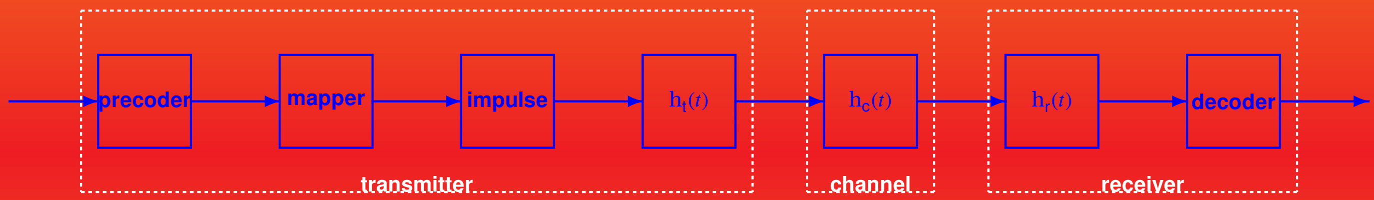
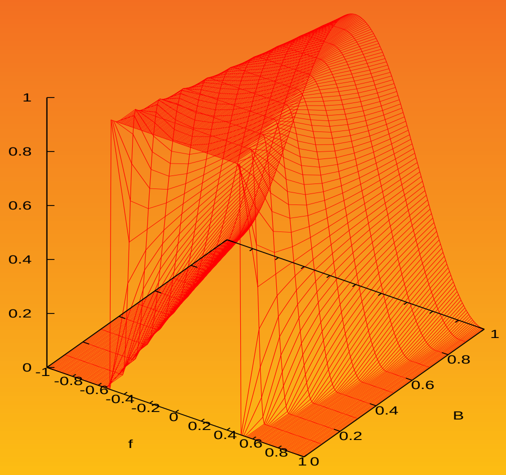
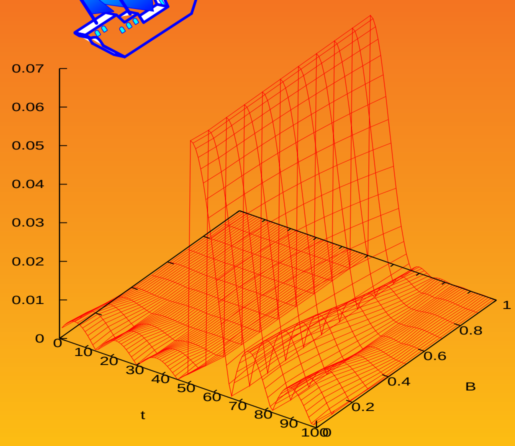
A Book Concerning Digital Communications

VERSION 0.02







Daniel J. Greenhoe

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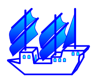
The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[1]



[1] Paine (2000) page 63 ⟨Golden Hind⟩

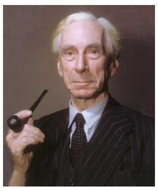
“Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?”

“Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.”

Alfred Edward Housman, English poet (1859–1936) [2]

“The uninitiated imagine that one must await inspiration in order to create. That is a mistake. I am far from saying that there is no such thing as inspiration; quite the opposite. It is found as a driving force in every kind of human activity, and is in no wise peculiar to artists. But that force is brought into action by an effort, and that effort is work. Just as appetite comes by eating so work brings inspiration, if inspiration is not discernible at the beginning.”

Igor Fyodorovich Stravinsky (1882–1971), Russian-born composer [3]



“As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.”

Bertrand Russell (1872–1970), British mathematician, in a 1962 November 23 letter to Dr. van Heijenoort. [4]

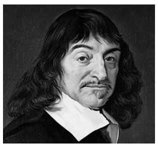
[2] quote: Housman (1936), page 64 ⟨"Smooth Between Sea and Land"⟩, Hardy (1940) ⟨section 7⟩
image: http://en.wikipedia.org/wiki/Image:Housman.jpg

[3] quote: Ewen (1961), page 408, Ewen (1950)
image: http://en.wikipedia.org/wiki/Image:Igor_Stravinsky.jpg

[4] quote: Heijenoort (1967), page 127
image: http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Russell.html

SYMBOLS

❝*rugula XVI. Quae vero praesentem mentis attentionem non requirunt, etiamsi ad conclusionem necessaria sint, illa melius est per brevissimas notas designare quam per integras figuras: ita enim memoria non poterit falli, nec tamen interim cogitatio distrahetur ad haec retinenda, dum aliis deducendis incumbit.*❞



❝*Rule XVI. As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.*❞

René Descartes (1596–1650), French philosopher and mathematician [5]

❝*In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished.*❞
Gottfried Leibniz (1646–1716), German mathematician, [6]

Symbol list

symbol	description	
numbers:		
$\mathbb{Z}$	integers	$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$
$\mathbb{W}$	whole numbers	$0, 1, 2, 3, \dots$
$\mathbb{N}$	natural numbers	$1, 2, 3, \dots$
$\mathbb{Z}^{\dashv}$	non-positive integers	$\dots, -3, -2, -1, 0$

...continued on next page...

[5] quote: Descartes (1684a) ⟨rugula XVI⟩, translation: Descartes (1684b) ⟨rule XVI⟩, image: Frans Hals (circa 1650), `http://en.wikipedia.org/wiki/Descartes`, public domain

[6] quote: Cajori (1993) ⟨paragraph 540⟩, image: `http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg`, public domain

symbol	description	
$\mathbb{Z}^-$	negative integers	$\ldots, -3, -2, -1$
$\mathbb{Z}_o$	odd integers	$\ldots, -3, -1, 1, 3, \ldots$
$\mathbb{Z}_e$	even integers	$\ldots, -4, -2, 0, 2, 4, \ldots$
$\mathbb{Q}$	rational numbers	$\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\backslash 0$
$\mathbb{R}$	real numbers	completion of $\mathbb{Q}$
$\mathbb{R}^{\vdash}$	non-negative real numbers	$[0, \infty)$
$\mathbb{R}^{\dashv}$	non-positive real numbers	$(-\infty, 0]$
$\mathbb{R}^+$	positive real numbers	$(0, \infty)$
$\mathbb{R}^-$	negative real numbers	$(-\infty, 0)$
$\mathbb{R}^*$	extended real numbers	$\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	arbitrary field	(often either $\mathbb{R}$ or $\mathbb{C}$)
∞	positive infinity	
$-\infty$	negative infinity	
π	pi	$3.14159265\ldots$
relations:		
®	relation	
Ⓐ	relational and	
$X \times Y$	Cartesian product of X and Y	
$(\triangle, \nabla)$	ordered pair	
$\lvert z\rvert$	absolute value of a complex number z	
$=$	equality relation	
$\triangleq$	equality by definition	
$\to$	maps to	
$\in$	is an element of	
$\notin$	is not an element of	
$\mathcal{D}$(®)	domain of a relation ®	
$\mathcal{I}$(®)	image of a relation ®	
$\mathcal{R}$(®)	range of a relation ®	
$\mathcal{N}$(®)	null space of a relation ®	
set relations:		
$\subseteq$	subset	
$\subsetneq$	proper subset	
$\supseteq$	super set	
$\supsetneq$	proper superset	
$\not\subseteq$	is not a subset of	
$\not\subset$	is not a proper subset of	
operations on sets:		
$A \cup B$	set union	
$A \cap B$	set intersection	
$A \triangle B$	set symmetric difference	
$A \backslash B$	set difference	
A^c	set complement	
$\lvert\cdot\rvert$	set order	
$\mathbb{1}_A(x)$	set indicator function or characteristic function	
logic:		
1	“true” condition	
0	“false” condition	
$\neg$	logical NOT operation	

...continued on next page...

symbol	description	
$\wedge$	logical AND operation	
$\vee$	logical inclusive OR operation	
$\oplus$	logical exclusive OR operation	
$\Longrightarrow$	"implies";	"only if"
$\Longleftarrow$	"implied by";	"if"
$\Longleftrightarrow$	"if and only if";	"implies and is implied by"
$\forall$	universal quantifier:	"for each"
$\exists$	existential quantifier:	"there exists"
order on sets:		
$\vee$	join or least upper bound	
$\wedge$	meet or greatest lower bound	
$\leq$	reflexive ordering relation	"less than or equal to"
$\geq$	reflexive ordering relation	"greater than or equal to"
$<$	irreflexive ordering relation	"less than"
$>$	irreflexive ordering relation	"greater than"
measures on sets:		
$\lvert X \rvert$	order or counting measure of a set X	
distance spaces:		
d	metric or distance function	
linear spaces:		
$\lVert\cdot\rVert$	vector norm	
$\lvert\lVert\cdot\rVert\rvert$	operator norm	
$\langle \triangle \mid \nabla \rangle$	inner-product	
$\operatorname{span}(\boldsymbol{V})$	span of a linear space $\boldsymbol{V}$	
algebras:		
$\Re$	real part of an element in a $*$-algebra	
$\Im$	imaginary part of an element in a $*$-algebra	
set structures:		
$\boldsymbol{T}$	a topology of sets	
$\boldsymbol{R}$	a ring of sets	
$\boldsymbol{A}$	an algebra of sets	
$\emptyset$	empty set	
2^X	power set on a set X	
sets of set structures:		
$\mathcal{T}(X)$	set of topologies on a set X	
$\mathcal{R}(X)$	set of rings of sets on a set X	
$\mathcal{A}(X)$	set of algebras of sets on a set X	
classes of relations/functions/operators:		
2^{XY}	set of *relations* from X to Y	
Y^X	set of *functions* from X to Y	
$\mathcal{S}_\mathrm{j}(X,Y)$	set of *surjective* functions from X to Y	
$\mathcal{I}_\mathrm{j}(X,Y)$	set of *injective* functions from X to Y	
$\mathcal{B}_\mathrm{j}(X,Y)$	set of *bijective* functions from X to Y	
$\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$	set of *bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	set of *linear bounded* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
$\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$	set of *continuous* functions/operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$	
specific transforms/operators:		
$\tilde{\mathbf{F}}$	*Fourier Transform* operator	
$\hat{\mathbf{F}}$	*Fourier Series* operator	

...continued on next page...

symbol	description
$\breve{\mathbf{F}}$	*Discrete Time Fourier Series* operator
$\mathbf{Z}$	*Z-Transform* operator
$\tilde{\mathsf{f}}(\omega)$	*Fourier Transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$
$\breve{\mathsf{x}}(\omega)$	*Discrete Time Fourier Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$
$\breve{\mathsf{x}}(z)$	*Z-Transform* of a sequence $(\!(x_n \in \mathbb{C})\!)_{n\in\mathbb{Z}}$

SYMBOL INDEX

CONTENTS

Part I

Modulation

CHAPTER 1

COMMUNICATION CHANNELS

1.1 System model

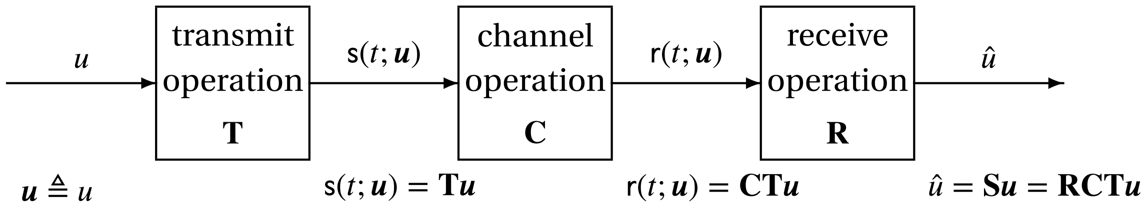


Figure 1.1: Communication system model

A communication system is an operator. $\mathbf{S}$ over an information sequence u that generates an estimated information sequence $\hat{u}$. The system operator factors into a receive operator $\mathbf{R}$, a channel operator $\mathbf{C}$, and a transmit operator $\mathbf{T}$ such that

$$\mathbf{S} = \mathbf{RCT}.$$

The transmit operator operates on an information sequence u to generate a channel signal $\mathsf{s}(t;\boldsymbol{u})$. The channel operator operates on the transmitted signal $\mathsf{s}(t;\boldsymbol{u})$ to generate the received signal $\mathsf{r}(t;\boldsymbol{u})$. The receive operator operates on the received signal $\mathsf{r}(t;\boldsymbol{u})$ to generate the estimate $\hat{u}$ (see Figure 1.1 (page 3)).

Definition 1.1. *Let U be the set of all sequences u and let*

DEF

$$\begin{array}{llll} \mathbf{S} & : U & \to U & \textit{(system operator)} \\ \mathbf{T} & : U & \to \mathbb{R}^\infty & \textit{(transmit operator)} \\ \mathbf{C} & : \mathbb{R}^\infty & \to \mathbb{R}^\infty & \textit{(channel operator)} \\ \mathbf{R} & : \mathbb{R}^\infty & \to U & \textit{(receive operator)} \end{array}$$

*be operators. A **digital communication system** is the operation $\mathbf{S}$ on the set of information sequences U such that $\mathbf{S} \triangleq \mathbf{RCT}$.*

Communication systems can be continuous or discrete valued in time and/or amplitude:

$s(t) = a(t)\psi(t)$	continuous time t	discrete time t
continuous amplitude $a(t)$	analog communications	discrete-time communications
discrete amplitude $a(t)$	—	digital communications

In this document, we normally take the approach that

1. **C** is stochastic
2. There is no structural constraint on **R**.
3. **R** is optimum with respect to the ML-criterion.

These characteristics are explained more fully below.

1.1.1 Channel operator

Real-world physical channels perform a number of operations on a signal. Often these operations are closely modeled by a channel operator **C**. Properties that characterize a particular channel operator associated with some physical channel include

- linear or non-linear
- time-invariant or time-variant
- memoryless or non-memoryless
- deterministic or stochastic.

Examples of physical channels include free space, air, water, soil, copper wire, and fiber optic cable. Information is carried through a channel using some physical process. These processes include:

Process	Example
electromagnetic waves	free space, air (Appendix A page 183)
acoustic waves	water, soil
electric field potential (voltage)	wire
light	fiber optic cable
quantum mechanics	

1.1.2 Receive operator

Let **I** be the *identity operator* (Definition G.3 page 254). Ideally, **R** is selected such that $\mathbf{RCT} = \mathbf{I}$. In this case we say that **R** is the *left inverse*[1] of **CT** and denote this left inverse by **C**. One example of a system where this inverse exists is the noiseless ISI system. While this is quite useful for mathematical analysis and system design, **C** does not actually exist for any real-world system.

When **C** does not exist, the "ideal" **R** is one that is optimum

1. with respect to some *criterion* (or cost function)
2. and sometimes under some structural *constraint*.

When a structural constraint is imposed on **R**, the solution is called *structured*; otherwise, it is called

[1] $\mathbf{X}^{-1}X$ is the *left inverse* of $\mathbf{X}$ if $\mathbf{X}^{-1}X\mathbf{X} = \mathbf{I}$.
$\mathbf{X}^{-1}X$ is the *right inverse* of $\mathbf{X}$ if $\mathbf{X}\mathbf{X}^{-1}X = \mathbf{I}$.
$\mathbf{X}^{-1}X$ is the *inverse* of $\mathbf{X}$ if $\mathbf{X}^{-1}X\mathbf{X} = \mathbf{X}\mathbf{X}^{-1}X = \mathbf{I}$.

non-structured.[2] A common example of a structured approach is the use of a transversal filter (FIR filter in DSP) in which optimal coefficients are found for the filter. A structured **R** is only optimal with respect to the imposed constraint. Even though **R** may be optimal with respect to this structure, **R** may not be optimal in general; that is, there may be another structure that would lead to a "better" solution. In a non-structured approach, **R** is free to take any form whatsoever (practical or impractical) and therefore leads to the best of the best solutions.

The nature of **R** depends heavily on the nature of **C**. If **C** does not exist, then the ideal **R** is one that is optimal with respect to some criterion (Chapter 6 page 63) If **C** is deterministic, then appropriate optimization criterion may include

- least square error (LSE) criterion
- minimum absolute error criterion
- minimum peak distortion criterion.

If **C** is stochastic then appropriate optimization criterion may include

Bayes:	pdf known and cost function defined
Maximum aposteriori probability (MAP):	pdf known and uniform cost function
Maximum likelihood (ML):	pdf known and no prior probability information
mini-max:	pdf not known but a cost function is defined
Neyman-Pearson:	pdf not known and no cost function defined.

Making **R** optimum with respect to one of these criterion leads to an *estimate* $\hat{u} = \mathbf{RCT}\boldsymbol{u}$ that is also optimum with respect to the same criterion (Definition 6.1 page 64).

1.2 Optimization in the case of additional operations

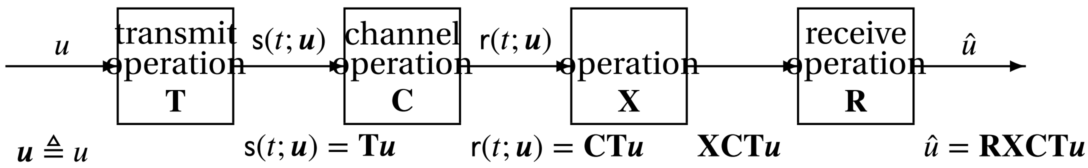


Figure 1.2: Theorem of reversibility

Often in communication systems, an additional operator **X** is inserted such that (see Figure 1.2 (page 5))

$$\mathbf{S} = \mathbf{RXCT}.$$

An example of such an operator **X** is a receive filter. Is it still possible to find an **R** that will perform as well as the case where **X** is not inserted? In general, the answer is "no". For example, if $\mathbf{X}r = 0$, then all received information is lost and obviously there is no **R** that can recover from this event. However, in the case where the right inverse $\mathbf{X}^{-1}X$ of **X** exists, then the answer to the question is "yes" and an optimum **R** still exists. That is, it doesn't matter if an **X** is inserted into system as long as **X** is invertible. This is stated formally in the next theorem.

Theorem 1.1 (Theorem of Reversibility). [3] *Let*

- $\hat{\theta}_=\mathbf{RCT}\boldsymbol{u}$ *be the optimum estimate of* $\boldsymbol{u}$

---

[2] Trees (2001) page 12
[3] Trees (2001) pages 289–290

$\mathbf{X}$ *be an operator with right inverse* $\mathbf{X}^{-1}X$.

Then there exists some $\mathbf{R}'$ *such that*

THM $\hat{\theta} = \mathbf{R}'\mathbf{X}\mathbf{C}\mathbf{T}\boldsymbol{u}$.

PROOF: Let $\mathbf{R}' = \mathbf{R}\mathbf{X}^{-1}X$. Then

$$\mathbf{R}'\mathbf{X}\mathbf{C}\mathbf{T}\boldsymbol{u} = \mathbf{R}\mathbf{X}^{-1}X\mathbf{C}\mathbf{T}\boldsymbol{u} = \mathbf{R}\mathbf{C}\mathbf{T}\boldsymbol{u} = \hat{\theta}.$$

1.3 Alternative system partitioning

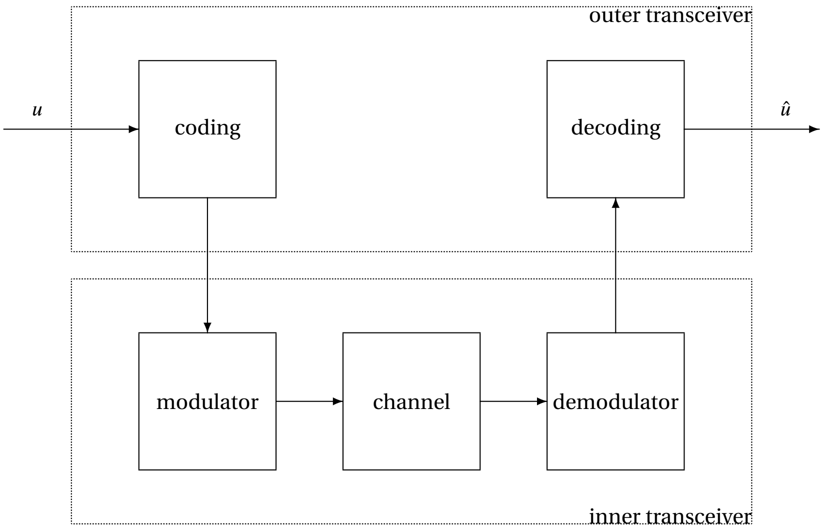


Figure 1.3: Inner/outer transceiver

A communication system can be partitioned into two parts (see Figure 1.3 (page 6)):[4]

1. outer transceiver: data encoding/decoding
2. inner transceiver: modulation/demodulation.

The outer transceiver can perform several types of coding on the data sequence to be transmitted:

1. source coding: compress data sequence size (lower limit is Shannon Entropy H)
2. channel coding: modify data sequence such that errors induced by the channel can be detected and corrected (all errors can be theoretically corrected if the data rate is at or below the Shannon channel capacity C).
3. modulation coding: make sequence "more suitable" for transmission through channel
4. encryption: increase the difficulty which an eavesdropper would need to be able to know the data sequence.

[4] Meyr et al. (1998), page 2

1.4 Channel Statistics

The receiver needs to make a decision as to what sequence $(\!(u)\!)$ the transmitter has sent. This decision should be optimal in some sense. Very often the optimization criterion is chosen to be the *maximal likelihood (ML)* criterion. The information that the receiver can use to make an optimal decision is the received signal $\mathsf{r}(t)$.

If the symbols in $\mathsf{r}(t)$ are statistically *independent*, then the optimal estimate of the current symbol depends only on the current symbol period of $\mathsf{r}(t)$. Using other symbol periods of $\mathsf{r}(t)$ has absolutely no additional benefit. Note that the AWGN channel is *memoryless*; that is, the way the channel treats the current symbol has nothing to do with the way it has treated any other symbol. Therefore, if the symbols sent by the transmitter into the channel are independent, the symbols coming out of the channel are also independent.

However, also note that the symbols sent by the transmitter are often very intentionally not independent; but rather a strong relationship between symbols is intentionally introduced. This relationship is called *channel coding*. With proper channel coding, it is theoretically possible to reduce the probability of communication error to any arbitrarily small value as long as the channel is operating below its *channel capacity*.

This chapter assumes that the received symbols are statistically independent; and therefore optimal decisions at the receiver for the current symbol are made only from the current symbol period of $\mathsf{r}(t)$.

The received signal $\mathsf{r}(t)$ over a single symbol period contains an uncountably infinite number of points. That is a lot. It would be nice if the receiver did not have to look at all those uncountably infinite number of points when making an optimal decision. And in fact the receiver does indeed not have to. As it turns out, a single finite set of *statistics* $\{\dot{r}_1,\ \dot{r}_2, \ldots, \dot{r}_N\}$ is sufficient (Theorem 7.1 page 71) for the receiver to make an optimal decision as to which value the transmitter sent.

CHAPTER 2

NARROWBAND SIGNALS

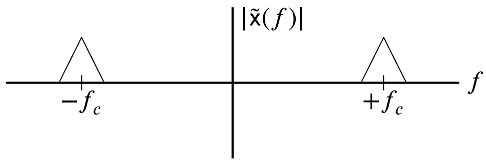


Figure 2.1: Narrowband signal

Communication systems are often assumed to be *narrowband* (next definition) meaning the bandwidth of the information carrying signal is "small" compared to the carrier frequency (Figure 2.1 page 9).

Definition 2.1. *Let* $\mathsf{x} : \mathbb{R} \to \mathbb{R}$ *be an information carrying waveform,* $\tilde{\mathsf{x}}(f) = [\tilde{\mathbf{F}}\mathsf{x}](f)$ *and* $\mathsf{f}_c \in \mathbb{R}$.

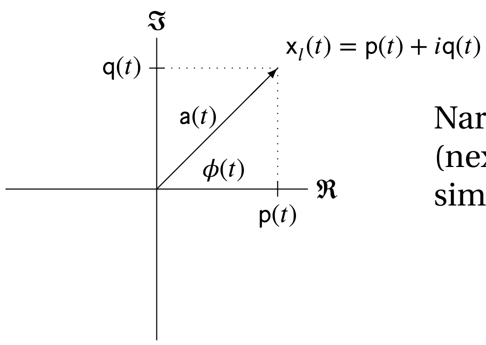
The signal $\mathsf{x}(t)$ *is* ***narrowband*** *if*

(1). The energy of $\tilde{\mathsf{x}}(f)$ *is located in the vicinity of frequency* $\pm f_c$ *and*

(2). the bandwidth of $\tilde{\mathsf{x}}(f)$ *is "small" compared to* f_c.

If $\mathsf{x}(t)$ *is the transmitted signal in a communication system* $\mathbf{S} = \mathbf{RCT}$ *such that* $\mathsf{x}(t) = \mathbf{T}\boldsymbol{u}$, *and* $\mathsf{x}(t)$ *is* NARROWBAND, *then* $\mathbf{S}$ *is a* ***narrowband system***.

2.1 Time representation



Narrowband signals have three common time representations (next definition). These three forms are equivalent under some simple relations (next proposition).

Definition 2.2. *Let* $\mathsf{x}(t)$ *be a* NARROWBAND SIGNAL *(Definition 2.1 page 9)*.

DEF

Let $\mathsf{x}(t) \triangleq \mathsf{a}(t)\cos\left[2\pi f_c t + \phi(t)\right]$. *Then*

$\mathsf{a}(t)$	*is the* ***amplitude***	*of* $\mathsf{x}(t)$	
$\phi(t)$	*is the* ***phase***	*of* $\mathsf{x}(t)$	
$\mathsf{p}(t)$	*is the* ***quadrature component***	*of* $\mathsf{x}(t)$ *where*	$\mathsf{p}(t) \triangleq \mathsf{a}(t)\cos[\phi(t)]$
$\mathsf{q}(t)$	*is the* ***inphase component***	*of* $\mathsf{x}(t)$ *where*	$\mathsf{q}(t) \triangleq \mathsf{a}(t)\sin[\phi(t)]$
$\mathsf{x}_l(t)$	*is the* ***complex envelope***	*of* $\mathsf{x}(t)$ *where*	$\mathsf{x}_l(t) \triangleq \mathsf{p}(t) + i\mathsf{q}(t)$

A narrowband signal $\mathsf{x} : \mathbb{R} \to \mathbb{R}$ can be represented by any of the following three **equivalent canonical forms**:

Proposition 2.1. *Let* $\mathsf{x}(t)$ *be defined as in Definition 2.3.*

PRP

$$
\begin{aligned}
\mathsf{x}(t) &\triangleq \mathsf{a}(t)\cos\left[2\pi f_c t + \phi(t)\right] && \text{(AMPLITUDE AND PHASE FORM)}\\
&= \mathsf{p}(t)\cos(2\pi f_c t) - \mathsf{q}(t)\sin(2\pi f_c t) && \text{(QUADRATURE FORM)}\\
&= \mathbf{R_e}\left[\mathsf{x}_l(t)e^{i2\pi f_c t}\right] && \text{(COMPLEX ENVELOPE FORM)}
\end{aligned}
$$

✎PROOF:

$$
\begin{aligned}
\mathsf{x}(t) &\triangleq \underbrace{\boxed{\mathsf{a}(t)\cos\left[2\pi f_c t + \phi(t)\right]}}_{\text{amplitude-phase form}} && \text{amplitude and phase form} && \text{(Definition 2.3 page 10)}\\
&= \underbrace{\mathsf{a}(t)\cos[\phi(t)]}_{\mathsf{p}(t)}\cos[2\pi f_c t] - \underbrace{\mathsf{a}(t)\sin[\phi(t)]}_{\mathsf{q}(t)}\sin[2\pi f_c t] && \text{by \textit{double angle formulas}}\\
&= \underbrace{\boxed{\mathsf{p}(t)\cos[2\pi f_c t] - \mathsf{q}(t)\sin[2\pi f_c t]}}_{\text{quadrature form}} && \text{quadrature form} && \text{(Definition 2.3 page 10)}\\
&= \mathbf{R_e}\left([\mathsf{p}(t) + i\mathsf{q}(t)][\cos(2\pi f_c t) + i\sin(2\pi f_c t)]\right) && \text{by definitions of } \mathbf{R_e} && \text{(Definition D.1 page 209)}\\
&= \underbrace{\boxed{\mathbf{R_e}\left[\mathsf{x}_l(t)e^{i2\pi f_c t}\right]}}_{\text{complex envelope form}} && \text{by \textit{Euler's identity}}
\end{aligned}
$$

✏

The three canonical forms in Proposition 2.1 (page 10) are now designated formally:

Definition 2.3. *Let* $\mathsf{x}(t)$ *be a* NARROWBAND SIGNAL.

DEF

$\mathsf{x}(t) \triangleq \mathsf{a}(t)\cos\left[2\pi f_c t + \phi(t)\right]$	*is the*	***amplitude and phase form***	*of* $\mathsf{x}(t)$
$\mathsf{x}(t) \triangleq \mathsf{p}(t)\cos(2\pi f_c t) - \mathsf{q}(t)\sin(2\pi f_c t)$	*is the*	***quadrature form***[1]	*of* $\mathsf{x}(t)$
$\mathsf{x}(t) \triangleq \mathbf{R_e}\left[\mathsf{x}_l(t)e^{i2\pi f_c t}\right]$	*is the*	***complex envelope form***	*of* $\mathsf{x}(t)$

Proposition 2.1 (page 10) gave the three canonical forms (Definition 2.3 page 10) in terms of a modulated narrowband signal $\mathsf{x}(t)$ and some quadrature components defined in Definition 2.2 (page 9). Proposition 2.2 (next) gives some relationships between these components.

Proposition 2.2.

PRP

$$
\begin{aligned}
\mathsf{x}_l(t) &= \mathsf{a}(t)e^{i\phi(t)}\\
\mathsf{a}(t) &= \sqrt{\mathsf{p}^2(t) + \mathsf{q}^2(t)} & \phi(t) &= \arctan\frac{\mathsf{q}(t)}{\mathsf{p}(t)}\\
\mathsf{p}(t) &= \mathsf{a}(t)\cos\phi & \mathsf{q}(t) &= \mathsf{a}(t)\sin\phi\\
\mathsf{p}(t) &= \mathbf{R_e}\left[\mathsf{x}_l(t)\right] & \mathsf{q}(t) &= \mathbf{I_m}\left[\mathsf{x}_l(t)\right]
\end{aligned}
$$

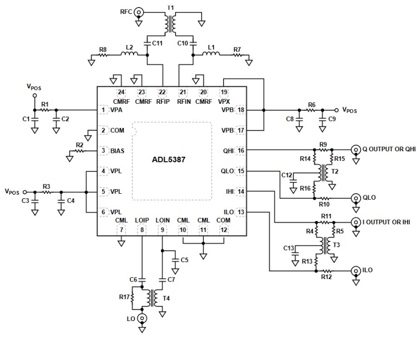
[1] $\mathsf{x}(t) = \mathsf{p}(t)\cos(2\pi f_c t) - \mathsf{q}(t)\sin(2\pi f_c t)$ is also known as *Rice's representation*.
Reference: Srinath et al. (1996), page 23

✎PROOF:

$$
\begin{aligned}
\mathsf{p} &= \mathbf{R_e}\left[\mathsf{p}+i\mathsf{q}\right] &&= \mathbf{R_e}\left[\mathsf{x}_l\right] \\
\mathsf{q} &= \mathbf{I_m}\left[\mathsf{p}+i\mathsf{q}\right] &&= \mathbf{I_m}\left[\mathsf{x}_l\right] \\
\mathsf{p} &= \mathbf{R_e}\left[\mathsf{p}+i\mathsf{q}\right] &&= \mathbf{R_e}\left[\mathsf{a}e^{i\phi}\right] &&= \mathbf{R_e}\left[\mathsf{a}\cos\phi + i\mathsf{a}\sin\phi\right] &&= \mathsf{a}\cos\phi \\
\mathsf{q} &= \mathbf{I_m}\left[\mathsf{p}+i\mathsf{q}\right] &&= \mathbf{I_m}\left[\mathsf{a}e^{i\phi}\right] &&= \mathbf{I_m}\left[\mathsf{a}\cos\phi + i\mathsf{a}\sin\phi\right] &&= \mathsf{a}\sin\phi \\
\mathsf{a}^2 &= \mathsf{a}^2(\cos^2\phi+\sin^2\phi) &&= (\mathsf{a}\cos\phi)^2+(\mathsf{a}\sin\phi)^2 &&= \mathsf{p}^2+\mathsf{q}^2 \\
\tan\phi &= \frac{\sin\phi}{\cos\phi} &&= \frac{\mathsf{a}\sin\phi}{\mathsf{a}\cos\phi} &&= \frac{\mathsf{q}}{\mathsf{p}}
\end{aligned}
$$

✏

Remark 2.1. In practice (with real hardware), you will likely first have access to the quadrature components $\mathsf{p}(t)$ and $\mathsf{q}(t)$. Take for example the *Analog Devices ADL5387 Quadrature Demodulator* and evaluation board, as illustrated to the right.[2] Note that *quadrature component* $\mathsf{p}(t)$ is available at connector "Q OUTPUT" and *in-phase component* $\mathsf{q}(t)$ is available at connector "I OUTPUT".



2.2 Frequency Representation

Any *real-valued* time signal $\mathsf{x} : \mathbb{R} \to \mathbb{R}$ is always *hermitian symmetric* in frequency such that $\tilde{\mathsf{x}}(f) = \tilde{\mathsf{x}}^*(-f)$ (Figure 2.2 page 11).

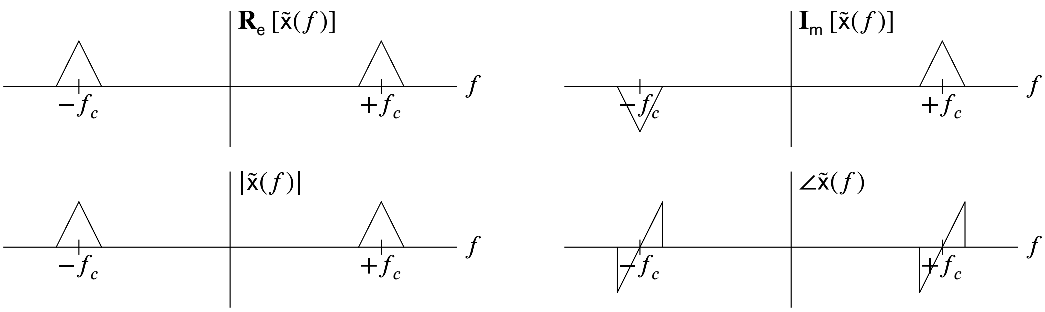


Figure 2.2: Frequency characteristics of any real-valued signal $\mathsf{x}(t)$

2.3 Lowpass representation

The *complex envelope* $\mathsf{x}_l : \mathbb{R} \to \mathbb{C}$ (Definition 2.3 page 10) of a narrowband signal $\mathsf{x} : \mathbb{R} \to \mathbb{R}$ is sometimes called the *lowpass representation* of $\mathsf{x}(t)$. Because all the information carried by $\mathsf{x}(t)$ is contained

[2] diagram copied from Devices (2016)

within a narrow band of $\tilde{x}(f)$, the lowpass representation $x_l(t)$ along with the parameter f_c is a sufficient representation of $x(t)$ and thus the high frequency factor $e^{j2\pi f_c t}$ may for engineering purposes often be ignored.

The sufficiency of the low-pass representation $x_l(t)$ is demonstrated in that

1. $x_l(t)$ together with f_c is sufficient to represent $x(t)$ in time (by Definition 2.3 (page 10))
2. $\tilde{x}_l(f)$ together with f_c is sufficient to represent $\tilde{x}(f)$ in frequency (Theorem 2.1 (page 12))
3. $x_l(t)$ is sufficient to calculate the energy in $x(t)$ (Theorem 2.1 (page 12))
4. $x_l(t)$ and the impulse response $h(t)$ of an LTI operation is sufficient to calculate the output of the LTI operation on $x(t)$ (Theorem 2.2 (page 13)).

Theorem 2.1. *Let $x : \mathbb{R} \to \mathbb{R}$ be a signal with center frequency $f_c \in \mathbb{R}$ and $x_l : \mathbb{R} \to \mathbb{C}$ the complex envelope of $x(t)$* (Definition 2.3 page 10).

THM

$$\left\{ \begin{array}{l} x(t) \textit{ is} \\ \text{NARROWBAND} \end{array} \right\} \implies \left\{ \begin{array}{rcl} \tilde{x}(f) &=& \frac{1}{2}\tilde{x}_l(f-f_c) + \frac{1}{2}\tilde{x}_l^*(-f-f_c) \\ \mathbf{E}x(t) &\approx& \frac{1}{2}\mathbf{E}x_l(t) \\ |\tilde{x}(f)|^2 &=& \frac{1}{4}\left|\tilde{x}_l(f-f_c)\right|^2 + \frac{1}{4}\left|\tilde{x}_l(-f-f_c)\right|^2 \\ \angle\tilde{x}(f) &=& \left. \begin{array}{rl} \angle\tilde{x}_l(f-f_c) & \textit{for } f \approx +f_c \\ -\angle\tilde{x}_l(f+f_c) & \textit{for } f \approx -f_c \end{array} \right\} \end{array} \right\}$$

✎PROOF:

1. Proof that $\mathbf{E}x(t) \approx \frac{1}{2}\mathbf{E}x_l(t)$:

$$\begin{aligned}
\mathbf{E}x(t) &\triangleq \|x(t)\|^2 \\
&= \left\|\mathbf{R_e}\left[x_l(t)e^{j2\pi f_c t}\right]\right\|^2 \\
&= \left\|\frac{1}{2}x_l(t)e^{j2\pi f_c t} + \frac{1}{2}x_l^*(t)e^{-j2\pi f_c t}\right\|^2 && \text{by \textit{Euler formulas} (Corollary D.2 page 217)} \\
&= \left\|\frac{1}{2}x_l(t)e^{j2\pi f_c t}\right\|^2 + \left\|\frac{1}{2}x_l^*(t)e^{-j2\pi f_c t}\right\|^2 + 2\mathbf{R_e}\left[\left\langle\frac{1}{2}x_l(t)e^{j2\pi f_c t} \,|\, \frac{1}{2}x_l^*(t)e^{-j2\pi f_c t}\right\rangle\right] && \text{by \textit{Polar Identity}} \\
&= \frac{1}{4}\left\|x_l(t)\right\|^2 + \frac{1}{4}\left\|x_l(t)\right\|^2 + \frac{1}{2}\underbrace{\mathbf{R_e}\left[\left\langle x_l(t)e^{j2\pi f_c t} \,|\, x_l^*(t)e^{-j2\pi f_c t}\right\rangle\right]}_{\approx 0 \text{ after \textit{low-pass filtering}}} \\
&\approx \frac{1}{2}\left\|x_l(t)\right\|^2 \\
&\triangleq \frac{1}{2}\mathbf{E}x_l(t)
\end{aligned}$$

2. lemma: $\tilde{x}(f) = \frac{1}{2}\tilde{x}_l(f-f_c) + \frac{1}{2}\tilde{x}_l^*(-f-f_c)$. Proof:

$$\begin{aligned}
\boxed{\tilde{x}(f)} &\triangleq [\tilde{\mathbf{F}}x(t)](f) && \text{by definition of } \tilde{x}(f) \\
&\triangleq \left\langle x(t) \,|\, e^{j2\pi f t}\right\rangle && \text{by definition of \textit{Fourier Transform}} \\
&\triangleq \left\langle \mathbf{R_e}\left[x_l(t)e^{j2\pi f_c t}\right] \,|\, e^{j2\pi f t}\right\rangle && \text{by definition of \textit{complex envelope} } x_l(t) \\
&= \left\langle \frac{1}{2}\left[x_l(t)e^{j2\pi f_c t} + x_l^*(t)e^{-j2\pi f_c t}\right] \,|\, e^{j2\pi f t}\right\rangle && \text{by \textit{Euler formulas} (Corollary D.2 page 217)} \\
&= \frac{1}{2}\left\langle x_l(t)e^{j2\pi f_c t} \,|\, e^{j2\pi f t}\right\rangle + \frac{1}{2}\left\langle x_l^*(t)e^{-j2\pi f_c t} \,|\, e^{j2\pi f t}\right\rangle && \text{by \textit{additive} property of } \langle\triangle \,|\, \triangledown\rangle \\
&\triangleq \frac{1}{2}\int_{t\in\mathbb{R}} x_l(t)e^{-j2\pi(f-f_c)t}\,dt + \frac{1}{2}\left[\int_{t\in\mathbb{R}} x_l(t)e^{-j2\pi(-f-f_c)t}\,dt\right]^* && \text{by definition of } \langle\triangle \,|\, \triangledown\rangle \\
&\triangleq \boxed{\frac{1}{2}\tilde{x}_l(f-f_c) + \frac{1}{2}\tilde{x}_l^*(-f-f_c)} && \text{by definition of \textit{Fourier Transform}}
\end{aligned}$$

3.

$$\begin{aligned}
\mathbf{R_e}\left[\tilde{x}(f)\right] &= \mathbf{R_e}\left[\frac{1}{2}\tilde{x}_l(f-f_c)+\frac{1}{2}\tilde{x}_l^*(-f-f_c)\right] && \text{by (2) lemma}\\
&= \frac{1}{2}\mathbf{R_e}\left[\tilde{x}_l(f-f_c)\right]+\frac{1}{2}\mathbf{R_e}\left[\tilde{x}_l^*(-f-f_c)\right]\\
&= \frac{1}{2}\mathbf{R_e}\left[\tilde{x}_l(f-f_c)\right]+\frac{1}{2}\mathbf{R_e}\left[\tilde{x}_l(-f-f_c)\right]\\
\mathbf{I_m}\left[\tilde{x}(f)\right] &= \mathbf{I_m}\left[\frac{1}{2}\tilde{x}_l(f-f_c)+\frac{1}{2}\tilde{x}_l^*(-f-f_c)\right] && \text{by (2) lemma}\\
&= \frac{1}{2}\mathbf{I_m}\left[\tilde{x}_l(f-f_c)\right]+\frac{1}{2}\mathbf{I_m}\left[\tilde{x}_l^*(-f-f_c)\right]\\
&= \frac{1}{2}\mathbf{I_m}\left[\tilde{x}_l(f-f_c)\right]-\frac{1}{2}\mathbf{I_m}\left[\tilde{x}_l(-f-f_c)\right]\\
|\tilde{x}(f)|^2 &= \left|\frac{1}{2}\tilde{x}_l(f-f_c)+\frac{1}{2}\tilde{x}_l^*(-f-f_c)\right|^2 && \text{by (2) lemma}\\
&= \frac{1}{4}\left[\tilde{x}_l(f-f_c)+\frac{1}{2}\tilde{x}_l^*(-f-f_c)\right]\left[\tilde{x}_l(f-f_c)+\frac{1}{2}\tilde{x}_l^*(-f-f_c)\right]^*\\
&= \frac{1}{4}\Big[\tilde{x}_l(f-f_c)\tilde{x}_l^*(f-f_c)+\tilde{x}_l(f-f_c)\tilde{x}_l(-f-f_c)+\\
&\qquad \tilde{x}_l^*(-f-f_c)\tilde{x}_l^*(f-f_c)+\tilde{x}_l^*(-f-f_c)\tilde{x}_l(-f-f_c)\Big]\\
&= \frac{1}{4}\left[\left|\tilde{x}_l(f-f_c)\right|^2+2\mathbf{R_e}\left[\tilde{x}_l(f-f_c)\tilde{x}_l(-f-f_c)\right]+\left|\tilde{x}_l^*(-f-f_c)\right|^2\right]\\
&= \frac{1}{4}\left[\left|\tilde{x}_l(f-f_c)\right|^2+\left|\tilde{x}_l(-f-f_c)\right|^2+0\right]\\
\angle\tilde{x}(f) &= \angle\left[\frac{1}{2}\tilde{x}_l(f-f_c)+\frac{1}{2}\tilde{x}_l^*(-f-f_c)\right]\\
&= \angle\left[\tilde{x}_l(f-f_c)+\tilde{x}_l^*(-f-f_c)\right]\\
&= \operatorname{atan}\frac{\mathbf{I_m}\left[\tilde{x}_l(f-f_c)+\tilde{x}_l^*(-f-f_c)\right]}{\mathbf{R_e}\left[\tilde{x}_l(f-f_c)+\tilde{x}_l^*(-f-f_c)\right]} && \text{by definition of } \angle\\
&= \operatorname{atan}\frac{\mathbf{I_m}\left[\tilde{x}_l(f-f_c)\right]+\mathbf{I_m}\left[\tilde{x}_l^*(-f-f_c)\right]}{\mathbf{R_e}\left[\tilde{x}_l(f-f_c)\right]+\mathbf{R_e}\left[\tilde{x}_l^*(-f-f_c)\right]}\\
&= \operatorname{atan}\frac{\mathbf{I_m}\left[\tilde{x}_l(f-f_c)\right]-\mathbf{I_m}\left[\tilde{x}_l(-f-f_c)\right]}{\mathbf{R_e}\left[\tilde{x}_l(f-f_c)\right]+\mathbf{R_e}\left[\tilde{x}_l(-f-f_c)\right]}\\
&= \begin{cases} \angle\tilde{x}_l(f-f_c) & : f\approx +f_c\\ -\angle\tilde{x}_l(f+f_c) & : f\approx -f_c\end{cases}
\end{aligned}$$

✏

Theorem 2.2. *Lowpass LTI theorem.*

1. *Let* $\mathsf{x}:\mathbb{R}\to\mathbb{R}$ *be a narrowband signal at center frequency* $f_c\in\mathbb{R}$*, with complex envelope* $\mathsf{x}_l:\mathbb{R}\to\mathbb{C}$*, and Fourier transform* $\tilde{\mathsf{x}}:\mathbb{R}\to\mathbb{C}$*.*
2. *Let* $\mathsf{h}:\mathbb{R}\to\mathbb{R}$ *be the narrowband impulse response of an LTI operation such that* $\mathsf{h}(t)$ *is located at center frequency* $f_c\in\mathbb{R}$*, has complex envelope* $\mathsf{h}_l:\mathbb{R}\to\mathbb{C}$*, and Fourier transform* $\tilde{\mathsf{h}}:\mathbb{R}\to\mathbb{C}$*.*
3. *Let* $\mathsf{y}:\mathbb{R}\to\mathbb{R}$ *be the response of the LTI operation on* $\mathsf{x}(t)$*. Let the complex envelope of* $\mathsf{y}(t)$ *be* $\mathsf{y}_l:\mathbb{R}\to\mathbb{C}$ *and the Fourier transform* $\tilde{\mathsf{y}}:\mathbb{R}\to\mathbb{C}$*.*

Then

THM
$$\begin{aligned}
\mathsf{y}_l(t) &= \frac{1}{2}\mathsf{h}_l(t)\star\mathsf{x}_l(t)\\
\tilde{\mathsf{y}}_l(f) &= \frac{1}{2}\tilde{\mathsf{h}}_l(f)\tilde{\mathsf{x}}_l(f)
\end{aligned}$$

✎PROOF:

$$
\begin{aligned}
\mathbf{R}_\mathsf{e}\left[\mathsf{y}_l(t)e^{i2\pi f_c t}\right] &= \mathsf{y}(t) \\
&= \mathsf{h}(t)\star\mathsf{x}(t) \\
&= \int_u \mathsf{h}(u)\mathsf{x}(t-u)\,\mathsf{d}u \\
&= \int_u \mathbf{R}_\mathsf{e}\left[\mathsf{h}_l(u)e^{i2\pi\mathsf{f}_c u}\right]\mathbf{R}_\mathsf{e}\left[\mathsf{x}_l(t-u)e^{i2\pi\mathsf{f}_c(t-u)}\right]\,\mathsf{d}u \\
&= \frac{1}{4}\int_u \left[\mathsf{h}_l(t)e^{i2\pi f_c t}+\mathsf{h}_l^*(t)e^{-i2\pi f_c t}\right]\left[\mathsf{x}_l(t-u)e^{i2\pi f_c(t-u)}+\mathsf{x}_l^*(t-u)e^{-i2\pi f_c(t-u)}\right]\,\mathsf{d}u \\
&= \frac{1}{4}\int_u \mathsf{h}_l(u)e^{i2\pi\mathsf{f}_c u}\mathsf{x}_l(t-u)e^{i2\pi f_c(t-u)}\,\mathsf{d}u+\frac{1}{4}\int_u \mathsf{h}_l(u)e^{i2\pi\mathsf{f}_c u}\mathsf{x}_l^*(t-u)e^{-i2\pi f_c(t-u)}\,\mathsf{d}u+ \\
&\quad\ \frac{1}{4}\int_u \mathsf{h}_l^*(u)e^{-i2\pi\mathsf{f}_c u}\mathsf{x}_l(t-u)e^{i2\pi f_c(t-u)}\,\mathsf{d}u+\frac{1}{4}\int_u \mathsf{h}_l^*(u)e^{-i2\pi\mathsf{f}_c u}\mathsf{x}_l^*(t-u)e^{-i2\pi f_c(t-u)}\,\mathsf{d}u \\
&= \frac{1}{4}e^{i2\pi f_c t}\int_u \mathsf{h}_l(u)\mathsf{x}_l(t-u)\,\mathsf{d}u+\frac{1}{4}e^{-i2\pi f_c t}\int_u \mathsf{h}_l(u)e^{i4\pi\mathsf{f}_c u}\mathsf{x}_l^*(t-u)\,\mathsf{d}u+ \\
&\quad\ \frac{1}{4}e^{i2\pi f_c t}\int_u \mathsf{h}_l^*(u)e^{-i4\pi\mathsf{f}_c u}\mathsf{x}_l(t-u)\,\mathsf{d}u+\frac{1}{4}e^{-i2\pi f_c t}\int_u \mathsf{h}_l^*(u)\mathsf{x}_l^*(t-u)\,\mathsf{d}u \\
&= \frac{1}{4}e^{i2\pi f_c t}\int_u \mathsf{h}_l(u)\mathsf{x}_l(t-u)\,\mathsf{d}u+\frac{1}{4}\left(e^{i2\pi f_c t}\int_u \mathsf{h}_l(u)\mathsf{x}_l(t-u)\,\mathsf{d}u\right)^*+ \\
&\quad\ \frac{1}{4}e^{i2\pi f_c t}\int_u \mathsf{h}_l^*(u)e^{-i4\pi\mathsf{f}_c u}\mathsf{x}_l(t-u)\,\mathsf{d}u+\frac{1}{4}\left(e^{i2\pi f_c t}\int_u \mathsf{h}_l^*(u)e^{-i4\pi\mathsf{f}_c u}\mathsf{x}_l(t-u)\,\mathsf{d}u\right)^* \\
&= \frac{1}{2}\mathbf{R}_\mathsf{e}\left[e^{i2\pi f_c t}\int_u \mathsf{h}_l(u)\mathsf{x}_l(t-u)\,\mathsf{d}u\right]+\frac{1}{2}\mathbf{R}_\mathsf{e}\left[e^{i2\pi f_c t}\int_u \mathsf{h}_l^*(u)e^{-i4\pi\mathsf{f}_c u}\mathsf{x}_l(t-u)\,\mathsf{d}u\right] \\
&= \frac{1}{2}\mathbf{R}_\mathsf{e}\left[e^{i2\pi f_c t}[\mathsf{h}_l\star\mathsf{x}_l](t)\right]+\frac{1}{2}\mathbf{R}_\mathsf{e}\left[e^{i2\pi f_c t}\int_u \mathsf{h}_l^*(u)\mathsf{x}_l(t-u)e^{-i4\pi\mathsf{f}_c u}\,\mathsf{d}u\right] \\
&\approx \frac{1}{2}\mathbf{R}_\mathsf{e}\left[e^{i2\pi f_c t}[\mathsf{h}_l\star\mathsf{x}_l](t)\right]+0?
\end{aligned}
$$

Note that convolving $\mathsf{x}_l(t)$ with $\mathsf{h}(t)$ directly does not work (we still need the factor $e^{i2\pi\mathsf{f}_c(t)}$).

$$
\begin{aligned}
\mathbf{R}_\mathsf{e}\left[\mathsf{y}_l(t)e^{i2\pi f_c t}\right] &= \mathsf{y}(t) \\
&= \mathsf{h}(t)\star\mathsf{x}(t) \\
&= \int_u \mathsf{h}(u)\mathsf{x}(t-u)\,\mathsf{d}u \\
&= \int_u \mathsf{h}(u)\mathbf{R}_\mathsf{e}\left[\mathsf{x}_l(t-u)e^{i2\pi\mathsf{f}_c(t-u)}\right]\,\mathsf{d}u \\
&= \mathbf{R}_\mathsf{e}\left[\int_u \mathsf{h}(u)\mathsf{x}_l(t-u)e^{i2\pi\mathsf{f}_c(t-u)}\,\mathsf{d}u\right] \\
&= \mathbf{R}_\mathsf{e}\left[\mathsf{h}(t)\star\left[\mathsf{x}_l(t)e^{i2\pi\mathsf{f}_c(t)}\right]\right]
\end{aligned}
$$

✏

2.4 Narrowband noise processes

A narrowband noise process $\mathsf{n}(t)$ can be represented in any of the three canonical forms presented in Definition 2.3 (page 10) (page 10):

$$\mathsf{n}(t) = \mathsf{a}(t)\cos[2\pi f_c t+\phi(t)] \qquad \text{(amplitude and phase)}$$

$$\begin{aligned}
&= \mathsf{p}(t)\cos(2\pi f_c t) - \mathsf{q}(t)\sin(2\pi f_c t) && \text{(quadrature)}\\
&= \mathbf{R_e}\left(\mathsf{n}_l(t)e^{j2\pi f_c t}\right) && \text{(complex envelope).}
\end{aligned}$$

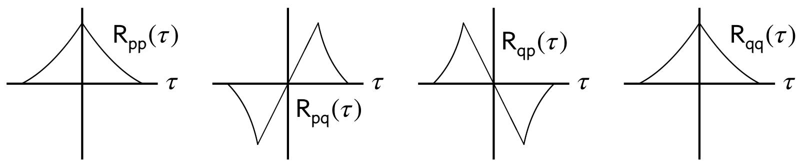


Figure 2.3: Correlations of inphase component $\mathsf{p}(t)$ and quadrature component $\mathsf{q}(t)$

Theorem 2.3. *Let* $\mathsf{n} : \mathbb{R} \to \mathbb{R}$ *be a narrowband noise process with quadrature components* $\mathsf{p} : \mathbb{R} \to \mathbb{R}$ *and* $\mathsf{q} : \mathbb{R} \to \mathbb{R}$ *and complex envelope* $\mathsf{z} : \mathbb{R} \to \mathbb{C}$ *such that*

$$\begin{aligned}
n(t) &= \mathsf{p}(t)\cos(2\pi f_c t) - \mathsf{q}(t)\sin(2\pi f_c t)\\
&= \mathbf{R_e}\left[\mathsf{z}(t)e^{i2\pi f_c t}\right]\\
\mathsf{R}_{\mathsf{xy}}(\tau) &\triangleq \mathsf{E}\left[\mathsf{x}(t+\tau)\mathsf{y}^*(t)\right].
\end{aligned}$$

Then (see Figure 2.3 (page 15))

THM

1. $\mathsf{E}[p(t)] = \mathsf{E}[q(t)] = 0$ *(component means are zero)*
2. $\mathsf{R}_{\mathsf{pp}}(\tau) = \mathsf{R}_{\mathsf{qq}}(\tau)$ *(autocorrelations are equal)*
3. $\mathsf{R}_{\mathsf{pq}}(\tau) = -\mathsf{R}_{\mathsf{qp}}(\tau)$ *(crosscorrelations are additive inverses)*
4. $\mathsf{R}_{\mathsf{pp}}(\tau) = \mathsf{R}_{\mathsf{pp}}(-\tau)$ *(autocorrelations are symmetric)*
5. $\mathsf{R}_{\mathsf{pq}}(\tau) = -\mathsf{R}_{\mathsf{pq}}(-\tau), \mathsf{R}_{\mathsf{qp}}(\tau) = -\mathsf{R}_{\mathsf{qp}}(-\tau)$ *(crosscorrelations are anti-symmetric)*
6. $\mathsf{R}_{\mathsf{pq}}(0) = 0$ *(components are uncorrelated for $\tau = 0$)*
7. $\mathsf{R}_{\mathsf{nn}}(\tau) = \mathsf{R}_{\mathsf{pp}}(\tau)\cos(2\pi f_c\tau) + \mathsf{R}_{\mathsf{pq}}(\tau)\sin(2\pi f_c\tau)$ *(noise autocorrelation)*
8. $\mathsf{R}_{\mathsf{zz}}(\tau) = 2\mathsf{R}_{\mathsf{pp}}(\tau) - 2i\mathsf{R}_{\mathsf{pq}}(\tau)$ *(complex envelope autocorrelation).*

✎PROOF:

$$\begin{aligned}
0 &= \mathsf{E}[n(t)]\\
&= \mathsf{E}\left[p(t)\cos(2\pi f_c t) - q(t)\sin(2\pi f_c t)\right]\\
&= \mathsf{E}\left[p(t)\cos(2\pi f_c t)\right] - \mathsf{E}\left[q(t)\sin(2\pi f_c t)\right]\\
&= \mathsf{E}[p(t)]\cos(2\pi f_c t) - \mathsf{E}[q(t)]\sin(2\pi f_c t)
\end{aligned}$$

$$\begin{aligned}
\mathsf{R}_{\mathsf{nn}}(\tau) &= \mathsf{E}[n(t+\tau)n(t)]\\
&= \mathsf{E}\left[\left(p(t+\tau)\cos(2\pi f_c t + 2\pi f_c\tau) - q(t)\sin(2\pi f_c t + 2\pi f_c\tau)\right)\left(p(t)\cos(2\pi f_c t) - q(t)\sin(2\pi f_c t)\right)\right]\\
&= \mathsf{E}\left[p(t+\tau)p(t)\cos(2\pi f_c t + 2\pi f_c\tau)\cos(2\pi f_c t)\right] - \mathsf{E}\left[p(t+\tau)q(t)\cos(2\pi f_c t + 2\pi f_c\tau)\sin(2\pi f_c t)\right]\\
&\quad -\mathsf{E}\left[q(t+\tau)p(t)\sin(2\pi f_c t + 2\pi f_c\tau)\cos(2\pi f_c t)\right] + \mathsf{E}\left[q(t+\tau)q(t)\sin(2\pi f_c t + 2\pi f_c\tau)\sin(2\pi f_c t)\right]\\
&= \mathsf{R}_{\mathsf{pp}}(\tau)\mathsf{E}\left[\cos(2\pi f_c t + 2\pi f_c\tau)\cos(2\pi f_c t)\right] - \mathsf{R}_{\mathsf{pq}}(\tau)\mathsf{E}\left[\cos(2\pi f_c t + 2\pi f_c\tau)\sin(2\pi f_c t)\right]\\
&\quad -\mathsf{R}_{\mathsf{qp}}(\tau)\mathsf{E}\left[\sin(2\pi f_c t + 2\pi f_c\tau)\cos(2\pi f_c t)\right] + \mathsf{R}_{\mathsf{qq}}(\tau)\mathsf{E}\left[\sin(2\pi f_c t + 2\pi f_c\tau)\sin(2\pi f_c t)\right]\\
&= \frac{1}{2}\mathsf{R}_{\mathsf{pp}}(\tau)\left[\cos(2\pi f_c\tau) + \cos(4\pi f_c t + 2\pi f_c\tau)\right] - \frac{1}{2}\mathsf{R}_{\mathsf{pq}}(\tau)\left[-\sin(2\pi f_c\tau) + \sin(4\pi f_c t + 2\pi f_c\tau)\right]\\
&\quad -\frac{1}{2}\mathsf{R}_{\mathsf{qp}}(\tau)\left[\sin(2\pi f_c\tau) + \sin(4\pi f_c t + 2\pi f_c\tau)\right] + \frac{1}{2}\mathsf{R}_{\mathsf{qq}}(\tau)\left[\cos(2\pi f_c\tau) - \cos(4\pi f_c t + 2\pi f_c\tau)\right]\\
&= \frac{1}{2}\left[\mathsf{R}_{\mathsf{pp}}(\tau) + \mathsf{R}_{\mathsf{qq}}(\tau)\right]\cos(2\pi f_c\tau) + \frac{1}{2}\left[\mathsf{R}_{\mathsf{pq}}(\tau) - \mathsf{R}_{\mathsf{qp}}(\tau)\right]\sin(2\pi f_c\tau)\\
&\quad +\frac{1}{2}\left[\mathsf{R}_{\mathsf{pp}}(\tau) - \mathsf{R}_{\mathsf{qq}}(\tau)\right]\cos(4\pi f_c t + 2\pi f_c\tau) - \frac{1}{2}\left[\mathsf{R}_{\mathsf{pq}}(\tau) + \mathsf{R}_{\mathsf{qp}}(\tau)\right]\sin(4\pi f_c t + 2\pi f_c\tau)
\end{aligned}$$

Because $\mathsf{R}_{nn}(\tau)$ is not a function of t, the last two terms must be zero for all t, which implies

$$\begin{aligned}\mathsf{R}_{pp}(\tau) &= \mathsf{R}_{qq}(\tau)\\ \mathsf{R}_{pq}(\tau) &= -\mathsf{R}_{qp}(\tau).\end{aligned}$$

From these we have

$$\begin{aligned}\mathsf{R}_{nn}(\tau) &= \frac{1}{2}\left[\mathsf{R}_{pp}(\tau)+\mathsf{R}_{qq}(\tau)\right]\cos(2\pi f_c\tau)+\frac{1}{2}\left[\mathsf{R}_{pq}(\tau)-\mathsf{R}_{qp}(\tau)\right]\sin(2\pi f_c\tau)\\ &\quad+\frac{1}{2}\left[\mathsf{R}_{pp}(\tau)-\mathsf{R}_{qq}(\tau)\right]\cos(4\pi f_c t+2\pi f_c\tau)-\frac{1}{2}\left[\mathsf{R}_{pq}(\tau)+\mathsf{R}_{qp}(\tau)\right]\sin(4\pi f_c t+2\pi f_c\tau)\\ &= \mathsf{R}_{pp}(\tau)\cos(2\pi f_c\tau)+\mathsf{R}_{pq}(\tau)\sin(2\pi f_c\tau)\end{aligned}$$

$$\begin{aligned}\mathsf{R}_{pq}(\tau) &= -\mathsf{R}_{qp}(\tau)\\ &\triangleq -\mathsf{E}\left[q(t+\tau)p(t)\right]\\ &= \mathsf{E}\left[p(t)q(t+\tau)\right]\\ &\triangleq -\mathsf{R}_{pq}(-\tau)\end{aligned}$$

This implies $\mathsf{R}_{pq}(\tau)$ is odd-symmetric.

$$\begin{aligned}&\mathsf{R}_{pq}(\tau) = -\mathsf{R}_{pq}(-\tau)\\ \Longrightarrow\quad &\mathsf{R}_{pq}(0) = -\mathsf{R}_{pq}(0)\\ \Longrightarrow\quad &\mathsf{R}_{pq}(0) = 0.\end{aligned}$$

$$\begin{aligned}\mathsf{R}_{zz}(\tau) &\triangleq \mathsf{E}\left[z(t+\tau)z^*(t)\right]\\ &= \mathsf{E}\left[(x(t+\tau)+iy(t+\tau))\,(x(t)+iy(t))^*\right]\\ &= \mathsf{E}\left[(x(t+\tau)+iy(t+\tau))\left(x^*(t)-iy^*(t)\right)\right]\\ &= \mathsf{E}\left[x(t+\tau)x^*(t)\right]-i\mathsf{E}\left[x(t+\tau)y^*(t)\right]+i\mathsf{E}\left[y(t+\tau)x^*(t)\right]+\mathsf{E}\left[y(t+\tau)y^*(t)\right]\\ &\triangleq \mathsf{R}_{pp}(\tau)-i\mathsf{R}_{pq}(\tau)+i\mathsf{R}_{qp}(\tau)+\mathsf{R}_{qq}(\tau)\\ &= \mathsf{R}_{pp}(\tau)-i\mathsf{R}_{pq}(\tau)-i\mathsf{R}_{pq}(\tau)+\mathsf{R}_{qq}(\tau)\\ &= 2\mathsf{R}_{pp}(\tau)-2i\mathsf{R}_{pq}(\tau)\end{aligned}$$

✏

CHAPTER 3

MODULATION

The transmission is performed by allowing the information sequence u to affect the behavior of a *carrier* signal. This technique is called *modulation* and we say that the information sequence *modulates* the carrier. There are two general types of modulation:

1. memoryless modulation: only depends on the current signal value
2. modulation with memory: depends on current and past signal values.

The *receiver* generates an estimate[1] $\hat{u}$ of the sent information sequence u from the received signal $r(t)$.

3.1 Memoryless Modulation

3.1.1 Definitions

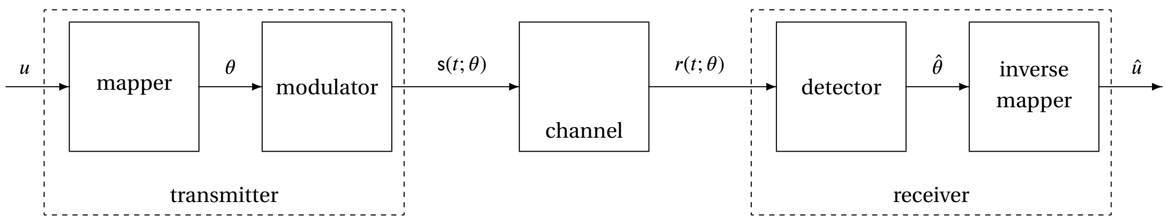


Figure 3.1: Memoryless modulation system model

Definition 3.1 (Digital modulation). *Let*

- $a_n \in \{0, 1, \dots, K-1\}$, $f_n \in \{0, 1, \dots, M-1\}$, *and* $\theta_n \in \{0, 1, \dots, N-1\}$
- $a_{\text{offset}}, f_{\text{offset}}, \theta_{\text{offset}} \in \mathbb{R}$
- $E, F \in \mathbb{R}^+$
- $T \in (0, \infty)$ *be the signalling period*
- $\{u_n\}$ *be an information sequence to be sent to a receiver*

[1] *estimation theory*: Section 7.4 page 76, Appendix 6 page 63

- g *be a function of the form*

$$(a_n, f_n, \theta_n) = \mathsf{g}(u_n).$$

- S *be a set of modulation waveforms*

DEF
$$S \triangleq \left\{ \mathsf{f}\,\mathsf{s}(t;u_n) = \left[a_n - a_{\text{offset}}\right] \left[\sqrt{\frac{2E}{T}} \cos\left[2\pi\left[f_c + F f_n - f_{\text{offset}}\right]t + \left[\theta_n \frac{2\pi}{N} - \theta_{\text{offset}}\right]\right]\right]\right\}$$

Then

- *A* ***memoryless digital modulation using sinusoidal carriers*** *(MDMSC) is the pair* (g, S).
- *A* ***Pulse Amplitude Modulation*** *(PAM) is MDMSC with*

$$f_n = f_{\text{offset}} = \theta_n = \theta_{\text{offset}} = 0$$

- *A* ***Phase Shift Keying*** *(PSK) is MDMSC with*

$$a_n = a_{\text{offset}} = f_n = f_{\text{offset}} = 0$$

- *A* ***Frequency Shift Keying*** *(FSK) is MDMSC with*

$$a_n = a_{\text{offset}} = \theta_n = \theta_{\text{offset}} = 0$$

- *A* ***Quadrature Amplitude Modulation*** *(QAM) is MDMSC with*

$$f_n = f_{\text{offset}} = 0$$

Theorem 3.1. *Let* (g, S) *be an MDMSC. The energy* $\mathbf{E}\mathsf{f}\,\mathsf{s}(t;n)$ *of* $\mathsf{f}\,\mathsf{s}(t;n) \in S$ *is*

THM
$$\mathbf{E}s_n \approx a_n^2 E$$

✎PROOF:

$$\begin{aligned}
\mathbf{E}\mathsf{f}\,\mathsf{s}(t;n) &\triangleq \left\| a_n \sqrt{\frac{2E}{T}} \cos(2\pi(f_c + \Delta f\, f_n)t + \theta_n) \right\|^2 \\
&= a_n^2 \frac{2E}{T} \left\| \cos(2\pi(f_c + \Delta f\, f_n)t + \theta_n) \right\|^2 \\
&= a_n^2 \frac{2E}{T} \int_0^T \cos^2(2\pi(f_c + \Delta f\, f_n)t + \theta_n)\, \mathrm{d}t \\
&= a_n^2 \frac{2E}{T} \frac{1}{2} \int_0^T 1 + \cos(4\pi(f_c + \Delta f\, f_n)t + 4\theta_n)\, \mathrm{d}t \\
&= a_n^2 \frac{E}{T} \left[\int_0^T 1\, \mathrm{d}t + \int_0^T \cos(4\pi(f_c + \Delta f\, f_n)t + 4\theta_n)\, \mathrm{d}t \right] \\
&\approx a_n^2 \frac{E}{T} \int_0^T 1\, \mathrm{d}t \\
&= a_n^2 E
\end{aligned}$$

✎

3.1.2 Orthogonality

Proposition 3.1. *Let* $(V, \langle \triangle \mid \triangledown \rangle, S)$ *be a modulation space and* $\mathsf{s}(t;m) \in S$.

PRP
$$\{(V, \langle \triangle \mid \triangledown \rangle, S) \text{ is PAM}\} \implies \left\{ \Psi \triangleq \left\{ \psi(t) = \frac{\sqrt{2}}{\|\lambda\|} \lambda(t) \cos(2\pi f_c t) \right\} \text{ is an orthonormal basis for } S. \right\}$$

✎PROOF:

1. Proof that Ψ spans S:

$$\begin{aligned}
\mathsf{s}(t;m) &\triangleq a_m \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) \\
&= a_m \psi(t)
\end{aligned}$$

2. Proof that Ψ is orthonormal with respect to $\langle \triangle \mid \nabla \rangle$.

$$\begin{aligned}
\langle \psi_c(t) \mid \psi_c(t) \rangle &= \left\langle \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) \mid \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) \right\rangle \\
&= \frac{\sqrt{2}}{\|\lambda\|}\frac{\sqrt{2}}{\|\lambda\|}\langle \lambda(t)\cos(2\pi f_c t) \mid \lambda(t)\cos(2\pi f_c t) \rangle \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\cos^2(2\pi f_c t)\; dt \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\frac{1}{2}\left[1+\cos(4\pi f_c t)\right]\; dt \\
&= \frac{2}{\|\lambda\|^2}\frac{1}{2}\int_0^T \lambda^2(t)\,[1]\; dt \\
&= \frac{1}{\|\lambda\|^2}\langle \lambda(t) \mid \lambda(t) \rangle \\
&= \frac{1}{\|\lambda\|^2}\|\lambda(t)\|^2 \\
&= 1
\end{aligned}$$

✏

Proposition 3.2. *Let $(V, \langle \triangle \mid \nabla \rangle, S)$ be a modulation space and $\mathsf{s}(t;m) \in S$.*

PRP

$$\{(V, \langle \triangle \mid \nabla \rangle, S) \text{ is PSK}\} \implies \left\{ \Psi \triangleq \left\{ \begin{array}{lcl} \psi_c(t) &=& \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t), \\ \psi_s(t) &=& -\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t) \end{array} \right\} \right\} \text{ is an orthonormal basis for } S.$$

✎PROOF:

1. Ψ spans S:

$$\begin{aligned}
\mathsf{s}(t;a_m,b_m) &\triangleq r\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t + \theta_m) \\
&= r\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\left[\cos\theta_m\cos(2\pi f_c t) - \sin\theta_m\sin(2\pi f_c t)\right] \\
&= r\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos\theta_m\cos(2\pi f_c t) - r\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin\theta_m\sin(2\pi f_c t) \\
&= r\cos\theta_m\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) - r\sin\theta_m\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t) \\
&= r\cos\theta_m\psi_c(t) + r_m\sin\theta_m\psi_s(t)
\end{aligned}$$

2. Proof that Ψ is orthonormal with respect to $\langle \triangle \mid \nabla \rangle$: See proof of Lemma 3.3 (page 21).

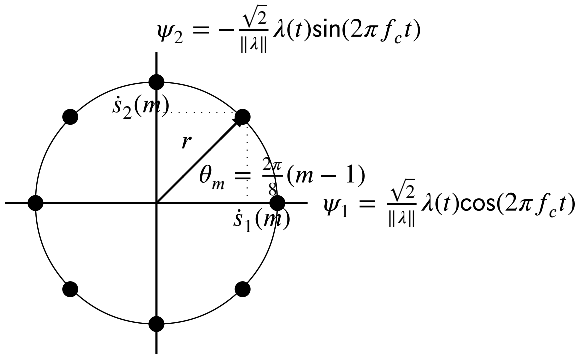


Figure 3.2: PSK vector representation, $M = 8$

✏

Theorem 3.2 (Orthogonality for FSK)**.** *Let (g, S) be an FSK modulation.*

1. *If $F \in \left\{n\frac{1}{2T} | k \in \mathbb{N}\right\}$, then $\mathsf{s}_m, \mathsf{s}_n \in S$ are orthogonal for $m \neq n$.*
2. *If $\mathsf{s}_1, \mathsf{s}_2 \in S$ possibly different phases and $F \in \left\{n\frac{1}{T} | k \in \mathbb{N}\right\}$, then $\mathsf{s}_m, \mathsf{s}_n \in S$ are orthogonal for $m \neq n$.*

✎PROOF:

1. Proof for identical phases:

$$\begin{aligned}
\langle \psi_m(t) \,|\, \psi_n(t) \rangle &= \left\langle \sqrt{\frac{2}{T}}\cos[2\pi(f_c + mf_d)t] \,\middle|\, \sqrt{\frac{2}{T}}\cos[2\pi(f_c + nf_d)t] \right\rangle \\
&= \frac{2}{T} \langle \cos[2\pi(f_c + mf_d)t] \,|\, \cos[2\pi(f_c + nf_d)t] \rangle \\
&= \frac{2}{T} \int_0^T \cos[2\pi(f_c + mf_d)t]\cos[2\pi(f_c + nf_d)t] \; dt \\
&= \frac{1}{2}\frac{2}{T} \int_0^T \cos[2\pi(f_c + mf_d)t - 2\pi(f_c + nf_d)t] + \cos[2\pi(f_c + mf_d)t + 2\pi(f_c + nf_d)t] \; dt \\
&= \frac{1}{T} \int_0^T \cos[2\pi(m-n)f_d t] + \cos[4\pi(f_c t + 2\pi(m+n)f_d t] \; dt \\
&\approx \frac{1}{T} \int_0^T \cos[2\pi(m-n)f_d t] \; dt \\
&= \frac{1}{T}\frac{1}{2\pi(m-n)f_d} \sin[2\pi(m-n)f_d t] \Big|_0^T \\
&= \frac{\sin[2\pi(m-n)f_d T]}{2\pi(m-n)f_d T} \\
&= \begin{cases} 1 & \text{for } m = n \\ \frac{\sin[2\pi(m-n)f_d T]}{2\pi(m-n)f_d T} & \text{for } m \neq n. \end{cases} \\
&= \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \text{ and } f_d = \frac{k}{2T}, k = 1, 2, 3, \ldots. \end{cases}
\end{aligned}$$

2. Proof for different phase:

$$
\begin{aligned}
\langle \psi_m(t;\phi) \,|\, \psi_n(t) \rangle &= \mathbf{L} \langle \cos(2\pi f_m t + \phi) \,|\, \cos(2\pi f_n t) \rangle \\
&= \mathbf{L} \int_t^{t+T} \cos(2\pi f_m t + \phi)\cos(2\pi f_n t)\, dt \\
&= \int_t^{t+T} \cos\left[2\pi(f_m - f_n)t + \phi\right]\, \mathrm{d}t \\
&= \left. \frac{\sin[2\pi(f_m - f_n)t + \phi]}{2\pi(f_m - f_n)} \right|_t^{t+T} \\
&= \frac{\sin[2\pi(f_m - f_n)(t+T) + \phi] - \sin[2\pi(f_m - f_n)t + \phi]}{2\pi(f_m - f_n)}
\end{aligned}
$$

3. For orthogonality, this implies

$$
\begin{aligned}
2\pi(f_m - f_n)(t+T) + \phi &= 2\pi(f_m - f_n)t + \phi + k2\pi, k = 1, 2, 3, \ldots \\
2\pi(f_m - f_n)T &= k2\pi \\
(f_m - f_n)T &= k \\
f_m - f_n &= \frac{k}{T}
\end{aligned}
$$

✏

Proposition 3.3. *Let* $(V, \langle \triangle \,|\, \triangledown \rangle, S)$ *be a QAM modulation space and* $\mathsf{s}(t; a_m, b_m) \in S$*. Then the set*

$$
\Psi \triangleq \left\{ \begin{array}{lcl} \psi_c(t) &=& \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t), \\ \psi_s(t) &=& -\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t) \end{array} \right\}
$$

is an orthonormal basis for S.

✎PROOF:

1. Ψ spans S:

$$
\begin{aligned}
\mathsf{s}(t; a_m, b_m) &\triangleq a_m \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) + b_m \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t) \\
&= a_m \psi_c(t) + b_m \psi_s(t)
\end{aligned}
$$

2. Ψ is orthonormal with respect to $\langle \triangle \,|\, \triangledown \rangle$.

$$\begin{aligned}
\langle \psi_c(t) \,|\, \psi_c(t) \rangle &= \left\langle \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) \,|\, \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) \right\rangle \\
&= \frac{\sqrt{2}}{\|\lambda\|}\frac{\sqrt{2}}{\|\lambda\|}\langle \lambda(t)\cos(2\pi f_c t) \,|\, \lambda(t)\cos(2\pi f_c t) \rangle \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\cos^2(2\pi f_c t)\; dt \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\frac{1}{2}\left[1+\cos(4\pi f_c t)\right]\; dt \\
&= \frac{2}{\|\lambda\|^2}\frac{1}{2}\int_0^T \lambda^2(t)\,[1]\; dt \\
&= \frac{1}{\|\lambda\|^2}\langle \lambda(t) \,|\, \lambda(t) \rangle \\
&= \frac{1}{\|\lambda\|^2}\|\lambda(t)\|^2 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle \psi_s(t) \,|\, \psi_s(t) \rangle &= \left\langle \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t) \,|\, \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t) \right\rangle \\
&= \frac{\sqrt{2}}{\|\lambda\|}\frac{\sqrt{2}}{\|\lambda\|}\langle \lambda(t)\sin(2\pi f_c t) \,|\, \lambda(t)\sin(2\pi f_c t) \rangle \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\sin^2(2\pi f_c t)\; dt \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\frac{1}{2}\left[1-\cos(4\pi f_c t)\right]\; dt \\
&= \frac{2}{\|\lambda\|^2}\frac{1}{2}\int_0^T \lambda^2(t)\,[1]\; dt \\
&= \frac{1}{\|\lambda\|^2}\langle \lambda(t) \,|\, \lambda(t) \rangle \\
&= \frac{1}{\|\lambda\|^2}\|\lambda(t)\|^2 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle \psi_s(t) \,|\, \psi_c(t) \rangle &= \langle \psi_c(t) \,|\, \psi_s(t) \rangle \\
&= \left\langle \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t) \,|\, \frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t) \right\rangle \\
&= \frac{\sqrt{2}}{\|\lambda\|}\frac{\sqrt{2}}{\|\lambda\|}\langle \lambda(t)\cos(2\pi f_c t) \,|\, \lambda(t)\sin(2\pi f_c t) \rangle \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\cos(2\pi f_c t)\sin(2\pi f_c t)\; dt \\
&= \frac{2}{\|\lambda\|^2}\mathbf{L}\int_0^T \lambda^2(t)\frac{1}{2}\left[\sin(4\pi f_c t)-\sin(0)\right]\; dt \\
&= \frac{1}{\|\lambda\|^2}\int_0^T \lambda^2(t)\left[\mathbf{L}\sin(4\pi f_c t)-0\right]\; dt
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{\|\lambda\|^2}\int_0^T \lambda^2(t)\,[0-0]\;dt \\ &= 0 \end{aligned}$$

✎

Definition 3.1 represents elements of S in rectangular form (a_m, b_m). The elements of S can also be represented in polar form (r_m, θ_m) as shown below.

$$\begin{aligned} \mathsf{s}(t;m) &= \dot{s}_c(a_m)\psi_c(t) + \dot{s}_c(b_m)\psi_s(t) \\ &= r_m\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\left[a_m\cos(2\pi f_c t) - b_m\sin(2\pi f_c t)\right] \\ &= r_m\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\left[\cos\theta_m\cos(2\pi f_c t) - \sin\theta_m\sin(2\pi f_c t)\right] \\ &= r_m\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos\left[2\pi f_c t + \theta_m\right] \end{aligned}$$

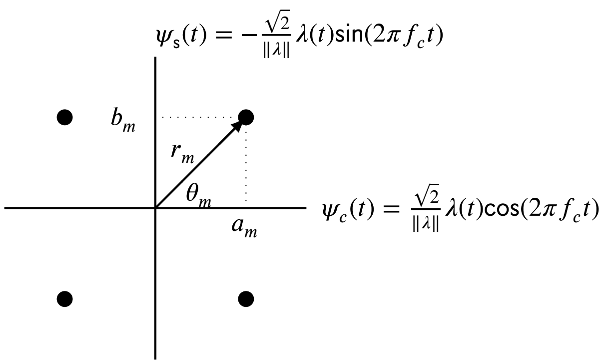


Figure 3.3: QAM rectangular (a_m, b_m) and polar (r_m, θ_m) representations

3.1.3 Measures

Measures

Theorem 3.3.

THM *The PAM modulation space has* ***energy*** *and* ***distance*** *measures*

$$\begin{aligned} \mathbf{E}\mathsf{s}(t;m) &= a_m^2 \\ \mathsf{d}(\mathsf{s}(t;m), \mathsf{s}(t;n)) &= |a_m - a_n|. \end{aligned}$$

✎PROOF: Because PAM is a modulation space,

- the energy measure follows from Theorem 3.4 page 23 (page 23)
- the distance measure from Theorem 3.5 page 24 (page 24).

