi \,|\, n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_m$: by Theorem 2.2 page 25.

✏

5.3 Orthonormal wavelet systems

5.3.1 Subspace properties

Definition 5.1 (page 67) requires that the scaling subspace $\boldsymbol{V}_0$ and the wavelet subspace $\boldsymbol{W}_0$ be orthogonal. However, Proposition 5.3 (next) shows that this constraint implies that for all scales n, $\boldsymbol{V}_n$

[6] Vidakovic (1999), page 71, Mallat (1989), page 72, Mallat (1999) page 225

is orthogonal to $\mathbf{W}_n$:

Proposition 5.3. [7] *Let* $\left(L^2_{\mathbb{R}}, (\!(\mathbf{V}_j)\!), (\!(\mathbf{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a wavelet system* (Definition 3.2 page 44).

PRP

$$\underbrace{\mathbf{V}_0 \perp \mathbf{W}_0}_{\text{orthogonal subspaces}} \implies \underbrace{\mathbf{V}_n \perp \mathbf{W}_n \quad \forall n \in \mathbb{Z}}_{\text{orthogonal subspaces}}$$

✎PROOF:

$$\begin{aligned}
0 &= \langle \phi(x) \,|\, \psi(x) \rangle && \text{because } \mathbf{V}_0 \perp \mathbf{W}_0 \\
&= \left\langle \phi(2^j u) \,|\, \psi(2^j u) \right\rangle && \text{where } u = 2^{-j} x \iff x = 2^j u \\
\implies \quad & \mathbf{V}_j \perp \mathbf{W}_j
\end{aligned}$$

⇰

5.3.2 Wavelet coefficients from wavelet and scaling functions

In an *orthonormal MRA system* (Definition 5.1 page 67), the scaling coefficients $(\!(h_n)\!)$ can be computed from ϕ (Proposition 5.1 page 67). In an *orthonormal wavelet system* (Definition 5.1 page 67), the wavelet coefficients $(\!(g_n)\!)$ can be computed from ϕ and ψ (next proposition).

Proposition 5.4. *Let* $\left(L^2_{\mathbb{R}}, (\!(\mathbf{V}_j)\!), (\!(\mathbf{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM (*Definition 3.2 page 44*).

PRP

$$\underbrace{\langle \phi \,|\, \mathbf{T}^n \phi \rangle = \bar{\delta}_n}_{\{\mathbf{T}^n\phi\} \text{ is ORTHONORMAL}} \implies g_n = \langle \psi \,|\, \mathbf{D}\mathbf{T}^n \phi \rangle \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\langle \psi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle &= \left\langle \sum_{m\in\mathbb{Z}} g_m \mathbf{D}\mathbf{T}^m \phi \,\middle|\, \mathbf{D}\mathbf{T}^n \phi \right\rangle && \text{by Theorem 3.1 page 44} \\
&= \sum_{m\in\mathbb{Z}} g_m \langle \mathbf{D}\mathbf{T}^m\phi \,|\, \mathbf{D}\mathbf{T}^n\phi \rangle \\
&= \sum_{m\in\mathbb{Z}} g_m \left\langle \sqrt{2}\phi(2x-m) \,|\, \sqrt{2}\phi(2x-n) \right\rangle && \text{by definitions of } \mathbf{T} \text{ and } \mathbf{D} \quad \text{(Definition 1.3 page 2)} \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{x\in\mathbb{R}} \sqrt{2}\phi(2x-m)\sqrt{2}\phi^*(2x-n) \, \mathrm{d}x && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ in } L^2_{\mathbb{R}} \quad \text{(Definition E.1 page 185)} \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{u\in\mathbb{R}} \sqrt{2}\phi(u-m)\sqrt{2}\phi^*(u-n)\tfrac{1}{2} \, \mathrm{d}u && u = 2x \qquad \mathrm{d}u = 2\,\mathrm{d}x \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n) \, \mathrm{d}u \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{u\in\mathbb{R}} \phi(u-m)\phi^*(u-n) \, \mathrm{d}u \\
&= \sum_{m\in\mathbb{Z}} g_m \int_{v\in\mathbb{R}} \phi(v)\phi^*(v-(n-m)) \, \mathrm{d}v && \text{where } v \triangleq u - m \implies u = v + m \\
&= \sum_{m\in\mathbb{Z}} g_m \langle \phi \,|\, \mathbf{T}^{n-m}\phi \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ in } L^2_{\mathbb{R}} \quad \text{(Definition E.1 page 185)}
\end{aligned}$$

[7] Jawerth and Sweldens (1994), pages 10–11

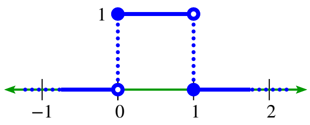
$$\begin{aligned}
&= \sum_{m\in\mathbb{Z}} g_m \bar{\delta}_{m-n} && \text{by left hypothesis}\\
&= \mathrm{g}_n
\end{aligned}$$

✏

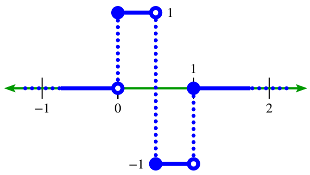
Example 5.2 (Haar wavelet).

EX

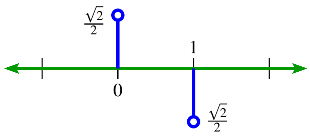
$$\text{Let } \phi(x) \triangleq \begin{cases} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise} \end{cases}$$



$$\text{and } \psi(x) \triangleq \begin{cases} 1 & \text{for } x \in \left[0:\frac{1}{2}\right) \\ -1 & \text{for } x \in \left[\frac{1}{2}:1\right) \\ 0 & \text{otherwise.} \end{cases}$$



$$\text{Then, } \mathrm{g}_n = \begin{cases} \frac{\sqrt{2}}{2} & \text{for } n=0 \\ -\frac{\sqrt{2}}{2} & \text{for } n=1 \\ 0 & \text{otherwise.} \end{cases}$$



✎PROOF:

$$\begin{aligned}
\mathrm{g}_n &= \langle \psi \,|\, \mathbf{DT}^n \phi \rangle && \text{by Proposition 5.4 page 80}\\
&= \left\langle \mathbb{1}_{[0:1)}(x) - 2\mathbb{1}_{\left[\frac{1}{2}:1\right)}(x) \,\middle|\, \sqrt{2}\mathbb{1}_{[0,1)}(2x-n) \right\rangle && \text{by definition of } \phi \text{ and } \psi\\
&= \sqrt{2}\left\langle \mathbb{1}_{[0:1)}(x) \,|\, \mathbb{1}_{[0,1)}(2x-n)\right\rangle - 2\sqrt{2}\left\langle \mathbb{1}_{\left[\frac{1}{2}:1\right)}(x) \,\middle|\, \mathbb{1}_{[0,1)}(2x-n)\right\rangle && \text{by property of } \langle \triangle \,|\, \nabla \rangle\\
&= \mathrm{h}_n - 2\sqrt{2}\int_{\mathbb{R}} \mathbb{1}_{\left[\frac{1}{2}:1\right)}(x)\mathbb{1}_{[0,1)}(2x-n)\,\mathrm{d}x && \text{by Example 5.1 page 68}\\
&= \mathrm{h}_n - 2\sqrt{2}\int_{\left[\frac{1}{2}:1\right)} (x)\mathbb{1}_{[0,1)}(2x-n)\,\mathrm{d}x && \text{by definition of } \mathbb{1} \quad \text{(Definition 1.2 page 1)}\\
&= \mathrm{h}_n - 2\sqrt{2}\tfrac{1}{2}\bar{\delta}_{n-1} && \text{by definition of } \mathbb{1} \quad \text{(Definition 1.2 page 1)}\\
&= \begin{cases} \frac{\sqrt{2}}{2} - 0 &= \frac{\sqrt{2}}{2} & \text{for } n=0 \\ \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} &= -\frac{\sqrt{2}}{2} & \text{for } n=1 \\ 0 &= 0 & \text{otherwise} \end{cases} && \text{by definition of } \bar{\delta} \quad \text{(Definition L.12 page 278)}
\end{aligned}$$

✏

5.3.3 Necessary conditions

Lemma 5.2 (orthonormal quadrature conditions). [8] *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system* (Definition 3.2 page 44). *Let* $\breve{\mathrm{h}}(\omega)$ *and* $\breve{\mathrm{g}}(\omega)$ *be the* DISCRETE TIME FOURIER TRANSFORMS

[8] Chui (1992), page 135, Mallat (1999) pages 229–238, Goswami and Chan (1999), page 110, Vaidyanathan (1990), page 65

(Definition P.1 page 355) *of* $(\!(\mathsf{h}_n)\!)$ *and* $(\!(\mathsf{g}_n)\!)$, *respectively.*

LEM

$$\underbrace{\left\{\begin{array}{lrcll} 1. & \langle\phi\,|\,\mathbf{T}^n\phi\rangle & = & \bar{\delta}_n & \text{and}\\ 2. & \langle\psi\,|\,\mathbf{T}^n\psi\rangle & = & \bar{\delta}_n & \text{and}\\ 3. & \langle\phi\,|\,\mathbf{T}^n\psi\rangle & = & 0 & \end{array}\right\}}_{\Omega \text{ is ORTHONORMAL (Definition 5.1 page 67)}} \implies \underbrace{\left\{\begin{array}{lrcll} A. & \sum_{m\in\mathbb{Z}} \mathsf{h}_m\mathsf{h}^*_{m-2n} & = & \bar{\delta}_n & \text{and}\\ B. & \sum_{m\in\mathbb{Z}} \mathsf{g}_m\mathsf{g}^*_{m-2n} & = & \bar{\delta}_n & \text{and}\\ C. & \sum_{m\in\mathbb{Z}} \mathsf{h}_m\mathsf{g}^*_{m-2n} & = & 0 & \end{array}\right\}}_{\text{ORTHONORMAL QUADRATURE CONDITIONS in “time”}} \quad \forall n\in\mathbb{Z}$$

$$\iff \underbrace{\left\{\begin{array}{lrcrcll} a. & |\breve{\mathsf{h}}(\omega)|^2 & + & |\breve{\mathsf{h}}(\omega+\pi)|^2 & = & 2 & \text{and}\\ b. & |\breve{\mathsf{g}}(\omega)|^2 & + & |\breve{\mathsf{g}}(\omega+\pi)|^2 & = & 2 & \text{and}\\ c. & \breve{\mathsf{h}}(\omega)\breve{\mathsf{g}}^*(\omega) & + & \breve{\mathsf{h}}(\omega+\pi)\breve{\mathsf{g}}^*(\omega+\pi) & = & 0 & \end{array}\right\}}_{\text{ORTHONORMAL QUADRATURE CONDITIONS in “frequency”}}$$

✎PROOF:

1. Proof that (1,2,3) $\implies$ (A): by Lemma 5.1 page 71.

2. Proof that (1,2,3) $\implies$ (B):

$$\begin{aligned} \sum_{m\in\mathbb{Z}} g_m\mathsf{g}^*_{m-2n} &= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} \mathsf{g}^*_k\bar{\delta}_{k-m+2n} && \text{by definition of } \bar{\delta}_n && \text{(Definition L.12 page 278)}\\ &= \sum_{m\in\mathbb{Z}} g_m \sum_{k\in\mathbb{Z}} \mathsf{g}^*_k \left\langle\phi\,|\,\mathbf{T}^{2n-m+k}\phi\right\rangle && \text{by hypothesis (1)}\\ &= \langle\psi\,|\,\mathbf{T}^n\psi\rangle && \text{by \textit{quadrature conditions in “time”}} && \text{(Theorem 3.3 page 47)}\\ &= \bar{\delta}_n && \text{by hypothesis (2)} \end{aligned}$$

3. Proof that (1,2,3) $\implies$ (C):

$$\begin{aligned} \sum_{m\in\mathbb{Z}} h_m\mathsf{g}^*_{m-2n} &= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} \mathsf{g}^*_k\bar{\delta}_{k-m+2n} && \text{by definition of } \bar{\delta}_n && \text{(Definition L.12 page 278)}\\ &= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} \mathsf{g}^*_k \left\langle\phi\,|\,\mathbf{T}^{2n-m+k}\phi\right\rangle && \text{by hypothesis (1)}\\ &= \langle\phi\,|\,\mathbf{T}^n\psi\rangle && \text{by \textit{quadrature conditions in “time”}} && \text{(Theorem 3.3 page 47)}\\ &= 0 && \text{by hypothesis (3)} \end{aligned}$$

4. Proof that (A,B,C) $\iff$ (a,b,c): by Lemma 5.2 page 81.

⇰

Proposition 5.5. [9] *Let* $\Omega \triangleq \left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM (Definition 3.2 page 44). *Let* $\breve{\mathsf{v}}(\omega)$ *be a* 2π PERIODIC *and* MEASURABLE *function in* $L^2_{\mathbb{R}}$ (Definition E.1 page 185).

PRP

Let $V_0 \perp W_0$ *and* $\{\mathbf{T}^n\phi\}$ *be* ORTHONORMAL. *Then*

$$\underbrace{\mathsf{f}\in W_0}_{(A)} \iff \left\{\begin{array}{ll} (1). & \mathsf{f}\in V_1 \qquad\qquad \text{and}\\ (2). & \text{There exists a MEASURABLE function } \breve{\mathsf{v}}(\omega)\in L^2_{\mathbb{R}} \text{ such that}\\ & \breve{\mathsf{v}}(\omega) = \breve{\mathsf{v}}(\omega+2\pi) \;(2\pi \text{ PERIODIC}) \text{ and}\\ & \tilde{\mathsf{f}}(\omega) = \frac{\sqrt{2}}{2}e^{i\frac{\omega}{2}}\breve{\mathsf{v}}(\omega)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}\right) \end{array}\right\}$$

[9] Wojtaszczyk (1997) pages 32–36 ⟨Proposition 2.18⟩, Hernández and Weiss (1996) pages 55–57 ⟨Lemma 2.11⟩

✎PROOF:

1. Proof that (A) $\Longrightarrow$ (1):

$$\begin{aligned} \mathsf{f} \in \boldsymbol{W}_0 &\Longrightarrow \mathsf{f} \in \boldsymbol{W}_0 \hat{+} \boldsymbol{V}_0 && \text{by definition of } \textit{Minkowski addition } \hat{+} && \text{(Definition U.2 page 389)} \\ &\Longleftrightarrow \mathsf{f} \in \boldsymbol{V}_1 && \text{because } \boldsymbol{V}_1 = \boldsymbol{V}_0 \hat{+} \boldsymbol{W}_0 && \text{(Definition 3.1 page 43)} \end{aligned}$$

2. Proof that (A) $\Longrightarrow$ (2):

(a) There exists a 2π *periodic* function $\tilde{a}(\omega)$ such that $\tilde{f}(\omega) = \tilde{a}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$. Proof:

$$\begin{aligned} \tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}\mathsf{f}(x) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)} \\ &= \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} a_n \mathbf{D}\mathbf{T}^n \phi && \text{by (A), } \mathsf{f} \in \boldsymbol{W}_0 \subseteq \boldsymbol{V}_1 \\ &= \sum_{n\in\mathbb{Z}} a_n \tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n \phi \\ &= \sum_{n\in\mathbb{Z}} a_n e^{i\frac{\omega}{2}} \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by Proposition 1.12 page 9} \\ &= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \textit{DTFT} && \text{(Definition P.1 page 355)} \end{aligned}$$

(b) $\tilde{a}(\omega)\breve{h}^*(\omega) + \tilde{a}(\omega+\pi)\breve{h}^*(\omega+\pi) = 0$. Proof:

$$\begin{aligned} 0 &= \mathsf{S}_{\mathsf{f}\phi}(\omega) && \text{by } \boldsymbol{V}_0 \perp \boldsymbol{W}_0 \text{ hypothesis and Theorem R.3 page 379} \\ &= \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\phi}^*(\omega+2\pi n) && \text{by Theorem R.1 page 373} \\ &= \sum_{n\in\mathbb{Z}} \underbrace{\tilde{\mathsf{a}}\left(\frac{\omega+2\pi n}{2}\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)}_{\tilde{\mathsf{f}}(\omega+2\pi n)} \underbrace{\breve{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}\right)\tilde{\phi}^*\left(\frac{\omega+2\pi n}{2}\right)}_{\tilde{\phi}^*(\omega+2\pi n)} && \text{by item (2a) page 83} \\ &= \underbrace{\sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}+2\pi n\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\right|^2}_{\text{even terms}} \\ &\quad + \underbrace{\sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}+2\pi n+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\right|^2}_{\text{odd terms}} \\ &= \sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\right|^2 + \sum_{n\in\mathbb{Z}} \tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\left|\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\right|^2 \\ &= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}\right) + \tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right) \\ &= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}\right) + \tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right) && \text{by } \textit{orthonormality} \text{ and Theorem R.3 page 379} \\ &\Longrightarrow \tilde{\mathsf{a}}(\omega)\breve{\mathsf{h}}^*(\omega) + \tilde{\mathsf{a}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega+\pi) = 0 \end{aligned}$$

(c) The vectors $[\tilde{a}(\omega), \tilde{a}(\omega+\pi)]$ and $\left[\breve{h}(\omega), \breve{h}(\omega+\pi)\right]$ are *orthogonal*. Proof:

$$\begin{aligned} \begin{bmatrix} \breve{\mathsf{h}}(\omega) \\ \breve{\mathsf{h}}(\omega+\pi) \end{bmatrix}^H \begin{bmatrix} \tilde{\mathsf{a}}(\omega) \\ \tilde{\mathsf{a}}(\omega+\pi) \end{bmatrix} &= \begin{bmatrix} \breve{\mathsf{h}}^*(\omega) & \breve{\mathsf{h}}^*(\omega+\pi) \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{a}}(\omega) \\ \tilde{\mathsf{a}}(\omega+\pi) \end{bmatrix} \\ &= \tilde{\mathsf{a}}(\omega)\breve{\mathsf{h}}^*(\omega) + \tilde{\mathsf{a}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega+\pi) \\ &= 0 && \text{by item (2b) page 83} \end{aligned}$$

(d) There exists a 2π *periodic* function $\tilde{\mathsf{b}}(\omega)$ such that $[\tilde{\mathsf{a}}(\omega), \tilde{\mathsf{a}}(\omega+\pi)] = \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega+\pi), -\breve{\mathsf{h}}^*(\omega)\right]$. Proof: Because the two vectors are *orthogonal* (item (2c) page 83), if one vector is rotated by 90°, then for fixed ω it is a scalar multiple of the other…

$$\begin{aligned}
[\tilde{\mathsf{a}}(\omega), \tilde{\mathsf{a}}(\omega+\pi)] &= \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega), \breve{\mathsf{h}}^*(\omega+\pi)\right]\underbrace{\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix}}_{\text{rotation matrix}} && \text{by item (2c) page 83}\\
&= \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega), \breve{\mathsf{h}}^*(\omega+\pi)\right]\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\\
&= \tilde{\mathsf{b}}(\omega)\left[\breve{\mathsf{h}}^*(\omega+\pi), -\breve{\mathsf{h}}^*(\omega)\right]
\end{aligned}$$

(e) By item (2d), $[\tilde{\mathsf{a}}(\omega+\pi), \tilde{\mathsf{a}}(\omega)] = \tilde{\mathsf{b}}(\omega+\pi)\left[\breve{\mathsf{h}}^*(\omega), -\breve{\mathsf{h}}^*(\omega+\pi)\right]$. Proof:

$$\begin{aligned}
[\tilde{\mathsf{a}}(\omega+\pi), \tilde{\mathsf{a}}(\omega)] &= [\tilde{\mathsf{a}}(\omega+\pi), \tilde{\mathsf{a}}(\omega+\pi+\pi)]\\
&= \tilde{\mathsf{b}}(\omega+\pi)\left[\breve{\mathsf{h}}^*(\omega+\pi+\pi), -\breve{\mathsf{h}}^*(\omega+\pi)\right] && \text{by item (2d) page 84}\\
&= \tilde{\mathsf{b}}(\omega+\pi)\left[\breve{\mathsf{h}}^*(\omega), -\breve{\mathsf{h}}^*(\omega+\pi)\right]
\end{aligned}$$

(f) By item (2d) and item (2e), $\tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi)$. Proof:

$$\underbrace{\begin{array}{lcll}
\tilde{\mathsf{a}}(\omega) & = & \tilde{\mathsf{b}}(\omega)\breve{\mathsf{h}}^*(\omega+\pi) & \text{by item (2d)}\\
\tilde{\mathsf{a}}(\omega) & = & -\tilde{\mathsf{b}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega+\pi) & \text{by item (2e)}
\end{array}}_{\Downarrow\\ \tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi)}
\qquad
\underbrace{\begin{array}{lcll}
\tilde{\mathsf{a}}(\omega+\pi) & = & -\tilde{\mathsf{b}}(\omega)\breve{\mathsf{h}}^*(\omega) & \text{by item (2d)}\\
\tilde{\mathsf{a}}(\omega+\pi) & = & \tilde{\mathsf{b}}(\omega+\pi)\breve{\mathsf{h}}^*(\omega) & \text{by item (2e)}
\end{array}}_{\Downarrow\\ \tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi)}$$

(g) Note that $\tilde{\mathsf{b}}(\omega) = -\tilde{\mathsf{b}}(\omega+\pi) \iff e^{-i(\omega+\pi)}\tilde{\mathsf{b}}(\omega+\pi) = e^{-i(\omega)}\tilde{\mathsf{b}}(\omega)$. Proof:

$$\begin{aligned}
-\tilde{\mathsf{b}}(\omega+\pi) &= -e^{i(\omega+\pi)}e^{-i(\omega+\pi)}\tilde{\mathsf{b}}(\omega+\pi)\\
&= -e^{i(\omega+\pi)}e^{-i\omega}\tilde{\mathsf{b}}(\omega) && \text{by right hypothesis}\\
&= e^{-i\pi}e^{i(\omega+\pi)}e^{-i\omega}\tilde{\mathsf{b}}(\omega)\\
&= e^{i\omega}e^{-i\omega}\tilde{\mathsf{b}}(\omega)\\
&= \tilde{\mathsf{b}}(\omega)
\end{aligned}$$

$$\begin{aligned}
e^{-i(\omega+\pi)}\tilde{\mathsf{b}}(\omega+\pi) &= -e^{-i\omega}\tilde{\mathsf{b}}(\omega+\pi)\\
&= e^{-i\omega}\tilde{\mathsf{b}}(\omega) && \text{by left hypothesis}
\end{aligned}$$

(h) Let $\tilde{\mathsf{v}}(\omega) \triangleq \sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right)$. Then (2) is true …

i. $\tilde{\mathsf{v}}(\omega)$ is 2π *periodic*. Proof:

$$\begin{aligned}
\tilde{\mathsf{v}}(\omega+2\pi) &\triangleq \sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right) && \text{by definition of } \tilde{\mathsf{v}}(\omega)\\
&= \sqrt{2}e^{-i\frac{\omega+2\pi}{2}}\tilde{\mathsf{b}}\left(\frac{\omega+2\pi}{2}\right)\\
&= \sqrt{2}e^{-i\frac{\omega}{2}+\pi}\tilde{\mathsf{b}}\left(\frac{\omega}{2}+\pi\right)\\
&= -\sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}+\pi\right)\\
&= -\sqrt{2}e^{-i\frac{\omega}{2}}\left[-\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right)\right] && \text{by item (2f) page 84}\\
&= \sqrt{2}e^{-i\frac{\omega}{2}}\tilde{\mathsf{b}}\left(\frac{\omega}{2}\right) && \text{by item (2f) page 84}\\
&\triangleq \tilde{\mathsf{v}}(\omega) && \text{by definition of } \tilde{\mathsf{v}}(\omega)
\end{aligned}$$

ii. $\tilde{\mathsf{f}}(\omega) = \frac{\sqrt{2}}{2} e^{i\frac{\omega}{2}} \tilde{\mathsf{v}}(\omega) \breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right) \tilde{\phi}\left(\frac{\omega}{2}\right)$. Proof:

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by item (2a) page 83} \\
&= \tilde{\mathsf{b}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by item (2d) page 84} \\
&= \frac{\sqrt{2}}{2} e^{i\frac{\omega}{2}} \tilde{\mathsf{v}}(\omega) \breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right) \tilde{\phi}\left(\frac{\omega}{2}\right) && \text{by definition of } \tilde{\mathsf{v}}(\omega)
\end{aligned}$$

3. lemma: proof that (2) $\Longrightarrow$ $\mathsf{f} \perp \boldsymbol{V}_0$:

$$\begin{aligned}
&\mathsf{S}_{\mathsf{f}\phi}(\omega) \\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\phi}(\omega+2\pi n)\tilde{\mathsf{f}}^*(\omega+2\pi n) && \text{by Theorem R.1 page 373} \\
&= 2\pi \sum_{n\in\mathbb{Z}} \tilde{\phi}(\omega+2\pi n)\left[\frac{\sqrt{2}}{2} e^{i\left(\frac{\omega+2\pi n}{2}\right)} \tilde{\mathsf{v}}(\omega+2\pi n)\breve{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right]^* && \text{by (A)} \\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega) \sum_{n\in\mathbb{Z}} \tilde{\phi}(\omega+2\pi n)\left[e^{-i\left(\frac{\omega}{2}+\pi n\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right] && \text{by (C)} \\
&= \underbrace{\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega) \sum_{n\in\mathbb{Z}_e} \tilde{\phi}(\omega+2\pi n)\left[e^{-i\left(\frac{\omega}{2}+\pi n\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right]}_{\text{even terms}} \\
&\quad + \underbrace{\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega) \sum_{n\in\mathbb{Z}_o} \tilde{\phi}(\omega+2\pi n)\left[e^{-i\left(\frac{\omega}{2}+\pi n\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right]}_{\text{odd terms}} \\
&= \underbrace{\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega) \sum_{n\in\mathbb{Z}} \tilde{\phi}(\omega+4\pi n)\left[e^{-i\left(\frac{\omega}{2}+2\pi n\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right)\right]}_{\text{even terms}} \\
&\quad + \underbrace{\sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega) \sum_{n\in\mathbb{Z}} \tilde{\phi}(\omega+4\pi n+2\pi)\left[e^{-i\left(\frac{\omega}{2}+2\pi n+\pi\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n+2\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\right]}_{\text{odd terms}} \\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)e^{-i\left(\frac{\omega}{2}\right)} \sum_{n\in\mathbb{Z}} \tilde{\phi}(\omega+4\pi n)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right) \\
&\quad - \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)e^{-i\left(\frac{\omega}{2}\right)} \sum_{n\in\mathbb{Z}} \tilde{\phi}(\omega+4\pi n+2\pi)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right) && \text{by Proposition P.1 page 355} \\
&= \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)e^{-i\left(\frac{\omega}{2}\right)} \sum_{n\in\mathbb{Z}} \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n\right) \\
&\quad - \sqrt{2}\pi\tilde{\mathsf{v}}^*(\omega)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)e^{-i\left(\frac{\omega}{2}\right)} \sum_{n\in\mathbb{Z}} \frac{\sqrt{2}}{2}\breve{\mathsf{h}}\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}^*\left(\frac{\omega}{2}+2\pi n+\pi\right) \\
&= \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2\right] \\
&\quad - \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+\pi+2\pi n\right)\right|^2\right] && \text{by Proposition P.1 page 355} \\
&= \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}\right) - \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\tilde{\mathsf{S}}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right) \\
&= \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right) - \frac{1}{4}\tilde{\mathsf{v}}^*(\omega)e^{-i\left(\frac{\omega}{2}\right)}\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}+\pi\right) && \text{by Theorem R.3 page 379}
\end{aligned}$$

$= 0$ by Lemma 5.2 page 81

$\iff \langle \mathsf{f} \,|\, \mathbf{T}^n\phi\rangle = \bar{\delta}_n$ by Theorem R.3 page 379

$\iff \mathsf{f} \perp \operatorname{span}\{\mathbf{T}^n\phi\}$

$\iff \mathsf{f} \perp V_0$

4. Proof that (A) $\Longleftarrow$ (1,2):
   (a) by (1), $\mathsf{f} \in V_1 = V_0 \dotplus W_0$
   (b) by hypothesis, $V_0 \perp W_0$
   (c) by item (3) page 85, $\mathsf{f} \perp V_0$
   (d) so $\mathsf{f} \in W_0$

✏

Theorem 5.4. [10] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM (Definition 3.2 page 44). *Let* $\tilde{\mathsf{v}}(\omega)$ *be a measurable function in* $L^2_{\mathbb{R}}$ (Definition E.1 page 185).

THM

If $\{\phi|\mathbf{T}^n\phi\} = \bar{\delta}_n$ *then*

$$\left\{\begin{array}{lll} (A). & \tilde{\psi}(\omega) = \underbrace{\frac{\sqrt{2}}{2}e^{i\frac{\omega}{2}}\tilde{\mathsf{v}}(\omega)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)}_{\tilde{\mathsf{g}}\left(\frac{\omega}{2}\right) \text{ (Definition 3.2 page 44)}}\tilde{\phi}\left(\frac{\omega}{2}\right) & \text{and} \\ (B). & \tilde{\mathsf{v}}(\omega) = \tilde{\mathsf{v}}(\omega+2\pi) & \text{and} \\ (C). & |\tilde{\mathsf{v}}(\omega)| = 1 & \end{array}\right\} \iff \underbrace{\left\{\begin{array}{llll} (1). & \langle\psi \,|\, \mathbf{T}^n\psi\rangle = \bar{\delta}_n & \text{and} \\ (2). & \langle\phi \,|\, \mathbf{T}^n\psi\rangle = 0 & \text{and} \\ (3). & \operatorname{span}\{\mathbf{T}^n\psi\} = W_0 & \end{array}\right\}}_{\substack{V_0 \perp W_0 \text{ and} \\ \psi \text{ is an ORTHONORMAL WAVELET for } L^2_{\mathbb{R}}}}$$

✎PROOF:

1. lemma: proof that (A) and $\{\mathbf{T}^n\phi\} = \bar{\delta}_n \implies \tilde{\mathsf{S}}_{\psi\psi}(\omega) = |\tilde{\mathsf{v}}(\omega)|^2$:

$$\begin{aligned}
&\tilde{\mathsf{S}}_{\psi\psi}(\omega) \\
&= 2\pi\sum_{n\in\mathbb{Z}}|\tilde{\psi}(\omega+2\pi n)|^2 && \text{by Theorem R.1 page 373} \\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\frac{\sqrt{2}}{2}e^{i\left(\frac{\omega+2\pi n}{2}\right)}\tilde{\mathsf{v}}(\omega+2\pi n)\breve{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right|^2 && \text{by (A)} \\
&= \pi\sum_{n\in\mathbb{Z}}\underbrace{\left|e^{i\left(\frac{\omega+2\pi n}{2}\right)}\right|^2}_{1}|\tilde{\mathsf{v}}(\omega+2\pi n)|^2\left|\breve{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right|^2 && \text{by property of } |\cdot| \\
&= \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}}\left|\breve{\mathsf{h}}^*\left(\frac{\omega+2\pi n}{2}+\pi\right)\tilde{\phi}\left(\frac{\omega+2\pi n}{2}\right)\right|^2 && \text{by (B) and (C)} \\
&= \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}_e}\underbrace{\left|\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+\pi n\right)\right|^2}_{\text{even terms}} + \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}_o}\underbrace{\left|\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+\pi n\right)\right|^2}_{\text{odd terms}} \\
&= \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}}\underbrace{\left|\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2}_{\text{even terms}} + \pi|\tilde{\mathsf{v}}(\omega)|^2\sum_{n\in\mathbb{Z}}\underbrace{\left|\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+2\pi n+\pi\right)\tilde{\phi}\left(\frac{\omega}{2}+2\pi n+\pi\right)\right|^2}_{\text{odd terms}}
\end{aligned}$$

[10] Wojtaszczyk (1997) pages 32–37 ⟨Section 2.4, Theorem 2.20⟩, Hernández and Weiss (1996) page 57 ⟨PROPOSITION 2.13⟩, MALLAT (1999) PAGES 236–238 ⟨THEOREM 7.3⟩, DAUBECHIES (1992) PAGE 133

$$= \frac{|\breve{v}(\omega)|^2}{2}\left\{\left|\breve{h}^*\left(\frac{\omega}{2}\right)\right|^2 \underbrace{\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2\right]}_{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}\right)} + \left|\breve{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2\underbrace{\left[2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}\left(\frac{\omega}{2}+\pi+2\pi n\right)\right|^2\right]}_{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)}\right\}$$

because *DTFT* is 2π *periodic* (Proposition P.1 page 355)

$$= \frac{|\breve{v}(\omega)|^2}{2}\left\{\left|\breve{h}^*\left(\frac{\omega}{2}\right)\right|^2 \tilde{S}_{\phi\phi}\left(\frac{\omega}{2}\right) + \left|\breve{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2 \tilde{S}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)\right\}$$

$$= \frac{|\breve{v}(\omega)|^2}{2}\left\{\left|\breve{h}^*\left(\frac{\omega}{2}\right)\right|^2 \cancelto{1}{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}\right)} + \left|\breve{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2 \cancelto{1}{\tilde{S}_{\phi\phi}\left(\frac{\omega}{2}+\pi\right)}\right\}$$ by Theorem R.3 page 379

$$= |\breve{v}(\omega)|^2$$ by Lemma 5.2 page 81

2. Proof that (A,B,C) $\Longrightarrow$ (1):

$$\tilde{S}_{\psi\psi}(\omega) = |\breve{v}(\omega)|^2$$ by item (1) page 86

$$= 1$$ by (B)

$$\iff \langle \psi \,|\, \mathbf{T}^n\psi\rangle = \bar{\delta}_n$$ by Theorem R.3 page 379

3. Proof that (A,B) $\Longrightarrow$ (2): by Proposition 5.5 page 82 with $\tilde{f}(\omega) \triangleq \tilde{\psi}(\omega)$.

4. lemma: Proof that $\{\mathbf{T}^n\phi\} = \bar{\delta}_n$ and (A,B,C) $\Longrightarrow$ for all $(\!(a_n)\!)$, there exists $(\!(b_n)\!)$ and $(\!(c_n)\!)$ such that $\tilde{a}\left(\frac{\omega}{2}\right) = \tilde{b}(\omega)\breve{h}\left(\frac{\omega}{2}\right) + \tilde{c}(\omega)\breve{g}\left(\frac{\omega}{2}\right)$

(a) First note the following relations hold in this case:

$$\begin{array}{rcrcll}
\left|\breve{h}(\omega)\right|^2 & + & \left|\breve{h}(\omega+\pi)\right|^2 & = & 2 & \text{by } \{\mathbf{T}^n\phi\} = \bar{\delta}_n \text{ and by Lemma 5.2 page 81}\\
|\breve{g}(\omega)|^2 & + & |\breve{g}(\omega+\pi)|^2 & = & 2 & \text{by (1) and by Lemma 5.2 page 81}\\
\breve{h}(\omega)\breve{g}^*(\omega) & + & \breve{h}(\omega+\pi)\breve{g}^*(\omega+\pi) & = & 0 & \text{by (2) and by Lemma 5.2 page 81}\\
\breve{g}\left(\frac{\omega}{2}\right) = e^{i\frac{\omega}{2}}\breve{v}(\omega)\breve{h}^*\left(\frac{\omega}{2}+\pi\right) & & & & & \text{by (A)}
\end{array}$$

(b) Then, the following values of $(\!(b_n)\!)$ and $(\!(c_n)\!)$ are solutions for all $(\!(a_n)\!)$

$$\begin{array}{rcl}
\mathsf{b}(\omega) & = & \frac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\breve{h}^*\left(\frac{\omega}{2}\right) + \frac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)\breve{h}^*\left(\frac{\omega}{2}+\pi\right)\\
\mathsf{c}(\omega) & = & \frac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\breve{g}^*\left(\frac{\omega}{2}\right) + \frac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)\breve{g}^*\left(\frac{\omega}{2}+\pi\right)
\end{array}$$

Proof:

$$\tilde{b}(\omega)\breve{h}\left(\frac{\omega}{2}\right) + \tilde{c}(\omega)\breve{g}\left(\frac{\omega}{2}\right)$$

$$= \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\left|\breve{h}\left(\frac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)\breve{h}\left(\frac{\omega}{2}\right)\breve{h}^*\left(\frac{\omega}{2}+\pi\right)$$

$$+ \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\left|\breve{g}\left(\frac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)\breve{g}\left(\frac{\omega}{2}\right)\breve{g}^*\left(\frac{\omega}{2}+\pi\right)$$ by item (4b) page 87

$$= \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\left|\breve{h}\left(\frac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)\breve{h}\left(\frac{\omega}{2}\right)\breve{h}^*\left(\frac{\omega}{2}+\pi\right)$$

$$+ \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\left|e^{i\frac{\omega}{2}}\breve{v}(\omega)\breve{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)e^{i\frac{\omega}{2}}\breve{v}(\omega)\breve{h}^*\left(\frac{\omega}{2}+\pi\right)e^{-i\left(\frac{\omega}{2}+\pi\right)}\breve{v}^*(\omega+2\pi)\breve{h}\left(\frac{\omega}{2}+2\pi\right)$$

$$= \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\left|\breve{h}\left(\frac{\omega}{2}\right)\right|^2 + \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)\breve{h}\left(\frac{\omega}{2}\right)\breve{h}^*\left(\frac{\omega}{2}+\pi\right)$$

$$+ \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\left|\breve{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2 - \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}+\pi\right)\breve{v}(\omega)\breve{h}^*\left(\frac{\omega}{2}+\pi\right)|\breve{v}(\omega)|^2\breve{h}\left(\frac{\omega}{2}\right)$$

$$= \tfrac{1}{2}\tilde{a}\left(\frac{\omega}{2}\right)\left[\left|\breve{h}\left(\frac{\omega}{2}\right)\right|^2 + \left|\breve{h}^*\left(\frac{\omega}{2}+\pi\right)\right|^2\right]$$

$$+\tfrac{1}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)\left[\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)-\breve{\mathsf{h}}^*\left(\frac{\omega}{2}+\pi\right)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\right]$$
$$=\tfrac{1}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)[2]+\tfrac{1}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}+\pi\right)[0] \qquad \text{by Lemma 5.2 page 81}$$
$$=\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)$$

5. Proof that (A,B,C) $\Longrightarrow$ (3):

$$\operatorname{span}\{\mathbf{T}^n\psi\}=W_0$$
$$\iff V_1=V_0\dotplus\underbrace{\operatorname{span}\{\mathbf{T}^n\psi\}}_{W_0} \qquad \text{by Definition 3.1 page 43}$$
$$\iff \underbrace{\operatorname{span}\{\mathbf{D}\mathbf{T}^n\phi\}}_{V_1}=\underbrace{\operatorname{span}\{\mathbf{T}^n\phi\}}_{V_0}\dotplus\underbrace{\operatorname{span}\{\mathbf{T}^n\psi\}}_{W_0} \qquad \text{by Definition 2.1 page 16}$$
$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \sum_{n\in\mathbb{Z}}a_n\mathbf{D}\mathbf{T}^n\phi=\sum_{n\in\mathbb{Z}}b_n\mathbf{T}^n\phi+\sum_{n\in\mathbb{Z}}c_n\mathbf{T}^n\psi \qquad \text{by Definition L.2 page 267}$$
$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \tilde{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}a_n\mathbf{D}\mathbf{T}^n\phi\right]=\tilde{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}b_n\mathbf{T}^n\phi\right]+\tilde{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}c_n\mathbf{T}^n\psi\right] \qquad \text{by Definition K.2 page 257}$$
$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \sum_{n\in\mathbb{Z}}a_n\tilde{\mathbf{F}}\mathbf{D}\mathbf{T}^n\phi=\sum_{n\in\mathbb{Z}}b_n\tilde{\mathbf{F}}\mathbf{T}^n\phi+\sum_{n\in\mathbb{Z}}c_n\tilde{\mathbf{F}}\mathbf{T}^n\psi$$
$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \sum_{n\in\mathbb{Z}}a_n\left[\frac{\sqrt{2}}{2}e^{-i\frac{\omega}{2}n}\tilde{\phi}\left(\frac{\omega}{2}\right)\right]=\sum_{n\in\mathbb{Z}}b_ne^{-i\omega n}\tilde{\phi}(\omega)+\sum_{n\in\mathbb{Z}}c_ne^{-i\omega n}\tilde{\psi}(\omega)$$

by Corollary 1.1 page 9

$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \frac{\sqrt{2}}{2}\tilde{\phi}\left(\frac{\omega}{2}\right)\sum_{n\in\mathbb{Z}}a_ne^{-i\frac{\omega}{2}n}=\tilde{\phi}(\omega)\sum_{n\in\mathbb{Z}}b_ne^{-i\omega n}+\tilde{\psi}(\omega)\sum_{n\in\mathbb{Z}}c_ne^{-i\omega n}$$
$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \frac{\sqrt{2}}{2}\tilde{\phi}\left(\frac{\omega}{2}\right)\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)=\tilde{\phi}(\omega)\tilde{\mathsf{b}}(\omega)+\tilde{\psi}(\omega)\tilde{\mathsf{c}}(\omega) \qquad \text{by def. of } \textit{DTFT} \text{ (Definition P.1 page 355)}$$
$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \text{(by Lemma 2.1 page 22)}$$
$$\frac{\sqrt{2}}{2}\tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)=\frac{\sqrt{2}}{2}\tilde{\mathsf{b}}(\omega)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)+\frac{\sqrt{2}}{2}\tilde{\mathsf{c}}(\omega)\breve{\mathsf{g}}\left(\frac{\omega}{2}\right)\tilde{\phi}\left(\frac{\omega}{2}\right)$$
$$\iff \overset{\forall(a_n)\exists(b_n),(c_n)}{\text{such that}}\quad \tilde{\mathsf{a}}\left(\frac{\omega}{2}\right)=\tilde{\mathsf{b}}(\omega)\breve{\mathsf{h}}\left(\frac{\omega}{2}\right)+\tilde{\mathsf{c}}(\omega)\breve{\mathsf{g}}\left(\frac{\omega}{2}\right)$$
$$\Longrightarrow (3) \qquad \text{by item (4) page 87}$$

6. Proof that (A,B) $\Longleftarrow$ (3):

$$\operatorname{span}\{\mathbf{T}^n\psi\}=W_0 \implies \psi\in W_0$$
$$\implies \text{(A) and (C)} \qquad \text{by Proposition 5.5 page 82}$$

7. Proof that (C) $\Longleftarrow$ (3) and $\{\mathbf{T}^n\phi\}=\bar{\delta}_n$:

$$1=\tilde{\mathsf{S}}_{\psi\psi}(\omega) \qquad \text{by right hypothesis and Theorem R.3 page 379}$$
$$=|\tilde{\mathsf{v}}(\omega)|^2 \qquad \text{by item (1) page 86}$$

5.3.4 Sufficient conditions

If we know the *scaling coefficient sequence* $(\!(h_n)\!)$ (Definition 2.3 page 25), then Theorem 5.4 page 86 gives a necessary and sufficient condition for the *wavelet coefficient sequence* $(\!(g_n)\!)$ (Definition 3.2 page 44) to generate a *wavelet system* (Definition 3.2 page 44). The actual value of $(\!(g_n)\!)$ depends on $(\!(h_n)\!)$ and the choice of $\tilde{v}(\omega)$ (Theorem 5.4 page 86). Here are some possible choices:[11]

1.	$\tilde{v}(\omega) \triangleq 1$	$\implies g_n = (-1)^n h^*_{-1-n}$	
2.	$\tilde{v}(\omega) \triangleq e^{i\omega}$	$\implies g_n = (-1)^n h^*_{-3-n}$	
3.	$\tilde{v}(\omega) \triangleq e^{-i\omega}$	$\implies g_n = (-1)^n h^*_{1-n}$	*conjugate mirror filter* (CMF)
4.	$\tilde{v}(\omega) \triangleq \pm(-1)^{N+1} e^{-i\frac{\omega}{2}(N+1)}$	$\implies g_n = \pm(-1)^n h^*_{N-n}$	*conjugate quadrature filter* (CQF), $N \in \mathbb{Z}$, (Definition O.9 page 349)

Theorem 5.5. [12] *Let* $\left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a* WAVELET SYSTEM *(Definition 5.1 page 67).*

THM

$$\left\{\begin{array}{lll} \text{(A).} & \underbrace{\langle \phi \,|\, \mathbf{T}^n\phi\rangle = \bar{\delta}_n}_{\{\mathbf{T}^n\phi\}\text{ is ORTHONORMAL}} & \text{and} \\ \text{(B).} & g_n = \underbrace{\pm(-1)^n h^*_{N-n}}_{\text{CQF condition}} & \forall n\in\mathbb{Z} \end{array}\right\} \implies \left\{\begin{array}{llll} \text{(1).} & \langle \phi \,|\, \mathbf{T}^n\psi\rangle & = 0 & \text{and} \\ \text{(2).} & \langle \psi \,|\, \mathbf{T}^n\psi\rangle & = \bar{\delta}_n & \text{and} \\ \text{(3).} & \operatorname{span}\{\mathbf{T}^n\psi \,|\, n\in\mathbb{Z}\} & = W_0 & \end{array}\right\} \forall n\in\mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
g_n = \pm(-1)^n h^*_{N-n} &\iff \tilde{g}(\omega) = \pm(-1)^N e^{-i\omega N}\breve{h}^*(\omega+\pi) && \text{by } \textit{CQF theorem} \text{ (Theorem O.5 page 349)}\\
&\implies \tilde{\psi}(2\omega) = \frac{\sqrt{2}}{2}\underbrace{\pm(-1)^N e^{-i\omega N}\breve{h}^*(\omega+\pi)}_{\tilde{g}(\omega)}\tilde{\phi}(\omega) && \text{by Proposition 3.2 page 48}\\
&\implies \tilde{\psi}(2\omega) = \frac{\sqrt{2}}{2}\underbrace{\left[\pm(-1)^N e^{-i\omega(N+1)}\right]}_{v(2\omega)} e^{i\omega}\breve{h}^*(\omega+\pi)\,\tilde{\phi}(\omega)\\
&\implies \left\{\begin{array}{llll} \text{(1).} & \langle \phi \,|\, \mathbf{T}^n\psi\rangle & = 0 & \text{and} \\ \text{(2).} & \langle \psi \,|\, \mathbf{T}^n\psi\rangle & = \bar{\delta}_n & \text{and} \\ \text{(3).} & \operatorname{span}\{\mathbf{T}^n\psi \,|\, n\in\mathbb{Z}\} & = W_0 & \end{array}\right\} && \text{by Theorem 5.4 page 86}
\end{aligned}$$

✏

Theorem 5.6. *Let* $\Omega \triangleq \left(L^2_{\mathbb{R}}, (\!(V_j)\!), (\!(W_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be a tuple.*

THM

$$\left\{\begin{array}{llll} \text{(A).} & \sum_n h_n = \sqrt{2} & \text{[ADMISSIBILITY (Theorem 2.3 page 27)]} & \text{and} \\ \text{(B).} & \sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n & \text{[ORTHO. QUADRATURE CONDITION (Lemma 5.1 page 71)]} & \text{and} \\ \text{(C).} & \sum_{k\in\mathbb{Z}} (-1)^k k^n h_k = 0 \quad \forall n=0,1,\dots,p-1 & \text{[}p\text{ VANISHING MOMENTS (Theorem 4.3 page 58)]} & \text{and} \\ \text{(D).} & \inf_{\omega\in[-\pi/2:\pi/2]} \left|\breve{h}(\omega)\right| > 0 & \text{[LOW-PASS RESPONSE]} & \text{and} \\ \text{(E).} & g_n = (-1)^n h^*_{N-n} & \text{[CQF CONDITION (Definition O.9 page 349)]} & \end{array}\right\}$$
$$\implies \left\{\begin{array}{ll} \text{(1).} & \Omega \text{ is an ORTHONORMAL WAVELET SYSTEM} \quad \text{and} \\ \text{(2).} & \psi \text{ has } p \text{ VANISHING MOMENTS} \end{array}\right\}$$

[11] Strang and Nguyen (1996) page 109 ⟨"A Brief History of h_1"⟩, Vidakovic (1999) page 59
[12] Mallat (1999) pages 236–238

✎PROOF:

1. lemma: $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ is an *orthonormal MRA system*. Proof:
   (a) If $p = 1$, then this is the *Haar MRA system*.
   (b) If $p \geq 2$, then proof follows from Theorem 5.2 page 74:
      i. Proof that $\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}$: by left hypothesis (A)
      ii. Proof that $\sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n$: by left hypothesis (B)
      iii. Proof that $\tilde{\phi}(\omega)$ is *continuous* at 0:

$$\begin{aligned}
(C) &\implies \sum_{k\in\mathbb{Z}} (-1)^k k^n \mathsf{h}_k = 0 \qquad \forall n=0,1,\ldots,p-1,\, p\geq 2 \\
&\implies \psi \text{ has } p \text{ vanishing moments} && \text{by Theorem 4.3 page 58} \\
&\implies \tilde{\phi}(\omega) \text{ is continuous at } \omega = 0 && \text{by Theorem 4.5 page 64}
\end{aligned}$$

      iv. Proof that $\inf_{\omega\in[-\pi/2:\pi/2]} \left|\breve{\mathsf{h}}(\omega)\right| > 0$: by left hypothesis (D)

2. Proof for (1): by (1) lemma, (E), and Theorem 5.5 page 89.

3. Proof for (2): by (C) and Theorem 4.3 page 58.

✏

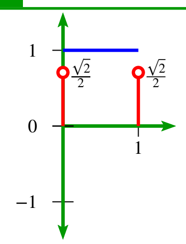
5.4 Examples

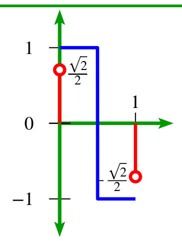
The scaling coefficients can be directly computed using simultaneous equations generated from the following necessary conditions.[13] They can be solved using a symbolic equation software package[14] or by hand.

Example 5.3 (2 coefficient case—Haar wavelet system). Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ be an *orthogonal* wavelet system with two non-zero scaling coefficients.

EX

$$\left\{\begin{array}{lll}
1.\ \operatorname{supp}\phi(x) = [0:1] & \text{(Theorem 3.6 page 51)} & \text{and} \\
2.\ \textit{admissibility condition} & \text{(Theorem 2.3 page 27)} & \text{and} \\
3.\ \{\mathbf{T}^n\phi\} \text{ is } \textit{orthonormal} & \text{(Lemma 5.2 page 81)} & \text{and} \\
4.\ \mathsf{g}_n = \pm(-1)^n \mathsf{h}^*_{N-n}\ \forall n\in\mathbb{Z} & \text{(Theorem 5.5 page 89)} &
\end{array}\right\} \implies \left\{\begin{array}{r|c|c}
n & \mathsf{h}_n & \mathsf{g}_n \\ \hline
0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\text{other} & 0 & 0
\end{array}\right\}$$





13 Vidakovic (1999), pages 92–93

14
Maxima™	`http://maxima.sourceforge.net/`	free and open source
Macsyma™	`http://www.scientek.com/macsyma/mxmain.htm`	a very non-free relative of *Maxima*
MuPAD™	`http://www.mupad.de/`	30 day free trial
Maple™	`http://www.maplesoft.com/`	
Mathematica™	`http://www.wolfram.com/`	

✎PROOF:

1. Proof that (1) $\implies$ that only h_0 and h_1 are non-zero: by Theorem 3.6 page 51.
2. Proof for values of h_0 and h_1:

$$\begin{aligned}
\mathsf{h}_0 + \mathsf{h}_1 &= \sqrt{2} && \text{by admissibility condition—Theorem 2.3 page 27}\\
\mathsf{h}_0^2 + \mathsf{h}_1^2 &= 1 && \text{orthogonal quadrature condition—Lemma 5.2 page 81}
\end{aligned}$$

$$\begin{aligned}
\mathsf{h}_0^2 + \left(\sqrt{2} - h_0\right)^2 &= 1\\
2h_0^2 - 2\sqrt{2}h_0 + 1 &= 0
\end{aligned}$$

$$\begin{aligned}
\mathsf{h}_0 &= \frac{2\sqrt{2} \pm \sqrt{\left(2\sqrt{2}\right)^2 - 4\cdot 2\cdot 1}}{2\cdot 2}\\
&= \tfrac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
\mathsf{h}_1 &= \sqrt{2} - \mathsf{h}_0\\
&= \sqrt{2} - \tfrac{\sqrt{2}}{2}\\
&= \tfrac{\sqrt{2}}{2}
\end{aligned}$$

3. Note: h_0 and h_1 are the *Haar scaling coefficients*. They can also be produced using other systems of equations including the following:
   (a) Admissibility condition and *partition of unity*—Example 2.3 (page 40)
   (b) *Daubechies-p*1 wavelets computed using spectral techniques—Example 6.3 (page 109)
4. Proof for values of g_0 and g_1: by (4) and Theorem 5.5 page 89.

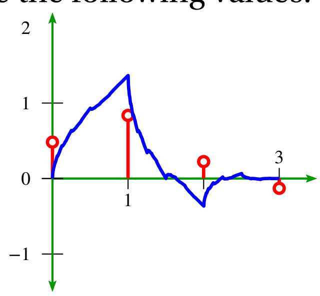
✏

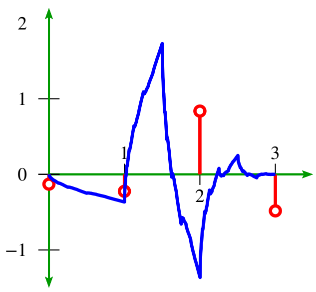
Example 5.4. [15] In this example, we constrain the system to have $p = 2$ vanishing moments. This same wavelet system is identical to the *Daubechies-p*2 wavelet system which is computed using a spectral factorization technique—Example 6.4 (page 110). Here are the equations:

$$\begin{array}{rcrcrcrcll}
\mathsf{h}_0 & + & \mathsf{h}_1 & + & \mathsf{h}_2 & + & \mathsf{h}_3 & = & \sqrt{2} & \text{(admissibility—Theorem 2.3 page 27)}\\
\mathsf{h}_0 & - & \mathsf{h}_1 & + & \mathsf{h}_2 & - & \mathsf{h}_3 & = & 0 & \text{(vanishing 0th moment/partition of unity—page 34)}\\
 & - & \mathsf{h}_1 & + & 2h_2 & - & 3h_3 & = & 0 & \text{(vanishing 1st moment—Theorem 4.3 page 58)}\\
\mathsf{h}_0^2 & + & \mathsf{h}_1^2 & + & \mathsf{h}_2^2 & + & \mathsf{h}_3^2 & = & 1 & (m = 0 \text{ ortho. quad. con.—Lemma 5.2 page 81})
\end{array}$$

Then the scaling coefficients have the following values:

EX	n	$\sqrt{2}\mathsf{h}_n$	$\sqrt{2}\mathsf{g}_n$
	0	$\frac{1+\sqrt{3}}{4}$	$\frac{1-\sqrt{3}}{4}$
	1	$\frac{3+\sqrt{3}}{4}$	$-\frac{3-\sqrt{3}}{4}$
	2	$\frac{3-\sqrt{3}}{4}$	$\frac{3+\sqrt{3}}{4}$
	3	$\frac{1-\sqrt{3}}{4}$	$-\frac{1+\sqrt{3}}{4}$





[15] Vidakovic (1999), page 92, Soman et al. (2010) page 83

✎PROOF:

1. Solve the 3 linear equations in terms of h_3:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} \\ 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} \\ 0 & -2 & 0 & -2 & -\sqrt{2} \\ 0 & -1 & 2 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & -2 & \sqrt{2} \\ 0 & 0 & -4 & 4 & -\sqrt{2} \\ 0 & 1 & -2 & 3 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 3 & -2 & \sqrt{2} \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & -1 & \frac{\sqrt{2}}{4} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & \frac{\sqrt{2}}{4} \\ 0 & 1 & 0 & 1 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & -1 & \frac{\sqrt{2}}{4} \end{bmatrix}$$

$$\begin{aligned} \mathsf{h}_0 &= \frac{\sqrt{2}}{4} - \mathsf{h}_3 \\ \mathsf{h}_1 &= \frac{\sqrt{2}}{2} - \mathsf{h}_3 \\ \mathsf{h}_2 &= \frac{\sqrt{2}}{4} + \mathsf{h}_3 \end{aligned}$$

2. Solve the forth quadratic equation in terms of h_3:

$$\begin{aligned} 1 &= \left(\frac{\sqrt{2}}{4} - h_3\right)^2 && + \left(\frac{\sqrt{2}}{2} - h_3\right)^2 && + \left(\frac{\sqrt{2}}{4} + h_3\right)^2 && + \mathsf{h}_3^2 \\ &= \left(\frac{1}{8} - \sqrt{2}2h_3 + h_3^2\right) && + \left(\frac{1}{2} - \sqrt{2}h_3 + h_3\right) && + \left(\frac{1}{8} + \frac{\sqrt{2}}{2} + h_3^2\right) && + \mathsf{h}_3^2 \\ &= 4h_3^2 - \sqrt{2}h_3 + \frac{3}{4} \end{aligned}$$

$$\begin{aligned} 4h_3^2 - \sqrt{2}h_3 - \frac{1}{4} &= 0 \\ \mathsf{h}_3 &= \frac{\sqrt{2} \pm \sqrt{2+4}}{8} \\ &= \frac{\sqrt{2} \pm \sqrt{6}}{8} \\ &= \frac{\sqrt{2}}{8}(1 \pm \sqrt{3}) \end{aligned}$$

3. Simplify the results:

$$\begin{aligned} \mathsf{h}_3 &= \tfrac{\sqrt{2}}{8}(1-\sqrt{3}) \\ \mathsf{h}_2 &= \tfrac{\sqrt{2}}{4} + \mathsf{h}_3 &&= \tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{8}(1-\sqrt{3}) &&= \tfrac{\sqrt{2}}{8}(2+1-\sqrt{3}) &&= \tfrac{\sqrt{2}}{8}(3-\sqrt{3}) \\ \mathsf{h}_1 &= \tfrac{\sqrt{2}}{2} - \mathsf{h}_3 &&= \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{8}(1-\sqrt{3}) &&= \tfrac{\sqrt{2}}{8}(4-1+\sqrt{3}) &&= \tfrac{\sqrt{2}}{8}(3+\sqrt{3}) \\ \mathsf{h}_0 &= \tfrac{\sqrt{2}}{4} - \mathsf{h}_3 &&= \tfrac{\sqrt{2}}{4} - \tfrac{\sqrt{2}}{8}(1-\sqrt{3}) &&= \tfrac{\sqrt{2}}{8}(2-1+\sqrt{3}) &&= \tfrac{\sqrt{2}}{8}(1+\sqrt{3}) \end{aligned}$$

4. Solution using *Maxima*™ (2 sets of solutions):

```
solve([h0+h1+h2+h3=sqrt(2),h0-h1+h2-h3=0,-h1+2*h2-3*h3=0,h0^2+h1^2+h2^2+h3^2=1],
      [h0,h1,h2,h3]);
```

$$\begin{array}{rcl} h_0 &=& -\frac{\sqrt{6}-\sqrt{2}}{8} \\ h_1 &=& -\frac{\sqrt{6}-3\sqrt{2}}{8} \\ h_2 &=& \frac{\sqrt{6}+3\sqrt{2}}{8} \\ h_3 &=& \frac{\sqrt{6}+\sqrt{2}}{8} \end{array} \qquad \begin{array}{rcl} h_0 &=& \frac{\sqrt{6}+\sqrt{2}}{8} \\ h_1 &=& \frac{\sqrt{6}+3\sqrt{2}}{8} \\ h_2 &=& -\frac{\sqrt{6}-3\sqrt{2}}{8} \\ h_3 &=& -\frac{\sqrt{6}-\sqrt{2}}{8} \end{array}$$

✏

Example 5.5 (4 scaling coefficient case). This example attempts to use the admissibility condition (Theorem 2.3 page 27) and orthonormal quadrature condition (Lemma 5.2 page 81) to design a 4 coefficient wavelet system. However, it fails due to the failure of the equations to be independent.

Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be an *orthogonal* wavelet system with four non-zero scaling coefficients.

1. We can use the following four equations to try to compute the values of the four scaling coefficients, and then compute the four wavelet coefficients from the four scaling coefficients. Note that the second, third, and fourth equations all assume orthonormality.

$$\begin{array}{rcrcrcrcrl} h_0 &+& h_1 &+& h_2 &+& h_3 &=& \sqrt{2} & \text{(admissibility—Theorem 2.3 page 27)} \\ h_0 &-& h_1 &+& h_2 &-& h_3 &=& 0 & \text{(zero at } -1 \text{ from orthonormality—page 73)} \\ h_0^2 &+& h_1^2 &+& h_2^2 &+& h_3^2 &=& 1 & (m=0 \text{ ortho. quad. con.—page 81}) \\ h_0h_2 &+& h_1h_3 & & & & &=& 0 & (m=1 \text{ ortho. quad. con.—page 81}) \end{array}$$

2. These equations are linearly independent, however they are still dependent [16] as shown next:

$$\begin{aligned} &\left[(h_0+h_1+h_2+h_3)+(h_0-h_1+h_2-h_3)\right]^2 \\ &\qquad+\left[(h_0+h_1+h_2+h_3)-(h_0-h_1+h_2-h_3)\right]^2 \\ &=[2h_0+2h_2]^2+[2h_1+2h_3]^2 \\ &=4[h_0+h_2]^2+4[h_1+h_3]^2 \\ &=4[h_0^2+2h_0h_2+h_2^2]+4[h_1^2+2h_1h_3+h_3^2] \\ &=4[h_0^2+h_1^2+h_2^2+h_3^2+2h_0h_2+2h_1h_3] \\ &=4(h_0^2+h_1^2+h_2^2+h_3^2)+8(h_0h_2+h_1h_3) \end{aligned}$$

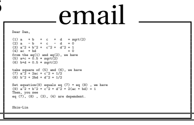
$\implies$ The last two equations are (non-linearly) dependent on the first two.

3. Because of this dependence, the system of equations has one degree of freedom.

4. Solutions using *Maxima*

```
solve([h0+h1+h2+h3=sqrt(2),h0-h1+h2-h3=0,h0^2+h1^2+h2^2+h3^2=1,h0*h2+h1*h3=0],
      [h0,h1,h2,h3]
     );
```

[16] email



Many many thanks to 謝欣霖 (Xiè Xīn Lín) for pointing this out to me in a 2005 October 13 email. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from 謝欣霖's 2005 October 13 email.)

(a) *Maxima* first solution:

$$
\begin{aligned}
h_1 &= \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2+4\sqrt{2}h_0+1}+1\right)\\
h_2 &= \frac{2h_0-\sqrt{2}}{2}\\
h_3 &= \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2+4\sqrt{2}h_0+1}-1\right)
\end{aligned}
$$

(b) *Maxima* second solution:

$$
\begin{aligned}
h_1 &= \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2+4\sqrt{2}h_0+1}-1\right)\\
h_2 &= -\frac{2h_0-\sqrt{2}}{2}\\
h_3 &= \frac{\sqrt{2}}{4}\left(\sqrt{-8h_0^2+4\sqrt{2}h_0+1}+1\right)
\end{aligned}
$$

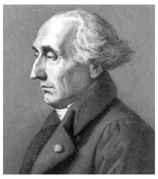
Example 5.6. [17] This example gives the equations for a 6 coefficient wavelet system. See also the six coefficient *Daubechies-p*3 wavelet system which is computed using a spectral factorization technique (Example 6.5 page 111).

$$
\begin{array}{rcrcrcrcrcrcrl}
h_0 & + & h_1 & + & h_2 & + & h_3 & + & h_4 & + & h_5 & = & \sqrt{2} & \text{(admissibility)}\\
h_0 & - & h_1 & + & h_2 & - & h_3 & + & h_4 & - & h_5 & = & 0 & \text{(vanishing 0th moment)}\\
 & - & h_1 & + & 2h_2 & - & 3h_3 & + & 4h_4 & - & 5h_5 & = & 0 & \text{(vanishing 1st moment)}\\
 & - & h_1 & + & 4h_2 & - & 9h_3 & + & 16h_4 & - & 25h_5 & = & 0 & \text{(vanishing 2nd moment)}\\
h_0^2 & + & h_1^2 & + & h_2^2 & + & h_3^2 & + & h_4^2 & + & h_5^2 & = & 1 & (m=0 \text{ orthonormal quadrature condition})\\
h_0h_2 & + & h_1h_3 & + & h_2h_4 & + & h_3h_5 & & & & & = & 0 & (m=1 \text{ orthonormal quadrature condition})
\end{array}
$$

[17] Soman et al. (2010) page 84

CHAPTER 6

MINIMUM PHASE CONSTRAINT



❝*I regard as quite useless the reading of large treatises of pure analysis: too large a number of methods pass at once before the eyes. It is in the works of applications that one must study them; one judges their ability there and one apprises the manner of making use of them.*❞

Joseph Louis Lagrange (1736-1813), mathematician [1]

6.1 General structure

This chapter introduces a wavelet design technique that represents the orthogonal quadrature condition in terms of a polynomial in z. It is the technique that was used by Ingrid Daubechies in her celebrated *Daubechies-p* class of wavelets (see Section 6.2 page 105). The technique can be summarized as follows:

1. **Design requirements.** Let $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \left(\!\left(\boldsymbol{W}_j\right)\!\right), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ be a wavelet system. We want to design this system to have two properties:
   (a) The scaling functions $(\!(\mathbf{T}^n\phi)\!)$ are **orthonormal** (Definition 5.1 page 67).
   (b) The wavelet function ψ has p *vanishing moments* (Definition 4.2 page 55).
2. **Form of solution.** Lemma 4.3 (page 65) demonstrates that the presence of the factor $\left(\frac{1+z^{-1}}{2}\right)^p$ in the z transform $\check{\mathsf{h}}(z)$ of the scaling coefficient sequence $(\!(h_n)\!)$ is a sufficient condition for ψ to have p vanishing moments. Therefore, we constrain $\check{\mathsf{h}}(z)$ to be of the following form:

$$\check{\mathsf{h}}(z) = \sqrt{2}\;\underbrace{\left(\frac{1+z^{-1}}{2}\right)^p}_{\substack{\text{provides } p \\ \text{vanishing moments} \\ \text{(Lemma 4.3 page 65)}}}\;\underbrace{\mathsf{Q}(z)}_{\text{provides orthogonality}}$$

[1] quote: Stopple (2003), page xi
image: http://en.wikipedia.org/wiki/Image:Langrange_portrait.jpg, public domain

So we need to compute the factor $\mathsf{Q}(z)$. Once we have $\mathsf{Q}(z)$, we can multiply the two factors, and the scaling coefficient sequence $(\!(h_n)\!)$ will simply be the coefficients of the polynomial $\check{\mathsf{h}}(z)$:

$$\check{\mathsf{h}}(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + \cdots + h_{2p-1} z^{-2p+1}$$

3. **Computing $\mathsf{Q}(z)$.**

   (a) The frequency representation of the orthogonal quadrature condition $\left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi)\right|^2 = 2$ (Lemma 5.1 page 71) is *periodic* in ω with period 2π and is even about 0. Because it's *periodic*, the quadrature condition can be expressed as a *Fourier Series* (Definition J.1 page 253). Because it's *even*, the quadrature condition can be expressed using only cosine terms.

   (b) Any *harmonic* cosine representation of a function may alternatively be expressed as a *trigonometric expansion* (Theorem I.2 page 233 with examples in Example I.1 page 235 and Example I.2 page 236). Therefore, the frequency orthogonal quadrature condition can be represented as a polynomial in $\cos\omega$.

   (c) As indicated by the trigonometric identity $\sin^2\theta \equiv \frac{1}{2}(1-\cos 2\theta)$ (Theorem H.11 page 228), any polynomial in $\cos\omega$ can alternatively be represented as a polynomial in $\sin^2\frac{\omega}{2}$. Therefore, the frequency orthogonal quadrature condition can be represented as a polynomial in $\sin^2\frac{\omega}{2}$.

   (d) With y defined as $y \triangleq \sin^2\frac{\omega}{2}$, the orthogonal quadrature condition together with the p vanishing moments constraint can be represented as (Lemma 6.2 page 98)

   $$(1-y)^p P(y) + y^p P(1-y) = 1 \quad \text{for some polynomial } P(y).$$

   (e) The forms $(1-y)^p P(y)$ and $y^p P(1-y)$ are also used in *Hermite interpolation* (Section N.2 page 333) with the result of the endpoints at $y=0$ and $y=1$ being increasingly "flatter" with increasing p. In particular, the first $p-1$ derivatives at the endpoints are 0 (Theorem N.1 page 334).

   (f) There are an infinite number of solutions for $\mathsf{P}(y)$. These solutions are of the form (Lemma 6.3 page 99)

   $$\mathsf{P}(y) \triangleq \mathsf{P}_m(y) + y^p \mathsf{R}\left(\frac{1}{2} - y\right) \quad \text{where} \quad \left\{ \begin{array}{l} \mathsf{P}_m(y) \triangleq \displaystyle\sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k \quad \text{and} \\ \underbrace{\mathsf{R}(y) = -\mathsf{R}(-y)}_{\mathsf{R}\text{ is any } odd \text{ polynomial}} \end{array} \right\}$$

   Furthermore, $\mathsf{P}(y)$ has minimum order when $\mathsf{R}(y) = 0$.

   (g) Examples of *odd polynomials* include the *Chebyshev polynomials* (Definition I.1 page 238) $\mathsf{T}_n(x)$ with n *odd* (Theorem I.3 page 238).

   (h) **Spectral factorization.** The quantity y can be expressed in terms of z as (Lemma 6.1 page 97)

   $$y = \left(\frac{1-z}{2}\right)\left(\frac{1-z^{-1}}{2}\right).$$

   This implies that for each root of a polynomial in y, there are two roots in the equivalent polynomial in z, and that these roots occur in recipricol pairs (if z is a root, then z^{-1} is also a root). Therefore, any polynomial $\mathsf{P}(y)$ in y can be factored into two factors in z such that $\mathsf{P}(y) = \mathsf{Q}(z)\mathsf{Q}(z^{-1})$. Once $\mathsf{P}(y)$ has been computed, we can then proceed to compute $\mathsf{Q}(z)$ with help from Theorem 6.1 (page 104).

Lemma 6.1.

LEM

$$\left.\begin{array}{rcl} y & \triangleq & \sin^2\left(\frac{\omega}{2}\right) \quad \textit{and} \\ z & \triangleq & e^{i\omega} \end{array}\right\} \implies \left\{\begin{array}{rcl} y & = & \frac{2-z-z^{-1}}{4} = \left(\frac{1-z}{2}\right)\left(\frac{1-z^{-1}}{2}\right) \\ 1-y & = & \cos^2\left(\frac{\omega}{2}\right) \\ & = & \frac{2+z+z^{-1}}{4} = \left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right) \\ \frac{1}{2}-y & = & \frac{1}{2}\cos\omega \\ z & = & 1-2y \pm \sqrt{y(y-1)} \end{array}\right.$$

✎PROOF:

$$\begin{aligned} y &\triangleq \sin^2\left(\frac{\omega}{2}\right) && \text{by definition of } y \\ &= \frac{1}{2} - \frac{1}{2}\cos(\omega) && \text{by \textit{half-angle formulas}} && \text{(Theorem H.11 page 228)} \\ &= \frac{1}{2} - \frac{1}{2}\frac{e^{i\omega}+e^{-i\omega}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)} \\ &= \frac{2}{4} - \frac{z+z^{-1}}{4} && \text{by definition of } z \\ &= \frac{2-z-z^{-1}}{4} \\ &= \left(\frac{1-z}{2}\right)\left(\frac{1-z^{-1}}{2}\right) \end{aligned}$$

$$\begin{aligned} 1-y &= 1-\sin^2\left(\frac{\omega}{2}\right) && \text{by definition of } y \\ &= \cos^2\left(\frac{\omega}{2}\right) && \text{by \textit{half-angle formulas}} && \text{(Theorem H.11 page 228)} \\ &= \frac{1}{2}(1+\cos\omega) && \text{by \textit{half-angle formulas}} && \text{(Theorem H.11 page 228)} \end{aligned}$$

$$\begin{aligned} 1-y &= 1-\frac{2-z-z^{-1}}{4} && \text{by previous result} \\ &= \frac{4-2+z+z^{-1}}{4} \\ &= \frac{2+z+z^{-1}}{4} \\ &= \left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right) \end{aligned}$$

$$\begin{aligned} \frac{1}{2}-y &= \frac{1}{2}-\sin^2\left(\frac{\omega}{2}\right) && \text{by definition of } y \\ &= \frac{1}{2}-\frac{1}{2}(1-\cos\omega) && \text{by \textit{half-angle formulas}} && \text{(Theorem H.11 page 228)} \\ &= \frac{1}{2}\cos\omega \end{aligned}$$

⇒

Lemma 5.1 (page 71) demonstrates that all orthogonal scaling functions satisfy the *orthonormal quadrature condition* $\left|\tilde{h}(\omega)\right|^2 + \left|\tilde{h}(\omega+\pi)\right|^2 = 2$. When designing orthogonal scaling functions with compact support, it is very useful to be able to express this quadrature condition as a polynomial in $\sin^2\frac{\omega}{2}$. Lemma 6.2 (next) does this by expressing the quadrature condition as a polynomial in y where $y = \sin^2\frac{\omega}{2}$.

Lemma 6.2. *Let* $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \left(\!\left(W_j\right)\!\right), \phi, \psi, \left(\!\left(h_n\right)\!\right), \left(\!\left(g_n\right)\!\right)\right)$ *be a wavelet system. Let* $y \triangleq \sin^2\left(\frac{\omega}{2}\right)$ *and* $P(y)$ *a polynomial in* y.

LEM

$$\left\{\begin{array}{ll} 1. & \underbrace{\left|\tilde{h}(\omega)\right|^2 + \left|\tilde{h}(\omega+\pi)\right|^2 = 2}_{\text{QUADRATURE CONDITION } in \text{ ``frequency''}} \quad and \\ 2. & \psi \textit{ has } p \text{ VANISHING MOMENTS} \end{array}\right\} \iff \left\{\begin{array}{ll} 1. & \textit{there exists } P(y) \textit{ such that} \\ & \left|\tilde{h}(\omega)\right|^2 = 2(1-y)^p P(y) \quad and \\ 2. & \underbrace{(1-y)^p P(y) + y^p P(1-y) = 1}_{\text{QUADRATURE CONDITION } as\ polynomial\ in\ y} \end{array}\right\}$$

✎PROOF:

1. Because $|Q(\omega)|^2$ is *periodic*, it has a *Fourier series* expansion (Definition J.1 page 253). Because it is real, it has no imaginary part. Therefore, $|Q(\omega)|^2$ can be expressed as a harmonic cosine polynomial as

$$|Q(\omega)|^2 = \sum_{n=0}^{\infty} a_n \cos\omega n$$

2. Proof for $\Longrightarrow$ part:

$$\begin{aligned}
\left|\tilde{h}(\omega)\right|^2 &= \left|\sqrt{2}\left(\frac{1+e^{i\omega}}{2}\right)^p Q(\omega)\right|^2 && \text{by Lemma 4.3 page 65}\\
&= 2\left|\left(\frac{1+e^{i\omega}}{2}\right)^p\right|^2 |Q(\omega)|^2\\
&= 2\left(\frac{1+e^{i\omega}}{2}\right)^p\left(\frac{1+e^{-i\omega}}{2}\right)^p |Q(\omega)|^2\\
&= 2\left(\frac{1+e^{i\omega}}{2}\right)^p\left(\frac{1+e^{-i\omega}}{2}\right)^p \sum_{n=0}^{\infty} c_n e^{-i\omega n} && \text{by item (1)}\\
&= 2\left(\frac{1+e^{i\omega}+e^{-i\omega}+1}{4}\right)^p \sum_{n=0}^{\infty} a_n \cos\omega n && \text{by item (1)}\\
&= 2\left(\frac{2+2\cos\omega}{4}\right)^p \sum_{n=0}^{\infty} a_n \cos\omega n\\
&= 2\left(\cos^2\frac{\omega}{2}\right)^p P'(\cos\omega) && \text{by } \textit{trigonometric expansion} \quad \text{(Theorem I.2 page 233)}\\
&= 2\left(1-\sin^2\frac{\omega}{2}\right)^p P'\left(1-2\sin^2\frac{\omega}{2}\right) && \text{by } \textit{half-angle formulas} \quad \text{(Theorem H.11 page 228)}\\
&= 2\left(1-\sin^2\frac{\omega}{2}\right)^p P\left(\sin^2\frac{\omega}{2}\right)\\
&= 2(1-y)^p P(y)
\end{aligned}$$

$$\begin{aligned}
\left|\tilde{h}(\omega+\pi)\right|^2 &= 2\left(1-\sin^2\frac{\omega+\pi}{2}\right)^p P\left(\sin^2\frac{\omega+\pi}{2}\right) && \text{by previous result}\\
&= 2\left(1-\cos^2\frac{\omega}{2}\right)^p P\left(\cos^2\frac{\omega}{2}\right) && \text{by Theorem H.7 page 222}\\
&= 2\left(\sin^2\frac{\omega}{2}\right)^p P\left(1-\sin^2\frac{\omega}{2}\right) && \text{by } \textit{half-angle formulas} \quad \text{(Theorem H.11 page 228)}\\
&= 2y^p P(1-y) && \text{by definition of } y
\end{aligned}$$

$$\begin{aligned}
2 &= \left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi))\right|^2 && \text{by left hypothesis}\\
&= (1-y)^p P(y) + y^p P(1-y) && \text{by previous results}
\end{aligned}$$

3. Proof for $\Longleftarrow$ part:

$$\begin{aligned}
1 &= (1-y)^p P(y) + y^p P(1-y) && \text{by right hypothesis}\\
&= \left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi))\right|^2 && \text{by previous results}
\end{aligned}$$

✏

In the polynomial quadrature condition, what are the possible solutions of $P(y)$? Lemma 6.3 (next) answers this question.

Lemma 6.3. [2] *Let* P, Q, *and* R *be polynomials over* $\mathbb{R}$.

LEM

$$\underbrace{(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) = 1}_{\text{quadrature condition as polynomial in } y} \iff \left\{\begin{array}{lll}
1. & \mathsf{P}(y) \triangleq \mathsf{P}_m(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right) & \text{where}\\
 & \mathsf{P}_m(y) \triangleq \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k & \text{and}\\
2. & \underbrace{\mathsf{R}(y) = -\mathsf{R}(-y)}_{\mathsf{R} \text{ is any \textbf{odd} polynomial.}} &
\end{array}\right.$$

Furthermore, $\mathsf{P}(y)$ *has minimum order when* $\mathsf{R}(y) = 0$.

✎PROOF:

1. Proof that $(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) = 1 \implies \mathsf{P}(y) = \mathsf{P}_m(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)$:

   (a) lemma: proof that $\frac{1}{(1-y)^p} = \sum_{k=0}^{\infty}\binom{p-1+k}{k}y^k$:

$$\begin{aligned}
\frac{1}{(1-y)^p} &= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}\frac{1}{(1-y)^p}\right]_{y=0} y^k && \text{by } \textit{Maclaurin series} \text{ (Theorem C.13 page 153)}\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}(1-y)^{-p}\right]_{y=0} y^k\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[(-p)(-1)\frac{\mathrm{d}^{k-1}}{\mathrm{d}y^{k-1}}(1-y)^{-p-1}\right]_{y=0} y^k\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[(p)(-p-1)(-1)\frac{\mathrm{d}^{k-2}}{\mathrm{d}y^{k-2}}(1-y)^{-p-2}\right]_{y=0} y^k\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[(p)(p+1)(p+2)\frac{\mathrm{d}^{k-3}}{\mathrm{d}y^{k-3}}(1-y)^{-p-3}\right]_{y=0} y^k\\
&\vdots\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[(p)(p+1)(p+2)\cdots(p+k-2)\frac{\mathrm{d}}{\mathrm{d}y}(1-y)^{-p-(k-1)}\right]_{y=0} y^k\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left[(p)(p+1)(p+2)\cdots(p+k-2)(p+k-1)(1-y)^{-p-k}\right]_{y=0} y^k
\end{aligned}$$

[2] Daubechies (1992), page 171, Pinsky (2002), pages 331–333, Chui (1992), pages 175–176

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{k!}(p)(p+1)(p+2)\cdots(p+k-2)(p+k-1)\; y^k \\
&= \sum_{k=0}^{\infty} \frac{(p+k-1)!}{k!(p-1)!}\; y^k \\
&= \sum_{k=0}^{\infty} \frac{(p+k-1)!}{k!(p+k-1-k)!}\; y^k \\
&= \sum_{k=0}^{\infty} \binom{p+k-1}{k} y^k \qquad \text{by definition of } \binom{n}{k} \quad \text{(Definition B.3 page 132)}
\end{aligned}
$$

(b) There are an infinite number of solutions for $\mathsf{P}(y)$. These solutions are here expressed as

$$
\mathsf{P}(y) \triangleq \underbrace{\mathsf{P}_m(y)}_{\text{order } p-1} + \underbrace{y^p \mathsf{R}\left(\frac{1}{2}-y\right)}_{\text{order} \ge p}
$$

(c) Note that $(1-y)^p$ and y^p are *relatively prime* and are both of order p. Therefore by *Bézout's Theorem* (Theorem C.4 page 147), there exists a unique solution for $\mathsf{P}(y)$ of order $p-1$. This unique solution is here labeled $\mathsf{P}_m(y)$. The more general solution (of order $\ge p$) is here expressed as $\mathsf{P}_m(y) + y^p \mathsf{R}\left(\frac{1}{2}-y\right)$.

$$
\underbrace{(1-y)^p}_{\text{order } p} \underbrace{\mathsf{P}_m(y)}_{\text{order } p-1} + \underbrace{y^p}_{\text{order } p} \underbrace{\mathsf{P}_m(1-y)}_{\text{order } p-1} = 1
$$

(d) We can compute the unique order $p-1$ solution $\mathsf{P}_m(y)$ of $\mathsf{P}(y)$ by using a Maclaurin expansion of $\mathsf{P}(y)$. Let $\mathbf{T}\mathsf{P}(y)$ represent a polynomial $\mathsf{P}(y)$ with all terms of order $\ge p$ truncated.

$$
\begin{aligned}
\underbrace{\mathsf{P}_m(y)}_{\text{order } p-1} &= \frac{1}{(1-y)^p} - \frac{y^p \mathsf{P}_m(1-y)}{(1-y)^p} && \text{by (1c) lemma} \\
&= \mathbf{T}\left\{ \frac{1}{(1-y)^p} - \underbrace{\cancelto{0}{\frac{y^p \mathsf{P}_m(1-y)}{(1-y)^p}}}_{\text{all terms order} \ge p} \right\} && \text{by \textit{Bézout's Theorem}} \quad \text{(Theorem C.4 page 147)} \\
&= \mathbf{T}\left\{ \frac{1}{(1-y)^p} \right\} && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 2)} \\
&= \mathbf{T}\left\{ \sum_{k=0}^{\infty} \binom{p+k-1}{k} y^k \right\} && \text{by item (1a) page 99} \\
&= \sum_{k=0}^{p-1} \binom{p+k-1}{k} y^k && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 2)}
\end{aligned}
$$

(e) Alternatively, specific cases of $\mathsf{P}_m(y)$ can also be calculated using the *Extended Euclid's Algorithm*:

i. $p=2$ case: See Example C.1 (page 146).
ii. $p=3$ case: See Example C.2 (page 146).
iii. $p=4$ case: See Example C.3 (page 147).

(f) Proof that $\mathsf{R}(y)$ must be an odd polynomial:

$$\begin{aligned}
&(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) \\
&= (1-y)^p\left[\mathsf{P}_m(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)\right] + y^p\left[\mathsf{P}_m(1-y) + (1-y)^p\mathsf{R}\left(\frac{1}{2}-[1-y]\right)\right] \\
&= (1-y)^p\mathsf{P}_m(y) + (1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + y^p\mathsf{P}_m(1-y) + y^p(1-y)^p\mathsf{R}\left(-\frac{1}{2}+y\right) \\
&= \underbrace{\left[(1-y)^p\mathsf{P}_m(y) + y^p\mathsf{P}_m(1-y)\right]}_{1} + \left[(1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + y^p(1-y)^p\mathsf{R}\left(y-\frac{1}{2}\right)\right] \\
&= 1 + (1-y)^p y^p\left[\mathsf{R}\left(\frac{1}{2}-y\right) + \mathsf{R}\left(y-\frac{1}{2}\right)\right] \\
&= 1 \\
&\iff \left[\mathsf{R}\left(\frac{1}{2}-y\right) + \mathsf{R}\left(y-\frac{1}{2}\right) = 0\right] \\
&\iff \mathsf{R}(y) = -\mathsf{R}(-y) \qquad (\mathsf{R}(y) \text{ is odd})
\end{aligned}$$

2. Proof that $(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) = 1 \iff \mathsf{P}(y) = \mathsf{P}_m(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)$:

(a) Examples of specific cases with $\mathsf{R}(y) = 0$:

$p = 1$ case:

$$\begin{aligned}
&(1-y)^1\mathsf{P}_m(y) + y^1\mathsf{P}_m(1-y) \\
&= (1-y)^1\sum_{k=0}^{k=1-1}\binom{1+k-1}{k}y^k + y^1\sum_{k=0}^{k=1-1}\binom{1+k-1}{k}(1-y)^k \\
&= (1-y)^1(1) + y^1(1) \\
&= 1
\end{aligned}$$

$p = 2$ case:

$$\begin{aligned}
&(1-y)^2\mathsf{P}_m(y) + y^2\mathsf{P}_m(1-y) \\
&= (1-y)^2\sum_{k=0}^{k=2-1}\binom{2+k-1}{k}y^k + y^2\sum_{k=0}^{k=2-1}\binom{2+k-1}{k}(1-y)^k \\
&= (1-y)^2(1+2y) + y^2[1+2(1-y)] \\
&= (1-2y+y^2)(1+2y) + y^2(3-2y) \\
&= \underbrace{1-2y+y^2+2y-4y^2+2y^3}_{(1-y)^p\mathsf{P}_m(y)} + \underbrace{3y^2-2y^3}_{y^p\mathsf{P}_m(1-y)} \\
&= \underbrace{1+0y-3y^2+2y^3}_{(1-y)^p\mathsf{P}_m(y)} + \underbrace{3y^2-2y^3}_{y^p\mathsf{P}_m(1-y)} \\
&= 1
\end{aligned}$$

$p = 3$ case:

$$\begin{aligned}
&(1-y)^3\mathsf{P}_m(y) + y^3\mathsf{P}_m(1-y) \\
&= (1-y)^3\sum_{k=0}^{k=3-1}\binom{3+k-1}{k}y^k + y^3\sum_{k=0}^{k=3-1}\binom{3+k-1}{k}(1-y)^k \\
&= (1-y)^3(1+3y+6y^2) + y^3\left[1+3(1-y)+6(1-y)^2\right] \\
&= \underbrace{(1-3y+3y^2-y^3)(1+3y+6y^2)}_{(1-y)^p\mathsf{P}_m(y)} + \underbrace{y^3\left[4-3y+6(1-2y+y^2)\right]}_{y^p\mathsf{P}_m(1-y)}
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{1 - 3y + 3y^2 - y^3 + 3y - 9y^2 + 9y^3 - 3y^4 + 6y^2 - 18y^3 + 18y^4 - 6y^5}_{(1-y)^p\mathsf{P}_m(y)} \\
&\quad \underbrace{+\, y^3\left[10 - 15y + 6y^2\right]}_{y^p\mathsf{P}_m(1-y)} \\
&= \underbrace{1 + 0y + 0y^2 - 10y^3 + 15y^4 - 6y^5}_{(1-y)^p\mathsf{P}_m(y)} + \underbrace{10y^3 - 15y^4 + 6y^5}_{y^p\mathsf{P}_m(1-y)} \\
&= 1
\end{aligned}
$$

(b) Proof for $\mathsf{R}(y) = 0$ case (using induction): [3]

i. Define $\mathsf{P}_p(y) \triangleq \sum_{k=0}^{p} \binom{p+k}{k} y^k$.

ii. lemma: proof that $(1-y)\mathsf{P}_p(y) = \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p(1-2y)$:

$$
\begin{aligned}
&(1-y)\mathsf{P}_p(y) \\
&= \mathsf{P}_p(y) - y\mathsf{P}_p(y) \\
&= \sum_{k=0}^{p} \binom{p+k}{k} y^k - \sum_{k=0}^{p} \binom{p+k}{k} y^{k+1} \\
&= \underbrace{1 + \sum_{k=1}^{p} \binom{p+k}{k} y^k}_{\sum_{k=0}^{p}\binom{p+k}{k} y^k} \underbrace{- \sum_{k=1}^{p} \binom{p+k-1}{k-1} y^k - \binom{2p}{p} y^p}_{-\sum_{k=0}^{p}\binom{p+k}{k} y^{k+1}} \\
&= 1 + \sum_{k=1}^{p} \left[\binom{p+k}{k} - \binom{p+k-1}{k-1}\right] y^k - \binom{2p}{p} y^{p+1} \\
&= 1 + \sum_{k=1}^{p} \left[\underbrace{\binom{p+k-1}{k} + \binom{p+k-1}{k-1}}_{\binom{p+k}{k}} - \binom{p+k-1}{k-1}\right] y^k - \binom{2p}{p} y^{p+1}
\end{aligned}
$$

by Pascal's Rule page 133

$$
\begin{aligned}
&= 1 + \sum_{k=1}^{p} \binom{p+k-1}{k} y^k - \binom{2p}{p} y^{p+1} \\
&= \sum_{k=0}^{p} \binom{p+k-1}{k} y^k - \binom{2p}{p} y^{p+1} \\
&= \sum_{k=0}^{p-1} \binom{p+k-1}{k} y^k + \binom{2p-1}{p} y^p - \binom{2p}{p} y^{p+1} \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p - \frac{(2p)!}{p!p!} y^{p+1} \qquad \text{by definition of } \mathsf{P}_p(y) \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p - \frac{2p}{p}\frac{(2p-1)!}{p!(p-1)!} y^{p+1} \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p - 2\binom{2p-1}{p} y^{p+1} \\
&= \mathsf{P}_{p-1}(y) + \binom{2p-1}{p} y^p(1-2y)
\end{aligned}
$$

[3] Many thanks to Chip Eastham for his extremely valuable help with this proof.

iii. lemma: proof that $y\mathsf{P}_p(1-y) = \mathsf{P}_{p-1}(1-y) - \binom{2p-1}{p}(1-y)^p(1-2y)$:

$$\begin{aligned}
y\mathsf{P}_p(1-y) &= \left[(1-u)\mathsf{P}_p(u)\right]_{u\triangleq 1-y} \\
&= \left[(1-u)\mathsf{P}_p(u)\right]_{u\triangleq 1-y} \\
&= \left[\mathsf{P}_{p-1}(u) + \binom{2p-1}{p}u^p(1-2u)\right]_{u\triangleq 1-y} && \text{by previous lemma} \\
&= \mathsf{P}_{p-1}(1-y) + \binom{2p-1}{p}(1-y)^p(-1+2y) \\
&= \mathsf{P}_{p-1}(1-y) - \binom{2p-1}{p}(1-y)^p(1-2y)
\end{aligned}$$

iv. Proof that $p-1$ case $\implies$ p case:

$$\begin{aligned}
&(1-y)^{p+1}\mathsf{P}_p(y) + y^{p+1}\mathsf{P}_p(1-y) \\
&= (1-y)^p(1-y)\mathsf{P}_p(y) + y^p y\mathsf{P}_p(1-y) \\
&= (1-y)^p\underbrace{\left[\mathsf{P}_{p-1}(y) + \binom{2p-1}{p}y^p(1-2y)\right]}_{\text{by item (2(b)ii) page 102}} + y^p\underbrace{\left[\mathsf{P}_{p-1}(1-y) - \binom{2p-1}{p}(1-y)^p(1-2y)\right]}_{\text{by item (2(b)iii) page 103}} \\
&= \underbrace{(1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y)}_{1} + \binom{2p-1}{p}(1-2y)\underbrace{\left[(1-y)^p y^p - y^p(1-y)^p\right]}_{0} \\
&= 1 \qquad \text{by induction hypothesis}
\end{aligned}$$

v. Therefore by induction, $(1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y) = 1$ is true for all $p \in \mathbb{N}$.

(c) Proof for $\mathsf{R}(y) \neq 0$ case:

$$\begin{aligned}
&(1-y)^p\mathsf{P}(y) + y^p\mathsf{P}(1-y) \\
&= (1-y)^p\left[\mathsf{P}_{p-1}(y) + y^p\mathsf{R}\left(\frac{1}{2}-y\right)\right] + y^p\left[\mathsf{P}_{p-1}(1-y) + (1-y)^p\mathsf{R}\left(\frac{1}{2}-[1-y]\right)\right] \\
&= (1-y)^p\mathsf{P}_{p-1}(y) + (1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + y^p\mathsf{P}_{p-1}(1-y) + y^p(1-y)^p\mathsf{R}\left(-\frac{1}{2}+y\right) \\
&= \left[(1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y)\right] + \underbrace{\left[(1-y)^p y^p\mathsf{R}\left(\frac{1}{2}-y\right) + -y^p(1-y)^p\mathsf{R}\left(y-\frac{1}{2}\right)\right]}_{0 \text{ because } \mathsf{R}(y) \text{ is an odd polynomial}} \\
&= (1-y)^p\mathsf{P}_{p-1}(y) + y^p\mathsf{P}_{p-1}(1-y) \\
&= 1 \qquad \text{by item (2b) page 102}
\end{aligned}$$

✑

Example 6.1. [4] Here are some examples of the minimal polynomial $\mathsf{P}_m(y)$ for varying number of vanishing moments p:

EX

p	$\mathsf{P}_m(y) = \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k$
1	1
2	$1 + 2y$
3	$1 + 3y + 6y^2$
4	$1 + 4y + 10y^2 + 20y^3$
5	$1 + 5y + 15y^2 + 35y^3 + 70y^4$
6	$1 + 6y + 21y^2 + 56y^3 + 126y^4 + 252y^5$
7	$1 + 7y + 28y^2 + 84y^3 + 210y^4 + 462y^5 + 924y^6$
8	$1 + 8y + 36y^2 + 120y^3 + 330y^4 + 792y^5 + 1716y^6 + 3432y^7$
9	$1 + 9y + 45y^2 + 165y^3 + 495y^4 + 1287y^5 + 3003y^6 + 6435y^7 + 12870y^8$
10	$1 + 10y + 55y^2 + 220y^3 + 715y^4 + 2002y^5 + 5005y^6 + 11440y^7 + 24310y^8 + 48620y^9$

[4] Pinsky (2002), page 333

Example 6.2. Here are some examples of the minimal polynomial $\mathsf{P}_m(y)$ evaluated at $y = [(-z+2-1/z)/4]$:

EX

p	$\mathsf{P}_m\left(\frac{z-2+z^{-1}}{-4}\right)$
1	1
2	$\frac{z^2-4z-1}{-2z}$
3	$\frac{3z^4-18z^3+38z^2-18z+3}{2^3z^2}$
4	$\frac{5z^6-40z^5+131z^4-208z^3+131z^2-40z+5}{-2^4z^3}$
5	$\frac{35z^8-350z^7+1520z^6-3650z^5+5018z^4-3650z^3+1520z^2-350z+35}{2^7z^4}$
6	$\frac{63z^{10}-756z^9+4067z^8-12768z^7+25374z^6-32216z^5+25374z^4-12768z^3+4067z^2-756z+63}{-2^8z^5}$

Theorem 6.1 (next) gives a method for computing scaling functions with compact support and p vanishing moments. This method results in an infinite number of possible solutions. For each value p, there are exactly $2^{\lfloor p/2\rfloor}$ solutions that are of minimum order $2p-1$. Of these $2^{\lfloor p/2\rfloor}$ solutions, there is one solution that is also *minimum phase*. This is the Daubechies-p scaling function. All the $2^{\lfloor p/2\rfloor}$ solutions occur when $\mathsf{R}(y)=0$. An infinite number of other solutions are also available when $\mathsf{R}(y)\neq 0$.

Theorem 6.1. *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *be an* ***orthonormal*** *wavelet system.*

THM

1. *If ψ has p vanishing moments, then $(\!(h_n)\!)_{n\in\mathbb{Z}}$ must be of a form that satisfies*

$$\begin{aligned}
i)\quad \sum_n h_n z^{-n} &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z)\\
ii)\quad \mathsf{Q}(z)\mathsf{Q}(z^{-1}) &= \left[\sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k + y^p R\left(\frac{1}{2}-y\right)\right]\Bigg|_{y=\frac{2-z-z^{-1}}{4}}
\end{aligned}$$

2. *The support of $(\!(h_n)\!)$ is minimized when $R(y)=0$.*
3. *If ψ has p vanishing moments, then the minimum support size of $(\!(h_n)\!)$ is $2p-1$ ($2p$ non-zero elements).*
4. *If $(\!(h_n)\!)$ has support size $2p-1$ ($2p$ non-zero coefficients) then ψ has maximum of p vanishing moments.*

✎PROOF:

1. Proof for the form of $(\!(h_n)\!)$:

   (a) For $y \triangleq \sin^2\frac{\omega}{2}$

$$\underbrace{\left|\tilde{\mathsf{h}}(\omega)\right|^2+\left|\tilde{\mathsf{h}}(\omega+\pi)\right|^2=2}_{\text{quadrature condition in frequency domain}} \iff \underbrace{(1-y)^p\ddot{\mathsf{Q}}(y)+y^p\ddot{\mathsf{Q}}(1-y)=1}_{\text{quadrature condition as a polynomial in } y}$$

   (b) The polynomial $\ddot{\mathsf{Q}}(\omega)$ is non-negative and therefore can be factorized into two factors:

$$\ddot{\mathsf{Q}}(y)\Big|_{y=\frac{2-z-z^{-1}}{4}} = \mathsf{Q}(z)\mathsf{Q}\left(\frac{1}{z^*}\right) \qquad \text{by } \textit{Fejér-Riesz factorization} \text{ (Theorem I.5 page 242)}$$

(c)

$$|\check{\mathsf{h}}(z)|^2 = 2\left(\frac{1+z^{-1}}{2}\right)^p \ddot{\mathsf{Q}}(z) \qquad = 2\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}\left(z\right)\mathsf{Q}\left(\frac{1}{z^*}\right)$$

$$\check{\mathsf{h}}(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z)$$

2. Proof that the support of $(\!(h_n)\!)$ is minimized when $R(y) = 0$:
   This follows directly from *Bezout's Theorem* (Lemma C.4 page 147).

$$\begin{aligned}
Q\left(z\right)Q\left(\frac{1}{z^*}\right) &= \ddot{\mathsf{Q}}(y)\Big|_{y=\frac{2-z-z^{-1}}{4}} && \text{by Theorem I.5 page 242}\\
&= \left[\sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k + \cancel{y^p R\left(\frac{1}{2}-y\right)}^{\,0}\right]_{y=\frac{2-z-z^{-1}}{4}} && \text{by Lemma 6.3 page 99}\\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}\left(\frac{2-z-z^{-1}}{4}\right)^k && \text{by Lemma 6.3 page 99}
\end{aligned}$$

3. Proof that the minimum support size of $(\!(h_n)\!)$ is $2p-1$ ($2p$ non-zero elements).

$$\begin{aligned}
\underbrace{\ddot{\mathsf{Q}}(y)\Big|_{y=\frac{2-z-z^{-1}}{4}}}_{\text{order } p-1 \text{ in } y} &= \underbrace{\ddot{\mathsf{Q}}\left(\frac{2-z-z^{-1}}{4}\right)}_{\text{order } 2p-2 \text{ in } z}\\
&= \underbrace{\mathsf{Q}\left(z\right)}_{\text{order } p-1}\underbrace{\mathsf{Q}\left(\frac{1}{z^*}\right)}_{\text{order } p-1} && \text{by *Fejér-Riesz factorization* (Theorem I.5 page 242)}\\
\underbrace{\check{\mathsf{h}}(z)}_{\text{order } 2p-1} &= \underbrace{\sqrt{2}\left(\frac{z+1}{2}\right)^p}_{\text{order } p}\underbrace{Q(z)}_{\text{order } p-1}
\end{aligned}$$

✐

6.2 Design details

Definition 6.1. *Let* $\check{\mathsf{h}}(z)$ *be the* Z-TRANSFORM *(Definition O.4 page 342) of a sequence* $(\!(h_n)\!)_{n\in\mathbb{Z}}$. *Let* $Q(z)$ *be a polynomial with real coefficients and*

$$y \triangleq \sin^2\left(\frac{\omega}{2}\right)$$

$$P(y) \triangleq \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k$$

$$\dot{Q} \triangleq \left\{Q(z)|Q(z)Q(z^{-1}) = P\left(\frac{2-z-z^{-1}}{4}\right\}\right) \qquad (\text{FEJÉR-RIESZ SPECTRAL FACTORIZATIONS } \textit{of } P)$$

$$\mathbf{R}Q \triangleq \{z_n | z_n \textit{ is a zero of } Q(z)\} \qquad (\textit{roots of } Q(z))$$

DEF $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *is a* ***Daubechies-p wavelet system*** *if*

1. $\check{\mathsf{h}}(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p Q(z)$ *and*
2. $Q(z)$ *is the polynomial in* $\dot{Q}$ such that $\forall z_n \in \mathbf{R}Q,\ |z_n| < 1$ *and*
3. $g_n = (-1)^n h_{2p-1-n}$

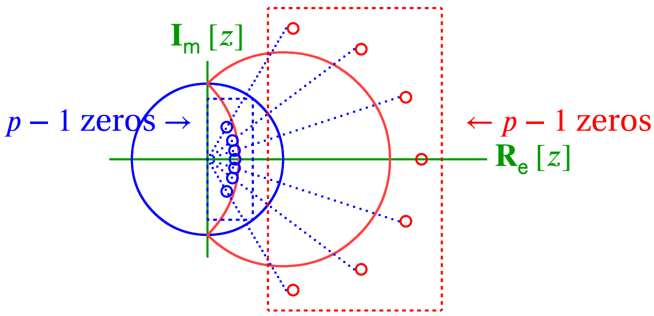
$\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$ *is also called a* D_p ***wavelet system***.

The Daubechies-p wavelets of Definition 6.1 page 105 can be calculated [5] by the following steps:

1. Compute the polynomial $P(y)$. This polynomial has $p-1$ roots in y.

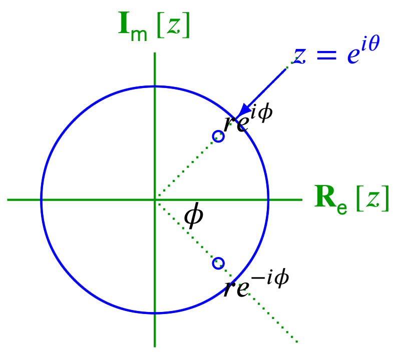
$$P(y) \triangleq \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k \qquad \text{where } \binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}$$

2. Compute $P\left(\frac{2-z-z^{-1}}{4}\right)$. This polynomial has $2p-2$ roots in z.



(a) Because the coefficients of $P\left(\frac{2-z-z^{-1}}{4}\right)$ are real, all of its roots occur in *complex conjugate pairs*:

$$\text{root at } z_1 = re^{i\theta} \iff \text{root at } z_1^* = \left(re^{i\theta}\right)^* = re^{-i\theta}$$



(b) All of the roots of $P\left(\frac{2-z-z^{-1}}{4}\right)$ occur in *conjugate reciprocal pairs*:

$$P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = re^{i\theta} \iff P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = \left(re^{i\theta}\right)^{-1} = \frac{1}{r}e^{-i\theta}$$

[5] For an actual implementation using *Octave*, see Section V.1 (page 399).

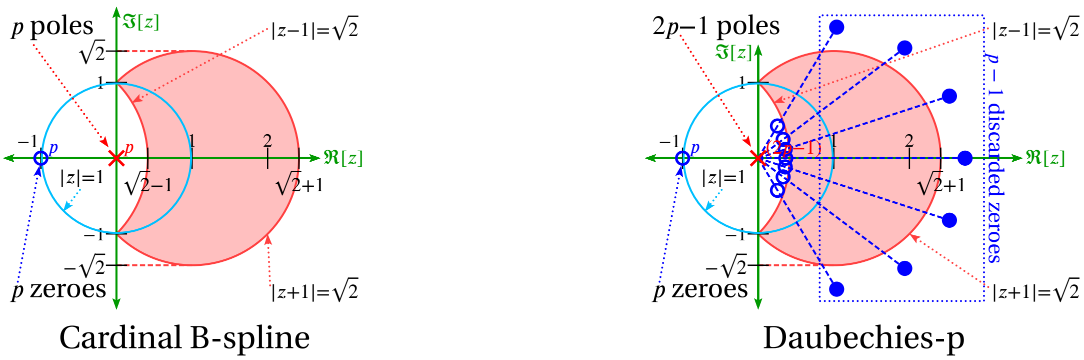
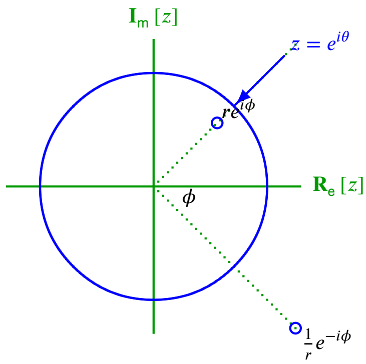


Figure 6.1: Zero locations for B-cardinal spline $N_p(x)$ (left) and Daubechies-p (right) scaling coefficients



3. By the *Fejér-Riesz spectral factorization theorem* (Theorem I.5 page 242), [6] $P\left(\frac{2-z-z^{-1}}{4}\right)$ can be factored in the form
$$P\left(\frac{2-z-z^{-1}}{4}\right) = Q(z)Q(z^{-1}).$$
4. Form $Q(z)$ from the $p-1$ zeros *inside* the unit circle.
5. The scaling coefficients $(\!(h_n)\!)$ are the coefficients of the polynomial
$$\check{h}(z) = \underbrace{\underbrace{\sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p}_{\text{order } p}\underbrace{Q(z)}_{\text{order } p-1}}_{\text{order } 2p-1 \text{ } (2p \text{ coefficients})}$$

As the number of vanishing moments approaches infinity ($p \to \infty$), the zeros of the Daubechies-p scaling coefficients approach two asymptotic arcs as described in the next theorem and illustrated in Figure 6.1 (page 107).

Lemma 6.4. [7] *Let* $\boldsymbol{\Omega} \triangleq \left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), (\!(\boldsymbol{W}_j)\!), \phi, \psi, (\!(h_n)\!), (\!(g_n)\!)\right)$.

LEM

$\boldsymbol{\Omega}$ is a DAUBECHIES-p WAVELET SYSTEM *(Definition 6.1 page 105)* $\Longrightarrow$
with increasing vanishing moments parameter p, the zeros of $Q(z)Q(z^{-1})$ asymptotically approach the area of the complex plain enclosed by the arcs
$$|z-1| = \sqrt{2} \qquad \textit{and} \qquad |z+1| = \sqrt{2}$$

[6] *Fejér-Riesz spectral factorization* (Theorem I.5 page 242)

[7] Strang and Nguyen (1996) page 170 ⟨0961408871⟩, Vidakovic (1999) page 78 ⟨Fig. 3.14⟩, Shen and Strang (1996) page 3823, Kateb and Lemarié-Rieusset (1997)

Theorem 6.2. *Let* $\boldsymbol{\Omega} \triangleq \left(L^2_{\mathbb{R}}, (\boldsymbol{V}_j), (\boldsymbol{W}_j), \phi, \psi, (h_n), (g_n)\right)$. *Let* $\operatorname{supp}(x_n)_{n\in\mathbb{Z}}$ *be defined as in Definition 2.4 page 31.*

THM

$\{\boldsymbol{\Omega}$ *is a* DAUBECHIES-p WAVELET SYSTEM (Definition 6.1 page 105)$\}$ $\Longrightarrow$

$$\left\{\begin{array}{lll}
(1). & \boldsymbol{\Omega} \text{ is an ORTHONORMAL WAVELET SYSTEM.} & \text{and}\\
(2). & \tilde{\mathsf{h}}(\omega) \text{ has MINIMUM PHASE.} & \text{and}\\
(3). & \operatorname{supp}(h_n) = 2p-1 \ \ ((h_n) \text{ has } 2p \text{ non-zero elements}) & \text{and}\\
(4). & \text{the support size of } (h_n) \text{ is the smallest possible} & \\
 & \text{for wavelets with } p \text{ vanishing moments.} &
\end{array}\right\}$$

✎PROOF:

1. proof for (1): For $p = 1$, this is the *Haar wavelet system*. For $p \geq 2$, this follows from Theorem 5.2 page 74.

   (a) Proof that (h_n) satisfies $\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}$:

$$\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z=1} &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p Q(z)\Big|_{z=1} && \text{by def. of } \textit{Daubechies-p wavelet system} \quad \text{(Definition 6.1 page 105)}\\
&= \sqrt{2}\left(\frac{1+1^{-1}}{2}\right)^p Q(1)\\
&= \sqrt{2}Q(1)\\
&= \sqrt{2}\sqrt{Q(1)Q^*(1^{-1})}\\
&= \sqrt{2}\sqrt{Q(z)Q^*\left(z^{-1}\right)}\Big|_{z=1}\\
&= \sqrt{2}\sqrt{P\left(\frac{2-z-z^{-1}}{4}\right)}\Bigg|_{z=1}\\
&= \sqrt{2}\sqrt{P(0)}\\
&= \sqrt{2}\sqrt{\sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k}\Bigg|_{y=0} && \text{by } \textit{orthonormal quadrature conditions} \quad \text{(Lemma 6.3 page 99)}\\
&= \sqrt{2}\sqrt{(1\cdot 0^0+0)} && \text{by definition of } \binom{n}{k} \quad \text{(Definition B.3 page 132)}\\
&= \sqrt{2}\sqrt{(1+0)}\\
&= \sqrt{2}\\
\Longrightarrow \sum_{n\in\mathbb{Z}} h_n &= \sqrt{2} && \text{by } \textit{orthonormal quadrature conditions} \quad \text{(Lemma 5.1 page 71)}
\end{aligned}$$

   (b) Proof that (h_n) satisfies $\sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n$:

$$\begin{aligned}
&\boldsymbol{\Omega} \text{ is a } D_p \textit{ wavelet system}\\
\Longrightarrow\ & (1-y)^p P(y) + y^p P(1-y) = 1 && \text{by Definition 6.1 page 105 and Lemma 6.3 page 99}\\
\Longleftrightarrow\ & \left|\tilde{\mathsf{h}}(\omega)\right|^2 + \left|\tilde{\mathsf{h}}(\omega+\pi)\right|^2 = 2 && \text{by Lemma 6.2 page 98}\\
\Longleftrightarrow\ & \sum_{m\in\mathbb{Z}} h_m h^*_{m-2n} = \bar{\delta}_n && \text{by } \textit{orthonormal quadrature conditions} \text{ (Lemma 5.1 page 71)}
\end{aligned}$$

(c) Proof that $\tilde{\phi}(\omega)$ is *continuous* at 0:

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p Q(z) && \text{by Definition 6.1 page 105}\\
&\implies \psi \text{ has } p \textit{ vanishing moments} && \text{by Lemma 4.3 page 65}\\
&\implies \tilde{\phi}(\omega) \text{ is } \textit{continuous} \text{ at } \omega = 0 && \text{by Theorem 4.5 page 64}
\end{aligned}$$

(d) Proof that $\inf_{\omega\in[-\pi/2:\pi/2]} \left|\check{\mathsf{h}}(\omega)\right| > 0$:
by Lemma 6.4 page 107—note that the zeros are never on the unit circle in the range $[-\pi/2 : \pi/2]$.

2. Proof for (2): The zeros are chosen to be *inside* the unit circle, thus giving $\check{\mathsf{h}}(\omega)$ *minimum phase*.

3. Proof for (3): See Theorem 6.1 page 104.

4. Proof for (4): See Theorem 6.1 page 104.

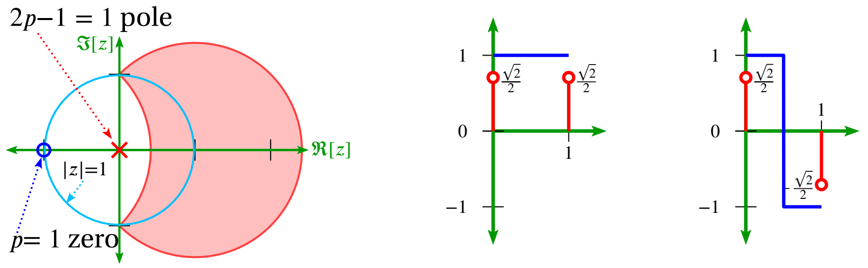
✏

6.3 Examples

Example 6.3. The **Daubechies-**1 wavelet system yields the following results.

EX

n	h_n	g_n
0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
1	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$



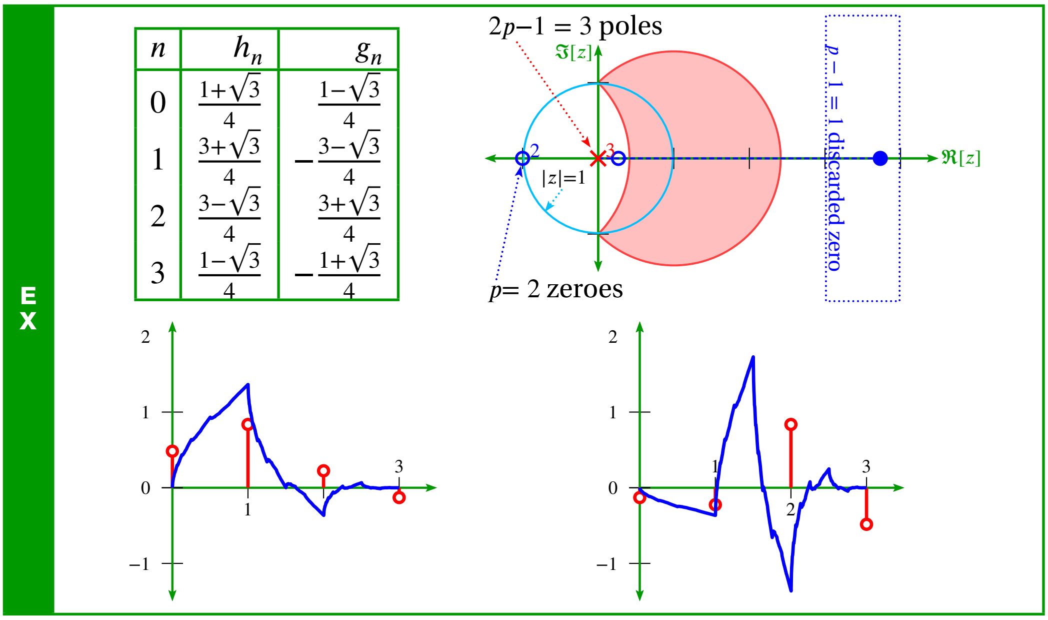
The **Daubechies-**1 wavelet system is equivalent to the *Haar wavelet system*, and it is possible to compute this wavelet system using "time-domain" techniques as well:

1. Admissibility condition and *orthonormality*—Example 5.3 (page 90)

2. Admissibility condition and *partition of unity*—Example 2.3 (page 40)

✎PROOF:

$$\begin{aligned}
\mathsf{Q}(z)\mathsf{Q}(z^{-1}) &= P(y)\big|_{y=\frac{2-z-z^{-1}}{4}}\\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}}\\
&= \sum_{k=0}^{0}\binom{1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}}\\
&= 1\\
&= \underbrace{1}_{\mathsf{Q}(z)}\cdot\underbrace{1}_{\mathsf{Q}(z^{-1})}
\end{aligned}$$



n	h_n	g_n
0	$\frac{1+\sqrt{3}}{4}$	$\frac{1-\sqrt{3}}{4}$
1	$\frac{3+\sqrt{3}}{4}$	$-\frac{3-\sqrt{3}}{4}$
2	$\frac{3-\sqrt{3}}{4}$	$\frac{3+\sqrt{3}}{4}$
3	$\frac{1-\sqrt{3}}{4}$	$-\frac{1+\sqrt{3}}{4}$

Figure 6.2: *Daubechies-2 wavelet system* (Example 6.4 page 110)

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^{p} \cancel{\mathsf{Q}(z)}^{\,1} \\
&= \underbrace{\frac{1}{\sqrt{2}}}_{h_0} + z^{-1}\underbrace{\frac{1}{\sqrt{2}}}_{h_1}
\end{aligned}$$

✏

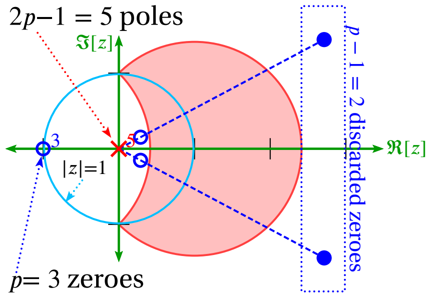
Example 6.4. The **Daubechies-2 wavelet system** yields the results illustrated in Figure 6.2 (page 110). It is also possible to compute this wavelet system using a "time-domain" technique as well—see Example 2.3 (page 40).

✎PROOF:

$$\begin{aligned}
\mathsf{Q}(z)\mathsf{Q}(z^{-1}) &= P(y)\Big|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \sum_{k=0}^{1}\binom{1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}} \\
&= (1+2y)\Big|_{y=\frac{2-z-z^{-1}}{4}} \\
&= 1+\frac{2-z-z^{-1}}{2} \\
&= \frac{4-z-z^{-1}}{2} \\
&= \frac{4z-z^2-1}{2z} \\
&= \frac{z^2-4z+1}{-2z} \\
&= \frac{1}{-2z}\left[z-(2-\sqrt{3})\right]\left[z-(2+\sqrt{3})\right] \\
&= \frac{1}{-2}\left[z-(2-\sqrt{3})\right]\left[1-(2+\sqrt{3})z^{-1}\right]
\end{aligned}$$

EX

n	h_n	g_n
0	0.3326705530	0.0352262919
1	0.8068915093	0.0854412739
2	0.4598775021	−0.1350110200
3	−0.1350110200	−0.4598775021
4	−0.0854412739	0.8068915093
5	0.0352262919	−0.3326705530



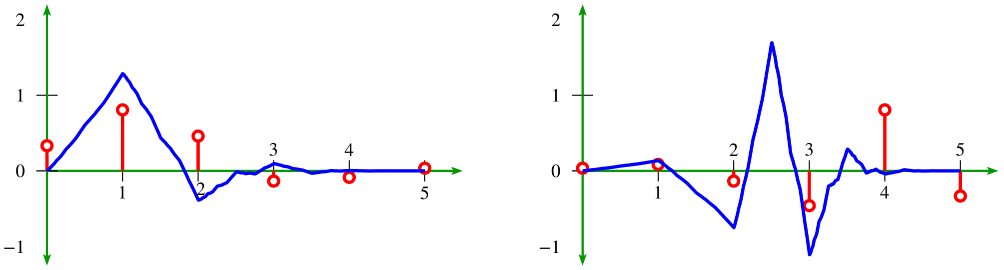


Figure 6.3: *Daubechies-3 wavelet system* (Example 6.5 page 111)

$$
\begin{aligned}
&= \frac{1}{2}\left[z-(2-\sqrt{3})\right]\left[(2+\sqrt{3})z^{-1}-1\right]\\
&= \frac{2+\sqrt{3}}{2}\left[z-(2-\sqrt{3})\right]\left[\frac{2+\sqrt{3}}{2+\sqrt{3}}z^{-1}-\frac{1}{2+\sqrt{3}}\right]\\
&= \frac{2+\sqrt{3}}{2}\left[z-(2-\sqrt{3})\right]\left[z^{-1}-(2-\sqrt{3})\right]\\
&= \underbrace{\frac{1+\sqrt{3}}{\sqrt{2}}\left[z-(2-\sqrt{3})\right]}_{\mathsf{Q}(z)}\underbrace{\frac{1+\sqrt{3}}{\sqrt{2}}\left[z^{-1}-(2-\sqrt{3})\right]}_{\mathsf{Q}(z^{-1})}
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z)\\
&= \frac{\sqrt{2}}{4}\left(\frac{z+1}{z}\right)^2 \mathsf{Q}(z)\\
&= \frac{\sqrt{2}}{4z^2}\frac{1+\sqrt{3}}{2}\left[z-(2-\sqrt{3})\right][z+1]^2\\
&= \frac{\sqrt{2}}{8z^2}(1+\sqrt{3})\left[z-(2-\sqrt{3})\right]\left[z^2+z+1\right]\\
&= \frac{\sqrt{2}}{8z^2}(1+\sqrt{3})[z^3+z^2(\sqrt{3})+z(-3+2\sqrt{2})+(-2+\sqrt{3})]\\
&= \frac{\sqrt{2}}{8z^2}\left[z^3(1+\sqrt{3})+z^2(3+\sqrt{3})+z(3-\sqrt{3})+(1-\sqrt{3})\right]\\
&= z\underbrace{\frac{\sqrt{2}}{8}(1+\sqrt{3})}_{h_0}+\underbrace{\frac{\sqrt{2}}{8}(3+\sqrt{3})}_{h_1}+z^{-1}\underbrace{\frac{\sqrt{2}}{8}(3-\sqrt{3})}_{h_2}+z^{-2}\underbrace{\frac{\sqrt{2}}{8}(1-\sqrt{3})}_{h_3}
\end{aligned}
$$

✏

Example 6.5. The **Daubechies-3 wavelet system** yields the results illustrated in Figure 6.3 (page 111).

✎PROOF:

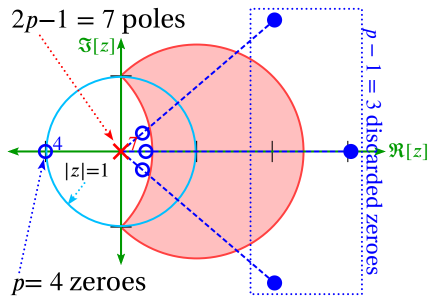
$$\begin{aligned}
\mathsf{Q}(z)\mathsf{Q}(z^{-1}) &= P(y)|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \left.\sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\right|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \left.\sum_{k=0}^{2}\binom{2+k}{k}y^k\right|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \left.\frac{2!}{0!(2-0)!}y^0+\frac{3!}{1!(3-1)!}y^1+\frac{4!}{2!(4-2)!}y^2\right|_{y=\frac{2-z-z^{-1}}{4}} \\
&= \left[1+3y+6y^2\right]_{y=\frac{2-z-z^{-1}}{4}} \\
&= 1+3\left(\frac{2-z-z^{-1}}{4}\right)+6\left(\frac{2-z-z^{-1}}{4}\right)^2 \\
&= 1+3\left(\frac{z^2-2z+1}{-4z}\right)+6\left(\frac{z^2-2z+1}{-4z}\right)^2 \\
&= 1+\left(\frac{-3z^2+6z-3}{4z}\right)+6\left(\frac{z^4-4z^3+6z^2-4z+1}{16z^2}\right) \\
&= \frac{16z^2}{16z^2}+\left(\frac{-12z^3+24z^2-12z}{16z^2}\right)+\left(\frac{6z^4-24z^3+36z^2-24z+6}{16z^2}\right) \\
&= \frac{6z^4-36z^3+76z^2-36z+6}{16z^2} \\
&= \frac{3}{8z^2}\left(z^4-6z^3+\frac{38}{3}z^2-6z+1\right) \\
&= \frac{3}{8z^2}\left(z^4-6z^3+\frac{38}{3}z^2-6z+1\right) \\
&= \frac{3}{8z^2}(z-r_1)(z-r_1^*)(z-r_2)(z-r_2^*) \\
&= \frac{3}{8}(z-r_1)(z-r_1^*)(1-z^{-1}r_2)(1-z^{-1}r_2^*) \\
&= \frac{3r_2r_2^*}{8}(z-r_1)(z-r_1^*)(r_2^{-1}-z^{-1})(r_2^{*-1}-z^{-1}) \\
&= \frac{3r_2r_2^*}{8}(z-r_1)(z-r_1^*)(\frac{1}{r_2}-z^{-1})(\frac{1}{r_2^*}-z^{-1}) \\
&= \frac{3|r_2|^2}{8}(z-r_1)(z-r_1^*)(r_1^*-z^{-1})(r_1-z^{-1}) \\
&= \frac{3|r_2|^2}{8}(z-r_1)(z-r_1^*)(z^{-1}-r_1^*)(z^{-1}-r_1) \\
&= \underbrace{\sqrt{\frac{3|r_2|^2}{8}}(z-r_1)(z-r_1^*)}_{\mathsf{Q}(z)}\underbrace{\sqrt{\frac{3|r_2|^2}{8}}(z^{-1}-r_1^*)(z^{-1}-r_1)}_{\mathsf{Q}(z^{-1})}
\end{aligned}$$

$$\begin{aligned}
r_1 &= 0.287251+i0.152892 = \frac{1}{r_2^*} \\
r_2 &= 2.712749+i1.443887 = \frac{1}{r_1^*}
\end{aligned}$$

$$\check{\mathsf{h}}(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p\mathsf{Q}(z)$$

EX

n	h_n	g_n
0	0.2303778133	−0.0105974018
1	0.7148465706	−0.0328830117
2	0.6308807679	0.0308413818
3	−0.0279837694	0.1870348117
4	−0.1870348117	−0.0279837694
5	0.0308413818	−0.6308807679
6	0.0328830117	0.7148465706
7	−0.0105974018	−0.2303778133



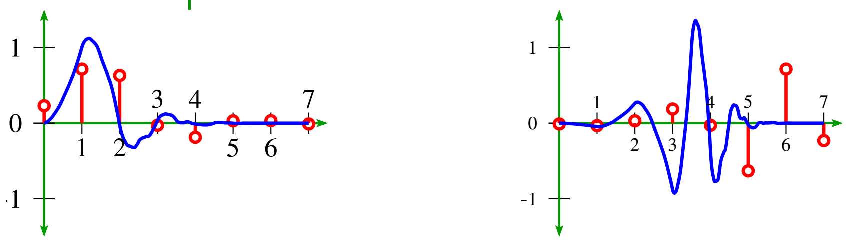


Figure 6.4: *Daubechies-4 wavelet system* (Example 6.6 page 113)

$$
\begin{aligned}
&= \sqrt{2}\left(\frac{z+1}{2z}\right)^3 \mathrm{Q}(z)\\
&= \sqrt{2}\sqrt{\frac{3|r_2|^2}{8}}\left(\frac{z+1}{2z}\right)^3 (z-r_1)(z-r_1^*)\\
&= \left(\frac{\sqrt{3|r_2|^2}}{16}\right)\frac{(z+1)^3(z-r_1)(z-r_1^*)}{z^3}\\
&= \left(\frac{\sqrt{3|r_2|^2}}{16}\right)\frac{(z^3+3z^2+3z+1)(z^2-2z\mathbf{R}_\mathrm{e}\left[r_1\right]+|r_1|^2)}{z^3}\\
&= \frac{h_0z^5+h_1z^4+h_2z^3+h_3z^2+h_4z+h_5}{z^3}
\end{aligned}
$$

$$
\begin{aligned}
h_0 &= +0.3326705529500830\\
h_1 &= +0.8068915093110932\\
h_2 &= +0.4598775021184915\\
h_3 &= -0.1350110200102552\\
h_4 &= -0.0854412738820268\\
h_5 &= +0.0352262918857096
\end{aligned}
$$

✏

Example 6.6. The **Daubechies-4 wavelet system** yields the results illustrated in Figure 6.4 (page 113).

✎PROOF:

$$
\begin{aligned}
P(y)\big|_{y=\frac{2-z-z^{-1}}{4}} &= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}}\\
&= \sum_{k=0}^{3}\binom{3+k}{k}y^k\Bigg|_{y=\frac{2-z-z^{-1}}{4}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3!}{0!(3-0)!}y^0 + \frac{4!}{1!(4-1)!}y^1 + \frac{5!}{2!(5-2)!}y^2 + \frac{6!}{3!(6-3)!}y^3\bigg|_{y=\frac{2-z-z^{-1}}{4}}\\
&= \left[1+4y+10y^2+20y^3\right]_{y=\frac{2-z-z^{-1}}{4}}\\
&= 1+4\left(\frac{2-z-z^{-1}}{4}\right)+10\left(\frac{2-z-z^{-1}}{4}\right)^2+20\left(\frac{2-z-z^{-1}}{4}\right)^3
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= +0.3288759177860292\\
r_2 &= +0.2840962981918215 + 0.2432282259103822i\\
r_3 &= +0.2840962981918215 - 0.2432282259103822i && = r_2^*\\
r_4 &= +3.0406604616474535 && = \frac{1}{r_1}\\
r_5 &= +2.0311355120914394 - 1.7389508076448230i && = \frac{1}{r_2}\\
r_6 &= +2.0311355120914394 + 1.7389508076448230i && = r_5^*
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z)\\
&= \sqrt{2}\left(\frac{z+1}{2z}\right)^4 \mathsf{Q}(z)\\
&= A_0\left[\frac{z+1}{2}\right]^4 \frac{(z-r_1)(z-r_2)(z-r_2^*)}{z^4}\\
&= \frac{h_0z^7+h_1^6+h_2z^5+h_3z^4+h_4z^3+h_5z^2+h_6z+h_7}{z^4}
\end{aligned}
$$

$$
\begin{aligned}
h_0 &= +0.2303778133088954\\
h_1 &= +0.7148465705529126\\
h_2 &= +0.6308807679298577\\
h_3 &= -0.0279837694168571\\
h_4 &= -0.1870348117190903\\
h_5 &= +0.0308413818355609\\
h_6 &= +0.0328830116668849\\
h_7 &= -0.0105974017850690
\end{aligned}
$$

✏

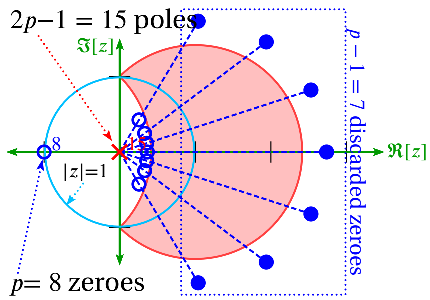
Example 6.7. The **Daubechies-**8 **wavelet system** yields the results illustrated in Figure 6.5 (page 115).

✎PROOF:

$$
\begin{aligned}
\mathsf{Q}(z)\mathsf{f}(Q^{-1}) &= P\left(\frac{2-z-z^{-1}}{4}\right)\bigg|_{y=\frac{2-z-z^{-1}}{4}}\\
&= \sum_{k=0}^{p-1}\binom{p-1+k}{k}y^k\bigg|_{y=\frac{2-z-z^{-1}}{4}}\\
&= \sum_{k=0}^{7}\binom{7+k}{k}y^k\bigg|_{y=\frac{2-z-z^{-1}}{4}}\\
&= \left[1+8y+36y^2+120y^3+330y^4+792y^5+1716y^6+3432y^7\right]_{y=\frac{2-z-z^{-1}}{4}}
\end{aligned}
$$

EX

n	h_n	g_n
0	0.0544158422	−0.0001174768
1	0.3128715909	−0.0006754494
2	0.6756307363	−0.0003917404
3	0.5853546837	0.0048703530
4	−0.0158291053	0.0087460940
5	−0.2840155430	−0.0139810279
6	0.0004724846	−0.0440882539
7	0.1287474266	0.0173693010
8	−0.0173693010	0.1287474266
9	−0.0440882539	−0.0004724846
10	0.0139810279	−0.2840155430
11	0.0087460940	0.0158291053
12	−0.0048703530	0.5853546837
13	−0.0003917404	−0.6756307363
14	0.0006754494	0.3128715909
15	−0.0001174768	−0.0544158422



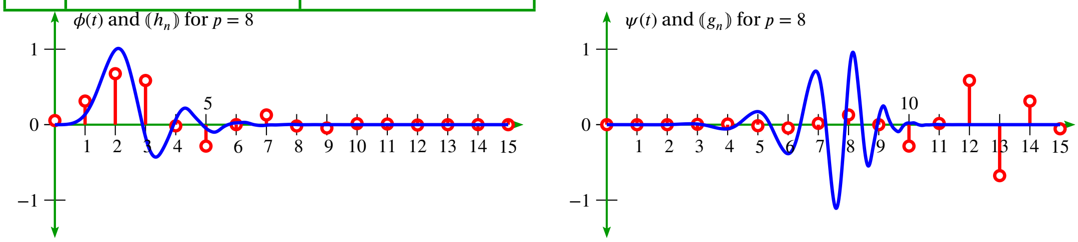


Figure 6.5: *Daubechies-8 wavelet system* (Example 6.7 page 114)

r_1	=	+0.3654035130742002		r_8	=	+2.7367005631344137	
r_2	=	+0.3577427639711839	+0.1159310245530727i	r_9	=	+1.0379714430713773	+1.7304352168829922i
r_3	=	+0.3577427639711839	−0.1159310245530727i	r_{10}	=	+1.0379714430713773	−1.7304352168829922i
r_4	=	+0.3298169959381100	+0.2476497421117944i	r_{11}	=	+1.9388494686323963	+1.4558242201373777i
r_5	=	+0.3298169959381100	−0.2476497421117944i	r_{12}	=	+1.9388494686323963	−1.4558242201373777i
r_6	=	+0.2549176775413241	+0.4249813706999120i	r_{13}	=	+2.5296496127413031	+0.8197646490830808i
r_7	=	+0.2549176775413241	−0.4249813706999120i	r_{14}	=	+2.5296496127413031	−0.8197646490830808i

$$
\begin{aligned}
\check{\mathsf{h}}(z) &= \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^p \mathsf{Q}(z) \\
&= \sqrt{2}\left(\frac{z+1}{2z}\right)^8 \mathsf{Q}(z) \\
&= A_0\,\frac{(z+1)^8(z-r_1)(z-r_2)(z-r_2^*)(z-r_4)(z-r_4^*)(z-r_6)(z-r_6^*)}{z^8} \\
&= \frac{h_0z^{15}+h_1z^{14}+h_2z^{13}+\cdots+h_{13}z^2+h_{14}z+h_{15}}{z^8}
\end{aligned}
$$

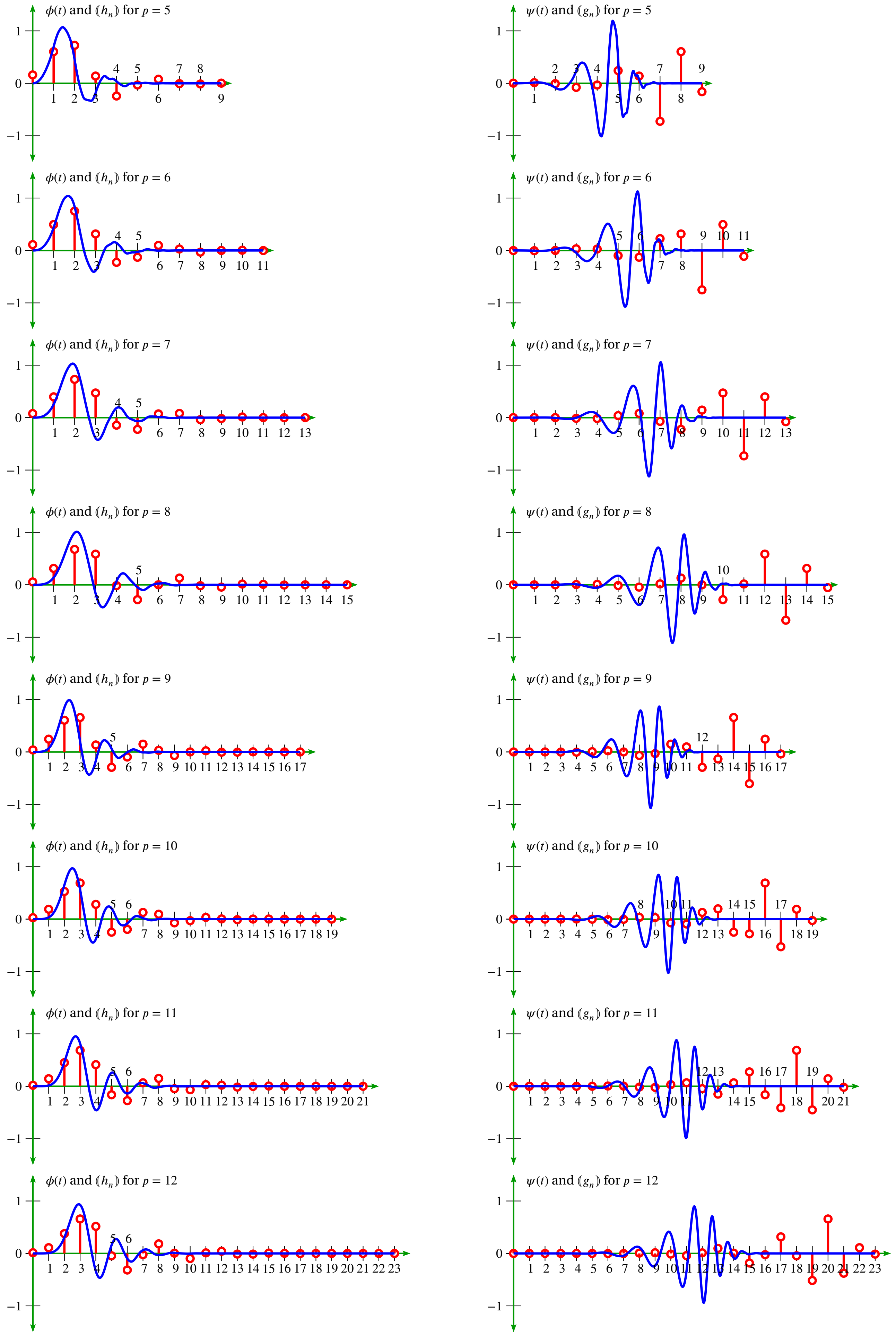
h_0	=	+0.0544158422430650	h_6	=	+0.0004724845739543	h_{11}	=	+0.0087460940474007
h_1	=	+0.3128715909140905	h_7	=	+0.1287474266204180	h_{12}	=	−0.0048703529934458
h_2	=	+0.6756307362969000	h_8	=	−0.0173693010017349	h_{13}	=	−0.0003917403733767
h_3	=	+0.5853546836540295	h_9	=	−0.0440882539307343	h_{14}	=	+0.0006754494064499
h_4	=	−0.0158291052560285	h_{10}	=	+0.0139810279173888	h_{15}	=	−0.0001174767841246
h_5	=	−0.2840155429611567						

✏

EX

p	n	h_n
5	0	0.1601023980
	1	0.6038292698
	2	0.7243085284
	3	0.1384281459
	4	−0.2422948871
	5	−0.0322448696
	6	0.0775714938
	7	−0.0062414902
	8	−0.0125807520
	9	0.0033357253
6	0	0.1115407434
	1	0.4946238904
	2	0.7511339080
	3	0.3152503517
	4	−0.2262646940
	5	−0.1297668676
	6	0.0975016056
	7	0.0275228655
	8	−0.0315820393
	9	0.0005538422
	10	0.0047772575
	11	−0.0010773011
7	0	0.0778520541
	1	0.3965393195
	2	0.7291320908
	3	0.4697822874
	4	−0.1439060039
	5	−0.2240361850
	6	0.0713092193
	7	0.0806126092
	8	−0.0380299369
	9	−0.0165745416
	10	0.0125509986
	11	0.0004295780
	12	−0.0018016407
	13	0.0003537138
8	0	0.0544158422
	1	0.3128715909
	2	0.6756307363
	3	0.5853546837
	4	−0.0158291053
	5	−0.2840155430
	6	0.0004724846
	7	0.1287474266
	8	−0.0173693010
	9	−0.0440882539
	10	0.0139810279
	11	0.0087460940
	12	−0.0048703530
	13	−0.0003917404
	14	0.0006754494
	15	−0.0001174768
9	0	0.0380779474
	1	0.2438346746
	2	0.6048231237
	3	0.6572880781
	4	0.1331973858
	5	−0.2932737833
	6	−0.0968407832
	7	0.1485407493
	8	0.0307256815
	9	−0.0676328291
	10	0.0002509471
	11	0.0223616621
	12	−0.0047232048
	13	−0.0042815037
	14	0.0018476469
	15	0.0002303858
	16	−0.0002519632
	17	0.0000393473
10	0	0.0266700579
	1	0.1881768001
	2	0.5272011889
	3	0.6884590395
	4	0.2811723437
	5	−0.2498464243
	6	−0.1959462744
	7	0.1273693403
	8	0.0930573646
	9	−0.0713941472
	10	−0.0294575368
	11	0.0332126741
	12	0.0036065536
	13	−0.0107331755
	14	0.0013953517
	15	0.0019924053
	16	−0.0006858567
	17	−0.0001164669
	18	0.0000935887
	19	−0.0000132642
11	0	0.0186942978
	1	0.1440670212
	2	0.4498997644
	3	0.6856867749
	4	0.4119643689
	5	−0.1622752450
	6	−0.2742308468
	7	0.0660435882
	8	0.1498120125
	9	−0.0464799551
	10	−0.0664387857
	11	0.0313350902
	12	0.0208409044
	13	−0.0153648209
	14	−0.0033408589
	15	0.0049284177
	16	−0.0003085929
	17	−0.0008930233
	18	0.0002491525
	19	0.0000544391
	20	−0.0000346350
	21	0.0000044943
12	0	0.0131122580
	1	0.1095662728
	2	0.3773551352
	3	0.6571987226
	4	0.5158864784
	5	−0.0447638857
	6	−0.3161784538
	7	−0.0237792573
	8	0.1824786059
	9	0.0053595697
	10	−0.0964321201
	11	0.0108491303
	12	0.0415462775
	13	−0.0122186491
	14	−0.0128408252
	15	0.0067114990
	16	0.0022486072
	17	−0.0021795036
	18	0.0000065451
	19	0.0003886531
	20	−0.0000885041
	21	−0.0000242415
	22	0.0000127770
	23	−0.0000015291

Table 6.1: Daubechies-p scaling coefficients $(\!(h_n)\!)$


$\phi(t)$ and $(\!(h_n)\!)$ for $p = 5$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 5$
$\phi(t)$ and $(\!(h_n)\!)$ for $p = 6$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 6$
$\phi(t)$ and $(\!(h_n)\!)$ for $p = 7$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 7$
$\phi(t)$ and $(\!(h_n)\!)$ for $p = 8$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 8$
$\phi(t)$ and $(\!(h_n)\!)$ for $p = 9$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 9$
$\phi(t)$ and $(\!(h_n)\!)$ for $p = 10$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 10$
$\phi(t)$ and $(\!(h_n)\!)$ for $p = 11$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 11$
$\phi(t)$ and $(\!(h_n)\!)$ for $p = 12$
$\psi(t)$ and $(\!(g_n)\!)$ for $p = 12$

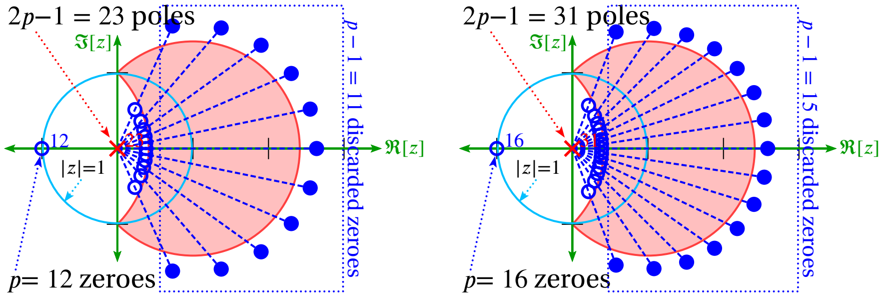


Figure 6.6: *Daubechies-p wavelet system* pole zero plots

CHAPTER 7

QUASI-SYMMETRY CONSTRAINT

7.1 Design details

The Daubechies-p wavelets (CHAPTER 6 page 95) are in general *asymmetric* because they are constructed under a *minimum phase* constraint—all zeros for polynomial $Q(z)$ are selected to be inside the unit circle. Suppose we want to design filters such that the number of vanishing moments is the same, but we are willing to lift the minimum phase condition to gain more symmetry. That is, instead of choosing the zeros from $Q(z)Q(z^{-1})$ that are inside the unit circle only, we choose some that are inside and some that are outside to give better balance. However, for zeros that are complex, we still select them in conjugate pairs so that the coefficients of $Q(z)$ are real.

Definition 7.1. *Let $Q(z)$ be a polynomial with real coefficients and*

$$\begin{aligned}
P(y) &\triangleq \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k \\
\dot{Q} &\triangleq \left\{ Q(z) | Q(z)Q(z^{-1}) = P\left(\frac{2-z-z^{-1}}{4}\right\}\right) && \textit{(Fejér-Riesz spectral factorizations of } P\textit{)} \\
\mathbf{R}Q &\triangleq (\!(r_n)\!)_{r_n \text{ is a root of } Q(z)} && \textit{(roots of } Q(z)\textit{)} \\
[\mathbf{A}Q](\omega) &\triangleq \operatorname{atan}\left(\frac{\mathbf{I}_{\mathsf{m}}\left[Q(e^{i\omega})\right]}{\mathbf{R}_{\mathsf{e}}\left[Q(e^{i\omega})\right]} \right) && \textit{(phase of } Q(e^{i\omega})\textit{)} \\
\mathbf{E}Q &\triangleq \inf_{m,c\in\mathbb{R}} \|[\mathbf{A}Q](\omega) - [m\omega + c]\|^2 && \textit{(linear phase estimation error of } Q(z)\textit{)} \\
\mathbf{M}Q &\triangleq \sum_{r_n \in \mathbf{R}Q} |r_n|^2 && \textit{(magnitude of roots of } Q(z)\textit{)} \\
\dot{E} &\triangleq \left\{ \varepsilon = \mathbf{E}Q | Q \in \dot{Q} \right\} && \textit{(set of all phase errors)} \\
\dot{M} &\triangleq \left\{ m = \mathbf{M}Q | Q \in \dot{Q} \right\} && \textit{(set of all root magnitudes)}
\end{aligned}$$

*Then the **Symlet-p scaling function** is the sequence $(\!(h_n)\!)_{n\in\mathbb{Z}}$ with z-transform $\check{\mathsf{h}}(z) = \sum_n h_n z^{-n}$ that satisfies*

DEF

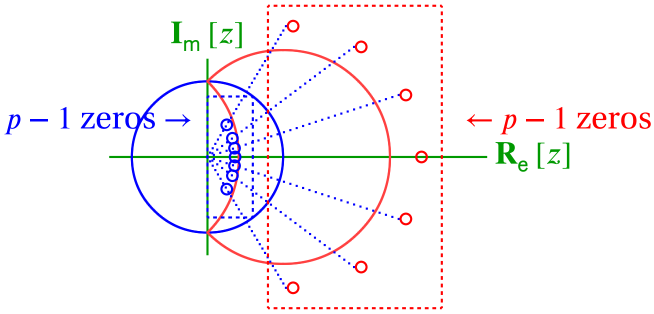
$$
\begin{array}{rrcl}
1. & \check{\mathsf{h}}(z) & = & \left(\dfrac{1+z^{-1}}{2}\right)^p Q(z) \\
2. & Q(z) & = & \arg\min_{Q(z)\in\dot{Q}} \left\{\mathbf{M}Q \in \dot{M} \,|\, \mathbf{E}Q = \min \dot{E}\right\}
\end{array}
$$

The Symlets of Definition 7.1 page 119 can be implemented [1] by the following steps: [2]

1. Compute the polynomial $P(y)$. This polynomial has $p-1$ roots in y.

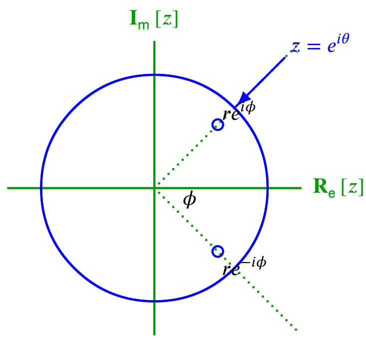
$$P(y) \triangleq \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k \qquad \text{where } \binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}$$

2. Compute $P\left(\frac{2-z-z^{-1}}{4}\right)$. This polynomial has $2p-2$ roots in z.



(a) Because the coefficients of $P\left(\frac{2-z-z^{-1}}{4}\right)$ are real, all of its roots occur in *complex conjugate pairs*:

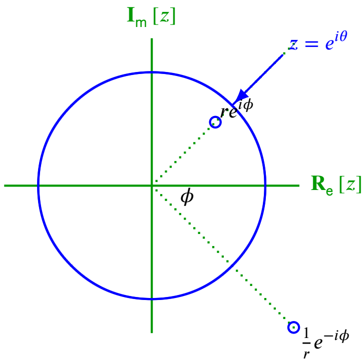
$$\text{root at } z_1 = re^{i\theta} \iff \text{root at } z_1^* = \left(re^{i\theta}\right)^* = re^{-i\theta}$$



(b) All of the roots of $P\left(\frac{2-z-z^{-1}}{4}\right)$ occur in *conjugate reciprocal pairs*:

$$P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = re^{i\theta} \iff P\left(\frac{2-z-z^{-1}}{4}\right) \text{ has a root at } z = \left(re^{i\theta}\right)^{-1} = \frac{1}{r}e^{-i\theta}$$

[1] For an actual implementation using *Octave*, see Section V.1 (page 399).
[2] Daubechies (1992), pages 254–257



3. By the *Fejér-Riesz spectral factorization* theorem (Theorem I.5 page 242) $P\left(\frac{2-z-z^{-1}}{4}\right)$ can be factored in the form
$$P\left(\frac{2-z-z^{-1}}{4}\right) = Q(z)Q(z^{-1}).$$
Find all such factors of $P\left(\frac{2-z-z^{-1}}{4}\right)$ under the following constraints:
   (a) The selected $p-1$ roots for $Q(z)$ occur in complex conjugate pairs (so $Q(z)$ will have real coefficients).
   (b) $Q(z)$ contains exactly one root from each conjugate reciprocal pair of $P\left(\frac{2-z-z^{-1}}{4}\right)$.

   The above two constraints imply that there are $2^{\lfloor p/2 \rfloor}$ choices of roots for $Q(z)$.
4. Find the two choices that result in a phase that is the closest (in the least square sense) to a straight line. There are two minimum error choices because for each choice, selecting the complementary choice (selecting the zeros that were discarded and discarding the zeros that were selected) will result in the exact same amount of phase error.
5. Of these two choices, select the one where the sum of the magnitude of the $p-1$ roots is least.[3]
6. The scaling coefficients $⦅h_n⦆$ are the coefficients of the polynomial
$$\check{\mathsf{h}}(z) = \underbrace{\underbrace{\left(\frac{1+z^{-1}}{2}\right)^p}_{\text{order } p} \underbrace{Q(z)}_{\text{order } p-1}}_{\text{order } 2p-1 \text{ } (2p \text{ coefficients})}$$

Theorem 7.1.

THM

Symlet-p wavelets have the following properties:

1. *Symlet-p wavelets have optimum linear phase (in the least squares sense) among all the Bezout polynomial factorization solutions.*
2. *$⦅h_n⦆$ has support size $2p-1$ ($⦅h_n⦆$) has $2p$ non-zero elements).*
3. *The support size of $⦅h_n⦆$ is the smallest possible for wavelets with p vanishing moments.*

[3] This step is arbitrary and not required by Daubechies' definition of Symlets.

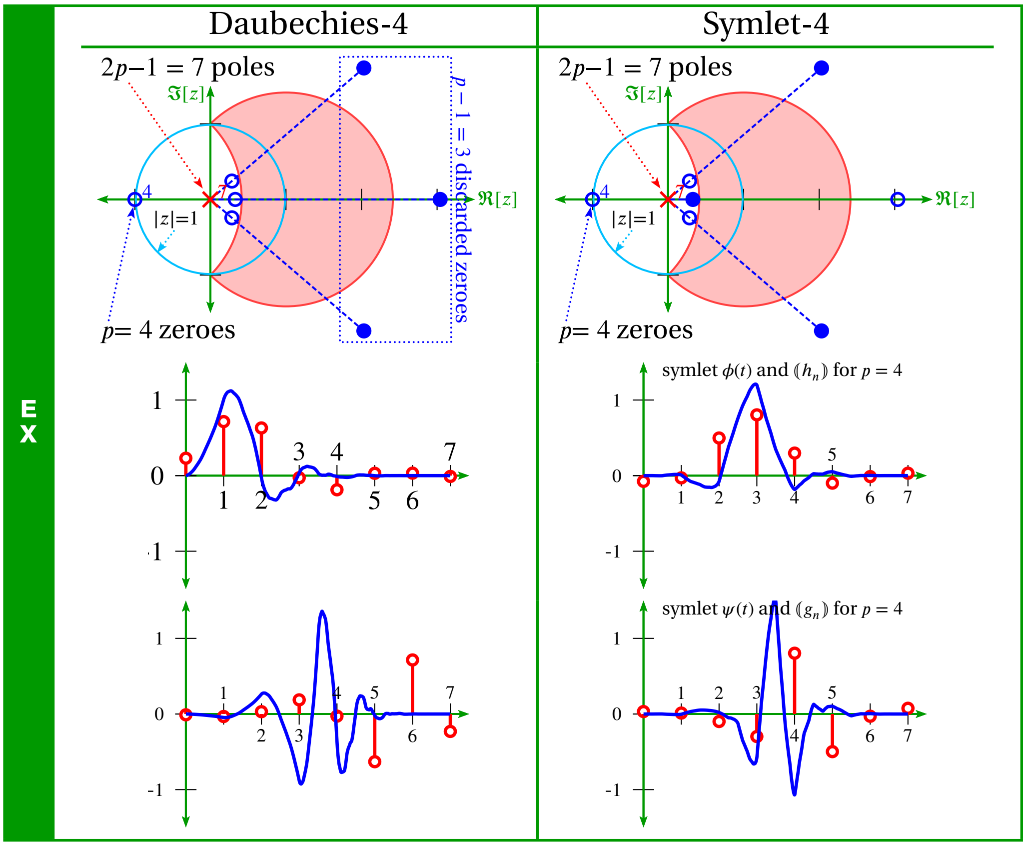


Figure 7.1: Daubechies-4 and Symlet-4 wavelet systems (Example 7.1 page 122)

✎PROOF:

1. Proof that $\tilde{\mathsf{h}}(\omega)$ has optimum **linear phase**: The optimization is performed exhaustively (all the phases are computed and linear error is measured). This is not a rigorous proof.
2. Proof that $⦅h_n⦆$ has support size $2p-1$: See Theorem 6.1 page 104.
3. Proof that the support size of $⦅h_n⦆$ is the smallest possible for wavelets with p vanishing moments. See Theorem 6.1 page 104.

⇨

7.2 Examples

Example 7.1. Figure 7.1 (page 122) compares the Daubechies-4 and Symlet-4 wavelet structures:

Example 7.2. Figure 7.2 (page 124) compares the Daubechies-8 and Symlet-8 wavelet structures.

Example 7.3. Figure 7.3 (page 124) compares the Daubechies-12 and Symlet-12 wavelet structures.

Example 7.4. Figure 7.4 (page 125) compares the Daubechies-16 and Symlet-16 wavelet structures.

Example 7.5. Figure 7.5 (page 125) and Figure 7.6 (page 126) show the pole-zero plots for Symlet-4, Symlet-5,…, Symlet-17 structures.

p	n	h_n
1	0	$\frac{1}{\sqrt{2}}$
	1	$\frac{1}{\sqrt{2}}$
2	0	$\frac{\sqrt{2}}{8}(1+\sqrt{3})$
	1	$\frac{\sqrt{2}}{8}(3+\sqrt{3})$
	2	$\frac{\sqrt{2}}{8}(3-\sqrt{3})$
	3	$\frac{\sqrt{2}}{8}(1-\sqrt{3})$
3	0	0.3326705530
	1	0.8068915093
	2	0.4598775021
	3	−0.1350110200
	4	−0.0854412739
	5	0.0352262919
4	0	−0.0757657148
	1	−0.0296355276
	2	0.4976186676
	3	0.8037387518
	4	0.2978577956
	5	−0.0992195436
	6	−0.0126039673
	7	0.0322231006
5	0	0.0273330683
	1	0.0295194909
	2	−0.0391342493
	3	0.1993975340
	4	0.7234076904
	5	0.6339789635
	6	0.0166021058
	7	−0.1753280899
	8	−0.0211018340
	9	0.0195388827
6	0	0.0154041093
	1	0.0034907121
	2	−0.1179901111
	3	−0.0483117426
	4	0.4910559419
	5	0.7876411410
	6	0.3379294217
	7	−0.0726375228
	8	−0.0210602925
	9	0.0447249018
	10	0.0017677119
	11	−0.0078007083
7	0	0.0120154193
	1	0.0172133763
	2	−0.0649080035
	3	−0.0641312898
	4	0.3602184609
	5	0.7819215933
	6	0.4836109157
	7	−0.0568044769
	8	−0.1010109209
	9	0.0447423495
	10	0.0204642076
	11	−0.0181266051
	12	−0.0032832978
	13	0.0022918340
8	0	−0.0033824160
	1	−0.0005421323
	2	0.0316950878
	3	0.0076074873
	4	−0.1432942384
	5	−0.0612733591
	6	0.4813596513
	7	0.7771857517
	8	0.3644418948
	9	−0.0519458381
	10	−0.0272190299
	11	0.0491371797
	12	0.0038087520
	13	−0.0149522583
	14	−0.0003029205
	15	0.0018899503
9	0	0.0014009155
	1	0.0006197809
	2	−0.0132719678
	3	−0.0115282102
	4	0.0302248789
	5	0.0005834627
	6	−0.0545689584
	7	0.2387609146
	8	0.7178970828
	9	0.6173384491
	10	0.0352724880
	11	−0.1915508313
	12	−0.0182337708
	13	0.0620777893
	14	0.0088592675
	15	−0.0102640640
	16	−0.0004731545
	17	0.0010694900
10	0	0.0008625782
	1	0.0007154205
	2	−0.0070567641
	3	0.0005956828
	4	0.0496861266
	5	0.0262403651
	6	−0.1215521056
	7	−0.0150192388
	8	0.5137098734
	9	0.7669548366
	10	0.3402160130
	11	−0.0878787115
	12	−0.0670899078
	13	0.0338423547
	14	−0.0008687521
	15	−0.0230054614
	16	−0.0011404298
	17	0.0050716492
	18	0.0003401493
	19	−0.0004101159
11	0	0.0006871194
	1	0.0013826742
	2	−0.0039185532
	3	−0.0027931771
	4	0.0372023572
	5	0.0509417072
	6	−0.0540827111
	7	−0.0286938383
	8	0.4078687490
	9	0.7685266798
	10	0.4520007834
	11	−0.0815151575
	12	−0.1499464788
	13	0.0182541524
	14	0.0237215478
	15	−0.0273470351
	16	−0.0085852863
	17	0.0098741222
	18	0.0024053043
	19	−0.0016456213
	20	−0.0002460505
	21	0.0001222747

Table 7.1: Symlet-p scaling coefficients (h_n)

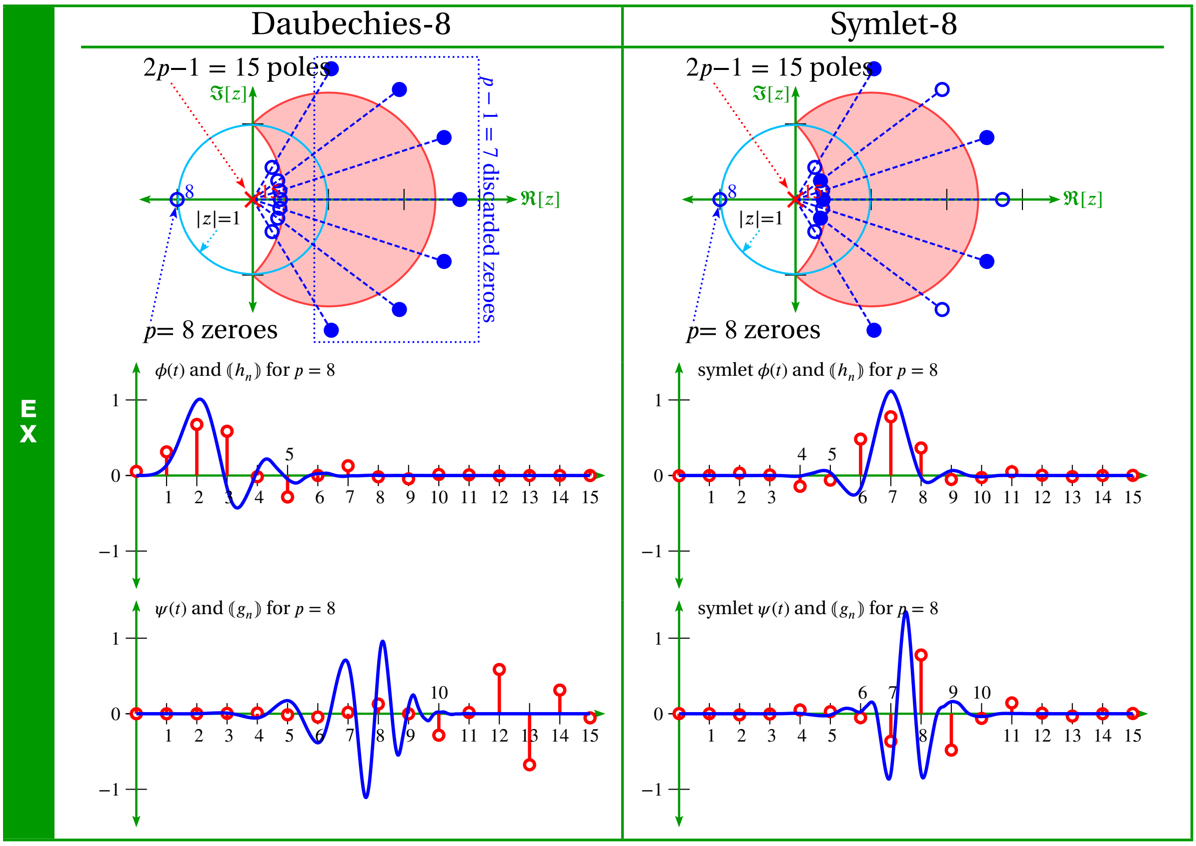


Figure 7.2: Daubechies-8 and Symlet-8 wavelet systems (Example 7.2 page 122)

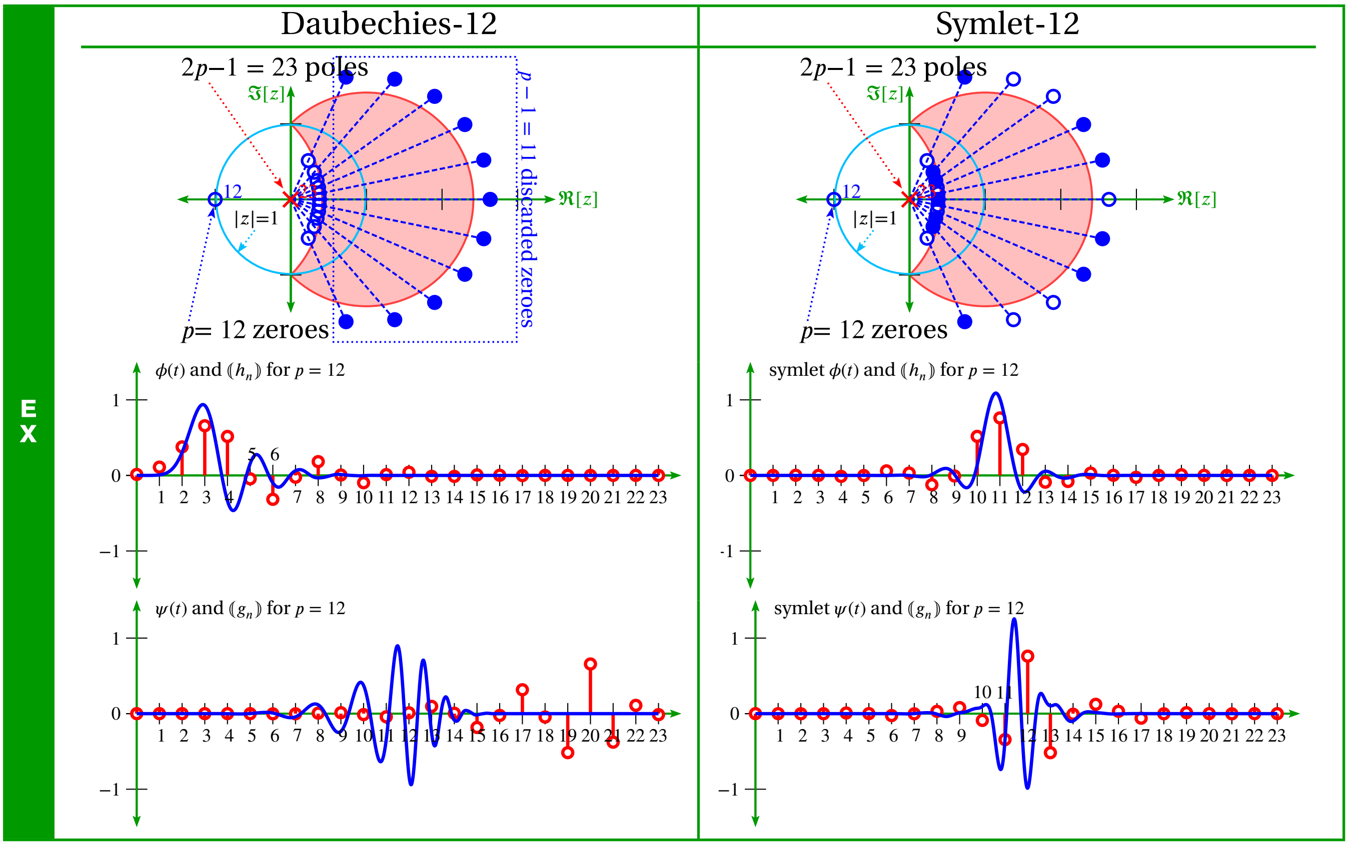


Figure 7.3: Daubechies-12 and Symlet-12 wavelet systems (Example 7.3 page 122)

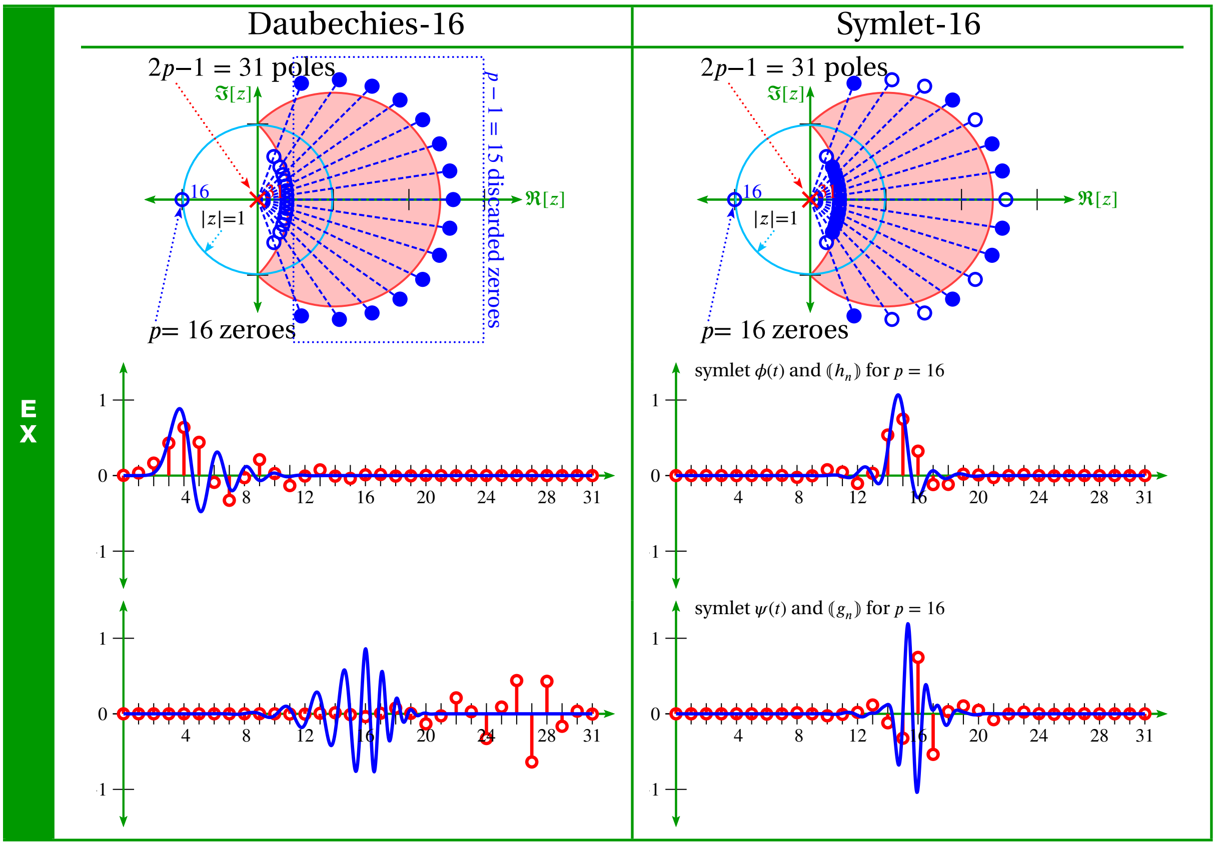


Figure 7.4: Daubechies-16 and Symlet-16 wavelet systems (Example 7.4 page 122)

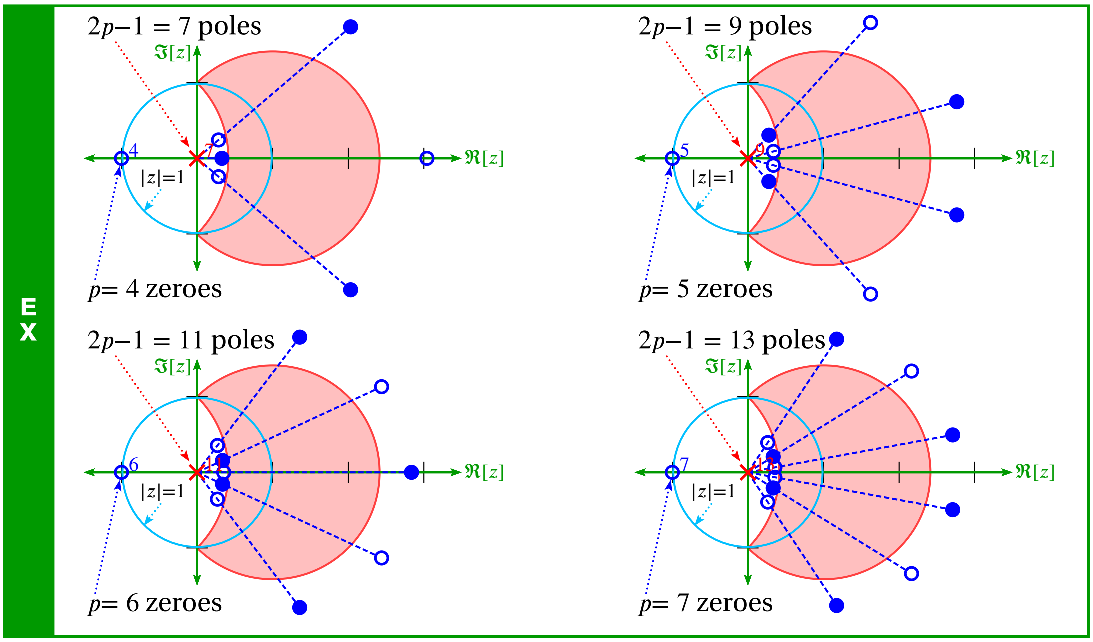


Figure 7.5: Pole-zero plots for Symlet-4, Symlet-5, Symlet-6, and Symlet-7 structures (Example 7.5 page 122)

EX

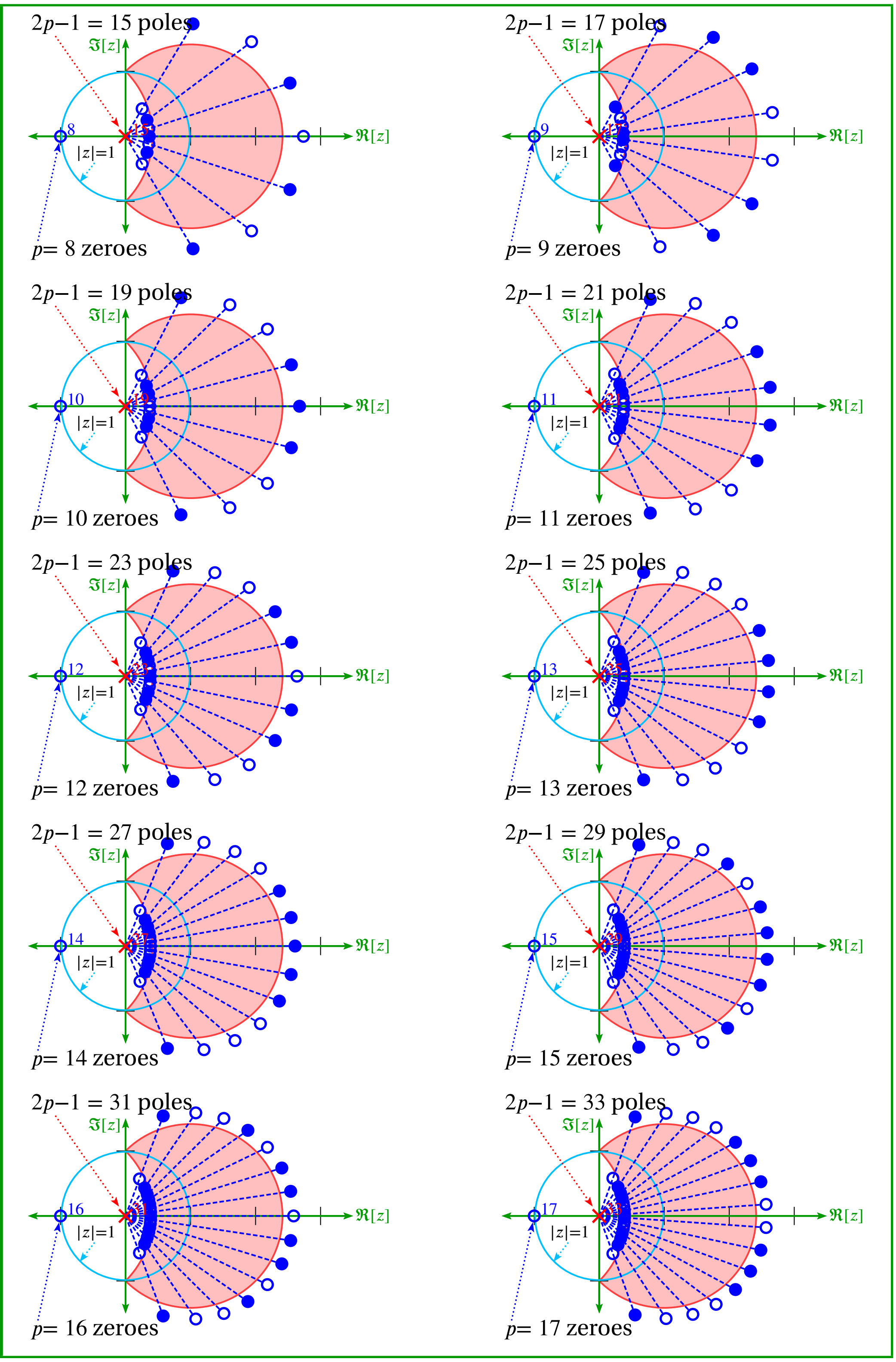


Figure 7.6: Pole-zero plots for Symlet-8, Symlet-9,..., Symlet-17 structures (Example 7.5 page 122)

Part I

Appendices

APPENDIX A

ALGEBRAIC STRUCTURES

❝In this book, learned reader, you have the rules of algebra… It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one.… this most abstruse and unsurpassed treasury of the entire arithmetic being brought to light and, as in a theater, exposed to the sight of all…❞

Gerolamo Cardano (1501–1576), Italian mathematician, physician, and astrologer [1]

A set together with one or more operations forms several standard mathematical structures:

group $\supseteq$ *ring* $\supseteq$ *commutative ring* $\supseteq$ *integral domain* $\supseteq$ *field*

Definition A.1. [2] *Let X be a set and $\diamond : X \times X \to X$ be an operation on X.*

DEF — *The pair $(X, \diamond)$ is a **group** if*

1. $\exists e \in X$ such that $e \diamond x = x \diamond e = x$ $\quad \forall x \in X$ $\quad$ (IDENTITY *element*) *and*
2. $\exists (-x) \in X$ such that $(-x) \diamond x = x \diamond (-x) = e$ $\quad \forall x \in X$ $\quad$ (INVERSE *element*) *and*
3. $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ $\quad \forall x,y,z \in X$ $\quad$ (ASSOCIATIVE)

Definition A.2. [3] *Let $+ : X \times X \to X$ and $* : X \times X \to X$ be operations on a set X. Furthermore, let the operation $*$ also be represented by juxtapostion as in $a * b \equiv ab$.*

DEF — *The triple $(X, +, *)$ is a **ring** if*

1. $(X, +)$ *is a group.* $\quad$ *(additive group)* *and*
2. $x(yz) = (xy)z$ $\quad \forall x,y,z \in X$ $\quad$ *(associative with respect to $*$)* *and*
3. $x(y+z) = (xy) + (xz)$ $\quad \forall x,y,z \in X$ $\quad$ *($*$ is left distributive over $+$)* *and*
4. $(x+y)z = (xz) + (yz)$ $\quad \forall x,y,z \in X$ $\quad$ *($*$ is right distributive over $+$).*

Definition A.3. [4]

---

[1] quote: Cardano (1545), page 1
image: http://en.wikipedia.org/wiki/Image:Cardano.jpg

[2] Durbin (2000), page 29

[3] Durbin (2000), pages 114–115

[4] Durbin (2000), page 118

DEF

A triple $(X, +, *)$ *is a* ***commutative ring*** *if*

1.	$(X, +, *)$ is a ring		(ring)	and
2.	$xy = yx$	$\forall x,y \in X$	(commutative).	

Definition A.4. [5] *Let R be a* COMMUTATIVE RING *(Definition A.3 page 129).*

DEF

A function $|\cdot|$ *in* $\mathbb{R}^{\mathbb{R}}$ *is an* ***absolute value*** *(or* ***modulus****) if*

1.	$\lvert x\rvert \geq 0$	$x \in \mathbb{R}$	(NON-NEGATIVE)	and
2.	$\lvert x\rvert = 0 \iff x = 0$	$x \in \mathbb{R}$	(NONDEGENERATE)	and
3.	$\lvert xy\rvert = \lvert x\rvert \cdot \lvert y\rvert$	$x,y \in \mathbb{R}$	(HOMOGENEOUS / SUBMULTIPLICATIVE)	and
4.	$\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$	$x,y \in \mathbb{R}$	(SUBADDITIVE / TRIANGLE INEQUALITY)	

Definition A.5. [6]

DEF

The structure $F \triangleq (X, +, \cdot, 0, 1)$ *is a* ***field*** *if*

1.	$(X, +, *)$ is a ring		(ring)	and
2.	$xy = yx$	$\forall x,y \in X$	(commutative with respect to $*$)	and
3.	$(X \setminus \{0\}, *)$ is a group		(group with respect to $*$).	

Definition A.6. [7] *Let* $\boldsymbol{V} = (F, +, \cdot)$ *be a vector space and* $\otimes : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ *be a vector-vector multiplication operator.*
An ***algebra*** *is any pair* $(\boldsymbol{V}, \otimes)$ *that satisfies* ($\otimes$ *is represented by juxtaposition)*

DEF

1.	$(\boldsymbol{ux})\boldsymbol{y} = \boldsymbol{u}(\boldsymbol{xy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$	(ASSOCIATIVE)	and
2.	$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}) = (\boldsymbol{ux}) + (\boldsymbol{uy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$	(LEFT DISTRIBUTIVE)	and
3.	$(\boldsymbol{u}+\boldsymbol{x})\boldsymbol{y} = (\boldsymbol{uy}) + (\boldsymbol{xy})$	$\forall \boldsymbol{u},\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$	(RIGHT DISTRIBUTIVE)	and
4.	$\alpha(\boldsymbol{xy}) = (\alpha\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x}(\alpha\boldsymbol{y})$	$\forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$ and $\alpha \in F$	(SCALAR COMMUTATIVE)	.

[5] Cohn (2002) page 312
[6] Durbin (2000), page 123, Weber (1893)
[7] Abramovich and Aliprantis (2002), page 3, Michel and Herget (1993), page 56

APPENDIX B

BINOMIAL RELATIONS

B.1 Factorials

Definition B.1 (factorial)**.**

DEF

*The **factorial** $n!$ is defined as*

$$n! \triangleq \begin{cases} n(n-1)(n-2)\cdots 1 & \text{for } n \in \mathbb{Z},\ n \geq 1 \\ 1 & \text{for } n \in \mathbb{Z},\ n = 0 \\ 0 & \text{for } n \in \mathbb{Z},\ n \leq -1 \end{cases}$$

Definition B.2. [1] *The quantities "x **to the** m **falling**", "x **to the** m **rising**", "x **to the** m **central**" are defined as follows:*

DEF

$$x^{\underline{m}} \triangleq \begin{cases} \underbrace{x(x-1)(x-2)\cdots(x-m+1)}_{m \text{ factors}} & \forall x\in\mathbb{C}, m\in\mathbb{N} \\ 1 & \forall x\in\mathbb{C}, m=0 \end{cases} \qquad \text{("}x\text{ to the } m \text{ falling")}$$

$$x^{\overline{m}} \triangleq \begin{cases} \underbrace{x(x+1)(x+2)\cdots(x+m-1)}_{m \text{ factors}} & \forall x\in\mathbb{C}, m\in\mathbb{N} \\ 1 & \forall x\in\mathbb{C}, m=0 \end{cases} \qquad \text{("}x\text{ to the } m \text{ rising")}$$

$$x^{\overline{\underline{m}}} \triangleq \begin{cases} \underbrace{x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1\right)}_{m \text{ factors}} & \forall x\in\mathbb{C}, m\in\mathbb{N} \\ 1 & \forall x\in\mathbb{C}, m=0 \end{cases} \qquad \text{("}x\text{ to the } m \text{ central")}$$

The rising and central expressions may be expressed in terms of the falling expression (next).

Proposition B.1. [2]

PRP

$$x^{\overline{m}} = (-1)^m x^{\underline{m}} \qquad\qquad x^{\overline{\underline{m}}} = x\left(x+\frac{m}{2}-1\right)^{\underline{(m-1)}}$$

---

[1] Graham et al. (1994) pages 47–48 ⟨equations (2.43), (2.44)⟩, Knuth (1992b), page 414 ⟨(2.11), (2.12)⟩, Aigner (2007) page 10, Steffensen (1950), page 8 ⟨descending, ascending, and central factorials⟩, Steffensen (1927), page 8 ⟨descending, ascending, and central factorials⟩

[2] Steffensen (1950), page 8 ⟨(3)⟩

✎PROOF:

$$\begin{aligned}
(-1)^m(-x)^{\underline{m}} &= (-1)^m[(-x)(-x-1)(-x-2)\cdots(-x-m+1)] && \text{by Definition B.2 page 131}\\
&= (-1)^m(-1)^m[(x)(x+1)(x+2)\cdots(x+m-1)]\\
&= x^{\overline{m}} && \text{by Definition B.2 page 131}
\end{aligned}$$

$$\begin{aligned}
x\left(x+\frac{m}{2}-1\right)^{\overline{(m-1)}} &= x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-1-1\right)\cdots\left(x+\frac{m}{2}-1-(m-1)+1\right) && \text{by Definition B.2 page 131}\\
&= x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1\right)\\
&= x^{\overline{m}}
\end{aligned}$$

✏

B.2 Binomial identities

Definition B.3 (Binomial coefficient).[3] *Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{Z}$ the set of integers. Let $x^{\underline{m}}$ represent "x to the m falling" (Definition B.2). Let $n!$ represent "n factorial" (Definition B.1).*

DEF

*The **binomial coefficient** $\binom{x}{k}$ is defined as*

$$\binom{x}{k} \triangleq \begin{cases} \dfrac{x^{\underline{k}}}{k!} & \forall x \in \mathbb{C} \quad k \in \mathbb{W} \quad (k = 0, 1, 2, 3, \ldots)\\ 0 & \forall x \in \mathbb{C} \quad k \in \mathbb{Z}^- \quad (k = -1, -2, -3, \ldots)\end{cases}$$

*The value x is called the **upper index** and the value k is called the **lower index**.*

Proposition B.2. *Let $\binom{n}{k}$ be the* BINOMIAL COEFFICIENT *(Definition B.3 page 132).*

PRP

1. $\binom{x}{0} = 1 \quad \forall x \in \mathbb{C}$	2. $\binom{n}{n} = 1 \quad \forall n \in \mathbb{W}$
3. $\binom{x}{1} = x \quad \forall x \in \mathbb{C}$	4. $\binom{x}{k} = 0 \quad \forall x \in \mathbb{C},\ x < k$

✎PROOF:

1. Proof that $\binom{x}{0} = 1$:

$$\begin{aligned}
\binom{x}{0} &= \frac{x^{\underline{0}}}{0!} && \text{by Definition B.3 page 132}\\
&= \frac{x^{\underline{0}}}{1} && \text{by Definition B.1 page 131}\\
&= 1 && \text{by Definition B.2 page 131}
\end{aligned}$$

[3] Graham et al. (1994) page 154 ⟨equation (5.1)⟩, Aigner (2007) page 10 ⟨(1)⟩, Coolidge (1949) pages 149–150, Stifel (1544)

2. Proof that $\binom{n}{n} = 1$:

$$\begin{aligned}
\binom{n}{n} &= \frac{n^{\underline{n}}}{n!} && \text{by Definition B.3 page 132}\\
&= \frac{n(n-1)\cdots(n-n+1)}{n!} && \text{by Definition B.2 page 131}\\
&= \frac{n(n-1)\cdots(1)}{n(n-1)\cdots(1)} && \text{by Definition B.1 page 131}\\
&= 1
\end{aligned}$$

3. Proof that $\binom{x}{1} = x$:

$$\begin{aligned}
\binom{x}{1} &= \frac{x^{\underline{1}}}{1!} && \text{by Definition B.3 page 132}\\
&= \frac{x^{\underline{1}}}{1} && \text{by Definition B.1 page 131}\\
&= x && \text{by Definition B.2 page 131}
\end{aligned}$$

4. Proof that $\binom{x}{k} = 0,\ \forall x < k$:

$$\begin{aligned}
\binom{x}{k} &= \frac{x^{\underline{k}}}{k!} && \text{by Definition B.3 page 132}\\
&= \frac{x(x-1)\cdots(0)\cdots(x-k+1)}{k!} && \text{by Definition B.2 page 131}\\
&= 0
\end{aligned}$$

✎

Theorem B.1. [4] *Let $\binom{n}{k}$ be the* BINOMIAL COEFFICIENT *(Definition B.3 page 132).*

THM

1.	$\binom{n}{k}$	$=$	$\dfrac{n!}{k!(n-k)!}$	$\forall n,k\in\mathbb{Z}, n\geq k\geq 0$	(FACTORIAL EXPANSION)
2.	$\binom{n}{k}$	$=$	$\binom{n}{n-k}$	$\forall n,k\in\mathbb{Z}, n\geq 0$	(SYMMETRY)
3.	$\binom{n+x+1}{n}$	$=$	$\binom{n+x}{n} + \binom{n+x}{n-1}$	$\forall n\in\mathbb{Z}, x\in\mathbb{C}$	(PASCAL'S RULE)
4.	$\binom{x+1}{k+1}$	$=$	$\binom{x}{k+1} + \binom{x}{k}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(PASCAL'S IDENTITY / STIFEL FORMULA)
5.	$\binom{x}{m}\binom{m}{k}$	$=$	$\binom{x}{k}\binom{x-k}{m-k}$	$\forall k,m\in\mathbb{Z}, x\in\mathbb{C}$	(TRINOMIAL REVISION)
6.	$\binom{x}{k}$	$=$	$\frac{x}{k}\binom{x-1}{k-1}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(ABSORPTION IDENTITY)
7.	$\binom{x}{k}$	$=$	$(-1)^k\binom{k-x-1}{k}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(UPPER NEGATION)
8.	$\binom{x}{k}$	$=$	$\binom{x-2}{k-2} + 2\binom{x-2}{k-1} + \binom{x-2}{k}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(SECOND-ORDER PASCAL'S IDENTITY)
9.	$\binom{x-1}{k-1}\binom{x}{k+1}\binom{x+1}{k}$	$=$	$\binom{x-1}{k}\binom{x}{k-1}\binom{x+1}{k+1}$	$\forall k\in\mathbb{Z}, x\in\mathbb{C}$	(HEXAGON IDENTITY)

✎PROOF:

[4] Graham et al. (1994) page 174 ⟨Table 174⟩, Gallier (2010) page 221, Gross (2008) page 227 ⟨Table 4.1.2⟩, Coolidge (1949), pages 149–150, Stifel (1544), Balakrishnan (1996) page 43 ⟨Pascal's Rule⟩, Harris et al. (2008) page 143 ⟨*hexagon identity*, (2.15)⟩, Ferland (2009) page 216 ⟨*second-order pascal identity*⟩

1. Proof for *factorial expansion*:

$$\begin{aligned}
\binom{n}{k} &\triangleq \frac{n^{\underline{k}}}{k!} && \forall n,k\in\mathbb{Z},\ n\ge k\ge 0 && \text{by Definition B.3}\\
&= \frac{n(n-1)(n-2)\cdot(n-k+1)}{k!} && \forall n,k\in\mathbb{Z},\ n\ge k\ge 0 && \text{by Definition B.2}\\
&= \frac{n(n-1)(n-2)\cdot(n-k+1)(n-k)(n-k-1)\cdots 1}{k!(n-k)!} && \forall n,k\in\mathbb{Z},\ n\ge k\ge 0 && \text{by Definition B.2}\\
&= \frac{n!}{k!(n-k)!} && \forall n,k\in\mathbb{Z},\ n\ge k\ge 0 && \text{by Definition B.1}
\end{aligned}$$

2. Proof for *symmetry* property:

(a) Proof for $n,k\in\mathbb{Z},\ n\ge k\ge 0$: (use item (1) page 134)

$$\begin{aligned}
\binom{n}{n-k} &= \frac{n!}{(n-k)!(n-(n-k)!} && \forall n,k\in\mathbb{Z},\ n\ge k\ge 0 && \text{by item (1) page 134}\\
&= \frac{n!}{k!(n-k)!} && \forall n,k\in\mathbb{Z},\ n\ge k\ge 0\\
&= \binom{n}{k} && \forall n,k\in\mathbb{Z},\ n\ge k\ge 0 && \text{by item (1) page 134}
\end{aligned}$$

(b) Proof for $n,k\in\mathbb{Z},\ n\ge 0>k$:

$$\begin{aligned}
\binom{n}{n-k} &= \frac{n^{\underline{n-k}}}{(n-k)!} && \forall n,k\in\mathbb{Z},\ n\ge 0>k && \text{by Definition B.3 page 132}\\
&= \frac{n(n-1)(n-2)\cdots 0\cdots(n-n+k+1)}{(n-k)!} && \forall n,k\in\mathbb{Z},\ n\ge 0>k && \text{by Definition B.2 page 131}\\
&= 0\\
&= \binom{n}{k} && \forall n,k\in\mathbb{Z},\ n\ge 0>k && \text{by Definition B.3 page 132}
\end{aligned}$$

(c) Proof for $n,k\in\mathbb{Z},\ n\ge 0>k$:

$$\begin{aligned}
\binom{n}{k} &= \frac{n^{\underline{k}}}{k!} && \forall n,k\in\mathbb{Z},\ k>n\ge 0 && \text{by Definition B.3 page 132}\\
&= \frac{n(n-1)(n-2)\cdots 0\cdots(n-k+1)}{(n-k)!} && \forall n,k\in\mathbb{Z},\ k>n\ge 0 && \text{by Definition B.2 page 131}\\
&= 0\\
&= \binom{n}{n-k} && \forall n,k\in\mathbb{Z},\ k>n\ge 0 && \text{by Definition B.3 page 132}
\end{aligned}$$

3. Proof for *Pascal's Rule*:

(a) Proof for $n<0,\ x\in\mathbb{C}$:

$$\begin{aligned}
\binom{n+x}{n}+\binom{n+x}{n-1} &= 0+0 && \text{by Definition B.3 page 132}\\
&= \binom{n+x+1}{n} && \text{by Definition B.3 page 132}
\end{aligned}$$

(b) Proof for $n=0,\ x\in\mathbb{C}$:

$$\begin{aligned}
\binom{n+x}{n}+\binom{n+x}{n-1} &= \binom{n+x}{0}+\binom{n+x}{-1} && \text{by } n=0 \text{ hypothesis}\\
&= 1+0 && \text{by Definition B.3 page 132}\\
&= \binom{n+x+1}{0} && \text{by Definition B.3 page 132}\\
&= \binom{n+x+1}{n} && \text{by } n=0 \text{ hypothesis}
\end{aligned}$$

(c) Proof for $n > 0,\ x \in \mathbb{C}$:

$$
\begin{aligned}
&\binom{n+x}{n} + \binom{n+x}{n-1} \\
&\triangleq \frac{n+x^{\underline{n}}}{n!} + \frac{n+x^{\underline{n-1}}}{(n-1)!} && \text{by Definition B.3 page 132} \\
&\triangleq \frac{(n+x)(n+x-1)\cdots(n+x-n+1)}{n!} \\
&\quad + \frac{(n+x)(n+x-1)\cdots(n+x-n+1+1)}{(n-1)!} && \text{by Definition B.2 page 131} \\
&= \frac{[(n+x)(n+x-1)\cdots(x+1)] + [(n+x)(n+x-1)\cdots(x+2)n]}{n!} \\
&= \frac{[(x+1)+n][(n+x)(n+x-1)\cdots(x+2)]}{n!} \\
&= \frac{(n+x+1)(n+x)(n+x-1)\cdots(x+2)}{n!} \\
&\triangleq \frac{(n+x+1)^{\underline{n}}}{n!} && \text{by Definition B.2 page 131} \\
&\triangleq \binom{n+x+1}{n} && \text{by Definition B.3 page 132}
\end{aligned}
$$

4. Proof for *Pascal's Identity*:

$$
\begin{aligned}
\binom{x+1}{k+1} &= \binom{k+y+1}{k+1} && \text{where } y \triangleq x-k \implies x = y+k \\
&= \binom{y+k}{k+1} + \binom{y+k}{k} && \text{by Pascal's Rule (item (3))} \\
&= \binom{x}{k+1} + \binom{x}{k} && \text{by definition of } m
\end{aligned}
$$

5. Proof for *Trinomial revision*:

(a) Proof for $k < 0$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= \binom{x}{m}0 && \text{by } k<0 \text{ hypothesis and Definition B.3 page 132} \\
&= \cancelto{0}{\binom{x}{k}}\binom{x-k}{m-k} && \text{by } k<0 \text{ hypothesis and Definition B.3 page 132}
\end{aligned}
$$

(b) Proof for $k \geq 0,\ m < 0$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= 0\binom{m}{k} && \text{by } m<0 \text{ hypothesis and Definition B.3 page 132} \\
&= \binom{x}{k}\cancelto{0}{\binom{x-k}{m-k}} && \text{by } k \geq 0, m<0 \text{ hypothesis and Definition B.3 page 132}
\end{aligned}
$$

(c) Proof for $m < k$ case:

$$
\begin{aligned}
\binom{x}{m}\binom{m}{k} &= \binom{x}{m}0 && \text{by Proposition B.2 page 132} \\
&= \binom{x}{k}\cancelto{0}{\binom{x-k}{m-k}} && \text{by } m<k \text{ hypothesis and Definition B.3 page 132}
\end{aligned}
$$

(d) Proof for remaining cases:

$$\begin{aligned}
&\binom{x}{m}\binom{m}{k} \\
&= \frac{x^{\underline{m}}}{m!}\frac{m^{\underline{k}}}{k!} && \text{by Definition B.3 page 132} \\
&= \frac{x(x-1)\cdots(x-m+1)}{m!}\frac{m(m-1)\cdots(m-k+1)}{k!} && \text{by Definition B.2 page 131} \\
&= \frac{x(x-1)\cdots(x-m+1)}{(m-k)!}\frac{1}{k!} \\
&= \frac{x(x-1)\cdots(x-k+1)}{k!}\frac{(x-k)(x-k-1)\cdots(x-m+1)}{(m-k)!} \\
&= \frac{x(x-1)\cdots(x-k+1)}{k!}\frac{(x-k)(x-k-1)\cdots((x-k)-(m-k)+1)}{(m-k)!} \\
&\triangleq \frac{x^{\underline{k}}}{k!}\frac{(x-k)^{\underline{m-k}}}{(m-k)!} && \text{by Definition B.2 page 131} \\
&\triangleq \binom{x}{k}\binom{x-k}{m-k} && \text{by Definition B.3 page 132}
\end{aligned}$$

6. Proof for *Absorption identity*:

$$\begin{aligned}
\frac{x}{k}\binom{x-1}{k-1} &= \frac{1}{k}\binom{x}{1}\binom{n-1}{k-1} && \text{by Proposition B.2 page 132} \\
&= \frac{1}{k}\binom{x}{k}\binom{k}{1} && \text{by Trinomial revision (item (5))} \\
&= \frac{1}{k}\binom{x}{k}k && \text{by Proposition B.2 page 132} \\
&= \binom{x}{k}
\end{aligned}$$

7. Proof for *Upper Negation*:

$$\begin{aligned}
(-1)^k\binom{k-x-1}{k} &\triangleq (-1)^k\frac{(k-x-1)^{\underline{k}}}{k!} && \text{by Definition B.3} \\
&\triangleq (-1)^k\frac{(k-x-1)(k-x-2)(k-x-3)\cdots(k-x-1-k+1)}{k!} && \text{by Definition B.2} \\
&= (-1)^k\frac{(k-x-1)(k-x-2)(k-x-3)\cdots(-x)}{k!} \\
&= (-1)^k(-1)^k\frac{(x)(x-1)\cdots(x(x-k+3)(x-k+2)(x-k+1)}{k!} \\
&\triangleq \frac{x^{\underline{k}}}{k!} && \text{by Definition B.2} \\
&\triangleq \binom{x}{k} && \text{by Definition B.3}
\end{aligned}$$

8. Proof for *2nd Order Pascal's Identity*:

$$\begin{aligned}
&\binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} \\
&\triangleq \frac{(x-2)^{\underline{(k-2)}}}{(k-2)!} + \frac{(x-2)^{\underline{(k-1)}}}{(k-1)!} + \frac{(x-2)^{\underline{k}}}{k!} \\
&\triangleq \frac{(x-2)(x-1)\cdots(x-k+2+1)}{(k-2)!} + 2\frac{(x-2)(x-1)\cdots(x-k+1+1)}{(k-1)!} + \frac{(x-2)\cdots(x-k+1)}{k!} \\
&= \frac{(x-2)\cdots(x-2-k+2+1)k(k-1)+2(x-2)\cdots(x-2-k+1+1)k+(x-2)\cdots(x-k-1)}{k!} \\
&= \frac{(x-2)(x-1)\cdots(x-k+1)k(k-1)+2(x-2)(x-1)\cdots(x-k)k+(n-2)(n-1)\cdots(x-k-1)}{k!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+2(x-k)k+(x-k)(x-k-1)]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+2(x-k)k-(x-k)k+(x-k)(x-1)]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)][k(k-1)+(x-k)k+(x-k)(x-1)]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)]\left[k^2-k+kx-k^2+x^2-x-kx+k\right]}{k!}\\
&= \frac{[(x-2)(x-1)\cdots(x-k+1)]\left[x^2-x\right]}{k!}\\
&= \frac{x(x-1)(x-2)(x-1)\cdots(x-k+1)}{k!}\\
&\triangleq \frac{n^{\underline{k}}}{k!}\\
&\triangleq \binom{n}{k}
\end{aligned}
$$

9. Proof for *Hexagon Identity*:

$$
\begin{aligned}
&\binom{x-1}{k-1}\binom{x}{k+1}\binom{x+1}{k}\\
&\triangleq \left[\frac{(x-1)^{\underline{(k-1)}}}{(k-1)!}\right]\left[\frac{x^{\underline{(k+1)}}}{(k+1)!}\right]\left[\frac{(x+1)^{\underline{k}}}{k!}\right]\\
&\triangleq \left[\frac{(x-1)\cdots(x-1-k+1+1)}{(k-1)!}\right]\left[\frac{x(x-1)\cdots(x-k-1+1)}{(k+1)!}\right]\left[\frac{(x+1)(x)\cdots(x+1-k+1)}{k!}\right]\\
&= \left[\frac{(x-1)\cdots(x-k+2)(x-k+1)}{(k-1)!}\right]\left[\frac{x(x-1)\cdots(x-k)}{(k+1)!}\right]\left[\frac{(x+1)(x)(x-1)\cdots(x-k+2)}{k!}\right]\\
&= \left[\frac{(x)(x-1)\cdots(x-k+2)}{(k-1)!}\right]\left[\frac{(x+1)x(x-1)\cdots(x-k)(x-k+1)}{(k+1)!}\right]\left[\frac{(x-1)\cdots(x-k)}{k!}\right]\\
&\triangleq \left[\frac{x^{\underline{(k-1)}}}{(k-1)!}\right]\left[\frac{(x+1)^{\underline{(k+1)}}}{(k+1)!}\right]\left[\frac{(x-1)^{\underline{k}}}{k!}\right]\\
&\triangleq \binom{x}{k-1}\binom{x+1}{k+1}\binom{x-1}{k}
\end{aligned}
$$

✎

From Pascal's Recursion we can construct *Pascal's Triangle*:[5]

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}\\
 & & & & \vdots & & & &
\end{array}
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1\\
 & & & & \vdots & & & &
\end{array}
$$

[5] Pascal (1655), Granville (1992), Granville (1997), Edwards (2002), Hall and Knight (1894), pages 320–321 ⟨article 393⟩

B.3 Binomial summations

Theorem B.2. [6] *Let* $\binom{n}{k}$ *be the* BINOMIAL COEFFICIENT *(Definition B.3 page 132).*

THM

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \qquad \text{(row sum)}$$

$$\sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m+1} \qquad \text{(upper sum / column sum)}$$

$$\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n} \qquad \text{(parallel summation formula / southeast diagonal)}$$

$$\sum_{k=0}^{m}\binom{n-k}{m-k} = \binom{n+1}{m} \qquad \text{(northwest diagonal)}$$

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k} \qquad \text{(Vandermonde's convolution)}$$

$$\sum_{i=-j}^{n-j}\binom{m}{j+i}\binom{n}{k-i} = \binom{m+n}{j+k} \qquad \text{(alternate Vandermonde's convolution)}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

✎PROOF:

1. Proof for *row sum* relation:

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}x^k &= \left.\sum_{k=0}^{n}\binom{n}{k}x^k\right|_{x=1} \\
&= (1+x)^n|_{x=1} && \text{by } \textit{Binomial Theorem} \quad \text{(Theorem C.14 page 153)} \\
&= (1+1)^n \\
&= 2^n
\end{aligned}$$

2. Proof for *upper sum* relation (proof by induction):

   (a) Proof for $(n,m) = (0,0)$ case:
   $$\sum_{0}^{0}\binom{k}{m} = \binom{0}{0} = 1 = \binom{0+1}{0+1}$$

   (b) Proof for $(n,m) = (1,0)$ case:
   $$\sum_{0}^{1}\binom{k}{m} = \binom{1}{0} + \binom{1}{1} = 2 = \binom{1+1}{0+1}$$

   (c) Proof for $(n,m) = (1,1)$ case:
   $$\sum_{0}^{1}\binom{k}{m} = \binom{1}{1} = 1 = \binom{1+1}{1+1}$$

[6] Graham et al. (1994) page 169 ⟨Table 169⟩, Gallier (2010) pages 218–223, Gross (2008) page 227 ⟨Table 4.1.2⟩, Harris et al. (2008) pages 137–142, Knuth (1992a), Vandermonde (1772), Zhū (1303)

(d) Proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
\sum_{k=m}^{n+1}\binom{k}{m} &= \binom{n+1}{m} + \sum_{k=m}^{n} \\
&= \binom{n+1}{m} + \binom{n+1}{m+1} && \text{by left hypothesis} \\
&= \binom{n+2}{m+1} && \text{by Pascal's recursion (Theorem B.1 page 133)}
\end{aligned}$$

3. Proof for *Parallel summation formula* (Proof by induction):

(a) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n}$ is true for $n=0$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{m+k}{k}\right|_{n=0} &= \binom{m+0}{0} \\
&= \frac{(m+0)!}{(m-0)!\,0!} && \text{by Definition B.3 page 132} \\
&= \frac{(m+1)!}{(m+1-0)!\,0!} \\
&= \binom{m+1}{0} && \text{by Definition B.3 page 132} \\
&= \left.\binom{n+m+1}{n}\right|_{n=0}
\end{aligned}$$

(b) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n}$ is true for $n=1$:

$$\begin{aligned}
\left.\sum_{k=0}^{n}\binom{m+k}{k}\right|_{n=1} &= \binom{m+0}{0} + \binom{m+1}{1} \\
&= \binom{m+1}{0} + \binom{m+1}{1} \\
&= \binom{m+1+1}{1} && \text{by Pascal's Rule page 133} \\
&= \left.\binom{n+m+1}{n}\right|_{n=1}
\end{aligned}$$

(c) Proof that $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n} \implies \sum_{k=0}^{n+1}\binom{m+k}{k} = \binom{(n+1)+m+1}{n+1}$:

$$\begin{aligned}
\sum_{k=0}^{n+1}\binom{m+k}{k} &= \binom{m}{0} + \sum_{k=1}^{n+1}\binom{m+k}{k} \\
&= \binom{m}{0} + \sum_{k=0}^{n}\binom{m+k+1}{k+1} \\
&= \binom{m}{0} + \sum_{k=0}^{n}\binom{m+k}{k} - \binom{m}{0} + \binom{m+n+1}{n+1} \\
&= \binom{n+m+1}{n} + \binom{m+n+1}{n+1} && \text{by left hypothesis} \\
&= \binom{n+m+2}{n+1} && \text{by Pascal's Rule page 133} \\
&= \binom{(n+1)+m+1}{(n+1)}
\end{aligned}$$

4. Proof for *Vandermonde's convolution*:

$$\begin{aligned}
\sum_{k=0}^{m+n} \binom{m+n}{k} x^k &= \sum_{k=0}^{m+n} \binom{m+n}{k} (1)^{m+n-k} x^k \\
&= (1+x)^{m+n} && \text{by \textit{Binomial Theorem}} && \text{(Theorem C.14 page 153)} \\
&= (1+x)^m (1+x)^n \\
&= \underbrace{\left[\sum_{k=0}^{m} \binom{m}{k} x^k\right]}_{(1+x)^m} \underbrace{\left[\sum_{j=0}^{n} \binom{n}{j} x^j\right]}_{(1+x)^n} && \text{by \textit{Binomial Theorem}} && \text{(Theorem C.14 page 153)} \\
&= \sum_{k=0}^{m+n} \left[\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}\right] x^k && \text{by Theorem G.3 page 207} \\
\implies \binom{m+n}{k} &= \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}
\end{aligned}$$

5. Proof for *alternate Vandermonde's convolution*:

$$\begin{aligned}
\binom{m+n}{j+k} &= \binom{m+n}{u} && \text{where } u \triangleq j+k \implies k = u-j \\
&= \sum_{v=0}^{n} \binom{m}{v}\binom{n}{u-v} \\
&= \sum_{v=0}^{n} \binom{m}{v}\binom{n}{j+k-v} \\
&= \sum_{i+j=0}^{i+j=n} \binom{m}{j+i}\binom{n}{k-i} && \text{where } i \triangleq v-j \implies v = i+j \\
&= \sum_{i=-j}^{i=n-j} \binom{m}{j+i}\binom{n}{k-i}
\end{aligned}$$

6. Proof that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$:

$$\begin{aligned}
\binom{2n}{n} &= \binom{n+n}{n} \\
&= \sum_{k=0} n \binom{n}{k}\binom{n}{n-k} && \text{by Vandermonde's convolution (item (4) page 140)} \\
&= \sum_{k=0} n \binom{n}{k}\binom{n}{k} && \text{by item (2)} \\
&= \sum_{k=0}^{n} \binom{n}{k}^2
\end{aligned}$$

✏

Theorem B.3. [7]

THM
$$\sum_{k=1}^{n} \frac{1}{k+1} < \ln(n+1) < \sum_{k=1}^{n} \frac{1}{k}$$

[7] Rivlin (1969), page 60

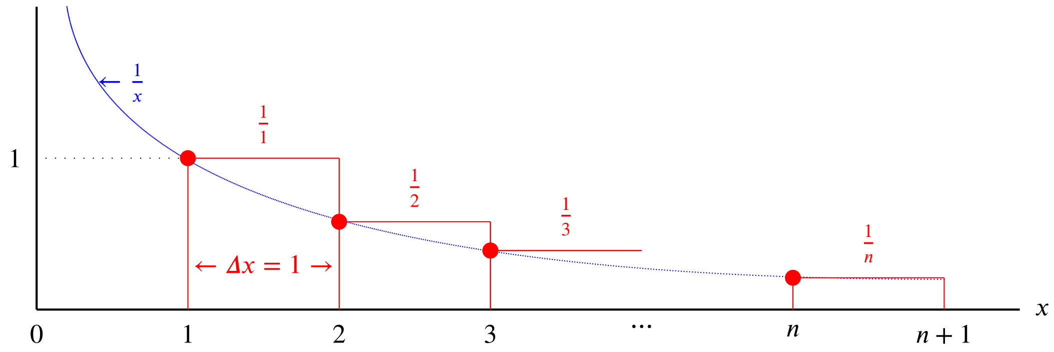


Figure B.1: $\ln(n+1)$

✎PROOF: The summations are simply lower and upper bounds of the integral of $\frac{1}{x}$ in the range $[1, \; n+1]$. This is illustrated in Figure B.1.

1. Proof that $\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k}$:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &> \int_1^{n+1} \frac{1}{x} \, \mathrm{d}x \\
&= \ln x \Big|_1^{n+1} \\
&= \ln(n+1) - \cancelto{0}{\ln(1)} \\
&= \ln(n+1)
\end{aligned}$$

2. Proof that $\sum_{k=1}^{n} \frac{1}{k+1} < \ln(n+1)$:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k+1} &< \int_1^{n+1} \frac{1}{x} \, \mathrm{d}x \\
&= \ln(n+1) - \cancelto{0}{\ln(1)} \\
&= \ln(n+1)
\end{aligned}$$

✏

APPENDIX C

POLYNOMIALS

C.1 Definitions

Definition C.1. [1] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

D E F

A function p *in* $\mathbb{F}^{\mathbb{F}}$ *is a* ***polynomial*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if it is of the form*

$$\mathsf{p}(x) \triangleq \sum_{n=0}^{N} \alpha_n x^n \qquad \alpha_n \in \mathbb{F},\ \alpha_N \neq 0.$$

The ***degree*** *of* p *is* N. *A* ***coefficient*** *of* p *is any element of* $\left(\!\left|\alpha_n\right|\!\right)_1^N$.

The ***leading coefficient*** *of* p *is* α_N.

Definition C.2. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

D E F

A polynomial p *of degree* N *over the field* $\mathbb{F}$ *and*
a polynomial q *of degree* M *over the field* $\mathbb{F}$ *are* ***equal*** *if*

1. $N = M$ *and*
2. $\alpha_n = \beta_n$ *for* $n = 0, 1, \ldots, N$.

The expression $\mathsf{p}(x) = \mathsf{q}(x)$ *(or* $\mathsf{p} = \mathsf{q}$*) denotes that* p *and* q *are* EQUAL.

[1] Barbeau (1989) page 1, Fuhrmann (2012) page 11, Borwein and Erdélyi (1995) page 2
[2] Fuhrmann (2012) page 11

C.2 Ring properties

C.2.1 Polynomial Arithmetic

Theorem C.1 (polynomial addition). [3] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

THM
$$\underbrace{\left(\sum_{n=0}^{N} \alpha_n x^n\right)}_{\mathsf{p}(x)} + \underbrace{\left(\sum_{n=0}^{M} \beta_n x^n\right)}_{\mathsf{q}(x)} = \underbrace{\sum_{n=0}^{\max(N,M)} \gamma_n x^n}_{\mathsf{p}(x)+\mathsf{q}(x)} \quad \textit{where} \quad \gamma_n \triangleq \begin{cases} \alpha_n + \beta_n & \textit{for } n \le \min(N,M) \\ \alpha_n & \textit{for } n > M \\ \beta_n & \textit{for } n > N \end{cases}$$
$$\textit{for all} \quad x,\ \alpha_n,\ \beta_n \in \mathbb{F}$$

Polynomial multiplication is equivalent to convolution (Definition O.3 page 341) of the coefficients (Definition C.1 page 143). [4]

Theorem C.2 (polynomial multiplication). [5] *Let* $(\!(\alpha_n \in \mathbb{C})\!)$, $(\!(b_n \in \mathbb{C})\!)$, *and* $x \in \mathbb{C}$.

THM
$$\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{n=0}^{M} \beta_n x^n\right) = \sum_{n=0}^{N+M} \underbrace{\left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right) x^n}_{\textit{Cauchy product}}$$

✎PROOF:

$$\begin{aligned}
\left(\sum_{n=0}^{N} \alpha_n x^n\right)\left(\sum_{m=0}^{M} \beta_m x^m\right) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n \beta_m x^{n+m} \\
&= \sum_{n=0}^{N}\sum_{k=n}^{M+n} \alpha_n \beta_{k-n} x^k && k \triangleq n+m \iff m = k-n \\
&= \sum_{n=0}^{N+M} \left(\sum_{k=\max(0,n-M)}^{\min(n,N)} \alpha_n \beta_{k-n}\right) x^n
\end{aligned}$$

Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	β_0	β_1	β_2	β_3	$\cdots$	β_M
α_0	$\alpha_0\beta_0$	$\alpha_0\beta_1 x$	$\alpha_0\beta_2 x^2$	$\alpha_0\beta_3 x^3$	$\cdots$	$\alpha_0\beta_M x^M$
α_1	$\alpha_1\beta_0 x$	$\alpha_1\beta_1 x^2$	$\alpha_1\beta_2 x^3$	$\alpha_1\beta_3 x^4$	$\cdots$	$\alpha_1\beta_M x^{1+M}$
α_2	$\alpha_2\beta_0 x^2$	$\alpha_2\beta_1 x^3$	$\alpha_2\beta_2 x^4$	$\alpha_2\beta_3 x^5$	$\cdots$	$\alpha_2\beta_M x^{2+M}$
α_3	$\alpha_3\beta_0 x^3$	$\alpha_3\beta_1 x^4$	$\alpha_3\beta_2 x^5$	$\alpha_3\beta_3 x^6$	$\cdots$	$\alpha_3\beta_M x^{3+M}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_N	$\alpha_N\beta_0 x^N$	$\alpha_N\beta_1 x^{N+1}$	$\alpha_N\beta_2 x^{N+2}$	$\alpha_N\beta_3 x^{N+3}$	$\cdots$	$\alpha_N\beta_M x^{N+M}$

1. The expression $\sum_{n=0}^{N}\sum_{m=0}^{M} \alpha_n\beta_m x^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

---

[3] Fuhrmann (2012) page 11

[4] *Convolution*: In fact, using *GNU Octave*™ or *MatLab*™, polynomial multiplication can be performed using convolution. For example, the operation $(x^3 + 5x^2 + 7x + 9)(4x^2 + 11)$ can be calculated in *GNU Octave*™ or *MatLab*™ with conv([1 5 7 9],[4 0 11])

[5] Apostol (1975), page 237

2. If we switched the order of summation to $\sum_{m=0}^{M}\sum_{n=0}^{N}\alpha_n\beta_m x^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

3. For $N = M = \infty$, the expression $\sum_{n=0}^{N+M}\left(\sum_{k=0}^{n}\alpha_k\beta_{n-k}\right)x^n$ is equivalent to adding *diagonally* starting from the upper left corner and proceeding towards the lower right.

4. For finite N and M...

   (a) The upper limit on the inner summation puts two constraints on k:
   $$\left\{\begin{array}{lcll} k & \le & n & \text{and} \\ k & \le & N & \end{array}\right\} \implies k \le \min(n, N)$$

   (b) The lower limit on the inner summation also puts two constraints on k:
   $$\left\{\begin{array}{lcll} k & \ge & 0 & \text{and} \\ k & \ge & n-M & \end{array}\right\} \implies k \ge \max(0, n-M)$$

✎

Polynomial division can be performed in a manner very similar to integer division (both integers and polynomials are *rings*).

Definition C.3 (Polynomial division). *The quantities of polynomial division are defined as follows:*

DEF
$$\frac{\mathsf{d}(x)}{\mathsf{p}(x)} = \mathsf{q}(x) + \frac{\mathsf{r}(x)}{\mathsf{p}(x)} \qquad \textit{where} \qquad \left\{\begin{array}{lll} \mathsf{d}(x) & \textit{is the } \textbf{\textit{dividend}} & \textit{and} \\ \mathsf{p}(x) & \textit{is the } \textbf{\textit{divisor}} & \textit{and} \\ \mathsf{q}(x) & \textit{is the } \textbf{\textit{quotient}} & \textit{and} \\ \mathsf{r}(x) & \textit{is the } \textbf{\textit{remainder}}. & \end{array}\right\}$$

C.2.2 Greatest common divisor

Theorem C.3 (Extended Euclidean Algorithm). [6]

Let $\mathsf{r}_1(x)$ *and* $\mathsf{r}_2(x)$ *be polynomials. The following algorithm computes their greatest common divisor* $\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$, *and factors* $\mathsf{a}(x)$ *and* $\mathsf{b}(x)$ *such that*

$$\mathsf{r}_1(x)\mathsf{a}(x) + \mathsf{r}_2(x)\mathsf{b}(x) = \gcd(\mathsf{r}_1, \mathsf{r}_2)$$

THM

n	*remainder* $\mathsf{r}_n = \mathsf{r}_{n-2} - \mathsf{q}_n\mathsf{r}_{n-1}$	*quotient* q_n	*factor* $\alpha_n = a_{n-2} - \mathsf{q}_n\alpha_{n-1}$	*factor* $\beta_n = b_{n-2} - \mathsf{q}_n\beta_{n-1}$
1	$\mathsf{r}_1(x)$	–	1	0
2	$\mathsf{r}_2(x)$	–	0	1
3	$\mathsf{r}_1 - q_3\mathsf{r}_2$	q_3	1	$-q_3$
4	$\mathsf{r}_2 - q_4 r_3$	q_4	$-\mathsf{q}_4$	$1 + \mathsf{q}_4\mathsf{q}_1$
5	$\mathsf{r}_1 - \mathsf{q}_5\mathsf{r}_2$	q_5	$1 + \mathsf{q}_5\mathsf{q}_4$	$-q_3 - \mathsf{q}_5(1 + \mathsf{q}_4\mathsf{q}_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\gcd(\mathsf{r}_1(x), \mathsf{r}_2(x))$	q_n	$\mathsf{a}(x) = a_{n-2} - q_n\alpha_{n-1}$	$\mathsf{b}(x) = b_{n-2} - q_n\beta_{n-1}$
$n+1$	0	q_{n+1}		

[6] Wicker (1995), page 53, Fuhrmann (2012) page 11

✎PROOF:

$$\begin{aligned} \mathsf{r}_1 &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3 \\ &= \mathsf{q}_3\mathsf{r}_2 + \mathsf{r}_3 \end{aligned}$$

✏

Example C.1. Let

$$\mathsf{u}(x) \triangleq (1-x)^2 \qquad \mathsf{v}(x) \triangleq x^2.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^2}_{\mathsf{u}(x)}\underbrace{(1+2x)}_{\mathsf{a}(x)} + \underbrace{(x^2)}_{\mathsf{v}(x)}\underbrace{(3-2x)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^2 = 1-2x+x^2 = \mathsf{u}(x)$	–	1	0
0	$x^2 = \mathsf{v}(x)$	–	0	1
1	$1-2x$	1	1	-1
2	$\frac{1}{2}x$	$-\frac{1}{2}x$	$\frac{1}{2}x$	$1-\frac{1}{2}x$
3	$1 = \gcd(\ (1-x)^2,\ x^2\ )$	-4	$1+2x = \mathsf{a}(x)$	$3-2x = \mathsf{b}(x)$
4	0	$\frac{1}{2}x$	–	–

✏

Example C.2. Let

$$\mathsf{u}(x) \triangleq (1-x)^3 \qquad \mathsf{v}(x) \triangleq x^3.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^3}_{\mathsf{u}(x)}\underbrace{(1+3x+6x^2)}_{\mathsf{a}(x)} + \underbrace{(x^3)}_{\mathsf{v}(x)}\underbrace{(10-15x+6x^2)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^3 = 1-3x+3x^2-x^3$	–	1	0
0	x^3	–	0	1
1	$1-3x+3x^2$	-1	1	1
2	$-\frac{1}{3}x+x^2$	$\frac{1}{3}x$	$-\frac{1}{3}x$	$1-\frac{1}{3}x$
3	$1-2x$	3	$1+x$	$-2+x$
4	$\frac{1}{6}x$	$-\frac{1}{2}x$	$\frac{1}{6}x+\frac{1}{2}x^2$	$1-\frac{4}{3}y+\frac{1}{2}x^2$
5	$1 = \gcd((1-x)^3, x^3)$	-12	$1+3x+6x^2 = \mathsf{a}(x)$	$10-15x+6x^2 = \mathsf{b}(x)$
6	0	$\frac{1}{6}x$		

✏

Example C.3. Let

$$\mathsf{u}(x) \triangleq (1-x)^4 \qquad \mathsf{v}(x) \triangleq x^4.$$

The greatest common divisor and factors of u and v are such that

$$\underbrace{(1-x)^4}_{\mathsf{u}(x)}\underbrace{(1+4x+10x^2+20x^3)}_{\mathsf{a}(x)} + \underbrace{(x^4)}_{\mathsf{v}(x)}\underbrace{(35-84x+70x^2-20x^3)}_{\mathsf{b}(x)} = \underbrace{1}_{\text{gcd}}$$

Because $\gcd(\mathsf{u},\mathsf{v}) = 1$, $\mathsf{u}(x)$ and $\mathsf{v}(x)$ are said to be *relatively prime*.

✎PROOF:

n	$\mathsf{r}_n = r_{n-2} - r_{n-1}q_n$	q_n	$\alpha_n = a_{n-2} - q_n\alpha_{n-1}$	$\beta_n = b_{n-2} - q_n\beta_{n-1}$
-1	$(1-x)^4 = 1 - 4x + 6^2 - 4x^3 + x^4$	–	1	0
0	x^4	–	0	1
1	$1 - 4x + 6x^2 - 4x^3$	1	1	-1
2	$\frac{1}{4}x - x^2 + \frac{3}{2}x^3$	$-\frac{1}{4}x$	$\frac{1}{4}x$	$1 - \frac{1}{4}x$
3	$1 - \frac{10}{3}x + \frac{10}{3}x^2$	$-\frac{8}{3}$	$1 + \frac{2}{3}x$	$\frac{5}{3} - \frac{2}{3}x$
4	$-\frac{1}{5}x + \frac{1}{2}x^2$	$\frac{3}{2}\cdot\frac{3}{10}x$	$-\frac{1}{5}x - \frac{3}{10}x^2$	$1 - x + \frac{3}{10}x^2$
5	$1 - 2x$	$\frac{20}{3}$	$1 + 2x + 2x^2$	$-5 + 6x - 2x^2$
6	$\frac{1}{20}x$	$-\frac{1}{4}x$	$\frac{1}{20}x + \frac{1}{5}x^2 + \frac{1}{2}x^3$	$1 - \frac{9}{4}x + \frac{18}{10}x^2 - \frac{1}{2}x^3$
7	$1 = \gcd((1-x)^4, x^4)$	-40	$1 + 4x + 10x^2 + 20x^3$	$35 - 84x + 70x^2 - 20x^3$
8	0	$\frac{1}{20}x$	–	–

✎

❝*Infinitesimal analysis was considered so attractive and important because of its numerous and useful applications; as such, it attracted upon itself all research attention and efforts. Concurrently, algebraic analysis appeared to be a field where nothing remained to be done, or where whatever remained to be done would have only been worthless speculation. …Nevertheless, the major contributors to infinitesimal analysis are well aware of the need to improve algebraic analysis: Their own progress depends upon it.*❞

Étienne Bézout, 1779 [7]

Theorem C.4 (Bézout's Identity). [8] [9] *Let* $\mathsf{p}_1(x)$ *be a polynomial of degree* n_1 *and* $\mathsf{p}_2(x)$ *be a polynomial of degree* n_2.

THM

$$\underbrace{\gcd(\mathsf{p}_1(x),\ \mathsf{p}_2(x)) = 1}_{\mathsf{p}_1(x) \text{ and } \mathsf{p}_2(x) \text{ are relatively prime}} \implies \left\{\begin{array}{ll} 1. & \exists \mathsf{q}_1(x), \mathsf{q}_2(x) \quad \text{such that} \\ & \underbrace{\overbrace{\mathsf{p}_1(x)}^{\text{degree } n_1}\underbrace{\mathsf{q}_1(x)}_{\text{degree } n_2-1}}{} + \underbrace{\overbrace{\mathsf{p}_2(x)}^{\text{degree } n_2}\underbrace{\mathsf{q}_2(x)}_{\text{degree } n_1-1}}{} = 1 \\ 2. & \text{order of } \mathsf{q}_1(x) = n_2 - 1 \\ 3. & \text{order of } \mathsf{q}_2(x) = n_1 - 1 \end{array}\right.$$

[7] quote: Bézout (1779a)
translation: Bézout (1779b), page xv
image: http://en.wikipedia.org/wiki/File:Etienne_Bezout2.jpg, public domain

[8] Bourbaki (2003b) page 2 ⟨Theorem 1 Chapter VII⟩, Fuhrmann (2012) pages 15–17 ⟨Corollary 1.31, Corollary 1.38⟩, Adhikari and Adhikari (2003) page 182, Warner (1990) page 381, Daubechies (1992), page 169, Mallat (1999), page 250

[9] Historical information: Bézout (1779a) ⟨???⟩, Bézout (1779b) ⟨???⟩, Bachet (1621) ⟨???⟩, Childs (2009) pages 37–46 ⟨some history on page 46⟩, http://serge.mehl.free.fr/chrono/Bachet.html, http://serge.mehl.free.fr/chrono/Bezout.html

✎PROOF: No proof at this time. ✏

C.3 Roots

❝*Neither the true nor the false roots are always real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation $x^3 - 6x^2 + 13x - 10 = 0$ as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remain always imaginary.*❞

René Descartes (1596–1650), French philosopher and mathematician[10]

Theorem C.5 (Fundamental Theorem of Algebra). [11] *Let* $\mathsf{p}(x)$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

THM

$$\left\{\text{degree of } \mathsf{p}(x) \text{ is } N\right\} \implies \left\{\begin{array}{l}\underbrace{\exists \left(\!\left| x_n \right|\!\right)_1^N \text{ such that } \mathsf{p}(x_n) = 0 \text{ for } n = 1, 2, \ldots, N}_{\mathsf{p}(x) \text{ has } N \text{ zeros}} \\ \text{where } x_n \text{ and } x_m \text{ are not necessarily distinct for } n \neq m.\end{array}\right\}$$

Corollary C.1. *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a polynomial over a field* $(\mathbb{F}, +, \cdot, 0, 1)$.

COR

$$\left\{\underbrace{\begin{array}{l}\text{There exists } \left(\!\left| x_n \right|\!\right)_1^N \\ \text{such that } \mathsf{p}(x_n) = 0 \text{ for } n = 0, 1, \ldots, N \\ \text{and where } x_n \text{ and } x_m \text{ are} \\ \text{not necessarily distinct for } n \neq m.\end{array}}_{N \text{ zeros of } \mathsf{p}(x)}\right\} \implies \left\{\mathsf{p}(x) = \left(\frac{\alpha_0}{\prod_{n=1}^{N}(-x_n)}\right) \underbrace{\prod_{n=1}^{N}(x - x_n)}_{N \text{ factors}}\right\}$$

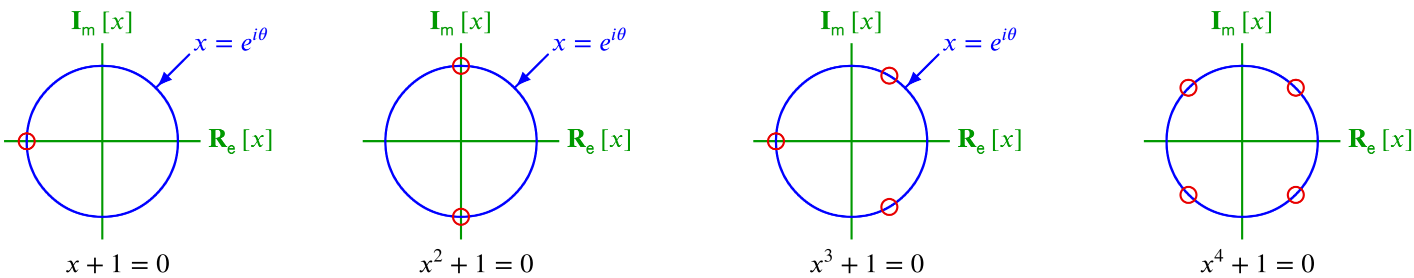


Figure C.1: Roots of $x^n + 1 = 0$

Lemma C.1.

LEM

$$x^N + 1 = 0 \qquad \implies \qquad x \in \left\{ e^{i\theta_n} | \theta_n = \frac{\pi}{N}(2n+1), n = 0, 1, \ldots, N-1 \right\}$$

---

[10] quote: Descartes (1637a)
English: Descartes (1954), page 175
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

[11] Prasolov (2004) pages 1–2 ⟨Section 1.1.1⟩, Borwein and Erdélyi (1995) page 11 ⟨Theorem 1.2.1⟩

✎PROOF:

$$\begin{aligned}
e^{iN\theta_n - i2\pi n} &= -1 && n \in \mathbb{Z}\\
N\theta_n - 2\pi n &= \pi && n = 0, 1, \ldots, N-1\\
N\theta_n &= 2\pi n + \pi\\
\theta_n &= \frac{\pi}{N}(2n+1)
\end{aligned}$$

✏

Theorem C.6. *Let* $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \le n \le N\}$ *and* $\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$.

THM

$$\underbrace{\alpha_n = \alpha^*_{-n} \quad \forall n \in I}_{(\alpha_n) \text{ is Hermitian symmetric}} \quad\iff\quad \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C}$$

✎PROOF:

1. Proof that $\alpha_n = \alpha^*_{-n} \implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\mathsf{p}(x) &\triangleq \sum_{n=-N}^{N} \alpha_n x^n && \text{by definition of } \mathsf{p}(x)\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha_{-n} x^{-n}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha_n x^n + \sum_{n=1}^{N} \alpha^*_n x^{-n} && \text{by left hypothesis}\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n x^{-n} + \sum_{n=1}^{N} \alpha_n x^n\\
&= \alpha_0 + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x}\right)^{-n}\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha^*_n \left(\frac{1}{x^*}\right)^{-n}\right]^*\\
&= \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n + \sum_{n=1}^{N} \alpha_{-n} \left(\frac{1}{x^*}\right)^{-n}\right]^* && \text{by left hypothesis}\\
&= \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^*\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by definition of } \mathsf{p}(x)
\end{aligned}$$

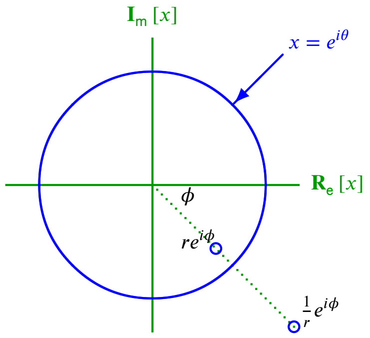


Figure C.2: Recipricol conjugate zero pairs

2. Proof that $\alpha_n = \alpha^*_{-n} \quad\Longleftarrow\quad \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right)$:

$$\begin{aligned}
\sum_{n=-N}^{N} \alpha_n x^n &\triangleq \mathsf{p}(x) && \text{by definition of } \mathsf{p}(x)\\
&= \mathsf{p}^*\left(\frac{1}{x^*}\right) && \text{by right hypothesis}\\
&\triangleq \left[\sum_{n=-N}^{N} \alpha_n \left(\frac{1}{x^*}\right)^n\right]^* && \text{by definition of } \mathsf{p}(x)\\
&= \sum_{n=-N}^{N} \alpha^*_n \left(\frac{1}{x}\right)^n\\
&= \sum_{n=-N}^{N} \alpha^*_{-n} x^n && \text{by symmetry of summation indices}\\
\implies \quad \alpha_n &= \alpha^*_{-n} && \text{by matching of polynomial coefficients}
\end{aligned}$$

⇰

Theorem C.7. *Let $N \in \mathbb{N}$, $I = \{n \in \mathbb{Z} | -N \le n \le N\}$ and*

$$\mathsf{p}(x) \triangleq \sum_{n=-N}^{N} \alpha_n x^n \qquad \forall x \in \mathbb{C}$$

T H M

$$\underbrace{\alpha_n = \alpha^*_{-n} \quad \forall n \in I}_{(\alpha_n) \text{ is Hermitian symmetric}} \quad\implies\quad \underbrace{\left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]}_{\text{roots occur in conjugate reciprocol pairs}}$$

✎PROOF:

$$\begin{aligned}
&\alpha_n = \alpha^*_{-n} \quad \forall n \in I && \text{by left hypothesis}\\
&\implies \mathsf{p}(x) = \mathsf{p}^*\left(\frac{1}{x^*}\right) \quad \forall x \in \mathbb{C} && \text{by Theorem C.6 page 149}\\
&\implies \left[\sigma \text{ is a root of } \mathsf{p}(x) \iff \frac{1}{\sigma^*} \text{ is a root of } \mathsf{p}(x)\right]
\end{aligned}$$

If σ is a zero of $\mathsf{p}(x)$, then so is $\frac{1}{\sigma^*}$ because

$$\mathsf{p}\left(\frac{1}{\sigma^*}\right) = \mathsf{p}^*(\sigma) = 0^* = 0.$$

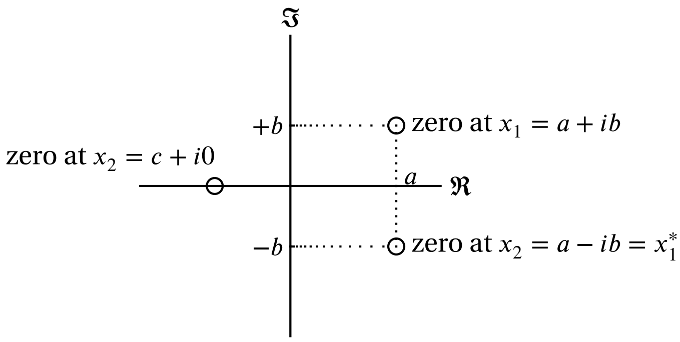


Figure C.3: Conjugate pairs of roots

Theorem C.8 page 151 (next) states that the roots of real polynomials occur in complex conjugate pairs. This is illustrated in Figure C.3.

Theorem C.8. [12] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial.*

THM

$$\left[\underbrace{(\alpha_n \in \mathbb{R})_{n=0,1,\ldots,N}}_{\text{coefficients are real}}\right] \implies \left[\underbrace{\mathsf{p}(x_0) = 0 \iff \mathsf{p}(x_0^*) = 0}_{\text{zeros occur in conjugate pairs}}\right]$$

Theorem C.9 (Routh-Hurwitz Criterion). [13] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$ *and*

$$d_0 \triangleq \alpha_0 \qquad d_1 \triangleq \alpha_1 \qquad d_2 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} \qquad d_3 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} \qquad d_4 \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 \end{vmatrix}$$

$$d_n \triangleq \begin{vmatrix} \alpha_1 & \alpha_0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ \alpha_{2n-3} & \alpha_{2n-4} & \cdots & \alpha_{n-2} \\ \alpha_{2n-1} & \alpha_{2n-2} & \cdots & \alpha_n \end{vmatrix}$$

Let $\mathbf{S}\,(x_n)$ *be the number of sign changes of some sequence* (x_n) *after eliminating all zero elements* $(x_n = 0)$*.*

THM

$$\begin{aligned} \underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Re[x_n] > 0\right\}\right|}_{\text{number of roots in right half plane}} &= \underbrace{\mathbf{S}\,(d_0,\ d_1,\ d_1 d_2,\ d_2 d_3,\ \ldots,\ d_{p-2} d_{p-1},\ \alpha_p)}_{\text{number of sign changes}} \\ &= \underbrace{\mathbf{S}\left(d_0,\ d_1,\ \frac{d_2}{d_1},\ \frac{d_3}{d_2},\ \ldots,\ \frac{d_p}{d_{p-1}}\right)}_{\text{number of sign changes}} \end{aligned}$$

Theorem C.10 (Descartes rule of signs). [14] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$*.*

[12] Korn and Korn (1968), page 17
[13] Korn and Korn (1968), page 17
[14] Korn and Korn (1968), page 17

THM

$$\underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Re[x_n] > 0\right\},\ \Im[x_n] = 0\right|}_{\textit{number of roots on right real axis}} = \underbrace{\mathbf{S}\left(\!\left(\alpha_n\right)\!\right) - 2m}_{\textit{number of sign changes -- even integer}} \qquad \textit{where } m \in \mathbb{W}$$

Theorem C.11. [15] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$.