✎

Proposition 3.4. *Let*

- $(V, \langle\triangle\,|\,\nabla\rangle, S)$ *be a modulation space and* $\mathsf{s}(t) \in S$

- $\Psi \triangleq \{\psi_n(t) : n = 1, 2, \ldots, N\}$ *be a set of orthonormal functions that span* S
- $\dot{s}_n \triangleq \langle \mathrm{s}(t) \,|\, \psi_n(t) \rangle$

PRP *The **energy** in* $\mathrm{s}(t)$ *is*

$$\mathbf{E}\mathrm{s}(t) = \sum_{n=1}^{N} |\dot{s}_n|^2$$

✎PROOF:

$$\begin{aligned}
\mathbf{E}\mathrm{s}(t) &\triangleq \|\mathrm{s}(t)\|^2 \\
&= \left\| \sum_{n=1}^{N} \dot{s}_n \psi_n(t) \right\|^2 \\
&= \sum_{n=1}^{N} |\dot{s}_n|^2
\end{aligned}$$

✏

Proposition 3.5. *Let*

- $(V, \langle \triangle \,|\, \triangledown \rangle, S)$ *be a modulation space and* $\mathrm{s}(t;m) \in S$
- $\Psi \triangleq \{\psi_n(t) : n = 1, 2, \ldots, N\}$ *be a set of orthonormal functions that span* S
- $\dot{s}_n(m) \triangleq \langle \mathrm{s}(t;m) \,|\, \psi_n(t) \rangle$

PRP *The **distance** between waveforms* $\mathrm{s}(t;m)$ *and* $\mathrm{s}(t;k)$ *is*

$$\mathrm{d}(\mathrm{s}(t;m), \mathrm{s}(t;k)) \triangleq \sqrt{\sum_{n=1}^{N} |\dot{s}_n(m) - \dot{s}_n(k)|^2}$$

✎PROOF:

$$\begin{aligned}
\mathrm{d}^2(\mathrm{s}(t;m),\ \mathrm{s}(t;k)) &\triangleq \|\mathrm{s}(t;m) - \mathrm{s}(t;k)\|^2 \\
&= \sum_{n=1}^{N} |\dot{s}_n(m) - \dot{s}_n(k)|^2 && \text{by Theorem ?? page ?? (page ??)}
\end{aligned}$$

✏

Theorem 3.4.

THM *The PSK modulation space has **energy** and **distance** measures*

$$\begin{aligned}
\mathbf{E}\mathrm{s}(t;m) &= r^2 \\
\mathrm{d}(\mathrm{s}(t;m), \mathrm{s}(t;n)) &= r\sqrt{2 - 2\cos\left(\theta_m - \theta_n\right)}.
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{E}\mathrm{s}(t;m) &\triangleq \|\mathrm{s}(t;m)\|^2 \\
&= \left\| \dot{s}_c(m)\psi_1(t) + \dot{s}_s(m)\psi_2(t) \right\|^2 \\
&= \dot{s}_c^2(m) + \dot{s}_s^2(m) \\
&= \left(r\cos\theta_m\right)^2 + \left(r\sin\theta_m\right)^2 \\
&= r^2\left(\cos^2\theta_m + \sin^2\theta_m\right) \\
&= r^2
\end{aligned}$$

$$\begin{aligned}
\mathsf{d}^2(\mathsf{s}(t;m),\ \mathsf{s}(t;n)) &= \|\mathsf{s}(t;m)-\mathsf{s}(t;n)\|^2 \\
&= \left\|[\dot{s}_c(m)\psi_1(t)+\dot{s}_s(m)\psi_2(t)]-[\dot{s}_c(n)\psi_1(t)+\dot{s}_s(n)\psi_2(t)]\right\|^2 \\
&= \left\|[\dot{s}_c(m)-\dot{s}_c(n)]\psi_1(t)+[\dot{s}_s(m)-\dot{s}_s(n)]\psi_2(t)\right\|^2 \\
&= [\dot{s}_c(m)-\dot{s}_c(n)]^2+[\dot{s}_s(m)-\dot{s}_s(n)]^2 && \text{by Theorem \textbf{??} page \textbf{??}} \\
&= [r\cos\theta_m - r\cos\theta_n]^2+[r\sin\theta_m + r\sin\theta_n]^2 \\
&= r^2\left([\cos\theta_m-\cos\theta_n]^2+[\sin\theta_m+\sin\theta_n]^2\right) \\
&= r^2\left([\cos^2\theta_m-2\cos\theta_m\cos\theta_n+\cos^2\theta_n]+[\sin^2\theta_m-2\sin\theta_m\sin\theta_n+\sin^2\theta_n]\right) \\
&= r^2\left([\cos^2\theta_m+\sin^2\theta_m]+[\cos^2\theta_n+\cos^2\theta_n]-2[\cos\theta_m\cos\theta_n+\sin\theta_m\sin\theta_n]\right) \\
&= r^2[1+1-2\cos(\theta_m-\theta_n)] \\
&= 2r^2[1-\cos(\theta_m-\theta_n)]
\end{aligned}$$

✏

Theorem 3.5.

THM *The FSK modulation space has **energy** and **distance** measures equivalent to*

$$\begin{aligned}
\mathbf{E}\mathsf{s}(t;m) &= \dot{s}^2 \\
\mathsf{d}(\mathsf{s}(t;m),\mathsf{s}(t;n)) &= \sqrt{2}\ \dot{s}
\end{aligned}$$

✎PROOF: The energy measure is a result of Theorem 3.4 page 23 (page 23).
For distance,

$$\begin{aligned}
\mathsf{d}^2(\mathsf{s}(t;m),\ \mathsf{s}(t;n)) &= \sum_{k=1}^{N}|\dot{s}_k(m)-\dot{s}_{nk}|^2 && \text{Theorem 3.5 page 24} \\
&= \sum_{k=1}^{N}|\dot{s}_k(m)-\dot{s}_{nk}|^2 \\
&= (\dot{s}-0)^2+(\dot{s}-0)^2 \\
&= 2\dot{s}^2.
\end{aligned}$$

✏

Theorem 3.6.

THM *The QAM modulation space has **energy** and **distance** measures equivalent to*

$$\begin{aligned}
\mathbf{E}\mathsf{s}(t;m) &= a_m^2+b_m^2 = r_m^2 \\
\mathsf{d}(\mathsf{s}(t;m),\mathsf{s}(t;n)) &= \sqrt{(a_m-a_n)^2+(b_m-b_n)^2}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathbf{E}\mathsf{s}(t;m) &\triangleq \|\mathsf{s}(t;m)\|^2 \\
&= \left\|a_m\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\cos(2\pi f_c t)+b_m\frac{\sqrt{2}}{\|\lambda\|}\lambda(t)\sin(2\pi f_c t)\right\|^2 \\
&= \left\|a_m\psi_c(t)+b_m\psi_s(t)\right\|^2 \\
&= a_m^2+b_m^2 \\
&= (r_m\cos\theta_m)^2+(r_m\sin\theta_m)^2 \\
&= r_m^2\left(\cos^2\theta_m+\sin^2\theta_m\right) \\
&= r_m^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}^2(\mathrm{s}(t;m),\ \mathrm{s}(t;n)) &\triangleq \|\mathrm{s}(t;m)-\mathrm{s}(t;n)\|^2 \\
&= \left\|\big(a_m\psi_c(t)+b_m\psi_s(t)\big)-\big(a_n\psi_c(t)+b_n\psi_s(t)\big)\right\|^2 \\
&= |a_m-a_n|^2+|b_m-b_n|^2 && \text{by Theorem ?? page ?? page ??}
\end{aligned}$$

3.2 Continuous Phase Modulation (CPM)

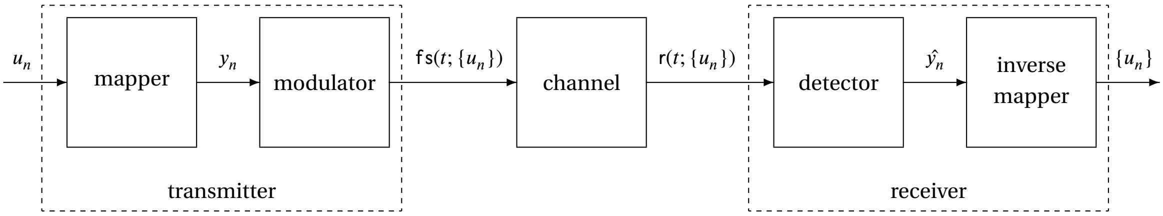


Figure 3.4: Continuous Phase Modulation system model

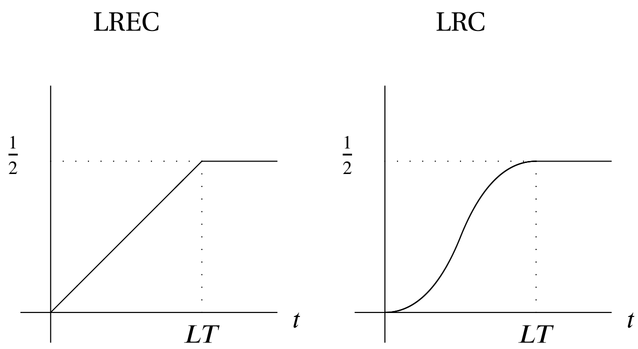


Figure 3.5: CPM phase pulses $\rho(t)$

Continuous modulation can be realized using *phase pulses* which are illustrated in Figure 3.5 (page 26) and defined in Definition 3.2 (next).

Definition 3.2. *Let $L \in \mathbb{N}$ be the **response length** and T the **signalling rate**. The function $\rho : \mathbb{R} \to \mathbb{R}$ is a **phase pulse** if*

1. *$\rho(t)$ is continuous*
2. *$\rho(t) = 0$ for $t \leq 0$*
3. *$\rho(t) = \frac{1}{2}$ for $t \geq LT$.*

Definition 3.3. *Let*

$$\begin{aligned}
n &= \left\lfloor \frac{t}{T} \right\rfloor \\
x_n &\in \{0, 1, \dots, M-1\} \\
y_n &= 2x_n - 1 \in \{\pm 1, \pm 2, \dots, \pm(M-1)\}.
\end{aligned}$$

*Then **Continuous Phase Modulation** (CPM) signalling waveforms are*

$$
\begin{aligned}
\mathsf{fs}(t;\ldots,u_{n-1},u_n) &= a\frac{2}{\sqrt{T}}\cos\left[2\pi f_c t + 2\pi\sum_{k=-\infty}^{n} y_k h_k \rho(t-kT)\right] \\
&= a\frac{2}{\sqrt{T}}\cos\left(\underbrace{2\pi f_c t}_{\text{carrier}} + \underbrace{\pi\sum_{k=-\infty}^{n-L} y_k h_k}_{\text{state}} + \underbrace{2\pi\sum_{k=n-L+1}^{n} y_k h_k \rho(t-kT)}_{\text{maintains continuous phase}}\right)
\end{aligned}
$$

3.2.1 Phase Pulse waveforms

$$\rho(t) = \int_t \rho'(t)\,dt$$

Rectangular (LREC)

$$\rho'(t) = \begin{cases} \frac{1}{2LT} & \text{for } 0 \le t \le LT \\ 0 & \text{otherwise} \end{cases} \qquad \rho(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2LT}t & \text{for } 0 \le t < LT \\ \frac{1}{2} & \text{for } t \ge LT \end{cases}$$

Raised Cosine (LRC)

$$\rho'(t) = \begin{cases} \frac{1}{2LT}\left[1-\cos\left(\frac{2\pi}{LT}t\right)\right] & \text{for } 0 \le t < LT \\ 0 & \text{otherwise} \end{cases} \qquad \rho(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2LT}\left[t-\frac{LT}{2\pi}\sin\left(\frac{2\pi}{LT}t\right)\right] & \text{for } 0 \le t < LT \\ \frac{1}{2} & \text{for } t \ge LT \end{cases}$$

Gaussian Minimum Shift Keying (GMSK)

$$\rho'(t) = \begin{cases} \mathsf{Q}\left[\frac{2\pi B(t-\frac{T}{2})}{\sqrt{ln 2}}\right] - \mathsf{Q}\left[\frac{2\pi B(t+\frac{T}{2})}{\sqrt{ln 2}}\right] & \text{for } 0 \le t < LT \\ 0 & \text{otherwise} \end{cases} \qquad \rho(t) = \int_{-\infty}^{t} \rho'(t)\,dt$$

3.2.2 Special Cases

Definition 3.4. ***Full response*** *CPM has response length $L = 1$.* ***Partial response*** *CPM has response length $L \ge 2$.*

In the case of Full Response CPM, the signalling waveform simplifies to

$$
\begin{aligned}
\mathsf{fs}(t;\ldots,u_{n-1},u_n) &= a\frac{2}{\sqrt{T}}\cos\left(2\pi f_c t + \pi\sum_{k=-\infty}^{n-L} y_k h_k + 2\pi\sum_{k=n-L+1}^{n} y_k h_k \rho(t-kT)\right)\\
&= a\frac{2}{\sqrt{T}}\cos\left(2\pi f_c t + \pi\sum_{k=-\infty}^{n-1} y_k h_k + 2\pi\sum_{k=n-1+1}^{n} y_k h_k \rho(t-kT)\right)\\
&= a\frac{2}{\sqrt{T}}\cos\left(\underbrace{2\pi f_c t}_{\text{carrier}} + \underbrace{\pi\sum_{k=-\infty}^{n-1} y_k h_k}_{\text{state}} + \underbrace{2\pi y_n h_n \rho(t-nT)}_{\text{maintains c.p.}}\right)
\end{aligned}
$$

Definition 3.5. ***Continuous Phase Frequency Shift Keying*** *(CPFSK) is full response CPM ($L=1$) with $h_n = h$ is constant and LREC phase pulse.*

In CPFSK, the signalling waveform is

$$
\begin{aligned}
\mathsf{fs}(t;\ldots,u_{n-1},u_n) &= a\frac{2}{\sqrt{T}}\cos\left(2\pi f_c t + \pi\sum_{k=-\infty}^{n-L} y_k h_k + 2\pi\sum_{k=n-L+1}^{n} y_k h_k \rho(t-kT)\right)\\
&= a\frac{2}{\sqrt{T}}\cos\left(2\pi f_c t + \pi\sum_{k=-\infty}^{n-1} y_k h + 2\pi y_n h\left(\frac{1}{2T}(t-nT)\right)\right)\\
&= a\frac{2}{\sqrt{T}}\cos\left(\underbrace{2\pi\left(f_c + \frac{h}{2T}y_n\right)t}_{\text{carrier}} + \underbrace{\pi h\sum_{k=-\infty}^{n-1} y_k}_{\text{state}} - \underbrace{\pi h n y_n}_{\text{maintains c.p.}}\right)
\end{aligned}
$$

Two sinusoidal waveforms are *coherent* if their frequency difference is $k\frac{1}{2T}$. The waveforms of CPFSK are therefore orthogonal if $h = m\frac{1}{2}$.

Definition 3.6. ***Orthogonal Continuous Phase Frequency Shift Keying*** *is full response CPM ($L=1$) with $h_n \in \left\{m\frac{1}{2} | m \in \mathbb{Z}\right\}$ and LREC phase pulse.*

For $m \in \mathbb{N}$, orthogonal CPFSK signalling waveforms are

$$
\begin{aligned}
\mathsf{fs}(t;\ldots,u_{n-1},u_n) &= a\frac{2}{\sqrt{T}}\cos\left(2\pi f_c t + \pi\sum_{k=-\infty}^{n-L} y_k h_k + 2\pi\sum_{k=n-L+1}^{n} y_k h_k \rho(t-kT)\right)\\
&= a\frac{2}{\sqrt{T}}\cos\left(2\pi\left(f_c + \frac{h}{2T}y_n\right)t + \pi\sum_{k=-\infty}^{n-1} y_k h - \pi h n y_n\right)\\
&= a\frac{2}{\sqrt{T}}\cos\left(\underbrace{2\pi\left(f_c + \frac{m}{4T}y_n\right)t}_{\text{carrier}} + \underbrace{\frac{m}{2}\pi\sum_{k=-\infty}^{n-1} y_k}_{\text{state}} - \underbrace{\frac{m}{2}\pi n y_n}_{\text{maintains c.p.}}\right)
\end{aligned}
$$

The minimum value of m in orthogonal CPFSK is 1. When $m = 1$ (the minimum value for orthogonality), the orthogonal CPFSK is also called *Minimum Shift Keying*.

Definition 3.7. ***Minimum Phase Shift Keying*** *(MSK) is is full response CPM ($L = 1$) with $h_n = \frac{1}{2}$ and LREC phase pulse.*

In MSK, the signalling waveform is

$$\begin{aligned}
\mathsf{fs}(t;\ldots,u_{n-1},u_n) &= a\frac{2}{\sqrt{T}}\cos\left[2\pi f_c t + \frac{\pi}{2}\left(\sum_{k=-\infty}^{n-1} y_k + \frac{t-nT}{T}\cdot y_n\right)\right]. \\
&= a\frac{2}{\sqrt{T}}\cos\left[2\pi\left(f_c + \frac{m}{4T}y_n\right)t + \frac{m}{2}\pi\sum_{k=-\infty}^{n-1} y_k - \frac{m}{\pi}ny_n\right] \\
&= a\frac{2}{\sqrt{T}}\cos\left[\underbrace{2\pi\left(f_c + \frac{1}{4T}y_n\right)t}_{\text{carrier}} + \underbrace{\frac{\pi}{2}\sum_{k=-\infty}^{n-1} y_k}_{\text{state}} - \underbrace{\frac{\pi}{2}ny_n}_{\text{maintains c.p.}}\right]
\end{aligned}$$

In summary:

Technique	$\rho(t)$	L	h_k
Continuous Phase Frequency Shift Keying (CPFSK)	LREC	1	h (constant)
Minimum Shift Keying (MSK)	LREC	1	$\frac{1}{2}$

3.2.3 Detection

The state of the signalling waveforms at intervals nT can be described by trellis diagrams.

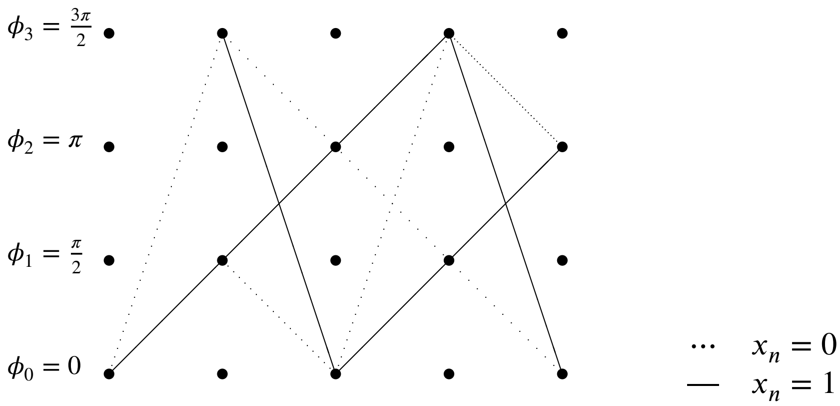


Figure 3.6: CPM $M = 2$, $h = 1/2$ (MSK-2) trellis diagram

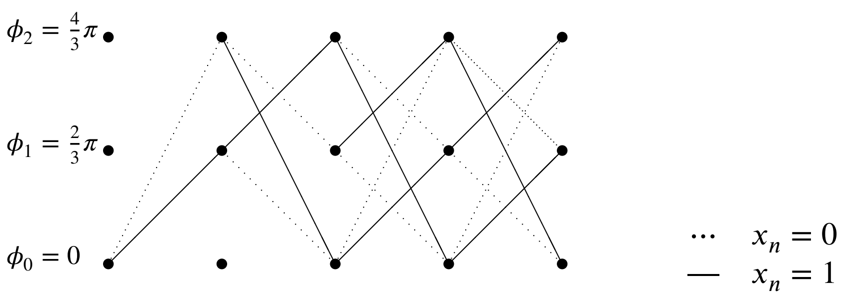


Figure 3.7: CPM $M = 2$, $h = 2/3$ trellis diagram

CHAPTER 4

SPREAD SPECTRUM

4.1 Introduction

Communication channel multiple access. A communication system provides the ability for a set of information to be sent from a transmitter to a receiver through a physical channel. If multiple sets of information need to be sent through the channel, then this channel must be shared. Multiple access of a channel can be achieved by separating the information sets in time, frequency, or code. These three multiple access techniques are referred to as

- TDMA Time Division Multiple Access: separation in time
- FDMA Frequency Division Multiple Access: separation in frequency
- CDMA Code Division Multiple Access: separation by code

CDMA Modulation Communication through a channel is typically performed by transmitted information *modulating* (affecting some parameter of) a *carrier* waveform. There are two basic types of CDMA modulation:

- DS Direct Sequence
- FH Frequency Hopping

In FH-CDMA modulation, an information sequence modulates the frequency of a sinusoidal carrier waveform. FH-CDMA will not be further discussed in this chapter.

In DS-CDMA modulation, an information sequence modulates a *pseudo-noise sequence* (pn-sequence). This pn-sequence and the information which modulates it are typically both binary sequences. The modulation operation itself is a simple *modulo 2 addition* operation in mathematics, which is equivalent to an *exclusive OR* operation in logic, which may be implemented with an *exclusive OR gate* in hardware.

Types of PN-Sequences Generating good PN-sequences is one of the keys to effective DS-CDMA communication system design. A sequence is simply a function f whose domain is the set of integers and range is some set R. This report is limited to *binary* pn-sequences, which are functions with range $\{0,1\}$ of the form

$$f : \mathbb{Z} \to \{0,1\}.$$

The most basic binary pn-sequence is the *m-sequence* (maximal length sequence). From this basic sequence, other sequences can be constructed such as *Gold* sequences.

4.2 Generating m-sequences mathematically

4.2.1 Definitions

An m-sequence can be represented as the coefficients of a *polynomial* over a *finite field*. Any *field* is defined by the triplet $(S, +, \cdot)$, where

S:	a set	
$+$:	addition operation in the form	$+ : \quad S \times S \to S$
$\cdot$:	multiplication operation in the form	$\cdot : \quad S \times S \to S$

Definition 4.1. ***Galois Field 2, GF(2)***
GF(2) is the field $(S, +, \cdot)$ *with members of the triplet defined as*

$$\begin{array}{rcl} S &=& \{0,1\} \\ + &:& \{0,1\}\times\{0,1\} \to \{0,1\} \\ \cdot &:& \{0,1\}\times\{0,1\} \to \{0,1\} \end{array} \quad \textit{such that} \quad \begin{array}{cc|c} a & b & a+b \\ \hline 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \quad \textit{and} \quad \begin{array}{cc|c} a & b & a\cdot b \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array}$$

M-sequences can be generated and represented as *polynomials over GF(2)*. A polynomial over GF(2) is a polynomial with coefficients selected from GF(2). An example of a polynomial over GF(2) is

$$1 + x^2 + x^5 + x^6 + x^7 + x^9.$$

The generation of an m-sequence is equivalent to polynomial division, which is very similar to integer division.

Definition 4.2. ***Polynomial division***
The quantities of polynomial division are identified as follows:

$$\frac{d(x)}{p(x)} = q(x) + \frac{r(x)}{p(x)} \quad \textit{where} \quad \begin{array}{ll} d(x) & \textit{is the dividend} \\ p(x) & \textit{is the divisor} \\ q(x) & \textit{is the quotient} \\ r(x) & \textit{is the remainder.} \end{array}$$

The ring of integers $\mathbb{Z}$ contains some special elements called *primes* which can only be divided[1] by themselves or 1. Rings of polynomials have a similar elements called *primitive polynomials*.

[1]The expression "a divides b" means that b/a has remainder 0.

Definition 4.3. *Primitive polynomial*
A primitive polynomial $p(x)$ of order n has the properties

1. *$p(x)$ cannot be factored*
2. *the smallest order polynomial that $p(x)$ can divide is $x^{2^n-1}+1=0$.*

Some examples[2] of primitive polynomials over $GF(2)$ are

order	primitive polynomial
2	$p(x) = x^2 + x + 1$
3	$p(x) = x^3 + x + 1$
4	$p(x) = x^4 + x + 1$
5	$p(x) = x^5 + x^2 + 1$
5	$p(x) = x^5 + x^4 + x^2 + x + 1$
16	$p(x) = x^{16} + x^{15} + x^{13} + x^4 + 1$
31	$p(x) = x^{31} + x^{28} + 1$

An m-sequence is the remainder when dividing any non-zero polynomial by a primitive polynomial. We can define an *equivalence relation*[3] on polynomials which defines two polynomials as *equivalent with respect to* $p(x)$ when their remainders are equal.

Definition 4.4. *Equivalence relation* $\equiv$

$$\textit{Let}\quad \frac{a_1(x)}{p(x)} = q_1(x) + \frac{r_1(x)}{p(x)} \quad \textit{and} \quad \frac{a_2(x)}{p(x)} = q_2(x) + \frac{r_2(x)}{p(x)}.$$

Then $a_1(x) \equiv a_2(x)$ with respect to $p(x)$ if $r_1(x) = r_2(x)$.

Using the equivalence relation of Definition 4.4, we can develop two very useful equivalent representations of polynomials over GF(2). We will call these two representations the *exponential* representation and the *polynomial* representation.

Example 4.1. By Definition 4.4 and under $p(x) = x^3 + x + 1$, we have the following equivalent representations:

---

2 Wicker (1995), pages 465–475

3 An equivalence relation $\equiv$ must satisfy three properties:

1. reflexivity: $a \equiv a$
2. symmetry: if $a \equiv b$ then $b \equiv a$.
3. transitivity: if $a \equiv b$ and $b \equiv c$ then $a \equiv c$.

reference: (Aliprantis and Burkinshaw, 1998, p.7)

$$\begin{array}{rcrcccccc}
\frac{x^0}{x^3+x+1} &=& 0 &+& \frac{1}{x^3+x+1} &\Longrightarrow& x^0 &\equiv& 1\\
\frac{x^1}{x^3+x+1} &=& 0 &+& \frac{x}{x^3+x+1} &\Longrightarrow& x^1 &\equiv& x\\
\frac{x^2}{x^3+x+1} &=& 0 &+& \frac{x^2}{x^3+x+1} &\Longrightarrow& x^2 &\equiv& x^2\\
\frac{x^3}{x^3+x+1} &=& 1 &+& \frac{x+1}{x^3+x+1} &\Longrightarrow& x^3 &\equiv& x+1\\
\frac{x^4}{x^3+x+1} &=& x &+& \frac{x^2+x}{x^3+x+1} &\Longrightarrow& x^4 &\equiv& x^2+x\\
\frac{x^5}{x^3+x+1} &=& x^2+1 &+& \frac{x^2+x+1}{x^3+x+1} &\Longrightarrow& x^5 &\equiv& x^2+x+1\\
\frac{x^6}{x^3+x+1} &=& x^3+x+1 &+& \frac{x^2+1}{x^3+x+1} &\Longrightarrow& x^6 &\equiv& x^2+1\\
\frac{x^7}{x^3+x+1} &=& x^4+x^2+x+1 &+& \frac{1}{x^3+x+1} &\Longrightarrow& x^7 &\equiv& 1
\end{array}$$

Notice that $x^7 \equiv x^0$, and so a cycle is formed with $2^3 - 1 = 7$ elements in the cycle. The monomials to the left of the $\equiv$ are the *exponential* representation and the polynomials to the right are the *polynomial* representation. Additionally, the polynomial representation may be put in a vector form giving a *vector* representation. The vectors may be interpreted as a binary number and represented as a decimal numeral.

exponential	polynomial	*vector*	*decimal*
x^0	1	[001]	1
x^1	x	[010]	2
x^2	x^2	[100]	4
x^3	$x+1$	[011]	3
x^4	x^2+x	[110]	6
x^5	x^2+x+1	[111]	7
x^6	x^2+1	[101]	5

4.2.2 Generating m-sequences using polynomial division

An m-sequence is generated by dividing any non-zero polynomial of order less than m by a primitive polynomial of order m. The m-sequence is the coefficients of the resulting polynomial. M-sequences will repeat every $2^m - 1$ values. This is the maximum sequence length possible when the sequence is generated by division in polynomials over GF(2).

Example 4.2. We can generate an m-sequence of length $2^3 - 1 = 7$ by dividing 1 by the primitive polynomial $x^3 + x + 1$.

$$
\begin{array}{r|l}
 & x^{-3}+x^{-5}+x^{-6}+\quad x^{-7}+x^{-10}+x^{-12}+x^{-13}+x^{-14}+x^{-17}+\cdots \\
\hline
x^3+x+1 & 1 \\
 & 1+x^{-2}+x^{-3} \\
\cline{2-2}
 & x^{-2}+x^{-3} \\
 & x^{-2}+x^{-4}+x^{-5} \\
\cline{2-2}
 & x^{-3}+x^{-4}+x^{-5} \\
 & x^{-3}+x^{-5}+x^{-6} \\
\cline{2-2}
 & x^{-4}+x^{-6} \\
 & x^{-4}+x^{-6}+x^{-7} \\
\cline{2-2}
 & x^{-7} \\
 & x^{-7}+x^{-9}+x^{-10} \\
\cline{2-2}
 & x^{-9}+x^{-10} \\
 & x^{-9}+x^{-11}+x^{-12} \\
\cline{2-2}
 & x^{-10}+x^{-11}+x^{-12} \\
 & x^{-10}+x^{-12}+x^{-13} \\
\cline{2-2}
 & x^{-11}+x^{-13} \\
 & x^{-11}+x^{-13}+x^{-14} \\
\cline{2-2}
 & x^{-14} \\
 & \vdots
\end{array}
$$

The coefficients, starting with the x^{-1} term, of the resulting polynomial form the m-sequence

$$0010111\ 0010111\ \cdots$$

which repeats every $2^3 - 1 = 7$ elements.

Note that the division operation in Example 4.2 can be performed using vector notation rather than polynomial notation.

Example 4.3. Generate an m-sequence of length $2^3 - 1 = 7$ by dividing 1 by the primitive polynomial $x^3 + x + 1$ using vector notation.

$$
\begin{array}{r|ccccccccccccccccccc}
 & & . & 0&0&1&0&1&1&1&0&0&1&0&1&1&1&0&\cdots \\
\hline
1011 & 1 & . & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0& \\
 & 1 & & 0&1&1 \\
 & 0 & & 0&1&1&0 \\
 & & & 0&0&0&0 \\
 & & & 0&1&1&0&0 \\
 & & & &1&0&1&1 \\
 & & & &0&1&1&1&0 \\
 & & & & &1&0&1&1 \\
 & & & & &0&1&0&1&0 \\
 & & & & & &1&0&1&1 \\
 & & & & & &0&0&0&1&0 \\
 & & & & & & &0&0&0&0 \\
 & & & & & & &0&0&1&0&0 \\
 & & & & & & & &0&0&0&0 \\
 & & & & & & & &0&1&0&0&0 \\
 & & & & & & & & &1&0&1&1 \\
 & & & & & & & & &0&0&1&1&0 \\
 & & & & & & & & & &0&0&0&0 \\
 & & & & & & & & & &0&1&1&0&0 \\
 & & & & & & & & & & &1&0&1&1 \\
 & & & & & & & & & & &0&1&1&1&0 \\
 & & & & & & & & & & & &0&0&0&0 \\
 & & & & & & & & & & & & &\vdots
\end{array}
$$

The coefficients, starting to the right of the binary point, is again the sequence

$$0010111\ 0010111\ \cdots.$$

4.2.3 Multiplication modulo a primitive polynomial

If $p(x)$ is a primitive polynomial, by Definition 4.4 the product of two polynomials is equivalent (with respect to $p(x)$) of the product *modulo* $p(x)$. The ability to multiplying two polynomials modulo a primitive polynomial is very useful for manipulating m-sequences.

In general, the product of two polynomials can be evaluated as follows. Let

$$\begin{aligned}
a(x) &\triangleq a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0\\
b(x) &\triangleq b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0
\end{aligned}$$

Then

$$\begin{aligned}
a(x)b(x) &= \left(a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0\right)\left(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0\right)\\
&= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots + a_mb_mx^{2m}\\
&= \left(\sum_{i=0}^{m-1} x^i \sum_{j=0}^{i} a_jb_{i-j}\right) + \left(\sum_{i=m}^{2m} x^i \sum_{j=0}^{2m-i} a_{i-m+j}b_{m-j}\right)
\end{aligned}$$

The product modulo $p(x)$ is obtained when the terms involving x^m, x^{m+1}, ..., x^{2m} are replaced by their equivalent polynomial representations (see Section 4.2.1).

Example 4.4. Suppose we want to find $(a_2x^2 + a_1x + a_0)(b_2x^2 + b_1x + b_0)$ modulo $x^3 + x + 1$.

$$\begin{aligned}
a(x)b(x) &= (a_2x^2 + a_1x + a_0)(b_2x^2 + b_1x + b_0)\\
&= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_1b_2 + a_2b_1)x^3 + a_2b_2x^4\\
&= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_1b_2 + a_2b_1)(x+1) + a_2b_2(x^2 + x)\\
&= (a_0b_0 + a_1b_2 + a_2b_1) + (a_0b_1 + a_1b_0 + a_1b_2 + a_2b_1 + a_2b_2)x + (a_0b_2 + a_1b_1 + a_2b_0 + a_2b_2)x^2
\end{aligned}$$

Notice that if the a_i and b_i coefficients are known, the resulting product has only three terms.

4.3 Generating m-sequences in hardware

Section 4.2 has already demonstrated how to generate m-sequences mathematically. If we further know how to implement each of those mathematical operations efficiently in hardware, we are done. That is what this section is about.

4.3.1 Field operations

The mapping tables for GF(2) addition and multiplication given in Definition 4.1 (page 32) are exactly the same as those for the hardware *exclusive OR (XOR)* gate and the *AND* gate, respectively.

4.3.2 Polynomial multiplication and division using DF1

Suppose we want to construct a circuit to compute the rational expression $f(x)\frac{b(x)}{a(x)}$. This is a common problem in *Digital Signal Processing (DSP)*; we can borrow results from there. DSP is generally concerned with polynomials over the field of real or complex numbers. However, a field is a field, and all fields (whether, real, complex, or GF(2)) support both addition and multiplication;[4] the rules change somewhat, but the basic structure is the same regardless. Alternatively, just as a typical digital filter operates over the real or complex field, **the m-sequence generator** described in this section **is a digital filter which operates over the field GF(2)**.

A sequential hardware multiplier-divider for polynomials is simple.

- Each x in $f(x)$, $b(x)$, and $a(x)$ represents a delay of one clock cycle. In DSP terminology, a delay of one clock cycle is represented by z^{-1}. Thus, $x = z^{-1}$.
- Let $f(x) = f_0 + f_1x + f_2x^2 + \cdots$.
  Then let $\dot{f}(n)$ be the sequence $\dot{f}(i) = f_i$, with $i \in \mathbb{Z}$.
- Let $b(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$.
  Let $\dot{b}(n)$ be the sequence $\dot{b}(i) = b_i$, with $i \in \mathbb{Z}$.
- Let $a(x) = 1 + a_1x + a_2x^2 + \cdots + a_mx^m$.
  Let $\dot{a}(n)$ be the sequence $\dot{a}(i) = a_i$, with $i \in \mathbb{Z}$.

Then the multiplier-divider (for any mathematical field) can be implemented as shown in Figure 4.1. This structure is called the *Direct Form I* implementation(Oppenheim and Schafer, 1999)344; it implements the rational expression

$$f(x)\frac{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0}{a_mx^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + 1}$$

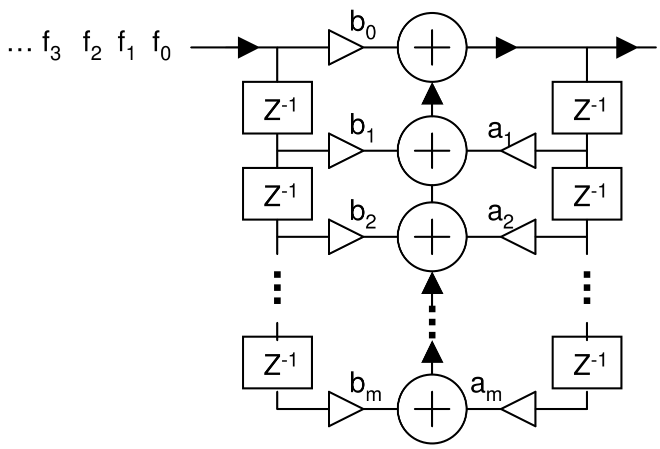


Figure 4.1: Direct Form I Implementation for $f(x)\dfrac{b(x)}{a(x)}$

In GF(2), the blocks in the figure can be implemented very simply:

- Each $x = z^{-1}$ element can be implemented as a simple D flip-flop.
- An $a_i = 1$ or $b_i = 1$ coefficient is implemented as a wire (closed circuit).
- An $a_i = 0$ or $b_i = 0$ coefficient is implemented as a no-connect (open circuit).

Example 4.5. Suppose we want to build a hardware circuit to generate an m-sequence specified by the rational expression

$$\frac{x^2 + x}{x^3 + x + 1}.$$

[4]**Fields:** Roughly speaking, a *group* is a set together with an operation on that set. An *additive group* is a set S with an addition operation $+ : S \times S \to S$. A *multiplicative group* is a set S with a multiplication operation $\cdot : S \times S \to S$. A *field* is constructed using two groups: An addition group and a multiplication group. See Appendix **??** page **??**. Reference: (**?**, p.123).

To do this we can set $f(x) = 1$, $b(x) = x^2 + x$ and $a(x) = x^3 + x + 1$. The resulting structure is shown in Figure 4.2. Notice that the two flip-flops on the left are for initialization only and are not used in

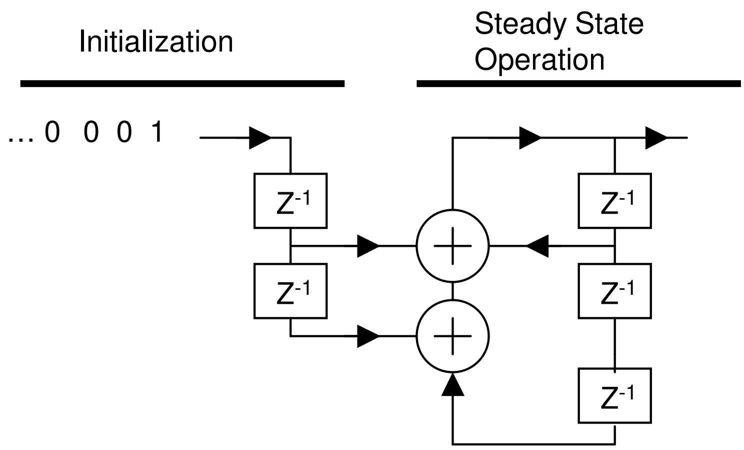


Figure 4.2: Direct Form I Implementation for Example 4.5

the steady state operation of the m-sequence generator. In fact, they can be eliminated altogether by proper initialization of the flip-flops on the right.

4.3.3 Polynomial multiplication and division using DF2

The Direct Form I structure shown in Figure 4.1 can be transformed to a new structure by transformation rules based on *Mason's Gain Formula*.[5] The resulting structure is known as Direct Form II(Oppenheim and Schafer, 1999)347 and is illustrated in Figure 4.3. Again, when using the DF2

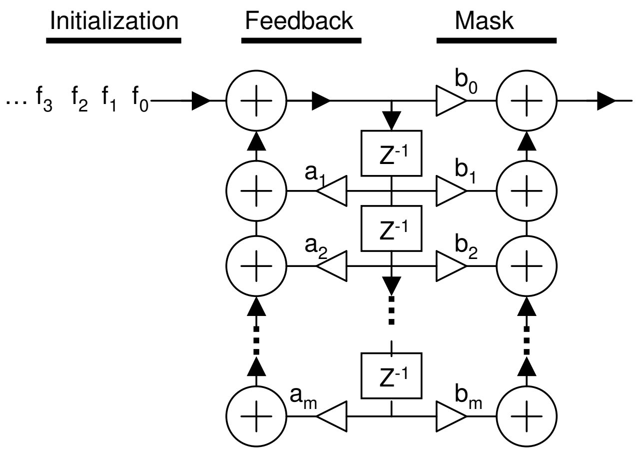


Figure 4.3: Direct Form II Implementation

structure for m-sequence generation, the $\dot{f}(n)$ sequence can be eliminated by proper initialization of the delay elements (flip flops).

[5]The transformation rules are as follows:

1. Reverse the direction of all signal paths.
2. Replace all nodes with addition operators.
3. Replace all addition operators with nodes.

(Oppenheim and Schafer, 1999, p.363)

4.3.4 Hardware polynomial modulo multiplier

The mathematics of polynomial multiplication modulo a primitive polynomial was already presented in Section 4.2.3 and demonstrated in Example 4.4 (page 36). It is straight forward to implement these equations in hardware:

- every $a_i b_j$ bitwise multiply operation is implemented with an AND gate
- every + between consecutive $a_i b_j$ terms is implemented with an XOR gate

Note that **the hardware modulo multiplier can be implemented using only combinatorial logic**(!); No sequential circuitry (such as flip-flops) are needed.

CHAPTER 5

LINE CODING

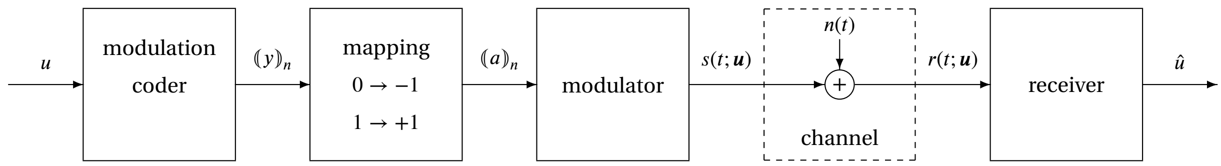


Figure 5.1: Modulation coding system model

This chapter discusses *modulation coding*. Modulation codes are also called *line codes* or *data translation codes*. (Proakis, 2001)579 Modulation coding is a transform $T : u \to ⦅y⦆_n$ from an input sequence u to an encoded sequence $⦅y⦆_n$ (see Figure 5.1). Modulation codes typically seek to accomplish two objectives:

1. time shaping: eliminate long strings of ones or zeros to improve synchronization or make media access more reliable.
2. spectral shaping: modify spectral characteristics such as reducing the DC component.

A particular modulation code may be specified using several methods including

1. state machine
2. transition matrix
3. algebraic equations.

5.1 Channel model

The modulation coding system model is illustrated in Figure 5.1.

The *modulation coding state machine* is a transform $T : (u_n) \to (y_n)$. Modulation coding can be modeled as a *state-space* with input u_n, output y_n, state $\boldsymbol{x}_n$ and state equations [1]

$$\begin{aligned} \boldsymbol{x}_{n+1} &= f_1(\boldsymbol{x}_n, u_n) \\ y_n &= f_2(\boldsymbol{x}_n, u_n). \end{aligned}$$

Other quantities appearing in Figure 5.1 can be expressed as

$$\begin{array}{llcl} \text{mapping output:} & a_n & = & 2y_n - 1 \\ \text{channel signal:} & s(t) & = & \sum_n a_n \lambda(t - nT) \\ \text{receive signal:} & r(t) & = & s(t) + n(t). \end{array}$$

The signaling waveform $\lambda(t)$ can be any of a number of waveforms. A common choice is the simple pulse function illustrated in Figure 5.2. But this assumes the channel supports an infinitely wide bandwidth signal. Bandlimited choices of signaling waveforms are described in Chapter 13 (page 143).

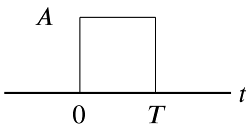


Figure 5.2: Pulse signaling waveform

5.2 Non-Return to Zero Modulation (NRZ)

5.2.1 Description

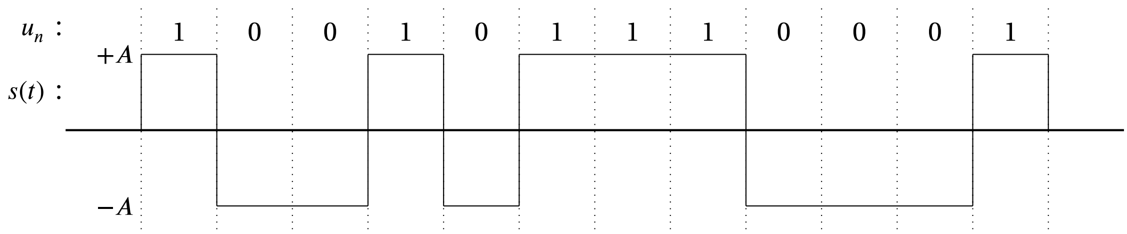


Figure 5.3: NRZ modulated waveform

The non-return to zero (NRZ) waveform is illustrated in Figure 5.3.

5.2.2 Statistics

Note that even if the data sequence u_n is an IID and WSS [2] sequence, the channel signal $s(t)$ is **not** WSS. Specifically, the autocorrelation $\mathsf{R}_{ss}(t+\tau, t)$ of $s(t)$ is not just a function of the time difference τ, but also a function of time t. This is due to the fact that within a bit period, if one point is known

[1] III ⟨\protect\char"2026\relax/State_Space_Models.html⟩
[2] IID: independently and identically distributed. WSS: wide sense stationary

then all the points in that bit period are known. Thus the points in a single bit period are certainly not independent and their autocorrelation is a function of time.

However, it is still possible to compute the time average of the autocorrelation and the Fourier transform of this average (similar to the spectral density). This is described in Theorem 5.1 and illustrated in Figure 5.4.

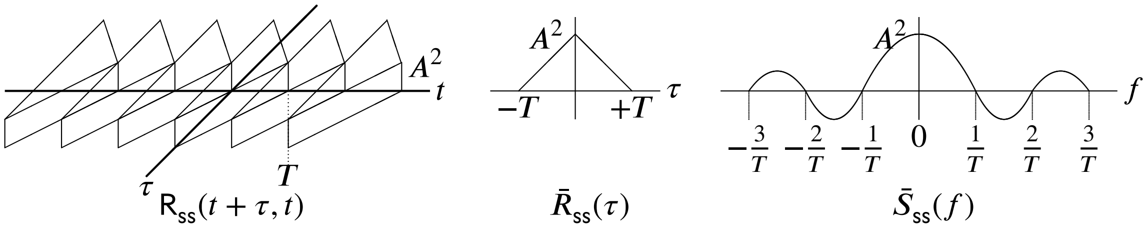


Figure 5.4: Statistics of NRZ modulated waveform

Theorem 5.1. *Let*

- $u_n : \mathbb{Z} \to \{0,1\}$ *be an IID WSS random process with probabilities*

$$\mathsf{P}\{u_n = 0\} = \mathsf{P}\{u_n = 1\} = \frac{1}{2} \qquad \textit{for all } n$$

- $s(t)$ *be the waveform NRZ modulated by* u_n
- $\mathsf{R}_{ss}(t+\tau,t)$ *be the autocorrelation of* $s(t)$ *such that*

$$\mathsf{R}_{ss}(t+\tau,t) \triangleq \mathsf{E}\left[s(t+\tau)s(t)\right]$$

- $\bar{R}_{ss}(\tau)$ *be the time average of* $\mathsf{R}_{ss}(t+\tau,t)$.

$$\bar{R}_{ss}(\tau) \triangleq \frac{1}{T}\int_0^T \mathsf{R}_{ss}(t+\tau,t)\, \mathrm{d}t$$

- $\bar{S}_{ss}(f)$ *be the Fourier transform of* $\bar{R}_{ss}(\tau)$ *such that*

$$\bar{S}_{ss}(f) \triangleq \int_\tau \bar{R}_{ss}(\tau) e^{-i2\pi f\tau}\, \mathrm{d}\tau.$$

Then

$$\begin{aligned}
\mathsf{R}_{ss}(t+\tau,t) &= \begin{cases} A^2 & : \tau \le (t \mod [T]) \le T \\ 0 & : \textit{otherwise} \end{cases} \\
\bar{R}_{ss}(\tau) &= \begin{cases} A^2\left(1-\frac{|\tau|}{T}\right) & : |\tau| \le T \\ 0 & : |\tau| > T. \end{cases} \\
\bar{S}_{xx}(f) &= A^2\left[\frac{\sin(\pi f T)}{\pi f T}\right]^2.
\end{aligned}$$

✎PROOF: For time intervals $\tau \le (t \mod [T]) \le T$, identical portions of $s(t+\tau)$ and $s(t)$ overlap and the resulting autocorrelation is

$$\begin{aligned}
\mathsf{R}_{ss}(t+\tau,t) &= \mathsf{E}\left[s(t+\tau)s(t)\right] \\
&= (-A)(-A)\mathsf{P}\left\{[s(t+\tau)=-A] \wedge [s(t)=-A]\right\} + (-A)(+A)\mathsf{P}\left\{[s(t+\tau)=-A] \wedge [s(t)=-A]\right\} + \\
&\quad (+A)(-A)\mathsf{P}\left\{[s(t+\tau)=-A] \wedge [s(t)=-A]\right\} + (+A)(+A)\mathsf{P}\left\{[s(t+\tau)=-A] \wedge [s(t)=-A]\right\} \\
&= (-A)(-A)\frac{1}{2} + (-A)(+A)\cdot 0 + (+A)(-A)\cdot 0 + (+A)(+A)\frac{1}{2} \\
&= A^2
\end{aligned}$$

For all other time intervals, especially $|\tau| > T$, $s(t+\tau)$ and $s(t)$ are statistically independent and hence

$$\mathsf{R}_{ss}(\tau) = \mathsf{E}\left[s(t+\tau)s(t)\right] = \mathsf{E}\left[s(t+\tau)\right]\mathsf{E}\left[s(t)\right] = 0\cdot 0 = 0.$$

Alternatively,

$$\begin{aligned}
\mathsf{R}_{ss}(t+\tau,t) &= \mathsf{E}\left[s(t+\tau)s(t)\right] \\
&= (-A)(-A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\} + (-A)(+A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\} + \\
&\quad (+A)(-A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\} + (+A)(+A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\} \\
&= (-A)(-A)\frac{1}{4} + (-A)(+A)\frac{1}{4} + (+A)(-A)\frac{1}{4} + (+A)(+A)\frac{1}{4} \\
&= A^2 - A^2 - A^2 + A^2 \\
&= 0.
\end{aligned}$$

✏

5.2.3 Detection

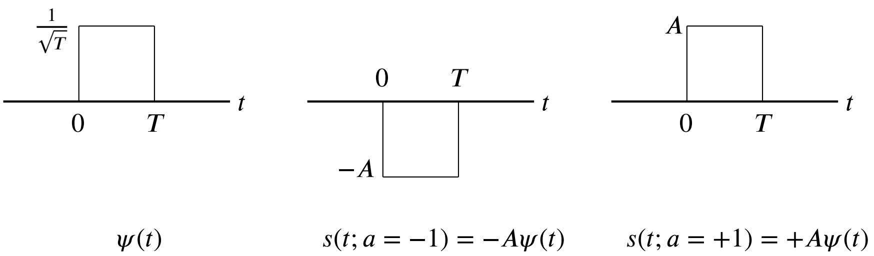


Figure 5.5: NRZ critical functions

Proposition 5.1. *The function*

$$\psi(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } 0 \le t < T \\ 0 & \text{otherwise.} \end{cases}$$

*forms an **orthonormal basis** for the NRZ signaling waveforms such that*

$$\begin{aligned}
s(t;a=-1) &= -A\psi(t) \\
s(t;a=+1) &= +A\psi(t).
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\langle \psi(t) \,|\, \psi(t) \rangle &= \left\langle \frac{1}{\sqrt{T}} \,|\, \frac{1}{\sqrt{T}} \right\rangle \\
&= \frac{1}{\sqrt{T}} \frac{1}{\sqrt{T}} \langle 1 \,|\, 1 \rangle \\
&= \frac{1}{T} \int_0^T 1 \cdot 1 \; dt \\
&= \frac{1}{T} t \big|_0^T \\
&= \frac{1}{T} (T - 0) \\
&= 1
\end{aligned}$$

✏

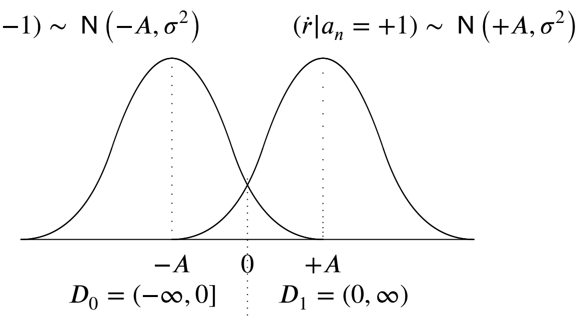


Figure 5.6: Decision statistics for NRZ modulation in AWGN channel

Proposition 5.2. *Let*

$$\begin{aligned}
\dot{r}(-1) &\triangleq \langle r(t) | s(t; a = -1) \textit{ was transmitted } | \psi(t) \rangle \\
\dot{r}(+1) &\triangleq \langle r(t) | s(t; a = +1) \textit{ was transmitted } | \psi(t) \rangle .
\end{aligned}$$

Then $\dot{r}(-1)$ and $\dot{r}(+1)$ are ***independent*** *random variables with marginal distributions*

$$\begin{aligned}
\dot{r}(-1) &\sim \mathsf{N}\left(-A, \sigma^2\right) \\
\dot{r}(+1) &\sim \mathsf{N}\left(+A, \sigma^2\right)
\end{aligned}$$

✎PROOF: This follows directly from Theorem 7.5 (page 76). ✏

Proposition 5.3. *The value*

$$\dot{r} \triangleq \langle r(t) \,|\, \psi(t) \rangle$$

is a sufficient statistic for optimal ML detection of the transmitted symbol a.

The optimal estimate $\hat{a}_{\mathsf{ml}}$ of a is

$$\hat{a} = \begin{cases} -1 & : \dot{r} \leq 0 \\ +1 & : \dot{r} > 0. \end{cases}$$

✎PROOF: This is a result of Theorem 7.6 (page 76). ✏

Proposition 5.4. *The probability of detection error in an NRZ modulation system*

$$\mathsf{P}\{error\} = \mathsf{Q}\left[\frac{a}{N_o}\right].$$

✎PROOF:

$$\begin{aligned}
\mathsf{P}\{error\} &= \mathsf{P}\left\{s_0(t) \text{ sent } \wedge \dot{r} > 0\right\} + \mathsf{P}\left\{s_1(t) \text{ sent } \wedge \dot{r} < 0\right\} \\
&= \mathsf{P}\left\{\dot{r} > 0 | s_0(t) \text{ sent }\right\} \mathsf{P}\left\{s_0(t) \text{ sent }\right\} + \mathsf{P}\left\{\dot{r} < 0 | s_1(t) \text{ sent }\right\} \mathsf{P}\left\{s_1(t) \text{ sent }\right\} \\
&= 2\mathsf{P}\left\{\dot{r} > 0 | s_0(t) \text{ sent }\right\} \frac{1}{2} \\
&= \mathsf{Q}\left[\frac{\mathsf{E}\dot{r}}{\sqrt{\operatorname{var}\dot{r}}}\right] \\
&= \mathsf{Q}\left[\frac{a}{N_o}\right]
\end{aligned}$$

✏

5.3 Return to Zero Modulation (RZ)

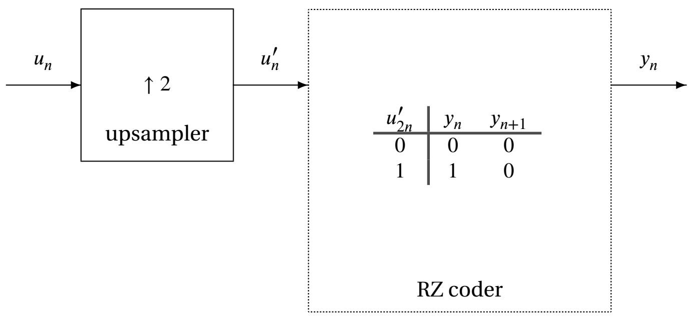


Figure 5.7: RZ modulation coder

The non-return to zero (RZ) modulation coder is illustrated in Figure 5.7. An example RZ modulated waveform is illustrated in Figure 5.8. An RZ modulated waveform $s(t)$ can be decomposed into a deterministic periodic waveform $d(t)$ and a random waveform $r(t)$ such that $s(t) = d(t) + r(t)$ (see Figure 5.9 page 47).[3]

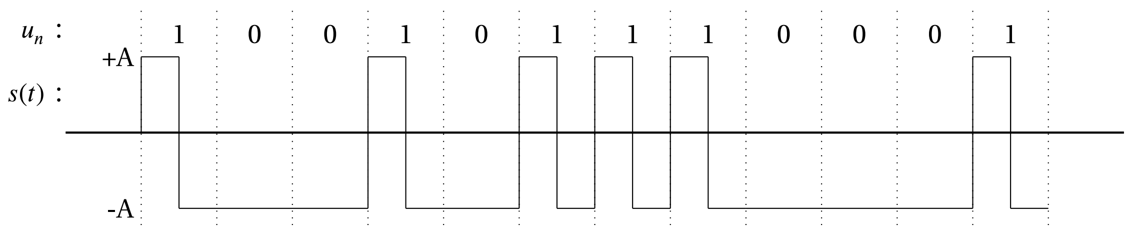


Figure 5.8: RZ waveform

Theorem 5.2. *Let*

[3] 🗐 Kao (2005)

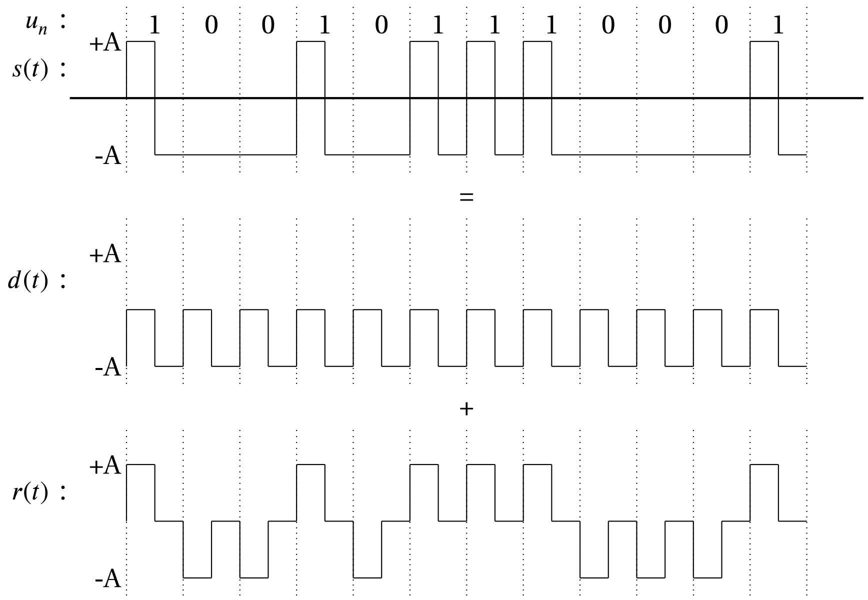


Figure 5.9: Decomposition of RZ modulated waveform

- $u_n : \mathbb{Z} \to \{0,1\}$ *be an IID WSS random process with probabilities*

$$\mathsf{P}\{u_n = 0\} = \mathsf{P}\{u_n = 1\} = \frac{1}{2} \qquad \textit{for all } n$$

- $s(t)$ *be the waveform RZ modulated by* u_n
- $d(t)$ *be the deterministic periodic waveform illustrated in Figure 5.9*
- $\mathsf{R}_{ss}(t+\tau,t)$ *be the autocorrelation of* $s(t)$ *such that*

$$\mathsf{R}_{ss}(t+\tau,t) \triangleq \mathsf{E}\left[s(t+\tau)s(t)\right]$$

- $\bar{R}_{ss}(\tau)$ *be the time average of* $\mathsf{R}_{ss}(t+\tau,t)$.

$$\bar{R}_{ss}(\tau) \triangleq \frac{1}{T}\int_0^T \mathsf{R}_{ss}(t+\tau,t)\,\mathrm{d}t$$

- $\bar{S}_{ss}(f)$ *be the Fourier transform of* $\bar{R}_{ss}(\tau)$ *such that*

$$\bar{S}_{ss}(f) \triangleq \int_\tau \bar{R}_{ss}(\tau)e^{-i2\pi f\tau}\,\mathrm{d}\tau.$$

Then

$$\mathsf{R}_{ss}(t+\tau,t) = \begin{cases} A^2 + d(t+\tau)d(t) & : \tau \le (t \mod [T]) \le \frac{T}{2} \\ d(t+\tau)d(t) & : \textit{otherwise} \end{cases}$$

$$\bar{R}_{ss}(\tau) = \frac{A^2}{2}\left(1-\frac{2|\tau|}{T}\right)\chi_{[-T/2,T/2]}(\tau) + \frac{A^2}{2}\sum_n\left(1-\frac{2|\tau-nT|}{T}\right)\chi_{[-T/2,T/2]}(\tau-nT)$$

$$\bar{S}_{xx}(f) = \frac{A^2T}{4}\left[\frac{\sin\left(\pi f\frac{T}{2}\right)}{\pi f\frac{T}{2}}\right]^2 + \frac{A^2T}{4}\sum_k\left[\frac{\sin\left(\pi k\frac{1}{2}\right)}{\pi k\frac{1}{2}}\right]^2\delta\left(f-\frac{k}{T}\right)$$

✎PROOF:

$$\begin{aligned}
\mathsf{R}_{ss}(t+\tau,t) &= \mathsf{E}\left[s(t+\tau)s(t)\right]\\
&= \mathsf{E}\left[[d(t+\tau)r(t+\tau)][d(t)+r(t)]\right]\\
&= \mathsf{E}\left[d(t+\tau)d(t)+d(t+\tau)r(t)+r(t+\tau)d(t)+r(t+\tau)r(t)\right]\\
&= d(t+\tau)d(t)+d(t+\tau)\mathsf{E}\left[r(t)\right]+d(t)\mathsf{E}\left[r(t+\tau)\right]+\mathsf{E}\left[r(t+\tau)r(t)\right]\\
&= \mathsf{R}_{rr}(t+\tau,t)+d(t+\tau)d(t)+d(t+\tau)\cdot 0+d(t)\cdot 0\\
&= \mathsf{R}_{rr}(t+\tau,t)+d(t+\tau)d(t)
\end{aligned}$$

For time intervals $\tau \le (t \mod [T]) \le T/2$, identical portions of $r(t+\tau)$ and $r(t)$ overlap and the resulting autocorrelation is

$$\begin{aligned}
\mathsf{R}_{rr}(t+\tau,t) &= (-A)(-A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\}+(-A)(+A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\}+\\
&\quad (+A)(-A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\}+(+A)(+A)\mathsf{P}\left\{[s(t+\tau)=-A]\wedge[s(t)=-A]\right\}\\
&= (-A)(-A)\frac{1}{2}+(-A)(+A)\cdot 0+(+A)(-A)\cdot 0+(+A)(+A)\frac{1}{2}\\
&= A^2
\end{aligned}$$

For all other time intervals, especially $|\tau| > T$, $r(t+\tau)$ and $r(t)$ are statistically independent and hence

$$\mathsf{R}_{rr}(\tau) = \mathsf{E}\left[r(t+\tau)r(t)\right] = \mathsf{E}\left[r(t+\tau)\right]\mathsf{E}\left[r(t)\right] = 0\cdot 0 = 0.$$

To compute the time average $\bar{R}_{ss}(\tau)$, we need to find the average of both $\mathsf{R}_{rr}(t+\tau,t)$ and $d(t+\tau)d(t)$.

$$\begin{aligned}
\frac{1}{T}\int_0^T d(t+\tau)d(t)\,\mathsf{d}t &= \frac{1}{T}\frac{A^2T}{2}\sum_n\left(1-\frac{|\tau-nT|}{T/2}\right)\chi_{[-T/2,T/2]}(\tau-nT)\\
&= \frac{A^2}{2}\sum_n\left(1-\frac{2|\tau-nT|}{T}\right)\chi_{[-T/2,T/2]}(\tau-nT)
\end{aligned}$$

$$\frac{1}{T}\int_0^T \mathsf{R}_{rr}(t+\tau,t)\,\mathsf{d}t = \begin{cases} \frac{A^2}{2}\left(1-\frac{2|\tau|}{T}\right) & : |\tau| \le \frac{T}{2}\\ 0 & : |\tau| > \frac{T}{2}. \end{cases}$$

$$\bar{R}_{ss}(\tau) = \frac{A^2}{2}\left(1-\frac{2|\tau|}{T}\right)\chi_{[-T/2,T/2]}(\tau)+\frac{A^2}{2}\sum_n\left(1-\frac{2|\tau-nT|}{T}\right)\chi_{[-T/2,T/2]}(\tau-nT)$$

$$\begin{aligned}
\bar{S}_{xx}(f) &= \frac{A^2T}{4}\left[\frac{\sin\left(\pi f\frac{T}{2}\right)}{\pi f\frac{T}{2}}\right]^2+\frac{A^2T}{4}\sum_k\left[\frac{\sin\left(\pi\frac{k}{T}\frac{T}{2}\right)}{\pi\frac{k}{T}\frac{T}{2}}\right]^2\delta\left(f-\frac{k}{T}\right)\\
&= \frac{A^2T}{4}\left[\frac{\sin\left(\pi f\frac{T}{2}\right)}{\pi f\frac{T}{2}}\right]^2+\frac{A^2T}{4}\sum_k\left[\frac{\sin\left(\pi k\frac{1}{2}\right)}{\pi k\frac{1}{2}}\right]^2\delta\left(f-\frac{k}{T}\right)
\end{aligned}$$

✏

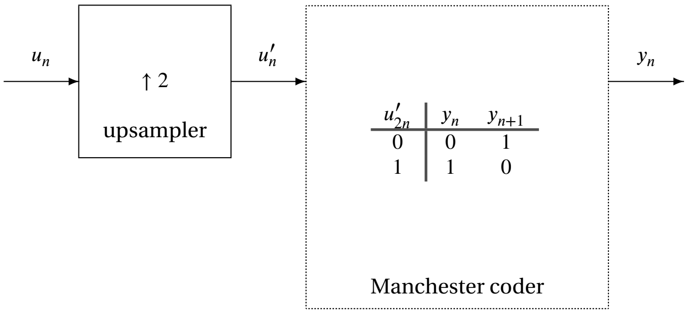


Figure 5.10: Manchester modulation coder

5.4 Manchester Modulation

The Manchester modulation coder is illustrated in Figure 5.10. An example RZ modulated waveform is illustrated in Figure 5.11.

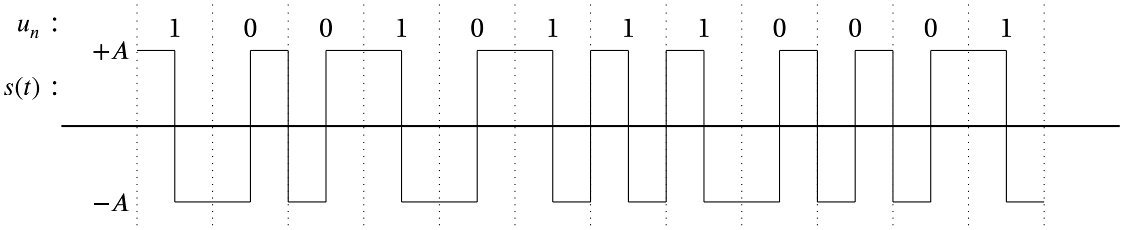


Figure 5.11: Manchester modulated waveform

Theorem 5.3. *Let*

- $u_n : \mathbb{Z} \to \{0, 1\}$ *be an IID WSS random process with probabilities*

$$\mathsf{P}\{u_n = 0\} = \mathsf{P}\{u_n = 1\} = \frac{1}{2} \qquad \text{for all } n$$

- $s(t)$ *be the waveform Manchester modulated by* u_n
- $\mathsf{R}_{ss}(t+\tau, t)$ *be the autocorrelation of* $s(t)$ *such that*

$$\mathsf{R}_{ss}(t+\tau, t) \triangleq \mathsf{E}\left[s(t+\tau)s(t)\right]$$

- $\bar{R}_{ss}(\tau)$ *be the time average of* $\mathsf{R}_{ss}(t+\tau, t)$.

$$\bar{R}_{ss}(\tau) \triangleq \frac{1}{T}\int_0^T \mathsf{R}_{ss}(t+\tau, t)\,\mathsf{d}t$$

- $\bar{S}_{ss}(f)$ *be the Fourier transform of* $\bar{R}_{ss}(\tau)$ *such that*

$$\bar{S}_{ss}(f) \triangleq \int_\tau \bar{R}_{ss}(\tau)e^{-i2\pi f\tau}\,\mathsf{d}\tau.$$

Then

$$\mathsf{R}_{ss}(t+\tau, t) = \begin{cases} 0 & : 0 \le (t \mod [T]) < \tau \\ +A^2 & : \tau \le (t \mod [T]) < \frac{T}{2} \\ -A^2 & : \frac{T}{2} \le (t \mod [T]) < \tau + \frac{T}{2} \\ +A^2 & : \tau + \frac{T}{2} \le (t \mod [T]) < T \end{cases}$$

$$\bar{R}_{ss}(\tau) = \begin{cases} A^2\left(1-3\frac{|\tau|}{T}\right) & : 0 \le |\tau| < \frac{T}{2} \\ -\frac{A^2}{2}\left(1-\frac{|\tau|}{T}\right) & : \frac{T}{2} \le |\tau| < T \end{cases}$$

$$\bar{S}_{xx}(f) \stackrel{?}{=} A^2T\frac{\sin^4 \pi f T/2}{\pi f T/2}$$

✎PROOF:

$$\begin{aligned}
\bar{S}_{ss}(f) &= \int_\tau \bar{R}_{ss}(\tau)e^{-i2\pi f\tau}\,\mathrm{d}\tau \\
&= \int_\tau \bar{R}_{ss}(\tau)\cos(2\pi f\tau)\,\mathrm{d}\tau - i\int_\tau \bar{R}_{ss}(\tau)\sin(2\pi f\tau)\,\mathrm{d}\tau \\
&= 2\int_0^T \bar{R}_{ss}(\tau)\cos(2\pi f\tau)\,\mathrm{d}\tau + 0 \\
&= 2\int_0^{T/2} A^2\left(1-3\frac{\tau}{T}\right)\cos(2\pi f\tau)\,\mathrm{d}\tau - 2\int_{T/2}^{T}\frac{A^2}{2}\left(1-\frac{\tau}{T}\right)\cos(2\pi f\tau)\,\mathrm{d}\tau \\
&= 2A^2\int_0^{T/2}\cos(2\pi f\tau)\,\mathrm{d}\tau - A^2\int_{T/2}^{T}\cos(2\pi f\tau)\,\mathrm{d}\tau - \frac{6A^2}{T}\int_0^{T/2}\tau\cos(2\pi f\tau)\,\mathrm{d}\tau + \frac{A^2}{T}\int_{T/2}^{T}\tau\cos(2\pi f\tau)\,\mathrm{d}\tau \\
&= A^2T\left(\frac{\sin\pi fT}{\pi fT}\right) - A^2T\left(\frac{\sin 2\pi fT}{2\pi fT}\right) + \frac{A^2T}{2}\left(\frac{\sin\pi fT}{\pi fT}\right) - \frac{6A^2T}{4}\frac{\sin\pi fT}{\pi fT} - \frac{6A^2}{4\pi f}\frac{\cos\pi fT}{\pi fT} \\
&\quad + \frac{6A^2}{(2\pi f)^2T} + A^2T\frac{\sin 2\pi fT}{2\pi fT} + \frac{A^2}{2\pi f}\frac{\cos 2\pi fT}{2\pi fT} - \frac{A^2T}{4}\frac{\sin\pi fT}{\pi fT} - \frac{T}{4\pi f}\frac{\cos\pi fT}{\pi fT} \\
&\stackrel{?}{=} A^2T\frac{\sin^4\pi fT/2}{\pi fT/2}
\end{aligned}$$

I have not been able to solve this well yet. The last line is taken from reference Kao (2005). ✏

5.5 Non-Return to Zero Modulation Inverted (NRZI)

NRZI is a modulation code, however it is *not* a runlength-limited code. NRZI has memory and is therefore a kind of state machine.

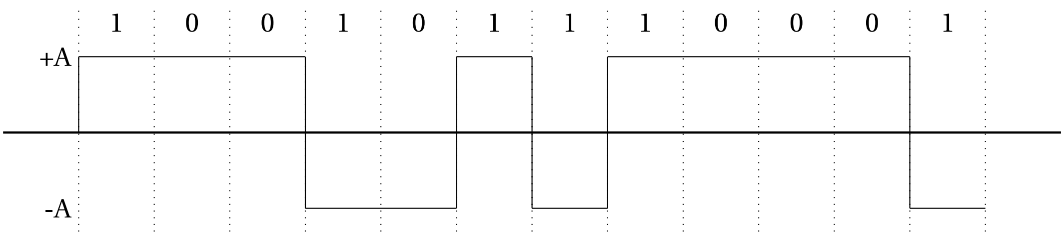


Figure 5.12: NRZI waveform

Definition 5.1. ***Non-Return to Zero Inverted (NRZI)*** *is a modulation code with input sequence u_n and output sequence y_n such that (see Figure 5.13)*

$$y_n = (y_{n-1} + u_n) \mod [2].$$

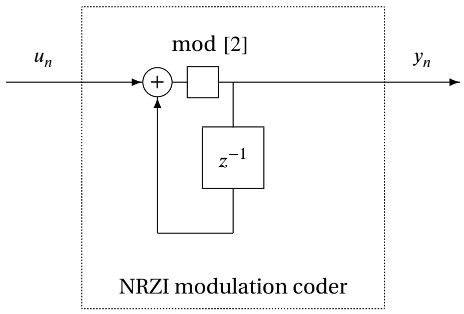


Figure 5.13: NRZI modulation coder

Detection. Detection in an AWGN channel can be performed using a trellis (see Figure 5.14) or single statistic decision regions. A very clean decision region approach is the *duobinary ISI solution* described in Section 13.3 (page 152).

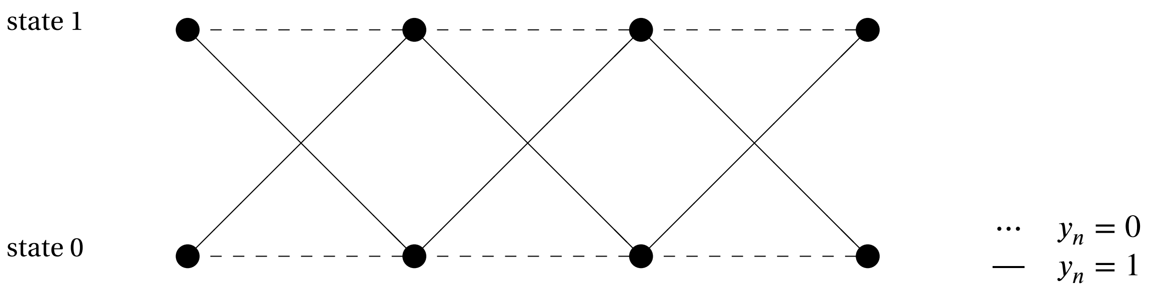


Figure 5.14: NRZI trellis diagram

5.6 Runlength-limited modulation codes

Definitions

Definition 5.2. *A* (d, k)-***coded sequence*** *is any binary sequence such that*

$$d \leq \text{(the number of 0s between any two consecutive 1s)} \leq k.$$

A $(d, k; n)$-***coded sequence*** *is a* (d, k)*-coded sequence of length* n.

Definition 5.3. ***Fixed length code set,*** $X(d, k; n)$**.**
The set $X(d, k; n)$ *is a set of* $(d, k; n)$*-coded sequences such that if*

$$(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in X(d, k; n)$$

then

$$(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

is also a (d, k)*-coded sequence.*

Definition 5.4. ***Variable length code set,*** $\bar{X}(d, k; n)$**.**
The set $\bar{X}(d, k; n)$ *is a set of* $(d, k; m)$*-coded sequences such that* $m \leq n$ *and if*

$$(a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_m) \in \bar{X}(d, k; n)$$

then

$$(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m)$$

is also a (d, k)*-coded sequence.*

State diagram. A (d, k) code can be modeled as a state diagram with $k + 1$ states such that the output y_n is

$$y_n = \begin{cases} 1 & : \text{state} = 0 \\ 0 & : \text{state} \neq 0. \end{cases}$$

and transitions between states are as shown in Figure 5.15.

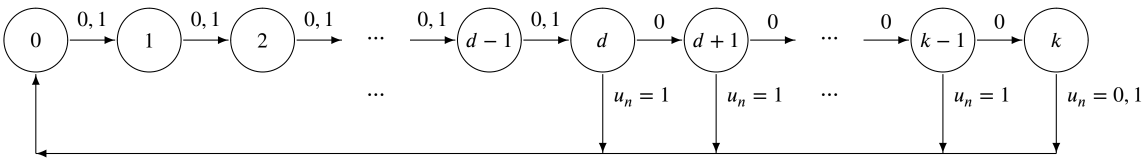


Figure 5.15: (d, k)-coded sequence state diagram

Definition 5.5. *The* ***transition matrix*** $\mathbf{D}_0$ *is the* $N \times N$ *square matrix with elements* a_{mn} *such that*

$$a_{mn} = \begin{cases} 1 & : \textit{coding state changes from } m \textit{ to } n \textit{ when input is } 0. \\ 0 & : \textit{coding state does not change when input is } 0. \end{cases}$$

The ***transition matrix*** $\mathbf{D}_1$ *is the* $N \times N$ *square matrix with elements* b_{mn} *such that*

$$b_{mn} = \begin{cases} 1 & : \textit{coding state changes from } m \textit{ to } n \textit{ when input is } 1. \\ 0 & : \textit{coding state does not change when input is } 1. \end{cases}$$

The ***transition matrix*** $\mathbf{D}$ *is the* $N \times N$ *square matrix with elements* d_{mn} *such that*

$$d_{mn} = a_{mn} \vee b_{mn}$$

where $\vee$ *is the* INCLUSIVE-OR OPERATION.

Transition matrices. The transition matrices for a (d, k) code are as follows:

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \mathbf{D}_1 = \begin{array}{c} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \text{row} \\ 0 \\ 1 \\ \vdots \\ d-1 \\ d \\ d+1 \\ \vdots \\ k-1 \end{array} \quad \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

k	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
2	0.8791	0.4057					
3	0.9468	0.5515	0.2878				
4	0.9752	0.6174	0.4057	0.2232			
5	0.9881	0.6509	0.4650	0.3218	0.1823		
6	0.9942	0.6690	0.4979	0.3746	0.2669	0.1542	
7	0.9971	0.6793	0.5174	0.4057	0.3142	0.2281	0.1335
8	0.9986	0.6853	0.5293	0.4251	0.3432	0.2709	0.1993
9	0.9993	0.6888	0.5369	0.4376	0.3630	0.2979	0.2382
10	0.9996	0.6909	0.5418	0.4460	0.3746	0.3158	0.2633
11	0.9998	0.6922	0.5450	0.4516	0.3833	0.3282	0.2804
12	0.9999	0.6930	0.5471	0.4555	0.3894	0.3369	0.2924
13	0.9999	0.6935	0.5485	0.4583	0.3937	0.3432	0.3011
14	0.9999	0.6938	0.5495	0.4602	0.3968	0.3478	0.3074
15	0.9999	0.6939	0.5501	0.4615	0.3991	0.3513	0.3122
∞	1.0000	0.6942	0.5515	0.4650	0.4057	0.3620	0.3282

Table 5.1: $C(d,k)$: Capacities of (d,k)-coded sequences

Characteristics

Symbol mapping. The symbols to be transmitted are mapped into the elements of $X(d,k;n)$. The maximum number of symbols that can be mapped is

$$\left\lfloor \log_2 |X(d,k;n)| \right\rfloor,$$

where $|\cdot| : X \to \mathbb{Z}$ represents the order of a set X.

Definition 5.6. *The **capacity** of a (d,k)-coded sequence is*

$$C(d,k) \triangleq \lim_{n\to\infty} \frac{1}{n} \left\lfloor \log_2 |X(d,k;n)| \right\rfloor.$$

Theorem 5.4. *Let*

- *$\mathbf{D}$ be the transition matrix of (d,k)*
- *$\lambda_{\max}$ be the largest eigenvalue of $\mathbf{D}$.*

Then the capacity $C(d,k)$ is

$$C(d,k) = \log_2 \lambda_{\max}.$$

The capacities for several $X(d,k)$-coded sequences are given in Table 5.1. (Proakis, 2001)582

Definition 5.7. *The **efficiency** of the $X(d,k;n)$ code set is*

$$\text{efficiency} \triangleq \frac{\textit{code rate of } X(d,k;n)}{C(d,k)}.$$

*The **efficiency** of the $\bar{X}(d,k;n)$ code set is*

$$\text{efficiency} \triangleq \frac{\textit{average code rate of } X(d,k;n)}{C(d,k)}.$$

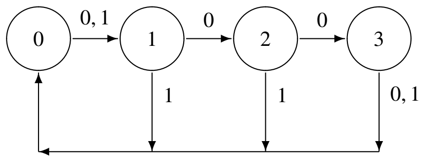


Figure 5.16: (1, 3)-coded sequence state diagram

Examples: fixed-length, no memory

Example 5.1. **Code set** $X(1,3;4)$**:**

Transition matrices:

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{D}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Capacity:

$$\begin{aligned}
|\mathbf{D} - \lambda \mathbf{I}| &= \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 1 & 0 \\ 1 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \\
&= -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} \\
&= -\lambda(-\lambda^3) - 1(\lambda^2 + \lambda + 1) \\
&= \lambda^4 - \lambda^2 - \lambda - 1
\end{aligned}$$

$$C(d,k) = \log_2(\lambda_{\max}) = \log_2(1.46557123) = 0.551463$$

There are multiple sets that are $X(1,3;4)$ code sets:

	$X(1,3;4)$ code sets		
u_n	set1	set2	set3
0	0010	1000	0100
1	0101	1010	0101

The efficiency for each of these sets is

$$\text{efficiency} = \frac{\text{code rate}}{C(d,k)} = \frac{1/4}{0.5515} = 0.4533$$

Example 5.2. **Code set** $X(2,\infty,4)$**:**

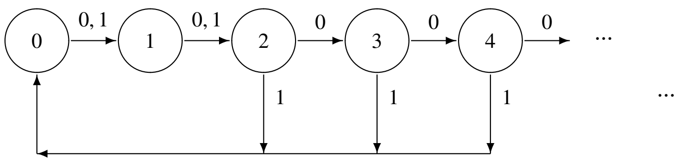


Figure 5.17: $(2,\infty)$-coded sequence state diagram

u_n	code
0	0001
1	0010

$$\text{efficiency} = \frac{\text{code rate}}{C(d,k)} = \frac{1/4}{0.5515} = 0.4533$$

Example 5.3. **Code set** $X(0,3,4)$**:**

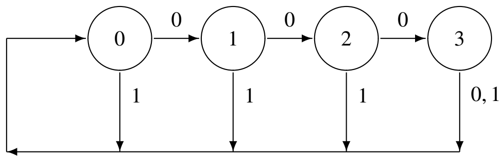


Figure 5.18: $(0,3)$-coded sequence state diagram

The state diagram is shown in Figure 5.18.

The transition matrices are

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{D}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{D} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

To find the channel capacity:

$$\begin{aligned}
|\mathbf{D} - \lambda\mathbf{I}| &= \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 1 & 0 \\ 1 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} \\
&= (1-\lambda)(-\lambda^3) - 1(\lambda^2 - (-\lambda - 1)) \\
&= \lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1
\end{aligned}$$

$$
\begin{aligned}
C(d,k) &= \log_2(\lambda_{\max}) \\
&= \log_2(1.927562) \\
&= 0.946777
\end{aligned}
$$

u_n	code
000	0100
001	0101
010	0110
011	1001
100	1010
101	1011
110	1100
111	1101

$$\text{efficiency} = \frac{\text{code rate}}{C(d,k)} = \frac{3/4}{0.9468} = 0.7921$$

Example: fixed-length, with memory

Example 5.4. **Code set** $X(1,3;2)$ **(Miller code):**

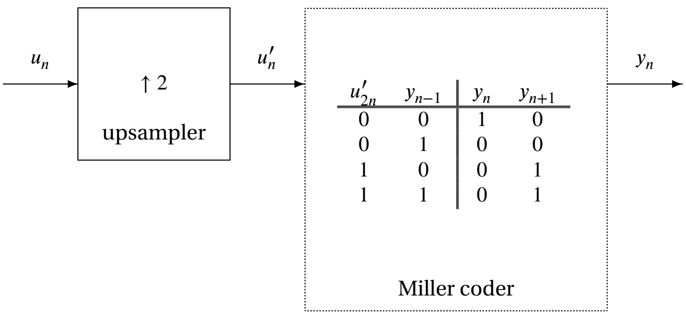


Figure 5.19: Miller modulation coder

The state diagram, transition matrices, and capacity for $(1,3)$-coded sequences is shown in Example 5.1 (page 54). The operation is illustrated in Figure 5.19 and described in the following table:

u'_{2n}	y_{n-1}	y_n	y_{n+1}
0	0	1	0
0	1	0	0
1	0	0	1
1	1	0	1

$$\text{efficiency} = \frac{\text{code rate}}{C(d,k)} = \frac{1/2}{0.5515} = 0.9066$$

Compare this to the memoryless $X(1,3,4)$ code which has efficiency 0.4533 (Example 5.1 page 54). In this case, allowing the code to have memory has doubled the efficiency.

Example: variable-length, no memory

Example 5.5. **Code set $\bar{X}(2,7)$:**
This code has both variable length input and variable length output. Many disk storage devices designed by IBM use this code.

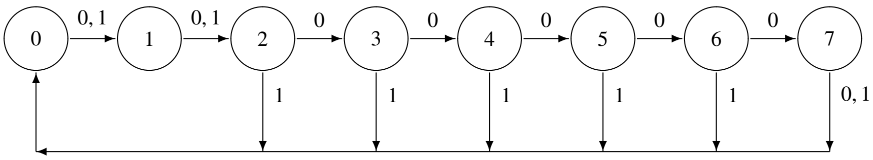


Figure 5.20: $(2,7)$-coded sequence state diagram

$$\mathbf{D}_0 = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0 \end{bmatrix} \quad \mathbf{D}_1 = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0 \end{bmatrix}$$

$$C(d,k) = \log_2(\lambda_{\max}) = \log_2(1.431343) = 0.517370$$

The code words are (Proakis, 2001)584

u_n	*code*
10	1000
11	0100
011	00100
010	001000
000	100100
0011	00100100
0010	00001000.

$$\text{efficiency} = \frac{\text{code rate}}{C(d,k)} = \frac{1/2}{0.517370} = 0.9664$$

5.7 Miller-NRZI modulation code

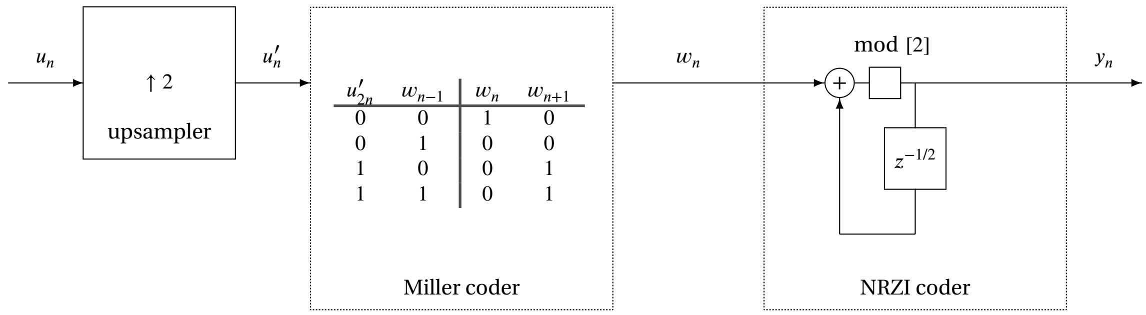


Figure 5.21: Miller-NRZI modulation coder

Miller-NRZI modulation coding is commonly called

- Miller coding
- Miller with precoding
- Delay modulation.

Miller-NRZI is a concatenation of a *Miller coder* (Example 5.4) and an NRZI coder (Section 5.5). Equations governing the operation of the coder include

$$\begin{aligned} y_n &= y_{n-1} \oplus w_n \\ y_{n+1} &= y_n \oplus w_{n+1}. \end{aligned}$$

The composition of the Miller and NRZI operations produces the following state table:

input	*state*				*output*	
u'_{2n}	w_{n-1}	y_{n-1}	w_n	w_{n+1}	y_n	y_{n+1}
0	0	0	1	0	1	1
0	0	1	1	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	0	1
1	0	1	0	1	1	0
1	1	0	0	1	0	1
1	1	1	0	1	1	0

For each input bit u_n, there are two new output bits (y_n, y_{n+1}) and two new state bits (w_{n+1}, y_{n+1}). Notice that because

$$\begin{array}{rcccccl} \text{old state} & \equiv & (w_{n-1}, y_{n-1}) & = & (y_{n-1} \oplus y_{n-2}, y_{n-1}) & \equiv & f(\text{old output}) \\ \text{current state} & \equiv & (w_{n+1}, y_{n+1}) & = & (y_{n+1} \oplus y_n, y_{n+1}) & \equiv & f(\text{current output}) \end{array}$$

the output pair (y_n, y_{n+1}) also contains the state information and can therefore also be used as the labels for the state of the system. This can be viewed as more convenient because then the output pair and the state pair are identical. In this case, state diagrams and trellises are easier to illustrate since we only have to label the states, while the outputs do not have to be labeled because the output pair (y_n, y_{n+1}) is identical to the state pair (y_n, y_{n+1}).

Conversion from the state pairs to the equivalent output pairs are as follows:

w_{n+1}	y_{n+1}	y_n	y_{n+1}
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	1

w_{n-1}	y_{n-1}	y_{n-2}	y_{n-1}
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	1

Using these conversions, a new equivalent state table is as follows:

input	old output		new output	
u'_{2n}	y_{n-2}	y_{n-1}	y_n	y_{n+1}
0	0	0	1	1
0	0	1	1	1
0	1	0	0	0
0	1	1	0	0
1	0	0	0	1
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0

A trellis diagram equivalent to this state table can be found in Figure 5.22. Notice the symmetry of the trellis. In particular, if we flip the trellis about an imaginary center axis while leaving the state labels undisturbed, the same trellis results.

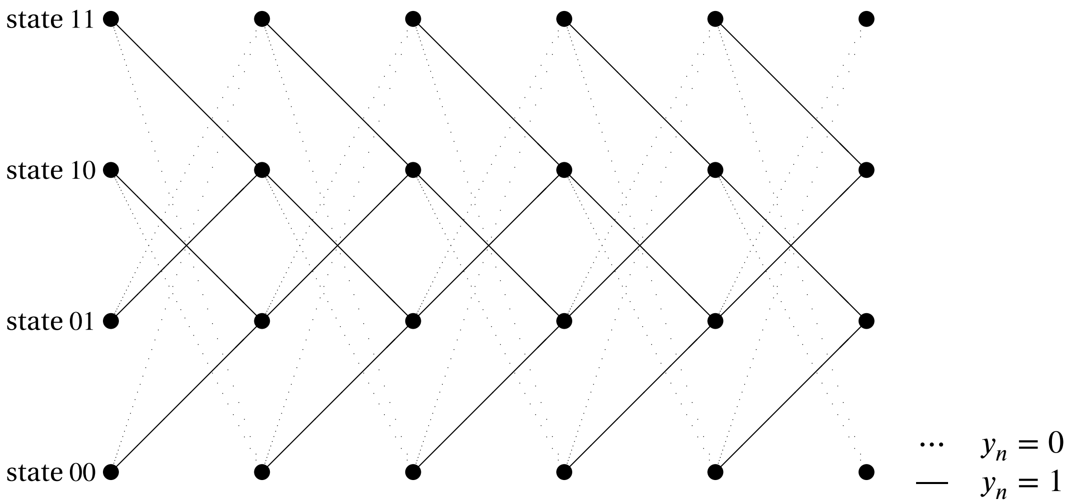


Figure 5.22: Miller-NRZI trellis diagram

Part II

Estimation

CHAPTER 6

ESTIMATION OVERVIEW

6.1 Estimation types

Estimation types. Let $\mathsf{x}(t;\theta)$ be a waveform with parameter θ. There are three basic types of estimation of x:

1. *detection*:
   - The waveform $\mathsf{x}(t;\theta_n)$ is known except for the value of parameter θ_n.
   - The parameter θ_n is one of a finite set of values.
   - Estimate θ_n and thereby also estimate $\mathsf{x}(t;\theta)$.
2. *parametric* estimation:
   - The waveform $\mathsf{x}(t;\theta)$ is known except for the value of parameter θ.
   - The parameter θ is one of an infinite set of values.
   - Estimate θ and thereby also estimate $\mathsf{x}(t;\theta)$.
3. *nonparametric* estimation:
   - The waveform $\mathsf{x}(t)$ is unknown and assumed without any parameter θ.
   - Estimate $\mathsf{x}(t)$.

Estimation criterion. Optimization requires a criterion against which the quality of an estimate is measured.[1] The most demanding and general criterion is the *Bayesian* criterion. The Bayesian criterion requires knowledge of the probability distribution functions and the definition of a *cost function*. Other criterion are special cases of the Bayesian criterion such that the cost function is defined in a special way, no cost function is defined, and/or the distribution is not known (Figure 6.2 page 66).

Estimation techniques. Estimation techniques can be classified into five groups (Figure 6.2 page 66):[2]

[1] Srinath et al. (1996) ⟨013125295X⟩.

[2] Nelles (2001) page 26 ⟨“Fig 2.2 Overview of linear and nonlinear optimization techniques”⟩, Nelles (2001) page 33 ⟨“Fig 2.5 The Bayes method is the most general approach but…”⟩, Nelles (2001) page 63 ⟨“Table 3.3 Relationship between linear recursive and nonlinear optimization techniques”⟩, Nelles (2001) page 66

1. sequential decoding
2. norm minimization
3. gradient search
4. inner product analysis
5. direct search

Sequential decoding is a non-linear estimation family. Perhaps the most famous of these is the Veterbi algorithm which uses a trellis to calculate the estimate. The Verterbi algorithm has been shown to yield an optimal estimate in the maximal likelihood (ML) sense. Norm minimization and gradient search algorithms are all linear algorithms. While this restriction to linear operations often simplifies calculations, it often yields an estimate that is not optimal in the ML sense.

6.2 Estimation criterion

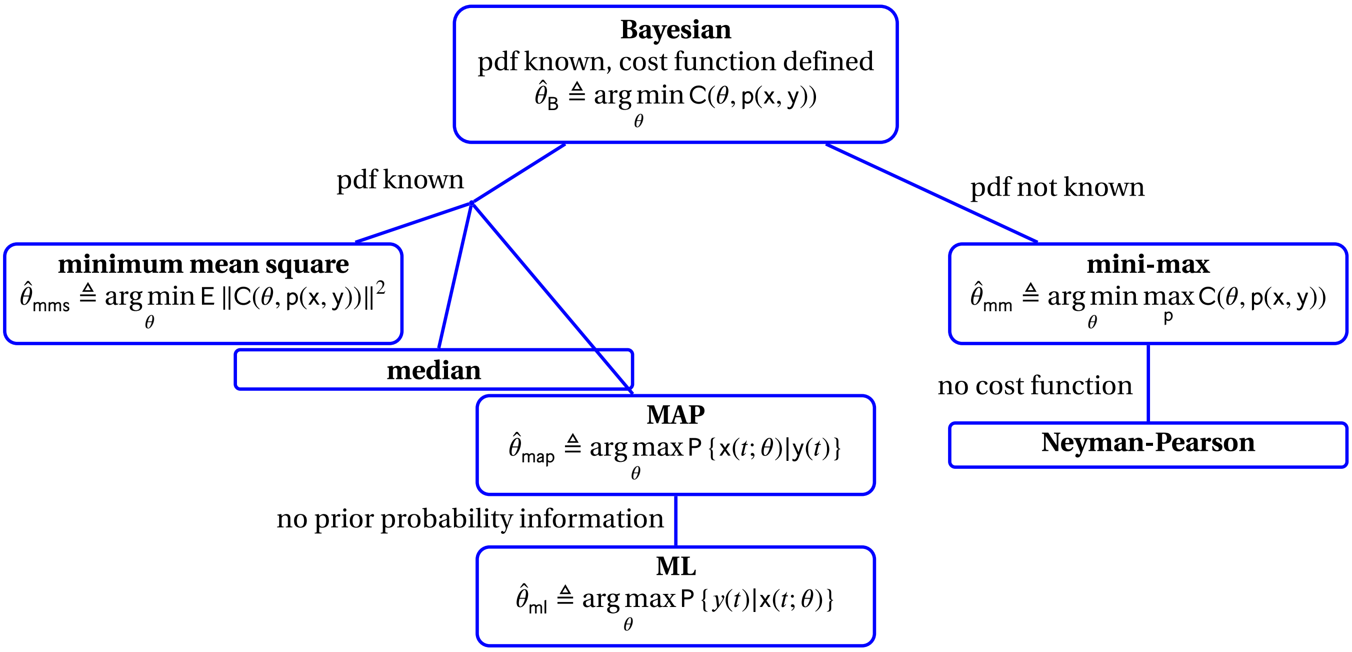


Figure 6.1: Estimation criterion

Definition 6.1. *Let*

- (A). $\mathsf{x}(t;\theta)$ *be a random process with unknown parameter* θ
- (B). $\mathsf{y}(t)$ *an observed random process which is statistically dependent on* $\mathsf{x}(t;\theta)$
- (C). $\mathsf{C}(\theta, \mathsf{p}(x, y))$ *be a cost function.*

Then the following ***estimates*** *are defined as follows:*

DEF

1. ***Bayesian estimate*** $\quad \hat{\theta}_{\mathsf{B}} \triangleq \arg\min_{\theta} C(\theta, \mathsf{p}(x, y))$
2. ***Mean square estimate*** ***("MS estimate")*** $\quad \hat{\theta}_{\mathsf{mms}} \triangleq \arg\min_{\theta} \mathsf{E}\,\|C(\theta, \mathsf{p}(x, y))\|^2$
3. ***mini-max estimate*** ***("MM estimate")*** $\quad \hat{\theta}_{\mathsf{mm}} \triangleq \arg\min_{\theta} \max_{\mathsf{p}} C(\theta, \mathsf{p}(x, y))$
4. ***maximum a-posteriori probability estimate*** ***("MAP estimate")*** $\quad \hat{\theta}_{\mathsf{map}} \triangleq \arg\max_{\theta} \mathsf{P}\{\mathsf{x}(t;\theta)|\mathsf{y}(t)\}$
5. ***maximum likelihood estimate*** ***("ML estimate")*** $\quad \hat{\theta}_{\mathsf{ml}} \triangleq \arg\max_{\theta} \mathsf{P}\{\mathsf{y}(t)|\mathsf{x}(t;\theta)\}$

Theorem 6.1. *Let* $\mathsf{x}(t;\theta)$ *be a random process with unknown parameter* θ.

THM $\{\mathsf{P}\{\theta\} = \text{CONSTANT}\} \implies \{\hat{\theta}_{\mathsf{map}} = \hat{\theta}_{\mathsf{ml}}\}$

✎PROOF:

$$\begin{aligned}
\hat{\theta}_{\mathsf{map}} &\triangleq \arg\max_{\theta} \mathsf{P}\{\mathsf{x}(t;\theta)|\mathsf{y}(t)\} && \text{by definition of } \hat{\theta}_{\mathsf{map}} && \text{(Definition 6.1 page 64)}\\
&\triangleq \arg\max_{\theta} \frac{\mathsf{P}\{\mathsf{x}(t;\theta)\wedge \mathsf{y}(t)\}}{\mathsf{P}\{r(t)\}} && \text{by definition of \textit{conditional probability}} && \text{(Definition ?? page ??)}\\
&\triangleq \arg\max_{\theta} \frac{\mathsf{P}\{r(t)|\mathsf{x}(t;\theta)\}\mathsf{P}\{\mathsf{x}(t;\theta)\}}{\mathsf{P}\{\mathsf{y}(t)\}} && \text{by definition of \textit{conditional probability}} && \text{(Definition ?? page ??)}\\
&= \arg\max_{\theta} \mathsf{P}\{\mathsf{y}(t)|\mathsf{x}(t;\theta)\}\mathsf{P}\{\mathsf{x}(t;\theta)\} && \text{because } \mathsf{y}(t) \text{ is independent of } \theta\\
&= \arg\max_{\theta} \mathsf{P}\{\mathsf{y}(t)|\mathsf{x}(t;\theta)\}\\
&\triangleq \hat{\theta}_{\mathsf{ml}} && \text{by definition of } \hat{\theta}_{\mathsf{ml}} && \text{(Definition 6.1 page 64)}
\end{aligned}$$

✏

6.3 Measures of estimator quality

Definition 6.2. [3]

DEF *The* ***mean square error*** $\mathsf{mse}(\hat{\theta})$ *of an estimate* $\hat{\theta}$ *of a parameter* θ *is defined as*

$$\mathsf{mse}(\hat{\theta}) \triangleq \mathsf{E}\left[\left(\hat{\theta}-\theta\right)^2\right]$$

Definition 6.3. [4]

DEF *The* ***normalized rms error*** $\epsilon(\hat{\theta})$ *of an estimate* $\hat{\theta}$ *of a parameter* θ *is defined as*

$$\epsilon(\hat{\theta}) \triangleq \frac{\sqrt{\mathsf{mse}(\hat{\theta})}}{\theta} \triangleq \frac{\sqrt{\mathsf{E}\left[\left(\hat{\theta}-\theta\right)^2\right]}}{\theta}$$

Definition 6.4. [5]

DEF *The* ***mean integrated square error*** $\mathsf{mse}(\hat{\theta})$ *of an estimate* $\hat{\theta}$ *of a parameter* θ *is defined as*

$$\mathsf{mse}(\hat{\theta}) \triangleq \mathsf{E}\int_{\theta\in\mathbb{R}} \left[\left(\hat{\theta}-\theta\right)^2\right]$$

The *mean square error* of $\hat{\theta}$ can be expressed as the sum of two components: the variance of $\hat{\theta}$ and the bias of $\hat{\theta}$ squared (next Theorem). For an example of Theorem 6.2 in action, see the proof for the $\mathsf{mse}(\hat{\mu})$ of the *arithmetic mean estimate* as provided in Theorem **??** (page **??**).

Theorem 6.2. [6] *Let* $\mathsf{mse}(\hat{\theta})$ *be the* MEAN SQUARE ERROR *(Definition 6.2 page 65) and* $\epsilon(\hat{\theta})$ *the* NORMALIZED

[3] Silverman (1986) page 35 ⟨§"1.3.2 Measures of discrepancy..."⟩, Bendat and Piersol (2010) ⟨§"1.4.3 Error Analysis Criteria"⟩, Bendat and Piersol (1966), page 183§"5.3 Statistical Errors for Parameter Estimates"

[4] Bendat and Piersol (2010) ⟨§"1.4.3 Error Analysis Criteria"⟩

[5] Silverman (1986) page 35 ⟨§"1.3.2 Measures of discrepancy..."⟩, Rosenblatt (1956) page 835 ⟨"integrated mean square error"⟩

[6] Choi (1978) page 76, Kay (1988) page 45 ⟨§"3.3 ESTIMATION THEORY"⟩, STUART AND ORD (1991) PAGE 629 ⟨"MINIUM MEAN-SQUARE-ERROR ESTIMATION"⟩, CLARKSON (1993) PAGE 51 ⟨§"2.6 ESTIMATION OF MOMENTS"⟩, BENDAT AND PIERSOL (2010) ⟨§"1.4.3 ERROR ANALYSIS CRITERIA"⟩, BENDAT AND PIERSOL (1966), PAGE 183§"5.3 STATISTICAL ERRORS FOR PARAMETER ESTIMATES", BENDAT AND PIERSOL (1980) PAGE 39 ⟨§"2.4.1 BIAS VERSUS RANDOM ERRORS"⟩

RMS ERROR (Definition 6.3 page 65) *of an estimator $\hat\theta$.*

THM

$$\operatorname{mse}(\hat\theta) = \underbrace{\mathsf{E}\left[\left(\hat\theta - \mathsf{E}\hat\theta\right)^2\right]}_{\text{variance of }\hat\theta} + \underbrace{\left[\mathsf{E}\hat\theta - \theta\right]^2}_{\text{bias of }\hat\theta\text{ squared}} \qquad \epsilon(\hat\theta) = \frac{\sqrt{\mathsf{E}\left[\left(\hat\theta - \mathsf{E}\hat\theta\right)^2\right] + \left[\mathsf{E}\hat\theta - \theta\right]^2}}{\theta}$$

✎PROOF:

$$\begin{aligned}
\operatorname{mse}(\hat\theta) &\triangleq \mathsf{E}\left[\left(\hat\theta - \theta\right)^2\right] && \text{by definition of mse} && \text{(Definition 6.2 page 65)}\\
&= \mathsf{E}\left[\left(\hat\theta \underbrace{- \mathsf{E}\hat\theta + \mathsf{E}\hat\theta}_{0} - \theta\right)^2\right] && \text{by \textit{additive identity} property of } (\mathbb{C}, +, \cdot, 0, 1)\\
&= \mathsf{E}\left[\left(\hat\theta - \mathsf{E}\hat\theta\right)^2 + \underbrace{\left(\mathsf{E}\hat\theta - \theta\right)^2}_{\text{constant}} - 2\left(\hat\theta - \mathsf{E}\hat\theta\right)\left(\mathsf{E}\hat\theta - \theta\right)\right] && \text{by \textit{Binomial Theorem}}\\
&= \mathsf{E}\left(\hat\theta - \mathsf{E}\hat\theta\right)^2 + \left(\mathsf{E}\hat\theta - \theta\right)^2 - 2\mathsf{E}\left[\hat\theta\mathsf{E}\hat\theta - \hat\theta\theta - \mathsf{E}\hat\theta\hat\theta + \mathsf{E}\hat\theta\theta\right] && \text{by \textit{linearity} of } \mathsf{E} && \text{(Theorem ?? page ??)}\\
&= \mathsf{E}\left(\hat\theta - \mathsf{E}\hat\theta\right)^2 + \left(\mathsf{E}\hat\theta - \theta\right)^2 - 2\underbrace{\left[\mathsf{E}\hat\theta\mathsf{E}\hat\theta - \mathsf{E}\hat\theta\mathsf{E}\theta - \mathsf{E}\hat\theta\mathsf{E}\hat\theta + \mathsf{E}\hat\theta\mathsf{E}\theta\right]}_{0} && \text{by \textit{linearity} of } \mathsf{E} && \text{(Theorem ?? page ??)}\\
&= \mathsf{E}\left(\hat\theta - \mathsf{E}\hat\theta\right)^2 + \left(\mathsf{E}\hat\theta - \theta\right)^2
\end{aligned}$$

✏

Definition 6.5. [7]

DEF *An estimate $\hat\theta$ of a parameter θ is a* ***minimum variance unbiased estimator*** *(MVUE) if*

(1). $\mathsf{E}\hat\theta = 0$ (UNBIASED) *and*

(2). *no other unbiased estimator $\hat\phi$ has smaller variance* $\operatorname{var}(\hat\phi)$

6.4 Estimation techniques

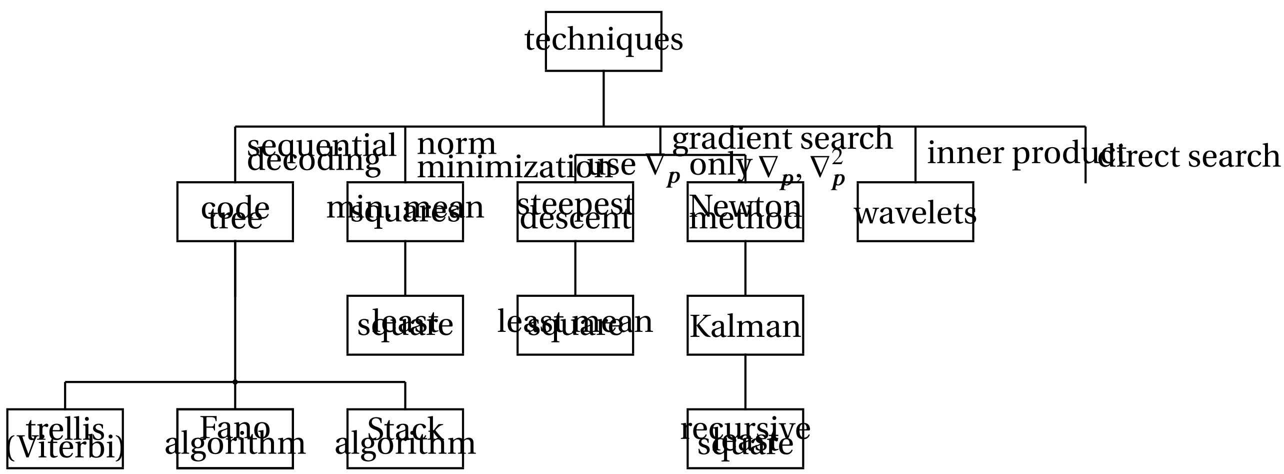


Figure 6.2: Estimation techniques

[7] Choi (1978) page 76, Shao (2003) page 161 ⟨§"The UMVUE"⟩, Bolstad (2007) page 164 ⟨§"Minimum Variance Unbiased Estimator"⟩,

6.5 Sequential decoding

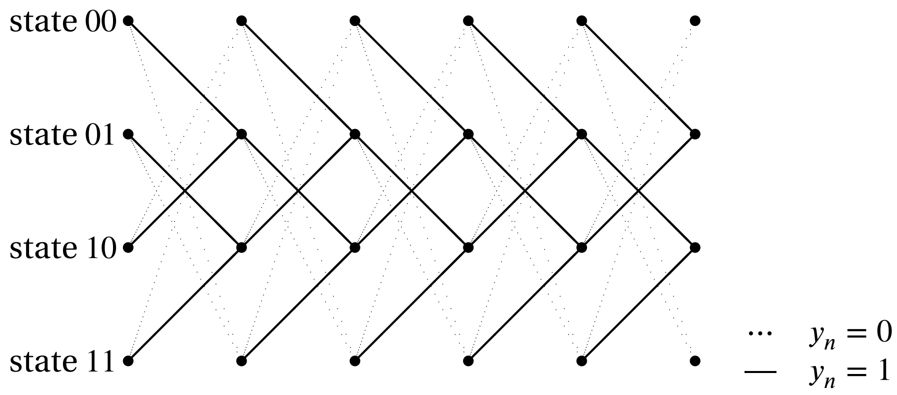


Figure 6.3: Viterbi algorithm trellis

It has been shown that the Viterbi algorithm (trellis) produces an optimal estimate in the maximal likelihood (ML) sense. A Verterbi trellis is shown in Figure 6.3 (page 67).

CHAPTER 7

PROJECTION STATISTICS FOR ADDITIVE NOISE SYSTEMS

7.1 Projection Statistics

Theorem 7.1 (page 71) (next) shows that the finite set $Y \triangleq \{\dot{y}_n | n = 1, 2, \ldots, N\}$ (a finite number of values) provides just as good an estimate as having the entire $\mathsf{y}(t;\theta)$ waveform (an uncountably infinite number of values) with respect to the following cases:

1. the conditional probability of $\mathsf{x}(t;\theta)$ given $\mathsf{y}(t;\theta)$
2. the *MAP estimate* of the sequence
3. the *ML estimate* of the sequence.

That is, even with a drastic reduction in the number of statistics from uncountably infinite to finite N, no quality is lost with respect to the estimators listed above. This amazing result is very useful in practical system implementation and also for proving other theoretical results (notably estimation and detection theorems).

But first, some definitions (next) that are used repeatedly in this chapter.

Definition 7.1. *Let* $\Psi \triangleq \{\psi_n | n = 1, 2, \ldots, N\}$ *be an* ORTHONORMAL BASIS *for a parameterized function* $\mathsf{x}(t;\theta)$ *with parameter* θ*. Let* $\mathsf{y}(t;\theta)$ *be* $\mathsf{x}(t;\theta)$ *plus a* RANDOM PROCESS $\mathsf{v}(t)$ *such that*

$$\mathsf{y}(t;\theta) \triangleq \mathsf{x}(t;\theta) + \mathsf{v}(t)$$

Let $\dot{y}_n$*,* $\dot{x}_n$*, and* $\dot{v}_n$ *be* PROJECTIONS *onto the* BASIS VECTOR $\psi_n(t)$ *such that*

DEF

$$\begin{aligned}
\dot{y}_n(\theta) &\triangleq \mathbf{P}_n \mathsf{y}(t;\theta) &&\triangleq \langle \mathsf{y}(t;\theta) \,|\, \psi_n(t) \rangle &&\triangleq \int_{t\in\mathbb{R}} \mathsf{y}(t;\theta)\psi_n(t)\, \mathrm{d}t \\
\dot{x}_n(\theta) &\triangleq \mathbf{P}_n \mathsf{x}(t) &&\triangleq \langle \mathsf{x}(t;\theta) \,|\, \psi_n(t) \rangle &&\triangleq \int_{t\in\mathbb{R}} \mathsf{x}(t;\theta)\psi_n(t)\, \mathrm{d}t \\
\dot{v}_n &\triangleq \mathbf{P}_n \mathsf{v}(t) &&\triangleq \langle \mathsf{v}(t) \,|\, \psi_n(t) \rangle &&\triangleq \int_{t\in\mathbb{R}} \mathsf{v}(t)\psi_n(t)\, \mathrm{d}t
\end{aligned}$$

Let the set Y *be defined as* $Y \triangleq \{\dot{y}_n(\theta) | 1, 2, \ldots, N\}$ *Let* $\hat{\theta}_{\text{map}}$ *be the* MAP ESTIMATE *and* $\hat{\theta}_{\text{ml}}$ *be the* ML ESTIMATE *(Definition 6.1 page 64) of* θ*.*

Lemma 7.1. *Let* Ψ*,* $\mathsf{v}(t)$*,* $\dot{v}_n$*, and* Y *be defined as in Definition 7.1 (page 69).*

LEM

$$\left\{ \mathsf{E}\mathsf{v}(t) = 0 \quad \text{(ZERO-MEAN)} \right\} \implies \left\{ \mathsf{E}\dot{v}_n = 0 \quad \text{(ZERO-MEAN)} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{E}\dot{v}_n &= \mathsf{E}\langle \mathsf{v}(t)\,|\,\psi_n(t)\rangle && \text{by definition of } \dot{v}_n && \text{(Definition 7.1 page 69)}\\
&= \langle \mathsf{E}\mathsf{v}(t)\,|\,\psi_n(t)\rangle && \text{by } \textit{linearity} \text{ of } \langle\triangle\,|\,\nabla\rangle\\
&= \langle 0\,|\,\psi_n(t)\rangle && \text{by } \textit{zero-mean} \text{ hypothesis}\\
&= 0
\end{aligned}$$

✏

Lemma 7.2. *Let Ψ, $\mathsf{v}(t)$, $\dot{v}_n$, and Y be defined as in Definition 7.1 (page 69).*

LEM $\left\{\ \mathsf{v}(t) \sim \mathsf{N}\left(0,\sigma^2\right) \quad \text{(GAUSSIAN)}\ \right\} \implies \left\{\ \dot{v}_n \sim \mathsf{N}\left(0,\sigma^2\right) \quad \text{(GAUSSIAN)}\ \right\}$

✎PROOF: The distribution follows because it is a linear operation on a Gaussian process. ✏

Lemma 7.3. *Let Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, and Y be defined as in Definition 7.1 (page 69).*

LEM $\left\{\begin{array}{lrcll} \text{(A).} & \mathsf{E}[\mathsf{v}(t)] &=& 0 & \text{and}\\ \text{(B).} & \mathsf{cov}\,[\mathsf{v}(t),\mathsf{v}(u)] &=& \sigma^2\delta(t-u) & \end{array}\right\} \implies \left\{\begin{array}{lrcll} \text{(1).} & \mathsf{E}\dot{v}_n &=& 0 & \text{(ZERO-MEAN)}\\ \text{(2).} & \mathsf{cov}\left[\dot{v}_n,\dot{v}_m\right] &=& \sigma^2\bar{\delta}_{n-m} & \text{(UNCORRELATED)} \end{array}\right\}$

✎PROOF:

1.

$$\mathsf{E}\dot{v}_n = 0 \qquad \text{by } \textit{additive} \text{ property and Theorem 7.2 page 73}$$

2.

$$\begin{aligned}
\mathsf{cov}\left[\dot{v}_m,\dot{v}_n\right] &= \mathsf{cov}\left[\langle \mathsf{v}(t)\,|\,\psi_m(t)\rangle, \langle \mathsf{v}(t)\,|\,\psi_n(t)\rangle\right] && \text{by def. of } \dot{v}_n && \text{(Definition 7.1 page 69)}\\
&= \mathsf{cov}\left[\left(\int_{t\in\mathbb{R}} \mathsf{v}(t)\psi_m(t)\,\mathsf{d}t\right), \left(\int_{u\in\mathbb{R}} \mathsf{v}(u)\psi_n(u)\,\mathsf{d}u\right)\right] && \text{by def. of } \langle\triangle\,|\,\nabla\rangle && \text{(Definition 7.1 page 69)}\\
&= \mathsf{E}\left[\left(\int_{t\in\mathbb{R}} \mathsf{v}(t)\psi_m(t)\,\mathsf{d}t\right)\left(\int_{u\in\mathbb{R}} \mathsf{v}(u)\psi_n(u)\,\mathsf{d}u\right)\right] && \text{by def. of } \mathsf{Cov}\\
&= \mathsf{E}\left[\int_{t\in\mathbb{R}}\int_{u\in\mathbb{R}} \mathsf{v}(t)\mathsf{v}(u)\psi_m(t)\psi_n(u)\,\mathsf{d}t\,\mathsf{d}u\right]\\
&= \int_{t\in\mathbb{R}}\int_{u\in\mathbb{R}} \mathsf{E}[\mathsf{v}(t)\mathsf{v}(u)]\psi_m(t)\psi_n(u)\,\mathsf{d}t\,\mathsf{d}u\\
&= \int_{t\in\mathbb{R}}\int_{u\in\mathbb{R}} \sigma^2\delta(t-u)\psi_m(t)\psi_n(u)\,\mathsf{d}t\,\mathsf{d}u && \text{by } \textit{white} \text{ hyp.} && \text{(B)}\\
&= \sigma^2\int_{t\in\mathbb{R}} \psi_m(t)\psi_n(u)\,\mathsf{d}t\\
&= \sigma^2\langle \psi_m(t)\,|\,\psi_n(u)\rangle && \text{by def. of } \langle\triangle\,|\,\nabla\rangle && \text{(Definition 7.1 page 69)}\\
&= \begin{cases} \sigma^2 & \text{for } n=m\\ 0 & \text{for } n\neq m. \end{cases} && \text{by } \textit{orthonormal} \text{ prop.} && \text{(Definition 7.1 page 69)}
\end{aligned}$$

✏

Lemma 7.4. *Let Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, and Y be defined as in Definition 7.1 (page 69).*

LEM $\left.\begin{array}{lrcll} \text{(A).} & \mathsf{cov}\,[\mathsf{v}(t),\mathsf{v}(u)] &=& \sigma^2\delta(t-u) & \text{and}\\ \text{(B).} & \mathsf{v}(t) &\sim& \mathsf{N}\left(0,\sigma^2\right) & \text{and}\\ \text{(C).} & \langle\psi_n\,|\,\psi_m\rangle &=& \bar{\delta}_{mn} & \end{array}\right\} \implies \left\{\begin{array}{lrcll} \text{(1).} & \dot{v}_n &\sim& \mathsf{N}\left(0,\sigma^2\right) & \text{(GAUSSIAN)}\\ \text{(2).} & \mathsf{cov}\left[\dot{v}_n,\dot{v}_m\right] &=& \sigma^2\bar{\delta}_{nm} & \text{(UNCORRELATED)}\\ \text{(3).} & \mathsf{P}\left\{\dot{v}_n \wedge \dot{v}_m\right\} &=& \mathsf{P}\left\{\dot{v}_n\right\}\mathsf{P}\left\{\dot{v}_m\right\} & \text{(INDEPENDENT)} \end{array}\right.$

✎PROOF:

1. Because the operations are *linear* on processes are *Gaussian* (hypothesis C).
2.

$$\begin{aligned} \mathsf{E}\dot{v}_n &= 0 && \text{by } \textit{AWN} \text{ properties and Theorem 7.4 page 75}\\ \operatorname{cov}\left[\dot{v}_m, \dot{v}_n\right] &= \sigma^2\bar{\delta}_{mn} && \text{by } \textit{AWN} \text{ properties and Lemma 7.3 page 70}\end{aligned}$$

3. Because the processes are *Gaussian*, *uncorrelated* implies *independent*.

✏

7.2 Sufficient Statistics

Theorem 7.1 (Sufficient Statistic Theorem). [1] *Let* Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, *and* Y *be defined as in Definition 7.1 (page 69). Let* $\hat{\theta}_{\text{map}}$ *be the* MAP ESTIMATE *and* $\hat{\theta}_{\text{ml}}$ *be the* ML ESTIMATE (Definition 6.1 page 64) *of* θ.

THM

$$\left\{\begin{array}{lll} \text{(A).} & \mathsf{v}(t) \textit{ is } \text{ZERO-MEAN} & \textit{and}\\ \text{(B).} & \mathsf{v}(t) \textit{ is } \text{WHITE} & \textit{and}\\ \text{(C).} & \mathsf{v}(t) \textit{ is } \text{GAUSSIAN} & \end{array}\right\} \implies \underbrace{\left\{\begin{array}{llll} \text{(1).} & \mathsf{P}\{\mathsf{x}(t;\theta)|\mathsf{y}(t;\theta)\} = \mathsf{P}\{\mathsf{x}(t;\theta)|Y\} & & \textit{and}\\ \text{(2).} & \hat{\theta}_{\text{map}} & = \arg\max_{\hat{\theta}} \mathsf{P}\{\mathsf{x}(t;\theta)|Y\} & \textit{and}\\ \text{(3).} & \hat{\theta}_{\text{ml}} & = \arg\max_{\hat{\theta}} \mathsf{P}\{Y|\mathsf{x}(t;\theta)\} & \end{array}\right\}}_{\textit{the } N \textit{ element set } Y \textit{ is a } \text{SUFFICIENT STATISTIC} \textit{ for estimating } \mathsf{x}(t;\theta)}$$

✎PROOF:

1. definition: Let $\mathsf{v}'(t) \triangleq \mathsf{v}(t) - \sum_{n=1}^{N} \dot{v}_n\psi_n(t)$.
2. lemma: The relationship between Y and $\mathsf{v}'(t)$ is given by

$$\begin{aligned}
&\boxed{\mathsf{y}(t;\theta)}\\
&= \sum_{n=1}^{N}\langle \mathsf{y}(t;\theta)\,|\,\psi_n(t)\rangle\,\psi_n(t) + \left[\mathsf{y}(t;\theta) - \sum_{n=1}^{N}\langle \mathsf{y}(t;\theta)\,|\,\psi_n(t)\rangle\,\psi_n(t)\right] && \text{by \textit{additive identity} property of } (\mathbb{C},+,\cdot,0,1)\\
&\triangleq \sum_{n=1}^{N}\langle \mathsf{y}(t;\theta)\,|\,\psi_n(t)\rangle\,\psi_n(t) + \left[\mathsf{y}(t;\theta) - \sum_{n=1}^{N}\langle \mathsf{x}(t)+\mathsf{v}(t)\,|\,\psi_n(t)\rangle\,\psi_n(t)\right] && \text{by definition of } \mathsf{y}(t;\theta)\\
&= \sum_{n=1}^{N}\dot{y}_n\psi_n(t) + \underbrace{\mathsf{x}(t)+\mathsf{v}(t)}_{\mathsf{y}(t;\theta)} - \underbrace{\sum_{n=1}^{N}\langle \mathsf{x}(t)\,|\,\psi_n(t)\rangle\,\psi_n(t)}_{\mathsf{x}(t)} - \underbrace{\sum_{n=1}^{N}\langle \mathsf{v}(t)\,|\,\psi_n(t)\rangle\,\psi_n(t)}_{\mathsf{v}(t)-\mathsf{v}'(t)} && \text{by definition of } \dot{y}_n \text{ and \textit{additive} property of } \langle\triangle\,|\,\triangledown\rangle \text{ (Definition G.9 page 266)}\\
&= \sum_{n=1}^{N}\dot{y}_n\psi_n(t) + \mathsf{x}(t) + \mathsf{v}(t) - \mathsf{x}(t) - \left[\mathsf{v}(t) - \mathsf{v}'(t)\right]
\end{aligned}$$

[1] Fisher (1922) page 316 ⟨"Criterion of Sufficiency"⟩

$$= \boxed{\sum_{n=1}^{N} \dot{y}_n \psi_n(t) + \mathsf{v}'(t)}$$

3. lemma: $\mathsf{E}\left[\dot{v}_n \mathsf{v}(t)\right] = N_o \psi_n(t)$. Proof:

$$\begin{aligned}
&\mathsf{E}\left[\dot{v}_n \mathsf{v}(t)\right] \\
&\triangleq \mathsf{E}\left[\left(\int_{t\in\mathbb{R}} \mathsf{v}(u)\psi_n(u)\, \mathrm{d}u\right)\mathsf{v}(t)\right] && \text{by definition of } \dot{v}_n(t) && \text{(Definition 7.1 page 69)} \\
&= \mathsf{E}\left[\int_{t\in\mathbb{R}} \mathsf{v}(u)\mathsf{v}(t)\psi_n(u)\, \mathrm{d}u\right] && \text{by } \textit{linearity} \text{ of } \int \mathrm{d}u \text{ operator} \\
&= \int_{t\in\mathbb{R}} \mathsf{E}[\mathsf{v}(u)\mathsf{v}(t)]\psi_n(u)\, \mathrm{d}u && \text{by } \textit{linearity} \text{ of } \mathsf{E} && \text{(Theorem ?? page ??)} \\
&= \int_{t\in\mathbb{R}} N_o\delta(u-t)\psi_n(u)\, \mathrm{d}u && \text{by } \textit{white} \text{ hypothesis} \\
&= N_o\psi_n(t) && \text{by property of } \textit{Dirac delta } \delta(t)
\end{aligned}$$

4. lemma: Y and $\mathsf{v}'(t)$ are *uncorrelated*: Proof:

$$\begin{aligned}
&\mathsf{E}\left[\dot{y}_n \mathsf{v}'(t)\right] \\
&\triangleq \mathsf{E}\left[\langle \mathsf{y}(t;\theta) \,|\, \psi_n(t)\rangle \left(\mathsf{v}(t) - \sum_{n=1}^{N} \langle \mathsf{v}(t) \,|\, \psi_n(t)\rangle\, \psi_n(t)\right)\right] && \text{by definitions of } \dot{y}_n \text{ and } \mathsf{v}'(t) \\
&\triangleq \mathsf{E}\left[\langle \mathsf{x}(t) + \mathsf{v}(t) \,|\, \psi_n(t)\rangle \left(\mathsf{v}(t) - \sum_{n=1}^{N} \langle \mathsf{v}(t) \,|\, \psi_n(t)\rangle\, \psi_n(t)\right)\right] && \text{by definition of } \mathsf{y}(t;\theta) \\
&= \mathsf{E}\left[\left(\langle \mathsf{x}(t) \,|\, \psi_n(t)\rangle + \langle \mathsf{v}(t) \,|\, \psi_n(t)\rangle\right)\left(\mathsf{v}(t) - \sum_{n=1}^{N} \langle \mathsf{v}(t) \,|\, \psi_n(t)\rangle\, \psi_n(t)\right)\right] && \text{by } \textit{additive} \text{ property of } \langle \triangle \,|\, \triangledown \rangle \text{ (Definition G.9 page 266)} \\
&= \mathsf{E}\left[\left(\dot{x}_n + \dot{v}_n\right)\left(\mathsf{v}(t) - \sum_{n=1}^{N} \dot{v}_n\psi_n(t)\right)\right] && \text{by definitions of } \dot{x}_n \text{ and } \dot{v}_n \text{ (Definition 7.1 page 69)} \\
&= \mathsf{E}\left[\dot{x}_n\mathsf{v}(t) - \dot{x}_n\sum_{n=1}^{N} \dot{v}_n\psi_n(t) + \dot{v}_n\mathsf{v}(t) - \dot{v}_n\sum_{n=1}^{N} \dot{v}_n\psi_n(t)\right] \\
&= \mathsf{E}\left[\dot{x}_n\mathsf{v}(t)\right] - \mathsf{E}\left[\dot{x}_n\sum_{n=1}^{N} \dot{v}_n\psi_n(t)\right] + \mathsf{E}\left[\dot{v}_n\mathsf{v}(t)\right] - \mathsf{E}\left[\sum_{m=1}^{N} \dot{v}_n\dot{v}_m\psi_m(t)\right] && \text{by } \textit{linearity} \text{ of } \mathsf{E} \text{ (Theorem ?? page ??)} \\
&= \dot{x}_n\cancelto{0}{\mathsf{E}\mathsf{v}(t)} - \dot{x}_n\sum_{n=1}^{N} \cancelto{0}{\mathsf{E}\left[\dot{v}_n\right]}\psi_n(t) + \mathsf{E}\left[\dot{v}_n\mathsf{v}(t)\right] - \sum_{m=1}^{N} \mathsf{E}\left[\dot{v}_n\dot{v}_m\right]\psi_m(t) && \text{by } \textit{linearity} \text{ of } \mathsf{E} \text{ (Theorem ?? page ??)} \\
&= 0 - 0 + \mathsf{E}\left[\dot{v}_n\mathsf{v}(t)\right] - \sum_{m=1}^{N} N_o\bar{\delta}_{mn}\psi_m(t) && \text{by } \textit{white} \text{ hypothesis} \\
&= N_o\psi_n(t) - N_o\psi_n(t) && \text{by (3) lemma} \\
&= 0 \\
&\implies \textbf{uncorrelated}
\end{aligned}$$

5. lemma: Y and $\mathsf{v}'(t)$ are *independent*. Proof: By (4) lemma, $\dot{y}_n$ and $\mathsf{v}'(t)$ are *uncorrelated*. By hypothesis, they are *Gaussian*, and thus are also **independent**.

6. Proof that $\mathsf{P}\{\mathsf{x}(t;\theta)|\mathsf{y}(t;\theta)\} = \mathsf{P}\left\{\mathsf{x}(t;\theta)|\dot{y}_1,\ \dot{y}_2, \ldots, \dot{y}_N\right\}$:

$$
\begin{aligned}
\mathsf{P}\{\mathsf{x}(t;\theta)|\mathsf{y}(t;\theta)\} &= \mathsf{P}\left\{\mathsf{x}(t;\theta)\Big|\sum_{n=1}^{N}\dot{y}_n\psi_n(t)+\mathsf{v}'(t)\right\} \\
&= \mathsf{P}\left\{\mathsf{x}(t;\theta)|Y,\mathsf{v}'(t)\right\} && \text{because } Y \text{ and } \mathsf{v}'(t) \text{ can be extracted by } \langle\cdots\,|\,\psi_n(t)\rangle \\
&= \frac{\mathsf{P}\left\{Y,\mathsf{v}'(t)|\mathsf{x}(t;\theta)\right\}P\{\mathsf{x}(t;\theta)\}}{\mathsf{P}\{Y,\mathsf{v}'(t)\}} \\
&= \frac{\mathsf{P}\{Y|\mathsf{x}(t;\theta)\}\,\mathsf{P}\left\{\mathsf{v}'(t)|\mathsf{x}(t;\theta)\right\}\mathsf{P}\{\mathsf{x}(t;\theta)\}}{\mathsf{P}\{Y\}\mathsf{P}\{\mathsf{v}'(t)\}} && \text{by } \textit{independence} \text{ of } Y \text{ and } \mathsf{v}'(t) \text{ ((5) lemma page 72)} \\
&= \frac{\mathsf{P}\{Y|\mathsf{x}(t;\theta)\}\,\mathsf{P}\left\{\mathsf{v}'(t)\right\}\mathsf{P}\{\mathsf{x}(t;\theta)\}}{\mathsf{P}\{Y\}\mathsf{P}\{\mathsf{v}'(t)\}} && \text{by } \textit{independence} \text{ of } \mathsf{x} \text{ and } \mathsf{v} \\
&= \frac{\mathsf{P}\{Y|\mathsf{x}(t;\theta)\}\,\mathsf{P}\{\mathsf{x}(t;\theta)\}}{\mathsf{P}\{Y\}} \\
&= \frac{\mathsf{P}\{Y,\mathsf{x}(t;\theta)\}}{\mathsf{P}\{Y\}} \\
&= \mathsf{P}\{\mathsf{x}(t;\theta)|Y\} && \text{by definition of } \textit{conditional probability} \text{ (Definition ?? page ??)}
\end{aligned}
$$

7. Proof that Y is a *sufficient statistic* for the *MAP estimate*:

$$
\begin{aligned}
\hat{\theta}_{\text{map}} &\triangleq \arg\max_{\hat{\theta}} \mathsf{P}\{\mathsf{x}(t;\theta)|\mathsf{y}(t;\theta)\} && \text{by definition of } \textit{MAP estimate} \text{ (Definition 6.1 page 64)} \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{\mathsf{x}(t;\theta)|Y\} && \text{by item (6)}
\end{aligned}
$$

8. Proof that Y is a *sufficient statistic* for the *ML estimate*:

$$
\begin{aligned}
\hat{\theta}_{\text{ml}} &\triangleq \arg\max_{\hat{\theta}} \mathsf{P}\{\mathsf{y}(t;\theta)|\mathsf{x}(t;\theta)\} && \text{by definition of } \textit{ML estimate} \text{ (Definition 6.1 page 64)} \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\left\{\sum_{n=1}^{N}\dot{y}_n\psi_n(t)+\mathsf{v}'(t)|\mathsf{x}(t;\theta)\right\} \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\left\{Y,\mathsf{v}'(t)|\mathsf{x}(t;\theta)\right\} && \text{because } Y \text{ and } \mathsf{v}'(t) \text{ can be extracted by } \langle\cdots\,|\,\psi_n(t)\rangle \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{Y|\mathsf{x}(t;\theta)\}\,\mathsf{P}\left\{\mathsf{v}'(t)\right\}\mathsf{x}(t;\theta) && \text{by } \textit{independence} \text{ of } Y \text{ and } \mathsf{v}'(t) \text{ ((5) lemma page 72)} \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{Y|\mathsf{x}(t;\theta)\}\,\mathsf{P}\left\{\mathsf{v}'(t)\right\} && \text{by } \textit{independence} \text{ of } \mathsf{x}(t) \text{ and } \mathsf{v}'(t) \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\{Y|\mathsf{x}(t;\theta)\} && \text{by } \textit{independence} \text{ of } \mathsf{v}'(t) \text{ and } \theta
\end{aligned}
$$

✏

7.3 Additive noise

Theorem 7.2 (Additive noise projection statistics). *Let Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, and Y be defined as in Definition 7.1 (page 69).*

THM

$$
\left\{\begin{array}{llcll}
\text{(A).} & \mathsf{y}(t;\theta) & \triangleq & \mathsf{x}(t;\theta)+\mathsf{v}(t) & (\textbf{additive}) \quad \text{and} \\
\text{(B).} & \mathsf{E}[\mathsf{v}(t)] & = & 0 & (\text{ZERO-MEAN}) \quad \text{and} \\
\text{(C).} & \mathsf{x}(t) & \subseteq & \operatorname{span}\Psi & (\Psi \text{ SPANS } \mathsf{x}(t)) \quad \text{and} \\
\text{(D).} & \langle\psi_n\,|\,\psi_m\rangle & = & \bar{\delta}_{mn} & (\text{ORTHONORMAL})
\end{array}\right\}
\implies
\left\{\ \mathsf{E}\left[\dot{y}_n(\theta)\right] = \dot{x}_n(\theta)\ \right\}
$$

✎PROOF:

$$
\begin{array}{rlll}
\mathsf{E}\left[\dot{y}_n(\theta)\right] &\triangleq \mathsf{E}\left[\langle \mathsf{y}(t;\theta) \,|\, \psi_n(t)\rangle\right] & \text{by definition of } \dot{y}_n & \text{(Definition 7.1 page 69)}\\
&= \mathsf{E}\langle \mathsf{x}(t;\theta) + \mathsf{v}(t) \,|\, \psi_n(t)\rangle & \text{by } \textit{additive} \text{ hypothesis} & \text{hypothesis (A)}\\
&= \mathsf{E}\left[\langle \mathsf{x}(t;\theta)\psi_n(t) \,|\, +\rangle \, \langle \mathsf{v}(t) \,|\, \psi_n(t)\rangle\right] & \text{by } \textit{additive} \text{ property of } \langle \triangle \,|\, \nabla \rangle & \text{(Definition G.9 page 266)}\\
&= \mathsf{E}\left[\left\langle \sum_{k=1}^{N} \dot{x}_k(\theta)\psi_k(t) \,\middle|\, \psi_n(t) \right\rangle + \dot{v}_n\right] & \text{by } \textit{basis} \text{ hypothesis} & \text{(C)}\\
&= \mathsf{E}\left[\sum_{k=1}^{N} \dot{x}_k(\theta) \, \langle \psi_k(t) \,|\, \psi_n(t)\rangle + \dot{v}_n\right] & \text{by } \textit{additive} \text{ property of } \langle \triangle \,|\, \nabla \rangle & \text{(Definition G.9 page 266)}\\
&= \mathsf{E}\left[\sum_{k=1}^{N} \dot{x}_k(\theta)\bar{\delta}_{k-n}(t) + \dot{v}_n\right] & \text{by } \textit{orthonormal} \text{ hypothesis} & \text{(D)}\\
&= \mathsf{E}\left[\dot{x}_n(\theta) + \dot{v}_n\right] & \text{by definition of } \bar{\delta} & \\
&= \mathsf{E}\dot{x}_n(\theta) + \cancelto{0}{\mathsf{E}\dot{v}_n} & \text{by } \textit{linearity} \text{ of } \mathsf{E} & \text{(Theorem ?? page ??)}\\
&= \dot{x}_n(\theta) & \text{by (B) and Lemma 7.1 page 69} &
\end{array}
$$

✏

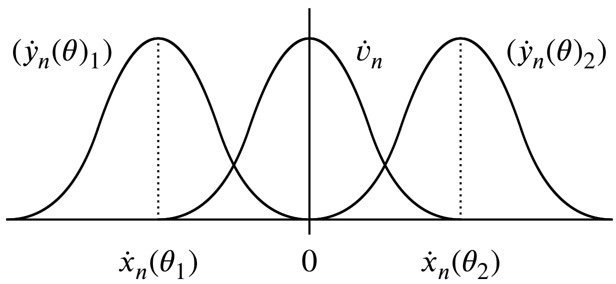


Figure 7.1: Additive *Gaussian* noise channel Statistics

Theorem 7.3 (Additive Gaussian noise projection statistics). *Let* Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, *and* Y *be defined as in Definition 7.1 (page 69).*

THM

$$
\underbrace{\left\{\begin{array}{lrcll}
\text{(A).} & \mathsf{y}(t;\theta) & \triangleq & \mathsf{x}(t) + \mathsf{v}(t) & (\textbf{\textit{additive}}) \quad \textit{and}\\
\text{(B).} & \mathsf{v}(t) & \sim & \mathsf{N}\left(0, \sigma^2\right) & (\textbf{\textit{Gaussian}}) \quad \textit{and}\\
\text{(C).} & \mathsf{x}(t) & \subseteq & \operatorname{span}\Psi & (\Psi \text{ SPANS } \mathsf{x}(t)) \quad \textit{and}\\
\text{(D).} & \langle \psi_n \,|\, \psi_m \rangle & = & \bar{\delta}_{mn} & (\text{ORTHONORMAL})
\end{array}\right\}}_{\text{ADDITIVE GAUSSIAN } \textit{system}}
\implies
\left\{ \dot{y}_n(\theta) \sim \mathsf{N}\left(\dot{x}_n(\theta), \sigma^2\right) \quad (\text{GAUSSIAN}) \right\}
$$

✎PROOF:

1. Proof for (1): By hypothesis (B) and Lemma 7.1 page 69.
2. Proof for (2):

$$
\begin{array}{rlll}
\mathsf{E}\left[\dot{y}_n(\theta)\right] &\triangleq \mathsf{E}\left[\langle \mathsf{y}(t;\theta) \,|\, \psi_n(t)\rangle \,|\theta\right] & \text{by definition of } \dot{y}_n & \text{(Definition 7.1 page 69)}\\
&= \mathsf{E}\left[\langle \mathsf{x}(t;\theta) + \mathsf{v}(t) \,|\, \psi_n(t)\rangle\right] & \text{by } \textit{additive} \text{ hypothesis} & \text{hypothesis (A)}\\
&= \mathsf{E}\left[\langle \mathsf{x}(t;\theta) \,|\, \psi_n(t)\rangle\right] + \mathsf{E}\left[\langle \mathsf{v}(t) \,|\, \psi_n(t)\rangle\right] & \text{by } \textit{additive} \text{ property of } \langle \triangle \,|\, \nabla \rangle & \text{(Definition G.9 page 266)}\\
&= \mathsf{E}\left\langle \sum_{k=1}^{N} \dot{x}_k(\theta)\psi_k(t) \,\middle|\, \psi_n(t) \right\rangle + \mathsf{E}\dot{v}_n & \text{by } \textit{basis} \text{ hypothesis} & \text{(C)}\\
&= \sum_{k=1}^{N} \mathsf{E}\left[\dot{x}_k(\theta)\right] \langle \psi_k(t) \,|\, \psi_n(t)\rangle + \mathsf{E}\dot{v}_n & \text{by } \textit{additive} \text{ property of } \langle \triangle \,|\, \nabla \rangle & \text{(Definition G.9 page 266)}
\end{array}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} \mathsf{E}\left[\dot{x}_k(\theta)\right]\bar{\delta}_{k-n}(t) + \mathsf{E}\dot{v}_n && \text{by } \textit{orthonormal} \text{ hypothesis} \quad \text{(D)}\\
&= \mathsf{E}\dot{x}_n(\theta) + \mathsf{E}\dot{v}_n && \text{by definition of } \bar{\delta}\\
&= \dot{x}_n(\theta) + 0 && \text{by Lemma 7.1 page 69}
\end{aligned}
$$

3. Proof for (3): The distribution follows because the process is a linear operations on a Gaussian process.

✏

Theorem 7.4 (Additive white noise projection statistics). *Let Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, and Y be defined as in Definition 7.1 (page 69).*

THM

$$
\underbrace{\left.\begin{array}{llcll}
\text{(A).} & \mathsf{y}(t;\theta) & \triangleq & \mathsf{x}(t) + \mathsf{v}(t) & \textit{and}\\
\text{(B).} & \operatorname{cov}[\mathsf{v}(t), \mathsf{v}(u)] & = & \sigma^2\delta(t-u) & \textit{and}\\
\text{(C).} & \mathsf{E}[\mathsf{v}(t)] & = & 0 & \textit{and}\\
\text{(E).} & \mathsf{x}(t) & \subseteq & \operatorname{span}\Psi & \textit{and}\\
\text{(E).} & \langle \psi_n \,|\, \psi_m \rangle & = & \bar{\delta}_{mn} &
\end{array}\right\}}_{\text{ADDITIVE WHITE } \textit{system}}
\implies
\left\{\begin{array}{lllll}
\text{(1).} & \mathsf{E}\dot{v}_n & = & 0 & \text{(ZERO-MEAN)}\\
\text{(2).} & \mathsf{E}(\dot{y}_n(\theta)) & = & \dot{x}_n(\theta) & \\
\text{(3).} & \operatorname{cov}\left[\dot{v}_n, \dot{v}_m\right] & = & \sigma^2\bar{\delta}_{nm} & \text{(UNCORRELATED)}\\
\text{(4).} & \operatorname{cov}\left[\dot{y}_n(\theta), \dot{y}_m|\theta\right] & = & \sigma^2\bar{\delta}_{nm} & \text{(UNCORRELATED)}
\end{array}\right.
$$

✎PROOF:

1. Because the noise is *additive* (hypothesis A)...

$$
\begin{aligned}
\mathsf{E}\dot{v}_n &= 0 && \text{by } \textit{additive} \text{ property and Theorem 7.2 page 73}\\
(\dot{y}_n(\theta)) &= \dot{x}_n(\theta) + \dot{v}_n && \text{by } \textit{additive} \text{ property and Theorem 7.2 page 73}\\
\mathsf{E}(\dot{y}_n|\theta) &= \dot{x}_n(\theta) && \text{by } \textit{additive} \text{ property and Theorem 7.2 page 73}
\end{aligned}
$$

2. Proof for (4):

$$
\begin{aligned}
\operatorname{cov}\left[\dot{y}_n(\theta), \dot{y}_m|\theta\right] &= \mathsf{E}\left[\dot{y}_n\dot{y}_m|\theta\right] - [\mathsf{E}\dot{y}_n(\theta)][\mathsf{E}\dot{y}_m|\theta]\\
&= \mathsf{E}\left[(\dot{x}_n(\theta) + \dot{v}_n)(\dot{x}_m(\theta) + \dot{v}_m)\right] - \dot{x}_n(\theta)\dot{x}_m(\theta)\\
&= \mathsf{E}\left[\dot{x}_n(\theta)\dot{x}_m(\theta) + \dot{x}_n(\theta)\dot{v}_m + \dot{v}_n\dot{x}_m(\theta) + \dot{v}_n\dot{v}_m\right] - \dot{x}_n(\theta)\dot{x}_m(\theta)\\
&= \dot{x}_n(\theta)\dot{x}_m(\theta) + \dot{x}_n(\theta)\mathsf{E}\left[\dot{v}_m\right] + \mathsf{E}\left[\dot{v}_n\right]\dot{x}_m(\theta) + \mathsf{E}\left[\dot{v}_n\dot{v}_m\right] - \dot{x}_n(\theta)\dot{x}_m(\theta)\\
&= 0 + \dot{x}_n(\theta)\cdot 0 + 0\cdot\dot{x}_m(\theta) + \operatorname{cov}\left[\dot{v}_n, \dot{v}_m\right] + [\mathsf{E}\dot{v}_n][\mathsf{E}\dot{v}_m]\\
&= \sigma^2\bar{\delta}_{nm} + 0\cdot 0 && \text{by Lemma 7.3}\\
&= \begin{cases} \sigma^2 & \text{for } n = m\\ 0 & \text{for } n \neq m. \end{cases}
\end{aligned}
$$

✏

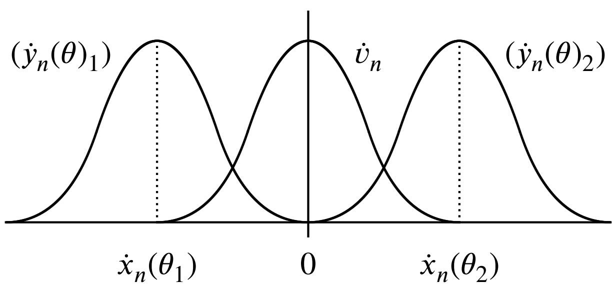


Figure 7.2: Additive white *Gaussian* noise channel statistics

Theorem 7.5 (AWGN projection statistics). *Let Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, and Y be defined as in Definition 7.1 (page 69).*

THM

$$\underbrace{\left.\begin{array}{llcll}
(A). & \mathsf{y}(t;\theta) & \triangleq & \mathsf{x}(t)+\mathsf{v}(t) & \text{and}\\
(B). & \operatorname{cov}[\mathsf{v}(t),\mathsf{v}(u)] & = & \sigma^2\delta(t-u) & \text{and}\\
(C). & \mathsf{v}(t) & \sim & \mathsf{N}\left(0,\sigma^2\right) & \text{and}\\
(D). & \mathsf{x}(t) & \subseteq & \operatorname{span}\Psi & \text{and}\\
(E). & \langle\psi_n\,|\,\psi_m\rangle & = & \bar{\delta}_{mn} &
\end{array}\right\}}_{\text{additive white Gaussian } system}
\implies
\left\{\begin{array}{lll}
(1). & \dot{y}_n(\theta)\sim\mathsf{N}\left(\dot{x}_n(\theta),\sigma^2\right) & (\text{Gaussian})\\
(2). & \operatorname{cov}\left[\dot{y}_n,\dot{y}_m\right]=\sigma^2\bar{\delta}_{nm} & (\text{uncorrelated})\\
(3). & \mathsf{P}\left\{\dot{y}_n\wedge\dot{y}_m\right\}=\mathsf{P}\left\{\dot{y}_n\right\}\mathsf{P}\left\{\dot{y}_m\right\} & (\text{independent})
\end{array}\right.$$

✎PROOF:

1. Proof for (1) follow because the operations are *linear* on processes are *Gaussian* (hypothesis C).
2.

$$\begin{aligned}
\mathsf{E}\dot{v}_n &= 0 && \text{by } AWN \text{ properties and Theorem 7.4 page 75}\\
\dot{y}_n &= \dot{x}_n+\dot{v}_n && \text{by } AWN \text{ properties and Theorem 7.4 page 75}\\
\mathsf{E}\dot{y}_n &= \dot{x}_n && \text{by } AWN \text{ properties and Theorem 7.4 page 75}\\
\operatorname{cov}\left[\dot{y}_n,\dot{y}_m\right] &= \sigma^2\bar{\delta}_{mn} && \text{by } AWN \text{ properties and Theorem 7.4 page 75}
\end{aligned}$$

3. Because the processes are *Gaussian*, *uncorrelated* implies *independent*.

✏

7.4 ML estimates

The AWGN projection statistics provided by Theorem 7.5 (page 76) help generate the optimal ML-estimates for a number of communication systems. These ML-estimates can be expressed in either of two standard forms:

- **Spectral decompostion**: The optimal estimate is expressed in terms of *projections* of signals onto orthonormal basis functions.
- **Matched signal**: The optimal estimate is expressed in terms of the (noisy) received signal correlated with ("matched" with) the (noiseless) transmitted signal.

Theorem 7.6 (page 76) (next) expresses the general optimal *ML estimate* in both of these forms.

Parameter detection is a special case of parameter estimation. In parameter detection, the estimate is a member of an finite set. In parameter estimation, the estimate is a member of an infinite set (Section 7.4 page 76).

Theorem 7.6 (General ML estimation). *Let Ψ, $\mathsf{y}(t;\theta)$, $\mathsf{x}(t)$, $\mathsf{v}(t)$, $\dot{y}_n$, $\dot{x}_n$, $\dot{v}_n$, and Y be defined as in Definition 7.1 (page 69). Let $\hat{\theta}_{\mathsf{ml}}$ be the* ML ESTIMATE (Definition 6.1 page 64) *of θ.*

THM

$$\begin{aligned}
\hat{\theta}_{\mathsf{ml}} &= \arg\min_{\hat{\theta}}\left[\sum_{n=1}^{N}[\dot{y}_n-\dot{x}_n(\hat{\theta})]^2\right] && (\textit{spectral decomposition})\\
&= \arg\max_{\hat{\theta}}\left[2\left\langle\mathsf{y}(t;\theta)\,|\,\mathsf{x}(t;\theta)\right\rangle-\|\mathsf{x}(t;\theta)\|^2\right] && (\textit{matched signal})
\end{aligned}$$

✎PROOF:

$$
\begin{aligned}
\hat{\theta}_{\text{ml}} &= \arg\max_{\hat{\theta}} \mathsf{P}\left\{\mathsf{y}(t;\theta)|\mathsf{x}(t;\theta)\right\} \\
&= \arg\max_{\hat{\theta}} \mathsf{P}\left\{\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n|\mathsf{x}(t;\theta)\right\} && \text{by Theorem 7.1 (page 71)} \\
&= \arg\max_{\hat{\theta}} \prod_{n=1}^{N} \mathsf{P}\left\{\dot{y}_n|\mathsf{x}(t;\theta)\right\} \\
&= \arg\max_{\hat{\theta}} \prod_{n=1}^{N} \mathsf{p}\left[\dot{\mathsf{y}}_n|\mathsf{x}(\mathsf{t};\theta)\right] \\
&= \arg\max_{\hat{\theta}} \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\frac{[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2}{-2\sigma^2} && \text{by Theorem 7.5 (page 76)} \\
&= \arg\max_{\hat{\theta}} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2 \\
&= \arg\max_{\hat{\theta}} \left[-\sum_{n=1}^{N}[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2\right] \\
&= \arg\max_{\hat{\theta}} \left[-\lim_{N\to\infty}\sum_{n=1}^{N}[\dot{y}_n - \dot{x}_n(\hat{\theta})]^2\right] && \text{by Theorem 7.1 (page 71)} \\
&= \arg\max_{\hat{\theta}} \left[-\|\mathsf{y}(t;\theta) - \mathsf{x}(t;\theta)\|^2\right] && \text{by \textit{Plancheral's formula} (Theorem ?? page ??)} \\
&= \arg\max_{\hat{\theta}} \left[-\|\mathsf{y}(t;\theta)\|^2 + 2\mathbf{R}_{\mathsf{e}}\langle \mathsf{y}(t;\theta) \,|\, \mathsf{x}(t;\theta)\rangle - \|\mathsf{x}(t;\theta)\|^2\right] \\
&= \arg\max_{\hat{\theta}} \left[2\langle \mathsf{y}(t;\theta) \,|\, \mathsf{x}(t;\theta)\rangle - \|\mathsf{x}(t;\theta)\|^2\right] && \text{because } \mathsf{y}(t;\theta) \text{ \textit{independent} of } \hat{\theta}
\end{aligned}
$$

✏

Theorem 7.7 (ML amplitude estimation). [2] *Let* **S** *be an additive white gaussian noise system.*

THM

$$
\left\{\begin{array}{llcll}
\text{(A).} & \mathsf{v}(t) \text{ is AWGN} & & & \textit{and} \\
\text{(B).} & \mathsf{y}(t;a) & = & \mathsf{x}(t;a) + \mathsf{v}(t) & \textit{and} \\
\text{(C).} & \mathsf{x}(t;a) & \triangleq & a\lambda(t). &
\end{array}\right\}
\implies
\left\{\begin{array}{lrcll}
\text{(1).} & \hat{a}_{\text{ml}} & = & \dfrac{1}{\|\lambda(t)\|^2}\displaystyle\sum_{n=1}^{N}\dot{y}_n\dot{\lambda}_n & \\
\text{(2).} & \mathsf{E}\hat{a}_{\text{ml}} & = & a & (\text{UNBIASED}) \\
\text{(3).} & \operatorname{var}\hat{a}_{\text{ml}} & = & \dfrac{\sigma^2}{\|\lambda(t)\|^2} & \\
\text{(4).} & \operatorname{var}\hat{a}_{\text{ml}} & = & \textit{CR lower bound} & (\text{EFFICIENT})
\end{array}\right\}
$$

✎PROOF:

1. *ML estimate* in "matched signal" form:

$$
\begin{aligned}
\hat{a}_{\text{ml}} &= \arg\max_{a} \left[2\langle \mathsf{y}(t;\theta) \,|\, \mathsf{x}(t;\theta)\rangle - \|\mathsf{x}(t;\phi)\|^2\right] && \text{by Theorem 7.6 (page 76)} \\
&= \arg\max_{a} \left[2\langle \mathsf{y}(t;\theta) \,|\, a\lambda(t)\rangle - \|a\lambda(t)\|^2\right] && \text{by hypothesis}
\end{aligned}
$$

[2] Srinath et al. (1996) pages 158–159

$$
\begin{aligned}
&= \arg_a \left[\frac{\partial}{\partial a} 2a \langle \mathsf{y}(t;\theta) \mid \lambda(t) \rangle - \frac{\partial}{\partial a} a^2 \|\lambda(t)\|^2 = 0 \right] \\
&= \arg_a \left[2 \langle \mathsf{y}(t;\theta) \mid \lambda(t) \rangle - 2a \|\lambda(t)\|^2 = 0 \right] \\
&= \arg_a \left[\langle \mathsf{y}(t;\theta) \mid \lambda(t) \rangle = a \|\lambda(t)\|^2 \right] \\
&= \frac{1}{\|\lambda(t)\|^2} \langle \mathsf{y}(t;\theta) \mid \lambda(t) \rangle
\end{aligned}
$$

2. *ML estimate* in "spectral decomposition" form:

$$
\begin{aligned}
\hat{a}_{\mathsf{ml}} &= \arg\min_a \left(\sum_{n=1}^{N} \left[\dot{y}_n - \dot{x}_n(a) \right]^2 \right) && \text{by Theorem 7.6 (page 76)} \\
&= \arg_a \left(\frac{\partial}{\partial a} \sum_{n=1}^{N} \left[\dot{y}_n - \dot{x}_n(a) \right]^2 = 0 \right) \\
&= \arg_a \left(2 \sum_{n=1}^{N} \left[\dot{y}_n - \dot{x}_n(a) \right] \frac{\partial}{\partial a} \dot{x}_n(a) = 0 \right) \\
&= \arg_a \left(\sum_{n=1}^{N} \left[\dot{y}_n - \langle a\lambda(t) \mid \psi_n(t) \rangle \right] \frac{\partial}{\partial a} \langle a\lambda(t) \mid \psi_n(t) \rangle = 0 \right) \\
&= \arg_a \left(\sum_{n=1}^{N} \left[\dot{y}_n - a \langle \lambda(t) \mid \psi_n(t) \rangle \right] \frac{\partial}{\partial a} (a \langle \lambda(t) \mid \psi_n(t) \rangle) = 0 \right) \\
&= \arg_a \left(\sum_{n=1}^{N} \left[\dot{y}_n - a\dot{\lambda}_n \right] \langle \lambda(t) \mid \psi_n(t) \rangle = 0 \right) \\
&= \arg_a \left(\sum_{n=1}^{N} \left[\dot{y}_n - a\dot{\lambda}_n \right] \dot{\lambda}_n = 0 \right) \\
&= \arg_a \left(\sum_{n=1}^{N} \dot{y}_n \dot{\lambda}_n = \sum_{n=1}^{N} a\dot{\lambda}_n^2 \right) \\
&= \left(\frac{1}{\sum_{n=1}^{N} \dot{\lambda}_n^2} \right) \sum_{n=1}^{N} \dot{y}_n \dot{\lambda}_n \\
&= \frac{1}{\|\lambda(t)\|^2} \sum_{n=1}^{N} \dot{y}_n \dot{\lambda}_n
\end{aligned}
$$

3. Prove that the estimate $\hat{a}_{\mathsf{ml}}$ is **unbiased**:

$$
\begin{aligned}
\mathsf{E}\hat{a}_{\mathsf{ml}} &= \mathsf{E} \frac{1}{\|\lambda(t)\|^2} \int_{t\in\mathbb{R}} \mathsf{y}(t;\theta)\lambda(t)\, \mathrm{d}t && \text{by previous result} \\
&= \mathsf{E} \frac{1}{\|\lambda(t)\|^2} \int_{t\in\mathbb{R}} [a\lambda(t) + \mathsf{v}(t)]\lambda(t)\, \mathrm{d}t && \text{by hypothesis} \\
&= \frac{1}{\|\lambda(t)\|^2} \int_{t\in\mathbb{R}} \mathsf{E}[a\lambda(t) + \mathsf{v}(t)]\lambda(t)\, \mathrm{d}t && \text{by linearity of } \int \cdot\, \mathrm{d}t \text{ and } \mathsf{E} \\
&= \frac{1}{\|\lambda(t)\|^2} a \int_{t\in\mathbb{R}} \lambda^2(t)\, \mathrm{d}t && \text{by } \mathsf{E} \text{ operation} \\
&= \frac{1}{\|\lambda(t)\|^2} a \|\lambda(t)\|^2 && \text{by definition of } \|\cdot\|^2 \\
&= a
\end{aligned}
$$

4. Compute the variance of $\hat{a}_{\mathrm{ml}}$:

$$\begin{aligned}
\mathsf{E}\hat{a}^2_{\mathrm{ml}} &= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^2}\int_{t\in\mathbb{R}} \mathsf{y}(t;\theta)\lambda(t)\,\mathsf{d}t\right]^2 \\
&= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}} \mathsf{y}(t;\theta)\lambda(t)\,\mathsf{d}t \int_{v} \mathsf{y}(v)\lambda(v)\,\mathsf{d}v\right] \\
&= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_{v}[a\lambda(t)+\mathsf{v}(t)][a\lambda(v)+\mathsf{v}(v)]\lambda(t)\lambda(v)\,\mathsf{d}v\,\mathsf{d}t\right] \\
&= \mathsf{E}\left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_{v}[a^2\lambda(t)\lambda(v)+a\lambda(t)\mathsf{v}(v)+a\lambda(v)\mathsf{v}(t)+\mathsf{v}(t)\mathsf{v}(v)]\lambda(t)\lambda(v)\,\mathsf{d}v\,\mathsf{d}t\right] \\
&= \left[\frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_{v}[a^2\lambda(t)\lambda(v)+0+0+\sigma^2\delta(t-v)]\lambda(t)\lambda(v)\,\mathsf{d}v\,\mathsf{d}t\right] \\
&= \frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_{v} a^2\lambda^2(t)\lambda^2(v)\,\mathsf{d}v\,\mathsf{d}t + \frac{1}{\|\lambda(t)\|^4}\int_{t\in\mathbb{R}}\int_{v}\sigma^2\delta(t-v)\lambda(t)\lambda(v)\,\mathsf{d}v\,\mathsf{d}t \\
&= \frac{1}{\|\lambda(t)\|^4}a^2\int_{t\in\mathbb{R}}\lambda^2(t)\,\mathsf{d}t\int_{v}\lambda^2(v)\,\mathsf{d}v + \frac{1}{\|\lambda(t)\|^4}\sigma^2\int_{t\in\mathbb{R}}\lambda^2(t)\,\mathsf{d}t \\
&= a^2\frac{1}{\|\lambda(t)\|^4}\|\lambda(t)\|^2\,\|\lambda(v)\|^2 + \frac{1}{\|\lambda(t)\|^4}\sigma^2\,\|\lambda(t)\|^2 \\
&= a^2 + \frac{\sigma^2}{\|\lambda(t)\|^2}
\end{aligned}$$

$$\begin{aligned}
\operatorname{var}\hat{a}_{\mathrm{ml}} &= \mathsf{E}\hat{a}^2_{\mathrm{ml}} - (\mathsf{E}\hat{a}_{\mathrm{ml}})^2 \\
&= \left(a^2+\frac{\sigma^2}{\|\lambda(t)\|^2}\right) - \left(a^2\right) \\
&= \frac{\sigma^2}{\|\lambda(t)\|^2}
\end{aligned}$$

5. Compute the Cramér-Rao Bound:

$$\begin{aligned}
\mathsf{p}\left[\mathsf{y}(\mathsf{t};\theta)|\mathsf{x}(\mathsf{t};\mathsf{a})\right] &= \mathsf{p}\left[\dot{\mathsf{y}}_1,\dot{\mathsf{y}}_2,\ldots,\dot{\mathsf{y}}_N|\mathsf{x}(\mathsf{t};\mathsf{a})\right] \\
&= \prod_{n=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\frac{(\dot{y}_n-a\dot{\lambda}_n)^2}{-2\sigma^2} \\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n-a\dot{\lambda}_n)^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial a}\ln\mathsf{p}\left[\mathsf{y}(\mathsf{t};\theta)|\mathsf{x}(\mathsf{t};\mathsf{a})\right] &= \frac{\partial}{\partial a}\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n-a\dot{\lambda}_n)^2 \\
&= \frac{\partial}{\partial a}\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N + \frac{\partial}{\partial a}\ln\exp\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n-a\dot{\lambda}_n)^2 \\
&= \frac{\partial}{\partial a}\left[\frac{1}{-2\sigma^2}\sum_{n=1}^{N}(\dot{y}_n-a\dot{\lambda}_n)^2\right] \\
&= \frac{1}{-2\sigma^2}\sum_{n=1}^{N}2(\dot{y}_n-a\dot{\lambda}_n)(-\dot{\lambda}_n) \\
&= \frac{1}{\sigma^2}\sum_{n=1}^{N}\dot{\lambda}_n(\dot{y}_n-a\dot{\lambda}_n)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;a)\right] &= \frac{\partial}{\partial a}\frac{\partial}{\partial a}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;a)\right]\\
&= \frac{\partial}{\partial a}\frac{1}{\sigma^2}\sum_{n=1}^{N}\dot{\lambda}_n(\dot{y}_n - a\dot{\lambda}_n)\\
&= \frac{1}{\sigma^2}\sum_{n=1}^{N}\dot{\lambda}_n(-\dot{\lambda}_n)\\
&= \frac{-1}{\sigma^2}\sum_{n=1}^{N}\dot{\lambda}_n^2\\
&= \frac{-\left\|\lambda(t)\right\|^2}{\sigma^2}
\end{aligned}$$

$$\begin{aligned}
\operatorname{var}\hat{a}_{\mathsf{ml}} &\triangleq \mathsf{E}\left[\hat{a}_{\mathsf{ml}} - \mathsf{E}\hat{a}_{\mathsf{ml}}\right]^2\\
&= \mathsf{E}\left[\hat{a}_{\mathsf{ml}} - a\right]^2\\
&\geq \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;a)\right]\right)}\\
&= \frac{-1}{\mathsf{E}\left(\frac{-\left\|\lambda(t)\right\|^2}{\sigma^2}\right)}\\
&= \frac{\sigma^2}{\left\|\lambda(t)\right\|^2} \qquad \text{(Cramér-Rao lower bound of the variance)}
\end{aligned}$$

6. Proof that $\hat{a}_{\mathsf{ml}}$ is an *efficient* estimate:

An estimate is *efficient* if $\operatorname{var}\hat{a}_{\mathsf{ml}} =$ CR lower bound. We have already proven this, so $\hat{a}_{\mathsf{ml}}$ is an *efficient* estimate.