THM

$$\underbrace{\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1} \geq 0}_{\textit{first } k \textit{ coefficients are nonnegative}} \implies \left\{\begin{array}{l} \underbrace{\left|\left\{x_n | \mathsf{p}(x_n) = 0,\ \Im[x_n] = 0\right\}\right|}_{\textit{number of real roots}} < \underbrace{1 + \left(\frac{q}{\alpha_0}\right)^{\frac{1}{k}}}_{\textit{upper bound}} \\ \textit{where } q \triangleq \underbrace{\max\left\{\ |\alpha_n|\ | \alpha_n < 0\right\}}_{\textit{largest negative coefficient}} \end{array}\right.$$

Theorem C.12 (Rolle's Theorem). [16] *Let* $\mathsf{p}(x) = \sum_{n=0}^{N} \alpha_n x^n$ *be a* N*th order polynomial with* $\alpha_n \in \mathbb{R}$. *The number of real zeros of* $\mathsf{p}'(x)$ *between any two real consecutive real zeros of* $\mathsf{p}(x)$ *is* ***odd***.

Definition C.4. [17] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD.

DEF

$\dfrac{\mathsf{p}(x)}{\mathsf{q}(x)}$ *is a* ***rational function***
if $\mathsf{p}(x)$ *and* $\mathsf{q}(x)$ *are* POLYNOMIALS *over* $(\mathbb{F}, +, \cdot, 0, 1)$.

Example C.4.

EX

An example of a rational function using polynomials in x^{-1} is

$$A(x) = \frac{b_0 + \beta_1 x^{-1} + \beta_2 x^{-2} + \beta_3 x^{-3}}{1 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \alpha_3 x^{-3}}$$

This can be expressed as a rational function using polynomials in x by multiplying numerator and denominator by x^3:

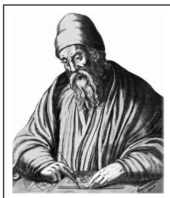
$$A(x) = \frac{x^3}{x^3} A(x) = \frac{b_0 x^3 + \beta_1 x^2 + \beta_2 x + \beta_3}{x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3}$$

Definition C.5.

DEF

The ***zeros*** *of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $B(x)$.
The ***poles*** *of a rational function* $H(x) = \frac{B(x)}{A(x)}$ *are the roots of* $A(x)$.

C.4 Polynomial expansions



❝*Thus, if a straight-line is cut at random, then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces.*❞
Euclid (~300BC), Greek mathematician, demonstrating the *Binomial theorem* for exponent $n = 2$ as in $(x + y)^2 = x^2 + 2xy + y^2$. [18]

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[15] Korn and Korn (1968), page 18
[16] Korn and Korn (1968), page 18
[17] Fuhrmann (2012) page 22
[18] quote: Euclid (circa 300BC) ⟨Book II, Proposition 4⟩, Coolidge (1949), page 147
image: http://commons.wikimedia.org/wiki/File:Euklid-von-Alexandria_1.jpg, public domain

Theorem C.13 (Taylor Series). [19] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathrm{d}}{\mathrm{d}x}$ in $\boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator.*

THM

$$\mathsf{f}(x) = \sum_{n=0}^{\infty} \frac{\left[\frac{\mathrm{d}}{\mathrm{d}x}^n \mathsf{f}\right](a)}{n!}(x-a)^n \qquad \forall a \in \mathbb{R}, \mathsf{f} \in \boldsymbol{C} \qquad \text{(Taylor series } \textit{about the point } a\text{)}$$

*A **Maclaurin series** is a* Taylor series *about the point* $a = 0$.

Theorem C.14 (Binomial Theorem). [20]

THM

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad \textit{where} \qquad \binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$$

✎Proof: This theorem is proven using two different techniques. Either is sufficient. The first requires the Maclaurin series resulting in a more compact proof, but requires the additional (here unproven) Maclaurin series. The second proof uses induction resulting in a longer proof, but does not require any external theorem.

1. Proof using Maclaurin series:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}y^k}\Big[(x+y)^n\Big]_{y=0} y^k && \text{by Maclaurin series (Theorem C.13 page 153)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\Big[n(n-1)(n-2)\cdots(n-k+1)(x+y)^{n-k}\Big]_{y=0} y^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{n!}{(n-k)!} x^{n-k} y^k \\
&= \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k && \text{by definition of } \binom{n}{k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k + \cancelto{0}{\sum_{k=n+1}^{\infty} \binom{n}{k} x^{n-k} y^k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k && \text{because } (x+y)^n \text{ has order } n
\end{aligned}$$

2. Proof using induction:

   (a) Proof that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ is true for $n = 0$:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\Bigg|_{n=0} &= \binom{0}{0} x^0 y^{0-0} \\
&= 1 \\
&= (x+y)^n\big|_{n=0}
\end{aligned}$$

[19] Flanigan (1983) page 221 ⟨Theorem 15⟩, Strichartz (1995) page 281, Sohrab (2003) page 317 ⟨Theorem 8.4.9⟩, Taylor (1715), Maclaurin (1742)

[20] Graham et al. (1994) page 162 ⟨(5.12)⟩, Rotman (2010) page 84 ⟨Proposition 2.5⟩, Bourbaki (2003a) page 99 ⟨Corollary 1⟩, Warner (1990) pages 189–190 ⟨Theorem 21.1⟩, Metzler et al. (1908), page 169 ⟨any real exponent⟩, Coolidge (1949)

(b) Proof that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ is true for $n=1$:

$$\begin{aligned}
\left.\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right|_{n=1} &= \binom{1}{0} x^0 y^{1-0} + \binom{1}{1} x^1 y^{1-1} \\
&= y + x \\
&= (x+y)^n|_{n=1}
\end{aligned}$$

(c) Proof that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \implies (x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n}{k} x^k y^{n+1-k}$:

$$\begin{aligned}
&\sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] x^k y^{n+1-k} && \text{by \textit{Pascal's Rule} (Theorem B.1 page 133)} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&= x^{n+1} + y^{n+1} + \left[\sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n+1-(k+1)} - x^{n+1}\right] + \left[\sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} - y^{n+1}\right] \\
&= x \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= x(x+y)^n + y(x+y)^n && \text{by left hypothesis} \\
&= (x+y)(x+y)^n \\
&= (x+y)^{n+1}
\end{aligned}$$

✎

APPENDIX D

OPERATORS ON LINEAR SPACES

❝*And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.*❞

Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

D.1 Operators on linear spaces

D.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition D.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD *(Definition A.5 page 130). Let* X *be a set, let* $+$ *be an* OPERATOR *(Definition D.2 page 156) in* X^{X^2}*, and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

[1] quote: Leibniz (1679) pages 248–249
image: `http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg`, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991), page 41, Halmos (1948), pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b), pages 119–120, Banach (1922) pages 134–135

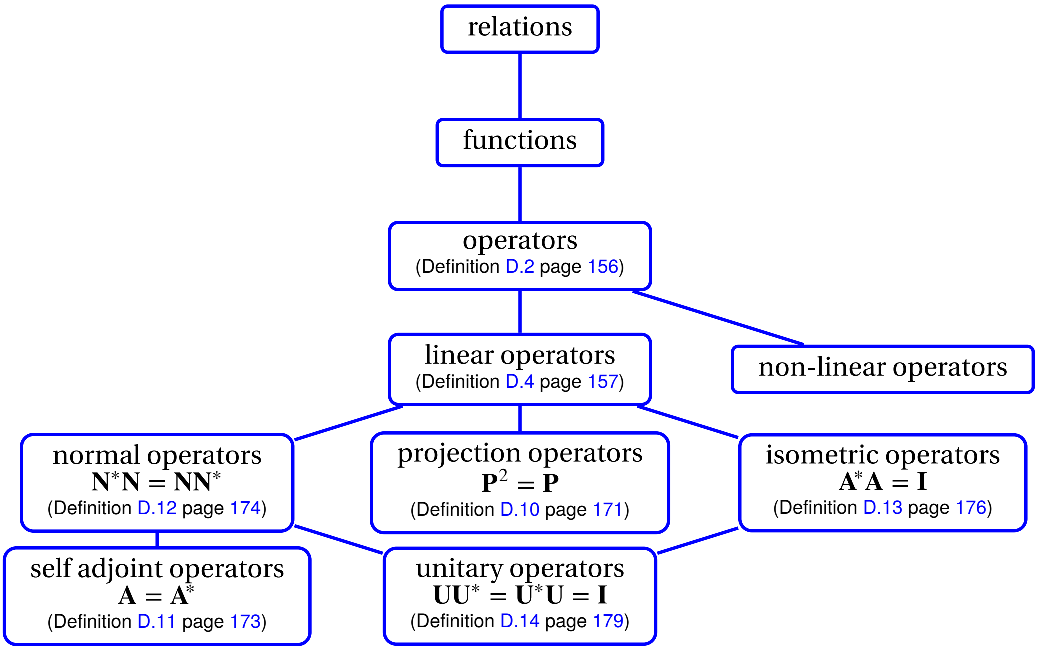


Figure D.1: Some operator types

DEF *The structure* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1. $\exists \mathbb{0} \in X$ such that $\boldsymbol{x} + \mathbb{0} = \boldsymbol{x}$ $\quad \forall \boldsymbol{x} \in X$ (+ IDENTITY)
2. $\exists \boldsymbol{y} \in X$ such that $\boldsymbol{x} + \boldsymbol{y} = \mathbb{0}$ $\quad \forall \boldsymbol{x} \in X$ (+ INVERSE)
3. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ $\quad \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ (+ *is* ASSOCIATIVE)
4. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ $\quad \forall \boldsymbol{x}, \boldsymbol{y} \in X$ (+ *is* COMMUTATIVE)
5. $1 \cdot \boldsymbol{x} = \boldsymbol{x}$ $\quad \forall \boldsymbol{x} \in X$ ($\cdot$ IDENTITY)
6. $\alpha \cdot (\beta \cdot \boldsymbol{x}) = (\alpha \cdot \beta) \cdot \boldsymbol{x}$ $\quad \forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$ ($\cdot$ ASSOCIATES *with* $\cdot$)
7. $\alpha \cdot (\boldsymbol{x} + \boldsymbol{y}) = (\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y})$ $\quad \forall \alpha \in S$ *and* $\boldsymbol{x}, \boldsymbol{y} \in X$ ($\cdot$ DISTRIBUTES *over* +)
8. $(\alpha + \beta) \cdot \boldsymbol{x} = (\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x})$ $\quad \forall \alpha, \beta \in S$ *and* $\boldsymbol{x} \in X$ ($\cdot$ PSEUDO-DISTRIBUTES *over* +)

The set X *is called the* ***underlying set***. *The elements of* X *are called* ***vectors***. *The elements of* $\mathbb{F}$ *are called* ***scalars***. *A linear space is also called a* ***vector space***. *If* $\mathbb{F} \triangleq \mathbb{R}$, *then* $\boldsymbol{\Omega}$ *is a* ***real linear space***. *If* $\mathbb{F} \triangleq \mathbb{C}$, *then* $\boldsymbol{\Omega}$ *is a* ***complex linear space***.

Definition D.2. [3]

DEF *A function* $\mathbf{A}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *is an* ***operator*** *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *if*
$\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both* LINEAR SPACES *(Definition D.1 page 155)*.

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}\boldsymbol{x} = \mathbf{B}\boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{X}\boldsymbol{Y}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}\boldsymbol{x} = \mathbf{I}$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

Definition D.3. [4] *Let* $\boldsymbol{X}^{\boldsymbol{X}}$ *be the set of all operators with from a* LINEAR SPACE $\boldsymbol{X}$ *to* $\boldsymbol{X}$. *Let* $\mathbf{I}$ *be an operator in* $\boldsymbol{X}^{\boldsymbol{X}}$. *Let* $\textcircled{1}(\boldsymbol{X})$ *be the* IDENTITY ELEMENT *in* $\boldsymbol{X}^{\boldsymbol{X}}$.

DEF $\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\qquad \mathbf{I} = \textcircled{1}(\boldsymbol{X})$.

[3] Heil (2011) page 42
[4] Michel and Herget (1993) page 411

D.1.2 Linear operators

Definition D.4. [5] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be linear spaces.*

DEF

An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *is **linear** if*

1. $\mathbf{L}(x+y) = \mathbf{L}x + \mathbf{L}y \quad \forall x,y\in\boldsymbol{X}$ (ADDITIVE) *and*
2. $\mathbf{L}(\alpha x) = \alpha\mathbf{L}x \quad \forall x\in\boldsymbol{X},\ \forall\alpha\in\mathbb{F}$ (HOMOGENEOUS).

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ *such that*

$$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \left\{\mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}} \middle| \mathbf{L} \text{ is linear}\right\}.$$

Theorem D.1. [6] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$.

THM

$$\mathbf{L} \text{ is LINEAR} \implies \left\{\begin{array}{llll}
1. & \mathbf{L}\mathbb{0} = \mathbb{0} & & \textit{and}\\
2. & \mathbf{L}(-x) = -(\mathbf{L}x) & \forall x\in\boldsymbol{X} & \textit{and}\\
3. & \mathbf{L}(x-y) = \mathbf{L}x - \mathbf{L}y & \forall x,y\in\boldsymbol{X} & \textit{and}\\
4. & \mathbf{L}\left(\sum_{n=1}^{N}\alpha_n x_n\right) = \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}x_n\right) & x_n\in\boldsymbol{X}, \alpha_n\in\mathbb{F} &
\end{array}\right.$$

✎PROOF:

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned}
\mathbf{L}\mathbb{0} &= \mathbf{L}(0\cdot\mathbb{0}) && \text{by additive identity property}\\
&= 0\cdot(\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \text{ (Definition D.4 page 157)}\\
&= \mathbb{0} && \text{by additive identity property}
\end{aligned}$$

2. Proof that $\mathbf{L}(-x) = -(\mathbf{L}x)$:

$$\begin{aligned}
\mathbf{L}(-x) &= \mathbf{L}(-1\cdot x) && \text{by additive inverse property}\\
&= -1\cdot(\mathbf{L}x) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} \text{ (Definition D.4 page 157)}\\
&= -(\mathbf{L}x) && \text{by additive inverse property}
\end{aligned}$$

3. Proof that $\mathbf{L}(x-y) = \mathbf{L}x - \mathbf{L}y$:

$$\begin{aligned}
\mathbf{L}(x-y) &= \mathbf{L}(x+(-y)) && \text{by additive inverse property}\\
&= \mathbf{L}(x) + \mathbf{L}(-y) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \text{ (Definition D.4 page 157)}\\
&= \mathbf{L}x - \mathbf{L}y && \text{by 2.}
\end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n x_n\right) = \sum_{n=1}^{N}\alpha_n\left(\mathbf{L}x_n\right)$:

 (a) Proof for $N = 1$:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n x_n\right) &= \mathbf{L}\left(\alpha_1 x_1\right) && \text{by } N=1 \text{ hypothesis}\\
&= \alpha_1\left(\mathbf{L}x_1\right) && \text{by } \textit{homogeneous} \text{ property of Definition D.4 page 157}
\end{aligned}$$

[5] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[6] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

(b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1}\alpha_n\boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1}+\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right)\\
&= \mathbf{L}(\alpha_{N+1}\boldsymbol{x}_{N+1})+\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) && \text{by \textit{linearity} property of Definition D.4 page 157}\\
&= \alpha_{N+1}\mathbf{L}(\boldsymbol{x}_{N+1})+\sum_{n=1}^{N}\mathbf{L}(\alpha_n\boldsymbol{x}_n) && \text{by left } N+1 \text{ hypothesis}\\
&= \sum_{n=1}^{N+1}\mathbf{L}(\alpha_n\boldsymbol{x}_n)
\end{aligned}$$

✏

Theorem D.2. [7] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathcal{N}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{X}$*	$\forall\mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathcal{I}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{Y}$*	$\forall\mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	

✎PROOF:

1. Proof that $\mathcal{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0}\in\mathcal{N}(\mathbf{L}) \implies \mathcal{N}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{N}(\mathbf{L}) \triangleq \{\boldsymbol{x}\in\boldsymbol{X}|\mathbf{L}\boldsymbol{x}=\mathbb{0}\}\subseteq\boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{N}(\mathbf{L}) \implies \mathbb{0}=\mathbf{L}(\boldsymbol{x}+\boldsymbol{y})=\mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\boldsymbol{X} \implies \mathbb{0}=\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
2. Proof that $\mathcal{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0}\in\mathcal{I}(\mathbf{L}) \implies \mathcal{I}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{I}(\mathbf{L}) \triangleq \{\boldsymbol{y}\in\boldsymbol{Y}|\exists\boldsymbol{x}\in\boldsymbol{X}$ such that $\boldsymbol{y}=\mathbf{L}\boldsymbol{x}\}\subseteq\boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{v}\in\boldsymbol{X}$ such that $\mathbf{L}\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{x}\in\boldsymbol{X}$ such that $\boldsymbol{y}=\mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y}=\alpha\mathbf{L}\boldsymbol{x}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$

✏

Example D.1. [8] Let $C([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX
$C([a:b],\mathbb{R})$ is a linear space.

Theorem D.3. [9] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of a linear operator $\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM		
$\mathbf{L}\boldsymbol{x}=\mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y}\in\mathcal{N}(\mathbf{L})$
$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathcal{N}(\mathbf{L})=\{\mathbb{0}\}$

[7] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩
[8] Eidelman et al. (2004) page 3
[9] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

✎PROOF:

1. Proof that $\mathbf{L}x = \mathbf{L}y \implies x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}(x-y) &= \mathbf{L}x - \mathbf{L}y && \text{by Theorem D.1 page 157}\\
&= \mathbb{0} && \text{by left hypothesis}\\
&\implies x - y \in \mathcal{N}(\mathbf{L}) && \text{by definition of } \textit{null space}
\end{aligned}$$

2. Proof that $\mathbf{L}x = \mathbf{L}y \impliedby x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}y &= \mathbf{L}y + \mathbb{0} && \text{by definition of linear space (Definition D.1 page 155)}\\
&= \mathbf{L}y + \mathbf{L}(x-y) && \text{by right hypothesis}\\
&= \mathbf{L}y + (\mathbf{L}x - \mathbf{L}y) && \text{by Theorem D.1 page 157}\\
&= (\mathbf{L}y - \mathbf{L}y) + \mathbf{L}x && \text{by \textit{associative} and \textit{commutative} properties (Definition D.1 page 155)}\\
&= \mathbf{L}x
\end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned}
\mathbf{L} \text{ is \textit{injective}} &\iff \{(\mathbf{L}x = \mathbf{L}y \iff x = y) \quad \forall x, y \in X\}\\
&\iff \left\{\left[\mathbf{L}x - \mathbf{L}y = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \left\{\left[\mathbf{L}(x-y) = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}
\end{aligned}$$

✏

Theorem D.4. [10] *Let W, X, Y, and Z be linear spaces over a field $\mathbb{F}$.*

THM					
1.	$\mathbf{L}(\mathbf{M}\mathbf{N})$	$=$ $(\mathbf{L}\mathbf{M})\mathbf{N}$	$\forall \mathbf{L}\in\mathcal{L}(Z,W), \mathbf{M}\in\mathcal{L}(Y,Z), \mathbf{N}\in\mathcal{L}(X,Y)$	(ASSOCIATIVE)
2.	$\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})$	$=$ $(\mathbf{L}\mathbf{M}) \dotplus (\mathbf{L}\mathbf{N})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(X,Y), \mathbf{N}\in\mathcal{L}(X,Y)$	(LEFT DISTRIBUTIVE)
3.	$(\mathbf{L} \dotplus \mathbf{M})\mathbf{N}$	$=$ $(\mathbf{L}\mathbf{N}) \dotplus (\mathbf{M}\mathbf{N})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(Y,Z), \mathbf{N}\in\mathcal{L}(X,Y)$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{L}\mathbf{M})$	$=$ $(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(X,Y), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{M}\mathbf{N}) = (\mathbf{L}\mathbf{M})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}(\mathbf{M} \dotplus \mathbf{N}) = (\mathbf{L}\mathbf{M}) \dotplus (\mathbf{L}\mathbf{N})$:

$$\begin{aligned}
[\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})]x &= \mathbf{L}[(\mathbf{M} \dotplus \mathbf{N})x]\\
&= \mathbf{L}[(\mathbf{M}x) \dotplus (\mathbf{N}x)]\\
&= [\mathbf{L}(\mathbf{M}x)] \dotplus [\mathbf{L}(\mathbf{N}x)] && \text{by \textit{additive} property Definition D.4 page 157}\\
&= [(\mathbf{L}\mathbf{M})x] \dotplus [(\mathbf{L}\mathbf{N})x]
\end{aligned}$$

3. Proof that $(\mathbf{L} \dotplus \mathbf{M})\mathbf{N} = (\mathbf{L}\mathbf{N}) \dotplus (\mathbf{M}\mathbf{N})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{L}\mathbf{M}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{L}\mathbf{M}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned}
[\alpha(\mathbf{L}\mathbf{M})]x &= \alpha[(\mathbf{L}\mathbf{M})x]\\
&= \mathbf{L}[\alpha(\mathbf{M}x)] && \text{by \textit{homogeneous} property Definition D.4 page 157}\\
&= \mathbf{L}[(\alpha\mathbf{M})x]\\
&= [\mathbf{L}(\alpha\mathbf{M})]x
\end{aligned}$$

[10] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

Theorem D.5 (Fundamental theorem of linear equations). *Michel and Herget (1993) page 99 Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ (Definition ?? page ??).*

THM

$$\dim \mathcal{I}(\mathbf{L}) + \dim \mathcal{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1, 2, \ldots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for $\mathcal{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{ \boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x} \} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \mathbb{0} \right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ⇨

D.2 Operators on Normed linear spaces

D.2.1 Operator norm

Definition D.5. [11] *Let $\boldsymbol{V} = (X, \mathbb{F}, \dot{+}, \cdot)$ be a linear space and $\mathbb{F}$ be a field with absolute value function $|\cdot| \in \mathbb{R}^{\mathbb{F}}$ (Definition A.4 page 130).*

DEF

*A **norm** is any functional $\|\cdot\|$ in $\mathbb{R}^X$ that satisfies*

1.	$\|\boldsymbol{x}\| \geq 0$	$\forall \boldsymbol{x} \in X$	(STRICTLY POSITIVE)	and
2.	$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	(NONDEGENERATE)	and
3.	$\|a\boldsymbol{x}\| = \lvert a \rvert \, \|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X,\ a \in \mathbb{C}$	(HOMOGENEOUS)	and
4.	$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(SUBADDITIVE/*triangle inquality*).	

*A **normed linear space** is the pair $(\boldsymbol{V}, \|\cdot\|)$.*

[11] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

Definition D.6. [12] [13] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.*

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} \qquad \forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition D.1 (next) shows that the functional defined in Definition D.6 (previous) is a *norm* (Definition D.5 page 160).

Proposition D.1. [14] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1. $|||\mathbf{A}||| \geq 0$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NON-NEGATIVE)	*and*
2. $|||\mathbf{A}||| = 0 \iff \mathbf{A} \stackrel{\circ}{=} \mathbb{0}$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(NONDEGENERATE)	*and*
3. $|||\alpha\mathbf{A}||| = |\alpha|\, |||\mathbf{A}|||$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), \alpha \in \mathbb{F}$	(HOMOGENEOUS)	*and*
4. $\left|\left|\left|\mathbf{A} \stackrel{\circ}{+} \mathbf{B}\right|\right|\right| \leq |||\mathbf{A}||| + |||\mathbf{B}|||$	$\forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \neq \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 161)} \\
&> 0
\end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \stackrel{\circ}{=} \mathbb{0}$:

$$\begin{aligned}
|||\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 161)} \\
&= \sup_{x \in \boldsymbol{X}} \{\|\mathbb{0}x\| \mid \|x\| \leq 1\} \\
&= 0
\end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha|\, |||\mathbf{A}|||$:

$$\begin{aligned}
|||\alpha\mathbf{A}||| &\triangleq \sup_{x \in \boldsymbol{X}} \{\|\alpha\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 161)} \\
&= \sup_{x \in \boldsymbol{X}} \{|\alpha|\, \|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 161)} \\
&= |\alpha| \sup_{x \in \boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of sup} \\
&= |\alpha|\ |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 161)}
\end{aligned}$$

[12] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225
[13] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990), page 290
[14] Rudin (1991) page 93

4. Proof that $\|\mathbf{A} \mathring{+} \mathbf{B}\| \le \|\mathbf{A}\| + \|\mathbf{B}\|$:

$$\begin{aligned}
\|\mathbf{A} \mathring{+} \mathbf{B}\| &\triangleq \sup_{x\in \boldsymbol{X}} \left\{ \left\|(\mathbf{A} \mathring{+} \mathbf{B})\boldsymbol{x}\right\| \mid \|\boldsymbol{x}\| \le 1 \right\} && \text{by definition of } \|\cdot\| \text{ (Definition D.6 page 161)}\\
&= \sup_{x\in \boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\}\\
&\le \sup_{x\in \boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x}\| + \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\} && \text{by definition of } \|\cdot\| \text{ (Definition D.6 page 161)}\\
&\le \sup_{x\in \boldsymbol{X}} \left\{ \|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\} + \sup_{x\in \boldsymbol{X}} \left\{ \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \right\}\\
&\triangleq \|\mathbf{A}\| + \|\mathbf{B}\| && \text{by definition of } \|\cdot\| \text{ (Definition D.6 page 161)}
\end{aligned}$$

⇨

Lemma D.1. *Let* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \|\cdot\|)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *and* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.

LEM

$$\|\mathbf{L}\| = \sup_{\boldsymbol{x}} \left\{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \right\} \qquad \forall \boldsymbol{x} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$$

✎Proof: [15]

1. Proof that $\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \ge \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$:

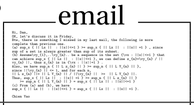
$$\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \ge \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ \boldsymbol{y} \in X \,\middle|\, \begin{array}{ll} 1. & \|\mathbf{L}\boldsymbol{y}\| = \sup_{x\in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \text{ and} \\ 2. & 0 < \|\boldsymbol{y}\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} \le \sup_{\boldsymbol{x}} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$:

$$\begin{aligned}
\sup_{x\in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} &= \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of set } Y\\
&= \frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \|\mathbf{L}\boldsymbol{y}\|\\
&= \|\boldsymbol{y}\| \left\| \frac{1}{\|\boldsymbol{y}\|} \mathbf{L}\boldsymbol{y} \right\| && \text{by homogeneous property (page 160)}\\
&= \|\boldsymbol{y}\| \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| && \text{by homogeneous property (page 157)}\\
&\le \|\boldsymbol{y}\| \sup_{y\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \right\} && \text{by definition of supremum}\\
&= \|\boldsymbol{y}\| \sup_{y\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \text{ for all } \boldsymbol{y} \in Y\\
&\le \sup_{y\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \,\middle|\, \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } 0 < \|\boldsymbol{y}\| \le 1\\
&\le \sup_{x\in X} \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{because } \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in X \quad \forall \boldsymbol{y} \in Y
\end{aligned}$$

[15] email

Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

4. By (1) and (3),

$$\sup_{x \in X} \{\|\mathbf{L}x\| \mid \|x\| \le 1\} = \sup_{x \in X} \{\|\mathbf{L}x\| \mid \|x\| = 1\}$$

✏

Proposition D.2. [16] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{X}), |||\cdot|||)$.

PRP: $|||\mathbf{I}||| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}x\| \mid \|x\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition D.6 page 161)}\\
&= \sup\{\|x\| \mid \|x\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition D.3 page 156)}\\
&= 1
\end{aligned}$$

✏

Theorem D.6. [17] *Let* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM:
$$\begin{array}{rcll}
\|\mathbf{L}x\| & \le & |||\mathbf{L}|||\ \|x\| & \forall \mathbf{L} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}),\ x \in \boldsymbol{X}\\
|||\mathbf{K}\mathbf{L}||| & \le & |||\mathbf{K}|||\ |||\mathbf{L}||| & \forall \mathbf{K}, \mathbf{L} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})
\end{array}$$

✎PROOF:

1. Proof that $\|\mathbf{L}x\| \le |||\mathbf{L}|||\ \|x\|$:

$$\begin{aligned}
\|\mathbf{L}x\| &= \frac{\|x\|}{\|x\|}\|\mathbf{L}x\|\\
&= \|x\| \left\| \frac{1}{\|x\|}\mathbf{L}x \right\| && \text{by property of norms}\\
&= \|x\| \left\| \mathbf{L}\frac{x}{\|x\|} \right\| && \text{by property of linear operators}\\
&\triangleq \|x\|\ \|\mathbf{L}y\| && \text{where } y \triangleq \frac{x}{\|x\|}\\
&\le \|x\| \sup_y \|\mathbf{L}y\| && \text{by definition of supremum}\\
&= \|x\| \sup_y \{\|\mathbf{L}y\| \mid \|y\| = 1\} && \text{because } \|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1\\
&\triangleq \|x\|\ \|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

[16] Michel and Herget (1993) page 410
[17] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

2. Proof that $|||\mathbf{KL}||| \le |||\mathbf{K}|||\ |||\mathbf{L}|||$:

$$\begin{aligned}
|||\mathbf{KL}||| &\triangleq \sup_{x\in X}\{\|(\mathbf{KL})x\| \mid \|x\| \le 1\} && \text{by Definition D.6 page 161 } (|||\cdot|||)\\
&= \sup_{x\in X}\{\|\mathbf{K}(\mathbf{L}x)\| \mid \|x\| \le 1\}\\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ \|\mathbf{L}x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ \|x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&= \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ 1 \mid \|x\| \le 1\} && \text{by definition of sup}\\
&= |||\mathbf{K}|||\ |||\mathbf{L}||| && \text{by definition of sup}
\end{aligned}$$

✏

D.2.2 Bounded linear operators

Definition D.7. [18] *Let $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ be a normed space of linear operators.*

DEF

An operator $\mathbf{B}$ *is* ***bounded*** *if* $|||\mathbf{B}||| < \infty$.
The quantity $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ *is the set of all* ***bounded linear operators*** *on* $(\boldsymbol{X},\boldsymbol{Y})$ *such that*

$$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y}) \triangleq \{\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \mid |||\mathbf{L}||| < \infty\}.$$

Theorem D.7. [19] *Let $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ be the set of linear operators over normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

THM

The following conditions are all EQUIVALENT:

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $x_0 \in X$ — $\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ $\iff$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point $x \in X$)* — $\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ $\iff$
3. $|||\mathbf{L}||| < \infty$ *($\mathbf{L}$ is* BOUNDED*)* — $\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ $\iff$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}x\| \le M\,\|x\|$ — $\forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), x\in X$

✎PROOF:

1. Proof that 1 $\implies$ 2:

$$\begin{aligned}
\epsilon &> \|\mathbf{L}x - \mathbf{L}x_0\| && \text{by hypothesis 1}\\
&= \|\mathbf{L}(x - x_0)\| && \text{by linearity (Definition D.4 page 157)}\\
&= \|\mathbf{L}(x + y - x_0 - y)\|\\
&= \|\mathbf{L}(x + y) - \mathbf{L}(x_0 + y)\| && \text{by linearity (Definition D.4 page 157)}\\
&\implies \mathbf{L} \text{ is continuous at point } x + y\\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}$$

2. Proof that 2 $\implies$ 1: obvious:

[18] Rudin (1991) pages 92–93
[19] Aliprantis and Burkinshaw (1998) page 227

3. Proof that 4 $\implies$ 2:[20]

$$\begin{aligned}
|||\mathbf{L}\boldsymbol{x}||| \le M\,\|\boldsymbol{x}\| &\implies |||\mathbf{L}(\boldsymbol{x}-\boldsymbol{y})||| \le M\,\|\boldsymbol{x}-\boldsymbol{y}\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le M\,\|\boldsymbol{x}-\boldsymbol{y}\| && \text{by linearity of } \mathbf{L} \text{ (Definition D.4 page 157)}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } M\,\|\boldsymbol{x}-\boldsymbol{y}\| < \epsilon\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } \|\boldsymbol{x}-\boldsymbol{y}\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that 3 $\implies$ 4:

$$\begin{aligned}
\|\mathbf{L}\boldsymbol{x}\| &\le \underbrace{|||\mathbf{L}|||}_{M}\;\|\boldsymbol{x}\| && \text{by Theorem D.6 page 163}\\
&= M\;\|\boldsymbol{x}\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that 1 $\implies$ 3:[21]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} = \infty\\
&\implies \exists\,(\boldsymbol{x}_n) \quad \text{such that} \quad \|\boldsymbol{x}_n\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\|\mathbf{L}\boldsymbol{x}_n\| \mid \|\boldsymbol{x}_n\| \le 1\right\} = \infty\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \|\mathbf{L}\boldsymbol{x}_n\|\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \|\mathbf{L}\boldsymbol{x}_n\| \ge n\\
&\implies \frac{1}{n}\|\boldsymbol{x}_n\| = \frac{1}{n} \text{ and } \frac{1}{n}\|\mathbf{L}\boldsymbol{x}_n\| \ge 1\\
&\implies \left\|\frac{\boldsymbol{x}_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{\boldsymbol{x}_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

✑

D.2.3 Adjoints on normed linear spaces

Definition D.8. *Let $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *of $\boldsymbol{X}$.*

DEF: $\mathbf{B}^*$ *is the* ***adjoint*** *of an operator* $\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ *if*

$$\mathsf{f}(\mathbf{B}\boldsymbol{x}) = \left[\mathbf{B}^*\mathsf{f}\right](\boldsymbol{x}) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ \boldsymbol{x} \in \boldsymbol{X}$$

Theorem D.8. [22] *Let $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *$\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM:
$$\begin{aligned}
(\mathbf{A} \dotplus \mathbf{B})^* &= \mathbf{A}^* \dotplus \mathbf{B}^* && \forall \mathbf{A},\mathbf{B}\in\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\lambda\mathbf{A})^* &= \lambda\mathbf{A}^* && \forall \mathbf{A},\mathbf{B}\in\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\mathbf{A}\mathbf{B})^* &= \mathbf{B}^*\mathbf{A}^* && \forall \mathbf{A},\mathbf{B}\in\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})
\end{aligned}$$

[20] Bollobás (1999), page 29
[21] Aliprantis and Burkinshaw (1998), page 227
[22] Bollobás (1999), page 156

✎PROOF:

$$\begin{aligned}
\left[\mathbf{A} \overset{\circ}{+} \mathbf{B}\right]^* \mathsf{f}(\boldsymbol{x}) &= \mathsf{f}\left(\left[\mathbf{A} \overset{\circ}{+} \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition D.8 page 165)} \\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition D.4 page 157)} \\
&= \mathsf{f}(\mathbf{A}\boldsymbol{x}) + \mathsf{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} \\
&= \mathbf{A}^*\mathsf{f}(\boldsymbol{x}) + \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 165)} \\
&= \left[\mathbf{A}^* + \mathbf{B}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{linear functional}
\end{aligned}$$

$$\begin{aligned}
[\lambda\mathbf{A}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 165)} \\
&= \lambda\mathsf{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of } \textit{linear functional} \\
&= \left[\lambda\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 165)}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{A}\mathbf{B}]^*\mathsf{f}(\boldsymbol{x}) &= \mathsf{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 165)} \\
&= \mathsf{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of } \textit{linear operators} && \text{(Definition D.4 page 157)} \\
&= \left[\mathbf{A}^*\mathsf{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 165)} \\
&= \mathbf{B}^*\left[\mathbf{A}^*\mathsf{f}\right](\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 165)} \\
&= \left[\mathbf{B}^*\mathbf{A}^*\right]\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition D.8 page 165)}
\end{aligned}$$

✏

Theorem D.9. [23] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^*$ be the adjoint of an operator $\mathbf{B}$.*

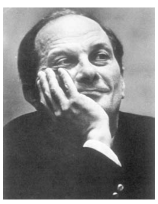
THM
$$|||\mathbf{B}||| = \left|\left|\left|\mathbf{B}^*\right|\right|\right| \qquad \forall \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF:

$$\begin{aligned}
|||\mathbf{B}||| &\triangleq \sup\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by Definition D.6 page 161} \\
&\overset{?}{=} \sup\left\{|\mathsf{g}(\mathbf{B}\boldsymbol{x}; \boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&= \sup\left\{|\mathsf{f}(\boldsymbol{x}; \mathbf{B}^*\boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&\triangleq \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&= \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \left\|\boldsymbol{y}^*\right\| \le 1\right\} \\
&\triangleq \left|\left|\left|\mathbf{B}^*\right|\right|\right| && \text{by Definition D.6 page 161}
\end{aligned}$$

✏

D.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞

Stanislaus M. Ulam (1909–1984), Polish mathematician [24]

[23] Rudin (1991) page 98

Theorem D.10 (Mazur-Ulam theorem). [25] *Let $\phi \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be a function on normed linear spaces $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. Let $\mathbf{I} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$ be the identity operator on $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.*

THM

$$\left.\begin{array}{ll} 1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} \quad \text{and} \\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall x,y \in \boldsymbol{X}}_{\text{isometric}} \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of z in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$
   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$
2. Let $\lambda \triangleq \sup_{g} \{\|g\boldsymbol{z} - \boldsymbol{z}\|\}$
3. Proof that $g \in W \implies g^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq g^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq g^{-1}\boldsymbol{y}$.

$$\begin{aligned}
\left\|g^{-1}\boldsymbol{x} - g^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\
&= \|g\hat{\boldsymbol{x}} - g\hat{\boldsymbol{y}}\| && \text{by left hypothesis} \\
&= \left\|gg^{-1}\boldsymbol{x} - gg^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } g^{-1}
\end{aligned}$$

4. Proof that $g\boldsymbol{z} = \boldsymbol{z}$:

$$\begin{aligned}
2\lambda &= 2\sup\{\|g\boldsymbol{z} - \boldsymbol{z}\|\} && \text{by definition of } \lambda \text{ item (2)} \\
&\le 2\|g\boldsymbol{z} - \boldsymbol{z}\| && \text{by definition of sup} \\
&= \|2\boldsymbol{z} - 2g\boldsymbol{z}\| \\
&= \|\psi g\boldsymbol{z} - g\boldsymbol{z}\| && \text{by definition of } \psi \text{ item (1)} \\
&= \left\|g^{-1}\psi g\boldsymbol{z} - g^{-1}g\boldsymbol{z}\right\| && \text{by item (3)} \\
&= \left\|g^{-1}\psi g\boldsymbol{z} - \boldsymbol{z}\right\| && \text{by definition of } g^{-1} \\
&= \left\|\psi g^{-1}\psi g\boldsymbol{z} - \boldsymbol{z}\right\| \\
&= \left\|g^{*}\boldsymbol{z} - \boldsymbol{z}\right\| \\
&\le \lambda && \text{by definition of } \lambda \text{ item (2)} \\
&\implies 2\lambda \le \lambda \\
&\implies \lambda = 0 \\
&\implies g\boldsymbol{z} = \boldsymbol{z}
\end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) = \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}$:

$$\begin{aligned}
\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) &= \\
&= \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}
\end{aligned}$$

[24] quote: Ulam (1991), page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[25] Oikhberg and Rosenthal (2007), page 598, Väisälä (2003), page 634, Giles (2000), page 11, Dunford and Schwartz (1957), page 91, Mazur and Ulam (1932)

6. Proof that $\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}$:

$$\begin{aligned}\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) &= \\ &= [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}\end{aligned}$$

✏

Theorem D.11 (Neumann Expansion Theorem). [26] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{ll} 1. & \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) \quad (\mathbf{A}\text{ is bounded}) \\ 2. & |||\mathbf{A}||| < 1 \end{array}\right\} \implies \left\{\begin{array}{llll} 1. & (\mathbf{I}-\mathbf{A})^{-1} & & \text{exists} \\ 2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \le & \dfrac{1}{1-|||\mathbf{A}|||} \\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \displaystyle\sum_{n=0}^{\infty}\mathbf{A}^n \\ & \multicolumn{3}{l}{\text{with uniform convergence}} \end{array}\right.$$

D.3 Operators on Inner product spaces

D.3.1 General Results

Definition D.9. [27] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space.*

DEF

A function $\langle \triangle \,|\, \triangledown \rangle \in \mathbb{F}^{X\times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

1.	$\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle$	$\ge$	0	$\forall \boldsymbol{x}\in X$	(non-negative)	and
2.	$\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle$	$=$	$0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x}\in X$	(nondegenerate)	and
3.	$\langle \alpha\boldsymbol{x} \,|\, \boldsymbol{y} \rangle$	$=$	$\alpha\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y}\in X,\ \forall\alpha\in\mathbb{C}$	(homogeneous)	and
4.	$\langle \boldsymbol{x}+\boldsymbol{y} \,|\, \boldsymbol{u} \rangle$	$=$	$\langle \boldsymbol{x} \,|\, \boldsymbol{u} \rangle + \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle$	$\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{u}\in X$	(additive)	and
5.	$\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle$	$=$	$\langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^*$	$\forall \boldsymbol{x},\boldsymbol{y}\in X$	(conjugate symmetric).	

An inner product is also called a ***scalar product***.
An ***inner product space*** *is the pair* $(\boldsymbol{\Omega}, \langle \triangle \,|\, \triangledown \rangle)$.

Theorem D.12. [28] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \triangledown \rangle\right)$.

THM

$$\begin{array}{llllllll} \langle \mathbf{B}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle & = & 0 & \forall \boldsymbol{x}\in X & \iff & \mathbf{B}\boldsymbol{x} & = & \mathbb{0} \quad \forall \boldsymbol{x}\in X \\ \langle \mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle & = & \langle \mathbf{B}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle & \forall \boldsymbol{x}\in X & \iff & \mathbf{A} & = & \mathbf{B} \end{array}$$

✎PROOF:

[26] Michel and Herget (1993) page 415
[27] Haaser and Sullivan (1991), page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72
[28] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

1. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x+\mathbf{B}x) \mid (x+\mathbf{B}x)\rangle + i\,\langle \mathbf{B}(x+i\mathbf{B}x) \mid (x+i\mathbf{B}x)\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}x+\mathbf{B}^2x) \mid x+\mathbf{B}x\right\rangle\right\} + i\left\{\left\langle \mathbf{B}x+i\mathbf{B}^2x) \mid x+i\mathbf{B}x\right\rangle\right\} && \text{by Definition D.4 page 157}\\
&= \left\{\langle \mathbf{B}x \mid x\rangle + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + \left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\} && \text{by Definition D.9 page 168}\\
&\quad + i\left\{\langle \mathbf{B}x \mid x\rangle - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2\left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle\right\} + \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle - \left\langle \mathbf{B}^2x \mid x\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle\\
&= 2\,\|\mathbf{B}x\|^2\\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition D.5 page 160}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products.

3. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \implies \mathbf{A} \triangleq \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \mid x\rangle - \langle \mathbf{B}x \mid x\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}x - \mathbf{B}x \mid x\rangle && \text{by \textit{additivity} property of } \langle \triangle \mid \nabla\rangle \text{ (Definition D.9 page 168)}\\
&= \langle (\mathbf{A}-\mathbf{B})x \mid x\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A}-\mathbf{B})x &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \impliedby \mathbf{A} \triangleq \mathbf{B}$:

$$\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \qquad \text{by } \mathbf{A} \triangleq \mathbf{B} \text{ hypothesis}$$

✏

D.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition D.3 page 169). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem D.13 page 170).
- Both support decomposition into "real" and "imaginary" parts (Theorem ?? page ??).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *null space* of an operator (Theorem D.14 page 171).

Proposition D.3. [29] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *(Definition D.7 page 164) on a* HILBERT SPACE $\boldsymbol{H}$.

PRP: *An operator $\mathbf{B}^*$ is the **adjoint** of $\mathbf{B} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ if*
$$\langle \mathbf{B}x \mid y\rangle = \langle x \mid \mathbf{B}^* y\rangle \qquad \forall x, y \in \boldsymbol{H}.$$

✎PROOF:

[29] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000), page 182, von Neumann (1929) page 49, Stone (1932) page 41

1. For fixed y, $\mathsf{f}(\boldsymbol{x}) \triangleq \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is a *functional* in $\mathbb{F}^X$.
2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}
\langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle &\triangleq \mathsf{f}(\mathbf{B}\boldsymbol{x}) \\
&\triangleq \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of \textit{operator adjoint}} && \text{(Definition D.8 page 165)} \\
&= \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle
\end{aligned}$$

✏

Example D.2.

EX

In matrix algebra ("linear algebra")

- The inner product operation $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$ is represented by $\boldsymbol{y}^H \boldsymbol{x}$.
- The linear operator is represented as a matrix $\boldsymbol{A}$.
- The operation of $\boldsymbol{A}$ on a vector $\boldsymbol{x}$ is represented as $\boldsymbol{A}\boldsymbol{x}$.
- The adjoint of matrix $\boldsymbol{A}$ is the Hermitian matrix $\boldsymbol{A}^H$.

✎PROOF:

$$\langle \boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H \boldsymbol{A}\boldsymbol{x} = [(\boldsymbol{A}\boldsymbol{x})^H \boldsymbol{y}]^H = [\boldsymbol{x}^H \boldsymbol{A}^H \boldsymbol{y}]^H = (\boldsymbol{A}^H \boldsymbol{y})^H \boldsymbol{x} \triangleq \left\langle \boldsymbol{x} \mid \boldsymbol{A}^H \boldsymbol{y} \right\rangle$$

✏

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition **??** page **??**). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[30]

Theorem D.13 (operator star-algebra). [31] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM

The pair $(\boldsymbol{H}, *)$ *is a* *-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \dotplus \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(INVOLUTARY)	

✎PROOF:

$$\begin{aligned}
\left\langle \boldsymbol{x} \mid (\mathbf{A} \dotplus \mathbf{B})^* \boldsymbol{y} \right\rangle &= \left\langle (\mathbf{A} \dotplus \mathbf{B})\boldsymbol{x} \mid \boldsymbol{y} \right\rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)} \\
&= \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle + \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 168)} \\
&= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle + \langle \boldsymbol{x} \mid \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of operator addition} \\
&= \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} + \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 168)} \\
&= \langle \boldsymbol{x} \mid (\mathbf{A}^* + \mathbf{B}^*)\boldsymbol{y} \rangle && \text{by definition of operator addition}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \mid (\alpha\mathbf{A})^* \boldsymbol{y} \rangle &= \langle (\alpha\mathbf{A})\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)} \\
&= \langle \alpha(\mathbf{A}\boldsymbol{x}) \mid \boldsymbol{y} \rangle && \text{by definition of scalar multiplication} \\
&= \alpha \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 168)} \\
&= \alpha \langle \boldsymbol{x} \mid \mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)} \\
&= \langle \boldsymbol{x} \mid \alpha^*\mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 168)}
\end{aligned}$$

[30] Sakai (1998) page 1
[31] Halmos (1998), pages 39–40, Rudin (1991) page 311

$$
\begin{aligned}
\langle \boldsymbol{x} \,|\, (\mathbf{AB})^* \boldsymbol{y} \rangle &= \langle (\mathbf{AB})\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)} \\
&= \langle \mathbf{A}(\mathbf{B}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle && \text{by definition of operator multiplication} \\
&= \langle (\mathbf{B}\boldsymbol{x}) \,|\, \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{B}^* \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)}
\end{aligned}
$$

$$
\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{A}^{**} \boldsymbol{y} \rangle &= \langle \mathbf{A}^* \boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)} \\
&= \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle^* && \text{by definition of inner product} && \text{(Definition D.9 page 168)} \\
&= \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* && \text{by definition of adjoint} && \text{(Proposition D.3 page 169)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A} \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition D.9 page 168)}
\end{aligned}
$$

⇨

Theorem D.14. [32] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\mathcal{N}(\mathbf{A}) = \mathcal{I}(\mathbf{A}^*)^\perp$$
$$\mathcal{N}(\mathbf{A}^*) = \mathcal{I}(\mathbf{A})^\perp$$

✎PROOF:

$$
\begin{aligned}
\mathcal{I}(\mathbf{A}^*)^\perp &= \left\{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}^*) \right\} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} \\
&= \{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A}\boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathbf{A}^* && \text{(Proposition D.3 page 169)} \\
&= \{ y \in \boldsymbol{H} \,|\, \mathbf{A}\boldsymbol{y} = 0 \} \\
&= \mathcal{N}(\mathbf{A}) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}(\mathbf{A})^\perp &= \{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}) \} \\
&= \{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \mathbf{A}\boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathcal{I} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A}^* \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition D.3 page 169)} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \mathbf{A}^* \boldsymbol{y} = 0 \right\} \\
&= \mathcal{N}(\mathbf{A}^*) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}
$$

⇨

D.4 Special Classes of Operators

D.4.1 Projection operators

Definition D.10. [33] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF

$\mathbf{P}$ *is a **projection** operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[32] Rudin (1991) page 312

[33] Rudin (1991) page 133 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 6, Halmos (1958) page 73 ⟨§41. Projections⟩

Theorem D.15. [34] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ with* NULL SPACE $\mathcal{N}(\mathbf{P})$ *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

$$\left.\begin{array}{lrcll} 1. & \mathbf{P}^2 & = & \mathbf{P} & (\mathbf{P} \textit{ is a projection operator}) \quad \textit{and} \\ 2. & \boldsymbol{\Omega} & = & \boldsymbol{X} \dot{+} \boldsymbol{Y} & (\boldsymbol{Y} \textit{ compliments } \boldsymbol{X} \textit{ in } \boldsymbol{\Omega}) \quad \textit{and} \\ 3. & \mathbf{P}\boldsymbol{\Omega} & = & \boldsymbol{X} & (\mathbf{P} \textit{ projects onto } \boldsymbol{X}) \end{array}\right\} \implies \left\{\begin{array}{lrcll} 1. & \mathcal{I}(\mathbf{P}) & = & \boldsymbol{X} & \textit{and} \\ 2. & \mathcal{N}(\mathbf{P}) & = & \boldsymbol{Y} & \textit{and} \\ 3. & \boldsymbol{\Omega} & = & \mathcal{I}(\mathbf{P}) \dot{+} \mathcal{N}(\mathbf{P}) & \end{array}\right.$$

✎PROOF:

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega} \\ &= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) \\ &= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_1 + \{\mathbb{0}\} \\ &= \boldsymbol{\Omega}_1 \end{aligned}$$

$$\begin{aligned} \mathcal{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\} \\ &= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} \\ &= \{\mathbb{0}\} + \boldsymbol{\Omega}_2 \\ &= \boldsymbol{\Omega}_2 \end{aligned}$$

✐

Theorem D.16. [35] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

THM

$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P} \textit{ is a projection operator}} \iff \underbrace{(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})}_{(\mathbf{I} - \mathbf{P}) \textit{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} (\mathbf{I} - \mathbf{P})^2 &= (\mathbf{I} - \mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I}(\mathbf{I} - \mathbf{P}) + (-\mathbf{P})(\mathbf{I} - \mathbf{P}) \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2 \\ &= \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} && \text{by left hypothesis} \\ &= \mathbf{I} - \mathbf{P} \end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned} \mathbf{P}^2 &= \underbrace{\mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P}^2}_{(\mathbf{I} - \mathbf{P})^2} - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P})^2 - (\mathbf{I} - \mathbf{P} - \mathbf{P}) \\ &= (\mathbf{I} - \mathbf{P}) - (\mathbf{I} - \mathbf{P} - \mathbf{P}) && \text{by right hypothesis} \\ &= \mathbf{P} \end{aligned}$$

✐

[34] Michel and Herget (1993) pages 120–121

[35] Michel and Herget (1993) page 121

Theorem D.17. [36] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathcal{N}(\mathbf{P})$, *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.[37]

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

1. $\mathbf{P}^* = \mathbf{P}$ ($\mathbf{P}$ *is* SELF-ADJOINT) $\iff$
2. $\mathbf{P}^*\mathbf{P} = \mathbf{P}\mathbf{P}^*$ ($\mathbf{P}$ *is* NORMAL) $\iff$
3. $\mathcal{I}(\mathbf{P}) = \mathcal{N}(\mathbf{P})^\perp$ $\iff$
4. $\langle \mathbf{P}x \mid x \rangle = \|\mathbf{P}x\|^2 \quad \forall x \in X$

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\Longrightarrow$ (2):

$$\begin{aligned} \mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)} \\ &= \mathbf{P}\mathbf{P}^* && \text{by Theorem D.13 page 170} \end{aligned}$$

Proof that (1) $\Longrightarrow$ (3):

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathcal{N}(\mathbf{P}^*)^\perp && \text{by Theorem D.14 page 171} \\ &= \mathcal{N}(\mathbf{P})^\perp && \text{by (1)} \end{aligned}$$

Proof that (3) $\Longrightarrow$ (4):

Proof that (4) $\Longrightarrow$ (1):

✎

D.4.2 Self Adjoint Operators

Definition D.11. [38] *Let* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be* ***self-adjoint*** *or* ***hermitian*** *if* $\mathbf{B} \doteq \mathbf{B}^*$.

Example D.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation

$$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}.$$

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \displaystyle\int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}} \mathsf{f}(u)\,\mathsf{d}u.$

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

[36] Rudin (1991) page 314

[37] *null space*: Definition ?? page ??
image set: Definition ?? page ??

[38] Historical works regarding self-adjoint operators: von Neumann (1929), page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932), page 50 ⟨"self-adjoint transformations"⟩

Theorem D.18. [39] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.

THM

$$\left\{\underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S}\text{ is self adjoint}}\right\} \implies \left\{\begin{array}{lll} 1. & \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle \in \mathbb{R} & \text{(the hermitian quadratic form of } \mathbf{S} \text{ is REAL-VALUED)} \\ 2. & \lambda_n \in \mathbb{R} & \text{(eigenvalues of } \mathbf{S} \text{ are REAL-VALUED)} \\ 3. & \lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0 & \text{(eigenvectors are ORTHOGONAL)} \end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle \in \mathbb{R}$:

$$\begin{aligned}
\langle \boldsymbol{x} \mid \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&= \langle \boldsymbol{x} \mid \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition D.9 page 168}
\end{aligned}$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$\begin{aligned}
\lambda_n \|\psi_n\|^2 &= \lambda_n \langle \psi_n \mid \psi_n \rangle && \text{by definition} \\
&= \langle \lambda_n\psi_n \mid \psi_n \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition D.9 page 168} \\
&= \langle \mathbf{S}\psi_n \mid \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \mid \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \mid \lambda_n\psi_n \rangle && \text{by definition of eigenpairs} \\
&= \lambda_n^* \langle \psi_n \mid \psi_n \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition D.9 page 168} \\
&= \lambda_n^* \|\psi_n\|^2 && \text{by definition}
\end{aligned}$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \mid \psi_m \rangle &= \langle \lambda_n\psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition D.9 page 168} \\
&= \langle \mathbf{S}\psi_n \mid \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \mid \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \mid \lambda_m\psi_m \rangle && \text{by definition of eigenpairs} \\
&= \lambda_m^* \langle \psi_n \mid \psi_m \rangle && \text{by definition of } \langle \triangle \mid \nabla \rangle \text{ Definition D.9 page 168} \\
&= \lambda_m \langle \psi_n \mid \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \mid \psi_m \rangle = 0$.

✏

D.4.3 Normal Operators

Definition D.12. [40] *Let* $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Let* $\mathbf{N}^*$ *be the adjoint of an operator* $\mathbf{N} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.

DEF

$\mathbf{N}$ *is* ***normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

[39] Lax (2002), pages 315–316, Keener (1988), pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩, Bertero and Boccacci (1998) page 225 ⟨§"9.2 SVD of a matrix …If all eigenvectors are normalized…"⟩

[40] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969), page 167, Frobenius (1878), Frobenius (1968), page 391

Theorem D.19. [41] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ *and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \iff \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\| \qquad \forall \boldsymbol{x} \in \boldsymbol{H}$$

✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \mid \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition D.3 page 169 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \mid \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (} \mathbf{N} \text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition D.3 page 169 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition D.3 page 169 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem D.13 page 170 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis (} \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Proposition D.3 page 169 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem D.20. [42] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ *and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same null space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \left\{\boldsymbol{x} \mid \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{(definition of } \mathcal{N})\\
&= \left\{\boldsymbol{x} \mid \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition D.5 page 160)}\\
&= \left\{\boldsymbol{x} \mid \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\}\\
&= \left\{\boldsymbol{x} \mid \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\right\} && \text{by definition of } \|\cdot\| \text{ (Definition D.5 page 160)}\\
&= \mathcal{N}(\mathbf{N}) && \text{(definition of } \mathcal{N})
\end{aligned}$$

✏

Theorem D.21. [43] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ *and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of* $\mathbf{N}$ *in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$. [44]

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m\rangle = 0}_{\text{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

---

[41] Rudin (1991) pages 312–313
[42] Rudin (1991) pages 312–313
[43] Rudin (1991) pages 312–313
[44] *image set*: Definition ?? page ??

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. [Rudin] claims both to be true.(Rudin, 1991)313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\quad\;\; = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\quad\;\; = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem D.13 page 170} \\
&\quad\;\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem D.13 page 170} \\
&\quad\;\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\Longrightarrow \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\iff \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n\psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition D.9 page 168} \\
&= \langle \mathbf{N}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{N}^*\psi_m \rangle && \text{by Proposition D.3 page 169 (definition of adjoint)} \\
&= \langle \psi_n \,|\, \lambda_m^*\psi_m \rangle && \text{by (4.)} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition D.9 page 168}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

D.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition D.13. *Let* $(X, +, \cdot, (\mathbb{F}, \dot+, \dot\times), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot+, \dot\times), \|\cdot\|)$ *be* NORMED LINEAR SPACES *(Definition D.5 page 160).*

DEF *An operator* $\mathbf{M} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is* ***isometric*** *if*
$$\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \qquad \forall x \in X.$$

Theorem D.22. [45] *Let* $(X, +, \cdot, (\mathbb{F}, \dot+, \dot\times), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot+, \dot\times), \|\cdot\|)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM
$$\underbrace{\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall x \in X}_{\text{isometric in length}} \iff \underbrace{\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X}_{\text{isometric in distance}}$$

✎PROOF:

[45] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

1. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \implies \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| &= \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\| && \text{by definition of linear operators (Definition D.4 page 157)}\\
&= \|\mathbf{M}\boldsymbol{u}\| && \text{let } \boldsymbol{u} \triangleq \boldsymbol{x} - \boldsymbol{y}\\
&= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by left hypothesis}
\end{aligned}$$

2. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \impliedby \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \|\mathbf{M}(\boldsymbol{x} - \mathbb{0})\|\\
&= \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition D.4 page 157)}\\
&= \|\boldsymbol{x} - \mathbb{0}\| && \text{by right hypothesis}\\
&= \|\boldsymbol{x}\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition D.13 page 176) in the more general normed linear spaces, while Theorem D.22 (page 176) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \iff \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem D.23 (next) demonstrates two additional equivalent properties.

Theorem D.23. [46] *Let $\mathcal{B}(\boldsymbol{X},\boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathcal{L}(\boldsymbol{X},\boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X},\boldsymbol{X})$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle}$.*

THM *The following conditions are all* ***equivalent****:*

1.	$\mathbf{M}^*\mathbf{M} = \mathbf{I}$				$\iff$
2.	$\langle \mathbf{M}\boldsymbol{x} \,	\, \mathbf{M}\boldsymbol{y}\rangle = \langle \boldsymbol{x} \,	\, \boldsymbol{y}\rangle$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	*($\mathbf{M}$ is surjective)*	$\iff$
3.	$\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|x - y\|$	$\forall \boldsymbol{x},\boldsymbol{y} \in X$	*(isometric in distance)*	$\iff$
4.	$\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X$	*(isometric in length)*	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y}\rangle &= \langle \boldsymbol{x} \,|\, \mathbf{M}^*\mathbf{M}\boldsymbol{y}\rangle && \text{by Proposition D.3 page 169 (definition of adjoint)}\\
&= \langle \boldsymbol{x} \,|\, \mathbf{I}\boldsymbol{y}\rangle && \text{by (1)}\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle && \text{by Definition D.3 page 156 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \sqrt{\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{x}\rangle} && \text{by definition of } \|\cdot\|\\
&= \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle} && \text{by right hypothesis}\\
&= \|\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{y}\rangle &= \|\mathbf{M}\boldsymbol{x} + \mathbf{M}\boldsymbol{y}\|^2 - \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|^2 + i\,\|\mathbf{M}\boldsymbol{x} + i\mathbf{M}\boldsymbol{y}\|^2 - i\,\|\mathbf{M}\boldsymbol{x} - i\mathbf{M}\boldsymbol{y}\|^2 && \text{by polarization id.}\\
&= \|\mathbf{M}(\boldsymbol{x} + \boldsymbol{y})\|^2 - \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\|^2 + i\,\|\mathbf{M}(\boldsymbol{x} + i\boldsymbol{y})\|^2 - i\,\|\mathbf{M}(\boldsymbol{x} - i\boldsymbol{y})\|^2 && \text{by Definition D.4}\\
&= \|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 + i\,\|\boldsymbol{x} + i\boldsymbol{y}\|^2 - i\,\|\boldsymbol{x} - i\boldsymbol{y}\|^2 && \text{by left hypothesis}
\end{aligned}$$

[46] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

4. Proof that (3) $\iff$ (4): by Theorem D.22 page 176

5. Proof that (4) $\implies$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition D.3 page 169 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem D.13 page 170 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y = 0\\
&= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by Definition D.3 page 156 (definition of } \mathbf{I})\\
\implies \quad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}$$

✎

Theorem D.24. [47] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M} \text{ is isometric}} \implies \left\{\begin{array}{lcll} |||\mathbf{M}||| &=& 1 & \text{(UNIT LENGTH)} \quad \textit{and}\\ |\lambda| &=& 1 & \forall \lambda \in \Lambda \end{array}\right.$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \implies |||\mathbf{M}||| = 1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{\boldsymbol{x}\in X}\{\|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Definition D.6 page 161}\\
&= \sup_{\boldsymbol{x}\in X}\{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\} && \text{by Theorem D.23 page 177}\\
&= \sup_{\boldsymbol{x}\in X}\{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x}, \lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|}\ \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem D.23 page 177}\\
&= \frac{1}{\|\boldsymbol{x}\|}\ \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|}\ |\lambda|\ \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

✎

Example D.4 (One sided shift operator). [48] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ (0, 1, 2, ...) and shift operators defined as

$$\begin{array}{lllll}
1. & \mathbf{S}_r\,(\!(x_0, x_1, x_2, \ldots)\!) & \triangleq & (\!(0, x_0, x_1, x_2, \ldots)\!) & \text{(right shift operator)}\\
2. & \mathbf{S}_l\,(\!(x_0, x_1, x_2, \ldots)\!) & \triangleq & (\!(x_1, x_2, x_3, \ldots)\!) & \text{(left shift operator)}
\end{array}$$

[47] Michel and Herget (1993) page 432
[48] Michel and Herget (1993) page 441

EX
1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \mid \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \rangle &= \langle \left(\!\left(0, x_0, x_1, x_2, \ldots \right)\!\right) \mid \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\, y_n^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\, y_{n+1}^* \\
&= \langle \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \mid \left(\!\left(y_1, y_2, y_3, \ldots \right)\!\right) \rangle \\
&= \left\langle \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \mid \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^*\mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^*\mathbf{S}_r &= \mathbf{S}_l\mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✏

D.4.5 Unitary operators

Definition D.14. [49] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF
*The operator $\mathbf{U}$ is **unitary** if* $\mathbf{U}^*\mathbf{U} \;=\; \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition D.4. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP
$$\left.\begin{array}{l} \mathbf{U} \text{ is UNITARY} \quad \text{and} \\ \mathbf{V} \text{ is UNITARY} \end{array}\right\} \implies (\mathbf{U}\mathbf{V}) \text{ is UNITARY.}$$

[49] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\left(\mathbf{V}^*\mathbf{U}^*\right) && \text{by Theorem D.8 page 165}\\
&= \mathbf{U}\left(\mathbf{VV}^*\right)\mathbf{U}^* && \text{by associative property}\\
&= \mathbf{UIU}^* && \text{by definition of unitary operators—Definition D.14 page 179}\\
&= \mathbf{I} && \text{by definition of unitary operators—Definition D.14 page 179}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \left(\mathbf{V}^*\mathbf{U}^*\right)(\mathbf{UV}) && \text{by Theorem D.8 page 165}\\
&= \mathbf{V}^*\left(\mathbf{U}^*\mathbf{U}\right)\mathbf{V} && \text{by associative property}\\
&= \mathbf{V}^*\mathbf{IV} && \text{by definition of unitary operators—Definition D.14 page 179}\\
&= \mathbf{I} && \text{by definition of unitary operators—Definition D.14 page 179}
\end{aligned}$$

✏

Theorem D.25. [50] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathbf{U}$ *be a bounded linear operator in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, *and* $\mathcal{I}(\mathbf{U})$ *the* IMAGE SET *of* $\mathbf{U}$.

THM

The following conditions are ***equivalent****:*

1. $\mathbf{UU}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ *(unitary)* $\iff$
2. $\langle \mathbf{U}x \mid \mathbf{U}y\rangle = \langle \mathbf{U}^*x \mid \mathbf{U}^*y\rangle = \langle x \mid y\rangle$ *and* $\mathcal{I}(\mathbf{U}) = X$ *(surjective)* $\iff$
3. $\|\mathbf{U}x - \mathbf{U}y\| = \left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x-y\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ *(isometric in distance)* $\iff$
4. $\|\mathbf{U}x\| = \|x\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ *(isometric in length)*

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) $\langle \mathbf{U}x \mid \mathbf{U}y\rangle = \langle \mathbf{U}^*x \mid \mathbf{U}^*y\rangle = \langle x \mid y\rangle$ by Theorem D.23 (page 177).
   (b) Proof that $\mathcal{I}(\mathbf{U}) = X$:

$$\begin{aligned}
X &\supseteq \mathcal{I}(\mathbf{U}) && \text{because } \mathbf{U}\in X^X\\
&\supseteq \mathcal{I}(\mathbf{UU}^*)\\
&= \mathcal{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{UU}^* = \mathbf{I})\\
&= X && \text{by Definition D.3 page 156 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\iff$ (3) $\iff$ (4): by Theorem D.23 page 177.
3. Proof that (3) $\implies$ (1):
   (a) Proof that $\|\mathbf{U}x - \mathbf{U}y\| = \|x-y\| \implies \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem D.23 page 177
   (b) Proof that $\left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x-y\| \implies \mathbf{UU}^* = \mathbf{I}$:

$$\begin{aligned}
\left\|\mathbf{U}^*x - \mathbf{U}^*y\right\| = \|x-y\| \implies \quad \mathbf{U}^{**}\mathbf{U}^* &= \mathbf{I} && \text{by Theorem D.23 page 177}\\
\mathbf{UU}^* &= \mathbf{I} && \text{by Theorem D.13 page 170}
\end{aligned}$$

✏

Theorem D.26. *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathbf{U}$ *be a bounded linear operator in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathcal{N}(\mathbf{U})$ *the* NULL SPACE *of* $\mathbf{U}$, *and* $\mathcal{I}(\mathbf{U})$ *the* IMAGE SET *of* $\mathbf{U}$.

THM

$$\underbrace{\mathbf{UU}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U}\text{ is unitary}} \implies \left\{\begin{array}{lllll}
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \\
\mathcal{I}(\mathbf{U}) & = & \mathcal{I}(\mathbf{U}^*) & = & X\\
\mathcal{N}(\mathbf{U}) & = & \mathcal{N}(\mathbf{U}^*) & = & \{\mathbb{0}\}\\
|||\mathbf{U}||| & = & \left|\left|\left|\mathbf{U}^*\right|\right|\right| & = & 1 \quad (\text{UNIT LENGTH})
\end{array}\right.$$

[50] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both **isometric** and **normal**:

$$\begin{array}{lllllll} \mathbf{U}^*\mathbf{U} & & & = & \mathbf{I} & \implies & \mathbf{U} \text{ is isometric} \\ \mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & \mathbf{I} & \implies & \mathbf{U}^* \text{ is isometric} \\ \mathbf{U}^{-1} & = & \mathbf{U}^* & & & \implies & \mathbf{U}^{-1} \text{ is isometric} \end{array}$$

$$\begin{array}{llllll} \mathbf{U}^*\mathbf{U} & = & \mathbf{U}\mathbf{U}^* & = & \implies & \mathbf{U} \text{ is normal} \\ \mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & \implies & \mathbf{U}^* \text{ is normal} \\ \mathbf{U}^{-1} & = & \mathbf{U}^* & & \implies & \mathbf{U}^{-1} \text{ is normal} \end{array}$$

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem D.25 page 180.

3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}^{-1})$:

$$\begin{aligned} \mathcal{N}(\mathbf{U}^*) &= \mathcal{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both normal and by Theorem D.21 page 175} \\ &= \mathcal{I}(\mathbf{U})^\perp && \text{by Theorem D.14 page 171} \\ &= X^\perp && \text{by above result} \\ &= \{\mathbb{0}\} \end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem D.24 page 178.

✏

Example D.5. Examples of *Fredholm integral operators* include

1.	Fourier Transform	$[\tilde{\mathbf{F}}\mathrm{x}](f)$	$=$	$\int_t \mathrm{x}(t)e^{-i2\pi ft}\,\mathrm{d}t$	$\kappa(t,f) =$	$e^{-i2\pi ft}$
2.	Inverse Fourier Transform	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathrm{x}}](t)$	$=$	$\int_f \tilde{\mathrm{x}}(f)e^{i2\pi ft}\,\mathrm{d}f$	$\kappa(f,t) =$	$e^{i2\pi ft}$
3.	Laplace operator	$[\mathbf{L}\mathrm{x}](s)$	$=$	$\int_t \mathrm{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t,s) =$	e^{-st}

Example D.6 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathrm{f}(x) \triangleq \mathrm{f}(x-1) \qquad \forall \mathrm{f} \in \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

EX

1. $\mathbf{T}^{-1}\mathrm{f}(x) = \mathrm{f}(x+1) \qquad \forall \mathrm{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ (inverse translation operator)
2. $\mathbf{T}^* = \mathbf{T}^{-1}$ ($\mathbf{T}$ is invertible)
3. $\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^* = \mathbf{I}$ ($\mathbf{T}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathrm{f}(x) = \mathrm{f}(x+1)$:

$$\begin{aligned} \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} \\ \mathbf{T}\mathbf{T}^{-1} &= \mathbf{I} \end{aligned}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned} \langle \mathbf{T}\mathrm{f}(x) \,|\, \mathrm{g}(x) \rangle &= \langle \mathrm{f}(x-1) \,|\, \mathrm{g}(x) \rangle && \text{by definition of } \mathbf{T} \\ &= \int_x \mathrm{f}(x-1)\mathrm{g}^*(x)\,\mathrm{d}x \\ &= \int_x \mathrm{f}(x)\mathrm{g}^*(x+1)\,\mathrm{d}x \\ &= \langle \mathrm{f}(x) \,|\, \mathrm{g}(x+1) \rangle \\ &= \left\langle \mathrm{f}(x) \,\middle|\, \underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathrm{g}(x) \right\rangle && \text{by 1.} \end{aligned}$$

Example D.7 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX

1. $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right) \quad \forall \mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$ (inverse dilation operator)
2. $\mathbf{D}^* = \mathbf{D}^{-1}$ ($\mathbf{D}$ is invertible)
3. $\mathbf{D}^*\mathbf{D} = \mathbf{D}\mathbf{D}^* = \mathbf{I}$ ($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x) \,|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathsf{d}x\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathsf{d}u && \text{let } u \triangleq 2x \implies \mathsf{d}x = \frac{1}{2}\,\mathsf{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\mathsf{d}u\\
&= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

Example D.8 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n-1}\right)\!\right)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \triangleq \left(\!\left(x_{n+1}\right)\!\right)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\left(\!\left(x_n\right)\!\right) \,|\, \left(\!\left(y_n\right)\!\right)\rangle &= \langle \left(\!\left(x_{n-1}\right)\!\right) \,|\, \left(\!\left(y_n\right)\!\right)\rangle && \text{by definition of } \mathbf{D}\\
&= \sum_n x_{n-1}\, y_n^*\\
&= \sum_n x_n\, y_{n+1}^*\\
&= \left\langle \left(\!\left(x_n\right)\!\right) \,|\, \left(\!\left(y_{n+1}\right)\!\right)\right\rangle\\
&= \left\langle \left(\!\left(x_n\right)\!\right) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\left(\!\left(y_n\right)\!\right)\right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary.

Example D.9 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator (Theorem J.1 page 254)

$$[\tilde{\mathbf{F}}\mathsf{x}](f) \triangleq \int_t \mathsf{x}(t)\underbrace{e^{-i2\pi ft}}_{\kappa(t,f)}\,\mathrm{d}t \qquad\qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](t) \triangleq \int_f \tilde{\mathsf{x}}(f)\underbrace{e^{i2\pi ft}}_{\kappa^*(t,f)}\,\mathrm{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{x} \mid \tilde{\mathsf{y}}\right\rangle &= \left\langle \int_t \mathsf{x}(t)e^{-i2\pi ft}\,\mathrm{d}t \mid \tilde{\mathsf{y}}(f)\right\rangle \\
&= \int_t \mathsf{x}(t)\left\langle e^{-i2\pi ft} \mid \tilde{\mathsf{y}}(f)\right\rangle\,\mathrm{d}t \\
&= \int_t \mathsf{x}(t)\int_f e^{-i2\pi ft}\tilde{\mathsf{y}}^*(f)\,\mathrm{d}f\,\mathrm{d}t \\
&= \int_t \mathsf{x}(t)\left[\int_f e^{i2\pi ft}\tilde{\mathsf{y}}(f)\,\mathrm{d}f\right]^*\,\mathrm{d}t \\
&= \left\langle \mathsf{x}(t) \mid \int_f \tilde{\mathsf{y}}(f)e^{i2\pi ft}\,\mathrm{d}f\right\rangle \\
&= \left\langle \mathsf{x} \mid \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*}\tilde{\mathsf{y}}\right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✏

Example D.10 (Rotation matrix). [51] Let the rotation matrix $\mathbf{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

EX
1. $\mathbf{R}^{-1}{}_\theta = \mathbf{R}_{-\theta}$
2. $\mathbf{R}^*{}_\theta = \mathbf{R}^{-1}{}_\theta$ ($\mathbf{R}$ is unitary)

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of Hermetian transpose operator } H \\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{by Theorem H.2 page 217} \\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R} \\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✏

[51] Noble and Daniel (1988), page 311

APPENDIX E

CALCULUS

Definition E.1. *Let $\mathbb{R}$ be the set of real numbers, $\mathscr{B}$ the set of* BOREL SETS *on $\mathbb{R}$, and μ the standard* BOREL MEASURE *on $\mathscr{B}$. Let $\mathbb{R}^{\mathbb{R}}$ be as in Definition 1.1 page 1.*

DEF

*The **space of Lebesgue square-integrable functions** $L^2_{(\mathbb{R},\mathscr{B},\mu)}$ (or $L^2_{\mathbb{R}}$) is defined as*

$$L^2_{\mathbb{R}} \triangleq L^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \,|\, \left(\int_{\mathbb{R}} |\mathsf{f}|^2 \right)^{\frac{1}{2}} \mathsf{d}\mu < \infty \right\}.$$

*The **standard inner product** $\langle \triangle \,|\, \nabla \rangle$ on $L^2_{\mathbb{R}}$ is defined as*

$$\langle \mathsf{f}(x) \,|\, \mathsf{g}(x) \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x)\, \mathsf{d}x.$$

*The **standard norm** $\|\cdot\|$ on $L^2_{\mathbb{R}}$ is defined as $\|\mathsf{f}(x)\| \triangleq \langle \mathsf{f}(x) \,|\, \mathsf{f}(x) \rangle^{\frac{1}{2}}$*

Definition E.2. *Let $\mathsf{f}(x)$ be a* FUNCTION *in $\mathbb{R}^{\mathbb{R}}$.*

DEF

$$\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x) \triangleq \mathsf{f}'(x) \triangleq \lim_{\varepsilon \to 0} \frac{\mathsf{f}(x+\varepsilon) - \mathsf{f}(x)}{\varepsilon}$$

Proposition E.1.

PRP

$$\left\{ \begin{array}{lll} (1). & \mathsf{f}(x) \text{ is CONTINUOUS} & \text{and} \\ (2). & \underbrace{\mathsf{f}(a+x) = \mathsf{f}(a-x)}_{\text{SYMMETRIC about a point } a} & \end{array} \right\} \implies \left\{ \begin{array}{llcll} (1). & \mathsf{f}'(a+x) & = & -\mathsf{f}'(a-x) & (\text{ANTI-SYMMETRIC about } a) \\ (2). & \mathsf{f}'(a) & = & 0 & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{f}'(a+x) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a+x+\varepsilon) - \mathsf{f}(a+x-\varepsilon)] \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x-\varepsilon) - \mathsf{f}(a-x+\varepsilon)] && \text{by hpothesis (2)} \\
&= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\mathsf{f}(a-x+\varepsilon) - \mathsf{f}(a-x-\varepsilon)] \\
&= -\mathsf{f}(a-x)
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}'(a) &= \frac{1}{2}\mathsf{f}'(a+0) + \frac{1}{2}\mathsf{f}'(a-0) \\
&= \frac{1}{2}\left[\mathsf{f}'(a+0) - \mathsf{f}'(a+0)\right] && \text{by previous result}
\end{aligned}$$

$$= 0$$

✏

Lemma E.1.

LEM $\mathsf{f}(x)$ *is* INVERTIBLE $\implies \left\{ \frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) = \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} \right\}$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}y}\mathsf{f}^{-1}(y) &\triangleq \lim_{\varepsilon\to 0}\frac{\mathsf{f}^{-1}(y+\varepsilon)-\mathsf{f}^{-1}(y)}{\varepsilon} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}y} && \text{(Definition E.2 page 185)}\\
&= \lim_{\delta\to 0}\left.\frac{1}{\left[\frac{\mathsf{f}(x+\delta)-\mathsf{f}(x)}{\delta}\right]}\right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{because in the limit, } \frac{\Delta y}{\Delta x} = \left(\frac{\Delta x}{\Delta y}\right)^{-1}\\
&\triangleq \left.\frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}(x)}\right|_{x\triangleq \mathsf{f}^{-1}(y)} && \text{by definition of } \frac{\mathsf{d}}{\mathsf{d}x} && \text{(Definition E.2 page 185)}\\
&= \frac{1}{\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\left[\mathsf{f}^{-1}(y)\right]} && \text{because } x \triangleq \mathsf{f}^{-1}(y)
\end{aligned}$$

✏

Theorem E.1. [1] *Let* f *be a continuous function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* n*th derivative of* f.

THM $\displaystyle\int_{[0:1)^n} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n} x_k\right)\mathsf{d}x_1\,\mathsf{d}x_2\cdots\,\mathsf{d}x_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) \qquad \forall n\in\mathbb{N}$

✎PROOF: Proof by induction:

1. Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\int_{[0:1)} \mathsf{f}^{(1)}(x)\,\mathsf{d}x &= \mathsf{f}(1)-\mathsf{f}(0) && \text{by \textit{Fundamental theorem of calculus}}\\
&= (-1)^{1+1}\binom{1}{1}\mathsf{f}(1) + (-1)^{1+0}\binom{1}{0}\mathsf{f}(0)\\
&= \sum_{k=0}^{1}(-1)^{n-k}\binom{n}{k}\mathsf{f}(k)
\end{aligned}$$

[1] Chui (1992) page 86 ⟨item (ii)⟩, Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2 (b)⟩

2. Induction step ...proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{[0:1)^{n+1}} \mathsf{f}^{(n+1)}\left(\sum_{k=1}^{n+1} x_k\right) \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_{n+1} \\
&= \int_{[0:1)^n} \left[\int_0^1 \mathsf{f}^{(n+1)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right) \mathrm{d}x_{n+1}\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\Bigg|_{x_{n+1}=0}^{x_{n+1}=1}\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n && \text{by \textit{Fundamental theorem of calculus}} \\
&= \int_{[0:1)^n} \left[\mathsf{f}^{(n)}\left(1 + \sum_{k=1}^{n} x_k\right) - \mathsf{f}^{(n)}\left(0 + \sum_{k=1}^{n} x_k\right)\right] \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k+1) - \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{by induction hypothesis} \\
&= \sum_{m=1}^{m=n+1} (-1)^{n-m+1}\binom{n}{m-1}\mathsf{f}(m) + \sum_{k=0}^{n} (-1)(-1)^{n-k}\binom{n}{k}\mathsf{f}(k) && \text{where } m \triangleq k+1 \implies k = m-1 \\
&= \left[\mathsf{f}(n+1) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k-1}\mathsf{f}(k)\right] + \left[(-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k}\mathsf{f}(k)\right] \\
&= \mathsf{f}(n+1) + (-1)^{n+1}\mathsf{f}(0) + \sum_{k=1}^{n} (-1)^{n-k+1}\underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\text{use \textit{Stifel formula}}}\mathsf{f}(k) \\
&= (-1)^0\binom{n+1}{n+1}\mathsf{f}(n+1) + (-1)^{n+1}\binom{n+1}{0}\mathsf{f}(0) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by \textit{Stifel formula} (Theorem B.1 page 133)} \\
&= \sum_{k=0}^{n+1} (-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)
\end{aligned}$$

✏

Some proofs invoke differentiation multiple times. This is simplified thanks to the *Leibniz rule*, also called the *generalized product rule* (*GPR*, next lemma). The Leibniz rule is remarkably similar in form to the *binomial theorem*.

Lemma E.2 (Leibniz rule / generalized product rule). [2] *Let* $\mathsf{f}(x), \mathsf{g}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *with derivatives* $\mathsf{f}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{f}(x)$ *and* $\mathsf{g}^{(n)}(x) \triangleq \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathsf{g}(x)$ *for* $n = 0, 1, 2, \ldots,$ *and* $\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$ *(binomial coefficient). Then*

LEM
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathsf{f}(x)\mathsf{g}(x)] = \sum_{k=0}^{n}\binom{n}{k}\mathsf{f}^{(k)}(x)\mathsf{g}^{(n-k)}(x)$$

Example E.1.

EX
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\mathsf{f}(x)\mathsf{g}(x)\right] = \mathsf{f}'''(x)\mathsf{g}(x) + 3\mathsf{f}''(x)\mathsf{g}'(x) + 3\mathsf{f}'(x)\mathsf{g}''(x) + \mathsf{f}(x)\mathsf{g}'''(x)$$

Theorem E.2 (Leibniz integration rule). [3]

THM
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathsf{a}(x)}^{\mathsf{b}(x)} \mathsf{g}(t)\,\mathrm{d}t = \mathsf{g}[\mathsf{b}(x)]\mathsf{b}'(x) - \mathsf{g}[\mathsf{a}(x)]\mathsf{a}'(x)$$

[2] Ben-Israel and Gilbert (2002) page 154, Leibniz (1710)

[3] Flanders (1973) page 615 ⟨(1.1)⟩, Talvila (2001), Knapp (2005b) page 389 ⟨Chapter VII⟩, ? page 422 ⟨Leibniz Rule. Theorem 1.⟩, http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

APPENDIX F

FINITE SUMS

❝ *I think that it was Harald Bohr who remarked to me that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."* ❞
G.H. Hardy (1877–1947) in his "Presidential Address" to the London Mathematical Society on November 8, 1928, about a remark that he suggested was from Harald Bohr (1887–1951), Danish mathematician pictured to the left. [1]

F.1 Summation

Definition F.1. [2] *Let + be an addition operator on a tuple* $\left(x_n\right)_m^N$.

DEF

*The **summation** of* $\left(x_n\right)$ *from index m to index N with respect to + is*

$$\sum_{n=m}^{N} x_n \triangleq \begin{cases} 0 & \text{for } N < m \\ \left(\sum_{n=m}^{N-1} x_n\right) + x_N & \text{for } N \geq m \end{cases}$$

Theorem F.1 (Generalized associative property). [3] *Let + be an addition operator on a tuple* $\left(x_n\right)_m^N$.

THM

$+$ *is* ASSOCIATIVE $\Longrightarrow$

$$\underbrace{\sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right) = \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n \qquad \text{for } m < L < M \leq N}_{\sum_{n=m}^{N} \text{ is ASSOCIATIVE}}$$

✎PROOF:

[1] quote: Hardy (1929), page 64
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Bohr_Harald.html
[2] reference: Berberian (1961) page 8 ⟨Definition I.3.1⟩
"$\sum$" notation: Fourier (1820) page 280
[3] Berberian (1961) pages 9–10 ⟨Theorem I.3.1⟩

1. Proof for $N < m$ case: $\sum_{n=m}^{N} x_n = 0$.

2. Proof for $N = m$ case: $\sum_{n=m}^{m} x_n = \left(\sum_{n=m}^{m-1} x_n\right) + x_m = 0 + x_m = x_m$.

3. Proof for $N = m+1$ case: $\sum_{n=m}^{m+1} x_n = \left(\sum_{n=m}^{m} x_n\right) + x_{m+1} = x_m + x_{m+1}$

4. Proof for $N = m+2$ case:

$$\begin{aligned}
\sum_{n=m}^{m+2} x_n &= \left(\sum_{n=m}^{m+1} x_n\right) + x_{m+2} && \text{by Definition F.1 page 189} \\
&= \left(x_m + x_{m+1}\right) + x_{m+2} && \text{by item (3)} \\
&= x_m + \left(x_{m+1} + x_{m+2}\right) && \text{by left hypothesis}
\end{aligned}$$

5. Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\sum_{n=m}^{N+1} x_n &= \underbrace{\left(\sum_{n=m}^{N} x_n\right)}_{\textit{associative}} + x_{N+1} && \text{by Definition F.1 page 189} \\
&= \left(\sum_{n=m}^{L} x_n + \left(\sum_{n=L+1}^{M} x_n + \sum_{n=M+1}^{N} x_n\right)\right) + x_{N+1} &&= \left(\left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \sum_{n=M+1}^{N} x_n\right) + x_{N+1} \\
&= \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N} x_n + x_{N+1}\right) &&= \left(\sum_{n=m}^{L} x_n + \sum_{n=L+1}^{M} x_n\right) + \left(\sum_{n=M+1}^{N+1} x_n\right)
\end{aligned}$$

✎

F.2 Means

F.2.1 Weighted φ-means

Definition F.2. [4]

DEF

*The $\left(\lambda_n\right)_1^N$ weighted ϕ-**mean** of a tuple $\left(x_n\right)_1^N$ is defined as*

$$\mathsf{M}_\phi\left(\left(x_n\right)\right) \triangleq \phi^{-1}\left(\sum_{n=1}^{N} \lambda_n \phi(x_n)\right)$$

where ϕ is a CONTINUOUS *and* STRICTLY MONOTONIC *function in $\mathbb{R}^{\mathbb{R}}$ and $\left(\lambda_n\right)_{n=1}^N$ is a sequence of weights for which $\sum_{n=1}^{N} \lambda_n = 1$.*

[4] Bollobás (1999) page 5

Lemma F.1. [5] *Let* $\mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis)$ *be the* $\llparenthesis \lambda_n \rrparenthesis_1^N$ *weighted* ϕ*-mean of a tuple* $\llparenthesis x_n \rrparenthesis_1^N$.

LEM

$$\begin{array}{llll}
\phi\psi^{-1} \text{ is CONVEX} & \text{and} \quad \phi \text{ is INCREASING} & \implies & \mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) \geq \mathsf{M}_\psi(\llparenthesis x_n \rrparenthesis)\\
\phi\psi^{-1} \text{ is CONVEX} & \text{and} \quad \phi \text{ is DECREASING} & \implies & \mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) \leq \mathsf{M}_\psi(\llparenthesis x_n \rrparenthesis)\\
\phi\psi^{-1} \text{ is CONCAVE} & \text{and} \quad \phi \text{ is INCREASING} & \implies & \mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) \leq \mathsf{M}_\psi(\llparenthesis x_n \rrparenthesis)\\
\phi\psi^{-1} \text{ is CONCAVE} & \text{and} \quad \phi \text{ is DECREASING} & \implies & \mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) \geq \mathsf{M}_\psi(\llparenthesis x_n \rrparenthesis)
\end{array}$$

✎PROOF:

1. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *increasing*:

$$\begin{aligned}
\mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) &\triangleq \phi^{-1}\left(\sum_{n=1}^N \lambda_n \phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi && \text{(Definition F.2 page 190)}\\
&= \phi^{-1}\left(\sum_{n=1}^N \lambda_n \phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\geq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^N \lambda_n \psi(x_n)\right) && \text{by \textit{Jensen's Inequality}}\\
&= \left(\psi^{-1}\sum_{n=1}^N \lambda_n \psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\triangleq \mathsf{M}_\psi(\llparenthesis x_n \rrparenthesis) && \text{by definition of } \mathsf{M}_\psi && \text{(Definition F.2 page 190)}
\end{aligned}$$

2. Case where $\phi\psi^{-1}$ is *convex* and ϕ is *decreasing*:

$$\begin{aligned}
\mathsf{M}_\phi(\llparenthesis x_n \rrparenthesis) &\triangleq \phi^{-1}\left(\sum_{n=1}^N \lambda_n \phi(x_n)\right) && \text{by definition of } \mathsf{M}_\phi && \text{(Definition F.2 page 190)}\\
&= \phi^{-1}\left(\sum_{n=1}^N \lambda_n \phi\psi^{-1}\psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\leq \phi^{-1}\left(\phi\psi^{-1}\sum_{n=1}^N \lambda_n \psi(x_n)\right) && \text{by \textit{Jensen's Inequality} and because } \phi^{-1} \text{ is \textit{decreasing}} && \text{(by hypothesis)}\\
&= \left(\psi^{-1}\sum_{n=1}^N \lambda_n \psi(x_n)\right) && \text{by definition of } \psi^{-1}\\
&\triangleq \mathsf{M}_\psi(\llparenthesis x_n \rrparenthesis) && \text{by definition of } \mathsf{M}_\psi && \text{(Definition F.2 page 190)}
\end{aligned}$$

✑

One of the most well known inequalities in mathematics is *Minkowski's Inequality* (1910, Theorem F.5 page 197). In 1946, H.P. Mulholland submitted a result[6] that generalizes Minkowski's Inequality to an equal weighted ϕ-mean. And Milovanović and Milovanovć (1979) generalized this even further to a *weighted* ϕ-mean (Theorem F.2, next).

Theorem F.2. [7]

THM

$$\left\{\begin{array}{lll|lll}
(1). & \phi \text{ is CONVEX} & \text{and} & (3). & \phi(0) = 0 & \text{and}\\
(2). & \phi \text{ is STRICTLY MONOTONIC} & \text{and} & (4). & \log\circ\phi\circ\exp \text{ is CONVEX} &
\end{array}\right\} \implies$$

$$\left\{\phi^{-1}\left(\sum_{n=1}^N \lambda_n \phi(x_n + y_n)\right) \leq \phi^{-1}\left(\sum_{n=1}^N \lambda_n \phi(x_n)\right) + \phi^{-1}\left(\sum_{n=1}^N \lambda_n \phi(y_n)\right)\right\}$$

[5] Pečarić et al. (1992) page 107, Bollobás (1999) page 5, Hardy et al. (1952) page 75

[6] Mulholland (1950)

[7] Milovanović and Milovanović (1979), Bullen (2003) page 306 ⟨Theorem 9⟩

F.2.2 Power means

Definition F.3. [8] *Let $\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle)$ be the $\langle \lambda_n \rangle_1^N$ weighted ϕ-mean of a* NON-NEGATIVE *tuple $\langle x_n \rangle_1^N$ (Definition F.2 page 190).*

DEF

A mean $\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle)$ is a ***power mean*** *with parameter r if $\phi(x) \triangleq x^r$. That is,*

$$\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle) = \left(\sum_{n=1}^{N} \lambda_n (x_n)^r \right)^{\frac{1}{r}}$$

Theorem F.3. [9] *Let $\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle)$ be* POWER MEAN *with parameter r of an N-tuple $\langle x_n \rangle_1^N$. Let $\mathbb{R}^*$ be the set of extended real numbers $(\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\})$.*[10]

THM

$$\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle) \triangleq \left(\sum_{n=1}^{N} \lambda_n (x_n)^r \right)^{\frac{1}{r}} \quad \text{is CONTINUOUS and STRICTLY INCREASING in } \mathbb{R}^*.$$

$$\mathsf{M}_{\phi(x;r)}(\langle x_n \rangle) = \begin{cases} \min\limits_{n=1,2,\ldots,N} \langle x_n \rangle & \text{for } r = -\infty \\ \prod\limits_{n=1}^{N} x_n^{\lambda_n} & \text{for } r = 0 \\ \max\limits_{n=1,2,\ldots,N} \langle x_n \rangle & \text{for } r = +\infty \end{cases}$$

✎PROOF:

1. Proof that $M_{\phi(x;r)}$ is *strictly increasing* in r:
   (a) Let r and s be such that $-\infty < r < s < \infty$.
   (b) Let $\phi_r \triangleq x^r$ and $\phi_s \triangleq x^s$. Then $\phi_r \phi_s^{-1} = x^{\frac{r}{s}}$.
   (c) The composite function $\phi_r \phi_s^{-1}$ is *convex* or *concave* depending on the values of r and s:

	$r < 0$ (ϕ_r decreasing)	$r > 0$ (ϕ_r increasing)
$s < 0$	*convex*	(not possible)
$s > 0$	*convex*	*concave*

   (d) Therefore by Lemma F.1 (page 191),
   $$-\infty < r < s < \infty \implies \mathsf{M}_{\phi(x;r)}(\langle x_n \rangle) < \mathsf{M}_{\phi(x;s)}(\langle x_n \rangle).$$

2. Proof that $\mathsf{M}_{\phi(x;r)}$ is continuous in r for $r \in \mathbb{R}\backslash 0$: The sum of continuous functions is continuous. For the cases of $r \in \{-\infty, 0, \infty\}$, see the items that follow.

3. Lemma: $\mathsf{M}_{\phi(x;-r)}(\langle x_n \rangle) = \left\{ \mathsf{M}_{\phi(x;r)}(\langle x_n^{-1} \rangle) \right\}^{-1}$. Proof:

$$\begin{aligned} \left\{ \mathsf{M}_{\phi(x;r)}(\langle x_n^{-1} \rangle) \right\}^{-1} &= \left\{ \left(\sum_{n=1}^{N} \lambda_n (x_n^{-1})^r \right)^{\frac{1}{r}} \right\}^{-1} && \text{by definition of } \mathsf{M}_\phi \\ &= \left(\sum_{n=1}^{N} \lambda_n (x_n)^{-r} \right)^{\frac{1}{-r}} \\ &= \mathsf{M}_{\phi(x;-r)}(\langle x_n \rangle) && \text{by definition of } \mathsf{M}_\phi \end{aligned}$$

[8] Bullen (2003) page 175, Bollobás (1999) page 6
[9] Bullen (2003) pages 175–177 ⟨see also page 203⟩, Bollobás (1999) pages 6–8, Besso (1879), Bienaymé (1840) page 68
[10] Rana (2002) pages 385–388 ⟨Appendix A⟩

4. Proof that $\lim_{r\to\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) = \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$:

   (a) Let $x_m \triangleq \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$

   (b) Note that $\lim_{r\to\infty} \mathsf{M}_\phi \leq \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$ because

$$\begin{aligned}
\lim_{r\to\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi \\
&\leq \lim_{r\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x) \triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing} \\
&= \lim_{r\to\infty} \left(x_m^r \underbrace{\sum_{n=1}^{N} \lambda_n}_{1}\right)^{\frac{1}{r}} && \text{because } x_m \text{ is a constant} \\
&= \lim_{r\to\infty} \left(x_m^r \cdot 1\right)^{\frac{1}{r}} \\
&= x_m \\
&= \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}$$

   (c) But also note that $\lim_{r\to\infty} \mathsf{M}_\phi \geq \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$ because

$$\begin{aligned}
\lim_{r\to\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_n^r\right)^{\frac{1}{r}} && \text{by definition of } \mathsf{M}_\phi \\
&\geq \lim_{r\to\infty} \left(w_m x_m^r\right)^{\frac{1}{r}} && \text{by definition of } x_m \text{ in item (4a) and because } \phi(x) \triangleq x^r \text{ and } \phi^{-1} \text{ are both increasing or both decreasing} \\
&= \lim_{r\to\infty} w_m^{\frac{1}{r}} x_m^{\frac{r}{r}} \\
&= x_m \\
&= \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis && \text{by definition of } x_m \text{ in item (4a)}
\end{aligned}$$

   (d) Combining items (b) and (c) we have $\lim_{r\to\infty} \mathsf{M}_\phi = \max_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$.

5. Proof that $\lim_{r\to-\infty} \mathsf{M}_\phi\left(\llparenthesis x_n \rrparenthesis\right) = \min_{n\in\mathbb{Z}} \llparenthesis x_n \rrparenthesis$:

$$\begin{aligned}
\lim_{r\to-\infty} \mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n \rrparenthesis\right) &= \lim_{r\to\infty} \mathsf{M}_{\phi(x;-r)}\left(\llparenthesis x_n \rrparenthesis\right) && \text{by change of variable } r \\
&= \lim_{r\to\infty} \left\{\mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)\right\}^{-1} && \text{by Lemma in item (3) page 192} \\
&= \lim_{r\to\infty} \frac{1}{\mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)} \\
&= \frac{\lim_{r\to\infty} 1}{\lim_{r\to\infty} \mathsf{M}_{\phi(x;r)}\left(\llparenthesis x_n^{-1} \rrparenthesis\right)} && \text{by property of lim [11]} \\
&= \frac{1}{\max_{n\in\mathbb{Z}} \llparenthesis x_n^{-1} \rrparenthesis} && \text{by item (4)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\left(\min_{n\in\mathbb{Z}} \lparen x_n \rparen\right)^{-1}} \\
&= \min_{n\in\mathbb{Z}} \lparen x_n \rparen
\end{aligned}$$

6. Proof that $\lim_{r\to 0} \mathsf{M}_\phi\left(\lparen x_n \rparen\right) = \prod_{n=1}^{N} x_n^{\lambda_n}$:

$$\begin{aligned}
\lim_{r\to 0} \mathsf{M}_\phi\left(\lparen x_n \rparen\right) &= \lim_{r\to 0} \exp\left\{\ln\left\{\mathsf{M}_\phi\left(\lparen x_n \rparen\right)\right\}\right\} \\
&= \lim_{r\to 0} \exp\left\{\ln\left\{\left(\sum_{n=1}^{N} \lambda_n \left(x_n^r\right)\right)^{\frac{1}{r}}\right\}\right\} && \text{by definition of } \mathsf{M}_\phi \\
&= \exp\left\{\frac{\frac{\partial}{\partial r} \ln\left(\sum_{n=1}^{N} \lambda_n \left(x_n^r\right)\right)}{\frac{\partial}{\partial r} r}\right\}_{r=0} && \text{by l'Hôpital's rule[12]} \\
&= \exp\left\{\frac{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r}\left(x_n^r\right)}{\sum_{n=1}^{N} \lambda_n \left(x_n^r\right)}\right\}_{r=0} && = \exp\left\{\frac{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r} \exp\left(\ln\left(x_n^r\right)\right)}{\sum_{n=1}^{N} \lambda_n}\right\}_{r=0} \\
&= \exp\left\{\frac{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r} \exp\left(r \ln\left(x_n\right)\right)}{1}\right\}_{r=0} && = \exp\left\{\sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial r} \exp\left(r \ln\left(x_n\right)\right)\right\}_{r=0} \\
&= \exp\left\{\sum_{n=1}^{N} \lambda_n \exp\left\{r \ln x_n\right\} \ln\left(x_n\right)\right\}_{r=0} && = \exp\left\{\sum_{n=1}^{N} \lambda_n \ln\left(x_n\right)\right\} \\
&= \exp\left\{\sum_{n=1}^{N} \ln\left(x_n^{\lambda_n}\right)\right\} && = \exp\left\{\ln \prod_{n=1}^{N} x_n^{\lambda_n}\right\} = \prod_{n=1}^{N} x_n^{\lambda_n}
\end{aligned}$$

✏

Definition F.4. *Let $\lparen x_n \rparen_1^N$ be a tuple. Let $\lparen \lambda_n \rparen_1^N$ be a tuple of weighting values.*

[11] Rudin (1976) page 85 ⟨4.4 Theorem⟩
[12] Rudin (1976) page 109 ⟨5.13 Theorem⟩

DEF

*The **harmonic mean** of* $(\!(x_n)\!)$ *is defined as* $\mu_h \triangleq \left(\sum_{n=1}^{N} \lambda_n \frac{1}{x_n}\right)^{-1}$ *where* $\sum_{n=1}^{N} \lambda_n = 1$

*The **geometric mean** of* $(\!(x_n)\!)$ *is defined as* $\mu_g \triangleq \sum_{n=1}^{N} x_n^{\lambda_n}$ *where* $\sum_{n=1}^{N} \lambda_n = 1$

*The **arithmetic mean** of* $(\!(x_n)\!)$ *is defined as* $\mu_a \triangleq \sum_{n=1}^{N} \lambda_n x_n$ *where* $\sum_{n=1}^{N} \lambda_n = 1$

*The **average** of* $(\!(x_n)\!)$ *is defined as* $\mu_a \triangleq \frac{1}{N} \sum_{n=1}^{N} x_n$

F.3 Inequalities on power means

Corollary F.1. [13] *Let* $(\!(x_n)\!)_1^N$ *be a tuple. Let* $(\!(\lambda_n)\!)_1^N$ *be a tuple of weighting values.*

COR

$$\min(\!(x_n)\!) \le \underbrace{\left(\sum_{n=1}^{N} \lambda_n \frac{1}{x_n}\right)^{-1}}_{\text{harmonic mean}} \le \underbrace{\prod_{n=1}^{N} x_n^{\lambda_n}}_{\text{geometric mean}} \le \underbrace{\sum_{n=1}^{N} \lambda_n x_n}_{\text{arithmetic mean}} \le \max(\!(x_n)\!) \qquad \textit{where} \quad \sum_{n=1}^{N} \lambda_n = 1$$

✎PROOF:

1. These five means are all special cases of the *power mean* $\mathsf{M}_{\phi(x:r)}$ (Definition F.3 page 192):
   - $r = \infty$: $\max(\!(x_n)\!)$
   - $r = 1$: arithmetic mean
   - $r = 0$: geometric mean
   - $r = -1$: harmonic mean
   - $r = -\infty$: $\min(\!(x_n)\!)$
2. The inequalities follow directly from Theorem F.3 (page 192).
3. Generalized AM-GM inequality: If one is only concerned with the arithmetic mean and geometric mean, their relationship can be established directly using *Jensen's Inequality*:

$$\begin{aligned}
\sum_{n=1}^{N} \lambda_n x_n &= b^{\log_b\left(\sum_{n=1}^{N} \lambda_n x_n\right)} \ge b^{\left(\sum_{n=1}^{N} \lambda_n \log_b x_n\right)} && \text{by } \textit{Jensen's Inequality}\\
&= \prod_{n=1}^{N} b^{(\lambda_n \log_b x_n)} = \prod_{n=1}^{N} b^{(\log_b x_n)\lambda_n} = \prod_{n=1}^{N} x_n^{\lambda_n}
\end{aligned}$$

✏

Lemma F.2 (Young's Inequality). [14]

[13] Bullen (2003) page 71, Bollobás (1999) page 5, Cauchy (1821), pages 457–459 ⟨Note II, theorem 17⟩, Jensen (1906) page 183

[14] Carothers (2000), page 43, Tolsted (1964), page 5, Maligranda (1995), page 257, Hardy et al. (1952) ⟨Theorem 24⟩, Young (1912) page 226

LEM

$$xy < \frac{x^p}{p} + \frac{y^q}{q} \quad \textit{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \textit{but } y \neq x^{p-1}$$

$$xy = \frac{x^p}{p} + \frac{y^q}{q} \quad \textit{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \textit{and } y = x^{p-1}$$

✎PROOF:

1. Proof that $\frac{1}{p-1} = q - 1$:

$$\begin{aligned}
\frac{1}{p} + \frac{1}{q} = 1 &\iff \frac{q}{q} + \frac{q}{p} = q \\
&\iff q\left(1 - \frac{1}{p}\right) = 1 \\
&\iff q = \frac{1}{1 - \frac{1}{p}} \\
&\iff q = \frac{p}{p-1} \\
&\iff q - 1 = \frac{p}{p-1} - \frac{p-1}{p-1} \\
&\iff q - 1 = \frac{p - (p-1)}{p-1} \\
&\iff q - 1 = \frac{1}{p-1}
\end{aligned}$$

2. Proof that $v = u^{p-1} \iff u = v^{q-1}$:

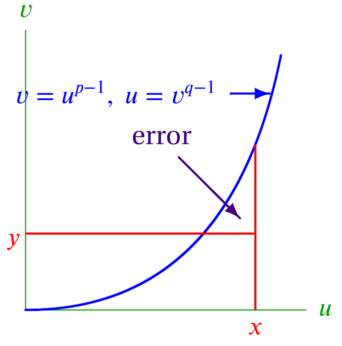
$$\begin{aligned}
u &= v^{\frac{1}{p-1}} && \text{by left hypothesis} \\
&= v^{q-1} && \text{by item (1)}
\end{aligned}$$

3. Proof that $v = u^{p-1}$ is propemonotonically increasing in u and $u = v^{q-1}$ is propemonotonically increasing in v:

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u} u^{p-1} && = (p-1)u^{p-2} && > 0 \\
\frac{\mathrm{d}u}{\mathrm{d}v} &= \frac{\mathrm{d}}{\mathrm{d}v} v^{q-1} && = (q-1)v^{q-2} && > 0
\end{aligned}$$

4. Proof that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$:

$$\begin{aligned}
xy &\leq \int_0^x u^{p-1}\ \mathrm{d}u + \int_0^y v^{q-1}\ \mathrm{d}v \\
&= \left.\frac{u^p}{p}\right|_0^x + \left.\frac{v^q}{q}\right|_0^y \\
&= \frac{x^p}{p} + \frac{y^q}{q}
\end{aligned}$$



Theorem F.4 (Hölder's Inequality). [15] *Let* $(\!(x_n \in \mathbb{C})\!)_1^N$ *and* $(\!(y_n \in \mathbb{C})\!)_1^N$ *be complex* N*-tuples.*

THM

$$\underbrace{\sum_{n=1}^{N} |x_n y_n|}_{\|\boldsymbol{x}\cdot\boldsymbol{y}\|_1} \le \underbrace{\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}}_{\|\boldsymbol{x}\|_p} \underbrace{\left(\sum_{n=1}^{N} |y_n|^q\right)^{\frac{1}{q}}}_{\|\boldsymbol{y}\|_q} \qquad \text{with} \qquad \frac{1}{p}+\frac{1}{q}=1 \qquad \forall\, 1<p<\infty$$

✎PROOF: Let $\|x_n\|_p \triangleq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$.

$$\begin{aligned}
\sum_{n=1}^{N} |x_n y_n| &= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \sum_{n=1}^{N} \frac{|x_n y_n|}{\|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q} \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \sum_{n=1}^{N} \frac{|x_n|}{\|(\!(x_n)\!)\|_p} \frac{|y_n|}{\|(\!(y_n)\!)\|_q} \\
&\le \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \sum_{n=1}^{N} \left(\frac{1}{p}\frac{|x_n|^p}{\|(\!(x_n)\!)\|_p^p} + \frac{1}{q}\frac{|y_n|^q}{\|(\!(y_n)\!)\|_q^q}\right) && \text{by } \textit{Young's Inequality} \quad \text{(Lemma F.2 page 195)} \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \left(\frac{1}{p}\cdot\frac{\sum |x_n|^p}{\|(\!(x_n)\!)\|_p^p} + \frac{1}{q}\cdot\frac{\sum |y_n|^q}{\|(\!(y_n)\!)\|_q^q}\right) \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \left(\frac{1}{p}\frac{\|(\!(x_n)\!)\|_p^p}{\|(\!(x_n)\!)\|_p^p} + \frac{1}{q}\frac{\|(\!(y_n)\!)\|_q^q}{\|(\!(y_n)\!)\|_q^q}\right) && \text{by definition of } \|\cdot\| \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q \underbrace{\left(\frac{1}{p}+\frac{1}{q}\right)}_{1} \\
&= \|(\!(x_n)\!)\|_p \|(\!(y_n)\!)\|_q && \text{by } \frac{1}{p}+\frac{1}{q}=1 \text{ constraint}
\end{aligned}$$

✏

Theorem F.5 (Minkowski's Inequality for sequences). [16] *Let* $(\!(x_n \in \mathbb{C})\!)_1^N$ *and* $(\!(y_n \in \mathbb{C})\!)_1^N$ *be complex* N*-tuples.*

THM

$$\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{\frac{1}{p}} \qquad \forall\, 1<p<\infty$$

✎PROOF:

1. Define $q \triangleq \dfrac{p}{p-1}$
2. Define $\|\boldsymbol{x}\|_p \triangleq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$.

[15] Bullen (2003) page 178 ⟨2.1⟩, Carothers (2000), page 44, Tolsted (1964), page 6, Maligranda (1995), page 257, Hardy et al. (1952) ⟨Theorem 11⟩, Hölder (1889)

[16] Bullen (2003) page 179, Carothers (2000), page 44, Tolsted (1964), page 7, Maligranda (1995), page 258, Hardy et al. (1952) ⟨Theorem 24⟩, Minkowski (1910) page 115

3. Proof that $\|x_n + y_n\|_p \le \|x_n\|_p + \|y_n\|_p$:

$$\begin{aligned}
&\boxed{\|x_n + y_n\|_p^p} \\
&= \sum_{n=1}^{N} |x_n + y_n|^p && \text{by definition of } \|\cdot\|_p && \text{(definition 2 page 197)} \\
&= \sum_{n=1}^{N} |x_n + y_n||x_n + y_n|^{p-1} && \text{by } \textit{homogeneous} \text{ property of } |\cdot| \\
&\boxed{\le} \sum_{n=1}^{N} |x_n||x_n + y_n|^{p-1} + \sum_{n=1}^{N} |y_n||x_n + y_n|^{p-1} && \text{by } \textit{subadditive} \text{ property of } |\cdot| \\
&= \sum_{n=1}^{N} \left|x_n(x_n + y_n)^{p-1}\right| + \sum_{n=1}^{N} \left|y_n(x_n + y_n)^{p-1}\right| && \text{by } \textit{homogeneous} \text{ property of } |\cdot| \\
&\boxed{\le} \|x_n\|_p \left\|(x_n + y_n)^{p-1}\right\|_q + \|y_n\|_p \left\|(x_n + y_n)^{p-1}\right\|_q && \text{by } \textit{Hölder's Inequality} && \text{(Theorem F.4 page 197)} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left\|(x_n + y_n)^{p-1}\right\|_q \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left(\sum_{n=1}^{N} \left|(x_n + y_n)^{p-1}\right|^q\right)^{\frac{1}{q}} && \text{by definition of } \|\cdot\|_p && \text{(definition 2 page 197)} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left(\sum_{n=1}^{N} \left|(x_n + y_n)^{p-1}\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} && \text{by definition 1} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \left(\sum_{n=1}^{N} \left|(x_n + y_n)\right|^p\right)^{\frac{p-1}{p}} \\
&= \left(\|x_n\|_p + \|y_n\|_p\right) \|x_n + y_n\|^{p-1} && \text{by definition of } \|\cdot\|_p && \text{(definition 2 page 197)} \\
&\implies \boxed{\|x_n + y_n\|_p \le \|x_n\|_p + \|y_n\|_p}
\end{aligned}$$

✏

❝*Cauchy is the only one occupied with pure mathematics: Poisson, Fourier, Ampere, etc., busy themselves exclusively with magnetism and other physical subjects. Mr. Laplace writes nothing now, I believe.*❞

Niels Henrik Abel in an 1826 letter [17]

Theorem F.6 (Cauchy-Schwarz Inequality for sequences). [18] *Let* $(x_n \in \mathbb{C})_1^N$ *and* $(y_n \in \mathbb{C})_1^N$ *be complex N-tuples.*

[17] quote: Bell (1986) page 318 ⟨Chapter 17. "Genius and Poverty" "Abel (1802–1829)"⟩, Boyer and Merzbach (2011) page 462 ⟨without "Mr. Laplace..." portion⟩. image: http://en.wikipedia.org/wiki/File:Augustin-Louis_Cauchy_1901.jpg, public domain

[18] Aliprantis and Burkinshaw (1998), page 278, Scharz (1885), Bouniakowsky (1859), Hardy et al. (1952) page 25 ⟨Theorem 11⟩, Cauchy (1821), page 455 ⟨???⟩

THM

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 \le \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \qquad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$$

$$\left|\sum_{n=1}^{N} x_n y_n^*\right|^2 = \left(\sum_{n=1}^{N} |x_n|^2\right)\left(\sum_{n=1}^{N} |y_n|^2\right) \iff \exists a \in \mathbb{C} \text{ such that } \boldsymbol{y} = a\boldsymbol{x} \qquad \forall \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$$

✎PROOF:

1. The *Cauchy-Schwarz Inequality for sequences* is a special case of the *Hölder inequality* (Theorem F.4 page 197) for $p = q = 2$.

2. Alternatively, the *Cauchy-Schwarz inequality for sequences* is a special case of the *Cauchy-Schwarz inequality for inner-product spaces*:

   (a) $\langle x_n \mid y_n \rangle \triangleq \sum_{n=1}^{N} x_n y_n$ is an inner-product and $\left(\llparenthesis x_n \rrparenthesis, \langle \triangle \mid \triangledown \rangle\right)$ is an inner-product space.

   (b) By the more general *Cauchy-Schwarz Inequality for inner-product spaces*,

$$\begin{aligned}
\left(\sum_{n=1}^{N} a_n \lambda_n\right)^2 &\triangleq \langle a_n \mid \lambda_n \rangle^2 && \text{by definition of } \langle x_n \mid y_n \rangle \\
&\boxed{\le} \|x_n\|^2 \|y_n\|^2 && \text{by } \textit{Cauchy-Schwarz Inequality for inner-product spaces} \text{ (Theorem ?? page ??)} \\
&\triangleq \left(\sum_{n=1}^{N} x_n^2\right)\left(\sum_{n=1}^{N} y_n^2\right) && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Not only does the *Hölder inequality* imply the *Cauchy-Schwarz inequality*, but somewhat surprisingly, the converse is also true: The Cauchy-Schwarz inequality implies the Hölder inequality.[19]

✎

Proposition F.1. [20]

PRP

$$(x+y)^p \le 2^p(x^p + y^p) \qquad \forall x, y \ge 0,\ 1 < p < \infty$$

✎PROOF:

$$\begin{aligned}
(x+y)^p &\le (2\max\{x,y\})^p \\
&= 2^p(\max\{x,y\})^p \\
&= 2^p(\max\{x^p,y^p\}) \\
&\le 2^p(x^p + y^p)
\end{aligned}$$

✎

[19] Bullen (2003) pages 183–185 ⟨Theorem 5⟩
[20] Carothers (2000), page 43

F.4 Power Sums

Theorem F.7 (Geometric Series). [21]

THM
$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \qquad \forall r \in \mathbb{C}\backslash 0$$

✎PROOF:

$$\boxed{\sum_{k=0}^{n-1} r^k} = \left(\frac{1}{1-r}\right)\left[(1-r)\sum_{k=0}^{n-1} r^k\right] = \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - r\sum_{k=0}^{n-1} r^k\right] = \left(\frac{1}{1-r}\right)\left[\sum_{k=0}^{n-1} r^k - \left(\sum_{k=0}^{n-1} r^k - 1 + r^n\right)\right]$$
$$= \left(\frac{1}{1-r}\right)\left[1-r^n\right] = \boxed{\frac{1-r^n}{1-r}}$$

⇨

Lemma F.3. *Let* $\mathsf{f}(x)$ *be a function.*

LEM
$$\mathsf{S}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) = \mathsf{S}(x+\tau) \qquad (\mathsf{S}(x) \textit{ is PERIODIC with period } \tau)$$

✎PROOF:

$$\mathsf{S}(x+\tau) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+\tau+n\tau) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+(n+1)\tau) = \sum_{m\in\mathbb{Z}} \mathsf{f}(x+m\tau) \qquad (\text{where } m \triangleq n+1)$$
$$\triangleq \mathsf{S}(x)$$

⇨

Proposition F.2 (Power Sums). [22]

PRP
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \forall n\in\mathbb{N} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \quad \forall n\in\mathbb{N}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n\in\mathbb{N} \qquad \sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \quad \forall n\in\mathbb{N}$$

✎PROOF:

1. Proof that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$: (proof by induction)

$$\sum_{k=1}^{n=1} k = 1 = \frac{1(1+1)}{2} = \left.\frac{n(n+1)}{2}\right|_{n=1}$$
$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + \left(n+1\right) = \underbrace{\left(\frac{n(n+1)}{2}\right) + \left(n+1\right)}_{\text{by left hypothesis}} = \left(n+1\right)\left(\frac{n}{2}+1\right)$$
$$= \left(n+1\right)\left(\frac{n+2}{2}\right) = \frac{(n+1)(n+2)}{2}$$

[21] Hall and Knight (1894), page 39 ⟨article 55⟩
[22] Amann and Escher (2008) pages 51–57, Menini and Oystaeyen (2004) page 91 ⟨Exercises 5.36–5.39⟩

2. Proof that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$: (proof by induction)

$$\sum_{k=1}^{n=1} k^2 = 1 = \frac{1(1+1)(2+1)}{6} = \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1}$$

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + \left(n+1\right)^2 = \underbrace{\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(n+1\right)^2}_{\text{by left hypothesis}} = \left(n+1\right)\left(\frac{n(2n+1)+6(n+1)}{6}\right)$$

$$= \left(n+1\right)\left(\frac{2n^2+7n+6}{6}\right) = \left(n+1\right)\left(\frac{(n+2)(2n+3)}{6}\right) = \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}$$

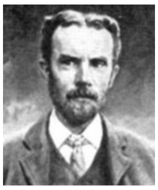
✐

APPENDIX G

INFINITE SUMS

❝*The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes…*❞

Niels Henrik Abel, in a January 16, 1826 letter to Holmboé [1]



❝*The series is divergent; therefore we may be able to do something with it.*❞
Oliver Heaviside (1850–1925) [2]

❝*Some modern appraisals of the cavalier style of 18th-century mathematicians in handling infinite series conveys the impression that these poor men set their brains aside when confronted by them.*❞

Ivor Grattan-Guinness (1990)[3]

G.1 Convergence

An infinite summation $\sum_{n=1}^{\infty} x_n$ is meaningless outside some topological space (e.g. metric space, normed space, etc.). The sum $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ is an abbreviation for $\lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{x}_n$ (next definition); and the concept of *limit* is also itself meaningless outside of a *topological space*.