Also, even without explicitly computing the variance of $\hat{a}_{\mathsf{ml}}$, the variance equals the *Cramér-Rao lower bound* (and hence $\hat{a}_{\mathsf{ml}}$ is an *efficient* estimate) if and only if

$$\hat{a}_{\mathsf{ml}} - a = \left(\frac{-1}{\mathsf{E}\left[\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;a)\right]\right]}\right)\left(\frac{\partial}{\partial a}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;a)\right]\right)$$

$$\begin{aligned}
\left(\frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial a^2}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;a)\right]\right)}\right)\left(\frac{\partial}{\partial a}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;a)\right]\right) &= \left(\frac{\sigma^2}{\left\|\lambda(t)\right\|^2}\right)\left(\frac{1}{\sigma^2}\sum_{n=1}^{N}\dot{\lambda}(\dot{y} - a\dot{\lambda})\right)\\
&= \frac{1}{\left\|\lambda(t)\right\|^2}\sum_{n=1}^{N}\dot{\lambda}\dot{y} - \frac{1}{\left\|\lambda(t)\right\|^2}\sum_{n=1}^{N}\dot{\lambda}^2\\
&= \hat{a}_{\mathsf{ml}} - a
\end{aligned}$$

✎

Theorem 7.8 (ML phase estimation). [3]

THM
$$\left\{\begin{array}{lllll}
(A). & \mathsf{v}(t) \text{ is AWGN} & & & and\\
(B). & \mathsf{y}(t;\phi) & = & \mathsf{x}(t;\phi) + \mathsf{v}(t) & and\\
(C). & \mathsf{x}(t;\phi) & \triangleq & A\cos(2\pi f_c t + \phi) &
\end{array}\right\} \implies \left\{\hat{\phi}_{\mathsf{ml}} = -\operatorname{atan}\left(\frac{\langle \mathsf{y}(t;\theta)\,|\,\sin(2\pi f_c t)\rangle}{\langle \mathsf{y}(t;\theta)\,|\,\cos(2\pi f_c t)\rangle}\right)\right\}$$

[3] Srinath et al. (1996) pages 159–160

✎PROOF:

$$\begin{aligned}
\hat{\phi}_{\mathsf{ml}} &= \arg\max_{\phi}\left[2\langle \mathsf{y}(t;\phi)\,|\,\mathsf{x}(t;\phi)\rangle - \|\mathsf{x}(t;\phi)\|^2\right] && \text{by Theorem 7.6 (page 76)}\\
&= \arg\max_{\phi}\left[2\langle \mathsf{y}(t;\phi)\,|\,\mathsf{x}(t;\phi)\rangle\right] && \text{because } \|\mathsf{x}(t;\phi)\| \text{ does not depend on } \phi\\
&= \arg_{\phi}\left[\frac{\partial}{\partial\phi}\langle \mathsf{y}(t;\phi)\,|\,\mathsf{x}(t;\phi)\rangle = 0\right]\\
&= \arg_{\phi}\left[\left\langle \mathsf{y}(t;\phi)\,|\,\frac{\partial}{\partial\phi}\mathsf{x}(t;\phi)\right\rangle = 0\right] && \text{because } \langle\triangle\,|\,\triangledown\rangle \text{ is } \textit{linear}\\
&= \arg_{\phi}\left[\left\langle \mathsf{y}(t;\phi)\,|\,\frac{\partial}{\partial\phi}A\cos(2\pi f_c t+\phi)\right\rangle = 0\right] && \text{by definition of } \mathsf{x}(t;\phi)\\
&= \arg_{\phi}\left[\langle \mathsf{y}(t;\phi)\,|\,-A\sin(2\pi f_c t+\phi)\rangle = 0\right] && \text{because } \frac{\partial}{\partial\phi}\cos(x) = -\sin(x)\\
&= \arg_{\phi}\left[-A\langle \mathsf{y}(t;\phi)\,|\,\cos(2\pi f_c t)\sin\phi + \sin(2\pi f_c t)\cos\phi\rangle = 0\right] && \text{by } \textit{double angle formulas}\\
&= \arg_{\phi}\left[\sin\phi\langle \mathsf{y}(t;\phi)\,|\,\cos(2\pi f_c t)\rangle = -\cos\phi\langle \mathsf{y}(t;\phi)\,|\,\sin(2\pi f_c t)\rangle\right]\\
&= \arg_{\phi}\left[\frac{\sin\phi}{\cos\phi} = -\frac{\langle \mathsf{y}(t;\phi)\,|\,\sin(2\pi f_c t)\rangle}{\langle \mathsf{y}(t;\phi)\,|\,\cos(2\pi f_c t)\rangle}\right]\\
&= \arg_{\phi}\left[\tan\phi = -\frac{\langle \mathsf{y}(t;\phi)\,|\,\sin(2\pi f_c t)\rangle}{\langle \mathsf{y}(t;\phi)\,|\,\cos(2\pi f_c t)\rangle}\right]\\
&= -\operatorname{atan}\left(\frac{\langle \mathsf{y}(t;\phi)\,|\,\sin(2\pi f_c t)\rangle}{\langle \mathsf{y}(t;\phi)\,|\,\cos(2\pi f_c t)\rangle}\right)
\end{aligned}$$

✏

Theorem 7.9 (ML estimation of a function of a parameter). [4] *Let* **S** *be an additive white gaussian noise system such that*

$$\begin{aligned}\mathsf{y}(t;\theta) &= \mathsf{x}(t;\theta) + \mathsf{v}(t)\\ \mathsf{x}(t;\theta) &= \mathsf{g}(\theta)\end{aligned}$$

and g *is* ONE-TO-ONE AND ONTO *(*INVERTIBLE*).*

THM

Then the optimal ML-estimate of parameter θ is $\hat{\theta}_{\mathsf{ml}} = \mathsf{g}^{-1}\left(\frac{1}{N}\sum_{n=1}^{N}\dot{y}_n\right)$.

If an ML ESTIMATE $\hat{\theta}_{\mathsf{ml}}$ *is unbiased* ($\mathsf{E}\hat{\theta}_{\mathsf{ml}} = \theta$) *then* $\operatorname{var}\hat{\theta}_{\mathsf{ml}} \geq \frac{\sigma^2}{N}\frac{1}{\left[\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right]^2}$.

If $\mathsf{g}(\theta) = \theta$ *then* $\hat{\theta}_{\mathsf{ml}}$ *is an* ***efficient*** *estimate such that* $\operatorname{var}\hat{\theta}_{\mathsf{ml}} = \frac{\sigma^2}{N}$.

✎PROOF:

$$\begin{aligned}
\hat{\theta}_{\mathsf{ml}} &= \arg\min_{\theta}\left[\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)]^2\right] && \text{by Theorem 7.6 page 76}\\
&= \arg_{\theta}\left[\frac{\partial}{\partial\theta}\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)]^2 = 0\right] && \text{because form is } \textit{quadratic}\\
&= \arg_{\theta}\left[2\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)]\frac{\partial}{\partial\theta}\mathsf{g}(\theta) = 0\right]\\
&= \arg_{\theta}\left[2\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)] = 0\right]
\end{aligned}$$

[4] Srinath et al. (1996) pages 142–143

$$
\begin{aligned}
&= \arg_\theta \left[\sum_{n=1}^{N} \dot{y}_n = N\mathsf{g}(\theta)\right] \\
&= \arg_\theta \left[\mathsf{g}(\theta) = \frac{1}{N}\sum_{n=1}^{N} \dot{y}_n\right] \\
&= \arg_\theta \left[\theta = \mathsf{g}^{-1}\left(\frac{1}{N}\sum_{n=1}^{N} \dot{y}_n\right)\right] \\
&= \mathsf{g}^{-1}\left(\frac{1}{N}\sum_{n=1}^{N} \dot{y}_n\right)
\end{aligned}
$$

If $\hat{\theta}_{\text{ml}}$ is unbiased ($\mathsf{E}\hat{\theta}_{\text{ml}} = \theta$), we can use the *Cramér-Rao bound* to find a lower bound on the variance:

$$
\begin{aligned}
\operatorname{var}\hat{\theta}_{\text{ml}} &\triangleq \mathsf{E}\left[\hat{\theta}_{\text{ml}} - \mathsf{E}\hat{\theta}_{\text{ml}}\right]^2 \\
&= \mathsf{E}\left[\hat{\theta}_{\text{ml}} - \theta\right]^2 \\
&\geq \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;\theta)\right]\right)} && \text{by \textit{Cramér-Rao Inequality}} \\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\dot{\mathsf{y}}_1,\dot{\mathsf{y}}_2,\ldots,\dot{\mathsf{y}}_N|\mathsf{x}(t;\theta)\right]\right)} && \text{by \textit{Sufficient Statistic Theorem} (Theorem 7.1 page 71)} \\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln\left[\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)]^2\right)\right]\right)} && \text{by \textit{AWGN} hypothesis and Theorem 7.5 page 76} \\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln\left[\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N\right] + \frac{\partial^2}{\partial\theta^2}\ln\left[\exp\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)]^2\right]\right)} \\
&= \frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\left(\frac{-1}{2\sigma^2}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)]^2\right)\right)} \\
&= \frac{2\sigma^2}{\mathsf{E}\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)]^2\right)} \\
&= \frac{2\sigma^2}{\mathsf{E}\left(-2\frac{\partial}{\partial\theta}\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)]\right)} && \text{by \textit{Chain Rule}} \\
&= \frac{-\sigma^2}{\mathsf{E}\left(\frac{\partial \mathsf{g}^2(\theta)}{\partial\theta^2}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)] + \frac{\partial \mathsf{g}(\theta)}{\partial\theta}\frac{\partial}{\partial\theta}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)]\right)} && \text{by \textit{Product Rule}} \\
&= \frac{-\sigma^2}{\mathsf{E}\left(\frac{\partial \mathsf{g}^2(\theta)}{\partial\theta^2}\sum_{n=1}^{N}[\dot{y}_n-\mathsf{g}(\theta)] - N\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\sigma^2}{\frac{\partial \mathsf{g}^2(\theta)}{\partial\theta^2}\sum_{n=1}^{N}\mathsf{E}[\dot{y}_n - \mathsf{g}(\theta)] - N\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\frac{\partial \mathsf{g}(\theta)}{\partial\theta}} \\
&= \frac{-\sigma^2}{-N\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\frac{\partial \mathsf{g}(\theta)}{\partial\theta}} && \text{because derivative of constant} = 0 \\
&= \frac{\sigma^2}{N}\frac{1}{\left[\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right]^2}
\end{aligned}
$$

The inequality becomes equality (an *efficient* estimate) if and only if

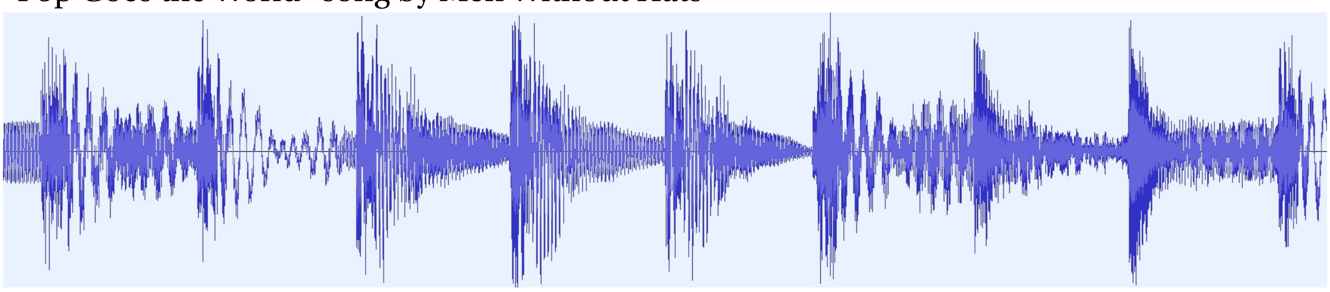
$$
\hat{\theta}_{\mathsf{ml}} - \theta = \left(\frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;\theta)\right]\right)}\right)\left(\frac{\partial}{\partial\theta}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;\theta)\right]\right).
$$

$$
\begin{aligned}
\left(\frac{-1}{\mathsf{E}\left(\frac{\partial^2}{\partial\theta^2}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;\theta)\right]\right)}\right)\left(\frac{\partial}{\partial\theta}\ln \mathsf{p}\left[\mathsf{y}(t;\theta)|\mathsf{x}(t;\theta)\right]\right)
&= \left(\frac{\sigma^2}{N}\frac{1}{\left[\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\right]^2}\right)\left(\frac{-1}{2\sigma^2}(2)\frac{\partial \mathsf{g}(\theta)}{\partial\theta}\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)]\right) \\
&= -\frac{1}{N}\frac{1}{\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}\left(\sum_{n=1}^{N}[\dot{y}_n - \mathsf{g}(\theta)]\right) \\
&= -\frac{1}{\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}\left(\frac{1}{N}\sum_{n=1}^{N}\dot{y}_n - \mathsf{g}(\theta)\right) \\
&= -\frac{1}{\frac{\partial \mathsf{g}(\theta)}{\partial\theta}}\left(\hat{\theta}_{\mathsf{ml}} - \mathsf{g}(\theta)\right) \\
&= -(\hat{\theta}_{\mathsf{ml}} - \theta)
\end{aligned}
$$

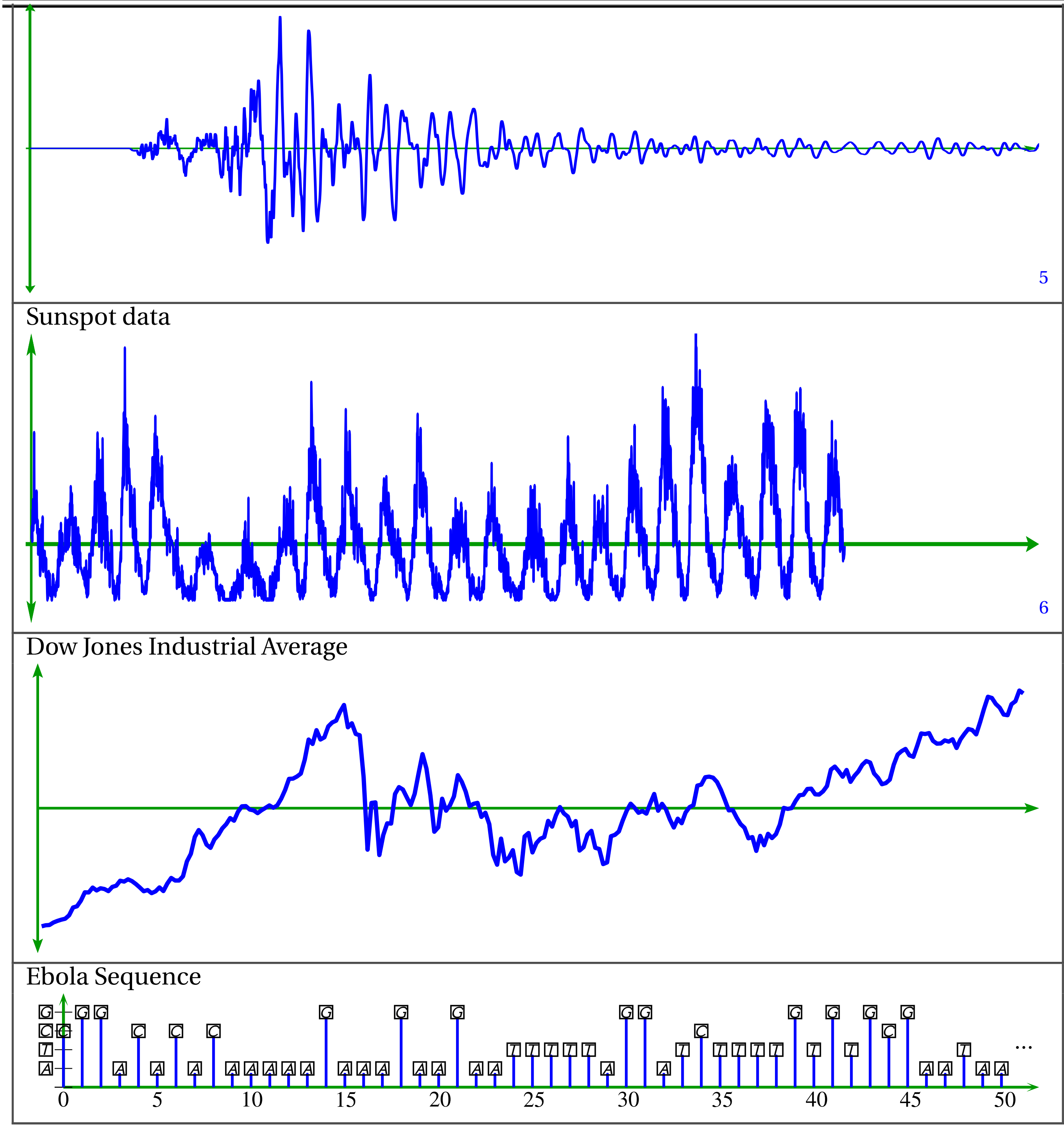
✏

7.5 Example data

"Pop Goes the World" song by Men Without Hats



Earthquake data



7.6 Colored noise

This chapter presented several theorems whose results depended on the noise being white. However if the noise is **colored**, then these results are invalid. But there is still hope for colored noise. Processing colored signals can be accomplished using two techniques:

1. Karhunen-Loève basis functions (Section C.1 page 203)

[5] https://www.iris.edu/wilber3/find_stations/10953070

[6] https://d32ogoqmya1dw8.cloudfront.net/files/introgeo/teachingwdata/examples/GreenwichSSNvstime.txt

2. whitening filter [7]

Karhunen-Loève. If the noise is *white*, the set $\{\langle y(t;\theta)\,|\,\psi_n(t)\rangle\,|n=1,2,\ldots,N\}$ is a *sufficient statistic* regardless of which set $\{\psi_n(t)\}$ of orthonormal basis functions are used. If the noise is *colored*, and if $\{\psi_n(t)\}$ satisfy the Karhunen-Loève criterion

$$\int_{t_2} \mathsf{R}_{xx}(t,u)\psi_n(u)\,du = \lambda_n\psi_n(t)$$

then the set $\{\langle y(t;\theta)\,|\,\psi_n(t)\rangle\}$ is still a *sufficient statistic*.

Whitening filter. The whitening filter makes the received signal $y(t;\theta)$ statistically white (uncorrelated in time). In this case, any orthonormal basis set can be used to generate sufficient statistics.

Wavelets. Wavelets have the property that they tend to whiten data. For more information, see Walter and Shen (2001) pages 329–350 ⟨"Chapter 14 Orthogonal Systems and Stochastic Processes"⟩, Mallat (1999), Johnstone and Silverman (1997), Wornell and Oppenheim (1992), and Vidakovic (1999) pages 10–14 ⟨"Example 1.2.5 Wavelets whiten data"⟩ (first four references cited by B. Vidakovic).

[7] *Continuous data whitening*: Section ?? page ??
Discrete data whitening: Section ?? page ??

CHAPTER 8

ESTIMATION USING MATCHED FILTER

Let S be the set of transmitted waveforms and Y be a set of orthonormal basis functions that span S. *Signal matching* computes the innerproducts of a received signal $\mathsf{y}(t;\theta)$ with each signal from S. *Orthonormal decomposition* computes the innerproducts of $\mathsf{y}(t;\theta)$ with each signal from the set Y.

In the case where $|S|$ is large, often $|Y| << |S|$ making orthonormal decomposition much easier to implement. For example, in a QAM-64 modulation system, signal matching requires $|S| = 64$ innerproduct calculations, while orthonormal decomposition only requires $|Y| = 2$ innerproduct calculations because all 64 signals in S can be spanned by just 2 orthonormal basis functions.

Maximizing SNR. Theorem 7.1 (page 71) shows that the innerproducts of $\mathsf{y}(t;\theta)$ with basis functions of Y is *sufficient* for optimal detection. Theorem 8.1 (page 87) (next) shows that a receiver can maximize the SNR of a received signal when signal matching is used.

Theorem 8.1. *Let* $\mathsf{x}(t)$ *be a transmitted signal,* $\mathsf{v}(t)$ *noise, and* $\mathsf{y}(t;\theta)$ *the received signal in an AWGN channel. Let the* SIGNAL TO NOISE RATIO *SNR be defined as*

$$\mathrm{SNR}[\mathsf{y}(t;\theta)] \triangleq \frac{|\langle \mathsf{x}(t) \,|\, \mathsf{x}(t)\rangle|^2}{\mathsf{E}\left[|\langle \mathsf{v}(t) \,|\, \mathsf{x}(t)\rangle|^2\right]}.$$

THM

$$\mathrm{SNR}[\mathsf{y}(t;\theta)] \le \frac{2\,\|\mathsf{x}(t)\|^2}{N_o}$$ *and is maximized (equality) when* $\mathsf{x}(t) = a\mathsf{x}(t)$, *where* $a \in \mathbb{R}$.

✎PROOF:

$$\begin{aligned}
\mathrm{SNR}[\mathsf{y}(t;\theta)] &\triangleq \frac{|\langle \mathsf{x}(t) \,|\, \mathsf{x}(t)\rangle|^2}{\mathsf{E}\left[|\langle \mathsf{v}(t) \,|\, \mathsf{x}(t)\rangle|^2\right]} \\
&= \frac{|\langle \mathsf{x}(t) \,|\, f(t)\rangle|^2}{\mathsf{E}\left[\left[\int_{t\in\mathbb{R}} \mathsf{v}(t)\mathsf{x}^*(t)\; dt\right]\left[\int_{\hat{\theta}} n(\hat{\theta}) f^*(\hat{\theta})\; du\right]^*\right]} \\
&= \frac{|\langle \mathsf{x}(t) \,|\, \mathsf{x}(t)\rangle|^2}{\mathsf{E}\left[\int_{t\in\mathbb{R}} \int_{\hat{\theta}} \mathsf{v}(t) n^*(\hat{\theta})\mathsf{x}^*(t)\mathsf{x}(\hat{\theta})\;\; \mathrm{d}t\; \mathrm{d}u\right]} \\
&= \frac{|\langle \mathsf{x}(t) \,|\, f(t)\rangle|^2}{\int_{t\in\mathbb{R}} \int_{\hat{\theta}} \mathsf{E}\left[\mathsf{v}(t) n^*(\hat{\theta})\right]\mathsf{x}^*(t)\mathsf{x}(\hat{\theta})\;\; \mathrm{d}t\; \mathrm{d}u}
\end{aligned}$$

$$= \frac{|\langle \mathsf{x}(t) \,|\, \mathsf{x}(t)\rangle|^2}{\int_{t\in\mathbb{R}} \int_{\hat{\theta}} \frac{1}{2} N_o \delta(t-\hat{\theta})\mathsf{x}^*(t)\mathsf{x}(\hat{\theta}) \;\; \mathrm{d}t \; \mathrm{d}u}$$

$$= \frac{|\langle \mathsf{x}(t) \,|\, \mathsf{x}(t)\rangle|^2}{\frac{1}{2} N_o \int_{t\in\mathbb{R}} \mathsf{x}^*(t)\mathsf{x}(t) \;\mathrm{d}t}$$

$$= \frac{|\langle \mathsf{x}(t) \,|\, \mathsf{x}(t)\rangle|^2}{\frac{1}{2} N_o \,\|\mathsf{x}(t)\|^2}$$

$$\le \frac{|\|\mathsf{x}(t)\| \;\; \|\mathsf{x}(t)\||^2}{\frac{1}{2} N_o \,\|\mathsf{x}(t)\|^2} \qquad \text{by } \textit{Cauchy-Schwarz Inequality}$$

$$= \frac{2\,\|\mathsf{x}(t)\|^2}{N_o}$$

The Cauchy-Schwarz Inequality becomes an equality (SNR is maximized) when $\mathsf{x}(t) = a\mathsf{x}(t)$. ✏

Implementation. The innerproduct operations can be implemented using either

1. a correlator or
2. a matched filter.

A correlator is simply an integrator of the form $\langle \mathsf{y}(t;\theta) \,|\, f(t)\rangle = \int_0^T \mathsf{y}(t;\theta) f(t) \;\mathrm{d}t$.

A matched filter introduces a function $\mathsf{h}(t)$ such that $\mathsf{h}(t) = \mathsf{x}(T-t)$ (which implies $\mathsf{x}(t) = h(T-t)$) giving

$$\underbrace{\langle \mathsf{y}(t;\theta) \,|\, \mathsf{x}(t)\rangle = \int_0^T \mathsf{y}(t;\theta)\mathsf{x}(t) \;\mathrm{d}t}_{\text{correlator}} = \underbrace{\left.\int_0^\infty \mathsf{x}(\tau)h(t-\tau) \;\mathrm{d}\tau\right|_{t=T} = \mathsf{x}(t) \star \mathsf{h}(t)|_{t=T}}_{\text{matched filter}}.$$

This shows that $\mathsf{h}(t)$ is the impulse response of a filter operation sampled at time τ. By Theorem 8.1 (page 87), the optimal impulse response is $\mathsf{h}(\tau - t) = \mathsf{f}(t) = \mathsf{x}(t)$. That is, the optimal $\mathsf{h}(t)$ is just a "flipped" and shifted version of $\mathsf{x}(t)$.

CHAPTER 9

PHASE ESTIMATION

9.1 Phase Estimation

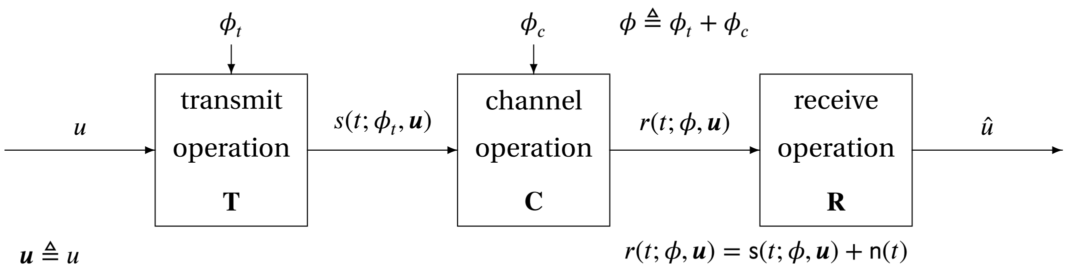


Figure 9.1: Phase estimation system model

In a narrowband communication system, the modulation sinusoid used by the transmitter generally has a different phase than the demodulation sinusoid used by the receiver. In many systems the receiver must estimate the phase of the received carrier.

Estimation types. The phase estimate may be *explicit* or *implicit*:

① explicit: compute an actual value for the phase estimate.
② implicit: generate a sinusoid with the same estimated phase as the carrier.

Algorithm classifications Synchronization algorithms can be classified in two ways. In the first, algorithms are classified according to whether the transmitted information is assumed to be known (*decision directed*) or unknown (*non-decision directed*) to the receiver. [1]

[1] *Decision/non-decision directed* is the classification used by Proakis (2001).

1. decision directed: transmitted information symbols are assumed to be known to the receiver.
2. non-decision directed: compute the expected value of a likelihood function with respect to probability distribution of the information symbols.

In the second, algorithms are classified according to whether or not they use feedback. [2]

① error tracking: with feedback – resembles the PLL operation
② feedforward: no feedback – uses bandpass filter

Hardware implementation. Implicit phase computation can be accomplished by using a *phase-lock loop* (*PLL*). Explicit phase computation algorithms often require the computation of the atan : $\mathbb{R} \to \mathbb{R}$ function.

9.1.1 ML estimate

Theorem 9.1. *In an AWGN channel with received signal* $r(t) = s(t;\phi) + n(t)$ *Let*

- $r(t) = s(t;\phi) + n(t)$ *be the received signal in an AWGN channel*
- $n(t)$ *a Gaussian white noise process*
- $s(t;\phi)$ *the transmitted signal such that*

$$s(t;\phi) = \sum_n a_n \lambda(t-nT)\cos(2\pi f_c t + \theta_n + \phi).$$

Then the optimal ML estimate of ϕ is either of the two equivalent expressions

THM

$$\begin{aligned}
\hat{\phi}_{\mathrm{ml}} &= -\operatorname{atan}\left[\frac{\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n)\,\mathrm{d}t}{\sum_n a_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t + \theta_n)\,\mathrm{d}t}\right] \\
&= \arg_\phi\left(\sum_n a_n \int_t r(t)\left[\lambda(t-nT)\sin(2\pi f_c t + \theta_n + \phi)\right]\,\mathrm{d}t = 0\right).
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\hat{\phi}_{\mathrm{ml}} &= \arg_\phi\left(2\int_t r(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathrm{d}t = \frac{\partial}{\partial\phi}\int_t s^2(t;\phi)\,\mathrm{d}t\right) && \text{by Theorem 7.6 page 76} \\
&= \arg_\phi\left(2\int_t r(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathrm{d}t = \frac{\partial}{\partial\phi}\|s(t;\phi)\|^2\,\mathrm{d}t\right) \\
&= \arg_\phi\left(2\int_t r(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathrm{d}t = 0\right) \\
&= \arg_\phi\left(\int_t r(t)\left[\frac{\partial}{\partial\phi}\sum_n a_n\lambda(t-nT)\cos(2\pi f_c t + \theta_n + \phi)\right]\,\mathrm{d}t = 0\right) \\
&= \arg_\phi\left(-\sum_n a_n\int_t r(t)\left[\lambda(t-nT)\sin(2\pi f_c t + \theta_n + \phi)\right]\,\mathrm{d}t = 0\right)
\end{aligned}$$

[2] *error tracking/feedforward* is the classification preferred by Meyr et al. (1998).

$$
\begin{aligned}
&= \arg_\phi\left(\sum_n a_n \int_t r(t)\lambda(t-nT)\left[\sin(2\pi f_c t+\theta_n)\cos(\phi)+\sin(\phi)\cos(2\pi f_c t+\theta_n)\right]\,\mathrm{d}t = 0\right)\\
&= \arg_\phi\left(\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(\phi)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t = -\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\cos(\phi)\,\mathrm{d}t\right)\\
&= \arg_\phi\left(\sin(\phi)\sum_n a_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t = -\cos(\phi)\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t\right)\\
&= \arg_\phi\left(\frac{\sin(\phi)}{\cos(\phi)} = -\frac{\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_n a_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)\\
&= \arg_\phi\left(\tan(\phi) = -\frac{\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_n a_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)\\
&= \arg_\phi\left(\phi = -\operatorname{atan}\left(\frac{\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_n a_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)\right)\\
&= -\operatorname{atan}\left(\frac{\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_n a_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)
\end{aligned}
$$

✏

9.1.2 Decision directed estimate

In this architecture (see Figure 9.2) the phase estimate $\hat{\phi}_{\mathsf{ml}}$ is explicitly computed in accordance with the equation

$$
\begin{aligned}
\hat{\phi}_{\mathsf{ml}} &= -\operatorname{atan}\left(\frac{\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_n a_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right) && \text{by Theorem 12.1 page 131}\\
&= -\operatorname{atan}\left(\frac{\sum_n a_n \int_t r(t)\lambda(t-nT)[\sin(2\pi f_c t)\cos\theta_n+\cos(2\pi f_c t)\sin\theta_n]\,\mathrm{d}t}{\sum_n a_n \int_t r(t)\lambda(t-nT)[\cos(2\pi f_c t)\cos\theta_n-\sin(2\pi f_c t)\sin\theta_n]\,\mathrm{d}t}\right)\\
&= -\operatorname{atan}\left(\frac{\sum_n a_n\cos\theta_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t)\,\mathrm{d}t + \sum_n a_n\sin\theta_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t)\,\mathrm{d}t}{\sum_n a_n\cos\theta_n \int_t r(t)\lambda(t-nT)\cos(2\pi f_c t)\,\mathrm{d}t - \sum_n a_n\sin\theta_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t)\,\mathrm{d}t}\right)
\end{aligned}
$$

Decision directed implicit estimation implementation

In this architecture (see Figure 9.3 page 92) the phase estimate $\hat{\phi}_{\mathsf{ml}}$ is not explicitly computed. Rather, a sinusoid that has the estimated phase $\hat{\phi}_{\mathsf{ml}}$ is generated using a *voltage controlled oscillator* (VCO). The entire structure which includes the VCO is called a **phase-lock loop** (*PLL*). The PLL operates in accordance with the equation

$$
\sum_n a_n \int_t r(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n+\hat{\phi}_{\mathsf{ml}})\,\mathrm{d}t = 0.
$$

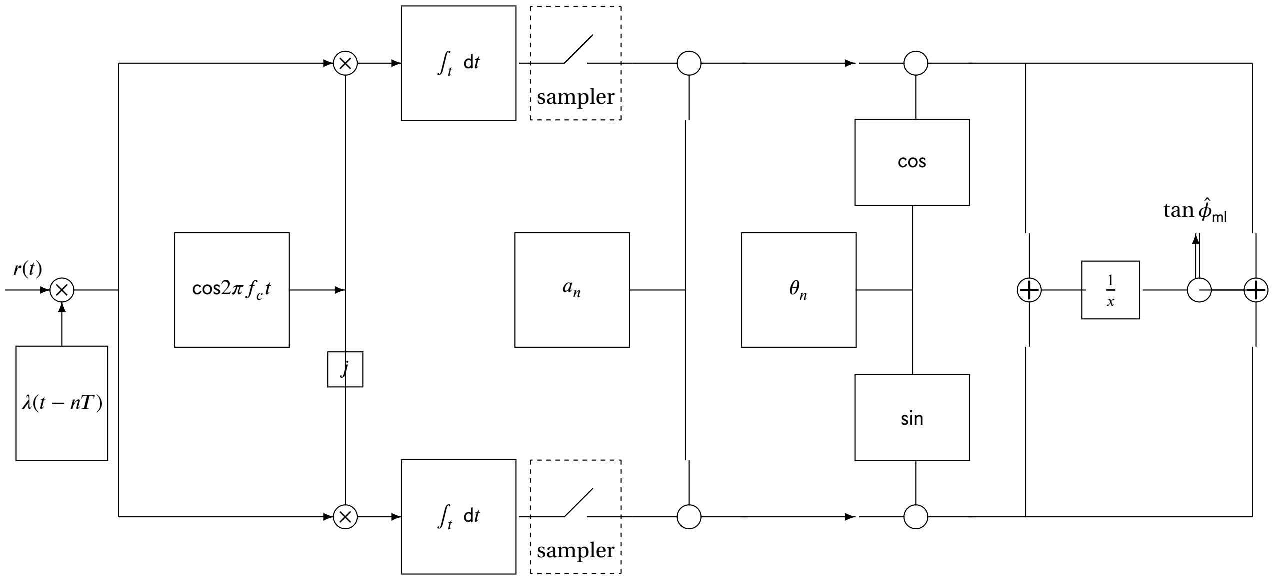


Figure 9.2: Explicit phase estimation implementation

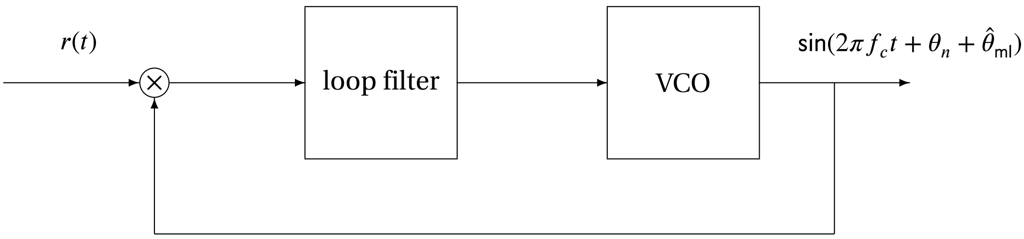


Figure 9.3: Implicit phase estimation implementation

9.1.3 Non-decision directed phase estimation

Definition 9.1.

$$\mathsf{E}_m \hat{\phi}_{ml} = \arg\max_{\phi} \mathsf{E}_m \int_t r(t) s_m(t;\phi)\, \mathrm{d}t.$$

$$\sum_{n=0}^{K-1} \int_{nT}^{(n+1)T} r(t)\cos(2\pi f_c t + \hat{\phi}_{ml})\, \mathrm{d}t \int_{nT}^{(n+1)T} r(t)\sin(2\pi f_c t + \hat{\phi}_{ml})\, \mathrm{d}t = 0$$

9.2 Phase Lock Loop

Reference: Kao (2005)

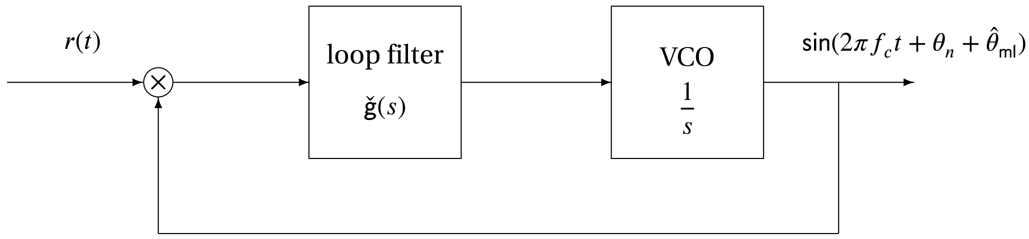


Figure 9.4: Implicit phase estimation implementation

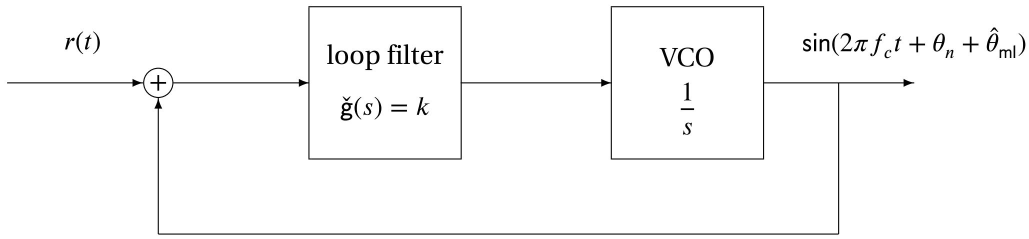


Figure 9.5: Implicit phase estimation implementation

9.2.1 First order response

Loop response

Eventhough the filter response is zero order ($\breve{g}(s) = k$), the total loop response ($\breve{h}(s)$) is first order. A causal first order filter has an exponential impulse response.

$$
\begin{aligned}
\breve{h}(s) &= \frac{\breve{g}(s)\frac{1}{s}}{1+\breve{g}(s)\frac{1}{s}} = \frac{\breve{g}(s)}{s+\breve{g}(s)} = \frac{k}{s+k} = \frac{1}{1+\frac{s}{k}} \\
\breve{h}(s)\big|_{s=i\omega} &= \tilde{h}(\omega) = \frac{1}{1+i\frac{\omega}{k}} \\
|\tilde{h}(\omega)|^2 &= \left|\frac{1}{1+i\frac{\omega}{k}}\right|^2 = \left(\frac{1}{1+i\frac{\omega}{k}}\right)\left(\frac{1}{1+i\frac{\omega}{k}}\right)^* = \frac{1}{1+\left(\frac{\omega}{k}\right)^2} \\
\left[\mathbf{L}ae^{-bt}\mu(t)\right](s) &= \int_t ae^{-bt}\mu(t)e^{-st}\,\mathrm{d}t \\
&= \int_0^\infty ae^{-(s+b)t}e^{-st}\,\mathrm{d}t \\
&= \frac{a}{-(s+b)}e^{-bt}\bigg|_0^\infty \\
&= \frac{a}{s+b} \\
h(t) &= ke^{-kt}\mu(t)
\end{aligned}
$$

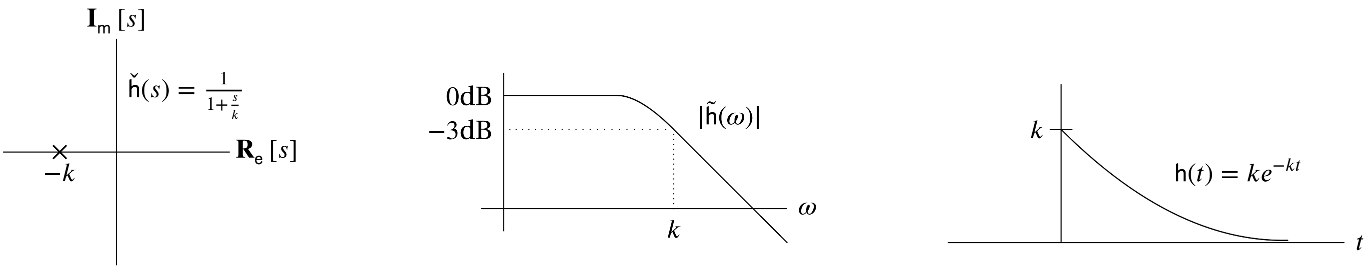


Figure 9.6: First Order Loop response

Phase step response

In Phase Shift Keying (PSK) modulation, the phase of the signal changes abruptly. Thus we are interested in the response of the PLL to a "phase step".

$$\theta_{in} = \theta_0 + \Delta\theta\mu(t)$$

$$\begin{aligned}
\theta_{vco} &= h(t) \star \theta_{in} \\
&= h(t) \star [\theta_0 + \Delta\theta\mu(\tau)] \\
&= h(t) \star \theta_0 + h(t) \star \Delta\theta\mu(\tau) \\
&= \int_\tau h(t-\tau)\theta_0 \, d\tau + \int_\tau h(t-\tau)\Delta\theta\mu(\tau) \, d\tau \\
&= \theta_0 \int_\tau h(t-\tau) \, d\tau + \Delta\theta \int_0^\infty h(t-\tau) \, d\tau \\
&= \theta_0 \int_\tau ke^{-k(t-\tau)}\mu(t-\tau) \, d\tau + \Delta\theta \int_0^\infty ke^{-k(t-\tau)}\mu(t-\tau) \, d\tau \\
&= \theta_0 ke^{-kt} \int_\tau e^{k\tau}\mu(t-\tau) \, d\tau + \Delta\theta ke^{-kt} \int_0^\infty e^{k\tau}\mu(t-\tau) \, d\tau \\
&= \theta_0 ke^{-kt} \int_{-\infty}^t e^{k\tau} \, d\tau + \Delta\theta ke^{-kt}\mu(t) \int_0^t e^{k\tau} \, d\tau \\
&= \theta_0 ke^{-kt} \left. \frac{1}{k}e^{k\tau} \right|_{-\infty}^t + \Delta\theta ke^{-kt}\mu(t) \left. \frac{1}{k}e^{k\tau} \right|_0^t \\
&= \theta_0 ke^{-kt}\frac{1}{k}(e^{kt} - 0) + \Delta\theta ke^{-kt}\frac{1}{k}(e^{kt} - 1)\mu(t) \\
&= \theta_0 + \Delta\theta(1 - e^{-kt})\mu(t)
\end{aligned}$$

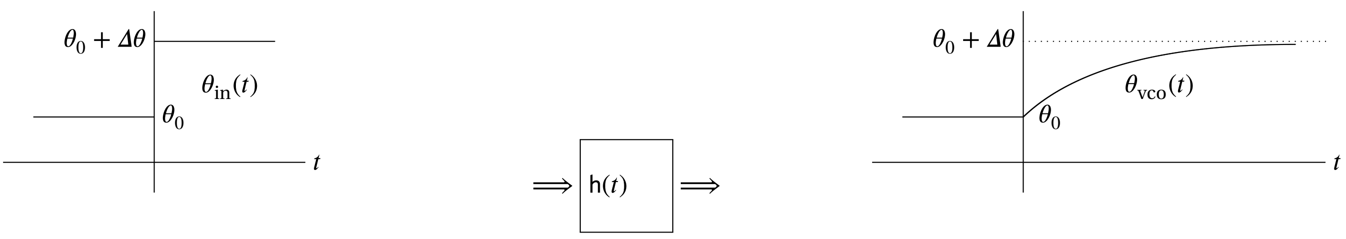


Figure 9.7: First Order Loop phase step response

Frequency step response

In Frequency Shift Keying (FSK) modulation, the frequency of the signal changes abruptly. Thus we are interested in the response of the PLL to a "frequency step". The change in frequency will be modelled as part of the phase.

$$\theta_{\text{in}} = \theta_0 + \Delta\omega t\mu(t)$$

$$\begin{aligned}
\theta_{\text{vco}} &= \mathsf{h}(t) \star \theta_{\text{in}} \\
&= \mathsf{h}(t) \star [\theta_0 + \Delta\omega t\mu(t)] \\
&= \mathsf{h}(t) \star \theta_0 + \mathsf{h}(t) \star \Delta\omega t\mu(t) \\
&= \int_\tau \mathsf{h}(t-\tau)\theta_0 \, \mathrm{d}\tau + \int_\tau \mathsf{h}(t-\tau)\Delta\omega\tau\mu(\tau) \, \mathrm{d}\tau \\
&= \theta_0 \int_\tau \mathsf{h}(t-\tau) \, \mathrm{d}\tau + \Delta\omega \int_0^\infty \mathsf{h}(t-\tau)\tau \, \mathrm{d}\tau \\
&= \theta_0 \int_\tau ke^{-k(t-\tau)}\mu(t-\tau) \, \mathrm{d}\tau + \Delta\omega \int_0^\infty ke^{-k(t-\tau)}\mu(t-\tau)\tau \, \mathrm{d}\tau \\
&= \theta_0 ke^{-kt} \int_\tau e^{k\tau}\mu(t-\tau) \, \mathrm{d}\tau + \Delta\omega ke^{-kt} \int_0^\infty e^{k\tau}\mu(t-\tau)\tau \, \mathrm{d}\tau \\
&= \theta_0 ke^{-kt} \int_{-\infty}^t e^{k\tau} \, \mathrm{d}\tau + \Delta\omega ke^{-kt}\mu(t) \int_0^t \tau e^{k\tau} \, \mathrm{d}\tau \\
&= \theta_0 ke^{-kt} \left.\frac{1}{k}e^{k\tau}\right|_{-\infty}^t + \Delta\omega ke^{-kt}\mu(t) \left[\left.\tau\frac{1}{k}e^{k\tau}\right|_0^t - \int_0^t \frac{1}{k}e^{k\tau} \, \mathrm{d}\tau \right] \\
&= \theta_0 ke^{-kt}\frac{1}{k}(e^{kt}-0) + \Delta\omega ke^{-kt}\mu(t) \left[\frac{1}{k}(te^{kt}-0) - \left.\frac{1}{k^2}e^{k\tau}\right|_0^t \right] \\
&= \theta_0 + \Delta\omega e^{-kt}\mu(t) \left[te^{kt} - \frac{1}{k}(e^{kt}-1) \right] \\
&= \theta_0 + \Delta\omega t\mu(t) - \frac{\Delta\omega}{k}(1-e^{-kt})\mu(t)
\end{aligned}$$

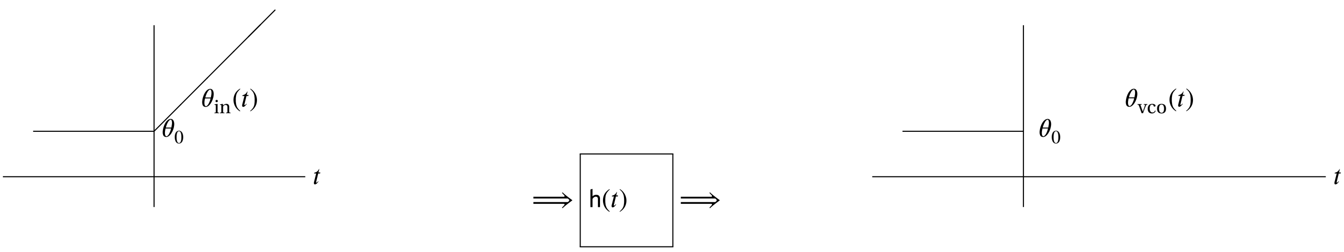


Figure 9.8: First Order Loop phase frequency response

CHAPTER 10

NETWORK DETECTION

10.1 Detection

For detection, we need

1. Cost function: for hard decisions, its range must be linearly ordered. For soft decisions, it can be a lattice.

2. system joint and marginal probabilities (for Bayesian detection)

10.2 Bayesian Estimation

Definition 10.1.

DEF

$$\begin{array}{lcll} H & \triangleq & \{h_1, h_2, h_3, \dots\} & \textit{set of hypotheses} \\ \boldsymbol{D} & \triangleq & \{D_1, D_2, D_3, \dots\} & \textit{partition—decision regions} \\ X & \triangleq & \{X_1, X_2, X_3, \dots\} & \textit{set of sensor inputs} \end{array}$$

$$\begin{aligned} \mathsf{C}(h; P) &= \min_{\boldsymbol{D}} \sum_i \mathsf{P}\left\{ \left[X \in D_i \right] \wedge \left[H \neq h_i \right] \right\} \\ &= \min_{\boldsymbol{D}} \sum_i \mathsf{P}\left\{ X \in D_i \mid H \neq h_i \right\} \mathsf{P}\left\{ H \neq h_i \right\} \\ &= \min_{\boldsymbol{D}} \sum_i \sum_{j \neq i} \left[1 - \mathsf{P}\left\{ X \in D_i \mid H = h_i \right\} \right] \sum_{j \neq i} \left[1 - \mathsf{P}\left\{ H = h_i \right\} \right] \end{aligned}$$

$$\hat{h} = \arg_h \mathsf{C}(h; P)$$

10.3 Joint Gaussian Model

Assume convexity …

$$
\begin{aligned}
\boldsymbol{D} &= \arg\min_{\boldsymbol{D}} \mathsf{C}(h;\boldsymbol{P}) \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) \underbrace{\mathsf{p}(H \neq h_i)}_{c} d\boldsymbol{x} = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} c \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) d\boldsymbol{x} = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} \sum_i \left[1 - \sum_{j\neq i} \int_{D_i} \mathsf{p}(\boldsymbol{x}|H = h_i) d\boldsymbol{x} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \frac{\partial}{\partial \boldsymbol{D}} \sum_i \left[1 - \sum_{j\neq i} \int_{D_i} \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{M}|}} \exp -\frac{1}{2}(\boldsymbol{x} - \mathsf{E}\boldsymbol{x})^T \boldsymbol{M}^{-1} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x}) d\boldsymbol{z} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \sum_i \left[1 - \sum_{j\neq i} \frac{\partial}{\partial \boldsymbol{D}} \int_{D_i} \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{M}|}} \exp -\frac{1}{2}(\boldsymbol{x} - \mathsf{E}\boldsymbol{x})^T \boldsymbol{M}^{-1} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x}) d\boldsymbol{z} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \sum_i \left[1 - \sum_{j\neq i} \begin{bmatrix} \frac{\partial}{\partial D_1} \\ \frac{\partial}{\partial D_2} \\ \vdots \; \frac{\partial}{\partial D_n} \end{bmatrix} \int_{D_i} \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{M}|}} \exp -\frac{1}{2}(\boldsymbol{x} - \mathsf{E}\boldsymbol{x})^T \boldsymbol{M}^{-1} (\boldsymbol{x} - \mathsf{E}\boldsymbol{x}) d\boldsymbol{z} \right] = 0 \right\} \\
&= \arg_{\boldsymbol{D}} \left\{ \sum_i \left[1 - \sum_{j\neq i} \underbrace{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}}_{\text{Jacobian matrix}} \right] \right\}
\end{aligned}
$$

For two variable Gaussian …

$$
\begin{aligned}
\mathsf{C} &= \min_{\boldsymbol{D}} \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) \underbrace{\mathsf{p}(H \neq h_i)}_{c} d\boldsymbol{x} \\
&= \min_{\boldsymbol{D}} c \sum_i \int_{D_i} \mathsf{p}(\boldsymbol{x}|H \neq h_i) d\boldsymbol{x} \\
&= \min_{\boldsymbol{D}} c \sum_i \left[1 - \sum_{j\neq i} \int_{D_i} \mathsf{p}(\boldsymbol{x}|H = h_i) d\boldsymbol{x} \right]
\end{aligned}
$$

$$= \min_D c \sum_i \left[1 - \sum_{j \neq i} \int_{D_i} \frac{1}{2\pi\sqrt{|M|}} \exp\left(\frac{z_1^2 \mathsf{E}[z_2 z_2] - 2 z_1 z_2 \mathsf{E}[z_1 z_2] + z_2^2 \mathsf{E}[z_1 z_1]}{-2|M|} \right) d\boldsymbol{z} \right]$$

10.4 2 hypothesis, 2 sensor detection

Theorem 10.1 (centralized case). *Let $(\Omega, \mathbb{E}, \mathsf{P})$ be a probability space. Let $D \subsetneq \mathbb{E}$ be the* DECISION REGION *indicating hypothesis $H = h_1$. Let $\pi_0 \triangleq \mathsf{P}\{H = h_0\}$ and $\pi_1 \triangleq \mathsf{P}\{H = h_1\}$.*

THM

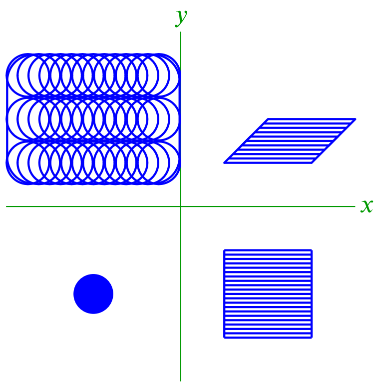
$$\begin{aligned} D &= \arg\min_D \left[\underbrace{\mathsf{P}\left\{(x,y) \in D | H = h_0\right\} \pi_0}_{\text{error for } H = h_0} + \underbrace{\mathsf{P}\left\{(x,y) \in D^c | H = h_1\right\} \pi_1}_{\text{error for } H = h_1} \right] \\ &= \arg\min_D \left[\underbrace{\pi_0 \int_D \mathsf{p}_0(x,y)\ \mathsf{d}x\ \mathsf{d}y}_{\text{error for } H = h_0} + \underbrace{\pi_1 \int_D \mathsf{p}_1(x,y)\ \mathsf{d}x\ \mathsf{d}y}_{\text{error for } H = h_1} \right] \end{aligned}$$

✎PROOF:

$$\begin{aligned} D &= \arg\min_D [\mathsf{P}\{\text{error}\}] && \text{by definition of decision region } D \\ &= \arg\min_D \left[\mathsf{P}\{\text{error} \wedge H = h_0\} + \mathsf{P}\{\text{error} \wedge H = h_1\}\right] \\ &= \arg\min_D \left[\mathsf{P}\left\{\text{error} | H = h_0\right\} \pi_0 + \mathsf{P}\left\{\text{error} | H = h_1\right\} \pi_1\right] \\ &= \arg\min_D \left[\mathsf{P}\left\{(x,y) \in D | H = h_0\right\} \pi_0 + \mathsf{P}\left\{(x,y) \in D^c | H = h_1\right\} \pi_1\right] \\ &= \arg\min_D \left[\pi_0 \int_D \mathsf{p}_0(x,y)\ \mathsf{d}x\ \mathsf{d}y + \pi_1 \int_D \mathsf{p}_1(x,y)\ \mathsf{d}x\ \mathsf{d}y\right] \end{aligned}$$

✏

Example 10.1. In the centralized case, the decision regions D in the xy-plane can be any arbitrary shape, as illustrated to the right.



Definition 10.2.

DEF

*Let $\mathbf{P}_x$ and $\mathbf{P}_y$ be **set projection operators** such that* $\begin{aligned} D_x &\triangleq \mathbf{P}_x D \\ D_y &\triangleq \mathbf{P}_y D \end{aligned}$

Proposition 10.1. *Let $+$ represent* MINKOWSKI ADDITION.

PRP

$$\mathbf{D} = \mathbf{D}_x + \mathbf{D}_y$$

Theorem 10.2 (distributed AND case). *Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a probability space. Let* $D \subsetneq \mathbb{E}$ *be the* DECISION REGION *indicating hypothesis* $H = h_1$. *Let* $\pi_0 \triangleq \mathsf{P}\{H = h_0\}$ *and* $\pi_1 \triangleq \mathsf{P}\{H = h_1\}$. *Let* $E \triangleq D^c$.

THM

$$D = \arg\min_D \begin{pmatrix} \mathsf{P}\{ & x \in E, & y \in E & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in E, & y \in D & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D, & y \in E & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D, & y \in D & \}\{H = h_0\}\pi_0 & \end{pmatrix}$$

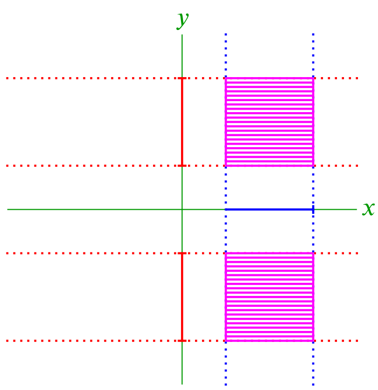
✎PROOF:

x	y	H	$x \wedge y$	
0	0	0	0	
0	1	0	0	
1	0	0	0	
1	1	0	1	error
0	0	1	0	error
0	1	1	0	error
1	0	1	0	error
1	1	1	1	

$$\begin{aligned} D &= \arg\min_D [\mathsf{P}\{\text{error}\}] && \text{by definition of decision region } D \\ &= \arg\min_D \left[\mathsf{P}\{\text{error} \wedge H = h_0\} + \mathsf{P}\{\text{error} \wedge H = h_1\}\right] \\ &= \arg\min_D \left[\mathsf{P}\{\text{error}|H = h_0\}\pi_0 + \mathsf{P}\{\text{error}|H = h_1\}\pi_1\right] \\ &= \arg\min_D \begin{pmatrix} \mathsf{P}\{ & x \in E_x, & y \in E_y & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D_x, & y \in E_y & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in E_x, & y \in D_y & \}\{H = h_1\}\pi_1 & + \\ \mathsf{P}\{ & x \in D_x, & y \in D_y & \}\{H = h_0\}\pi_0 & \end{pmatrix} \end{aligned}$$

✏

Example 10.2. In the distributed AND case, the decision regions D in the xy-plane are only simple rectangular shapes, as illustrated to the right.



Proposition 10.2.

PRP

In general, distributed AND *detection is suboptimal.*

✎PROOF: Because only rectangular decision regions are possible, detection is suboptimal. ✏

Theorem 10.3. [1]

THM

For the distributed AND *detection*

$$D_x = \left\{ x | \pi_0 \int_{D_y} \mathsf{p}_0(x, y)\, \mathrm{d}x\, \mathrm{d}y \le \pi_1 \int_{D_y} \mathsf{p}_1(x, y)\, \mathrm{d}x\, \mathrm{d}y \right\}$$

[1] Willett et al. (2000), page 3268

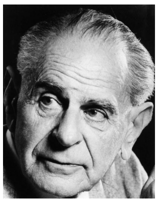
✎PROOF:

$$D_x = \left\{x | y \in D_y \quad \Longrightarrow \quad \mathsf{P}\left\{(x,y)\,|H = h_0\right\}\pi_0 \leq \mathsf{P}\left\{(x,y)\,|H = h_1\right\}\pi_1\right\}$$
$$= \left\{x | \pi_0 \int_{D_y} \mathrm{p}_0\,(x,y)\ \mathsf{d}x\ \mathsf{d}y \leq \pi_1 \int_{D_y} \mathrm{p}_1\,(x,y)\ \mathsf{d}x\ \mathsf{d}y\right\}$$

✏

CHAPTER 11

SYSTEM IDENTIFICATION



❝ *I can therefore gladly admit that falsificationists like myself much prefer an attempt to solve an interesting problem by a bold conjecture, even (and especially) if it so turns out to be false, to any recital of a sequence of irrelevant truisms. We prefer this because we believe that this is the way in which we can learn from our mistakes and that in finding that our conjecture was false we shall have learned much about the truth, and shall have got nearer to the truth.* ❞

Karl R. Popper (1902–1994) [1]

11.1 Estimation techniques

Let **S** be a *system* with *impulse response* $\mathsf{h}(n)$ with with *DTFT* $\tilde{\mathsf{H}}(\omega)$, input $\mathsf{x}(n)$, and output $\mathsf{y}(n)$. Often in the field of "digital signal processing" (DSP), **S** is a "filter" with known $\mathsf{h}(n)$ and $\tilde{\mathsf{H}}(\omega)$ because the filter **S** was designed by a designer who had direct control over $\mathsf{h}(n)$.

However in many other practical situations, **S** is some other system for which $\mathsf{h}(n)$ and $\tilde{\mathsf{H}}(\omega)$ are *not* known...but which we may want to *estimate*. Examples of such **S** is a device on an industrial shaker table, a communication channel, or the entire earth.

Determining $\mathsf{h}(n)$ and/or $\tilde{\mathsf{H}}(\omega)$ is part of an operation called "*system identication*". Determining $\tilde{\mathsf{H}}(\omega)$ in particular is referred to as "*Frequency Response Identification*"[2] or as "*Frequency Response Function*" ("*FRF*") estimation.[3] *FRF* estimation is a challenging problem and one that many have devoted much effort to. This chapter describes some of that effort.

In the early days, people used a rather obvious technique for determining $\tilde{\mathsf{H}}(\omega)$—the humble *sine sweep*. That is, they drove the input with a sine wave with slowly increasing (or decreasing) frequency while measuring the resulting output. This technique, although effective, was "very slow".[4]

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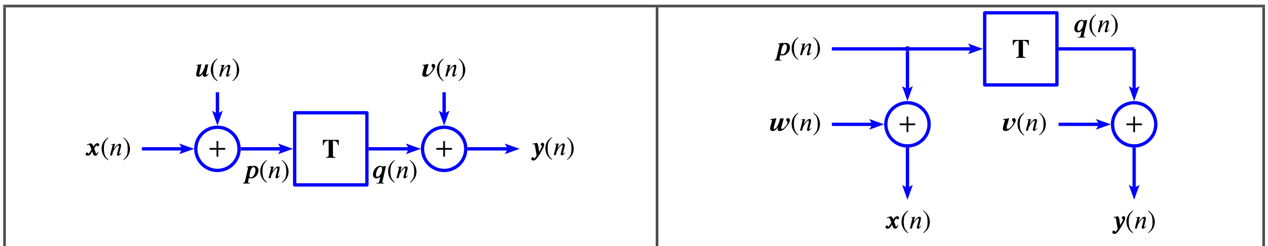
[1] quote: Popper (1962), page 231, Popper (1963) page 313
image: `https://en.wikipedia.org/wiki/File:Karl_Popper.jpg`, "no known copyright restrictions"
[2] Shin and Hammond (2008) page 292
[3] Cobb (1988) page 1 ⟨FRF "measurement"⟩
[4] Leuridan et al. (1986)911"Stepped Sine Testing", Cobb (1988) page 1 ⟨Chapter 1—Introduction⟩, Ewins



(A) **communications additive noise model**
The "input signal" is $x(n)$.

(B) **measurement additive noise model**
The "input signal" is $p(n)$.

In each model, $x(n)$ and $y(n)$ are "known", and $u(n)$, $v(n)$, and $w(n)$ are *not*.
In definition, the two models are **equivalent** under the relation $u(n) = -w(n)$.
In practice, they are **different**:

- in (A), x and u would be typically *uncorrelated*;
- in (B), x and $w = -u$ are very much *correlated* (x is a function of u).

Figure 11.1: Additive noise systems with *linear*/**non-linear** operator **T**

And there is another problem here—we don't always have control over the input signal. Examples of this include earthquake and volcanic activity analysis.

An alternative to the sine-sweep input is *random sequence* input. All the techniques that follow in this chapter are of this type. A problem with using random sequences directly for estimating $\tilde{H}(\omega)$ is that the estimate $\hat{H}(\omega)$ is itself also random. This is not what we want. We want an estimate that we can actually write down on paper or at least plot on paper.

A solution to this is to not use the random sequences directly to estimate $\tilde{H}(\omega)$, but instead to first use the *expectation* operator E (Definition **??** page **??**). The expectation operator takes a quantity X that is inherently "random" (with some probability distribution $p(x)$) and turns it into a deterministic "constant" EX.

The operator E is also used by the spectral density functions $\tilde{S}_{xx}(\omega)$ and $\tilde{S}_{xy}(\omega)$ (Definition **??** page **??**). And $\tilde{S}_{xx}(\omega)$ and $\tilde{S}_{xy}(\omega)$ are what are typically used to calculate an estimate $\hat{H}(\omega)$.

11.2 Additive noise system models

Consider the additive noise systems illustrated in Figure 11.1 (page 104).

- The illustration on the left is suitable for modeling a communications system where x is the transmitted signal, y is the received signal, u and v are thermal noise, and the "transfer function" **H** is the communications channel (air, water, wires, etc.) that one wishes to estimate.
- The illustration on the right is suitable for modeling a testing system where p is an input test signal (from an industrial shaker or from a naturally occurring signal originating from geophysical activity), w is measurement noise, x is the measured input contaminated by noise, and **H** is the device under test (a piece of equipment, a building, or the entire earth).

Note that the two models are an equivalent system **S** under the relation $u = -w$. But although one might expect such a sign difference to wreak mathematical havoc in resulting equations, this is

(1986) pages 125–140 ⟨3.7 Use of different excitation types⟩

simply not the case here because

$$\tilde{\mathsf{S}}_{ww} = \tilde{\mathsf{F}}\mathsf{E}[\mathsf{w}(m)\mathsf{w}*(0)] = \tilde{\mathsf{F}}\mathsf{E}\big[(-\mathsf{u}(m))\big(-\mathsf{u}^*(0)\big)\big] = \tilde{\mathsf{F}}\mathsf{E}\big[(\mathsf{u}(m))\big(\mathsf{u}^*(0)\big)\big] = \tilde{\mathsf{S}}_{uu}$$

So the sign difference is not that big of a difference after all. But there are some key differences in practice:

- In the communications model (on the left), the "input signal" is $\mathsf{x}(n)$ and the frequency-domain input *signal-to-noise ratio* (*SNR*) is $\tilde{\mathsf{S}}_{xx}(\omega)/\tilde{\mathsf{S}}_{uu}(\omega)$. In the measurement model (on the right), the "input signal" is $\mathsf{p}(n)$ and the frequency-domain input *signal-to-noise ratio* (*SNR*) is $\tilde{\mathsf{S}}_{pp}(\omega)/\tilde{\mathsf{S}}_{ww}(\omega) = \tilde{\mathsf{S}}_{pp}(\omega)/\tilde{\mathsf{S}}_{uu}(\omega)$.
- On the left, x and u would be typically *uncorrelated*; on the right, x and $\mathsf{w} = -\mathsf{u}$ are very much *correlated* (x is a function of u).

11.3 Transfer function estimate definitions and interpretation

As a first attempt at estimating the transfer function $\mathbf{H}$ of $\mathbf{S}$, or at least the magnitude squared of $\mathbf{H}$, we might assume $\mathbf{H}$ to be *LTI*, take a cue from the relation $\tilde{\mathsf{S}}_{yy} = \tilde{\mathsf{S}}_{xx}|\tilde{\mathsf{H}}|^2$ of Corollary **??** (page **??**), and arrive at a function called "*transmissibility*" (next definition).

Definition 11.1. [5] *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathsf{x}(n)$ *and output* $\mathsf{y}(n)$.

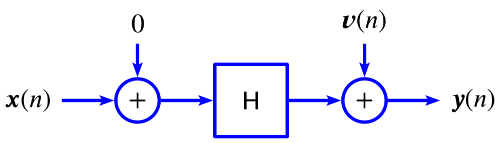
DEF: ***transmissibility*** $\tilde{\mathsf{T}}_{xy}(\omega)$ *is defined as* $$\tilde{\mathsf{T}}_{xy}(\omega) \triangleq \sqrt{\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}}$$

Transmissibility is in essence the ratio of "*spectral power*" (Remark **??** page **??**) output to *spectral power* input. Note that it is a real-valued function (because $\tilde{\mathsf{S}}_{xx}$ and $\tilde{\mathsf{S}}_{yy}$ are real-valued). We might suspect that we could attain better estimates of $\mathbf{H}$ by allowing the estimates to be complex-valued. And in fact, all the remaining estimates in this section are in general complex-valued.

And so to start (again), and in the very special (a.k.a unrealistic) case of $\mathbf{S}$ having *zero measurement noise* (*zero measurement error*) ($\mathsf{v} = \mathsf{u} = \mathsf{w} = 0$), $\mathsf{h}(n)$ being *linear time invariant* (*LTI*), and input $\mathsf{x}(n)$ being *wide sense stationary*...then we can determine (a.k.a "identify") $\mathsf{h}(n)$ or $\tilde{\mathsf{H}}(\omega)$ exactly by $\tilde{\mathsf{H}}(\omega) = \tilde{\mathsf{S}}_{yx}(\omega)/\tilde{\mathsf{S}}_{xx}(\omega)$ (Corollary **??** page **??**).

However, in practical situations, there is measurement noise/error. Examples may include "road noise" from a test being performed in a moving vehicle or *quantization noise* from an *analog-to-digital converter* (*ADC*).

If the measurement error is at the output only (and under the assumptions of *LTI* and *WSS*) then $\hat{\mathsf{H}}_1$ (next definition) is the ideal estimator in the sense that $\hat{\mathsf{H}}_1 = \tilde{\mathsf{H}}$ (Corollary 11.4 page 123).



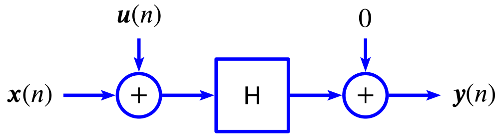
Definition 11.2. [6] *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathsf{x}(n)$ *and output* $\mathsf{y}(n)$.

[5] Bendat and Piersol (2010) page 469 ⟨$|H(f)| = \big[G_{yy}(f)/G_{xx}(f)\big]^{1/2}$⟩, Yan and Ren (2012) page 204 ⟨(1) $\big[G_{YY}(s)\big] = [H(s)]\big[G_{FF}(s)\big]\big[H^*(s)\big]^T$⟩, Goldman (1999) page 179 ⟨Transmissibility …$H'_{ab} = G_{bb}/G_{aa}$ (note: differs by $\sqrt{\cdot}$ from Bendat and Piersol⟩, Zhang et al. (2016), Zhou and Wahab (2018) page 824, https://link.springer.com/chapter/10.1007/978-3-319-54109-9_4

[6] Bendat and Piersol (1993) pages 106–109 ⟨5.1.1 Optimality of Calculations⟩, Bendat and Piersol (2010) page 185 ⟨$H_1(f) = G_{xy}(f)/G_{xx}(f)$ (6.37)⟩, Shin and Hammond (2008) page 293 ⟨ $H_1(f) = \tilde{\mathsf{S}}_{xy}(f)/\tilde{\mathsf{S}}_{xx}(f)$ (9.63); which dif-

DEF *The **Least Squares transfer function estimate** $\hat{\mathsf{H}}_1(\omega)$ of **S** is defined as* $\hat{\mathsf{H}}_1(\omega) \triangleq \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}$

The estimator $\hat{\mathsf{H}}_1$ is a good start. However in the early 1980s, L. D. Mitchell pointed out that in the presence of input noise, $\hat{\mathsf{H}}_1$ is far from ideal in that it is *biased* with respect to $\tilde{\mathsf{H}}$; in fact, $\hat{\mathsf{H}}_1$ *under estimates* $\tilde{\mathsf{H}}$ (Corollary 11.4 page 123). Mitchell proposed a new estimator $\hat{\mathsf{H}}_2$ (next definition).



This estimator has the special property that when there is input noise but no output noise (and under *LTI*, *WSS*, and *uncorrelated* assumptions), then it is ideal in the sense that $\hat{\mathsf{H}}_2(\omega) = \tilde{\mathsf{H}}(\omega)$ (Corollary 11.4 page 123).
Note also that in the case of both no input and no output noise, then $\hat{\mathsf{H}}_1 = \hat{\mathsf{H}}_2$ (Corollary **??** page **??**).

Definition 11.3. [7] *Let* **S** *be a* SYSTEM *with input* $\mathsf{x}(n)$ *and output* $\mathsf{y}(n)$.

DEF *The **Inverse Method transfer function estimate** $\hat{\mathsf{H}}_2(\omega)$ of **S** is defined as* $\hat{\mathsf{H}}_2(\omega) \triangleq \dfrac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}$

Mitchell's $\hat{\mathsf{H}}_2$ contribution "generated a flurry of activity"[8] and soon more $\tilde{\mathsf{H}}$ estimators appeared. So far we have

- $\hat{\mathsf{H}}_1$ which is ideal when there is no input noise but *under estimate*s $\tilde{\mathsf{H}}$ when there is (Corollary 11.4 page 123)
- $\hat{\mathsf{H}}_2$ which is ideal when there is no output noise but *over estimate*s $\tilde{\mathsf{H}}$ when there is (Corollary 11.4 page 123).

But what about estimators for when there is noise on both input and output? Armed with two estimators that between them account for both input and output noise, an "ad hoc" solution might be to somehow take mean values of $\hat{\mathsf{H}}_1$ and $\hat{\mathsf{H}}_2$ to induce new estimators—this approach summarizes the next three definitions. An arguably more mature approach is to find estimators that are optimal with respect to least squares measures—and this approach summarizes Definition 11.9 – Definition 11.7 (page 109).

Definition 11.4. *Let* **S** *be a* SYSTEM *with input* $\mathsf{x}(n)$ *and output* $\mathsf{y}(n)$.

DEF *The **Arithmetic Mean transfer function estimate** $\hat{\mathsf{H}}_{am}(\omega)$ of **S** is defined as*

$$\hat{\mathsf{H}}_{am}(\omega) \triangleq \frac{\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|^2 + \tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{yy}(\omega)}{2\tilde{\mathsf{S}}_{xx}(\omega)\tilde{\mathsf{S}}_{xy}(\omega)}$$

fers from Definition 11.2, but see APPENDIX **??** page **??** ⟩, Bendat (1978)cited by Cobb(1988)—variance estimate for $\hat{\mathsf{H}}_1$, Allemang et al. (1979) ⟨cited by Shin(2008)⟩, Leuridan et al. (1986) page 910 ⟨*Least Squares Technique*; (8) $[G_{xx}](H) = [G_{xy}]$⟩, Abom (1986)cited by Cobb(1988)—variance estimate for $\hat{\mathsf{H}}_1$, Allemang et al. (1987) pages 54–55 ⟨5.3.1 H_1 Technique; $[H] = [G_{XF}][G_{FF}]^{-1}$ (11)⟩, Cobb (1988) page 2 ⟨$^1\hat{H}(f) = \hat{G}_{yx}(f)/\hat{G}_{xx}(f)$ (1)⟩, Goyder (1984) page 438 ⟨$H(i\omega) = S_{qp}/S_{pp}$ (3)⟩, Pintelon and Schoukens (2012) page 233 ⟨$\hat{G}(\Omega_k) = S_{yu}(j\omega_k)S_{uu}^{-1}(j\omega_k)$ (7-30)⟩, White et al. (2006) page 678 ⟨$H_1(f) = \hat{S}_{x_m y_m}(f)/\hat{S}_{x_m x_m}(f)$ (1) which differs by conjugate, references Bendat and Piersol⟩,

[7] Shin and Hammond (2008) page 293 ⟨$H_2(f) = \tilde{\mathsf{S}}_{yy}(f)/\tilde{\mathsf{S}}_{yx}(f)$ (9.65); which differs from Definition 11.3, but see APPENDIX **??** page **??** ⟩, Bendat and Piersol (2010) page 186 ⟨$H_2(f) = G_{yy}(f)/G_{yx}(f)$ (6.42)⟩, Mitchell (1980) ⟨cited by Cobb(1988)⟩, Mitchell (1982) page 278 ⟨"Define what will be called an inverse method for calculation of a FRF as…"; $H_2(f) = G_{yy}/G_{yx}$ (6); Note this differs with Definition 11.3 by a conjugate, but note that Mitchell seems to follow Bendat (see his [3] and [4]), which would explain this difference (APPENDIX **??** page **??**)⟩, Cobb (1988) page 3 ⟨$^2\hat{H}(f) = \hat{G}_{yy}(f)/\hat{G}_{xy}(f)$ (1)⟩, White et al. (2006) page 678 ⟨$H_2(f) = \hat{S}_{y_m y_m}(f)/\hat{S}_{y_m x_m}(f)$ (2) which differs by conjugate, references Bendat and Piersol⟩

[8] Cobb (1988) page 3

Proposition 11.1. [9] *Let* **S** *be a* SYSTEM *with input* x(n) *and output* y(n).

PRP
$$\hat{\mathsf{H}}_{\mathsf{am}}(\omega) = \frac{\hat{\mathsf{H}}_1(\omega)+\hat{\mathsf{H}}_2(\omega)}{2} \qquad \text{(\textbf{arithmetic mean} of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2)$$

✎PROOF:

$$\begin{aligned}
\hat{\mathsf{H}}_{\mathsf{am}}(\omega) &\triangleq \frac{\left|\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)\right|^2+\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{2\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)} && \text{by definition of } \hat{\mathsf{H}}_{\mathsf{am}} && \text{(Definition 11.4 page 106)}\\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)\tilde{\mathsf{S}}^*_{\mathsf{xy}}(\omega)}{2\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)}+\frac{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{2\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)} = \frac{\frac{\tilde{\mathsf{S}}^*_{\mathsf{xy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}+\frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)}}{2}\\
&= \frac{\hat{\mathsf{H}}_1(\omega)+\hat{\mathsf{H}}_2(\omega)}{2} && \text{by definitions of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2 && \text{(Definition 11.2 page 105, Definition 11.3 page 106)}
\end{aligned}$$

✏

Definition 11.5. *Let* **S** *be a* SYSTEM *with input* x(n) *and output* y(n).

DEF
The ***Geometric mean transfer function estimate*** $\hat{\mathsf{H}}_{\mathsf{gm}}(\omega)$ *of* **S** *is defined as*
$$\hat{\mathsf{H}}_{\mathsf{gm}}(\omega) \triangleq \frac{\tilde{\mathsf{S}}^*_{\mathsf{xy}}(\omega)}{\left|\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)\right|}\sqrt{\frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}}$$

Proposition 11.2. [10] *Let* **S** *be a* SYSTEM *with input* x(n) *and output* y(n).

PRP
$$\pm\hat{\mathsf{H}}_{\mathsf{gm}}(\omega) = \sqrt{\hat{\mathsf{H}}_1(\omega)\hat{\mathsf{H}}_2(\omega)} \qquad \text{(\textbf{geometric mean} of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2)$$

✎PROOF:

$$\begin{aligned}
\pm\hat{\mathsf{H}}_{\mathsf{gm}}(\omega) &\triangleq \pm\frac{\tilde{\mathsf{S}}^*_{\mathsf{xy}}(\omega)}{\left|\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)\right|}\sqrt{\frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}} && \text{by definition of } \hat{\mathsf{H}}_{\mathsf{gm}} && \text{(Definition 11.5 page 107)}\\
&= \sqrt{\frac{\left[\tilde{\mathsf{S}}^*_{\mathsf{xy}}(\omega)\right]^2}{\left|\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)\right|^2}\frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}} = \sqrt{\frac{\tilde{\mathsf{S}}^*_{\mathsf{xy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)}\frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}} = \sqrt{\frac{\tilde{\mathsf{S}}^*_{\mathsf{xy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}\frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)}}\\
&= \sqrt{\hat{\mathsf{H}}_1(\omega)\hat{\mathsf{H}}_2(\omega)} && \text{by definitions of } \hat{\mathsf{H}}_1 \text{ and } \hat{\mathsf{H}}_2 && \text{(Definition 11.2 page 105, Definition 11.3 page 106)}\\
&= \textbf{Geometric mean} \text{ of } \hat{\mathsf{H}}_1(\omega) \text{ and } \hat{\mathsf{H}}_2(\omega)
\end{aligned}$$

Note that for a complex number $z \triangleq |z|e^{i\phi}$, $\sqrt{z}$ has two solutions:[11]
$$\sqrt{z} = \sqrt{|z|e^{i\phi}} = \{z_1, z_2\} = \left\{\sqrt{|z|}e^{i(\phi/2)}, \sqrt{|z|}e^{i(\phi/2+\pi)}\right\} = \pm\sqrt{|z|}e^{i(\phi/2)}$$
because $z_1^2 = z$ and $z_2^2 = z$. ✏

Note that the *geometric mean estimator* (Definition 11.5 page 107) and *transmissibility* (Definition 11.1 page 105) are closely related (next).

---

[9] Mitchell (1982) page 279 ⟨"Frequency Response Calculation: The Average Method"⟩, Zheng et al. (2002) page 918 ⟨"1.3 Arithmetic Mean Estimator H_3"⟩

[10] Zheng et al. (2002) page 918 ⟨"1.4 Geometric Mean Estimator H_4"⟩

[11] Many many thanks to Ben Cleveland for his help with this!!!

Proposition 11.3. *Let $\phi(\omega)$ be the* PHASE *of* $\tilde{S}_{xy}(\omega)$ *such that* $\tilde{S}_{xy}(\omega) \triangleq \left|\tilde{S}_{xy}(\omega)\right| e^{-i\phi(\omega)}$

PRP
$$\hat{H}_{gm}(\omega) = \tilde{T}_{xy}(\omega)\, e^{-i\phi(\omega)} \quad \left(\begin{array}{rcll} \left|\hat{H}_{gm}(\omega)\right| &=& \tilde{T}_{xy}(\omega) & \text{is the MAGNITUDE of } \hat{H}_{gm}(\omega) \quad \text{and} \\ \angle\hat{H}_{gm}(\omega) &=& -\angle\tilde{S}_{xy}(\omega) & \text{is the PHASE of } \hat{H}_{gm}(\omega) \end{array}\right)$$

✎PROOF: Let $\phi(\omega)$ be the *phase* of

$$\begin{aligned}
\hat{H}_{gm}(\omega) &\triangleq \sqrt{\hat{H}_1(\omega)\hat{H}_2(\omega)} && \text{by definition of } \hat{H}_{gm} && \text{(Definition 11.5 page 107)} \\
&\triangleq \sqrt{\frac{\tilde{S}^*_{xy}(\omega)}{\tilde{S}_{xx}(\omega)}\frac{\tilde{S}_{yy}(\omega)}{\tilde{S}_{xy}(\omega)}} && \text{by definitions of } \hat{H}_1 \text{ and } \hat{H}_2 && \text{(Definition 11.2 page 105, Definition 11.3 page 106)} \\
&= \sqrt{\frac{\tilde{S}_{yy}(\omega)}{\tilde{S}_{xx}(\omega)}\frac{\tilde{S}^*_{xy}(\omega)}{\tilde{S}_{xy}(\omega)}} \\
&= \tilde{T}_{xy}(\omega)\sqrt{\frac{\tilde{S}^*_{xy}(\omega)}{\tilde{S}_{xy}(\omega)}} && \text{by definition of } \tilde{T}_{xy} && \text{(Definition 11.1 page 105)} \\
&= \tilde{T}_{xy}(\omega)\sqrt{\frac{\left|\tilde{S}_{xy}(\omega)\right|e^{-i\phi(\omega)}}{\left|\tilde{S}_{xy}(\omega)\right|e^{i\phi(\omega)}}} && \text{where } \tilde{S}_{xy}(\omega) \triangleq \left|\tilde{S}_{xy}(\omega)\right|e^{-i\phi(\omega)} \\
&= \tilde{T}_{xy}(\omega)\sqrt{e^{-i2\phi(\omega)}} \\
&= \tilde{T}_{xy}(\omega)\, e^{-i\phi(\omega)}
\end{aligned}$$

✏

Remark 11.1. *Transmissibility* $\tilde{T}_{xy}$ is a kind of "black sheep" of the system identification function family. All the other members of this family ($\hat{H}_1$, $\hat{H}_2$, $\hat{H}_v$, $\hat{H}_s$) are *complex-valued*, but $\tilde{T}_{xy}$ is only *real-valued*—a seemingly ordinary Joe born into a super-hero family. But Proposition 11.3 suggests that $\tilde{T}_{xy}$ is not simply a "black sheep", but rather a "dark horse" with abilities that can easily be unleashed by slight redefinition. In particular, Proposition 11.3 demonstrates that $\tilde{T}_{xy}$ is the *magnitude* of the geometric mean of $\hat{H}_1$ and $\hat{H}_2$. We can thus justifiably define a **complex transmissibility** function as $\hat{H}_{gm}$…and the magnitude of this *complex transmissibility* function is the *ordinary transmissibility* function of Definition 11.1 (page 105).