1 quote: Kline (1972) page 973 ⟨Chapter 47⟩
image: http://en.wikipedia.org/wiki/File:Niels_Henrik_Abel.jpg, public domain

2 quote: Kline (1972) page 1096 ⟨Chapter 47⟩
image: http://en.wikipedia.org/wiki/File:Oliver_Heaviside2.jpg, public domain

3 Grattan-Guinness (1990) page 163

Definition G.1. [4] *Let (X, T) be a topological space and* lim *be the limit generated by the topology T.*

DEF
$$\sum_{n=1}^{\infty} x_n \triangleq \sum_{n\in\mathbb{N}} x_n \triangleq \lim_{N\to\infty} \sum_{n=1}^{N} x_n$$
$$\sum_{n=-\infty}^{\infty} x_n \triangleq \sum_{n\in\mathbb{Z}} x_n \triangleq \lim_{N\to\infty} \left(\sum_{n=0}^{N} x_n\right) + \left(\lim_{N\to-\infty} \sum_{n=-1}^{N} x_n\right)$$

In general, the order of summation of an infinite series *does* matter.

Definition G.2. [5] *Let P be the set of all* PERMUTATIONS *in* $\mathbb{N}^{\mathbb{N}}$.

DEF
A series $\sum_{n=1}^{\infty} x_n$ *is* ***absolutely convergent*** *if* $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} x_{\mathsf{p}(n)} \quad \forall \mathsf{p} \in P$

A series is ***conditionally convergent*** *if it is* CONVERGENT *but not* ABSOLUTELY CONVERGENT.

Theorem G.1 (Riemann Series Theorem). [6] *Let $\mathsf{p}(n)$ be a permutation on $\mathbb{N}$. Let $(\!(a_n)\!)_{n\in\mathbb{Z}}$ be a sequence of real numbers.*

THM
$$\left\{\begin{array}{l}\sum_{n=1}^{\infty} a_n \text{ is} \\ \text{CONDITIONALLY CONVERGENT}\end{array}\right\} \implies \left\{\begin{array}{l}\text{For every } x \in \mathbb{R} \text{ there exists } \mathsf{p} \\ \text{such that } \sum_{n=1}^{\infty} a_{\mathsf{p}(n)} = x \\ \text{or such that } \sum_{n=1}^{\infty} a_{\mathsf{p}(n)} \text{ is DIVERGENT}\end{array}\right\}$$

Theorem G.2. [7]

THM
$$\sum_{n=1}^{\infty} |x_n| < \infty \implies \left\{ \sum_{n=1}^{\infty} x_n \quad \text{is ABSOLUTELY CONVERGENT.} \right\}$$

Example G.1 (Logarithmic Series). [8] Consider the sum $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n}$. To which value this sum converges, or whether it even converges at all, depends on the order in which the terms are summed. This is demonstrated by the following series:

[4] Klauder (2010) page 4, Kubrusly (2001) page 43, Bachman and Narici (1966) pages 3–4

[5] Kadets and Kadets (1997) page 5 ⟨Theorem 1.1.1 (Riemann's theorem)⟩, Bromwich (1908), page 64 ⟨IV. Absolute Convergence.⟩, Szász and Barlaz (1952), page 2

[6] Kadets and Kadets (1997) page 5 ⟨Theorem 1.1.1 (Riemann's theorem)⟩, Bromwich (1908), page 68 ⟨Article 28. Riemann's Theorem⟩

[7] Kadets and Kadets (1997) page 5 ⟨Theorem 1.1.1 (Riemann's theorem)⟩

[8] Bromwich (1908), pages 51–52 ⟨Article 21 Example 1⟩, Hall and Knight (1894), page 191 ⟨article 223⟩, Jolley (1961) pages 14–15 ⟨item (71)⟩, Sloane (2014) ⟨http://oeis.org/A002939⟩⟨$2n(2n-1)$⟩, Graham et al. (1994) page 99 ⟨n.w. diagonal of *spiral function*⟩, Many many thanks to Po-Ning Chen (Chinese: ???, pinyin: Chén Bó Níng) for his consultation regarding this series.

EX

If the series is added in the given order, the result is $\ln 2$:

$$\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} \triangleq \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{(n-1)}\frac{1}{n} = \ln 2$$

But if the order is changed, the sum can be any real value:

Let x be any real value (even an irrational one such as π or $\sqrt{2}$).

$$\text{Let} \qquad \mathsf{p}(N) \triangleq \begin{cases} \text{next unused odd value} & \text{if } \sum_{n=1}^{N-1}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} \le x \\ \text{next unused even value} & \text{if } \sum_{n=1}^{N-1}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} > x \end{cases}$$

$$\sum_{n=1}^{\infty}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} \triangleq \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} = x$$

The series can even be summed in such a way that it does not converge at all:

Let $\mathsf{q}(n)$ be a permutation that partitions the natural numbers into odd and even values such that $\left(\!\left(x_{\mathsf{q}(n)}\right)\!\right) = (\!(1, 3, 5, \ldots, 2, 4, 6, \ldots)\!)$.

$$\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} = \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n-1}}_{\infty} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n}}_{\infty} \implies \text{diverges}$$

✎PROOF:

1. Proof that $\displaystyle\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n} = \ln 2$ using polynomial expansion:[9]

   (a) Lemma: Proof that $\frac{1}{1+x} = \sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}$:

$$\begin{aligned}(1+x)\left(\sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}\right) &= \sum_{k=0}^{2n-1}(-1)^k x^k + \sum_{k=0}^{2n-1}(-1)^k x^{k+1} + x^{2n} \\ &= 1 + \sum_{k=1}^{2n-1}(-1)^k x^k - \sum_{k=1}^{2n-1}(-1)^k x^k + x^{2n} \\ &= 1 + x^{2n}\end{aligned}$$

   (b) Lemma: Proof that $\lim_{n\to\infty}\int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x = 0$:

$$\begin{aligned}0 &< \lim_{n\to\infty}\int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x \\ &< \lim_{n\to\infty}\int_0^1 x^{2n}\,\mathrm{d}x \\ &= \lim_{n\to\infty}\left.\frac{x^{2n+1}}{2n+1}\right|_0^1 \\ &= \lim_{n\to\infty}\frac{1}{2n+1} \\ &= 0\end{aligned}$$

[9] Bromwich (1908), pages 51–52 ⟨Article 21 Example 1⟩

(c) Proof that sum = ln 2:

$$\begin{aligned}
\ln 2 &= \ln 2 - \ln 1 \\
&= \int_1^2 \frac{1}{x}\,\mathrm{d}x \\
&= \int_0^1 \frac{1}{x+1}\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_0^1 \left\{ \sum_{k=0}^{2n-1} (-1)^k x^k + \frac{x^{2n}}{1+x} \right\} \mathrm{d}x && \text{by item (1a)} \\
&= \sum_{k=0}^{2n-1} (-1)^k \int_0^1 x^k\,\mathrm{d}x + \lim_{n\to\infty} \int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{k=0}^{2n-1} (-1)^k \frac{1}{k} + 0 && \text{by item (1b)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^{k-1} \frac{1}{k} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}
\end{aligned}$$

2. Proof that $\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{1}{n} = \ln 2$ using Taylor expansion:[10]

(a) Lemma: Proof that $\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n} x^n$:

$$\begin{aligned}
\ln(x+1) &= \sum_{n=0}^{\infty} \frac{\left[\mathbf{D}^n \ln(x+1)\right](0)}{n!} x^n && \text{by } \textit{Taylor series expansion} \text{ (Theorem C.13 page 153)} \\
&= \frac{\ln(1+0)}{0!}x^0 + \frac{\frac{1}{1+0}}{1!}x^1 - \frac{\frac{1}{(1+0)^2}}{2!}x^2 + \frac{\frac{2}{(1+0)^3}}{3!}x^3 - \frac{\frac{6}{(1+0)^4}}{4!}x^4 + \frac{\frac{24}{(1+0)^5}}{5!}x^5 - \frac{\frac{120}{(1+0)^6}}{6!}x^6 + \cdots \\
&= 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n} x^n
\end{aligned}$$

(b) Proof that $\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{1}{n} = \ln 2$:

$$\begin{aligned}
&= \underbrace{\left(1 - \frac{1}{2}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{\frac{1}{12}} + \underbrace{\left(\frac{1}{5} - \frac{1}{6}\right)}_{\frac{1}{30}} + \underbrace{\left(\frac{1}{7} - \frac{1}{8}\right)}_{\frac{1}{56}} + \dots \\
&= \left.\sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n} x^n \right|_{x=1} \\
&= \ln 2 && \text{by Lemma in item 1} \\
&\approx 0.693147
\end{aligned}$$

[10] Hall and Knight (1894), page 191 ⟨article 223⟩

3. Proof that $\sum_{n=1}^{\infty}(-1)^{(\mathsf{p}(n)-1)}\frac{1}{\mathsf{p}(n)} = x$: If the partial sum is less than x, positive values are added. If the partial sum is greater than x, negative values are added. The limit is x.

4. Proof that $\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} = \infty - \infty$:

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{(\mathsf{q}(n)-1)}\frac{1}{\mathsf{q}(n)} &= \underbrace{\sum_{n=1}^{\infty}(-1)^{(2n-1-1)}\frac{1}{2n-1}}_{\text{odd indices}} + \underbrace{\sum_{n=1}^{\infty}(-1)^{(2n-1)}\frac{1}{2n}}_{\text{even indices}} \\
&= \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n-1}}_{\infty} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{2n}}_{\infty} \\
&\to \text{(diverges)}
\end{aligned}$$

✏

Divergent series could even result in decisions that may be considered extremely irrational, as demonstrated by *St. Petersberg Paradox* (next).

Example G.2 (St. Petersburg Paradox). [11] There is a lottery with a prize pot of \$1. A coin is tossed. If the coin is a tail, the money in the lottery is doubled (\$2, \$4, \$8, \$16, ...). If the coin is a head, you win the money and the game is finished.

How much money would you be willing to play this game? The answer to this question for some people may depend on the expected value of how much money would be won. But the expected value of the amount of money you would win is

$$\tfrac{1}{2}\times\$1 + \tfrac{1}{4}\times\$2 + \tfrac{1}{8}\times\$4 + \tfrac{1}{16}\times\$8 + \ldots = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots = \infty$$

Since the expected value of the win is infinity, you should be willing to pay any finite amount of money to play this game (even trillions of dollars). But yet common sense would tell most people that this would be an unwise investment.

G.2 Multiplication

Theorem G.3. [12] *Let* $\left(x_n\right)_1^N$ *and* $\left(y_n\right)_1^N$ *be sequences over a ring* $(\mathbb{X}, +, \times)$.

THM

$$\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) = \sum_{n=0}^{p+q}\underbrace{\left(\sum_{k=\max(0,n-q)}^{\min(n,p)} x_k y_{n-k}\right)}_{\textit{Cauchy product}}$$

✎PROOF:

[11] Székely (1986) pages 27–28, Bernoulli (1783) pages 31–32 ⟨§17⟩, de Montmort (1713) page 402 ⟨1713 letter from Nicolas Bernoulli⟩

[12] Apostol (1975) page 204 ⟨Definition 8.45⟩

1.

$$\begin{aligned}
\left(\sum_{n=0}^{p} x_n\right)\left(\sum_{m=0}^{q} y_m\right) &= \sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m} \\
&= \sum_{n=0}^{p}\sum_{k=n}^{q+n} x_n y_{k-n} && k = n+m \qquad m = k-n \\
&\vdots \\
&= \sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)
\end{aligned}$$

2. Perhaps the easiest way to see the relationship is by illustration with a matrix of product terms:

	y_0	y_1	y_2	y_3	$\cdots$	y_q
x_0	x_0y_0	x_0y_1	x_0y_2	x_0y_3	$\cdots$	x_0y_q
x_1	x_1y_0	x_1y_1	x_1y_2	x_1y_3	$\cdots$	x_1y_q
x_2	x_2y_0	x_2y_1	x_2y_2	x_2y_3	$\cdots$	x_2y_q
x_3	x_3y_0	x_3y_1	x_3y_2	x_3y_3	$\cdots$	x_3y_q
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_p	x_py_0	x_py_1	x_py_2	x_py_3	$\cdots$	x_py_q

(a) The expression $\sum_{n=0}^{p}\sum_{m=0}^{q} x_n y_m z^{n+m}$ is equivalent to adding *horizontally* from left to right, from the first row to the last.

(b) If we switched the order of summation to $\sum_{m=0}^{q}\sum_{n=0}^{p} x_n y_m z^{n+m}$, then it would be equivalent to adding *vertically* from top to bottom, from the first column to the last.

(c) However the final result expression $\sum_{n=0}^{p+q}\left(\sum_{k=0}^{n} x_k y_{n-k}\right)$ is equivalent to adding *diagonally* starting from the upper left corner and proceding to the lower right.

(d) Upper limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:

$$\left.\begin{array}{rcl} k & \le & n \\ k & \le & p \end{array}\right\} \implies k \le \min(n,p)$$

(e) Lower limit on inner summation: Looking at the x_k terms, we see that there are two constraints on k:

$$\left.\begin{array}{rcl} k & \ge & 0 \\ k & \ge & n-q \end{array}\right\} \implies k \ge \max(0, n-q)$$

✏

Corollary G.1. *Let* $(\!|x_n \in \mathbb{C}|\!)$ *and* $(\!|y_n \in \mathbb{C}|\!)$.

COR

$$\left(\sum_{n=0}^{\infty} x_n\right)\left(\sum_{m=0}^{\infty} y_m\right) = \sum_{n=0}^{\infty}\underbrace{\left(\sum_{k=0}^{n} x_k y_{n-k}\right)}_{\text{Cauchy product}}$$

✎PROOF:

$$\begin{aligned}
\left(\sum_{n=0}^{p=\infty} x_n\right)\left(\sum_{m=0}^{q=\infty} y_m\right) &= \sum_{n=0}^{\infty}\left(\sum_{k=\max(0,n-\infty)}^{\min(n,\infty)} x_k y_{n-k}\right) && \text{by Theorem G.3 page 207} \\
&= \sum_{n=0}^{\infty}\left((\sum_{k=0}^{n} x_k y_{n-k}\right)
\end{aligned}$$

✏

Theorem G.4. [13] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM
$$\left\{\begin{array}{ll} X \text{ is ABSOLUTELY CONVERGENT} & \text{and} \\ Y \text{ is ABSOLUTELY CONVERGENT} & \end{array}\right\} \implies \left\{\begin{array}{ll} Z \text{ is ABSOLUTELY CONVERGENT} & \text{and} \\ Z = XY. & \end{array}\right\}$$

Theorem G.5. [14] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM
$$\left\{\begin{array}{lll} 1. & X \text{ is ABSOLUTELY CONVERGENT} & \text{and} \\ 2. & Y \text{ is CONVERGENT} & \end{array}\right\} \implies \left\{\begin{array}{lll} 1. & Z \text{ is CONVERGENT} & \text{and} \\ 2. & Z = XY & \end{array}\right\}$$

Theorem G.6. [15] *Let* $X \triangleq \sum_{n=0}^{\infty} x_n$, $Y \triangleq \sum_{n=0}^{\infty} y_n$, *and* $Z \triangleq \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}\right)$.

THM
$$\left\{\begin{array}{lll} 1. & X \text{ is CONVERGENT} & \text{and} \\ 2. & Y \text{ is CONVERGENT} & \text{and} \\ 3. & Z \text{ is CONVERGENT} & \end{array}\right\} \implies \{Z = XY\}$$

G.3 Summability

Cauchy and Abel, the 19th century champions of rigour in analysis, firmly rejected any and all divergent sums. However in more recent times, certain classes of divergent sums have been found to be extremely useful. Often such sums are ones that are said to be *summable*.

Definition G.3. [16]

DEF

The series $\sum_{n=0}^{\infty} x_n$ *is* ***summable by the*** k***-th arithmetic mean of Cesàro to limit*** x, *or* ***summable*** (C,k) ***to the limit*** x, *if*

$$\lim_{n\to\infty} \frac{S_n^k}{A_n^k} = x \quad \text{for } n \in \mathbb{W} \text{ and where}$$

$$S_n^k \triangleq \begin{cases} \sum_{m=0}^{n} x_m & \text{for } k=0 \\ \sum_{m=0}^{n} S_m^{k-1} & \text{for } k=1,2,3,\ldots \end{cases} \quad \text{and} \quad A_n^k \triangleq \begin{cases} 1 & \text{for } k=0 \\ \sum_{m=0}^{n} A_m^{k-1} & \text{for } k=1,2,3,\ldots \end{cases}$$

Proposition G.1. [17]

[13] Hardy (1949), pages 227–228 ⟨Theorem 160⟩, Bromwich (1908), page 66 ⟨Article 27.⟩, Cauchy (1821) pages 147–148 ⟨6.ᵉ Théorème⟩

[14] Hardy (1949), page 228 ⟨Theorem 161⟩, Bromwich (1908), pages 85–86 ⟨Article 35.⟩, Mertens (1875)

[15] Hardy (1949), page 228 ⟨Theorem 162⟩, Abel (1826)

[16] Zygmund (2002a) pages 75–76, Hardy (1949), page 96 ⟨5.4 Cesàro means⟩, Whittaker and Watson (1920), pages 155–156 ⟨8.43, 8.431⟩, Cesàro (1890)

[17] Zygmund (2002a) pages 75–76, Thomson et al. (2008) page 129 ⟨Definition 3.54⟩, Szász and Barlaz (1952), page 13

P R P

$\sum_{n=0}^{\infty} x_n$ *is summable* $(C,0)$ *to the limit* x *if* $\lim_{N\to\infty} \sum_{n=0}^{N} x_n = x$ *(normal convergence)*

$\sum_{n=0}^{\infty} x_n$ *is summable* $(C,1)$ *to the limit* x *if* $\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} s_n = x$ *(arthimetic mean)*

where $s_n \triangleq \sum_{m=0}^{n} x_m$

Definition G.4. [18]

D E F

The series $\sum_{n=0}^{\infty} a_n$ *is* ***summable by Euler's method to limit*** a *if*

$$\lim_{x\to 1-0} \sum_{n=0}^{\infty} a_n x^n = a$$

Example G.3. [19]

E X

The series $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ is *divergent*
(it is *not* summable $(C,0)$),
but yet it *is* summable $(C,1)$ to the limit $\frac{1}{2}$.
It is also summable by Euler's method to the limit $\frac{1}{2}$.

✎PROOF:

1. Proof for Cesàro summability:

   (a) Note that the sequence of partial sums s_n is $s_0 = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0$, $s_4 = 1$, ... That is $s_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

   (b) Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} s_k &= \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=0}^{2n} s_k \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \left(\underbrace{\sum_{k=0}^{n} s_{2k}}_{\text{even terms}} + \underbrace{\sum_{k=0}^{n-1} s_{2k+1}}_{\text{odd terms}} \right) \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \left(\sum_{k=0}^{n} 1 + \sum_{k=0}^{n-1} 0 \right) \\
&= \lim_{n\to\infty} \frac{n+1}{2n+1} \\
&= \frac{1}{2}
\end{aligned}$$

[18] Whittaker and Watson (1920), page 155 ⟨8.42⟩

[19] Thomson et al. (2008) page 130 ⟨Example 3.56⟩, Whittaker and Watson (1920), page 155 ⟨8.42⟩

2. Proof for Euler summability:

$$\begin{aligned}
\lim_{x\to 1-0}\sum_{n=0}^{\infty}(-1)^n &= \lim_{x\to 1-0}\lim_{n\to\infty}\left(\frac{1}{1+x} = \sum_{k=0}^{2n-1}(-1)^k x^k + \frac{x^{2n}}{1+x}\right) \\
&= \lim_{x\to 1-0}\frac{1}{1+x} && \text{by item (1a) (page 205) of Example G.1} \\
&= \frac{1}{2}
\end{aligned}$$

✏

G.4 Convergence in Banach spaces

The properties of *strong convergence* and *weak convergence* are defined on *sequences*. An infinite sum $\sum_{n=1}^{\infty}\boldsymbol{x}_n$ in a Banach space is the limit of a sequence of partial sums $\left(\!\left(\sum_{n=1}^{N}\boldsymbol{x}_n\right)\!\right)$, so the properties of strong and weak convergence apply to infinite sums as well. Definition G.5 (next) assigns special equality symbols for these sums.

Definition G.5. *Let* $\boldsymbol{B} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *be a Banach space.*

DEF

The expression $\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty}\boldsymbol{x}_n$ *denotes that the sum* ***converges strongly*** *to* $\boldsymbol{x}$.

The expression $\boldsymbol{x} \doteq \sum_{n=1}^{\infty}\boldsymbol{x}_n$ *denotes that the sum* ***converges weakly*** *to* $\boldsymbol{x}$.

G.5 Convergence tests for real sequences

Theorem G.7 (*comparison test*). [20]

THM

$$\left\{\begin{array}{lll} 1. & \sum_{n=1}^{\infty}(\!(y_n)\!) \text{ CONVERGES} & \textit{and} \\ 2. & x_n \le y_n & \forall n\in\mathbb{N} \end{array}\right\} \implies \sum_{n=1}^{\infty}(\!(x_n)\!) \text{ CONVERGES}$$

[20] Bonar et al. (2006) page 26 ⟨Theorem 1.53 (Limit Comparison Test Strengthened)⟩, Heinbockel (2010) page 152 ⟨Comparison Tests⟩

APPENDIX H

TRIGONOMETRIC FUNCTIONS

H.1 Definition Candidates

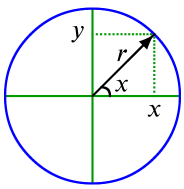
Definition H.1 (Hermitian components). [1] *Let* $(\mathbb{F}, *)$ *be a* $*$*-algebra a* (STAR ALGEBRA).

DEF

*The **real part** of* x *is defined as*	$\Re x$	$\triangleq$	$\frac{1}{2}\left(x + x^*\right)$	$\forall x \in \mathbb{F}$
*The **imaginary part** of* x *is defined as*	$\Im x$	$\triangleq$	$\frac{1}{2i}\left(x - x^*\right)$	$\forall x \in \mathbb{F}$

There are several ways of defining the sine and cosine functions, including the following:[2]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[3]



$$\cos x \triangleq \frac{x}{r} \qquad \sin x \triangleq \frac{y}{r}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[4]

$$\cos x \triangleq \mathbf{R}_\mathrm{e} e^{ix} \qquad\qquad \sin x \triangleq \mathbf{I}_\mathrm{m}\left(e^{ix}\right)$$

3. **Polynomial:** Let $\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space (Definition G.1 page 204). The sine

[1] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964) page 242

[2] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means "bay" or "inlet"—thus resulting in the Latin translation *sinus*, which also means "bay" or "inlet". Reference: Boyer and Merzbach (1991) page 252

[3] Abramowitz and Stegun (1972), page 78

[4] Euler (1748)

and cosine functions can be defined in terms of *Taylor expansions* such that[5]

$$\begin{aligned}
\cos(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\sin(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}$$

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \prod_{n=0}^{N} x_n$ in some topological space (Definition G.1 page 204). The sine and cosine functions can be defined in terms of a product of factors such that[6]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1 - \left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x \prod_{n=1}^{\infty}\left[1 - \left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[7]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - (n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - (n\pi)^2}\right)}_{\cot(x)} \sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ such that

$$\begin{aligned}
\cos(x) &\triangleq \mathsf{f}(x) \quad \text{where} \quad \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} = 0}_{\text{differential equation}} \quad \underbrace{\mathsf{f}(0) = 1}_{\text{1st initial condition}} \quad \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) = 0}_{\text{2nd initial condition}} \\
\sin(x) &\triangleq \mathsf{g}(x) \quad \text{where} \quad \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{g} + \mathsf{g} = 0}_{\text{differential equation}} \quad \underbrace{\mathsf{g}(0) = 0}_{\text{1st initial condition}} \quad \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{g}\right](0) = 1}_{\text{2nd initial condition}}
\end{aligned}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[8]

$$\begin{aligned}
\cos(x) &\triangleq \mathsf{f}^{-1}(x) \quad \text{where} \quad \mathsf{f}(x) \triangleq \underbrace{\int_x^1 \sqrt{\frac{1}{1-y^2}}\,\mathrm{d}y}_{\arccos(x)} \\
\sin(x) &\triangleq \mathsf{g}^{-1}(x) \quad \text{where} \quad \mathsf{g}(x) \triangleq \underbrace{\int_0^x \sqrt{\frac{1}{1-y^2}}\,\mathrm{d}y}_{\arcsin(x)}
\end{aligned}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ (Definition H.2 page 215). Support for such an approach includes the following:

[5] Rosenlicht (1968), page 157, Abramowitz and Stegun (1972), page 74
[6] Abramowitz and Stegun (1972), page 75
[7] Abramowitz and Stegun (1972), page 75
[8] Abramowitz and Stegun (1972), page 79

- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ (Theorem H.1 page 217).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem H.3 page 218).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem H.4 page 219).
- The complex exponential function is a solution of a second order homogeneous differential equation (Definition H.5 page 220).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section H.6 page 228).

H.2 Definitions

Definition H.2. [9] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator.*

DEF

*The function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ is the **cosine** function $\cos(x) \triangleq \mathsf{f}(x)$ if*

$$\begin{array}{llcll}
1. & \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} & = & 0 & \text{(second order homogeneous differential equation) and} \\
2. & \mathsf{f}(0) & = & 1 & \text{(first initial condition) and} \\
3. & \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) & = & 0 & \text{(second initial condition).}
\end{array}$$

Definition H.3. [10] *Let $\boldsymbol{C}$ and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ be defined as in definition of $\cos(x)$ (Definition H.2 page 215).*

DEF

*The function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ is the **sine** function $\sin(x) \triangleq \mathsf{f}(x)$ if*

$$\begin{array}{llcll}
1. & \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} & = & 0 & \text{(second order homogeneous differential equation) and} \\
2. & \mathsf{f}(0) & = & 0 & \text{(first initial condition) and} \\
3. & \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0) & = & 1 & \text{(second initial condition).}
\end{array}$$

Definition H.4. [11]

DEF

Let π ("pi") be defined as the element in $\mathbb{R}$ such that

$$\begin{array}{llcll}
(1). & \cos\left(\frac{\pi}{2}\right) & = & 0 & \text{and} \\
(2). & \pi & > & 0 & \text{and}
\end{array}$$

(3). *π is the **smallest** of all elements in $\mathbb{R}$ that satisfies (1) and (2).*

H.3 Basic properties

Lemma H.1. [12] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator.*

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[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229

[10] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229

[11] Rosenlicht (1968) page 158

[12] Rosenlicht (1968), page 156, Liouville (1839)

LEM

$$\left\{\frac{d^2}{dx^2}f + f = 0\right\} \iff \left\{\begin{aligned} f(x) &= \underbrace{[f](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{d}{dx}f\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} \\ &= \left(f(0) + \left[\frac{d}{dx}f\right](0)x\right) - \left(\frac{f(0)}{2!}x^2 + \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3\right) + \left(\frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5\right)\cdots \end{aligned}\right\}$$

✎PROOF: Let $f'(x) \triangleq \frac{d}{dx}f(x)$.

$$f'''(x) = -\left[\frac{d}{dx}f\right](x)$$

$$f^{(4)}(x) = -\left[\frac{d}{dx}f\right](x) \qquad = -\left[\frac{d^2}{dx^2}f\right](x) = f(x)$$

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \implies f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n && \text{by Taylor expansion (Theorem C.13 page 153)} \\ &= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{\left[\frac{d^2}{dx^2}f\right](0)}{2!}x^2 - \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 - \cdots \\ &= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{f(0)}{2!}x^2 - \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3 + \frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5 - \cdots \\ &= f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \end{aligned}$$

2. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \impliedby f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned} \left[\frac{d^2}{dx^2}f\right](x) &= \frac{d}{dx}\frac{d}{dx}[f(x)] \\ &= \frac{d}{dx}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] && \text{by right hypothesis} \\ &= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)f(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n-1}\right] \\ &= \sum_{n=1}^{\infty}(-1)^n\left[\frac{f(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n-1)!}x^{2n-1}\right] \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \\ &= -f(x) && \text{by right hypothesis} \end{aligned}$$

✏

Theorem H.1 (Taylor series for cosine/sine). [13]

THM

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \forall x\in\mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma H.1 page 215}\\
&= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by cos initial conditions (Definition H.2 page 215)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
\sin(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma H.1 page 215}\\
&= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by sin initial conditions (Definition H.3 page 215)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

✏

Theorem H.2. [14]

THM

$\cos(0) = 1$	$\cos(-x) = \cos(x)$	$\forall x\in\mathbb{R}$
$\sin(0) = 0$	$\sin(-x) = -\sin(x)$	$\forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\cos(0) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=0} && \text{by \textit{Taylor series for cosine} (Theorem H.1 page 217)}\\
&= 1\\
\sin(0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big|_{x=0} && \text{by \textit{Taylor series for sine} (Theorem H.1 page 217)}\\
&= 0\\
\cos(-x) &= 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \cdots && \text{by \textit{Taylor series for cosine} (Theorem H.1 page 217)}\\
&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\
&= \cos(x) && \text{by \textit{Taylor series for cosine} (Theorem H.1 page 217)}\\
\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots && \text{by \textit{Taylor series for sine} (Theorem H.1 page 217)}
\end{aligned}$$

[13] Rosenlicht (1968), page 157
[14] Rosenlicht (1968), page 157

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$
$$= \sin(x) \qquad \text{by } \textit{Taylor series for sine} \qquad \text{(Theorem H.1 page 217)}$$

⇒

Lemma H.2. [15]

$\cos(1) > 0$, $\cos(2) < 0$	$x \in (0:2) \implies \sin(x) > 0$

✎PROOF:

$$\cos(1) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=1} \qquad \text{by } \textit{Taylor series for cosine} \qquad \text{(Theorem H.1 page 217)}$$
$$= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots$$
$$> 0$$
$$\cos(2) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=2} \qquad \text{by } \textit{Taylor series for cosine} \qquad \text{(Theorem H.1 page 217)}$$
$$= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots$$
$$< 0$$

$$x \in (0:2) \implies \text{each term in the sequence } \left(\!\!\left(\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots\right)\!\!\right) \text{ is} > 0$$
$$\implies \sin(x) > 0$$

⇒

Proposition H.1. *Let π be defined as in Definition H.4 (page 215).*

(A). *The value π **exists** in $\mathbb{R}$.*
(B). $2 < \pi < 4$.

✎PROOF:

$$\cos(1) > 0 \qquad \text{by Lemma H.2 page 218}$$
$$\cos(2) < 0 \qquad \text{by Lemma H.2 page 218}$$
$$\implies 1 < \frac{\pi}{2} < 2$$
$$\implies 2 < \pi < 4$$

⇒

Theorem H.3. [16] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let $\mathrm{f}'(0) \triangleq \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\right](0)$.*

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f} + \mathrm{f} = 0\right\} \iff \left\{\mathrm{f}(x) = \mathrm{f}(0)\cos(x) + \mathrm{f}'(0)\sin(x)\right\} \qquad \forall \mathrm{f}\in\boldsymbol{C}, \forall x\in\mathbb{R}$$

[15] Rosenlicht (1968), page 158

[16] Rosenlicht (1968), page 157. The general solution for the *non-homogeneous* equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = \mathrm{g}(x)$ with initial conditions $\mathrm{f}(a) = 1$ and $\mathrm{f}'(a) = \rho$ is $\mathrm{f}(x) = \cos(x) + \rho\sin(x) + \int_a^x \mathrm{g}(y)\sin(x-y)\,\mathrm{d}y$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992), page 371, Liouville (1839). Volterra equation references: Pedersen (2000), page 99, Lalescu (1908), Lalescu (1911)

✎PROOF:

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) = -f(x) \implies f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$\begin{aligned} f(x) &= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by left hypothesis and Lemma H.1 page 215} \\ &= f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x && \text{by definitions of cos and sin (Definition H.2 page 215, Definition H.3 page 215)} \end{aligned}$$

2. Proof that $\frac{d^2}{dx^2}f = -f \impliedby f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$\begin{aligned} f(x) &= f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x && \text{by right hypothesis} \\ &= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} \\ \implies & \frac{d^2}{dx^2}f + f = 0 && \text{by Lemma H.1 page 215} \end{aligned}$$

✏

Theorem H.4. [17] *Let $\frac{d}{dx} \in C^C$ be the differentiation operator.*

THM			
	$\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x \in \mathbb{R}$	$\frac{d}{dx}\sin(x) = \cos(x) \quad \forall x \in \mathbb{R}$	$\cos^2(x) + \sin^2(x) = 1 \quad \forall x \in \mathbb{R}$

✎PROOF:

$$\begin{aligned} \frac{d}{dx}\cos(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} && \text{by *Taylor series* (Theorem H.1 page 217)} \\ &= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!} \\ &= -\sin(x) && \text{by *Taylor series* (Theorem H.1 page 217)} \\ \frac{d}{dx}\sin(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by *Taylor series* (Theorem H.1 page 217)} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\ &= \cos(x) && \text{by *Taylor series* (Theorem H.1 page 217)} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\left[\cos^2(x) + \sin^2(x)\right] &= -2\cos(x)\sin(x) + 2\sin(x)\cos(x) \\ &= 0 \\ &\implies \cos^2(x) + \sin^2(x) \text{ is constant} \\ &\implies \cos^2(x) + \sin^2(x) \\ &= \cos^2(0) + \sin^2(0) \\ &= 1 + 0 = 1 && \text{by Theorem H.2 page 217} \end{aligned}$$

✏

[17] Rosenlicht (1968), page 157

Proposition H.2.

PRP $\sin\left(\frac{\pi}{2}\right) = 1$

✎PROOF:

$$\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2) + 0} \\
&= \pm\sqrt{\sin^2(\pi/2) + \cos^2(\pi/2)} && \text{by definition of } \pi && \text{(Definition H.4 page 215)} \\
&= \pm\sqrt{1} && \text{by Theorem H.4 page 219} \\
&= \pm 1 \\
&= 1 && \text{by Lemma H.2 page 218}
\end{aligned}$$

✏

H.4 The complex exponential

Definition H.5.

DEF *The function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq \mathsf{f}(x)$ *if*

1. $\frac{\mathsf{d}^2}{\mathsf{d}x^2}\mathsf{f} + \mathsf{f} = 0$ *(second order homogeneous differential equation)* *and*
2. $\mathsf{f}(0) = 1$ *(first initial condition)* *and*
3. $\left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0) = i$ *(second initial condition).*

Theorem H.5 (Euler's identity). [18]

THM $e^{ix} = \cos(x) + i\sin(x) \qquad \forall x \in \mathbb{R}$

✎PROOF:

$$\begin{aligned}
\exp(ix) &= \mathsf{f}(0)\cos(x) + \left[\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sin(x) && \text{by Theorem H.3 page 218} \\
&= \cos(x) + i\sin(x) && \text{by Definition H.5 page 220}
\end{aligned}$$

✏

Proposition H.3.

PRP $e^{-i\pi/2} = -i \quad \big| \quad e^{i\pi/2} = i$

✎PROOF:

$$\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem H.5 page 220)} \\
&= 0 + i && \text{by Theorem H.2 (page 217) and Proposition H.2 (page 220)} \\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem H.5 page 220)} \\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem H.2 page 217} \\
&= 0 - i && \text{by Theorem H.2 (page 217) and Proposition H.2 (page 220)}
\end{aligned}$$

✏

[18] Euler (1748), Bottazzini (1986), page 12

Corollary H.1.

COR $$e^{ix} = \sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\boxed{e^{ix}} = \cos(x) + i\sin(x) \qquad \text{by } \textit{Euler's identity} \qquad \text{(The}$$

$$= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} \qquad \text{by } \textit{Taylor series} \qquad \text{(The}$$

$$= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!} = \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}$$

✏

Corollary H.2 (Euler formulas). [19]

COR $$\cos(x) = \mathbf{R_e}\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \sin(x) = \mathbf{I_m}\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\boxed{\mathbf{R_e}\left(e^{ix}\right)} \triangleq \frac{e^{ix}+\left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} \qquad \text{by definition of } \Re$$

$$= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} \qquad \text{by } \textit{Euler's identity} \qquad \text{(Theorem H.5 page 220)}$$

$$= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} = \frac{\cos(x)}{2} + \frac{\cos(x)}{2} = \boxed{\cos(x)}$$

$$\boxed{\mathbf{I_m}\left(e^{ix}\right)} \triangleq \frac{e^{ix}-\left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} \qquad \text{by definition of } \Im$$

$$= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} \qquad \text{by } \textit{Euler's identity} \qquad \text{(Theorem H.5 page 220)}$$

$$= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} = \frac{i\sin(x)}{2i} + \frac{i\sin(x)}{2i} = \boxed{\sin(x)}$$

✏

Theorem H.6. [20]

THM $$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

[19] Euler (1748), Bottazzini (1986), page 12
[20] Rudin (1987) page 1

✎PROOF:

$$\begin{aligned}
e^{\alpha}\, e^{\beta} &= \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) && \text{by Corollary H.1 page 221}\\
&= \sum_{n\in\mathbb{W}}\sum_{k=0}^{n} \frac{\alpha^k}{k!}\,\frac{\beta^{n-k}}{(n-k)!} && \text{by Corollary G.1 page 208}\\
&= \sum_{n\in\mathbb{W}}\sum_{k=0}^{n} \frac{n!}{n!}\,\frac{\alpha^k}{k!}\,\frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\,\alpha^k\,\beta^{n-k}\\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k}\alpha^k\,\beta^{n-k}\\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the \textit{Binomial Theorem}} && \text{(Theorem C.14 page 153)}\\
&= e^{\alpha+\beta} && \text{by Corollary H.1 page 221}
\end{aligned}$$

✏

H.5 Trigonometric Identities

Theorem H.7 (shift identities).

THM							
$\cos\left(x+\frac{\pi}{2}\right)$	$=$	$-\sin x$	$\forall x\in\mathbb{R}$	$\sin\left(x+\frac{\pi}{2}\right)$	$=$	$\cos x$	$\forall x\in\mathbb{R}$
$\cos\left(x-\frac{\pi}{2}\right)$	$=$	$\sin x$	$\forall x\in\mathbb{R}$	$\sin\left(x-\frac{\pi}{2}\right)$	$=$	$-\cos x$	$\forall x\in\mathbb{R}$

✎PROOF:

$$\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)}+e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)}\\
&= \frac{e^{ix}e^{i\frac{\pi}{2}}+e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta}=e^{\alpha}e^{\beta}\text{ result} && \text{(Theorem H.6 page 221)}\\
&= \frac{e^{ix}(i)+e^{-ix}(-i)}{2} && \text{by Proposition H.3 page 220}\\
&= \frac{e^{ix}-e^{-ix}}{-2i}\\
&= -\sin x && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)}\\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)}+e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)}\\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}}+e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta}=e^{\alpha}e^{\beta}\text{ result} && \text{(Theorem H.6 page 221)}\\
&= \frac{e^{ix}(-i)+e^{-ix}(i)}{2} && \text{by Proposition H.3 page 220}\\
&= \frac{e^{ix}-e^{-ix}}{2i}\\
&= \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)}
\end{aligned}$$

$$\begin{aligned}
\sin\left(x+\frac{\pi}{2}\right) &= \cos\left(\left[x+\frac{\pi}{2}\right]-\frac{\pi}{2}\right) && \text{by previous result}\\
&= \cos(x)\\
\sin\left(x-\frac{\pi}{2}\right) &= -\cos\left(\left[x-\frac{\pi}{2}\right]+\frac{\pi}{2}\right) && \text{by previous result}\\
&= -\cos(x)
\end{aligned}$$

✏

Theorem H.8 (product identities)**.**

THM	*(A).*	$\cos x \cos y$	$=$	$\tfrac{1}{2}\cos(x-y)$	$+$	$\tfrac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$
	(B).	$\cos x \sin y$	$=$	$-\tfrac{1}{2}\sin(x-y)$	$+$	$\tfrac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
	(C).	$\sin x \cos y$	$=$	$\tfrac{1}{2}\sin(x-y)$	$+$	$\tfrac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
	(D).	$\sin x \sin y$	$=$	$\tfrac{1}{2}\cos(x-y)$	$-$	$\tfrac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary H.2 page 221)
(algebraic method requiring *complex number system* $\mathbb{C}$):

$$\begin{aligned}
\cos x\cos y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right) && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)}\\
&= \frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(-x+y)}+e^{i(-x-y)}}{4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{4}+\frac{e^{i(x-y)}+e^{-i(x-y)}}{4}\\
&= \frac{2\cos(x+y)}{4}+\frac{2\cos(x-y)}{4} && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)}\\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem H.3 page 218)
(differential equation method requiring only *real number system* $\mathbb{R}$):

$$\begin{aligned}
&\mathsf{f}(x) \triangleq \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)\\
&\implies \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) = -\tfrac{1}{2}\sin(x-y)-\tfrac{1}{2}\sin(x+y) && \text{by Theorem H.4 page 219}\\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x) = -\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y) && \text{by Theorem H.4 page 219}\\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x)+\mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property}\\
&\implies \underbrace{\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)}_{\mathsf{f}(x)} = \underbrace{[\tfrac{1}{2}\cos(0-y)+\tfrac{1}{2}\cos(0+y)]}_{\mathsf{f}''(0)}\cos(x)+\underbrace{[-\tfrac{1}{2}\sin(0-y)-\tfrac{1}{2}\sin(0+y)]}_{\mathsf{f}'(0)}\sin(x)\\
&\implies \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y) = \cos y\cos x+0\sin(x)\\
&\implies \cos x\cos y = \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)
\end{aligned}$$

3. Proof for (B) using *Euler formulas* (Corollary H.2 page 221):

$$\begin{aligned}
\sin x \sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary H.2 page 221}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary H.2 page 221}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x \sin y &= \cos(x)\cos\left(y-\frac{\pi}{2}\right) && \text{by \textit{shift identities}} && \text{(Theorem H.7 page 222)}\\
&= \frac{1}{2}\cos\left(x+y-\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(x-y+\frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y)-\frac{1}{2}\sin(x-y) && \text{by \textit{shift identities}} && \text{(Theorem H.7 page 222)}\\
\sin x \cos y &= \cos y \sin x\\
&= \frac{1}{2}\sin(y+x)-\frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y)+\frac{1}{2}\sin(x-y) && \text{by Theorem H.2 page 217}
\end{aligned}$$

✎

Proposition H.4.

PRP			
	(A). $\cos(\pi) = -1$	(C). $\cos(2\pi) = 1$	(E). $e^{i\pi} = -1$
	(B). $\sin(\pi) = 0$	(D). $\sin(2\pi) = 0$	(F). $e^{i2\pi} = 0$

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1+1+\cos(\pi)\\
&= -1+2[{}^1\!/\!_2\cos({}^\pi\!/\!_2-{}^\pi\!/\!_2)+{}^1\!/\!_2\cos({}^\pi\!/\!_2+{}^\pi\!/\!_2)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem H.2 page 217)}\\
&= -1+2\cos({}^\pi\!/\!_2)\cos({}^\pi\!/\!_2) && \text{by \textit{product identities}} && \text{(Theorem H.8 page 223)}\\
&= -1+2(0)(0) && \text{by definition of } \pi && \text{(Definition H.4 page 215)}\\
&= -1\\
\sin(\pi) &= 0+\sin(\pi)\\
&= 2[-{}^1\!/\!_2\sin({}^\pi\!/\!_2-{}^\pi\!/\!_2)+{}^1\!/\!_2\sin({}^\pi\!/\!_2+{}^\pi\!/\!_2)] && \text{by } \sin(0)=0 \text{ result} && \text{(Theorem H.2 page 217)}\\
&= 2\cos({}^\pi\!/\!_2)\sin({}^\pi\!/\!_2) && \text{by \textit{product identities}} && \text{(Theorem H.8 page 223)}\\
&= 2(0)\sin({}^\pi\!/\!_2) && \text{by definition of } \pi && \text{(Definition H.4 page 215)}\\
&= 0\\
\cos(2\pi) &= 1+\cos(2\pi)-1\\
&= 2[{}^1\!/\!_2\cos(\pi-\pi)+{}^1\!/\!_2\cos(\pi+\pi)]-1 && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem H.2 page 217)}\\
&= 2\cos(\pi)\cos(\pi)-1 && \text{by \textit{product identities}} && \text{(Theorem H.8 page 223)}\\
&= 2(-1)(-1)-1 && \text{by (A)}\\
&= 1
\end{aligned}$$

$$\begin{aligned}
\sin(2\pi) &= 0 + \sin(2\pi) \\
&= 2[{}^1\!/\!{}_2\sin(\pi - \pi) + {}^1\!/\!{}_2\sin(\pi + \pi)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem H.2 page 217)} \\
&= 2\sin(\pi)\cos(\pi) && \text{by \textit{product identities}} && \text{(Theorem H.8 page 223)} \\
&= 2(0)(-1) && \text{by (A) and (B)} \\
&= 0 \\
e^{i\pi} &= \cos(\pi) + i\sin(\pi) && \text{by \textit{Euler's identity}} && \text{(Theorem H.5 page 220)} \\
&= -1 + 0 && \text{by (A) and (B)} \\
&= -1 \\
e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) && \text{by \textit{Euler's identity}} && \text{(Theorem H.5 page 220)} \\
&= 1 + 0 && \text{by (C) and (D)} \\
&= 1
\end{aligned}$$

⇒

Theorem H.9 (double angle formulas). [21]

THM

$$\begin{array}{llcll}
\text{(A).} & \cos(x+y) & = & \cos x\cos y - \sin x\sin y & \forall x,y\in\mathbb{R} \\
\text{(B).} & \sin(x+y) & = & \sin x\cos y + \cos x\sin y & \forall x,y\in\mathbb{R} \\
\text{(C).} & \tan(x+y) & = & \dfrac{\tan x + \tan y}{1 - \tan x\tan y} & \forall x,y\in\mathbb{R}
\end{array}$$

✎PROOF:

1. Proof for (A) using *product identities* (Theorem H.8 page 223).

$$\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x+y)}_{\cos(x+y)} + \underbrace{\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x-y)}_{0} \\
&= \left[\frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\right] - \left[\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\right] \\
&= \cos x\cos y - \sin x\sin y && \text{by Theorem H.8 page 223}
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem H.3 page 218):

$$\begin{aligned}
\mathrm{f}(x) \triangleq \cos(x+y) &\implies \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x) = -\sin(x+y) && \text{by Theorem H.4 page 219} \\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) = -\cos(x+y) && \text{by Theorem H.4 page 219} \\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = 0 && \text{by \textit{additive inverse} property} \\
&\implies \cos(x+y) = \cos y\cos x - \sin y\sin x && \text{by Theorem H.3 page 218} \\
&\implies \cos(x+y) = \cos x\cos y - \sin x\sin y && \text{by \textit{commutative} property}
\end{aligned}$$

[21] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x - \frac{\pi}{2} + y\right) && \text{by \textit{shift identities} (Theorem H.7 page 222)}\\
&= \cos\left(x-\frac{\pi}{2}\right)\cos(y) - \sin\left(x-\frac{\pi}{2}\right)\sin(y) && \text{by (A)}\\
&= \sin(x)\cos(y) + \cos(x)\sin(y) && \text{by \textit{shift identities} (Theorem H.7 page 222)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)}\\
&= \frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} && \text{by (A)}\\
&= \left(\frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}\right)\left(\frac{\cos x\cos y}{\cos x \cos y}\right)\\
&= \frac{\frac{\sin x\cos y}{\cos x\cos y} + \frac{\cos x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y} - \frac{\sin x\sin y}{\cos x\cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}
\end{aligned}$$

✎

Theorem H.10 (trigonometric periodicity).

THM

(A).	$\cos(x + M\pi)$	$=$	$(-1)^M\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$	(D).	$\cos(x + 2M\pi)$	$=$	$\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
(B).	$\sin(x + M\pi)$	$=$	$(-1)^M\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$	(E).	$\sin(x + 2M\pi)$	$=$	$\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
(C).	$e^{i(x+M\pi)}$	$=$	$(-1)^M e^{ix}$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$	(F).	$e^{i(x+2M\pi)}$	$=$	e^{ix}	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$

✎PROOF:

1. Proof for (A):

   (a) $M = 0$ case: $\cos(x + 0\pi) = \cos(x) = (-1)^0\cos(x)$

   (b) Proof for $M > 0$ cases (by induction):

      i. Base case $M = 1$:

$$\begin{aligned}
\cos(x+\pi) &= \cos x \cos\pi - \sin x \sin\pi && \text{by \textit{double angle formulas}} && \text{(Theorem H.9 page 225)}\\
&= \cos x(-1) - \sin x(0) && \text{by } \cos\pi = -1 \text{ result} && \text{(Proposition H.4 page 224)}\\
&= (-1)^1\cos x
\end{aligned}$$

      ii. Inductive step…Proof that M case $\implies$ $M+1$ case:

$$\begin{aligned}
\cos(x + [M+1]\pi) &= \cos([x+\pi] + M\pi)\\
&= (-1)^M\cos(x+\pi) && \text{by induction hypothesis (M case)}\\
&= (-1)^M(-1)\cos(x) && \text{by base case (item 1(b)i) page 226)}\\
&= (-1)^{M+1}\cos(x)\\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\cos(x + M\pi) &\triangleq \cos(x - N\pi) && \text{by definition of } N \\
&= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem H.9 page 225)} \\
&= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem H.2 page 217} \\
&= \cos(x)\cos(0 + N\pi) + \sin(x)\sin(0 + N\pi) \\
&= \cos(x)(-1)^N\cos(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (1b) page 226)} \\
&= (-1)^N\cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem H.2 page 217)} \\
&\triangleq (-1)^{-M}\cos(x) && \text{by definition of } N \\
&= (-1)^M\cos(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\cos(x + M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)} \\
&= e^{iM\pi}\left[\frac{e^{ix} + e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^\alpha e^\beta \text{ result} && \text{(Theorem H.6 page 221)} \\
&= \left(e^{i\pi}\right)^M \cos x && \text{by \textit{Euler formulas}} && \text{(Corollary H.2 page 221)} \\
&= (-1)^M \cos x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition H.4 page 224)}
\end{aligned}$$

2. Proof for (B):

(a) $M = 0$ case: $\sin(x + 0\pi) = \sin(x) = (-1)^0\sin(x)$

(b) Proof for $M > 0$ cases (by induction):

i. Base case $M = 1$:

$$\begin{aligned}
\sin(x + \pi) &= \sin x \cos\pi + \cos x \sin\pi && \text{by \textit{double angle formulas}} && \text{(Theorem H.9 page 225)} \\
&= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} && \text{(Proposition H.4 page 224)} \\
&= (-1)^1 \sin x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies M + 1$ case:

$$\begin{aligned}
\sin(x + [M + 1]\pi) &= \sin([x + \pi] + M\pi) \\
&= (-1)^M \sin(x + \pi) && \text{by induction hypothesis } (M \text{ case}) \\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 227)} \\
&= (-1)^{M+1}\sin(x) \\
&\implies M + 1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\sin(x + M\pi) &\triangleq \sin(x - N\pi) && \text{by definition of } N \\
&= \sin(x)\sin(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem H.9 page 225)} \\
&= \sin(x)\sin(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem H.2 page 217} \\
&= \sin(x)\sin(0 + N\pi) + \sin(x)\sin(0 + N\pi) \\
&= \sin(x)(-1)^N\sin(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (2b) page 227)} \\
&= (-1)^N\sin(x) && \text{by } \sin(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem H.2 page 217)} \\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N \\
&= (-1)^M\sin(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)} - e^{-i(x+M\pi)}}{2i} && \text{by } \textit{Euler formulas} && \text{(Corollary H.2 page 221)}\\
&= e^{iM\pi}\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem H.6 page 221)}\\
&= \left(e^{i\pi}\right)^M \sin x && \text{by } \textit{Euler formulas} && \text{(Corollary H.2 page 221)}\\
&= (-1)^M \sin x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition H.4 page 224)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem H.6 page 221)}\\
&= \left(e^{i\pi}\right)^M\left(e^{ix}\right)\\
&= (-1)^M e^{ix} && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition H.4 page 224)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):
$$\begin{aligned}
\cos(i(x+2M\pi)) &= (-1)^{2M}\cos(ix) &&= \cos(ix) && \text{by (A)}\\
\sin(i(x+2M\pi)) &= (-1)^{2M}\sin(ix) &&= \sin(ix) && \text{by (B)}\\
e^{i(x+2M\pi)} &= (-1)^{2M}e^{ix} &&= e^{ix} && \text{by (C)}
\end{aligned}$$

✑

Theorem H.11 (half-angle formulas/squared identities)**.**

THM		
	(A). $\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$	(C). $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$
	(B). $\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$	

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x) + \tfrac{1}{2}\cos(x+x) && \text{by } \textit{product identities} && \text{(Theorem H.8 page 223)}\\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem H.2 page 217)}\\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x) - \tfrac{1}{2}\cos(x+x) && \text{by } \textit{product identities} && \text{(Theorem H.8 page 223)}\\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem H.2 page 217)}\\
\cos^2 x + \sin^2 x &= \tfrac{1}{2}[1+\cos(2x)] + \tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)}\\
& && \text{note: see also} && \text{Theorem H.4 page 219}
\end{aligned}$$

✑

H.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

$$\begin{aligned}
\langle \cos(x) \,|\, \sin(x)\rangle &= \int_{-\pi}^{+\pi} \cos(x)\sin(x)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x-x)\,\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(x+x)\,\mathrm{d}x && \text{by Theorem H.8 page 223}\\
&= \frac{1}{2}\int_{-\pi}^{+\pi} \cancelto{0}{\sin(0)}\,\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{+\pi} \sin(2x)\,\mathrm{d}x
\end{aligned}$$

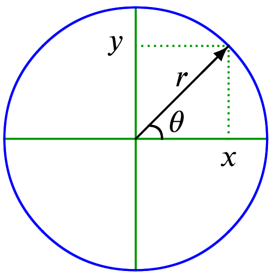
$$
\begin{aligned}
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi}\cos(2x)\\
&= -\frac{1}{4}[\cos(2\pi)-\cos(-2\pi)]\\
&= 0
\end{aligned}
$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

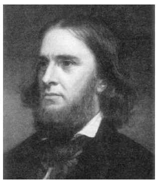
We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$



$$
\begin{aligned}
\cos\theta &\triangleq \frac{x}{r} & \sec\theta &\triangleq \frac{r}{x}\\
\sin\theta &\triangleq \frac{y}{r} & \csc\theta &\triangleq \frac{r}{y}\\
\tan\theta &\triangleq \frac{y}{x} & \cot\theta &\triangleq \frac{x}{y}
\end{aligned}
$$

H.7 The power of the exponential



❝*Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.*❞
Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [22]

❝*Young man, in mathematics you don't understand things. You just get used to them.*❞
John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [23]

[22] quote: Kasner and Newman (1940), page 104
image: `http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html`

[23] quote: Zukav (1980), page 208
image: `http://en.wikipedia.org/wiki/John_von_Neumann`
The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary H.3. [24]

COR $\boxed{e^{i\pi} + 1 = 0}$

✎PROOF:

$$\begin{aligned} e^{ix}\Big|_{x=\pi} &= [\cos x + i\sin x]_{x=\pi} && \text{by Euler's identity (Theorem H.5 page 220)} \\ &= -1 + i \cdot 0 && \text{by Proposition H.4 page 224} \\ &= -1 \end{aligned}$$

✏

There are many transforms available, several of them integral transforms $[\mathbf{A}f](s) \triangleq \int_t f(s)\kappa(t,s)\,ds$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:

① The *Laplace Transform* with kernel $\kappa(t, s) \triangleq e^{st}$

② The *Fourier Transform* with kernel $\kappa(t, \omega) \triangleq e^{i\omega t}$.

Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem H.12 page 230).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem H.12. [25] *Let* $\mathbf{L}$ *be an operator with kernel* $h(t,\omega)$ *and*
$\check{h}(s) \triangleq \langle h(t,\omega) \,|\, e^{st} \rangle$ (LAPLACE TRANSFORM).

THM
$$\left\{\begin{array}{ll} 1. & \mathbf{L} \text{ is LINEAR and} \\ 2. & \mathbf{L} \text{ is TIME-INVARIANT} \end{array}\right\} \implies \left\{ \mathbf{L}e^{st} = \underbrace{\check{h}^*(-s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenvector}} \right\}$$

✎PROOF:

[24] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

[25] Mallat (1999), page 2, ...page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

$$\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \,|\, \mathsf{h}((t;u),s)\rangle && \text{by linear hypothesis} \\
&= \langle e^{su} \,|\, \mathsf{h}((t-u),s)\rangle && \text{by time-invariance hypothesis} \\
&= \left\langle e^{s(t-v)} \,|\, \mathsf{h}(v,s)\right\rangle && \text{let } v = t-u \implies u = t-v \\
&= e^{st}\ \langle e^{-sv} \,|\, \mathsf{h}(v,s)\rangle && \text{by additivity of } \langle \triangle \,|\, \nabla \rangle \\
&= \langle \mathsf{h}(v,s) \,|\, e^{-sv}\rangle^*\ e^{st} && \text{by conjugate symmetry of } \langle \triangle \,|\, \nabla \rangle \\
&= \left\langle \mathsf{h}(v,s) \,|\, e^{(-s)v}\right\rangle^*\ e^{st} \\
&= \check{\mathsf{h}}^*(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}$$

✏

APPENDIX I

TRIGONOMETRIC POLYNOMIALS

❝ *I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives.*❞

Charles Hermite (1822 – 1901), French mathematician, in an 1893 letter to Stieltjes, in response to the "pathological" everywhere continuous but nowhere differentiable *Weierstrass functions* $\mathsf{f}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$.[1]

I.1 Trigonometric expansion

Theorem I.1 (DeMoivre's Theorem)**.**

THM
$$\left(re^{ix}\right)^n = r^n(\cos nx + i\sin nx) \qquad \forall r,x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\left(re^{ix}\right)^n &= r^n e^{inx} \\
&= r^n\left(\cos nx + i\sin nx\right) && \text{by Euler's identity (Theorem H.5 page 220)}
\end{aligned}$$

⊞

The cosine with argument nx can be expanded as a polynomial in $\cos(x)$ (next).

Theorem I.2 (trigonometric expansion)**.** [2]

---

[1] quote: Hermite (1893)
translation: Lakatos (1976), page 19
image: http://www-groups.dcs.sx-and.ac.uk/~history/PictDisplay/Hermite.html

[2] Rivlin (1974) page 3 ⟨(1.8)⟩

THM

$$\cos(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

$$\sin(nx) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\sin x)^{n-2(k-m)} \qquad \forall n\in\mathbb{W} \text{ and } x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(nx) &= \Re\left(\cos nx + i\sin nx\right) \\
&= \Re\left(e^{inx}\right) \\
&= \Re\left[\left(e^{ix}\right)^n\right] \\
&= \Re\left[(\cos x + i\sin x)^n\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n}\binom{n}{k}(\cos x)^{n-k}(i\sin x)^k\right] \\
&= \Re\left[\sum_{k\in\mathbb{Z}}^{n} i^k\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \Re\left[\sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + i\sum_{k\in\{1,5,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right. \\
&\quad \left. - \sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x + -i\sum_{k\in\{3,7,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x\right] \\
&= \sum_{k\in\{0,4,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x - \sum_{k\in\{2,6,\ldots,n\}}\binom{n}{k}\cos^{n-k}x\sin^k x \\
&= \sum_{k\in\{0,2,\ldots,n\}}\binom{n}{k}(-1)^{\frac{k}{2}}\cos^{n-k}x\sin^k x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\sin^{2k}x \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x(1-\cos^2 x)^k \\
&= \left[\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^k\cos^{n-2k}x\right]\left[\sum_{m=0}^{k}\binom{k}{m}(-1)^m\cos^{2m}x\right] \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}x
\end{aligned}$$

$$\begin{aligned}
\sin(nx) &= \cos\left(nx - \frac{\pi}{2}\right) \\
&= \cos\left(n\left[x - \frac{\pi}{2n}\right]\right) \\
&= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}\cos^{n-2(k-m)}\left(n\left[x - \frac{\pi}{2n}\right]\right)
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \cos^{n-2(k-m)} \left(nx - \frac{\pi}{2} \right)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} \sin^{n-2(k-m)} (nx)$$

✏

Example I.1.

EX

$$\begin{aligned} \cos 5x &= 16\cos^5 x - 20\cos^3 x + 5\cos x \\ \sin 5x &= 16\sin^5 x - 20\sin^3 x + 5\sin x. \end{aligned}$$

✎PROOF:

1. Proof using *DeMoivre's Theorem* (Theorem I.1 page 233):

$$\begin{aligned}
&\cos 5x + i\sin 5x \\
&= e^{i5x} \\
&= \left(e^{ix}\right)^5 \\
&= (\cos x + i\sin x)^5 \\
&= \sum_{k=0}^{5} \binom{5}{k} [\cos x]^{5-k} [i\sin x]^k \\
&= \binom{5}{0} [\cos x]^{5-0} [i\sin x]^0 + \binom{5}{1} [\cos x]^{5-1} [i\sin x]^1 + \binom{5}{2} [\cos x]^{5-2} [i\sin x]^2 + \\
&\binom{5}{3} [\cos x]^{5-3} [i\sin x]^3 + \binom{5}{4} [\cos x]^{5-4} [i\sin x]^4 + \binom{5}{5} [\cos x]^{5-5} [i\sin x]^5 \\
&= 1\cos^5 x + i5\cos^4 x \sin x - 10\cos^3 x \sin^2 x - i10\cos^2 x \sin^3 x + 5\cos x \sin^4 x + i1\sin^5 x \\
&= \left[\cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x\right] + i\left[5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10\cos^3 x(1 - \cos^2 x) + 5\cos x(1 - \cos^2 x)(1 - \cos^2 x)\right] + \\
&i\left[5(1 - \sin^2 x)(1 - \sin^2 x)\sin x - 10(1 - \sin^2 x)\sin^3 x + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5\cos x(1 - 2\cos^2 x + \cos^4 x)\right] + \\
&i\left[5(1 - 2\sin^2 x + \sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \left[\cos^5 x - 10(\cos^3 x - \cos^5 x) + 5(\cos x - 2\cos^3 x + \cos^5 x)\right] + \\
&i\left[5(\sin x - 2\sin^3 x + \sin^5 x) - 10(\sin^3 x - \sin^5 x) + \sin^5 x\right] \\
&= \underbrace{\left[16\cos^5 x - 20\cos^3 x + 5\cos x\right]}_{\cos 5x} + i\underbrace{\left[16\sin^5 x - 20\sin^3 x + 5\sin x\right]}_{\sin 5x}
\end{aligned}$$

2. Proof using trigonometric expansion (Theorem I.2 page 233):

$$\begin{aligned}
\cos 5x &= \sum_{k=0}^{\lfloor \frac{5}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{n-2(k-m)} \\
&= \sum_{k=0}^{2} \sum_{m=0}^{k} (-1)^{k+m} \binom{n}{2k} \binom{k}{m} (\cos x)^{5-2(k-m)} \\
&= (-1)^0 \binom{5}{0} \binom{0}{0} \cos^5 x + (-1)^1 \binom{5}{2} \binom{1}{0} \cos^3 x + (-1)^2 \binom{5}{2} \binom{1}{1} \cos^5 x + \\
&(-1)^2 \binom{5}{4} \binom{2}{0} \cos^1 x + (-1)^3 \binom{5}{4} \binom{2}{1} \cos^3 x + (-1)^4 \binom{5}{4} \binom{2}{2} \cos^5 x
\end{aligned}$$

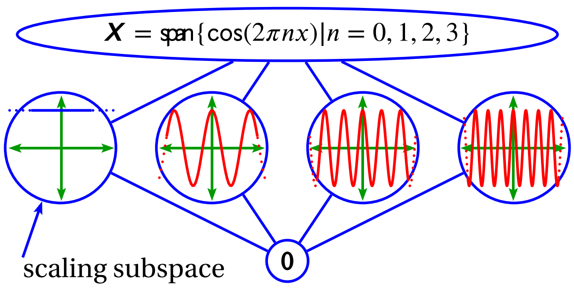


Figure I.1: Lattice of harmonic cosines $\{\cos(nx) | n = 0, 1, 2, \ldots\}$

$$\begin{aligned}
&= +(1)(1)\cos^5 x - (10)(1)\cos^3 x + (10)(1)\cos^5 x + (5)(1)\cos x - (5)(2)\cos^3 x + (5)(1)\cos^5 x \\
&= +(1 + 10 + 5)\cos^5 x + (-10 - 10)\cos^3 x + 5\cos x \\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}$$

✏

Example I.2. [3]

EX

n	$\cos nx$		polynomial in $\cos x$	n	$\cos nx$		polynomial in $\cos x$
0	$\cos 0x$	$=$	1	4	$\cos 4x$	$=$	$8\cos^4 x - 8\cos^2 x + 1$
1	$\cos 1x$	$=$	$\cos^1 x$	5	$\cos 5x$	$=$	$16\cos^5 x - 20\cos^3 x + 5\cos x$
2	$\cos 2x$	$=$	$2\cos^2 x - 1$	6	$\cos 6x$	$=$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
3	$\cos 3x$	$=$	$4\cos^3 x - 3\cos x$	7	$\cos 7x$	$=$	$64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

✎PROOF:

$$\begin{aligned}
\cos 2x &= \sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{2-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^2 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^0 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^2 x \\
&= +(1)(1)\cos^2 x - (1)(1) + (1)(1)\cos^2 x \\
&= 2\cos^2 x - 1
\end{aligned}$$

$$\begin{aligned}
\cos 3x &= \sum_{k=0}^{\lfloor \frac{3}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{3}{2k} \binom{k}{m} (\cos x)^{3-2(k-m)} \\
&= (-1)^0 \binom{3}{0}\binom{0}{0} \cos^3 x + (-1)^1 \binom{3}{2}\binom{1}{0} \cos^1 x + (-1)^2 \binom{3}{2}\binom{1}{1} \cos^3 x \\
&= +\binom{3}{0}\binom{0}{0} \cos^3 x - \binom{3}{2}\binom{1}{0} \cos^1 x + \binom{3}{2}\binom{1}{1} \cos^3 x \\
&= +(1)(1)\cos^3 x - (3)(1)\cos^1 x + (3)(1)\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}$$

$$\cos 4x = \sum_{k=0}^{\lfloor \frac{4}{2} \rfloor} \sum_{m=0}^{k} (-1)^{k+m} \binom{4}{2k} \binom{k}{m} (\cos x)^{4-2(k-m)}$$

[3] Abramowitz and Stegun (1972), page 795, Guillemin (1957), page 593 ⟨(21)⟩, Sloane (2014) ⟨http://oeis.org/A039991⟩, Sloane (2014) ⟨http://oeis.org/A028297⟩

$$
\begin{aligned}
&= \sum_{k=0}^{2}\sum_{m=0}^{k}(-1)^{k+m}\binom{4}{2k}\binom{k}{m}(\cos x)^{4-2(k-m)}\\
&= (-1)^{0+0}\binom{4}{2\cdot 0}\binom{0}{0}(\cos x)^{4-2(0-0)} + (-1)^{1+0}\binom{4}{2\cdot 1}\binom{1}{0}(\cos x)^{4-2(1-0)}\\
&\quad + (-1)^{1+1}\binom{4}{2\cdot 1}\binom{1}{1}(\cos x)^{4-2(1-1)} + (-1)^{2+0}\binom{4}{2\cdot 2}\binom{2}{0}(\cos x)^{4-2(2-0)}\\
&\quad + (-1)^{2+1}\binom{4}{2\cdot 2}\binom{2}{1}(\cos x)^{4-2(2-1)} + (-1)^{2+2}\binom{4}{2\cdot 2}\binom{2}{2}(\cos x)^{4-2(2-2)}\\
&= (1)(1)\cos^4 x - (6)(1)\cos^2 x + (6)(1)\cos^4 x + (1)(1)\cos^0 x - (1)(2)\cos^2 x + (1)(1)\cos^4 x\\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}
$$

$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ see Example I.1 page 235

$$
\begin{aligned}
\cos 6x &= \sum_{k=0}^{\left\lfloor\frac{6}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{6}{2k}\binom{k}{m}(\cos x)^{6-2(k-m)}\\
&= (-1)^0\binom{6}{0}\binom{0}{0}\cos^6 x + (-1)^1\binom{6}{2}\binom{1}{0}\cos^4 x + (-1)^2\binom{6}{2}\binom{1}{1}\cos^6 x + (-1)^2\binom{6}{4}\binom{2}{0}\cos^2 x +\\
&(-1)^3\binom{6}{4}\binom{2}{1}\cos^4 x + (-1)^4\binom{6}{4}\binom{2}{2}\cos^6 x + (-1)^3\binom{6}{6}\binom{3}{0}\cos^0 x + (-1)^4\binom{6}{6}\binom{3}{1}\cos^2 x +\\
&(-1)^5\binom{6}{6}\binom{3}{2}\cos^4 x + (-1)^6\binom{6}{6}\binom{3}{3}\cos^6 x\\
&= +(1)(1)\cos^6 x - (15)(1)\cos^4 x + (15)(1)\cos^6 x + (15)(1)\cos^2 x - (15)(2)\cos^4 x + (15)(1)\cos^6 x\\
&\quad - (1)(1)\cos^0 x + (1)(3)\cos^2 x - (1)(3)\cos^4 x + (1)(1)\cos^6 x\\
&= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1
\end{aligned}
$$

$$
\begin{aligned}
\cos 7x &= \sum_{k=0}^{\left\lfloor\frac{7}{2}\right\rfloor}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{n-2(k-m)}\\
&= \sum_{k=0}^{3}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{2k}\binom{k}{m}(\cos x)^{7-2(k-m)}\\
&= (-1)^0\binom{7}{0}\binom{0}{0}\cos^7 x + (-1)^1\binom{7}{2}\binom{1}{0}\cos^5 x + (-1)^2\binom{7}{2}\binom{1}{1}\cos^7 x + (-1)^2\binom{7}{4}\binom{2}{0}\cos^3 x\\
&\quad + (-1)^3\binom{7}{4}\binom{2}{1}\cos^5 x + (-1)^4\binom{7}{4}\binom{2}{2}\cos^7 x + (-1)^3\binom{7}{6}\binom{3}{0}\cos^1 x + (-1)^4\binom{7}{6}\binom{3}{1}\cos^3 x\\
&\quad + (-1)^5\binom{7}{6}\binom{3}{2}\cos^5 x + (-1)^6\binom{7}{6}\binom{3}{3}\cos^7 x\\
&= (1)(1)\cos^7 x - (21)(1)\cos^5 x + (21)(1)\cos^7 x + (35)(1)\cos^3 x\\
&\quad - (35)(2)\cos^5 x + (35)(1)\cos^7 x - (7)(1)\cos^1 x + (7)(3)\cos^3 x\\
&\quad - (7)(3)\cos^5 x + (7)(1)\cos^7 x\\
&= (1 + 21 + 35 + 7)\cos^7 x - (21 + 70 + 21)\cos^5 x + (35 + 21)\cos^3 x - (7)\cos^1 x\\
&= 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x
\end{aligned}
$$

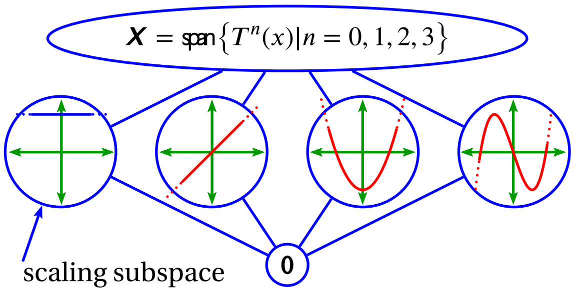


Figure I.2: Lattice of Chebyshev polynomials $\{T_n(x)|n = 0, 1, 2, 3\}$

Note: Trigonometric expansion of $\cos(nx)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[4]

```
trigexpand(cos(2*x));
trigexpand(cos(3*x));
trigexpand(cos(4*x));
trigexpand(cos(5*x));
trigexpand(cos(6*x));
trigexpand(cos(7*x));
```

✏

Definition I.1.

DEF: *The nth **Chebyshev polynomial of the first kind** is defined as*

$$T_n(x) \triangleq \cos nx \qquad \textit{where} \qquad \cos x \triangleq x$$

Theorem I.3. [5] *Let $T_n(x)$ be a* Chebyshev polynomial *with $n \in \mathbb{W}$.*

THM:

$$\begin{array}{lcl} n \text{ is EVEN} & \Longrightarrow & T_n(x) \text{ is EVEN.} \\ n \text{ is ODD} & \Longrightarrow & T_n(x) \text{ is ODD.} \end{array}$$

Example I.3. Let $T_n(x)$ be a *Chebyshev polynomial* with $n \in \mathbb{W}$.

EX:

$T_0(x) = 1$	$T_4(x) = 8x^4 - 8x^2 + 1$
$T_1(x) = x$	$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_2(x) = 2x^2 - 1$	$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
$T_3(x) = 4x^3 - 3x$	

✎Proof: Proof of these equations follows directly from Example I.2 (page 236). ✏

I.2 Trigonometric reduction

Theorem I.2 (page 233) showed that $\cos nx$ can be expressed as a polynomial in $\cos x$. Conversely, Theorem I.4 (next) shows that a polynomial in $\cos x$ can be expressed as a linear combination of $(\!(\cos nx)\!)_{n\in\mathbb{Z}}$.

Theorem I.4 (trigonometric reduction)**.**

[4] maxima, pages 157–158 ⟨10.5 Trigonometric Functions⟩

[5] Rivlin (1974) page 5 ⟨(1.13)⟩, Süli and Mayers (2003) page 242 ⟨Lemma 8.2⟩, Davidson and Donsig (2010) page 222 ⟨exercise 10.7.A(a)⟩

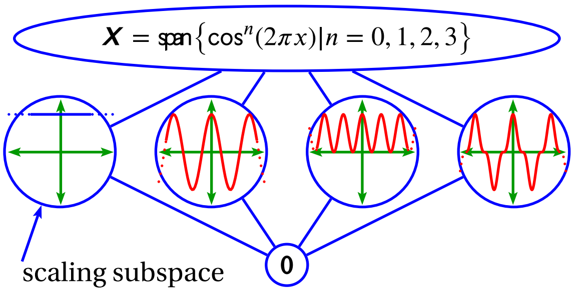


Figure I.3: Lattice of exponential cosines $\left\{\cos^n x | n = 0, 1, 2, 3\right\}$

THM

$$\begin{aligned}\cos^n x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\ &= \begin{cases} \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & \text{for } n \text{ even} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & \text{for } n \text{ odd}\end{cases}\end{aligned}$$

✎PROOF:

$$\begin{aligned}\cos^n x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n \\ &= \mathbf{R}_e\left[\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\right] \\ &= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)x}e^{-ikx}\right] \\ &= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}\right] \\ &= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\cos[(n-2k)x] + i\sin[(n-2k)x]\right)\right] \\ &= \mathbf{R}_e\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] + i\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sin[(n-2k)x]\right] \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos[(n-2k)x] \\ &= \begin{cases} \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos[(n-2k)x] & : \; n \text{ even} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\cos[(n-2k)x] & : \; n \text{ odd}\end{cases}\end{aligned}$$

⇛

Example I.4. [6]

[6] Abramowitz and Stegun (1972), page 795, Sloane (2014) ⟨http://oeis.org/A100257⟩, Sloane (2014) ⟨http://oeis.org/A008314⟩

EX

n	$\cos^n x$	trigonometric reduction	n	$\cos^n x$	trigonometric reduction
0	$\cos^0 x$	$= 1$	4	$\cos^4 x$	$= \dfrac{\cos 4x + 4\cos 2x + 3}{2^3}$
1	$\cos^1 x$	$= \cos x$	5	$\cos^5 x$	$= \dfrac{\cos 5x + 5\cos 3x + 10\cos x}{2^4}$
2	$\cos^2 x$	$= \dfrac{\cos 2x + 1}{2}$	6	$\cos^6 x$	$= \dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{2^5}$
3	$\cos^3 x$	$= \dfrac{\cos 3x + 3\cos x}{2^2}$	7	$\cos^7 x$	$= \dfrac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{2^6}$

✎PROOF:

$$\begin{aligned}
\cos^0 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\bigg|_{n=0} \\
&= \frac{1}{2^0}\sum_{k=0}^{0}\binom{0}{k}\cos[(0-2k)x] \\
&= \binom{0}{0}\cos[(0-2\cdot 0)x] \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\cos^1 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\bigg|_{n=1} \\
&= \frac{1}{2^1}\sum_{k=0}^{1}\binom{1}{k}\cos[(1-2k)x] \\
&= \frac{1}{2}\left[\binom{1}{0}\cos[(1-2\cdot 0)x] + \binom{1}{1}\cos[(1-2\cdot 1)x]\right] \\
&= \frac{1}{2}\left[1\cos x + 1\cos(-x)\right] \\
&= \frac{1}{2}\left(\cos x + \cos x\right) \\
&= \cos x
\end{aligned}$$

$$\begin{aligned}
\cos^2 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\bigg|_{n=2} \\
&= \frac{1}{2^2}\sum_{k=0}^{2}\binom{2}{k}\cos\left([2-2k]x\right) \\
&= \frac{1}{2^2}\left[\binom{2}{0}\cos\left([2-2\cdot 0]x\right) + \binom{2}{1}\cos\left([2-2\cdot 1]x\right) + \binom{2}{2}\cos\left([2-2\cdot 2]x\right) + \right] \\
&= \frac{1}{2^2}\left[1\cos(2x) + 2\cos(0x) + 1\cos(-2x)\right] \\
&= \frac{1}{2^2}\left[\cos(2x) + 2 + \cos(2x)\right] \\
&= \frac{1}{2}\left[\cos(2x) + 1\right]
\end{aligned}$$

$$\begin{aligned}
\cos^3 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left([n-2k]x\right)\bigg|_{n=3} \\
&= \frac{1}{2^3}\sum_{k=0}^{3}\binom{3}{k}\cos\left([3-2k]x\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^3}\left[1\cos(3x) + 3\cos(1x) + 3\cos(-1x) + 1\cos(-3x)\right] \\
&= \frac{1}{2^3}\left[\cos(3x) + 3\cos(x) + 3\cos(x) + \cos(3x)\right] \\
&= \frac{1}{2^2}\left[\cos(3x) + 3\cos(x)\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^4 x &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos([n-2k]x)\bigg|_{n=4} \\
&= \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}\cos([4-2k]x) \\
&= \frac{1}{2^4}\left[1\cos(4x) + 4\cos(2x) + 6\cos(0x) + 4\cos(-2x) + 1\cos(-4x)\right] \\
&= \frac{1}{2^3}\left[\cos(4x) + 4\cos(2x) + 3\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^5 x &= \frac{1}{2^{5-1}}\sum_{k=0}^{\left\lfloor\frac{5}{2}\right\rfloor}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\sum_{k=0}^{2}\binom{5}{k}\cos[(5-2k)x] \\
&= \frac{1}{16}\left[\binom{5}{0}\cos 5x + \binom{5}{1}\cos 3x + \binom{5}{2}\cos x\right] \\
&= \frac{1}{16}\left[\cos 5x + 5\cos 3x + 10\cos x\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^6 x &= \frac{1}{2^6}\binom{6}{\frac{6}{2}} + \frac{1}{2^{6-1}}\sum_{k=0}^{\frac{6}{2}-1}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{2^6}\binom{6}{3} + \frac{1}{2^5}\sum_{k=0}^{2}\binom{6}{k}\cos[(6-2k)x] \\
&= \frac{1}{64}20 + \frac{1}{32}\left[\binom{6}{0}\cos 6x + \binom{6}{1}\cos 4x\binom{6}{2}\cos 2x\right] \\
&= \frac{1}{32}\left[\cos 6x + 6\cos 4x + 15\cos 2x + 10\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos^7 x &= \frac{1}{2^{7-1}}\sum_{k=0}^{\left\lfloor\frac{7}{2}\right\rfloor}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\sum_{k=0}^{2}\binom{7}{k}\cos[(7-2k)x] \\
&= \frac{1}{64}\left[\binom{7}{0}\cos 7x + \binom{7}{1}\cos 5x + \binom{7}{2}\cos 3x + \binom{7}{3}\cos x\right] \\
&= \frac{1}{64}\left[\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right]
\end{aligned}
$$

Note: Trigonometric reduction of $\cos^n(x)$ for particular values of n can also be performed with the free software package *Maxima*™ using the syntax illustrated to the right:[7]

```
trigreduce((cos(x))^2);
trigreduce((cos(x))^3);
trigreduce((cos(x))^4);
trigreduce((cos(x))^5);
trigreduce((cos(x))^6);
trigreduce((cos(x))^7);
```

[7] http://maxima.sourceforge.net/docs/manual/en/maxima_15.html
maxima, page 158 ⟨10.5 Trigonometric Functions⟩

✏

I.3 Spectral Factorization

Theorem I.5 (Fejér-Riesz spectral factorization).[8] *Let* $[0,\infty) \subsetneq \mathbb{R}$ *and*

$$\mathsf{p}\left(e^{ix}\right) \triangleq \sum_{n=-N}^{N} a_n e^{inx} \qquad \textit{(Laurent trigonometric polynomial order } 2N\textit{)}$$

$$\mathsf{q}\left(e^{ix}\right) \triangleq \sum_{n=1}^{N} b_n e^{inx} \qquad \textit{(standard trigonometric polynomial order } N\textit{)}$$

THM $$\mathsf{p}\left(e^{ix}\right) \in [0,\infty) \quad \forall x \in [0,2\pi] \quad \Longrightarrow \quad \begin{cases} \exists \left(\!\left(b_n\right)\!\right)_{n\in\mathbb{Z}} & \text{such that} \\ \mathsf{p}\left(e^{ix}\right) = \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right) & \forall x \in \mathbb{R} \end{cases}$$

✎PROOF:

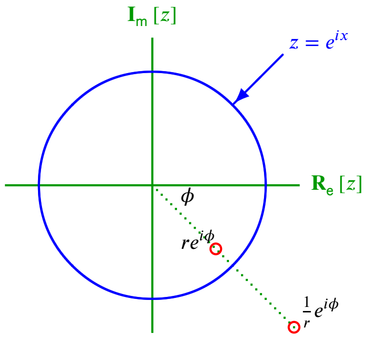
1. Proof that $a_n = a^*_{-n}$ ($\left(\!\left(a_n\right)\!\right)_{n\in\mathbb{Z}}$ is *Hermitian symmetric*):
Let $a_n \triangleq r_n e^{i\phi_n}$, $r_n, \phi_n \in \mathbb{R}$. Then

$$\begin{aligned}
\mathsf{p}\left(e^{inx}\right) &\triangleq \sum_{n=-N}^{N} a_n e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{i\phi_n} e^{inx} \\
&= \sum_{n=-N}^{N} r_n e^{inx+\phi_n} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \sum_{n=-N}^{N} r_n \sin(nx+\phi_n) \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(0x+\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) + \sum_{n=1}^{N} r_{-n} \sin(-nx+\phi_{-n}) \right]}_{\text{imaginary part must equal 0 because } p(x)\in\mathbb{R}} \\
&= \sum_{n=-N}^{N} r_n \cos(nx+\phi_n) + i \underbrace{\left[r_0 \sin(\phi_0) + \sum_{n=1}^{N} r_n \sin(nx+\phi_n) - \sum_{n=1}^{N} r_{-n} \sin(nx-\phi_{-n}) \right]}_{\implies r_n = r_{-n},\ \phi_n = -\phi_{-n} \quad \implies a_n = a^*_{-n},\ a_0\in\mathbb{R}}
\end{aligned}$$

2. Because the coefficients $\left(\!\left(c_n\right)\!\right)_{n\in\mathbb{Z}}$ are *Hermitian symmetric* and by Theorem C.7 (page 150), the zeros of $P(z)$ occur in *conjugate reciprocol pairs*. This means that if $\sigma \in \mathbb{C}$ is a zero of $P(z)$ ($P(\sigma) = 0$), then $\frac{1}{\sigma^*}$ is also a zero of $P(z)$ ($P\left(\frac{1}{\sigma^*}\right) = 0$). In the complex z plane, this relationship means zeros are reflected across the unit circle such that

$$\frac{1}{\sigma^*} = \frac{1}{(re^{i\phi})^*} = \frac{1}{r}\frac{1}{e^{-i\phi}} = \frac{1}{r}e^{i\phi}$$

[8] Pinsky (2002), pages 330–331



3. Because the zeros of $\mathsf{p}(z)$ occur in conjugate recipricol pairs, $\mathsf{p}\left(e^{ix}\right)$ can be factored:

$$\begin{aligned}
\mathsf{p}\left(e^{ix}\right) &= \mathsf{p}(z)|_{z=e^{ix}} \\
&= z^{-N} C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z - \frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N} z^{-1}\left(z - \frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(1 - \frac{1}{\sigma_n^*}z^{-1}\right)\Bigg|_{z=e^{ix}} \\
&= C \prod_{n=1}^{N}(z-\sigma_n) \prod_{n=1}^{N}\left(z^{-1}-\sigma_n^*\right)\left(-\frac{1}{\sigma_n^*}\right)\Bigg|_{z=e^{ix}} \\
&= \left[C \prod_{n=1}^{N}\left(-\frac{1}{\sigma_n^*}\right)\right]\left[\prod_{n=1}^{N}(z-\sigma_n)\right]\left[\prod_{n=1}^{N}\left(\frac{1}{z^*}-\sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \left[C_2 \prod_{n=1}^{N}(z-\sigma_n)\right]\left[C_2 \prod_{n=1}^{N}\left(\frac{1}{z^*}-\sigma_n\right)\right]^*\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}(z)\mathsf{q}^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{ix}} \\
&= \mathsf{q}\left(e^{ix}\right)\mathsf{q}^*\left(e^{ix}\right)
\end{aligned}$$

✏

I.4 Dirichlet Kernel

❝*Dirichlet alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous proof is. Rather we learn it first from him. When Gauss says he has proved something it is clear; when Cauchy says it, one can wager as much pro as con; when Dirichlet says it, it is certain.*❞

Carl Gustav Jacob Jacobi (1804–1851), Jewish-German mathematician [9]

[9] quote: Schubring (2005), page 558
image: http://en.wikipedia.org/wiki/File:Carl_Jacobi.jpg, public domain

The *Dirichlet Kernel* is critical in proving what is not immediately obvious in examining the Fourier Series—that for a broad class of periodic functions, a function can be recovered from (with uniform convergence) its Fourier Series analysis.

Definition I.2. [10]

DEF: *The **Dirichlet Kernel** $\mathsf{D}_n \in \mathbb{R}^{\mathbb{W}}$ with period τ is defined as*

$$\mathsf{D}_n(x) \triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}kx}$$

Proposition I.1. [11] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ *(Definition I.2 page 244).*

PRP:
$$\mathsf{D}_n(x) = \frac{1}{\tau}\frac{\sin\left(\frac{\pi}{\tau}[2n+1]x\right)}{\sin\left(\frac{\pi}{\tau}x\right)}$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &\triangleq \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \mathsf{D}_n \quad \text{(Definition I.2 page 244)}\\
&= \frac{1}{\tau}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}(k-n)x} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\sum_{k=0}^{2n} e^{i\frac{2\pi}{\tau}kx} = \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\sum_{k=0}^{2n}\left(e^{i\frac{2\pi}{\tau}x}\right)^k\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\frac{1-\left(e^{i\frac{2\pi}{\tau}x}\right)^{2n+1}}{1-e^{i\frac{2\pi}{\tau}x}} && \text{by \textit{geometric series}} \quad \text{(Theorem F.7 page 200)}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi}{\tau}nx}\frac{1-e^{i\frac{2\pi}{\tau}(2n+1)x}}{1-e^{i\frac{2\pi}{\tau}x}} = \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(\frac{e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{i\frac{\pi}{\tau}x}}\right)\frac{e^{-i\frac{\pi}{\tau}(2n+1)x}-e^{i\frac{\pi}{\tau}(2n+1)x}}{e^{-i\frac{\pi}{\tau}x}-e^{i\frac{\pi}{\tau}x}}\\
&= \frac{1}{\tau}e^{-i\frac{2\pi n}{\tau}x}\left(e^{i\frac{2\pi n}{\tau}x}\right)\frac{-2i\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{-2i\sin\left[\frac{\pi}{\tau}x\right]} = \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}x\right]}
\end{aligned}$$

✎

Proposition I.2. [12] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ *(Definition I.2 page 244).*

PRP:
$$\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x = 1$$

✎PROOF:

$$\begin{aligned}
\int_0^\tau \mathsf{D}_n(x)\,\mathrm{d}x &\triangleq \int_0^\tau \frac{1}{\tau}\sum_{k=-n}^{n} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \mathsf{D}_n \text{ (Definition I.2 page 244)}\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right) + i\sin\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x
\end{aligned}$$

[10] Katznelson (2004) page 14, Heil (2011) pages 443–444, Folland (1992), pages 33–34
[11] Katznelson (2004) page 14, Heil (2011) page 444, Folland (1992), page 34
[12] Bruckner et al. (1997) pages 620–621

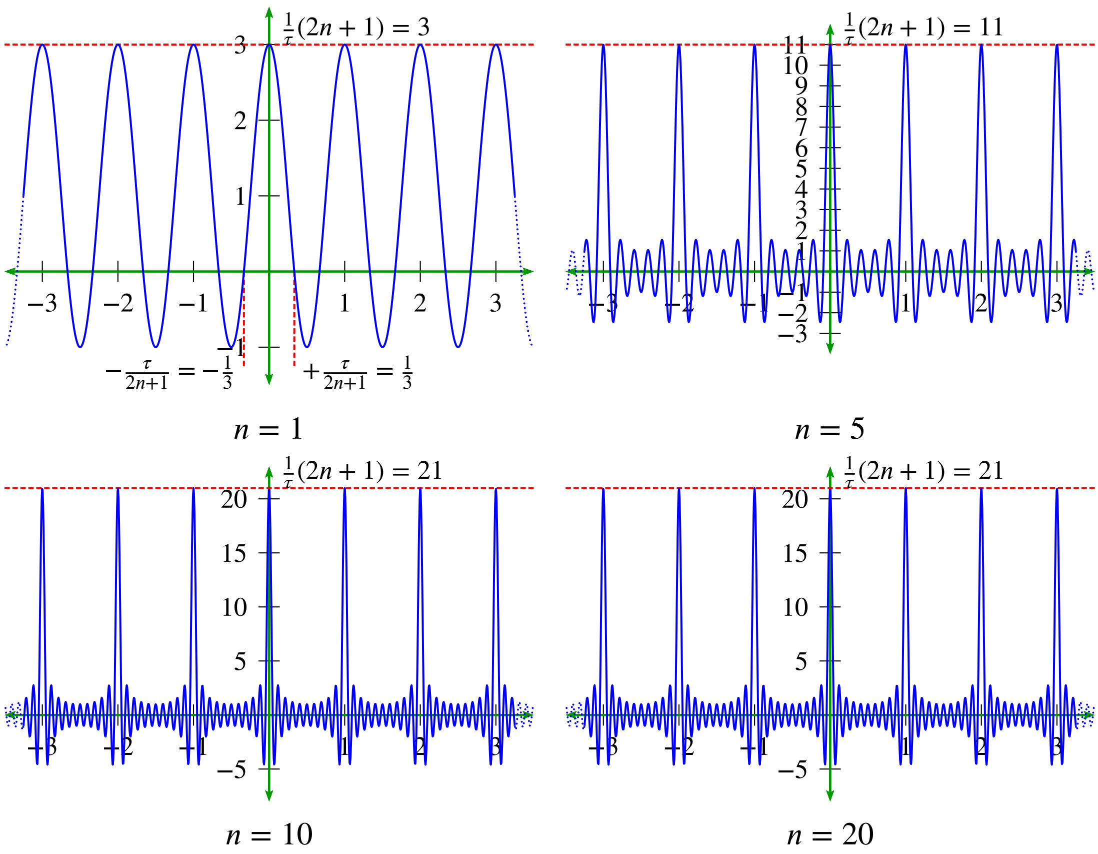


Figure I.4: D_n function for $N = 1, 5, 10, 20$. $\mathsf{D}_n \to$ comb. (See Proposition I.1 page 244).

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{k=-n}^{n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\left(\frac{2\pi}{\tau}nx\right)\,\mathrm{d}x \\
&= \frac{1}{\tau}\sum_{k=-n}^{n} \left.\frac{\sin\left(\frac{2\pi}{\tau}nx\right)}{\frac{2\pi}{\tau}n}\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \frac{1}{\tau}\sum_{k=-n}^{n}\left[\frac{\sin\left(\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n} - \frac{\sin\left(-\frac{2\pi}{\tau}n\frac{\tau}{2}\right)}{\frac{2\pi}{\tau}n}\right] \\
&= \frac{1}{\tau}\frac{\tau}{2}\sum_{k=-n}^{n}\left[\frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi n)}{\pi n}\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n)}{\pi n}\right]_{k=0} \\
&= 1
\end{aligned}
$$

✎

Proposition I.3. *Let D_n be the* Dirichlet Kernel *with period τ* (Definition I.2 page 244). *Let w_N (the "*width*" of $\mathsf{D}_n(x)$) be the distance between the two points where the center pulse of $\mathsf{D}_n(x)$ intersects the x axis.*

PRP

$$
\begin{aligned}
\mathsf{D}_n(0) &= \frac{1}{\tau}(2n+1) \\
w_n &= \frac{2\tau}{2n+1}
\end{aligned}
$$

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(0) &= \mathsf{D}_n(x)\big|_{t=0} \\
&= \frac{1}{\tau}\left.\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by Proposition I.1 page 244} \\
&= \frac{1}{\tau}\left.\frac{\frac{\mathsf{d}}{\mathsf{d}x}\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\frac{\mathsf{d}}{\mathsf{d}x}\sin\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \frac{1}{\tau}\left.\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{\cos\left[\frac{\pi}{\tau}(2n+1)x\right]}{\cos\left[\frac{\pi}{\tau}t\right]}\right|_{t=0} \\
&= \frac{1}{\tau}\frac{\frac{\pi}{\tau}(2n+1)}{\frac{\pi}{\tau}}\frac{1}{1} \\
&= \frac{1}{\tau}(2n+1)
\end{aligned}$$

The center pulse of kernel $\mathsf{D}_n(x)$ intersects the x axis at

$$t = \pm\frac{\tau}{(2n+1)}$$

which implies

$$w_n = \frac{\tau}{2n+1} + \frac{\tau}{2n+1} = \frac{2\tau}{(2n+1)}.$$

✏

Proposition I.4. [13] *Let* D_n *be the* DIRICHLET KERNEL *with period* τ (Definition I.2 page 244).

PRP: $\mathsf{D}_n(x) = \mathsf{D}_n(-x)$ (D_n *is an* EVEN *function*)

✎PROOF:

$$\begin{aligned}
\mathsf{D}_n(x) &= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)x\right]}{\sin\left[\frac{\pi}{\tau}t\right]} && \text{by Proposition I.1 page 244} \\
&= \frac{1}{\tau}\frac{-\sin\left[-\frac{\pi}{\tau}(2n+1)x\right]}{-\sin\left[-\frac{\pi}{\tau}t\right]} && \text{because } \sin x \text{ is an } \textit{odd} \text{ function} \\
&= \frac{1}{\tau}\frac{\sin\left[\frac{\pi}{\tau}(2n+1)(-x)\right]}{\sin\left[\frac{\pi}{\tau}(-x)\right]} \\
&= \mathsf{D}_n(-x) && \text{by Proposition I.1 page 244}
\end{aligned}$$

✏

[13] Bruckner et al. (1997) pages 620–621

I.5 Trigonometric summations

Theorem I.6 (Lagrange trigonometric identities). [14]

T H M

$$\sum_{n=0}^{N-1}\cos(nx) = \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)+\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx) = \frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)+\cos\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=0}^{N-1}\Re e^{inx} = \Re\sum_{n=0}^{N-1}e^{inx} = \Re\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Re\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by \textit{geometric series}} && \text{(Theorem F.7 page 200)}\\
&= \Re\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-i\frac{1}{2}x}-e^{i\frac{1}{2}x}}\right)\right]\\
&= \Re\left[\left(e^{i\frac{1}{2}(N-1)x}\right)\left(\frac{-i\frac{1}{2}\sin\left(\frac{1}{2}Nx\right)}{-i\frac{1}{2}\sin\left(\frac{1}{2}x\right)}\right)\right]\\
&= \cos\left(\frac{1}{2}(N-1)x\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right)\\
&= \frac{-\frac{1}{2}\sin\left(-\frac{1}{2}x\right)+\frac{1}{2}\sin\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by \textit{product identities}} && \text{(Theorem H.8 page 223)}\\
&= \frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=0}^{N-1}\Im e^{inx} = \Im\sum_{n=0}^{N-1}e^{inx} = \Im\sum_{n=0}^{N-1}\left(e^{ix}\right)^n \\
&= \Im\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] && \text{by \textit{geometric series}} && \text{(Theorem F.7 page 200)}\\
&= \Im\left[\left(\frac{e^{i\frac{1}{2}Nx}}{e^{i\frac{1}{2}x}}\right)\left(\frac{e^{-i\frac{1}{2}Nx}-e^{i\frac{1}{2}Nx}}{e^{-ix/2}-e^{i\frac{1}{2}x}}\right)\right]\\
&= \Im\left[\left(e^{i(N-1)x/2}\right)\left(\frac{-\frac{1}{2}i\sin\left(\frac{1}{2}Nx\right)}{-\frac{1}{2}i\sin\left(\frac{1}{2}x\right)}\right)\right]
\end{aligned}$$

[14] Muniz (1953) page 140 ⟨"Lagrange's Trigonometric Identities"⟩, Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (14), (13)⟩

$$
\begin{aligned}
&= \sin\left(\frac{(N-1)x}{2}\right)\left(\frac{\sin\left(\frac{1}{2}Nx\right)}{\sin\left(\frac{1}{2}x\right)}\right) \\
&= \frac{\frac{1}{2}\cos\left(-\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\left[N-\frac{1}{2}\right]x\right)}{\sin\left(\frac{1}{2}x\right)} && \text{by *product identities*} \quad \text{(Theorem H.8 page 223)} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}
\end{aligned}
$$

Note that these results (summed with indices from $n = 0$ to $n = N - 1$) are compatible with Muniz (1953) page 140 (summed with indices from $n = 1$ to $n = N$) as demonstrated next:

$$
\begin{aligned}
\sum_{n=0}^{N-1}\cos(nx) &= \sum_{n=1}^{N}\cos(nx) + [\cos(0x) - \cos(Nx)] \\
&= \left[-\frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] + [\cos(0x) - \cos(Nx)] && \text{by Muniz (1953) page 140} \\
&= \left(1 - \frac{1}{2}\right) + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\cos(Nx)}{2\sin\left(\frac{1}{2}x\right)} \\
&= \frac{1}{2} + \frac{\sin\left(\left[N+\frac{1}{2}\right]x\right) - 2\left[\sin\left(\left[\frac{1}{2}-N\right]x\right) + \sin\left[\left(\frac{1}{2}+N\right)x\right]\right]}{2\sin\left(\frac{1}{2}x\right)} && \text{by Theorem H.8 page 223} \\
&= \frac{1}{2} + \frac{\sin\left(\frac{1}{2}[2N-1]x\right)}{2\sin\left(\frac{1}{2}x\right)} && \Longrightarrow \text{above result} \\
\sum_{n=0}^{N-1}\sin(nx) &= \sum_{n=1}^{N}\sin(nx) + [\sin(0x) - \sin(Nx)] \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} + [0 - \sin(Nx)] && \text{by Muniz (1953) page 140} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - 2\sin\left(\frac{1}{2}x\right)\sin(Nx)}{2\sin\left(\frac{1}{2}x\right)} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) - \frac{\cos\left(\left[N+\frac{1}{2}\right]x\right) - \left[\cos\left(\left[\frac{1}{2}-N\right]x\right) - \cos\left(\left[\frac{1}{2}+N\right]x\right)\right]}{2\sin\left(\frac{1}{2}x\right)} \\
&= \frac{1}{2}\cot\left(\frac{1}{2}x\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)} && \Longrightarrow \text{above result}
\end{aligned}
$$

⇨

Theorem I.7. [15]

[15] Jeffrey and Dai (2008) pages 128–130 ⟨2.4.1.6 Sines, Cosines, and Tagents of Multiple Angles; (16) and (17)⟩

THM

$$\sum_{n=0}^{N-1}\cos(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]-\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

$$\sum_{n=0}^{N-1}\sin(nx+y) = \cos(y)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right]+\sin(y)\left[\frac{1}{2}\cot\left(\frac{1}{2}x\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]x\right)}{2\sin\left(\frac{1}{2}x\right)}\right] \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}\sum_{n=0}^{N-1}\cos(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)-\sin(nx)\sin(y)] && \text{by } \textit{double angle formulas} \text{ (Theorem H.9 page 225)}\\ &= \cos(y)\sum_{n=0}^{N-1}\cos(nx)-\sin(y)\sum_{n=0}^{N-1}\sin(nx)\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{N-1}\sin(nx+y) &= \sum_{n=0}^{N-1}[\cos(nx)\cos(y)+\sin(nx)\sin(y)] && \text{by } \textit{double angle formulas} \text{ (Theorem H.9 page 225)}\\ &= \cos(y)\sum_{n=0}^{N-1}\cos(nx)+\sin(y)\sum_{n=0}^{N-1}\sin(nx)\end{aligned}$$

✎

Corollary I.1 (Summation around unit circle).

THM

$$\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\sin\left(\theta+\frac{2nM\pi}{N}\right)=0 \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

$$\sum_{n=0}^{N-1}\cos^2\left(\theta+\frac{2nM\pi}{N}\right)=\sum_{n=0}^{N-1}\sin^2\left(\theta+\frac{2nM\pi}{N}\right)=\frac{N}{2} \quad \forall\theta\in\mathbb{R}\ \forall M\in\mathbb{N}$$

✎PROOF:

$$\begin{aligned}&\sum_{n=0}^{N-1}\cos\left(\theta+\frac{2nM\pi}{N}\right)\\
&=\cos(\theta)\sum_{n=0}^{N-1}\cos\left(\frac{2nM\pi}{N}\right)-\sin(\theta)\sum_{n=0}^{N-1}\sin\left(\frac{2nM\pi}{N}\right) && \text{by Theorem H.9 page 225}\\
&=\cos(\theta)\left[\frac{1}{2}+\frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{2M\pi}{N}\right)+\frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{2M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] && \text{by Theorem I.6 page 247}\\
&=\cos(\theta)\left[\frac{1}{2}-\frac{\sin\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{\cos\left(\frac{M\pi}{N}-2M\pi\right)}{2\sin\left(\frac{M\pi}{N}\right)}\right]\\
&=\cos(\theta)\left[\frac{1}{2}-\frac{1}{2}\frac{\sin\left(\frac{M\pi}{N}\right)}{\sin\left(\frac{M\pi}{N}\right)}\right]-\sin(\theta)\left[\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)-\frac{1}{2}\cot\left(\frac{M\pi}{N}\right)\right] && \text{by } \textit{trigonometric periodicity} \text{ (Theorem H.10 page 226)}\\
&=\cos(\theta)[0]-\sin(\theta)[0]\\
&=0\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \sin\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by \textit{shift identities}} && \text{(Theorem H.7 page 222)}\\
&= \sum_{n=0}^{N-1} \cos\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2}\\
&= 0 && \text{by previous result}
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{N-1} \cos\left(\theta + \frac{2nM\pi}{N}\right)\sin\left(\theta + \frac{2nM\pi}{N}\right)\\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] - \left[\theta + \frac{2nM\pi}{N}\right]\right) + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(\left[\theta + \frac{2nM\pi}{N}\right] + \left[\theta + \frac{2nM\pi}{N}\right]\right) && \text{by Theorem H.8 page 223}\\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \cancelto{0}{\sin(0)} + \frac{1}{2}\sum_{n=0}^{N-1} \sin\left(2\theta + \frac{4nM\pi}{N}\right)\\
&= \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1} \sin\left(\frac{4nM\pi}{N}\right) && \text{by Theorem H.9 page 225}\\
&= \cos(2\theta)\left[\frac{1}{2} + \frac{\sin\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{1}{2}\frac{4M\pi}{N}\right) + \frac{\cos\left(\left[N-\frac{1}{2}\right]\frac{4M\pi}{N}\right)}{2\sin\left(\frac{1}{2}\frac{4M\pi}{N}\right)}\right] && \text{by Theorem I.6 page 247}\\
&= \cos(2\theta)\left[\frac{1}{2} - \frac{\sin\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(2\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{\cos\left(\frac{2M\pi}{N} - 4M\pi\right)}{2\sin\left(\frac{2M\pi}{N}\right)}\right]\\
&= \cos(\theta)\left[\frac{1}{2} - \frac{1}{2}\frac{\sin\left(\frac{2M\pi}{N}\right)}{\sin\left(\frac{2M\pi}{N}\right)}\right] - \sin(\theta)\left[\frac{1}{2}\cot\left(\frac{2M\pi}{N}\right) - \frac{1}{2}\cot\left(\frac{2M\pi}{N}\right)\right] && \text{by \textit{trigonometric periodicity}} \text{ (Theorem H.10 page 226)}\\
&= \cos(\theta)[0] - \sin(\theta)[0]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2nM\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos\left(2\theta + \frac{4nM\pi}{N}\right)\right] && \text{by Theorem H.11 page 228}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}\left[1 + \cos(2\theta)\cos\left(\frac{4nM\pi}{N}\right) - \sin(2\theta)\sin\left(\frac{4nM\pi}{N}\right)\right] && \text{by Theorem H.9 page 225}\\
&= \frac{1}{2}\sum_{n=0}^{N-1} 1 + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\cos\left(\frac{4nM\pi}{N}\right) - \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin\left(\frac{4nM\pi}{N}\right)\\
&= \left[\frac{1}{2}\sum_{n=0}^{N-1} 1\right] + \frac{1}{2}\cos(2\theta)0 - \frac{1}{2}\sin(2\theta)0 && \text{by previous results}\\
&= \frac{N}{2}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2nM\pi}{N}\right) &= \sum_{n=0}^{N-1} \cos^2\left(\theta - \frac{\pi}{2} + \frac{2nM\pi}{N}\right) && \text{by \textit{shift identities} (Theorem H.7 page 222)} \\
&= \sum_{n=0}^{N-1} \cos^2\left(\phi + \frac{2nM\pi}{N}\right) && \text{where } \phi \triangleq \theta - \frac{\pi}{2} \\
&= \frac{N}{2} && \text{by previous result}
\end{aligned}$$

✏

APPENDIX J

FOURIER SERIES

❝ ...*et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur.*❞


MÉMOIRE
SUR LA
PROPAGATION DE LA CHALEUR
DANS LES CORPS SOLIDES.
PAR M. FOURIER (¹).

L'auteur de ce Mémoire s'est proposé de soumettre la Théorie de la chaleur à l'Analyse mathématique et de vérifier, par l'expérience, les résultats du calcul. Pour exposer l'état de la question, supposons une barre de métal, cylindrique et d'une longueur indéterminée, plongée par une de ses extrémités dans un fluide entretenu à une température constante : la chaleur se répandra successivement dans la barre; et, sans la perte qui a lieu à sa surface et à son autre extrémité, elle prendroit dans toute son étendue la température constante du foyer; mais, à cause de cette perte, la chaleur ne s'étendra d'une manière sensible que jusqu'à une distance du foyer dépendante de la grosseur de la barre, de la conductibilité du métal et de son degré de poli, qui influe sur le rayonnement; de sorte que des thermomètres placés dans l'éten-

❝ ...*and the innovation of the subject, together with its importance, convinced the committee to crown this work. By observing however that the way in which the author arrives at his equations is not free from difficulties, and the analysis of which, to integrate them, still leaves something to be desired, either relative to generality, or even on the side of rigour.*❞

A competition awards committee consisting of the mathematical giants Lagrange, Laplace, Legendre, and others, commenting on Fourier's 1807 landmark paper *Dissertation on the propagation of heat in solid bodies* that introduced the *Fourier Series*. [1]

J.1 Definition

The *Fourier Series* expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

Definition J.1. [2]

DEF *The **Fourier Series operator** $\hat{\mathbf{F}} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{R}}$ is defined as*

$$\left[\hat{\mathbf{F}}\mathsf{f}\right](n) \triangleq \frac{1}{\sqrt{\tau}} \int_0^\tau \mathsf{f}(x) e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x \qquad \forall \mathsf{f} \in \left\{\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}} | \mathsf{f} \textit{ is periodic with period } \tau\right\}$$

[1] quote: Lagrange et al. (1812b), page 374, Lagrange et al. (1812a), page 112, Kahane (2008) page 199
translation: assisted by Google Translate, Castanedo (2005) ⟨chapter 2 footnote 5⟩
paper: Fourier (1807)

[2] Katznelson (2004) page 3

J.2 Inverse Fourier Series operator

Theorem J.1. *Let $\hat{\mathbf{F}}$ be the Fourier Series operator.*

THM: *The **inverse Fourier Series** operator $\hat{\mathbf{F}}^{-1}$ is given by*

$$\left[\hat{\mathbf{F}}^{-1}\left(\!\left(\tilde{x}_n\right)\!\right)_{n\in\mathbb{Z}}\right](x) \triangleq \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}} \tilde{x}_n e^{i\frac{2\pi}{\tau}nx} \qquad \forall \left(\!\left(\tilde{x}_n\right)\!\right)\in\ell^2_{\mathbb{R}}$$

✎PROOF: The proof of the pointwise convergence of the Fourier Series is notoriously difficult. It was conjectured in 1913 by Nokolai Luzin that the Fourier Series for all square summable periodic functions are pointwise convergent: Luzin (1913)

Fifty-three years later (1966) at a conference in Moscow, Lennart Axel Edvard Carleson presented one of the most spectacular results ever in mathematics; he demonstrated that the Luzin conjecture is indeed correct. Carleson formally published his result that same year: Carleson (1966)

Carleson's proof is expounded upon in Reyna's (2002) 175 page book: de Reyna (2002)

Interestingly enough, Carleson started out trying to disprove Luzin's conjecture. Carleson said this in an interview published in 2001:[3]
"...the problem of course presents itself already when you are a student and I was thinking about the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I realised why there should be a counter-example and how you should produce it. I thought I really understood what was the background and then to my amazement I could prove that this "correct" counter-example couldn't exist and I suddenly realised that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so."

For now, if you just want some intuitive justification for the Fourier Series, and you can somehow imagine that the Dirichlet kernel generates a *comb function* of *Dirac delta* functions, then perhaps what follows may help (or not). It is certainly not mathematically rigorous and is by no means a real proof (but at least it is less than 175 pages).

$$\begin{aligned}
\left[\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{x}\right](x) &= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(x)e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}\mathsf{x}} && \text{by definition of } \hat{\mathbf{F}} \text{ Definition J.1 page 253}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}\,\mathrm{d}u\right]e^{i\frac{2\pi}{\tau}nx} && \text{by definition of } \hat{\mathbf{F}}^{-1}\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{-i\frac{2\pi}{\tau}nu}e^{i\frac{2\pi}{\tau}nx}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\tau}}\int_0^\tau \mathsf{x}(u)e^{i\frac{2\pi}{\tau}n(x-u)}\,\mathrm{d}u
\end{aligned}$$

[3] Carleson and Engquist (2001), http://www.gap-system.org/~history/Biographies/Carleson.html

$$
\begin{aligned}
&= \int_0^\tau \mathsf{x}(u) \underbrace{\frac{1}{\tau}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi}{\tau}n(x-u)}}_{\lim_{N\to\infty}\mathsf{D}_n(x)} \, \mathrm{d}u \\
&= \int_0^\tau \mathsf{x}(u)\left[\sum_{n\in\mathbb{Z}} \delta(x-u-n\tau)\right] \mathrm{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{u=0}^{u=\tau} \mathsf{x}(u)\delta(x-u-n\tau)\,\mathrm{d}u \\
&= \sum_{n\in\mathbb{Z}} \int_{v-n\tau=0}^{v-n\tau=\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathrm{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v-n\tau)\delta(x-v)\,\mathrm{d}v && \text{where } v \triangleq u+n\tau \\
&= \sum_{n\in\mathbb{Z}} \int_{v=n\tau}^{v=(n+1)\tau} \mathsf{x}(v)\delta(x-v)\,\mathrm{d}v && \text{because x is periodic with period } \tau \\
&= \int_{\mathbb{R}} \mathsf{x}(v)\delta(x-v)\,\mathrm{d}v \\
&= \mathsf{x}(x) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} \text{ (Definition D.3 page 156)}
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](n) &= \hat{\mathbf{F}}\left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] && \text{by definition of } \hat{\mathbf{F}}^{-1} \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau \left[\frac{1}{\sqrt{\tau}}\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}kx}\right] e^{-i\frac{2\pi}{\tau}nx}\,\mathrm{d}x && \text{by definition of } \hat{\mathbf{F}} \text{ (Definition J.1 page 253)} \\
&= \frac{1}{\tau}\int_0^\tau \left[\sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)e^{i\frac{2\pi}{\tau}(k-n)x}\right] \mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\left[\frac{1}{\tau}\int_0^\tau e^{i\frac{2\pi}{\tau}(k-n)x}\,\mathrm{d}x\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\frac{1}{\tau}\left[\frac{1}{i\frac{2\pi}{\tau}(k-n)}e^{i\frac{2\pi}{\tau}(k-n)x}\right]_0^\tau \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\frac{1}{i2\pi(k-n)}\left[e^{i2\pi(k-n)}-1\right] \\
&= \sum_{k\in\mathbb{Z}} \tilde{\mathsf{x}}(k)\,\bar{\delta}(k-n)\lim_{x\to 0}\left[\frac{e^{i2\pi x}-1}{i2\pi x}\right] \\
&= \tilde{\mathsf{x}}(n)\left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{i2\pi x}-1\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(i2\pi x)}\right|_{x=0} && \text{by } \textit{l'Hôpital's rule} \\
&= \tilde{\mathsf{x}}(n)\left.\frac{i2\pi e^{i2\pi x}}{i2\pi}\right|_{x=0} \\
&= \tilde{\mathsf{x}}(n) \\
&= \mathbf{I}\tilde{\mathsf{x}}(n) && \text{by definition of } \mathbf{I} \text{ (Definition D.3 page 156)}
\end{aligned}
$$

✏

Theorem J.2.

THM *The **Fourier Series adjoint** operator* $\hat{\mathbf{F}}^*$ *is given by*

$$\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}\mathsf{x}(x) \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} &= \left\langle \frac{1}{\sqrt{\tau}} \int_0^{\tau} \mathsf{x}(x) e^{-i\frac{2\pi}{\tau}nx}\, \mathrm{d}x \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} && \text{by definition of } \hat{\mathbf{F}} \text{ Definition J.1 page 253}\\
&= \frac{1}{\sqrt{\tau}} \int_0^{\tau} \mathsf{x}(x) \left\langle e^{-i\frac{2\pi}{\tau}nx} \,|\, \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{Z}} \mathrm{d}x && \text{by additivity property of } \langle \triangle \,|\, \nabla \rangle\\
&= \int_0^{\tau} \mathsf{x}(x) \frac{1}{\sqrt{\tau}} \left\langle \tilde{\mathsf{y}}(n) \,|\, e^{-i\frac{2\pi}{\tau}nx} \right\rangle_{\mathbb{Z}}^* \mathrm{d}x && \text{by property of } \langle \triangle \,|\, \nabla \rangle\\
&= \int_0^{\tau} \mathsf{x}(x) \left[\hat{\mathbf{F}}^{-1} \tilde{\mathsf{y}}(n)\right]^* \mathrm{d}x && \text{by definition of } \hat{\mathbf{F}}^{-1} \text{ page 254}\\
&= \left\langle \mathsf{x}(x) \,|\, \underbrace{\hat{\mathbf{F}}^{-1}}_{\hat{\mathbf{F}}^*} \tilde{\mathsf{y}}(n) \right\rangle_{\mathbb{R}}
\end{aligned}$$

✏

The Fourier Series operator has several nice properties:

- $\hat{\mathbf{F}}$ is *unitary*[4] (Corollary J.1 page 256).
- Because $\hat{\mathbf{F}}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancheral's formula*, and more (Corollary J.2 page 256).

Corollary J.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\hat{\mathbf{F}}$ *be the Fourier Series operator with adjoint* $\hat{\mathbf{F}}^*$.

COR

$$\hat{\mathbf{F}}\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^*\hat{\mathbf{F}} = \mathbf{I} \qquad (\hat{\mathbf{F}} \text{ is unitary...and thus also normal and isometric})$$

✎PROOF: This follows directly from the fact that $\hat{\mathbf{F}}^* = \hat{\mathbf{F}}^{-1}$ (Theorem J.2 (page 255)). ✏

Corollary J.2. *Let* $\hat{\mathbf{F}}$ *be the Fourier series operator,* $\hat{\mathbf{F}}^*$ *be its adjoint, and* $\hat{\mathbf{F}}^{-1}$ *be its inverse.*

COR

$$\begin{array}{rclcll}
\mathcal{R}(\hat{\mathbf{F}}) &=& \mathcal{R}(\hat{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
|||\hat{\mathbf{F}}||| &=& |||\hat{\mathbf{F}}^{-1}||| &=& 1 & (\text{UNITARY})\\
\left\langle \hat{\mathbf{F}}\boldsymbol{x} \,|\, \hat{\mathbf{F}}\boldsymbol{y} \right\rangle &=& \left\langle \hat{\mathbf{F}}^{-1}\boldsymbol{x} \,|\, \hat{\mathbf{F}}^{-1}\boldsymbol{y} \right\rangle &=& \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle & (\text{PARSEVAL'S EQUATION})\\
\|\hat{\mathbf{F}}\boldsymbol{x}\| &=& \|\hat{\mathbf{F}}^{-1}\boldsymbol{x}\| &=& \|\boldsymbol{x}\| & (\text{PLANCHEREL'S FORMULA})\\
\|\hat{\mathbf{F}}\boldsymbol{x} - \hat{\mathbf{F}}\boldsymbol{y}\| &=& \|\hat{\mathbf{F}}^{-1}\boldsymbol{x} - \hat{\mathbf{F}}^{-1}\boldsymbol{y}\| &=& \|\boldsymbol{x} - \boldsymbol{y}\| & (\text{ISOMETRIC})
\end{array}$$

✎PROOF: These results follow directly from the fact that $\hat{\mathbf{F}}$ is unitary (Corollary J.1 page 256) and from the properties of unitary operators (Theorem D.26 page 180). ✏

J.3 Fourier series for compactly supported functions

Theorem J.3.

THM *The set*

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i\frac{2\pi}{\tau}nx} \,\middle|\, n \in \mathbb{Z} \right\}$$

is an ORTHONORMAL BASIS *for all functions* $\mathsf{f}(x)$ *with support in* $[0 : \tau]$.