REM
$$\textbf{complex transmissibility } \tilde{T}'_{xy}(\omega) \triangleq \hat{H}_{gm}(\omega)$$

Definition 11.6. *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $x(n)$ *and output* $y(n)$.

DEF
The ***Harmonic mean transfer function estimate*** $\hat{H}_{hm}(\omega)$ *of* $\mathbf{S}$ *is defined as*
$$\hat{H}_{hm}(\omega) \triangleq \frac{2\tilde{S}_{yy}(\omega)\tilde{S}^*_{xy}(\omega)}{\tilde{S}_{xx}(\omega)\tilde{S}_{yy}(\omega) + \left|\tilde{S}_{xy}(\omega)\right|^2}$$

Proposition 11.4. [12] *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $x(n)$ *and output* $y(n)$.

PRP
$$\hat{H}_{hm}(\omega) = \frac{2}{\frac{1}{\hat{H}_1(\omega)} + \frac{1}{\hat{H}_2(\omega)}} \qquad \textit{(Harmonic mean of } \hat{H}_1 \textit{ and } \hat{H}_2\textit{)}$$

[12] Carne and Dohrmann (2006) $\langle H_C = [H_A^{-1} + H_B^{-1}]^{-1}\rangle$

✎PROOF:

$$\begin{aligned}
\hat{H}_{hm}(\omega) &\triangleq \frac{2\tilde{S}_{yy}(\omega)\tilde{S}^*_{xy}(\omega)}{\tilde{S}_{xx}(\omega)\tilde{S}_{yy}(\omega)+\left|\tilde{S}_{xy}(\omega)\right|^2} && \text{by definition of } \hat{H}_{hm} && \text{(Definition 11.6 page 108)}\\
&= \frac{2}{\dfrac{\tilde{S}_{xx}(\omega)\tilde{S}_{yy}(\omega)+\left|\tilde{S}_{xy}(\omega)\right|^2}{\tilde{S}_{yy}(\omega)\tilde{S}^*_{xy}(\omega)}} = \frac{2}{\dfrac{\tilde{S}_{xx}(\omega)}{\tilde{S}^*_{xy}(\omega)}+\dfrac{\tilde{S}_{xy}(\omega)}{\tilde{S}_{yy}(\omega)}}\\
&= \frac{2}{\dfrac{1}{\hat{H}_1(\omega)}+\dfrac{1}{\hat{H}_2(\omega)}} && \text{by definitions of } \hat{H}_1 \text{ and } \hat{H}_2 && \text{(Definition 11.2 page 105, Definition 11.3 page 106)}\\
&= \textbf{Harmonic mean of } \hat{H}_1(\omega) \text{ and } \hat{H}_2(\omega)
\end{aligned}$$

✏

A bit of review reveals $\hat{H}_1$ at the low end of the estimation problem, $\hat{H}_2$ at the high end, and $\hat{H}_{hm}$, $\hat{H}_{gm}$, and $\hat{H}_{am}$ somewhere between. But these three "between" estimates are not shown to be optimal in any sense—they are just conceptually interesting. What we might really like is a family of estimators that

- include $\hat{H}_1$ and $\hat{H}_2$ as limiting cases
- include the between cases
- are optimal in some sense

The estimator $\hat{H}_\kappa(\omega;\kappa)$ is one such estimator (next definition) that

- has $\hat{H}_1$ and $\hat{H}_2$ as limiting cases (Theorem 11.1 page 111),
- is optimal in the least squares sense (Theorem 11.6 page 124), and
- allows for a system designer to specify an output-input spectral noise ratio $\kappa(\omega)$ that can vary with frequency ω.

Moreover, $\hat{H}_\kappa(\omega)$ includes some special cases:

- In the case of constant κ, $\hat{H}_\kappa$ simplifies to the *Scaling transfer function estimate* $\hat{H}_s$ (Definition 11.8 page 109).
- In the case of $\kappa = 1$, $\hat{H}_\kappa$ and $\hat{H}_s$ simplify to the *Total least squares transfer function estimate* $\hat{H}_v$ (Definition 11.9 page 110).

Definition 11.7. [13] *Let* **S** *be a* SYSTEM *with input* $x(n)$ *and output* $y(n)$.

DEF: *The* ***transfer function estimate*** $\hat{H}_\kappa(\omega;\kappa)$ *with* ***scaling function*** $\kappa(\omega)$ *is defined as*

$$\hat{H}_\kappa(\omega;\kappa) \triangleq \frac{\tilde{S}_{yy}(\omega)-\kappa(\omega)\tilde{S}_{xx}(\omega)+\sqrt{\left[\tilde{S}_{yy}(\omega)-\kappa(\omega)\tilde{S}_{xx}(\omega)\right]^2+4\kappa(\omega)\left|\tilde{S}_{xy}(\omega)\right|^2}}{2\tilde{S}_{xy}(\omega)}$$

Definition 11.8. [14] *Let* **S** *be a* SYSTEM *with input* $x(n)$ *and output* $y(n)$.

DEF: *The* ***Scaling transfer function estimate*** $\hat{H}_s(\omega;s)$ *of* **S** *with* ***scaling parameter*** $s \in [0:\infty)$ *is defined as* $\hat{H}_s(\omega;s) \triangleq \hat{H}_\kappa(\omega;\kappa)$ *with* $\kappa(\omega) \triangleq s^2$

[13] White et al. (2006) page 679 ⟨(6)⟩, Shin and Hammond (2008) page 293 ⟨(9.67)⟩

[14] Shin and Hammond (2008) page 293 ⟨(9.67) with $\kappa(\omega) = s^2$⟩, White et al. (2006) page 679 ⟨(6) with $\kappa(\omega) = s^2$⟩, Leclere et al. (2014) ⟨(10) $\kappa(f) = 1/s^2$ and x and y swapped⟩, Wicks and Vold (1986) page 898 ⟨has additional s in denominator⟩, Zheng et al. (2002) page 918 ⟨(10), seems to differ⟩

Definition 11.9. [15] *Let* $\mathbf{S}$ *be a* SYSTEM *with input* $\mathsf{x}(n)$ *and output* $\mathsf{y}(n)$.

DEF *The **Total Least Squares transfer function estimate*** $\hat{\mathsf{H}}_{\mathsf{v}}(\omega)$ *of* $\mathbf{S}$ *is defined as*

$$\hat{\mathsf{H}}_{\mathsf{v}}(\omega) \triangleq \hat{\mathsf{H}}_{\kappa}(\omega;\kappa) \qquad \text{with } \kappa(\omega) = 1$$

The previous estimators all assumed two signals: an input $\mathsf{x}(n)$ and an output $\mathsf{y}(n)$. However, in many practical systems, there is a third signal that is "driving" the system. In 1984 Goyder proposed an estimator (next definition) that is based on three signals.

Definition 11.10 (Three channel estimate)**.** [16] *Let* $\mathbf{S}$ *be a system with input* $\mathsf{x}(n)$, *output* $\mathsf{y}(n)$, *and a driver* $\mathsf{p}(n)$.

DEF *The **transfer function estimate*** $\hat{\mathsf{H}}_{\mathsf{c}}(\omega)$ *is defined as*

$$\hat{\mathsf{H}}_{\mathsf{c}}(\omega) \triangleq \frac{\tilde{\mathsf{S}}_{\mathsf{py}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{px}}(\omega)}$$

11.4 Estimator relationships

Lemma 11.1.

LEM

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] = \frac{\tilde{\mathsf{S}}_{\mathsf{xx}}(p\tilde{\mathsf{S}}_{\mathsf{xx}}-\tilde{\mathsf{S}}_{\mathsf{yy}})+2|\tilde{\mathsf{S}}_{\mathsf{xy}}|^2-\tilde{\mathsf{S}}_{\mathsf{xx}}\sqrt{(p\tilde{\mathsf{S}}_{\mathsf{xx}}-\tilde{\mathsf{S}}_{\mathsf{yy}})^2+4p|\tilde{\mathsf{S}}_{\mathsf{xy}}|^2}}{\sqrt{(p\tilde{\mathsf{S}}_{\mathsf{xx}}-\tilde{\mathsf{S}}_{\mathsf{yy}})^2+4p|\tilde{\mathsf{S}}_{\mathsf{xy}}|^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] = \frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(p\tilde{\mathsf{S}}_{\mathsf{yy}}-\tilde{\mathsf{S}}_{\mathsf{xx}})+2|\tilde{\mathsf{S}}_{\mathsf{xy}}|^2+\tilde{\mathsf{S}}_{\mathsf{yy}}\sqrt{(p\tilde{\mathsf{S}}_{\mathsf{yy}}-\tilde{\mathsf{S}}_{\mathsf{xx}})^2+4p|\tilde{\mathsf{S}}_{\mathsf{xy}}|^2}}{\sqrt{(p\tilde{\mathsf{S}}_{\mathsf{yy}}-\tilde{\mathsf{S}}_{\mathsf{xx}})^2+4p|\tilde{\mathsf{S}}_{\mathsf{xy}}|^2}}$$

✎PROOF:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\left[\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] &= -\tilde{\mathsf{S}}_{\mathsf{xx}} + \frac{-2\tilde{\mathsf{S}}_{\mathsf{xx}}\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right) + 4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}{2\sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}} \\
&= \frac{4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2 - 2\tilde{\mathsf{S}}_{\mathsf{xx}}\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right) - 2\tilde{\mathsf{S}}_{\mathsf{xx}}\sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{2\sqrt{\left(\tilde{\mathsf{S}}_{\mathsf{yy}} - p\tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}} \\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{xx}}\left(p\tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{yy}}\right) + 2\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2 - \tilde{\mathsf{S}}_{\mathsf{xx}}\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{yy}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{yy}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\left[p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}} + \sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}\right] &= \tilde{\mathsf{S}}_{\mathsf{yy}} + \frac{2\tilde{\mathsf{S}}_{\mathsf{yy}}\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right) + 4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}{2\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}} \\
&= \frac{4\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2 + 2\tilde{\mathsf{S}}_{\mathsf{yy}}\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right) + 2\tilde{\mathsf{S}}_{\mathsf{yy}}\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}{2\sqrt{\left(p\tilde{\mathsf{S}}_{\mathsf{yy}} - \tilde{\mathsf{S}}_{\mathsf{xx}}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{\mathsf{xy}}\right|^2}}
\end{aligned}$$

[15] White et al. (2006) page 679 ⟨(6)⟩, Shin and Hammond (2008) page 294 ⟨(9.69)⟩

[16] Shin and Hammond (2008) page 297 ⟨$H_3(f) = S_{ry}(f)/S_{rx}(f)$ (9.78)⟩, Cobb (1988) page 4 ⟨${}^{c}\hat{H}(f) = \hat{G}_{ys}(f)/\hat{G}_{xs}(f)$ (1.4)⟩, Goyder (1984) page 440 ⟨$H(i\omega) = S_{qz}/S_{pz}$ (5)⟩, Cobb and Mitchell (1990) page 450 ⟨(1)⟩, Pintelon and Schoukens (2012) page 241 ⟨$\hat{G}(\Omega_k) = \hat{G}_{ry}(\Omega_k)\hat{G}_{ru}^{-1}(\Omega_k)$ (7-49)⟩

$$= \frac{\tilde{S}_{yy}\left(p\tilde{S}_{yy} - \tilde{S}_{xx}\right) + 2\left|\tilde{S}_{xy}\right|^2 + \tilde{S}_{yy}\sqrt{\left(p\tilde{S}_{yy} - \tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}{\sqrt{\left(p\tilde{S}_{yy} - \tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2}}$$

✑

Lemma 11.2.

LEM

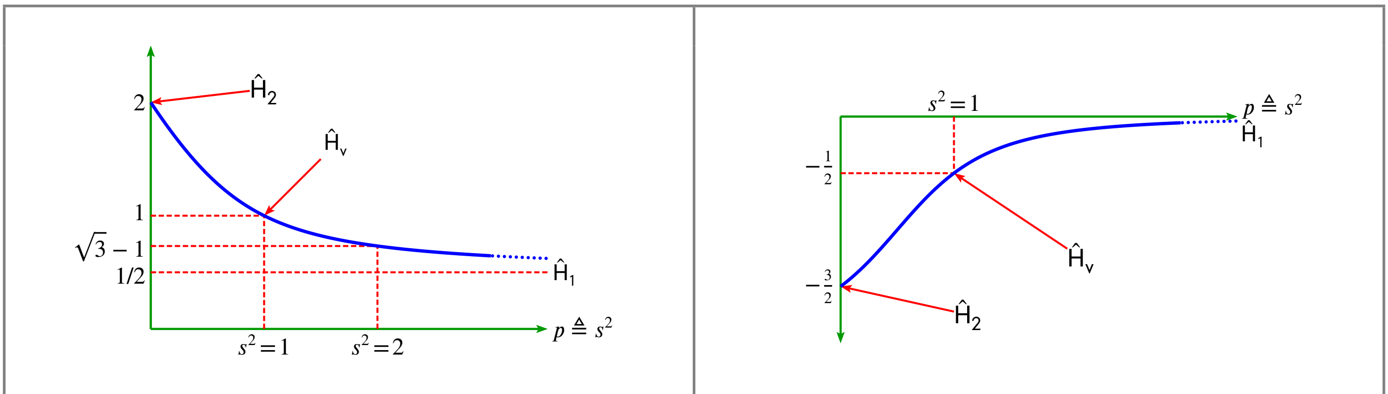
$$\begin{aligned}
\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{\left(\tilde{S}_{yy} - p\tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2} &\geq 0 \\
p\tilde{S}_{yy} - \tilde{S}_{xx} + \sqrt{\left(p\tilde{S}_{yy} - \tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2} &\geq 0
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
&\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{\left(\tilde{S}_{yy} - p\tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2} && \geq 0 \\
\iff &\sqrt{\left(p\tilde{S}_{xx} - \tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2} && \geq p\tilde{S}_{xx} - \tilde{S}_{yy} \\
\iff &\left(p\tilde{S}_{xx} - \tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2 && \geq \left(p\tilde{S}_{xx} - \tilde{S}_{yy}\right)^2 \\
\iff &4p\left|\tilde{S}_{xy}\right|^2 && \geq 0 \\
\iff &\left|\tilde{S}_{xy}\right| && \geq 0
\end{aligned}$$

$$\begin{aligned}
&p\tilde{S}_{yy} - \tilde{S}_{xx} + \sqrt{\left(p\tilde{S}_{yy} - \tilde{S}_{xx}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2} && \geq 0 \\
\iff &\sqrt{\left(\tilde{S}_{xx} - p\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2} && \geq \tilde{S}_{xx} - p\tilde{S}_{yy} \\
\iff &\left(\tilde{S}_{xx} - p\tilde{S}_{yy}\right)^2 + 4p\left|\tilde{S}_{xy}\right|^2 && \geq \left(\tilde{S}_{xx} - p\tilde{S}_{yy}\right)^2 \\
\iff &4p\left|\tilde{S}_{xy}\right|^2 && \geq 0 \\
\iff &\left|\tilde{S}_{xy}\right| && \geq 0
\end{aligned}$$

✑



$\hat{H}_s$ as a function of $p \triangleq s^2$ | $\frac{d}{dp}\hat{H}_s$ where $p \triangleq s^2$

Figure 11.2: $\hat{H}_s$ with $\tilde{S}_{xx} = \tilde{S}_{yy} = 1$ and $\tilde{S}_{xy} = \frac{1}{2}$

Theorem 11.1. *Let $\hat{H}_s$ be defined as in Definition 11.8 (page 109).*

THM

$$\begin{array}{lcll}
\{s_1 < s_2\} & \Longrightarrow & \left|\hat{H}_s(\omega; s_2)\right| \leq \hat{H}_s(\omega; s_1) & (\hat{H}_s \text{ is } \textbf{monotonically decreasing} \text{ in } s) \\
\left|\hat{H}_1(\omega)\right| & \leq \left|\hat{H}_s(\omega; s)\right| \leq & \left|\hat{H}_2(\omega)\right| & \\
\hat{H}_s(\omega; s)\big|_{s=0} & = & \hat{H}_2(\omega) & \\
\hat{H}_s(\omega; s)\big|_{s=1} & = & \hat{H}_v(\omega) & \\
\lim_{s\to\infty} \hat{H}_s(\omega; s) & = & \hat{H}_1(\omega) &
\end{array}$$

✎PROOF: I. Proofs for equalities:

$$\begin{aligned}
\hat{\mathsf{H}}_s(\omega;s)\big|_{s=0} &\triangleq \left.\frac{\tilde{\mathsf{S}}_{yy} - s^2\tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - s^2\tilde{\mathsf{S}}_{xx}\right]^2 + 4s^2\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}}\right|_{s=0} && \text{by def. of } \hat{\mathsf{H}}_s && \text{(Definition 11.8 page 109)}\\
&= \frac{\tilde{\mathsf{S}}_{yy} - 0 + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - 0\right]^2 + 0}}{2\tilde{\mathsf{S}}_{xy}} = \frac{\tilde{\mathsf{S}}_{yy}}{\tilde{\mathsf{S}}_{xy}}\\
&\triangleq \hat{\mathsf{H}}_2 && \text{by def. of } \hat{\mathsf{H}}_2 && \text{(Definition 11.3 page 106)}\\
\hat{\mathsf{H}}_s(\omega;s)\big|_{s=1} &\triangleq \left.\frac{\tilde{\mathsf{S}}_{yy} - s^2\tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - s^2\tilde{\mathsf{S}}_{xx}\right]^2 + 4s^2\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}}\right|_{s=1} && \text{by def. of } \hat{\mathsf{H}}_s && \text{(Definition 11.8 page 109)}\\
&= \frac{\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx}\right]^2 + 4\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}}\\
&\triangleq \hat{\mathsf{H}}_v && \text{by def. of } \hat{\mathsf{H}}_v && \text{(Definition 11.9 page 110)}\\
\lim_{s\to\infty}\hat{\mathsf{H}}_s(\omega;s) &\triangleq \lim_{s\to\infty}\frac{\tilde{\mathsf{S}}_{yy} - s^2\tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - s^2\tilde{\mathsf{S}}_{xx}\right]^2 + 4s^2\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}} && \text{by def. of } \hat{\mathsf{H}}_s && \text{(Definition 11.8 page 109)}\\
&\triangleq \lim_{p\to0}\frac{\tilde{\mathsf{S}}_{yy} - {}^1\!/\!_p\tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - {}^1\!/\!_p\tilde{\mathsf{S}}_{xx}\right]^2 + 4{}^1\!/\!_p\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}} && \text{where } p \triangleq \frac{1}{s^2}\\
&= \lim_{p\to0}\frac{p\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx} + \sqrt{\left[p\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx}\right]^2 + 4p\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2p\tilde{\mathsf{S}}_{xy}} && \text{by mult. by } 1 = \frac{p}{p}\\
&= \lim_{p\to0}\frac{\frac{\mathsf{d}}{\mathsf{d}p}\left[p\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx} + \sqrt{\left[p\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx}\right]^2 + 4p\left|\tilde{\mathsf{S}}_{xy}\right|^2}\right]}{\frac{\mathsf{d}}{\mathsf{d}p}\left[2p\tilde{\mathsf{S}}_{xy}\right]} && \text{by \textit{l'Hôpital's rule}}\\
&= \lim_{p\to0}\frac{\dfrac{\tilde{\mathsf{S}}_{yy}\left(p\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx}\right) + 2\left|\tilde{\mathsf{S}}_{xy}\right|^2 + \tilde{\mathsf{S}}_{yy}\sqrt{\left(p\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{\sqrt{\left(p\tilde{\mathsf{S}}_{yy} - \tilde{\mathsf{S}}_{xx}\right)^2 + 4p\left|\tilde{\mathsf{S}}_{xy}\right|^2}}}{2\tilde{\mathsf{S}}_{xy}} && \text{by Lemma 11.1 page 110}\\
&= \frac{\tilde{\mathsf{S}}_{yy}\left(-\tilde{\mathsf{S}}_{xx}\right) + 2\left|\tilde{\mathsf{S}}_{xy}\right|^2 + \tilde{\mathsf{S}}_{yy}\sqrt{\left(-\tilde{\mathsf{S}}_{xx}\right)^2}}{2\tilde{\mathsf{S}}_{xy}\sqrt{\left(-\tilde{\mathsf{S}}_{xx}\right)^2}}\\
&= \frac{2\left|\tilde{\mathsf{S}}_{xy}\right|^2}{2\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{xy}} = \frac{\tilde{\mathsf{S}}^*_{xy}}{\tilde{\mathsf{S}}_{xx}} \triangleq \hat{\mathsf{H}}_1 && \text{by def. of } \hat{\mathsf{H}}_1 && \text{(Definition 11.2 page 105)}
\end{aligned}$$

II. Proof for monoticity:

1. Let $p \triangleq s^2$
2. lemma:

$$\boxed{\left[2\left|\tilde{\mathsf{S}}_{xy}\right|^2 + \tilde{\mathsf{S}}_{xx}\left(p\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy}\right)\right]^2}$$
$$= 4\left|\tilde{\mathsf{S}}_{xy}\right|^4 + 4\tilde{\mathsf{S}}_{xx}\left|\tilde{\mathsf{S}}_{xy}\right|^2\left(p\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy}\right) + \tilde{\mathsf{S}}^2_{xx}\left(p\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy}\right)^2$$

$$\boxed{\le} 4|\tilde{S}_{xy}|^2\tilde{S}_{xx}\tilde{S}_{yy} + 4p\tilde{S}_{xx}|\tilde{S}_{xy}|^2\tilde{S}_{xx} - 4\tilde{S}_{xx}|\tilde{S}_{xy}|^2\tilde{S}_{yy} + \tilde{S}_{xx}^2(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 \qquad \left(\text{by } \textit{Cauchy Schwartz inequality} \text{ (Theorem ?? page ??)}\right)$$

$$= \cancelto{0}{4\tilde{S}_{xx}\tilde{S}_{yy}|\tilde{S}_{xy}|^2} + 4p\tilde{S}_{xx}|\tilde{S}_{xy}|^2\tilde{S}_{xx} - \cancelto{0}{4\tilde{S}_{xx}\tilde{S}_{yy}|\tilde{S}_{xy}|^2} + \tilde{S}_{xx}^2(p\tilde{S}_{xx} - \tilde{S}_{yy})^2$$

$$= \tilde{S}_{xx}^2\left[(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2\right]$$

$$= \boxed{\left[\tilde{S}_{xx}\sqrt{(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2}\right]^2}$$

3. lemma: $2|\tilde{S}_{xy}|^2 + \tilde{S}_{xx}(p\tilde{S}_{xx} - \tilde{S}_{yy}) \le \tilde{S}_{xx}\sqrt{(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2}$. Proof:

$$2|\tilde{S}_{xy}|^2 + \tilde{S}_{xx}(p\tilde{S}_{xx} - \tilde{S}_{yy}) \le \tilde{S}_{xx}\sqrt{(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2}$$

$$\iff \left[2|\tilde{S}_{xy}|^2 + \tilde{S}_{xx}(p\tilde{S}_{xx} - \tilde{S}_{yy})\right]^2 \le \left[\tilde{S}_{xx}\sqrt{(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2}\right]^2 \qquad \left(\text{because } \mathsf{f}(x) \triangleq x^2 \text{ is } \textit{strictly monotonic increasing}\right)$$

The previous inequality is true by (2) lemma, so (3) lemma also true.

4. Proof that $\frac{d}{dp}|\hat{H}_s| \le 0$:

$$\begin{aligned}
\frac{d}{dp}|\hat{H}_s| &\triangleq \frac{d}{dp}\left|\frac{\tilde{S}_{yy} - s^2\tilde{S}_{xx} + \sqrt{(\tilde{S}_{yy} - s^2\tilde{S}_{xx})^2 + 4s^2|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}}\right| && \text{by def. of } \hat{H}_s \text{ (Definition 11.8 page 109)}\\
&\triangleq \frac{d}{dp}\left[\frac{\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{(\tilde{S}_{yy} - p\tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}{2\tilde{S}_{xy}}\right] && \text{by definition of } p \text{ (item (1) page 112)}\\
&= \frac{d}{dp}\left(\frac{\left|\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{(\tilde{S}_{yy} - p\tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}\right|}{|2\tilde{S}_{xy}|}\right)\\
&= \frac{d}{dp}\left(\frac{\tilde{S}_{yy} - p\tilde{S}_{xx} + \sqrt{(\tilde{S}_{yy} - p\tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}{2|\tilde{S}_{xy}|}\right) && \text{by Lemma 11.2 page 111}\\
&= \frac{\dfrac{\tilde{S}_{xx}(p\tilde{S}_{xx} - \tilde{S}_{yy}) + 2|\tilde{S}_{xy}|^2 - \tilde{S}_{xx}\sqrt{(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2}}{\sqrt{(\tilde{S}_{yy} - p\tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}}{2|\tilde{S}_{xy}|} && \text{by Lemma 11.1 page 110}\\
&= \frac{\tilde{S}_{xx}(p\tilde{S}_{xx} - \tilde{S}_{yy}) + 2|\tilde{S}_{xy}|^2 - \tilde{S}_{xx}\sqrt{(p\tilde{S}_{xx} - \tilde{S}_{yy})^2 + 4p|\tilde{S}_{xy}|^2}}{2|\tilde{S}_{xy}|\sqrt{(\tilde{S}_{yy} - p\tilde{S}_{xx})^2 + 4p|\tilde{S}_{xy}|^2}}\\
&\le 0 && \text{by (3) lemma}
\end{aligned}$$

✏

Theorem 11.2. *Let* **S** *be a* SYSTEM *with input* $x(n)$ *and output* $y(n)$.

THM
$$|\hat{H}_1(\omega)| \le |\hat{H}_{hm}(\omega)| \le |\hat{H}_{gm}(\omega)| \le |\hat{H}_{am}(\omega)| \le |\hat{H}_2(\omega)|$$

✎PROOF:

1. lemma: $\hat{\mathsf{H}}_1(\omega) \le \hat{\mathsf{H}}_2(\omega)$. Proof:

$$\begin{aligned}
\left|\hat{\mathsf{H}}_1\right| &\triangleq \left|\frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}\right| && \text{by definition of } \hat{\mathsf{H}}_1 && \text{(Definition 11.2 page 105)}\\
&= \left|\frac{\langle \mathsf{y} \mid \mathsf{x}\rangle}{\|\mathsf{x}\|^2}\right| = \frac{|\langle \mathsf{y} \mid \mathsf{x}\rangle|}{\|\mathsf{x}\|^2}\\
&\le \frac{|\langle \mathsf{y} \mid \mathsf{x}\rangle|}{\|\mathsf{x}\|^2}\left|\frac{\|\mathsf{x}\|\,\|\mathsf{y}\|}{\langle \mathsf{y} \mid \mathsf{x}\rangle}\right|^2 && \text{by \textit{Cauchy Schwartz inequality}} && \text{Theorem ?? page ??}\\
&= \frac{\|\mathsf{y}\|^2}{|\langle \mathsf{y} \mid \mathsf{x}\rangle|} = \left|\frac{\|\mathsf{y}\|^2}{\langle \mathsf{x} \mid \mathsf{y}\rangle}\right| = \left|\frac{\tilde{\mathsf{S}}_{yy}(\omega)}{\tilde{\mathsf{S}}_{xy}(\omega)}\right|\\
&= \left|\hat{\mathsf{H}}_2\right| && \text{by definition of } \hat{\mathsf{H}}_2 && \text{(Definition 11.3 page 106)}
\end{aligned}$$

2. remainder of the proof:

$$\begin{aligned}
\left|\hat{\mathsf{H}}_1(\omega)\right| &= \min\left\{\hat{\mathsf{H}}_1(\omega), \hat{\mathsf{H}}_2(\omega)\right\} && \text{by (1) lemma}\\
&\le \left|\hat{\mathsf{H}}_{hm}(\omega)\right| && \text{by Corollary ?? page ??} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&\le \left|\hat{\mathsf{H}}_{gm}(\omega)\right| && \text{by Corollary ?? page ??} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&\le \left|\hat{\mathsf{H}}_{am}(\omega)\right| && \text{by Corollary ?? page ??} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&\le \max\left\{\hat{\mathsf{H}}_1(\omega), \hat{\mathsf{H}}_2(\omega)\right\} && \text{by Corollary ?? page ??} && \text{with } \lambda_1 = \lambda_2 = 1/2\\
&= \left|\hat{\mathsf{H}}_2(\omega)\right| && \text{by (1) lemma}
\end{aligned}$$

✏

Theorem 11.2 (page 113) compared the magnitudes of several transfer function estimates and demonstrated a simple *linear* relationship. What about phase? The phase of those estimates is even simpler than the magnitude, as demonstrated next.

Proposition 11.5 (Estimator phase). *Let $z \triangleq |z|e^{i\phi}$ be a* COMPLEX *number in the set of complex numbers $\mathbb{C}$. Let $\angle z \triangleq \phi$ be the* PHASE *of z.*

PRP

$$\begin{aligned}
\angle\hat{\mathsf{H}}_1(\omega) &= \angle\hat{\mathsf{H}}_{hm}(\omega) = \angle\hat{\mathsf{H}}_{gm}(\omega) = \angle\hat{\mathsf{H}}_{am}(\omega) = \angle\hat{\mathsf{H}}_2(\omega) = \angle\hat{\mathsf{H}}_s(\omega) = \angle\hat{\mathsf{H}}_v(\omega) = \angle\hat{\mathsf{H}}_\kappa(\omega)\\
&= \angle\mathsf{C}_{xy}(\omega)\\
&= -\angle\tilde{\mathsf{S}}_{xy}(\omega)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\angle\hat{\mathsf{H}}_1 &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yx}}{\tilde{\mathsf{S}}_{xx}} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.2 page 105)}\\
\angle\hat{\mathsf{H}}_{hm} &\triangleq \angle\frac{2\tilde{\mathsf{S}}_{yy}\tilde{\mathsf{S}}^*_{xy}}{\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{yy}+\left|\tilde{\mathsf{S}}_{xy}\right|^2} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.6 page 108)}\\
\angle\hat{\mathsf{H}}_{gm} &\triangleq \angle\frac{\tilde{\mathsf{S}}^*_{xy}}{\left|\tilde{\mathsf{S}}_{xy}\right|}\sqrt{\frac{\tilde{\mathsf{S}}_{yy}}{\tilde{\mathsf{S}}_{xx}}} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.5 page 107)}\\
\angle\hat{\mathsf{H}}_{am} &\triangleq \angle\frac{\left|\tilde{\mathsf{S}}_{xy}\right|^2+\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{yy}}{2\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.4 page 106)}\\
\angle\hat{\mathsf{H}}_2 &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}}{\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.3 page 106)}\\
\angle\hat{\mathsf{H}}_s &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}-s^2\tilde{\mathsf{S}}_{xx}+\sqrt{\left[\tilde{\mathsf{S}}_{yy}-s^2\tilde{\mathsf{S}}_{xx}\right]^2+4s^2\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.8 page 109)}\\
\angle\hat{\mathsf{H}}_v &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}-\tilde{\mathsf{S}}_{xx}+\sqrt{\left[\tilde{\mathsf{S}}_{yy}-\tilde{\mathsf{S}}_{xx}\right]^2+4\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.9 page 110)}\\
\angle\hat{\mathsf{H}}_\kappa &\triangleq \angle\frac{\tilde{\mathsf{S}}_{yy}-\kappa\tilde{\mathsf{S}}_{xx}+\sqrt{\left[\tilde{\mathsf{S}}_{yy}-\kappa\tilde{\mathsf{S}}_{xx}\right]^2+4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}} && = \angle\frac{1}{\tilde{\mathsf{S}}_{xy}} && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.7 page 109)}\\
\angle\mathsf{C}_{xy} &\triangleq \angle\frac{\tilde{\mathsf{S}}^*_{xy}}{\sqrt{\tilde{\mathsf{S}}_{xx}\tilde{\mathsf{S}}_{yy}}} && = && = \angle\tilde{\mathsf{S}}^*_{xy} = -\angle\tilde{\mathsf{S}}_{xy}\\
&\uparrow \text{(Definition 11.12 page 126)}
\end{aligned}$$

✐

11.5 Alternate forms

Any standard kit of algebraic tricks should arguably always include the ability to swap the location of a square root between numerator and denominator. If you are of this persuasion, after traveling from the definition of $\hat{\mathsf{H}}_s$ on page 109, you won't be dissappointed when arriving at the next proposition (Proposition 11.6 page 115). But it has more use than just allowing you to entertain friends at social occasions. It also makes it very easy to see (using only algebra) what previously employed *l'Hôpital's rule* (using calculus) in the proof of Theorem 11.1—that $\lim_{s\to\infty}\hat{\mathsf{H}}_s = \hat{\mathsf{H}}_1$.

Proposition 11.6. [17] *Let $\hat{\mathsf{H}}_\kappa(\omega;\kappa)$ be defined as in Definition 11.7 (page 109).*

[17] Shin and Hammond (2008) page 293 ⟨(9.67)⟩, Leclere et al. (2014) ⟨(10) $\kappa(f) = 1/s^2$ and x and y swapped⟩

PRP

$$\hat{\mathsf{H}}_\kappa(\omega;s) = \frac{2\kappa(\omega)\tilde{\mathsf{S}}_{yx}(\omega)}{\kappa(\omega)\tilde{\mathsf{S}}_{xx}(\omega) - \tilde{\mathsf{S}}_{yy}(\omega) + \sqrt{\left[\kappa(\omega)\tilde{\mathsf{S}}_{xx}(\omega) - \tilde{\mathsf{S}}_{yy}(\omega)\right]^2 + 4\kappa(\omega)\left|\tilde{\mathsf{S}}_{xy}(\omega)\right|^2}}$$
$$= \frac{2\tilde{\mathsf{S}}^*_{xy}}{\tilde{\mathsf{S}}_{xx} - \frac{1}{\kappa(\omega)}\tilde{\mathsf{S}}_{yy} + \sqrt{\left[\tilde{\mathsf{S}}_{xx} - \frac{1}{\kappa(\omega)}\tilde{\mathsf{S}}_{yy}\right]^2 + \frac{4}{\kappa(\omega)}\left|\tilde{\mathsf{S}}_{xy}\right|^2}}$$

✎PROOF: We can transform $\hat{\mathsf{H}}_\kappa$ from that found in Definition 11.8 (page 109) to the forms in this proposition by the technique of "*rationalizing the denominator*"[18]

$$\hat{\mathsf{H}}_\kappa \triangleq \frac{\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 + 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}}{2\tilde{\mathsf{S}}_{xy}} \qquad \text{by definition of } \hat{\mathsf{H}}_\kappa \text{ (Definition 11.8 page 109)}$$

$$= \frac{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 + 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}\right]\overbrace{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx} - \sqrt{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 + 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}\right]}^{\text{"rationalizing factor"}}}{2\tilde{\mathsf{S}}_{xy}\underbrace{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx} - \sqrt{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 + 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}\right]}_{\text{"rationalizing factor"}}}$$

$$= \frac{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 - \left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 - 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}{2\tilde{\mathsf{S}}_{xy}\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx} - \sqrt{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 + 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}\right]} = \frac{-4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}{2\tilde{\mathsf{S}}_{xy}\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx} - \sqrt{\left[\tilde{\mathsf{S}}_{yy} - \kappa\tilde{\mathsf{S}}_{xx}\right]^2 + 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}\right]}$$

$$= \boxed{\frac{2\kappa\tilde{\mathsf{S}}^*_{xy}}{\kappa\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy} + \sqrt{\left[\kappa\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy}\right]^2 + 4\kappa\left|\tilde{\mathsf{S}}_{xy}\right|^2}}} = \frac{2\frac{\kappa}{\kappa}\tilde{\mathsf{S}}^*_{xy}}{\frac{\kappa}{\kappa}\tilde{\mathsf{S}}_{xx} - \frac{1}{\kappa}\tilde{\mathsf{S}}_{yy} + \sqrt{\frac{1}{s^4}\left[\kappa\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy}\right]^2 + \frac{4\kappa}{s^4}\left|\tilde{\mathsf{S}}_{xy}\right|^2}}$$

$$= \frac{2\tilde{\mathsf{S}}^*_{xy}}{\tilde{\mathsf{S}}_{xx} - \frac{1}{\kappa}\tilde{\mathsf{S}}_{yy} + \sqrt{\left[\frac{\kappa}{\kappa}\tilde{\mathsf{S}}_{xx} - \frac{1}{\kappa}\tilde{\mathsf{S}}_{yy}\right]^2 + \frac{4}{\kappa}\left|\tilde{\mathsf{S}}_{xy}\right|^2}} = \boxed{\frac{2\tilde{\mathsf{S}}^*_{xy}}{\tilde{\mathsf{S}}_{xx} - \frac{1}{\kappa}\tilde{\mathsf{S}}_{yy} + \sqrt{\left[\tilde{\mathsf{S}}_{xx} - \frac{1}{\kappa}\tilde{\mathsf{S}}_{yy}\right]^2 + \frac{4}{\kappa}\left|\tilde{\mathsf{S}}_{xy}\right|^2}}}$$

Integrity check for $s = 0$ and $s \to \infty$ cases: Let $p \triangleq \kappa$.

$$\lim_{p\to\infty} \hat{\mathsf{H}}_\kappa = \lim_{p\to\infty} \frac{2\tilde{\mathsf{S}}_{yx}}{\tilde{\mathsf{S}}_{xx} - \frac{1}{p}\tilde{\mathsf{S}}_{yy} + \sqrt{\left[\tilde{\mathsf{S}}_{xx} - \frac{1}{p}\tilde{\mathsf{S}}_{yy}\right]^2 + \frac{4}{p}\left|\tilde{\mathsf{S}}_{xy}\right|^2}} = \frac{2\tilde{\mathsf{S}}_{yx}}{\tilde{\mathsf{S}}_{xx} + \sqrt{\left[\tilde{\mathsf{S}}_{xx}\right]^2}}$$

$$= \frac{\tilde{\mathsf{S}}_{yx}}{\tilde{\mathsf{S}}_{xx}} \triangleq \hat{\mathsf{H}}_1 \qquad \text{by def. of } \hat{\mathsf{H}}_1 \text{ (Definition 11.2 page 105)}$$

$$\lim_{p\to 0} \hat{\mathsf{H}}_\kappa = \lim_{p\to 0} \frac{2p\tilde{\mathsf{S}}_{yx}}{p\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy} + \sqrt{\left[p\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy}\right]^2 + 4p\left|\tilde{\mathsf{S}}_{xy}\right|^2}}$$

$$= \lim_{p\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}p}\left(2p\tilde{\mathsf{S}}_{yx}\right)}{\frac{\mathrm{d}}{\mathrm{d}p}\left(p\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy} + \sqrt{\left[p\tilde{\mathsf{S}}_{xx} - \tilde{\mathsf{S}}_{yy}\right]^2 + 4p\left|\tilde{\mathsf{S}}_{xy}\right|^2}\right)} \qquad \text{by } \textit{l'Hôpital's rule}$$

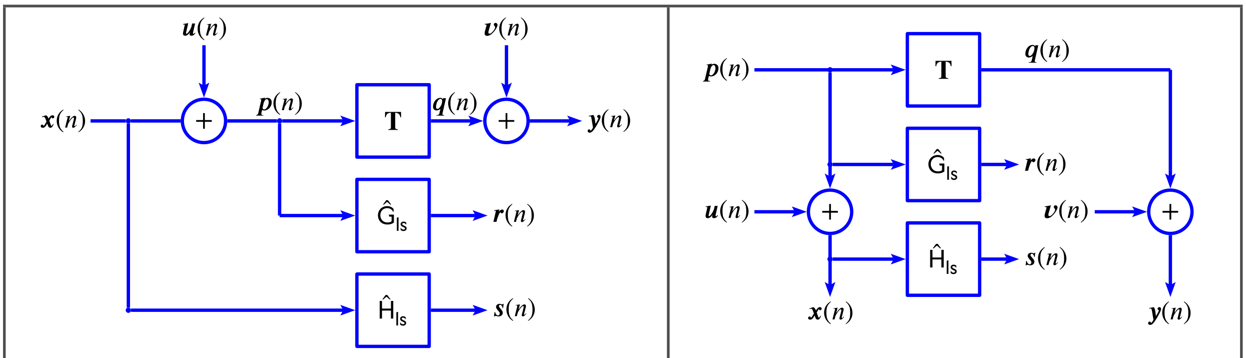
[18] Slaught and Lennes (1915), page 274 ⟨"197. Rationalizing the Denominator."⟩ https://archive.org/details/elementaryalgebr00slaurich/page/274 Note that the operation in the proof of Proposition 11.6 is being performed essentially in reverse…or rather "rationalizing the numerator".

$$
\begin{aligned}
&= \lim_{p\to 0} \frac{2\tilde{S}_{yx}}{\dfrac{\tilde{S}_{xx}\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)+2\left|\tilde{S}_{xy}\right|^2+\tilde{S}_{xx}\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2+4p\left|\tilde{S}_{xy}\right|^2}}{\sqrt{\left(p\tilde{S}_{xx}-\tilde{S}_{yy}\right)^2+4p\left|\tilde{S}_{xy}\right|^2}}} && \text{by Lemma 11.1 page 110} \\
&= \frac{2\tilde{S}_{yy}\tilde{S}_{yx}}{-\tilde{S}_{xx}\tilde{S}_{yy}+2\left|\tilde{S}_{xy}\right|^2+\tilde{S}_{xx}\tilde{S}_{yy}} = \frac{2\tilde{S}_{yy}\tilde{S}_{yx}}{2\left|\tilde{S}_{xy}\right|^2} = \frac{\tilde{S}_{yy}}{\tilde{S}_{xy}} \\
&\triangleq \hat{H}_2 && \text{by def. of } \hat{H}_2 \text{ (Definition 11.3 page 106)}
\end{aligned}
$$

✏

11.6 Least squares estimates of non-linear systems

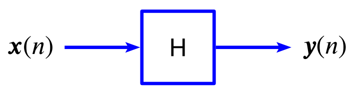
> *"The legendary Hungarian mathematician John von Neumann once referred to the the theory of nonequilibrium systems as the "theory of non-elephants," ...Nevertheless, such a theory of non-elephants will be attempted here."*
>
> Per Bak, in *"how nature works..."* [19]



(A) *Least squares estimation* for *communications additive noise model* (Corollary 11.1 page 120)

(B) *Least squares estimations* for *measurement additive noise model* (Corollary 11.2 page 121)

Figure 11.3: Least Square estimation (Theorem 11.3 page 118)



Let **S** be the *system* illustrated to the right. **If** there is no measurement noise on the input and output and **if H** is *linear time invariant*, then $\tilde{H} = \tilde{S}_{xy}/\tilde{S}_{xx}$ (Corollary **??** page **??**). But what if there is output measurement noise? And what if **H** is not *LTI*? What is the best least-squares estimate of $\tilde{H}$? The answer depends on how you define "the best".

The defintion of "best" or "optimal" is given by a *cost function* $C(\hat{H})$. There are several possible cost functions. Definition 11.11 provides some. Theorem 11.3 then demonstrate optimal solutions with respect to these definitions.

Definition 11.11. *Let* **S** *be a system defined as in Figure 11.3 (page 117) (A) or (B). Define the following* COST FUNCTIONS *for spectral* LEAST-SQUARES *estimates:*

DEF

$$
\begin{aligned}
C_{rq}(\hat{G}) &\triangleq \tilde{\mathbf{F}}\left\|r(n)-q(n)\right\|^2 \triangleq \tilde{\mathbf{F}}\left\langle r(n)-q(n)\,|\,r(0)-q(0)\right\rangle \triangleq \tilde{F}E\left(\left[r(n)-q(n)\right]\left[r(0)-q(0)\right]^*\right) \\
C_{sy}(\hat{H}) &\triangleq \tilde{\mathbf{F}}\left\|s(n)-y(n)\right\|^2 \triangleq \tilde{\mathbf{F}}\left\langle s(n)-y(n)\,|\,s(0)-y(0)\right\rangle \triangleq \tilde{F}E\left(\left[s(n)-y(n)\right]\left[s(0)-y(0)\right]^*\right)
\end{aligned}
$$

[19] Bak (2013) page 29 ⟨§ Systems in Balance Are Not Complex⟩

Lemma 11.3. *Let $\mathsf{C}_{rq}(\hat{\mathsf{G}})$ and $\mathsf{C}_{sy}(\hat{\mathsf{H}})$ be defined as in Definition 11.11 (page 117).*

LEM

$$\begin{aligned}
\mathsf{C}_{rq}(\hat{\mathsf{G}}) &= \tilde{\mathsf{S}}_{pp}(\omega)\left|\hat{\mathsf{G}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{py}(\omega)\hat{\mathsf{G}}(\omega) - \tilde{\mathsf{S}}^*_{py}(\omega)\hat{\mathsf{G}}^*(\omega) + \tilde{\mathsf{S}}_{qq}(\omega)\\
\mathsf{C}_{sy}(\hat{\mathsf{H}}) &= \tilde{\mathsf{S}}_{xx}(\omega)\left|\hat{\mathsf{H}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{xy}(\omega)\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}^*_{xy}(\omega)\hat{\mathsf{H}}^*(\omega) + \tilde{\mathsf{S}}_{yy}(\omega)
\end{aligned}$$

✎PROOF:

$\boxed{\mathsf{C}_{rq}(\hat{\mathsf{G}})}$

$$\begin{aligned}
&\triangleq \breve{\mathbf{F}}\mathsf{E}\left(\left[\mathsf{r}(n)-\mathsf{q}(n)\right]\left[\mathsf{r}(0)-\mathsf{q}(0)\right]^*\right) && \text{by definition of } \mathsf{C}_{rq} && \text{(Definition 11.11 page 117)}\\
&= \tilde{\mathbf{F}}\left[\mathsf{E}\left[\mathsf{r}(n)\mathsf{r}^*(0)\right] - \mathsf{E}\left[\mathsf{r}(n)\mathsf{q}^*(0)\right] - \mathsf{E}\left[\mathsf{q}(n)\mathsf{r}^*(0)\right] + \mathsf{E}\left[\mathsf{q}(n)\mathsf{q}^*(0)\right]\right] && \text{by } \textit{linearity} \text{ of } \mathsf{E} && \text{(Theorem ?? page ??)}\\
&\triangleq \tilde{\mathbf{F}}\left[\mathsf{R}_{rr}(m) - \mathsf{R}_{rq}(m) - \mathsf{R}_{qr}(m) + \mathsf{R}_{qq}(m)\right] && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition ?? page ??)}\\
&\triangleq \left[\tilde{\mathsf{S}}_{rr}(\omega) - \tilde{\mathsf{S}}_{rq}(\omega) - \tilde{\mathsf{S}}_{qr}(\omega) + \tilde{\mathsf{S}}_{qq}(\omega)\right] && \text{by definition of } \tilde{\mathsf{S}}_{xy} && \text{(Definition ?? page ??)}\\
&= \boxed{\tilde{\mathsf{S}}_{pp}(\omega)\left|\hat{\mathsf{G}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{py}(\omega)\hat{\mathsf{G}}(\omega) - \tilde{\mathsf{S}}^*_{py}(\omega)\hat{\mathsf{G}}^*(\omega) + \tilde{\mathsf{S}}_{qq}(\omega)} && \text{by (A)–(D) and} && \text{Corollary ?? page ??}
\end{aligned}$$

$\boxed{\mathsf{C}_{sy}(\hat{\mathsf{H}})}$

$$\begin{aligned}
&\triangleq \breve{\mathbf{F}}\mathsf{E}\left(\left[\mathsf{s}(n)-\mathsf{y}(n)\right]\left[\mathsf{s}(0)-\mathsf{y}(0)\right]^*\right) && \text{by definition of } \mathsf{C}_{sy} && \text{(Definition 11.11 page 117)}\\
&= \tilde{\mathbf{F}}\left[\mathsf{E}\left[\mathsf{s}(n)\mathsf{s}^*(0)\right] - \mathsf{E}\left[\mathsf{s}(n)\mathsf{y}^*(0)\right] - \mathsf{E}\left[\mathsf{y}(n)\mathsf{s}^*(0)\right] + \mathsf{E}\left[\mathsf{y}(n)\mathsf{y}^*(0)\right]\right] && \text{by } \textit{linearity} \text{ of } \mathsf{E} && \text{(Theorem ?? page ??)}\\
&\triangleq \tilde{\mathbf{F}}\left[\mathsf{R}_{ss}(m) - \mathsf{R}_{sy}(m) - \mathsf{R}_{ys}(m) + \mathsf{R}_{yy}(m)\right] && \text{by definition of } \mathsf{R}_{xy} && \text{(Definition ?? page ??)}\\
&\triangleq \left[\tilde{\mathsf{S}}_{ss}(\omega) - \tilde{\mathsf{S}}_{sy}(\omega) - \tilde{\mathsf{S}}_{ys}(\omega) + \tilde{\mathsf{S}}_{yy}(\omega)\right] && \text{by definition of } \tilde{\mathsf{S}}_{xy} && \text{(Definition ?? page ??)}\\
&= \boxed{\tilde{\mathsf{S}}_{xx}(\omega)\left|\hat{\mathsf{H}}(\omega)\right|^2 - \tilde{\mathsf{S}}_{xy}(\omega)\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}^*_{xy}(\omega)\hat{\mathsf{H}}^*(\omega) + \tilde{\mathsf{S}}_{yy}(\omega)} && \text{by (A)–(D) and} && \text{Corollary ?? (page ??)}
\end{aligned}$$

✏

Theorem 11.3. *Let $\mathbf{S}$ be the SYSTEM illustrated in Figure 11.3 page 117 (A) or (B).*

THM

$$\left.\begin{array}{llll}
\text{(A).} & \mathsf{x}, \mathsf{u}, \textit{ and } \mathsf{v} \textit{ are} & \text{WSS} & \textit{and}\\
\text{(B).} & \mathsf{x}, \mathsf{u}, \textit{ and } \mathsf{v} \textit{ are} & \text{UNCORRELATED} & \textit{and}\\
\text{(C).} & \mathsf{E}\mathsf{u} = \mathsf{E}\mathsf{v} = 0 & \text{(ZERO-MEAN)} & \textit{and}\\
\text{(D).} & \hat{\mathsf{G}}_{ls} \textit{ and } \hat{\mathsf{H}}_{ls} \textit{ are} & \text{LTI} &
\end{array}\right\} \implies \left\{\begin{array}{llll}
\text{(1).} & \arg\min\limits_{\hat{\mathsf{G}}} \mathsf{C}_{rq}(\hat{\mathsf{G}}) & = & \dfrac{\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)}\\
\text{(2).} & \arg\min\limits_{\hat{\mathsf{H}}} \mathsf{C}_{sy}(\hat{\mathsf{H}}) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}
\end{array}\right.$$

✎PROOF:

1. Define shorthand function names $\hat{\mathsf{G}} \triangleq \hat{\mathsf{G}}_{ls}$ and $\hat{\mathsf{H}} \triangleq \hat{\mathsf{H}}_{ls}$.
2. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{G}}_R}\mathsf{C}_{rq}(\hat{\mathsf{G}})\\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_R}\left(\tilde{\mathsf{S}}_{pp}\left|\hat{\mathsf{G}}\right|^2 - \hat{\mathsf{G}}\tilde{\mathsf{S}}_{py} - \hat{\mathsf{G}}^*\tilde{\mathsf{S}}^*_{py} + \tilde{\mathsf{S}}_{qq}\right) && \text{by Lemma 11.3 page 118}\\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_R}(\tilde{\mathsf{S}}_{pp}[\hat{\mathsf{G}}_R^2 + \hat{\mathsf{G}}_I^2] - (\hat{\mathsf{G}}_R + i\hat{\mathsf{G}}_I)\tilde{\mathsf{S}}_{py} - (\hat{\mathsf{G}}_R + i\hat{\mathsf{G}}_I)^*\tilde{\mathsf{S}}^*_{py} + \tilde{\mathsf{S}}_{qq})\\
&= 2\hat{\mathsf{G}}_R\tilde{\mathsf{S}}_{pp} - \tilde{\mathsf{S}}_{py} - \tilde{\mathsf{S}}^*_{py} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{G}}_R}\tilde{\mathsf{S}}_{qq}} && \text{because } \mathsf{q} \text{ does not vary with } \hat{\mathsf{G}}\\
&= 2\hat{\mathsf{G}}_R\tilde{\mathsf{S}}_{pp} - 2\mathbf{R}_e\tilde{\mathsf{S}}_{py}\\
&= 2\hat{\mathsf{G}}_R\tilde{\mathsf{S}}_{pp} - 2\mathbf{R}_e\tilde{\mathsf{S}}_{yp} && \text{by Corollary ?? page ??}\\
\implies &\boxed{\hat{\mathsf{G}}_R(\omega) = \frac{\mathbf{R}_e\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)}}
\end{aligned}$$

3. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{G}}_I} \mathsf{C}_{\mathsf{rq}}(\hat{\mathsf{G}}) \\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_I} \left(\tilde{\mathsf{S}}_{\mathsf{pp}} |\hat{\mathsf{G}}|^2 - \hat{\mathsf{G}}\tilde{\mathsf{S}}_{\mathsf{py}} - \hat{\mathsf{G}}^*\tilde{\mathsf{S}}^*_{\mathsf{py}} + \tilde{\mathsf{S}}_{\mathsf{qq}} \right) && \text{by Lemma 11.3 page 118} \\
&= \frac{\partial}{\partial \hat{\mathsf{G}}_I} \left[\tilde{\mathsf{S}}_{\mathsf{pp}} [\hat{\mathsf{G}}_R^2 + \hat{\mathsf{G}}_I^2] - (\hat{\mathsf{G}}_R + i\hat{\mathsf{G}}_I)\tilde{\mathsf{S}}_{\mathsf{py}} - (\hat{\mathsf{G}}_R - i\hat{\mathsf{G}}_I)\tilde{\mathsf{S}}^*_{\mathsf{py}} + \tilde{\mathsf{S}}_{\mathsf{qq}} \right] \\
&= 2\hat{\mathsf{G}}_I \tilde{\mathsf{S}}_{\mathsf{pp}} - i\tilde{\mathsf{S}}_{\mathsf{py}} + i\tilde{\mathsf{S}}^*_{\mathsf{py}} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{G}}_I} \tilde{\mathsf{S}}_{\mathsf{qq}}} && \text{because q does not vary with } \hat{\mathsf{G}} \\
&= 2\hat{\mathsf{G}}_I \tilde{\mathsf{S}}_{\mathsf{pp}} - 2i(i\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{py}}) \\
&= 2\hat{\mathsf{G}}_I \tilde{\mathsf{S}}_{\mathsf{pp}} + 2i(i\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{yp}}) && \text{by Corollary ?? page ??} \\
\implies & \boxed{\hat{\mathsf{G}}_I(\omega) = \frac{\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}}
\end{aligned}$$

4. Proof for $\hat{\mathsf{G}} \triangleq \hat{\mathsf{G}}_{\mathsf{ls}}$ expression:

$$\begin{aligned}
\boxed{\hat{\mathsf{G}}(\omega)} &= \hat{\mathsf{G}}_R(\omega) + i\hat{\mathsf{G}}_I(\omega) \\
&= \frac{\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)} + \frac{i\mathbf{I}_{\mathsf{m}}\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)} && \text{by (2) lemma and (3) lemma} \\
&= \boxed{\frac{\tilde{\mathsf{S}}_{\mathsf{yp}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}}
\end{aligned}$$

5. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{H}}_R} \mathsf{C}_{\mathsf{sy}}(\hat{\mathsf{H}}) \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_R} \left(\tilde{\mathsf{S}}_{\mathsf{xx}} |\hat{\mathsf{H}}|^2 - \hat{\mathsf{H}}\tilde{\mathsf{S}}_{\mathsf{xy}} - \hat{\mathsf{H}}^*\tilde{\mathsf{S}}^*_{\mathsf{xy}} + \tilde{\mathsf{S}}_{\mathsf{yy}} \right) && \text{by Lemma 11.3 page 118} \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_R} \left(\tilde{\mathsf{S}}_{\mathsf{xx}} [\hat{\mathsf{H}}_R^2 + \hat{\mathsf{H}}_I^2] - (\hat{\mathsf{H}}_R + i\hat{\mathsf{H}}_I)\tilde{\mathsf{S}}_{\mathsf{xy}} - (\hat{\mathsf{H}}_R + i\hat{\mathsf{H}}_I)^*\tilde{\mathsf{S}}^*_{\mathsf{xy}} + \tilde{\mathsf{S}}_{\mathsf{yy}} \right) \\
&= 2\hat{\mathsf{H}}_R \tilde{\mathsf{S}}_{\mathsf{xx}} - \tilde{\mathsf{S}}_{\mathsf{xy}} - \tilde{\mathsf{S}}^*_{\mathsf{xy}} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{H}}_R} \tilde{\mathsf{S}}_{\mathsf{yy}}} && \text{because y does not vary with } \hat{\mathsf{H}} \\
&= 2\hat{\mathsf{H}}_R \tilde{\mathsf{S}}_{\mathsf{xx}} - 2\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{xy}} \\
&= 2\hat{\mathsf{H}}_R \tilde{\mathsf{S}}_{\mathsf{xx}} - 2\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{yx}} && \text{by Corollary ?? page ??} \\
\implies & \boxed{\hat{\mathsf{H}}_R(\omega) = \frac{\mathbf{R}_{\mathsf{e}}\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}}
\end{aligned}$$

6. lemma:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\mathsf{H}}_I} \mathsf{C}_{\mathsf{sy}}(\hat{\mathsf{H}}) \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_I} \left(\tilde{\mathsf{S}}_{\mathsf{xx}} |\hat{\mathsf{H}}|^2 - \tilde{\mathsf{S}}_{\mathsf{xy}}\hat{\mathsf{H}} - \tilde{\mathsf{S}}^*_{\mathsf{xy}}\hat{\mathsf{H}}^* + \tilde{\mathsf{S}}_{\mathsf{yy}} \right) && \text{by Lemma 11.3 page 118} \\
&= \frac{\partial}{\partial \hat{\mathsf{H}}_I} \left[\tilde{\mathsf{S}}_{\mathsf{xx}} [\hat{\mathsf{H}}_R^2 + \hat{\mathsf{H}}_I^2] - \tilde{\mathsf{S}}_{\mathsf{xy}}(\hat{\mathsf{H}}_R + i\hat{\mathsf{H}}_I) - \tilde{\mathsf{S}}^*_{\mathsf{xy}}(\hat{\mathsf{H}}_R - i\hat{\mathsf{H}}_I) + \tilde{\mathsf{S}}_{\mathsf{yy}} \right] \\
&= 2\hat{\mathsf{H}}_I \tilde{\mathsf{S}}_{\mathsf{xx}} - i\tilde{\mathsf{S}}_{\mathsf{xy}} + i\tilde{\mathsf{S}}^*_{\mathsf{xy}} + \cancelto{0}{\frac{\partial}{\partial \hat{\mathsf{H}}_R} \tilde{\mathsf{S}}_{\mathsf{yy}}} && \text{because q does not vary with } \hat{\mathsf{H}}
\end{aligned}$$

$$
\begin{aligned}
&= 2\hat{\mathsf{H}}_I\tilde{\mathsf{S}}_{xx} - 2i(i\mathbf{I}_m\tilde{\mathsf{S}}_{xy}) \\
&= 2\hat{\mathsf{H}}_I\tilde{\mathsf{S}}_{xx} + 2i(i\mathbf{I}_m\tilde{\mathsf{S}}_{yx}) && \text{by Corollary ?? page ??} \\
&= 2\hat{\mathsf{H}}_I\tilde{\mathsf{S}}_{xx} - 2\mathbf{I}_m\tilde{\mathsf{S}}_{yx} \\
\implies & \boxed{\hat{\mathsf{H}}_I(\omega) = \frac{\mathbf{I}_m\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}}
\end{aligned}
$$

7. Proof for $\hat{\mathsf{H}} \triangleq \hat{\mathsf{H}}_{ls}$ expression:

$$
\begin{aligned}
\boxed{\hat{\mathsf{H}}(\omega)} &= \hat{\mathsf{H}}_R(\omega) + i\hat{\mathsf{H}}_I(\omega) \\
&= \frac{\mathbf{R}_e\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} + \frac{i\mathbf{I}_m\tilde{\mathsf{S}}_{yp}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by (5) lemma and (6) lemma} \\
&= \frac{\mathbf{R}_e\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} + \frac{i\mathbf{I}_m\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Theorem ?? page ??} \\
&= \boxed{\frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}}
\end{aligned}
$$

✑

Using Theorem 11.3 (previous) we can see that the optimal **least-squares** operators $\hat{\mathsf{G}}_{ls}$ and $\hat{\mathsf{H}}_{ls}$ for the **non-linear** operator $\mathbf{T}$ in Figure 11.3 (page 117) (A) and (B) are (next two corollaries)

$$
\boxed{
\begin{array}{llcll}
(1). & \hat{\mathsf{G}}_{ls}(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)+\tilde{\mathsf{S}}_{uu}(\omega)} & \text{for (A)—\textit{communication system}} \\
(2). & \hat{\mathsf{G}}_{ls}(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)-\tilde{\mathsf{S}}_{uu}(\omega)} & \text{for (B)—\textit{measurement system}} \\
(3). & \hat{\mathsf{H}}_{ls}(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} & \text{for either (A) or (B)}
\end{array}
}
$$

Corollary 11.1. *Let* $\mathbf{S}$ *be the* SYSTEM *illustrated in Figure 11.3 page 117 (A).*

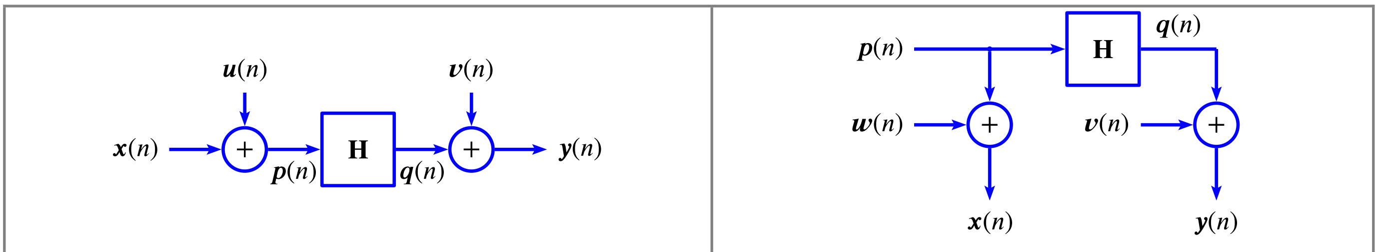
THM

$$
\left\{ \text{\textit{hypotheses of Theorem 11.3 page 118}} \right\} \implies
\left\{
\begin{array}{llcl}
(1). & \arg\min\limits_{\hat{\mathsf{G}}_{ls}} \mathsf{C}_{rq}(\hat{\mathsf{G}}_{ls}) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)+\tilde{\mathsf{S}}_{uu}(\omega)} \\
(2). & \arg\min\limits_{\hat{\mathsf{H}}_{ls}} \mathsf{C}_{sy}(\hat{\mathsf{H}}_{ls}) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)}
\end{array}
\right\}
$$

✎PROOF:

$$
\begin{aligned}
\hat{\mathsf{G}}_{ls} &= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)} && \text{by Theorem 11.3 page 118} \\
&= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)+\tilde{\mathsf{S}}_{uu}(\omega)} && \text{by Theorem ?? page ??} \\
\hat{\mathsf{H}}_{ls} &= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Theorem 11.3 page 118}
\end{aligned}
$$

✑



(A) **communications LTI additive noise model** (B) **measurement LTI additive noise model**

Figure 11.4: Additive noise systems with **LTI** operator **H**

Corollary 11.2. *Let* **S** *be the* SYSTEM *illustrated in Figure 11.3 page 117 (B).*

THM
$$\left\{\textit{hypotheses of Theorem 11.3 page 118}\right\} \implies \left\{\begin{array}{lrcl} (1). & \arg\min_{\hat{\mathsf{G}}_{ls}} \mathsf{C}_{rq}(\hat{\mathsf{G}}_{ls}) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)-\tilde{\mathsf{S}}_{uu}(\omega)} \\ (2). & \arg\min_{\hat{\mathsf{H}}_{ls}} \mathsf{C}_{sy}(\hat{\mathsf{H}}_{ls}) & = & \dfrac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} \end{array}\right\}$$

✎PROOF:

$$\begin{aligned} \hat{\mathsf{G}} &= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{pp}(\omega)} && \text{by Theorem 11.3 page 118} \\ &= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)-\tilde{\mathsf{S}}_{uu}(\omega)} && \text{by Theorem ?? page ??} \\ \hat{\mathsf{H}} &= \frac{\tilde{\mathsf{S}}_{yx}(\omega)}{\tilde{\mathsf{S}}_{xx}(\omega)} && \text{by Theorem 11.3 page 118} \end{aligned}$$

⇒

It follows immediately from Corollary 11.1 (page 120) and Corollary 11.2 (page 121) that, in the special case of no input noise ($\mathsf{u}(n)=0$), $\hat{\mathsf{H}}_1$ is the optimal least-squares estimate of $\tilde{\mathsf{H}}$ (next corollary).

Corollary 11.3. [20] *Let* **S** *be the* SYSTEM *illustrated in Figure 11.3 page 117 (A) or (B).*

COR
$$\left\{\begin{array}{ll} (1). & \textit{hypotheses of Theorem 11.3} \quad \textit{and} \\ (2). & \mathsf{u}(n)=0 \end{array}\right\} \implies \left\{\hat{\mathsf{G}}_{ls}(\omega)=\hat{\mathsf{H}}_{ls}(\omega)=\hat{\mathsf{H}}_1(\omega)\right\}$$

11.7 Least squares estimates of linear systems

The previous section did assume the estimates $\hat{\mathsf{H}}_1$ and $\hat{\mathsf{H}}_2$ to be *linear time invariant* (*LTI*), but it did *not* assume that the system transfer function **T** itself to be *LTI*. But making the LTI assumption on **H** yields some interesting and insightful results, such as those in this section.

Theorem 11.4 (Estimating H in communication additive noise system)**.** *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 page 121 (A).*

[20] Bendat and Piersol (1980) pages 98–100 ⟨5.1.1 Optimal Character of Calculations; note: proof minimizing $\tilde{\mathsf{S}}_{vv}$ but yields same result⟩, Bendat and Piersol (1993) pages 106–109 ⟨5.1.1 Optimality of Calculations⟩, Bendat and Piersol (2010) pages 187–190 ⟨6.1.4 Optimum Frequency Response Functions⟩

THM

$$\left\{\begin{array}{lll} \text{(A).} & \mathbf{H} \textit{ is} & \text{LINEAR TIME INVARIANT (LTI)} \quad \textit{and} \\ \text{(B).} & \mathsf{x}(n) \textit{ is} & \text{WIDE-SENSE STATIONARY (WSS)} \quad \textit{and} \\ \text{(C).} & \mathsf{x}(n), \mathsf{u}(n), \textit{and } \mathsf{v}(n) \textit{ are} & \text{UNCORRELATED} \end{array}\right\} \implies \left\{\begin{array}{llll} \text{(1).} & \hat{\mathsf{H}}_1(\omega) & = & \tilde{\mathsf{H}}(\omega) \quad \textit{and} \\ \text{(2).} & \hat{\mathsf{H}}_2(\omega) & = & \dfrac{\tilde{\mathsf{S}}_{\mathsf{vv}}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} + \tilde{\mathsf{H}}(\omega)\left[1 + \dfrac{\tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}\right] \end{array}\right\}$$

✎PROOF:

$$\begin{aligned}
\hat{\mathsf{H}}_1(\omega) &\triangleq \frac{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} && \text{by definition of } \hat{\mathsf{H}}_1 && \text{(Definition 11.2 page 105)} \\
&= \frac{\tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} && \text{by Corollary ?? page ??} \\
&= \tilde{\mathsf{H}}(\omega) \\
\hat{\mathsf{H}}_2(\omega) &\triangleq \frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)} && \text{by definition of } \hat{\mathsf{H}}_2 && \text{(Definition 11.3 page 106)} \\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} && \text{by Corollary ?? page ??} \\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{vv}}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} && \text{by Theorem ?? page ??} \\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{vv}}(\omega) + \tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{H}}(\omega)\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} && \text{by Corollary ?? page ??} \\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{vv}}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} + \frac{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{H}}(\omega)\left[\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) + \tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)\right]}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} \\
&= \frac{\tilde{\mathsf{S}}_{\mathsf{vv}}(\omega)}{\tilde{\mathsf{H}}^*(\omega)\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} + \tilde{\mathsf{H}}(\omega)\left[1 + \frac{\tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)}\right]
\end{aligned}$$

✏

Theorem 11.5 (Estimating H in measurement additive noise system). [21] *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 page 121 (B).*

THM

$$\left\{\begin{array}{lll} \text{(A).} & \mathbf{H} \textit{ is} & \text{LINEAR TIME INVARIANT} \quad \textit{and} \\ \text{(B).} & \mathsf{x}(n) \textit{ is} & \text{WIDE-SENSE STATIONARY} \quad \textit{and} \\ \text{(C).} & \mathsf{x}(n), \mathsf{u}(n), \textit{and } \mathsf{v}(n) \textit{ are} & \text{UNCORRELATED} \end{array}\right\} \implies \left\{\begin{array}{lllll} \text{(1).} & \hat{\mathsf{H}}_1(\omega) & = & \tilde{\mathsf{H}}(\omega)\left[\dfrac{1}{1 + \dfrac{\tilde{\mathsf{S}}_{\mathsf{ww}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}}\right] & \text{(UNDER-ESTIMATED)} \quad \textit{and} \\ \text{(2).} & \hat{\mathsf{H}}_2(\omega) & = & \tilde{\mathsf{H}}(\omega)\left[1 + \dfrac{\tilde{\mathsf{S}}_{\mathsf{vv}}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)}\right] & \text{(OVER-ESTIMATED)} \end{array}\right\}$$

[21] Shin and Hammond (2008) page 294 ⟨$H_1(f) = H(f)$ (9.70); $H_2(f) = H(f)\left(1 + S_{n_y n_y}(f)/S_{yy}(f)\right)$ (9.71)⟩, Shin and Hammond (2008) page 294 ⟨$H_1(f) = H(f)/\left(1 + S_{n_x n_x}/S_{xx}(f)\right)$ (9.72); $H_2(f) = H(f)$ (9.73)⟩, Mitchell (1982) page 277 ⟨$H_1(f) = H_0(f)/\left(1 + G_{nn}/G_{uu}\right)$⟩ Mitchell (1982) page 278 ⟨$H_2(f) = H_0(f)\left(1 + G_{mm}/G_{vv}\right)$⟩

✎PROOF:

$$\begin{aligned}
\hat{H}_1(\omega) &\triangleq \frac{\tilde{S}_{yx}(\omega)}{\tilde{S}_{xx}(\omega)} && \text{by definition of } \hat{H}_1 && \text{(Definition 11.2 page 105)}\\
&= \frac{\tilde{S}_{pp}(\omega)\tilde{H}(\omega)}{\tilde{S}_{xx}(\omega)} && \text{by Corollary ?? page ??}\\
&= \frac{\tilde{S}_{pp}(\omega)\tilde{H}(\omega)}{\tilde{S}_{pp}(\omega)+\tilde{S}_{ww}(\omega)} && \text{by hypothesis (A)} && \text{and Corollary ?? page ??}\\
&= \tilde{H}(\omega)\left[\frac{1}{1+\frac{\tilde{S}_{ww}(\omega)}{\tilde{S}_{pp}(\omega)}}\right]\\
\hat{H}_2(\omega) &\triangleq \frac{\tilde{S}_{yy}(\omega)}{\tilde{S}_{xy}(\omega)} && \text{by definition of } \hat{H}_2 && \text{(Definition 11.3 page 106)}\\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{xy}(\omega)} && \text{by hypothesis (C)} && \text{and Corollary ?? page ??}\\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{xq}(\omega)} && \text{by hypothesis (C)} && \text{and Theorem ?? page ??}\\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{pq}(\omega)} && \text{by hypothesis (C)} && \text{and Lemma ?? page ??}\\
&= \frac{\tilde{S}_{qq}(\omega)+\tilde{S}_{vv}(\omega)}{\tilde{S}_{qq}(\omega)/\tilde{H}(\omega)} && \text{by } \textit{LTI} \text{ hypothesis (A)} && \text{and Corollary ?? page ??}\\
&= \tilde{H}(\omega)\left[1+\frac{\tilde{S}_{vv}(\omega)}{\tilde{S}_{qq}(\omega)}\right] && \text{by hypotheses (A) and (B)} && \text{and Corollary ?? page ??}
\end{aligned}$$

✏

Corollary 11.4. *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 (page 121).*

COR

$$\left\{\begin{array}{ll}\text{(A).} & \textit{hypotheses of Theorem 11.5 and}\\ \text{(B).} & u(n)=u(n)=0 \quad (\text{NO INPUT NOISE})\end{array}\right\} \implies \left\{\hat{H}_1(\omega) = \tilde{H}(\omega) \quad (\text{UNBIASED})\right\}$$

$$\left\{\begin{array}{ll}\text{(A).} & \textit{hypotheses of Theorem 11.5 and}\\ \text{(B).} & v(n)=0 \quad (\text{NO OUTPUT NOISE})\end{array}\right\} \implies \left\{\hat{H}_2(\omega) = \tilde{H}(\omega) \quad (\text{UNBIASED})\right\}$$

Lemma 11.4. *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 (page 121).*

LEM

$$\left\{\textit{There exists } \kappa(\omega) \textit{ such that } \tilde{S}_{vv}(\omega)=\kappa(\omega)\tilde{S}_{uu}(\omega)\right\}$$
$$\implies \left\{\tilde{S}_{uu}(\omega) = \frac{\left|\hat{H}(\omega)\right|^2\tilde{S}_{xx}(\omega)-\hat{H}(\omega)\tilde{S}_{xy}(\omega)-\hat{H}^*(\omega)\tilde{S}^*_{xy}(\omega)+\tilde{S}_{yy}(\omega)}{\kappa(\omega)+\left|\hat{H}(\omega)\right|^2}\right\}$$

✎PROOF:

1. Development based on results of previous chapters:

$$\begin{aligned}
\tilde{S}_{vv} &= \tilde{S}_{yy}-\tilde{S}_{qq} && \text{by Corollary ?? page ??}\\
&= \tilde{S}_{yy}-\tilde{S}_{pq}\hat{H} && \text{by Corollary ?? page ??}\\
&= \tilde{S}_{yy}-\tilde{S}_{xy}\hat{H} && \text{by Theorem ?? page ??}\\
\tilde{S}_{uu} &= \tilde{S}_{xx}-\tilde{S}_{pp} && \text{by Corollary ?? page ??}
\end{aligned}$$

$$= \tilde{S}_{xx} - \frac{\tilde{S}_{qp}}{\hat{H}} \qquad \text{by Corollary ?? page ??}$$

$$= \tilde{S}_{xx} - \frac{\tilde{S}_{yx}}{\hat{H}} \qquad \text{by Theorem ?? page ??}$$

$$\tilde{S}_{uu}\left[\left|\hat{H}\right|^2 + \kappa\right] = \left|\hat{H}\right|^2\tilde{S}_{uu} + \kappa\tilde{S}_{uu}$$

$$\triangleq \tilde{S}_{uu}\left|\hat{H}\right|^2 + \tilde{S}_{vv} \qquad \text{by definition of } \kappa(\omega)$$

$$= \left|\hat{H}\right|^2\left[\tilde{S}_{xx} - \frac{\tilde{S}_{yx}}{\hat{H}}\right] + \left[\tilde{S}_{yy} - \tilde{S}_{xy}\hat{H}\right]$$

$$= \left|\hat{H}\right|^2\tilde{S}_{xx} - \hat{H}^*\tilde{S}_{yx} - \tilde{S}_{xy}\hat{H} + \tilde{S}_{yy}$$

$$\implies \boxed{\tilde{S}_{uu}(\omega) = \frac{\left|\hat{H}(\omega)\right|^2\tilde{S}_{xx}(\omega) - \hat{H}(\omega)\tilde{S}_{xy}(\omega) - \hat{H}^*(\omega)\tilde{S}^*_{xy}(\omega) + \tilde{S}_{yy}(\omega)}{\kappa(\omega) + \left|\hat{H}(\omega)\right|^2}}$$

2. Development of Wicks and Vold (Wicks and Vold (1986)):

$$\tilde{Y} - \tilde{V} = \tilde{Q} = \hat{H}\tilde{P} = \hat{H}(\tilde{X} - \tilde{U}) \qquad \text{by definition of } \hat{H}$$

$$\hat{H}\tilde{U} - \tilde{V} = \hat{H}\tilde{X} - \tilde{Y} \qquad \text{by } \textit{left distributive} \text{ prop.} \quad \text{(Theorem G.4 page 257)}$$

$$E\left(\left[\hat{H}\tilde{U} - \tilde{V}\right]\left[\hat{H}\tilde{U} - \tilde{V}\right]^*\right) = E\left(\left[\hat{H}\tilde{X} - \tilde{Y}\right]\left[\hat{H}\tilde{X} - \tilde{Y}\right]^*\right)$$

$$\left|\hat{H}\right|^2\tilde{S}_{uu} - \hat{H}\cancelto{0}{\tilde{S}_{uv}} - \hat{H}^*\cancelto{0}{\tilde{S}_{vu}} + \tilde{S}_{vv} = \left|\hat{H}\right|^2\tilde{S}_{xx} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy} \qquad \text{because u and v are } \textit{uncorrelated}$$

$$\left|\hat{H}\right|^2\tilde{S}_{uu} + \kappa\tilde{S}_{uu} = \left|\hat{H}\right|^2\tilde{S}_{xx} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy} \qquad \text{by hypothesis}$$

✏

Theorem 11.6. [22] *Let* **S** *be the* SYSTEM *illustrated in Figure 11.4 (page 121). Let* $\hat{H}_\kappa(\omega)$ *be the transfer function estimate defined in Definition 11.7 (page 109).*

THM
$$\left\{\begin{array}{ll} (1). & \textit{There exists } \kappa(\omega) \textit{ such that} \\ & \tilde{S}_{vv}(\omega) = \kappa(\omega)\tilde{S}_{uu}(\omega) \qquad \textit{and} \\ (2). & C(\hat{H}_{ls}) = \tilde{S}_{uu}(\omega) \end{array}\right\} \implies \left\{\begin{array}{l} \arg\min_{\hat{H}} C(\hat{H}) \;=\; \hat{H}_\kappa(\omega) \\ (\hat{H}_\kappa \textit{ is the "optimal" estimator for minimizing system noise}) \end{array}\right\}$$

✎PROOF:

1. Let
$$F \triangleq \left|\hat{H}(\omega)\right|^2\tilde{S}_{xx}(\omega) - \hat{H}(\omega)\tilde{S}_{xy}(\omega) - \hat{H}^*(\omega)\tilde{S}^*_{xy}(\omega) + \tilde{S}_{yy}(\omega) \quad \text{(numerator in Lemma 11.4)} \quad \text{and}$$
$$G \triangleq \kappa(\omega) + \left|\hat{H}(\omega)\right|^2 \quad \text{(denominator in Lemma 11.4)}$$

2. lemma $\left(\frac{\partial}{\partial\hat{H}_R}\tilde{S}_{uu}\right)$:

$$\boxed{0} = \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_R}\tilde{S}_{uu} \qquad \text{set } \frac{\partial}{\partial\hat{H}_R}\tilde{S}_{uu} = 0 \text{ to find optimum } \hat{H}_R$$

$$= \frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_R}\frac{F}{G} \qquad \text{by Lemma 11.4 page 123}$$

$$= \frac{1}{2}G^2\frac{(F'G - G'F)}{G^2} \qquad \text{by } \textit{Quotient Rule}$$

$$= \frac{1}{2}(F'G - G'F)$$

[22] Wicks and Vold (1986) page 898 ⟨has additional s in denominator⟩, Shin and Hammond (2008) page 293 ⟨(9.67)⟩, White et al. (2006) page 679 ⟨(6)⟩

$$
\begin{aligned}
&= \tfrac{1}{2}\left[2\hat{H}_R\tilde{S}_{xx} - \tilde{S}_{xy} - \tilde{S}^*_{xy}\right]G - \tfrac{1}{2}2\hat{H}_R F && \text{by definition of } F, G && \text{(item (1) page 124)}\\
&= \boxed{\hat{H}_R\tilde{S}_{xx}G - G\mathbf{R}_e\tilde{S}_{xy} - \hat{H}_R F}
\end{aligned}
$$

3. lemma $\left(\frac{\partial}{\partial \hat{H}_I}\tilde{S}_{uu}\right)$:

$$
\begin{aligned}
\boxed{0} &= \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_I}\tilde{S}_{uu} && \text{set } \frac{\partial}{\partial\hat{H}_I}\tilde{S}_{uu} = 0 \text{ to find optimum } \hat{H}_I\\
&= \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_I}\frac{F}{G} && \text{by Lemma 11.4 page 123}\\
&= \frac{1}{2}G^2\frac{(F'G - G'F)}{G^2} && \text{by \textit{Quotient Rule}}\\
&= \frac{1}{2}(F'G - G'F)\\
&= \frac{1}{2}\left[2\hat{H}_I\tilde{S}_{xx} - i\tilde{S}_{xy} + i\tilde{S}^*_{xy}\right]G - \frac{1}{2}2\hat{H}_I F && \text{by definition of } F, G && \text{(item (1) page 124)}\\
&= \boxed{\hat{H}_I\tilde{S}_{xx}G + G\mathbf{I}_m\tilde{S}_{xy} - \hat{H}_I F}
\end{aligned}
$$

4. Solve for $\hat{H}$ ...

$$
\begin{aligned}
0 = 0 + i0 &= \frac{1}{2}G^2 0 + \frac{1}{2}G^2 0 = \frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_R}\tilde{S}_{uu} + i\frac{1}{2}G^2\frac{\partial}{\partial\hat{H}_I}\tilde{S}_{uu}\\
&= \left[\hat{H}_R\tilde{S}_{xx}G - G\mathbf{R}_e\tilde{S}_{xy} - \hat{H}_R F\right] + i\left[\hat{H}_I\tilde{S}_{xx}G + G\mathbf{I}_m\tilde{S}_{xy} - \hat{H}_I F\right] && \text{by (2) lemma and (3) lemma}\\
&= \hat{H}\tilde{S}_{xx}G - \tilde{S}^*_{xy}G - \hat{H}F \quad \text{because } \mathbf{R}_e(z) + i\mathbf{I}_m(z) = z \text{ and } \mathbf{R}_e(z) - i\mathbf{I}_m(z) = z^*\\
&= \hat{H}\tilde{S}_{xx}G - \tilde{S}_{yx}G - \hat{H}F && \text{by Corollary ?? page ??}\\
&= \hat{H}\tilde{S}_{xx}\left(\kappa + |\hat{H}|^2\right) - \tilde{S}_{yx}\left(\kappa + |\hat{H}|^2\right) - \hat{H}\left(|\hat{H}|^2\tilde{S}_{xx} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy}\right) && \text{by } F, G \text{ defs.}\\
&= \hat{H}\tilde{S}_{xx}\left(\kappa + \cancel{|\hat{H}|^2}\right) - \tilde{S}_{yx}\left(\kappa + |\hat{H}|^2\right) - \hat{H}\left(\cancel{|\hat{H}|^2\tilde{S}_{xx}} - \hat{H}\tilde{S}_{xy} - \hat{H}^*\tilde{S}^*_{xy} + \tilde{S}_{yy}\right)\\
&= \kappa\hat{H}\tilde{S}_{xx} - \tilde{S}_{yx}\left(\kappa + |\hat{H}|^2\right) + \left(\hat{H}^2\tilde{S}_{xy} + |\hat{H}|^2\tilde{S}^*_{xy} - \hat{H}\tilde{S}_{yy}\right)\\
&= \kappa\hat{H}\tilde{S}_{xx} - \kappa\tilde{S}_{yx} - \cancel{\tilde{S}_{yx}|\hat{H}|^2} + \left(\hat{H}^2\tilde{S}_{xy} + \cancel{|\hat{H}|^2\tilde{S}^*_{xy}} - \hat{H}\tilde{S}_{yy}\right)\\
&= \hat{H}^2\tilde{S}_{xy} + \hat{H}\left[\kappa\tilde{S}_{xx} - \tilde{S}_{yy}\right] - \kappa\tilde{S}^*_{xy}\\
\implies & \boxed{\hat{H} = \frac{\left(\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right) \pm \sqrt{\left(\tilde{S}_{yy} - \kappa\tilde{S}_{xx}\right)^2 + 4\kappa\left|\tilde{S}_{xy}\right|^2}}{2\tilde{S}_{xy}}} && \text{by \textit{Quadratic Equation}}
\end{aligned}
$$

✏

11.8 Coherence

11.8.1 Application

Coherence has two basic purposes:

1. The *coherence* of x and y is a measure of how closely x and y are statistically related. That is, it is an indication of how much x and y "cohere" or "stick" together

2. The *coherence* of x and y is a measure of how reliable are the estimates $\hat{H}_1$ and $\hat{H}_2$ (Definition 11.2 page 105, Definition 11.3 page 106). If the coherence is 0.70 or above, then we can have high confidence that the estimates $\hat{H}_1$ and $\hat{H}_2$ are "good" estimates.[23]

11.8.2 Definitions

Definition 11.12. [24] *Let* **S** *be a* SYSTEM *with input* $x(n)$ *and output* $y(n)$.

DEF

*The **complex coherence** function is defined as* $$C_{xy}(\omega) \triangleq \frac{\tilde{S}^*_{xy}(\omega)}{\sqrt{\tilde{S}_{xx}(\omega)\tilde{S}_{yy}(\omega)}}$$

*The **ordinary coherence** function is defined as* $$\gamma^2_{xy}(\omega) \triangleq \frac{\left|\tilde{S}_{xy}(\omega)\right|^2}{\tilde{S}_{xx}(\omega)\tilde{S}_{yy}(\omega)}$$

Proposition 11.7.

PRP

$$\gamma^2_{xy}(\omega) = \frac{\hat{H}_1(\omega)}{\hat{H}_2(\omega)}$$

✎PROOF:

$$\boxed{\gamma^2_{xy}(\omega)} \triangleq \frac{\left|\tilde{S}_{xy}\right|^2}{\tilde{S}_{xx}\tilde{S}_{yy}}$$ by definition of γ^2_{xy} (Definition 11.12 page 126)

$$= \frac{\tilde{S}^*_{xy}/\tilde{S}_{xx}}{\tilde{S}_{yy}/\tilde{S}_{xy}} \triangleq \boxed{\frac{\hat{H}_1}{\hat{H}_2}}$$ by definitions of $\hat{H}_1$ and $\hat{H}_2$ (Definition 11.2 page 105, Definition 11.3 page 106)

⇨

Remark 11.2. Note that the *complex transmissibility* $\tilde{T}'_{xy}$ of Remark 11.1 provides a nice mathematical symmetry (always a good sign of good direction) with *coherence* in the system identification family tree. In particular, note that the following:

REM

$$C_{xy} \triangleq \sqrt{\frac{\hat{H}^*_1}{\hat{H}_2}} \text{ whereas } \tilde{T}'_{xy} \triangleq \sqrt{\hat{H}_1\hat{H}_2}$$

✎PROOF:

$$\sqrt{\frac{\hat{H}^*_1(\omega)}{\hat{H}_2(\omega)}}$$ by definition of $\hat{H}_{gm}$ (Definition 11.5 page 107)

⇨

11.8.3 A warning

Estimators yield, as the name implies, estimates. These estimates in general contain some error.

[23] Liang and Lee (2015) pages 363–365 ⟨7.4.2COHERECE FUNCTION⟩

[24] Chen et al. (2012) page 4699(1), (2), Liang and Lee (2015) pages 363–365 ⟨7.4.2 Coherence function⟩, Ewins (1986) page 131 ⟨$\gamma^2 = H_1(\omega)/H_2(\omega)$ (3.8)⟩

Example 11.1 (The K=1 Welch estimate of coherence). Suppose we have two *uncorrelated* stationary sequences $\mathsf{x}(n)$ and $\mathsf{y}(n)$. Then, there CSD $\mathsf{S}_{\mathsf{xy}}(\omega)$ should be 0 because

$$\begin{aligned}
\mathsf{S}_{\mathsf{xy}}(\omega) &\triangleq \breve{\mathbf{F}}\mathsf{E}\mathsf{R}_{\mathsf{xy}}(m) \\
&= \breve{\mathbf{F}}\mathsf{E}[x(n)y[n+m]] \\
&= \breve{\mathbf{F}}\left[\mathsf{E}_{\mathsf{x}}(n)\right]\left[\mathsf{E}_{\mathsf{y}}[n+m]\right] \\
&= \breve{\mathbf{F}}[0][0] \\
&= 0
\end{aligned}$$

This will give a coherence of 0 also:

$$C(\omega) = \frac{\mathsf{S}_{\mathsf{xy}}}{\sqrt{\mathsf{S}_{\mathsf{xx}}\mathsf{S}_{\mathsf{yy}}}} = 0$$

However, the Welch estimate with $K = 1$ will yield

$$\begin{aligned}
|C(\omega)| &= \left|\frac{\mathsf{S}_{\mathsf{xy}}}{\sqrt{\mathsf{S}_{\mathsf{xx}}\mathsf{S}_{\mathsf{yy}}}}\right| \\
&= \left|\frac{(\tilde{\mathbf{F}}x)(\tilde{\mathbf{F}}y)^*}{\sqrt{\left|\tilde{\mathbf{F}}x\right|^2\left|\tilde{\mathbf{F}}y\right|^2}}\right| \\
&= 1
\end{aligned}$$

Part III

Channel Distortion

CHAPTER 12

OPTIMAL SYMBOL DETECTION

12.1 ML Estimation

Theorem 12.1. *In an AWGN channel with received signal* $\mathsf{r}(t) = s(t;\phi) + n(t)$ *Let*

- $\mathsf{r}(t) = s(t;\phi) + n(t)$ *be the received signal in an AWGN channel*
- $n(t)$ *a Gaussian white noise process*
- $\mathsf{s}(t;\phi)$ *the transmitted signal such that*

$$s(t;\phi) = \sum_{n\in\mathbb{Z}} a_n \lambda(t-nT)\cos(2\pi f_c t + \theta_n + \phi).$$

Then the optimal ML estimate of ϕ *is either of the two equivalent expressions*

THM

$$\begin{aligned}
\hat{\phi}_{\mathrm{ml}} &= -\operatorname{atan}\left[\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t + \theta_n)\,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\cos(2\pi f_c t + \theta_n)\,\mathrm{d}t}\right] \\
&= \arg_\phi\left(\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\lambda(t-nT)\sin(2\pi f_c t + \theta_n + \phi)\right]\,\mathrm{d}t = 0\right).
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\hat{\phi}_{\mathrm{ml}} &= \arg_\phi\left(2\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathrm{d}t = \frac{\partial}{\partial\phi}\int_{t\in\mathbb{R}} s^2(t;\phi)\,\mathrm{d}t\right) && \text{by Theorem 7.6 page 76} \\
&= \arg_\phi\left(2\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathrm{d}t = \frac{\partial}{\partial\phi}\|s(t;\phi)\|^2\,\mathrm{d}t\right) \\
&= \arg_\phi\left(2\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}s(t;\phi)\right]\,\mathrm{d}t = 0\right) \\
&= \arg_\phi\left(\int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\frac{\partial}{\partial\phi}\sum_{n\in\mathbb{Z}} a_n\lambda(t-nT)\cos(2\pi f_c t + \theta_n + \phi)\right]\,\mathrm{d}t = 0\right)
\end{aligned}$$

$$= \arg_\phi\left(-\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\left[\lambda(t-nT)\sin(2\pi f_c t+\theta_n+\phi)\right]\,\mathrm{d}t = 0\right)$$
$$= \arg_\phi\left(\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\left[\sin(2\pi f_c t+\theta_n)\cos(\phi)+\sin(\phi)\cos(2\pi f_c t+\theta_n)\right]\,\mathrm{d}t = 0\right)$$
$$= \arg_\phi\left(\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(\phi)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t = -\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\cos(\phi)\,\mathrm{d}t\right)$$
$$= \arg_\phi\left(\sin(\phi)\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t = -\cos(\phi)\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t\right)$$
$$= \arg_\phi\left(\frac{\sin(\phi)}{\cos(\phi)} = -\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)$$
$$= \arg_\phi\left(\tan(\phi) = -\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)$$
$$= \arg_\phi\left(\phi = -\operatorname{atan}\left(\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)\right)$$
$$= -\operatorname{atan}\left(\frac{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\sin(2\pi f_c t+\theta_n)\,\mathrm{d}t}{\sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} \mathsf{r}(t)\lambda(t-nT)\cos(2\pi f_c t+\theta_n)\,\mathrm{d}t}\right)$$

✎

12.2 Generalized coherent modulation

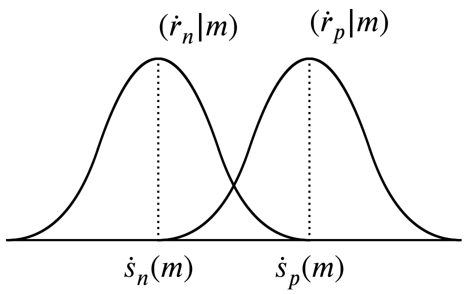


Figure 12.1: Distributions of orthonormal components

Theorem 12.2. *Let*

- $(V, \langle\cdot\,|\,\cdot\rangle, S)$ *be a modulation space*
- $\Psi \triangleq \{\psi_n(t) : n = 1, 2, \ldots, N\}$ *be a set of orthonormal functions that span* S
- $\dot{r}_n \triangleq \langle \mathsf{r}(t)\,|\,\psi_n(t)\rangle$
- $R \triangleq \{\dot{r}_n : n = 1, 2, \ldots, N\}$
- $\dot{s}_n(m) \triangleq \langle s(t;m)\,|\,\psi_n(t)\rangle$

and let V *be partitioned into* ***decision regions***

$$\{D_m : m = 1, 2, \ldots, |S|\}$$

such that

$$\mathsf{r}(t) \in D_{\hat{m}} \iff \hat{m} = \arg\max_m \mathsf{P}\{s(t;m)|\mathsf{r}(t)\}.$$

Then the ***probability of detection error*** *is*

THM
$$\mathsf{P}\{error\} = 1 - \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \sum_m \mathsf{P}\{m\ sent\} \int_{\boldsymbol{r}\in D_m} \exp\frac{-1}{2\sigma^2}\sum_{n=1}^N [\dot{r}_n - \dot{s}_n(m)]^2\, d\boldsymbol{r}.$$

✎PROOF:

$$\begin{aligned}
\mathsf{P}\{\text{error}\} &= 1 - \mathsf{P}\{\text{no error}\}\\
&= 1 - \sum_m \mathsf{P}\left\{(m\text{ sent}) \wedge (\hat{m} = m\text{ detected})\right\}\\
&= 1 - \sum_m \mathsf{P}\left\{(\hat{m} = m\text{ detected})|(m\text{ sent})\right\}\mathsf{P}\{m\text{ sent}\}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\,\mathsf{P}\{\boldsymbol{r}|(m\text{ sent})\}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \mathsf{p}\,[\boldsymbol{r}|(\mathsf{m\ sent})]\,d\boldsymbol{r}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \prod_n \mathsf{p}\left[\dot{\mathsf{r}}_n|\mathsf{m}\right]\,d\boldsymbol{r}\\
&= 1 - \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \prod_{n=1}^N \frac{1}{\sqrt{2\pi\sigma^2}}\exp\frac{-[\dot{r}_n - \mathsf{E}\dot{r}_n]^2}{2\sigma^2}\,d\boldsymbol{r}\\
&= 1 - \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \sum_m \mathsf{P}\{m\text{ sent}\}\int_{\boldsymbol{r}\in D_m} \exp\frac{-1}{2\sigma^2}\sum_{n=1}^N [\dot{r}_n - \dot{s}_n(m)]^2\,d\boldsymbol{r}
\end{aligned}$$

✏

12.3 Frequency Shift Keying (FSK)

Theorem 12.3. *In an FSK modulation space, the optimal ML estimator of m is*

THM
$$\hat{m} = \arg\max_m \dot{r}_m.$$

✎PROOF:

$$\begin{aligned}
\hat{m} &= \arg\max_m \mathsf{P}\{\mathsf{r}(t)|s(t;m)\}\\
&= \arg\min_m \sum_{n=1}^N [\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 7.6 (page 76)}\\
&= \arg\min_m \sum_{n=1}^N [\dot{r}_n^2 - 2\dot{r}_n\dot{s}_n(m) + \dot{s}_n^2(m)]\\
&= \arg\min_m \sum_{n=1}^N [-2\dot{r}_n\dot{s}_n(m) + \dot{s}_n^2(m)] && \dot{r}_n^2 \text{ is independent of } m\\
&= \arg\min_m \sum_{n=1}^N [-2\dot{r}_n a\bar{\delta}_{mn} + a^2\bar{\delta}_{mn}]\\
&= \arg\min_m [-2a\dot{r}_m + a^2]
\end{aligned}$$

$$= \arg\min_m [-\dot{r}_m] \qquad a \text{ and } 2 \text{ independent of } m$$
$$= \arg\max_m [\dot{r}_m]$$

✏

Theorem 12.4. *If an FSK modulation space let*

$$\begin{array}{rcl|lcl|l}
z_2 & \triangleq & \dot{r}_1(1) - \dot{r}_2(1) & z_2 > 0 & \implies & \dot{r}_1 > \dot{r}_2 & m = 1 \\
z_3 & \triangleq & \dot{r}_1(1) - \dot{r}_3(1) & z_3 > 0 & \implies & \dot{r}_1 > \dot{r}_3 & m = 1 \\
 & \vdots & & & & & \\
z_M & \triangleq & \dot{r}_1(1) - \dot{r}_M(1) & z_M > 0 & \implies & \dot{r}_1 > \dot{r}_M & m = 1
\end{array}$$

Then the ***probability of detection error*** *is*

THM
$$\boxed{\mathsf{P}\{\textit{error}\} = 1 - \frac{M-1}{M}\int_0^\infty \int_0^\infty \cdots \int_0^\infty \mathsf{p}(z_2, z_3, \ldots, z_M)\, dz_2 dz_3 \cdots dz_M} \quad \textit{where}$$

$$\mathsf{p}(z_2, z_3, \ldots, z_M) = \frac{1}{(2\pi)^{\frac{M-1}{2}}\sqrt{\det R}} \exp -\frac{1}{2} \begin{bmatrix} z_2 - \dot{s} \\ z_3 - \dot{s} \\ \vdots \\ z_M - \dot{s} \end{bmatrix}^T R^{-1} \begin{bmatrix} z_2 - \dot{s} \\ z_3 - \dot{s} \\ \vdots \\ z_M - \dot{s} \end{bmatrix}$$

and

$$R = \begin{bmatrix} \operatorname{cov}[z_2, z_2] & \operatorname{cov}[z_2, z_3] & \cdots & \operatorname{cov}[z_2, z_M] \\ \operatorname{cov}[z_3, z_2] & \operatorname{cov}[z_3, z_3] & \cdots & \operatorname{cov}[z_3, z_M] \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cov}[z_M, z_2] & \operatorname{cov}[z_M, z_3] & \cdots & \operatorname{cov}[z_M, z_M] \end{bmatrix} = N_o \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$$

The inverse matrix R^{-1} is equivalent to (????)

$$R^{-1} \stackrel{?}{=} \frac{1}{M N_o} \begin{bmatrix} M-1 & -1 & \cdots & -1 \\ -1 & M-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & M-1 \end{bmatrix}$$

✎PROOF:

$$\begin{aligned} \mathsf{E} z_k &= \mathsf{E}\left[\dot{r}_{11} - \dot{r}_{1k}\right] \\ &= \mathsf{E}\dot{r}_{11} - \mathsf{E}\dot{r}_{1k} \\ &= \dot{s} - 0 \\ &= \dot{s} \end{aligned}$$

$$
\begin{aligned}
\operatorname{cov}\left[z_m, z_n\right] &= \mathsf{E}\left[z_m z_n\right] - [\mathsf{E} z_m][\mathsf{E} z_n] \\
&= \mathsf{E}\left[(\dot{r}_{11} - \dot{r}_{1m})(\dot{r}_{11} - \dot{r}_{1n})\right] - \dot{s}^2 \\
&= \mathsf{E}\left[\dot{r}_{11}^2 - \dot{r}_{11}\dot{r}_{1n} - \dot{r}_{1m}\dot{r}_{11}\dot{r}_{1m}\dot{r}_{1n}\right] - \dot{s}^2 \\
&= [\operatorname{var}\dot{r}_{11} + (\mathsf{E}\dot{r}_{11})^2] - \mathsf{E}\left[\dot{r}_{11}\right]\mathsf{E}\left[\dot{r}_{1n}\right] - \mathsf{E}\left[\dot{r}_{1m}\right]\mathsf{E}\left[\dot{r}_{11}\right] + [\operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] + (\mathsf{E}\dot{r}_{1m})(\mathsf{E}\dot{r}_{1n})] - \dot{s}^2 \\
&= [\operatorname{var}\dot{r}_{11} + \dot{s}^2] - a\cdot 0 - 0\cdot a + [\operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] + 0\cdot 0] - \dot{s}^2 \\
&= \operatorname{var}\dot{r}_{11} + \operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] \\
&= N_o + \operatorname{cov}\left[\dot{r}_{1m}, \dot{r}_{1n}\right] \\
&= \begin{cases} 2N_o & \text{for } m = n \\ N_o & \text{for } m \neq n. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
P\{\text{error}\} &= 1 - P\{\text{no error}\} \\
&= 1 - \sum_{m=1}^{M} P\{\text{m transmitted}) \wedge (\forall k \neq m, \dot{r}_m > \dot{r}_k\} \\
&= 1 - (M-1)P\{1 \text{ transmitted}) \wedge (\dot{r}_{11} > \dot{r}_{12}) \wedge (\dot{r}_{11} > \dot{r}_{13}) \wedge \cdots \wedge (\dot{r}_{11} > \dot{r}_{1M})\} \\
&= 1 - (M-1)P\{(\dot{r}_{11} - \dot{r}_{12} > 0) \wedge (\dot{r}_{11} - \dot{r}_{13} > 0) \wedge \cdots \wedge (\dot{r}_{11} - \dot{r}_{1M} > 0) | 1 \text{ transmitted})\} P\{1 \text{ transmitted})\} \\
&= 1 - \frac{M-1}{M} P\{(z_2 > 0) \wedge (z_3 > 0) \wedge \cdots \wedge (z_M > 0) | 1 \text{ transmitted})\} \\
&= 1 - \frac{M-1}{M} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \mathsf{p}(z_2, z_3, \ldots, z_M)\, dz_2 dz_3 \cdots dz_M.
\end{aligned}
$$

✏

12.4 Quadrature Amplitude Modulation (QAM)

12.4.1 Receiver statistics

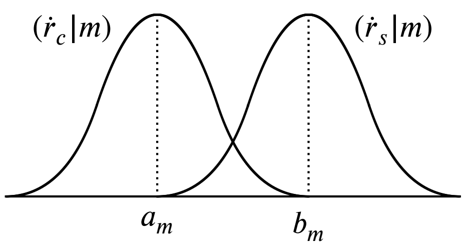


Figure 12.2: Distributions of QAM components

Theorem 12.5. *Let $(V, \langle \cdot | \cdot \rangle$ be a QAM modulation space such that*

$$
\begin{aligned}
\mathsf{r}(t) &= \mathsf{s}(t; m) + \mathsf{n}(t) \\
\dot{r}_c &\triangleq \langle \mathsf{r}(t) \,|\, \psi_c(t) \rangle \\
\dot{r}_s &\triangleq \langle \mathsf{r}(t) \,|\, \psi_s(t) \rangle .
\end{aligned}
$$

Then $(\dot{r}_c|m)$ *and* $(\dot{r}_s|m)$ *are* ***independent*** *and have* ***marginal distributions***

$$\begin{aligned}(\dot{r}_c|m) &\sim \mathsf{N}\left(a_m, \sigma^2\right) = \mathsf{N}\left(r_m \cos\theta_m, \sigma^2\right)\\ (\dot{r}_s|m) &\sim \mathsf{N}\left(b_m, \sigma^2\right) = \mathsf{N}\left(r_m \sin\theta_m, \sigma^2\right).\end{aligned}$$

✎PROOF: See Theorem 7.5 (page 76) page 76. ✏

12.4.2 Detection

Theorem 12.6. *Let* $(V, \langle\cdot\,|\,\cdot\rangle, S)$ *be a QAM modulation space with*

$$\begin{aligned}\mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t)\\ \dot{r}_c &\triangleq \langle \mathsf{r}(t)\,|\,\psi_c(t)\rangle\\ \dot{r}_s &\triangleq \langle \mathsf{r}(t)\,|\,\psi_s(t)\rangle\,.\end{aligned}$$

Then $\{\dot{r}_c, \dot{r}_s\}$ *are sufficient statistics for optimal ML detection and the optimal ML estimate of* m *is*

$$\hat{u}_{\mathsf{ml}}[m] = \arg\min_m \left[(\dot{r}_c - a_m)^2 + (\dot{r}_s - b_m)^2\right].$$

✎PROOF:

$$\begin{aligned}\hat{u}_{\mathsf{ml}}[m] &= \arg\max_m \mathsf{P}\{\mathsf{r}(t)|s(t;m)\} && \text{by Definition 6.1 (page 64)}\\ &= \arg\min_m \sum_{n=1}^{N} [\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 7.6 (page 76)}\\ &= \arg\min_m \left[(\dot{r}_c - a_m)^2 + (\dot{r}_s - b_m)^2\right]\end{aligned}$$

✏

12.4.3 Probability of error

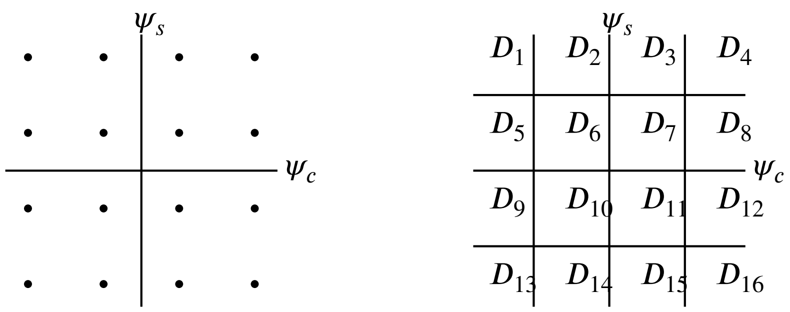


Figure 12.3: QAM-16 cosstellation and decision regions

Theorem 12.7. *In a QAM-16 constellation as shown in Figure 12.3 (page 136), the probability of error is*

$$\mathsf{P}\{error\} = \frac{9}{4}Q^2\left(\frac{\dot{s}_{21} - \dot{s}_{11}}{2N_o}\right).$$

✎PROOF: Let

$$d \triangleq \dot{s}_{21} - \dot{s}_{11}.$$

Then

$$\begin{aligned}
\mathsf{P}\{\text{error}\} &= \sum_{m=1}^{M} \mathsf{P}\left\{[s(t;m) \text{ transmitted }] \wedge [(\dot{r}_1, \dot{r}_2) \notin D_m]\right\} \\
&= \sum_{m=1}^{M} \mathsf{P}\left\{[(\dot{r}_1, \dot{r}_2) \notin D_m] | [s(t;m) \text{ transmitted }]\right\} \mathsf{P}\left\{[s(t;m) \text{ transmitted }]\right\} \\
&= \frac{1}{M} \sum_{m=1}^{M} \mathsf{P}\left\{[(\dot{r}_1, \dot{r}_2) \notin D_m] | [s(t;m) \text{ transmitted }]\right\} \\
&= \frac{1}{M} \left[4\mathsf{P}\left\{(\dot{r}_1, \dot{r}_2) \notin D_1 | s_1(t)\right\} + 8\mathsf{P}\left\{(\dot{r}_1, \dot{r}_2) \notin D_2 | s_2(t)\right\} + 4\mathsf{P}\left\{(\dot{r}_1, \dot{r}_2) \notin D_6 | s_6(t)\right\}\right] \\
&= \frac{1}{M} \left[4 \int\int_{(x,y)\notin D_1} \mathsf{p}_{xy|1}(x,y)\, \mathsf{d}x\, \mathsf{d}y + 8 \int\int_{(x,y)\notin D_2} \mathsf{p}_{xy|2}(x,y)\, \mathsf{d}x\, \mathsf{d}y + \right. \\
&\quad \left. 4 \int\int_{(x,y)\notin D_6} \mathsf{p}_{xy|6}(x,y)\, \mathsf{d}x\, \mathsf{d}y \right] \\
&= \frac{1}{M} \left[4 \int\int_{(x,y)\notin D_1} \mathsf{p}_{x|1}(x)\mathsf{p}_{y|1}(y)\, \mathsf{d}x\, \mathsf{d}y + 8 \int\int_{(x,y)\notin D_2} \mathsf{p}_{x|2}(x)\mathsf{p}_{y|2}(y)\, \mathsf{d}x\, \mathsf{d}y + \right. \\
&\quad \left. 4 \int\int_{(x,y)\notin D_6} \mathsf{p}_{x|6}(x)\mathsf{p}_{y|6}(y)\, \mathsf{d}x\, \mathsf{d}y \right] \\
&= \frac{1}{M} \left[4Q\left(\frac{d}{2N_o}\right) Q\left(\frac{d}{2N_o}\right) + 8Q\left(\frac{d}{2N_o}\right) 2Q\left(\frac{d}{2N_o}\right) + 4 \cdot 2Q\left(\frac{d}{2N_o}\right) 2Q\left(\frac{d}{2N_o}\right)\right] \\
&= \frac{9}{4} Q^2\left(\frac{d}{2N_o}\right)
\end{aligned}$$

✏

12.5 Phase Shift Keying (PSK)

12.5.1 Receiver statistics

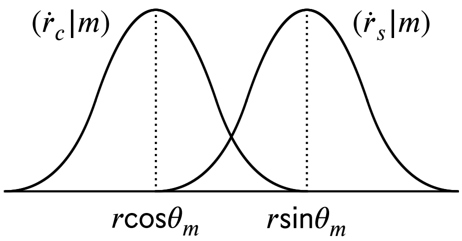


Figure 12.4: Distributions of PSK components

Theorem 12.8. *Let*

$$\begin{aligned}
\dot{r}_c &\triangleq \langle \mathsf{r}(t) \,|\, \psi_c(t) \rangle \\
\dot{r}_s &\triangleq \langle \mathsf{r}(t) \,|\, \psi_s(t) \rangle \\
\theta_m &\triangleq \operatorname{atan}\left[\frac{\dot{r}_s(m)}{\dot{r}_c(m)}\right].
\end{aligned}$$

The statistics $(\dot{r}_c|m)$ *and* $(\dot{r}_s|m)$ *are **independent** with marginal distributions*

$$\begin{aligned}
(\dot{r}_c|m) &\sim \mathsf{N}\left(r\cos\theta_m, \sigma^2\right) \\
(\dot{r}_s|m) &\sim \mathsf{N}\left(r\sin\theta_m, \sigma^2\right) \\
\mathsf{p}_{\theta_m}(\theta|m) &= \int_0^\infty x \mathsf{p}_{\dot{r}_c}(x|m) \mathsf{p}_{\dot{r}_s}(x\tan\theta|m) dx.
\end{aligned}$$

✎PROOF:

Indepence and marginal distributions of $\dot{r}_1(m)$ and $\dot{r}_2(m)$ follow directly from Theorem 7.5 (page 76) (page 76).

Let $X \triangleq \dot{r}_1(m)$, $Y \triangleq \dot{r}_2(m)$ and $\Theta \triangleq \theta_m$. Then[1]

$$\begin{aligned}
\mathsf{p}_\theta(\theta)d\theta &\triangleq \mathsf{P}\{\theta < \Theta \le \theta + d\theta\} \\
&= \mathsf{P}\left\{\theta < \operatorname{atan}\frac{Y}{X} \le \theta + d\theta\right\} \\
&= \mathsf{P}\left\{\tan(\theta) < \frac{Y}{X} \le \tan(\theta + d\theta)\right\} \\
&= \mathsf{P}\left\{\tan(\theta) < \frac{Y}{X} \le \tan\theta + (1+\tan^2\theta)\, \mathsf{d}\theta\right\} \\
&= \int_0^\infty \mathsf{P}\left\{\left[\tan\theta < \frac{Y}{X} \le \tan\theta + (1+\tan^2\theta)\, \mathsf{d}\theta\right] \wedge \left[(x < X \le x + \mathsf{d}x)\right]\right\} \\
&= \int_0^\infty \mathsf{P}\left\{\tan\theta < \frac{Y}{x} \le \tan\theta + (1+\tan^2\theta)\, \mathsf{d}\theta \,\middle|\, x < X \le x + dx\right\} \mathsf{P}\{x < X \le x + \mathsf{d}x\} \\
&= \int_0^\infty \mathsf{P}\left\{x\tan\theta < Y \le x\tan\theta + x(1+\tan^2\theta)\, \mathsf{d}\theta) \,\middle|\, X = x\right\} \mathsf{p}_\mathsf{x}(x)\, \mathsf{d}x \\
&= \int_0^\infty [\mathsf{p}_\mathsf{Y}(x\tan\theta)x(1+\tan^2\theta)]\mathsf{p}_\mathsf{x}(x)\, \mathsf{d}x\, \mathsf{d}\theta \\
&= (1+\tan^2\theta)\int_0^\infty x\mathsf{p}_\mathsf{Y}(x\tan\theta)\mathsf{p}_\mathsf{x}(x)\, \mathsf{d}x\, \mathsf{d}\theta
\end{aligned}$$

$$\implies$$

$$\mathsf{p}_\theta(\theta)d\theta = (1+\tan^2\theta)\int_0^\infty x\mathsf{p}_\mathsf{Y}(x\tan\theta)\mathsf{p}_\mathsf{x}(x)\, \mathsf{d}x$$

✏

[1] A similar example is in Papoulis (1991), page 138

12.5.2 Detection

Theorem 12.9. *Let* $(V, \langle\cdot\,|\,\cdot\rangle, S)$ *be a PSK modulation space with*

$$\begin{aligned} \mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t) \\ \dot{r}_c &\triangleq \langle \mathsf{r}(t)\,|\,\psi_c(t)\rangle \\ \dot{r}_s &\triangleq \langle \mathsf{r}(t)\,|\,\psi_s(t)\rangle . \end{aligned}$$

Then $\{\dot{r}_c, \dot{r}_s\}$ *are sufficient statistics for optimal ML detection and the optimal ML estimate of* m *is*

$$\hat{u}_{\mathsf{ml}}[m] = \arg\min_m \left[(\dot{r}_1 - r\cos\theta_m)^2 + (\dot{r}_2 - r\sin\theta_m)^2\right].$$

✎PROOF:

$$\begin{aligned} \hat{u}_{\mathsf{ml}}[m] &= \arg\max_m \mathsf{P}\{\mathsf{r}(t)|s(t;m)\} && \text{by Definition 6.1 (page 64)} \\ &= \arg\min_m \sum_{n=1}^{N} [\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 7.6 (page 76)} \\ &= \arg\min_m \left[(\dot{r}_1 - r\cos\theta_m)^2 + (\dot{r}_2 - r\sin\theta_m)^2\right]. \end{aligned}$$

✏

12.5.3 Probability of error

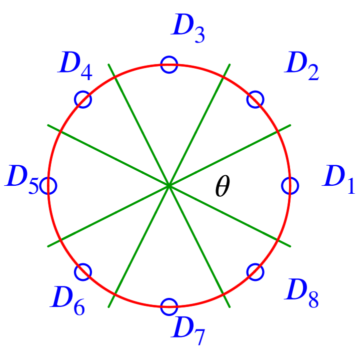


Figure 12.5: PSK-8 Decision regions

Theorem 12.10. *The probability of error using PSK modulation is*

$$\mathsf{P}\{error\} = M\left[1 - \int_{\frac{2\pi}{M}\left(m-\frac{3}{2}\right)}^{\frac{2\pi}{M}\left(m-\frac{1}{2}\right)} \mathsf{p}_{\theta_1}(\theta)\, d\theta\right].$$

✎PROOF: See Figure 12.5 (page 139).

$$\begin{aligned}
\mathsf{P}\{\text{error}\} &= \sum_{m=1}^{M} \mathsf{P}\left\{\text{error}|s(t;m) \text{ was transmitted}\right\} \\
&= M\mathsf{P}\left\{\text{error}|s_1(t) \text{ was transmitted}\right\} \\
&= M\left[1 - \int_{\frac{2\pi}{M}\left(m-\frac{3}{2}\right)}^{\frac{2\pi}{M}\left(m-\frac{1}{2}\right)} \mathsf{p}_{\theta_1}(\theta)\, d\theta\right].
\end{aligned}$$

⇨

12.6 Pulse Amplitude Modulation (PAM)

12.6.1 Receiver statistics

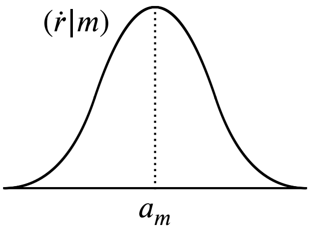


Figure 12.6: Distribution of PAM component

Theorem 12.11. *Let $(V, \langle\cdot\,|\,\cdot\rangle$ be a PAM modulation space such that*

$$\begin{aligned}
\mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t) \\
\dot{r}_c &\triangleq \langle \mathsf{r}(t)\,|\,\psi_c(t)\rangle \\
\dot{r}_s &\triangleq \langle \mathsf{r}(t)\,|\,\psi_s(t)\rangle\,.
\end{aligned}$$

*Then $(\dot{r}|m)$ has **distribution***

$$\dot{r}(m) \sim \mathsf{N}\left(a_m, \sigma^2\right).$$

✎PROOF: This follows directly from Theorem 7.5 (page 76) (page 76). ⇨

12.6.2 Detection

Theorem 12.12. *Let $(V, \langle\cdot\,|\,\cdot\rangle, S)$ be a PAM modulation space with*

$$\begin{aligned}
\mathsf{r}(t) &= \mathsf{s}(t;m) + \mathsf{n}(t) \\
\dot{r} &\triangleq \langle \mathsf{r}(t)\,|\,\psi(t)\rangle\,.
\end{aligned}$$

Then $\dot{r}$ is a sufficient statistic for the optimal ML detection of m and the optimal ML estimate of m is

$$\hat{u}_{\mathsf{ml}}[m] = \arg\min_m |\dot{r} - a_m|.$$

✎PROOF:

$$\begin{aligned}
\hat{u}_{\mathsf{ml}}[m] &= \arg\max_m \mathsf{P}\left\{\mathsf{r}(t)|a_m\right\} && \text{by Definition 6.1 (page 64)}\\
&= \arg\min_m \sum_{n=1}^{N}[\dot{r}_n - \dot{s}_n(m)]^2 && \text{by Theorem 7.6 (page 76)}\\
&= \arg\min_m [\dot{r} - \dot{s}(m)]^2\\
&= \arg\min_m |\dot{r} - \dot{s}(m)|
\end{aligned}$$

✏

12.6.3 Probability of error

Theorem 12.13. *The probability of detection error in a PAM modulation space is*

$$\mathsf{P}\{error\} = 2\frac{M-1}{M}\mathsf{Q}\left[\frac{a_2 - a_1}{2\sqrt{N_o}}\right].$$

✎PROOF: Let $d \triangleq a_2 - a_1$ and $\sigma \triangleq \sqrt{\operatorname{var}\dot{r}} = \sqrt{N_o}$. Also, let the decision regions D_m be as illustrated in Figure 12.7 (page 141). Then

$$\begin{aligned}
\mathsf{P}\{error\} &= \sum_{m=1}^{M} \mathsf{P}\left\{s(t;m)\text{ sent } \wedge r \notin D_m\right\}\\
&= \sum_{m=1}^{M} \mathsf{P}\left\{\dot{r} \notin D_m | \mathsf{s}(t;m)\text{ sent }\right\} \mathsf{P}\left\{s(t;m)\text{ sent }\right\}\\
&= \sum_{m=1}^{M} \mathsf{P}\left\{\dot{r}_m \notin D_m\right\} \frac{1}{M}\\
&= \frac{1}{M}\left(\mathsf{Q}\left[\frac{d}{2\sigma}\right] + 2\mathsf{Q}\left[\frac{d}{2\sigma}\right] + \ldots 2\mathsf{Q}\left[\frac{d}{2\sigma}\right] + \mathsf{Q}\left[\frac{d}{2\sigma}\right]\right)\\
&= 2\frac{M-1}{M}\mathsf{Q}\left[\frac{d}{2\sigma}\right]\\
&= 2\frac{M-1}{M}\mathsf{Q}\left[\frac{\dot{s}_2 - \dot{s}_1}{2\sqrt{N_o}}\right]
\end{aligned}$$

✏

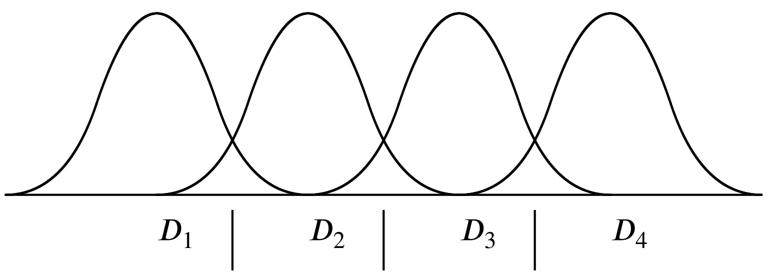


Figure 12.7: 4-ary PAM in AWGN channel

CHAPTER 13

BANDLIMITED CHANNEL (ISI)

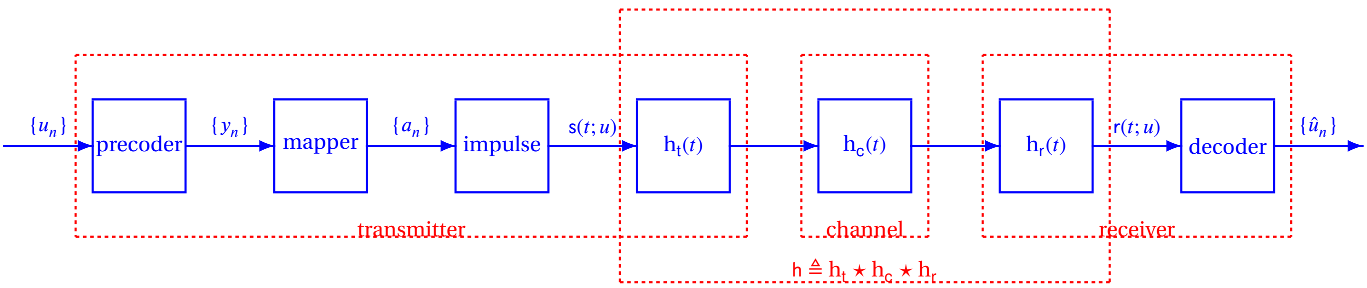


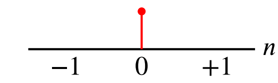
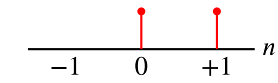
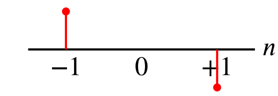
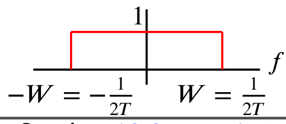
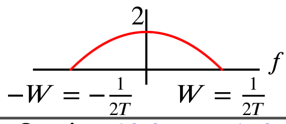
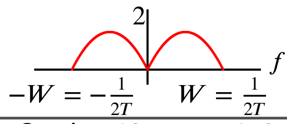
Figure 13.1: ISI system model

System disturbances. There are two fundamental disturbances in any communication system which increase the probability of communication error:

1. noise
2. intersymbol interference (ISI)

Noise is produced by a number of sources; one of them being *thermal noise* and therefore can never be eliminated in any system which operates above -273° C (absolute zero). ISI is produced as a result of band-limited communication channels. Unlike noise, it is possible to completely eliminate ISI by the proper selection of the symbol waveform used to carry information through the channel.

This chapter describes the cause of ISI in a communication system and discusses techniques of designing signaling waveforms with no ISI. Three solutions are presented and are summarized in the following table:

zero ISI solution	duobinary solution	modified duobinary solution
$\mathsf{h}(nT) = \begin{cases} 1 & : \; n=0 \\ 0 & : \; \text{otherwise} \end{cases}$	$\mathsf{h}(nT) = \begin{cases} 1 & : \; n=0,1 \\ 0 & : \; \text{otherwise} \end{cases}$	$\mathsf{h}(nT) = \begin{cases} 1 & : \; n=-1 \\ -1 & : \; n=+1 \\ 0 & : \; \text{otherwise} \end{cases}$
		
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
$\frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f+\frac{n}{T}\right) = 1$	$\frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f+\frac{n}{T}\right) = 2e^{-i\pi fT}\cos(\pi fT)$	$\frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f+\frac{n}{T}\right) = i2\sin(2\pi fT)$
		
Section 13.2 page 145	Section 13.3 page 152	Section 13.4 page 159

13.1 Description of ISI

The channel model is illustrated in Figure 13.1 (page 143). The signal received at the decoder is

$$\mathsf{r}(t;u) = \sum_n a_n \mathsf{h}(t-nT).$$

We arbitrarily scale $\mathsf{h}(t)$ such that

$$\mathsf{h}(0) = 1.$$

If this signal is sampled at intervals T, we have

$$\begin{aligned}
\mathsf{r}(nT) &= \mathsf{r}(t)|_{t=nT} \\
&= \left.\sum_m a_m \mathsf{h}(t-mT)\right|_{t=nT} \\
&= \sum_m a_m \mathsf{h}(nT-mT) \\
&= a_n\mathsf{h}(0) + \sum_{m\neq n} a_m \mathsf{h}(nT-mT) \\
&= \underbrace{a_n}_{\text{desired}} + \underbrace{\sum_{m\neq n} a_m \mathsf{h}(nT-mT)}_{\text{ISI (not wanted)}}
\end{aligned}$$

At the sampling intervals, we only want a_n, not the other terms. These other terms are referred to as *Intersymbol Interference* (ISI).

Definition 13.1. ***Intersymbol interference*** *(ISI) is a communication system characteristic such that a received signal sample* $\mathsf{r}(nT)$ *is a function of one or more information values* $a_m, m \neq n$. *If* $\mathsf{r}(nT)$ *is a function of* a_n *alone, then we say the system has* ***zero ISI***.

If $\mathsf{h}(t)$ is properly designed, the communication system will have zero ISI.

13.2 Zero-ISI solution

13.2.1 Constraints

Previously we stated that for zero ISI,

$$\underbrace{a_n}_{\text{desired}} + \underbrace{\sum_{m\neq n} a_m \mathsf{h}(nT - mT)}_{\text{ISI (not wanted)}}$$

This equation is satisfied if and only if

$$\mathsf{h}(nT) = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n \neq 0 \end{cases}$$

Also, the channel imposes a band-width constraint W. These considerations can be combined into two fundamental constraints on the signaling pulse $\mathsf{h}(t)$:

$$\begin{array}{lllcl} ① & \textbf{sampling constraint}: & \mathsf{h}(nT) & = & \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n \neq 0 \end{cases} \\ ② & \textbf{bandwidth constraint}: & [\tilde{\mathbf{F}}\mathsf{h}](f) & = & 0 \text{ for } |f| \geq W. \end{array}$$

These two constraints are in conflict with each other. The sampling constraint is quite easy to satisfy by designing h with support (region on t where $\mathsf{h}(t) \neq 0$) only within $[0, T)$. However, giving h small support makes $\tilde{\mathsf{h}}$ have large bandwidth, violating the bandwidth constraint. However, Theorem 13.1 (next) gives a criterion which allows both constraints to be satisfied simultaneously.

Theorem 13.1 (Partition of unity criterion). [1] *Let $\tilde{\mathsf{h}}(f)$ be the Fourier Transform of a function $\mathsf{h}(t)$ and $T \in \mathbb{R}$ a constant. Then*

THM

$$\left[\mathsf{h}(nT) = \begin{cases} 1 & : \quad n = 0 \\ 0 & : \quad n \neq 0 \end{cases}\right] \iff \left[\frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) = 1.\right]$$

✎PROOF: This theorem is easily proven using the *Inverse Poisson's Summation Formula* (*IPSF*) (Theorem F.3 page 248) which states

$$\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) = T\sum_n \mathsf{h}(nT)e^{-i2\pi f nT}$$

1. Prove "only if" case ($\Longrightarrow$):

$$\begin{aligned} \frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) &= \sum_n \mathsf{h}(nT)e^{-i2\pi f nT} && \text{by IPSF} \\ &= \mathsf{h}(0) + \sum_{n\neq 0} \mathsf{h}(nT)e^{-i2\pi f nT} \\ &= 1 && \text{by left hypothesis} \end{aligned}$$

[1] Proakis (2001), page 557

2. Prove "if" case ($\Longleftarrow$):

$$\begin{aligned}
1 &= \frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f+\frac{n}{T}\right) && \text{by right hypothesis}\\
&= \sum_n \mathsf{h}(nT)e^{-i2\pi f nT} && \text{by IPSF}\\
&= \mathsf{h}(0) + \sum_{n\neq 0} \mathsf{h}(nT)e^{-i2\pi f nT}\\
&= \mathsf{h}(0) + \sum_{n\neq 0} \mathsf{h}(nT)\cos(2\pi f nT) - i\sum_{n\neq 0}\mathsf{h}(nT)\sin(2\pi f nT)\\
\Longrightarrow \mathsf{h}(nT) &= \begin{cases} 1 & : \quad n=0\\ 0 & : \quad n\neq 0\end{cases} && \text{because "1" is real for all } f
\end{aligned}$$

✑

13.2.2 Signaling rate limits

Definition 13.2. [2] *The **characteristic function** $\chi_A : X \to \{0,1\}$ of set A is defined as*

DEF
$$\chi_A(x) \triangleq \begin{cases} 1 & \textit{for} \quad x\in A\subseteq X\\ 0 & \textit{for} \quad x\notin A\subseteq X\end{cases}$$

Next are two complimentary theorems; both of which are closely related to the partition of unity criterion:

1. Nyquist signaling theorem (Theorem 13.2 (page 146)) A signal may be transmitted with zero-ISI if the signaling rate is less than or equal to $2W$.
2. Shannon sampling theorem (Theorem 13.3 (page 147)) Perfect reconstruction of a sampled signal is possible if the sampling rate is greater than or equal to $2W$.

Theorem 13.2 (Nyquist signaling theorem)**.** [3] *Let $s(t)$ be a signal of the form*

$$s(t) = \sum_n a_n \mathsf{h}(t-nT_1)$$

and with bandwidth

$$[\tilde{\mathbf{F}}s](f) = 0 \textit{ for } |f| \geq W.$$

Then there exists $\mathsf{h}(t)$ such that if

$$\frac{1}{T_1} \leq 2W$$

[2] Aliprantis and Burkinshaw (1998), page 126
[3] Proakis (2001), page 13

then

$$s(t) = \sum_n s(nT_1)\mathsf{h}(t - nT_1).$$

Furthermore, if

$$\frac{1}{T_1} = 2W$$

then

$$s(t) = \sum_n s(nT_1)\frac{\sin\left[\frac{\pi}{T_1}(t - nT_1)\right]}{\frac{\pi}{T_1}(t - nT_1)}.$$

✎PROOF: The upper signaling rate bound (equality) is proven by the partition of unity criterion. Given a signaling rate $1/T$, the pulse shape with the smallest bandwidth that forms a partition of unity in the frequency domain is the sync function in the time domain, which is a rectangular pulse in frequency domain given by

$$\frac{1}{2W}\chi_{[-W,+W]}(f).$$

✏

Theorem 13.3 (Shannon sampling theorem). [4] *Let $r(t)$ be a signal with bandwidth*

$$[\tilde{\mathbf{F}}r](f) = 0 \textit{ for } |f| \geq W$$

and sampled at time intervals T_2.

Then there exists $\mathsf{h}(t)$ such that if

$$\frac{1}{T_2} \geq 2W$$

then

$$s(t) = \sum_n s(nT_2)\mathsf{h}(t - nT_2).$$

Furthermore, if

$$\frac{1}{T_2} = 2W$$

then

$$s(t) = \sum_n s(nT_2)\frac{\sin\left[\frac{\pi}{T_2}(t - nT_2)\right]}{\frac{\pi}{T_2}(t - nT_2)}.$$

13.2.3 Zero-ISI system impulse responses

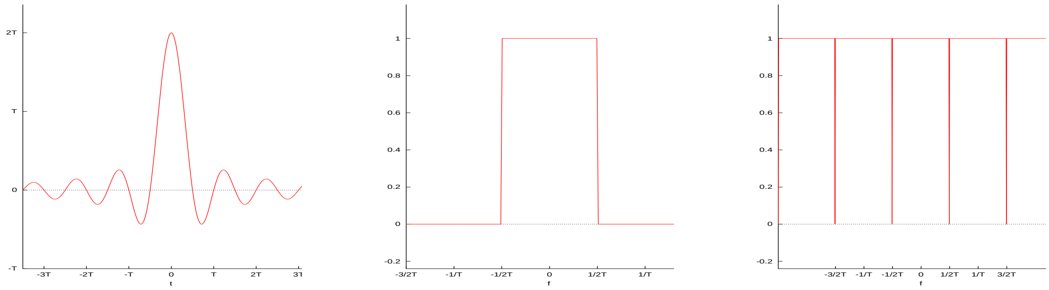
Using Partition of Unity Theorem 13.1, we can design ISI waveforms in the frequency domain and thus easily satisfy both the constraints given in Section 13.2.

[4] Proakis (2001), page 13

Nyquist Rate zero-ISI waveform

The maximum signaling rate is $1/T = 2W$ (Nyquist Signaling Theorem). If we signal at this maximum rate, there is only one waveform $\tilde{\mathsf{h}}$ which satisfies the partition of unity condition: $\tilde{\mathsf{h}}(f) = \chi_{[-1/2T,1/2T)}(f)$. In the time domain this is the sinc function

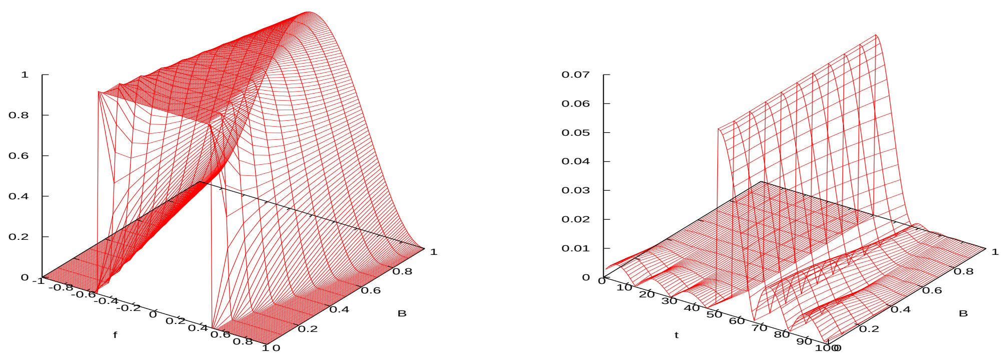
$$\mathsf{h}(t) = \frac{1}{T}\frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t}$$



$\mathsf{h}(t) = \frac{1}{T}\frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t}$ $\qquad$ $\tilde{\mathsf{h}}(f) = \chi_{[-1/2T,1/2T)}(f)$ $\qquad$ $\sum_n \tilde{\mathsf{h}}\left(f - \frac{n}{T}\right)$

Figure 13.2: Nyquist rate zero-ISI signaling waveform

Raised cosine zero-ISI waveforms



$\tilde{\mathsf{h}}(f)$ $\qquad$ $\mathsf{h}(t)$

Figure 13.3: Raised cosine for various roll-off factors β

The **Raised Cosine** is the Fourier Transform of one of the most widely used signaling waveforms.[5]

[5]Note: The raised cosine is similar to the *Meyer wavelet*. ref: (Vidakovic, 1999, page 65)

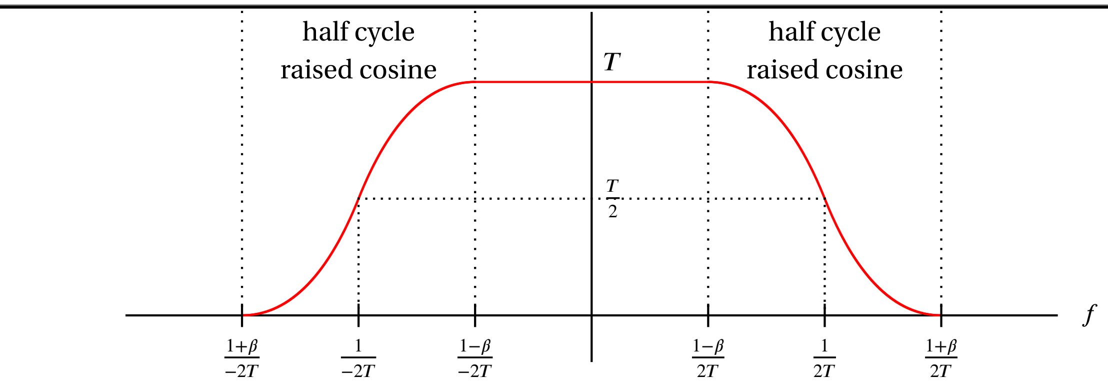


Figure 13.4: Raised cosine

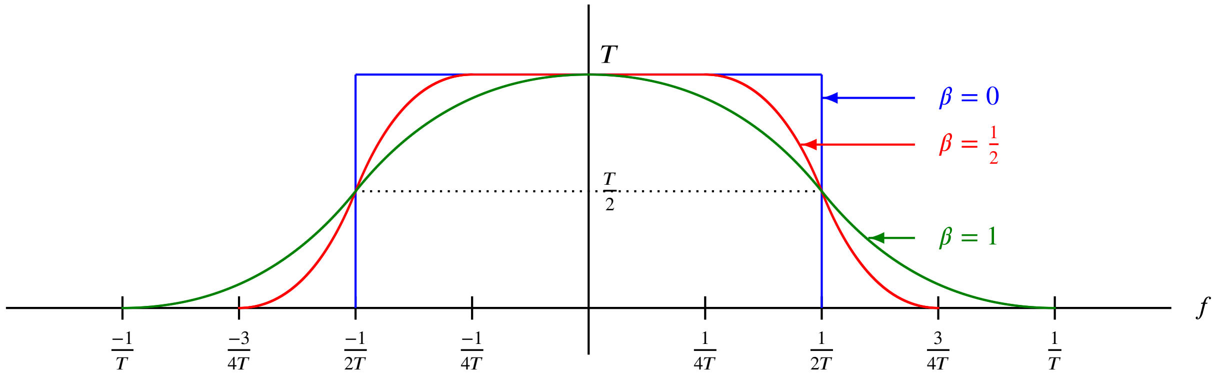


Figure 13.5: Raised cosine for various β values

In the frequency domain it has the form[6]

$$\tilde{\mathsf{h}}(f) = \begin{cases} T & : \quad 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left[1+\cos\left(\frac{\pi T}{\beta}\left[|f| - \frac{1-\beta}{2T}\right]\right)\right] & : \quad \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & : \quad |f| > \frac{1+\beta}{2T} \end{cases}$$

The value $\beta \in [0,1]$ is the *roll-off factor*. The raised cosine for various roll-off factors β is illustrated in Figure 13.3.

Shifted versions of $\tilde{\mathsf{h}}(f)$ sum to unity because the cosine regions sum to unity:

$$\frac{1}{2}[1+\cos(\theta)] + \frac{1}{2}[1+\cos(\theta+\pi)] = \frac{1}{2}[1+\cos(\theta)] + \frac{1}{2}[1-\cos(\theta)] = 1$$

The inverse Fourier transform of the raised cosine filter is illustrated in Figure 13.3. These waveforms are the signaling waveforms h. Notice how they becoming smoother in frequency but wider in time with increasing β;

B-Spline zero-ISI waveforms

B-Splines are formed by repeatedly convolving the χ function with itself.

[6] Proakis (2001), page 560

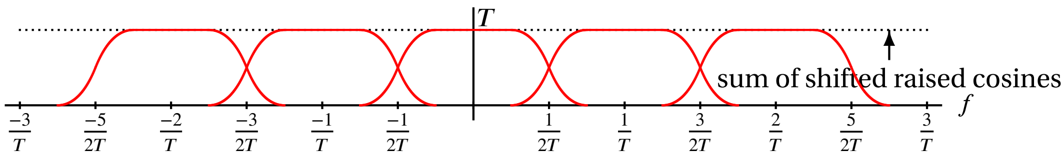


Figure 13.6: Sum of shifted raised cosines

Definition 13.3. *A* ***B-spline*** $\beta_m(f)$ *of order* m *is the characteristic function* $\theta = \chi(f)_{[-1/2T, 1/2T)}$ *convolved with itself* m *times. That is, if* $*$ *is the convolution operation, then*

$$
\begin{aligned}
\beta_0 &\triangleq \theta \\
\beta_1 &\triangleq \theta * \theta && = \beta_0 * \theta \\
\beta_2 &\triangleq \theta * \theta * \theta && = \beta_1 * \theta \\
\beta_3 &\triangleq \theta * \theta * \theta * \theta && = \beta_2 * \theta \\
&\vdots
\end{aligned}
$$

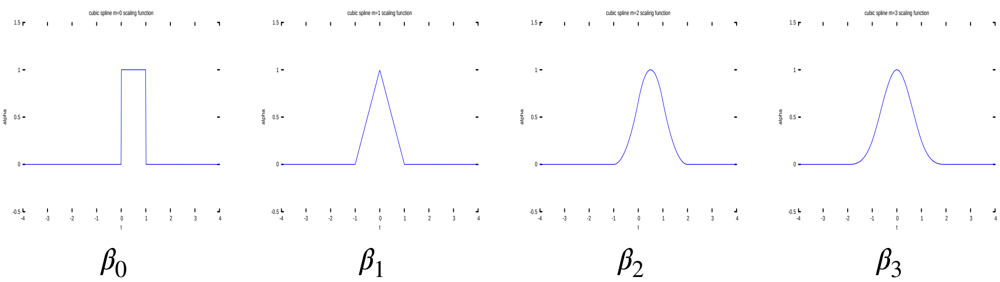


Figure 13.7: B-Splines of order 0,1,2,3

All B-Splines form a partition of unity.[7] and their inverse Fourier Transforms may therefore be used as signaling waveforms $\mathsf{h}(t)$.

Theorem 13.4. *All B-Splines* β_m *of order* $m \in \{0, 1, 2, \ldots\}$ *form a partition of unity.*

✎PROOF:

1. A B-Spline $\tilde{\beta}_m$ of order m is the χ function convolved with itself m times.
2. This implies that the inverse Fourier Transform β_m is

$$\beta_m(t) = \left[\frac{2}{T} \frac{\sin\left(\frac{2\pi}{T}t\right)}{\frac{2\pi}{T}t}\right]^{m+1}$$

3. This equation satisfies the Partition of Unity criterion (Theorem 13.1).

$$\beta_m(nT) = \left[\frac{2}{T} \frac{\sin(2\pi n)}{2\pi n}\right]^{m+1} = \begin{cases} (2/T)^m & : \quad n = 0 \\ 0 & : \quad n \neq 0 \end{cases}$$

4. Therefore, β_m forms a partition of unity for all $m = 0, 1, 2, \ldots$.

[7] Goswami and Chan (1999), page 46

Because β_m form a partition of unity, we can use their inverse Fourier transforms as signaling waveforms h_m. That is, if $\tilde{\mathsf{h}}_m = \beta_m$ then

$$\mathsf{h}_m \triangleq \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{h}}_m \triangleq \tilde{\mathbf{F}}^{-1}\beta_m = \left[\frac{2}{T}\frac{\sin\left(\frac{2\pi}{T}t\right)}{\frac{2\pi}{T}t}\right]^{m+1}$$

are valid signaling waveforms.

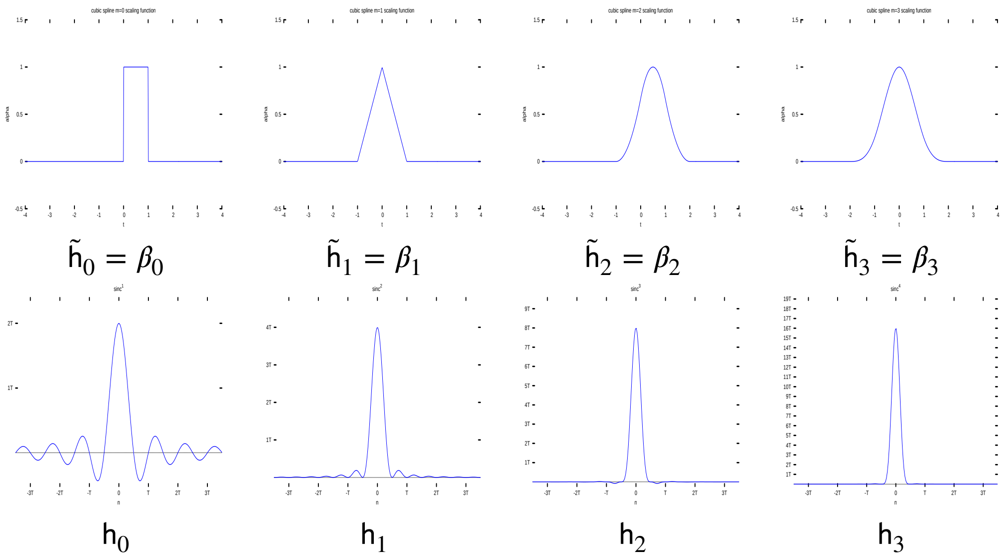


Figure 13.8: B-Splines signaling waveforms in frequency and time domains

Wavelet scaling function zero-ISI waveforms

Wavelets consists of two families of functions: the *scaling functions* $\phi_{m,n}(t)$ and the *wavelet functions* $\psi_{m,n}(t)$. Each member of the family may be scaled by 2^m and translated by n. There are many scaling and wavelet functions available. Most scaling functions ϕ satisfy the partition of unity criterion[8]. The inverse Fourier Transform of scaling functions may therefore be used as signaling waveforms.

One advantage of using wavelet zero-ISI waveforms is that a *fast wavelet transform* (FWT) is available requiring only order $\log n$ operations, even faster than the fast fourier transform's $n \log n$ operations. The availability of the FWT in addition to the wavelet's natural signal analysis capability, may allow the system to make further use of the incoming waveforms for channel estimation, channel equalization, and symbol detection.

[8] Jawerth and Sweldens (1994), page 8 ⟨???⟩

13.3 Duobinary ISI solution

13.3.1 Constraints

The received waveform $r(t)$ is of the form

$$r(t) = \sum_m a_m \mathsf{h}(t - mT).$$

At sampling instants $t = nT$, $r(t)$ has the form

$$\begin{aligned}
r(nT) &= r(t)|_{t=nT} \\
&= \sum_m a_m \mathsf{h}(nT - mT) \\
&= a_m \mathsf{h}(nT - mT)\big|_{m=n} + a_m \mathsf{h}(nT - mT)\big|_{m=n-1} + \sum_{m \neq n, n-1} a_m \mathsf{h}(nT - mT) \\
&= a_n \mathsf{h}(nT - nT) + a_{n-1} \mathsf{h}(nT - (n-1)T) + \sum_{m \neq n, n-1} a_m \mathsf{h}(nT - mT) \\
&= a_n \mathsf{h}(0) + a_{n-1} \mathsf{h}(T) + \sum_{m \neq n, n-1} a_m \mathsf{h}(nT - mT)
\end{aligned}$$

We place the following constraints on the signaling waveform $\mathsf{h}(t)$:

① **sampling constraint:** $\mathsf{h}(nT) = \begin{cases} 1 & \text{for } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

② **bandwidth constraint:** $[\tilde{\mathbf{F}}\mathsf{h}](f) = 0 \text{ for } |f| \geq W.$

These two constraints are in conflict with each other. However, they are both satisfied if the criterion in Theorem 13.5 (page 152) is met.

13.3.2 Criterion

Theorem 13.5. *Let $\tilde{\mathsf{h}}(f)$ be the Fourier Transform of a function $\mathsf{h}(t)$ and $T \in \mathbb{R}$ a constant. Then*

THM
$$\left[\mathsf{h}(nT) = \begin{cases} 1 & : \quad n = 0, 1 \\ 0 & : \quad \textit{otherwise} \end{cases} \right] \iff \left[\frac{1}{T} \sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) = 2e^{-i\pi f T} \cos(\pi f T). \right]$$

✎PROOF: This theorem is easily proven using the *Inverse Poisson's Summation Formula*(IPSF) (Theorem F.3 page 248) which states

$$\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) = T \sum_n \mathsf{h}(nT) e^{-i2\pi f nT}$$

1. Prove "only if" case ($\Longrightarrow$):

$$\begin{aligned}
\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) &= T \sum_n \mathsf{h}(nT) e^{-i2\pi f nT} && \text{by IPSF}\\
&= T\left[1 + e^{-i2\pi f T}\right] && \text{by left hypothesis}\\
&= 2T e^{-i\pi f T}\left(\frac{1}{2}e^{i\pi f T} + \frac{1}{2}e^{-i\pi f T}\right)\\
&= 2T e^{-i\pi f T}\cos(\pi f T) && \text{by } \textit{Euler formulas} \text{ Corollary D.2 page 217}
\end{aligned}$$

2. Prove "if" case ($\Longleftarrow$):

$$\begin{aligned}
2e^{-i\pi f T}\cos(\pi f T) &= \frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) && \text{by right hypothesis}\\
&= \frac{1}{T} T \sum_n \mathsf{h}(nT) e^{-i2\pi f nT} && \text{by IPSF}\\
&= 2e^{-i\pi f T}\sum_n \mathsf{h}(nT)\frac{1}{2}e^{i\pi f T}e^{-i2\pi f nT}\\
&= 2e^{-i\pi f T}\sum_n \mathsf{h}(nT)\frac{1}{2}e^{-i\pi f T(2n-1)}\\
&= 2e^{-i\pi f T}\left[\mathsf{h}(0)\frac{1}{2}e^{i\pi f T} + \mathsf{h}(T)\frac{1}{2}e^{-i\pi f T} + \sum_{n\neq 0,1}\mathsf{h}(nT)\frac{1}{2}e^{-i\pi f T(2n-1)}\right]\\
&\Longrightarrow\\
\mathsf{h}(nT) &= \begin{cases} 1 & : \quad n = 0, 1\\ 0 & : \quad \text{otherwise}\end{cases} && \text{because } \cos(\pi f T) \text{ has no in}
\end{aligned}$$

✎

13.3.3 Signaling waveform

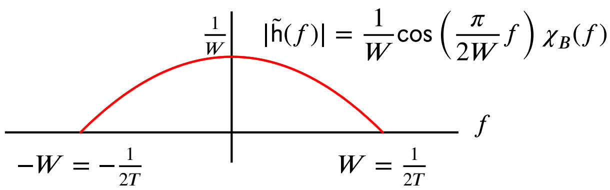


Figure 13.9: Duobinary waveform $\tilde{\mathsf{h}}(f)$ at Nyquist rate

The next theorem specifies a signaling waveform which satisfies the criterion at the Nyquist rate

$$W = \frac{1}{2T}.$$

Unlike the zero-ISI Nyquist rate signaling waveform (Figure 13.2 (page 148)), the duobinary Nyquist rate signaling waveform (Figure 13.9 (page 153)) can be easily approximated in real systems.

Theorem 13.6. *The waveform* $\mathsf{h}(t)$ *with Fourier transform* $\tilde{\mathsf{h}}(f)$ *(see Figure 13.9 (page 153)) satisfies the criterion stated in Theorem 13.5 (page 152), where*

$$\tilde{\mathsf{h}}(f) = \begin{cases} 2Te^{-i\pi Tf}\cos(\pi Tf) & : \frac{-1}{2T} \le f < \frac{1}{2T} \\ 0 & : otherwise \end{cases}$$

$$\begin{aligned}
\mathsf{h}(t) &= \frac{\sin\left[\frac{\pi}{T}t\right]}{\frac{\pi}{T}t} + \frac{\sin\left[\frac{\pi}{T}(t-T)\right]}{\frac{\pi}{T}(t-T)} \\
&\triangleq \operatorname{sinc}\frac{\pi}{T}t + \operatorname{sinc}\frac{\pi}{T}(t-T)
\end{aligned}$$

✎PROOF: Let $B = [-1/2T, +1/2T)$ such that

$$\chi_B(f) \triangleq \begin{cases} 1 & : f \in [-1/2T, +1/2T) \\ 0 & : \text{otherwise.} \end{cases}$$

Then First, observe that $\tilde{\mathsf{h}}(f)$ satisfies the criterion of Theorem 13.5 (page 152):

$$\begin{aligned}
\sum_n \tilde{\mathsf{h}}\left(f+\frac{n}{T}\right) &= \sum_n 2Te^{-i\pi T\left(f+\frac{n}{T}\right)}\cos\left[\pi T(f+\frac{n}{T})\right]\chi_B\left(f+\frac{n}{T}\right) \\
&= 2T\sum_n e^{-i\pi Tf}e^{-i\pi n}\left[\cos(\pi Tf)\cos(\pi n) - \sin(\pi Tf)\sin(\pi n)\right]\chi_B\left(f+\frac{n}{T}\right) \\
&= 2Te^{-i\pi Tf}\sum_n (-1)^n\left[\cos(\pi Tf)(-1)^n - \sin(\pi Tf)\cdot 0\right]\chi_B\left(f+\frac{n}{T}\right) \\
&= 2Te^{-i\pi Tf}\sum_n \cos(\pi Tf)\chi_B\left(f+\frac{n}{T}\right) \\
&= 2Te^{-i\pi Tf}\cos(\pi Tf)\sum_n \chi_B\left(f+\frac{n}{T}\right) \\
&= 2Te^{-i\pi Tf}\cos(\pi Tf)
\end{aligned}$$

The signaling waveform $\mathsf{h}(t)$ can be found by taking the inverse Fourier Transform of $\tilde{\mathsf{h}}(f)$:

$$\begin{aligned}
\mathsf{h}(t) &= [\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{h}}](t) \\
&= \int_f \mathsf{h}(f)e^{i2\pi ft}\,\mathrm{d}f \\
&= \int_{\frac{-1}{2T}}^{\frac{1}{2T}} 2Te^{-i\pi Tf}\cos(\pi Tf)e^{i2\pi ft}\,\mathrm{d}f \\
&= 2T\int_{\frac{-1}{2T}}^{\frac{1}{2T}} e^{-i\pi Tf}\frac{1}{2}\left[e^{i\pi Tf}+e^{-i\pi Tf}\right]e^{i2\pi ft}\,\mathrm{d}f \\
&= T\int_{\frac{-1}{2T}}^{\frac{1}{2T}} \left[1+e^{-i2\pi Tf}\right]e^{i2\pi ft}\,\mathrm{d}f \\
&= T\int_{\frac{-1}{2T}}^{\frac{1}{2T}} e^{i2\pi ft}+e^{i2\pi(t-T)f}\,\mathrm{d}f
\end{aligned}$$

$$
\begin{aligned}
&= T\frac{e^{i2\pi ft}}{i2\pi t}\bigg|_{\frac{-1}{2T}}^{\frac{1}{2T}} + T\frac{e^{i2\pi f(t-T)}}{i2\pi(t-T)}\bigg|_{\frac{-1}{2T}}^{\frac{1}{2T}} \\
&= \frac{e^{i\frac{\pi}{T}t} - e^{-i\frac{\pi}{T}t}}{i2\frac{\pi}{T}t} + \frac{e^{i\frac{\pi}{T}(t-T)} - e^{-i\frac{\pi}{T}(t-T)}}{i2\frac{\pi}{T}(t-T)} \\
&= \frac{\sin[\frac{\pi}{T}t]}{\frac{\pi}{T}t} + \frac{\sin[\frac{\pi}{T}(t-T)]}{\frac{\pi}{T}(t-T)}
\end{aligned}
$$

✏

13.3.4 Detection

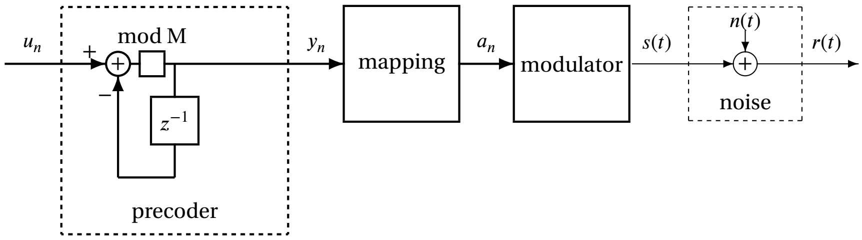


Figure 13.10: Duobinary Detection Model

Detection of a received signal using duobinary modulation presents a special problem because each received symbol at time period n is a function of both the time n and $n-1$ transmitted symbols (has single symbol ISI). In this case and if channel noise is zero, detection can still be performed without error using the algorithm described below and illustrated in Figure 13.10 (page 155).

Lemma 13.1.

LEM $(a+b) \mod = (a \mod + b \mod) \mod$

✎PROOF:

$$
\begin{array}{rcl@{\quad}c@{\quad}rcl}
a &=& Mq_1 + r_1 & \iff & a \mod &=& r_1 \\
b &=& Mq_2 + r_2 & \iff & b \mod &=& r_2 \\
a+b &=& Mq_3 + r_3 & \iff & (a+b) \mod &=& r_3 \\
r_1+r_2 &=& Mq_4 + r_4 & \iff & (r_1+r_2) \mod &=& r_4
\end{array}
$$

✏

Theorem 13.7. *Let* $u_n \in \{0, 1, \dots, M-1\}$ *be the data transmitted using* DUOBINARY *symbol signaling. Let*

$$
\begin{aligned}
\mathsf{r}(t) &\triangleq \mathsf{s}(t;u) + \mathsf{n}(t) \\
r_n &\triangleq r(t)|_{t=nT} = r(nT) \\
y_n &\triangleq (u_n - y_{n-1}) \mod \\
a_n &\triangleq 2y_n - M + 1 \\
n_n &\triangleq n(t)|_{t=nT} = n(nT) \\
S_n &\triangleq \sum_{k=-\infty}^{n} (-1)^{n-k} u_k.
\end{aligned}
$$

Then

THM $r_n|u_n, S_{n-1} = 2\Big[[u_n \mod +(-S_{n-1}) \mod] \mod + S_{n-1} \mod -(M-1)\Big] + n_n$

If $\mathsf{n}(t)$ *is a white Gaussian random process, then*

THM $r_n \sim \mathsf{N}\Big(2\Big[[u_n \mod +(-S_{n-1}) \mod] \mod + S_{n-1} \mod -(M-1)\Big], \sigma^2\Big)$

✎PROOF:

The sequence $\{y_n\}$ is the precoded sequence:

$$\begin{aligned}
y_n &= (u_n - y_{n-1}) \mod \\
&= [u_n - (u_{n-1} - y_{n-2})] \mod \\
&= (u_n - u_{n-1} + u_{n-2} - y_{n-3}) \mod \\
&= (u_n - u_{n-1} + u_{n-2} - u_{n-3} + y_{n-4}) \mod \\
&= \left(\sum_{k=-\infty}^{n} (-1)^{n-k} u_k\right) \mod \\
&= S_n \mod
\end{aligned}$$

A mapping is performed on each y_n to produce a_n:

$$a_n = 2y_n - M + 1.$$

The modulator uses the duobinary signaling waveform $\mathsf{h}(t)$ and a_n to produce the transmitted signal $s(t)$ at signaling rate $1/T$:

$$s(t) = \sum_n a_n \mathsf{h}(t - nT).$$

Before going further, here is a useful relation:

$$\begin{aligned}
S_n &\triangleq \sum_{k=-\infty}^{n} (-1)^{n-k} u_k \\
&= u_n + \sum_{k=-\infty}^{n-1} (-1)^{n-k} u_k \\
&= u_n - \sum_{k=-\infty}^{n-1} (-1)(-1)^{n-k} u_k \\
&= u_n - \sum_{k=-\infty}^{n-1} (-1)^{-1}(-1)^{n-k} u_k \\
&= u_n - \sum_{k=-\infty}^{n-1} (-1)^{n-1-k} u_k \\
&\triangleq u_n - S_{n-1}
\end{aligned}$$

The received signal samples r_n are as follows:

$$\begin{aligned}
\mathsf{r}_n &= \mathsf{r}(t)|_{t=nT} \\
&= [\mathsf{s}(t) + n(t)]_{t=nT} \\
&= \left[\sum_m a_n \mathsf{h}(t - mT) + n(t)\right]_{t=nT} \\
&= \sum_m a_m \mathsf{h}(nT - mT) + n(nT) \\
&= a_n \mathsf{h}(0) + a_{n-1}\mathsf{h}(T) + n_n \\
&= a_n + a_{n-1} + n_n \\
&= \left(2y_n - M + 1\right) + \left(2y_{n-1} - M + 1\right) + n_n \\
&= 2\left(y_n + y_{n-1} - M + 1\right) + n_n \\
&= 2\left[\left(\sum_{k=-\infty}^{n} (-1)^{n-k} u_k\right) \mod + \left(\sum_{k=-\infty}^{n-1} (-1)^{n-1-k} u_k\right) \mod - M + 1\right] + n_n \\
&= 2\left[S_n \mod + S_{n-1} \mod - M + 1\right] + n_n \\
&= 2\left[(u_n - S_{n-1}) \mod + S_{n-1} \mod - (M-1)\right] + n_n \\
&= 2\left[[u_n \mod + (-S_{n-1}) \mod] \mod + S_{n-1} \mod - (M-1)\right] + n_n
\end{aligned}$$

Thus, $(r_n|u_n, S_{n-1})$ have Gaussian distribution with means

$$\mathsf{E}\left[r_n|u_n, S_n\right] = 2\left[\left(u_n + S_{n-1}\right) \mod + \left(M - S_{n-1}\right) \mod - (M-1)\right].$$

✏

That is the good news. The bad news is that in general we don't know S_n. However, the additional good news is that it doesn't matter what S_{n-1} is because the values $\mathsf{E}\left[r_n|u_n\right]$ are always distinct from the values $\mathsf{E}\left[r_m|v_m\right]$ if $u_n \neq v_n$. That is

$$\begin{aligned}
&(u_n \neq v_n) \implies \\
&\mathsf{E}\left[r_n|u_n, S_{n-1}\right] \neq \mathsf{E}\left[r_n|v_n, S_{n-1}\right] \qquad \forall S_{n-1}
\end{aligned}$$

For ML optimization, we are interested in the distributions $\mathsf{p}(r_n|u_n)$. However, what we conveniently have is $\mathsf{p}(r_n|u_n, S_{n-1})$. If we assume that all values of $S_{n-1} \in \{0, 1, \ldots, M-1\}$ are equally likely, we can convert from the latter to the former by the relation:

$$\begin{aligned}
\mathsf{p}(r_n|u_n) &= \frac{\mathsf{p}(r_n, u_n)}{\mathsf{p}(u_n)} \\
&= \frac{\mathsf{p}(u_n|r_n)\mathsf{p}(r_n)}{\mathsf{p}(u_n)} \\
&= \frac{\mathsf{p}(u_n|r_n)\mathsf{p}(r_n)}{\mathsf{p}(u_n)} \\
&= \frac{\sum_{s=0}^{M-1} \mathsf{p}(u_n, S_{n-1} = s|r_n)\mathsf{p}(r_n)}{\mathsf{p}(u_n)} \\
&= \frac{\sum_{s=0}^{M-1} \mathsf{p}(r_n|u_n, S_{n-1} = s)\mathsf{p}(r_n)\mathsf{p}(u_n, S_{n-1})}{\mathsf{p}(u_n)\mathsf{p}(r_n)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sum_{s=0}^{M-1} \mathsf{p}(r_n|u_n, S_{n-1} = s)\mathsf{p}(u_n)\mathsf{p}(S_{n-1})}{\mathsf{p}(u_n)} \\
&= \sum_{m=0}^{M-1} \mathsf{p}(r_n|u_n, S_{n-1} = m)\mathsf{p}(S_{n-1}) \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \mathsf{p}(r_n|u_n, S_{n-1} = m)
\end{aligned}
$$

Detection in the case $M = 2$

For the case $M = 2$, we have the following mean values:

u_n	$S_{n-1} \mod [2]$	$\mathsf{E}\left[r_n|u_n, S_{n-1}\right]$
0	0	−2
0	1	2
1	0	0
1	1	0

This gives distributions (see Figure 13.11 (page 158))

$$
\begin{aligned}
(r_n|u_n = 0) &\sim \frac{1}{2}\mathsf{N}\left(-2, \sigma^2\right) + \frac{1}{2}\mathsf{N}\left(2, \sigma^2\right) \\
(r_n|u_n = 1) &\sim \mathsf{N}\left(0, \sigma^2\right).
\end{aligned}
$$

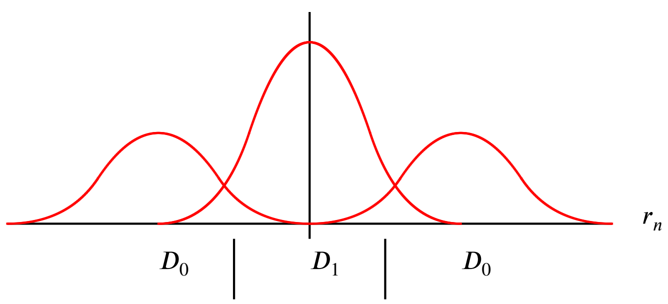


Figure 13.11: Duobinary receiver distributions for $M = 2$

Detection in the case $M = 4$

For the case $M = 4$, we have the following mean values:

u_n	$S_{n-1} \mod [4]$	$\mathsf{E}\left[r_n \vert u_n, S_{n-1}\right]$
0	0	−6
0	1	2
0	2	2
0	3	2
1	0	−4
1	1	−4
1	2	4
1	3	4
2	0	−2
2	1	−2
2	2	−2
2	3	6
3	0	0
3	1	0
3	2	0
3	3	0

This gives distributions (see Figure 13.12 (page 159))

$$
\begin{aligned}
(r_n|u_n = 0) &\sim \frac{1}{4}\mathsf{N}\left(-6, \sigma^2\right) + \frac{3}{4}\mathsf{N}\left(2, \sigma^2\right) \\
(r_n|u_n = 1) &\sim \frac{1}{2}\mathsf{N}\left(-4, \sigma^2\right) + \frac{1}{2}\mathsf{N}\left(4, \sigma^2\right) \\
(r_n|u_n = 2) &\sim \frac{1}{4}\mathsf{N}\left(6, \sigma^2\right) + \frac{3}{4}\mathsf{N}\left(-2, \sigma^2\right) \\
(r_n|u_n = 3) &\sim \mathsf{N}\left(0, \sigma^2\right).
\end{aligned}
$$

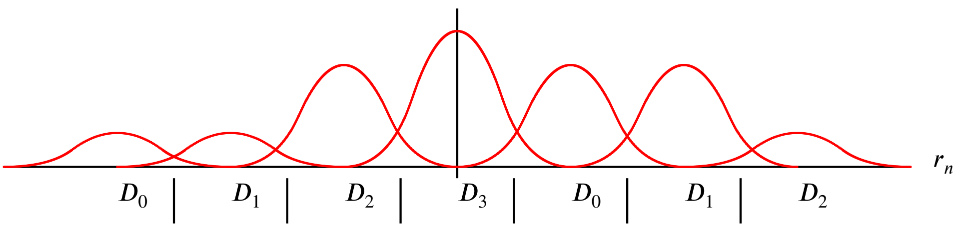


Figure 13.12: Duobinary receiver distributions for $M = 4$

13.4 Modified Duobinary ISI solution

13.4.1 Constraints

The received waveform $r(t)$ is of the form

$$r(t) = \sum_m a_m \mathsf{h}(t - mT).$$

At sampling instants $t = nT$, $r(t)$ has the form

$$\begin{aligned}
r(nT) &= r(t)|_{t=nT} \\
&= \sum_m a_m \mathsf{h}(nT - mT) \\
&= a_m \mathsf{h}(nT - mT)\Big|_{m=n} + a_m \mathsf{h}(nT - mT)\Big|_{m=n-1} + \sum_{m\neq n-1,n+1} a_m \mathsf{h}(nT - mT) \\
&= a_{n-1}\mathsf{h}(nT - (n-1)T) + a_{n+1}\mathsf{h}(nT - (n+1)T) + \sum_{m\neq n-1,n,n+1} a_m \mathsf{h}(nT - mT) \\
&= a_{n+1}\mathsf{h}(-T) + a_{n-1}\mathsf{h}(T) + \sum_{m\neq n-1,n+1} a_m \mathsf{h}(nT - mT)
\end{aligned}$$

We place the following constraints on the signaling waveform $\mathsf{h}(t)$:

We place the following constraints on the signaling waveform $\mathsf{h}(t)$:

1. **sampling constraint:** $\mathsf{h}(nT) = \begin{cases} +1 & \text{for } n = -1 \\ -1 & \text{for } n = +1 \\ 0 & \text{otherwise} \end{cases}$
2. **bandwidth constraint:** $[\tilde{\mathbf{F}}\mathsf{h}](f) = 0 \text{ for } |f| \geq W.$

These two constraints are in conflict with each other. However, they are both satisfied if the criterion in Theorem 13.8 (page 160) is met.