[4] *unitary operators*: Definition D.14 page 179

APPENDIX K

FOURIER TRANSFORM

❝*The analytical equations . . . extend to all general phenomena. There cannot be a language more universal and more simple, more free from errors and from obscurities, . . . mathematical analysis is as extensive as nature itself; . . . it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them.*❞

Joseph Fourier (1768–1830) [1]

K.1 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on , and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} \middle| \int_{\mathbb{R}} |\mathsf{f}|^2 \, \mathrm{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \mid \nabla \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \mathrm{d}\mu$ such that

$$\langle \mathsf{f} \mid \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, \mathrm{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \mid \nabla \rangle \right)$ is a *Hilbert space*.

Definition K.1. *Let κ be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF | *The function κ is the **Fourier kernel** if* $\kappa(x,\omega) \triangleq e^{i\omega x} \qquad \forall x,\omega \in \mathbb{R}$

Definition K.2. [2] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

---

[1] quote: Fourier (1878), pages 7–8 ⟨Preliminary Discourse⟩
image: http://en.wikipedia.org/wiki/File:Fourier2.jpg, public domain

[2] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965), page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

DEF *The* ***Fourier Transform*** *operator* $\tilde{\mathbf{F}}$ *is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \qquad \forall \mathsf{f} \in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

This definition of the Fourier Transform is also called the ***unitary Fourier Transform****.*

Remark K.1 (**Fourier transform scaling factor**). [3] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B \int_{\mathbb{R}} \mathsf{F}(\omega) e^{i\omega x}\, \mathrm{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x}\, \mathrm{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x}\, \mathrm{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int \cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem K.2 page 258)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

K.2 Operator properties

Theorem K.1 (Inverse Fourier transform). [4] *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator (Definition K.2 page 257). The inverse* $\tilde{\mathbf{F}}^{-1}$ *of* $\tilde{\mathbf{F}}$ *is*

THM
$$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) e^{i\omega x}\, \mathrm{d}\omega \qquad \forall \tilde{\mathsf{f}} \in L^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem K.2. *Let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with inverse* $\tilde{\mathbf{F}}^{-1}$ *and adjoint* $\tilde{\mathbf{F}}^*$*.*

THM
$$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \mathsf{g} \right\rangle &= \left\langle \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \,|\, \mathsf{g}(\omega) \right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 257} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left\langle e^{-i\omega x} \,|\, \mathsf{g}(\omega) \right\rangle \mathrm{d}x && \text{by \textit{additive property} of } \langle \triangle \,|\, \triangledown \rangle \\
&= \int_{\mathbb{R}} \mathsf{f}(x) \frac{1}{\sqrt{2\pi}} \left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle^* \mathrm{d}x && \text{by \textit{conjugate symmetric property} of } \langle \triangle \,|\, \triangledown \rangle
\end{aligned}$$

[3] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005b) pages 374–375
[4] Chorin and Hald (2009) page 13

$$= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2\pi}} \left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle \right\rangle \qquad \text{by definition of } \langle \triangle \,|\, \nabla \rangle$$

$$= \left\langle \mathsf{f} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*} \mathsf{g} \right\rangle \qquad \text{by Theorem K.1 page 258}$$

✏

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary*[5] (Corollary K.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem K.3 page 259).

Corollary K.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

COR
$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem K.2 page 258). ✏

Theorem K.3. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $\left(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \nabla \rangle\right)$. *Let* $\mathcal{R}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

THM
$$\begin{array}{rclcll}
\mathcal{R}(\mathbf{F}\tau) &=& \mathcal{R}(\tilde{\mathbf{F}}^{-1}) &=& \boldsymbol{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| &=& |||\tilde{\mathbf{F}}^{-1}||| &=& 1 & \text{(UNITARY)}\\
\left\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g} \right\rangle &=& \left\langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g} \right\rangle &=& \langle \mathsf{f} \,|\, \mathsf{g} \rangle & \text{(PARSEVAL'S EQUATION)}\\
\left\|\tilde{\mathbf{F}}\mathsf{f}\right\| &=& \left\|\tilde{\mathbf{F}}^{-1}\mathsf{f}\right\| &=& \|\mathsf{f}\| & \text{(PLANCHEREL'S FORMULA)}\\
\left\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\right\| &=& \left\|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\right\| &=& \|\mathsf{f} - \mathsf{g}\| & \text{(ISOMETRIC)}
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary K.1 page 259) and from the properties of unitary operators (Theorem D.26 page 180). ✏

Theorem K.4 (Shift relations)**.** *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator.*

THM
$$\begin{array}{rcl}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) &=& e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega)\\
\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) &=& \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v)
\end{array}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x-u) e^{-i\omega x}\, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257))}\\
&= \frac{1}{\sqrt{2\pi}} \int_{v\in\mathbb{R}} \mathsf{f}(v) e^{-i\omega(u+v)}\, \mathrm{d}v && \text{where } v \triangleq x - u \implies t = u + v\\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{v\in\mathbb{R}} \mathsf{f}(v) e^{-i\omega v}\, \mathrm{d}v\\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x && \text{by change of variable } t = v\\
&= e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257))}
\end{aligned}$$

$$\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ivx}\mathsf{g}(x) e^{-i\omega x}\, \mathrm{d}x \qquad \text{by definition of } \tilde{\mathbf{F}} \qquad \text{(Definition K.2 page 257))}$$

[5] *unitary operators*: Definition D.14 page 179

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{g}(x) e^{-i(\omega-v)x}\, \mathrm{d}x$$

$$= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v) \qquad \text{by definition of } \tilde{\mathbf{F}} \qquad \text{(Definition K.2 page 257)}$$

✎

Theorem K.5 (Complex conjugate). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

THM

$$\begin{array}{rrcll} & \tilde{\mathbf{F}}\mathsf{f}^*(-x) & = & -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* & \forall \mathsf{f} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \\ \mathsf{f} \textit{ is real} \implies & \tilde{\mathsf{f}}(-\omega) & = & \left[\tilde{\mathsf{f}}(\omega)\right]^* & \forall \omega \in \mathbb{R} \qquad \text{REALITY CONDITION} \end{array}$$

✎PROOF:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}} \int \mathsf{f}^*(-x) e^{-i\omega x}\, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)}\\
&= \frac{1}{\sqrt{2\pi}} \int \mathsf{f}^*(u) e^{i\omega u}(-1)\, \mathrm{d}u && \text{where } u \triangleq -x \implies \mathrm{d}x = -\,\mathrm{d}u\\
&= -\left[\frac{1}{\sqrt{2\pi}} \int \mathsf{f}(u) e^{-i\omega u}\, \mathrm{d}u\right]^*\\
&\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)}\\
\tilde{\mathsf{f}}(-\omega) &\triangleq \frac{1}{\sqrt{2\pi}} \int \mathsf{f}(x) e^{-i(-\omega)x}\, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)}\\
&= \left[\frac{1}{\sqrt{2\pi}} \int \mathsf{f}^*(x) e^{-i\omega x}\, \mathrm{d}x\right]^*\\
&= \left[\frac{1}{\sqrt{2\pi}} \int \mathsf{f}(x) e^{-i\omega x}\, \mathrm{d}x\right]^* && \text{by } \mathsf{f} \text{ is real hypothesis}\\
&\triangleq \tilde{\mathsf{f}}^*(\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)}
\end{aligned}$$

✎

K.3 Convolution

Definition K.3. [6]

DEF

*The **convolution operation** is defined as*

$$\left[\mathsf{f} \star \mathsf{g}\right](x) \triangleq \mathsf{f}(x) \star \mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\, \mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g} \in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem O.2 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem K.6 (convolution theorem). [7] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $\star$ the convolution operator.*

[6] Bachman (1964), page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩
[7] Bracewell (1978) page 110

THM

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g} \in \mathscr{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

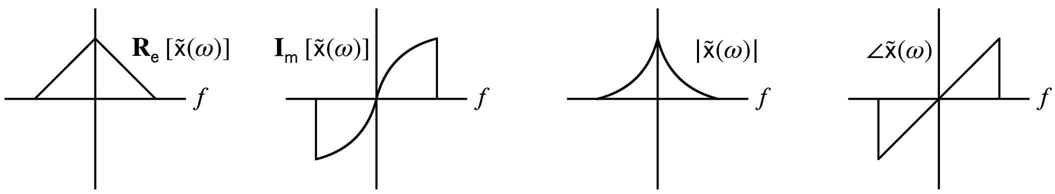
$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g} \in \mathscr{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x) \star \mathsf{g}(x)\right](\omega) &= \tilde{\mathbf{F}}\left[\int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathsf{d}u\right](\omega) && \text{by definition of } \star \text{ (Definition K.3 page 260)} \\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\tilde{\mathbf{F}}\mathsf{g}(x-u)\right](\omega)\,\mathsf{d}u \\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega)\,\mathsf{d}u && \text{by Theorem K.4 page 259} \\
&= \sqrt{2\pi}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\omega u}\,\mathsf{d}u\right)}_{[\tilde{\mathbf{F}}\mathsf{f}](\omega)}\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) \\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition K.2 page 257)} \\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \tilde{\mathbf{F}}\left[\left(\tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathsf{f}(x)\right)\mathsf{g}(x)\right](\omega) && \text{by definition of operator inverse (page 156)} \\
&= \tilde{\mathbf{F}}\left[\left(\frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)e^{ivx}\,\mathsf{d}v\right)\mathsf{g}(x)\right](\omega) && \text{by Theorem K.1 page 258} \\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\left(e^{ivx}\,\mathsf{g}(x)\right)\right](\omega,v)\,\mathsf{d}v \\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-v)\,\mathsf{d}v && \text{by Theorem K.4 page 259} \\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \star \left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \star \text{ (Definition K.3 page 260)}
\end{aligned}$$

✏

K.4 Real valued functions



(symmetric) (anti-symmetric) (symmetric) (anti-symmetric)

Figure K.1: Fourier transform components of real-valued signal

Theorem K.7. *Let* $\mathsf{f}(x)$ *be a function in* $\mathscr{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM

$$\left\{\begin{array}{l}\mathsf{f}(x) \text{ is REAL-VALUED} \\ (\mathsf{f} \in \mathbb{R}^{\mathbb{R}})\end{array}\right\} \implies \left\{\begin{array}{rcll}
\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & \text{(HERMITIAN SYMMETRIC)} \\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(SYMMETRIC)} \\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(ANTI-SYMMETRIC)} \\
\left|\tilde{\mathsf{f}}(\omega)\right| &=& \left|\tilde{\mathsf{f}}(-\omega)\right| & \text{(SYMMETRIC)} \\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & \text{(ANTI-SYMMETRIC)}.
\end{array}\right\}$$

✎PROOF:

$$\begin{array}{lllllll}
\tilde{\mathsf{f}}(\omega) & \triangleq & [\tilde{\mathbf{F}}\mathsf{f}(x)](\omega) & \triangleq & \langle \mathsf{f}(x) \,|\, e^{i\omega x} \rangle & = & \langle \mathsf{f}(x) \,|\, e^{i(-\omega)x} \rangle^* & \triangleq & \tilde{\mathsf{f}}^*(-\omega) \\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] & = & \mathbf{R_e}\left[\tilde{\mathsf{f}}^*(-\omega)\right] & = & \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] & = & \mathbf{I_m}\left[\tilde{\mathsf{f}}^*(-\omega)\right] & = & -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] \\
|\tilde{\mathsf{f}}(\omega)| & = & |\tilde{\mathsf{f}}^*(-\omega)| & = & |\tilde{\mathsf{f}}(-\omega)| \\
\angle\tilde{\mathsf{f}}(\omega) & = & \angle\tilde{\mathsf{f}}^*(-\omega) & = & -\angle\tilde{\mathsf{f}}(-\omega)
\end{array}$$

✏

K.5 Moment properties

Definition K.4. [8]

DEF

*The quantity M_n is the n**th moment** of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\,\mathrm{d}x \qquad \text{for } n \in \mathbb{W}.$$

Lemma K.1. [9] *Let M_n be the nTH MOMENT (Definition K.4 page 262) and $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ the FOURIER TRANSFORM (Definition K.2 page 257) of a function $\mathsf{f}(x)$ in $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185).*

LEM

$$\begin{array}{rcll}
\mathsf{M}_n & = & \sqrt{2\pi}(i)^n \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} & \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \\
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} & = & \frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n & \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}
\end{array}$$

✎PROOF:

$$\begin{aligned}
\sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition K.2 page 257)} \\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\right] \mathrm{d}x \Big|_{\omega=0} \\
&= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathrm{d}x \Big|_{\omega=0} \\
&= (-i^2)^n \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\
&\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \text{ (Definition K.4 page 262)}
\end{aligned}$$

✏

Lemma K.2. [10] *Let M_n be the nTH MOMENT (Definition K.4 page 262) and $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ the FOURIER TRANSFORM (Definition K.2 page 257) of a function $\mathsf{f}(x)$ in $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185).*

LEM

$$\mathsf{M}_n = 0 \quad \Longleftrightarrow \quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

[8] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83

[9] Goswami and Chan (1999), pages 38–39

[10] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

1. Proof for ($\implies$) case:

$$\begin{aligned}
0 &= \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by left hypothesis} \\
&= \sqrt{2\pi}(-i)^{-n} \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by Lemma K.1 page 262} \\
\implies & \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} = 0
\end{aligned}$$

2. Proof for ($\impliedby$) case:

$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by right hypothesis} \\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x} \,\mathsf{d}x \bigg|_{\omega=0} && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega x} \,\mathsf{d}x \bigg|_{\omega=0} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \,\mathsf{d}x \bigg|_{\omega=0} \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) x^n \,\mathsf{d}x \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by definition of } \langle \cdot \,|\, \cdot \rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \text{ (Definition E.1 page 185)}
\end{aligned}$$

✏

Lemma K.3 (Strang-Fix condition).[11] *Let $\mathsf{f}(x)$ be a function in $\boldsymbol{L}^2_{\mathbb{R}}$ and M_n the nTH MOMENT (Definition K.4 page 262) of $\mathsf{f}(x)$. Let $\mathbf{T}$ be the TRANSLATION OPERATOR (Definition 1.3 page 2).*

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "time"}} \iff \underbrace{\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n \bar{\delta}_k \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "frequency"}}$$

✎PROOF:

1. Proof for ($\implies$) case:

$$\begin{aligned}
\left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi k x} \bar{\delta}_k \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x} \,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi k x} \bar{\delta}_k && \text{by Definition K.2 page 257} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x} \,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi k x} \bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x} \,\mathsf{d}x\right]_{\omega=2\pi k} e^{i2\pi k x} \bar{\delta}_k \\
&= (-i)^n \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) \bar{\delta}_k && \text{by } \textit{PSF} \text{ (Theorem 1.2 page 10)} \\
&= \frac{1}{\sqrt{2\pi}} (-i)^n \bar{\delta}_k \mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

[11] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83, Mallat (1999), pages 241–243, Fix and Strang (1969)

2. Proof for ($\Longleftarrow$) case:

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left[(-i)^n \bar{\delta}_k \mathsf{M}_n\right] e^{-i2\pi kx} && \text{by definition of } \bar{\delta} && \text{(Definition L.12 page 278)}\\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\Bigg|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis}\\
&= \sum_{k\in\mathbb{Z}} \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]\Bigg|_{\omega=2\pi k} e^{-i2\pi kx}\\
&= \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathsf{d}x\right]\Bigg|_{\omega=2\pi k} e^{-i2\pi kx}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathsf{d}x\right]\Bigg|_{\omega=2\pi k} e^{-i2\pi kx}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } \textit{PSF} && \text{(Theorem 1.2 page 10)}
\end{aligned}
$$

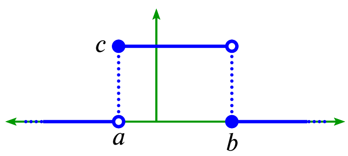
⇨

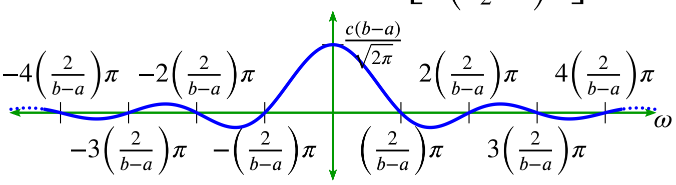
K.6 Examples

Example K.1 (rectangular pulse). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$
\mathsf{f}(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases}
\qquad
\tilde{\mathsf{f}}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right]
$$





✎PROOF:

$$
\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega)\\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[\mathsf{f}\left(x-\frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} && \text{(Theorem K.4 page 259)}\\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x-\frac{a+b}{2}\right)\right](\omega) && \text{by definition of } \mathsf{f}(x)\\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 1)}\\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x) e^{-i\omega x}\,\mathsf{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition K.2 page 257)}\\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} c e^{-i\omega x}\,\mathsf{d}x && \text{by definition of } \mathbb{1} && \text{(Definition 1.2 page 1)}\\
&= \frac{c}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \frac{1}{-i\omega} e^{-i\omega x}\Bigg|_{-\frac{b-a}{2}}^{\frac{b-a}{2}}\\
&= \frac{2c}{\sqrt{2\pi}\omega} e^{-i\left(\frac{a+b}{2}\right)\omega} \left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i}\right]
\end{aligned}
$$

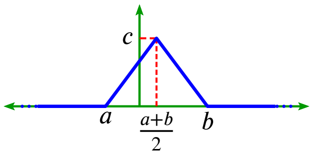
$$= \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right] \qquad \text{by } \textit{Euler formulas} \qquad \text{(Corollary H.2 page 221)}$$

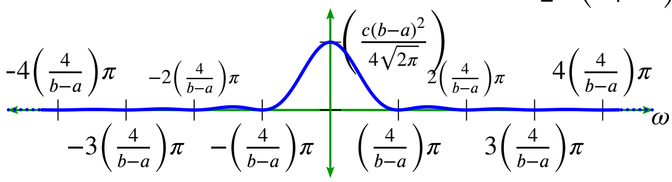
✏

Example K.2 (triangle). Let $\tilde{f}(\omega)$ be the *Fourier transform* of a function $f(x) \in L^2_{\mathbb{R}}$.

EX

$$f(x) = \begin{cases} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$





✎PROOF:

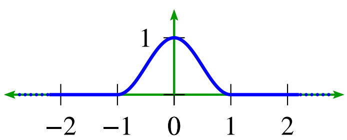
$$\begin{aligned}
\tilde{f}(\omega) &= \tilde{\mathbf{F}}[f(x)](\omega) && \text{by definition of } \tilde{f}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[f\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} \quad \text{(Theorem K.4 page 259)} \\
&= \tilde{\mathbf{F}}\left[c\left(1 - \frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) && \text{by definition of } f(x) \\
&= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x) \star \mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega) \\
&= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] && \text{by } \textit{convolution theorem} \quad \text{(Theorem O.2 page 344)} \\
&= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2 \\
&= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 && \text{by } \textit{Rectangular pulse} \text{ ex.} \quad \text{Example K.1 page 264} \\
&= \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2
\end{aligned}$$

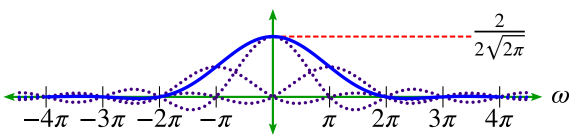
✏

Example K.3. Let a function f be defined in terms of the cosine function (Definition H.2 page 215) as follows:

EX

$$f(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\text{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\text{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\text{sinc}(\omega-\pi)}\right]$$





✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 1) on a set A.

$$\begin{aligned}
\tilde{\mathsf{f}}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x} \,\mathrm{d}x && \text{by definition of } \tilde{\mathsf{f}}(\omega) \text{ (Definition K.2)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos^2\left(\frac{\pi}{2}x\right) \mathbb{1}_{[-1:1]}(x) e^{-i\omega x} \,\mathrm{d}x && \text{by definition of } \mathsf{f}(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right) e^{-i\omega x} \,\mathrm{d}x && \text{by definition of } \mathbb{1} \text{ (Definition 1.2)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left[\frac{e^{i\frac{\pi}{2}x} + e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x} \,\mathrm{d}x && \text{by Corollary H.2 page 221} \\
&= \frac{1}{4\sqrt{2\pi}} \int_{-1}^{1} \left[2 + e^{i\pi x} + e^{-i\pi x}\right] e^{-i\omega x} \,\mathrm{d}x \\
&= \frac{1}{4\sqrt{2\pi}} \int_{-1}^{1} 2e^{-i\omega x} + e^{-i(\omega+\pi)x} + e^{-i(\omega-\pi)x} \,\mathrm{d}x \\
&= \frac{1}{4\sqrt{2\pi}} \left[2\frac{e^{-i\omega x}}{-i\omega} + \frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)} + \frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1} \\
&= \frac{1}{2\sqrt{2\pi}} \left[2\frac{e^{-i\omega} - e^{+i\omega}}{-2i\omega} + \frac{e^{-i(\omega+\pi)} - e^{+i(\omega+\pi)}}{-2i(\omega+\pi)} + \frac{e^{-i(\omega-\pi)} - e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1} \\
&= \frac{1}{2\sqrt{2\pi}} \Bigg[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\Bigg]
\end{aligned}$$

✏

APPENDIX L

LINEAR COMBINATIONS

L.1 Linear combinations in linear spaces

A *linear space* (Definition D.1 page 155) in general is not equipped with a *topology*. Without a topology, it is not possible to determine whether an *infinite sum* (Definition G.1 page 204) of vectors converges. Therefore in this section (dealing with linear spaces), all definitions related to sums of vectors will be valid for *finite* sums(Definition F.1 page 189) only (finite "N").

Definition L.1. [1] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF: *A vector* $\boldsymbol{x} \in X$ *is a* ***linear combination*** *of the vectors in* $\{\boldsymbol{x}_n\}$ *if there exists* $\{\alpha_n \in \mathbb{F} \mid n=1,2,\ldots,N\}$ *such that* $\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$.

Definition L.2. [2] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space and* Y *be a subset of* X.

DEF: *The* ***linear span*** *of* Y *is defined as* $\operatorname{span} Y \triangleq \left\{ \sum_{\gamma\in\Gamma} \alpha_\gamma \boldsymbol{y}_\gamma \,\middle|\, \alpha_\gamma \in \mathbb{F},\ \boldsymbol{y}_\gamma \in Y \right\}$.
The set Y ***spans*** *a set* A *if* $A \subseteq \operatorname{span} Y$.

Proposition L.1. [3] *Let* $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

PRP:
1. $\operatorname{span}\{\boldsymbol{x}_n\}$ *is a* LINEAR SPACE (Definition D.1 page 155) *and*
2. $\operatorname{span}\{\boldsymbol{x}_n\}$ *is a* LINEAR SUBSPACE *of* $\boldsymbol{L}$.

Definition L.3. [4] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

DEF: *The set* $Y \triangleq \{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ *is* ***linearly independent*** *in* $\boldsymbol{L}$ *if*
$$\left\{ \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = 0 \right\} \implies \{\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0\}.$$
The set Y *is* ***linearly dependent*** *in* $\boldsymbol{L}$ *if* Y *is not linearly independent in* $\boldsymbol{L}$.

[1] Berberian (1961) page 11 ⟨Definition I.4.1⟩, Kubrusly (2001) page 46
[2] Michel and Herget (1993) page 86 ⟨3.3.7 Definition⟩, Kurdila and Zabarankin (2005) page 44, Searcóid (2002) page 71 ⟨Definition 3.2.5—more general definition⟩
[3] Kubrusly (2001) page 46
[4] Bachman and Narici (1966) pages 3–4, Christensen (2003) page 2, Heil (2011) page 156 ⟨Definition 5.7⟩

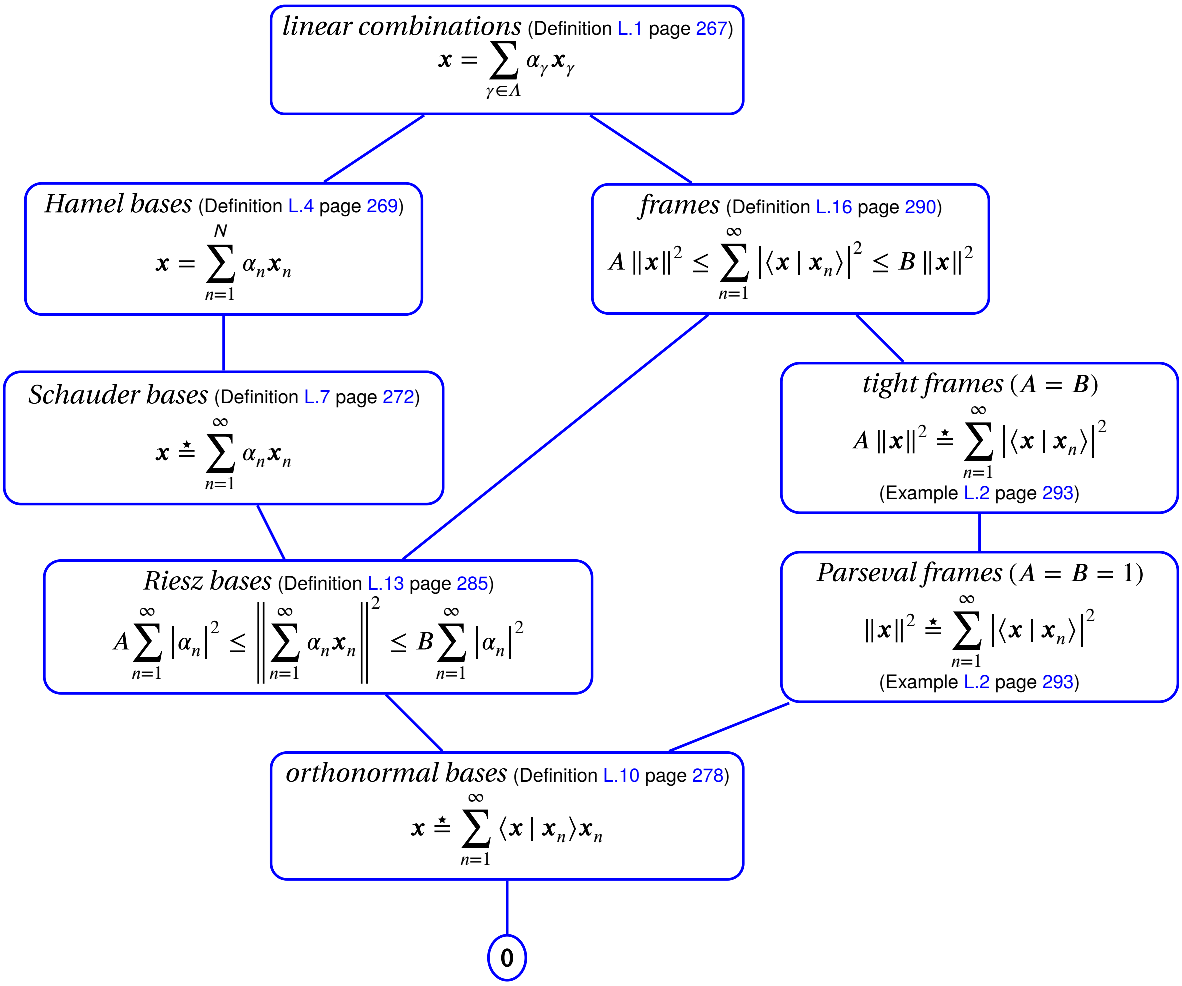


Figure L.1: Lattice of *linear combinations*

Definition L.4. [5] *Let* $\{x_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in a* LINEAR SPACE $L \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

DEF

The set $\{x_n\}$ *is a* ***Hamel basis*** *for* L *if*

1. $\{x_n\}$ SPANS L (Definition L.2 page 267) *and*
2. $\{x_n\}$ *is* LINEARLY INDEPENDENT *in* L (Definition L.1 page 267) .

A HAMEL BASIS *is also called a* ***linear basis***.

Definition L.5. [6] *Let* $L \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE. *Let* x *be a* VECTOR *in* L *and* $Y \triangleq \{x_n \in X \mid n=1,2,\ldots,N\}$ *be a set of vectors in* L.

DEF

The expression $\sum_{n=1}^{N} \alpha_n x_n$ *is the* ***expansion*** *of* x *on* Y *in* L *if* $x = \sum_{n=1}^{N} \alpha_n x_n$.

In this case, the sequence $(\alpha_n)_{n=1}^{N}$ *is the* ***coordinates*** *of* x *with respect to* Y *in* L.

If $\alpha_N \neq 0$, *then* N *is the* ***dimension*** $\dim L$ *of* L.

Theorem L.1. [7] *Let* $\{x_n \mid n=1,2,\ldots,N\}$ *be a* HAMEL BASIS (Definition L.4 page 269) *for a* LINEAR SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$.

THM

$$\left\{ x = \sum_{n=1}^{N} \alpha_n x_n = \sum_{n=1}^{N} \beta_n x_n \right\} \implies \underbrace{\alpha_n = \beta_n \quad \forall n=1,2,\ldots,N}_{\text{coordinates of } x \text{ are UNIQUE}} \qquad \forall x \in X$$

✎PROOF:

$$\begin{aligned}
\mathbb{0} &= x - x \\
&= \sum_{n=1}^{N} \alpha_n x_n - \sum_{n=1}^{N} \beta_n x_n \\
&= \sum_{n=1}^{N} (\alpha_n - \beta_n) x_n \\
&\implies \{x_n\} \text{ is } \textit{linearly dependent} \text{ if } (\alpha_n - \beta_n) \neq 0 \qquad \forall n = 1,2,\ldots,N \\
&\implies (\alpha_n - \beta_n) = 0 \qquad \forall n = 1,2,\ldots,N \qquad \text{(because } \{x_n\} \text{ is a } \textit{basis} \text{ and therefore must be } \textit{linearly independent}\text{)} \\
&\implies \alpha_n = \beta_n \text{ for } n = 1,2,\ldots,N
\end{aligned}$$

✏

Theorem L.2. [8] *Let* $L \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a* LINEAR SPACE.

THM

$$\left\{ \begin{array}{ll} 1. & \{x_n \in X \mid n=1,2,\ldots,N\} \text{ is a HAMEL BASIS for } L \quad \text{and} \\ 2. & \{y_n \in X \mid n=1,2,\ldots,M\} \text{ is a set of LINEARLY INDEPENDENT vectors in } L \end{array} \right\}$$
$$\implies \left\{ \begin{array}{lll} 1. & M \leq N & \text{and} \\ 2. & M = N \implies \{y_n \mid n=1,2,\ldots,M\} \text{ is a BASIS for } L & \text{and} \\ 3. & M \neq N \implies \{y_n \mid n=1,2,\ldots,M\} \text{ is NOT a basis for } L & \end{array} \right\}$$

✎PROOF:

1. Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ is a *basis* for L:

[5] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[6] Hamel (1905), Bachman and Narici (1966) page 4, Kubrusly (2001) pages 48–49 ⟨Section 2.4⟩, Young (2001) page 1, Carothers (2005) page 25, Heil (2011) page 125 ⟨Definition 4.1⟩

[7] Michel and Herget (1993) pages 89–90 ⟨Theorem 3.3.25⟩

[8] Michel and Herget (1993) pages 90–91 ⟨Theorem 3.3.26⟩

(a) Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ *spans* $\mathbf{L}$:

i. Because $\{x_n | n=1,2,\ldots,N\}$ is a *basis* for $\mathbf{L}$, there exists $\beta \in \mathbb{F}$ and $\{\alpha_n \in \mathbb{F} | n=1,2,\ldots,N\}$ such that $\beta y_1 + \sum_{n=1}^{N} \alpha_n x_n = 0.$

ii. Select an n such that $\alpha_n \neq 0$ and renumber (if necessary) the above indices such that $x_n = -\frac{\beta}{\alpha_n} y_1 + \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n} x_n.$

iii. Then, for any $y \in X$, we can write

$$\begin{aligned}
y &= \sum_{n=1}^{N} \gamma_{n\in\mathbb{Z}} x_n \\
&= \left(\sum_{n=1}^{N-1} \gamma_{n\in\mathbb{Z}} x_n\right) + \gamma_{n\in\mathbb{Z}}\left(-\frac{\beta}{\alpha_n} y_1 - \sum_{n=1}^{N-1} \frac{\alpha_n}{\alpha_n} x_n\right) \\
&= -\frac{\beta\gamma_n}{\alpha_n} y_1 + \sum_{n=1}^{N-1}\left(\gamma_n - \frac{\alpha_n\gamma_n}{\alpha_n}\right) x_n \\
&= \delta y_1 + \sum_{n=1}^{N-1} \delta_{n\in\mathbb{Z}} x_n
\end{aligned}$$

iv. This implies that $\{y_1, x_1, \ldots, x_{N-1}\}$ *spans* $\mathbf{L}$:

(b) Proof that $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly independent*:

i. If $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly dependent*, then there exists $\{\epsilon, \epsilon_1, \ldots, \epsilon_{N-1}\}$ such that $\epsilon y_1 + \left(\sum_{n=1}^{N-1} \epsilon_{n\in\mathbb{Z}} x_n\right) + 0x_n = 0.$

ii. item (1(b)i) implies that the coordinate of y_1 associated with x_n *is* 0. $y_1 = -\left(\sum_{n=1}^{N-1} \frac{\epsilon_n}{\epsilon} x_n\right) + 0x_n = 0.$

iii. item (1(a)i) implies that the coordinate of y_1 associated with x_n is *not* 0. $y_1 = -\sum_{n=1}^{N} \frac{\alpha_n}{\beta} x_n.$

iv. This implies that item (1(b)i) (that the set is linearly dependent) is *false* because item (1(b)ii) and item (1(b)iii) *contradict* each other.

v. This implies $\{y_1, x_1, \ldots, x_{N-1}\}$ is *linearly independent*.

2. Proof that $\{y_1, y_2, x_1, \ldots, x_{N-2}\}$ is a *basis*: Repeat item (1).

3. Suppose $m = n$. Proof that $\{y_1, y_2, \ldots, y_M\}$ is a *basis*: Repeat item (1) $M-1$ times.

4. Proof that $M \not> N$:

(a) Suppose that $M = N + 1$.

(b) Then because $\{y_n | n=1,2,\ldots,N\}$ is a *basis*, there exists $\{\zeta_n | n=1,2,\ldots,N+1\}$ such that $\sum_{n=1}^{N+1} \zeta_{n\in\mathbb{Z}} y_{n\in\mathbb{Z}} = 0.$

(c) This implies that $\{y_n | n=1,2,\ldots,N+1\}$ is *linearly dependent*.

(d) This implies that $\{y_n | n=1,2,\ldots,N+1\}$ is *not* a basis.

(e) This implies that $M \not> N$.

5. Proof that $M \neq N \implies \{y_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$:

   (a) Proof that $M > N \implies \{y_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$: same as in item (4).

   (b) Proof that $M < N \implies \{y_n|_{n=1,2,\ldots,M}\}$ is *not* a basis for $\boldsymbol{L}$:

      i. Suppose $m = N - 1$.

      ii. Then $\{y_n|_{n=1,2,\ldots,N-1}\}$ is a *basis* and there exists λ such that
      $$\sum_{n=1}^{N} \lambda_{n\in\mathbb{Z}} y_{n\in\mathbb{Z}} = 0.$$

      iii. This implies that $\{y_n|_{n=1,2,\ldots,N}\}$ is *linearly dependent* and is *not* a basis.

      iv. But this contradicts item (3), therefore $M \neq N - 1$.

      v. Because $M = N$ yields a basis but $M = N - 1$ does not, $M < N - 1$ also does not yield a basis.

✎

Corollary L.1. [9] *Let* $\boldsymbol{L} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space.*

COR
$$\underbrace{\left\{\begin{array}{ll} 1. & \{x_n \in X|_{n=1,2,\ldots,N}\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \quad \text{and} \\ 2. & \{y_n \in X|_{n=1,2,\ldots,M}\} \text{ is a HAMEL BASIS for } \boldsymbol{L} \end{array}\right\} \implies \{N = M\}}_{\text{(all Hamel bases for } \boldsymbol{L} \text{ have the same number of vectors)}}$$

✎PROOF: This follows from Theorem L.2 (page 269). ✎

L.2 Bases in topological linear spaces

A linear space supports the concept of the *span* of a set of vectors (Definition L.2 page 267). In a topological linear space $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$, a set A is said to be *total* in $\boldsymbol{\Omega}$ if the span of A is *dense* in $\boldsymbol{\Omega}$. In this case, A is said to be a *total set* or a *complete set*. However, this use of "complete" in a "*complete set*" is not equivalent to the use of "complete" in a "*complete metric space*" .[10] In this text, except for these comments and Definition L.6, "complete" refers to the metric space definition only.

If a set is both *total* and *linearly independent* (Definition L.3 page 267) in $\boldsymbol{\Omega}$, then that set is a *Hamel basis* (Definition L.4 page 269) for $\boldsymbol{\Omega}$.

Definition L.6. [11] *Let A^- be the* CLOSURE *of a A in a* TOPOLOGICAL LINEAR SPACE $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \boldsymbol{T})$. *Let* $\operatorname{span} A$ *be the* SPAN *(Definition L.2 page 267) of a set A.*

DEF
*A set of vectors A is **total** (or **complete** or **fundamental**) in $\boldsymbol{\Omega}$ if*
$$(\operatorname{span} A)^- = \boldsymbol{\Omega} \qquad (\text{SPAN of } A \text{ is DENSE in } \boldsymbol{\Omega}).$$

[9] Kubrusly (2001) page 52 ⟨Theorem 2.7⟩, Michel and Herget (1993) page 91 ⟨Theorem 3.3.31⟩

[10] Haaser and Sullivan (1991) pages 296–297 ⟨6·Orthogonal Bases⟩, Rynne and Youngson (2008) page 78 ⟨Remark 3.50⟩, Heil (2011) page 21 ⟨Remark 1.26⟩

[11] Young (2001) page 19 ⟨Definition 1.5.1⟩, Sohrab (2003) page 362 ⟨Definition 9.2.3⟩, Gupta (1998) page 134 ⟨Definition 2.4⟩, Bachman and Narici (1966) pages 149–153 ⟨Definition 9.3, Theorems 9.9 and 9.10⟩

L.3 Schauder bases in Banach spaces

Definition L.7. [12] *Let* $\boldsymbol{B} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* BANACH SPACE. *Let* $\stackrel{\star}{=}$ *represent* STRONG CONVERGENCE *(Definition G.5 page 211) in* $\boldsymbol{B}$.

DEF

The countable set $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ *is a* ***Schauder basis*** *for* $\boldsymbol{B}$ *if for each* $\boldsymbol{x} \in X$

1. $\exists (\alpha_n \in \mathbb{F})_{n\in\mathbb{N}}$ such that $\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ (STRONG CONVERGENCE *in* $\boldsymbol{B}$) *and*
2. $\left\{\sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \stackrel{\star}{=} \sum_{n=1}^{\infty} \beta_n \boldsymbol{x}_n\right\} \implies \{(\alpha_n) = (\beta_n)\}$ (COEFFICIENT FUNCTIONALS *are* UNIQUE)

In this case, $\sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n$ *is the* ***expansion*** *of* $\boldsymbol{x}$ *on* $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ *and*
the elements of (α_n) *are the* ***coefficient functionals*** *associated with the basis* $\{\boldsymbol{x}_n\}$. *Coefficient functionals are also called* ***coordinate functionals***.

In a Banach space, the existence of a Schauder basis implies that the space is *separable* (Theorem L.3 page 272). The question of whether the converse is also true was posed by Banach himself in 1932,[13] and became know as "*The basis problem*". This remained an open question for many years. The question was finally answered some 41 years later in 1973 by Per Enflo (University of California at Berkley), with the answer being "no". Enflo constructed a counterexample in which a separable Banach space does *not* have a Schauder basis.[14] Life is simpler in Hilbert spaces where the converse *is* true: a Hilbert space has a Schauder basis *if and only if* it is separable (Theorem L.11 page 285).

Theorem L.3. [15] *Let* $\boldsymbol{B} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be a* BANACH SPACE. *Let* $\mathbb{Q}$ *be the field of rational numbers.*

THM

$$\left\{\begin{array}{ll} 1. & \boldsymbol{B} \text{ has a SCHAUDER BASIS and} \\ 2. & \mathbb{Q} \text{ is DENSE in } \mathbb{F}. \end{array}\right\} \implies \{\boldsymbol{B} \text{ is SEPARABLE}\}$$

✎PROOF:

1. lemma:

$$\begin{aligned}
\left|\left\{\boldsymbol{x} | \exists (\alpha_n \mathbb{Q})_{n\in\mathbb{N}} \text{ such that } \lim_{N\to\infty}\left\|\boldsymbol{x} - \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right\| = 0\right\}\right| &= |\mathbb{Q}\times\mathbb{N}| \\
&= |\mathbb{Z}\times\mathbb{Z}| \\
&= |\mathbb{Z}| \\
&= \textit{countably infinite}
\end{aligned}$$

---

[12] Carothers (2005) pages 24–25, Christensen (2003) pages 46–49 ⟨Definition 3.1.1 and page 49⟩, Young (2001) page 19 ⟨Section 6⟩, Singer (1970), page 17, Schauder (1927), Schauder (1928)
[13] Banach (1932a), page 111
[14] Enflo (1973), Lindenstrauss and Tzafriri (1977) pages 84–95 ⟨Section 2.d⟩
[15] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Giles (2000) page 17, Heil (2011) page 21 ⟨Theorem 1.27⟩

2. remainder of proof:

$\boldsymbol{B}$ has a *Schauder basis* $(\!(\boldsymbol{x}_n)\!)_{n\in\mathbb{N}}$

$\Longrightarrow$ for every $\boldsymbol{x}\in\boldsymbol{B}$, there exists $(\!(\alpha_n\in\mathbb{F})\!)_{n\in\mathbb{N}}$ such that $\boldsymbol{x}\doteq\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n$ by Definition L.7 page 272

$\Longrightarrow$ for every $\boldsymbol{x}\in\boldsymbol{B}$, there exists $(\!(\alpha_n\in\mathbb{F})\!)_{n\in\mathbb{N}}$ such that $\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\|=0$ by Definition G.5

$\Longrightarrow$ for every $\boldsymbol{x}\in\boldsymbol{B}$, there exists $(\!(\alpha_n\in\mathbb{Q})\!)_{n\in\mathbb{N}}$ such that $\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\|=0$ because $\mathbb{Q}^-=\mathbb{F}$

$\Longrightarrow$ $\boldsymbol{B}=\left\{\boldsymbol{x}|\exists(\!(\alpha_n\mathbb{Q})\!)_{n\in\mathbb{N}}\text{ such that }\lim_{N\to\infty}\left\|\boldsymbol{x}-\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\|=0\right\}$

$\Longrightarrow$ $\boldsymbol{B}=\left\{\boldsymbol{x}|\exists(\!(\alpha_n\mathbb{Q})\!)_{n\in\mathbb{N}}\text{ such that }\boldsymbol{x}=\lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right\}^-$

$\Longrightarrow$ $\boldsymbol{B}$ is *separable* by (1) lemma page 272

✏

Definition L.8. [16] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\|\cdot\|)$.

DEF $\{\boldsymbol{x}_n\}$ ***is equivalent*** *to* $\{\boldsymbol{y}_n\}$
if there exists a BOUNDED INVERTIBLE *operator* $\mathbf{R}$ *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *such that* $\mathbf{R}\boldsymbol{x}_n=\boldsymbol{y}_n\quad\forall n\in\mathbb{Z}$

Theorem L.4. [17] *Let* $\{\boldsymbol{x}_n|\,n\in\mathbb{N}\}$ *and* $\{\boldsymbol{y}_n|\,n\in\mathbb{N}\}$ *be* SCHAUDER BASES *of a* BANACH SPACE $(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\|\cdot\|)$.

THM $\left\{\{\boldsymbol{x}_n\}\text{ is EQUIVALENT to }\{\boldsymbol{y}_n\}\right\}\iff\left\{\sum_{n=1}^{\infty}\alpha_n\boldsymbol{x}_n\text{ is CONVERGENT}\iff\sum_{n=1}^{\infty}\alpha_n\boldsymbol{y}_n\text{ is CONVERGENT}\right\}$

Lemma L.1. [18] *Let* $\boldsymbol{\Omega}\triangleq(X,+,\cdot,(\mathbb{F},\dot{+},\dot{\times}),\boldsymbol{T})$ *be a topological linear space. Let* $\operatorname{span}A$ *be the* SPAN *of a set* A (Definition L.2 page 267). *Let* $\tilde{\mathsf{f}}(\omega)$ *and* $\tilde{\mathsf{g}}(\omega)$ *be the* FOURIER TRANSFORMS (Definition K.2 page 257) *of the functions* $\mathsf{f}(x)$ *and* $\mathsf{g}(x)$, *respectively, in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition E.1 page 185). *Let* $\breve{\mathsf{a}}(\omega)$ *be the* DTFT (Definition P.1 page 355) *of a sequence* $(\!(a_n)\!)_{n\in\mathbb{Z}}$ *in* $\boldsymbol{\ell}^2_{\mathbb{R}}$ (Definition O.2 page 341).

LEM $\left\{\begin{array}{ll}(1). & \{\mathbf{T}^n\mathsf{f}|\,n\in\mathbb{Z}\}\text{ is a SCHAUDER BASIS for }\boldsymbol{\Omega}\text{ and}\\ (2). & \{\mathbf{T}^n\mathsf{g}|\,n\in\mathbb{Z}\}\text{ is a SCHAUDER BASIS for }\boldsymbol{\Omega}\end{array}\right\}\Longrightarrow\left\{\begin{array}{l}\exists(\!(a_n)\!)_{n\in\mathbb{Z}}\text{ such that}\\ \tilde{\mathsf{f}}(\omega)=\breve{\mathsf{a}}(\omega)\tilde{\mathsf{g}}(\omega)\end{array}\right\}$

✎PROOF: Let V_0' be the space spanned by $\{\mathbf{T}^n\phi|\,n\in\mathbb{Z}\}$.

$$\begin{aligned}\tilde{\mathsf{f}}(\omega)&\triangleq\tilde{\mathbf{F}}\mathsf{f} &&\text{by definition of }\tilde{\mathbf{F}}\quad\text{(Definition K.2 page 257)}\\ &=\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}}a_n\mathbf{T}\mathsf{g} &&\text{by (2)}\\ &=\sum_{n\in\mathbb{Z}}a_n\tilde{\mathbf{F}}\mathbf{T}\mathsf{g}\end{aligned}$$

[16] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[17] Young (2001) page 25 ⟨Definition 1.8.1, Theorem 1.8.7⟩
[18] Daubechies (1992), page 140

$$= \underbrace{\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}}_{\breve{a}(\omega)} \underbrace{\tilde{\mathbf{F}}\mathsf{g}}_{\tilde{\mathsf{g}}(\omega)}$$ by Corollary 1.1 page 9

$$= \breve{a}(\omega)\tilde{\mathsf{g}}(\omega)$$ by definition of $\check{\mathbf{F}}$ and $\tilde{\mathbf{F}}$ by (Definition P.1 page 355, Definition K.2 page 257)

$$\begin{aligned}
V_0 &\triangleq \left\{ \mathsf{f}(x) | \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathbf{F}}\mathsf{f}(x) = \tilde{\mathbf{F}} \sum_{n\in\mathbb{Z}} b_n \mathbf{T}^n \mathsf{g}(x) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{b}}(\omega)\tilde{\mathsf{g}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{b}}(\omega)\breve{\mathsf{a}}(\omega)\tilde{\mathsf{f}}(\omega) \right\} \\
&= \left\{ \mathsf{f}(x) | \tilde{\mathsf{f}}(\omega) = \tilde{\mathsf{c}}(\omega)\tilde{\mathsf{f}}(\omega) \right\} && \text{where } \tilde{\mathsf{c}}(\omega) \triangleq \tilde{\mathsf{b}}(\omega)\breve{\mathsf{a}}(\omega) \\
&= \left\{ \mathsf{f}(x) | \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{c}_n \mathsf{f}(x-n) \right\} \\
&\triangleq V_0'
\end{aligned}$$

⇒

L.4 Linear combinations in inner product spaces

Definition L.9. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *be an* INNER PRODUCT SPACE *(Definition D.9 page 168).*

DEF

Two vectors $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* X *are* ***orthogonal*** *if*

$$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \begin{cases} 0 & \text{for } \boldsymbol{x} \neq \boldsymbol{y} \\ c \in \mathbb{F}\backslash 0 & \text{for } \boldsymbol{x} = \boldsymbol{y} \end{cases}$$

In an *inner product space, orthogonality* is a special case of *linear independence*; or alternatively, linear independence is a generalization of orthogonality (next theorem).

Theorem L.5. [19] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 168)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM

$$\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is ORTHOGONAL} \\ \text{(Definition L.9 page 274)} \end{array} \right\} \implies \left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is LINEARLY INDEPENDENT} \\ \text{(Definition L.1 page 267)} \end{array} \right\}$$

✎PROOF:

1. Proof using *Pythagorean theorem* :
   Let $(\!(\alpha_n)\!)_{n\in\mathbb{N}}$ be a sequence with at least one nonzero element.

[19] Aliprantis and Burkinshaw (1998) page 283 ⟨Corollary 32.8⟩, Kubrusly (2001) page 352 ⟨Proposition 5.34⟩

$$\left\| \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \right\|^2 = \sum_{n=1}^{N} \| \alpha_n \boldsymbol{x}_n \|^2 \qquad \text{by left hypoth. and } \textit{Pythagorean Theorem}$$

$$= \sum_{n=1}^{N} |\alpha_n|^2 \, \|\boldsymbol{x}_n\|^2 \qquad \text{by definition of } \|\cdot\| \qquad \text{(Definition D.5 page 160)}$$

$$> 0$$

$$\implies \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \neq 0$$

$$\implies (\boldsymbol{x}_n)_{n\in\mathbb{N}} \text{ is } \textit{linearly independent} \qquad \text{by definition of } \textit{linear independence} \qquad \text{(Definition L.3 page 267)}$$

2. Alternative proof:

$$\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n = \mathbb{0} \implies \left\langle \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n \,\middle|\, \boldsymbol{x}_m \right\rangle = \langle \mathbb{0} \,|\, \boldsymbol{x}_m \rangle$$

$$\implies \sum_{n=1}^{N} \alpha_n \langle \boldsymbol{x}_n \,|\, \boldsymbol{x}_m \rangle = 0$$

$$\implies \sum_{n=1}^{N} \alpha_n \bar{\delta}(k-m) = 0$$

$$\implies \alpha_m = 0 \qquad \text{for } m = 1, 2, \ldots, N$$

✏

Theorem L.6 (Bessel's Equality). [20] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n=1,2,\ldots,N \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 168)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ *and with* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle}$.

THM

$$\left\{ \begin{array}{c} \{\boldsymbol{x}_n\} \text{ is ORTHONORMAL} \\ \text{(Definition L.9 page 274)} \end{array} \right\} \implies \left\{ \underbrace{\left\| \boldsymbol{x} - \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2}_{\text{approximation error}} = \|\boldsymbol{x}\|^2 - \sum_{n=1}^{N} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 \quad \forall \boldsymbol{x} \in X \right\}$$

✎PROOF:

$$\left\| \boldsymbol{x} - \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2$$

$$= \|\boldsymbol{x}\|^2 + \left\| \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 - 2\Re \left\langle \boldsymbol{x} \,\middle|\, \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\rangle \qquad \text{by } \textit{polar identity}$$

$$= \|\boldsymbol{x}\|^2 + \left\| \sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right)^* \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right] \qquad \text{by property of } \langle \triangle \,|\, \nabla \rangle \qquad \text{(Definition D.9 page 168)}$$

$$= \|\boldsymbol{x}\|^2 + \sum_{n=1}^{N} \| \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \|^2 - 2\Re \left[\left(\sum_{n=1}^{N} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right)^* \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right] \qquad \text{by } \textit{Pythagorean Theorem}$$

[20] Bachman et al. (2000) page 103, Pedersen (2000) pages 38–39

$$
\begin{aligned}
&= \|x\|^2 + \sum_{n=1}^{N} \|\langle x \,|\, x_n\rangle x_n\|^2 - 2\Re\left(\sum_{n=1}^{N} \langle x \,|\, x_n\rangle^* \,\langle x \,|\, x_n\rangle\right) \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \underbrace{\|x_n\|^2}_{1} - 2\Re\left(\sum_{n=1}^{N} \langle x \,|\, x_n\rangle^* \,\langle x \,|\, x_n\rangle\right) && \text{by property of } \|\cdot\| && \text{(Definition D.5 page 160)} \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \cdot 1 - 2\Re\left(\sum_{n=1}^{N} \langle x \,|\, x_n\rangle^* \,\langle x \,|\, x_n\rangle\right) && \text{by def. of \textit{orthonormality}} && \text{(Definition L.9 page 274)} \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 - 2\Re \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 - 2 \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 && \text{because } |\cdot| \text{ is real} \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2
\end{aligned}
$$

⇰

Theorem L.7 (Bessel's inequality). [21] *Let* $\left\{ x_n \in X \,\middle|\, {\scriptstyle n=1,2,\ldots,N} \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 168)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ *and with* $\|x\| \triangleq \sqrt{\langle x \,|\, x\rangle}$.

THM
$$
\left\{ \begin{array}{c} \{x_n\} \text{ is ORTHONORMAL} \\ \text{(Definition L.9 page 274)} \end{array} \right\} \implies \left\{ \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \le \|x\|^2 \quad \forall x \in X \right\}
$$

✎PROOF:

$$
\begin{aligned}
0 &\le \left\| x - \sum_{n=1}^{N} \langle x \,|\, x_n\rangle x_n \right\|^2 && \text{by definition of } \|\cdot\| && \text{(Definition D.5 page 160)} \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 && \text{by \textit{Bessel's Equality}} && \text{(Theorem L.6 page 275)}
\end{aligned}
$$

⇰

The *Best Approximation Theorem* (next) shows that

- the best sequence for representing a vector is the sequence of projections of the vector onto the sequence of basis functions
- the error of the projection is orthogonal to the projection.

Theorem L.8 (Best Approximation Theorem). [22] *Let* $\left\{ x_n \in X \,\middle|\, {\scriptstyle n=1,2,\ldots,N} \right\}$ *be a set of vectors in an* INNER PRODUCT SPACE *(Definition D.9 page 168)* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \,|\, \nabla \rangle)$ *and with* $\|x\| \triangleq \sqrt{\langle x \,|\, x\rangle}$.

[21] Giles (2000) pages 54–55 ⟨3.13 Bessel's inequality⟩, Bollobás (1999) page 147, Aliprantis and Burkinshaw (1998) page 284

[22] Walter and Shen (2001), pages 3–4, Pedersen (2000), page 39, Edwards (1995), pages 94–100, Weyl (1940)

THM

$$\left\{\begin{array}{c}\{x_n\} \text{ is} \\ \text{ORTHONORMAL} \\ \text{(Definition L.9 page 274)}\end{array}\right\} \implies \left\{\begin{array}{lll} 1. & \underset{(\alpha_n)_{n=1}^N}{\arg\min}\left\|x - \sum_{n=1}^N \alpha_n x_n\right\| = \underbrace{(\langle x \mid x_n\rangle)_{n=1}^N}_{\text{best } \alpha_n = \langle x \mid x_n\rangle} & \forall x \in X \quad \text{and} \\ 2. & \underbrace{\left(\sum_{n=1}^N \langle x \mid x_n\rangle x_n\right)}_{\text{approximation}} \perp \underbrace{\left(x - \sum_{n=1}^N \langle x \mid x_n\rangle x_n\right)}_{\text{approximation error}} & \forall x \in X \end{array}\right\}$$

✎PROOF:

1. Proof that $(\langle x \mid x_n\rangle)$ is the best sequence:

$$\begin{aligned}
&\left\|x - \sum_{n=1}^N \alpha_n x_n\right\|^2 \\
&= \|x\|^2 - 2\Re\left\langle x \mid \sum_{n=1}^N \alpha_n x_n\right\rangle + \left\|\sum_{n=1}^N \alpha_n x_n\right\|^2 \\
&= \|x\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^* \langle x \mid x_n\rangle\right) + \sum_{n=1}^N \|\alpha_n x_n\|^2 && \text{by } \textit{Pythagorean Theorem} \\
&= \|x\|^2 - 2\Re\left(\sum_{n=1}^N \alpha_n^* \langle x \mid x_n\rangle\right) + \sum_{n=1}^N |\alpha_n|^2 + \underbrace{\left[\sum_{n=1}^N |\langle x \mid x_n\rangle|^2 - \sum_{n=1}^N |\langle x \mid x_n\rangle|^2\right]}_{0} \\
&= \left[\|x\|^2 - \sum_{n=1}^N |\langle x \mid x_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle x \mid x_n\rangle|^2 - 2\mathbf{R_e}\left[\alpha_n^* \langle x \mid x_n\rangle\right] + |\alpha_n|^2\right] \\
&= \left[\|x\|^2 - \sum_{n=1}^N |\langle x \mid x_n\rangle|^2\right] + \sum_{n=1}^N \left[|\langle x \mid x_n\rangle|^2 - \alpha_n^* \langle x \mid x_n\rangle - \alpha_n \langle x \mid x_n\rangle^* + |\alpha_n|^2\right] \\
&= \left\|x - \sum_{n=1}^N \langle x \mid x_n\rangle x_n\right\|^2 + \sum_{n=1}^N |\langle x \mid x_n\rangle - \alpha_n|^2 && \text{by } \textit{Bessel's Equality} \quad \text{(Theorem L.6 page 275)} \\
&\geq \left\|x - \sum_{n=1}^N \langle x \mid x_n\rangle x_n\right\|^2
\end{aligned}$$

2. Proof that the approximation and approximation error are orthogonal:

$$\begin{aligned}
\left\langle \sum_{n=1}^N \langle x \mid x_n\rangle x_n \mid x - \sum_{n=1}^N \langle x \mid x_n\rangle x_n\right\rangle &= \left\langle \sum_{n=1}^N \langle x \mid x_n\rangle x_n \mid x\right\rangle - \left\langle \sum_{n=1}^N \langle x \mid x_n\rangle x_n \mid \sum_{n=1}^N \langle x \mid x_n\rangle x_n\right\rangle \\
&= \sum_{n=1}^N \langle x \mid x_n\rangle^* \langle x \mid x_n\rangle - \sum_{n=1}^N \sum_{m=1}^N \langle x \mid x_n\rangle \langle x \mid x_m\rangle^* \langle x_n \mid x_m\rangle \\
&= \sum_{n=1}^N |\langle x \mid x_n\rangle|^2 - \sum_{n=1}^N \sum_{m=1}^N \langle x \mid x_n\rangle \langle x \mid x_m\rangle^* \bar{\delta}_{nm} \\
&= \sum_{n=1}^N |\langle x \mid x_n\rangle|^2 - \sum_{n=1}^N |\langle x \mid x_n\rangle|^2 \\
&= 0
\end{aligned}$$

L.5 Orthonormal bases in Hilbert spaces

Definition L.10. *Let* $\{x_n \in X \mid_{n=1,2,\ldots,N}\}$ *be a set of vectors in an* INNER PRODUCT SPACE (Definition D.9 page 168) $\boldsymbol{\Omega} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

DEF
The set $\{x_n\}$ *is an* ***orthogonal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{x_n\}$ *is* ORTHOGONAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.
The set $\{x_n\}$ *is an* ***orthonormal basis*** *for* $\boldsymbol{\Omega}$ *if* $\{x_n\}$ *is* ORTHONORMAL *and is a* SCHAUDER BASIS *for* $\boldsymbol{\Omega}$.

Definition L.11. [23] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ *be a Hilbert space.*

DEF
Suppose there exists a set $\{x_n \in X \mid n\in\mathbb{N}\}$ *such that* $x \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n$.
Then the quantities $\langle x \mid x_n \rangle$ *are called the* ***Fourier coefficients*** *of* x *and the sum* $\sum_{n=1}^{\infty} \langle x \mid x_n \rangle x_n$ *is called the* ***Fourier expansion*** *of* x *or the* ***Fourier series*** *for* x.

Definition L.12.

DEF
The ***Kronecker delta function*** $\bar{\delta}_n$ *is defined as* $\bar{\delta}_n \triangleq \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$ and $\forall n\in\mathbb{Z}$

Lemma L.2 (Perfect reconstruction). *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

LEM
$$\left\{ \begin{array}{lll} (1). & (x_n) \text{ is a BASIS for } \boldsymbol{H} & \text{and} \\ (2). & (x_n) \text{ is ORTHONORMAL} & \end{array} \right\} \implies \overbrace{x \stackrel{\star}{=} \sum_{n=1}^{\infty} \underbrace{\langle x \mid x_n \rangle}_{\text{Fourier coefficient}} x_n}^{\text{Fourier expansion}} \quad \forall x\in X$$

✎PROOF:

$$\begin{aligned}
\langle x \mid x_n \rangle &= \left\langle \sum_{m\in\mathbb{Z}} \alpha_m x_m \mid x_n \right\rangle && \text{by left hypothesis (1)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \langle x_m \mid x_n \rangle && \text{by \textit{homogeneous} property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition D.9 page 168)} \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \bar{\delta}_{n-m} && \text{by left hypothesis (2)} && \text{(Definition L.9 page 274)} \\
&= \alpha_n
\end{aligned}$$

Proposition L.2. [24] *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

[23] Fabian et al. (2010) page 27 ⟨Theorem 1.55⟩, Young (2001) page 6, Young (1980) page 6
[24] Han et al. (2007) pages 93–94 ⟨Proposition 3.11⟩

PRP

$$\underbrace{\|\boldsymbol{x}\|^2 \stackrel{\star}{=} \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2}_{\text{PARSEVAL FRAME}} \iff \underbrace{\boldsymbol{x} \stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n}_{\text{FOURIER EXPANSION (Definition L.11 page 278)}} \qquad \forall x \in X$$

✎PROOF:

1. Proof that *Parseval frame* $\Longleftarrow$ *Fourier expansion*

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &\triangleq \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by definition of } \|\cdot\| \\
&= \left\langle \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x} \,\middle|\, \boldsymbol{x}_n \right\rangle && \text{by right hypothesis} \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle^* && \text{by property of } \langle \triangle \,|\, \nabla \rangle \\
&\stackrel{\star}{=} \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by property of } \mathbb{C}
\end{aligned}$$

2. Proof that *Parseval frame* $\Longrightarrow$ *Fourier expansion*

   (a) Let $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ be the *standard othornormal basis* such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.

   (b) Let $\mathbf{M}$ be an operator in $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{x} \triangleq \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{e}_n$.

   (c) lemma: $\mathbf{M}$ is *isometric*. Proof:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \sum_{n=1}^{\infty} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{e}_n \right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2b) page 279)} \\
&= \sum_{n=1}^{\infty} \|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{e}_n\|^2 && \text{by \textit{Pythagorean Theorem}} \\
&= \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 \|\boldsymbol{e}_n\|^2 && \text{by \textit{homogeneous} property of } \|\cdot\| && \text{(Definition D.5 page 160)} \\
&= \sum_{n=1}^{\infty} |\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle|^2 && \text{by definition of \textit{orthonormal}} && \text{(Definition L.9 page 274)} \\
&= \|\boldsymbol{x}\|^2 && \text{by Parseval frame hypothesis} \\
\Longrightarrow &\quad \mathbf{M} \text{ is \textit{isometric}} && \text{by definition of \textit{isometric}} && \text{(Definition D.13 page 176)}
\end{aligned}$$

   (d) Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthornormal basis* for $\boldsymbol{H}$.

(e) Proof for *Fourier expansion*:

$$
\begin{aligned}
\boldsymbol{x} &= \sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by } \textit{Fourier expansion} \text{ (Proposition L.3 page 282)} \\
&= \sum_{n=1}^{\infty} \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by (2c) lemma page 279 and Theorem D.23 page 177} \\
&= \sum_{n=1}^{\infty} \left\langle \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \boldsymbol{e}_m \,\middle|\, \sum_{k=1}^{\infty} \langle \boldsymbol{u}_n \mid \boldsymbol{x}_k \rangle \boldsymbol{e}_k \right\rangle \boldsymbol{u}_n && \text{by item (2b) page 279} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \sum_{k=1}^{\infty} \langle \boldsymbol{u}_n \mid \boldsymbol{x}_k \rangle^* \langle \boldsymbol{e}_m \mid \boldsymbol{e}_k \rangle \boldsymbol{u}_n && \text{by Definition D.9 page 168} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \langle \boldsymbol{u}_n \mid \boldsymbol{x}_m \rangle^* \boldsymbol{u}_n && \text{by item (2a) page 279 and Definition L.9 page 274} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \langle \boldsymbol{x}_m \mid \boldsymbol{u}_n \rangle \boldsymbol{u}_n && \text{by Definition D.9 page 168} \\
&= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \sum_{n=1}^{\infty} \langle \boldsymbol{x}_m \mid \boldsymbol{u}_n \rangle \boldsymbol{u}_n \\
&= \sum_{m=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_m \rangle \boldsymbol{x}_m && \text{by item (2d) page 279}
\end{aligned}
$$

✏

When is a set of orthonormal vectors in a Hilbert space $\boldsymbol{H}$ *total*? Theorem L.9 (next) offers some help.

Theorem L.9 (The Fourier Series Theorem). [25] *Let* $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ *and let* $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.

THM

(A) $\{\boldsymbol{x}_n\}$ *is* ORTHONORMAL *in* $\boldsymbol{H}$ $\implies$

$$
\left\{
\begin{array}{llrcll}
 & (1). & (\operatorname{span}\{\boldsymbol{x}_n\})^- & = & \boldsymbol{H} & & (\{\boldsymbol{x}_n\} \textit{ is TOTAL in } \boldsymbol{H}) \\
\iff & (2). & \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \langle \boldsymbol{y} \mid \boldsymbol{x}_n \rangle^* & \forall \boldsymbol{x},\boldsymbol{y}\in X & (\text{GENERALIZED PARSEVAL'S IDENTITY}) \\
\iff & (3). & \|\boldsymbol{x}\|^2 & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 & \forall \boldsymbol{x}\in X & (\text{PARSEVAL'S IDENTITY}) \\
\iff & (4). & \boldsymbol{x} & \stackrel{\star}{=} & \displaystyle\sum_{n=1}^{\infty} \langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n & \forall \boldsymbol{x}\in X & (\text{FOURIER SERIES EXPANSION})
\end{array}
\right\}
$$

✎PROOF:

[25] Bachman and Narici (1966) pages 149–155 ⟨Theorem 9.12⟩, Kubrusly (2001) pages 360–363 ⟨Theorem 5.48⟩, Aliprantis and Burkinshaw (1998) pages 298–299 ⟨Theorem 34.2⟩, Christensen (2003) page 57 ⟨Theorem 3.4.2⟩, Berberian (1961) pages 52–53 ⟨Theorem II§8.3⟩, Heil (2011) pages 34–35 ⟨Theorem 1.50⟩, Bracewell (1978) page 112 ⟨Rayleigh's theorem⟩

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle x \,|\, y\rangle &\stackrel{*}{=} \left\langle \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle x_n \,\middle|\, \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle x_m \right\rangle && \text{by (A) and (1)}\\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \left\langle x_n \,\middle|\, \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle x_m \right\rangle && \text{by property of } \langle\cdot\,|\,\cdot\rangle && \text{(Definition D.9 page 168)}\\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle^* \langle x_n \,|\, x_m\rangle && \text{by property of } \langle\cdot\,|\,\cdot\rangle && \text{(Definition D.9 page 168)}\\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \sum_{m=1}^{\infty} \langle y \,|\, x_m\rangle^* \bar{\delta}_{mn} && \text{by (A)}\\
&\stackrel{*}{=} \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \langle y \,|\, x_n\rangle^* && \text{by definition of } \bar{\delta}_n && \text{(Definition L.12 page 278)}
\end{aligned}$$

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\|x\|^2 &\triangleq \langle x \,|\, x\rangle && \text{by definition of } \textit{induced norm}\\
&= \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle \langle x \,|\, x_n\rangle^* && \text{by (2)}\\
&= \sum_{n=1}^{\infty} |\langle x \,|\, x_n\rangle|^2
\end{aligned}$$

3. Proof that (3) $\iff$ (4) *not* using (A): by Proposition L.2 page 278

4. Proof that (3) $\implies$ (1) (proof by contradiction):

   (a) Suppose $\{x_n\}$ is *not total*.

   (b) Then there must exist a vector y in H such that the set $B \triangleq \{x_n\} \cup y$ is *orthonormal*.

   (c) Then $1 = \|y\|^2 \neq \sum_{n=1}^{\infty} |\langle y \,|\, x_n\rangle|^2 = 0$.

   (d) But this contradicts (3), and so $\{x_n\}$ must be *total* and (3) $\implies$ (1).

5. Extraneous proof that (3) $\implies$ (4) (this proof is not really necessary here):

$$\begin{aligned}
\left\| x - \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle x_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x \,|\, x_n\rangle|^2 && \text{by } \textit{Bessel's Equality} && \text{(Theorem L.6 page 275)}\\
&= 0 && \text{by (3)}\\
\implies \quad & x \stackrel{*}{=} \sum_{n=1}^{\infty} \langle x \,|\, x_n\rangle x_n && \text{by definition of } \stackrel{*}{=} && \text{(Definition G.5 page 211)}
\end{aligned}$$

6. Extraneous proof that (A) $\implies$ (4) (this proof is not really necessary here)

   (a) The sequence $\sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2$ is *monotonically increasing* in n.

   (b) By Bessel's inequality (page 276), the sequence is upper bounded by $\|x\|^2$:

$$\sum_{n=1}^{N} |\langle x \,|\, x_n\rangle|^2 \leq \|x\|^2$$

(c) Because this sequence is both monotonically increasing and bounded in n, it must equal its bound in the limit as n approaches infinity:

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left|\langle x \mid x_n\rangle\right|^2 = \|x\|^2 \tag{L.1}$$

(d) If we combine this result with *Bessel's Equality* (Theorem L.6 page 275) we have

$$\begin{aligned}
\lim_{N\to\infty}\left\|x-\sum_{n=1}^{N}\langle x \mid x_n\rangle x_n\right\|^2 &= \|x\|^2 - \lim_{N\to\infty}\sum_{n=1}^{N}|\langle x \mid x_n\rangle|^2 && \text{by } \textit{Bessel's Equality} \text{ (Theorem L.6 page 275)}\\
&= \|x\|^2-\|x\|^2 && \text{by equation (L.1) page 282}\\
&= 0
\end{aligned}$$

✎

Proposition L.3 (Fourier expansion). *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown\rangle)$.

PRP
$$\underbrace{\{x_n\} \textit{ is an } \text{ORTHONORMAL BASIS} \textit{ for } \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{x \stackrel{\star}{=} \sum_{n=1}^{\infty}\alpha_n x_n}_{(1)} \iff \underbrace{\alpha_n = \langle x \mid x_n\rangle}_{(2)}\right\}$$

✎PROOF:

1. Proof that (1) $\implies$ (2): by Lemma L.2 page 278
2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\left\|x-\sum_{n=1}^{\infty}\alpha_n x_{n\in\mathbb{Z}}\right\|^2 &= \left\|x-\sum_{n=1}^{\infty}\langle x \mid x_n\rangle x_{n\in\mathbb{Z}}\right\|^2 && \text{by right hypothesis}\\
&= \|x\|^2-\sum_{n=1}^{\infty}|\langle x \mid x_n\rangle|^2 && \text{by Bessel's equality} && \text{(Theorem L.6 page 275)}\\
&= 0 && \text{by } \textit{Parseval's Identity} && \text{(Theorem L.9 page 280)}\\
&\stackrel{\text{def}}{\iff} x \stackrel{\star}{=} \sum_{n=1}^{\infty}\langle x \mid x_n\rangle x_n && \text{by definition of } \textit{strong convergence} && \text{(Definition G.5 page 211)}
\end{aligned}$$

✎

Proposition L.4 (Riesz-Fischer Theorem). [26] *Let* $\{x_n \in X \mid n\in\mathbb{N}\}$ *be a set of vectors in a* HILBERT SPACE $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown\rangle)$.

PRP
$$\underbrace{\{x_n\} \textit{ is an } \text{ORTHONORMAL BASIS} \textit{ for } \boldsymbol{H}}_{(A)} \implies \left\{\underbrace{\sum_{n=1}^{\infty}|\alpha_n|^2 < \infty}_{(1)} \iff \underbrace{\exists x\in\boldsymbol{H} \text{ such that } \alpha_n = \langle x \mid x_n\rangle}_{(2)}\right\}$$

✎PROOF:

[26] Young (2001) page 6

1. Proof that (1) $\Longrightarrow$ (2):

   (a) If (1) is true, then let $\boldsymbol{x} \triangleq \sum_{n\in\mathbb{N}} \alpha_n \boldsymbol{x}_n$.

   (b) Then

$$
\begin{aligned}
\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle &= \left\langle \sum_{m\in\mathbb{N}} \alpha_m \boldsymbol{x}_m \mid \boldsymbol{x}_n \right\rangle && \text{by definition of } \boldsymbol{x} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \langle \boldsymbol{x}_m \mid \boldsymbol{x}_n \rangle && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \nabla \rangle && \text{(Definition D.9 page 168)} \\
&= \sum_{m\in\mathbb{N}} \alpha_m \bar{\delta}_{mn} && \text{by (A)} \\
&= \sum_{m\in\mathbb{N}} \alpha_n && \text{by definition of } \bar{\delta} && \text{(Definition L.12 page 278)}
\end{aligned}
$$

2. Proof that (1) $\Longleftarrow$ (2):

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} |\alpha_n|^2 &= \sum_{n\in\mathbb{N}} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 && \text{by (2)} \\
&\le \|\boldsymbol{x}\|^2 && \text{by } \textit{Bessel's Inequality} && \text{(Theorem L.7 page 276)} \\
&\le \infty
\end{aligned}
$$

✏

Theorem L.10. [27]

THM *All* SEPARABLE HILBERT SPACES *are* ISOMORPHIC. *That is,*

$$
\left\{ \begin{array}{l} \boldsymbol{X} \text{ is a separable} \\ \text{Hilbert space} \quad \text{and} \\ \boldsymbol{Y} \text{ is a separable} \\ \text{Hilbert space} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lllll} \multicolumn{5}{l}{\text{there is a BIJECTIVE operator } \mathbf{M} \in \boldsymbol{Y}^{\boldsymbol{X}} \text{ such that}} \\ (1). & \boldsymbol{y} & = \mathbf{M}\boldsymbol{x} & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} & \text{and} \\ (2). & \|\mathbf{M}\boldsymbol{x}\| & = \|\boldsymbol{x}\| & \forall x\in\boldsymbol{X} & \text{and} \\ (3). & \langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y} \rangle & = \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle & \forall x\in\boldsymbol{X}, y\in\boldsymbol{Y} & \end{array} \right\}
$$

✎PROOF:

1. Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{ \boldsymbol{x}_n | n\in\mathbb{N} \}$.
   Let $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be a *separable Hilbert space* with *orthonormal basis* $\{ \boldsymbol{y}_n | n\in\mathbb{N} \}$.

2. Proof that there exists *bijective* operator $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$:

   (a) Let $\mathbf{M}$ be defined such that $\boldsymbol{y}_n \triangleq \mathbf{M}\boldsymbol{x}_n$.

   (b) Thus $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$.

   (c) Because $\mathbf{M}$ is a *bijection* between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$, $\mathbf{M}$ has an inverse operator $\mathbf{M}^{-1}$ between $\{\boldsymbol{x}_n\}$ and $\{\boldsymbol{y}_n\}$ such that $\boldsymbol{x}_n = \mathbf{M}^{-1}\boldsymbol{y}_n$.

3. Proof that $\mathbf{M}$ and $\mathbf{M}^{-1}$ are *bijective* operators between $\boldsymbol{X}$ and $\boldsymbol{Y}$:

[27] Young (2001) page 6

(a) Proof that $\mathbf{M}$ maps $\boldsymbol{X}$ *into* $\boldsymbol{Y}$:

$$\begin{aligned}
\boldsymbol{x} \in \boldsymbol{X} \iff \boldsymbol{x} &\stackrel{\star}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem L.9 page 280)}\\
&\implies \exists \boldsymbol{y} \in \boldsymbol{Y} \quad \text{such that} \quad \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle = \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition L.4 page 282)}\\
&\implies\\
\boldsymbol{y} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem L.9 page 280)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{y}_n && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition L.4 page 282)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n && \text{by definition of } \mathbf{M} && \text{(item (2a) page 283)}\\
&= \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by prop. of linear ops.} && \text{(Theorem D.1 page 157)}\\
&= \mathbf{M}\boldsymbol{x} && \text{by definition of } \boldsymbol{x}
\end{aligned}$$

(b) Proof that $\mathbf{M}^{-1}$ maps $\boldsymbol{Y}$ *into* $\boldsymbol{X}$:

$$\begin{aligned}
\boldsymbol{y} \in \boldsymbol{Y} \iff \boldsymbol{y} &\stackrel{\star}{=} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem L.9 page 280)}\\
&\implies \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle = \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition L.4 page 282)}\\
&\implies\\
\boldsymbol{x} &= \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n && \text{by \textit{Fourier expansion}} && \text{(Theorem L.9 page 280)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{x}_n && \text{by \textit{Riesz-Fischer Thm.}} && \text{(Proposition L.4 page 282)}\\
&= \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \mathbf{M}^{-1}\boldsymbol{y}_n && \text{by definition of } \mathbf{M}^{-1} && \text{(item (2c) page 283)}\\
&= \mathbf{M}^{-1} \sum_{n\in\mathbb{N}} \langle \boldsymbol{y} \,|\, \boldsymbol{y}_n \rangle \boldsymbol{y}_n && \text{by prop. of \textit{linear ops.}} && \text{(Theorem D.1 page 157)}\\
&= \mathbf{M}^{-1}\boldsymbol{y} && \text{by definition of } \boldsymbol{y}
\end{aligned}$$

4. Proof for (2):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\|^2 &= \left\| \mathbf{M} \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem L.9 page 280)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \mathbf{M}\boldsymbol{x}_n \right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem D.1 page 157)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{y}_n \right\|^2 && \text{by definition of } \mathbf{M} && \text{(item (2a) page 283)}\\
&= \sum_{n\in\mathbb{N}} \left| \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \right|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition L.4 page 282)}\\
&= \left\| \sum_{n\in\mathbb{N}} \langle \boldsymbol{x} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \right\|^2 && \text{by \textit{Parseval's Identity}} && \text{(Proposition L.4 page 282)}\\
&= \|\boldsymbol{x}\|^2 && \text{by \textit{Fourier expansion}} && \text{(Theorem L.9 page 280)}
\end{aligned}$$

5. Proof for (3): by (2) and Theorem D.23 page 177

Theorem L.11. [28] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: *$\boldsymbol{H}$ has a* SCHAUDER BASIS $\iff$ *$\boldsymbol{H}$ is* SEPARABLE

Theorem L.12. [29] *Let $\boldsymbol{H}$ be a* HILBERT SPACE.

THM: *$\boldsymbol{H}$ has an* ORTHONORMAL BASIS $\iff$ *$\boldsymbol{H}$ is* SEPARABLE

L.6 Riesz bases in Hilbert spaces

Definition L.13. [30] *Let $\{x_n \in X \mid n\in\mathbb{N}\}$ be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dotplus,$

DEF: *$\{x_n\}$ is a **Riesz basis** for $\boldsymbol{H}$ if $\{x_n\}$ is* EQUIVALENT (Definition L.8 page 273) *to some* ORTHONORMAL BASIS (Definition L.10 page 278) *in $\boldsymbol{H}$.*

Definition L.14. [31] *Let $(x_n \in X)_{n\in\mathbb{N}}$ be a sequence of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$.

DEF: *The sequence (x_n) is a **Riesz sequence** for $\boldsymbol{H}$ if*

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 \qquad \forall (\alpha_n)\in\ell^2_{\mathbb{F}}.$$

Definition L.15. *Let $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$ be an* INNER PRODUCT SPACE (Definition D.9 page 168).

DEF: *The sequences $(x_n \in X)_{n\in\mathbb{Z}}$ and $(y_n \in X)_{n\in\mathbb{Z}}$ are **biorthogonal** with respect to each other in $\boldsymbol{X}$ if* $\langle x_n \mid y_m\rangle = \bar{\delta}_{nm}$

Lemma L.3. [32] *Let $\{x_n \mid n\in\mathbb{N}\}$ be a sequence in a* HILBERT SPACE $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$. *Let $\{y_n \mid n\in\mathbb{N}\}$ be a sequence in a* HILBERT SPACE $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dotplus, \dot{\times}), \langle \triangle \mid \triangledown \rangle)$. *Let*

LEM:

$$\left\{\begin{array}{lll}
\text{(i).} & \{x_n\} \text{ is TOTAL in } \boldsymbol{X} & \text{and}\\
\text{(ii).} & \text{There exists } A>0 \text{ such that } A\sum_{n\in C}|a_n|^2 \le \left\|\sum_{n\in C} a_n x_n\right\|^2 \text{ for finite } C & \text{and}\\
\text{(iii).} & \text{There exists } B>0 \text{ such that } \left\|\sum_{n=1}^{\infty} b_n y_n\right\|^2 \le B\sum_{n=1}^{\infty}|b_n|^2 \quad \forall (b_n)_{n\in\mathbb{N}}\in\ell^2_{\mathbb{F}} &
\end{array}\right\} \implies$$

$$\left\{\begin{array}{lll}
\text{(1).} & \mathbf{R}^\circ \text{ is a linear bounded operator that maps from } \operatorname{span}\{x_n\} \text{ to } \operatorname{span}\{y_n\} & \\
& \quad \text{where } \mathbf{R}^\circ \sum_{n\in C} c_n x_n \triangleq \sum_{n\in C} c_n y_n, \text{ for some sequence } (c_n) \text{ and finite set } C & \text{and}\\
\text{(2).} & \mathbf{R} \text{ has a unique extension to a bounded operator } \mathbf{R} \text{ that maps from } \boldsymbol{X} \text{ to } \boldsymbol{Y} & \text{and}\\
\text{(3).} & |||\mathbf{R}^\circ||| \le \frac{B}{A} & \text{and}\\
\text{(4).} & |||\mathbf{R}||| \le \frac{B}{A} &
\end{array}\right\}$$

[28] Bachman et al. (2000) page 112 ⟨3.4.8⟩, Berberian (1961) page 53 ⟨Theorem II§8.3⟩
[29] Kubrusly (2001) page 357 ⟨Proposition 5.43⟩
[30] Young (2001) page 27 ⟨Definition 1.8.2⟩, Christensen (2003) page 63 ⟨Definition 3.6.1⟩, Heil (2011) page 196 ⟨Definition 7.9⟩
[31] Christensen (2003) pages 66–68 ⟨page 68 and (3.24) on page 66⟩, Wojtaszczyk (1997) page 20 ⟨Definition 2.6⟩
[32] Christensen (2003) pages 65–66 ⟨Lemma 3.6.5⟩

Theorem L.13. [33] *Let* $\left\{ \boldsymbol{x}_n \in X \,\middle|\, n\in\mathbb{N} \right\}$ *be a set of vectors in a* SEPARABLE HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM

$$\left\{ \begin{array}{l} \{\boldsymbol{x}_n\} \text{ is a RIESZ BASIS} \\ \text{for } \boldsymbol{H} \end{array} \right\} \iff \left\{ \begin{array}{lll} (1). & \{\boldsymbol{x}_n\} \text{ is TOTAL in } \boldsymbol{H} & \text{and} \\ (2). & \exists A,B\in\mathbb{R}^+ \text{ such that } & \forall (\alpha_n)\in\ell^2_{\mathbb{F}}, \\ & A\displaystyle\sum_{n=1}^{\infty}|\alpha_n|^2 \le \left\| \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \right\|^2 \le B\sum_{n=1}^{\infty}|\alpha_n|^2 & \end{array} \right\}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:
   (a) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (1): all bases for $\boldsymbol{H}$ are *total* in $\boldsymbol{H}$.
   (b) Proof that *Riesz basis* hypothesis $\Longrightarrow$ (2):
      i. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be an *orthonormal basis* for $\boldsymbol{H}$.
      ii. Let $\mathbf{R}$ be a *bounded bijective* operator such that $\boldsymbol{x}_n = \mathbf{R}\boldsymbol{u}_n$.
      iii. Proof for upper bound B:

$$\begin{aligned}
\left\| \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \right\|^2 &= \left\| \sum_{n=1}^{\infty} \alpha_n \mathbf{R}\boldsymbol{u}_n \right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii))} \\
&= \left\| \mathbf{R} \sum_{n=1}^{\infty} \alpha_n \boldsymbol{u}_n \right\|^2 && \text{by Theorem D.1 page 157} \\
&\le \|\!|\mathbf{R}|\!\|^2 \left\| \sum_{n=1}^{\infty} \alpha_n \boldsymbol{u}_n \right\|^2 && \text{by Theorem D.6 page 163} \\
&= \|\!|\mathbf{R}|\!\|^2 \sum_{n=1}^{\infty} \left\| \alpha_n \boldsymbol{u}_n \right\|^2 && \text{by \textit{Pythagorean Theorem}} \\
&= \|\!|\mathbf{R}|\!\|^2 \sum_{n=1}^{\infty} |\alpha|^2 \left\| \boldsymbol{u}_n \right\|^2 && \text{by \textit{homogeneous} property of norms} && \text{(Definition D.5 page 160)} \\
&= \underbrace{\|\!|\mathbf{R}|\!\|^2}_{B} \sum_{n=1}^{\infty} |\alpha|^2 && \text{by definition of \textit{orthonormality}} && \text{(Definition L.9 page 274)}
\end{aligned}$$

      iv. Proof for lower bound A:

$$\begin{aligned}
\left\| \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \right\|^2 &= \frac{\|\!|\mathbf{R}^{-1}|\!\|^2}{\|\!|\mathbf{R}^{-1}|\!\|^2} \left\| \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \right\|^2 && \text{because } \|\!|\mathbf{R}^{-1}|\!\| > 0 && \text{(Proposition D.1 page 161)} \\
&\ge \frac{1}{\|\!|\mathbf{R}^{-1}|\!\|^2} \left\| \mathbf{R}^{-1} \sum_{n=1}^{\infty} \alpha_n \boldsymbol{x}_n \right\|^2 && \text{by Theorem D.6 page 163} \\
&= \frac{1}{\|\!|\mathbf{R}^{-1}|\!\|^2} \left\| \mathbf{R}^{-1} \sum_{n=1}^{\infty} \alpha_n \mathbf{R}\boldsymbol{u}_n \right\|^2 && \text{by definition of } \mathbf{R} && \text{(item (1(b)ii) page 286)} \\
&= \frac{1}{\|\!|\mathbf{R}^{-1}|\!\|^2} \left\| \mathbf{R}^{-1}\mathbf{R} \sum_{n=1}^{\infty} \alpha_n \boldsymbol{u}_n \right\|^2 && \text{by property of \textit{linear operators}} && \text{(Theorem D.1 page 157)}
\end{aligned}$$

[33] Young (2001) page 27 ⟨Theorem 1.8.9⟩, Christensen (2003) page 66 ⟨Theorem 3.6.6⟩, Heil (2011) pages 197–198 ⟨Theorem 7.13⟩, Christensen (2008) pages 61–62 ⟨Theorem 3.3.7⟩

$$= \underbrace{\frac{1}{\|\mathbf{R}^{-1}\|^2}}_{A} \left\| \sum_{n=1}^{\infty} \alpha_n \boldsymbol{u}_n \right\|^2 \quad \text{by definition of inverse op.} \quad \text{(Definition D.3 page 156)}$$

$$= \frac{1}{\|\mathbf{R}^{-1}\|^2} \sum_{n=1}^{\infty} \|\alpha_n \boldsymbol{u}_n\|^2 \quad \text{by } \textit{Pythagorean Theorem}$$

$$= \frac{1}{\|\mathbf{R}^{-1}\|^2} \sum_{n=1}^{\infty} |\alpha_n|^2 \|\boldsymbol{u}_n\|^2 \quad \text{by } \|\cdot\| \textit{ homogeneous} \text{ prop.} \quad \text{(Definition D.5 page 160)}$$

$$= \underbrace{\frac{1}{\|\mathbf{R}^{-1}\|^2}}_{A} \sum_{n=1}^{\infty} |\alpha_n|^2 \quad \text{by def. of } \textit{orthonormality} \quad \text{(Definition L.9 page 274)}$$

2. Proof for ($\Longrightarrow$) case:
   (a) Let $\{\boldsymbol{u}_n | _{n\in\mathbb{N}}\}$ be an *orthonormal basis* for $\boldsymbol{H}$.
   (b) Using (2) and Lemma L.3 (page 285), construct an bounded extension operator $\mathbf{R}$ such that $\mathbf{R}\boldsymbol{u}_n = \boldsymbol{x}_n$ for all $n \in \mathbb{N}$.
   (c) Using (2) and Lemma L.3 (page 285), construct an bounded extension operator $\mathbf{S}$ such that $\mathbf{S}\boldsymbol{x}_n = \boldsymbol{u}_n$ for all $n \in \mathbb{N}$.
   (d) Then, $\mathbf{RV}\boldsymbol{x} = \mathbf{VR}\boldsymbol{x} \implies \mathbf{V} = \mathbf{R}^{-1}$, and so $\mathbf{R}$ is a bounded invertible operator
   (e) and $\{\boldsymbol{x}_n\}$ is a *Riesz sequence*.

✏

Theorem L.14. [34] *Let* $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle | \nabla \rangle)$ *be a* SEPARABLE HILBERT SPACE.

THM

$$\left\{ \begin{array}{l} (\boldsymbol{x}_n \in \boldsymbol{H})_{n\in\mathbb{Z}} \textit{ is a} \\ \text{RIESZ BASIS } \textit{for } \boldsymbol{H} \end{array} \right\} \implies \left\{ \begin{array}{l} \textit{There exists } (\boldsymbol{y}_n \in \boldsymbol{H})_{n\in\mathbb{Z}} \textit{ such that} \\ \text{(1). } (\boldsymbol{x}_n) \textit{ and } (\boldsymbol{y}_n) \textit{ are } \text{BIORTHOGONAL} \quad \textit{and} \\ \text{(2). } (\boldsymbol{y}_n) \textit{ is also a } \text{RIESZ BASIS } \textit{for } \boldsymbol{H} \quad \textit{and} \\ \text{(3). } \exists B>A>0 \quad \text{such that} \\ \quad A\sum_{n=1}^{\infty}|a_n|^2 \le \left\|\sum_{n=1}^{\infty} a_n \boldsymbol{x}_n\right\|^2 = \left\|\sum_{n=1}^{\infty} a_n \boldsymbol{y}_n\right\|^2 \le B\sum_{n=1}^{\infty}|a_n|^2 \\ \quad \forall (a_n)_{n\in\mathbb{N}} \in \ell^2_{\mathbb{F}} \end{array} \right\}$$

✎PROOF:

1. Proof for (1):
   (a) Let $\boldsymbol{e}_n$ be the *unit vector* in $\boldsymbol{H}$ such that the nth element of $\boldsymbol{e}_n$ is 1 and all other elements are 0.
   (b) Let $\mathbf{M}$ be an operator on $\boldsymbol{H}$ such that $\mathbf{M}\boldsymbol{e}_n = \boldsymbol{x}_n$.
   (c) Note that $\mathbf{M}$ is *isometric*, and as such $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \quad \forall \boldsymbol{x}\in\boldsymbol{H}$.
   (d) Let $\boldsymbol{y}_n \triangleq (\mathbf{M}^{-1})^*$.
   (e) Then,

$$\begin{aligned} \langle \boldsymbol{y}_n | \boldsymbol{x}_m \rangle &= \left\langle (\mathbf{M}^{-1})^* \boldsymbol{e}_n \,|\, \mathbf{M}\boldsymbol{e}_m \right\rangle \\ &= \left\langle \boldsymbol{e}_n \,|\, \mathbf{M}^{-1}\mathbf{M}\boldsymbol{e}_m \right\rangle \\ &= \langle \boldsymbol{e}_n | \boldsymbol{e}_m \rangle \\ &= \bar{\delta}_{nm} \\ &\implies \{\boldsymbol{x}_n\} \text{ and } \{\boldsymbol{y}_n\} \text{ are } \textit{biorthogonal} \qquad \text{by Definition L.9 page 274} \end{aligned}$$

[34] Wojtaszczyk (1997) page 20 ⟨Lemma 2.7(a)⟩

2. Proof for (3):

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{y}_n\right\| &= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \left(\mathbf{M}^{-1}\right)^* \boldsymbol{e}_n\right\| && \text{by definition of } \boldsymbol{y}_n && \text{(Proposition 1d page 287)}\\
&= \left\|\left(\mathbf{M}^{-1}\right)^* \sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{by property of } \textit{linear ops.}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \left(\mathbf{M}^{-1}\right)^* \text{ is } \textit{isometric} && \text{(Definition D.13 page 176)}\\
&= \left\|\mathbf{M}\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{e}_n\right\| && \text{because } \mathbf{M} \text{ is } \textit{isometric} && \text{(Definition D.13 page 176)}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \mathbf{M}\boldsymbol{e}_n\right\| && \text{by property of } \textit{linear operators}\\
&= \left\|\sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{x}_n\right\| && \text{by definition of } \mathbf{M}\\
\implies & \left\{\boldsymbol{y}_n\right\} \text{ is a } \textit{Riesz basis} && \text{by left hypothesis}
\end{aligned}$$

3. Proof for (2): by (3) and definition of *Riesz basis* (Definition L.13 page 285)

✏

Proposition L.5. [35] *Let* $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ *be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \triangledown \rangle\right)$.

PRP

$$\left\{\begin{array}{l}\left\{\boldsymbol{x}_n\right\} \textit{ is a } \text{RIESZ BASIS} \textit{ for } \boldsymbol{H} \textit{ with}\\ A\displaystyle\sum_{n=1}^{\infty}\left|a_n\right|^2 \le \left\|\sum_{n=1}^{\infty} a_n \boldsymbol{x}_n\right\|^2 \le B\sum_{n=1}^{\infty}\left|a_n\right|^2\\ \forall\left\{a_n\right\}\in\ell^2_{\mathbb{F}}\end{array}\right\} \implies \left\{\begin{array}{l}\left\{\boldsymbol{x}_n\right\} \textit{ is a } \text{FRAME} \textit{ for } \boldsymbol{H} \textit{ with}\\ \underbrace{\dfrac{1}{B}\|\boldsymbol{x}\|^2 \le \displaystyle\sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \le \dfrac{1}{A}\|\boldsymbol{x}\|^2}_{\text{STABILITY CONDITION}}\\ \forall \boldsymbol{x}\in\boldsymbol{H}\end{array}\right\}$$

✎PROOF:

1. Let $\left\{\boldsymbol{y}_n \middle| n\in\mathbb{N}\right\}$ be a *Riesz basis* that is *biorthogonal* to $\left\{\boldsymbol{x}_n \middle| n\in\mathbb{N}\right\}$ (Theorem L.14 page 287).

2. Let $\boldsymbol{x} \triangleq \displaystyle\sum_{n=1}^{\infty} a_n \boldsymbol{y}_n$.

3. lemma:

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 &= \sum_{n=1}^{\infty}\left|\left\langle \sum_{m=1}^{\infty} a_n \boldsymbol{y}_m \,\middle|\, \boldsymbol{x}_n\right\rangle\right|^2 && \text{by definition of } \boldsymbol{x} && \text{(item (2) page 288)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \langle \boldsymbol{y}_m \mid \boldsymbol{x}_n\rangle\right|^2 && \text{by } \textit{homogeneous} \text{ property of } \langle \triangle \mid \triangledown \rangle && \text{(Definition D.9 page 168)}\\
&= \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_n \bar{\delta}_{mn}\right|^2 && \text{by definition of } \textit{biorthogonal} && \text{(Definition L.15 page 285)}\\
&= \sum_{n=1}^{\infty}\left|a_n\right|^2 && \text{by definition of } \bar{\delta} && \text{(Definition L.12 page 278)}
\end{aligned}$$

[35] Igari (1996) page 220 ⟨Lemma 9.8⟩, Wojtaszczyk (1997) pages 20–21 ⟨Lemma 2.7(a)⟩

4. Then

$$\begin{array}{rlllll}
 & A\sum_{n=1}^{\infty}|a_n|^2 & \le \left\|\sum_{n=1}^{\infty} a_n \boldsymbol{x}_n\right\|^2 & \le B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} & \text{(item (1) page 288)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le \left\|\sum_{n=1}^{\infty} a_n \boldsymbol{y}_n\right\| & \le B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \{\boldsymbol{y}_n\} & \text{(item (1) page 288)}\\
\implies & A\sum_{n=1}^{\infty}|a_n|^2 & \le \|\boldsymbol{x}\|^2 & \le B\sum_{n=1}^{\infty}|a_n|^2 & \text{by definition of } \boldsymbol{x} & \text{(item (2) page 288)}\\
\implies & A\sum_{n=1}^{\infty}|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle|^2 & \le \|\boldsymbol{x}\|^2 & \le B\sum_{n=1}^{\infty}|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle|^2 & \text{by (3) lemma} & \\
\implies & \frac{1}{B}\|\boldsymbol{x}\|^2 & \le \sum_{n=1}^{\infty}|\langle \boldsymbol{x} \,|\, \boldsymbol{x}_n\rangle|^2 & \le \frac{1}{A}\|\boldsymbol{x}\|^2 & &
\end{array}$$

✏

Theorem L.15 (Battle-Lemarié orthogonalization).[36] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition K.2 page 257) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$.

THM
$$\left\{\begin{array}{ll} 1. & \{\mathbf{T}^n\mathsf{g} \,|\, n\in\mathbb{Z}\} \text{ is a RIESZ BASIS for } \boldsymbol{L}^2_{\mathbb{R}} \quad \text{and} \\ 2. & \tilde{\mathsf{f}}(\omega) \triangleq \dfrac{\tilde{\mathsf{g}}(\omega)}{\sqrt{2\pi\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2}} \end{array}\right\} \implies \left\{\begin{array}{l} \{\mathbf{T}^n\mathsf{f} \,|\, n\in\mathbb{Z}\} \\ \text{is an ORTHONORMAL BASIS for } \boldsymbol{L}^2_{\mathbb{R}} \end{array}\right\}$$

✎PROOF:

1. Proof that $\{\mathbf{T}^n\mathsf{f} \,|\, n\in\mathbb{Z}\}$ is orthonormal:

$$\begin{aligned}
\tilde{S}_{\phi\phi}(\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem R.1 page 373}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{2\pi\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi(n-m))|^2}}\right|^2 && \text{by left hypothesis}\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{\tilde{\mathsf{g}}(\omega+2\pi n)}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2\\
&= \sum_{n\in\mathbb{Z}}\left|\frac{1}{\sqrt{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}}\right|^2 |\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= \frac{1}{\sum_{m\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi m)|^2}\sum_{n\in\mathbb{Z}}|\tilde{\mathsf{g}}(\omega+2\pi n)|^2\\
&= 1\\
\implies & \{\mathsf{f}_n \,|\, n\in\mathbb{Z}\} \text{ is orthonormal} && \text{by Theorem R.3 page 379}
\end{aligned}$$

[36] Wojtaszczyk (1997) page 25 ⟨Remark 2.4⟩, Vidakovic (1999), page 71, Mallat (1989), page 72, Mallat (1999), page 225, Daubechies (1992) page 140 ⟨(5.3.3)⟩

2. Proof that $\{\mathbf{T}^n\mathsf{f} \mid n\in\mathbb{Z}\}$ is a basis for $\boldsymbol{V}_0$: by Lemma L.1 page 273.

⇰

L.7 Frames in Hilbert spaces

Definition L.16. [37] *Let $\{\boldsymbol{x}_n \in X \mid n\in\mathbb{N}\}$ be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

DEF

*The set $\{\boldsymbol{x}_n\}$ is a **frame** for $\boldsymbol{H}$ if* (STABILITY CONDITION)

$$\exists A, B \in \mathbb{R}^+ \quad \text{such that} \quad A\,\|\boldsymbol{x}\|^2 \le \sum_{n=1}^{\infty} \left|\langle \boldsymbol{x} \mid \boldsymbol{x}_n\rangle\right|^2 \le B\,\|\boldsymbol{x}\|^2 \qquad \forall x\in X.$$

*The quantities A and B are **frame bounds**.*
*The quantity A' is the **optimal lower frame bound** if*
$A' = \sup\{A \in \mathbb{R}^+ | A \textit{ is a lower frame bound}\}$.
*The quantity B' is the **optimal upper frame bound** if*
$B' = \inf\{B \in \mathbb{R}^+ | B \textit{ is an upper frame bound}\}$.
*A frame is a **tight frame** if $A = B$.*
*A frame is a **normalized tight frame** (or a **Parseval frame**) if $A = B = 1$.*
*A frame $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\}$ is an **exact frame** if for some $m \in \mathbb{Z}$, $\{\boldsymbol{x}_n \mid n\in\mathbb{N}\} \setminus \{\boldsymbol{x}_m\}$ is* NOT *a frame.*

A frame is a *Parseval frame* (Definition L.16) if it satisfies *Parseval's Identity* (Theorem L.9 page 280). All orthonormal bases are Parseval frames (Theorem L.9 page 280); but not all Parseval frames are orthonormal bases.

Definition L.17. *Let $\{\boldsymbol{x}_n\}$ be a **frame** (Definition L.16 page 290) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$. *Let* $\mathbf{S}$ *be an* OPERATOR *on* $\boldsymbol{H}$.

DEF

$\mathbf{S}$ *is a **frame operator** for $\{\boldsymbol{x}_n\}$ if* $\qquad \mathbf{S}\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n\rangle \boldsymbol{x}_n \qquad \forall \mathsf{f}\in\boldsymbol{H}.$

Theorem L.16. [38] *Let* $\mathbf{S}$ *be a* FRAME OPERATOR *(Definition L.17 page 290) of a* FRAME $\{\boldsymbol{x}_n\}$ *(Definition L.16 page 290) for the* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM

(1). $\mathbf{S}$ *is* INVERTIBLE. *and*

(2). $\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \mathbf{S}^{-1}\boldsymbol{x}_n\rangle \boldsymbol{x}_n = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n\rangle \mathbf{S}^{-1}\boldsymbol{x}_n \quad \forall \mathsf{f} \in \boldsymbol{H}$

Theorem L.17. [39] *Let $\{\boldsymbol{x}_n \in X \mid n=1,2,\ldots,N\}$ be a set of vectors in a* HILBERT SPACE $\boldsymbol{H} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$.

THM

$\{\boldsymbol{x}_n\}$ *is a* FRAME *for* $\operatorname{span}\{\boldsymbol{x}_n\}$.

✎PROOF:

[37] Young (2001) pages 154–155, Christensen (2003) page 88 ⟨Definitions 5.1.1, 5.1.2⟩, Heil (2011) pages 204–205 ⟨Definition 8.2⟩, Jørgensen et al. (2008) page 267 ⟨Definition 12.22⟩, Duffin and Schaeffer (1952) page 343, Daubechies et al. (1986), page 1272

[38] Christensen (2008) pages 100–102 ⟨Theorem 5.1.7⟩

[39] Christensen (2003) page 3

1. Upper bound: Proof that there exists B such that $\sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 \le B \|\boldsymbol{x}\|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

$$\begin{aligned}
\sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 &\le \sum_{n=1}^{N} \langle \boldsymbol{x}_n \mid \boldsymbol{x}_n \rangle \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle && \text{by } \textit{Cauchy-Schwarz inequality} \\
&= \underbrace{\left\{ \sum_{n=1}^{N} \|\boldsymbol{x}_n\|^2 \right\}}_{B} \|\boldsymbol{x}\|^2
\end{aligned}$$

2. Lower bound: Proof that there exists A such that $A \|\boldsymbol{x}\|^2 \le \sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{H}$:

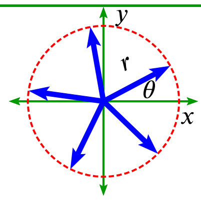
$$\begin{aligned}
\sum_{n=1}^{N} |\langle \boldsymbol{x} \mid \boldsymbol{x}_n \rangle|^2 &= \sum_{n=1}^{N} \left| \left\langle \boldsymbol{x}_n \mid \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right\rangle \right|^2 \|\boldsymbol{x}\|^2 \\
&\ge \underbrace{\left(\inf_{\boldsymbol{y}} \left\{ \sum_{n=1}^{N} |\langle \boldsymbol{x}_n \mid \boldsymbol{y} \rangle|^2 \mid \|\boldsymbol{y}\| = 1 \right\} \right)}_{A} \|\boldsymbol{x}\|^2
\end{aligned}$$

✏

Example L.1. Let $(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle)$ be an inner product space with $\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \mid \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle \triangleq x_1 x_2 + y_1 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition L.17 page 290) with *inverse* $\mathbf{S}^{-1}$.

EX

Let $N \in \{3, 4, 5, \ldots\}$, $\theta \in \mathbb{R}$, and $r \in \mathbb{R}^+$ $(r > 0)$.
Let $\boldsymbol{x}_n \triangleq r \begin{bmatrix} \cos(\theta + 2n\pi/N) \\ \sin(\theta + 2n\pi/N) \end{bmatrix} \quad \forall n \in \{0, 1, \ldots, N-1\}$.
Then, $(\!(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})\!)$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{Nr^2}{2}$.



Moreover, $\quad \mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{S}^{-1} = \frac{2}{Nr^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad$ and $\quad \boldsymbol{v} = \frac{2}{Nr^2} \sum_{n=0}^{N-1} \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \quad \forall \boldsymbol{v} \in \mathbb{R}^2$.

✎PROOF:

1. Proof that $(\!(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})\!)$ is a *tight frame* with *frame bound* $A = \frac{Nr^2}{2}$: Let $\boldsymbol{v} \triangleq (x, y) \in \mathbb{R}^2$.

$$\begin{aligned}
\sum_{n=0}^{N-1} |\langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle|^2 &\triangleq \sum_{n=0}^{N-1} \left| \boldsymbol{v}^{\mathbf{H}} r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \right|^2 && \text{by definitions of } \boldsymbol{v} \text{ of } \langle \boldsymbol{y} \mid \boldsymbol{x} \rangle \\
&\triangleq \sum_{n=0}^{N-1} r^2 \left| x\cos\left(\theta + \frac{2n\pi}{N}\right) + y\sin\left(\theta + \frac{2n\pi}{N}\right) \right|^2 && \text{by definition of } \boldsymbol{y}^{\mathbf{H}}\boldsymbol{x} \text{ operation} \\
&= r^2 x^2 \sum_{n=0}^{N-1} \cos^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 y^2 \sum_{n=0}^{N-1} \sin^2\left(\theta + \frac{2n\pi}{N}\right) + r^2 xy \sum_{n=0}^{N-1} \cos\left(\theta + \frac{2n\pi}{N}\right) \sin\left(\theta + \frac{2n\pi}{N}\right) \\
&= r^2 x^2 \frac{N}{2} + r^2 y^2 \frac{N}{2} + r^2 xy0 && \text{by Corollary I.1 page 249} \\
&= (x^2 + y^2) \frac{Nr^2}{2} = \left(\frac{Nr^2}{2} \right) \boldsymbol{v}^{\mathbf{H}} \boldsymbol{v} \triangleq \underbrace{\left(\frac{Nr^2}{2} \right)}_{A} \|\boldsymbol{v}\|^2 && \text{by definition of } \|\boldsymbol{v}\|
\end{aligned}$$

2. Proof that $\mathbf{S} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

(a) Let $\boldsymbol{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) lemma: $\mathbf{S}\boldsymbol{e}_1 = \frac{Nr^2}{2}\boldsymbol{e}_1$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_1 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\cos\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \cos^2\left(\theta + \frac{2n\pi}{N}\right) \\ \cos\left(\theta + \frac{2n\pi}{N}\right)\sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} N/2 \\ 0 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_1 \qquad \text{by *Summation around unit circle* (Corollary I.1 page 249)}
\end{aligned}$$

(c) lemma: $\mathbf{S}\boldsymbol{e}_2 = \frac{Nr^2}{2}\boldsymbol{e}_2$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_2 &= \sum_{n=0}^{N-1} \langle \boldsymbol{e}_2 \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n \\
&= \sum_{n=0}^{N-1} r\sin\left(\theta + \frac{2n\pi}{N}\right) r \begin{bmatrix} \cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} = r^2 \sum_{n=0}^{N-1} \begin{bmatrix} \sin\left(\theta + \frac{2n\pi}{N}\right)\cos\left(\theta + \frac{2n\pi}{N}\right) \\ \sin^2\left(\theta + \frac{2n\pi}{N}\right) \end{bmatrix} \\
&= r^2 \begin{bmatrix} 0 \\ N/2 \end{bmatrix} = \frac{Nr^2}{2}\boldsymbol{e}_2 \qquad \text{by *Summation around unit circle* (Corollary I.1 page 249)}
\end{aligned}$$

(d) Complete the proof of item (2) using *Eigendecomposition* $\mathbf{S} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$:

$$\begin{aligned}
\mathbf{S}\boldsymbol{e}_1 &= \frac{Nr^2}{2}\boldsymbol{e}_1 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_1 \text{ is an } \textit{eigenvector} \text{ of } \mathbf{S} \text{ with } \textit{eigenvalue } \frac{Nr^2}{2} \\
\mathbf{S}\boldsymbol{e}_2 &= \frac{Nr^2}{2}\boldsymbol{e}_2 && \text{by (2c) lemma} \\
&\implies \boldsymbol{e}_2 \text{ is an } \textit{eigenvector} \text{ of } \mathbf{S} \text{ with } \textit{eigenvalue } \frac{Nr^2}{2}
\end{aligned}$$

$$\mathbf{S} = \overbrace{\underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} | & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ | & | \end{bmatrix}^{-1}}_{\mathbf{Q}^{-1}}}^{\textit{Eigendecomposition} \text{ of } \mathbf{S}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Nr^2}{2} & 0 \\ 0 & \frac{Nr^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{Nr^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Proof that $\mathbf{S}^{-1} = \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$\begin{aligned}
\mathbf{S}\mathbf{S}^{-1} &= \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)} \\
\mathbf{S}^{-1}\mathbf{S} &= \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{Nr^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq \mathbf{I}_2 && \text{by item (2)}
\end{aligned}$$

4. Proof that $\boldsymbol{v} = \frac{2}{Nr^2}\sum_{n=0}^{N-1} \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n\right)\boldsymbol{x}_n$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n = \sum_{n=0}^{N-1} \left\langle \boldsymbol{v} \,|\, \frac{2}{Nr^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n && \text{by item (3)} \\
&= \frac{2}{Nr^2}\sum_{n=0}^{N-1} \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \frac{2}{Nr^2}\sum_{n=0}^{N-1} \left(\boldsymbol{v}^{\mathbf{H}}\boldsymbol{x}_n\right)\boldsymbol{x}_n && \text{by definition of } \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle
\end{aligned}$$

Example L.2 (Peace Frame/Mercedes Frame). [40] Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ be an inner product space with $\left\langle (x_1, y_1) \mid (x_2, y_2) \right\rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition L.17 page 290) with *inverse* $\mathbf{S}^{-1}$.

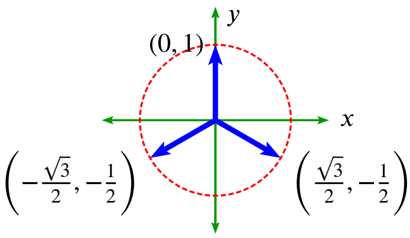
EX

Let $\boldsymbol{x}_1 \triangleq \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\boldsymbol{x}_2 \triangleq \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}$, and $\boldsymbol{x}_3 \triangleq \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$.

Then, $(\!(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\!)$ is a **tight frame** for $\mathbb{R}^2$ with *frame bound* $A = \frac{3}{2}$.

Moreover, $\mathbf{S} = \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{S}^{-1} = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

and $$\boldsymbol{v} = \frac{2}{3}\sum_{n=1}^{3} \langle \boldsymbol{v} \mid \boldsymbol{x}_n \rangle \boldsymbol{x}_n \triangleq \frac{2}{3}\sum_{n=1}^{3} \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_n\right) \boldsymbol{x}_n \qquad \forall \boldsymbol{v} \in \mathbb{R}^2.$$



✎PROOF:

1. This frame is simply a special case of the frame presented in Example L.1 (page 291) with $r = 1$, $N = 3$, and $\theta = \pi/2$.

2. Let's give it a try! Let $\boldsymbol{v} \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned}
\sum_{n=1}^{3} \left\langle \boldsymbol{v} \mid \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n &= \frac{2}{3}\sum_{n=1}^{3} \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_n\right) \boldsymbol{x}_n && \text{by Example L.1 page 291} \\
&= \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_1\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_2\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}} \boldsymbol{x}_3\right)\boldsymbol{x}_3 \\
&= \frac{2}{3}\left(\left(\boldsymbol{v}^{\mathbf{H}} \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}} \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}} \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{2}{3}\cdot\frac{1}{2}\left(\left(\boldsymbol{v}^{\mathbf{H}} \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\boldsymbol{x}_1 + \left(\boldsymbol{v}^{\mathbf{H}} \begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_2 + \left(\boldsymbol{v}^{\mathbf{H}} \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right)\boldsymbol{x}_3\right) \\
&= \frac{1}{3}\left((2)\frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \left(-\sqrt{3}-1\right)\frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix} + \left(\sqrt{3}-1\right)\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}\right) \\
&= \frac{1}{6}\begin{bmatrix} 2(0) & + & (-\sqrt{3}-1)(-\sqrt{3}) & + & (\sqrt{3}-1)(\sqrt{3}) \\ 2(2) & + & (-\sqrt{3}-1)(-1) & + & (\sqrt{3}-1)(-1) \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 0 & + & (3+\sqrt{3}) & + & (3-\sqrt{3}) \\ 4 & + & (1+\sqrt{3}) & + & (1-\sqrt{3}) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \triangleq \boldsymbol{v}
\end{aligned}$$

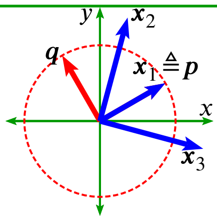
In Example L.1 (page 291) and Example L.2 (page 293), the frame operator $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ were computed. In general however, it is not always necessary or even possible to compute these, as illustrated in Example L.3 (next).

Example L.3. [41] Let $\left(\mathbb{R}^2, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$ be an inner product space with $\left\langle (x_1, y_1) \mid (x_2, y_2) \right\rangle \triangleq x_1 y_1 + x_2 y_2$. Let $\mathbf{S}$ be the *frame operator* (Definition L.17 page 290) with *inverse* $\mathbf{S}^{-1}$.

[40] Heil (2011) pages 204–205 ⟨$r = 1$ case⟩, Byrne (2005) page 80 ⟨$r = 1$ case⟩, Han et al. (2007) page 91 ⟨Example 3.9, $r = \sqrt{2/3}$ case⟩

[41] Christensen (2003) pages 7–8 ⟨?⟩

EX Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be *orthonormal* vectors in $\boldsymbol{X} \triangleq \operatorname{span}\{\boldsymbol{p}, \boldsymbol{q}\}$.
Let $\boldsymbol{x}_1 \triangleq \boldsymbol{p}$, $\boldsymbol{x}_2 \triangleq \boldsymbol{p}+\boldsymbol{q}$, and $\boldsymbol{x}_3 \triangleq \boldsymbol{p}-\boldsymbol{q}$.
Then, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, , \boldsymbol{x}_3\}$ is a **frame** for $\boldsymbol{X}$ with *frame bounds* $A=0$ and $B=5$.



Moreover,
$\mathbf{S}^{-1}\boldsymbol{x}_1 = {}^1\!/\!_3\boldsymbol{p}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_2 = {}^1\!/\!_3\boldsymbol{p} + {}^1\!/\!_2\boldsymbol{q}$ and
$\mathbf{S}^{-1}\boldsymbol{x}_3 = {}^1\!/\!_3\boldsymbol{p} - {}^1\!/\!_2\boldsymbol{q}$.

✎PROOF:

1. Proof that $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is a *frame* with *frame bounds* $A=0$ and $B=5$:

$$\begin{aligned}
\sum_{n=1}^{3} \left|\langle \boldsymbol{v} \,|\, \boldsymbol{x}_n\rangle\right|^2 &\triangleq |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle|^2 && \text{by definitions of } \boldsymbol{x}_1, \boldsymbol{x}_2, \text{ and } \boldsymbol{x}_3\\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 && \text{by } \textit{additivity} \text{ of } \langle \triangle \,|\, \nabla\rangle \text{ (Definition D.9 page 168)}\\
&= |\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 + \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* + \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right)\\
&\quad + \left(|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + |\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2 - \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle^* - \langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle \langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle^*\right)\\
&= 3|\langle \boldsymbol{v} \,|\, \boldsymbol{p}\rangle|^2 + 2|\langle \boldsymbol{v} \,|\, \boldsymbol{q}\rangle|^2\\
&\boxed{\leq} 3\,\|\boldsymbol{v}\|\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{v}\|\,\|\boldsymbol{q}\| && \text{by } \textit{CS Inequality}\\
&= \|\boldsymbol{v}\|\,(3\,\|\boldsymbol{p}\| + 2\,\|\boldsymbol{q}\|)\\
&= \boxed{5}\,\|\boldsymbol{v}\| && \text{by } \textit{orthonormality} \text{ of } \boldsymbol{p} \text{ and } \boldsymbol{q}
\end{aligned}$$

2. lemma: $\mathbf{S}\boldsymbol{p} = 3\boldsymbol{p}$, $\mathbf{S}\boldsymbol{q} = 2\boldsymbol{q}$, $\mathbf{S}^{-1}\boldsymbol{p} = \frac{1}{3}\boldsymbol{p}$, and $\mathbf{S}^{-1}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}$. Proof:

$$\begin{aligned}
\mathbf{S}\boldsymbol{p} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{p} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n\\
&= \langle \boldsymbol{p} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{p} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\, (\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{p} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\, (\boldsymbol{p}-\boldsymbol{q})\\
&= (1)\boldsymbol{p} + (1+0)(\boldsymbol{p}+\boldsymbol{q}) + (1-0)(\boldsymbol{p}-\boldsymbol{q})\\
&= 3\boldsymbol{p}\\
\implies \mathbf{S}^{-1}\boldsymbol{p} &= \frac{1}{3}\boldsymbol{p}\\
\mathbf{S}\boldsymbol{q} &\triangleq \sum_{n=1}^{3} \langle \boldsymbol{q} \,|\, \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n\\
&= \langle \boldsymbol{q} \,|\, \boldsymbol{p}\rangle\, \boldsymbol{p} + \langle \boldsymbol{q} \,|\, \boldsymbol{p}+\boldsymbol{q}\rangle\, (\boldsymbol{p}+\boldsymbol{q}) + \langle \boldsymbol{q} \,|\, \boldsymbol{p}-\boldsymbol{q}\rangle\, (\boldsymbol{p}-\boldsymbol{q})\\
&= (0)\boldsymbol{q} + (0+1)(\boldsymbol{p}+\boldsymbol{q}) + (0-1)(\boldsymbol{p}-\boldsymbol{q})\\
&= 2\boldsymbol{q}\\
\implies \mathbf{S}^{-1}\boldsymbol{q} &= \frac{1}{2}\boldsymbol{q}
\end{aligned}$$

3. Remark: Without knowing $\boldsymbol{p}$ and $\boldsymbol{q}$, from (2) lemma it follows that it is not possible to compute $\mathbf{S}$ or $\mathbf{S}^{-1}$ explicitly.

4. Proof that $\mathbf{S}^{-1}\boldsymbol{x}_1 = {}^1\!/\!_3\boldsymbol{p}$, $\mathbf{S}^{-1}\boldsymbol{x}_2 = {}^1\!/\!_3\boldsymbol{p} + {}^1\!/\!_2\boldsymbol{q}$ and $\mathbf{S}^{-1}\boldsymbol{x}_3 = {}^1\!/\!_3\boldsymbol{p} - {}^1\!/\!_2\boldsymbol{q}$:

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_1 &\triangleq \mathbf{S}^{-1}\boldsymbol{p} && \text{by definition of } \boldsymbol{x}_1\\
&= \frac{1}{3}\boldsymbol{p} && \text{by (2) lemma}\\
\mathbf{S}^{-1}\boldsymbol{x}_2 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2\\
&= \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

$$\begin{aligned}
\mathbf{S}^{-1}\boldsymbol{x}_3 &\triangleq \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) && \text{by definition of } \boldsymbol{x}_2 \\
&= \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q} && \text{by (2) lemma}
\end{aligned}$$

5. Check that $\boldsymbol{v} = \sum_n \langle \boldsymbol{v} \,|\, \boldsymbol{x}_n \rangle \boldsymbol{x}_n = \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}$:

$$\begin{aligned}
\boldsymbol{v} &= \sum_{n=1}^{3} \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{x}_n \right\rangle \boldsymbol{x}_n \\
&= \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}+\boldsymbol{q}) \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \mathbf{S}^{-1}(\boldsymbol{p}-\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q}) \\
&= \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} \right\rangle \boldsymbol{p} + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q} \right\rangle (\boldsymbol{p}+\boldsymbol{q}) + \left\langle \boldsymbol{v} \,|\, \frac{1}{3}\boldsymbol{p} - \frac{1}{2}\boldsymbol{q}) \right\rangle (\boldsymbol{p}-\boldsymbol{q}) \\
&= \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \left(\frac{1}{3}-\frac{1}{3}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{q} + \left(\frac{1}{2}-\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{p} + \left(\frac{1}{2}+\frac{1}{2}\right) \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q} \\
&= \langle \boldsymbol{v} \,|\, \boldsymbol{p} \rangle \boldsymbol{p} + \langle \boldsymbol{v} \,|\, \boldsymbol{q} \rangle \boldsymbol{q}
\end{aligned}$$

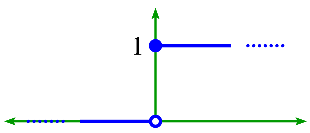
✏

APPENDIX M

B-SPLINES

M.1 Definitions

Definition M.1. *Let X be a set.*



DEF *The **step function** $\sigma \in \mathbb{R}^{\mathbb{R}}$ is defined as*

$$\sigma(x) \triangleq \mathbb{1}_{[0:\infty)}(x) \qquad \forall x \in \mathbb{R}.$$

Lemma M.1. *Let $\sigma(x)$ be the* STEP FUNCTION (Definition M.1 page 297).

LEM
$$\{\mathsf{g}(x) > 0\} \implies \left\{\sigma\left[\mathsf{g}(x)\mathsf{f}(x)\right] = \sigma[\mathsf{f}(x)]\right\} \qquad \forall \mathsf{f},\mathsf{g} \in \mathbb{R}^{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\sigma\left[\mathsf{g}(x)\mathsf{f}(x)\right] &\triangleq \mathbb{1}_{[0:\infty)}\left[\mathsf{g}(x)\mathsf{f}(x)\right] && \text{by definition of } \sigma(x) && \text{(Definition M.1 page 297)}\\
&\triangleq \begin{cases} 1 & \text{for } \mathsf{g}(x)\mathsf{f}(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 1)}\\
&= \begin{cases} 1 & \text{for } \mathsf{f}(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} && \text{by } \mathsf{g}(x) > 0 \text{ hypothesis} &&\\
&\triangleq \mathbb{1}_{[0:\infty)}[\mathsf{f}(x)] && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 1)}\\
&\triangleq \sigma[\mathsf{f}(x)] && \text{by definition of } \sigma(x) && \text{(Definition M.1 page 297)}
\end{aligned}$$

✏

Definition M.2. [1] *Let $\mathbb{1}$ be the* SET INDICATOR *function* (Definition 1.2 page 1). *Let $\mathsf{f}(x) \star \mathsf{g}(x)$ represent the* CONVOLUTION *operation* (Definition K.3 page 260).

DEF *The n**th order cardinal B-spline** $\mathsf{N}_n(x)$ for $n \in \mathbb{W}$ is defined as*

$$\mathsf{N}_n(x) \triangleq \left\{ \begin{array}{ll} \mathbb{1}_{[0:1)}(x) & \text{for } n = 0 \\ \mathsf{N}_{n-1}(x) \star \mathsf{N}_0(x) & \text{for } n \in \mathbb{W}\backslash 0 \end{array} \right\} \qquad \forall x \in \mathbb{R}$$

Lemma M.2. [2]

[1] Chui (1992) page 85 ⟨(4.2.1)⟩, Christensen (2008) page 140, Chui (1988) page 1

[2] Christensen (2008) page 140, Chui (1992) page 85 ⟨(4.2.1)⟩, Chui (1988) page 1, Prasad and Iyengar (1997) page 145

LEM
$$\mathsf{N}_n(x) = \int_{\tau=0}^{\tau=1} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau \qquad \forall n \in \{1,2,3,\ldots\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{N}_n(x) &\triangleq \mathsf{N}_{n-1}(x) \star \mathsf{N}_0(x) && \text{by definition of } \mathsf{N}_n(x) && \text{(Definition M.2 page 297)}\\
&\triangleq \int_{\mathbb{R}} \mathsf{N}_{n-1}(x-\tau)\mathsf{N}_0(\tau)\,\mathrm{d}\tau && \text{by definition of convolution operation } \star && \text{(Definition K.3 page 260)}\\
&\triangleq \int_{\mathbb{R}} \mathsf{N}_{n-1}(x-\tau)\mathbb{1}_{[0:1)}(\tau)\,\mathrm{d}\tau && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition M.2 page 297)}\\
&= \int_{[0:1)} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 1)}\\
&= \int_{[0:1]} \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau\\
&\triangleq \int_0^1 \mathsf{N}_{n-1}(x-\tau)\,\mathrm{d}\tau
\end{aligned}$$

✏

Lemma M.3. *Let* $\mathsf{f}(x)$ *be a* FUNCTION *in* $\mathbb{R}^{\mathbb{R}}$. *Let* $\mathsf{F}(x)$ *be the* ANTI-DERIVATIVE *of* $\mathsf{f}(x)$. *Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition M.1 page 297)*.

LEM
$$\begin{aligned}
&\int_{y=a}^{y=b} \mathsf{f}(x-y)\sigma(x-y)\,\mathrm{d}y\\
&= \left\{\begin{array}{ll} -\int_{y=x-a}^{y=x-b} \mathsf{f}(y)\,\mathrm{d}y & \text{for } x \ge b\\ -\int_{y=x-a}^{y=0} \mathsf{f}(y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} = \left\{\begin{array}{ll} \mathsf{F}(x-a)-\mathsf{F}(x-b) & \text{for } x \ge b\\ \mathsf{F}(x-a)-\mathsf{F}(0) & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\}\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(0)-\mathsf{F}(x-b)]\sigma(x-b)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\int_{y=a}^{y=b} \mathsf{f}(x-y)\sigma(x-y)\,\mathrm{d}y &= \left\{\begin{array}{ll} \int_{y=a}^{y=b} \mathsf{f}(x-y)\,\mathrm{d}y & \text{for } x \ge b\\ \int_{y=a}^{y=x} \mathsf{f}(x-y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by definition of } \sigma \text{ (Definition M.1 page 297)}\\
&= \left\{\begin{array}{ll} -\int_{u=x-a}^{u=x-b} \mathsf{f}(u)\,\mathrm{d}u & \text{for } x \ge b\\ -\int_{u=x-a}^{u=0} \mathsf{f}(u)\,\mathrm{d}u & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{where } u \triangleq x-y \implies y = x-u\\
&= \left\{\begin{array}{ll} -\int_{y=x-a}^{y=x-b} \mathsf{f}(y)\,\mathrm{d}y & \text{for } x \ge b\\ -\int_{y=x-a}^{y=0} \mathsf{f}(y)\,\mathrm{d}y & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by change of dummy variable } (u \to y)\\
&= \left\{\begin{array}{ll} \mathsf{F}(x-a)-\mathsf{F}(x-b) & \text{for } x \ge b\\ \mathsf{F}(x-a)-\mathsf{F}(0) & \text{for } a \le x \le b\\ 0 & \text{for } x \le a \end{array}\right\} && \text{by \textit{Fundamental Theorem of Calculus}}\\
&= [\mathsf{F}(x-a)-\mathsf{F}(x-b)]\sigma(x-b) + [\mathsf{F}(x-a)-\mathsf{F}(0)][\sigma(x-a)-\sigma(x-b)]\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(x-a)-\mathsf{F}(x-b)-\mathsf{F}(x-a)+\mathsf{F}(0)]\sigma(x-b)\\
&= [\mathsf{F}(x-a)-\mathsf{F}(0)]\sigma(x-a) + [\mathsf{F}(0)-\mathsf{F}(x-b)]\sigma(x-b)
\end{aligned}$$

✏

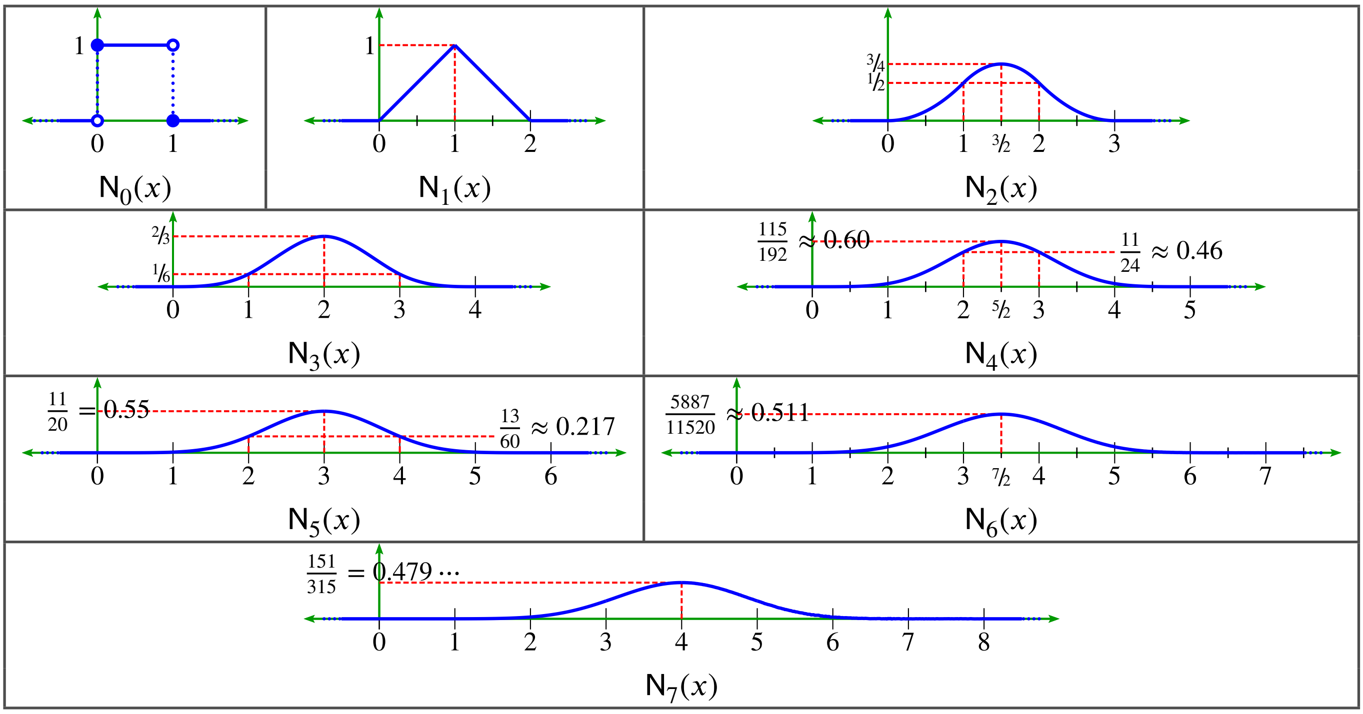


Figure M.1: some low order B-splines (Example M.1 page 299)

Lemma M.4. *Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition M.1 page 297).*

LEM

$$\int_{\tau=0}^{\tau=1} (x-\tau-k)^n \sigma(x-\tau-k)\,\mathrm{d}\tau = \frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k) - (x-k-1)^{n+1}\sigma(x-k-1)\right]$$

✎PROOF:

$$\begin{aligned}
&\int_{\tau=0}^{\tau=1} (x-\tau-k)^n \sigma(x-\tau-k)\,\mathrm{d}\tau \\
&= \int_{y=k}^{y=k+1} (x-y)^n \sigma(x-y)\,\mathrm{d}y && \text{where } y \triangleq \tau+k \implies \tau = y-k \\
&= [\mathsf{F}(x-k)-\mathsf{F}(0)]\sigma(x-k) + [\mathsf{F}(0)-\mathsf{F}(x-k-1)]\sigma(x-k-1) && \text{by Lemma M.3 (page 298), where } \mathsf{f}(x) \triangleq x^n \\
&= \frac{\left[(x-k)^{n+1}-0\right]\sigma(x-k) + \left[0-(x-k-1)^{n+1}\right]\sigma(x-k-1)}{n+1} && \text{because } \mathsf{F}(x) \triangleq \int \mathsf{f}(x)\,\mathrm{d}x = \frac{x^{n+1}}{n+1}+c \\
&= \frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k) - (x-k-1)^{n+1}\sigma(x-k-1)\right]
\end{aligned}$$

✏

Example M.1. [3] Let $\sigma(x)$ be the *step function* (Definition M.1 page 297). Let $\binom{n}{k}$ be the *binomial coefficient* (Definition B.3 page 132). The 0*th order B-spline* (Definition M.2 page 297) $\mathsf{N}_0(x)$ can be expressed as follows:

EX

$$\mathsf{N}_0(x) = \left\{\begin{array}{ll} 1 & \text{for } x \in [0:1) \\ 0 & \text{otherwise} \end{array}\right\} = \left\{\sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-k)^0\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_0(x)$ is illustrated in Figure M.1 (page 299).

[3] Schumaker (2007) page 136 ⟨Table 1⟩

✎PROOF:

$$\begin{aligned}
\mathsf{N}_0(x) &= \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition M.2 page 297)}\\
&= \sigma(x) - \sigma(x-1) && \text{by definition of } \sigma(x) && \text{(Definition M.1 page 297)}\\
&= \left[\binom{1}{0}\sigma(x) - \binom{1}{1}\sigma(x-1)\right] && \text{by definition of \textit{binomial coefficient} } \binom{n}{k} && \text{(Definition B.3 page 132)}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-k)^0\sigma(x-k) && \text{by definition of } \sum \text{ operator}
\end{aligned}$$

✏

Example M.2. [4] Let $\sigma(x)$ be the *step function*. Let $\binom{n}{k}$ be the *binomial coefficient*. The 1*st order B-spline* $\mathsf{N}_1(x)$ can be expressed as follows:

EX

$$\mathsf{N}_1(x) = \left\{\begin{array}{rl} x & \text{for } x \in [0:1] \\ -x+2 & \text{for } x \in [1:2] \\ 0 & \text{otherwise} \end{array}\right\} = \left\{\sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_1(x)$ is illustrated in Figure M.1 (page 299).

✎PROOF:

$$\begin{aligned}
\mathsf{N}_1(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_0(x-\tau)\,\mathrm{d}\tau && \text{by Lemma M.2 page 297}\\
&= \int_{\tau=0}^{\tau=1} \sum_{k=0}^{1}(-1)^k\binom{1}{k}(x-\tau-k)^0\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by Example M.1 page 299}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}\int_{\tau=0}^{\tau=1}(x-\tau-k)^0\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by \textit{linearity} of } \int \mathrm{d}\tau \text{ operator}\\
&= \sum_{k=0}^{1}(-1)^k\binom{1}{k}\frac{1}{0+1}\left[(x-k)^{0+1}\sigma(x-k)-(x-k-1)^{0+1}\sigma(x-k-1)\right] && \text{by Lemma M.4 page 299}\\
&= \begin{pmatrix} 1\{(x-0)\sigma(x-0)-(x-1)\sigma(x-1)\} \\ -1\{(x-1)\sigma(x-1)-(x-2)\sigma(x-2)\} \end{pmatrix}\\
&= x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-k)\sigma(x-k) && \text{by def. of } \binom{n}{k} \text{ (Definition B.3 page 132)}\\
&= \left\{\begin{array}{ll} x & \text{for } x\in[0:1) \\ -x+2 & \text{for } x\in[1:2] \\ 0 & \text{otherwise} \end{array}\right\} && \text{by def. of } \sigma(x) \text{ (Definition M.1 page 297)}
\end{aligned}$$

✏

Example M.3. [5] Let $\sigma(x)$ be the *step function*. Let $\binom{n}{k}$ be the *binomial coefficient*. The 2*nd order B-spline* $\mathsf{N}_2(x)$ can be expressed as follows:

EX

$$\mathsf{N}_2(x) = \frac{1}{2}\left\{\begin{array}{rl} x^2 & \text{for } x\in[0:1) \\ -2x^2+6x-3 & \text{for } x\in[1:2] \\ x^2-6x+9 & \text{for } x\in[2:3] \\ 0 & \text{otherwise} \end{array}\right\} = \left\{\frac{1}{2}\sum_{k=0}^{3}(-1)^k\binom{3}{k}(x-k)^2\sigma(x-k) \quad \forall x\in\mathbb{R}\right\}$$

The B-spline $\mathsf{N}_2(x)$ is illustrated in Figure M.1 (page 299).

[4] Christensen (2008) page 148 ⟨Exercise 6.2⟩, Christensen (2010) page 212 ⟨Exercise 10.2⟩, Heil (2011) pages 142–143 ⟨Definition 4.22 (The Schauder System)⟩, Schumaker (2007) page 136 ⟨Table 1⟩, Stoer and Bulirsch (2002) page 124

[5] Christensen (2008) page 148 ⟨Exercise 6.2⟩, Christensen (2010) page 212 ⟨Exercise 10.2⟩, Schumaker (2007) page 136 ⟨Table 1⟩, Stoer and Bulirsch (2002) page 124

✎PROOF:

$$\begin{aligned}
\mathsf{N}_2(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_1(x-\tau)\,\mathrm{d}\tau && \text{by Lemma M.2 page 297}\\
&= \int_{\tau=0}^{\tau=1} \sum_{k=0}^{2}(-1)^k\binom{2}{k}(x-\tau-k)\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by Example M.2 page 300}\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\int_{\tau=0}^{\tau=1}(x-\tau-k)\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by } \textit{linearity} \text{ of } \int \mathrm{d}\tau \text{ operator}\\
&= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\frac{1}{1+1}\left[(x-k)^{1+1}\sigma(x-k)-(x-k-1)^{1+1}\sigma(x-k-1)\right] && \text{by Lemma M.4 page 299}\\
&= \frac{1}{2}\left(\begin{array}{ll} 1 & \{(x-0)^2\sigma(x-0)-(x-1)^2\sigma(x-1)\}\\ -2 & \{(x-1)^2\sigma(x-1)-(x-2)^2\sigma(x-2)\}\\ +1 & \{(x-2)^2\sigma(x-2)-(x-3)^2\sigma(x-3)\}\end{array}\right)\\
&= \frac{1}{2}\left[x^2\sigma(x)-3(x-1)^2\sigma(x-1)+3(x-2)^2\sigma(x-2)-(x-3)^2\sigma(x-3)\right]\\
&= \frac{1}{2}\sum_{k=0}^{3}(-1)^k\binom{3}{k}(x-k)^2\sigma(x-k) && \text{by def. of } \binom{n}{k} \text{ (Definition B.3 page 132)}\\
&= \frac{1}{2}\left\{\begin{array}{rl} x^2 & \text{for } x\in[0:1)\\ -2x^2+6x-3 & \text{for } x\in[1:2]\\ x^2-6x+9 & \text{for } x\in[2:3]\\ 0 & \text{otherwise}\end{array}\right\} && \text{by def. of } \sigma(x) \text{ (Definition M.1 page 297)}
\end{aligned}$$

The final steps of this proof can be calculated "by hand" or by using the free and open source software package *Maxima* along with the script file listed in Section V.2 (page 409). ✏

M.2 Algebraic properties

Theorem M.1 (next) presents a closed form expression for an *nth order B-spline* $\mathsf{N}_n(x)$ based on the definition of $\mathsf{N}_n(x)$ given in Definition M.2 (page 297). Alternatively, Theorem M.1 could serve as the definition and Definition M.2 as a property.

Theorem M.1. [6] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition M.2 page 297). Let* $\sigma(x)$ *be the* STEP FUNCTION *(Definition M.1 page 297).*

THM
$$\mathsf{N}_n(x) = \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n\sigma(x-k) \qquad \forall n\in\{0,1,2,\ldots\}=\mathbb{W}$$

✎PROOF: Proof follows by induction:

1. base case (choose one):
   - Proof for $n=0$ case: by Example M.1 (page 299).
   - Proof for $n=1$ case: by Example M.2 (page 300).
   - Proof for $n=2$ case: by Example M.3 (page 300).

[6] Christensen (2008) page 142 ⟨Theorem 6.1.3⟩, Chui (1992) page 84 ⟨(4.1.12)⟩

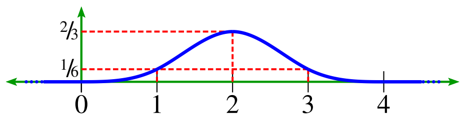
2. inductive step—proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\mathsf{N}_{n+1}(x)\\
&= \int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma M.2 page 297}\\
&= \int_0^1 \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-\tau-k)^n\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by induction hypothesis}\\
&= \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\int_0^1(x-\tau-k)^n\sigma(x-\tau-k)\,\mathrm{d}\tau && \text{by \textit{linearity} of } \int \mathrm{d}\tau \text{ operator}\\
&= \frac{1}{n!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\frac{1}{n+1}\left[(x-k)^{n+1}\sigma(x-k)-(x-k-1)^{n+1}\sigma(x-k-1)\right] && \text{by Lemma M.4 page 299}\\
&= \frac{1}{(n+1)!}\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}\left[(x-k)^{n+1}\sigma(x-k)-(x-k-1)^{n+1}\sigma(x-k-1)\right]\\
&= \frac{1}{(n+1)!}\left[\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)-\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k-1)^{n+1}\sigma(x-k-1)\right]\\
&= \frac{1}{(n+1)!}\left[\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)-\sum_{m=1}^{m=n+2}(-1)^{m-1}\binom{n+1}{m-1}(x-m)^{n+1}\sigma(x-m)\right]\\
&\qquad \text{where } m\triangleq k+1 \implies k=m-1\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l}\displaystyle\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)\\ \displaystyle-\sum_{m=1}^{m=n+2}(-1)^{m-1}\left[\binom{n+2}{m}-\binom{n+1}{m}\right](x-m)^{n+1}\sigma(x-m)\end{array}\right) && \begin{array}{l}\text{by \textit{Pascal's identity / Stifel formula}}\\ \text{(Theorem B.1 page 133)}\end{array}\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l}\displaystyle\sum_{m=1}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m)\\ \displaystyle-\sum_{m=1}^{m=n+2}(-1)^m\binom{n+1}{m}(x-m)^{n+1}\sigma(x-m)\\ \displaystyle+\sum_{k=0}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k)\end{array}\right) && \text{note } (-1)^{m-1}=-(-1)^m\\
&= \frac{1}{(n+1)!}\left(\begin{array}{l|ll}\displaystyle\sum_{m=0}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m) & \text{(A)} & \text{desired } n+1 \text{ case}\\ \displaystyle-\qquad(-1)^0\binom{n+2}{0}(x-0)^{n+1}\sigma(x-0) & \text{(B)} & \text{cancelled by (F)}\\ \displaystyle-\sum_{m=1}^{m=n+1}(-1)^m\binom{n+1}{m}(x-m)^{n+1}\sigma(x-m) & \text{(C)} & \text{cancelled by (E)}\\ \displaystyle-\qquad(-1)^{n+2}\binom{n+1}{n+2}(x-n-2)^{n+1}\sigma(x-n-2) & \text{(D)} & \binom{n+1}{n+2}=0 \text{ by Proposition B.2 page 132}\\ \displaystyle+\sum_{k=1}^{k=n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1}\sigma(x-k) & \text{(E)} & \text{cancelled by (C)}\\ \displaystyle+\qquad(-1)^0\binom{n+1}{0}(x-0)^{n+1}\sigma(x-0) & \text{(F)} & \binom{n+2}{0}=\binom{n+1}{0}=1\text{, so (F) is cancelled by (B)}\end{array}\right)\\
&= \frac{1}{(n+1)!}\sum_{m=0}^{m=n+2}(-1)^m\binom{n+2}{m}(x-m)^{n+1}\sigma(x-m) && (n+1 \text{ case})
\end{aligned}$$

Example M.4. [7] Let $\mathsf{N}_3(x)$ be the 3*rd order B-spline* (Definition M.2 page 297).[8]

EX

$$\mathsf{N}_3(x) = \frac{1}{6}\begin{cases} x^3 & \text{for } 0 \le x \le 1 \\ -3x^3+12x^2-12x+4 & \text{for } 1 \le x \le 2 \\ 3x^3-24x^2+60x-44 & \text{for } 2 \le x \le 3 \\ -x^3+12x^2-48x+64 & \text{for } 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$



✎PROOF: This expression can be calculated "by hand" using Theorem M.1 (page 301) or by using the free and open source software package *Maxima* along with the script file listed in Section V.2 (page 409). ⇨

Example M.5. Let $\mathsf{N}_4(x)$ be the 4*th order B-spline* (Definition M.2 page 297).

EX

$$\mathsf{N}_4(x) = \frac{1}{24}\begin{cases} x^4 & \text{for } 0 \le x \le= 1 \\ -4x^4+20x^3-30x^2+20x-5 & \text{for } 1 \le x \le= 2 \\ 6x^4-60x^3+210x^2-300x+155 & \text{for } 2 \le x \le= 3 \\ -4x^4+60x^3-330x^2+780x-655 & \text{for } 3 \le x \le= 4 \\ x^4-20x^3+150x^2-500x+625 & \text{for } 4 \le x \le= 5 \\ 0 & \text{otherwise} \end{cases}$$

✎PROOF: This expression can be calculated "by hand" using Theorem M.1 (page 301) or by using the free and open source software package *Maxima* along with the script file listed in Section V.2 (page 409). ⇨

Example M.6. Let $\mathsf{N}_5(x)$ be the 5*th order B-spline* (Definition M.2 page 297).

EX

$$\mathsf{N}_5(x) = \frac{1}{120}\begin{cases} x^5 & \text{for } 0 \le x \le= 1 \\ -5x^5+30x^4-60x^3+60x^2-30x+6 & \text{for } 1 \le x \le= 2 \\ 10x^5-120x^4+540x^3-1140x^2+1170x-474 & \text{for } 2 \le x \le= 3 \\ -10x^5+180x^4-1260x^3+4260x^2-6930x+4386 & \text{for } 3 \le x \le= 4 \\ 5x^5-120x^4+1140x^3-5340x^2+12270x-10974 & \text{for } 4 \le x \le= 5 \\ -x^5+30x^4-360x^3+2160x^2-6480x+7776 & \text{for } 5 \le x \le= 6 \\ 0 & \text{otherwise} \end{cases}$$

The 5*th order B-spline* $\mathsf{N}_5(x)$ is illustrated in Figure M.1 (page 299).

✎PROOF: This expression can be calculated "by hand" using Theorem M.1 (page 301) or by using the free and open source software package *Maxima* along with the script file listed in Section V.2 (page 409). ⇨

Example M.7. Let $\mathsf{N}_6(x)$ be the 6*th order B-spline* (Definition M.2 page 297).

EX

$$\mathsf{N}_6(x) = \frac{1}{720}\begin{cases} x^6 & \text{for } 0 \le x \le 1 \\ -6x^6+42x^5-105x^4+140x^3-105x^2+42x-7 & \text{for } 1 \le x \le 2 \\ 15x^6-210x^5+1155x^4-3220x^3+4935x^2-3990x+1337 & \text{for } 2 \le x \le 3 \\ -20x^6+420x^5-3570x^4+15680x^3-37590x^2+47040x-24178 & \text{for } 3 \le x \le 4 \\ 15x^6-420x^5+4830x^4-29120x^3+96810x^2-168000x+119182 & \text{for } 4 \le x \le 5 \\ -6x^6+210x^5-3045x^4+23380x^3-100065x^2+225750x-208943 & \text{for } 5 \le x \le 6 \\ x^6-42x^5+735x^4-6860x^3+36015x^2-100842x+117649 & \text{for } 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

The 6*th order B-spline* $\mathsf{N}_6(x)$ is illustrated in Figure M.1 (page 299).

✎PROOF: This expression can be calculated "by hand" using Theorem M.1 (page 301) or by using the free and open source software package *Maxima* along with the script file listed in Section V.2 (page 409). ⇨

[7] Schumaker (2007) page 136 ⟨Table 1⟩, Shizgal (2015) page 92 ⟨(2.199)⟩, Szabó and Horváth (2004) page 146 ⟨(4)⟩, Wei and Billings (2006) page 578 ⟨Table 1⟩, Maleknejad et al. (2013) ⟨(9)⟩

[8] For help with plotting *B-spline*s, see APPENDIX V (page 399).

Example M.8. Let $\mathsf{N}_7(x)$ be the 7*th order B-spline* (Definition M.2 page 297).

EX

$$7!\mathsf{N}_7(x) = 5040\mathsf{N}_7(x) =$$

$$\begin{cases}
x^7 & \text{for } 0 \le x \le 1 \\
-7x^7 + 56x^6 - 168x^5 + 280x^4 - 280x^3 + 168x^2 - 56x + 8 & \text{for } 1 \le x \le 2 \\
21x^7 - 336x^6 + 2184x^5 - 7560x^4 + 15400x^3 - 18648x^2 + 12488x - 3576 & \text{for } 2 \le x \le 3 \\
-35x^7 + 840x^6 - 8400x^5 + 45360x^4 - 143360x^3 + 267120x^2 - 273280x + 118896 & \text{for } 3 \le x \le 4 \\
35x^7 - 1120x^6 + 15120x^5 - 111440x^4 + 483840x^3 - 1238160x^2 + 1733760x - 1027984 & \text{for } 4 \le x \le 5 \\
-21x^7 + 840x^6 - 14280x^5 + 133560x^4 - 741160x^3 + 2436840x^2 - 4391240x + 3347016 & \text{for } 5 \le x \le 6 \\
7x^7 - 336x^6 + 6888x^5 - 78120x^4 + 528920x^3 - 2135448x^2 + 4753336x - 4491192 & \text{for } 6 \le x \le 7 \\
-x^7 + 56x^6 - 1344x^5 + 17920x^4 - 143360x^3 + 688128x^2 - 1835008x + 2097152 & \text{for } 7 \le x \le 8 \\
0 & \text{otherwise}
\end{cases}$$

The 7*th order B-spline* $\mathsf{N}_7(x)$ is illustrated in Figure M.1 (page 299).

✎PROOF: This expression can be calculated "by hand" using Theorem M.1 (page 301) or by using the free and open source software package *Maxima* along with the script file listed in Section V.2 (page 409). ✏

Example M.9. [9] The $(n+1)^2$ coefficients of the order $n, n-1, \ldots, 0$ monomials of each *B-spline* $\mathsf{N}_n(x)$ multiplied by $n!$ induce an *integer sequence*

$\mathbb{x} \triangleq (\!(1, 1, 0, -1, 2, 1, 0, 0, -2, 6, -3, 1, -6, 9, 1, 0, 0, 0, -3, 12, -12, 4, 3, -24, 60, -44, -1, 12, -48, 64, \ldots)\!)$

as more fully listed in Table M.1 (page 328). In this sequence $\mathbb{x} \triangleq (\!(x_0, x_1, x_2, \ldots)\!)$, the coefficients for the *order* n *B-spline* $\mathsf{N}_n(x)$ begin at the sequence index value

$$p \triangleq \sum_{k=0}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \text{and end at index value } p + (n+1)^2 - 1.$$

For example, the coefficients for $\mathsf{N}_3(x)$ begin at index value $p \triangleq 0 + 1 + 4 + 9 = 14$ and end at index value $p + 4^2 - 1 = 29$. Using these coefficients gives the following expression for $\mathsf{N}_3(x)$:

$$\mathsf{N}_3(x) = \left[\begin{array}{rrrr|l} 1 & 0 & 0 & 0 & \text{for } 0 \le x < 1 \\ -3 & 12 & -12 & 4 & \text{for } 1 \le x < 2 \\ 3 & -24 & 60 & -44 & \text{for } 2 \le x < 3 \\ -1 & 12 & -48 & 64 & \text{for } 3 \le x < 4 \end{array}\right] \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{cases} x^3 & \text{for } 0 \le x < 1 \\ -3x^3 + 12x^2 - 12x + 4 & \text{for } 1 \le x < 2 \\ 3x^3 - 24x^2 + 60x - 44 & \text{for } 2 \le x < 3 \\ -x^3 + 12x^2 - 48x + 64 & \text{for } 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

...which agrees with the result presented in Example M.4 (page 303).

✎PROOF:

1. The coefficients for the sequence $\mathbb{x}$ may be computed with assistance from *Maxima* together with the script file listed in Section V.2 (page 409).

2. Proof that $\sum_{k=0}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$: The summation is a *power sum*. The relation may be proved using *induction*.[10]

   (a) Base case: n=0 case ...

$$\sum_{k=0}^{k=0} k^2 = 0$$
$$= \frac{0(0+1)(2\cdot 0+1)}{6}$$
$$= \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=0}$$

[9] Greenhoe (2017b)

[10] Greenhoe (2017a), pages 186–187 ⟨Proposition 11.2 (Power Sums)⟩

(b) Base case: n=1 case ...

$$\begin{aligned}\sum_{k=0}^{k=1} k^2 &= 0+1\\ &= \frac{1(1+1)(2\cdot 1+1)}{6}\\ &= \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1}\end{aligned}$$

(c) inductive step—proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}\sum_{k=0}^{n+1} k^2 &= \left(\sum_{k=0}^{n} k^2\right) + \left(n+1\right)^2\\ &= \left(\frac{n(n+1)(2n+1)}{6}\right) + \left(n+1\right)^2 && \text{by } n \text{ case hypothesis}\\ &= \left(n+1\right)\left(\frac{n(2n+1)+6(n+1)}{6}\right)\\ &= \left(n+1\right)\left(\frac{2n^2+7n+6}{6}\right)\\ &= \left(n+1\right)\left(\frac{(n+2)(2n+3)}{6}\right)\\ &= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}\end{aligned}$$

✏

Theorem M.2. [11]

THM $$\frac{d}{dx}\mathsf{N}_n(x) = \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1) \qquad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}$$

✎PROOF:

1. Proof using Lemma M.2 (page 297) and the *Fundamental Theorem of Calculus*:

$$\begin{aligned}\frac{d}{dx}\mathsf{N}_n(x) &= \frac{d}{dx}\int_0^1 \mathsf{N}_{n-1}(x-\tau)\,d\tau && \text{by Lemma M.2 page 297}\\ &= \frac{d}{dx}\int_{x-u=0}^{x-u=1} \mathsf{N}_{n-1}(u)(-1)\,du && \text{where } u \triangleq x-\tau \implies \tau = x-u\\ &= \frac{d}{dx}\int_{u=x-1}^{u=x} \mathsf{N}_{n-1}(u)\,du\\ &= \left\{\frac{d}{dx}\int \mathsf{N}_{n-1}(u)\,du\Big|_{u=x}\right\} - \left\{\frac{d}{dx}\int \mathsf{N}_{n-1}(u)\,du\Big|_{u=x-1}\right\} && \text{by \textit{Fundamental Theorem of Calculus}[12]}\\ &= \left\{\mathsf{N}_{n-1}(x)\frac{d}{dx}(x)\right\} - \left\{\mathsf{N}_{n-1}(x-1)\frac{d}{dx}(x-1)\right\} && \text{by \textit{Chain Rule}[13]}\\ &= \mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)\end{aligned}$$

[11] Höllig (2003) page 25 ⟨(3.2)⟩, Schumaker (2007) page 121 ⟨Theorem 4.16⟩
[12] Hijab (2011) page 163 ⟨Theorem 4.4.3⟩
[13] Hijab (2011) pages 73–74 ⟨Theorem 3.1.2⟩

2. Proof using Lemma M.2 (page 297) and *induction*:

(a) Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\mathsf{N}_0(x) - \mathsf{N}_0(x-1) &= \underbrace{\sigma(x) - \sigma(x-1)}_{\mathsf{N}_0(x)} - \underbrace{[\sigma(x-1) - \sigma(x-2)]}_{\mathsf{N}_0(x-1)} && \text{by Example M.1 page 299} \\
&= \sigma(x) - 2\sigma(x-1) + \sigma(x-2) \\
&= \sum_{k=0}^{2} (-1)^k \binom{2}{k} \sigma(x-k) \\
&= \frac{\mathsf{d}}{\mathsf{d}x} \sum_{k=0}^{2} (-1)^k \binom{2}{k} (x-k)\sigma(x-k) \\
&= \frac{\mathsf{d}}{\mathsf{d}x} \mathsf{N}_1(x) && \text{by Example M.2 page 300}
\end{aligned}$$

(b) Base case ...proof for $n = 2$ case:

$$\begin{aligned}
&\mathsf{N}_1(x) - \mathsf{N}_1(x-1) \\
&= \underbrace{x\sigma(x) - 2(x-1)\sigma(x-1) + (x-2)\sigma(x-2)}_{\mathsf{N}_1(x)} \\
&\quad - \underbrace{[(x-1)\sigma(x-1) - 2(x-2)\sigma(x-2) + (x-3)\sigma(x-3)]}_{\mathsf{N}_1(x-1)} && \text{by Example M.2 page 300} \\
&= x\sigma(x) + [-2x + 2 - x + 1]\sigma(x-1) + [x - 2 + 2x - 4]\sigma(x-2) + [-x+3]\sigma(x-3) \\
&= x\sigma(x) + [-3x + 3]\sigma(x-1) + [3x - 6]\sigma(x-2) + [-x+3]\sigma(x-3) \\
&= \frac{\mathsf{d}}{\mathsf{d}x} \left\{ \begin{array}{l} \frac{1}{2}x^2\sigma(x) + \left[-\frac{3}{2}x^2 + 3x - \frac{1}{2}\right]\sigma(x-1) + \left[\frac{3}{2}x^2 - 6x + 3\right]\sigma(x-2) \\ \qquad + \left[-\frac{1}{2}x^2 + 3x - \frac{5}{2}\right]\sigma(x-3) \end{array} \right\} \\
&= \frac{\mathsf{d}}{\mathsf{d}x} \mathsf{N}_2(x) && \text{by Example M.3 page 300}
\end{aligned}$$

(c) Proof that n case $\implies$ $n + 1$ case:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}x} \mathsf{N}_{n+1}(x) &= \frac{\mathsf{d}}{\mathsf{d}x} \int_0^1 \mathsf{N}_n(x-\tau)\, \mathsf{d}\tau && \text{by Lemma M.2 page 297} \\
&= \int_0^1 \frac{\partial}{\partial\tau} \mathsf{N}_n(x-\tau)\, \mathsf{d}\tau && \text{by } \textit{Leibniz Integration Rule} \text{ (Theorem E.2 page 187)} \\
&= \int_0^1 \left[\mathsf{N}_{n-1}(x-\tau) - \mathsf{N}_{n-1}(x-1-\tau)\right] \mathsf{d}\tau && \text{by left hypothesis} \\
&= \int_0^1 \mathsf{N}_{n-1}(x-\tau)\, \mathsf{d}\tau - \int_0^1 \mathsf{N}_{n-1}(x-1-\tau)\, \mathsf{d}\tau \\
&= \mathsf{N}_n(x) - \mathsf{N}_n(x-1) && \text{by Lemma M.2 page 297}
\end{aligned}$$

✏

Theorem M.3 (*B-spline recursion*). [14] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition M.2 page 297)*.

THM

$$\mathsf{N}_n(x) = \frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1) \qquad \forall n \in \{1, 2, 3, \ldots\},\ \forall x \in \mathbb{R}$$

[14] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972)

✎PROOF:

1. Base case ...proof for $n = 1$ case:

$$\begin{aligned}
\frac{x}{1}\mathsf{N}_0(x) + \frac{1+1-x}{1}\mathsf{N}_0(x-1) &= \frac{x}{1}\underbrace{[\sigma(x)-\sigma(x-1)]}_{\mathsf{N}_0(x)} + \frac{1+1-x}{1}\underbrace{[\sigma(x-1)-\sigma(x-2)]}_{\mathsf{N}_0(x-1)}\\
&= x\sigma(x) + [-x-x+2]\sigma(x-1) + [x-2]\sigma(x-2)\\
&= \mathsf{N}_1(x) \qquad \text{by Example M.2 page 300}
\end{aligned}$$

2. Induction step ...proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1) + c_1\\
&= \int \frac{\mathsf{d}}{\mathsf{d}x}\left\{\frac{x}{n+1}\mathsf{N}_n(x) + \frac{n+2-x}{n+1}\mathsf{N}_n(x-1)\right\}\,\mathsf{d}x\\
&= \int \underbrace{\frac{1}{n+1}\mathsf{N}_n(x) + \frac{x}{n+1}\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_n(x)}_{\frac{\mathsf{d}}{\mathsf{d}x}\frac{x}{n+1}\mathsf{N}_n(x)} + \underbrace{\frac{-1}{n+1}\mathsf{N}_n(x-1) + \frac{n+2-x}{n}\frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_n(x-1)}_{\frac{\mathsf{d}}{\mathsf{d}x}\frac{n+2-x}{n+1}\mathsf{N}_n(x-1)}\,\mathsf{d}x\\
&\qquad \text{by product rule}\\
&= \int \frac{1}{n+1}\underbrace{\left[\frac{x}{n}\mathsf{N}_{n-1}(x) + \frac{n+1-x}{n}\mathsf{N}_{n-1}(x-1)\right]}_{\text{by } n \text{ hypothesis}} + \frac{x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x) - \mathsf{N}_{n-1}(x-1)\right]}_{\text{by Theorem M.2 page 305}}\\
&\qquad - \underbrace{\left[\frac{x-1}{n^2+n}\mathsf{N}_{n-1}(x-1) + \frac{n-x+2}{n(n+1)}\mathsf{N}_{n-1}(x-2)\right]}_{\text{by induction hypothesis}}\\
&\qquad + \frac{n+2-x}{n+1}\underbrace{\left[\mathsf{N}_{n-1}(x-1) - \mathsf{N}_{n-1}(x-2)\right]}_{\text{by Theorem M.2 page 305}}\,\mathsf{d}x\\
&= \int \left[\frac{x}{n(n+1)} + \frac{x}{n+1}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n-x+1}{n(n+1)} - \frac{x-1}{n(n+1)} + \frac{n+2-2x}{n+1}\right]\mathsf{N}_{n-1}(x-1)\\
&\qquad + \left[\frac{-n-2+x}{n(n+1)} + \frac{-n-2+x}{n+1}\right]\mathsf{N}_{n-1}(x-2)\,\mathsf{d}x\\
&= \int \left[\frac{x+nx}{n(n+1)}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x+n(n+2-2x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-1)\\
&\qquad + \left[\frac{-n-2+x+n(-n-2+x)}{n(n+1)}\right]\mathsf{N}_{n-1}(x-2)\,\mathsf{d}x\\
&= \int \left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+2-2x}{n}\right]\mathsf{N}_{n-1}(x-1) + \left[\frac{-n-2+x}{n}\right]\mathsf{N}_{n-1}(x-2)\,\mathsf{d}x\\
&= \int \underbrace{\left[\frac{x}{n}\right]\mathsf{N}_{n-1}(x) + \left[\frac{n+1-x}{n}\right]\mathsf{N}_{n-1}(x-1)}_{\mathsf{N}_n(x)}\\
&\qquad \underbrace{-\left[\frac{x-1}{n}\right]\mathsf{N}_{n-1}(x-1) - \left[\frac{n+2-x}{n}\right]\mathsf{N}_{n-1}(x-2)}_{\mathsf{N}_{n-1}(x-1)}\,\mathsf{d}x\\
&= \int \mathsf{N}_n(x) - \mathsf{N}_n(x-1)\,\mathsf{d}x \qquad \text{by } n \text{ hypothesis}\\
&= \int \frac{\mathsf{d}}{\mathsf{d}x}\mathsf{N}_{n+1}(x)\,\mathsf{d}x \qquad \text{by Theorem M.2 page 305}\\
&= \mathsf{N}_{n+1}(x) + c_2
\end{aligned}$$

Proof that $c_1 = c_2$: By item (2) (page 308), $\mathsf{N}_n(x) = 0$ for $x < 0$. Therefore, $c_1 = c_2$.

⇨

Theorem M.4 (*B-spline general form*). [15] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition M.2 page 297). Let* $\operatorname{supp} \mathsf{f}$ *be the* SUPPORT *of a function* $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$.

THM

1.	$\mathsf{N}_n(x) \ge 0$	$\forall n \in \mathbb{W},\ \forall x \in \mathbb{R}$	(NON-NEGATIVE)
2.	$\operatorname{supp} \mathsf{N}_n(x) = [0:n+1]$	$\forall n \in \mathbb{W}$	(CLOSED SUPPORT)
3.	$\displaystyle\int_{\mathbb{R}} \mathsf{N}_n(x)\,\mathrm{d}x = 1$	$\forall n \in \mathbb{W}$	(UNIT AREA)
4.	$\mathsf{N}_n\left(\frac{n+1}{2} - x\right) = \mathsf{N}_n\left(\frac{n+1}{2} + x\right)$	$\forall n \in \mathbb{N}\quad \forall x \in \mathbb{R}$	(SYMMETRIC *about* $x = \frac{n+1}{2}$)

✎PROOF:

1. Proof that $\mathsf{N}_n(x) \ge 0$ (proof by induction):

   (a) base case…proof that $\mathsf{N}_0(x) \ge 0$:

$$\begin{aligned}
\mathsf{N}_0(x) &\triangleq \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition M.2 page 297)}\\
&\ge 0 && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 1)}
\end{aligned}$$

   (b) inductive step—proof that $\{\mathsf{N}_n(x) \ge 0\} \implies \{\mathsf{N}_{n+1}(x) \ge 0\}$:

$$\begin{aligned}
\mathsf{N}_{n+1}(x) &= \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma M.2 page 297}\\
&\ge 0 && \text{by induction hypothesis } (\mathsf{N}_n(x) \ge 0)
\end{aligned}$$

2. Proof that $\operatorname{supp} \mathsf{N}_n(x) = [0:n+1]$ (proof by induction):

   (a) Base case …proof that $\operatorname{supp} \mathsf{N}_0 = [0:1]$:

$$\begin{aligned}
\operatorname{supp} \mathsf{N}_0 &\triangleq \operatorname{supp} \mathbb{1}_{[0:1)} && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition M.2 page 297)}\\
&= \{[0:1)\}^- && \text{by definition of } \textit{support} \text{ operator}\\
&= [0:1] && \text{by definition of } \textit{closure} \text{ operator}
\end{aligned}$$

   (b) Induction step …proof that $\{\operatorname{supp} \mathsf{N}_n = [0:n+1]\} \implies \{\operatorname{supp} \mathsf{N}_{n+1} = [0:n+2]\}$:

$$\begin{aligned}
\operatorname{supp} \mathsf{N}_{n+1}(x) &= \operatorname{supp} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma M.2 page 297}\\
&= \operatorname{supp} \int_{[0:1]} \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by def. of } \textit{Lebesgue integration}\\
&= \left\{x \in \mathbb{R} | (x-\tau) \in [0:n+1] \text{ for some } \tau \in [0:1]\right\}^- && \text{by induction hypothesis}\\
&= [0:n+1] \cup [0+1:n+1+1]^-\\
&= [0:n+2]^-\\
&= [0:n+2] && \text{by property of } \textit{closure} \text{ operator}
\end{aligned}$$

3. Proof that $\int_{\mathbb{R}} \mathsf{N}_n(x)\,\mathrm{d}x = 1$ (proof by induction):

[15] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972), Prasad and Iyengar (1997) pages 145–146 ⟨Theorem 6.2⟩

(a) Base case …proof that $\int_{\mathbb{R}} \mathsf{N}_0(x) = 1$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{N}_0(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \mathbb{1}_{[0:1)}\,\mathrm{d}x && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition M.2 page 297)}\\
&= \int_{[0:1)} 1\,\mathrm{d}x && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 1)}\\
&= \int_{[0:1]} 1\,\mathrm{d}x && \text{by property of \textit{Lebesgue integration}}\\
&= 1
\end{aligned}$$

(b) Induction step …proof that $\left\{\int_{\mathbb{R}} \mathsf{N}_n(x) = 1\right\} \implies \left\{\int_{\mathbb{R}} \mathsf{N}_{n+1} = 1\right\}$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{N}_{n+1}(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau\,\mathrm{d}x && \text{by Lemma M.2 page 297}\\
&= \int_0^1\int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&= \int_0^1\int_{\mathbb{R}} \mathsf{N}_n(u)\,\mathrm{d}u\,\mathrm{d}\tau && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \int_0^1 1\,\mathrm{d}\tau && \text{by induction hypothesis}\\
&= 1
\end{aligned}$$

4. Proof that $\mathsf{N}_n(x)$ is *symmetric* for $n \in \{1,2,3,\ldots\}$:

(a) Note that $\mathsf{N}_0(x)$ $(n=0)$ is *not symmetric* (in particular it fails at $x = 1/2$) because
$\mathsf{N}_0\left(\frac{0+1}{2} - \frac{1}{2}\right) = \mathsf{N}_0(0) = 1 \neq 0 = \mathsf{N}_1(1) = \mathsf{N}_0\left(\frac{0+1}{2} + \frac{1}{2}\right)$

(b) Base case …proof for $n = 1$ case:

$$\begin{aligned}
\mathsf{N}_1\left(\frac{1+1}{2} - x\right) &= \mathsf{N}_1(1-x)\\
&= \left\{\begin{array}{rl} (1-x) & \text{for } 1-x \in [0:1]\\ -(1-x)+2 & \text{for } 1-x \in [1:2]\\ 0 & \text{otherwise}\end{array}\right\} && \text{by Example M.2 page 300}\\
&= \left\{\begin{array}{rl} -x+1 & \text{for } -x \in [-1:0]\\ x+1 & \text{for } -x \in [0:1]\\ 0 & \text{otherwise}\end{array}\right\}\\
&= \left\{\begin{array}{rl} x+1 & \text{for } x \in [-1:0]\\ -x+1 & \text{for } x \in [0:1]\\ 0 & \text{otherwise}\end{array}\right\}\\
&= \left\{\begin{array}{rl} (1+x) & \text{for } 1+x \in [0:1]\\ -(1+x)+2 & \text{for } 1+x \in [1:2]\\ 0 & \text{otherwise}\end{array}\right\}\\
&= \mathsf{N}_1(1+x) && \text{by Example M.2 page 300}\\
&= \mathsf{N}_1\left(\frac{1+1}{2} + x\right)
\end{aligned}$$

(c) Induction step …proof that $n-1$ case $\implies$ n case:

$$\begin{aligned}
&\mathsf{N}_n\left(\frac{n+1}{2}+x\right)\\
&= \frac{\frac{n+1}{2}+x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}+x\right) + \frac{n+1-\left(\frac{n+1}{2}+x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}+x-1\right) && \text{by Theorem M.3 page 306}\\
&= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x+\frac{1}{2}\right]\right) + \frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}+\left[x-\frac{1}{2}\right]\right)\\
&= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x+\frac{1}{2}\right]\right) + \frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n}{2}-\left[x-\frac{1}{2}\right]\right) && \text{by induction hypothesis}\\
&= \frac{n+1-\left(\frac{n+1}{2}-x\right)}{n}\mathsf{N}_{n-1}\left(\left[\frac{n+1}{2}-x\right]-1\right) + \frac{\frac{n+1}{2}-x}{n}\mathsf{N}_{n-1}\left(\frac{n+1}{2}-x\right)\\
&= \mathsf{N}_n\left(\frac{n+1}{2}-x\right) && \text{by Theorem M.3 page 306}
\end{aligned}$$

✏

M.3 Projection properties

In the case where $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is to be used as a basis in some subspace of $\boldsymbol{L}^2_{\mathbb{R}}$, one may want to *project* a function $\mathsf{f}(x)$ onto a basis function $\mathsf{N}_n(x-k)$. This is especially true when $(\!(\mathsf{N}_n(x-k))\!)$ is *orthogonal*; but in the case of *B-spline*s this is only true when $n=0$ (Theorem M.8 page 320). Nevertheless, projection of a function onto $\mathsf{N}_n(x-k)$, or the projection of $\mathsf{N}_n(x)$ onto another basis fucntion (such as the complex exponential in the case of *Fourier analysis* as in Lemma M.5 page 312), is still useful. Projection in an *inner product space* is typically performed using the *inner product* $\langle \mathsf{f}(x) \,|\, \mathsf{N}_n(x-k)\rangle$; and in the space $\boldsymbol{L}^2_{\mathbb{R}}$, this inner product is typically defined as an *integral* such that

$$\langle \mathsf{f}(x) \,|\, \mathsf{N}_n(x-k)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_n(x-k)\,\mathrm{d}x.$$

As it turns out, there is a way to compute this inner product that only involves the function $\mathsf{f}(x)$ and the order parameter n (next theorem).

Theorem M.5. [16] *Let* f *be a* CONTINUOUS *function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathsf{f}^{(n)}$ *the* nTH DERIVATIVE *of* $\mathsf{f}(x)$.

THM

(1). $$\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_n(x)\,\mathrm{d}x = \int_{[0:1)^{n+1}} \mathsf{f}(x_1+x_2+\cdots+x_{n+1})\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n+1}$$

(2). $$\int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathrm{d}x = \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k)$$

✎PROOF:

1. Proof for (1) (proof by induction):

(a) Base case …proof for $n=0$ case:

$$\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_0(x)\,\mathrm{d}x = \int_{[0:1)} \mathsf{f}(x)\,\mathrm{d}x \qquad \text{by definition of } \mathsf{N}_0(x) \qquad \text{(Definition M.2 page 297)}$$

[16] Chui (1992) page 85 ⟨(4.2.2), (4.2.3)⟩, Christensen (2008) page 140 ⟨Theorem 6.1.1⟩

(b) Inductive step—proof that n case $\implies$ $n+1$ case:

$$\begin{aligned}
&\int_{\mathbb{R}} \mathsf{f}(x)\mathsf{N}_{n+1}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left[\int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau\right]\mathsf{f}(x)\,\mathrm{d}x && \text{by Lemma M.2 page 297} \\
&= \int_{[0:1)} \int_{\mathbb{R}} \mathsf{N}_n(x-\tau)\mathsf{f}(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_{[0:1)} \int_{\mathbb{R}} \mathsf{N}_n(u)\mathsf{f}(u+\tau)\,\mathrm{d}u\,\mathrm{d}\tau && \text{where } u \triangleq x-\tau \implies x = u+\tau \\
&= \int_{[0:1)} \int_{[0:1)^{n+1}} \mathsf{f}(u_1+u_2+\cdots+u_{n+1}+\tau)\,\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_{n+1}\,\mathrm{d}\tau && \text{by induction hypothesis} \\
&= \int_{[0:1)^{n+2}} \mathsf{f}(u_1+u_2+\cdots+u_{n+1}+u_{n+2})\,\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_{n+2}\,\mathrm{d}\tau \\
&= \int_{[0:1)^{n+2}} \mathsf{f}(x_1+x_2+\cdots+x_{n+1}+x_{n+2})\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n+2} && \text{by change of variables } u_k \to x_k
\end{aligned}$$

2. Proof for (2):

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{f}^{(n)}(x)\mathsf{N}_n(x)\,\mathrm{d}x &= \int_{[0:1)^{n+1}} \mathsf{f}^{(n)}\left(\sum_{k=1}^{n+1} x_k\right)\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n+1} && \text{by (1)} \\
&= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\mathsf{f}(k) && \text{by Theorem E.1 page 186}
\end{aligned}$$

✏

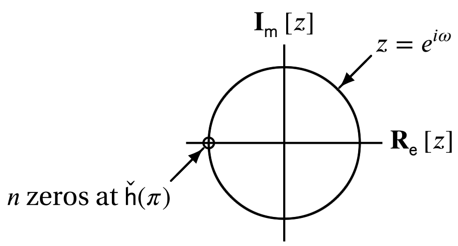


Figure M.2: Zero locations for B-cardinal spline $N_n(x)$ scaling coefficients

M.4 Fourier analysis

Simply put, no matter what new and fancy basis sequences are discovered, the *Fourier transform* never goes out of style. This is largely because the *kernel* of the Fourier transform—the *complex exponential* function—has two properties that makes it extremely special:

- The complex exponential is an eigenvalue of any *linear time invariant* (*LTI*) operator (Theorem H.12 page 230).
- The complex exponential generates a *continuous point spectrum* for the *differential operator*.

Thus, we might expect the projection of the *B-spline* function $\mathsf{N}_n(x)$ onto the complex exponential (essentially the *Fourier transform* of $\mathsf{N}_n(x)$,...next lemma) to be useful. Such a hunch would be confirmed because it is useful for proving that

- the sequence $(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$ is a *Riesz basis* (Lemma M.6 page 315, Theorem M.8 page 320) and
- the sequence $(\mathsf{N}_n(x-k))_{k\in\mathbb{Z}}$ is a *multiresolution analysis* (Theorem M.10 page 323).

Lemma M.5. [17] *Let $\tilde{\mathbf{F}}$ be the* Fourier Transform *operator (Definition K.2 page 257).*

$$\text{LEM}\quad \tilde{\mathbf{F}}\mathsf{N}_n(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1} = \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1} \triangleq \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1}$$

✎Proof:

1. Proof using Theorem M.5 page 310:

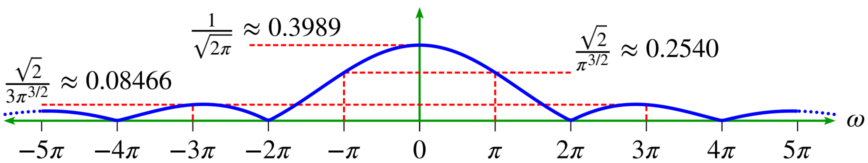
$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{N}_n(x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition K.2 page 257)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{[0:1)^{n+1}} e^{-i\omega(x_1+x_2+\cdots+x_{n+1})}\,\mathrm{d}x_1\;\mathrm{d}x_2\;\cdots,\;\mathrm{d}x_{n+1} && \text{by Theorem M.5}\\
&= \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{n+1}\left(\int_{[0:1)} e^{-i\omega x_k}\,\mathrm{d}x_k\right) && \text{because } e^{x+y}=e^xe^y\\
&= \frac{1}{\sqrt{2\pi}}\left(\int_0^1 e^{-i\omega x}\,\mathrm{d}x\right)^{n+1} = \frac{1}{\sqrt{2\pi}}\left(\left.\frac{e^{-i\omega x}}{-i\omega}\right|_0^1\right)^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1} = \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}}{i\omega}\right)\right]^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}\left[e^{-i\frac{\omega}{2}}\left(\frac{2i\sin\left(\frac{\omega}{2}\right)}{\frac{2i\omega}{2}}\right)\right]^{n+1} && \text{by \textit{Euler formulas}} \quad \text{(Corollary H.2 page 221)}\\
&= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1}
\end{aligned}$$

2. Proof using *rectangular pulse* example (Example K.1 page 264) and *Convolution Theorem* (Theorem O.2 page 344):

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \left[\sqrt{2\pi}\right]^n\left[\tilde{\mathbf{F}}\mathsf{N}_0\right]^{n+1} && \text{by \textit{Convolution Theorem}} \quad \text{(Theorem O.2 page 344)}\\
&= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\frac{c(b-a)}{\sqrt{2\pi}}e^{-i\left(\frac{a+b}{2}\omega\right)}\left(\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)}\right)\right]^{n+1} && \text{by \textit{rectangular pulse} example with } a=0,\ b=c=1 \quad \text{(Example K.1 page 264)}\\
&= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{\omega}{2}\right)}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{(\omega/2)}\right)\right]^{n+1}\\
&= \frac{1}{\sqrt{2\pi}}e^{-i\left(\frac{(n+1)\omega}{2}\right)}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{(\omega/2)}\right)^{n+1}
\end{aligned}$$

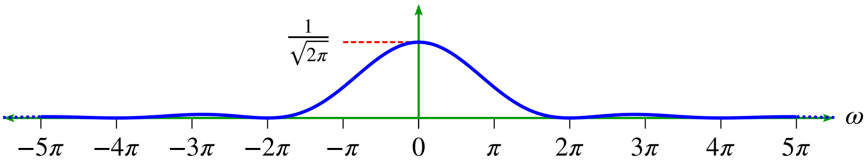
✏

Example M.10. The *Fourier transform magnitude* $\left|\left[\tilde{\mathbf{F}}\mathsf{N}_0\right](\omega)\right|$ of the *0 order B-spline* $\mathsf{N}_0(x)$ is illustrated to the right.

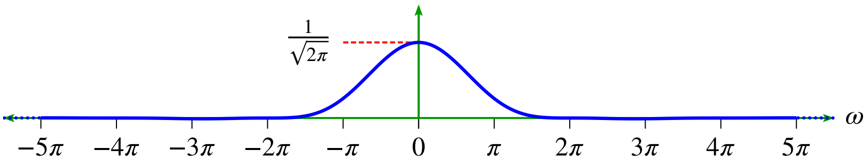


[17] Christensen (2008) page 142 ⟨Corollary 6.1.2⟩

Example M.11. The *Fourier transform magnitude* $\left|[\tilde{\mathbf{F}}\mathsf{N}_1](\omega)\right|$ of the *1st order B-spline* $\mathsf{N}_1(x)$ is illustrated to the right.



Example M.12. The *Fourier transform magnitude* $\left|[\tilde{\mathbf{F}}\mathsf{N}_2](\omega)\right|$ of the *2nd order B-spline* $\mathsf{N}_2(x)$ is illustrated to the right.



M.5 Basis properties

M.5.1 Uniqueness properties

Coefficients of a *basis sequence* are not always *unique*. Take for example a very trivial sequence (α_1, α_2) in which the coefficients are summed. If $\mathsf{f}(x) \triangleq \alpha_1 + \alpha_2$ and $\mathsf{g}(x) \triangleq \beta_1 + \beta_2$,

$$\begin{array}{lll} \text{then} & \{(\alpha_1,\alpha_2) = (\beta_1,\beta_2)\} & \implies \mathsf{f}(x) = \mathsf{g}(x) \\ \text{but} & \mathsf{f}(x) = \mathsf{g}(x) & \centernot\implies \{(\alpha_1,\alpha_2) = (\beta_1,\beta_2)\}, \end{array}$$

because for example if $(\alpha_1, \alpha_2) = (1, 2)$ and $(\beta_1, \beta_2) = (-6, 9)$, then $\mathsf{f}(x) = \mathsf{g}(x)$, but $(\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)$. This example demonstrates that the "if and only if" condition $\iff$ does not hold and coefficients are not unique in all *basis sequences*. But arguably a minimal requirement for any practical basis sequence is that the coefficients are *unique* (the "if and only if" condition $\iff$ holds). And indeed, in a *B-spline* basis sequence $\left(\!\left(\mathsf{N}_n(x-k)\right)\!\right)_{k\in\mathbb{Z}}$, the coefficients $\left(\!\left(\alpha_k\right)\!\right)_{k\in\mathbb{Z}}$ are *unique*, as demonstrated by Theorem M.6 (next).

Theorem M.6. [18] *Let* $\mathsf{N}_n(x)$ *be the* nTH-ORDER B-SPLINE (Definition M.2 page 297). *Let*

$$\mathsf{f}(x) \triangleq \sum_{k\in\mathbb{Z}} \alpha_k \mathsf{N}_n(x-k) \quad \textit{and} \quad \mathsf{g}(x) \triangleq \sum_{k\in\mathbb{Z}} \beta_k \mathsf{N}_n(x-k).$$

THM

$$\underbrace{\left\{ \mathsf{f}(x) = \mathsf{g}(x) \quad \forall x \in \mathbb{R} \right\} \iff \left\{ \left(\!\left(\alpha_k\right)\!\right)_{k\in\mathbb{Z}} = \left(\!\left(\beta_k\right)\!\right)_{k\in\mathbb{Z}} \right\}}_{\textit{coefficients are } \textsc{unique}}$$

✎PROOF:

1. Proof that $\impliedby$ condition holds:

$$\begin{aligned} \mathsf{f}(x) &\triangleq \sum_{k\in\mathbb{Z}} \alpha_k \mathsf{N}_n(x-k) && \text{by definition of } \mathsf{f}(x) \\ &= \sum_{k\in\mathbb{Z}} \beta_k \mathsf{N}_n(x-k) && \text{by right hypothesis} \\ &\triangleq \mathsf{g}(x) && \text{by definition of } \mathsf{g}(x) \end{aligned}$$

2. Proof that $\implies$ condition holds (proof by contradiction):

   (a) Suppose it does *not* hold.

[18] Wojtaszczyk (1997) page 55 ⟨Theorem 3.11⟩

(b) Then there exists sequences $(\!(\alpha_k)\!)_{k\in\mathbb{Z}}$ and $(\!(\beta_k)\!)_{k\in\mathbb{Z}}$ such that
$(\!(\alpha_k)\!) - (\!(\beta_k)\!) \triangleq (\!(alpha_k - \beta_k)\!) \neq (\!(0, 0, 0, \cdots)\!)$
but also such that $\mathsf{f}(x) - \mathsf{g}(x) = 0\ \forall x \in \mathbb{R}$.

(c) If this were possible, then

$$\begin{aligned}
0 &= \mathsf{f}(x) - \mathsf{g}(x) \\
&= \sum_{m\in\mathbb{Z}} \alpha_m \mathsf{N}_n(x-m) - \sum_{m\in\mathbb{Z}} \beta_m \mathsf{N}_n(x-m) \\
&= \sum_{m\in\mathbb{Z}} \left(\alpha_m - \beta_m\right) \mathsf{N}_n(x-m) \\
&= \sum_{m=0}^{m=n} \left(\alpha_m - \beta_m\right) \frac{1}{n!}\left[\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x-k)^n \sigma(x-k)\right] && \text{by Theorem M.1 page 301}
\end{aligned}$$

(d) But this is *impossible* because $\mathsf{N}(x)$ is *non-negative* (Theorem M.4 page 308).

(e) Therefore, there is a contradiction, and the $\Longrightarrow$ condition *does* hold.

✎

M.5.2 Partition of unity properties

In the case in which a sequence of *B-splines* $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is to be used as a *basis* for some subspace of $\boldsymbol{L}^2_{\mathbb{R}}$, arguably one of the most important properties for the sequence to have is the *partition of unity* property such that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1$. This allows for convenient representation of the most basic functions, such as constants.[19] As it turns out, B-splines *do* have this property (next theorem).

Theorem M.7 (*B-spline partition of unity*). [20] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition M.2 page 297)*.

THM	
	$\displaystyle\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1 \quad \forall n \in \mathbb{W}$ (PARTITION OF UNITY)

✎PROOF:

1. lemma: $\sum_{k\in\mathbb{Z}} \mathsf{N}_0(x-k) = 1$. Proof:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{N}_0(x-k) &= \sum_{k\in\mathbb{Z}} \mathbb{1}_{[0:1)}(x-k) && \text{by definition of } \mathsf{N}_0(x) && \text{(Definition M.2 page 297)} \\
&= 1 && \text{by definition of } \mathbb{1}_A(x) && \text{(Definition 1.2 page 1)}
\end{aligned}$$

2. Proof for this theorem follows from the $n = 0$ case ((1) lemma page 314), the definition of $\mathsf{N}_n(x)$ (Definition M.2 page 297), and Corollary Q.1 (page 367).

3. Alternatively, this theorem can be proved by *induction*:

   (a) Base case ($n = 0$ case): by (1) lemma.

[19] Jawerth and Sweldens (1994) page 8

[20] Christensen (2008) page 140 ⟨Theorem 6.1.1⟩, Höllig (2003) page 27 ⟨3.4⟩, Schumaker (2007) page 120 ⟨Theorem 4.15⟩, de Boor (2001) page 90 ⟨B-Spline Property (i)⟩, Chui (1988) page 2 ⟨Theorem 1.1⟩, Wojtaszczyk (1997) page 53 ⟨Theorem 3.7⟩, Cox (1972), de Boor (1972)

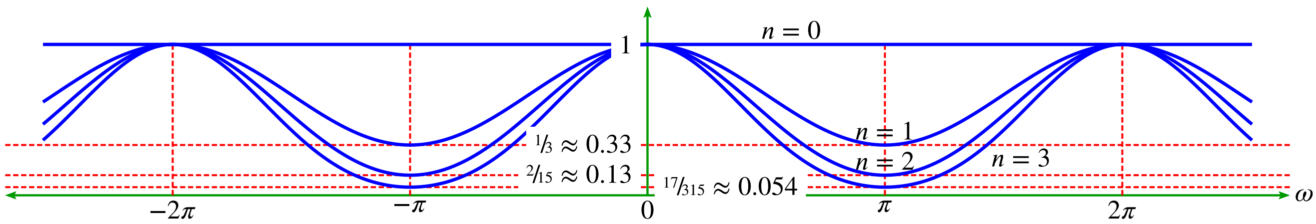


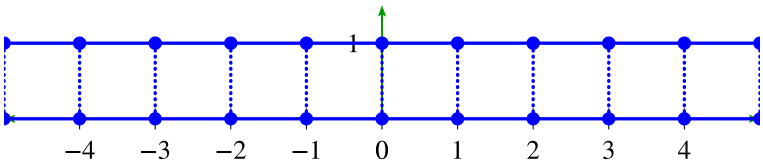
Figure M.3: *auto-power spectrum* $\tilde{S}_n(\omega)$ plots of *B-splines* $\mathsf{N}_n(x)$ (Lemma M.6 page 315) For C and LaTeX source code to generate such a plot, see Section V.3 (page 412).

(b) Inductive step—proof that $\sum_{k\in\mathbb{Z}} \mathsf{N}_n(x-k) = 1 \implies \sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) = 1$:

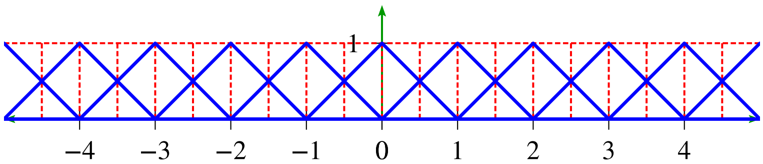
$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{N}_{n+1}(x-k) &= \sum_{k\in\mathbb{Z}} \int_{\tau=0}^{\tau=1} \mathsf{N}_n(x-k-\tau)\,\mathrm{d}\tau && \text{by Lemma M.2 page 297}\\
&= \sum_{k\in\mathbb{Z}} \int_{x-u=0}^{x-u=1} \mathsf{N}_n(u-k)(-1)\,\mathrm{d}u && \text{where } u \triangleq x-\tau \implies \tau = x-u\\
&= \sum_{k\in\mathbb{Z}} \int_{u=x-1}^{u=x} \mathsf{N}_n(u-k)\,\mathrm{d}u\\
&= \int_{u=x-1}^{u=x} \left(\sum_{k\in\mathbb{Z}} \mathsf{N}_n(u-k)\right)\mathrm{d}u\\
&= \int_{u=x-1}^{u=x} 1\,\mathrm{d}u && \text{by induction hypothesis}\\
&= 1
\end{aligned}$$

✏

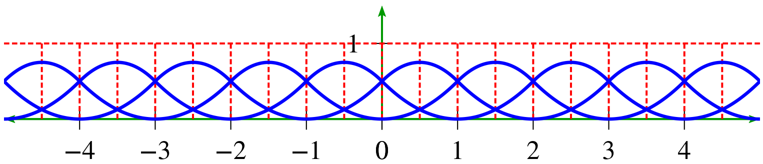
Example M.13. The *partition of unity* property for the 0 *order B-spline* $\mathsf{N}_0(x)$ (Example M.1 page 299) is illustrated to the right.



Example M.14. The *partition of unity* property for the 1*st order B-spline* $\mathsf{N}_1(x)$ (Example M.2 page 300) is illustrated to the right.



Example M.15. The *partition of unity* property for the 2*nd order B-spline* $\mathsf{N}_2(x)$ (Example M.3 page 300) is illustrated to the right.



M.5.3 Riesz basis properties

Lemma M.6. *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE *(Definition M.2 page 297).*
Let $\tilde{S}_n(\omega) \triangleq 2\pi \sum_{k\in\mathbb{Z}} \left|\tilde{\mathbf{F}}\mathsf{N}_n(\omega - 2\pi k)\right|^2$ *be the* AUTO-POWER SPECTRUM *(Definition R.3 page 373) of* $\mathsf{N}_n(x)$.

LEM

(1). $0 < \tilde{S}_n(\omega) \le 1 \quad \forall \omega \in \mathbb{R} \quad , \quad \forall n \in \mathbb{W}$	(3). $\tilde{S}_n(0) = 1 \quad \forall n \in \mathbb{W}$	*Note: see illustration in Figure M.3 page 315.*
(2). $\tilde{S}_n(\omega) = 1 \quad \forall \omega \in \mathbb{R} \quad , \quad \text{for } n = 0$	(4). $\tilde{S}_n(\pi) \le \frac{1}{3} \quad \forall n \in \mathbb{W}\backslash\{0\}$	

✎PROOF:

1. lemma: $\tilde{S}_n(\omega) = \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}}\right]^{2(n+1)}$. Proof:

$$\begin{aligned}
\tilde{S}_n(\omega) &\triangleq 2\pi \sum_{k\in\mathbb{Z}} \left|\tilde{\mathbf{F}}\mathrm{N}_n(\omega - 2\pi k)\right|^2 && \text{by Definition R.3 page 373}\\
&= 2\pi \sum_{k\in\mathbb{Z}} \left| \frac{1}{\sqrt{2\pi}} e^{-i\frac{(n+1)(\omega-2\pi k)}{2}} \left(\frac{\sin\left(\frac{\omega-2\pi k}{2}\right)}{\frac{\omega-2\pi k}{2}} \right)^{n+1} \right|^2 && \text{by Lemma M.5 page 312}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega-2\pi k}{2}\right)}{\frac{\omega-2\pi k}{2}} \right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2} - k\pi\right)}{k\pi - \frac{\omega}{2}} \right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{(-1)^k \sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}} \right]^{2(n+1)}\\
&= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}} \right]^{2(n+1)}
\end{aligned}$$

2. lemma (one sided series form):

$$\begin{aligned}
\tilde{S}_n(\omega) &= \sum_{k\in\mathbb{Z}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi - \frac{\omega}{2}} \right]^{2(n+1)} && \text{by (1) lemma}\\
&= \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)} \left(\sum_{k=1}^{\infty} \left[\frac{1}{2k - \frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty} \left[\frac{1}{2k + \frac{\omega}{\pi}}\right]^{2(n+1)} \right)
\end{aligned}$$

3. lemma: $\tilde{S}_n(\omega)$ is *continuous* for all $\omega \in \mathbb{R}$.
   Proof: $\sin(\omega/2)$ and $\omega/2$ are *continuous*, so $\tilde{S}_n(\omega)$ is *continuous* as well.

4. lemma: $\tilde{S}_n(\omega)$ is *periodic* with period 2π (and so we only need to examine $\tilde{S}_n(\omega)$ for $\omega \in [0 : 2\pi)$). Proof of *periodicity*: This follows directly from Proposition R.2 (page 374).

5. lemma: $\tilde{S}_n(-\omega) = \tilde{S}_n(\omega)$ (*symmetric* about 0) and $\tilde{S}_n(\pi - \omega) = \tilde{S}_n(\pi + \omega)$ (*symmetric* about π). Proof: This follows directly from Proposition R.3 (page 375).

6. Proof that $\tilde{S}_n(0) = 1$:

$$\begin{aligned}
\tilde{S}_n(0) &= \lim_{\omega\to 0} \tilde{S}_n(\omega) && \text{by (3) lemma}\\
&= \lim_{\omega\to 0}\left[\left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right)\right] && \text{by (2) lemma}\\
&= \lim_{\omega\to 0}\left[\frac{\cos\left(\frac{\omega}{2}\right)}{-\frac{1}{2}}\right]^{2(n+1)} + 0 \qquad \text{by } \textit{l'Hôpital's rule}\\
&= (-1)^{2(n+1)} = 1
\end{aligned}$$

7. Proof that $\tilde{S}_n(\pi)$ converges to some value > 0:

(a) Proof that $\tilde{S}_n(\pi) > 0$:

$$\begin{aligned}
\tilde{S}_n(\pi) &= \left[\frac{\sin(\pi/2)}{(\pi/2)}\right]^{2(n+1)} + \left[\frac{\sin(\pi/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\pi}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\pi}{\pi}}\right]^{2(n+1)}\right) && \text{by (2) lemma}\\
&= \left(\frac{2}{\pi}\right)^{2(n+1)}\left[1 + \left(\frac{1}{1}\right)^{2(n+1)} + \left(\frac{1}{3}\right)^{2(n+1)} + \left(\frac{1}{3}\right)^{2(n+1)} + \left(\frac{1}{5}\right)^{2(n+1)} + \left(\frac{1}{5}\right)^{2(n+1)} + \cdots\right]\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \underbrace{\sum_{k=1}^{\infty}\left[\frac{1}{2k-1}\right]^{2(n+1)}}_{\text{Dirichlet Lambda function } \lambda(2n+2)}\\
&> 0 \qquad \text{because } x^2 > 0 \text{ for all } x \in \mathbb{R}\backslash\{0\}
\end{aligned}$$

(b) Proof that $\tilde{S}_n(\pi)$ *converges*:

$$\begin{aligned}
\tilde{S}_n(\pi) &= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^{2(n+1)} && \text{by item (7a)}\\
&\le 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{2(n+1)}\\
&\le 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{2}\\
&\implies \textit{convergence} && \text{by } \textit{comparison test}
\end{aligned}$$

(c) Tighter bounds for $\tilde{S}_n(\pi)$ for certain values of $n \in \{0, 1, 2, 3, 4\}$:

$$\begin{aligned}
\tilde{S}_n(\pi) &= 2\left(\frac{2}{\pi}\right)^{2(n+1)} \sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^{2(n+1)} && \text{by item (7a)}\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)} U_{2(n+1)} && \text{by Jolley (1961), pages 56–57 } \langle(307)\rangle\\
&= 2\left(\frac{2}{\pi}\right)^{2(n+1)}\left[\frac{\pi^{2(n+1)}\alpha_{n+1}}{(4)[(2n+2)!]}\right] && \text{by Jolley (1961), pages 56–57 } \langle(307)\rangle\\
&= \frac{2^{2n+1}\alpha_{n+1}}{(2n+2)!}
\end{aligned}$$

$$= \begin{cases} \frac{2^1(1)}{2!} & \text{for } n=0 \quad (\alpha_1 = 1) \\ \frac{2^3(1)}{4!} & \text{for } n=1 \quad (\alpha_2 = 1) \\ \frac{2^5(3)}{6!} & \text{for } n=2 \quad (\alpha_3 = 3) \\ \frac{2^7(17)}{8!} & \text{for } n=3 \quad (\alpha_4 = 17) \\ \frac{2^9(155)}{10!} & \text{for } n=4 \quad (\alpha_5 = 155) \end{cases} \qquad \text{by Jolley (1961), page 234 } \langle(1130)\rangle$$

$$= \begin{cases} 1 & \text{for } n=0 \\ \frac{1}{3} & \text{for } n=1 \\ \frac{2}{15} & \text{for } n=2 \\ \frac{17}{315} & \text{for } n=3 \\ \frac{62}{2835} & \text{for } n=4 \end{cases} = \begin{cases} 1 & \text{for } n=0 \\ 0.3333333333333\cdots & \text{for } n=1 \\ 0.1333333333333\cdots & \text{for } n=2 \\ 0.0539682539682\cdots & \text{for } n=3 \\ 0.0218694885361\cdots & \text{for } n=4 \end{cases}$$

(d) Being important for the $n = 0$ case, note that[21]

$$\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^2}_{\textit{Dirichlet Lambda function } \lambda(2)} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

(e) Proof that $\tilde{S}_n(\pi) \le \frac{1}{3}$: because $\tilde{S}_n(\pi) = \frac{1}{3}$ for $n = 1$ (item (7c) page 317) and because $\tilde{S}_n(\pi)$ is decreasing for increasing n.

8. lemma: $\tilde{S}_n(\omega)$ *converges* to some value $> 0\ \forall \omega \in \mathbb{R}$. Proof:

(a) For $\omega = 0$, $\tilde{S}_n(\omega) = 1$ by item (6).

(b) Proof that $\tilde{S}_n(\omega) > 0$ for $\omega \in (0 : 2\pi)$:

$$\begin{aligned} \tilde{S}_n(\omega) &= \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)} \left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right) && \text{by (2) lemma} \\ &> 0 \end{aligned}$$

(c) Proof that $\tilde{S}_n(\omega)$ *converges*:

i. lemma: $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{2k \pm \frac{\omega}{\pi}}\right]^{2(n+1)}$ *converges*. Proof:

$$\begin{aligned} \lim_{b\to\infty}\int_1^b \left[\frac{1}{2y \pm \frac{\omega}{\pi}}\right]^{2(n+1)} \mathrm{d}y &= \lim_{b\to\infty}\int_1^b \left[2y \pm \frac{\omega}{\pi}\right]^{-2n-2} \mathrm{d}y \\ &= \lim_{b\to\infty} \left.\frac{\left[2y \pm \frac{\omega}{\pi}\right]^{-2n-1}}{2(-2n-1)}\right|_1^b \\ &= \lim_{b\to\infty}\left(\frac{-1}{2(2n+1)}\right)\left[\frac{1}{\left[2b \pm \frac{\omega}{\pi}\right]^{2n+1}} - \frac{1}{\left[2 \pm \frac{\omega}{\pi}\right]^{2n+1}}\right] \\ &= 0 + \frac{1}{2(2n+1)\left[2 \pm \frac{\omega}{\pi}\right]^{2n+1}} \\ &< \infty && \forall \omega \in [0 : 2\pi) \\ \implies \sum_{k=1}^{\infty}\left[\frac{1}{2k \pm \frac{\omega}{\pi}}\right]^{2(n+1)} & \textit{ converges} && \text{by } \textit{integral test} \end{aligned}$$

[21] Nahin (2011) page 153, Bailey et al. (2013) page 334 ⟨Catalan's Constant⟩, Bailey et al. (2011) ⟨(15)⟩, Wells (1987) page 36 ⟨Dictionary entry for π: pages 31–37⟩, Heinbockel (2010) page 94 ⟨(2.27) *Dirichlet Lambda function*⟩

ii. completion of proof using (8(c)i) lemma ...

$$\begin{aligned}
&\tilde{S}_n(\omega)\\
&= \left[\frac{\sin(\omega/2)}{(\omega/2)}\right]^{2(n+1)} + \left[\frac{\sin(\omega/2)}{(\pi/2)}\right]^{2(n+1)}\left(\sum_{k=1}^{\infty}\left[\frac{1}{2k-\frac{\omega}{\pi}}\right]^{2(n+1)} + \sum_{k=1}^{\infty}\left[\frac{1}{2k+\frac{\omega}{\pi}}\right]^{2(n+1)}\right) && \text{by (2) lemma}\\
&\implies \tilde{S}_n(\omega) \textit{ converges } \forall\omega\in(0:2\pi) && \text{by (8(c)i) lemma}
\end{aligned}$$

9. lemma (an expression for $\tilde{S}_n'(\omega)$):

$$\begin{aligned}
\tilde{S}_n'(\omega) &\triangleq \frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{S}_n(\omega)\\
&= \frac{\mathrm{d}}{\mathrm{d}\omega}\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2(n+1)} && \text{by (1) lemma page 316}\\
&= \sum_{k\in\mathbb{Z}}\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2(n+1)} && \text{by } \textit{linearity} \text{ of } \frac{\mathrm{d}}{\mathrm{d}\omega} \text{ operator}\\
&= \sum_{k\in\mathbb{Z}}2(n+1)\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right] && \text{by } \textit{power rule}\\
&= 2(n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\frac{1}{2}\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)-\sin\left(\frac{\omega}{2}\right)\left(-\frac{1}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by } \textit{quotient rule}\\
&= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right]
\end{aligned}$$

10. lemma: $\tilde{S}_n'(0) = \tilde{S}_n'(\pi) = 0$. Proof: This follows from Proposition R.3 (page 375). Here is alternate proof:

$$\begin{aligned}
\tilde{S}_n'(0) &= \lim_{\omega\to 0}\tilde{S}_n'(\omega)\\
&= \lim_{\omega\to 0}(n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by (9) lemma}\\
&= \lim_{\omega\to 0}(n+1)\left[\frac{\sin\left(\frac{\omega}{2}\right)}{-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(-\frac{\omega}{2}\right)^2}\right]\\
&= (n+1)\lim_{\omega\to 0}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{-\frac{\omega}{2}}\right]^{2n+1}\lim_{\omega\to 0}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(-\frac{\omega}{2}\right)^2}\right]\\
&= (n+1)[-1]^{2n+1}\lim_{\omega\to 0}\left[\frac{-\frac{1}{2}\sin\left(\frac{\omega}{2}\right)\left(-\frac{\omega}{2}\right)+\cos\left(\frac{\omega}{2}\right)\left(-\frac{1}{2}\right)+\cos\left(\frac{\omega}{2}\right)\left(\frac{1}{2}\right)}{-\frac{2}{2}\left(-\frac{\omega}{2}\right)}\right] && \text{by } \textit{l'Hôpital's rule}\\
&= (1)(0)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\tilde{S}'_n(\pi) &= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\pi}{2}\right)}{k\pi-\frac{\pi}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\pi}{2}\right)\left(k\pi-\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)}{\left(k\pi-\frac{\pi}{2}\right)^2}\right] && \text{by (9) lemma}\\
&= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{1}{k\pi-\frac{\pi}{2}}\right]^{2n+1}\left[\frac{0\left(k\pi-\frac{\pi}{2}\right)+1}{\left(k\pi-\frac{\pi}{2}\right)^2}\right]\\
&= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\sum_{k\in\mathbb{Z}}\left[\frac{1}{2k-1}\right]^{2n+3}\\
&= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\left[\left(\frac{1}{1}\right)^{2n+3}+\left(\frac{1}{-1}\right)^{2n+3}+\left(\frac{1}{3}\right)^{2n+3}+\left(\frac{1}{-3}\right)^{2n+3}+\ldots\right]\\
&= (n+1)\left(\frac{2}{\pi}\right)^{2n+3}\sum_{k=1}^{\infty}(-1)^{k+1}\alpha_k \qquad \text{where } \alpha_k \triangleq \begin{cases}\left(\frac{1}{k}\right)^{2n+3} & \text{for } k \text{ odd}\\ \left(\frac{1}{k-1}\right)^{2n+3} & \text{for } k \text{ even}\end{cases}\\
&= 0 \qquad \text{because } \lim_{k\to\infty}\alpha_k = 0 \text{ and by \textit{Alternating Series Test}}
\end{aligned}$$

11. lemma: $\tilde{S}_n(\omega)$ is *decreasing* with respect to $\omega \in [0 : \pi]$. Proof:

$$\begin{aligned}
\tilde{S}'_n(\omega) &= (n+1)\sum_{k\in\mathbb{Z}}\left[\frac{\sin\left(\frac{\omega}{2}\right)}{k\pi-\frac{\omega}{2}}\right]^{2n+1}\left[\frac{\cos\left(\frac{\omega}{2}\right)\left(k\pi-\frac{\omega}{2}\right)+\sin\left(\frac{\omega}{2}\right)}{\left(k\pi-\frac{\omega}{2}\right)^2}\right] && \text{by (9) lemma page 319}\\
&= \underbrace{(n+1)\left(\sin\frac{\omega}{2}\right)^{2n+1}}_{\geq 0 \text{ for } \omega\in[0:2\pi)}\underbrace{\sum_{k\in\mathbb{Z}}\left[\frac{1}{k\pi-\frac{\omega}{2}}\right]^{2n+2}}_{>0}\left[\underbrace{\left(\cos\frac{\omega}{2}\right)}_{\text{sign change at } \omega=\pi}+\frac{\overbrace{\sin\frac{\omega}{2}}^{>0 \text{ for } \omega\in(0:2\pi)}}{\underbrace{k\pi-\frac{\omega}{2}}_{\textit{decreasing} \text{ w.r.t. } \omega\in\mathbb{R}}}\right]
\end{aligned}$$

12. lemma: $\tilde{S}_n(\omega)$ is *increasing* with respect to $\omega \in [\pi : 2\pi)$. Proof: This is true because $\tilde{S}_n(\omega)$ is *decreasing* in $[0 : \pi]$ ((11) lemma) and because $\tilde{S}_n(\omega)$ is *symmetric* about $\omega = \pi$ ((5) lemma).

13. Proof that $0 < \tilde{S}_n(\omega) \leq 1$:
    (a) $\tilde{S}_n(\omega) > 0$ by (8) lemma and
    (b) $\tilde{S}_n(0) = 1$ by item (6) and
    (c) $\tilde{S}_n(\omega)$ is *decreasing* from $\omega = 0$ to $\omega = \pi$ by (11) lemma and
    (d) $\tilde{S}_n(\omega)$ is *increasing* from $\omega = \pi$ to $\omega = 2\pi$ by (12) lemma and
    (e) $\tilde{S}_n(2\pi) = 1$ because $\tilde{S}_n(2\pi) = \tilde{S}_n(0)$ by (4) lemma.

⇒

Theorem M.8. [22]

1. $(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ *is a*	RIESZ BASIS	*for* $\operatorname{span}(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$		$\forall n \in \mathbb{W}$
2. $(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ *is an*	ORTHONORMAL BASIS	*for* $\operatorname{span}(\!(\mathrm{N}_n(x-k))\!)_{k\in\mathbb{Z}}$	$\iff$	$n = 0$

✎PROOF:

[22] Wojtaszczyk (1997) page 56 ⟨Proposition 3.12⟩, Prasad and Iyengar (1997) page 148 ⟨Theorem 6.3⟩, Forster and Massopust (2009) page 66 ⟨Theorem 2.25⟩

1. Proof that $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is a *Riesz basis* for $\operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$:

$$\begin{aligned}
&0 < \tilde{S}_n(\omega) \le 1 && \text{by Lemma M.6 page 315 (1)}\\
&\implies (\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}} \text{ is a } \textit{Riesz basis} \text{ for } \operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}} && \text{by Theorem R.2 page 376}
\end{aligned}$$

2. Proof that $\{n=0\} \iff (\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is an *orthonormal basis* for $\operatorname{span}(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$:

$$\begin{aligned}
n=0 &\iff \tilde{S}_n(\omega) = 1 && \text{by Lemma M.6 page 315 (2), (4)}\\
&\iff (\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}} \text{ is an } \textit{orthonormal basis} \text{ for } \operatorname{span}(\!(\mathsf{N}_n(x-k))\!) && \text{by Theorem R.3 page 379}
\end{aligned}$$

✑

M.6 Mutiresolution properties

M.6.1 Introduction

In 1989, Stéphane G. Mallat introduced the *Multiresolution Analysis* (MRA) structure (Definition 2.1 page 16) An MRA is very powerful because it can be used to approximate functions at incrementally increasing "scales" of resolution, and furthermore induces a *wavelet*. In fact, the MRA has become the dominate wavelet construction method. Moreover, P.G. Lemarié has proved that all wavelets with *compact support* are generated by an MRA. [23]

M.6.2 B-spline dyadic decomposition

One key feature of an MRA is *dyadic decomposition* such that $\mathsf{N}_n(x) = \sum_k \alpha_n \mathsf{N}_n(2x-n)$ for some sequence $(\!(\alpha_n)\!)$. As it turns out, *B-spline*s also have this property (next theorem).

Theorem M.9 (*B-spline dyadic decomposition*). [24] *Let* $\mathsf{N}_n(x)$ *be the* nTH ORDER B-SPLINE.

THM
$$\mathsf{N}_n(x) = \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-k) \qquad \forall n\in\mathbb{W},\ \forall x\in\mathbb{R}$$

✎PROOF:

1. Base case ...proof for $n=0$ case:

$$\begin{aligned}
\mathsf{N}_0(x) &= \mathbb{1}_{[0:1)}(x) && \text{by definition of } \mathbb{1}_A(x) \quad \text{(Definition 1.2 page 1)}\\
&= \mathbb{1}_{[0:\frac{1}{2})}(x) + \mathbb{1}_{[\frac{1}{2}:1)}(x)\\
&= \mathbb{1}_{[2\times 0:2\times\frac{1}{2})}(2x) + \mathbb{1}_{[2\times\frac{1}{2}-1:2\times 1-1)}(2x-1)\\
&= \mathbb{1}_{[0:1)}(2x) + \mathbb{1}_{[0:1)}(2x-1)\\
&= \frac{1}{2^0}\sum_{k=0}^{0+1}\binom{0+1}{k}\mathsf{N}_n(2x-k)
\end{aligned}$$

[23] Mallat (1999) page 240, Definition 2.1 (page 16)
[24] Prasad and Iyengar (1997) pages 151–152 ⟨proof using Fourier transform⟩

2. Induction step…proof that n case $\implies$ $n+1$ case:

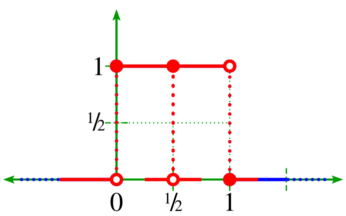
$$
\begin{aligned}
\mathsf{N}_{n+1}(x) &= \int_0^1 \mathsf{N}_n(x-\tau)\,\mathrm{d}\tau && \text{by Lemma M.2 page 297}\\
&= \int_0^1 \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-2\tau-k)\,\mathrm{d}\tau && \text{by induction hypothesis}\\
&= \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\int_{\tau=0}^{\tau=1}\mathsf{N}_n(2x-2\tau-k)\,\mathrm{d}\tau && \text{by } \textit{linearity} \text{ of } \textstyle\sum \text{ operator}\\
&= \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\int_{u=0}^{u=2}\mathsf{N}_n(2x-u-k)\frac{1}{2}\,\mathrm{d}u && \text{where } u \triangleq 2\tau \implies \tau=\tfrac{1}{2}u\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\int_{u=0}^{u=1}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u+\int_{u=1}^{u=2}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u\right]\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\int_{u=0}^{u=1}\mathsf{N}_n(2x-k-u)\,\mathrm{d}u+\int_{v=0}^{v=1}\mathsf{N}_n(2x-k-v-1)\,\mathrm{d}v\right] && \text{where } v\triangleq u-1 \implies u=v+1\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left[\mathsf{N}_n(2x-k)+\mathsf{N}_n(2x-k-1)\right] && \text{by Lemma M.2 page 297}\\
&= \frac{1}{2^{n+1}}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}\mathsf{N}_n(2x-k)+\sum_{m=1}^{n+2}\binom{n+1}{m-1}\mathsf{N}_n(2x-m)\right] && \text{where } m\triangleq k+1 \implies k=m-1\\
&= \frac{1}{2^{n+1}}\left[\underbrace{\sum_{k=1}^{n+1}\left[\binom{n+1}{k}+\binom{n+1}{k-1}\right]\mathsf{N}_n(2x-k)}_{\text{common indices of above two summations}}+\underbrace{\binom{n+1}{0}\mathsf{N}_n(2x-0)}_{k=0\text{ term}}+\underbrace{\binom{n+2}{n+2}\mathsf{N}_n(2x-n-2)}_{m=n+2\text{ term}}\right]\\
&= \frac{1}{2^{n+1}}\left[\underbrace{\sum_{k=1}^{n+1}\binom{n+2}{k}\mathsf{N}_n(2x-k)}_{\text{by } \textit{Stifel formual} \text{ (Theorem B.1 page 133)}}+\underbrace{\binom{n+2}{0}\mathsf{N}_n(2x-0)}_{\text{because }\binom{n+1}{0}=1=\binom{n+2}{0}}+\binom{n+2}{n+2}\mathsf{N}_n(2x-n-2)\right]\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+2}\binom{n+2}{k}\mathsf{N}_n(2x-k)
\end{aligned}
$$

✎

Example M.16. [25]The 0 *order B-spline dyadic decomposition*

$$\mathsf{N}_0(x) = \frac{1}{1}\sum_{k=0}^{k=1}\binom{1}{k}\mathsf{N}_0(2x-k)$$

is illustrated to the right.

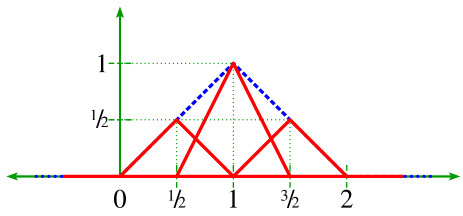


[25] Strang (1989) page 615 ⟨Box function⟩, Strang and Nguyen (1996) page 441 ⟨Box function⟩

Example M.17. [26]The 1*st order B-spline dyadic decomposition*

$$\mathsf{N}_1(x) = \frac{1}{2}\sum_{k=0}^{k=2}\binom{2}{k}\mathsf{N}_1(2x-k)$$

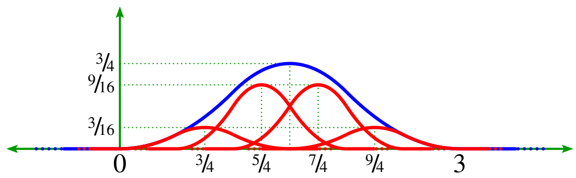
is illustrated to the right.



Example M.18. The 2*nd order B-spline dyadic decomposition*

$$\mathsf{N}_2(x) = \frac{1}{4}\sum_{k=0}^{k=3}\binom{3}{k}\mathsf{N}_2(2x-k)$$

is illustrated to the right.



M.6.3 B-spline MRA scaling functions

Theorem M.10. *Let f $N_n(x)$ be the nTH ORDER B-SPLINE* (Definition M.2 page 297).
Let $\boldsymbol{V}_j \triangleq \operatorname{span}\left(\!\left(\mathsf{N}_n\left(2^j x - k\right)\right)\!\right)_{k\in\mathbb{Z}}$.

THM	$\left(\!\left(\boldsymbol{V}_j\right)\!\right)_{j\in\mathbb{Z}}$ *is a* MULTIRESOLUTION ANALYSIS *on* $\boldsymbol{L}^2_{\mathbb{R}}$ *with* SCALING FUNCTION $\phi(x) \triangleq \mathsf{N}_n(x)$

✎PROOF:

1. lemma: $(\!(\mathsf{N}_n(x-k))\!)_{k\in\mathbb{Z}}$ is a *Riesz sequence*. Proof: by Theorem M.8 (page 320).
2. lemma: $\exists\,(\!(h_k)\!)$ such that $\mathsf{N}_n(x) = \sum_{k\in\mathbb{Z}} h_k \mathsf{N}_n(2x-k)$. Proof: by Theorem M.9 (page 321). In fact, note that $h_k = \frac{1}{2^n\sqrt{2}}\binom{n+1}{k}$
3. lemma: $\tilde{\mathbf{F}}\mathsf{N}_n(\omega)$ is *continuous* at 0. Proof:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(\omega) &= \tfrac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1} && \text{by Lemma M.5 page 312}\\
&\implies \textit{continuous} \text{ at } 0 && \text{by known property of sinc function}
\end{aligned}$$

4. lemma: $\tilde{\phi}(0) \neq 0$. Proof:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathsf{N}_n(0) &= \tfrac{1}{\sqrt{2\pi}}e^{-i\frac{(n+1)\omega}{2}}\left(\operatorname{sinc}\frac{\omega}{2}\right)^{n+1}\Bigg|_{\omega=0} && \text{by Lemma M.5 page 312}\\
&= 1\cdot\frac{1}{1/2} = 2 && \text{by } \textit{l'Hôpital's rule}\\
&\neq 0
\end{aligned}$$

5. The completion of this proof follows directly from (1) lemma, (2) lemma, (3) lemma, (4) lemma, and Theorem 2.6 (page 31).

✏

[26] Strang (1989) page 615 ⟨Hat function⟩, Strang and Nguyen (1996) page 442 ⟨Hat function⟩, Heil (2011) page 380 ⟨Fig. 12.10⟩

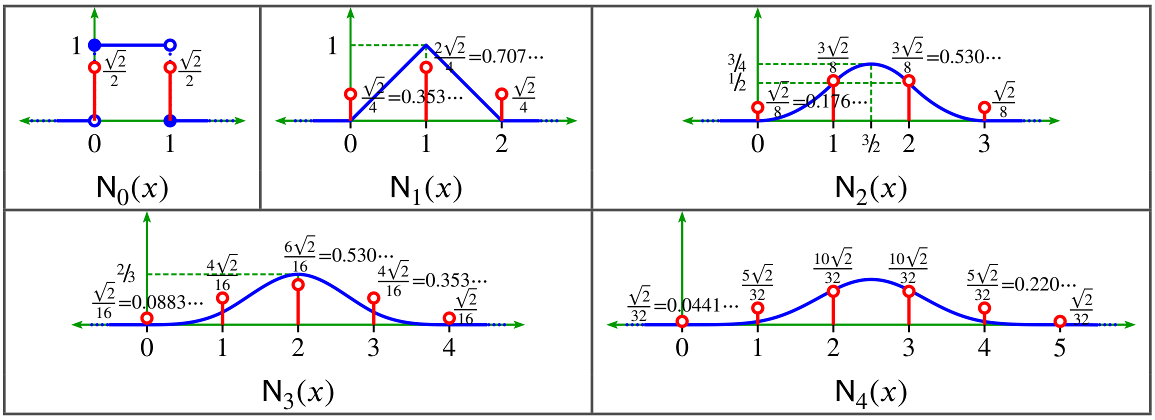


Figure M.4: *dilation equation* demonstrations for selected B-splines (Example M.19 page 324)

M.6.4 B-spline MRA coefficient sequences

Because each *B-spline* $\mathsf{N}_n(x)$ is the *scaling function* for an *MRA* (Theorem M.10 page 323), each *B-spline* also satisfies the *dilation equation* (Theorem 2.1 page 22) such that

$$\mathsf{N}_n(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \mathsf{N}(2x-k) \quad \text{where} \quad h_k = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2^{n+1}}\binom{n+1}{k} & \text{for } n = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{array}\right\}$$

The resulting *sequence* $(\!(h_k)\!)_{k\in\mathbb{Z}}$ is the *order n B-spline MRA coefficient sequence* induced by the *order n B-spline MRA scaling sequence* $\phi(x) \triangleq \mathsf{N}_n(x)$.[27]

Example M.19. See Figure M.4 (page 324) for some *dilation equation* demonstrations of selected B-splines.

Theorem M.11 (B-spline scaling coefficients). *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be an* MRA SYSTEM *(Definition 2.3 page 25). Let* $\mathsf{N}_n(x)$ *be a* nTH ORDER B-SPLINE *(Definition M.2 page 297).*

THM

$$\underbrace{\phi(x) \triangleq \mathsf{N}_n(x)}_{\text{(1) B-spline scaling function}} \implies (\!(h_k)\!) = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2^{n+1}}\binom{n}{k} & \text{for } k = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{array}\right. \quad \text{(2) scaling sequence in “time”}$$

$$\iff \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} = \frac{\sqrt{2}}{2^n}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}} \quad \text{(3) scaling sequence in “z domain”}$$

$$\iff \tilde{\mathsf{h}}(\omega) = 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} \quad \text{(4) scaling sequence in “frequency”}$$

✎PROOF:

1. Proof that (1) $\implies$ (3): By Theorem M.10 page 323 we know that $\mathsf{N}_n(x)$ is a *scaling function* (Definition 2.1 page 16). So then we know that we can use Lemma 2.1 page 22.

$$\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \sqrt{2}\frac{\tilde{\phi}(2\omega)}{\tilde{\phi}(\omega)} && \text{by Lemma 2.1 page 22} \\
&= \sqrt{2}\frac{\tilde{\mathsf{N}}_n(2\omega)}{\tilde{\mathsf{N}}_n(\omega)} && \text{by (1)} \\
&= \sqrt{2}\frac{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i2\omega}}{2i\omega}\right)^{n+1}}{\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1}} && \text{by Lemma M.5 page 312}
\end{aligned}$$

[27] For *Octave*/ *MatLab* code useful for plotting a function given a sequence of coefficients $(\!(h_k)\!)$, see Section V.1 (page 399).

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{1-z^{-2}}{1-z^{-1}}\right)^{n+1}\bigg|_{z=e^{i\omega}} \\
&= \frac{\sqrt{2}}{2^{n+1}}\left[\left(\frac{1-z^{-2}}{1-z^{-1}}\right)\left(\frac{1+z^{-1}}{1+z^{-1}}\right)\right]^{n+1}\bigg|_{z=e^{i\omega}} \\
&= \frac{\sqrt{2}}{2^{n+1}}\left(\frac{\left(1-z^{-2}\right)\left(1+z^{-1}\right)}{1-z^{-2}}\right)^{n+1}\bigg|_{z=e^{i\omega}} \\
&= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\Big|_{z\triangleq e^{i\omega}}
\end{aligned}
$$

2. Proof that (3) $\iff$ (2):

$$
\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\bigg|_{z\triangleq e^{i\omega}} && \text{by (3)} \\
&= \frac{\sqrt{2}}{2^{n}}\left(\sum_{k=0}^{n+1}\binom{n}{k}z^{-k}\right)\bigg|_{z\triangleq e^{i\omega}} && \text{by \textit{binomial theorem}} \\
\iff \qquad h_k &= \frac{\sqrt{2}}{2^{n+1}}\binom{n}{k} && \text{by definition of \textit{Z transform} (Definition O.4 page 342)}
\end{aligned}
$$

3. Proof that (3) $\implies$ (4):

$$
\begin{aligned}
\tilde{\mathsf{h}}(\omega) &= \check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} && \text{by definition of \textit{DTFT} (Definition P.1 page 355)} \\
&= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\bigg|_{z\triangleq e^{i\omega}} && \text{by (3)} \\
&= \frac{\sqrt{2}}{2^{n}}\left(1+e^{-i\omega}\right)^{n+1} && \text{by definition of } z \\
&= \frac{\sqrt{2}}{2^{n}}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1} \\
&= \frac{\sqrt{2}}{2^{n}}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1} \\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1}
\end{aligned}
$$

4. Proof that (3) $\impliedby$ (4):

$$
\begin{aligned}
\check{\mathsf{h}}(z)\Big|_{z\triangleq e^{i\omega}} &= \check{\mathsf{h}}\left(e^{i\omega}\right) \\
&= \tilde{\mathsf{h}}(\omega) \\
&= 2\sqrt{2}e^{-i\frac{n+1}{2}\omega}\left[\cos\left(\frac{\omega}{2}\right)\right]^{n+1} && \text{by (4)} \\
&= \frac{\sqrt{2}}{2^{n}}e^{-i\frac{n+1}{2}\omega}\left[2\cos\left(\frac{\omega}{2}\right)\right]^{n+1} \\
&= \frac{\sqrt{2}}{2^{n}}\left[e^{-i\frac{1}{2}\omega}\left(e^{i\frac{\omega}{2}}+e^{-i\frac{\omega}{2}}\right)\right]^{n+1} \\
&= \frac{\sqrt{2}}{2^{n}}\left(1+e^{-i\omega}\right)^{n+1} \\
&= \frac{\sqrt{2}}{2^{n}}\left(1+z^{-1}\right)^{n+1}\bigg|_{z\triangleq e^{i\omega}}
\end{aligned}
$$

Example M.20 (2 coefficient case). [28] Let $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$ be an *MRA system* (Definition 2.3 page 25).

EX

$$\underbrace{\left\{\begin{array}{ll} 1. & \operatorname{supp}\phi(x) = [0:1] \quad \text{and} \\ 2. & \left(\!\left(\phi(x-k)\right)\!\right) \text{ forms a} \\ & \textit{partition of unity} \end{array}\right\}}_{\text{(A)}} \iff \underbrace{h_n = \left\{\begin{array}{ll} \frac{\sqrt{2}}{2} & \text{for } n=0 \\ \frac{\sqrt{2}}{2} & \text{for } n=1 \\ 0 & \text{otherwise} \end{array}\right\}}_{\text{(B)}} \iff \underbrace{\left\{\phi(x)=\mathsf{N}_0(x)\right\}}_{\text{(C)}}$$

✎PROOF:

1. Proof that (A) $\implies$ (B):

   (a) lemma: Only h_0 and h_1 are *non-zero*; All other coefficients h_k are 0. Proof: This follows from $\operatorname{supp}\phi(x) = [0:1]$ (Definition 2.4 page 31) and by Theorem 2.7 page 32.

   (b) lemma (equations for $\left(\!\left(h_k\right)\!\right)$): Because $\left(\!\left(h_k\right)\!\right)$ is a *scaling coefficient sequence* (Definition 2.1 page 16), it must satisfy the *admissibility equation* (Theorem 2.3 page 27). And because $\left(\!\left(\phi(x-k)\right)\!\right)$ forms a *partition of unity*, it must satisfy the equations given by Theorem 2.8 (page 34). (1a) lemma and these two constraints yield two simultaneous equations and two unknowns:

$$\begin{array}{rcll} h_0 + h_1 &=& \sqrt{2} & (\textit{admissibility condition}) \\ h_0 - h_1 &=& 0 & (\textit{partition of unity / zero at } -1 \textit{ / vanishing 0th moment}) \end{array}$$

   (c) lemma: The equations provided by (1b) lemma can be expressed in matrix algebra form as follows...

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

   (d) lemma: The *inverse* $\mathbf{A}^{-1}$ of $\mathbf{A}$ can be expressed as demonstrated below...

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & -1 & -1/2 & 1/2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right]$$
$$\implies \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (e) Proof for the values of $\left(\!\left(h_k\right)\!\right)$ (B):

$$\begin{aligned} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} &= \mathbf{A}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} && \text{by (1c) lemma} \\ &= \mathbf{A}^{-1}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} && \text{by (1c) lemma} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} && \text{by (1d) lemma} \\ &= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

[28] Haar (1910), Wojtaszczyk (1997) pages 14–15 ⟨"Sources and comments"⟩

2. Proof that (B) $\implies$ (C):

$$\begin{aligned}
\text{(B)} \implies \phi(x) &= \sum_{k\in\mathbb{Z}} h_k \sqrt{2}\phi(2x-k) && \textit{dilation equation} && \text{(Theorem 2.1 page 22)}\\
&= \sum_{k=0}^{k=1} \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\phi(2x-k) && \text{by item (1e) page 326}\\
&= \sum_{k=0}^{k=1} \phi(2x-k)\\
&= \sum_{k=0}^{k=1} \binom{1}{k}\phi(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition B.3 page 132)}\\
&\implies \text{(D)} && \text{by } \textit{B-spline dyadic decomposition} && \text{(Theorem M.9 page 321)}
\end{aligned}$$

3. Proof that (B) $\impliedby$ (C):

$$\begin{aligned}
\text{(C)} \implies \mathsf{N}_0(x) &= \sum_{k=0}^{k=1} \binom{1}{k}\mathsf{N}_0(2x-k) && \text{by } \textit{B-spline dyadic decomposition} && \text{(Theorem M.9 page 321)}\\
&= \sum_{k=0}^{k=1} \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{N}_0(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition B.3 page 132)}\\
&= \sum_{k\in\mathbb{Z}} h_k\sqrt{2}\mathsf{N}_0(2x-k) && \text{by definition of } \binom{n}{k} && \text{(Definition B.3 page 132)}\\
&\implies \text{(B)}
\end{aligned}$$

4. Proof that (A) $\impliedby$ (C):
   1. Proof that (C) $\implies \operatorname{supp}\phi(x) = [0:1]$: by Theorem M.4 (page 308)
   2. Proof that (C) $\implies \langle\!\langle\phi(x-k)\rangle\!\rangle$ forms a *partition of unity*: by Theorem M.7 (page 314)

✏

EX

n=0,	(÷0!)	1;								
n=1,	(÷1!)	1,	0;							
		−1,	2;							
n=2,	(÷2!)	1,	0,	0;						
		−2,	6,	−3;						
		1,	−6,	9;						
n=3,	(÷3!)	1,	0,	0,	0;					
		−3,	12,	−12,	4;					
		3,	−24,	60,	−44;					
		−1,	12,	−48,	64;					
n=4,	(÷4!)	1,	0,	0,	0,	0;				
		−4,	20,	−30,	20,	−5;				
		6,	−60,	210,	−300,	155;				
		−4,	60,	−330,	780,	−655;				
		1,	−20,	150,	−500,	625;				
n=5,	(÷5!)	1,	0,	0,	0,	0,	0;			
		−5,	30,	−60,	60,	−30,	6;			
		10,	−120,	540,	−1140,	1170,	−474;			
		−10,	180,	−1260,	4260,	−6930,	4386;			
		5,	−120,	1140,	−5340,	12270,	−10974;			
		−1,	30,	−360,	2160,	−6480,	7776;			
n=6,	(÷6!)	1,	0,	0,	0,	0,	0,	0;		
		−6,	42,	−105,	140,	−105,	42,	−7;		
		15,	−210,	1155,	−3220,	4935,	−3990,	1337;		
		−20,	420,	−3570,	15680,	−37590,	47040,	−24178;		
		15,	−420,	4830,	−29120,	96810,	−168000,	119182;		
		−6,	210,	−3045,	23380,	−100065,	225750,	−208943;		
		1,	−42,	735,	−6860,	36015,	−100842,	117649;		
n=7,	(÷7!)	1,	0,	0,	0,	0,	0,	0,	0;	
		−7,	56,	−168,	280,	−280,	168,	−56,	8;	
		21,	−336,	2184,	−7560,	15400,	−18648,	12488,	−3576;	
		−35,	840,	−8400,	45360,	−143360,	267120,	−273280,	118896;	
		35,	−1120,	15120,	−111440,	483840,	−1238160,	1733760,	−1027984;	
		−21,	840,	−14280,	133560,	−741160,	2436840,	−4391240,	3347016;	
		7,	−336,	6888,	−78120,	528920,	−2135448,	4753336,	−4491192;	
		−1,	56,	−1344,	17920,	−143360,	688128,	−1835008,	2097152;	
n=8,	(÷8!)	1,	0,	0,	0,	0,	0,	0,	0,	0;
		−8,	72,	−252,	504,	−630,	504,	−252,	72,	−9
		28,	−504,	3780,	−15624,	39690,	−64008,	64260,	−36792,	9207
		−56,	1512,	−17388,	111384,	−436590,	1079064,	−1650348,	1432872,	−541917
		70,	−2520,	39060,	−340200,	1821330,	−6146280,	12800340,	−15082200,	7715619
		−56,	2520,	−49140,	541800,	−3691170,	15903720,	−42324660,	63667800,	−41503131
		28,	−1512,	35532,	−474264,	3929310,	−20674584,	67410252,	−124449192,	99584613
		−8,	504,	−13860,	217224,	−2121210,	13208328,	−51179940,	112731192,	−107948223
		1,	−72,	2268,	−40824,	459270,	−3306744,	14880348,	−38263752,	43046721

Table M.1: Coefficients of the *B-spline*s $\mathsf{N}_n(x)$ multiplied by $n!$ (Example M.9 page 304)

APPENDIX N

INTERPOLATION

N.1 Polynomial interpolation

Definition N.1. [1] *The **Lagrange polynomial** $L_{P,n}(x)$ with respect to the $n+1$ points $P=\{(x_k,y_k)|k=0,1,2,\dots,n\}$ is defined as*

DEF
$$L_{P,n}(x) \triangleq \sum_{k=0}^{n} y_k \prod_{m\neq n} \frac{x-x_m}{x_k-x_m}$$

Proposition N.1. *Let $L_{P,n}(x)$ be the Lagrange polynomial with respect to the points $P=\{(x_k,y_k)|k=0,1,2,\dots,n\}$.*

PRP
1. *$L_{P,n}(x)$ is an nth order polynomial.*
2. *$L_{P,n}(x)$ intersects all $n+1$ points in P.*

Example N.1 (Lagrange interpolation). The Lagrange polynomial $L_{P,3}(x)$ with respect to the 4 points $P=\{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is

EX
$$L_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$\begin{aligned}
L_{P,3}(x) &= \sum_{k=0}^{n} y_k \prod_{m\neq n} \frac{x-x_m}{x_k-x_m} && \text{by Definition N.1}\\
&= y_0\frac{(x+1)(x-3)(x-5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1\frac{(x+2)(x-3)(x-5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\\
&\quad + y_2\frac{(x+2)(x+1)(x-5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3\frac{(x+2)(x+1)(x-3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\\
&= 1\frac{(x+1)(x-3)(x-5)}{(-2+1)(-2-3)(-2-5)} + 3\frac{(x+2)(x-3)(x-5)}{(-1+2)(-1-3)(-1-5)}\\
&\quad + 2\frac{(x+2)(x+1)(x-5)}{(3+2)(3+1)(3-5)} + 4\frac{(x+2)(x+1)(x-3)}{(5+2)(5+1)(5-3)}
\end{aligned}$$

[1] Matthews and Fink (1992), page 206

$$
\begin{aligned}
&= 1\underbrace{\frac{x^3-7x^2+7x+15}{-35}}_{\text{roots}=-1,3,5} + 3\underbrace{\frac{x^3-6x^2-x+30}{24}}_{\text{roots}=-2,3,5} + 2\underbrace{\frac{x^3-2x^2-13x-10}{-40}}_{\text{roots}=-2,-1,5} + 4\underbrace{\frac{x^3-7x-6}{84}}_{\text{roots}=-2,-1,3} \\
&= -\frac{x^3-7x^2+7x+15}{35} + \frac{x^3-6x^2-x+30}{8} - \frac{x^3-2x^2-13x-10}{20} + \frac{x^3-7x-6}{21} \\
&= x^3\left(\frac{-8\cdot 20\cdot 21+35\cdot 20\cdot 21-35\cdot 8\cdot 21+35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right) \\
&\quad + x^2\left(\frac{7\cdot 8\cdot 20\cdot 21-6\cdot 35\cdot 20\cdot 21+2\cdot 35\cdot 8\cdot 21+0\cdot 35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right) \\
&\quad + x\left(\frac{-7\cdot 8\cdot 20\cdot 21-35\cdot 20\cdot 21+13\cdot 35\cdot 8\cdot 21-7\cdot 35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right) \\
&\quad + \left(\frac{-15\cdot 8\cdot 20\cdot 21+30\cdot 35\cdot 20\cdot 21+10\cdot 35\cdot 8\cdot 21-6\cdot 35\cdot 8\cdot 20}{35\cdot 8\cdot 20\cdot 21}\right) \\
&= \frac{11060}{117600}x^3 + \frac{-52920}{117600}x^2 + \frac{-980}{117600}x + \frac{415800}{117600} \\
&= \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}
\end{aligned}
$$

✏

Definition N.2. [2] *The* ***Newton polynomial*** $N_{P,n}(x)$ *with respect to the* $n+1$ *points* $P = \left\{(x_k, y_k) | k = 0, 1, 2, \ldots, n\right\}$ *is defined as*

DEF
$$N_{P,n}(x) \triangleq \sum_{k=0}^{n} \alpha_k \prod_{m=0}^{k} (x - x_m)$$

Proposition N.2. *Let* $N_{P,n}(x)$ *be the Newton polynomial with respect to the points* $P = \left\{(x_k, y_k) | k = 0, 1, 2, \ldots, n\right\}$.