13.4.2 Criterion

Theorem 13.8. *Let $\tilde{\mathsf{h}}(f)$ be the Fourier Transform of a function $\mathsf{h}(t)$ and $T \in \mathbb{R}$ a constant.*

THM

$$\left[\mathsf{h}(nT) = \begin{cases} +1 & : & n = -1 \\ -1 & : & n = +1 \\ 0 & : & \textit{otherwise} \end{cases}\right] \iff \left[\frac{1}{T}\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) = i2\sin(2\pi fT).\right]$$

✎Proof: This theorem is easily proven using the *Inverse Poisson's Summation Formula* (IPSF) which states

$$\sum_{n\in\mathbb{Z}} \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) = T\sum_n \mathsf{h}(nT)e^{-i2\pi fnT}$$

1. "Only if" case ($\Longrightarrow$):

$$\begin{aligned}
\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) &= T\sum_n \mathsf{h}(nT)e^{-i2\pi fnT} && \text{by IPSF} \\
&= T\left[\mathsf{h}(-1T)e^{-i2\pi f(-1)T} + \mathsf{h}(1T)e^{-i2\pi f1T} + \sum_{n\neq n-1,n+1}\mathsf{h}(nT)e^{-i2\pi fnT}\right] \\
&= T\left[(1)e^{-i2\pi f(-1)T}(-1)e^{-i2\pi f1T}\right] && \text{by left hypothesis} \\
&= T\left[e^{i2\pi fT} - e^{-i2\pi fT}\right] \\
&= i2T\sin(2\pi fT) && \text{by \textit{Euler formulas} Corollary D.2}
\end{aligned}$$

2. "If" case ($\Longleftarrow$):

$$
\begin{aligned}
i2T\sin(2\pi fT) &= \sum_n \tilde{\mathsf{h}}\left(f+\frac{n}{T}\right) && \text{by right hypothesis}\\
&= T\sum_n \mathsf{h}(nT)e^{-i2\pi fnT} && \text{by IPSF}\\
&= i2T\sum_n \mathsf{h}(nT)\frac{1}{2i}e^{-i2\pi fnT}\\
&= i2T\left[\frac{\mathsf{h}(-T)e^{i2\pi fT}+\mathsf{h}(T)e^{-i2\pi fT}}{2i}+\sum_{n\neq -1,1}\mathsf{h}(nT)\frac{1}{2i}e^{-i2\pi fnT}\right]
\end{aligned}
$$

$$\Longrightarrow$$

$$
\mathsf{h}(nT) = \begin{cases} 1 & : \; n = -1\\ -1 & : \; n = 1\\ 0 & : \; \text{otherwise}\end{cases} \qquad \text{because } \sin(2\pi fT) \text{ has no imagina}
$$

✏

13.4.3 Signaling waveform

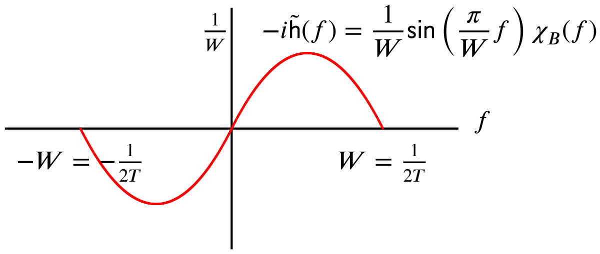


Figure 13.13: Modified duobinary waveform $\tilde{\mathsf{h}}(f)$ at Nyquist rate

The next theorem specifies a signaling waveform which satisfies the criterion at the Nyquist rate

$$W = \frac{1}{2T}.$$

Like the duobinary Nyquist rate signaling waveform (Figure 13.9 (page 153)), the modified duobinary Nyquist rate signaling waveform (Figure 13.13 (page 161)) can be easily approximated in real systems. Unlike the duobinary Nyquist rate signaling waveform, the modified duobinary Nyquist rate signaling waveform has no DC component making it a better candidate for channels that attenuate DC (for example, capacitively coupled channels).

Theorem 13.9. *The waveform* $\mathsf{h}(t)$ *with Fourier transform* $\tilde{\mathsf{h}}(f)$ *(see Figure 13.13 (page 161)) satisfies the criterion stated in Theorem 13.8 (page 160), where*

THM

$$
\begin{aligned}
\tilde{\mathsf{h}}(f) &= \begin{cases} i2T\sin(2\pi fT) & : \frac{-1}{2T} \le f < \frac{1}{2T}\\ 0 & : \textit{otherwise.}\end{cases}\\
\mathsf{h}(t) &= \frac{\sin[\frac{\pi}{T}(t+T)]}{\frac{\pi}{T}(t+T)} - \frac{\sin[\frac{\pi}{T}(t-T)]}{\frac{\pi}{T}(t-T)}\\
&= \operatorname{sinc}\frac{\pi}{T}(t+T) - \operatorname{sinc}\frac{\pi}{T}(t-T)
\end{aligned}
$$

✎PROOF: Let $B = [-1/2T, +1/2T)$ such that

$$\chi_B(f) \triangleq \begin{cases} 1 & : f \in [-1/2T, +1/2T) \\ 0 & : \text{otherwise.} \end{cases}$$

Then First, observe that $\tilde{\mathsf{h}}(f)$ satisfies the criterion of Theorem 13.8 (page 160):

$$\begin{aligned}
\sum_n \tilde{\mathsf{h}}\left(f + \frac{n}{T}\right) &= \sum_n i2T\sin[2\pi(f + \frac{n}{T})T]\chi_B\left(f + \frac{n}{T}\right) \\
&= i2T\sum_n \sin(2\pi fT + 2\pi n)\chi_B\left(f + \frac{n}{T}\right) \\
&= i2T\sum_n [\sin(2\pi fT)\cos(2\pi n) + \cos(2\pi fT)\sin(2\pi n)]\,\chi_B\left(f + \frac{n}{T}\right) \\
&= i2T\sum_n [\sin(2\pi fT)\cdot 1 + \cos(2\pi fT)\cdot 0]\,\chi_B\left(f + \frac{n}{T}\right) \\
&= i2T\sum_n \sin(2\pi fT)\chi_B\left(f + \frac{n}{T}\right) \\
&= i2T\sin(2\pi fT)\sum_n \chi_B\left(f + \frac{n}{T}\right) \\
&= i2T\sin(2\pi fT)
\end{aligned}$$

The signaling waveform $\mathsf{h}(t)$ can be found by taking the inverse Fourier Transform of $\tilde{\mathsf{h}}(f)$:

$$\begin{aligned}
\mathsf{h}(t) &= [\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{h}}](t) \\
&= \int_f \mathsf{h}(f)e^{i2\pi ft}\,\mathrm{d}f \\
&= \int_{\frac{-1}{2T}}^{\frac{1}{2T}} i2T\sin(2\pi Tf)e^{i2\pi ft}\,\mathrm{d}f \\
&= i2T\int_{\frac{-1}{2T}}^{\frac{1}{2T}} \frac{1}{2i}\left[e^{i2\pi Tf} - e^{-i2\pi Tf}\right]e^{i2\pi ft}\,\mathrm{d}f \\
&= T\int_{\frac{-1}{2T}}^{\frac{1}{2T}} \left[e^{i2\pi Tf} - e^{-i2\pi Tf}\right]e^{i2\pi ft}\,\mathrm{d}f \\
&= T\int_{\frac{-1}{2T}}^{\frac{1}{2T}} \left[e^{i2\pi f(t+T)} - e^{i2\pi f(t-T)}\right]\,\mathrm{d}f \\
&= T\left.\frac{e^{i2\pi f(t+T)}}{i2\pi(t+T)}\right|_{\frac{-1}{2T}}^{\frac{1}{2T}} - T\left.\frac{e^{i2\pi f(t-T)}}{i2\pi(t-T)}\right|_{\frac{-1}{2T}}^{\frac{1}{2T}} \\
&= \frac{e^{i\frac{\pi}{T}(t+T)} - e^{-i\frac{\pi}{T}(t+T)}}{2i\frac{\pi}{T}(t+T)} - \frac{e^{i\frac{\pi}{T}(t-T)} - e^{-i\frac{\pi}{T}(t-T)}}{2i\frac{\pi}{T}(t-T)} \\
&= \frac{2i\sin[\frac{\pi}{T}(t+T)]}{2i\frac{\pi}{T}(t+T)} - \frac{2i\sin[\frac{\pi}{T}(t-T)]}{2i\frac{\pi}{T}(t-T)} \\
&= \frac{\sin[\frac{\pi}{T}(t+T)]}{\frac{\pi}{T}(t+T)} - \frac{\sin[\frac{\pi}{T}(t-T)]}{\frac{\pi}{T}(t-T)}
\end{aligned}$$

⇨

CHAPTER 14

DISTORTED FREQUENCY RESPONSE CHANNEL

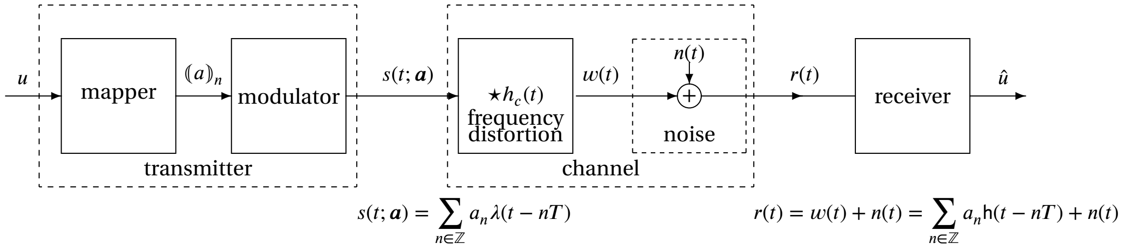


Figure 14.1: Equalization system model

14.1 Channel Model

In this chapter, the channel model includes both deterministic and random distortion.

①	linear deterministic distortion	(convolution with $h_c(t)$)
②	linear stochastic distortion	(additive white Gaussian noise).

Let

u	be the information sequence
$(\!(a)\!)_n$	be a mapped sequence under a one to one function $a_n = f(u_n)$
$\lambda(t)$	be the *modulation waveform*
$s(t;\boldsymbol{a})$	be the *transmitted waveform*
$h_c(t)$	be the *channel impulse response*
$n(t)$	be the *channel noise* with distribution $n(t) \sim \mathsf{N}\left(0, \sigma^2\right)$.

The following definitions apply throughout this chapter:

DEF

$$
\begin{aligned}
\mathsf{s}(t;(\!(a)\!)_n) &\triangleq \sum_{n\in\mathbb{Z}} a_n \lambda(t-nT) \\
\mathsf{h}(t) &\triangleq \lambda(t) \star \mathsf{h}_c(t) = \int_\tau \mathsf{h}(\tau)\lambda(t-\tau)\,\mathsf{d}\tau \\
w(t) &\triangleq \int_\tau h_c(\tau)s(t-\tau)\,d\tau \\
r(t) &\triangleq w(t) + n(t).
\end{aligned}
$$

Under these definitions the received signal can be expressed as follows:

$$\begin{aligned}
r(t) &= w(t) + n(t) \\
&= \int_\tau h_c(\tau) s(t-\tau)\,\mathrm{d}\tau + n(t) \\
&= \int_\tau h_c(\tau) \sum_{n\in\mathbb{Z}} a_n \lambda(t-\tau-nT)\,\mathrm{d}\tau + n(t) \\
&= \sum_{n\in\mathbb{Z}} a_n \int_\tau \mathsf{h}_c(\tau)\lambda(t-\tau-nT)\,\mathrm{d}\tau + n(t) \\
&= \sum_{n\in\mathbb{Z}} a_n \mathsf{h}(t-nT) + n(t)
\end{aligned}$$

14.2 Sufficient statistic sequence

14.2.1 Receiver statistics

Define the innerproduct quantities as

DEF

$$\begin{aligned}
\dot{r}_n &\triangleq \langle r(t) \,|\, \psi_n(t) \rangle \\
\dot{n}_n &\triangleq \langle n(t) \,|\, \psi_n(t) \rangle \\
\dot{h}_n(m) &\triangleq \langle \mathsf{h}(t-mT) \,|\, \psi_n(t) \rangle
\end{aligned}$$

The quantity $\dot{r}_n$ is a random variable with form

$$\begin{aligned}
\dot{r}_n &\triangleq \langle r(t) \,|\, \psi_n(t) \rangle \\
&= \langle w(t) + n(t) \,|\, \psi_n(t) \rangle \\
&= \langle w(t) \,|\, \psi_n(t) \rangle + \langle n(t) \,|\, \psi_n(t) \rangle \\
&= \left\langle \sum_m a_m \mathsf{h}(t-mT) \,|\, \psi_n(t) \right\rangle + \langle n(t) \,|\, \psi_n(t) \rangle \\
&= \sum_m a_m \langle \mathsf{h}(t-mT) \,|\, \psi_n(t) \rangle + \langle n(t) \,|\, \psi_n(t) \rangle \\
&= \sum_m a_m \dot{h}_n(m) + \dot{n}_n.
\end{aligned}$$

By Theorem 7.5 (page 76), the quantity $\dot{r}_n$ given $\boldsymbol{a}$ has Gaussian distribution

$$(\dot{r}_n|\boldsymbol{a}) \sim \mathsf{N}\left(\sum_m a_m \dot{h}_n(m), \sigma^2 \right)$$

and $\dot{r}_n|\boldsymbol{a}$ and $\dot{r}_m|\boldsymbol{a}$ are independent for $n \neq m$.

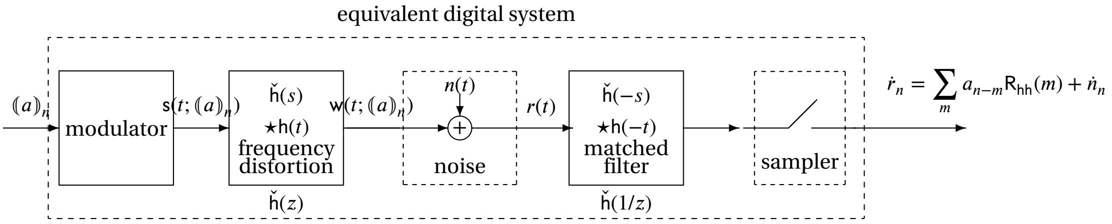


Figure 14.2: Sufficient statistic sequence $(\dot{r}_n)$ for ML estimation

14.2.2 ML estimate and sufficient statistic

Definition 14.1.

DEF

$$\begin{aligned}
R_{hh}(m) &\triangleq \langle h(t+mT) \,|\, h(t)\rangle \triangleq \int_{t\in\mathbb{R}} h(t+mT)h^*(t)\,dt && \textit{(autocorrelation)}\\
\dot{r}_n &\triangleq \langle r(t) \,|\, h(t-nT)\rangle \triangleq \int_{t\in\mathbb{R}} r(t)h^*(t-nT)\,dt && \textit{(receive statistic)}\\
\dot{n}_n &\triangleq \langle n(t) \,|\, h(t-nT)\rangle \triangleq \int_{t\in\mathbb{R}} n(t)h^*(t-nT)\,dt && \textit{(noise statistic)}
\end{aligned}$$

Under these definitions, the receive statistic can be represented as follows (see Figure 14.2 page 165):

$$\begin{aligned}
\dot{r}_n &\triangleq \langle r(t) \,|\, h(t-nT)\rangle\\
&= \left\langle \sum_m a_n h(t-mT) + n(t) \,\middle|\, h(t-nT) \right\rangle\\
&= \left\langle \sum_m a_n h(t-mT) \,\middle|\, h(t-nT) \right\rangle + \langle n(t) \,|\, h(t-nT)\rangle\\
&= \sum_m a_m \langle h(t-mT) \,|\, h(t-nT)\rangle + \langle n(t) \,|\, h(t-nT)\rangle\\
&= \sum_m a_m R_{hh}(n-m) + \dot{n}_n\\
&= \sum_k a_{n-k} R_{hh}(k) + \dot{n}_n && \text{where } k \triangleq n-m \implies m = n-k\\
&= \sum_m a_{n-m} R_{hh}(m) + \dot{n}_n && \text{by change of free variable}
\end{aligned}$$

Theorem 14.1. *Under Definitions 14.1,*

1. *The sequence $(\dot{r}_n)$ is a* ***sufficient statistic*** *for determining the maximum likelihood (ML) estimate of $\boldsymbol{a}$.*
2. *The ML estimate of $\boldsymbol{a}$ is*

THM

$$\hat{\boldsymbol{a}}_{ml} = \arg\max_{\boldsymbol{a}} \left(2\sum_{n\in\mathbb{Z}} a_n \dot{r}_n - \sum_{n\in\mathbb{Z}} \sum_m a_n a_{m+n} R_{hh}(m) \right).$$

✎PROOF:

$$\hat{\boldsymbol{a}}_{ml} \triangleq \arg\max_{\boldsymbol{a}} P\left\{ r(t) | s(t;(a)_n) \right\}$$

$$= \arg\max_{a} \left[2 \int_{t\in\mathbb{R}} r(t)\mathsf{w}(t;(\hat{a})_n) - \int_{t\in\mathbb{R}} \mathsf{w}^2(t;(\hat{a})_n)\, \mathrm{d}t \right] \qquad \text{by Theorem 7.6 page 76}$$

$$= \arg\max_{a} \left[2 \int_{t\in\mathbb{R}} r(t) \sum_{n\in\mathbb{Z}} a_n \mathsf{h}(t-nT)\, \mathrm{d}t - \int_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}} a_n \mathsf{h}(t-nT) \sum_{m} a_m \mathsf{h}(t-mT)\, \mathrm{d}t \right]$$

$$= \arg\max_{a} \left[2 \sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} r(t)\mathsf{h}(t-nT)\, \mathrm{d}t - \sum_{n\in\mathbb{Z}} \sum_{m} a_n a_m \int_{t\in\mathbb{R}} \mathsf{h}(t-nT)\mathsf{h}(t-mT)\, \mathrm{d}t \right]$$

$$= \arg\max_{a} \left[2 \sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} r(t)\mathsf{h}(t-nT)\, \mathrm{d}t - \sum_{n\in\mathbb{Z}} \sum_{m} a_n a_m \mathsf{R}_{\mathsf{hh}}(m-n) \right]$$

$$= \arg\max_{a} \left[2 \sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} r(t)\mathsf{h}(t-nT)\, \mathrm{d}t - \sum_{n\in\mathbb{Z}} \sum_{k} a_n a_k \mathsf{R}_{\mathsf{hh}}(k-n) \right]$$

$$= \arg\max_{a} \left[2 \sum_{n\in\mathbb{Z}} a_n \int_{t\in\mathbb{R}} r(t)\mathsf{h}(t-nT)\, \mathrm{d}t - \sum_{n\in\mathbb{Z}} \sum_{m} a_n a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) \right]$$

$$= \arg\max_{a} \left[2 \sum_{n\in\mathbb{Z}} a_n \dot{r}_n - \sum_{n\in\mathbb{Z}} \sum_{m} a_n a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) \right]$$

✏

If the autocorrelation is zero for $|n| > L$, then Theorem 14.1 (page 165) reduces to the simpler form stated in Corollary 14.1 (next).

Corollary 14.1. *If*

$$\mathsf{R}_{\mathsf{hh}}(n) = 0 \textit{ for } |n| > L$$

then

COR

$$\boxed{\hat{\boldsymbol{a}}_{\mathsf{ml}} = \arg\max_{a} \left(2 \sum_{n\in\mathbb{Z}} a_n \dot{r}_n - \sum_{n\in\mathbb{Z}} a_n \left[a_n \mathsf{R}_{\mathsf{hh}}(0) + 2 \sum_{m=1}^{L} a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) \right] \right)}$$

✎PROOF: First note that

$$\sum_{n\in\mathbb{Z}} \sum_{m=-L}^{L} a_{m+n} \mathsf{R}_{\mathsf{hh}}(m)$$

is maximized when a_{m+n} is symmetric about n (??????). Then

$$\begin{aligned}
\hat{\boldsymbol{a}}_{\mathsf{ml}} &= \arg\max_{a} \left(2 \sum_{n\in\mathbb{Z}} a_n \dot{r}_n - \sum_{n\in\mathbb{Z}} \sum_{m} a_n a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) \right) \\
&= \arg\max_{a} \left(2 \sum_{n\in\mathbb{Z}} a_n \dot{r}_n - \sum_{n\in\mathbb{Z}} a_n \sum_{m=-L}^{L} a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) \right) \\
&= \arg\max_{a} \left(2 \sum_{n\in\mathbb{Z}} a_n \dot{r}_n - \sum_{n\in\mathbb{Z}} a_n \left[a_n \mathsf{R}_{\mathsf{hh}}(0) + \sum_{m=-L}^{1} a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) + \sum_{m=1}^{L} a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) \right] \right) \\
&= \arg\max_{a} \left(2 \sum_{n\in\mathbb{Z}} a_n \dot{r}_n - \sum_{n\in\mathbb{Z}} a_n \left[a_n \mathsf{R}_{\mathsf{hh}}(0) + 2 \sum_{m=1}^{L} a_{m+n} \mathsf{R}_{\mathsf{hh}}(m) \right] \right)
\end{aligned}$$

✏

14.2.3 Statistics of sufficient statistic sequence $(\dot{r}_n)$

The elements of the ML sufficient sequence $(\dot{r}_n|\boldsymbol{a})$ have Gaussian distribution, however the sequence is **colored**. That is $\dot{r}_n$ is correlated with $\dot{r}_m$ (and therefore also not independent). To whiten the sequence $(\dot{r}_n)$, a whitening filter may be used. Whitening filters can be implemented in analog (Section **??** page **??**) or digitally (Section **??** page **??**).

Theorem 14.2.

THM

$$\begin{aligned}
\mathsf{E}\dot{n}_n &= 0\\
\operatorname{cov}\left[\dot{n}_n, \dot{n}_m\right] &= N_o \mathsf{R}_{\mathsf{hh}}(n-m)\\
\mathsf{E}\dot{r}_n|\boldsymbol{a} &= \sum_m a_{n-m}\mathsf{R}_{\mathsf{hh}}(m)\\
\dot{r}_n|\boldsymbol{a} &\sim \mathsf{N}\left(\sum_m a_{n-m}\mathsf{R}_{\mathsf{hh}}(m),\, N_o\mathsf{R}_{\mathsf{hh}}(0)\right)\\
\operatorname{cov}\left[\dot{r}_n|\boldsymbol{a}, \dot{r}_m|\boldsymbol{a}\right] &= N_o \mathsf{R}_{\mathsf{hh}}(n-m)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{E}\dot{n}_n &= \mathsf{E}\left\langle n(t)\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \left\langle \mathsf{E}n(t)\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \left\langle 0\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\operatorname{cov}\left[\dot{n}_n, \dot{n}_m\right] &= \mathsf{E}\left[\dot{n}_n\dot{n}_m\right] - \mathsf{E}\left[\dot{n}_n\right]\mathsf{E}\left[\dot{n}_m\right]\\
&= \mathsf{E}\left[\dot{n}_n\dot{n}_m\right] - 0\cdot 0\\
&= \mathsf{E}\left[\left\langle n(t)\,|\,\mathsf{h}(t-nT)\right\rangle\left\langle n(t)\,|\,\mathsf{h}(t-mT)\right\rangle\right]\\
&= \mathsf{E}\left[\left\langle n(t)\,|\,\mathsf{h}(t-nT)\right\rangle\left\langle n(u)\,|\,\mathsf{h}(u-mT)\right\rangle\right]\\
&= \mathsf{E}\left[\left\langle n(t)\left\langle n(u)\,|\,\mathsf{h}(u-mT)\right\rangle\,|\,\mathsf{h}(t-nT)\right\rangle\right]\\
&= \mathsf{E}\left[\left\langle\left\langle n(t)n(u)\,|\,\mathsf{h}(u-mT)\right\rangle\,|\,\mathsf{h}(t-nT)\right\rangle\right]\\
&= \left\langle\left\langle \mathsf{E}\left[n(t)n(u)\right]\,|\,\mathsf{h}(u-mT)\right\rangle\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \left\langle\left\langle N_o\delta(t-u)\,|\,\mathsf{h}(u-mT)\right\rangle\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= N_o\left\langle \mathsf{h}(t-mT)\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= N_o\mathsf{R}_{\mathsf{hh}}(n-m)
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}\dot{r}_n &\triangleq \mathsf{E}\left\langle r(t)\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \mathsf{E}\left\langle \sum_k a_k\mathsf{h}(t-kT) + n(t)\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \left\langle \sum_k a_k\mathsf{h}(t-kT) + \mathsf{E}n(t)\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \left\langle \sum_k a_k\mathsf{h}(t-kT) + 0\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \sum_k a_k\left\langle \mathsf{h}(t-kT)\,|\,\mathsf{h}(t-nT)\right\rangle\\
&= \sum_k a_k\mathsf{R}_{\mathsf{hh}}(n-k)\\
&= \sum_m a_{n-m}\mathsf{R}_{\mathsf{hh}}(m) && \text{where } m\triangleq n-k \implies k=n-m
\end{aligned}$$

$$\begin{aligned}
\operatorname{cov}\left[\dot{r}_n, \dot{r}_m\right] &= \mathsf{E}\left[\left(\dot{r}_n - \mathsf{E}\dot{r}_n\right)\left(\dot{r}_m - \mathsf{E}\dot{r}_m\right)\right] \\
&= \mathsf{E}\left[\dot{n}_n \dot{n}_m\right] \\
&= \operatorname{cov}\left[\dot{n}_n, \dot{n}_m\right] \\
&= N_o \mathsf{R}_{\mathsf{hh}}(n-m)
\end{aligned}$$

✎

14.2.4 Spectrum of sufficient statistic sequence $(\dot{r}_n)$

The Fourier Transform cannot be used to evaluate the spectrum of the sequences $(\dot{r}_n)$, $\mathsf{R}_{\mathsf{hh}}(m)$, and $(\dot{n}_n)$ directly because the sequences are not functions of a continuous variable. Instead we compute the spectral content of their sampled continuous equivalents as defined next:

DEF

$$\begin{aligned}
R_s(t) &\triangleq \langle \mathsf{h}(u+t) \,|\, \mathsf{h}(u) \rangle \sum_{n\in\mathbb{Z}} \delta(t-nT) \\
\dot{r}_s(t) &\triangleq \langle r(u) \,|\, \mathsf{h}(u-t) \rangle \sum_{n\in\mathbb{Z}} \delta(t-nT) \\
\dot{n}_s(t) &\triangleq \langle n(u) \,|\, \mathsf{h}(u-t) \rangle \sum_{n\in\mathbb{Z}} \delta(t-nT) \\
a_s(t) &\triangleq a(t) \sum_{n\in\mathbb{Z}} \delta(t-nT).
\end{aligned}$$

Note that under these definitions

PRP

$$\begin{aligned}
\mathsf{R}_{\mathsf{hh}}(m) &= R_s(t)|_{t=mT} \\
\dot{r}_n &= \dot{r}_s(t)|_{t=nT} \\
\dot{n}_n &= \dot{n}_s(t)|_{t=nT} \\
a_n &= a_s(t)|_{t=nT}.
\end{aligned}$$

$$\begin{aligned}
S_s(f) &\triangleq [\tilde{\mathbf{F}} R_s](f) \\
&= \left[\tilde{\mathbf{F}} \langle \mathsf{h}(u+t) \,|\, \mathsf{h}(u) \rangle \sum_{n\in\mathbb{Z}} \delta(t-nT)\right](f) \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \left[\tilde{\mathbf{F}}\langle \mathsf{h}(u+t) \,|\, \mathsf{h}(u)\rangle\right]\left(f-\frac{n}{T}\right) && \text{by Theorem ?? (page ??)} \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \int_{t\in\mathbb{R}} \langle \mathsf{h}(u+t) \,|\, \mathsf{h}(u)\rangle e^{-i2\pi\left(f-\frac{n}{T}\right)t} \, \mathrm{d}t \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \int_{t\in\mathbb{R}} \int_u \mathsf{h}(u+t) h^*(u) e^{-i2\pi\left(f-\frac{n}{T}\right)t} \, \mathrm{d}t \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \int_{t\in\mathbb{R}} \int_u \mathsf{h}(u+t) h^*(u) e^{-i2\pi\left(f-\frac{n}{T}\right)t} \, \mathrm{d}t && \text{where } v \triangleq u+t \iff t = v-u \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \int_v \int_u \mathsf{h}(v)\mathsf{h}^*(u) e^{-i2\pi\left(f-\frac{n}{T}\right)(v-u)} \, \mathrm{d}u \, \mathrm{d}v \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \int_u \mathsf{h}^*(u) e^{i2\pi\left(f-\frac{n}{T}\right)u} \, \mathrm{d}u \int_v \mathsf{h}(v) e^{-i2\pi\left(f-\frac{n}{T}\right)v} \, \mathrm{d}v \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \left(\int_u \mathsf{h}(u) e^{-i2\pi\left(f-\frac{n}{T}\right)u} \, \mathrm{d}u\right)^* \int_v \mathsf{h}(v) e^{-i2\pi\left(f-\frac{n}{T}\right)v} \, \mathrm{d}v
\end{aligned}$$

$$= \frac{1}{T}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{h}}^*\left(f-\frac{n}{T}\right)\tilde{\mathsf{h}}\left(f-\frac{n}{T}\right)$$
$$= \frac{1}{T}\sum_{n\in\mathbb{Z}} \left|\tilde{\mathsf{h}}\left(f-\frac{n}{T}\right)\right|^2$$

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\dot{n}_s\right](f) &= \left[\tilde{\mathbf{F}}\langle n(u)\,|\,\mathsf{h}(u-t)\rangle \sum_{n\in\mathbb{Z}}\delta(t-nT)\right](f)\\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}}[\tilde{\mathbf{F}}\langle n(u)\,|\,\mathsf{h}(u-t)\rangle]\left(f-\frac{n}{T}\right)\\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}}\int_{t\in\mathbb{R}} \langle n(u)\,|\,\mathsf{h}(u-t)\rangle\, e^{-i2\pi\left(f-\frac{n}{T}\right)t}\,\mathsf{d}t\\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}}\int_{t\in\mathbb{R}}\int_u n(u)\mathsf{h}^*(u-t)e^{-i2\pi\left(f-\frac{n}{T}\right)t}\,\mathsf{d}u\,\mathsf{d}t\\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}}\int_v\int_u n(u)\mathsf{h}^*(v)e^{-i2\pi\left(f-\frac{n}{T}\right)(u-v)}\,\mathsf{d}u\,\mathsf{d}v && \text{where } v \triangleq u-t \iff t=u-v\\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}}\int_u n(u)e^{-i2\pi\left(f-\frac{n}{T}\right)u}\,\mathsf{d}u\int_v \mathsf{h}^*(v)e^{i2\pi\left(f-\frac{n}{T}\right)v}\,\mathsf{d}v\\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}}\int_u n(u)e^{-i2\pi\left(f-\frac{n}{T}\right)u}\,\mathsf{d}u\left[\int_v \mathsf{h}(v)e^{i2\pi\left(f-\frac{n}{T}\right)v}\,\mathsf{d}v\right]^*\\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{n}}\left(f-\frac{n}{T}\right)\tilde{\mathsf{h}}^*\left(f-\frac{n}{T}\right)
\end{aligned}$$

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\dot{r}\right](f) &= \tilde{\mathsf{a}}_s(f)S_s(f)+\tilde{\mathsf{n}}_s(f)\\
&= \tilde{\mathsf{a}}_s(f)S_s(f)+\tilde{\mathsf{n}}_s(f)\\
&= \tilde{\mathsf{a}}_s(f)\frac{1}{T}\sum_{n\in\mathbb{Z}}\left|\tilde{\mathsf{h}}\left(f-\frac{n}{T}\right)\right|^2+\frac{1}{T}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{n}}\left(f-\frac{n}{T}\right)\tilde{\mathsf{h}}^*\left(f-\frac{n}{T}\right)
\end{aligned}$$

Note that the Fourier Transform $\tilde{\mathsf{n}}(f)$ only exists if it has finite energy (such as with most bandlimited noise). Thus, if $n(t)$ is a true white noise process, $\tilde{\mathsf{n}}(f)$ does not exist.

14.3 Implementations

14.3.1 Trellis

The ML estimate can be computed by the use of a trellis. The distance metrics $\mu(n;\boldsymbol{a},L)$ for the trellis can be computed recursively.

Theorem 14.3. *Let a metric $\mu(n;\boldsymbol{a},L)$ be defined such that*

$$\mathsf{R}_{\mathsf{hh}}(n) \;=\; 0 \text{ for } |n|>L.$$

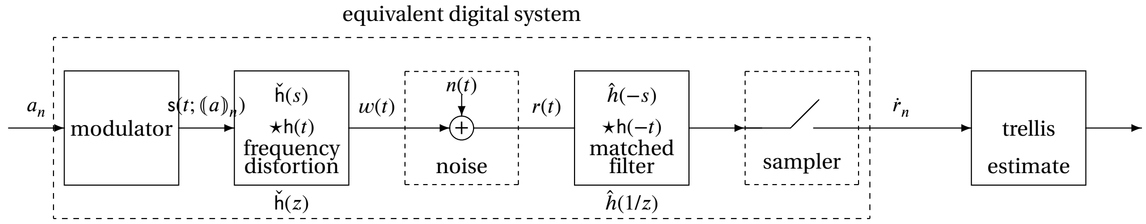


Figure 14.3: Trellis implementation

$$\mu(n;\boldsymbol{a},L) \triangleq 2\sum_{k=-\infty}^{n} a_k \dot{r}_k - \sum_{k=-\infty}^{n} a_k \left[a_k \mathsf{R}_{hh}(0) + 2\sum_{m=1}^{L} a_{m+k}\mathsf{R}_{hh}(m) \right]$$

Then

THM

$$\boxed{\mu(n;\boldsymbol{a},L) = \mu(n-1;\boldsymbol{a},L) + 2a_n\dot{r}_n - a_n^2\mathsf{R}_{hh}(0) - 2a_n\sum_{m=1}^{L} a_{m+n}\mathsf{R}_{hh}(m)}$$

✎PROOF:

$$\begin{aligned}
&\mu(n;\boldsymbol{a},L) - \mu(n-1;\boldsymbol{a},L)\\
&= \left(2\sum_{k=-\infty}^{n} a_k\dot{r}_k - \sum_{k=-\infty}^{n} a_k\left[a_k\mathsf{R}_{hh}(0) + 2\sum_{m=1}^{L} a_{m+k}\mathsf{R}_{hh}(m)\right]\right) - \\
&\left(2\sum_{k=-\infty}^{n-1} a_k\dot{r}_k - \sum_{k=-\infty}^{n-1} a_k\left[a_k\mathsf{R}_{hh}(0) + 2\sum_{m=1}^{L} a_{m+k}\mathsf{R}_{hh}(m)\right]\right)\\
&= 2a_n\dot{r}_n - a_n\left[a_n\mathsf{R}_{hh}(0) + 2\sum_{m=1}^{L} a_{m+n}\mathsf{R}_{hh}(m)\right]\\
&= 2a_n\dot{r}_n - a_n^2\mathsf{R}_{hh}(0) - 2a_n\sum_{m=1}^{L} a_{m+n}\mathsf{R}_{hh}(m)
\end{aligned}$$

✏

Example 14.1. Let $L = 2$ in a binary ($M = 2$) communications channel. Then

$$\begin{aligned}
\mu(n;\boldsymbol{a},L) &= \mu(n-1;\boldsymbol{a},L) + 2a_n\dot{r}_n - a_n^2\mathsf{R}_{hh}(0) - 2a_n\sum_{m=1}^{L} a_{m+n}\mathsf{R}_{hh}(m)\\
&= \mu(n-1;\boldsymbol{a},2) + 2a_n\dot{r}_n - a_n^2\mathsf{R}_{hh}(0) - 2a_na_{n+1}\mathsf{R}_{hh}(1) - 2a_na_{n+2}\mathsf{R}_{hh}(2)
\end{aligned}$$

The metric $\mu(n;\boldsymbol{a},1)$ is controlled by three binary variables (a_{n-1}, a_n, a_{n+1}) and therefore the can be represented with an $2^{3-1} = 4$ state trellis. At each time interval n, each of the 8 path metrics in the set

$$\left\{\mu(n;(a_n,a_{n+1},a_{n+2}),2) \,:\, a_i \in \{-1,+1\}\right\}$$

are computed and the "shortest path" through the trellis is selected.

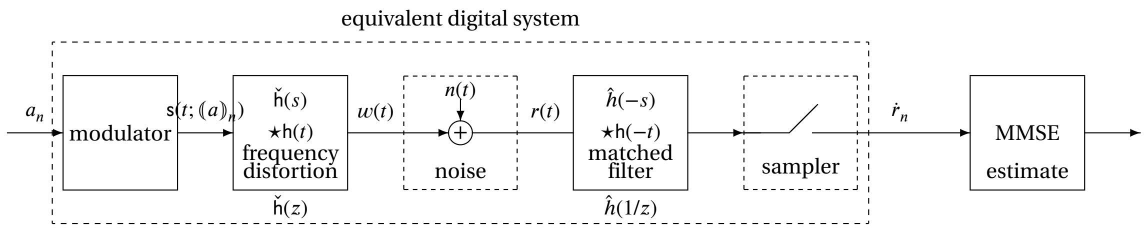


Figure 14.4: Minimum Mean Square Estimate Implementation

14.3.2 Minimum mean square estimate

Theorem 14.1 (page 165) guarantees that the sequence $(\dot{r}_n)$ is a sufficient statistic for computing the ML estimate of information sequence (a_n). Using $(\dot{r}_n)$, Section 14.3.1 shows that the ML estimate can be computed using a trellis. However, the trellis calculations can be very computationally demanding. A simpler approach is to use minimum mean square estimation (MMSE). MMSE can be computationally less demanding, but yields an estimate that is not equal to the ML estimate (MMSE is suboptimal). Minimum mean square estimation is presented in Section **??** (page **??**). Let

M: estimate order (M is odd)

N: parameter order (N is odd).

Then an estimate $\hat{\boldsymbol{a}}$ of the transmitted symbols can be calculated as follows.

$$\hat{\boldsymbol{a}} \triangleq \begin{bmatrix} \hat{a}_{n-\frac{M-1}{2}} \\ \vdots \\ \hat{a}_{n-1} \\ \hat{a}_n \\ \hat{a}_{n+1} \\ \vdots \\ \hat{a}_{n+\frac{M-1}{2}} \end{bmatrix} = U^H \boldsymbol{p} \qquad \boldsymbol{p} \triangleq \begin{bmatrix} \boldsymbol{p}_{n-\frac{N-1}{2}} \\ \vdots \\ \boldsymbol{p}_{n-1} \\ \boldsymbol{p}_n \\ \boldsymbol{p}_{n+1} \\ \vdots \\ \boldsymbol{p}_{n+\frac{N-1}{2}} \end{bmatrix}$$

$$U^H \triangleq \begin{bmatrix} \dot{r}_{n-\left(\frac{M-1}{2}\right)+\left(\frac{N-1}{2}\right)} & \dot{r}_{n-\left(\frac{M-1}{2}\right)+\left(\frac{N-1}{2}-1\right)} & \cdots & \dot{r}_{n-\left(\frac{M-1}{2}\right)-\left(\frac{N-1}{2}\right)} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{r}_{n-(1)+\left(\frac{N-1}{2}\right)} & \dot{r}_{n-(1)+\left(\frac{N-1}{2}-1\right)} & \cdots & \dot{r}_{n-(1)-\left(\frac{N-1}{2}\right)} \\ \dot{r}_{n+(0)+\left(\frac{N-1}{2}\right)} & \dot{r}_{n+(0)+\left(\frac{N-1}{2}-1\right)} & \cdots & \dot{r}_{n+(0)-\left(\frac{N-1}{2}\right)} \\ \dot{r}_{n+(1)+\left(\frac{N-1}{2}\right)} & \dot{r}_{n+(1)+\left(\frac{N-1}{2}-1\right)} & \cdots & \dot{r}_{n+(1)-\left(\frac{N-1}{2}\right)} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{r}_{n+\left(\frac{M-1}{2}\right)+\left(\frac{N-1}{2}\right)} & \dot{r}_{n+\left(\frac{M-1}{2}\right)+\left(\frac{N-1}{2}-1\right)} & \cdots & \dot{r}_{n+\left(\frac{M-1}{2}\right)-\left(\frac{N-1}{2}\right)} \end{bmatrix}$$

Let

$$\begin{aligned} \hat{\boldsymbol{a}}(\boldsymbol{p}) &\triangleq U^H \boldsymbol{p} \\ \boldsymbol{e}(\boldsymbol{p}) &\triangleq \hat{\boldsymbol{a}} - \boldsymbol{a} \\ \mathsf{C}(\boldsymbol{p}) &\triangleq \mathsf{E}\,\|\boldsymbol{e}\|^2 \triangleq \mathsf{E}\left[\boldsymbol{e}^T \boldsymbol{e}\right] \\ \hat{\theta}_{\text{mms}} &\triangleq \arg\min_{\boldsymbol{p}} \mathsf{C}(\boldsymbol{p}) \\ R &\triangleq \mathsf{E}\left[U U^H\right] \\ W &\triangleq \mathsf{E}\left[U \boldsymbol{y}\right]. \end{aligned}$$

Then

$$\begin{aligned}
\mathsf{C}(\boldsymbol{p}) &= \boldsymbol{p}^H R\boldsymbol{p} - (W^H\boldsymbol{p})^* - W^H\boldsymbol{p} + \mathsf{E}\left[\boldsymbol{a}^H\boldsymbol{a}\right] \\
\nabla_{\boldsymbol{p}}\mathsf{C}(\boldsymbol{p}) &= 2\mathbf{R}_{\mathsf{e}}[R]\boldsymbol{p} - 2\Re W \\
\hat{\theta}_{\mathsf{mms}} &= (\Re R)^{-1}(\Re W) \\
\mathsf{C}(\hat{\theta}_{\mathsf{mms}}) &= (\Re W^H)(\Re R)^{-1}R(\Re R)^{-1}(\Re W) - 2(\Re W^H)(\Re R)^{-1}(\Re W) + \mathsf{E}\left[\boldsymbol{a}^H\boldsymbol{a}\right] \\
\mathsf{C}(\hat{\theta}_{\mathsf{mms}})|_{R\text{ real}} &= \mathsf{E}\left[\boldsymbol{a}^H\boldsymbol{a}\right] - (\Re W^H)R^{-1}(\Re W).
\end{aligned}$$

14.3.3 Minimum peak distortion estimate

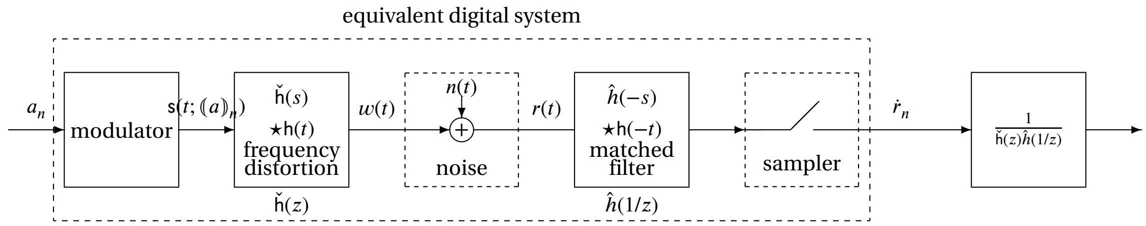


Figure 14.5: Peak distortion estimation

Peak distortion is achieved when there is **no** ISI. This means that the impulse response of the channel and post-channel processing must be only an impulse. Ideally this can be achieved by filtering $\dot{r}_n$ with the inverse of the equivalent system digital filters. See Figure 14.5 (page 172).

CHAPTER 15

MULTIPATH FADING CHANNEL

15.1 Channel model

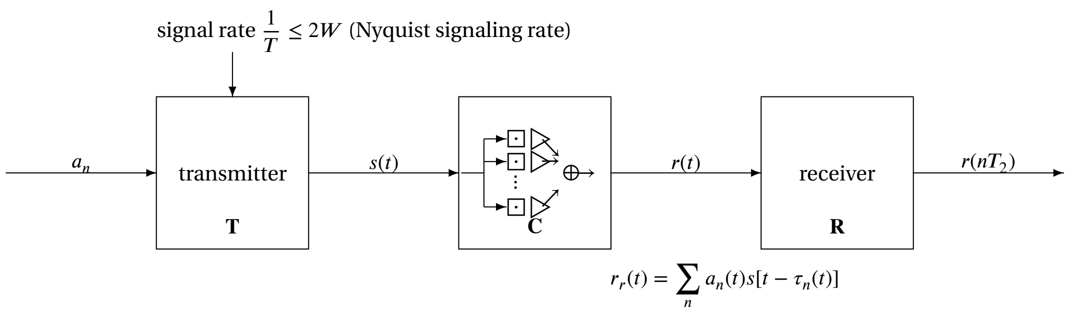


Figure 15.1: Multipath system model

Sources of interference. In the multipath-fading channel, there are two sources of interference: *multipath* and *fading*. These are briefly described next and illustrated in Figure 15.2 (page 174).

- **multipath**: Multipath is a process caused by multiple signal paths in a channel. Each path n is characterized by a scaling coefficient α_n and a delay τ_n.
  These weighted delays create a filter with some frequency response at time t.
  The stochastic bandwidth of this filter is the *coherence bandwidth* $(\Delta f)_c$.
  We would like the bandwidth W of the transmitted signal $s(t)$ to fit comfortably within the coherence bandwidth such that $W \ll (\Delta f)_c$. In this case we say that the channel is *frequency non-selective.*
- **fading**: Fading is a process caused by the values of the scaling coefficients and delays changing with time t. When the path n scaling coefficient α_n tends to zero, the signal traversing that path is attenuated and we say that it "fades". A measure of how fast paths change is the *coherence time* $(\Delta t)_c$. We would like the paths to remain stable for at least as long as a symbol period T such that $T \ll (\Delta t)_c$. In this case we say that the channel is *slowly fading.*

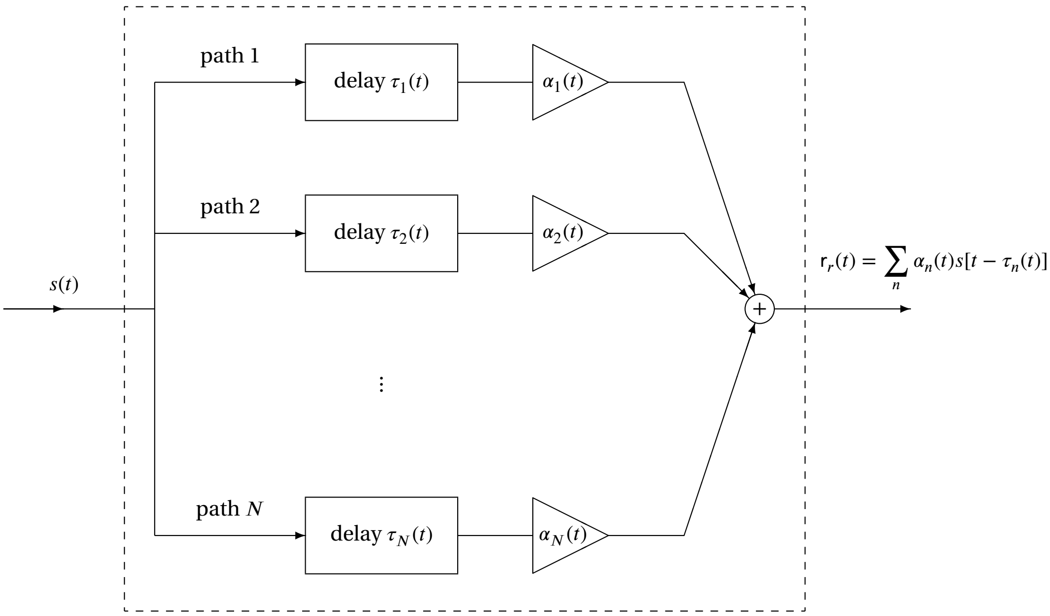


Figure 15.2: Multipath system model

Channel operator space. Many communication systems can be modeled as illustrated in Figure 15.2 (page 174). The system may be *discrete* (finite N) or *continuous* (infinite N); The system response may characterized by its *real-time response* or by its *instantaneous response*. These four possibilities are given in the following table:

$\mathsf{r}(t)$	discrete	continuous
real-time	$\mathsf{r}_r(t) = \sum_n \alpha_n(t)s[t-\tau_n(t)]$	$\mathsf{r}_{rc}(t) = \int_y \alpha(t;y)s[t-\tau(t;y)]\,\mathsf{d}y$
instantaneous	$\mathsf{r}(\tau;t) = \sum_n \alpha_n(t)s[\tau-\tau_n(t)]$	$\mathsf{r}_c(\tau;t) = \int_y \alpha(t;y)s[\tau-\tau(t;y)]\,\mathsf{d}y$

In the instantaneous response, the values of the system parameters $\alpha_n(t)$ and $\tau_n(t)$ are "frozen" at time instant t, the system response is then given as a function of τ. In this chapter, analysis will be performed using the discrete instantaneous response.

Definition 15.1. *Let channel operator* $\mathbf{C} : \{s : \mathbb{R} \to \mathbb{R}\} \to \{r : \mathbb{R} \to \mathbb{R}\}$ *be such that*

$$[\mathbf{C}s](\tau;t) = \sum_n \alpha_n(t)s[\tau - \tau_n(t)]$$

and under the constraints

1. $\alpha_n(t)$ *is zero mean*
2. $\alpha_n(t)$ *and* $\alpha_m(t)$ *are uncorrelated for* $n \neq m$.
3. $\tau_n(t)$ *and* $\tau_m(t)$ *are uncorrelated for* $n \neq m$.
4. $\alpha(t)$ *and* $\tau(t)$ *are uncorrelated.*
5. *the impulse response of* $\mathbf{C}$ *is WSS with respect to real-time* t.
6. $\tau(t)$ *are continuous with respect to real-time* t.

Let $\mathsf{h} : \mathbb{R}^2 \to \mathbb{R}$ *be the impulse response of* $\mathbf{C}$ *such that*

$$\mathsf{h}(\tau;t) = [\mathbf{C}\delta](\tau;t) = \sum_n \alpha_n(t)\delta[\tau - \tau_n(t)].$$

The following terms apply to the listed quantities:

t: *real-time*
τ: *response-time*
α_n: *reflection coefficient*
τ_n: *path delay*

Justification in real-world environments for the constraints of Definition 15.1 (page 174) is as follows:

1. This is just for mathematical convenience. We make the DC value equal to "0".
2. The amount of energy reflected from two different surfaces (α_n and α_m) are uncorrelated.
3. The length of two signal paths (τ_n and τ_m) are uncorrelated.
4. The amount of energy reflected from a surface ($\alpha(t)$) and the length of the signal path ($\tau(t)$) are uncorrelated.
5. The statistical properties of the channel do not change with time.
6. The continuity constraint is especially important in the real-time case when $s(t)$ is a very short pulse, or even an impulse $\delta(t)$. For example, in the impulse case, $\delta[t-\tau(t)]$ is only non-zero when $t = \tau(t)$. But if $\tau(t)$ is not continuous, it may never equal t and the impulse is completely lost even when $\alpha(t) \neq 0$. Having the continuity constraint helps fix the problem.

15.2 Receiver statistics

Proposition 15.1.

$$\mathsf{E}\left[\mathsf{r}(\tau;t)\right] = 0$$

✎PROOF:

$$\mathsf{E}\left[\mathsf{r}(\tau;t)\right] = \mathsf{E}\left[\sum_n \alpha_n(t)s[\tau-\tau_n(t)]\right] = \sum_n \mathsf{E}\left[\alpha_n(t)\right]s[\tau-\tau_n(t)] = \sum_n 0\cdot\mathsf{E}\left[s[\tau-\tau_n(t)]\right] = 0.$$

✏

Proposition 15.2. *Operation* **C** *is uncorrelated with respect to* τ *(***C** *is white with respect to* τ*).*

✎PROOF: By Definition 15.1 (page 174), $\tau_n(t)$ and $\tau_m(t)$ are uncorrelated for $m \neq n$. Different values of τ correspond to different path delays $\tau_n(t), \tau_m(t)$. Thus **C** is uncorrelated with respect to τ. ✏

Suppose $\mathsf{R}'_{\mathsf{hh}}(\tau_1,\tau_2;t_1,t_2) \triangleq \mathsf{E}\left[\mathsf{h}(\tau_1;t_1)\mathsf{h}(\tau_2;t_2)\right]$ is the autocorrelation function of the impulse response $\mathsf{h}(\tau;t)$. We already have two key characteristics of $\mathsf{h}(\tau;t)$:

1. $\mathsf{h}(\tau;t)$ is uncorrelated with respect to τ (by Proposition 15.2 page 175).
   So we only care about the case $\tau = \tau_1 = \tau_2$.
2. $\mathsf{h}(\tau;t)$ is WSS with respect to t (by Definition 15.1 (page 174)).
   So we only care about the case $\Delta t = t_1 - t_2$.

Because of these two characteristics, the autocorrelation function can be simplified to

$$\mathsf{R}_{\mathsf{hh}}(\tau;\Delta t) = \mathsf{R}_{\mathsf{hh}}(\tau;t_1-t_2) = \mathsf{R}'_{\mathsf{hh}}(\tau_1,\tau_2;t_1,t_2).$$

Definition 15.2. *Let* $\mathsf{R}_{\mathsf{hh}} : \mathbb{R}^2 \to \mathbb{R}$ *be the* ***autocorrelation*** *function of impulse response* $\mathsf{h} : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\mathsf{R}_{\mathsf{hh}}(\tau; \Delta t) \triangleq \mathsf{E}\left[\mathsf{h}(\tau; t + \Delta t)\mathsf{h}^*(\tau; t)\right].$$

15.3 Multipath measurement functions

The Fourier transform can operate over $\mathsf{R}_{\mathsf{hh}}(\tau; \Delta t)$ with respect to τ, Δt, or both to generate three new functions $\mathrm{R}^{R}_{\mathsf{hh}}(f)$, $\mathrm{R}^{L}_{\mathsf{hh}}(f)$, and $\mathrm{R}^{\bowtie}_{\mathsf{hh}}(f)$. This provides a total of four equivalent functions for measuring multipath. These four functions are formally defined in Definition 15.3 (page 176) and illustrated in Figure 15.3 (page 176).

Definition 15.3. *Let* $\mathsf{R}_{\mathsf{hh}} : \mathbb{R}^2 \to \mathbb{R}$, $\mathrm{R}^{R}_{\mathsf{hh}} : \mathbb{R}^2 \to \mathbb{R}$, $\mathrm{R}^{L}_{\mathsf{hh}} : \mathbb{R}^2 \to \mathbb{R}$, *and* $\mathrm{R}^{\bowtie}_{\mathsf{hh}} : \mathbb{R}^2 \to \mathbb{R}$ *be defined as*

DEF

1.	*autocorrelation function*	$\mathrm{R}_{\mathsf{hh}}(\tau; \Delta t)$	$\triangleq$	$\mathsf{E}\left[\mathsf{h}(\tau; t + \Delta t)\mathsf{h}^*(\tau; t)\right]$
2.	*spaced-frequency spaced-time function*	$\mathrm{R}^{R}_{\mathsf{hh}}(\Delta f; \Delta t)$	$\triangleq$	$\tilde{\mathbf{F}}_{\tau}\mathrm{R}_{\mathsf{hh}}(\tau; \Delta t)$
3.	*scattering function*	$\mathrm{R}^{L}_{\mathsf{hh}}(\tau; \lambda)$	$\triangleq$	$\tilde{\mathbf{F}}_{\Delta t}\mathrm{R}_{\mathsf{hh}}(\tau; \Delta t)$
4.	*Doppler function*	$\mathrm{R}^{\bowtie}_{\mathsf{hh}}(\Delta f; \lambda)$	$\triangleq$	$\tilde{\mathbf{F}}_{\tau}\tilde{\mathbf{F}}_{\Delta t}\mathrm{R}_{\mathsf{hh}}(\tau; \Delta t)$

The arguments of these functions are designated as

τ	*delay*
Δf	*frequency difference*
Δt	*time difference*
λ	*Doppler frequency.*

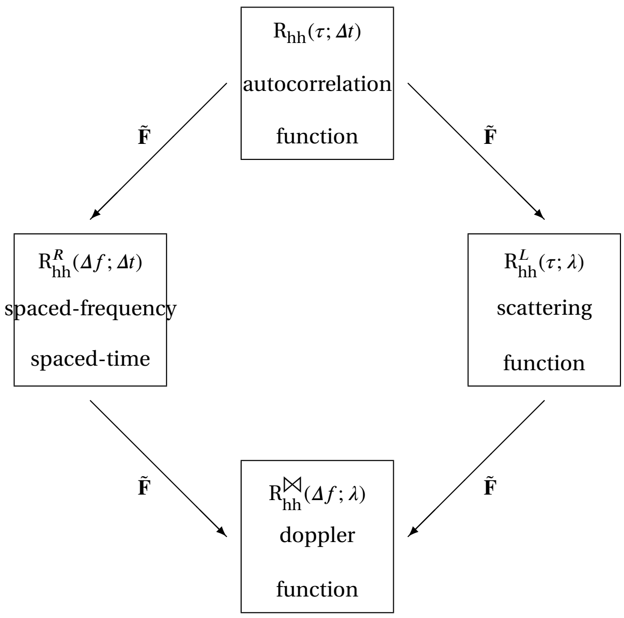


Figure 15.3: Multipath measurement functions

The Fourier transform of a random process (in time) is also a random process (in "frequency"). The Fourier transform of the random process $\mathsf{h}(\tau; t)$ with respect to τ is therefore a random process and has an autocorrelation function. This autocorrelation function is equivalent to the spaced-frequency-spaced-time function $\mathrm{R}^{R}_{\mathsf{hh}}(\Delta f; \Delta t)$ as shown next.

Proposition 15.3. *Let* $\tilde{\mathsf{h}} : \mathbb{R}^2 \to \mathbb{C}$ *be the Fourier transform of* $\mathsf{h} : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\tilde{\mathsf{h}}(f;t) \triangleq [\tilde{\mathbf{F}}\mathsf{h}(\tau;t)](f;t) \triangleq \int_\tau \mathsf{h}(\tau;t)e^{-i2\pi f\tau}\,\mathrm{d}\tau.$$

Then

$$\mathsf{E}\left[\tilde{\mathsf{h}}(f_1;t+\Delta t)\tilde{\mathsf{h}}^*(f_2;t)\right] = \mathrm{R}^R_{\mathsf{hh}}(\Delta f;\Delta t).$$

✎PROOF:

$$\begin{aligned}
\mathsf{E}\left[\tilde{\mathsf{h}}(f_1;t+\Delta t)\tilde{\mathsf{h}}^*(f_2;t)\right] &= \mathsf{E}\left[\int_{\tau_1}\mathsf{h}(\tau_1;t+\Delta t)e^{-i2\pi f_1\tau_1}\,\mathrm{d}\tau_1\left(\int_{\tau_2}\mathsf{h}(\tau_2;t)e^{-i2\pi f_2\tau_2}\,d\tau_2\right)^*\right]\\
&= \mathsf{E}\left[\int_{\tau_1}\int_{\tau_2}\mathsf{h}(\tau_1;t+\Delta t)e^{-i2\pi f_1\tau_1}\mathsf{h}^*(\tau_2;t)e^{i2\pi f_2\tau_2}\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\right]\\
&= \int_{\tau_1}\int_{\tau_2}\mathsf{E}\left[\mathsf{h}(\tau_1;t+\Delta t)\mathsf{h}^*(\tau_2;t)\right]e^{-i2\pi f_1\tau_1}e^{i2\pi f_2\tau_2}\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\\
&= \int_\tau \mathsf{E}\left[\mathsf{h}(\tau;t+\Delta t)\mathsf{h}^*(\tau;t)\right]e^{-i2\pi(f_1-f_2)\tau}\,\mathrm{d}\tau\\
&= \int_\tau \mathrm{R}_{\mathsf{hh}}(\tau;\Delta t)e^{-i2\pi\Delta f\tau}\,\mathrm{d}\tau\\
&= \tilde{\mathbf{F}}_\tau\mathrm{R}_{\mathsf{hh}}(\tau;\Delta t)\\
&= \mathrm{R}^R_{\mathsf{hh}}(\Delta f;\Delta t)
\end{aligned}$$

The following proof fails (diverges). However I still include it here anyway. Maybe someone can show me what I did wrong:

$$\begin{aligned}
\mathsf{E}\left[\tilde{\mathsf{h}}(\tau;\lambda_1)\tilde{\mathsf{h}}^*(\tau;\lambda_2)\right] &= \mathsf{E}\left[\tilde{\mathsf{h}}(\tau;\lambda_1)\tilde{\mathsf{h}}^*(\tau;\lambda_2)\right]\\
&= \mathsf{E}\left[\int_t \mathsf{h}(\tau;t)e^{-i2\pi\lambda_1 t}\,\mathrm{d}t\left(\int_u \mathsf{h}(\tau;u)e^{-i2\pi\lambda_2 u}\,\mathrm{d}u\right)^*\right]\\
&= \mathsf{E}\left[\int_t \mathsf{h}(\tau;t)e^{-i2\pi\lambda_1 t}\,\mathrm{d}t\int_u \mathsf{h}^*(\tau;u)e^{i2\pi\lambda_2 u}\,\mathrm{d}u\right]\\
&= \int_t\int_u \mathsf{E}\left[\mathsf{h}(\tau;t)\mathsf{h}^*(\tau;u)\right]e^{-i2\pi\lambda_1 t}e^{i2\pi\lambda_2 u}\,\mathrm{d}u\,\mathrm{d}t\\
&= \int_t\int_u \mathsf{E}\left[\mathsf{h}(\tau;u+\Delta t)\mathsf{h}^*(\tau;u)\right]e^{-i2\pi\lambda_1(u+\Delta t)}e^{i2\pi\lambda_2 u}\,\mathrm{d}u\,\mathrm{d}t && \Delta t = t-u \iff t = u+\Delta t\\
&= \int_u\int_{\Delta t}\mathrm{R}_{\mathsf{hh}}(\tau;\Delta t)e^{-i2\pi\lambda_1(u+\Delta t)}e^{i2\pi\lambda_2 u}\,\mathrm{d}\Delta t\,\mathrm{d}u\\
&= \int_u e^{-i2\pi(\lambda_1-\lambda_2)u}\,\mathrm{d}u\int_{\Delta t}\mathrm{R}_{\mathsf{hh}}(\tau;\Delta t)e^{-i2\pi\lambda_1\Delta t}\,\mathrm{d}\Delta t\\
&= \delta(\lambda_1-\lambda_2)\mathrm{R}^L_{\mathsf{hh}}(\tau;\lambda_1)
\end{aligned}$$

✏

15.4 Profile functions

Setting one of the two inputs in each measurement function of Definition 15.3 (page 176) to zero generates four new "profile" functions. The width of these four profile functions are four critical parameters. The four profile functions and four critical parameters are defined in Definition 15.4 (page 178) and illustrated in Figure 15.4 (page 178).

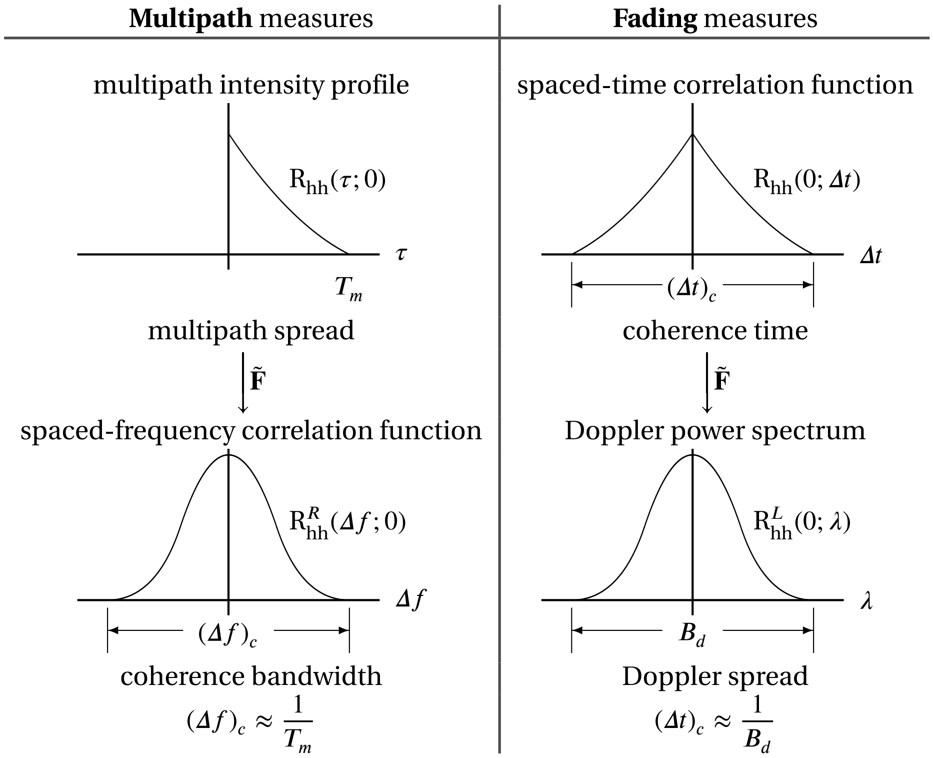


Figure 15.4: Profile functions with critical parameters

Definition 15.4. *The following four* ***profile functions*** *are defined as*

DEF

1.	*multipath intensity profile*	$R_{hh}(\tau;0)$
2.	*spaced-time correlation function*	$R^R_{hh}(0;\Delta t)$
3.	*Doppler power spectrum*	$R^L_{hh}(0;\lambda)$
4.	*spaced-frequency correlation function*	$R^R_{hh}(\Delta f;0)$

The following four ***critical parameters*** *are defined as*

DEF

1.	*multipath spread*	T_m	*is the width of* $R_{hh}(\tau;0)$
2.	*coherence time*	$(\Delta t)_c$	*is the width of* $R^R_{hh}(0;\Delta t)$
3.	*Doppler spread*	B_d	*is the width of* $R^L_{hh}(0;\lambda)$
4.	*coherence bandwidth*	$(\Delta f)_c$	*is the width of* $R^R_{hh}(\Delta f;0)$

Multipath intensity profile $R_{hh}(\tau;0)$

Power. The *multipath intensity profile* $R_{hh}(\tau;0)$ is a measure of the power (the "intensity") of a signal as a function of the path delay τ (each path has a delay τ). This is demonstrated by

$$\begin{aligned}
R_{hh}(\tau;0) &\triangleq \mathsf{E}\left[\mathsf{h}(\tau;t+0)\mathsf{h}^*(\tau;t)\right] \\
&= \mathsf{E}|\mathsf{h}(\tau;t)|^2 \\
&= \mathsf{E}|\mathsf{h}(\tau;0)|^2 \qquad \text{(because } \mathsf{h}(\tau;t) \text{ is WSS with respect to } t\text{)}.
\end{aligned}$$

Path correlation. As a signal traverses two paths where one is longer and longer paths relative to the other, the resulting two signals are less and less correlated. If they are delayed by more than the *multipath spread* T_m, then they are uncorrelated.

Spaced-time correlation profile $\mathrm{R}^R_{\mathrm{hh}}(0; \Delta t)$

The *spaced-time correlation profile* $\mathrm{R}^R_{\mathrm{hh}}(0; \Delta t)$ measures the time auto-correlation of a signal traveling through a single path. A signal is uncorrelated with a delayed version of itself if the delay is greater than the *coherence time* $(\Delta t)_c$.

Doppler power spectrum $\mathrm{R}^L_{\mathrm{hh}}(0; \lambda)$

The *Doppler power spectrum* $\mathrm{R}^L_{\mathrm{hh}}(\tau; 0)$ is a measure of signal power density as a function of λ.

Spaced-frequency correlation function $\mathrm{R}^R_{\mathrm{hh}}(\Delta f; 0)$

The *spaced-frequency correlation function* $\mathrm{R}^R_{\mathrm{hh}}(\Delta f; 0)$ measures the correlation of two sinusoids. If two sinusoids are separted in fequency by more than the *coherence bandwidth* $(\Delta f)_c$, then they are uncorrelated.

15.5 Channel classification

Definition 15.5. *For a signal $s(t)$ in a multipath channel let*

- *T be the signalling period*
- *W be the bandwidth.*

Then $s(t)$ is

DEF

frequency non-selective channel	*if*	$W \ll (\Delta f)_c$	*or*	$W \gg T_m$	$\star$
frequency selective channel	*if*	$W \gg (\Delta f)_c$	*or*	$W \ll T_m$	
slowly fading channel	*if*	$T \ll (\Delta t)_c$	*or*	$T \gg B_d$	$\star$
fast fading channel	*if*	$T \gg (\Delta t)_c$	*or*	$T \ll B_d$.	
underspread channel	*if*	$T_m B_d < 1$			
overspread channel	*if*	$T_m B_d > 1$			

The "underspead/overspread" definitions are related to the *Nyquist signaling rate*.[1] The Nyquist signaling theorem states the signaling rate $1/T$ is related to the transmitted signal bandwidth W by

[1] *Nyquist signaling theorem*: Theorem 13.2 page 146.

$1/T \le 2W$. So at the maximum rate, $TW = 1/2 \approx 1$.

$$\begin{array}{rcll} TW & \approx & 1 & \text{(by Nyquist signaling theorem)} \\ B_d & \ll & T & \text{(for slowly fading channel)} \\ T_m & \ll & W & \text{(for frequency non-selective channel)} \\ T_m B_d & < & TW \approx 1 & \text{(for slowly fading, fequency non-selective channel).} \end{array}$$

15.6 Multipath-fading countermeasures

There are two general classes of multipath-fading countermeasures:

1. diversity techniques
2. Rake receiver.

Diversity techniques for compensating for multipath are[2]

1. frequency diversity
2. time diversity
3. antenna diversity
4. path diversity
5. angle of arrival diversity
6. polarization diversity

The rake receiver is a transversal filter with coefficients optimized for channel operation.

[2] Proakis (2001), pages 821–822

Part IV

Appendices

APPENDIX A

ELECTROMAGNETICS

Physics involves the study of principles which govern the natural world. Some of these governing principles can be described using a concept called a "field". Three naturally occuring fields have been identified:

- gravitational field
- electric field
- magnetic field

Thus far no set of equations has been found that show the relationship between all three of these fields. However, James Maxwell has successfully constructed a set of four equations which demonstrate the relationship between the electric and magnetic fields. These equations show that electric and magnetic fields are intimately related and thus the joint study of these fields is called *electromagnetic field* theory.

A.1 Identities

The following identities are useful in working with differential operators. Identities will be distinguished from equations[1] by using the assignment "≡" rather than "=".

Theorem A.1 (Stokes' Theorem)**.**

THM
$$\int_s (\nabla \times A) \cdot d\mathbf{s} \equiv \oint_l \mathbf{A} \cdot d\mathbf{L}$$

Theorem A.2 (Divergence Theorem)**.**

THM
$$\int_v (\nabla \cdot A)\, dv \equiv \oint_s \mathbf{A} \cdot d\mathbf{s}$$

[1] An *identity* is a special case of an *equation*; And in this sense an identity is different from an equation. An identity is true over the entire domain of the free variable. However, an equation may only be true over a portion of the domain or may even be always false. For example, suppose $\theta \in \mathbb{R}$. Then $\sin^2\theta + \cos^2\theta \equiv 1$ is an **identity** because it is true for all $\theta \in \mathbb{R}$. The expression $\cos^2\theta = 1$ is only an **equation** (not an identity) because it is only true at integer multiples of 2π. The expression $\cos^2\theta = 2$ is an **equation** which is not true for any value in the domain ($\theta \in \mathbb{R}$). Reference: Smith (1999/2000)

Theorem A.3 (Laplacian Identity).

THM
$$\nabla\times\nabla\times\boldsymbol{A} \equiv \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

A.2 Electromagnetic Field Definitions

A.2.1 Vector quantities

Maxwell's equations describe electromagnetic properties in terms of four vector quantities: E, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$.

Definition A.1.

DEF
*The **electric field** E describes the force per unit charge exerted by the field.*

$$\mathsf{E} \triangleq \frac{\mathbf{F}}{Q} \qquad \text{where } \mathbf{F} \text{ is force exerted on a charge } Q.$$

Definition A.2.

DEF
*The **electric flux density** $\mathbf{D}$ specifies the equivalent charge per unit area.*

Definition A.3.

DEF
*The **magnetic field** $\mathbf{H}$ specifies the force generated by the movement of a charged particle.*

Definition A.4.

DEF
*The **magnetic flux density** $\mathbf{B}$ specifies the equivalent force of movement of charge per unit area exerted by a magnetic field $\mathbf{H}$.*

A.2.2 Operators

The relationship between the electric flux density $\mathbf{D}$ and electric field E is described by the *permittivity operator* $\mathcal{E}$ as defined Definition A.5 (next definition).

Remark A.1. [2] For a very wide class of media, the relation between $\mathbf{D}$ and E can be described very accurately as $\mathbf{D} = \mathcal{E}\mathsf{E}$. However in general, $\mathbf{D}$ is a function of both E and $\mathbf{H}$ such that $\mathbf{D} = f(\mathsf{E}, \mathbf{H})$. One such class of media is *bianisotropic media.*

Definition A.5.

DEF
*The **permittivity operator** $\mathcal{E}$ is defined as* $\mathbf{D} = \mathcal{E}\mathsf{E}$

If the operation $\mathcal{E}$ is INVERTIBLE *then* $\mathsf{E} = \mathcal{E}^{-1}\mathbf{D}$ *where $\mathcal{E}^{-1}$ is the inverse operation of $\mathcal{E}$*

The relationship between the magnetic flux density $\mathbf{B}$ and magnetic field $\mathbf{H}$ is described by the *permeability operator* $\mathcal{U}$ as defined in Definition A.6 (next definition).

Remark A.2. Similar to Remark A.1, for an very wide class of media, the relation between $\mathbf{B}$ and $\mathbf{H}$ can be described very accurately as $\mathbf{B} = \mathcal{U}\mathbf{H}$. However in general, $\mathbf{B}$ is a function of both $\mathbf{H}$ and E such that $\mathbf{B} = \mathsf{g}(\mathbf{H}, \mathsf{E})$ for some function g.

[2] Kong (1990), page 5

Definition A.6.

DEF *The* ***permeability operator*** $\mathcal{U}$ *is defined as* $\mathbf{B} = \mathcal{U}\mathbf{H}$
If the operation $\mathcal{U}$ *is* INVERTIBLE *then* $\mathbf{H} = \mathcal{U}^{-1}\mathbf{B}$ *where* $\mathcal{U}^{-1}$ *is the inverse operation of* $\mathcal{U}$

A.2.3 Types of Media

Electromagnetic waves propagate through a *media*. A media may be classified according to whether it is **linear**, **homogeneous**, **isotropic**, **time-invariant**, or **simple**.

Definition A.7.

DEF *A media is* ***simple*** *if the operators* $\mathcal{E}$ *and* $\mathcal{U}$ *are multiplicative constants* ϵ *and* μ *such that*

$$\begin{aligned}\mathbf{D} &= \epsilon\mathsf{E} \quad \text{and}\\ \mathbf{B} &= \mu\mathbf{H}\end{aligned}$$

A.3 Electromagnetic Field Axioms

The fundamentals of electromagnetic theory are at their core based largely on empirical results rather than on mathematical analysis. Since they are based on experiment rather than analysis, we present them here as "axioms", which of course require no proof.

Axiom A.1 (Maxwell-Faraday Axiom)**.**

AX $\nabla\times\mathsf{E} = -\frac{\partial}{\partial t}\mathbf{B}$

Axiom A.2 (Maxwell-Ampere Axiom)**.**

AX $\nabla\times\mathbf{H} = \frac{\partial}{\partial t}\mathbf{D} + \mathbf{J}$ *where* $\mathbf{J}$ *is electric current density*

Axiom A.3 (Maxwell-Gauss-D Axiom)**.**

AX $\nabla\cdot\mathbf{D} = \rho$ *where* ρ *is electric charge density*

Axiom A.4 (Maxwell-Gauss-B Axiom)**.**

AX $\nabla\cdot\mathbf{B} = 0$

A.4 Wave Equations

In a simple media, electric and magnetic fields propagate in the form of waves. This can be shown using two theorems.

- In a *linear* media, the time/space relationships between E and $\mathbf{H}$ can be described using second order differential equations (Theorem A.4 page 186).
- In a *simple* media, the solution to these equations are waves propagating in both time and location (Theorem A.5 page 188).

Theorem A.4 (Electric field wave equation).