PRP
1. $N_{P,n}(x)$ *is an* n*th order polynomial.*
2. $N_{P,n}(x)$ *intersects all* $n+1$ *points in* P.

Example N.2 (Newton polynomial interpolation). The Newton polynomial $N_{P,3}(x)$ with respect to the 4 points
$P = \{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is

EX
$$N_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$
\begin{aligned}
N_{P,3}(x) &= \sum_{k=0}^{n} \alpha_k \prod_{m=1}^{k} (x - x_m) \\
&= \alpha_0 + \alpha_1(x-x_0) + \alpha_2(x-x_0)(x-x_1) + \alpha_3(x-x_0)(x-x_1)(x-x_2) \\
&= \alpha_0 + \alpha_1(x+2) + \alpha_2(x+2)(x+1) + \alpha_3(x+2)(x+1)(x-3) \\
&= \alpha_0 + \alpha_1(x+2) + \alpha_2(x^2+3x+2) + \alpha_3(x^3-7x-6) \\
&= x^3(\alpha_3) + x^2(\alpha_2) + x(-7\alpha_3+3\alpha_2+\alpha_1) + (-6\alpha_3+2\alpha_2+2\alpha_1+\alpha_0) \\
&= \begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -6 & -7 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}
\end{aligned}
$$

[2] Matthews and Fink (1992), page 220

$$
\begin{aligned}
\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & 0 \\ 1 & (x_3 - x_0) & (x_3 - x_0)(x_3 - x_1) & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (-1+2) & 0 & 0 \\ 1 & (3+2) & (3+2)(3+1) & 0 \\ 1 & (5+2) & (5+2)(5+1) & (5+2)(5+1)(5-3) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 \\ 1 & 7 & 42 & 84 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 & 0 & 0 & 1 & 0 \\ 1 & 7 & 42 & 84 & 0 & 0 & 0 & 1 \end{array}\right]
&= \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 5 & 20 & 0 & -1 & 0 & 1 & 0 \\ 0 & 7 & 42 & 84 & -1 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 20 & 0 & 4 & -5 & 1 & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & 6-\frac{42}{5} & -7+\frac{42}{4} & -\frac{42}{20} & 1 \end{array}\right] \\
&= \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{12}{5} & \frac{14}{4} & -\frac{42}{20} & 1 \end{array}\right] \\
&= \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{24}{10} & \frac{35}{10} & -\frac{21}{10} & \frac{10}{10} \end{array}\right] \\
&= \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 1 & -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840} \end{array}\right]
\end{aligned}
$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -\frac{9}{20} \\ \frac{79}{840} \end{bmatrix}$$

$$N_{P,3}(x) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -6 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & | & 2 & | & -\frac{9}{20} & | & \frac{79}{840} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -6 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 4 - \frac{9}{10} - \frac{79}{140} & | & 2 - \frac{27}{20} - \frac{79}{120} & | & -\frac{9}{20} & | & \frac{79}{840} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

$$= \frac{79}{840}x^3 - \frac{378}{840}x^2 - \frac{7}{840}x + \frac{2970}{840}$$

✏

Example N.3 (Least squares polynomial interpolation). [3] The best 3rd order polynomial in the **least squares** $S_{P,3}(x)$ sense with respect to the 4 points
$P = \{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is

EX $$S_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 5 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

We want to find a third order polynomial

$$dx^3 + cx^2 + bx + a$$

that best approximates the 4 points in the least squares sense. We define the matrix U (known) and vector $\hat{\theta}$ (to be computed) as follows:

$$U^H \triangleq \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \qquad \hat{\theta} \triangleq \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

[3] van Overschee and de Moor (2012) page 14 ⟨Table 1.1; historical perspective as relates to "subspace identification"⟩

EX

p	$(1-y)^p\mathsf{P}_m(y) = (1-y)^p \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k$
1	$1 - y$
2	$1 - 3y^2 + 2y^3$
3	$1 - 10y^3 + 15y^4 - 6y^5$
4	$1 - 35y^4 + 84y^5 - 70y^6 + 20y^7$
5	$1 - 126y^5 + 420y^6 - 540y^7 + 315y^8 - 70y^9$
6	$1 - 462y^6 + 1980y^7 - 3465y^8 + 3080y^9 - 1386y^{10} + 252y^{11}$

Table N.1: Low-pass term $(1-y)^p\mathsf{P}_m(y)$

Then, using *Least squares*, the best coefficients for the polynomial are

$$\begin{aligned}
\hat{\theta} &= \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
&= R^{-1}W \\
&= (UU^H)^{-1}(U\mathbf{y}) \\
&= \left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}^H \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}^H \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix}^H \begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & (-2) & (-2)^2 & (-2)^3 \\ 1 & (-1) & (-1)^2 & (-1)^3 \\ 1 & (3) & (3)^2 & (3)^3 \\ 1 & (5) & (5)^2 & (5)^3 \end{bmatrix}^H \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix}^H \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix}^H \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \right) \\
&= \begin{bmatrix} 2970 \\ -7 \\ -378 \\ 79 \end{bmatrix}
\end{aligned}$$

✏

N.2 Hermite interpolation

The quadrature condition can be expressed as a polynomial in $y = \sin^2\frac{\omega}{2}$ (Lemma 6.2 page 98). The first term in this polynomial quadrature condition is a low-pass response and the second term is a high pass; and they meet in the middle at $\omega = \frac{\pi}{2}$.

$$\underbrace{(1-y)^p\mathsf{P}(y)}_{\text{low-pass}} + \underbrace{y^p\mathsf{P}(1-y)}_{\text{high-pass}} = 1$$

The low-pass and high-pass terms are especially smooth at $\omega = 0$ ($y = 0$) and $\omega = \pi$ ($y = 1$) in that the first $p-1$ derivatives at both points are zero for both terms. This is illustrated in Figure N.1 (page

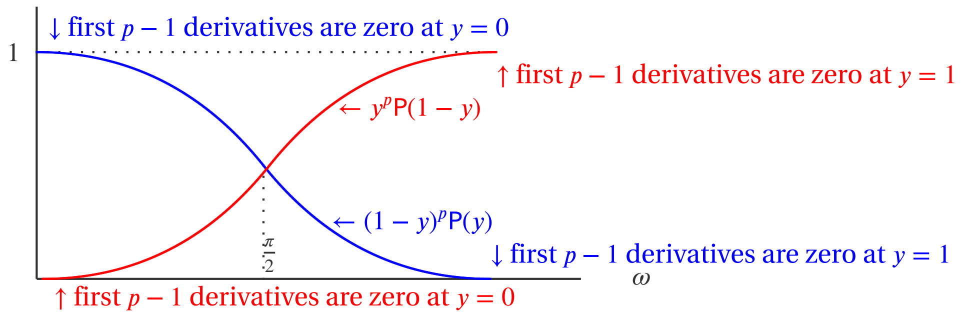


Figure N.1: Polynomial quadrature condition low-pass and high-pass terms

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Theorem N.1 (*Hermite Interpolation*).

THM

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}y^k\right]\Bigg|_{y=0} &= \bar{\delta}_n && \text{for } n=0,1,2,\ldots,p-1\\
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}y^k\right]\Bigg|_{y=1} &= 0 && \text{for } n=0,1,2,\ldots,p-1\\
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[y^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}(1-y)^k\right]\Bigg|_{y=0} &= 0 && \text{for } n=0,1,2,\ldots,p-1\\
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[y^p\sum_{k=0}^{p-1}\binom{p+k-1}{k}(1-y)^k\right]\Bigg|_{y=1} &= \bar{\delta}_n && \text{for } n=0,1,2,\ldots,p-1
\end{aligned}$$

✎PROOF: Let

$$\begin{aligned}
\mathsf{f}(y) &\triangleq (1-y)^p\sum_{n=0}^{p-1}\binom{p-1+n}{n}y^n\\
\mathsf{g}(y) &\triangleq y^p\sum_{n=0}^{p-1}\binom{p-1+n}{n}(1-y)^n\\
q &\triangleq p-1
\end{aligned}$$

1. Proof that $\mathsf{f}(0)=1$:

$$\begin{aligned}
\mathsf{f}(0) &= (1-y)^p\sum_{m=0}^{p-1}\binom{p-1+m}{m}y^m\Bigg|_{y=0}\\
&= (1-y)^p\left[\binom{p-1}{0}+\sum_{m=1}^{p-1}\binom{p-1+m}{m}y^m\right]\Bigg|_{y=0}\\
&= 1
\end{aligned}$$

2. Proof that $\mathsf{f}(y) = p\sum_{n=0}^{2p-1}\left[\sum_{k=\max(0,n-q)}^{\min(n,p)} (-1)^k \frac{(p+n-k-1)!}{(p-k)!(n-k)!\,k!}\right] y^n$:

$$\begin{aligned}
(1-y)^p \mathsf{P}_m(y) &= \sum_{n=0}^{p}\binom{p}{n}(-1)^n y^n \sum_{m=0}^{p-1}\binom{p-1+m}{m} y^m \\
&= \sum_{n=0}^{2p-1}\sum_{k=\max(0,n-q)}^{\min(n,p)}\binom{p}{k}(-1)^k\binom{p-1+n-k}{n-k} y^n && \text{by Theorem C.2 page 144} \\
&= \sum_{n=0}^{2p-1}\sum_{k=\max(0,n-q)}^{\min(n,p)}(-1)^k\frac{p!}{(p-k)!k!}\frac{(p-1+n-k)!}{(p-1)!(n-k)!} y^n \\
&= p\sum_{n=0}^{2p-1}\left[\sum_{k=\max(0,n-q)}^{\min(n,p)}(-1)^k\frac{(p+n-k-1)!}{(p-k)!(n-k)!\,k!}\right] y^n
\end{aligned}$$

3. Proof that $\mathsf{f}^{(n)}(0) = \bar{\delta}_n$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\mathsf{P}_m(y)\right]\Big|_{y=0} &= \frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[p\sum_{m=0}^{2p-1}\left[\sum_{k=\max(0,m-q)}^{\min(m,p)}(-1)^k\frac{(p+m-k-1)!}{(p-k)!(m-k)!\,k!}\right]y^m\right]\Bigg|_{y=0} && \text{by 1.} \\
&= p\sum_{m=n}^{2p-1}\sum_{k=\max(0,m-q)}^{\min(m,p)}(-1)^k\frac{(p-1+m-k)!}{(p-k)!(m-k)!k!}\frac{m!}{(m-n)!}y^{m-n}\Bigg|_{y=0} \\
&= p\sum_{k=\max(0,n-q)}^{\min(n,p)}(-1)^k\frac{(p-1+n-k)!}{(p-k)!}\frac{n!}{(n-k)!k!} \\
&= p\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(p+n-k-1)!}{(p-k)!} \\
&\overset{?}{=} \bar{\delta}_n \qquad \text{for } n = 0, 1, 2, \ldots, p-1
\end{aligned}$$

4. Proof that $\mathsf{f}^{(n)}(0) = \bar{\delta}_n$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
&\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p\mathsf{P}_m(y)\right]\Big|_{y=0} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\mathrm{d}^{n-k}}{\mathrm{d}y^{n-k}}(1-y)^p\right]\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}\mathsf{P}_m(y)\right]\Bigg|_{y=0} && \text{by Lemma E.2 (Leibnitz rule)} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\mathrm{d}^{n-k}}{\mathrm{d}y^{n-k}}(1-y)^p\right]\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}\sum_{m=0}^{p-1}\binom{p-1+m}{m}y^m\right]\Bigg|_{y=0} && \text{by definition of } \mathsf{P}_m(y) \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}(1-y)^{(p-n+k)}\right]\left[\sum_{m=k}^{p-1}\binom{p-1+m}{m}\frac{m!}{(m-k)!}y^{m-k}\right]\Bigg|_{y=0} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}\right]\left[\binom{p-1+k}{k}k!\right] \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^{n-k}\frac{p!}{(p-n+k)!}\right]\left[\frac{(p-1+k)!}{(p-1)!k!}\,k!\right] \\
&= (-1)^n p\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(p+k-1)!}{(p+k-n)!} \\
&\overset{?}{=} \bar{\delta}_n \qquad \text{for } k = 0, 1, 2, \ldots, p-1
\end{aligned}$$

5. Proof that $\mathsf{f}^{(n)}(1) = 0$ for $n = 0, 1, 2, \ldots, p-1$:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[(1-y)^p \mathsf{P}_m(y)\right]\Big|_{y=1} &= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k}(1-y)^p\right]\mathsf{P}_m^{(n-k)}(y)\Big|_{y=1} && \text{by Lemma E.2 (Leibnitz rule)}\\
&= \sum_{k=0}^{n}\binom{n}{k}\left[(-1)^k\frac{p!}{(p-k)!}(1-y)^{p-k}\right]\mathsf{P}_m^{(n-k)}(y)\Big|_{y=1}\\
&= \sum_{k=0}^{n}\binom{n}{k}\,0\cdot\mathsf{P}_m^{(n-k)}(y)\Big|_{y=1} && \text{by Lemma E.2}\\
&= 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1
\end{aligned}$$

✎

N.3 Cardinal Series and Sampling

N.3.1 Cardinal series basis

The *Paley-Wiener* class of functions (next definition) are those with a bandlimited Fourier transform. The cardinal series forms an orthogonal basis for such a space (Theorem N.3 page 337). In a *frame* $(\boldsymbol{x}_n)_{n\in\mathbb{Z}}$ with *frame operator* $\mathbf{S}$ on a *Hilbert Space* $\boldsymbol{H}$ with *inner product* $\langle \triangle \mid \nabla\rangle$, a function $\mathsf{f}(x)$ in the space spanned by the frame can be represented by

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \underbrace{\left\langle \mathsf{f} \mid \mathbf{S}^{-1}\boldsymbol{x}_n\right\rangle}_{\text{“Fourier coefficient”}} \boldsymbol{x}_n.$$

If the frame is *orthonormal* (giving an *orthonormal basis*), then $\mathbf{S} = \mathbf{S}^{-1} = \mathbf{I}$ and

$$\mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \langle \mathsf{f} \mid \boldsymbol{x}_n\rangle\, \boldsymbol{x}_n.$$

In the case of the cardinal series, the *Fourier coefficients* (Definition L.11 page 278) are particularly simple—these coefficients are samples of f taken at regular intervals (Theorem N.4 page 337). In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \mid \delta(x-n\tau)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n\tau)\,\mathrm{d}t \triangleq \mathsf{f}(n\tau)$$

Definition N.3. [4]

DEF: *A function* $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is in the* ***Paley-Wiener*** *class of functions* $\boldsymbol{PW}_\sigma^p$ *if there exists* $\mathsf{F} \in \boldsymbol{L}^p(-\sigma : \sigma)$ *such that*

$$\mathsf{f}(x) = \int_{-\sigma}^{\sigma} \mathsf{F}(\omega)e^{ix\omega}\,\mathrm{d}\omega \qquad (\mathsf{f} \text{ has a BANDLIMITED Fourier transform } \mathsf{F} \text{ with bandwidth } \sigma)$$

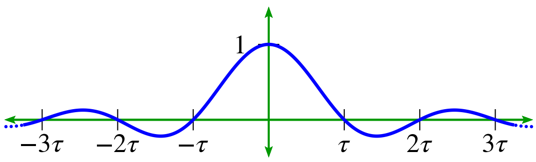
for $p \in [1:\infty)$ *and* $\sigma \in (0:\infty)$.

Theorem N.2 (Paley-Wiener Theorem for Functions). [5] *Let* f *be an* ENTIRE FUNCTION *(the domain of* f *is the entire complex plane* $\mathbb{C}$*). Let* $\sigma \in \mathbb{R}^+$.

THM:
$$\left\{\mathsf{f} \in \boldsymbol{PW}_\sigma^2\right\} \iff \left\{\begin{array}{lll} 1. & \exists C \in \mathbb{R}^+ \text{ such that } |\mathsf{f}(z)| \le Ce^{\sigma|z|} & (\text{EXPONENTIAL TYPE}) \text{ and}\\ 2. & \mathsf{f} \in \boldsymbol{L}_{\mathbb{R}}^2 & \end{array}\right\}$$

[4] Higgins (1996) page 52 ⟨Definition 6.15⟩

[5] Boas (1954) page 103 ⟨6.8.1 Theorem of Paley and Wiener⟩, Katznelson (2004) page 212 ⟨7.4 Theorem⟩, Zygmund (2002b) pages 272–273 ⟨(7·2) Theorem of Paley-Wiener⟩, Yosida (1980) page 161, Rudin (1987) page 375 ⟨19.3 Theorem⟩, Young (2001) page 85 ⟨Theorem 18⟩



Theorem N.3 (Cardinal sequence). [6]

THM

$$\left\{\frac{1}{\tau} \geq 2\sigma\right\} \implies \textit{The sequence } \left(\!\!\left(\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)}\right)\!\!\right)_{n\in\mathbb{Z}} \textit{ is an ORTHONORMAL BASIS for } \boldsymbol{PW}_\sigma^2.$$

Theorem N.4 (Sampling Theorem). [7]

THM

$$\left\{\begin{array}{ll} 1. & \mathsf{f} \in \boldsymbol{PW}_\sigma^2 \quad \textit{and} \\ 2. & \frac{1}{\tau} \geq 2\sigma \end{array}\right\} \implies \mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)}}_{\text{CARDINAL SERIES}}.$$

✎PROOF:

$$\text{Let} \quad \mathsf{s}(x) \triangleq \frac{\sin\left[\frac{\pi}{\tau}x\right]}{\frac{\pi}{\tau}x} \iff \tilde{\mathsf{s}}(\omega) = \begin{cases} \tau & : |f| \leq \frac{1}{2\tau} \\ 0 & : \text{otherwise} \end{cases}$$

1. Proof that the set is *orthonormal*: see Hardy (1941)
2. Proof that the set is a *basis*:

$$\begin{aligned}
\mathsf{f}(x) &= \int_\omega \tilde{\mathsf{f}}(\omega)e^{i\omega t}\, d\omega && \text{by \textit{inverse Fourier transform}} && \text{(Theorem K.1 page 258)}\\
&= \int_\omega \mathbf{T}\tilde{\mathsf{f}}_{\mathsf{d}}(\omega)\tilde{\mathsf{s}}(\omega)e^{i\omega t}\, d\omega && \text{if } W \leq \frac{1}{2T}\\
&= \mathbf{T}\mathsf{f}_{\mathsf{d}}(x) \star \mathsf{s}(x) && \text{by \textit{Convolution theorem}} && \text{(Theorem O.2 page 344)}\\
&= \mathbf{T}\int_u [\mathsf{f}_{\mathsf{d}}(u)]\mathsf{s}(x-u)\, du && \text{by \textit{convolution} definition} && \text{(Definition K.3 page 260)}\\
&= \mathbf{T}\int_u \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(u)\delta(u-n\tau)\right] \mathsf{s}(x-u)\, du && \text{by \textit{sampling} definition} && \text{(Theorem N.5 page 338)}\\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \int_u \mathsf{f}(u)\mathsf{s}(x-u)\delta(u-n\tau)\, du\\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\mathsf{s}(x-n\tau) && \text{by prop. of \textit{Dirac delta}}\\
&= \mathbf{T}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)\frac{\sin\left[\frac{\pi}{\tau}(x-n\tau)\right]}{\frac{\pi}{\tau}(x-n\tau)} && \text{by definition of } \mathsf{s}(x)
\end{aligned}$$

✏

[6] Higgins (1996) page 52 ⟨Definition 6.15⟩, Hardy (1941) ⟨*orthonormality*⟩, Higgins (1985), page 56 ⟨H1.; historical notes⟩

[7] Whittaker (1915), Kotelnikov (1933), Whittaker (1935), Shannon (1948) ⟨Theorem 13⟩, Shannon (1949) page 11 II (1991) page 1, Nashed and Walter (1991), Higgins (1996) page 5, Young (2001) pages 90–91 ⟨THE PALEY-WIENER SPACE⟩, Papoulis (1980) pages 418–419 ⟨The Sampling Theorem⟩. The *sampling theorem* was "discovered" and published by multiple people: Nyquist in 1928 (DSP?), Whittaker in 1935 (interpolation theory), and Shannon in 1949 (communication theory). references: Mallat (1999), page 43, Oppenheim and Schafer (1999), page 143.

N.3.2 Sampling

Definition N.4. [8] *Let $\delta(x)$ be the* Dirac delta *distribution.*

DEF *The **Shah Function** $\text{Ш}(x)$ is defined as* $\text{Ш}(x) \triangleq \sum_{n\in\mathbb{Z}} \delta(x-n)$

If $\mathsf{f}_\mathsf{d}(x)$ is the function $\mathsf{f}(x)$ sampled at rate $1/T$, then $\tilde{\mathsf{f}}_\mathsf{d}(\omega)$ is simply $\tilde{\mathsf{f}}(\omega)$ *replicated* every $1/T$ Hertz and *scaled* by $1/T$. This is proven in Theorem N.5 (next) and illustrated in Figure N.2 (page 338).

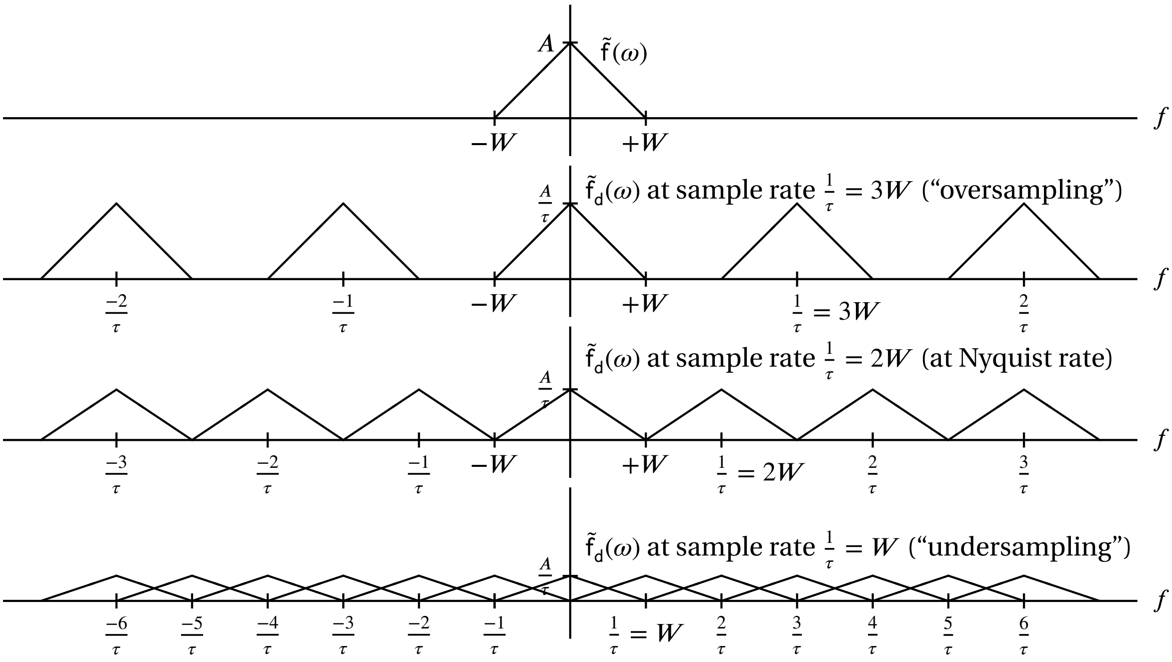


Figure N.2: Sampling in frequency domain

Theorem N.5. *Let $\mathsf{f}, \mathsf{f}_\mathsf{d} \in \boldsymbol{L}^2_\mathbb{R}$ and $\tilde{\mathsf{f}}, \tilde{\mathsf{f}}_\mathsf{d} \in \boldsymbol{L}^2_\mathbb{R}$ be their respective fourier transforms. Let $\mathsf{f}_\mathsf{d}(x)$ be the **sampled** $\mathsf{f}(x)$ such that*

$$\mathsf{f}_\mathsf{d}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x-n\tau).$$

THM
$$\left\{ \mathsf{f}_\mathsf{d}(x) \triangleq \mathsf{f}(x)\text{Ш}(x) \triangleq \mathsf{f}(x)\sum_{n\in\mathbb{Z}} \delta(x-n\tau) \right\} \implies \left\{ \tilde{\mathsf{f}}_\mathsf{d}(\omega) = \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right) \right\}$$

✎Proof:

$$\begin{aligned}
\tilde{\mathsf{f}}_\mathsf{d}(\omega) &\triangleq \int_t \mathsf{f}_\mathsf{d}(x)e^{-i\omega t}\,\mathsf{d}t \\
&= \int_t \left[\sum_{n\in\mathbb{Z}} \mathsf{f}(x)\delta(x-n\tau)\right] e^{-i\omega t}\,\mathsf{d}t \\
&= \sum_{n\in\mathbb{Z}} \int_t \mathsf{f}(x)\delta(x-n\tau)e^{-i\omega t}\,\mathsf{d}t
\end{aligned}$$

[8] Bracewell (1978) page 77 ⟨The sampling or replicating symbol Ш(x)⟩, Córdoba (1989)191. Note: The symbol Ш is the Cyrillic upper case "sha" character, which has been assigned Unicode location U+0428. Reference: http://unicode.org/cldr/utility/character.jsp?a=0428

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau} && \text{by definition of } \delta \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega + \frac{2\pi}{\tau}n\right) && \text{by } \textit{IPSF} \qquad \text{(Theorem 1.3 page 10)} \\
&= \frac{2\pi}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\omega - \frac{2\pi}{\tau}n\right)
\end{aligned}
$$

✏

Suppose a waveform $\mathsf{f}(x)$ is sampled at every time T generating a sequence of sampled values $\mathsf{f}(n\tau)$. Then in general, we can *approximate* $\mathsf{f}(x)$ by using interpolation between the points $\mathsf{f}(n\tau)$. Interpolation can be performed using several interpolation techniques.

In general all techniques lead only to an approximation of $\mathsf{f}(x)$. However, if $\mathsf{f}(x)$ is *bandlimited* with bandwidth $W \le \frac{1}{2T}$, then $\mathsf{f}(x)$ is *perfectly reconstructed* (not just approximated) from the sampled values $\mathsf{f}(n\tau)$ (Theorem N.4 page 337).

APPENDIX O

OPERATIONS ON SEQUENCES

O.1 Convolution operator

Definition O.1. [1] *Let X^Y be the set of all functions from a set Y to a set X. Let $\mathbb{Z}$ be the set of integers.*

DEF *A function f in X^Y is a* ***sequence*** *over X if $Y = \mathbb{Z}$.*
A sequence may be denoted in the form $(\!(x_n)\!)_{n\in\mathbb{Z}}$ or simply as $(\!(x_n)\!)$.

Definition O.2. [2] *Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a* FIELD *(Definition A.5 page 130).*

DEF *The* ***space of all absolutely square summable sequences*** *$\boldsymbol{\ell}^2_{\mathbb{F}}$ over $\mathbb{F}$ is defined as*

$$\boldsymbol{\ell}^2_{\mathbb{F}} \triangleq \left\{ (\!(x_n)\!)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}$$

The space $\boldsymbol{\ell}^2_{\mathbb{R}}$ is an example of a *separable Hilbert space*. In fact, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is the *only* separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, $\boldsymbol{\ell}^2_{\mathbb{R}}$ is isomorphic to $\boldsymbol{L}^2_{\mathbb{R}}$, the *space of all absolutely square Lebesgue integrable functions*.

Definition O.3.

DEF *The* ***convolution*** *operation $\star$ is defined as*

$$(\!(x_n)\!) \star (\!(y_n)\!) \triangleq \left(\!\!\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m} \right)\!\!\right)_{n\in\mathbb{Z}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$$

Proposition O.1. *Let $\star$ be the* CONVOLUTION OPERATOR *(Definition O.3 page 341).*

PRP $$(\!(x_n)\!) \star (\!(y_n)\!) = (\!(y_n)\!) \star (\!(x_n)\!) \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}} \qquad (\star \text{ is COMMUTATIVE})$$

[1] Bromwich (1908), page 1, Thomson et al. (2008) page 23 ⟨Definition 2.1⟩, Joshi (1997) page 31
[2] Kubrusly (2011) page 347 ⟨Example 5.K⟩

✎PROOF:

$$\begin{aligned}
[x \star y](n) &\triangleq \sum_{m\in\mathbb{Z}} x_m y_{n-m} && \text{by Definition O.3 page 341}\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{where } k = n-m \iff m = n-k\\
&= \sum_{k\in\mathbb{Z}} x_{n-k} y(k) && \text{by change commutivity of addition}\\
&= \sum_{m\in\mathbb{Z}} x_{n-m} y_m && \text{by change of variables}\\
&= \sum_{m\in\mathbb{Z}} y_m x_{n-m} && \text{by commutative property of the field over } \mathbb{C}\\
&\triangleq (y \star x)_n && \text{by Definition O.3 page 341}
\end{aligned}$$

✏

Proposition O.2. *Let* $\star$ *be the* CONVOLUTION OPERATOR *(Definition O.3 page 341). Let* $\ell^2_{\mathbb{R}}$ *be the set of* ABSOLUTELY SUMMABLE *sequences (Definition O.2 page 341).*

PRP
$$\left\{\begin{array}{lllll} \text{(A).} & \mathsf{x}(n) & \in & \ell^2_{\mathbb{R}} & \text{and}\\ \text{(B).} & \mathsf{y}(n) & \in & \ell^2_{\mathbb{R}} & \end{array}\right\} \implies \left\{ \sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] = \mathsf{x}[-n] \star \mathsf{y}(n) \right\}$$

✎PROOF:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathsf{x}[k]\mathsf{y}[n+k] &= \sum_{-p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{where } p \triangleq -k && \implies k = -p\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}[-p]\mathsf{y}[n-p] && \text{by } \textit{absolutely summable} \text{ hypothesis} && \text{(Definition O.2 page 341)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{x}'[p]\mathsf{y}[n-p] && \text{where } \mathsf{x}'[n] \triangleq \mathsf{x}[-n] && \implies \mathsf{x}[-n] = \mathsf{x}'[n]\\
&\triangleq \mathsf{x}'[n] \star \mathsf{y}[n] && \text{by definition of } \textit{convolution} \star && \text{(Definition O.3 page 341)}\\
&\triangleq \mathsf{x}[-n] \star \mathsf{y}[n] && \text{by definition of } \mathsf{x}'[n]
\end{aligned}$$

✏

O.2 Z-transform

Definition O.4. [3]

DEF
The ***z-transform*** $\mathbf{Z}$ *of* $(x_n)_{n\in\mathbb{Z}}$ *is defined as*

$$[\mathbf{Z}(x_n)](z) \triangleq \underbrace{\sum_{n\in\mathbb{Z}} x_n z^{-n}}_{\text{Laurent series}} \qquad \forall (x_n) \in \ell^2_{\mathbb{R}}$$

Theorem O.1. *Let* $X(z) \triangleq \mathbf{Z}\mathsf{x}[n]$ *be the* Z-TRANSFORM *of* $\mathsf{x}[n]$.

THM
$$\left\{ \check{\mathsf{x}}(z) \triangleq \mathbf{Z}(\mathsf{x}[n]) \right\} \implies \left\{\begin{array}{lrcll}
\text{(1).} & \mathbf{Z}(\alpha\mathsf{x}[n]) & = & \alpha\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(2).} & \mathbf{Z}(\mathsf{x}[n-k]) & = & z^{-k}\check{\mathsf{x}}(z) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(3).} & \mathbf{Z}(\mathsf{x}[-n]) & = & \check{\mathsf{x}}\left(\frac{1}{z}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(4).} & \mathbf{Z}(\mathsf{x}^*[n]) & = & \check{\mathsf{x}}^*(z^*) & \forall (x_n)\in\ell^2_{\mathbb{R}} \text{ and}\\
\text{(5).} & \mathbf{Z}(\mathsf{x}^*[-n]) & = & \check{\mathsf{x}}^*\left(\frac{1}{z^*}\right) & \forall (x_n)\in\ell^2_{\mathbb{R}}
\end{array}\right\}$$

[3] *Laurent series*: Abramovich and Aliprantis (2002) page 49

✎PROOF:

$$\begin{aligned}
\alpha\mathbb{Z}\check{x}(z) &\triangleq \alpha\mathbf{Z}\left(\!\left(x[n]\right)\!\right) && \text{by definition of } \check{x}(z)\\
&\triangleq \alpha\sum_{n\in\mathbb{Z}} x[n]z^{-n} && \text{by definition of } \mathbf{Z} \text{ operator}\\
&\triangleq \sum_{n\in\mathbb{Z}} (\alpha x[n])z^{-n} && \text{by \textit{distributive} property}\\
&\triangleq \mathbf{Z}\left(\!\left(\alpha x[n]\right)\!\right) && \text{by definition of } \mathbf{Z} \text{ operator}\\
z^{-k}\check{x}(z) &= z^{-k}\mathbf{Z}\left(\!\left(x[n]\right)\!\right) && \text{by definition of } \check{x}(z) && \text{(left hypothesis)}\\
&\triangleq z^{-k}\sum_{n=-\infty}^{n=+\infty} x[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
&= \sum_{n=-\infty}^{n=+\infty} x[n]z^{-n-k}\\
&= \sum_{m-k=-\infty}^{m-k=+\infty} x[m-k]z^{-m} && \text{where } m \triangleq n+k && \implies n = m-k\\
&= \sum_{m=-\infty}^{m=+\infty} x[m-k]z^{-m}\\
&= \sum_{n=-\infty}^{n=+\infty} x[n-k]z^{-n} && \text{where } n \triangleq m\\
&\triangleq \mathbf{Z}\left(\!\left(x[n-k]\right)\!\right) && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
\mathbf{Z}\left(\!\left(x^*[n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
&\triangleq \left(\sum_{n\in\mathbb{Z}} x[n](z^*)^{-n}\right)^* && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
&\triangleq \check{x}^*(z^*) && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
\mathbf{Z}\left(\!\left(x[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
&= \sum_{-m\in\mathbb{Z}} x[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition O.2 page 341)}\\
&= \sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition O.2 page 341)}\\
&\triangleq \check{x}\left(\frac{1}{z}\right) && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
\mathbf{Z}\left(\!\left(x^*[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]z^{-n} && \text{by definition of } \mathbf{Z} && \text{(Definition O.4 page 342)}\\
&= \sum_{-m\in\mathbb{Z}} x^*[m]z^{m} && \text{where } m \triangleq -n && \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x^*[m]z^{m} && \text{by \textit{absolutely summable} property} && \text{(Definition O.2 page 341)}\\
&= \sum_{m\in\mathbb{Z}} x^*[m]\left(\frac{1}{z}\right)^{-m} && \text{by \textit{absolutely summable} property} && \text{(Definition O.2 page 341)}\\
&= \left(\sum_{m\in\mathbb{Z}} x[m]\left(\frac{1}{z^*}\right)^{-m}\right)^* && \text{by \textit{absolutely summable} property} && \text{(Definition O.2 page 341)}
\end{aligned}$$

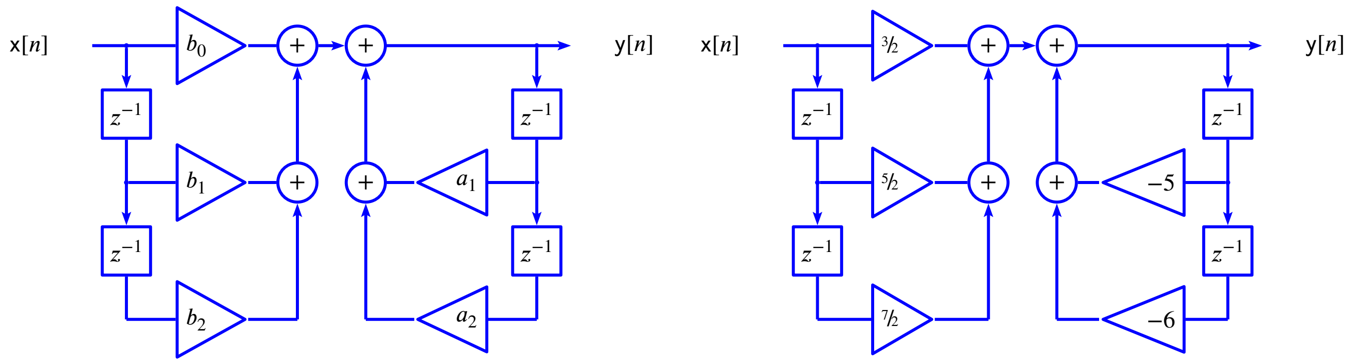


Figure O.1: Direct form 1 order 2 IIR filters

$$\triangleq \check{x}^*\left(\frac{1}{z^*}\right) \qquad \text{by definition of } \mathbf{Z} \qquad \text{(Definition O.4 page 342)}$$

Theorem O.2 (convolution theorem). *Let $\star$ be the convolution operator* (Definition O.3 page 341).

THM

$$\underbrace{\mathbf{Z}\left(\langle\!\langle x_n\rangle\!\rangle \star \langle\!\langle y_n\rangle\!\rangle\right)}_{\text{sequence convolution}} = \underbrace{\left(\mathbf{Z}\,\langle\!\langle x_n\rangle\!\rangle\right)\left(\mathbf{Z}\,\langle\!\langle y_n\rangle\!\rangle\right)}_{\text{series multiplication}} \qquad \forall \langle\!\langle x_n\rangle\!\rangle_{n\in\mathbb{Z}}, \langle\!\langle y_n\rangle\!\rangle_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
[\mathbf{Z}(x \star y)](z) &\triangleq \mathbf{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by Definition O.3 page 341}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by Definition O.4 page 342}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k = n-m \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right]\\
&\triangleq \left(\mathbf{Z}\,\langle\!\langle x_n\rangle\!\rangle\right)\left(\mathbf{Z}\,\langle\!\langle y_n\rangle\!\rangle\right) && \text{by Definition O.4 page 342}
\end{aligned}$$

O.3 From z-domain back to time-domain

$$\check{y}(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) - a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Example O.1. See Figure O.1 (page 344)

$$\frac{3z^2+5z+7}{2z^2+10z+12} = \frac{3z^2+5z+7}{2\left(z^2+5z+6\right)} = \frac{\left(3/2 z^2+5/2 z+7/2\right)}{z^2+5z+6} = \frac{\left(3/2+5/2 z^{-1}+7/2 z^{-2}\right)}{1+5z^{-1}+6z^{-2}}$$

O.4 Zero locations

The system property of *minimum phase* is defined in Definition O.5 (next) and illustrated in Figure O.2 (page 345).

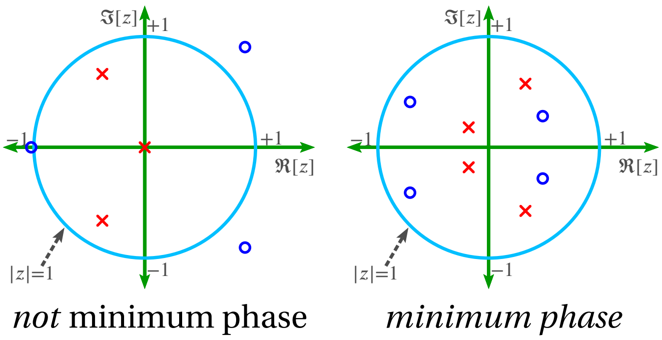


Figure O.2: Minimum Phase filter

Definition O.5. [4] *Let* $\check{x}(z) \triangleq \mathbf{Z}\left(\!\left(x_n\right)\!\right)$ *be the* Z TRANSFORM (Definition O.4 page 342) *of a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$. *Let* $\left(\!\left(z_n\right)\!\right)_{n\in\mathbb{Z}}$ *be the* ZEROS *of* $\check{x}(z)$.

DEF *The sequence* $\left(\!\left(x_n\right)\!\right)$ *is* ***minimum phase*** *if*

$$\underbrace{\left|z_n\right| < 1 \qquad \forall n \in \mathbb{Z}}_{\check{x}(z)\text{ has all its ZEROS inside the unit circle}}$$

The impulse response of a minimum phase filter has most of its energy concentrated near the beginning of its support, as demonstrated next.

Theorem O.3 (Robinson's Energy Delay Theorem). [5] *Let* $\mathsf{p}(z) \triangleq \sum_{n=0}^{N} a_n z^{-n}$ *and* $\mathsf{q}(z) \triangleq \sum_{n=0}^{N} b_n z^{-n}$ *be polynomials.*

THM

$$\left\{\begin{array}{ll} \mathsf{p} & \text{is MINIMUM PHASE} \quad \text{and} \\ \mathsf{q} & \text{is NOT } \textit{minimum phase} \end{array}\right\} \implies \underbrace{\sum_{n=0}^{m-1}\left|a_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{p}(z)} \geq \underbrace{\sum_{n=0}^{m-1}\left|b_n\right|^2}_{\text{“energy” of the first } m \text{ coefficients of } \mathsf{q}(z)} \qquad \forall 0 \leq m \leq N$$

But for more *symmetry*, put some zeros inside and some outside the unit circle.

Example O.2. An example of a minimum phase polynomial is the Daubechies-4 scaling function. The minimum phase polynomial causes most of the energy to be concentrated near the origin, making it very *asymmetric*. In contrast, the Symlet-4 has a design very similar to that of Daubechies-4, but the selected zeros are not all within the unit circle in the complex z plane. This results in a

[4] Farina and Rinaldi (2000) page 91, Dumitrescu (2007) page 36
[5] Dumitrescu (2007) page 36, Robinson (1962), Robinson (1966) ⟨???⟩, Claerbout (1976), pages 52–53

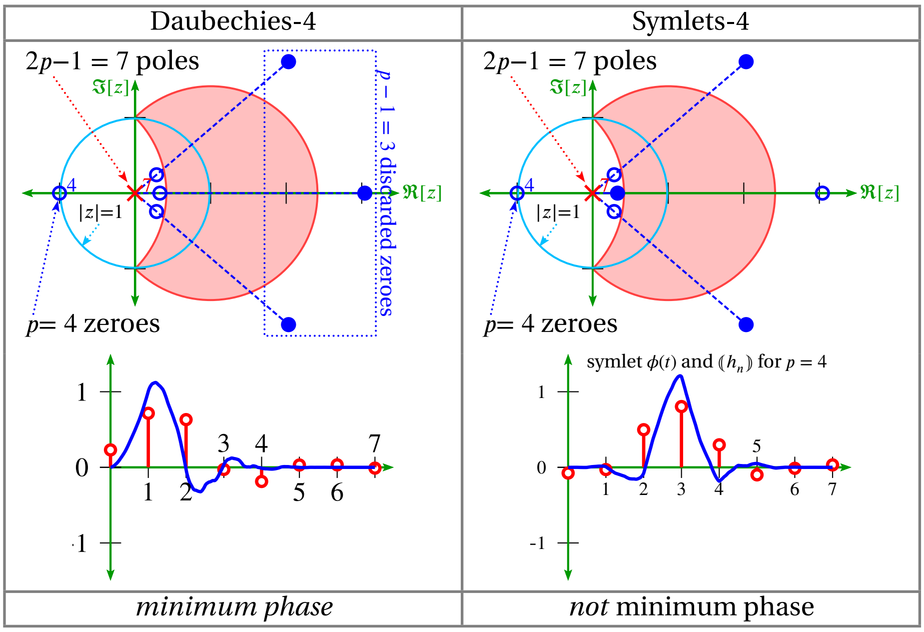


Figure O.3: Daubechies-4 and Symlet-4 scaling functions pole-zero plots

scaling function that is more symmetric and less contrated near the origin. Both scaling functions are illustrated in Figure O.3 (page 346).

O.5 Pole locations

Definition O.6.

> **DEF** *A filter (or system or operator)* $\mathbf{H}$ *is* ***causal***
> *if its current output does not depend on future inputs.*

Definition O.7.

> **DEF** *A filter (or system or operator)* $\mathbf{H}$ *is* ***time-invariant***
> *if the mapping it performs does not change with time.*

Definition O.8.

> **DEF** *An operation* $\mathbf{H}$ *is* ***linear*** *if any output* y_n *can be described as a linear combination of inputs* x_n *as in*
> $$y_n = \sum_{m\in\mathbb{Z}} \mathsf{h}(m)\mathsf{x}(n-m)\,.$$

For a filter to be *stable*, place all the poles *inside* the unit circle.

Theorem O.4. *A causal LTI filter is* ***stable*** *if all of its poles are* ***inside*** *the unit circle.*

Example O.3. Stable/unstable filters are illustrated in Figure O.4 (page 347).

True or False? This filter has no poles:

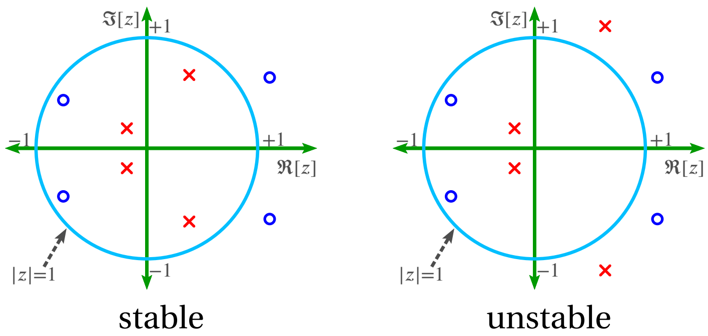
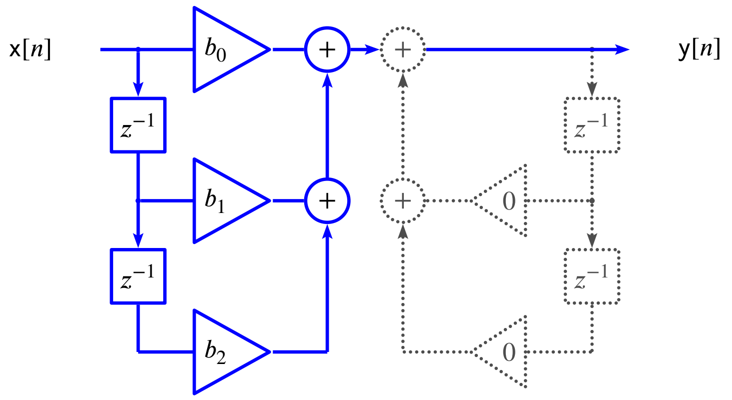
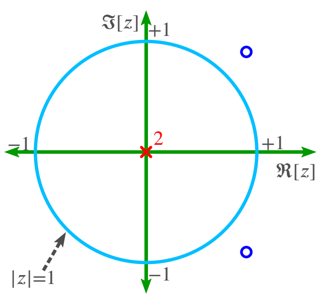


Figure O.4: Pole-zero plot stable/unstable causal LTI filters (Example O.3 page 346)

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$



$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{z^2}{z^2} \times \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2}$$



O.6 Mirroring for real coefficients

If you want real coefficients, choose poles and zeros in conjugate pairs (next).

Proposition O.3.

PRP $\left\{ \begin{array}{l} \text{ZEROS } and \text{ POLES} \\ occur\ in \text{ CONJUGATE PAIRS} \end{array} \right\} \implies \left\{ \begin{array}{l} \text{COEFFICIENTS} \\ are \text{ REAL.} \end{array} \right\}$

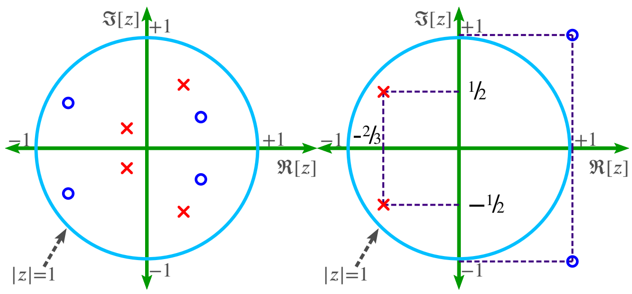


Figure O.5: Conjugate pair structure yielding real coefficients

✎PROOF:

$$\begin{aligned}
\left(z-p_1\right)\left(z-p_1^*\right) &= [z-(a+ib)][z-(a-ib)]\\
&= z^2+[-a+ib-ib-a]z-[ib]^2\\
&= z^2-2az+b^2
\end{aligned}$$

✏

Example O.4. See Figure O.5 (page 348).

$$\begin{aligned}
H(z) &= G\frac{\left[z-z_1\right]\left[z-z_2\right]}{\left[z-p_1\right]\left[z-p_2\right]} = G\frac{[z-(1+i)][z-(1-i)]}{[z-(-\tfrac{2}{3}+i\tfrac{1}{2})][z-(-\tfrac{2}{3}-i\tfrac{1}{2})]}\\
&= G\frac{z^2-z[(1-i)+(1+i)]+(1-i)(1+i)}{z^2-z[(-\tfrac{2}{3}+i\tfrac{1}{2})+(-\tfrac{2}{3}+i\tfrac{1}{2})]+(-\tfrac{2}{3}+i\tfrac{1}{2})(-\tfrac{2}{3}+i\tfrac{1}{2})}\\
&= G\frac{z^2-2z+2}{z^2-\tfrac{4}{3}z+(\tfrac{4}{3}+\tfrac{1}{4})} = G\frac{z^2-2z+2}{z^2-\tfrac{4}{3}z+\tfrac{19}{12}}
\end{aligned}$$

O.7 Rational polynomial operators

A digital filter is simply an operator on $\ell^2_{\mathbb{R}}$. If the digital filter is a causal LTI system, then it can be expressed as a rational polynomial in z as shown next.

Lemma O.1. *A causal LTI operator* $\mathbf{H}$ *can be expressed as a rational expression* $\check{\mathsf{h}}(z)$.

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Nz^{-N}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}\\
&= \frac{\displaystyle\sum_{n=0}^{N} b_nz^{-n}}{1+\displaystyle\sum_{n=1}^{N} a_nz^{-n}}
\end{aligned}$$

A filter operation $\check{\mathsf{h}}(z)$ can be expressed as a product of its roots (poles and zeros).

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Nz^{-N}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}\\
&= \alpha\frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}
\end{aligned}$$

where α is a constant, z_i are the zeros, and p_i are the poles. The poles and zeros of such a rational expression are often plotted in the z-plane with a unit circle about the origin (representing $z = e^{i\omega}$). Poles are marked with $\times$ and zeros with $\bigcirc$. An example is shown in Figure O.6 page 349. Notice that in this figure the zeros and poles are either real or occur in complex conjugate pairs.

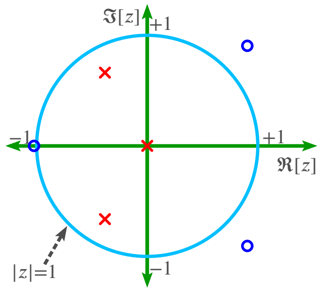


Figure O.6: Pole-zero plot for rational expression with real coefficients

O.8 Filter Banks

Conjugate quadrature filters (next definition) are used in *filter banks*. If $\check{x}(z)$ is a *low-pass filter*, then the conjugate quadrature filter of $\check{y}(z)$ is a *high-pass filter*.

Definition O.9. [6] *Let* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *be* SEQUENCES (Definition O.1 page 341) *in* $\ell^2_{\mathbb{R}}$ (Definition O.2 page 341).

DEF

The sequence $(\!(y_n)\!)$ *is a* ***conjugate quadrature filter*** *with shift* N *with respect to* $(\!(x_n)\!)$ *if*

$$y_n = \pm(-1)^n x^*_{N-n}$$

A CONJUGATE QUADRATURE FILTER *is also called a* ***CQF*** *or a* ***Smith-Barnwell filter***.
Any triple $\left((\!(x_n)\!), (\!(y_n)\!), N\right)$ *in this form is said to satisfy the* ***conjugate quadrature filter condition*** *or the* ***CQF condition***.

Theorem O.5 (CQF theorem). [7] *Let* $\check{y}(\omega)$ *and* $\check{x}(\omega)$ *be the* DTFT*s* (Definition P.1 page 355) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$, *respectively, in* $\ell^2_{\mathbb{R}}$ (Definition O.2 page 341).

THM

$$\begin{array}{rcrcll}
\underbrace{y_n = \pm(-1)^n x^*_{N-n}}_{\text{(1) CQF in "time"}} & \iff & \check{y}(z) & = & \pm(-1)^N z^{-N}\check{x}^*\left(\frac{-1}{z^*}\right) & \text{(2) CQF in "z-domain"}\\
& \iff & \check{y}(\omega) & = & \pm(-1)^N e^{-i\omega N}\check{x}^*(\omega+\pi) & \text{(3) CQF in "frequency"}\\
& \iff & x_n & = & \pm(-1)^N(-1)^n y^*_{N-n} & \text{(4) "reversed" CQF in "time"}\\
& \iff & \check{x}(z) & = & \pm z^{-N}\check{y}^*\left(\frac{-1}{z^*}\right) & \text{(5) "reversed" CQF in "z-domain"}\\
& \iff & \check{x}(\omega) & = & \pm e^{-i\omega N}\check{y}^*(\omega+\pi) & \text{(6) "reversed" CQF in "frequency"}
\end{array}$$

$\forall N \in \mathbb{Z}$

✎PROOF:

[6] Strang and Nguyen (1996) page 109, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩, Smith and Barnwell (1984a), Smith and Barnwell (1984b), Mintzer (1985)

[7] Strang and Nguyen (1996) page 109, Mallat (1999) pages 236–238 ⟨(7.58),(7.73)⟩, Haddad and Akansu (1992) pages 256–259 ⟨section 4.5⟩, Vaidyanathan (1993) page 342 ⟨(7.2.7), (7.2.8)⟩

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{y}(z) &= \sum_{n\in\mathbb{Z}} y_n z^{-n} && \text{by definition of } z\text{-transform} && \text{(Definition O.4 page 342)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{(\pm)(-1)^n x^*_{N-n}}_{CQF} z^{-n} && \text{by (1)}\\
&= \pm\sum_{m\in\mathbb{Z}} (-1)^{N-m} x^*_m z^{-(N-m)} && \text{where } m \triangleq N-n \implies && n = N-m\\
&= \pm(-1)^N z^{-N} \sum_{m\in\mathbb{Z}} (-1)^{-m} x^*_m \left(z^{-1}\right)^{-m}\\
&= \pm(-1)^N z^{-N} \sum_{m\in\mathbb{Z}} x^*_m \left(-\frac{1}{z}\right)^{-m}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(-\frac{1}{z^*}\right)^{-m}\right]^*\\
&= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by definition of } z\text{-transform} && \text{(Definition O.4 page 342)}
\end{aligned}$$

2. Proof that (1) $\impliedby$ (2):

$$\begin{aligned}
\check{y}(z) &= \pm(-1)^N z^{-N} \check{x}^*\left(\frac{-1}{z^*}\right) && \text{by (2)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x_m \left(\frac{-1}{z^*}\right)^{-m}\right]^* && \text{by definition of } z\text{-transform} && \text{(Definition O.4 page 342)}\\
&= \pm(-1)^N z^{-N} \left[\sum_{m\in\mathbb{Z}} x^*_m \left(-z^{-1}\right)^{-m}\right] && \text{by definition of } z\text{-transform} && \text{(Definition O.4 page 342)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^{N-m} x^*_m z^{-(N-m)}\\
&= \sum_{m\in\mathbb{Z}} (\pm)(-1)^n x^*_{N-n} z^{-n} && \text{where } n = N-m \implies && m \triangleq N-n\\
&\implies x_n = \pm(-1)^n x^*_{N-n}
\end{aligned}$$

3. Proof that (1) $\implies$ (3):

$$\begin{aligned}
\check{y}(\omega) &\triangleq \check{x}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT \text{ (Definition P.1 page 355)}\\
&= \left[\pm(-1)^N z^{-N} \check{x}\left(\frac{-1}{z^*}\right)\right]_{z=e^{i\omega}} && \text{by (2)}\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i\pi} e^{i\omega}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}\left(e^{i(\omega+\pi)}\right)\\
&= \pm(-1)^N e^{-i\omega N} \check{x}(\omega+\pi) && \text{by definition of } DTFT \text{ (Definition P.1 page 355)}
\end{aligned}$$

4. Proof that (1) $\implies$ (6):

$$\begin{aligned}
\check{x}(\omega) &= \sum_{n\in\mathbb{Z}} y_n e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition P.1 page 355)}\\
&= \sum_{n\in\mathbb{Z}} \underbrace{\pm(-1)^n x^*_{N-n}}_{CQF} e^{-i\omega n} && \text{by (1)}\\
&= \sum_{m\in\mathbb{Z}} \pm(-1)^{N-m} x^*_m e^{-i\omega(N-m)} && \text{where } m \triangleq N-n \implies && n = N-m\\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} (-1)^m x^*_m e^{i\omega m}
\end{aligned}$$

$$
\begin{aligned}
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} e^{i\pi m} x_m^* e^{i\omega m} \\
&= \pm(-1)^N e^{-i\omega N} \sum_{m\in\mathbb{Z}} x_m^* e^{i(\omega+\pi)m} \\
&= \pm(-1)^N e^{-i\omega N} \left[\sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+\pi)m}\right]^* \\
&= \pm(-1)^N e^{-i\omega N} \breve{x}^*(\omega+\pi) && \text{by definition of } \textit{DTFT} && \text{(Definition P.1 page 355)}
\end{aligned}
$$

5. Proof that (1) $\impliedby$ (3):

$$
\begin{aligned}
y_n &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{y}(\omega)e^{i\omega n}\,\mathrm{d}\omega && \text{by } \textit{inverse DTFT} && \text{(Theorem P.3 page 361)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\pm(-1)^N e^{-iN\omega}\breve{x}^*(\omega+\pi)}_{\text{right hypothesis}} e^{i\omega n}\,\mathrm{d}\omega && \text{by right hypothesis} \\
&= \pm(-1)^N \frac{1}{2\pi}\int_{-\pi}^{+\pi} \breve{x}^*(\omega+\pi)e^{i\omega(n-N)}\,\mathrm{d}\omega && \text{by right hypothesis} \\
&= \pm(-1)^N \frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{i(v-\pi)(n-N)}\,\mathrm{d}v && \text{where } v \triangleq \omega+\pi \implies && \omega = v-\pi \\
&= \pm(-1)^N e^{-i\pi(n-N)} \frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}^*(v)e^{iv(n-N)}\,\mathrm{d}v \\
&= \pm(-1)^N \underbrace{(-1)^N}_{e^{i\pi N}} \underbrace{(-1)^n}_{e^{-i\pi n}} \left[\frac{1}{2\pi}\int_{0}^{2\pi} \breve{x}(v)e^{iv(N-n)}\,\mathrm{d}v\right]^* \\
&= \pm(-1)^n x_{N-n}^* && \text{by } \textit{inverse DTFT} && \text{(Theorem P.3 page 361)}
\end{aligned}
$$

6. Proof that (1) $\iff$ (4):

$$
\begin{aligned}
y_n = \pm(-1)^n x_{N-n}^* &\iff (\pm)(-1)^n y_n = (\pm)(\pm)(-1)^n(-1)^n x_{N-n}^* \\
&\iff \pm(-1)^n y_n = x_{N-n}^* \\
&\iff \left(\pm(-1)^n y_n\right)^* = \left(x_{N-n}^*\right)^* \\
&\iff \pm(-1)^n y_n^* = x_{N-n} \\
&\iff x_{N-n} = \pm(-1)^n y_n^* \\
&\iff x_m = \pm(-1)^{N-m} y_{N-m}^* && \text{where } m \triangleq N-n \implies && n = N-m \\
&\iff x_m = \pm(-1)^{N-m} y_{N-m}^* \\
&\iff x_m = \pm(-1)^N(-1)^m y_{N-m}^* \\
&\iff x_n = \pm(-1)^N(-1)^n y_{N-n}^* && \text{by change of free variables}
\end{aligned}
$$

7. Proofs for (5) and (6): not included. See proofs for (2) and (3).

✏

Theorem O.6. [8] *Let* $\breve{y}(\omega)$ *and* $\breve{x}(\omega)$ *be the* DTFT*s* (Definition P.1 page 355) *of the sequences* $(\!(y_n)\!)_{n\in\mathbb{Z}}$ *and* $(\!(x_n)\!)_{n\in\mathbb{Z}}$*, respectively, in* $\ell^2_{\mathbb{R}}$ (Definition O.2 page 341)*.*

THM

Let $y_n = \pm(-1)^n x_{N-n}^*$ (CQF CONDITION, *Definition O.9 page 349*)*. Then*

$$
\left\{
\begin{array}{llcllll}
\text{(A)} & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} = 0 & \iff & \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} & = & 0 & \text{(B)} \\
& & \iff & \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k & = & 0 & \text{(C)} \\
& & \iff & \sum_{k\in\mathbb{Z}} k^n y_k & = & 0 & \text{(D)}
\end{array}
\right\} \quad \forall n\in\mathbb{W}
$$

[8] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

✎PROOF:

1. Proof that (A) $\implies$ (B):

$$\begin{aligned}
0 &= \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{y}(\omega)\Big|_{\omega=0} && \text{by (A)}\\
&= \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n (\pm)(-1)^N e^{-i\omega N}\breve{x}^*(\omega+\pi)\Big|_{\omega=0} && \text{by \textit{CQF theorem}} \quad \text{(Theorem O.5 page 349)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{\ell} \left[e^{-i\omega N}\right] \cdot \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma E.2 page 187)}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} e^{-i\omega N} \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0}\\
&= (\pm)(-1)^N \cancel{e^{-i0N}} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{x}^*(\omega+\pi)\right]\Big|_{\omega=0}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{x}^{(0)}(\pi) &= 0\\
&\implies & \breve{x}^{(1)}(\pi) &= 0\\
&\implies & \breve{x}^{(2)}(\pi) &= 0\\
&\implies & \breve{x}^{(3)}(\pi) &= 0\\
&\implies & \breve{x}^{(4)}(\pi) &= 0\\
&\ \ \vdots & \vdots\quad &\\
&\implies & \breve{x}^{(n)}(\pi) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

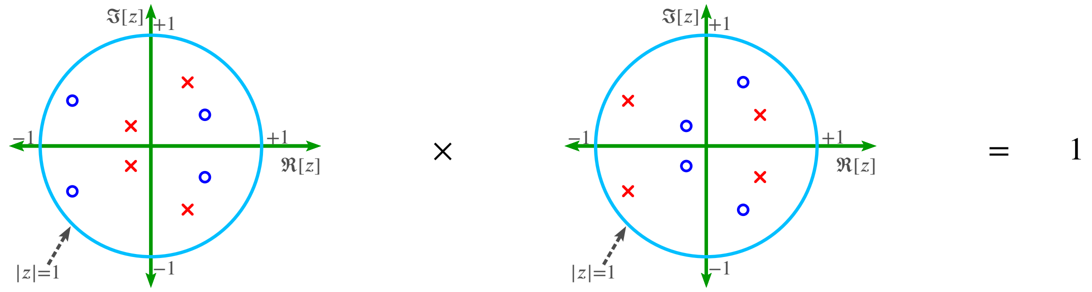
2. Proof that (A) $\impliedby$ (B):

$$\begin{aligned}
0 &= \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{x}(\omega)\Big|_{\omega=\pi} && \text{by (B)}\\
&= \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^n (\pm)e^{-i\omega N}\breve{y}^*(\omega+\pi)\Big|_{\omega=\pi} && \text{by \textit{CQF theorem}} \quad \text{(Theorem O.5 page 349)}\\
&= (\pm) \sum_{\ell=0}^{n} \binom{n}{\ell} \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{\ell} \left[e^{-i\omega N}\right] \cdot \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi} && \text{by \textit{Leibnitz GPR}} \quad \text{(Lemma E.2 page 187)}\\
&= (\pm) \sum_{\ell=0}^{n} \binom{n}{\ell} (-iN)^{\ell} e^{-i\omega N} \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}\\
&= (\pm) \cancel{e^{-i\pi N}} \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}\\
&= (\pm)(-1)^N \sum_{\ell=0}^{n} \binom{n}{\ell} -iN^{\ell} \left[\tfrac{\mathsf{d}}{\mathsf{d}\omega}\right]^{n-\ell} \left[\breve{y}^*(\omega+\pi)\right]\Big|_{\omega=\pi}
\end{aligned}$$

$$\begin{aligned}
&\implies & \breve{y}^{(0)}(0) &= 0\\
&\implies & \breve{y}^{(1)}(0) &= 0\\
&\implies & \breve{y}^{(2)}(0) &= 0\\
&\implies & \breve{y}^{(3)}(0) &= 0\\
&\implies & \breve{y}^{(4)}(0) &= 0\\
&\ \ \vdots & \vdots\quad &\\
&\implies & \breve{y}^{(n)}(0) &= 0\\
&\implies & \breve{y}^{(n)}(0) &= 0 \quad \text{for } n = 0, 1, 2, \ldots
\end{aligned}$$

3. Proof that (B) $\iff$ (C): by Theorem P.5 page 363

4. Proof that (A) $\iff$ (D): by Theorem P.5 page 363



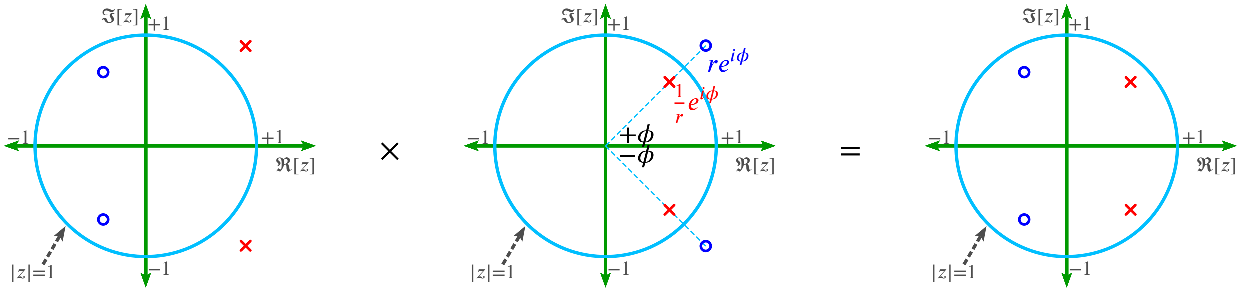
$$\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)} \times \frac{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 1$$

5. Proof that (CQF) $\not\Leftarrow$ (A): Here is a counterexample: $\check{y}(\omega) = 0$.

✏

O.9 Inverting non-minimum phase filters

Minimum phase filters are easy to invert: each *zero* becomes a *pole* and each *pole* becomes a *zero*.



unstable | all-pass | stable!

$$\begin{array}{ll}
|A(z)|_{z=e^{i\omega}} = \dfrac{1}{r}\left|\dfrac{z-re^{i\phi}}{z-\frac{1}{r}e^{i\phi}}\right|_{z=e^{i\omega}} & = \left|\dfrac{z-re^{i\phi}}{rz-e^{i\phi}}\right|_{z=e^{i\omega}} \\
= \left|e^{i\phi}\left(\dfrac{e^{-i\phi}z-r}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} & = \left|z\left(\dfrac{e^{-i\phi}-rz^{-1}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} \\
= \left|-z\left(\dfrac{rz^{-1}-e^{-i\phi}}{rz-e^{i\phi}}\right)\right|_{z=e^{i\omega}} & = \left|\underbrace{e^{i\pi}}_{-1}e^{i\omega}\left(\dfrac{re^{-i\omega}-e^{-i\phi}}{re^{i\omega}-e^{i\phi}}\right)\right| \\
= \left|\dfrac{1}{e^{-iv}}\left(\dfrac{re^{-i\omega}-e^{-i\phi}}{\left(re^{i\omega}-e^{i\phi}\right)^*}\right)\right| & = \left|\dfrac{re^{-i\omega}-e^{-i\phi}}{re^{-i\omega}-e^{-i\phi}}\right| \\
= 1 &
\end{array}$$

APPENDIX P

DISCRETE TIME FOURIER TRANSFORM

P.1 Definition

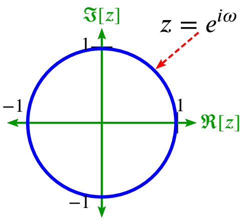
Definition P.1.

DEF

*The **discrete-time Fourier transform** $\breve{\mathbf{F}}$ of $(\!(x_n)\!)_{n\in\mathbb{Z}}$ is defined as*

$$\left[\breve{\mathbf{F}}\,(\!(x_n)\!)\right](\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

If we compare the definition of the *Discrete Time Fourier Transform* (Definition P.1 page 355) to the definition of the Z-transform (Definition O.4 page 342), we see that the DTFT is just a special case of the more general Z-Transform, with $z = e^{i\omega}$. If we imagine $z \in \mathbb{C}$ as a complex plane, then $e^{i\omega}$ is a unit circle in this plane. The "frequency" ω in the DTFT is the unit circle in the much larger z-plane, as illustrated to the right.



P.2 Properties

Proposition P.1 (DTFT periodicity). *Let* $\breve{\mathsf{x}}(\omega) \triangleq \breve{\mathbf{F}}\left[(\!(x_n)\!)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition P.1 page 355) of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

PRP

$$\underbrace{\breve{\mathsf{x}}(\omega) = \breve{\mathsf{x}}(\omega + 2\pi n)}_{\text{PERIODIC } with\ period\ 2\pi} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathsf{x}}(\omega + 2\pi n) &= \sum_{m\in\mathbb{Z}} x_m e^{-i(\omega+2\pi n)m} && = \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} \cancelto{1}{e^{-i2\pi nm}} \\
&= \sum_{m\in\mathbb{Z}} x_m e^{-i\omega m} && = \breve{\mathsf{x}}(\omega)
\end{aligned}$$

✏

Theorem P.1. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[\left(\!\left(x[n]\right)\!\right)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition P.1 page 355) of a sequence* $\left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM

$$\left\{ \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\left(\!\left(x[n]\right)\!\right) \right\} \implies \left\{ \begin{array}{llcll} (1). & \breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &=& \tilde{x}(-\omega) & \text{and} \\ (2). & \breve{\mathbf{F}}\left(\!\left(x^*[n]\right)\!\right) &=& \tilde{x}^*(-\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}\left(\!\left(x^*[-n]\right)\!\right) &=& \tilde{x}^*(\omega) & \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition P.1 page 355)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}\\
&\triangleq \tilde{x}(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x^*[n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition P.1 page 355)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\textbf{algebras} && \text{(Definition ?? page ??)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[n]e^{-i(-\omega)n}\right)^*\\
&\triangleq \tilde{x}^*(-\omega) && \text{by left hypothesis}
\end{aligned}$$

$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x^*[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x^*[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition P.1 page 355)}\\
&= \left(\sum_{n\in\mathbb{Z}} x[-n]e^{i\omega n}\right)^* && \text{by } \textit{distributive} \text{ property of } *\text{-}\textbf{algebras} && \text{(Definition ?? page ??)}\\
&= \left(\sum_{m\in\mathbb{Z}} x[m]e^{-i\omega m}\right)^* && \text{where } m \triangleq -n \implies n = -m\\
&\triangleq \tilde{x}^*(\omega) && \text{by left hypothesis}
\end{aligned}$$

⇰

Theorem P.2. *Let* $\breve{x}(\omega) \triangleq \breve{\mathbf{F}}\left[\left(\!\left(x[n]\right)\!\right)\right](\omega)$ *be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition P.1 page 355) of a sequence* $\left(\!\left(x[n]\right)\!\right)_{n\in\mathbb{Z}}$ *in* $\ell^2_{\mathbb{R}}$.

THM

$$\left\{ \begin{array}{lll} (1). & \tilde{x}(\omega) \triangleq \breve{\mathbf{F}}\left(\!\left(x[n]\right)\!\right) & \text{and} \\ (2). & \left(\!\left(x[n]\right)\!\right) \text{ is REAL-VALUED} & \end{array} \right\} \implies \left\{ \begin{array}{llcllll} (1). & \breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &=& \tilde{x}(-\omega) & & & \text{and} \\ (2). & \breve{\mathbf{F}}\left(\!\left(x^*[n]\right)\!\right) &=& \tilde{x}^*(-\omega) &=& \tilde{x}(\omega) & \text{and} \\ (3). & \breve{\mathbf{F}}\left(\!\left(x^*[-n]\right)\!\right) &=& \tilde{x}^*(\omega) &=& \tilde{x}(-\omega) & \end{array} \right\}$$

✎PROOF:

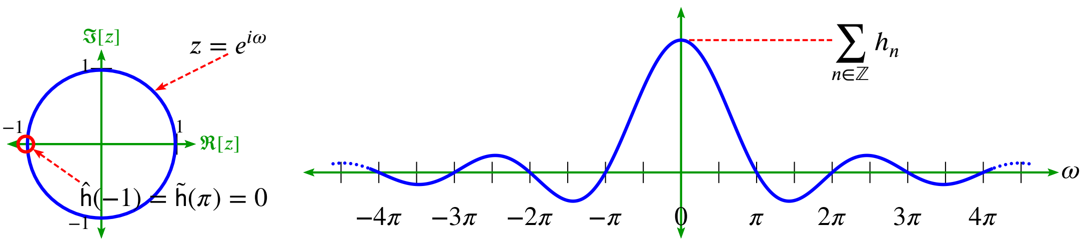
$$\begin{aligned}
\breve{\mathbf{F}}\left(\!\left(x[-n]\right)\!\right) &\triangleq \sum_{n\in\mathbb{Z}} x[-n]e^{-i\omega n} && \text{by definition of } DTFT && \text{(Definition P.1 page 355)}\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{i\omega m} && \text{where } m \triangleq -n \implies n = -m\\
&= \sum_{m\in\mathbb{Z}} x[m]e^{-i(-\omega)m}
\end{aligned}$$

$$\triangleq \tilde{\mathsf{x}}(-\omega) \qquad \text{by left hypothesis}$$

$$\begin{aligned}
\boxed{\tilde{\mathsf{x}}^*(-\omega)} &= \boxed{\breve{\mathbf{F}}\left(\!\left(\mathsf{x}^*[n]\right)\!\right)} && \text{by Theorem P.1 page 356}\\
&= \breve{\mathbf{F}}\left(\!\left(\mathsf{x}[n]\right)\!\right) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(\omega)} && \text{by definition of } \tilde{\mathsf{x}}(\omega) \quad \text{(Definition P.1 page 355)}
\end{aligned}$$

$$\begin{aligned}
\boxed{\tilde{\mathsf{x}}^*(\omega)} &= \boxed{\breve{\mathbf{F}}\left(\!\left(\mathsf{x}^*[-n]\right)\!\right)} && \text{by Theorem P.1 page 356}\\
&= \breve{\mathbf{F}}\left(\!\left(\mathsf{x}[-n]\right)\!\right) && \text{by } \textit{real-valued} \text{ hypothesis}\\
&= \boxed{\tilde{\mathsf{x}}(-\omega)} && \text{by result (1)}
\end{aligned}$$

✑



Proposition P.2. *Let* $\check{\mathsf{x}}(z)$ *be the* Z-TRANSFORM *(Definition O.4 page 342) and* $\breve{\mathsf{x}}(\omega)$ *the* DISCRETE-TIME FOURIER TRANSFORM *(Definition P.1 page 355) of* $\left(\!\left(x_n\right)\!\right)$.

PRP

$$\underbrace{\left\{\sum_{n\in\mathbb{Z}} x_n = c\right\}}_{\text{(1) time domain}} \iff \underbrace{\left\{\check{\mathsf{x}}(z)\Big|_{z=1} = c\right\}}_{\text{(2) z domain}} \iff \underbrace{\left\{\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} = c\right\}}_{\text{(3) frequency domain}} \qquad \forall \left(\!\left(x_n\right)\!\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}, c \in \mathbb{R}$$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\check{\mathsf{x}}(z)\Big|_{z=1} &= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} && \text{by definition of } \check{\mathsf{x}}(z) \text{ (Definition O.4 page 342)}\\
&= \sum_{n\in\mathbb{Z}} x_n && \text{because } z^n = 1 \text{ for all } n \in \mathbb{Z}\\
&= c && \text{by hypothesis (1)}
\end{aligned}$$

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\breve{\mathsf{x}}(\omega)\Big|_{\omega=0} &= \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\Big|_{\omega=0} && \text{by definition of } \breve{\mathsf{x}}(\omega) \quad \text{(Definition P.1 page 355)}\\
&= \sum_{n\in\mathbb{Z}} x_n z^{-n}\Big|_{z=1} \\
&= \check{\mathsf{x}}(z)\Big|_{z=1} && \text{by definition of } \check{\mathsf{x}}(z) \quad \text{(Definition O.4 page 342)}\\
&= c && \text{by hypothesis (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=0} \\
&= \breve{\mathsf{x}}(\omega) && \text{by definition of } \breve{\mathsf{x}}(\omega) && \text{(Definition P.1 page 355)} \\
&= c && \text{by hypothesis (3)}
\end{aligned}$$

⇨

Proposition P.3. *If the coefficients are* ***real****, then the magnitude response (MR) is* ***symmetric****.*

✎PROOF:

$$\begin{aligned}
\left|\tilde{\mathsf{h}}(-\omega)\right| &\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]z^{-m}\right|_{z=e^{-i\omega}} \\
&= \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{i\omega m}\right| && = \left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}^*[m]e^{-i\omega m}\right)^*\right| \\
&= \underbrace{\left|\left(\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right)^*\right|}_{\text{if } x[m] \text{ is } real} && = \left|\sum_{m\in\mathbb{Z}} \mathsf{x}[m]e^{-i\omega m}\right| \\
&\triangleq \left|\check{\mathsf{h}}(z)\right|_{z=e^{-i\omega}} && \triangleq \left|\tilde{\mathsf{h}}(\omega)\right|
\end{aligned}$$

⇨

Proposition P.4. [1]

PRP

$$\underbrace{\sum_{n\in\mathbb{Z}}(-1)^n x_n = c}_{\text{(1) in "time"}} \iff \underbrace{\breve{\mathsf{x}}(z)|_{z=-1} = c}_{\text{(2) in "z domain"}} \iff \underbrace{\breve{\mathsf{x}}(\omega)|_{\omega=\pi} = c}_{\text{(3) in "frequency"}}$$

$$\iff \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{2n}, \sum_{n\in\mathbb{Z}} h_{2n+1}\right) = \left(\frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n + c\right), \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} h_n - c\right)\right)}_{\text{(4) sum of even, sum of odd}}$$

$\forall c\in\mathbb{R}, (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\breve{\mathsf{x}}(z)|_{z=-1} &= \left.\sum_{n\in\mathbb{Z}} x_n z^{-n}\right|_{z=-1} \\
&= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= c && \text{by (1)}
\end{aligned}$$

[1] Chui (1992) page 123

2. Proof that (2) $\implies$ (3):

$$\begin{aligned}
\left.\sum_{n\in\mathbb{Z}} x_n e^{-i\omega n}\right|_{\omega=\pi} &= \sum_{n\in\mathbb{Z}} (-1)^n x_n \\
&= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n && = \left.\sum_{n\in\mathbb{Z}} z^{-n} x_n\right|_{z=-1} \\
&= c && \text{by (2)}
\end{aligned}$$

3. Proof that (3) $\implies$ (1):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (-1)^n x_n &= \sum_{n\in\mathbb{Z}} (-1)^{-n} x_n \\
&= \left.\sum_{n\in\mathbb{Z}} e^{-i\omega n} x_n\right|_{\omega=\pi} \\
&= c && \text{by (3)}
\end{aligned}$$

4. Proof that (2) $\implies$ (4):

(a) Define $A \triangleq \sum_{n\in\mathbb{Z}} h_{2n}$ $\qquad B \triangleq \sum_{n\in\mathbb{Z}} h_{2n+1}$.

(b) Proof that $A - B = c$:

$$\begin{aligned}
c &= \sum_{n\in\mathbb{Z}} (-1)^n x_n && \text{by (2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_e} (-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o} (-1)^n x_n}_{\text{odd terms}} \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}} (-1)^{2n+1} x_{2n+1} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} - \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&\triangleq A - B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(c) Proof that $A + B = \sum_{n\in\mathbb{Z}} x_n$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n &= \sum_{n \text{ even}} x_n + \sum_{n \text{ odd}} x_n \\
&= \underbrace{\sum_{n\in\mathbb{Z}} x_{2n}}_{A} + \underbrace{\sum_{n\in\mathbb{Z}} x_{2n+1}}_{B} \\
&= A + B && \text{by definitions of } A \text{ and } B
\end{aligned}$$

(d) This gives two simultaneous equations:

$$\begin{aligned}
A - B &= c \\
A + B &= \sum_{n\in\mathbb{Z}} x_n
\end{aligned}$$

(e) Solutions to these equations give

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_{2n} &\triangleq A && = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right)\\
\sum_{n\in\mathbb{Z}} x_{2n+1} &\triangleq B && = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right)
\end{aligned}$$

5. Proof that (2) $\Longleftarrow$ (4):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(-1)^n x_n &= \underbrace{\sum_{n\in\mathbb{Z}_e}(-1)^n x_n}_{\text{even terms}} + \underbrace{\sum_{n\in\mathbb{Z}_o}(-1)^n x_n}_{\text{odd terms}}\\
&= \sum_{n\in\mathbb{Z}}(-1)^{2n} x_{2n} + \sum_{n\in\mathbb{Z}}(-1)^{2n+1} x_{2n+1}\\
&= \sum_{n\in\mathbb{Z}} x_{2n} - \sum_{n\in\mathbb{Z}} x_{2n+1}\\
&= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n + c\right) - \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} x_n - c\right) && \text{by (3)}\\
&= c
\end{aligned}$$

✏

Lemma P.1. *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* DTFT (Definition P.1 page 355) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

LEM
$$\underbrace{(x_n \in \mathbb{R})_{n\in\mathbb{Z}}}_{\text{REAL-VALUED sequence}} \implies \underbrace{|\breve{\mathsf{x}}(\omega)|^2 = |\breve{\mathsf{x}}(-\omega)|^2}_{\text{EVEN}} \qquad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
|\breve{\mathsf{x}}(\omega)|^2 &= |\breve{\mathsf{x}}(z)|^2\Big|_{z=e^{i\omega}}\\
&= \breve{\mathsf{x}}(z)\breve{\mathsf{x}}^*(z)\Big|_{z=e^{i\omega}}\\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m z^{-n}\right]^*\Bigg|_{z=e^{i\omega}}\\
&= \left[\sum_{n\in\mathbb{Z}} x_n z^{-n}\right]\left[\sum_{m\in\mathbb{Z}} x_m^*(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n x_m^* z^{-n}(z^*)^{-m}\Bigg|_{z=e^{i\omega}}\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m^* z^{-n}(z^*)^{-m} + \sum_{m<n} x_n x_m^* z^{-n}(z^*)^{-m}\right]\Bigg|_{z=e^{i\omega}}\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m<n} x_n x_m e^{i\omega(m-n)}\right]\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m e^{i\omega(m-n)} + \sum_{m>n} x_n x_m e^{-i\omega(m-n)}\right]\\
&= \sum_{n\in\mathbb{Z}}\left[|x_n|^2 + \sum_{m>n} x_n x_m \left(e^{i\omega(m-n)} + e^{-i\omega(m-n)}\right)\right]
\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} \left[|x_n|^2 + \sum_{m>n} x_n x_m 2\cos[\omega(m-n)] \right]$$
$$= \sum_{n\in\mathbb{Z}} |x_n|^2 + 2\sum_{n\in\mathbb{Z}}\sum_{m>n} x_n x_m \cos[\omega(m-n)]$$

Since cos is real and even, then $|\check{x}(\omega)|^2$ must also be real and even. ✑

Theorem P.3 (inverse DTFT). [2] *Let $\check{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition P.1 page 355) of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\check{x}^{-1}$ be the inverse of $\check{x}$.*

THM
$$\underbrace{\left\{ \check{x}(\omega) \triangleq \sum_{n\in\mathbb{Z}} x_n e^{-i\omega n} \right\}}_{\check{x}(\omega)\triangleq\check{\mathbf{F}}(x_n)} \implies \underbrace{\left\{ x_n = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \check{x}(\omega)e^{i\omega n}\, d\omega \quad \forall\alpha\in\mathbb{R} \right\}}_{(x_n)=\check{\mathbf{F}}^{-1}\check{\mathbf{F}}(x_n)} \quad \forall (x_n)_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \check{x}(\omega)e^{i\omega n}\, d\omega &= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \underbrace{\left[\sum_{m\in\mathbb{Z}} x_m e^{-i\omega m}\right]}_{\check{x}(\omega)} e^{i\omega n}\, d\omega && \text{by definition of } \check{x}(\omega)\\
&= \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \sum_{m\in\mathbb{Z}} x_m e^{-i\omega(m-n)}\, d\omega\\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \int_{\alpha-\pi}^{\alpha+\pi} e^{-i\omega(m-n)}\, d\omega\\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x_m \left[2\pi\bar{\delta}_{m-n}\right]\\
&= x_n
\end{aligned}$$

✑

Theorem P.4 (orthonormal quadrature conditions). [3] *Let $\check{x}(\omega)$ be the* DISCRETE-TIME FOURIER TRANSFORM *(Definition P.1 page 355) of a sequence $(x_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\bar{\delta}_n$ be the* KRONECKER DELTA FUNCTION *at n (Definition L.12 page 278).*

THM
$$\begin{aligned}
\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} &= 0 &\iff \check{x}(\omega)\check{y}^*(\omega) + \check{x}(\omega+\pi)\check{y}^*(\omega+\pi) &= 0 & \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}}\\
\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} &= \bar{\delta}_n &\iff |\check{x}(\omega)|^2 + |\check{x}(\omega+\pi)|^2 &= 2 & \forall n\in\mathbb{Z}, \forall (x_n),(y_n)\in\ell^2_{\mathbb{R}}
\end{aligned}$$

✎PROOF: Let $z \triangleq e^{i\omega}$.

[2] J.S.Chitode (2009) page 3-95 ⟨(3.6.2)⟩
[3] Daubechies (1992) pages 132–137 ⟨(5.1.20),(5.1.39)⟩

1. Proof that $2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)$:

$$\begin{aligned}
&2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-2n} z^{-2n}\\
&= 2\sum_{k\in\mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} z^{-n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n}\left(1+e^{i\pi n}\right)\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k\in\mathbb{Z}} x_k \sum_{n\in\mathbb{Z}} y^*_{k-n} z^{-n} e^{i\pi n}\\
&= \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m z^{-(k-m)} + \sum_{k\in\mathbb{Z}} x_k \sum_{m\in\mathbb{Z}} y^*_m e^{-i(\omega+\pi)(k-m)} && \text{where } m \triangleq k-n\\
&= \sum_{k\in\mathbb{Z}} x_k z^{-k} \sum_{m\in\mathbb{Z}} y^*_m z^m + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k} \sum_{m\in\mathbb{Z}} y^*_m e^{+i(\omega+\pi)m}\\
&= \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i\omega m}\right]^* + \sum_{k\in\mathbb{Z}} x_k e^{-i(\omega+\pi)k}\left[\sum_{m\in\mathbb{Z}} y_m e^{-i(\omega+\pi)m}\right]^*\\
&\triangleq \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi)
\end{aligned}$$

2. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \implies \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
0 &= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

3. Proof that $\sum_{m\in\mathbb{Z}} x_m y^*_{m-2n} = 0 \impliedby \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) = 0$:

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 0 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 0$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = 0$.

4. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \implies |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2 &= 2\sum_{n\in\mathbb{Z}} \bar{\delta}_{n\in\mathbb{Z}} e^{-i2\omega n}\\
&= 2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} && \text{by left hypothesis}\\
&= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}
\end{aligned}$$

5. Proof that $\sum_{m\in\mathbb{Z}} x_m x^*_{m-2n} = \bar{\delta}_n \impliedby |\breve{x}(\omega)|^2 + |\breve{x}(\omega'+\pi)|^2 = 2$:
Let $g_n \triangleq x_n$.

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} &= \breve{x}(\omega)\breve{y}^*(\omega) + \breve{x}(\omega+\pi)\breve{y}^*(\omega+\pi) && \text{by item (1)}\\
&= 2 && \text{by right hypothesis}
\end{aligned}$$

Thus by the above equation, $\sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n}\right] e^{-i2\omega n} = 1$. The only way for this to be true is if $\sum_{k\in\mathbb{Z}} x_k y^*_{k-2n} = \bar{\delta}_n$.

⇒

P.3 Derivatives

Theorem P.5. [4] *Let* $\breve{\mathsf{x}}(\omega)$ *be the* DTFT (Definition P.1 page 355) *of a sequence* $(x_n)_{n\in\mathbb{Z}}$.

THM

$$\text{(A)}\quad \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=0} = 0 \quad\iff\quad \sum_{k\in\mathbb{Z}} k^n x_k = 0 \quad \text{(B)} \quad \forall n\in\mathbb{W}$$

$$\text{(C)}\quad \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=\pi} = 0 \quad\iff\quad \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k = 0 \quad \text{(D)} \quad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof that $(A) \implies (B)$:

$$\begin{aligned}
0 &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=0} && \text{by hypothesis (A)}\\
&= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \breve{\mathsf{x}}(\omega) \text{ (Definition P.1 page 355)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega k}\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k
\end{aligned}$$

2. Proof that $(A) \impliedby (B)$:

$$\begin{aligned}
\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \breve{\mathsf{x}}(\omega)\bigg|_{\omega=0} &= \left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\bigg|_{\omega=0} && \text{by definition of } \tilde{g}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\left[\frac{\mathsf{d}}{\mathsf{d}\omega}\right]^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\bigg|_{\omega=0}\\
&= (-i)^n \sum_{k\in\mathbb{Z}} k^n x_k\\
&= 0 && \text{by hypothesis (B)}
\end{aligned}$$

[4] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

3. Proof that $(C) \implies (D)$:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{\mathsf{x}}(\omega)\Big|_{\omega=\pi} && \text{by hypothesis (C)}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=\pi} && \text{by definition of } \check{\mathsf{x}} \text{ (Definition P.1 page 355)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k
\end{aligned}$$

4. Proof that $(C) \impliedby (D)$:

$$\begin{aligned}
\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \check{\mathsf{x}}(\omega)\Big|_{\omega=\pi} &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \sum_{k\in\mathbb{Z}} x_k e^{-i\omega k}\Big|_{\omega=\pi} && \text{by definition of } \check{\mathsf{x}} \text{ (Definition P.1 page 355)}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega k}\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n e^{-i\omega k}\right]\Big|_{\omega=\pi}\\
&= \sum_{k\in\mathbb{Z}} x_k \left[(-i)^n k^n (-1)^k\right]\\
&= (-i)^n \sum_{k\in\mathbb{Z}} (-1)^k k^n x_k\\
&= 0 && \text{by hypothesis (D)}
\end{aligned}$$

✎

APPENDIX Q

PARTITION OF UNITY

Q.1 Definition and motivation

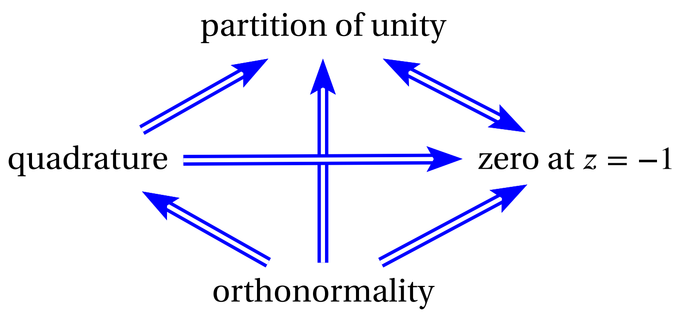


Figure Q.1: Implications of scaling function properties

A very common property of scaling functions (Definition 2.1 page 16) is the *partition of unity* property (Definition Q.1 page 366). The partition of unity is a kind of generalization of *orthonormality*; that is, *all* orthonormal scaling functions form a partition of unity (Theorem 5.1 page 73). But the partition of unity property is not just a consequence of orthonormality, but also a generalization of orthonormality, in that if you remove the orthonormality constraint, the partition of unity is still a reasonable constraint in and of itself.

There are two reasons why the partition of unity property is a reasonable constraint on its own:

- Without a partition of unity, it is difficult to represent a function as simple as a constant.[1]
- For a multiresolution system $\left(\boldsymbol{L}^2_{\mathbb{R}}, \left(\!\left(\boldsymbol{V}_j\right)\!\right), \phi, (\!(h_n)\!)\right)$, the partition of unity property is equivalent to $\sum_{n\in\mathbb{Z}}(-1)^n h_n = 0$ (Theorem 2.8 page 34). As viewed from the perspective of discrete time signal processing (Appendix O page 341), this implies that the scaling coefficients form a "*lowpass filter*"; lowpass filters provide a kind of "coarse approximation" of a function. And that is what the scaling function is "supposed" to do—to provide a coarse approximation at some resolution or "scale" (Definition 2.1 page 16).

[1] Jawerth and Sweldens (1994) page 8

Definition Q.1. [2]

DEF: *A function* $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ *forms a* ***partition of unity*** *if*

$$\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) = 1 \qquad \forall x \in \mathbb{R}.$$

Q.2 Results

Theorem Q.1. [3] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(\boldsymbol{V}_j)\!), \phi, (\!(h_n)\!)\right)$ *be a multiresolution system* (Definition 2.3 page 25). *Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* Fourier transform (Definition K.2 page 257) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* Kronecker delta function.

THM:

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f} = c}_{\text{PARTITION OF UNITY in "time"}} \qquad \Longleftrightarrow \qquad \underbrace{\left[\tilde{\mathbf{F}}\mathsf{f}\right](2\pi n) = \bar{\delta}_n}_{\text{PARTITION OF UNITY in "frequency"}}$$

✎Proof: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
c &= \sum_{m\in\mathbb{Z}} \mathbf{T}^m \mathsf{f}(x) && \text{by left hypothesis} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m) && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 2)} \\
&= \sqrt{2\pi} \sum_{m\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi m) e^{i2\pi m x} && \text{by } PSF \quad \text{(Theorem 1.2 page 10)} \\
&= \underbrace{\sqrt{2\pi}\, \tilde{\mathsf{f}}(2\pi n) e^{i2\pi n x}}_{\text{real and constant for } n=0} + \underbrace{\sqrt{2\pi} \sum_{m\in\mathbb{Z}\setminus n} \tilde{\mathsf{f}}(2\pi m) e^{i2\pi m x}}_{\text{complex and non-constant}} \\
\Longrightarrow &\quad \sqrt{2\pi}\tilde{\mathsf{f}}(2\pi n) = c\bar{\delta}_n && \text{because } c \text{ is real and constant for all } x
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) && \text{by definition of } \mathbf{T} \quad \text{(Definition 1.3 page 2)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi n) e^{-i2\pi n x} && \text{by } PSF \quad \text{(Theorem 1.2 page 10)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \frac{c}{\sqrt{2\pi}} \bar{\delta}_n e^{-i2\pi n x} && \text{by right hypothesis} \\
&= \sqrt{2\pi} \frac{c}{\sqrt{2\pi}} e^{-i2\pi 0 x} && \text{by definition of } \bar{\delta}_n \quad \text{(Definition L.12 page 278)} \\
&= c
\end{aligned}$$

✏

[2] Kelley (1955) page 171, Munkres (2000) page 225, Jänich (1984) page 116, Willard (1970), page 152 ⟨item 20C⟩, Willard (2004) page 152 ⟨item 20C⟩
[3] Jawerth and Sweldens (1994) page 8

Corollary Q.1.

COR
$$\left\{\begin{array}{l}\exists \mathsf{g} \in \boldsymbol{L}^2_{\mathbb{R}} \textit{ such that} \\ \mathsf{f}(x) = \mathbb{1}_{[-1:1)}(x) \star \mathsf{g}(x)\end{array}\right\} \implies \left\{\begin{array}{l}\mathsf{f}(x) \textit{ generates} \\ \textit{a } \text{PARTITION OF UNITY}\end{array}\right\}$$

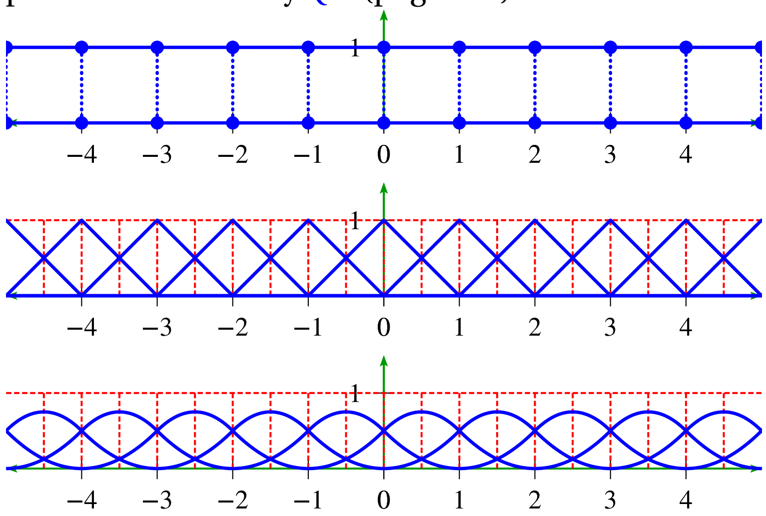
✎PROOF:

$$\begin{aligned}
\mathsf{f}(x) = \mathbb{1}_{[0:1)}(x) \star \mathsf{g}(x) &\implies \tilde{\mathsf{f}}(\omega) = \tilde{\mathbf{F}}\left[\mathbb{1}_{[-1:1)}\right](\omega)\tilde{\mathsf{g}}(\omega) && \text{by \textit{convolution theorem}} && \text{(Theorem O.2 page 344)} \\
&\iff \tilde{\mathsf{f}}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{\sin(\omega)}{\omega}\tilde{\mathsf{g}}(\omega) && \text{by \textit{rectangular pulse} ex.} && \text{(Example K.1 page 264)} \\
&\implies \tilde{\mathsf{f}}(2\pi n) = 0 \\
&\iff \mathsf{f}(x) \text{ generates a \textit{partition of unity}} && \text{by Theorem Q.1 page 366}
\end{aligned}$$

✏

Q.3 Examples

Example Q.1. All *B-splines* (Definition M.2 page 297) form a partition of unity (Theorem M.4 page 308). All B-splines of order $n = 1$ or greater can be generated by convolution with a *pulse* function, similar to that specified in Corollary Q.1 (page 367) and as illustrated below:



(Example M.13 page 315)

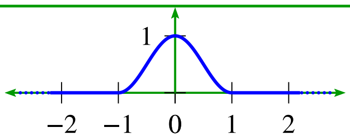
(Example M.14 page 315)

(Example M.15 page 315)

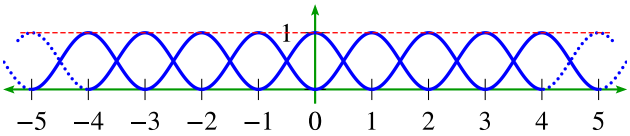
Example Q.2. Let a function f be defined in terms of the cosine function (Definition H.2 page 215) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

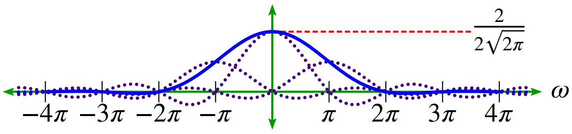


Then f induces a *partition of unity*:



Note that $$\tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)}\right]$$

and so $\tilde{\mathsf{f}}(2\pi n) = \frac{1}{\sqrt{2\pi}}\bar{\delta}_n$:



✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 1) on a set A.

1. Proof that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f} = 1$ (time domain proof):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\cos^2(x)\mathbb{1}_{[-1:1]}(x) && \text{by definition of } \mathsf{f}(x) \\
&= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\cos^2(x)\mathbb{1}_{[-1:1)}(x) && \text{because } \cos^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 1 \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n) && \text{by definition of } \mathbf{T} \text{ (Definition 1.3 page 2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_o} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n)}_{\text{odd part}} + \underbrace{\sum_{n\in\mathbb{Z}_e} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n)}_{\text{even part}} \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-2n)\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-2n-1)\right)\mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x - n\pi\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x - n\pi - \frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n}\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} (-1)^{2n}\cos^2\left(\frac{\pi}{2}x - \frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n-1) && \text{by Theorem H.11 page 228} \\
&= \cos^2\left(\frac{\pi}{2}x\right)\sum_{n\in\mathbb{Z}} \mathbb{1}_{[-1:1)}(x-2n) + \sin^2\left(\frac{\pi}{2}x\right)\sum_{n\in\mathbb{Z}} \mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \cos^2\left(\frac{\pi}{2}x\right)\cdot 1 + \sin^2\left(\frac{\pi}{2}x\right)\cdot 1 \\
&= 1 && \text{by \textit{square identity} (Theorem H.11 page 228)}
\end{aligned}$$

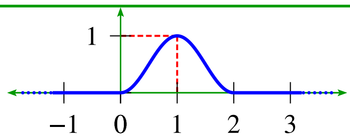
2. Proof that $\tilde{\mathsf{f}}(\omega) = \cdots$: by Example K.3 page 265

✎

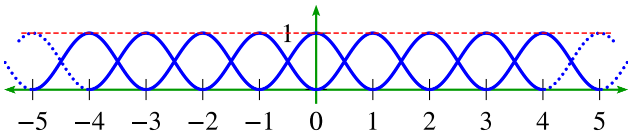
Example Q.3. Let a function f be defined in terms of the sine function (Definition H.3 page 215) as follows:

CNT

$$\mathsf{f}(x) \triangleq \begin{cases} \sin^2\left(\frac{\pi}{2}x\right) & \text{for } x \in [0:2] \\ 0 & \text{otherwise} \end{cases}$$



Then $\int_{\mathbb{R}} \mathsf{f}(x)\,\mathsf{d}x = 1$ and f induces a *partition of unity*



✎PROOF:

1. Proof that $\int_{\mathbb{R}} \mathsf{f}(x)\,\mathsf{d}x = 1$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{f}(x)\,\mathsf{d}x &= \int_{\mathbb{R}} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x)\,\mathsf{d}x && \text{by definition of } \mathsf{f}(x)\\
&= \int_0^2 \sin^2\left(\frac{\pi}{2}x\right)\mathsf{d}x && \text{by definition of } \mathbb{1}_{A(x)} \text{ (Definition 1.2 page 1)}\\
&= \int_0^2 \frac{1}{2}[1-\cos(\pi x)]\,\mathsf{d}x && \text{by Theorem H.11 page 228}\\
&= \frac{1}{2}\left[x - \frac{1}{\pi}\sin(\pi x)\right]_0^2\\
&= \frac{1}{2}[2-0-0-0]\\
&= 1
\end{aligned}$$

2. Proof that $\mathsf{f}(x)$ forms a *partition of unity*:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x) && \text{by definition of } \mathsf{f}(x)\\
&= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{because } \sin^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 2\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^{m-1}\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{where } m \triangleq n+1 \implies n = m-1\\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m+1)\right)\mathbb{1}_{[0:2)}(x-m+1) && \text{by definition of } \mathbf{T} \text{ (Definition 1.3 page 2)}\\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m)+\frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-m)\\
&= \sum_{m\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-m)\right)\mathbb{1}_{[-1:1)}(x-m) && \text{by Theorem H.11 page 228}\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x) && \text{by definition of } \mathbf{T} \text{ (Definition 1.3 page 2)}\\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x) && \text{because } \cos^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 1\\
&= 1 && \text{by Example Q.2 page 367}
\end{aligned}$$

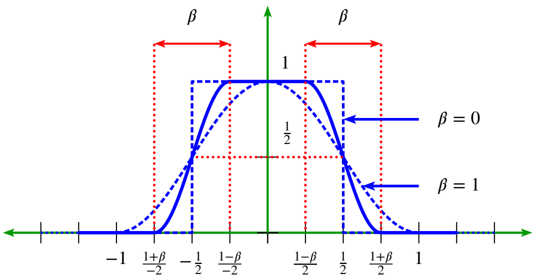
⇰

Example Q.4 (raised cosine). [4] Let a function f be defined in terms of the cosine function (Definition H.2 page 215) as follows:

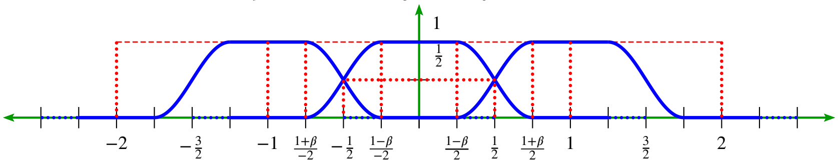
[4] Proakis (2001) pages 560–561

EX

Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } 0 \le |x| < \frac{1-\beta}{2} \\ \frac{1}{2}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x| - \frac{1-\beta}{2}\right)\right]\right\} & \text{for } \frac{1-\beta}{2} \le |x| < \frac{1+\beta}{2} \\ 0 & \text{otherwise} \end{cases}$



Then f induces a *partition of unity*:



✎PROOF:

1. definition: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition 1.2 page 1) on a set A.
   Let $A \triangleq \left[\frac{1+\beta}{-2} : \frac{1-\beta}{-2}\right)$, $\quad B \triangleq \left[\frac{1-\beta}{-2} : \frac{1-\beta}{2}\right)$, and $\quad C \triangleq \left[\frac{1-\beta}{2} : \frac{1+\beta}{2}\right)$

2. lemma: $\mathbb{1}_A(x-1) = \mathbb{1}_C(x)$. Proof:

$$\begin{aligned}
\mathbb{1}_A(x-1) &\triangleq \begin{cases} 1 & \text{if } -\frac{1+\beta}{2} \le x-1 < -\frac{1-\beta}{2} \\ 0 & \text{otherwise} \end{cases} && \text{by definition of } \mathbb{1} \text{ (Definition 1.2 page 1) and } A \text{ ((2) lemma page 370)} \\
&= \begin{cases} 1 & \text{if } 1-\frac{1+\beta}{2} \le x < 1-\frac{1-\beta}{2} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \frac{1-\beta}{2} \le x < \frac{1+\beta}{2} \\ 0 & \text{otherwise} \end{cases} \\
&\triangleq \mathbb{1}_C(x) && \text{by definition of } \mathbb{1} \text{ (Definition 1.2 page 1) and } C \text{ ((2) lemma page 370)}
\end{aligned}$$

3. lemma: $-1 + \frac{1-\beta}{2} = -\beta - \frac{1-\beta}{2}$. Proof:

$$-1 + \frac{1-\beta}{2} = \frac{-2+1-\beta}{2} \quad = \frac{-1-\beta}{2} = (-\beta+\beta) - \left(\frac{1+\beta}{2}\right) \quad = -\beta + \frac{2\beta-1-\beta}{2} = -\beta - \frac{1-\beta}{2}$$

4. Proof that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f} = 1$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) && \text{by Definition 1.3} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_A(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by definition 1 page 370} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) \\
&\quad + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n-1)\mathbb{1}_A(x-n-1) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by Proposition 1.1} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n-1)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by (2) lemma page 370}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n| - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n-1| - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) && \text{by definition of } \mathsf{f}(x) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left((x-n) - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(-(x-n-1) - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) && \text{by def. of } \mathbb{1}_C(x) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-1 + \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\beta - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) && \text{by (3) lemma page 370} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2} - \frac{\pi\beta}{\beta}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left\{1-\cos\left[\frac{\pi}{\beta}\left(x-n - \frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\mathbb{1}_C(x-n) + \frac{1}{2}\sum_{n\in\mathbb{Z}}\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \sum_{n\in\mathbb{Z}}\mathbb{1}_{B\cup C}(x-n) \\
&= 1
\end{aligned}
$$

✏

APPENDIX R

POWER SPECTRUM FUNCTIONS

R.1 Correlation

Definition R.1 and Definition R.2 define four quantities. In this document, the quantities' notation and terminology are similar to those used in the study of *random processes*.

Definition R.1. [1] *Let* $\langle \triangle \mid \triangledown \rangle$ *be the* STANDARD INNER PRODUCT *in* $L^2_{\mathbb{R}}$ (Definition E.1 page 185).

$$\begin{array}{lcllll} \mathsf{R}_{\mathsf{fg}}(n) & \triangleq & \langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{g}(x) \rangle, & n \in \mathbb{Z}; & \mathsf{f}, \mathsf{g} \in L^2_{\mathbb{F}}, & \text{is the \textbf{cross-correlation function} of f and g.} \\ \mathsf{R}_{\mathsf{ff}}(n) & \triangleq & \langle \mathsf{f}(x) \mid \mathbf{T}^n \mathsf{f}(x) \rangle, & n \in \mathbb{Z}; & \mathsf{f} \in L^2_{\mathbb{F}}, & \text{is the \textbf{autocorrelation function} of f.} \end{array}$$

Definition R.2. [2] *Let* $\mathsf{R}_{\mathsf{fg}}(n)$ *and* $\mathsf{R}_{\mathsf{ff}}(n)$ *be the sequences defined in Definition R.1 page 373. Let* $\mathbf{Z}\,(\!(x_n)\!)$ *be the* Z-TRANSFORM (Definition O.4 page 342) *of a sequence* $(\!(x_n)\!)_{n\in\mathbb{Z}}$.

$$\begin{array}{lclll} \check{\mathsf{S}}_{\mathsf{fg}}(z) & \triangleq & \mathbf{Z}\big[\mathsf{R}_{\mathsf{fg}}(n)\big], & \mathsf{f,g} \in L^2_{\mathbb{F}}, & \text{is the \textbf{complex cross-power spectrum} of f and g.} \\ \check{\mathsf{S}}_{\mathsf{ff}}(z) & \triangleq & \mathbf{Z}\big[\mathsf{R}_{\mathsf{ff}}(n)\big], & \mathsf{f,g} \in L^2_{\mathbb{F}}, & \text{is the \textbf{complex auto-power spectrum} of f.} \end{array}$$

R.2 Power Spectrum

Definition R.3. [3] *Let* $\check{\mathsf{S}}_{\mathsf{fg}}(z)$ *and* $\check{\mathsf{S}}_{\mathsf{ff}}(z)$ *be the functions defined in Definition R.2 page 373.*

$$\begin{array}{lclll} \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) & \triangleq & \check{\mathsf{S}}_{\mathsf{fg}}\left(e^{i\omega}\right), & \forall \mathsf{f,g} \in L^2_{\mathbb{F}}, & \text{is the \textbf{cross-power spectrum} of f and g.} \\ \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) & \triangleq & \check{\mathsf{S}}_{\mathsf{ff}}\left(e^{i\omega}\right), & \forall \mathsf{f} \in L^2_{\mathbb{F}}, & \text{is the \textbf{auto-power spectrum} of f.} \end{array}$$

Theorem R.1. [4] *Let* $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ *and* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be defined as in Definition R.3 (page 373).*
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition K.2 page 257) *of a function* $\mathsf{f}(x) \in L^2_{\mathbb{F}}$.

$$\begin{array}{lcll} \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) & = & 2\pi \displaystyle\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega + 2\pi n)\tilde{\mathsf{g}}^*(\omega + 2\pi n) & \forall \mathsf{f,g} \in L^2_{\mathbb{F}} \\ \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) & = & 2\pi \displaystyle\sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega + 2\pi n)\right|^2 & \forall \mathsf{f} \in L^2_{\mathbb{F}} \end{array}$$

[1] Chui (1992) page 134, Papoulis (1991) pages 294–332 ⟨(10-29), (10-169)⟩
[2] Chui (1992) page 134, Papoulis (1991) page 334 ⟨(10-178)⟩
[3] Chui (1992) page 134, Papoulis (1991) page 333 ⟨(10-179)⟩
[4] Chui (1992) page 135

✎PROOF: Let $z \triangleq e^{i\omega}$.

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) &\triangleq \check{\mathsf{S}}_{\mathsf{fg}}(z) && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition R.3 page 373)}\\
&= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n) z^{-n} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition R.2 page 373)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n)\rangle\, z^{-n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition R.3 page 373)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathbf{F}}[\mathsf{f}(x)] \,|\, \tilde{\mathbf{F}}\big[\mathsf{g}(x-n)\big] \right\rangle z^{-n} && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem K.3 page 259)}\\
&= \sum_{n\in\mathbb{Z}} \left\langle \tilde{\mathsf{f}}(v) \,|\, e^{-ivn}\tilde{\mathsf{g}}(v) \right\rangle z^{-n} && \text{by } \textit{shift relation} && \text{(Theorem K.4 page 259)}\\
&= \sum_{n\in\mathbb{Z}} \sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v)e^{ivu}\,\mathrm{d}v\right]_{u=n} z^{-n} && \text{by definition of } \boldsymbol{L}^2_{\mathbb{R}} && \text{(Definition E.1 page 185)}\\
&= \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \left[\tilde{\mathbf{F}}^{-1}\left(\sqrt{2\pi}\tilde{\mathsf{f}}(v)\tilde{\mathsf{g}}^*(v)\right)\right]_{u=n} e^{-i\omega n} && \text{by Theorem K.1 page 258}\\
&= 2\pi\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n) && \text{by } \textit{IPSF} \text{ with } \tau=1 && \text{(Theorem 1.3 page 10)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)\Big|_{\mathsf{g}=\mathsf{f}} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) && \text{(Definition R.3 page 373)}\\
&= 2\pi\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{g}}^*(\omega+2\pi n)\Bigg|_{\mathsf{g}\triangleq\mathsf{f}} && \text{by previous result}\\
&= 2\pi\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(\omega+2\pi n)\tilde{\mathsf{f}}^*(\omega+2\pi n)\\
&= 2\pi\sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{because } |z|^2 \triangleq zz^* \quad \forall z\in\mathbb{C}
\end{aligned}$$

✎

Proposition R.1. *Let $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition R.3 (page 373).*

PRP $\quad \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) \;\geq\; 0 \quad$ (NON-NEGATIVE)

✎PROOF:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) &= 2\pi\sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{f}}(\omega+2\pi n)\right|^2 && \text{by Theorem R.1 page 373}\\
&\geq 0 && \text{because } |z|\geq 0 \quad \forall z\in\mathbb{C}
\end{aligned}$$

✎

Proposition R.2. *Let $\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega)$ and $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ be defined as in Definition R.3 (page 373).*

PRP
$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) && (\text{PERIODIC } \textit{with period } 2\pi)\\
\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega+2\pi) &= \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) && (\text{PERIODIC } \textit{with period } 2\pi)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(\omega+2\pi) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\omega+2\pi+2\pi n)\tilde{g}^*(\omega+2\pi+2\pi n) && \text{by Theorem R.1 page 373}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}[\omega+2\pi(n+1)]\tilde{g}^*[\omega+2\pi(n+1)]\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{f}[\omega+2\pi m]\tilde{g}^*[\omega+2\pi m] && \text{where } m\triangleq n+1\\
&= \tilde{S}_{fg}(\omega) && \text{by Theorem R.1 page 373}\\
\tilde{S}_{ff}(\omega+2\pi) &= \tilde{S}_{fg}(\omega+2\pi)\big|_{g=f}\\
&= \tilde{S}_{fg}(\omega)\big|_{g=f} && \text{by previous result}\\
&= \tilde{S}_{ff}(\omega)
\end{aligned}$$

✏

Proposition R.3. *Let $\tilde{S}_{fg}(\omega)$ and $\tilde{S}_{ff}(\omega)$ be defined as in Definition R.3 (page 373).*

PRP

$$\begin{array}{lcrcll}
f,g \text{ are real} & \Longrightarrow & \tilde{S}_{fg}(-\omega) & = & \tilde{S}_{gf}(\omega) & \\
f \text{ is real} & \Longrightarrow & \tilde{S}_{ff}(-\omega) & = & \tilde{S}_{ff}(\omega) & (\text{SYMMETRIC about } 0)\\
f,g \text{ are real} & \Longrightarrow & \tilde{S}_{fg}(\pi-\omega) & = & \tilde{S}_{gf}(\pi+\omega) & \\
f \text{ is real} & \Longrightarrow & \tilde{S}_{ff}(\pi-\omega) & = & \tilde{S}_{ff}(\pi+\omega) & (\text{SYMMETRIC about } \pi)
\end{array}$$

✎PROOF:

$$\begin{aligned}
\tilde{S}_{fg}(-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(-\omega+2\pi n)\tilde{g}^*(-\omega+2\pi n) && \text{by Theorem R.1 page 373}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\omega-2\pi n)\tilde{g}(\omega-2\pi n) && \text{by hypothesis and Theorem K.5 page 260}\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\omega+2\pi m)\tilde{f}^*(\omega+2\pi m) && \text{where } m\triangleq -n\\
&= \tilde{S}_{gf}(\omega) && \text{by Theorem R.1 page 373}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{fg}(\pi-\omega) &= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}(\pi-\omega+2\pi n)\tilde{g}^*(\pi-\omega+2\pi n) && \text{by Theorem R.1 page 373}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(-\pi+\omega-2\pi n)\tilde{g}(-\pi+\omega-2\pi n) && \text{by hypothesis and Theorem K.5 page 260}\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega-2\pi-2\pi n)\tilde{g}(\pi+\omega-2\pi-2\pi n)\\
&= 2\pi\sum_{n\in\mathbb{Z}}\tilde{f}^*(\pi+\omega+2\pi(-n-1))\tilde{g}(\pi+\omega+2\pi(-n-1))\\
&= 2\pi\sum_{m\in\mathbb{Z}}\tilde{g}(\pi+\omega+2\pi m)\tilde{f}^*(\pi+\omega+2\pi m) && \text{where } m\triangleq -n-1\\
&= \tilde{S}_{gf}(\pi+\omega) && \text{by Theorem R.1 page 373}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{ff}(-\omega) &= \tilde{S}_{fg}(-\omega)\big|_{g\triangleq f}\\
&= \tilde{S}_{gf}(+\omega)\big|_{g\triangleq f} && \text{by previous result}\\
&= \tilde{S}_{ff}(+\omega) && \text{by definition of g } (g\triangleq f)
\end{aligned}$$

$$\tilde{S}_{ff}(\pi-\omega) = \tilde{S}_{fg}(\pi-\omega)\big|_{g\triangleq f}$$

$$= \tilde{\mathsf{S}}_{\mathsf{gf}}(\pi+\omega)\big|_{\mathsf{g}\triangleq\mathsf{f}} \qquad \text{by previous result}$$
$$= \tilde{\mathsf{S}}_{\mathsf{ff}}(\pi+\omega) \qquad \text{by definition of } \mathsf{g}\ (\mathsf{g}\triangleq\mathsf{f})$$

⇰

Proposition R.4. *Let* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be the* AUTO-POWER SPECTRUM *(Definition R.3 page 373) of a function* $\mathsf{f}(x)\in\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{S}}'_{\mathsf{ff}}(\omega)\triangleq\frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *(Definition E.2 page 185).*

PRP

$$\left\{\begin{array}{ll}(a). & \mathsf{f} \textit{ is } \text{REAL} \quad \textit{and}\\ (b). & \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) \textit{ is } \text{CONTINUOUS } \textit{at } \omega=0\end{array}\right\} \implies \left\{\begin{array}{ll}(1). & \tilde{\mathsf{S}}'_{\mathsf{ff}}(0)=0 \quad \textit{and}\\ (2). & \underbrace{\tilde{\mathsf{S}}'_{\mathsf{ff}}(\omega)=-\tilde{\mathsf{S}}'_{\mathsf{ff}}(-\omega) \quad \forall\omega\in\mathbb{R}}_{\text{ANTI-SYMMETRIC } \textit{about } 0}\end{array}\right\}$$

$$\left\{\begin{array}{ll}(c). & \mathsf{f} \textit{ is } \text{REAL} \quad \textit{and}\\ (d). & \tilde{\mathsf{S}}_{\mathsf{ff}}(\omega) \textit{ is } \text{CONTINUOUS } \textit{at } \omega=\pi\end{array}\right\} \implies \left\{\begin{array}{ll}(3). & \tilde{\mathsf{S}}'_{\mathsf{ff}}(\pi)=0 \quad \textit{and}\\ (4). & \underbrace{\tilde{\mathsf{S}}'_{\mathsf{ff}}(\pi+\omega)=-\tilde{\mathsf{S}}'_{\mathsf{ff}}(\pi-\omega) \quad \forall\omega\in\mathbb{R}}_{\text{ANTI-SYMMETRIC } \textit{about } \pi}\end{array}\right\}$$

✎PROOF: This follows from Proposition R.3 (page 375) and Proposition E.1 (page 185). ⇰

Theorem R.2 (next) is a major result and provides strong motivation for bothering with *power spectrum* functions in the first place. In particular, the *auto-power spectrum* being *bounded* provides a necessary and sufficient condition for a sequence of functions $\langle\!\langle\phi(x-n)\rangle\!\rangle_{n\in\mathbb{Z}}$ to be a *Riesz basis* (Definition L.13 page 285) for the *span* $\operatorname{span}\langle\!\langle\phi(x-n)\rangle\!\rangle$ of the sequence.

Theorem R.2. [5] *Let* $\tilde{\mathsf{S}}_{\mathsf{ff}}(\omega)$ *be defined as in Definition R.3 (page 373). Let* $\|\cdot\|$ *be defined as in Definition E.1 (page 185). Let* $0<A<B$.

THM

$$\underbrace{\left\{A\sum_{n\in\mathbb{N}}|a_n|^2 \le \left\|\sum_{n\in\mathbb{Z}}a_n\phi(x-n)\right\|^2 \le B\sum_{n\in\mathbb{N}}|\alpha_n|^2 \quad \forall\langle\!\langle a_n\rangle\!\rangle\in\ell^2_{\mathbb{F}}\right\}}_{\langle\!\langle\phi(x-n)\rangle\!\rangle \textit{ is a } \text{RIESZ BASIS} \textit{ for } \operatorname{span}\langle\!\langle\phi(x-n)\rangle\!\rangle \text{ (Theorem L.13 page 286)}} \iff \left\{A\le\tilde{\mathsf{S}}_{\phi\phi}(\omega)\le B\right\}$$

✎PROOF:

1. lemma:

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}}a_n\phi(x-n)\right\|^2 &= \left\|\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}}a_n\phi(x-n)\right\|^2 && \text{because } \tilde{\mathbf{F}} \text{ is } \textit{unitary} \text{ (Theorem K.2 page 258)}\\
&= \left\|\breve{a}(\omega)\tilde{\phi}(\omega)\right\|^2 && \text{by Proposition 1.13 page 9}\\
&= \int_{\mathbb{R}}\left|\breve{a}(\omega)\tilde{\phi}(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \|\cdot\|\\
&= \sum_{n\in\mathbb{Z}}\int_0^{2\pi}\left|\breve{a}(\omega+2\pi n)\tilde{\phi}(\omega+2\pi n)\right|^2 \mathrm{d}\omega\\
&= \int_0^{2\pi}\sum_{n\in\mathbb{Z}}|\breve{a}(\omega+2\pi n)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 \mathrm{d}\omega\\
&= \int_0^{2\pi}\sum_{n\in\mathbb{Z}}|\breve{a}(\omega)|^2\left|\tilde{\phi}(\omega+2\pi n)\right|^2 \mathrm{d}\omega && \text{by Proposition P.1 page 355}\\
&= \int_0^{2\pi}|\breve{a}(\omega)|^2\frac{1}{2\pi}2\pi\sum_{n\in\mathbb{Z}}\left|\tilde{\phi}(\omega+2\pi n)\right|^2 \mathrm{d}\omega
\end{aligned}$$

[5] Wojtaszczyk (1997) pages 22–23 ⟨Proposition 2.8⟩, Igari (1996) page 219 ⟨Lemma 9.6⟩, Pinsky (2002) page 306 ⟨Theorem 6.4.8⟩

$$= \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega)\, d\omega \qquad \text{by definition of } \tilde{S}_{\phi\phi}(\omega) \text{ (Theorem R.1 page 373)}$$

2. lemma:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2\, d\omega &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right|^2 d\omega && \text{by def. of } DTFT \text{ (Definition P.1 page 355)}\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\left[\sum_{m\in\mathbb{Z}} a_m e^{-i\omega m}\right]^* d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\left[\sum_{m\in\mathbb{Z}} a_m^* e^{i\omega m}\right] d\omega\\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n a_m^* \int_0^{2\pi} e^{-i\omega(n-m)}\, d\omega\\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n a_m^* 2\pi\bar{\delta}_{nm}\\
&= \sum_{n\in\mathbb{Z}} |a_n|^2 && \text{by definition of } \bar{\delta} \text{ (Definition L.12 page 278)}
\end{aligned}$$

3. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\boxed{A\sum_{n\in\mathbb{Z}} |a_n|^2} &= \frac{A}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2\, d\omega && \text{by (2) lemma page 377}\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 A\, d\omega\\
&\boxed{\leq} \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega)\, d\omega && \text{by right hypothesis}\\
&= \boxed{\left\|\sum_{n\in\mathbb{Z}} a_n \phi(x-n)\right\|^2} && \text{by (1) lemma page 376}\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega)\, d\omega && \text{by (1) lemma page 376}\\
&\boxed{\leq} \frac{1}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2 B\, d\omega && \text{by right hypothesis}\\
&= \frac{B}{2\pi}\int_0^{2\pi} |\breve{a}(\omega)|^2\, d\omega\\
&= \boxed{B\sum_{n\in\mathbb{Z}} |a_n|^2} && \text{by (2) lemma page 377}
\end{aligned}$$

4. Proof for ($\Longrightarrow$) case:

(a) Let $Y \triangleq \{\omega \in [0:2\pi] | \tilde{S}_{\phi\phi}(\omega) > \alpha\}$
and $X \triangleq \{\omega \in [0:2\pi] | \tilde{S}_{\phi\phi}(\omega) < \alpha\}$

(b) Let $\mathbb{1}_{A(x)}$ be the *set indicator* (Definition 1.2 page 1) of a set A.
Let $(b_n)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem P.3 page 361) of $\mathbb{1}_Y(\omega)$ such that
$$\mathbb{1}_Y(\omega) \triangleq \sum_{n\in\mathbb{N}} b_n e^{-i\omega n} \triangleq \breve{b}(\omega).$$
Let $(a_n)_{n\in\mathbb{Z}}$ be the *inverse DTFT* (Theorem P.3 page 361) of $\mathbb{1}_X(\omega)$ such that
$$\mathbb{1}_X(\omega) \triangleq \sum_{n\in\mathbb{N}} a_n e^{-i\omega n} \triangleq \breve{a}(\omega).$$

(c) Proof that $\alpha \le B$:
Let $\mu(A)$ be the *measure* of a set A.

$$
\begin{aligned}
\boxed{B} \sum_{n\in\mathbb{Z}} \left|b_n\right|^2 &\ge \left\| \sum_{n\in\mathbb{Z}} b_n \phi(x-n) \right\|^2 && \text{by left hypothesis} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left|\tilde{\mathsf{b}}(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by (1) lemma page 376} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left|\mathbb{1}_Y(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 377)} \\
&= \frac{1}{2\pi} \int_Y |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 377)} \\
&\boxed{\ge} \frac{\alpha}{2\pi} \mu(Y) && \text{by definition of } Y && \text{(item (4a) page 377)} \\
&= \int_0^{2\pi} \left|\mathbb{1}_Y(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \mathbb{1}_Y(\omega) && \text{(item (4b) page 377)} \\
&= \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}} b_n e^{-i\omega n} \right|^2 \mathrm{d}\omega && \text{by definition of } (\!(b_n)\!) && \text{(item (4b) page 377)} \\
&= \int_0^{2\pi} \left|\tilde{\mathsf{b}}(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \tilde{\mathsf{b}}(\omega) && \text{(item (4b) page 377)} \\
&= \boxed{\alpha} \sum_{n\in\mathbb{Z}} \left|b_n\right|^2 && \text{by (2) lemma page 377}
\end{aligned}
$$

(d) Proof that $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$:
(i). $\tilde{\mathsf{S}}_{\phi\phi}(\omega) > \alpha$ whenever $\omega \in Y$ (item (4a) page 377).
(ii). But even then, $\alpha \le B$ (item (4c) page 378).
(iii). So, $\tilde{\mathsf{S}}_{\phi\phi}(\omega) \le B$.

(e) Proof that $A \le \alpha$:
Let $\mu(A)$ be the *measure* of a set A.

$$
\begin{aligned}
\boxed{A} \sum_{n\in\mathbb{Z}} \left|a_n\right|^2 &\le \left\| \sum_{n\in\mathbb{Z}} a_n \phi(x-n) \right\|^2 && \text{by left hypothesis} \\
&= \frac{1}{2\pi} \int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by (1) lemma page 376} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left|\mathbb{1}_X(\omega)\right|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition 1.2 page 1)} \\
&= \frac{1}{2\pi} \int_X |1|^2 \tilde{\mathsf{S}}_{\phi\phi}(\omega)\, \mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition 1.2 page 1)} \\
&\boxed{\le} \frac{\alpha}{2\pi} \mu(X) && \text{by definition of } X && \text{(item (4a) page 377)} \\
&= \int_0^{2\pi} \left|\mathbb{1}_X(\omega)\right|^2 \mathrm{d}\omega && \text{by definition of } \mathbb{1}_X(\omega) && \text{(Definition 1.2 page 1)} \\
&= \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \right|^2 \mathrm{d}\omega && \text{by definition of } (\!(a_n)\!) && \text{((2) lemma page 377)} \\
&= \int_0^{2\pi} |\breve{\mathsf{a}}(\omega)|^2 \mathrm{d}\omega && \text{by definition of } \breve{\mathsf{a}}(\omega) && \text{((2) lemma page 377)} \\
&= \boxed{\alpha} \sum_{n\in\mathbb{Z}} \left|a_n\right|^2 && \text{by (2) lemma page 377}
\end{aligned}
$$

(f) Proof that $A \leq \tilde{S}_{\phi\phi}(\omega)$:

(i). $\tilde{S}_{\phi\phi}(\omega) < \alpha$ whenever $\omega \in X$ (item (4a) page 377).

(ii). But even then, $A \leq \alpha$ (item (4e) page 378).

(iii). So, $A \leq \tilde{S}_{\phi\phi}(\omega)$.

✏

In the case that f and g are *orthonormal*, the spectral density relations simplify considerably (next).

Theorem R.3. [6] *Let* $\tilde{S}_{ff}$ *and* $\tilde{S}_{fg}$ *be the* SPECTRAL DENSITY FUNCTIONS *(Definition R.3 page 373).*

THM

$$\begin{array}{rcll c rcll}
\langle f(x) \,|\, f(x-n) \rangle &=& \bar{\delta}_n & (\langle f(x-n)\rangle \text{ is ORTHONORMAL}) & \iff & \tilde{S}_{ff}(\omega) &=& 1 & \forall f \in L^2_{\mathbb{F}} \\
\langle f(x) \,|\, g(x-n) \rangle &=& 0 & (f(x) \text{ is ORTHOGONAL to } \langle g(x-n)\rangle) & \iff & \tilde{S}_{fg}(\omega) &=& 0 & \forall f, g \in L^2_{\mathbb{F}}
\end{array}$$

✎PROOF:

1. Proof that $\langle f(x) \,|\, f(x-n) \rangle = \bar{\delta}_n \iff \tilde{S}_{ff}(\omega) = 1$: This follows directly from Theorem R.2 (page 376) with $A = B = 1$ (by *Parseval's Identity* Theorem L.9 page 280 since $\{\mathbf{T}^n f\}$ is *orthonormal*)

2. Alternate proof that $\langle f(x) \,|\, f(x-n) \rangle = \bar{\delta}_n \implies \tilde{S}_{ff}(\omega) = 1$:

$$\begin{aligned}
\tilde{S}_{ff}(\omega) &= \sum_{n\in\mathbb{Z}} R_{ff}(n) e^{-i\omega n} && \text{by definition of } \tilde{S}_{fg} && \text{(Definition R.3 page 373)} \\
&= \sum_{n\in\mathbb{Z}} \langle f(x) \,|\, f(x-n) \rangle \, e^{-i\omega n} && \text{by definition of } R_{ff} && \text{(Definition R.1 page 373)} \\
&= \sum_{n\in\mathbb{Z}} \bar{\delta}_n e^{-i\omega n} && \text{by left hypothesis} \\
&= 1 && \text{by definition of } \bar{\delta} && \text{(Definition L.12 page 278)}
\end{aligned}$$

3. Alternate proof that $\langle f(x) \,|\, f(x-n) \rangle = \bar{\delta}_n \impliedby \tilde{S}_{ff}(\omega) = 1$:

$$\begin{aligned}
&\langle f(x) \,|\, f(x-n) \rangle \\
&= \left\langle \tilde{\mathbf{F}} f(x) \,|\, \tilde{\mathbf{F}} f(x-n) \right\rangle && \text{by } \textit{unitary} \text{ property of } \tilde{\mathbf{F}} && \text{(Theorem K.3 page 259)} \\
&= \left\langle \tilde{f}(\omega) \,|\, e^{-i\omega n} \tilde{f}(\omega) \right\rangle && \text{by } \textit{shift property} \text{ of } \tilde{\mathbf{F}} && \text{(Theorem K.4 page 259)} \\
&= \int_{\mathbb{R}} \tilde{f}(\omega) e^{i\omega n} \tilde{f}^*(\omega) \, d\omega && \text{by definition of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition E.1 page 185)} \\
&= \int_{\mathbb{R}} \left|\tilde{f}(\omega)\right|^2 e^{i\omega n} \, d\omega \\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \left|\tilde{f}(\omega)\right|^2 e^{i\omega n} \, d\omega \\
&= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \left|\tilde{f}(u + 2\pi n)\right|^2 e^{i(u+2\pi n)n} \, du && \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \left|\tilde{f}(u + 2\pi n)\right|^2 \right] e^{iun} \cancelto{1}{e^{i2\pi nn}} \, du \\
&= \frac{1}{2\pi} \int_0^{2\pi} \tilde{S}_{ff}(\omega) e^{iun} \, du && \text{by Theorem R.1 page 373} \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{iun} \, du && \text{by right hypothesis} \\
&= \bar{\delta}_n && \text{by definition of } \bar{\delta} && \text{(Definition L.12 page 278)}
\end{aligned}$$

[6] Hernández and Weiss (1996) page 50 ⟨Proposition 2.1.11⟩, Wojtaszczyk (1997) page 23 ⟨Corollary 2.9⟩, Igari (1996) pages 214–215 ⟨Lemma 9.2⟩, Pinsky (2002) page 306 ⟨Corollary 6.4.9⟩

4. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n) \rangle = 0 \implies \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

$$\begin{aligned}
\tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) &= \sum_{n\in\mathbb{Z}} \mathsf{R}_{\mathsf{fg}}(n) e^{-i\omega n} && \text{by definition of } \tilde{\mathsf{S}}_{\mathsf{fg}} && \text{(Definition R.3 page 373)}\\
&= \sum_{n\in\mathbb{Z}} \langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n) \rangle\, e^{-i\omega n} && \text{by definition of } \mathsf{R}_{\mathsf{fg}} && \text{(Definition R.1 page 373)}\\
&= \sum_{n\in\mathbb{Z}} 0 e^{-i\omega n} && \text{by left hypothesis}\\
&= 0
\end{aligned}$$

5. Proof that $\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n) \rangle = 0 \impliedby \tilde{\mathsf{S}}_{\mathsf{fg}}(\omega) = 0$:

$$\begin{aligned}
&\langle \mathsf{f}(x) \,|\, \mathsf{g}(x-n) \rangle\\
&= \left\langle \tilde{\mathbf{F}}\mathsf{f}(x) \,|\, \tilde{\mathbf{F}}\mathsf{g}(x-n) \right\rangle && \text{by \textit{unitary} property of } \tilde{\mathbf{F}} && \text{(Theorem K.3 page 259)}\\
&= \left\langle \tilde{\mathsf{f}}(\omega) \,|\, e^{-i\omega n}\tilde{\mathsf{g}}(\omega) \right\rangle && \text{by \textit{shift property} of } \tilde{\mathbf{F}} && \text{(Theorem K.4 page 259)}\\
&= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) e^{i\omega n} \tilde{\mathsf{g}}^*(\omega) \,\mathrm{d}\omega && \text{by definition of } \langle \triangle \,|\, \nabla \rangle && \text{(Definition E.1 page 185)}\\
&= \int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega) \tilde{\mathsf{g}}^*(\omega) e^{i\omega n} \,\mathrm{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} \tilde{\mathsf{f}}(\omega) \tilde{\mathsf{g}}^*(\omega) e^{i\omega n} \,\mathrm{d}\omega\\
&= \sum_{n\in\mathbb{Z}} \int_{0}^{2\pi} \tilde{\mathsf{f}}(u+2\pi n) \tilde{\mathsf{g}}^*(u+2\pi n) e^{i(u+2\pi n)n} \,\mathrm{d}u && \text{where } u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n\\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \left[2\pi \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(u+2\pi n) \tilde{\mathsf{g}}^*(u+2\pi n) \right] e^{iun} \cancelto{1}{e^{i2\pi nn}} \,\mathrm{d}u\\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{\mathsf{S}}_{\mathsf{fg}}(u) e^{iun} \,\mathrm{d}u && \text{by Theorem R.1 page 373}\\
&= \frac{1}{2\pi} \int_{0}^{2\pi} 0 \cdot e^{iun} \,\mathrm{d}u && \text{by right hypothesis}\\
&= 0
\end{aligned}$$

✏

APPENDIX S

POLLEN PARAMETERIZED WAVELETS

David Pollen showed that there are an uncountably infinite number of compactly supported scaling functions (and wavelets) available for coefficient length 4 and greater. This is stated in Theorem S.1, demonstrated in Example S.1 (page 381) and illustrated in Figure S.1 (page 382).

Theorem S.1 (Pollen parameterization theorem). [1]

THM

$$\check{\mathsf{h}}(z) = \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix} E(z^2)\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}$$

where

$$E(z) = \begin{cases} U_1 U_2 \cdots U_{2k} & \text{for } h_n \text{ of length } 4k+2,\ k \in \mathbb{W}\\ U_1 U_2 \cdots U_{2k+1} & \text{for } h_n \text{ of length } 4k+4,\ k \in \mathbb{W}\end{cases}$$

$$U(z) = \begin{bmatrix}\mathsf{u}(z) & \mathsf{v}(z)\\ -\mathsf{v}(z^{-1}) & \mathsf{u}(z^{-1})\end{bmatrix}$$

$$\mathsf{u}(z) = \frac{1}{2}\left[(1-\cos\theta)z + (1+\cos\theta)\right]$$

$$\mathsf{v}(z) = \frac{1}{2}\left[(-\sin\theta) + (\sin\theta)z^{-1}\right]$$

Example S.1 (Pollen length-4 scaling coefficients). [2]

EX

n	h_n
0	$\frac{\sqrt{2}}{4}(1-\cos\theta+\sin\theta)$
1	$\frac{\sqrt{2}}{4}(1+\cos\theta+\sin\theta)$
2	$\frac{\sqrt{2}}{4}(1+\cos\theta-\sin\theta)$
3	$\frac{\sqrt{2}}{4}(1-\cos\theta-\sin\theta)$

[1] Klappenecker (1996), page 3

[2] Klappenecker (1996), page 3, Burrus et al. (1998), page 66

✎PROOF:

$$\begin{aligned}
\check{\mathsf{h}}(z) &= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix} E(z^2)\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix} U(z^2)\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}\mathsf{u}(z^2) & \mathsf{v}(z^2)\\ -\mathsf{v}(z^{-2}) & \mathsf{u}(z^{-2})\end{bmatrix}\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\left[(1-\cos\theta)z^2+(1+\cos\theta)\right] & \frac{1}{2}\left[(-\sin\theta)+(\sin\theta)z^{-2}\right]\\ -\frac{1}{2}\left[(-\sin\theta)+(\sin\theta)z^2\right] & \frac{1}{2}\left[(1-\cos\theta)z^{-2}+(1+\cos\theta)\right]\end{bmatrix}\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{4}\begin{bmatrix}-\left[(1-\cos\theta)z^2+(1+\cos\theta)\right]+\left[(-\sin\theta)+(\sin\theta)z^2\right]\\ -\left[(-\sin\theta)+(\sin\theta)z^{-2}\right]+\left[(1-\cos\theta)z^{-2}+(1+\cos\theta)\right]\end{bmatrix}^T\begin{bmatrix}-z^{-1}\\ 1\end{bmatrix}\\
&= \frac{\sqrt{2}}{4}\Big[(1-\cos\theta)z+(1+\cos\theta)z^{-1}-(\sin\theta)z^{-1}+(\sin\theta)z\\
&\qquad +\sin\theta-(\sin\theta)z^{-2}+(1-\cos\theta)z^{-2}+(1+\cos\theta)\Big]\\
&= \frac{\sqrt{2}}{4}\Big[(1-\cos\theta+\sin\theta)z+(1+\cos\theta+\sin\theta)\\
&\qquad +(1+\cos\theta-\sin\theta)z^{-1}+(1-\cos\theta-\sin\theta)z^{-2}\Big]
\end{aligned}$$

✏

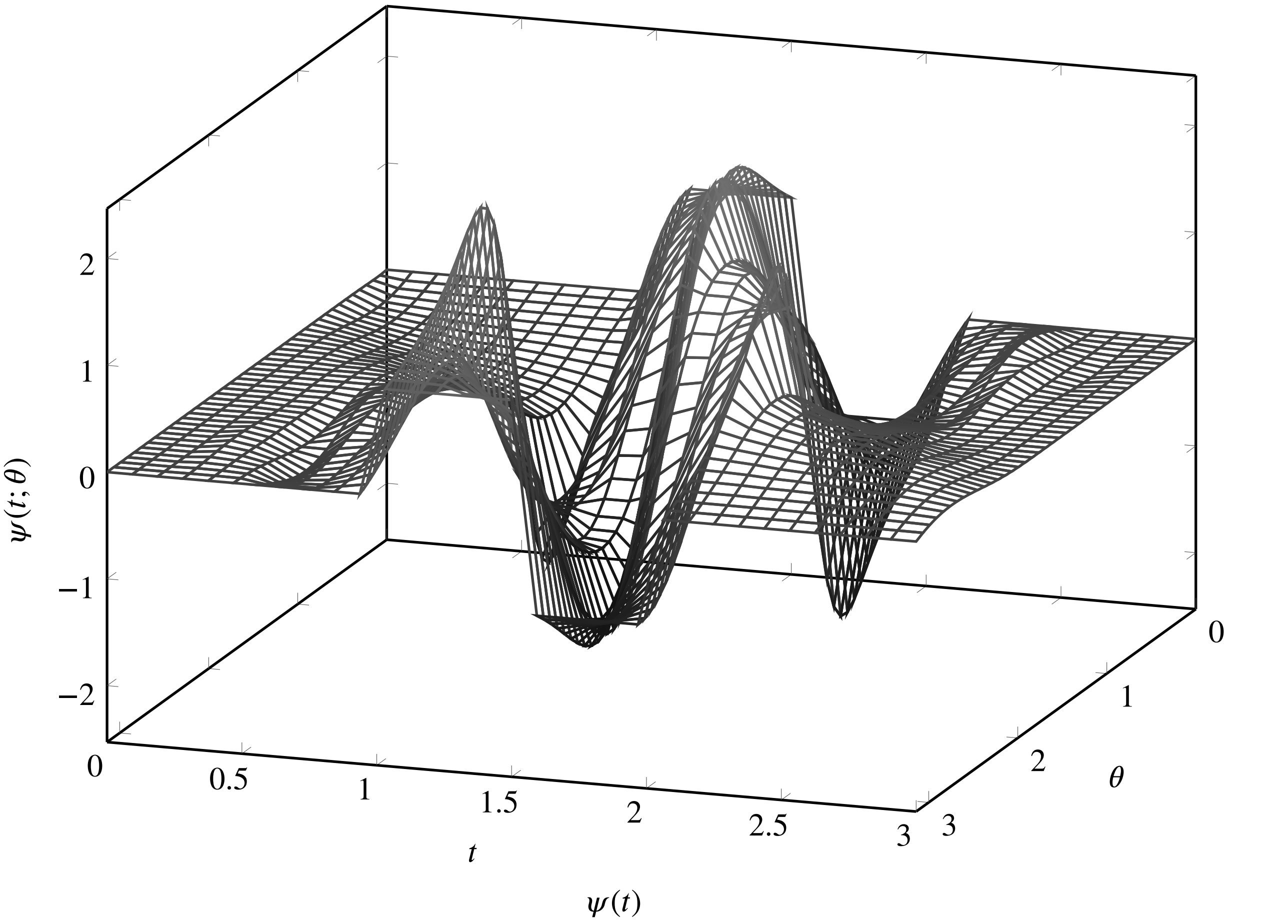


Figure S.1: Pollen length-4 wavelet function with varying parameter θ

Example S.2 (Pollen length-6 scaling coefficients). [3] Length-6 scaling coefficients have two parameters, here designated α and β. The scaling coefficients are as follows.

EX

n	h_n
0	$\frac{1}{4\sqrt{2}}[(1+\cos(\alpha)+\sin(\alpha))(1-\cos(\beta)-\sin(\beta))+2\sin(\beta)\cos(\alpha)]$
1	$\frac{1}{4\sqrt{2}}[(1-\cos(\alpha)+\sin(\alpha))(1+\cos(\beta)-\sin(\beta))-2\sin(\beta)\cos(\alpha)]$
2	$\frac{1}{2\sqrt{2}}[(1+\cos(\alpha-\beta)+\sin(\alpha-\beta)]$
3	$\frac{1}{2\sqrt{2}}[(1+\cos(\alpha-\beta)-\sin(\alpha-\beta)]$
4	$\frac{1}{\sqrt{2}}-h(0)-h(2)$
5	$\frac{1}{\sqrt{2}}-h(1)-h(3)$

[3] Burrus et al. (1998), page 66, Vidakovic (1999), page 95

APPENDIX T

FAST WAVELET TRANSFORM (FWT)

The Fast Wavelet Transform can be computed using simple discrete filter operations (as a conjugate mirror filter).

Definition T.1 (Wavelet Transform)**.** *Let the wavelet transform* $\mathbf{W} : \{f : \mathbb{R} \to \mathbb{C}\} \to \{w : \mathbb{Z}^2 \to \mathbb{C}\}$ *be defined as* [1]

DEF
$$[\mathbf{W}\mathsf{f}](j,n) \triangleq \left\langle \mathsf{f}(x) \,|\, \psi_{k,n}(x) \right\rangle$$

Definition T.2. *The following relations are defined as described below:*

DEF

scaling coefficients	$v_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$v_j(n)$	$\triangleq$	$\left\langle f(x) \,\vert\, \phi_{j,n}(x) \right\rangle$
wavelet coefficients	$w_j : \mathbb{Z} \to \mathbb{C}$	*such that*	$w_j(n)$	$\triangleq$	$\left\langle f(x) \,\vert\, \psi_{j,n}(x) \right\rangle$
scaling filter coefficients	$\bar{\mathsf{h}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{h}}(n)$	$\triangleq$	$\mathsf{h}(-n)$
wavelet filter coefficients	$\bar{\mathsf{g}} : \mathbb{Z} \to \mathbb{C}$	*such that*	$\bar{\mathsf{g}}(n)$	$\triangleq$	$\mathsf{g}(-n)$

The scaling and wavelet filter coefficients at scale j are equal to the filtered and downsampled (Theorem **??** page **??**) scaling filter coefficients at scale $j+1$:[2]

- The convolution (Definition O.3 page 341) of $v_{j+1}(n)$ with $\bar{\mathsf{h}}(n)$ and then downsampling by 2 produces $v_j(n)$.
- The convolution of $v_{j+1}(n)$ with $\bar{\mathsf{g}}(n)$ and then downsampling by 2 produces $w_j(n)$.

This is formally stated and proved in the next theorem.

---

[1] Notice that this definition is similar to the definition of transforms of other analysis systems:

	Laplace Transform	$\mathcal{L}f(s)$	$\triangleq$	$\langle \mathsf{f}(x) \,\vert\, e^{sx} \rangle$	$\triangleq$	$\int_x \mathsf{f}(x) e^{-sx} \,\mathrm{d}x$
	Continuous Fourier Transform	$\mathcal{F}f(\omega)$	$\triangleq$	$\left\langle \mathsf{f}(x) \,\vert\, e^{i\omega x} \right\rangle$	$\triangleq$	$\int_x \mathsf{f}(x) e^{-i\omega x} \,\mathrm{d}x$
	Fourier Series Transform	$\mathcal{F}_s f(k)$	$\triangleq$	$\left\langle \mathsf{f}(x) \,\vert\, e^{i\frac{2\pi}{T}kx} \right\rangle$	$\triangleq$	$\int_x \mathsf{f}(x) e^{-i\frac{2\pi}{T}kx} \,\mathrm{d}x$
	Z-Transform	$\mathcal{Z}f(z)$	$\triangleq$	$\langle \mathsf{f}(x) \,\vert\, z^n \rangle$	$\triangleq$	$\sum_n \mathsf{f}(x) z^{-n}$
	Discrete Fourier Transform	$\mathcal{F}_d f(k)$	$\triangleq$	$\left\langle f(n) \,\vert\, e^{i\frac{2\pi}{N}kn} \right\rangle$	$\triangleq$	$\sum_n \mathsf{f}(x) e^{-i\frac{2\pi}{N}kn}$

[2] Mallat (1999), page 257, Burrus et al. (1998), page 35

Theorem T.1.

$$\begin{aligned} v_j(n) &= [\bar{\mathsf{h}} \star v_{j+1}](2n) \\ w_j(n) &= [\bar{\mathsf{g}} \star v_{j+1}](2n) \end{aligned}$$

✎PROOF:

$$\begin{aligned}
v_j(n) &= \langle \mathsf{f}(x) \mid \phi_{j,n}(x) \rangle \\
&= \left\langle \mathsf{f}(x) \mid \sqrt{2^j}\phi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \mid \sqrt{2^j}\sqrt{2}\sum_m h(m)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \mid \sum_m h(j)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \mid \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m h(m) \left\langle \mathsf{f}(x) \mid \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m h(m) v_{j+1}(2n+m) \\
&= \sum_p h(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p-2n \\
&= \sum_p \bar{\mathsf{h}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{h}} \star v_{j+1}](2n)
\end{aligned}$$

$$\begin{aligned}
w_j(n) &= \langle \mathsf{f}(x) \mid \psi_{j,n}(x) \rangle \\
&= \left\langle \mathsf{f}(x) \mid \sqrt{2^j}\psi\left(2^j x - n\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \mid \sqrt{2^j}\sqrt{2}\sum_m \mathsf{g}(j)\phi\left(2(2^j x - n) - m\right) \right\rangle \\
&= \left\langle \mathsf{f}(x) \mid \sum_m \mathsf{g}(m)\sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \mid \sqrt{2^{j+1}}\phi\left(2^{j+1}x - 2n - m\right) \right\rangle \\
&= \sum_m \mathsf{g}(m) \left\langle \mathsf{f}(x) \mid \phi_{j+1,2n+m}(x) \right\rangle \\
&= \sum_m \mathsf{g}(m) v_{j+1}(2n+m) \\
&= \sum_p \mathsf{g}(p-2n) v_{j+1}(p) && \text{let } p = 2n+m \iff m = p-2n \\
&= \sum_p \bar{\mathsf{g}}(2n-p) v_{j+1}(p) \\
&= [\bar{\mathsf{g}} \star v_{j+1}](2n)
\end{aligned}$$

✏

These filtering and downsampling operations are equivalent to the operations performed by a filter bank. Therefore, a filter bank can be used to implement a Fast Wavelet Transform (FWT). This is illustrated in Figure T.1 page 387.

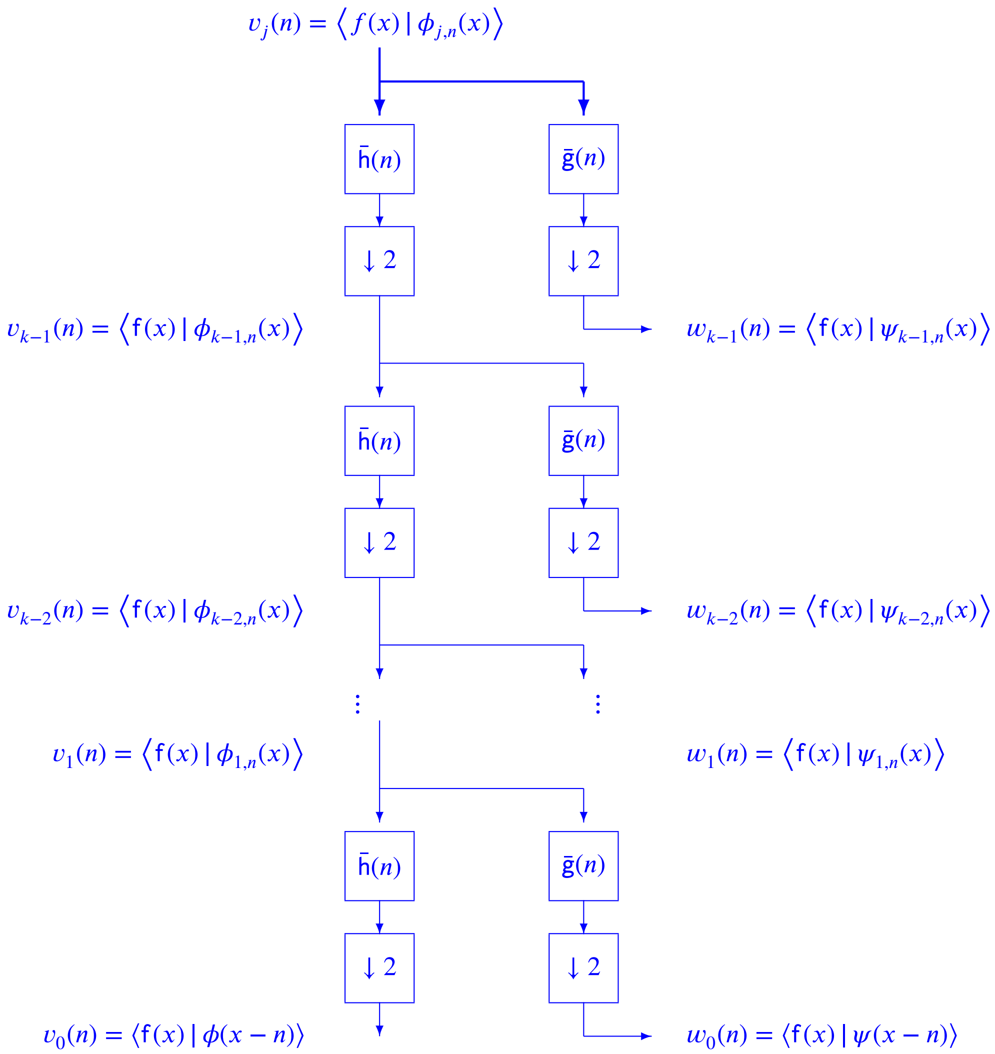


Figure T.1: k-Stage Fast Wavelet Transform

APPENDIX U

TRANSLATION SPACES

U.1 Translation

U.1.1 Definitions

Definition U.1. *Let X be a set and $\mathbf{I}$ be the identity operator on X.*

DEF

$\mathbf{T}_x$ *is a* ***translation operator*** *on X if*

1. $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$ $\quad \forall A \in 2^X$ (IDENTITY) and
2. $\mathbf{T}_x \mathbf{T}_y = \mathbf{T}_y \mathbf{T}_x$ $\quad \forall x,y \in X$ (COMMUTATIVE) and
3. $\mathbf{T}_x \bigcup_{i\in I} A_i = \bigcup_{i \in I} \mathbf{T}_x A_i$ $\quad \forall A,Y \in 2^X, x \in X$ (DISTRIBUTIVE over $\cup$) and
4. $\bigcup_{b\in B} \mathbf{T}_b A = \bigcup_{a \in A} \mathbf{T}_a B$ $\quad \forall A,B \in 2^X$ and
5. $\mathbf{T}_x (A \cap B) = (\mathbf{T}_x A) \cap (\mathbf{T}_x B)$ $\quad \forall A,B \in 2^X, x \in X$ and
6. $\mathbf{T}_x (A^{\mathsf{c}}) = \mathsf{c}(\mathbf{T}_x A)$ $\quad \forall A,B \in 2^X, x \in X$.

The pair $(X, \mathbf{T})$ is a ***translation space*** *on X.*

Definition U.2. [1] *Let X be a set on which is defined the translation opertor $\mathbf{T}_x$.* ***Minkowski addition*** *$\oplus$ and* ***Minkowski subtraction*** *$\ominus$ is defined as follows:*

DEF

$$A \oplus B = \bigcup_{b\in B} \mathbf{T}_b A \qquad \forall A,B \in 2^X \qquad \text{(MINKOWSKI ADDITION)}$$

$$A \ominus B = \bigcap_{b\in B} \mathbf{T}_b A \qquad \forall A,B \in 2^X \qquad \text{(MINKOWSKI SUBTRACTION)}$$

Theorem U.1 (next) shows a relationship between Minkowski addition and Minkowski subtraction.

Theorem U.1 (de Morgan relations). [2] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\mathsf{c}(A \oplus B) = A^{\mathsf{c}} \ominus B \qquad \forall A,B \in 2^X$$

$$\mathsf{c}(A \ominus B) = A^{\mathsf{c}} \oplus B \qquad \forall A,B \in 2^X$$

[1] Matheron (1975) page 17
Lay (1982) page 7

[2] Pitas and Venetsanopoulos (1991), page 159

✎PROOF:

$$\begin{aligned}
\mathsf{c}(A \oplus B) &= \mathsf{c}\left(\bigcup_{b\in B} \mathbf{T}_b A\right) && \text{by Definition U.2 page 389}\\
&= \bigcap_{b\in B} \mathsf{c}\left(\mathbf{T}_b A\right) && \text{by Demorgan relation page 389}\\
&= \bigcap_{b\in B} \mathbf{T}_b\left(A^{\mathsf{c}}\right) && \text{by Definition U.1 page 389}\\
&= A^{\mathsf{c}} \ominus B && \text{by Theorem U.2 page 392}
\end{aligned}$$

$$\begin{aligned}
\mathsf{c}(A \ominus B) &= \mathsf{c}\left(\bigcap_{b\in B} \mathbf{T}_b A\right) && \text{by Definition U.2 page 389}\\
&= \bigcup_{b\in B} \mathsf{c}\left(\mathbf{T}_b A\right) && \text{by Demorgan relation page 389}\\
&= \bigcup_{b\in B} \mathbf{T}_b\left(A^{\mathsf{c}}\right) && \text{by Definition U.1 page 389}\\
&= A^{\mathsf{c}} \oplus B && \text{by Theorem U.2 page 392}
\end{aligned}$$

✏

U.1.2 Examples

Example U.1 (Translation on groups). [3] Let $\oplus$ be the Minkowski addition operator defined in terms of the *translation operator* $\mathbf{T}$. Let $(X, +)$ be a *group*.

EX
$$\left\{ \mathbf{T}_x A \triangleq \{a + x | a \in A\} \quad \forall A \in 2^X \right\} \implies \left\{ \begin{array}{ll} \mathbf{T}_x \text{ is a translation operator} & \text{and} \\ A \oplus B = \left\{a + b | a \in A \text{ and } b \in B\right\} & \forall A, B \in 2^X \end{array} \right\}$$

✎PROOF:

1. Proof that $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$:

$$\begin{aligned}
\mathbf{T}_0 A &= \{a + 0 | a \in A\} && \text{by definition of } \mathbf{T}_x\\
&= \{a | a \in A\} && \text{by additive identity property of groups}\\
&= A
\end{aligned}$$

2. Proof that $\mathbf{T}_x\mathbf{T}_y = \mathbf{T}_y\mathbf{T}_x$:

$$\begin{aligned}
\mathbf{T}_x\mathbf{T}_y A &= \mathbf{T}_x \{a + y | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a + y + x | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a + x + y | a \in A\} && \text{by commutative property of groups}\\
&= \mathbf{T}_y \{a + x | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \mathbf{T}_y\mathbf{T}_x \{a | a \in A\} && \text{by definition of } \mathbf{T}_x
\end{aligned}$$

[3] Matheron (1975) pages 16–17
Pitas and Venetsanopoulos (1991) page 159
Lay (1982) page 7

3. Proof that $\mathbf{T}_x \bigcup_{i\in I} A_i = \bigcup_{i\in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcup_i A_i &= \left\{ y + x | y \in \bigcup_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y + x | \bigvee_i y \in A_i \right\} \\
&= \bigcup_i \{ y + x | y \in A_i \} \\
&= \bigcup_i \mathbf{T}_x \{ y | y \in A_i \} \\
&= \bigcup_i \mathbf{T}_x A
\end{aligned}$$

4. Proof that $\bigcup_{b\in B} \mathbf{T}_b A = \bigcup_{a\in A} \mathbf{T}_a B$:

$$\begin{aligned}
\bigcup_{b\in B} \mathbf{T}_b A &= \bigcup_{b\in B} \{ a + b | a \in A \} && \text{by definition of } \mathbf{T}_x \\
&= \{ a + b | a \in A \text{ and } b \in B \} \\
&= \{ b + a | b \in B \text{ and } a \in A \} \\
&= \bigcup_{a\in A} \{ b + a | b \in B \} \\
&= \bigcup_{a\in A} \mathbf{T}_a B
\end{aligned}$$

5. Proof that $\mathbf{T}_x \bigcap_{i\in I} A_i = \bigcap_{i\in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcap_i A_i &= \left\{ y + x | y \in \bigcap_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y + x | \bigwedge_i y \in A_i \right\} \\
&= \bigcap_i \{ y + x | y \in A_i \} \\
&= \bigcap_i \mathbf{T}_x \{ y | y \in A_i \} \\
&= \bigcap_i \mathbf{T}_x A
\end{aligned}$$

6. Proof that $\mathbf{T}_x \left(A^{\mathsf{c}} \right) = \mathsf{c} \left(\mathbf{T}_x A \right)$:

$$\begin{aligned}
\mathbf{T}_x \mathsf{c} A &= \mathbf{T}_x \{ a | a \in A^{\mathsf{c}} \} \\
&= \{ a + x | a \in A^{\mathsf{c}} \} \\
&= \{ a + x | a \notin A \} \\
&= \{ a + x | \neg(a \in A) \} \\
&= \mathsf{c} \{ a + x | a \in A \} \\
&= \mathsf{c} \mathbf{T}_x A
\end{aligned}$$

$$\begin{aligned}
A \oplus B &= \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition U.2 page 389} \\
&= \{ a + b | a \in A \text{ and } b \in B \} && \text{by Definition U.1 page 389}
\end{aligned}$$

✏

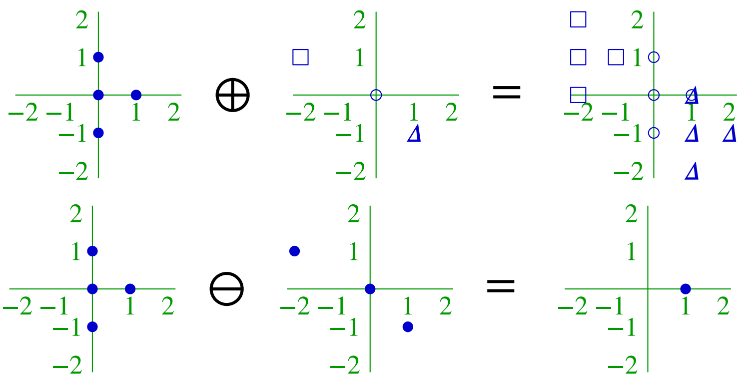


Figure U.1: Illustration for Example U.2 (page 392)

Example U.2. Let

$$\begin{aligned} A &\triangleq \{(0,0),\ (0,1),\ (0,-1),\ (1,1)\} \\ B &\triangleq \{(0,0),\ (-2,1),\ \ (1,-1)\} \end{aligned}$$

Then

$$\begin{aligned} A \oplus B &= \{(0,0),\ (0,1),\ (0,-1),\ (1,1),\ (-2,1),\ (-2,2),\ (-2,0),\ (-1,2),\ (1,-1),\ (1,-2),\ (2,0)\} \\ A \ominus B &= \{(1,0)\} \end{aligned}$$

These relationships are illustrated in Figure U.1 (page 392).

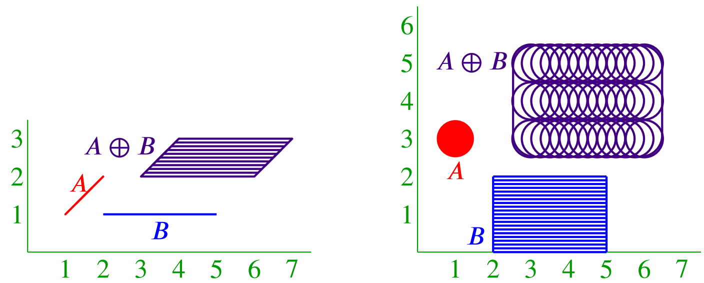


Figure U.2: Illustration for Example U.3 page 392

Example U.3. [4] Two more examples are illustrated in Figure U.2 (page 392).

U.1.3 Additive properties

Theorem U.2. [5] *Let $(X,+)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

THM

$$\begin{array}{rcllr} A \oplus \{0\} &=& A & \forall A\subseteq X & \\ A \oplus B &=& B \oplus A & \forall A,B\subseteq X & \text{(commutative)} \\ A \oplus (B \oplus C) &=& (A \oplus B) \oplus C & \forall A,B,C\subseteq X & \text{(associative)} \\ \mathbf{T}_x(A \oplus B) &=& \left(\mathbf{T}_x A\right) \oplus B & \forall A,B\subseteq X,\ x\in X & \text{(translation invariant)} \end{array}$$

[4] Lay (1982) page 7

[5] Pitas and Venetsanopoulos (1991), pages 163–164

✎PROOF:

$$\begin{aligned}
A \oplus \{0\} &= A \oplus B|_{B=\{0\}} \\
&= \left. \bigcup_{b\in B} \mathbf{T}_b A \right|_{B=\{0\}} && \text{by Definition U.2 page 389} \\
&= \mathbf{T}_0 A \\
&= A && \text{by Definition U.1 page 389}
\end{aligned}$$

$$\begin{aligned}
A \oplus B &= \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition U.2 page 389} \\
&= \bigcup_{a\in A} \mathbf{T}_a B && \text{by Definition U.1 page 389} \\
&= B \oplus A && \text{by Definition U.2 page 389}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \oplus C) &= \bigcup_{y\in B\oplus C} \mathbf{T}_y A && \text{by Definition U.2 page 389} \\
&= \bigcup_{a\in A} \mathbf{T}_a (B \oplus C) && \text{by Definition U.1 page 389} \\
&= \bigcup_{a\in A} \mathbf{T}_a \left(\bigcup_{c\in C} \mathbf{T}_c B \right) && \text{by Definition U.2 page 389} \\
&= \bigcup_{a\in A} \left(\bigcup_{c\in C} \mathbf{T}_a \mathbf{T}_c B \right) && \text{by Definition U.1 page 389} \\
&= \bigcup_{a\in A} \left(\bigcup_{c\in C} \mathbf{T}_c \mathbf{T}_a B \right) && \text{by Definition U.1 page 389} \\
&= \bigcup_{c\in C} \mathbf{T}_c \left(\bigcup_{a\in A} \mathbf{T}_a B \right) && \text{by Definition U.1 page 389} \\
&= \bigcup_{c\in C} \mathbf{T}_c \left(\bigcup_{b\in B} \mathbf{T}_b A \right) && \text{by Definition U.1 page 389} \\
&= \bigcup_{c\in C} \mathbf{T}_c (A \oplus B) && \text{by Definition U.2 page 389} \\
&= (A \oplus B) \oplus C && \text{by Definition U.2 page 389}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_x (A \oplus B) &= \mathbf{T}_x \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition U.2 page 389} \\
&= \bigcup_{b\in B} \mathbf{T}_x \mathbf{T}_b A && \text{by Definition U.1 page 389} \\
&= \bigcup_{b\in B} \mathbf{T}_b \mathbf{T}_x A && \text{by Definition U.1 page 389} \\
&= \left(\mathbf{T}_x A \right) \oplus B && \text{by Definition U.2 page 389}
\end{aligned}$$

✏

Theorem U.3. [6] *Let $(X, +)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

[6] Pitas and Venetsanopoulos (1991), page 163

THM

$$\begin{array}{rcll}
A \oplus (B \cup C) &=& (A \oplus B) \cup (A \oplus C) & \forall A,B,C \subseteq X \quad (\oplus \text{ is LEFT DISTRIBUTIVE } over \cup)\\
(A \cup B) \oplus C &=& (A \oplus C) \cup (B \oplus C) & \forall A,B,C \subseteq X \quad (\oplus \text{ RIGHT DISTRIBUTIVE } over \cup)\\
A \oplus (B \cap C) &\subseteq& (A \oplus B) \cap (A \oplus C) & \forall A,B,C \subseteq X\\
(A \cap B) \oplus C &\subseteq& (A \oplus C) \cap (B \oplus C) & \forall A,B,C \subseteq X
\end{array}$$

✎PROOF:

$$\begin{aligned}
(A \cup B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c (A \cup B) && \text{by Definition U.2 page 389}\\
&= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cup (\mathbf{T}_c B) \right] && \text{by Definition U.1 page 389}\\
&= \left(\bigcup_{c \in C} \mathbf{T}_c A \right) \cup \left(\bigcup_{c \in C} \mathbf{T}_c B \right)\\
&= (A \oplus C) \cup (B \oplus C) && \text{by Definition U.2 page 389}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \cup C) &= (B \cup C) \oplus A && \text{by Theorem U.2 page 392}\\
&= (B \oplus A) \cup (C \oplus A) && \text{by previous result}\\
&= (A \oplus B) \cup (A \oplus C) && \text{by Theorem U.2 page 392}
\end{aligned}$$

$$\begin{aligned}
(A \cap B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c (A \cap B) && \text{by Theorem U.2 page 392}\\
&= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cap (\mathbf{T}_c B) \right] && \text{by Definition U.1 page 389}\\
&\subseteq \left(\bigcup_{c \in C} \mathbf{T}_c A \right) \cap \left(\bigcup_{c \in C} \mathbf{T}_c B \right) && \text{by minimax inequality}\\
&= (A \oplus C) \cap (B \oplus C) && \text{by Theorem U.2 page 392}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \cap C) &= (B \cap C) \oplus A && \text{by Theorem U.2 page 392}\\
&\subseteq (B \oplus A) \cap (C \oplus A) && \text{by previous result}\\
&= (A \oplus B) \cap (A \oplus C) && \text{by Theorem U.2 page 392}
\end{aligned}$$

✏

U.1.4 Subtractive properties

Theorem U.4. [7] *Let $(X, +)$ be a group with with Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\begin{array}{rrcll}
 & A \ominus \{0\} &=& A & \forall A \subseteq X\\
 & A \ominus B &=& B^c \ominus A^c & \forall A,B \subseteq X\\
 & \mathbf{T}_x (A \ominus B) &=& (\mathbf{T}_x A) \ominus B & \forall A,B \subseteq X,\ x \in X \quad (\text{TRANSLATION INVARIANT})\\
A \subseteq B \implies & A \ominus C &\subseteq& B \ominus C & \forall A,B,C \subseteq X \quad (\text{INCREASING})
\end{array}$$

[7] Pitas and Venetsanopoulos (1991), pages 164–165

✎PROOF:

$$\begin{aligned}
A \ominus \{0\} &= \mathsf{c}(A^{\mathsf{c}} \oplus \{0\}) && \text{by Theorem U.1 page 389}\\
&= \mathsf{c}(A^{\mathsf{c}}) && \text{by Theorem U.2 page 392}\\
&= A
\end{aligned}$$

$$\begin{aligned}
A \ominus B &= \mathsf{cc}(A \ominus B)\\
&= \mathsf{c}(A^{\mathsf{c}} \oplus B) && \text{by Theorem U.1 page 389}\\
&= \mathsf{c}(B \oplus A^{\mathsf{c}}) && \text{by Theorem U.2 page 392}\\
&= B^{\mathsf{c}} \ominus A^{\mathsf{c}} && \text{by Theorem U.1 page 389}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_x(A \ominus B) &= \mathbf{T}_x\mathsf{c}\left(A^{\mathsf{c}} \oplus B\right) && \text{by Theorem U.1 page 389}\\
&= \mathsf{c}\mathbf{T}_x\left(A^{\mathsf{c}} \oplus B\right) && \text{by Definition U.1 page 389}\\
&= \mathsf{c}\left(\mathbf{T}_x A^{\mathsf{c}} \oplus B\right) && \text{by Theorem U.2 page 392}\\
&= \mathsf{c}\left(\mathsf{c}\mathbf{T}_x A \oplus B\right) && \text{by Definition U.1 page 389}\\
&= \mathbf{T}_x A \ominus B && \text{by Theorem U.1 page 389}
\end{aligned}$$

$$\begin{aligned}
A \ominus C &= \bigcap_{c\in C} A_c && \text{by Theorem U.2 page 392}\\
&\subseteq \bigcap_{c\in C} B_c && \text{by } A \subseteq B \text{ hypothesis}\\
&= B \ominus C && \text{by Definition U.2 page 389}
\end{aligned}$$

✏

Theorem U.5. [8] *Let $(X, +)$ be a group with with Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\begin{array}{rcllr}
A \ominus (B \cup C) &=& (A \ominus B) \cap (A \ominus C) & \forall A,B,C \subseteq X & (\ominus \text{ LEFT DISTRIBUTIVE } \textit{over} \cup)\\
(A \cap B) \ominus C &=& (A \ominus C) \cap (B \ominus C) & \forall A,B,C \subseteq X & (\ominus \text{ RIGHT DISTRIBUTIVE } \textit{over} \cap)\\
(A \cup B) \ominus C &\supseteq& (A \ominus C) \cup (B \ominus C) & \forall A,B,C \subseteq X &\\
A \ominus (B \cap C) &\supseteq& (A \ominus B) \cup (A \ominus C) & \forall A,B,C \subseteq X &
\end{array}$$

✎PROOF:

$$\begin{aligned}
A \ominus (B \cup C) &= \mathsf{cc}\Big[\, A \ominus (B \cup C) \,\Big]\\
&= \mathsf{c}\Big[\, A^{\mathsf{c}} \oplus (B \cup C) \,\Big] && \text{by Theorem U.1 page 389}\\
&= \mathsf{c}\Big[\, (A^{\mathsf{c}} \oplus B) \cup (A^{\mathsf{c}} \oplus C) \,\Big] && \text{by Theorem U.3 page 393}\\
&= \left[\mathsf{c}\left(A^{\mathsf{c}} \oplus B\right)\right] \cap \left[\mathsf{c}\left(A^{\mathsf{c}} \oplus C\right)\right] && \text{by Demorgan relation page 389}\\
&= (A \ominus B) \cap (A \ominus C) && \text{by Theorem U.1 page 389}
\end{aligned}$$

$$\begin{aligned}
(A \cap B) \ominus C &= \mathsf{c}[(A \cap B) \ominus C]\\
&= \mathsf{c}[\mathsf{c}(A \cap B) \oplus C] && \text{by Theorem U.1 page 389}\\
&= \mathsf{c}\left[(A^{\mathsf{c}} \cup B^{\mathsf{c}}) \oplus C\right]\\
&= \mathsf{c}\left[(A^{\mathsf{c}} \oplus C) \cup (B^{\mathsf{c}} \oplus C)\right] && \text{by Theorem U.3 page 393}\\
&= \mathsf{c}(A^{\mathsf{c}} \oplus C) \cap \mathsf{c}(B^{\mathsf{c}} \oplus C)\\
&= (A \ominus C) \cap (B \ominus C) && \text{by Theorem U.1 page 389}
\end{aligned}$$

[8] Pitas and Venetsanopoulos (1991), page 165

$$\begin{aligned}
A \ominus (B \cap C) &= \mathsf{cc}\left[\, A \ominus (B \cap C) \,\right] \\
&= \mathsf{c}\left[\, A^{\mathsf{c}} \oplus (B \cap C) \,\right] && \text{by Theorem U.1 page 389} \\
&\supseteq \mathsf{c}\left[(A^{\mathsf{c}} \oplus B) \cap (A^{\mathsf{c}} \oplus C)\right] && \text{by Theorem U.3 page 393} \\
&= \left[\mathsf{c}(A^{\mathsf{c}} \oplus B)\right] \cup \left[\mathsf{c}(A^{\mathsf{c}} \oplus C)\right] && \text{by Demorgan relation page 389} \\
&= (A \ominus B) \cup (A \ominus C) && \text{by Theorem U.1 page 389}
\end{aligned}$$

$$\begin{aligned}
(A \cup B) \ominus C &= \mathsf{cc}[(A \cup B) \ominus C] \\
&= \mathsf{c}[\mathsf{c}(A \cup B) \oplus C] && \text{by Theorem U.1 page 389} \\
&= \mathsf{c}\left[\left(A^{\mathsf{c}} \cap B^{\mathsf{c}}\right) \oplus C\right] && \text{by Demorgan relation page 389} \\
&\supseteq \mathsf{c}\left[\left(A^{\mathsf{c}} \oplus C\right) \cap \left(B^{\mathsf{c}} \oplus C\right)\right] \\
&= \mathsf{c}\left(A^{\mathsf{c}} \oplus C\right) \cup \mathsf{c}\left(B^{\mathsf{c}} \oplus C\right) \\
&= (A \ominus C) \cup (B \ominus C) && \text{by Theorem U.1 page 389}
\end{aligned}$$

⇨

Theorem U.6. [9] *Let $(X, +)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM	$A \ominus (B \oplus C)$	$=$	$(A \ominus B) \ominus C$	$\forall A,B,C \subseteq X$
	$A \oplus (B \ominus C)$	$\subseteq$	$(A \oplus B) \ominus C$	$\forall A,B,C \subseteq X$

✎PROOF:

$$\begin{aligned}
A \ominus (B \oplus C) &= \mathsf{cc}\left[\, A \ominus (B \oplus C) \,\right] \\
&= \mathsf{c}\left[\, A^{\mathsf{c}} \oplus (B \oplus C) \,\right] && \text{by Theorem U.1 page 389} \\
&= \mathsf{c}\left[\, (A^{\mathsf{c}} \oplus B) \oplus C \,\right] && \text{by Theorem U.2 page 392} \\
&= \mathsf{c}(A^{\mathsf{c}} \oplus B) \ominus C && \text{by Theorem U.1 page 389} \\
&= (A \ominus B) \ominus C && \text{by Theorem U.1 page 389}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \ominus C) &= A \oplus \left(\bigcap_{c \in C} \mathbf{T}_c B\right) && \text{by Definition U.2 page 389} \\
&= \left(\bigcap_{c \in C} \mathbf{T}_c B\right) \oplus A && \text{by Theorem U.2 page 392} \\
&= \bigcup_{a \in A} \mathbf{T}_a \left(\bigcap_{c \in C} \mathbf{T}_c B\right) && \text{by Definition U.2 page 389} \\
&= \bigcup_{a \in A} \bigcap_{c \in C} \mathbf{T}_a \mathbf{T}_c B && \text{by Definition U.1 page 389} \\
&\subseteq \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_a \mathbf{T}_c B && \text{by minimax inequality} \\
&= \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_c \mathbf{T}_a B && \text{by Definition U.1 page 389}
\end{aligned}$$

[9] Pitas and Venetsanopoulos (1991), page 166

$$
\begin{aligned}
&= \bigcap_{c\in C} \mathbf{T}_c\left(\bigcup_{a\in A} \mathbf{T}_a B\right) && \text{by Definition U.1 page 389}\\
&= \bigcap_{c\in C} \mathbf{T}_c(B \oplus A) && \text{by Definition U.2 page 389}\\
&= (B \oplus A) \ominus C && \text{by Definition U.2 page 389}\\
&= (A \oplus B) \ominus C && \text{by Theorem U.2 page 392}
\end{aligned}
$$

✏

U.2 Operations

Definition U.3. [10] *Let $(X,+)$ be a group.*

DEF: *The* ***symmetric set*** *of A is the set* $\check{A} \triangleq -A \qquad \forall A\subseteq X$

Definition U.4. [11] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and D^s be the symmetric set of set D.*

DEF:
The ***dilation*** *of A by D is the operation* $A \oplus \check{D} \qquad \forall A,D\subseteq X.$
The ***erosion*** *of A by E is the operation* $A \ominus \check{E} \qquad \forall A,E\subseteq X.$

Definition U.5. [12] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and B^s be the symmetric set of a set B.*

DEF:
The ***opening*** *of A with respect to B is the set* $A_B \triangleq \underbrace{\underbrace{(A \ominus \check{B})}_{\text{erosion}} \oplus B}_{\text{dilation}} \qquad \forall A,B\subseteq X.$

The ***closing*** *of A with respect to B is the set* $A^B \triangleq \underbrace{\underbrace{(A \oplus \check{B})}_{\text{dilation}} \ominus B}_{\text{erosion}} \qquad \forall A,B\subseteq X.$

Theorem U.7. [13] *Let $(X,+)$ be a group with A_B representing the opening of a set A with respect to a set B and A^B representing the closing of a set A with respect to a set B.*

THM:
(complement of the opening)→ $\mathsf{c}(A_B) = (A^{\mathsf{c}})^B$ *←(closing of the complement)* $\forall A,B\subseteq X$
(complement of the closing)→ $\mathsf{c}(A^B) = (A^{\mathsf{c}})_B$ *←(opening of the complement)* $\forall A,B\subseteq X$

✎PROOF:

$$
\begin{aligned}
\mathsf{c}(A_B) &= \mathsf{c}\left[\,(A \ominus \check{B}) \oplus B\,\right] && \text{by Definition U.5 page 397}\\
&= \mathsf{c}(A \ominus \check{B}) \ominus B && \text{by Theorem U.1 page 389}\\
&= \mathsf{c}(A \ominus \check{B}) \ominus B && \text{by Theorem U.1 page 389}\\
&= (A^{\mathsf{c}} \oplus \check{B}) \ominus B && \text{by Theorem U.1 page 389}\\
&= (A^{\mathsf{c}})^B && \text{by Definition U.5 page 397}
\end{aligned}
$$

---

[10] Matheron (1975), page 17
[11] Pitas and Venetsanopoulos (1991), page 161
[12] Serra (1982), page 50
[13] Serra (1982), page 51

$$
\begin{aligned}
\mathsf{c}(A^B) &= \mathsf{c}\left[\,(A \oplus \breve{B}) \ominus B\,\right] && \text{by Definition U.5 page 397}\\
&= \mathsf{c}(A \oplus \breve{B}) \oplus B && \text{by Theorem U.1 page 389}\\
&= \mathsf{c}(A \oplus \breve{B}) \oplus B && \text{by Theorem U.1 page 389}\\
&= (A^\mathsf{c} \ominus \breve{B}) \oplus B && \text{by Theorem U.1 page 389}\\
&= (A^\mathsf{c})_B && \text{by Definition U.5 page 397}
\end{aligned}
$$

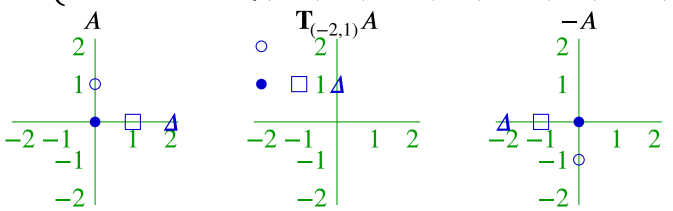
✏

Example U.4.

EX

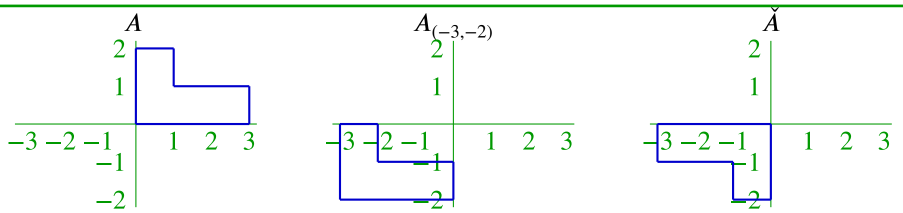
$$A \triangleq \{(0,0)\,,\ (1,0)\,,\ (2,0)\,,\ (0,1)\} \implies$$

$$\left\{\begin{array}{lcl} A_{(-2,1)} &=& \{(-2,1)\,,\ (-1,1)\,,\ (0,1)\,,\ (-2,2)\} \quad \text{and} \\ \breve{A} &=& \{(0,0)\,,\ (-1,0)\,,\ (-2,0)\,,\ (0,-1)\} \end{array}\right\}$$



Example U.5. An example similar to Example U.4 (page 398) but using solid shapes is illustrated next:

EX



APPENDIX V

SOURCE CODE

The source code in this appendix for *GNU Octave*. [1] Octave is similar to *MatLab* with some differences:

1. GNU Octave is free.
2. GNU Octave is open-source.
3. GNU Octave uses a separate graphics engine called *GNU-Plot* for all graphing.

Octave code can easily be adapted to MatLab code and vice-versa.

V.1 GNU Octave script file

```
printf(" ========================================================= \n");
printf("| Daniel J. Greenhoe                                       |\n");
printf("| http://banyan.cm.nctu.edu.tw/~dgreenhoe/wsd/index.html  |\n");
printf("| Experiments with odd polynomials R(1/2-y)               |\n");
printf(" ========================================================= \n");

%-----------------------------------------------------------
% References:
%
%    http://banyan.cm.nctu.edu.tw/~dgreenhoe/wsd/index.html
%
%    C. Sidney Burrus, Ramesh A. Gopinath, and Haitao Guo
%    Introduction to Wavelets and Wavelet Transforms; A Primer
%    Prentice Hall, 1998
%    ISBN 0-13-489600-9
%-----------------------------------------------------------

%======================================
% Initialization
%======================================
clear;

%======================================
% Function Declarations
%======================================

%--------------------------------------------------------------------------
%function: downsample x(n) by L
%--------------------------------------------------------------------------
```

[1] *GNU Octave*: http://www.octave.org/

```
function y = downSample(x,L)
   n = length(x);
   m = ceil(n/L);
   y = 0*x(1:m);
   for i=0:(m-1)
      y(i+1) = x(i*L+1);
   endfor
endfunction

%------------------------------------------------------------------------
%function: upsample x(n) by L
%------------------------------------------------------------------------
function y = upSample(x,L)
   n = length(x);
   m = n*L;
   y = zeros(1,m);
   for i=0:(n -1)
      y(i*L+1) = x(i+1);
   endfor
endfunction

%------------------------------------------------------------------------
%function: Generate scaling function from scaling filter coefficients h(n)
%   p = scaling function
%   h = scaling filter coefficients h(n)
%   k = number of iterations
%   d = density = number phi(t) samples per h(n) sample
%
% reference: Burrus page 67
%     MatLab code:    Appendix C page 258
%     Key equation:   page 67
%
% reference: Rao
%
%------------------------------------------------------------------------
function p = gen_phi(h,k,d)
   h = 2*h/sum(h);              %scale to sqrt(2)*sqrt(2)
   m =  length(h)-1;            %
   p = ones(1,d*m)/m;           %<p,1>=1    ref: Rao p.53
   hu = upSample(h,d);          %upsample h(n) to match phi(t) density
   printf('\n');
   for i = 0:(k-1)              %iterate
      printf('%d ',i);
      fflush(stdout);
      ph  = conv(hu,p);         %convolve
      p   = downSample(ph,2);   %downsample
   endfor
   p = p(1:m*d);
   printf('\n');
endfunction

%------------------------------------------------------------------------
%function: Generate wavelet function psi(t) from scaling function phi(t)
%          and wavelet filter coefficients g(n)
%   phi = scaling function phi(t)
%   g = wavelet filter coefficients g(n)
%   d = density = number phi(t) samples per g(n) sample
% reference: Burrus page 15
%
%------------------------------------------------------------------------
function psi = gen_psi(phi,g,d)
   g   = sqrt(2)*g;             %scale to sqrt(2)*sqrt(2)
   gu  = upSample(g,d);         %upsample h(n) to match phi(t) density
   pg  = conv(gu,phi);          %convolve
   psi = downSample(pg,2);      %downsample
   m =  length(g)-1;            %
   psi = psi(1:m*d);            %truncate
endfunction

%------------------------------------------------------------------------
%function: Generate wavelet coefficients g(n) from scaling coefficients h(n)
% references:
%     Mallat page 238
%     Burrus page  15
%     Burrus page  79
%------------------------------------------------------------------------
function g = h2g_coefs(h)
```

```
   g = h;
   N = length(h);
   for n=0:(N-1)
      g(n +1) = (-1)**(n)*h(N-1-n +1);
   endfor
endfunction

%----------------------------------------------------------------------------
%function: Generate filter coefficients from wavelet/scalar coefficients
%    _
%   h(n) = h(-n) with shift by N where N = length of h(n)
%    _
%   g(n) = g(-n) with shift by N
%----------------------------------------------------------------------------
function hbar = h2hbar(h)
   hbar = zeros(size(h));           %allocate memory
   N    = length(h)      ;          %N = length of h(n)
   for n = 0:(N-1)                  %
      hbar(n +1) = h(N-1-n +1);     %hbar(n) = h(N-1-n)
   endfor
endfunction

%----------------------------------------------------------------------------
%                                                       -z + 2 -1/z
% Convert polynomial in y to polynomial in y = -------------
%                                                            4
%                                   /              \
%          |                        |   -z + 2 -1/z  |
%     P(y)|                    = P|  -------------  |   = Q(z)Q(1/z)
%          |y=[(-z+2-1/z)/4]        \        4       /
%
% Input
% ------
%  P = [p_{n-1} ... p_2 p_1 p_0]
%     ==> P(y)=p_{n-1}y^{m-1}+...+p_2 y^2+p_1y+p_0
%
% Output
% ------
%  QQ = [q_{2n-2} ... q_2 q_1 q_0]
%
%                     q_{2n-2}z^{2n-2} +...+ q_2 z^2 + r_1 z + r_0
%      ==>  Q(z)Q(1/z)=------------------------------------------------
%                                          z^n
%
% Theory
% ------
% This conversion is useful when using polynomials in y = sin^2(w/2).
% Polynomials in sin^2(w/2) can be used to represent any even
% periodic funtion with period 2pi.
%    * sin^2(w/2) = (1/2)(1-cosw)
%
%                        1-cosw      1      e^{w}+e^{-w}      2 - z - 1/z     |
% y = sin^2(w/2) = --------- = --- - -------------- = -------------- |
%                           2        2           4                  4        |z=e^w
%
%----------------------------------------------------------------------------
function QQ = y2sin2(P)
   n=length(P);                     % number of terms in P(y). num 0s=n-1
   N=2*n-1;                         % number of terms in Q(z)Q(1/z)
   QQ= zeros(1,N);                  % init Q(z)Q(1/z) = 0+0z+...+0z^{N-1}
   q=1;                             % init q(z)       = 1
   for k=0:n-1                      %
     QQ = [QQ,0](2:N+1);            % z[-z + 2 -1/z] = -z^2 + 2z - 1
     QQ = QQ + \                    % Q(z)Q(1/z)
        P(n-k)*[zeros(1,N-2*k-1),q]/4^k;%   = SUM p1k_k*[(-z+2-z^-1)/4]^k
     q = conv(q,[-1 2 -1]);         % q(z) = q(z)[-z^2 + 2z - 1]
   endfor                           %
endfunction

%----------------------------------------------------------------------------
% Generate Daubechies class scaling coefficients
% - Daubechies-p: minimum support (2p-1, R(y)=0) and minimum phase
% - Symmlets-p:   minimum support (2p-1, R(y)=0) and quasi-linear phase
% - Experimental: non-minimum support (>2p-1, R(y)!=0)
%
%  Input
%  ------
```

```
%  p = number of vanishing moments
%  R = [r_{m-1} ... r_2 r_1 r_0] ==> R(y)=r_{m-1}y^{m-1}+...+r_2 y^2+r_1y+r_0
%
%  Output
%  ------
%  n  = p + m
%
%  QQ = [q_{2n-1} ... q_1 q_0]
%
%                    q_{2n-1}z^{2n-1}+...+r_2 z^2+r_1z+r_0
%    ==>  Q(z)Q(1/z)=--------------------------------------
%                                 z^n
%
%  rQQ= [ r_1 r_2 ... r_{n-1}  r_n r_{n+1}...r_{n-2} ]
%       |<--roots inside -->|<--roots outside  --->|
%       |    unit circle     |    unit circle        |
%
%  A  = [a_p ... a_2 a_1 a_0]
%
%                        (z+1)^p
%    ==>  A(z) = sqrt(2)(---)   = a_pz^p + ... + a_2z^2 + a_1z + a_0
%                        ( 2 )
%
%  rA = [ r_1 r_2 ... r_p ] = roots of A(z)
%
%
%  Theory
%  ------
%  h(n) = scaling coefficients
%
%  H(z) = SUM h(n) z^{-n}
%          n
%       = A(z)Q(z)
%
%                 (z+1)^p
%       = sqrt(2)(---)   Q(z)
%                 ( 2 )
%
%  Finding Q(z) involves factoring a polynomial P(...):
%
%  P(y) = Pm(y) + y^p R(1/2-y)
%
%  In general, R is any polynomial that satisfies R(1/2-y)=-R(y-1/2)
%    (R is an odd polynomial about y=1/2)
%  For Daubechies-p wavelets and symmlets, R(y)=0
%    (giving Daubechies-p and symmlets minimum support)
%
%          p-1  ( p-1+k )
%  Pm(y) = SUM  (       ) y^k
%          k=0  (   k   )
%
%  Q(z)Q(1/z) = P([2-z-1/z]/4)
%-----------------------------------------------------------------------------
function [n,A,QQ,rA,rQQ] = gen_Dclass(p,R)
                                        % Compute A(z) = sqrt(2)[ (z+1)/2 ]^p
                                        % ---------------------------------------
  A=1;                                  % A(z) = (z+1)^0 = 1
  for k=1:p                             %
    A=conv(A,[1 1]);                    % A(z) = (z+1)^k  k=1,2,3,...,p
  endfor                                %
  A = (sqrt(2)/2^p)*A;                  % A(z) = sqrt(2)[(z+1)/2]^p
  rA = sort(roots(A))';                 % rA   = roots of A(z)

                                        % Pm(y)
                                        % ---------------------------------------
  Pm = zeros(1,p);                      % init Pm(y) = 0y^{p-1}+...+0y^2+0y+0
  for k=0:p-1                           %
    Pm(p-k) = bincoeff(p-1+k,k);        % Pm(y) = SUM {p-1+k choose k} y^k
  endfor                                %

                                        % P( [2-z-1/z]/4 )
                                        % ---------------------------------------
  m=length(R);                          %
  P=[zeros(1,m),Pm] + [R,zeros(1,p)];   % P(y) = Pm(y) + y^p R(y)
  while P(1)==0                         % remove leading zero coefficient terms
    n=length(P);                        %   n = number of terms in P(y)
    P=P(2:n);                           %   remove leading zero coefficient
```

```
    endwhile                          %
    n=length(P);                      % number of terms in P(y). num 0s=n-1
    QQ = y2sin2(P);                   % Q(z)Q(1/z) = P(y)|_y={(-z+2-1/z)/4}

                                      % Compute H(z)=A(z)Q(z)
                                      % --------------------------------
    rQQ = sort(roots(QQ))';           % roots of Q(z)Q(1/z) (p-1 roots of Q(z), p-1 roots of Q(1/z))
    rQ  = rQQ(1:n-1);                 % roots of Q(z) (p-1 roots inside unit circle)
    Q   = poly(rQ);                   % convert roots into poly Q(z)
    Q   = sign(real(Q)).*abs(Q);      % eliminate any extraneous imag. components
    H   = conv(A,Q);                  % H(z) = A(z)Q(z)
    rH  = sort(roots(H))';            % roots of H(z)
    h   = (sqrt(2)/sum(H))*H;         % admissibility cond: SUM h_n = sqrt(2)
endfunction

%-----------------------------------------------------------------------------
%function: Generate Daubechies-p scaling coefficients
%
%  Input
%  ------
%  p:    number of vanishing moments
%
%  Output
%  ------
%  n   = p + m
%  h:    scaling coefficients
%  rQQ:  roots of Q(z)Q(z^-1)
%                 |    |_______ p-1 roots outside unit circle
%                 |____________ p-1 roots inside  unit circle
%  rH:   roots of H(z)          p roots at z=-1 and p-1 roots of Q(z)
%
%
%  Theory
%  ------
%  h(n)  = scaling coefficients
%
%  H(z)  = SUM h(n) z^{-n}
%               n
%        = A(z)Q(z)
%
%                 (z+1)^p
%        = sqrt(2)(---)    Q(z)
%                 ( 2 )
%
%-----------------------------------------------------------------------------
function [h,rQQ,rH] = gen_Dp(p)
  R = [0];                           % for Daubechies-p, R(y)=0
  [n,A,QQ,rA,rQQ] = gen_Dclass(p,R); %
  rQ  = rQQ(1:p-1);                  % roots of Q(z) (p-1 roots inside unit circle)
  Q   = poly(rQ);                    % convert roots into poly Q(z)
  Q   = sign(real(Q)).*abs(Q);       % eliminate any extraneous imag. components
  H   = conv(A,Q);                   % H(z) = A(z)Q(z)
  rH  = sort(roots(H))';             % roots of H(z)
  h   = (sqrt(2)/sum(H))*H;          % admissibility cond: SUM h_n = sqrt(2)
endfunction

%-----------------------------------------------------------------------------
%function: Generate Symmlets-p scaling coefficients
%
%  Input
%  ------
%  p:    number of vanishing moments
%
%  Output
%  ------
%  h:    scaling coefficients
%  rQQ:  roots of Q(z)Q(z^-1)
%                 |    |_______ p-1 roots outside unit circle
%                 |____________ p-1 roots inside  unit circle
%  rH:   roots of H(z)          p roots at z=-1 and p-1 roots of Q(z)
%
%-----------------------------------------------------------------------------
function [h,rQQ,rH] = gen_Sp(p)
                                      % Initialization
                                      % --------------------
  N = 1024;                           % number of data points for linear measure
  R = [0];                            % for Symmlets-p, R(y)=0
```

```
   [n,A,QQ,rA,rQQ]=gen_Dclass(p,R);     %
   Nquads     = floor(p/2);             % number of zero quads
   Nsol       = 2^Nquads;               % number of possible solutions
   rQQ_mat    = zeros(p-1,Nsol);        % allocate matrix of roots for all solutions
   mse        = zeros(1,   Nsol);       % allocate mean square error of each solution
   sqMag      = zeros(1,   Nsol);       % allocate square magnitude of each solution

                                        % Main loop: generate all root selections
                                        % --------------------------------------
   for n=0:Nsol-1                       % loop once for each possible solution
     t = conj(rQQ(1:2:p-1));            % extract one representative from each quad
     for m=0:Nquads-1                   % take conjugate recipricols of roots
       if( mod(floor(n/2^m),2) )        % if bit k of n is 1,
         t(m+1) = conj(1/t(m+1));       %   then take conjugate reciprical of root
       endif                            %
     endfor                             %

     rNew          = zeros(p-1,1);      % initialize new root column vector
     rNew(1:2:p-1) = t;                 % rNew=[r1 (real), r2, r2*, r3, r3*,...]'
     rNew(2:2:p-1) = conj(t(mod(p+1,2)+1:Nquads));
     rMat(:,n+1)   = rNew;              % store new root vector in root matrix

                                        % Linear phase measurements
                                        % ------------------------------
     h = real(poly(rNew));              % convert new roots into a poly in z
     [H,w] = freqz(h,[1],N);            % compute the frequency response
     phase = unwrap(angle(H));          % unwrapped phase of h
     [pCoefs,pVals]=polyfit(w,phase,1); % find the 1st order poly p(w) that best fits phase(w)
     mse(n+1) = (pVals.yf-phase)'*(pVals.yf-phase)/N;  % measure the error
     sqMag(n+1) = (rNew'*rNew);         % measure the magnitude and store
   endfor
                                        % Find best set of roots
                                        % ----------------------
   [err1,i1] = min(mse);                % find the 1st location of minimum error
   t=mse;                               % copy mse to temporary vector t
   t(i1) = max(t);                      % get 1st minimum out of the way to find 2nd
   [err2,i2] = min(t);                  % find the 2nd location of minimum error
   t= max(mse)*ones(1,Nsol);            % set t to all large values
   if( sqMag(i1)<sqMag(i2) )            % decide which min to use based on smaller magnitude
     iMinErr = i1;                      %   if the magnitude of first is smaller
   else                                 %
     iMinErr = i2;                      %   if the magnitude of the second is smaller
   endif                                %
   mmse = mse(iMinErr);                 % minimum mean square error
   rNew = rMat(:,iMinErr);              % extract min phase min magnitude vector

                                        % Compute H(z)=A(z)Q(z)
                                        % ------------------------------------
   rQQ = [rNew; conj(1./rNew)]';        % vector w/ both retained and discarded roots
   rQ  = rQQ(1:p-1);                    % roots of Q(z) (p-1 roots inside unit circle)
   Q   = poly(rQ);                      % convert roots into poly Q(z)
   Q   = sign(real(Q)).*abs(Q);         % eliminate any extraneous imag. components
   H   = conv(A,Q);                     % H(z) = A(z)Q(z)
   rH  = sort(roots(H))';               % roots of H(z)
   h   = (sqrt(2)/sum(H))*H;            % admissibility cond: SUM h_n = sqrt(2)
endfunction

%-----------------------------------------------------------------------------
% Generate experimental coefficients with R(y)!=0
%
%  Input
%  ------
%  p:   number of vanishing moments
%  R = [r_{m-1} ... r_2 r_1 r_0] ==> R(y)=r_{m-1}y^{m-1}+...+r_2 y^2+r_1y+r_0
%
%  Output
%  ------
%  h:   scaling coefficients
%  rQQ: roots of Q(z)Q(z^-1)
%                 |    |_______ p-1 roots outside unit circle
%                 |____________ p-1 roots inside  unit circle
%  rH:  roots of H(z)          p roots at z=-1 and p-1 roots of Q(z)
%
%
%  Theory
%  ------
%  h(n) = scaling coefficients
```

```
%
%  H(z)  = SUM h(n)  z^{-n}
%              n
%        = A(z)Q(z)
%
%                (z+1)^p
%        = sqrt(2)(---)    Q(z)
%                ( 2 )
%
%----------------------------------------------------------------------------
function [n,h,rQQ,rH] = gen_Rp(p,R)
  [n,A,QQ,rA,rQQ] = gen_Dclass(p,R);  %
  rQ  = rQQ(1:n-1);                   % roots of Q(z) (n-1 roots inside unit circle)
  Q   = poly(rQ);                     % convert roots into poly Q(z)
  Q   = sign(real(Q)).*abs(Q);        % eliminate any extraneous imag. components
  H   = conv(A,Q);                    % H(z) = A(z)Q(z)
  rH  = sort(roots(H))';              % roots of H(z)
  h   = (sqrt(2)/sum(H))*H;           % admissibility cond: SUM h_n = sqrt(2)
endfunction

#----------------------------------------------------------------------------
#function: Generate scaling coefficients h(n) from length-4 parameter alpha
#          Pollen wavelets
#  references:
#    http://www2.isye.gatech.edu/~brani/datasoft/DL.pdf  page 2
#    Burrus page  66
#----------------------------------------------------------------------------
function h = gen_pollen4(alpha)
  h =  (sqrt(2)/4)  * [\
       1 + cos(alpha) - sin(alpha) ;
       1 + cos(alpha) + sin(alpha) ;
       1 - cos(alpha) + sin(alpha) ;
       1 - cos(alpha) - sin(alpha) ;
      ];
%  h =  (sqrt(2)/4)  * [\   Burrus ?
%        1 - cos(alpha) + sin(alpha) ;
%        1 + cos(alpha) + sin(alpha) ;
%        1 + cos(alpha) - sin(alpha) ;
%        1 - cos(alpha) - sin(alpha) ;
%       ];
endfunction

%-----------------------------------
% Write data to file
%-----------------------------------
function data2file(h,g,rR,rH,filename,comment)
  data  = fopen(filename,"w");
  fprintf(data,'%%=====================================================================\n');
  fprintf(data,'%% Daniel J. Greenhoe \n');
  fprintf(data,'%% file: %s \n',filename);
  fprintf(data,'%% %s \n',comment);
  fprintf(data,'%% %s \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
  fprintf(data,'%%=====================================================================\n\n');

  p=length(h)/2;
  fprintf(data ,'scaling coefficients\n' );
  fprintf(data ,'-------------------------\n' );
  for n=1:length(h)
    fprintf(data ,'h_{%02d} = %13.10f  \n',n-1,h(n) );
  endfor

  fprintf(data ,'\n' );
  fprintf(data ,'wavelet coefficients\n');
  fprintf(data ,'-------------------------\n' );
  for n=1:length(g)
    fprintf(data ,'g_{%02d} = %13.10f  \n',n-1,g(n) );
  endfor

  fprintf(data ,'\n' );
  fprintf(data ,'roots of Q(z)\n' );
  fprintf(data ,'-------------------------\n' );
  for n=1:p-1
    fprintf(data ,'r_{%02d} = %13.10f  %13.10fi \n',n-1, real(rR(n)), imag(rR(n)) );
  endfor

  fprintf(data ,'\n' );
  fprintf(data ,'roots of Q(z^-1)\n' );
```

```
    fprintf(data ,'-------------------------\n');
    for n=p:length(rR)
      fprintf(data ,'r_{%02d} = %13.10f   %13.10fi  \n',n-1, real(rR(n)), imag(rR(n)) );
    endfor

    fprintf(data ,'\n');
    fprintf(data ,'roots of H(z)\n');
    fprintf(data ,'-------------------------\n');
    for n=1:length(rH)
      fprintf(data ,'r_{%02d} = %13.10f   %13.10fi  \n',n-1, real(rH(n)), imag(rH(n)) );
      if(n==p-1) fprintf(data,'\n'); endif
    endfor

    fprintf(data ,'\n' );
    fprintf(data ,'LaTeX drawing commands    \n');
    fprintf(data ,'-------------------------\n');
    for n=1:p-1
      fprintf(data ,'   \\put( %15.10f, %15.10f ) {\\circle {15}}  \n',real(rR(n))*100, imag(rR(n))*100 );
    endfor
    fprintf(data,"\n");
    for n=p:length(rR)
      fprintf(data ,'   \\put( %15.10f, %15.10f ) {\\circle*{15}}  \n',real(rR(n))*100, imag(rR(n))*100 );
    endfor

    fflush(data);
    fclose(data);
    printf('output file name = \"%s\"  \n',filename);
endfunction

%--------------------------------------------------------------------------
% Write data to plot file
% <filename> can be used by the command '\fileplot' in a TeX environment
% to generate a plot of (x,y).
% The command '\fileplot' is available in the PSTricks package of TeX.
% Reference: http://www.ctan.org/pkg/pstricks
%--------------------------------------------------------------------------
function data2plotfile(x,y,filename,comment)
    data   = fopen(filename,'w');
    fprintf(data,'%%==========================================================================\n');
    fprintf(data,'%% Daniel J. Greenhoe \n');
    fprintf(data,'%% file: %s  \n',filename);
    fprintf(data,'%% number of data points = %d\n',length(x));
    fprintf(data,'%% %s  \n',comment);
    fprintf(data,'%% %s  \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
    fprintf(data,'%% The command \\fileplot{%s} may be used in a LaTeX environment for plotting
             data.\n',filename);
    fprintf(data,'%% \\fileplot is available in the LaTeX PSTricks package.\n');
    fprintf(data,'%% Reference: http://www.ctan.org/pkg/pstricks\n');
    fprintf(data,'%%==========================================================================\n');

    fprintf(data ,'[\n');
    for n=1:length(x)
      fprintf(data ,'(%13.10f,%13.10f)\n',x(n),y(n));
    endfor
    fprintf(data,']\n');
    fprintf(data,'%%---end of file---' );
    fflush(data);
    fclose(data);
    printf("data written to \'%s\'  \n",filename);
endfunction

%--------------------------------------------------------------------------
% Demonstration of Daubechies-p wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%--------------------------------------------------------------------------
function demo_Dp(p,N,iterations)
                                              % calculate coefficient sequences
                                              % -------------------------
    [h,rQQ,rH] = gen_Dp(p);                   % Daubechies-p
    g    = h2g_coefs(h);                      % generate wavelet coefficients g(n)
    d    = round(N/length(h));                % density=round(samples/unit(1))
    phi  = gen_phi(h,   iterations,d);        % generate phi(x) from h(n)
    psi  = gen_psi(phi,g,d);                  % generate psi(x) from g(n)
   M    = length(phi);

                                              % Generate output data files
```

```
                                                    % ------------------------
   x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
   data2file(h,g,rQQ,rH,sprintf('d%d.dat',p),      sprintf('data for Daubechies-%d wavelets',p) );
   data2plotfile(x,phi,sprintf('d%d_phi.dat',p),sprintf('plot file for Daubechies-%d scaling
             function',p));
   data2plotfile(x,psi,sprintf('d%d_psi.dat',p),sprintf('plot file for Daubechies-%d wavelet
             function',p));
   plot(x,phi,x,psi);

endfunction

%------------------------------------------------------------------------------
% Demonstration of Symmlet-p wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%------------------------------------------------------------------------------
function demo_Symmlet_p(p,N,iterations)
                                                    % Generate scaling coefficients {h_n}
                                                    % ------------------------
   [h,rQQ,rH] = gen_Sp(p);                          % Symmlet-p
   g   = h2g_coefs(h);                              % generate wavelet coefficients g(n)
   d   = round(N/length(h));                        % density=round(samples/unit(1))
   phi = gen_phi(h,  iterations,d);                 % generate phi(x) from h(n)
   psi = gen_psi(phi,g,d);                          % generate psi(x) from g(n)
  M   = length(phi);

                                                    % Generate output data files
                                                    % ------------------------
   x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
   data2file(h,g,rQQ,rH, sprintf("s%d.dat",p),      sprintf("Symmlet-%d data file",p));
   data2plotfile(x,phi,  sprintf('s%d_phi.dat',p),sprintf('plot file for Symmlet-%d scaling
             function',p));
   data2plotfile(x,psi,  sprintf('s%d_psi.dat',p),sprintf('plot file for Symmlet-%d wavelet
             function',p));
   plot(x,phi,x,psi);
endfunction

%------------------------------------------------------------------------------
% Demonstration of Daubechies-p class wavelet with R(y)!=0
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%------------------------------------------------------------------------------
function demo_Ry_p(p,N,iterations)
                                                    % R(y)
                                                    % ------------------------
 R=[  3 0 5 0 7 0 3 0 1 0];                                       % R(y)
 ptitle=sprintf("p=%d     R(y)=3y^9 + 5y^7 + 7y^5 + 3y^3 + y     Daniel J. Greenhoe",p);% title

                                                    % Generate scaling coefficients {h_n}
                                                    % ------------------------
   [n,h,rQQ,rH]= gen_Rp(p,R);                       % Ry(p)
   g   = h2g_coefs(h);                              % generate wavelet coefficients g(n)
   d   = round(N/length(h));                        % density=round(samples/unit(1))
   phi = gen_phi(h,  iterations,d);                 % generate phi(x) from h(n)
   psi = gen_psi(phi,g,d);                          % generate psi(x) from g(n)

                                                    % Generate output
                                                    % -------------------------
   data2file(h,g,rQQ,rH, sprintf("R%d.dat",p), sprintf("R(y)-%d data file",p) );   pause(delay_sec);
endfunction

%------------------------------------------------------------------------------
% Demonstration of Pollen length 4 wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%  alpha:      parameter alpha
%  p4_phi.dat and p4_psi.dat can be used with LaTeX's \plotfile command.
%  \pltfile is available in the pstricks package
%   Reference: http://www.ctan.org/pkg/pstricks
%------------------------------------------------------------------------------
function demo_pollen4a(alpha,N,iterations)
                                                    % calculate coefficient sequences
                                                    % ------------------------
   h   = gen_pollen4(alpha);        % Pollen length-4
   g   = h2g_coefs(h);                              % generate wavelet coefficients g(n)
   d   = round(N/length(h));                        % density=round(samples/unit(1))
   phi = gen_phi(h,  iterations,d);                 % generate phi(x) from h(n)
```

```
  psi = gen_psi(phi,g,d);              % generate psi(x) from g(n)
  M   = length(phi);

                                       % Generate output data files
                                       % ------------------------
  x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
  %data2file(h,g,rQQ,rH,'p4.dat',    sprintf('data for Pollen-4 alpha=%13.10f wavelets',alpha));
  data2plotfile(x,phi,'p4_phi.dat',sprintf('plot file for Pollen-4 alpha=%13.10f scaling function',alpha));
  data2plotfile(x,psi,'p4_psi.dat',sprintf('plot file for Pollen-4 alpha=%13.10f wavelet function',alpha));
  plot(x,phi,x,psi);

endfunction

%----------------------------------------------------------------------------
% Demonstration of Pollen length 4 wavelets
%  iterations: number of iterations to use to generate phi(t) (e.g. 16)
%  N:          data size (e.g. 1024)
%  nalpha:     number of alpha values
%  a:          starting alpha value
%  b:          ending alpha value
%  p4_phi.dat and p4_psi.dat can be used with GNU Plot's 'splot' function
%  Reference: http://www.gnuplot.info
%----------------------------------------------------------------------------
function demo_pollen4(a,b,nalpha,N,iterations)
  filename1='p4_psi.dat';
  filename2='p4_phi.dat';
  data1  = fopen(filename1,'w');
  data2  = fopen(filename2,'w');
  fprintf(data1,'#==========================================================================\n');
  fprintf(data1,'# Daniel J. Greenhoe \n');
  fprintf(data1,'# file: %s \n',filename1);
  fprintf(data1,'# %s \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
  fprintf(data1,'# The command splot{%s} may be used in a GNU Plot environment for plotting data.\n',filename1);
  fprintf(data1,'# Reference: http://www.gnuplot.info\n');
  fprintf(data1,'#==========================================================================\n');

  fprintf(data2,'#==========================================================================\n');
  fprintf(data2,'# Daniel J. Greenhoe \n');
  fprintf(data2,'# file: %s \n',filename2);
  fprintf(data2,'# %s \n',strftime('%Y %B %d %A %r (%Z)',localtime(time)));
  fprintf(data2,'# The command splot{%s} may be used in a GNU Plot environment for plotting data.\n',filename2);
  fprintf(data2,'# Reference: http://www.gnuplot.info\n');
  fprintf(data2,'#==========================================================================\n');

  for i = 0:nalpha-1
                                       % calculate coefficient sequences
                                       % ------------------------
    alpha = a + i/(nalpha-1)*(b-a)     % calculate alpha
    h     = gen_pollen4(alpha);        % Pollen length-4
    g     = h2g_coefs(h);              % generate wavelet coefficients g(n)
    d     = round(N/length(h));        % density=round(samples/unit(1))
    phi = gen_phi(h,  iterations,d);   % generate phi(x) from h(n)
    psi = gen_psi(phi,g,d);            % generate psi(x) from g(n)
    M   = length(phi);

                                       % Generate output data files
                                       % ------------------------
    x = [0:length(phi)-1]*(length(h)-1)/(length(phi)-1);
    y1 = psi;
    y2 = phi;

    for n=1:length(x)
      fprintf(data1 ,'%13.10f  %13.10f  %13.10f\n',alpha,x(n),y1(n));
    endfor
    for n=1:length(x)
      fprintf(data2 ,'%13.10f  %13.10f  %13.10f\n',alpha,x(n),y2(n));
    endfor
    fprintf(data1 ,'\n');
    fprintf(data2 ,'\n');

  endfor
  fprintf(data1,'##---end of file---' );
  fprintf(data2,'##---end of file---' );
```

```
  fflush(data1);
  fflush(data2);
  fclose(data1);
  fclose(data2);
  printf("data written to \'%s\' \n",filename1);
  printf("data written to \'%s\' \n",filename2);

endfunction

%====================================
% Main
%====================================
                                    % parameters
                                    %-----------------------------------
N = 1024;                           % number of data points
iterations = 16;                    % number of iterations

                                    % demos
                                    %-----------------------------------
%for p=1:12
%  demo_Dp(p,N,iterations);
%endfor
%demo_Dp(16,N,iterations);

%demo_Symmlet_p( 4,N,iterations);
%demo_Symmlet_p( 8,N,iterations);
%demo_Symmlet_p(12,N,iterations);
%demo_Symmlet_p(16,N,iterations);
%demo_Ry_p(3);
%demo_pollen4a(pi/6,16,iterations)%
%demo_pollen4a(pi/4,16,iterations)%
%demo_pollen4a(pi/2,16,iterations)%
%demo_pollen4a(pi,16,iterations)%
%demo_pollen4(0,pi,256,256,iterations)%
demo_pollen4(0,pi,32,32,iterations)%
%function demo_pollen4(a,b,nalpha,N,iterations)

%====================================
% End Processing
%====================================
```

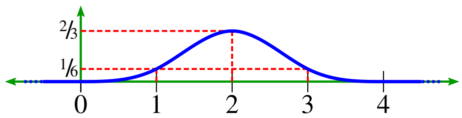
V.2 B-spline polynomial calculation and plotting

The polynomials for *B-spline*s as demonstrated in Example M.3 (page 300)–Example M.9 (page 304) can be calculated using the free and open source software package *Maxima* along with the following script file:

```
/*=====================================================================*
 * Daniel J. Greenhoe                                                  *
 * Maxima script file                                                  *
 * To execute this script, start Maxima in a command window            *
 * in the subdirectory containing this file (e.g. c:\math\maxima\)     *
 * and then after the (%i...) prompt enter                             *
 *   batchload("bspline.max")$                                         *
 * Data produced will be written to the file "bsplineout.txt".         *
 * reference: http://maxima.sourceforge.net/documentation.html         *
 *=====================================================================*/
/*--------------------------------------
 * initialize script
 *--------------------------------------*/
reset();
kill(all);
load(orthopoly);
display2d:false; /* 2-dimensional display */
writefile("bsplineout.txt");
/*--------------------------------------
 * n = B-spline order parameter
 * may be set to any value in {1,2,3,...}
```

```
 *-----------------------------------------*/
n:2;
print("====================================================================");
print("Daniel J. Greenhoe");
print("Output file for nth order B-spline Nn(x) calculation, n=",n," .");
print("Output produced using Maxima running the script file bspline.max");
print("====================================================================");
Nnx:(1/n!)*sum((-1)^k*binomial(n+1,k)*(x-k)^n*unit_step(x-k),k,0,n+1);
print("------------------------------------------------------------");
print("              n+1      k (n+1)      n                   ");
print(" n! Nn(x) = SUM (-1)   (    ) (x-k)  step(x-k)  ,n=",n);
print("              k=0        ( k )                           ");
print("              ",n+1,"      k (",n+1,")      ",n);
print(n,"! Nn(x) = SUM (-1)   (    ) (x-k)  step(x-k)");
print("              k=0        ( k )");
print("            = ",expand(n!*Nnx));
print("------------------------------------------------------------");
assume(x<=0);    print(n!,"N(x)= ",expand(n!*Nnx)," for x<=0");  forget(x<=0);
for i:0 thru n step 1 do(
   assume(x>i,x<(i+1)),
   print(n!,"N(x)= ",expand(n!*Nnx)," for ",i,"<x<",i+1),
   tex(expand(n!*Nnx),"djg.tex"),/*write output in TeX format to file "djg.tex"*/
   forget(x>i,x<(i+1))
   );
assume(x>(n+1)); print(n!,"N(x)= ",expand(n!*Nnx)," for x>",n+1); forget(x>(n+1));
print("------------------------------------------------------------");
print(" values at some specific points x:                         ");
print("------------------------------------------------------------");
y:Nnx,x=(n+1)/2;print("N(",(n+1)/2,")= ",y," (center value)");
y:Nnx,x=(n+2)/2;print("N(",(n+2)/2,")= ",y);
y:Nnx,x=(n+3)/2;print("N(",(n+3)/2,")= ",y);
/*-----------------------------------------
 * close output file
 *-----------------------------------------*/
closefile();
```

Once the polynomial expressions for a *B-spline* have been calculated, they can be plotted within a LaTeX environment using the pst-plot package along with a LaTeX source file such as the following:[2]

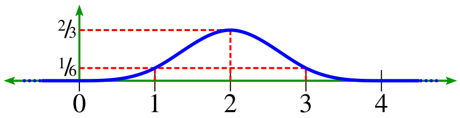



```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
% N_3(x) B-spline
% nominal unit = 10mm
%=====================================================================
\begin{pspicture}(-1,-0.5)(5,1)
  %---------------------------------------
  % parameters
  %---------------------------------------
  \psset{plotpoints=64,labelsep=1pt}%
  %---------------------------------------
  % axes
  %---------------------------------------
  \psaxes[linewidth=0.75pt,linecolor=axis,yAxis=false,ticks=x,labels=x]{<->}(0,0)(-1,0)(5,1)% x axis
  \psaxes[linewidth=0.75pt,linecolor=axis,xAxis=false,ticks=x,labels=x]{->}(0,0)(-1,0)(5,1)%  y axis
  %---------------------------------------
  % annotation
  %---------------------------------------
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](2,0)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.667)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](1,0)(1,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](3,0)(3,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.1667)(3,0.1667)%
  \uput[180](0,0.667){$\sfrac{2}{3}$}%
  \uput[180](0,0.1667){$\sfrac{1}{6}$}%
  %---------------------------------------
  % function plot
  %---------------------------------------
  \psplot{0}{1}{+1 x 3 exp mul                                    6 div}% for 0<=x<=1
```

[2] For help with PostScript® math operators, see Adobe (1999), pages 508–509 ⟨Arithmetic and Math Operators⟩.

```
  \psplot{1}{2}{-3 x 3 exp mul +12 x 2 exp mul add -12 x mul add  +4 add 6 div}% for 1<=x<=2
  \psplot{2}{3}{+3 x 3 exp mul -24 x 2 exp mul add +60 x mul add -44 add 6 div}% for 2<=x<=3
  \psplot{3}{4}{-1 x 3 exp mul +12 x 2 exp mul add -48 x mul add +64 add 6 div}% for 3<=x<=4
  \psline(0,0)(-0.5,0)\psline[linestyle=dotted](-0.5,0)(-0.75,0)%           % for x<=0
  \psline(4,0)(4.5,0)\psline[linestyle=dotted](4.5,0)(4.75,0)%              % for x>=4
\end{pspicture}%
```

Alternatively, one can plot $N_3(x)$ more or less directly from the equation given in Theorem M.1 (page 301) without first calculating the polynomial expressions:




```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% N_3(x) B-spline
% nominal unit = 10mm
%============================================================================
\begin{pspicture}(-1,-0.5)(5,1)
  %---------------------------------------
  % parameters
  %---------------------------------------
  \psset{plotpoints=64,labelsep=1pt}%
  %---------------------------------------
  % axes
  %---------------------------------------
  \psaxes[linewidth=0.75pt,linecolor=axis,yAxis=false,ticks=x,labels=x]{<->}(0,0)(-1,0)(5,1)% x axis
  \psaxes[linewidth=0.75pt,linecolor=axis,xAxis=false,ticks=x,labels=x]{->}(0,0)(-1,0)(5,1)%  y axis
  %---------------------------------------
  % annotation
  %---------------------------------------
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](2,0)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.667)(2,0.667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](1,0)(1,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](3,0)(3,0.1667)%
  \psline[linestyle=dashed,linewidth=0.75pt,linecolor=red](0,0.1667)(3,0.1667)%
  \uput[180](0,0.667){$\sfrac{2}{3}$}%
  \uput[180](0,0.1667){$\sfrac{1}{6}$}%
  %----------------------------------------------------------------------
  % for n=3
  %         1  n+1        (n+1)                     1   4           ( 4 )
  % N_n(x)--- SUM (-1)^k  (    )  (x-k)^n s(x-k)  = --- SUM  (-1)^k (   )  (x-k)^3 s(x-k)
  %         n! k=1        ( k )                     3! k=1          ( k )
  %
  % where s(x) = 0 for x<0 and 1 for x>=0  (step function)
  %----------------------------------------------------------------------
  \psplot{0}{1}   {1 x 0 sub 3 exp mul   6 div}%
  \psplot{1}{2}   {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 div}%
  \psplot{2}{3}   {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   6 div}%
  \psplot{3}{4}   {1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   4 x 3 sub
           3 exp mul sub   6 div}%
  \psplot{4}{4.5}{1 x 0 sub 3 exp mul   4 x 1 sub 3 exp mul sub   6 x 2 sub 3 exp mul add   4 x 3 sub
           3 exp mul sub   1 x 4 sub 3 exp mul add 6 div}%
  %    N_3(x) = [(4choose0)(x-0)^3    - (4choose1)(x-1)^3      + (4choose2)(x-2)^3      -
           (4choose3)(x-3)^3      + (4choose4)(x-4)^3      ]/3!
  %           = [       1 (x-0)^3    -         4 (x-1)^3      +         6 (x-2)^3      -         4
           (x-3)^3      +         1 (x-4)^3      ]/6
  \psline(0,0)(-0.5,0)%
  \psline[linestyle=dotted](-0.5,0)(-0.75,0)%
  \psline[linestyle=dotted](4.5,0)(4.75,0)%
\end{pspicture}%
```

The LaTeX source listed above can be made to output a pdf file with tight borders using the preview package and[3]

```
\usepackage[active,tightpage]{preview}%
\PreviewBorder=0pt%
\PreviewEnvironment{pspicture}%
```

[3] http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput

V.3 Auto-power spectrum plot

The plot provided in Figure M.3 (page 315) was generated using C and LaTeX code very similar to that displayed in this section:

1. C code for generating data points. This outputs to stdout (standard output terminal), but can be redirected into a file such as Snn_bspline1.dat, Snn_bspline2.dat, Snn_bspline3.dat, etc. In a *Windows* command environment ("DOS" window), this can be performed with something like this:

   C:\ mycprogram.exe > Snn_bspline1.dat

```c
int bspline_Sdat(void){
  const double n=3;         /* order of B-spline                          */
  const long M=1024;        /* approximate number of data points          */
  const long N=1000000;     /* number of iterations per data point        */
  double w=0,s=0;
  long k=0;
  printf("[\n");                                 /* LaTeX \fileplot support */
  for(w=-8.0;w<=8.0;w+=16./(double)M){
    s=0;
    for(k=1; k<=N; k++){
      s += pow(1.0/(2*k-w/PI),2.*n+2.);          /* 1st summation          */
      s += pow(1.0/(2*k+w/PI),2.*n+2.);          /* 2nd summation          */
      }
    s*=pow(sin(w/2.)/(PI/2.),2.*n+2.);           /* scaling factor         */
    if(w==0) s+=1;                               /* 1st term: w=0 case     */
    else     s+=pow(sin(w/2.)/(w/2.),2.*n+2.);   /* 1st term: w!=0 case    */
    printf("(%lf, %lf)\n",w,s);                  /* print one data point   */
    }
  printf("]\n");                                 /* LaTeX \fileplot support */
  return 0;
  }
```

2. The files produced by item (1) can be plotted within LaTeX using the pst-plot package.

```latex
%=======================================================================
% Daniel J. Greenhoe
% LaTeX file
% auto-power spectrum plots for B-splines of order n=0,1,2,3
% nominal xunit = 10mm
% nominal yunit = 20mm
%=======================================================================
\begin{pspicture}(-8.5,-0.25)(9,1.25)%
  %---------------------------------------
  % axes
  %---------------------------------------
  \psaxes[linecolor=axis,linewidth=0.75pt,yAxis=false,ticks=none,labels=none]{<->}(0,0)(-8.5,0)(8.5,1.25)% x-axis
  \psaxes[linecolor=axis,linewidth=0.75pt,xAxis=false,ticks=none,labels=none]{->}(0,0)(-8.5,0)(8.5,1.25)% y-axis
  %---------------------------------------
  % y-axis guide lines
  %---------------------------------------
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-8.5,0.333333)(8.5,0.333333)% min line for n=1 B-spline
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-8.5,0.133333)(8.5,0.133333)% min line for n=2 B-spline
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-8.5,0.053968)(8.5,0.053968)% min line for n=3 B-spline
  %---------------------------------------
  % x-axis labeling
  %---------------------------------------
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](3.1415927,0)(3.1415927,1)%
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-3.1415927,0)(-3.1415927,1)%
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](6.283185,0)(6.283185,1)%
  \psline[linecolor=red,linestyle=dashed,linewidth=0.75pt](-6.283185,0)(-6.283185,1)%
  \uput{2pt}[0](8.5,0){$\omega$}%
  \uput{2pt}[-90](0,0){$0$}%
  \uput{2pt}[-90](-6.283185307179586476925286766559,0){$-2\pi$}%
  \uput{2pt}[-90](-3.1415926535897932384626433832795,0){$-\pi$}%
  \uput{2pt}[-90](3.1415926535897932384626433832795,0){$\pi$}%
```

```
  \uput{2pt}[-90](6.283185307179586476925286766559,0){$2\pi$}%
  %--------------------------------------
  % data plot
  %--------------------------------------
  \psline(-8,1)(8,1)%                                     order n=0 B-spline data plot
  \fileplot{../../common/math/graphics/bsplines/Snn_bspline1.dat}% order n=1 B-spline data plot
  \fileplot{../../common/math/graphics/bsplines/Snn_bspline2.dat}% order n=2 B-spline data plot
  \fileplot{../../common/math/graphics/bsplines/Snn_bspline3.dat}% order n=3 B-spline data plot
  %--------------------------------------
  % data plot lables
  %--------------------------------------
  \uput{2pt}[90](1.9,1){$n=0$}%
  \uput{2pt}[90](3.1415297,0.333333){$n=1$}%
  \uput{2pt}[90](3.1415297,0.133333){$n=2$}%
  \uput{2pt}[-45](4.5,0.335){$n=3$}%
  %--------------------------------------
  % y-axis labels
  %--------------------------------------
  \uput*{2mm}[180](0,1){$1$}%
  \uput*{2mm}[180](0,0.333333){$\sfrac{1}{3}\approx0.33$}%   min value for n=1 B-spline
  \uput*{2mm}[180](0,0.133333){$\sfrac{2}{15}\approx0.13$}%  min value for n=2 B-spline
  \uput*{2mm}[0](0,0.053968){$\sfrac{17}{315}\approx0.054$}% min value for n=3 B-spline
\end{pspicture}%
```

3. The LaTeX source listed in item (2) can be made to output a pdf file with tight borders using the preview package and[4]

```
\usepackage[active,tightpage]{preview}%
\PreviewBorder=0pt%
\PreviewEnvironment{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% XeLaTeX file
% generate tight pdf graphics file for inclusion in a document
% nominal unit = 10mm
% nominal font size = \scriptsize
%============================================================================
\input{shelltop.tex}%
\begin{document}%
  %\gsize%
  %\scriptsize%
  %\psset{unit=1\latunit}%
  \psset{xunit=1.0\latunit}%
  \psset{yunit=2.0\latunit}%
  \input{../../common/math/graphics/bsplines/Snn_bspline.tex}%
\end{document}%
```

4. The source listed in item (3) uses a file called shelltop.tex:

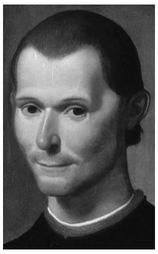
```
%============================================================================
% Daniel J. Greenhoe
% XeLaTeX file
% preamble packages for shell files to
% generate tight pdf graphics file for inclusion in a document
%============================================================================
\documentclass{article}%
%--------------------------------------
% Style Packages
%--------------------------------------
 \usepackage{../../common/sty/packages}%
 \usepackage{../../common/sty/switches}  %
 \usepackage{../../common/sty/fonts}%
 \usepackage{../../common/sty/dan}%
 \usepackage{../../common/sty/colors_rgb}%
%\usepackage{../../common/sty/colors_cmyk}%
%\usepackage{../../common/sty/colors_gray}%
 \usepackage{../../common/sty/math}%
 \usepackage{../../common/sty/wavelets}%
%\usepackage{../../common/sty/language}%
 \usepackage{../../common/sty/defaults}  % default values
```

[4] http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput

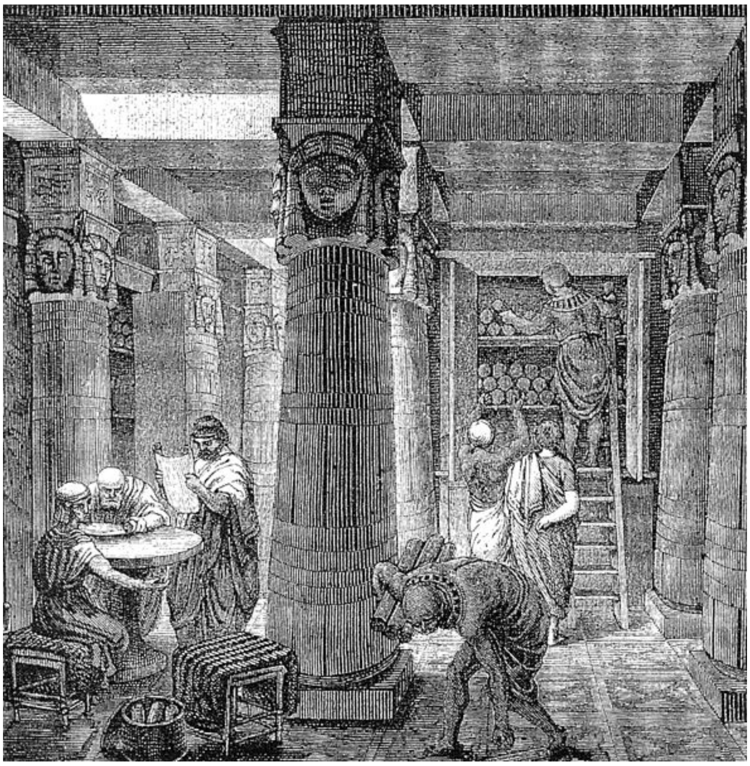
```
%--------------------------------------
% color space
%--------------------------------------
%\selectcolormodel{cmyk}% use cmyk color model
 \selectcolormodel{rgb}% use rgb color model
%--------------------------------------
% redefine some commands
%--------------------------------------
%\newcommand{\struct}[1]{{#1}\index{#1}\index{structures!#1}}% place here and in the index and under index structures! heading
%\newcommand{\structe}[1]{\emph{#1}\index{#1}\index{structures!#1}}% place here with emphasis and in the index and under index structures! heading
%\newcommand{\structb}[1]{\textbf{#1}\index{#1}\index{structures!#1}}% place here with emphasis and in the index and under index structures! heading
%\newcommand{\structd}[1]{\textbf{#1}\index{#1|textbf}\index{structures!#1|textbf}\index{definitions!#1|textb
   place here with emphasis and in the index and under index structures! heading
\renewcommand{\struct}[1]{{#1}}% place here and in the index and under index structures! heading
\renewcommand{\structe}[1]{\emph{#1}}% place here with emphasis and in the index and under index structures! heading
\renewcommand{\structb}[1]{\textbf{#1}}% place here with emphasis and in the index and under index structures! heading
\renewcommand{\structd}[1]{\textbf{#1}}% place here with emphasis and in the index and under index structures! heading
%--------------------------------------
% preview mode
% reference:
   http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput
%--------------------------------------
\usepackage[active,tightpage]{preview}%
\PreviewBorder=0pt%
\PreviewEnvironment{pspicture}%
```

Back Matter

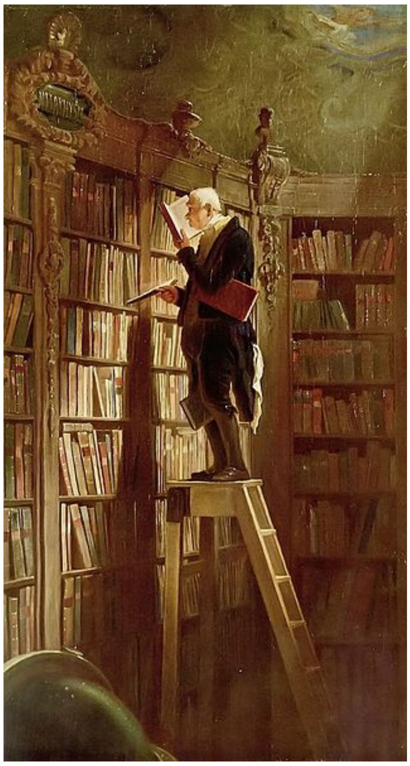
❝*It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.*❞
Niels Henrik Abel (1802–1829), Norwegian mathematician [5]



❝*When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.*❞
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [6]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850

[7]

❝*To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.*❞
Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [8]

5 quote: Simmons (2007), page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

6 quote: Machiavelli (1961), page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

7 http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

[8] quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

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