THM

$$\left\{\begin{array}{lllll}
(1). & \mathcal{E} \text{ and } \mathcal{U} \text{ are } \textbf{linear}. & \text{and} & & \\
(2). & \mathcal{E} \text{ and } \mathcal{U} \text{ are } \textbf{time-invariant} & \text{and} & & \\
(3). & \mathcal{E} \text{ and } \mathcal{U} \text{ are } \textbf{invertible} & (\mathcal{E}^{-1} \text{ and } \mathcal{U}^{-1} \text{ exist}) & \text{and} \\
(4). & \text{If } \mathsf{E} = 0, \text{ then } \mathbf{D} = 0 & (\mathbf{D} = \mathcal{E}0 = 0) & \text{and} \\
(5). & \text{If } \mathbf{H} = 0, \text{ then } \mathbf{B} = 0 & (\mathbf{B} = \mathcal{U}0 = 0) & \text{and} \\
(6). & \text{The charge density is constant in location} & (\nabla\rho = 0) & \text{and} \\
(7). & \text{Current flow is constant in location and time} & (\frac{\partial}{\partial t}\mathbf{J} = 0 \text{ and } \nabla\mathbf{J} = 0) &
\end{array}\right\} \implies \left\{\begin{array}{lcl}
\nabla^2\mathsf{E} & = & \mathcal{E}\mathcal{U} \\
\nabla^2 H & = & \mathcal{E}\mathcal{U}
\end{array}\right.$$

✎PROOF: The condition that $\mathcal{E}$ is linear and invertible implies $\mathcal{E}^{-1}$ is also linear. We now analyze the curl of the left hand side of the Maxwell-Faraday Axiom.

$$\begin{aligned}
\nabla\times\nabla\times E &= \nabla(\nabla\cdot E) - \nabla^2\mathsf{E} && \text{by Theorem A.3 page 184} \\
&= \nabla(\nabla\cdot\mathcal{E}^{-1}\mathbf{D}) - \nabla^2\mathsf{E} && \text{because } \mathcal{E} \text{ is invertible} \\
&= \nabla\mathcal{E}^{-1}(\nabla\cdot\mathbf{D}) - \nabla^2\mathsf{E} && \text{because } \mathcal{E}^{-1} \text{ is linear} \\
&= \mathcal{E}^{-1}\nabla(\nabla\cdot\mathbf{D}) - \nabla^2\mathsf{E} && \text{because } \mathcal{E}^{-1} \text{ is linear} \\
&= \mathcal{E}^{-1}\nabla\rho - \nabla^2\mathsf{E} && \text{by Axiom A.3 page 185} \\
&= \mathcal{E}^{-1}0 - \nabla^2\mathsf{E} && \text{by condition 6} \\
&= \mathcal{E}^{-1}\mathcal{E}0 - \nabla^2\mathsf{E} && \text{by condition 4} \\
&= 0 - \nabla^2\mathsf{E} && \text{because } \mathcal{E}^{-1}\mathcal{E} = I \text{ is the identity operator} \\
&= -\nabla^2\mathsf{E}
\end{aligned}$$

We now analyze the curl of the right side of the Maxwell-Faraday Axiom.

$$\begin{aligned}
\nabla\times\left(-\frac{\partial}{\partial t}\mathbf{B}\right) &= -\frac{\partial}{\partial t}\nabla\times\mathbf{B} && \text{by linearity of operators} \\
&= -\frac{\partial}{\partial t}\nabla\times\mathcal{U}H && \text{by Definition A.6 page 185} \\
&= -\frac{\partial}{\partial t}\mathcal{U}\nabla\times H && \text{by linearity of } \mathcal{U} \\
&= -\mathcal{U}\frac{\partial}{\partial t}\nabla\times H && \text{by time-invariance of } \mathcal{U} \\
&= -\mathcal{U}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\mathbf{D} + \mathbf{J}\right) && \text{by the Maxwell-Ampere Axiom} \\
&= -\mathcal{U}\left(\frac{\partial^2}{\partial t^2}\mathbf{D} + \frac{\partial}{\partial t}\mathbf{J}\right) \\
&= -\mathcal{U}\left(\frac{\partial^2}{\partial t^2}\mathbf{D} + 0\right) && \text{by condition 7} \\
&= -\mathcal{U}\left(\frac{\partial^2}{\partial t^2}\mathcal{E}\mathsf{E}\right) && \text{by Definition A.5 page 184} \\
&= -\mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathsf{E} && \text{by time-invariance of } \mathcal{E}
\end{aligned}$$

Starting with the Maxwell-Ampere Axiom and using the results of the previous two sets of equations, we can now prove the first equation of the theorem.

$$\begin{aligned}
\nabla\times\mathsf{E} &= -\frac{\partial}{\partial t}\mathbf{B} && \Rightarrow \\
\nabla\times\nabla\times\mathsf{E} &= \nabla\times(-\frac{\partial}{\partial t}\mathbf{B}) && \Leftrightarrow \\
-\nabla^2\mathsf{E} &= -\mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathsf{E} && \Leftrightarrow \\
\nabla^2\mathsf{E} &= \mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathsf{E}
\end{aligned}$$

The condition that $\mathcal{U}$ is linear and invertible implies $\mathcal{U}^{-1}$ is also linear.

We now analyze the curl of the left hand side of the Maxwell-Ampere Axiom.

$$\begin{aligned}
\nabla\times\nabla\times H &\equiv \nabla(\nabla\cdot H) - \nabla^2 H && \text{by Theorem A.3 page 184} \\
&= \nabla(\nabla\cdot\mathcal{U}^{-1}\mathbf{B}) - \nabla^2 H && \text{because } \mathcal{U} \text{ is invertible} \\
&= \nabla\mathcal{U}^{-1}(\nabla\cdot\mathbf{B}) - \nabla^2 H && \text{because } \mathcal{U}^{-1} \text{ is linear} \\
&= \mathcal{U}^{-1}\nabla(\nabla\cdot\mathbf{B}) - \nabla^2 H && \text{because } \mathcal{U}^{-1} \text{ is } \textit{linear} \\
&= \mathcal{U}^{-1}0 - \nabla^2 H && \text{by Axiom A.4 page 185} \\
&= \mathcal{U}^{-1}\mathcal{U}0 - \nabla^2 H && \text{by condition 5} \\
&= 0 - \nabla^2 H && \text{because } \mathcal{U}^{-1}\mathcal{U} = I \text{ is the identity operator} \\
&= -\nabla^2 H
\end{aligned}$$

We now analyze the curl of the right side of the *Maxwell-Faraday Axiom* (Axiom A.1 page 185).

$$\begin{aligned}
\nabla\times\left(\frac{\partial}{\partial t}\mathbf{D}+\mathbf{J}\right) &= \frac{\partial}{\partial t}\nabla\times\mathbf{D} + \nabla\times\mathbf{J} && \text{by linearity of operators} \\
&= \frac{\partial}{\partial t}\nabla\times\mathbf{D} && \text{by condition 7} \\
&= \frac{\partial}{\partial t}\nabla\times\mathcal{E}\mathsf{E} && \text{by Definition A.5 page 184} \\
&= \frac{\partial}{\partial t}\mathcal{E}\nabla\times\mathsf{E} && \text{by linearity of } \mathcal{E} \\
&= \mathcal{E}\frac{\partial}{\partial t}\nabla\times E && \text{by time-invariance of } \mathcal{E} \\
&= \mathcal{E}\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\mathbf{B}\right) && \text{by the Maxwell-Faraday Axiom} \\
&= -\mathcal{E}\frac{\partial^2}{\partial t^2}\mathbf{B} \\
&= -\mathcal{E}\frac{\partial^2}{\partial t^2}\mathcal{U}\mathbf{H} && \text{by Definition A.6 page 185} \\
&= -\mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathbf{H} && \text{by } \textit{time-invariance} \text{ of } \mathcal{U}
\end{aligned}$$

Starting with the Maxwell-Faraday Axiom and using the results of the previous two sets of equations, we can now prove the second part of the theorem.

$$
\begin{aligned}
\nabla\times\mathbf{H} &= \frac{\partial}{\partial t}\mathbf{D}+\mathbf{J} && \Rightarrow\\
\nabla\times\nabla\times\mathbf{H} &= \nabla\times(\frac{\partial}{\partial t}\mathbf{D}+\mathbf{J}) && \Leftrightarrow\\
-\nabla^2\mathbf{H} &= -\mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathbf{H} && \Leftrightarrow\\
\nabla^2\mathbf{H} &= \mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathbf{H}
\end{aligned}
$$

⇨

Theorem A.4 (page 186) shows that under Axioms Axiom A.1 – Axiom A.4 (page 185) and certain other general conditions, both the electric field and magnetic field can be represented as second order differential equations in location and time. The general solution to these equations is given in the next theorem.

Theorem A.5. [3] *In a simple media, the wave equation for the electric field* E *has the following general solution:*

THM

$$\boxed{\mathsf{E}(x,y,z,t) = p_1(\hat{k}\cdot r - vt) + p_2(\hat{k}\cdot r + vt)}$$

where p_1 *and* p_2 *are any vector functions,* $\hat{\mathbf{k}} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$ *is a unit vector in the direction of wave propagation,* r *is a position vector, and* $v = 1/\sqrt{\epsilon\mu}$.

✎PROOF: According to Theorem A.4 (page 186),

$$\nabla^2\mathsf{E} \quad = \quad \mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathsf{E}. \tag{A.1}$$

Since the media is simple, the operation $\mathcal{E}\mathcal{U}$ equivalent to multiplicatioin by $\epsilon\mu$ and so

$$\nabla^2\mathsf{E} = \epsilon\mu\frac{\partial^2}{\partial t^2}\mathsf{E}.$$

This equation is actually three equations.

$$
\begin{aligned}
\nabla^2 E_x &= \epsilon\mu\frac{\partial^2}{\partial t^2}E_x && \text{x component}\\
\nabla^2 E_y &= \epsilon\mu\frac{\partial^2}{\partial t^2}E_y && \text{y component}\\
\nabla^2 E_z &= \epsilon\mu\frac{\partial^2}{\partial t^2}E_z && \text{z component}
\end{aligned}
$$

Proving any one of them proves them all. We pick the first one. The term $\epsilon\mu\frac{\partial^2}{\partial t^2}E_x$ can be evaluated as follows:

$$
\begin{aligned}
\epsilon\mu\frac{\partial^2}{\partial t^2}E_x &= \epsilon\mu\frac{\partial^2}{\partial t^2}p_{1x}(\hat{\mathbf{k}}\cdot r - vt) + \epsilon\mu\frac{\partial^2}{\partial t^2}p_{2x}(\hat{\mathbf{k}}\cdot r + vt)\\
&= \epsilon\mu v^2 p''_{1x}(\hat{\mathbf{k}}\cdot r - vt) + \epsilon\mu v^2 p''_{2x}(\hat{\mathbf{k}}\cdot r + vt)\\
&= p''_{1x}(\hat{\mathbf{k}}\cdot r - vt) + p''_{2x}(\hat{\mathbf{k}}\cdot r + vt)
\end{aligned}
$$

[3] Inan and Inan (2000), page 21

The term $\nabla^2 E_x$ can be evaluated as follows:

$$\nabla^2 E_x = \nabla^2 p_{1x}(\hat{\mathbf{k}} \cdot r - vt) + \nabla^2 p_{2x}(\hat{\mathbf{k}} \cdot r - vt)$$

The two terms on the right can be simplified.

$$\begin{aligned}
\nabla^2 p_{1x}(\hat{\mathbf{k}} \cdot r - vt) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) p_{1x}(\hat{\mathbf{k}} \cdot r - vt) \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) p_{1x}(xk_x + yk_y + zk_z - vt) \\
&= \frac{\partial^2}{\partial x^2} p_{1x}(xk_x + yk_y + zk_z - vt) + \frac{\partial^2}{\partial y^2} p_{1x}(xk_x + yk_y + zk_z - vt) + \\
&\quad \frac{\partial^2}{\partial z^2} p_{1x}(xk_x + yk_y + zk_z - vt) \\
&= k_x^2 p_{1x}''(xk_x + yk_y + zk_z - vt) + k_y^2 p_{1x}''(xk_x + yk_y + zk_z - vt) + \\
&\quad k_z^2 p_{1x}''(xk_x + yk_y + zk_z - vt) \\
&= (k_x^2 + k_y^2 + k_z^2) p_{1x}''(xk_x + yk_y + zk_z - vt) \\
&= \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} p_{1x}''(\hat{\mathbf{k}} \cdot r - vt) \\
&= p_{1x}''(\hat{\mathbf{k}} \cdot r - vt)
\end{aligned}$$

$$\nabla^2 p_{2x}(\hat{\mathbf{k}} \cdot r + vt) = p_{2x}''(\hat{\mathbf{k}} \cdot r + vt)$$

The term $\nabla^2 E_x$ can now be expressed as

$$\begin{aligned}
\nabla^2 E_x &= \nabla^2 p_{1x}(\hat{\mathbf{k}} \cdot r - vt) + \nabla^2 p_{2x}(\hat{\mathbf{k}} \cdot r - vt) \\
&= p_{1x}''(\hat{\mathbf{k}} \cdot r - vt) + p_{2x}''(\hat{\mathbf{k}} \cdot r + vt) \\
&= \epsilon\mu \frac{\partial^2}{\partial t^2} E_x.
\end{aligned}$$

✑

A.5 Effect of objects on electromagnetic waves

The following are attributes of an electromagnetic wave. Some of these attributes can be affected by an object in the path of the wave. Because the attributes of the wave can be affected by the object, measurements of the attributes can be exploited to infer some information about the object.

- propagation
- polarization
- permittivity
- permeability

Propagation An object can affect electromagnetic wave propagation in the following ways.

- Reflection
- Refraction
- Diffraction

Reflection A single reflection is very useful for gaining information about a single surface of an object. This is used extensively by radar and sonar systems. Of course multiple refections could be used to gain more information about the object. This could involve several reflections over time or an array of transmitting and receiving antennas.

Refraction, permittivity, permeability Refraction is very useful for determining the internal composition of an object. The electric field wave equation tells us that

$$\nabla^2 \mathsf{E} = \mathcal{E}\mathcal{U}\frac{\partial^2}{\partial t^2}\mathsf{E}$$

where $\mathcal{E}$ is the *permittivity operator* and $\mathcal{U}$ the *permeability operator*. Using numerical techniques, it may be possible to "solve" (find the mapping for) the operation $\mathcal{E}\mathcal{U}$. In general the operation is *non-linear*. However in many cases it may be *linear* or approximately linear in which case $\mathcal{E}\mathcal{U}$ may be modeled as a matrix. One technique for analyzing the matrix is to perform a *singular value decomposition* (SVD) and then analyze the pseudo eigenvalues and eigenvectors of the decomposition to gain a clearer understanding of the properties of the object. The SVD of $\mathcal{E}\mathcal{U}$ can be expresed as

$$\mathcal{E}\mathcal{U} = U\Lambda V$$

where Λ is a diagonal matrix containing the pseudo-eigenvalues of $\mathcal{E}\mathcal{U}$ and U and V are matrices containing the pseudo-eigenvectors.

Diffraction An object may completely block a portion of an oncoming electromagnetic wave. However, due to diffraction, the wave may essentially reconstruct the hole the object made in the wave as the wave propagates farther and farther past the object. This effect is at least partly explained by *Huygen's principle.* Information gathered from a diffracted wave could perhaps give move information about the overall shape of an object than a single reflection could. This is because a reflected wave only carries information about a single surface, whereas a diffracted wave flows around an object and therefore may carry information about the entire outer surface of the object.

Polarization Qualitatively, polarization is the general "shape" of the electric field $\mathsf{E}(x, y, z, t)$. For example, FM radio uses linear polarization. Some radar systems use circular polarization. If $\mathsf{E}(x, y, z, t)$ is extremely random in magnitude and direction over time, then the wave is said to be *unpolarized.* Light from the sun is an example of a wave that is nearly unpolarized[4]. A more formal (quantitative) definition of polarization is presented next.

Definition A.8.

DEF

*Let a **polarization function** $p(x, y, z)$ be defined as*

$$p(x,y,z) \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathsf{E}(x,y,z,t)\,dt$$

*The shape of $p(x, y, z, t)$ is the **polarization** of $\mathsf{E}(x, y, z, t)$.*

Remark A.3. [5] An object can affect the polarization of a wave. This has been exploited in radar systems to distinguish a metal object from clouds and "clutter".

[4] Inan and Inan (2000), page 94
[5] Inan and Inan (2000), page 96

APPENDIX B

INFORMATION THEORY

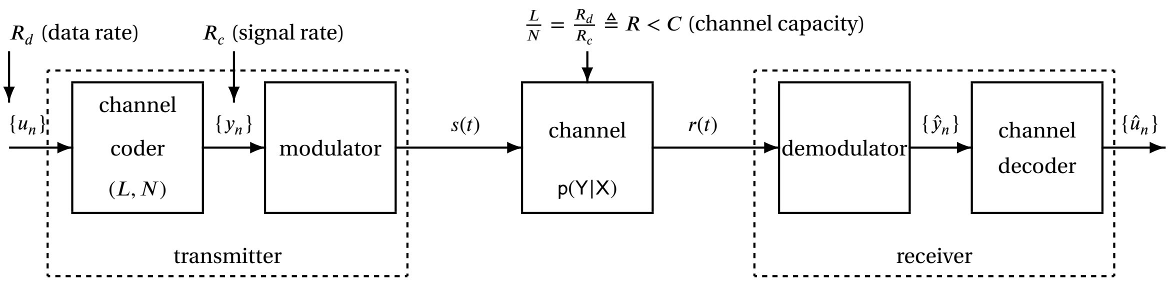


Figure B.1: Memoryless modulation system model

B.1 Information Theory

B.1.1 Definitions

The *Kullback Leibler distance* $\mathsf{D}(\mathsf{p}_1, \mathsf{p}_2)$ (Definition B.1 page 191) is a measure between two probability density functions p_1 and p_2. It is not a true distance measure[1] but it behaves in a similar manner. If $\mathsf{p}_1 = \mathsf{p}_2$, then the *KL distance* is 0. If p_1 is very different from p_2, then $|\mathsf{D}(\mathsf{p}_1, \mathsf{p}_2)|$ will be much larger.

Definition B.1. [2] *Let p_1 and p_2 be probability density functions. Then the* ***Kullback Leibler distance*** *(the* KL DISTANCE, *also called the* ***relative entropy****) of p_1 and p_2 is*

DEF $$\mathsf{D}(\mathsf{p}_1, \mathsf{p}_2) \triangleq \mathsf{E} \log_2 \frac{\mathsf{p}_1(\mathsf{X})}{\mathsf{p}_2(\mathsf{X})} \quad \textit{bits}$$

If the base of logarithm is e (the "natural logarithm") rather than 2, then the units are NATS *rather than* BITS.

The *mutual information* $\mathsf{I}(\mathsf{X}; \mathsf{Y})$ of random variable X and Y is the *KL distance* between their *joint distribution* $\mathsf{p}(\mathsf{X}, \mathsf{Y})$ and the product of their *marginal distributions* $\mathsf{p}(\mathsf{X})$ and $\mathsf{p}(\mathsf{Y})$. If X and Y are independent, then the *KL distance* between joint and marginal product is $\log 1 = 0$ and they have no *mutual information* $(\mathsf{I}(\mathsf{X}; \mathsf{Y}) = 0)$. If X and Y are highly correlated, then the *joint distribution* is

[1] *Distance measure*: Definition **??** (page **??**)

[2] Kullback and Leibler (1951), Csiszar (1961), ichi Amari (2012), Cover and Thomas (1991) page 18

much different than the product of the marginals making the *KL distance* greater and along with it the *mutual information* greater as well.

Definition B.2 (Mutual information). [3]

DEF
$$\mathsf{I}(X;Y) \triangleq \mathsf{D}\left(\mathsf{p}(X,Y),\mathsf{p}(X)\mathsf{p}(Y)\right) \triangleq \mathsf{E}_{xy}\log_2 \frac{\mathsf{p}(X,Y)}{\mathsf{p}(X)\mathsf{p}(Y)} \qquad \textit{bits}$$

The *self information* $\mathsf{I}(X;X)$ of random variable X is the *mutual information* between X and itself. That is, it is a measure of the information contained in X. Self information $\mathsf{I}(X;X)$ can also be viewed as the *KL distance* between the constant 1 (no information because 1 is completely known) and $\mathsf{p}(X)$.

Definition B.3 (Self information). [4]

DEF
$$\mathsf{I}(X;X) \triangleq \mathsf{D}\left(1,\mathsf{p}(X)\right) \triangleq \mathsf{E}_{x}\log_2 \frac{1}{\mathsf{p}(X)} \qquad \textit{bits}$$

The *entropy* $\mathsf{H}(X)$ of a random variable X is equivalent to the self information $\mathsf{I}(X;X)$ of X. That is, the entropy of X is a measure of the information contained in X.

Likewise, the *conditional entropy* $\mathsf{H}(X|Y)$ of X given Y is the information contained in X given Y has occurred. If X and Y are independent, then X does not care about the occurrence of Y. Thus in this case, the occurrence of $Y = y$ does not change the amount of information provided by X and $\mathsf{H}(X|Y) = \mathsf{H}(X)$. If X and Y are highly correlated, the occurrence of $Y = y$ tells us a lot about what the value of X might turn out to be. Thus in this case, the information provided by X given Y is greatly reduced and $\mathsf{H}(X|Y) << \mathsf{H}(X)$.

The *joint entropy* $\mathsf{H}(X,Y)$ of X and Y is the amount of information contained in the ordered pair (X,Y).

Definition B.4 (Entropy). [5]

DEF
$$\begin{array}{lrcll}
\textit{entropy of } X: & \mathsf{H}(X) & \triangleq & \mathsf{E}_x \log_2 \frac{1}{\mathsf{p}(X)} & \textit{bits}\\
\textit{joint entropy of } X,Y: & \mathsf{H}(X,Y) & \triangleq & \mathsf{E}_{xy} \log_2 \frac{1}{\mathsf{p}(X,Y)} & \textit{bits}\\
\textit{conditional entropy of } X \textit{ given } Y: & \mathsf{H}(X|Y) & \triangleq & \mathsf{E}_{xy} \log_2 \frac{1}{\mathsf{p}(X|Y)} & \textit{bits}
\end{array}$$

B.1.2 Relations

Theorem B.1.

THM
$$\mathsf{H}(X,Y) = \mathsf{H}(Y,X)$$

✎PROOF:

$$\begin{aligned}
\mathsf{H}(X,Y) &\triangleq \mathsf{E}_{xy}\log\frac{1}{\mathsf{p}_{xy}(X,Y)}\\
&= \mathsf{E}_{yx}\log\frac{1}{\mathsf{p}_{yx}(Y,X)}\\
&\triangleq \mathsf{H}(Y,X)
\end{aligned}$$

[3] Kullback (1959), Cover and Thomas (1991), pages 18–19
[4] Hartley (1928), Fano (1949), Cover and Thomas (1991), pages 18–19
[5] Cover and Thomas (1991), pages 15–17

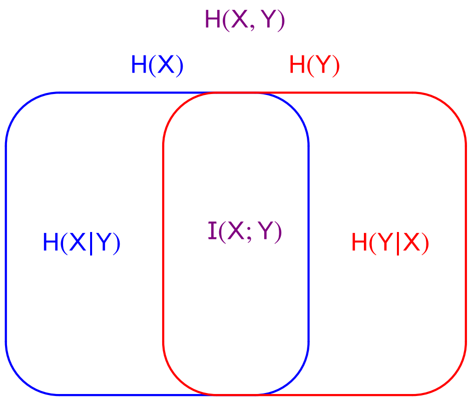


Figure B.2: Relationship between information and entropy

Theorem B.2 (Entropy chain rule).

THM

$$\begin{aligned}
\mathsf{H}(X,Y) &= \mathsf{H}(X|Y) + \mathsf{H}(Y) \\
&= \mathsf{H}(Y|X) + \mathsf{H}(X). \\
\mathsf{H}(X_1, X_2, \ldots, X_N) &= \sum_{n=1}^{N-1} \mathsf{H}(X_n | X_{n+1}, \ldots, X_N) + \mathsf{H}(X_N)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathsf{H}(X,Y) &\triangleq \mathsf{E}_{xy} \log \frac{1}{p(X,Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{p(X|Y)p(Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{p(X|Y)} + \mathsf{E}_{xy} \log \frac{1}{p(Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{p(X|Y)} + \mathsf{E}_{y} \log \frac{1}{p(Y)} \\
&= \mathsf{H}(X|Y) + \mathsf{H}(Y)
\end{aligned}$$

$$\begin{aligned}
\mathsf{H}(X,Y) &\triangleq \mathsf{E}_{xy} \log \frac{1}{p(X,Y)} \\
&= \mathsf{E}_{xy} \log \frac{1}{p(Y|X)p(X)} \\
&= \mathsf{E}_{xy} \log \frac{1}{p(Y|X)} + \mathsf{E}_{xy} \log \frac{1}{p(X)} \\
&= \mathsf{E}_{xy} \log \frac{1}{p(Y|X)} + \mathsf{E}_{y} \log \frac{1}{p(X)} \\
&= \mathsf{H}(Y|X) + \mathsf{H}(X)
\end{aligned}$$

$$\begin{aligned}
\mathsf{H}(X_1, X_2, \ldots, X_N) &= \mathsf{H}(X_1 | X_2, \ldots, X_N) + \mathsf{H}(X_2, \ldots, X_N) \\
&= \mathsf{H}(X_1 | X_2, \ldots, X_N) + \mathsf{H}(X_2 | X_3, \ldots, X_N) + \mathsf{H}(X_3, \ldots, X_N) \\
&= \mathsf{H}(X_1 | X_2, \ldots, X_N) + \mathsf{H}(X_2 | X_3, \ldots, X_N) + \mathsf{H}(X_3 | X_4, \ldots, X_N) + \mathsf{H}(X_4, \ldots, X_N)
\end{aligned}$$

$$= \sum_{n=1}^{N-1} \mathsf{H}(\mathsf{X}_n|\mathsf{X}_{n+1},\ldots,\mathsf{X}_n) + \mathsf{H}(\mathsf{X}_N)$$

✏

Theorem B.3.

THM

$$\begin{aligned}
\mathrm{I}(X;Y) &= \mathsf{H}(X) - \mathsf{H}(X|Y)\\
\mathrm{I}(X;Y) &= \mathsf{H}(Y) - \mathsf{H}(Y|X)\\
\mathrm{I}(X;Y) &= \mathsf{H}(X) + \mathsf{H}(Y) - \mathsf{H}(X,Y)\\
\mathrm{I}(X;Y) &= \mathrm{I}(Y;X)\\
\mathrm{I}(X;X) &= \mathsf{H}(X)
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\mathrm{I}(X;Y) &\triangleq \mathsf{E}_{xy}\log_2\frac{p(X,Y)}{p(X)p(Y)}\\
&= \mathsf{E}_{xy}\log_2\frac{p(X|Y)}{p(X)}\\
&= \mathsf{E}_{xy}\log_2\frac{1}{p(X)} + \mathsf{E}_{xy}\log_2 p(X|Y)\\
&= \mathsf{E}_{xy}\log_2\frac{1}{p(X)} - \mathsf{E}_{xy}\log_2\frac{1}{p(X|Y)}\\
&\triangleq \mathsf{H}(X) - \mathsf{H}(X|Y)
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(X;Y) &\triangleq \mathsf{E}_{xy}\log_2\frac{p(X,Y)}{p(X)p(Y)}\\
&= \mathsf{E}_{xy}\log_2\frac{p(Y|X)}{p(Y)}\\
&= \mathsf{E}_{xy}\log_2\frac{1}{p(Y)} + \mathsf{E}_{xy}\log_2 p(Y|X)\\
&= \mathsf{E}_{xy}\log_2\frac{1}{p(Y)} - \mathsf{E}_{xy}\log_2\frac{1}{p(Y|X)}\\
&\triangleq \mathsf{H}(Y) - \mathsf{H}(Y|X)
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(X;Y) &= \mathsf{H}(Y) - \mathsf{H}(Y|X)\\
&= \mathrm{I}(Y;X)
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(X;X) &\triangleq \mathsf{E}_{xy}\log_2\frac{p(X,X)}{p(X)p(X)}\\
&= \mathsf{E}_{xy}\log_2\frac{p(X)}{p(X)p(X)}\\
&= \mathsf{E}_{xy}\log_2\frac{1}{p(X)}\\
&\triangleq \mathsf{H}(X)
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(X;Y) &\triangleq \mathsf{H}(X) - \mathsf{H}(X|Y)\\
&= \mathsf{H}(X) - [\mathsf{H}(X,Y) - \mathsf{H}(Y)]\\
&= \mathsf{H}(X) + \mathsf{H}(Y) - \mathsf{H}(X,Y)
\end{aligned}$$

✏

Theorem B.4 (Information chain rule)**.**

THM
$$I(X_1, X_2, \ldots, X_N; Y) = \sum_{n=1}^{N-1} I(X_n|X_{n+1}, \ldots, X_N) + I(X_N)$$

✎PROOF:

$$\begin{aligned}
I(X_1, X_2, \ldots, X_N; Y) &= H(X_1, X_2, \ldots, X_N) - H(X_1, X_2, \ldots, X_N|Y) \\
&= \sum_{n=1}^{N-1} H(X_n|X_{n+1}, \ldots, X_N) + H(X_N) - \sum_{n=1}^{N-1} H(X_n|X_{n+1}, \ldots, X_N, Y) - H(X_N|Y) \\
&= \sum_{n=1}^{N-1} \left[H(X_n|X_{n+1}, \ldots, X_N) - H(X_n|X_{n+1}, \ldots, X_N, Y)\right] + \left[H(X_N) - H(X_N|Y)\right] \\
&= \sum_{n=1}^{N-1} I(X_n|X_{n+1}, \ldots, X_N) + I(X_N)
\end{aligned}$$

✏

B.1.3 Properties

Theorem B.5. [6]

THM
$$\begin{aligned}
D(p_1, p_2) &\geq 0 \\
I(X; Y) &\geq 0
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
D(p_1, p_2) &\triangleq E_x \log \frac{p_1(X)}{p_2(X)} \\
&= E_x \left[-\log \frac{p_2(X)}{p_1(X)}\right] \\
&\geq -\log E_x \left[\frac{p_2(X)}{p_1(X)}\right] && \text{by \textit{Jensen's Inequality} (Theorem ?? page ??)} \\
&= -\log \int_x p_1(x) \frac{p_2(x)}{p_1(x)}\, dx \\
&= -\log \int_x p_2(x)\, dx \\
&= -\log(1) \\
&= 0
\end{aligned}$$

✏

[6] Cover and Thomas (1991), page 26

B.2 Channel Capacity

Definition B.5. *Let (L, N) be a block coder with* N *output bits for each L input bits.*

$$\begin{array}{rcll} R & \triangleq & \frac{L}{N} & \textit{coding rate} \\ \mathsf{C} & \triangleq & \max \mathsf{I}(\mathsf{X};\mathsf{Y}) & \textit{channel capacity} \\ \mathsf{E}(R) & \triangleq & \max_\rho \max_Q [\mathsf{E}_0(\rho, Q) - \rho R] & \textit{random coding exponent} \end{array}$$

Theorem B.6 (noisy channel coding theorem). [7]

THM

If $R < \mathsf{C}$ then it is possible to construct an encoder and decoder such that the probability of error P_e is arbitrarily small. Specifically

$$P_e \le e^{-N\mathsf{E}(R)}$$

For $0 \le R \ge \mathsf{C}$, the function $\mathsf{E}(R)$ is POSITIVE, DECREASING, *and* CONVEX.

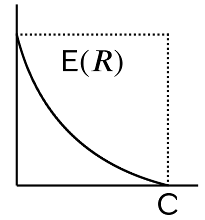


Figure B.3: Typical $\mathsf{E}(R)$

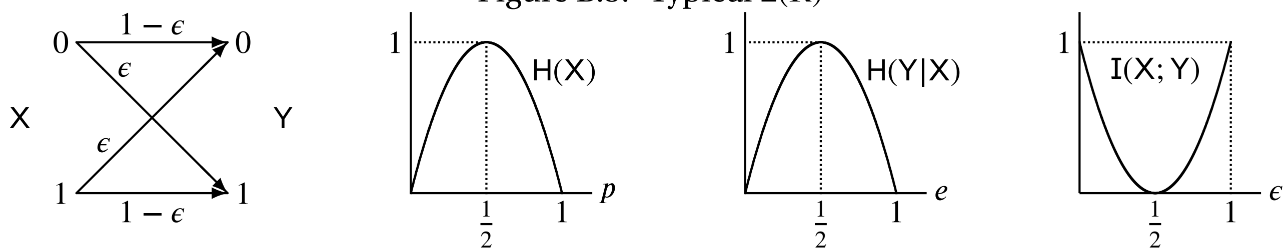


Figure B.4: Binary symmetric channel (BSC)

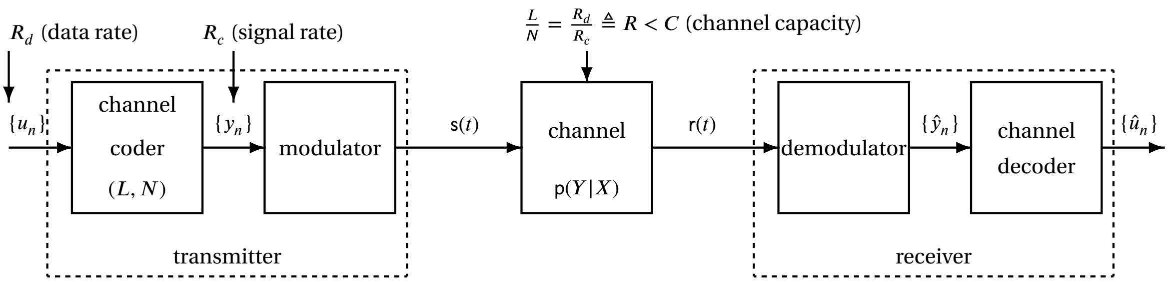


Figure B.5: Memoryless modulation system model

How much information can be reliably sent through the channel? The answer depends on the *channel capacity* C. As proven by the *Noisy Channel Coding Theorem* (NCCT), each transmitted symbol can carry up to C bits for any arbitrarily small probability of error greater than zero. The price for decreasing error is increasing the block code size.

Note that the NCCT does not say at what rate (in bits/second) you can send data through the AWGN channel. The AWGN channel knows nothing of time (and is therefore not a realistic channel). The NCCT channel merely gives a *coding rate*. That is, the number of information bits each symbol can carry. Channels that limit the rate (in bits/second) that can be sent through it are obviously aware of time and are often referred to as *bandlimited channels*.

[7] Gallager (1968), page 143

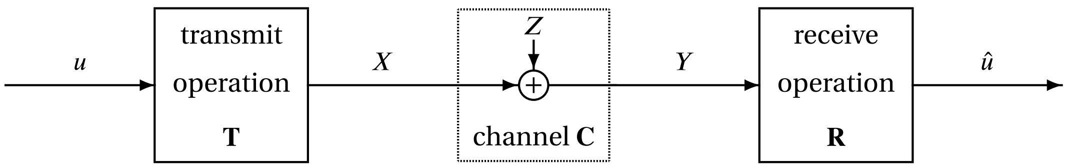


Figure B.6: Additive noise system model

Theorem B.7. *Let* $Z \sim \mathsf{N}\left(0, \sigma^2\right)$. *Then*

THM $\mathsf{H}(Z) = \dfrac{1}{2}\log_2 2\pi e\sigma^2$

✎PROOF:

$$
\begin{aligned}
\mathsf{H}(Z) &= \mathsf{E}_z \log \frac{1}{\mathsf{p}(Z)} \\
&= -\mathsf{E}_z \log \mathsf{p}(z) \\
&= -\mathsf{E}_z \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-z^2}{2\sigma^2}}\right] \\
&= -\mathsf{E}_z \left[-\frac{1}{2}\log(2\pi\sigma^2) + \frac{-z^2}{2\sigma^2}\log e\right] \\
&= \frac{1}{2}\mathsf{E}_z \left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2} z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}\mathsf{E}_z z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}(\sigma^2 + 0)\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \log e\right] \\
&= \frac{1}{2}\log(2\pi e\sigma^2)
\end{aligned}
$$

✏

Theorem B.8. *Let* $Y = X + Z$ *be a Gaussian channel with* $\mathsf{E}X^2 = P$ *and* $Z \sim \mathsf{N}\left(0, \sigma^2\right)$. *Then*

THM $\mathrm{I}(X;Y) \le \dfrac{1}{2}\log\left(1 + \dfrac{P}{\sigma^2}\right) = \mathsf{C}$

✎PROOF: No proof at this time.

Reference: (Cover and Thomas, 1991, page 241) ✏

Example B.1. 1. If there is no transmitted energy ($P = 0$), then the capacity of the channel to pass information is

$$
\begin{aligned}
\mathsf{C} &= \frac{1}{2}\log_2\left(1 + \frac{P}{\sigma^2}\right) \\
&= \frac{1}{2}\log_2\left(1 + \frac{0}{\sigma^2}\right) \\
&= 0
\end{aligned}
$$

That is, the symbols cannot carry any information.

2. If there is finite symbol energy and no noise ($\sigma^2 = 0$), then the capacity of the channel to pass information is

$$\begin{aligned} \mathsf{C} &= \frac{1}{2}\log_2\left(1+\frac{P}{0}\right) \\ &= \infty \end{aligned}$$

That is, each symbol can carry an infinite amount of information. That is, we can use a modulation scheme with an infinite number of of signaling waveforms (analog modulation) and thus each symbol can be represented by one of an infinite number of waveforms.

3. If the transmitted energy is ($P = 15\sigma^2$), then the capacity of the channel to pass information is

$$\begin{aligned} \mathsf{C} &= \frac{1}{2}\log_2\left(1+\frac{15\sigma^2}{\sigma^2}\right) \\ &= \frac{1}{2}\log_2(1+15) \\ &= \frac{1}{2}4 \\ &= 2 \end{aligned}$$

This means

$$2 = \mathsf{C} > \mathsf{R} \triangleq \frac{\text{information bits}}{\text{symbol}} = \frac{\text{information bits}}{\text{coded bits}} \times \frac{\text{coded bits}}{\text{symbol}} = \mathsf{r_c r_s}$$

This means that if the coding rate is $\mathsf{r_c} = 1/4$, then we must use a modulation with 256 ($\mathsf{r_s} = 8$ bits/symbol) or fewer waveforms.

Conversely, if the modulation scheme uses 4 waveforms, then $\mathsf{r_s} = 2$ bits/symbol and so the code rate $\mathsf{r_c}$ can be up to 1 (almost no coding redundancy is needed).

4. If there is the transmitted energy ($P = \sigma^2$), then the capacity of the channel to pass information is

$$\begin{aligned} \mathsf{C} &= \frac{1}{2}\log_2\left(1+\frac{\sigma^2}{\sigma^2}\right) \\ &= \frac{1}{2}\log_2(1+1) \\ &= \frac{1}{2} \end{aligned}$$

That is, each symbol can carry just under 1/2 bits of information. This means

$$\frac{1}{2} = \mathsf{C} > \mathsf{R} \triangleq \frac{\text{information bits}}{\text{symbol}} = \frac{\text{information bits}}{\text{coded bits}} \times \frac{\text{coded bits}}{\text{symbol}} = \mathsf{r_c r_s}$$

This means that if the coding rate is $\mathsf{r_c} = 1/4$, then we must use a modulation with 4 ($\mathsf{r_s} = 2$ bits/symbol) or fewer waveforms.

Conversely, if the modulation scheme uses 16 waveforms, then $\mathsf{r_s} = 4$ bits/symbol and so the code rate $\mathsf{r_c}$ must be less than 1/8.

B.3 Specific channels

B.3.1 Binary Symmetric Channel (BSC)

The properties of the *binary symmetric channel (BSC)* are illustrated in Figure B.4 (page 196) and stated in Theorem B.9 (next).

Theorem B.9 (Binary symmetric channel). *Let* $\mathbf{C} : \mathsf{X} \to \mathsf{Y}$ *be a channel operation with* $\mathsf{X}, \mathsf{Y} \in \{0, 1\}$ *and*

$$\begin{aligned} p &\triangleq \mathsf{P}\{\mathsf{X}=1\} \\ \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\} &= \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\} \triangleq \epsilon \end{aligned}$$

Then

T H M

$$\begin{aligned} \mathsf{P}\{\mathsf{Y}=1\} &= \epsilon + p - 2\epsilon p \\ \mathsf{P}\{\mathsf{Y}=0\} &= 1 - p - \epsilon + 2\epsilon p \\ \mathsf{H}(\mathsf{X}) &= p \log_2 \tfrac{1}{p} + (1-p)\log_2 \tfrac{1}{(1-p)} \\ \mathsf{H}(\mathsf{Y}) &= (1-p-\epsilon+2\epsilon p)\log_2 \tfrac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p)\log_2 \tfrac{1}{\epsilon+p-2\epsilon p} \\ \mathsf{H}(\mathsf{Y}|\mathsf{X}) &= (1-\epsilon)\log_2 \tfrac{1}{1-\epsilon} + \epsilon \log_2 \tfrac{1}{\epsilon} \\ \mathsf{I}(\mathsf{X};\mathsf{Y}) &= (1-p-\epsilon+2\epsilon p)\log_2 \tfrac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p)\log_2 \tfrac{1}{\epsilon+p-2\epsilon p} \\ &\quad -(1-\epsilon)\log_2 \tfrac{1}{1-\epsilon} + -\epsilon \log_2 \tfrac{1}{\epsilon} \\ \mathsf{C} &= 1 + \epsilon \log_2 \epsilon + (1-\epsilon)\log_2(1-\epsilon) \end{aligned}$$

✎PROOF:

$$\begin{aligned} \mathsf{P}\{\mathsf{X}=1\} &\triangleq p \\ \mathsf{P}\{\mathsf{X}=0\} &= 1-p \\ \mathsf{P}\{\mathsf{Y}=1\} &= \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\}\mathsf{P}\{\mathsf{X}=0\} + \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=1\}\mathsf{P}\{\mathsf{X}=1\} \\ &= \epsilon(1-p) + (1-\epsilon)p \\ &= \epsilon - \epsilon p + p - \epsilon p \\ &= \epsilon + p - 2\epsilon p \\ \mathsf{P}\{\mathsf{Y}=0\} &= \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=0\}\mathsf{P}\{\mathsf{X}=0\} + \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\}\mathsf{P}\{\mathsf{X}=1\} \\ &= (1-\epsilon)(1-p) + \epsilon p \\ &= 1 - p - \epsilon + \epsilon p + \epsilon p \\ &= 1 - p - \epsilon + 2\epsilon p \end{aligned}$$

$$\begin{aligned} \mathsf{H}(\mathsf{X}) &\triangleq \mathsf{E}_x \log_2 \frac{1}{\mathsf{p}(\mathsf{X})} \\ &= \sum_{n=0}^{1} \mathsf{P}\{\mathsf{X}=n\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{X}=n\}} \\ &= \mathsf{P}\{\mathsf{X}=0\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{X}=0\}} + \mathsf{P}\{\mathsf{X}=1\}\log_2 \frac{1}{\mathsf{P}\{\mathsf{X}=1\}} \\ &= p \log_2 \frac{1}{p} + (1-p)\log_2 \frac{1}{(1-p)} \end{aligned}$$

$$\mathsf{H}(\mathsf{Y}) \triangleq \mathsf{E}_y \log_2 \frac{1}{\mathsf{p}(\mathsf{Y})}$$

$$
\begin{aligned}
&= \sum_{n=0}^{1} \mathsf{P}\{\mathsf{Y}=n\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=n\}} \\
&= \mathsf{P}\{\mathsf{Y}=0\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=0\}} + \mathsf{P}\{\mathsf{Y}=1\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=1\}} \\
&= (1-p-\epsilon+2\epsilon p) \log_2 \frac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p) \log_2 \frac{1}{\epsilon+p-2\epsilon p}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{H}(\mathsf{Y}|\mathsf{X}) &\triangleq \mathsf{E}_{\mathsf{xy}} \log_2 \frac{1}{\mathsf{p}(\mathsf{Y}|\mathsf{X})} \\
&= \sum_{m=0}^{1}\sum_{n=0}^{1} \mathsf{P}\{\mathsf{X}=m, \mathsf{Y}=n\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=n|\mathsf{X}=m\}} \\
&= \sum_{m=0}^{1}\sum_{n=0}^{1} \mathsf{P}\{\mathsf{Y}=n|\mathsf{X}=m\}\, \mathsf{P}\{\mathsf{X}=m\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=n|\mathsf{X}=m\}} \\
&= \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=0\}\, \mathsf{P}\{\mathsf{X}=0\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=0\}} + \\
&\quad\; \mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\}\, \mathsf{P}\{\mathsf{X}=1\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=0|\mathsf{X}=1\}} + \\
&\quad\; \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\}\, \mathsf{P}\{\mathsf{X}=0\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=0\}} + \\
&\quad\; \mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=1\}\, \mathsf{P}\{\mathsf{X}=1\} \log_2 \frac{1}{\mathsf{P}\{\mathsf{Y}=1|\mathsf{X}=1\}} \\
&= (1-\epsilon)(1-p) \log_2 \frac{1}{1-\epsilon} + \epsilon p \log_2 \frac{1}{\epsilon} + \epsilon(1-p) \log_2 \frac{1}{\epsilon} + (1-\epsilon)p \log_2 \frac{1}{1-\epsilon} \\
&= (1-p-\epsilon+\epsilon p+p-\epsilon p) \log_2 \frac{1}{1-\epsilon} + (\epsilon p+\epsilon-\epsilon p) \log_2 \frac{1}{\epsilon} \\
&= (1-\epsilon) \log_2 \frac{1}{1-\epsilon} + \epsilon \log_2 \frac{1}{\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{I}(\mathsf{X};\mathsf{Y}) &= \mathsf{H}(\mathsf{Y}) - \mathsf{H}(\mathsf{Y}|\mathsf{X}) \\
&= (1-p-\epsilon+2\epsilon p) \log_2 \frac{1}{1-p-\epsilon+2\epsilon p} + (\epsilon+p-2\epsilon p) \log_2 \frac{1}{\epsilon+p-2\epsilon p} - (1-\epsilon) \log_2 \frac{1}{1-\epsilon} + -\epsilon \log_2 \frac{1}{\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{C} &\triangleq \max_p \mathsf{I}(\mathsf{X};\mathsf{Y}) \\
&= \mathsf{I}(\mathsf{X};\mathsf{Y})\big|_{p=\frac{1}{2}} \\
&= \frac{1}{2} \log_2 \frac{1}{\frac{1}{2}} + \frac{1}{2} \log_2 \frac{1}{\frac{1}{2}} - (1-\epsilon) \log_2 \frac{1}{1-\epsilon} + -\epsilon \log_2 \frac{1}{\epsilon} \\
&= 1 + \epsilon \log_2 \epsilon + (1-\epsilon) \log_2 (1-\epsilon)
\end{aligned}
$$

✎

Remark B.1.

REM

When $\epsilon = 0$	(noiseless channel), the channel capacity is 1 bit (maximum capacity).
When $\epsilon = 1$	(inverting channel), the channel capacity is still 1 bit.
When $\epsilon = 1/2$	(totally random channel), the channel capacity is 0.
When $p = 1$ (1	is always transmitted), the entropy of X is 0.
When $p = 0$ (0	is always transmitted), the entropy of X is 0.
When $p = 1/2$	(totally random transmission), the entropy of X is 1 bit (maximum entropy).

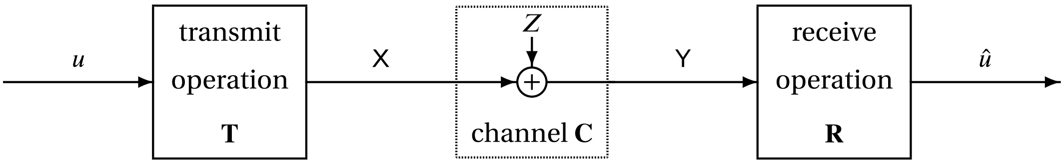


Figure B.7: Additive noise system model

B.3.2 Gaussian Noise Channel

Theorem B.10. *Let* $Z \sim \mathsf{N}\left(0, \sigma^2\right)$. *Then*

THM
$$\mathsf{H}(Z) = \frac{1}{2}\log_2 2\pi e\sigma^2$$

✎PROOF:

$$\begin{aligned}
\mathsf{H}(Z) &= \mathsf{E}_z \log \frac{1}{\mathsf{p}(Z)} \\
&= -\mathsf{E}_z \log \mathsf{p}(z) \\
&= -\mathsf{E}_z \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-z^2}{2\sigma^2}}\right] \\
&= -\mathsf{E}_z \left[-\frac{1}{2}\log(2\pi\sigma^2) + \frac{-z^2}{2\sigma^2}\log e\right] \\
&= \frac{1}{2}\mathsf{E}_z \left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2} z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}\mathsf{E}_z z^2\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{\log e}{\sigma^2}(\sigma^2 + 0)\right] \\
&= \frac{1}{2}\left[\log(2\pi\sigma^2) + \log e\right] \\
&= \frac{1}{2}\log(2\pi e\sigma^2)
\end{aligned}$$

✏

Theorem B.11. [8] *Let* $\mathsf{Y} = \mathsf{X} + Z$ *be a Gaussian channel with* $\mathsf{E}\mathsf{X}^2 = P$ *and* $Z \sim \mathsf{N}\left(0, \sigma^2\right)$. *Then*

THM
$$\mathsf{I}(\mathsf{X};\mathsf{Y}) \le \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right) = \mathsf{C} \qquad \textit{bits per usage}$$

Theorem B.12. [9] *Let* $\mathsf{Y} = \mathsf{X} + Z$ *be a bandlimited Gaussian channel with* $\mathsf{E}\mathsf{X}^2 = P$ *and* $Z \sim \mathsf{N}\left(0, \sigma^2\right)$ *and bandwidth* W. *Then*

THM
$$\mathsf{C} = W\log\left(1 + \frac{P}{\sigma^2 W}\right) \qquad \textit{bits per second}$$

[8] Cover and Thomas (1991), page 241
[9] Cover and Thomas (1991), page 250

APPENDIX C

RANDOM PROCESS EIGEN-ANALYSIS

C.1 Definitions

Definition C.1. *Let* $\mathsf{x}(t)$ *be random processes with* AUTO-CORRELATION *function (Definition ?? page ??)* $\mathsf{R}_{\mathsf{xx}}(t,u)$.

DEF	*The **auto-correlation operator** $\mathbf{R}$ of $\mathsf{x}(t)$ is defined as* $\mathbf{R}\mathsf{f} \triangleq \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{f}(u)\,\mathsf{d}u$

Definition C.2. *Let* $\mathsf{x}(t)$ *be a* RANDOM PROCESS *with* AUTO-CORRELATION $\mathsf{R}_{\mathsf{xx}}(\tau)$ *(Definition ?? page ??).*

DEF	*A* RANDOM PROCESS $\mathsf{x}(t)$ *is **white** if* $\mathsf{R}_{\mathsf{xx}}(\tau) = \delta(\tau)$

If a random process $\mathsf{x}(t)$ is *white* (Definition C.2 page 203) and the set $\Psi = \{\psi_1(t), \psi_2(t), \dots, \psi_N(t)\}$ is **any** set of orthonormal basis functions, then the innerproducts $\langle n(t) \,|\, \psi_n(t)\rangle$ and $\langle n(t) \,|\, \psi_m(t)\rangle$ are *uncorrelated* for $m \neq n$. However, if $\mathsf{x}(t)$ is **colored** (not white), then the innerproducts are not in general uncorrelated. But if the elements of Ψ are chosen to be the eigenfunctions of $\mathbf{R}$ such that $\mathbf{R}\psi_n = \lambda_n\psi_n$, then by Theorem **??** (page **??**), the set $\{\psi_n(t)\}$ are *orthogonal* and the innerproducts **are** *uncorrelated* even though $\mathsf{x}(t)$ is not white. This criterion is called the Karhunen-Loève criterion for $\mathsf{x}(t)$.

Theorem C.1. *Let* $\mathbf{R}$ *be an* AUTO-CORRELATION *operator.*

THM			
	$\langle \mathbf{R}\mathsf{x} \,|\, \mathsf{x}\rangle \geq 0$	$\forall \mathsf{x} \in \boldsymbol{X}$	(NON-NEGATIVE)
	$\langle \mathbf{R}\mathsf{x} \,|\, \mathsf{y}\rangle = \langle \mathsf{x} \,|\, \mathbf{R}\mathsf{y}\rangle$	$\forall \mathsf{x},\mathsf{y} \in \boldsymbol{X}$	(SELF-ADJOINT)

✎PROOF:

1. Proof that $\mathbf{R}$ is *non-negative*:

$$\begin{aligned}
\langle \mathbf{R}\mathsf{y} \,|\, \mathsf{y}\rangle &= \left\langle \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{y}(u)\,\mathsf{d}u \,\middle|\, \mathsf{y}(t)\right\rangle && \text{by definition of } \mathbf{R} && \text{(Definition C.1 page 203)}\\
&= \left\langle \int_{u\in\mathbb{R}} \mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u)\right]\mathsf{y}(u)\,\mathsf{d}u \,\middle|\, \mathsf{y}(t)\right\rangle && \text{by definition of } \mathsf{R}_{\mathsf{xx}}(t,u) && \text{(Definition ?? page ??)}\\
&= \mathsf{E}\left[\left\langle \int_{u\in\mathbb{R}} \mathsf{x}(t)\mathsf{x}^*(u)\mathsf{y}(u)\,\mathsf{d}u \,\middle|\, \mathsf{y}(t)\right\rangle\right] && \text{by } \textit{linearity} \text{ of } \langle \triangle \,|\, \nabla\rangle \text{ and } \int
\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{E}\left[\int_{u\in\mathbb{R}} \mathsf{x}^*(u)\mathsf{y}(u)\, \mathsf{d}u\, \langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle\right] && \text{by } \textit{additivity} \text{ property of } \langle \triangle \mid \nabla\rangle \\
&= \mathsf{E}\left[\langle \mathsf{y}(u) \mid \mathsf{x}(u)\rangle \langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle\right] && \text{by local definition of } \langle \triangle \mid \nabla\rangle \\
&= \mathsf{E}\left[\langle \mathsf{x}(u) \mid \mathsf{y}(u)\rangle^* \langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle\right] && \text{by } \textit{conjugate symmetry} \text{ prop.} \\
&= \mathsf{E}|\langle \mathsf{x}(t) \mid \mathsf{y}(t)\rangle|^2 && \text{by definition of } |\cdot| \quad \text{(Definition ?? page ??)} \\
&\geq 0
\end{aligned}
$$

2. Proof that $\mathbf{R}$ is self-adjoint:

$$
\begin{aligned}
\langle [\mathbf{R}\mathsf{x}](t) \mid \mathsf{y}\rangle &= \left\langle \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{x}(u)\, \mathsf{d}u \mid \mathsf{y}(t)\right\rangle && \text{by definition of } \mathbf{R} \quad \text{(Definition C.1 page 203)} \\
&= \int_{u\in\mathbb{R}} \mathsf{x}(u)\, \langle \mathsf{R}_{\mathsf{xx}}(t,u) \mid \mathsf{y}(t)\rangle\, \mathsf{d}u && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla\rangle \\
&= \int_{u\in\mathbb{R}} \mathsf{x}(u)\, \langle \mathsf{y}(t) \mid \mathsf{R}_{\mathsf{xx}}(t,u)\rangle^*\, \mathsf{d}u && \text{by } \textit{conjugate symmetry} \text{ prop.} \\
&= \langle \mathsf{x}(u) \mid \langle \mathsf{y}(t) \mid \mathsf{R}_{\mathsf{xx}}(t,u)\rangle\rangle && \text{by local definition of } \langle \triangle \mid \nabla\rangle \\
&= \left\langle \mathsf{x}(u) \mid \int_{t\in\mathbb{R}} \mathsf{y}(t)\mathsf{R}^*_{\mathsf{xx}}(t,u)\, \mathsf{d}t\right\rangle && \text{by local definition of } \langle \triangle \mid \nabla\rangle \\
&= \left\langle \mathsf{x}(u) \mid \int_{t\in\mathbb{R}} \mathsf{y}(t)\mathsf{R}_{\mathsf{xx}}(u,t)\, \mathsf{d}t\right\rangle && \text{by property of } \mathsf{R}_{\mathsf{xx}} \quad \text{(Theorem ?? page ??)} \\
&= \left\langle \mathsf{x}(u) \mid \underbrace{\mathbf{R}}_{\mathbf{R}^*}\mathsf{y}\right\rangle && \text{by definition of } \mathbf{R} \quad \text{(Definition C.1 page 203)} \\
\implies \quad & \mathbf{R} = \mathbf{R}^* \quad \implies \mathbf{R} \text{ is } \textit{self adjoint}
\end{aligned}
$$

C.2 Properties

Theorem C.2. [1] *Let* $(\lambda_n)_{n\in\mathbb{Z}}$ *be the eigenvalues and* $(\psi_n)_{n\in\mathbb{Z}}$ *be the eigenfunctions of operator* $\mathbf{R}$ *such that* $\mathbf{R}\psi_n = \lambda_n\psi_n$.

THM

1. $\lambda_n \in \mathbb{R}$ — *(eigenvalues of* $\mathbf{R}$ *are* REAL*)*
2. $\lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m\rangle = 0$ — *(eigenfunctions associated with distinct eigenvalues are* ORTHOGONAL*)*
3. $\|\psi_n(t)\|^2 > 0 \implies \lambda_n \geq 0$ — *(eigenvalues are* NON-NEGATIVE*)*
4. $\|\psi_n(t)\|^2 > 0, \langle \mathbf{R}\mathsf{f} \mid \mathsf{f}\rangle > 0 \implies \lambda_n > 0$ — *(if* $\mathbf{R}$ *is* POSITIVE DEFINITE*, then eigenvalues are* POSITIVE*)*

✎PROOF:

1. Proof that eigenvalues are *real-valued*: Because $\mathbf{R}$ is self-adjoint, its eigenvalues are real.

2. eigenfunctions associated with distinct eigenvalues are orthogonal: Because $\mathbf{R}$ is self-adjoint, this property follows.

[1] Keener (1988), pages 114–119

3. Proof that eigenvalues are *non-negative*:

$$\begin{aligned}
0 &\geq \langle \mathbf{R}\psi_n \,|\, \psi_n \rangle && \text{by definition of non-negative definite} \\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by hypothesis} \\
&= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of inner-products} \\
&= \lambda_n \left\| \psi_n \right\|^2 && \text{by definition of norm induced by inner-product}
\end{aligned}$$

4. Eigenvalues are *positive* if **R** is *positive definite*:

$$\begin{aligned}
0 &> \langle \mathbf{R}\psi_n \,|\, \psi_n \rangle && \text{by definition of } \textit{positive definite} \\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by hypothesis} \\
&= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of inner-products} \\
&= \lambda_n \left\| \psi_n \right\|^2 && \text{by definition of norm induced by inner-product}
\end{aligned}$$

✏

Theorem C.3 (Karhunen-Loève Expansion). [2] *Let* **R** *be the* AUTO-CORRELATION OPERATOR *(Definition C.1 page 203) of a* RANDOM PROCESS $\mathsf{x}(t)$. *Let* $(\lambda_n)_{n\in\mathbb{Z}}$ *be the eigenvalues of* **R** *and* $(\psi_n)_{n\in\mathbb{Z}}$ *are the eigenfunctions of* **R** *such that* $\mathbf{R}\psi_n = \lambda_n \psi_n$.

THM

$$\underbrace{\left\| \psi_n(t) \right\| = 1}_{\{\psi_n(t)\} \text{ are NORMALIZED}} \implies \underbrace{\mathsf{E}\left(\left| \mathsf{x}(t) - \sum_{n\in\mathbb{Z}} \langle \mathsf{x}(t) \,|\, \psi_n(t) \rangle \, \psi_n(t) \right|^2 \right) = 0}_{\text{CONVERGENCE IN PROBABILITY}} \qquad \left(\left\{ \psi_n(t) \right\} \text{ is a BASIS for } \mathsf{x}(t) \right)$$

✎PROOF:

1. Define $\dot{x}_n \triangleq \langle \mathsf{x}(t) \,|\, \psi_n(t) \rangle$

2. Define $\mathbf{R}\mathsf{x}(t) \triangleq \int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\mathsf{x}(u)\, \mathsf{d}u$

3. lemma: $\mathsf{E}[\mathsf{x}(t)\mathsf{x}(t)] = \sum_{n\in\mathbb{Z}} \lambda_n \left| \psi_n(t) \right|^2$. Proof:

$$\mathsf{E}[\mathsf{x}(t)\mathsf{x}(t)] = \sum_{n\in\mathbb{Z}} \lambda_n \left| \psi_n(t) \right|^2$$

by *non-negative* property (Theorem C.1 page 203)
and *Mercer's Theorem* (Theorem ?? page ??)

4. lemma:

$$\begin{aligned}
&\mathsf{E}\left[\mathsf{x}(t) \left(\sum_{n\in\mathbb{Z}} \dot{x}_n \psi_n(t) \right)^* \right] \\
&\triangleq \mathsf{E}\left[\mathsf{x}(t) \left(\sum_{n\in\mathbb{Z}} \int_{u\in\mathbb{R}} \mathsf{x}(u)\psi_n^*(u)\, \mathsf{d}u \psi_n(t) \right)^* \right] && \text{by definition of } \dot{x} && \text{(definition 1 page 205)} \\
&= \sum_{n\in\mathbb{Z}} \left(\int_{u\in\mathbb{R}} \mathsf{E}\left[\mathsf{x}(t)\mathsf{x}^*(u) \right] \psi_n(u)\, \mathsf{d}u \right) \psi_n^*(t) && \text{by } \textit{linearity} && \text{(Theorem ?? page ??)} \\
&\triangleq \sum_{n\in\mathbb{Z}} \left(\int_{u\in\mathbb{R}} \mathsf{R}_{\mathsf{xx}}(t,u)\psi_n(u)\, \mathsf{d}u \right) \psi_n^*(t) && \text{by definition of } \mathsf{R}_{\mathsf{xx}}(t,u) && \text{(Definition ?? page ??)}
\end{aligned}$$

[2] Keener (1988), pages 114–119

$$\triangleq \sum_{n\in\mathbb{Z}} \left(\mathbf{R}\psi_n(t)\psi_n^*(t)\right) \qquad \text{by definition of } \mathbf{R} \quad \text{(definition 2 page 205)}$$
$$= \sum_{n\in\mathbb{Z}} \lambda_n\psi_n(t)\psi_n^*(t) \qquad \text{by property of } \textit{eigen-system}$$
$$= \sum_{n\in\mathbb{Z}} \lambda_n\left|\psi_n(t)\right|^2$$

5. lemma:

$$\mathsf{E}\left[\sum_{n\in\mathbb{Z}} \dot{x}_n\psi_n(t)\left(\sum_{m\in\mathbb{Z}} \dot{x}_m\psi_m(t)\right)^*\right]$$
$$\triangleq \mathsf{E}\left[\sum_{n\in\mathbb{Z}}\int_{u\in\mathbb{R}} \mathsf{x}(u)\psi_n^*(u)\,\mathrm{d}u\psi_n(t)\left(\sum_{m\in\mathbb{Z}}\int_v \mathsf{x}(v)\psi_m^*(v)\,\mathrm{d}v\psi_m(t)\right)^*\right] \qquad \text{by definition of } \dot{x} \text{ (definition 1 page 205)}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\int_v \mathsf{E}\left[\mathsf{x}(u)\mathsf{x}^*(v)\right]\psi_m(v)\,\mathrm{d}v\right)\psi_n^*(u)\,\mathrm{d}u\psi_n(t)\psi_m^*(t) \qquad \text{by } \textit{linearity} \text{ (Theorem ?? page ??)}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u\left(\int_v \mathsf{R}_{\mathsf{xx}}(u,v)\psi_m(v)\,\mathrm{d}v\right)\psi_n^*(u)\,\mathrm{d}u\psi_n(t)\psi_m^*(t) \qquad \text{by definition of } \mathsf{R}_{\mathsf{xx}}(t,u) \text{ (Definition ?? page ??)}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u \left(\mathbf{R}\psi_m(u)\right)\psi_n^*(u)\,\mathrm{d}u\psi_n(t)\psi_m^*(t) \qquad \text{by definition of } \mathbf{R} \text{ (definition 2 page 205)}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_u \left(\lambda_m\psi_m(u)\right)\psi_n^*(u)\,\mathrm{d}u\psi_n(t)\psi_m^*(t) \qquad \text{by property of } \textit{eigen-system}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\left(\int_{u\in\mathbb{R}} \psi_m(u)\psi_n^*(u)\,\mathrm{d}u\right)\psi_n(t)\psi_m^*(t) \qquad \text{by } \textit{linearity}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\left\|\psi(t)\right\|^2\bar{\delta}_{mn}\psi_n(t)\psi_m^*(t) \qquad \text{by } \textit{orthogonal} \text{ property (Theorem C.2 page 204)}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\lambda_m\bar{\delta}_{mn}\psi_n(t)\psi_m^*(t) \qquad \text{by } \textit{normalized} \text{ hypothesis}$$
$$= \sum_{n\in\mathbb{Z}}\lambda_n\psi_n(t)\psi_n^*(t) \qquad \text{by definition of } \textit{Kronecker delta } \bar{\delta} \text{ (Definition ?? page ??)}$$
$$= \sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2$$

6. Proof that $\{\psi_n(t)\}$ is a *basis* for $\mathsf{x}(t)$:

$$\mathsf{E}\left(\left|\mathsf{x}(t)-\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right|^2\right)$$
$$= \mathsf{E}\left(\left[\mathsf{x}(t)-\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right]\left[\mathsf{x}(t)-\sum_{m\in\mathbb{Z}}\dot{x}_m\psi_m(t)\right]^*\right)$$
$$= \mathsf{E}\left(\mathsf{x}(t)\mathsf{x}^*(t)-\mathsf{x}(t)\left[\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right]^*-\mathsf{x}^*(t)\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)+\left[\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right]\left[\sum_{m\in\mathbb{Z}}\dot{x}_m\psi_m(t)\right]^*\right)$$
$$= \mathsf{E}\left(\mathsf{x}(t)\mathsf{x}^*(t)\right)-\mathsf{E}\left[\mathsf{x}(t)\left[\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right]^*\right]-\mathsf{E}\left[\mathsf{x}^*(t)\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\right]+\mathsf{E}\left[\sum_{n\in\mathbb{Z}}\dot{x}_n\psi_n(t)\left[\sum_{m\in\mathbb{Z}}\dot{x}_m\psi_m(t)\right]^*\right]$$

by *linearity* of E (Theorem ?? page ??)

$$= \underbrace{\sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2}_{\text{by (3) lemma}}-\underbrace{\sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2}_{\text{by (4) lemma}}-\underbrace{\left[\sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2\right]^*}_{\text{by (4) lemma}}+\underbrace{\sum_{n\in\mathbb{Z}}\lambda_n\left|\psi_n(t)\right|^2}_{\text{by (5) lemma}}$$

$$= 0$$

✏

Remark C.1. The *matrix* **R** is **Toeplitz**. For more information about the properties of *Toeplitz* matrices, see Grenander and Szegö (1958), Widom (1965), Gray (1971), Smylie et al. (1973) page 408 ⟨§"B. Properties of the Toeplitz Matrix"⟩, Grenander and Szegö (1984), Haykin and Kesler (1979), Haykin and Kesler (1983), Böttcher and Silbermann (1999), Gray (2006).

APPENDIX D

TRIGONOMETRIC FUNCTIONS

D.1 Definition Candidates

Definition D.1 (Hermitian components). [1] *Let* $(\mathbb{F}, *)$ *be a* $*$*-algebra a* (STAR ALGEBRA).

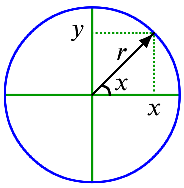
D E F

*The **real part** of* x *is defined as* $\Re x \triangleq \frac{1}{2}\left(x + x^*\right) \quad \forall x \in \mathbb{F}$

*The **imaginary part** of* x *is defined as* $\Im x \triangleq \frac{1}{2i}\left(x - x^*\right) \quad \forall x \in \mathbb{F}$

There are several ways of defining the sine and cosine functions, including the following:[2]

1. **Planar geometry:** Trigonometric functions are traditionally introduced as they have come to us historically—that is, as related to the parameters of triangles.[3]



$$\cos x \triangleq \frac{x}{r} \qquad \sin x \triangleq \frac{y}{r}$$

2. **Complex exponential:** The cosine and sine functions are the real and imaginary parts of the complex exponential such that[4]

$$\cos x \triangleq \mathbf{R}_e e^{ix} \qquad \sin x \triangleq \mathbf{I}_m\left(e^{ix}\right)$$

3. **Polynomial:** Let $\sum_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \sum_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions

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[1] Michel and Herget (1993) page 430, Rickart (1960) page 179, Gelfand and Naimark (1964) page 242

[2] The term *sine* originally came from the Hindu word *jiva* and later adapted to the Arabic word *jiba*. Abrabic-Latin translator Robert of Chester apparently confused this word with the Arabic word *jaib*, which means "bay" or "inlet"—thus resulting in the Latin translation *sinus*, which also means "bay" or "inlet". Reference: Boyer and Merzbach (1991) page 252

[3] Abramowitz and Stegun (1972), page 78

[4] Euler (1748)

can be defined in terms of *Taylor expansions* such that[5]

$$\begin{aligned}
\cos(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\sin(x) &\triangleq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}$$

4. **Product of factors:** Let $\prod_{n=0}^{\infty} x_n \triangleq \lim_{N\to\infty} \prod_{n=0}^{N} x_n$ in some topological space. The sine and cosine functions can be defined in terms of a product of factors such that[6]

$$\cos(x) \triangleq \prod_{n=1}^{\infty}\left[1 - \left(\frac{x}{(2n-1)\frac{\pi}{2}}\right)^2\right] \qquad \sin(x) \triangleq x \prod_{n=1}^{\infty}\left[1 - \left(\frac{x}{n\pi}\right)^2\right]$$

5. **Partial fraction expansion:** The sine function can be defined in terms of a partial fraction expansion such that[7]

$$\sin(x) \triangleq \frac{1}{\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - (n\pi)^2}} \qquad \cos(x) \triangleq \underbrace{\left(\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - (n\pi)^2}\right)}_{\cot(x)} \sin(x)$$

6. **Differential operator:** The sine and cosine functions can be defined as solutions to differential equations expressed in terms of the differential operator $\frac{d}{dx}$ such that

$$\begin{aligned}
\cos(x) &\triangleq \mathrm{f}(x) \quad \text{where} \quad \underbrace{\frac{d^2}{dx^2}\mathrm{f} + \mathrm{f} = 0}_{\text{differential equation}} \quad \underbrace{\mathrm{f}(0) = 1}_{\text{1st initial condition}} \quad \underbrace{\left[\frac{d}{dx}\mathrm{f}\right](0) = 0}_{\text{2nd initial condition}} \\
\sin(x) &\triangleq \mathrm{g}(x) \quad \text{where} \quad \underbrace{\frac{d^2}{dx^2}\mathrm{g} + \mathrm{g} = 0}_{\text{differential equation}} \quad \underbrace{\mathrm{g}(0) = 0}_{\text{1st initial condition}} \quad \underbrace{\left[\frac{d}{dx}\mathrm{g}\right](0) = 1}_{\text{2nd initial condition}}
\end{aligned}$$

7. **Integral operator:** The sine and cosine functions can be defined as inverses of integrals of square roots of rational functions such that[8]

$$\begin{aligned}
\cos(x) &\triangleq \mathrm{f}^{-1}(x) \quad \text{where} \quad \mathrm{f}(x) \triangleq \underbrace{\int_x^1 \sqrt{\frac{1}{1-y^2}}\, dy}_{\arccos(x)} \\
\sin(x) &\triangleq \mathrm{g}^{-1}(x) \quad \text{where} \quad \mathrm{g}(x) \triangleq \underbrace{\int_0^x \sqrt{\frac{1}{1-y^2}}\, dy}_{\arcsin(x)}
\end{aligned}$$

For purposes of analysis, it can be argued that the more natural approach for defining harmonic functions is in terms of the differentiation operator $\frac{d}{dx}$ (Definition D.2 page 211). Support for such an approach includes the following:

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[5] Rosenlicht (1968), page 157, Abramowitz and Stegun (1972), page 74
[6] Abramowitz and Stegun (1972), page 75
[7] Abramowitz and Stegun (1972), page 75
[8] Abramowitz and Stegun (1972), page 79

- Both sine and cosine are very easily represented analytically as polynomials with coefficients involving the operator $\frac{d}{dx}$ (Theorem D.1 page 213).
- All solutions of homogeneous second order differential equations are linear combinations of sine and cosine (Theorem D.3 page 214).
- Sine and cosine themselves are related to each other in terms of the differentiation operator (Theorem D.4 page 215).
- The complex exponential function is a solution of a second order homogeneous differential equation (Definition D.5 page 216).
- Sine and cosine are orthogonal with respect to an innerproduct generated by an integral operator—which is a kind of inverse differential operator (Section D.6 page 224).

D.2 Definitions

Definition D.2. [9] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{d}{dx} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator.*

DEF

The function $f \in \mathbb{C}^{\mathbb{C}}$ is the ***cosine*** *function $\cos(x) \triangleq f(x)$ if*

1. $\frac{d^2}{dx^2}f + f = 0$ *(second order homogeneous differential equation) and*
2. $f(0) = 1$ *(first initial condition) and*
3. $\left[\frac{d}{dx}f\right](0) = 0$ *(second initial condition).*

Definition D.3. [10] *Let $\boldsymbol{C}$ and $\frac{d}{dx} \in \boldsymbol{C}^{\boldsymbol{C}}$ be defined as in definition of $\cos(x)$ (Definition D.2 page 211).*

DEF

The function $f \in \mathbb{C}^{\mathbb{C}}$ is the ***sine*** *function $\sin(x) \triangleq f(x)$ if*

1. $\frac{d^2}{dx^2}f + f = 0$ *(second order homogeneous differential equation) and*
2. $f(0) = 0$ *(first initial condition) and*
3. $\left[\frac{d}{dx}f\right](0) = 1$ *(second initial condition).*

Definition D.4. [11]

DEF

Let π ("pi") be defined as the element in $\mathbb{R}$ such that

(1). $\cos\left(\frac{\pi}{2}\right) = 0$ *and*

(2). $\pi > 0$ *and*

(3). *π is the* ***smallest*** *of all elements in $\mathbb{R}$ that satisfies (1) and (2).*

D.3 Basic properties

Lemma D.1. [12] *Let $\boldsymbol{C}$ be the* SPACE OF ALL CONTINUOUSLY DIFFERENTIABLE REAL FUNCTIONS *and $\frac{d}{dx} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator.*

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[9] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[10] Rosenlicht (1968) page 157, Flanigan (1983) pages 228–229
[11] Rosenlicht (1968) page 158
[12] Rosenlicht (1968), page 156, Liouville (1839)

LEM

$$\left\{\frac{d^2}{dx^2}f + f = 0\right\} \iff \left\{\begin{aligned} f(x) &= \underbrace{[f](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\textit{even terms}} + \underbrace{\left[\frac{d}{dx}f\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\textit{odd terms}} \\ &= \left(f(0) + \left[\frac{d}{dx}f\right](0)x\right) - \left(\frac{f(0)}{2!}x^2 + \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3\right) + \left(\frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5\right)\cdots \end{aligned}\right\}$$

✎PROOF: Let $f'(x) \triangleq \frac{d}{dx}f(x)$.

$$f'''(x) = -\left[\frac{d}{dx}f\right](x)$$

$$f^{(4)}(x) = -\left[\frac{d}{dx}f\right](x) \qquad\qquad = -\left[\frac{d^2}{dx^2}f\right](x) = f(x)$$

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \implies f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n && \text{by Taylor expansion} \\ &= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{\left[\frac{d^2}{dx^2}f\right](0)}{2!}x^2 - \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 - \cdots \\ &= f(0) + \left[\frac{d}{dx}f\right](0)x - \frac{f(0)}{2!}x^2 - \frac{\left[\frac{d}{dx}f\right](0)}{3!}x^3 + \frac{f(0)}{4!}x^4 + \frac{\left[\frac{d}{dx}f\right](0)}{5!}x^5 - \cdots \\ &= f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \end{aligned}$$

2. Proof that $\left[\frac{d^2}{dx^2}f\right](x) + f(x) = 0 \impliedby f(x) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right]$:

$$\begin{aligned} \left[\frac{d^2}{dx^2}f\right](x) &= \frac{d}{dx}\frac{d}{dx}[f(x)] \\ &= \frac{d}{dx}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] && \text{by right hypothesis} \\ &= \sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n)(2n-1)f(0)}{(2n)!}x^{2n-2} + \frac{(2n+1)(2n)\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n-1}\right] \\ &= \sum_{n=1}^{\infty}(-1)^n\left[\frac{f(0)}{(2n-2)!}x^{2n-2} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n-1)!}x^{2n-1}\right] \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{f(0)}{(2n)!}x^{2n} + \frac{\left[\frac{d}{dx}f\right](0)}{(2n+1)!}x^{2n+1}\right] \\ &= -f(x) && \text{by right hypothesis} \end{aligned}$$

✏

Theorem D.1 (Taylor series for cosine/sine). [13]

THM

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \forall x\in\mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\cos(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma D.1 page 211}\\
&= 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by cos initial conditions (Definition D.2 page 211)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}
\end{aligned}$$

$$\begin{aligned}
\sin(x) &= \underbrace{\mathsf{f}(0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\text{even terms}} + \underbrace{\left[\tfrac{\mathsf{d}}{\mathsf{d}x}\mathsf{f}\right](0)\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms}} && \text{by Lemma D.1 page 211}\\
&= 0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + 1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by sin initial conditions (Definition D.3 page 211)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

✏

Theorem D.2. [14]

THM

$$\begin{array}{rcl|rcll}
\cos(0) &=& 1 & \cos(-x) &=& \cos(x) & \forall x\in\mathbb{R}\\
\sin(0) &=& 0 & \sin(-x) &=& -\sin(x) & \forall x\in\mathbb{R}
\end{array}$$

✎PROOF:

$$\begin{aligned}
\cos(0) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big|_{x=0} && \text{by \textit{Taylor series for cosine}} && \text{(Theorem D.1 page 213)}\\
&= 1\\
\sin(0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big|_{x=0} && \text{by \textit{Taylor series for sine}} && \text{(Theorem D.1 page 213)}\\
&= 0\\
\cos(-x) &= 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \cdots && \text{by \textit{Taylor series for cosine}} && \text{(Theorem D.1 page 213)}\\
&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\
&= \cos(x) && \text{by \textit{Taylor series for cosine}} && \text{(Theorem D.1 page 213)}\\
\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots && \text{by \textit{Taylor series for sine}} && \text{(Theorem D.1 page 213)}
\end{aligned}$$

[13] Rosenlicht (1968), page 157
[14] Rosenlicht (1968), page 157

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \sin(x) \qquad \text{by } \textit{Taylor series for sine} \qquad \text{(Theorem D.1 page 213)}$$

✎

Lemma D.2. [15]

LEM

$$\begin{array}{lcl|lcl} \cos(1) & > & 0 & x \in (0:2) & \Longrightarrow & \sin(x) > 0 \\ \cos(2) & < & 0 & & & \end{array}$$

✎PROOF:

$$\begin{aligned}
\cos(1) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=1} && \text{by } \textit{Taylor series for cosine} \quad \text{(Theorem D.1 page 213)}\\
&= 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \cdots \\
&> 0 \\
\cos(2) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big|_{x=2} && \text{by } \textit{Taylor series for cosine} \quad \text{(Theorem D.1 page 213)}\\
&= 1 - \frac{4}{2} + \frac{16}{24} - \frac{64}{720} + \cdots \\
&< 0
\end{aligned}$$

$$\begin{aligned}
x \in (0:2) &\Longrightarrow \text{each term in the sequence } \left(\left(x - \frac{x^3}{3!}\right), \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right), \left(\frac{x^9}{9!} - \frac{x^{11}}{11!}\right), \ldots\right) \text{ is } > 0 \\
&\Longrightarrow \sin(x) > 0
\end{aligned}$$

✎

Proposition D.1. *Let π be defined as in Definition D.4 (page 211).*

PRP

(A). *The value π **exists** in $\mathbb{R}$.*

(B). $2 < \pi < 4$.

✎PROOF:

$$\begin{aligned}
&\cos(1) > 0 && \text{by Lemma D.2 page 214}\\
&\cos(2) < 0 && \text{by Lemma D.2 page 214}\\
&\Longrightarrow 1 < \frac{\pi}{2} < 2 \\
&\Longrightarrow 2 < \pi < 4
\end{aligned}$$

✎

Theorem D.3. [16] *Let $\boldsymbol{C}$ be the space of all continuously differentiable real functions and $\frac{\mathrm{d}}{\mathrm{d}x} \in \boldsymbol{C}^{\boldsymbol{C}}$ the differentiation operator. Let* $\mathsf{f}'(0) \triangleq \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\right](0)$.

THM

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f} + \mathsf{f} = 0\right\} \iff \left\{\mathsf{f}(x) = \mathsf{f}(0)\cos(x) + \mathsf{f}'(0)\sin(x)\right\} \qquad \forall \mathsf{f} \in \boldsymbol{C}, \forall x \in \mathbb{R}$$

[15] Rosenlicht (1968), page 158

[16] Rosenlicht (1968), page 157. The general solution for the *non-homogeneous* equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x) + \mathsf{f}(x) = \mathsf{g}(x)$ with initial conditions $\mathsf{f}(a) = 1$ and $\mathsf{f}'(a) = \rho$ is $\mathsf{f}(x) = \cos(x) + \rho\sin(x) + \int_a^x \mathsf{g}(y)\sin(x-y)\,\mathrm{d}y$. This type of equation is called a *Volterra integral equation of the second type*. References: Folland (1992), page 371, Liouville (1839). Volterra equation references: Pedersen (2000), page 99, Lalescu (1908), Lalescu (1911)

✎PROOF:

1. Proof that $\left[\frac{d^2}{dx^2}f\right](x) = -f(x) \implies f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$\begin{aligned}
f(x) &= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by left hypothesis and Lemma D.1 page 211}\\
&= f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x && \text{by definitions of cos and sin (Definition D.2 page 211, Definition D.3 page 211)}
\end{aligned}$$

2. Proof that $\frac{d^2}{dx^2}f = -f \impliedby f(x) = f(0)\cos(x) + \left[\frac{d}{dx}f\right](0)\sin(x)$:

$$\begin{aligned}
f(x) &= f(0)\cos x + \left[\frac{d}{dx}f\right](0)\sin x && \text{by right hypothesis}\\
&= f(0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}_{\cos(x)} + \left[\frac{d}{dx}f\right](0)\underbrace{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}}_{\sin(x)}\\
\implies\quad & \frac{d^2}{dx^2}f + f = 0 && \text{by Lemma D.1 page 211}
\end{aligned}$$

✏

Theorem D.4. [17] *Let $\frac{d}{dx} \in C^C$ be the differentiation operator.*

THM

$$\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x\in\mathbb{R} \quad\Big|\quad \frac{d}{dx}\sin(x) = \cos(x) \quad \forall x\in\mathbb{R} \quad\Big|\quad \cos^2(x)+\sin^2(x) = 1 \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} && \text{by Taylor series (Theorem D.1 page 213)}\\
&= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}\\
&= -\sin(x) && \text{by Taylor series (Theorem D.1 page 213)}\\
\frac{d}{dx}\sin(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} && \text{by Taylor series (Theorem D.1 page 213)}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= \cos(x) && \text{by Taylor series (Theorem D.1 page 213)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left[\cos^2(x)+\sin^2(x)\right] &= -2\cos(x)\sin(x) + 2\sin(x)\cos(x)\\
&= 0\\
&\implies \cos^2(x)+\sin^2(x) \text{ is } \textit{constant}\\
&\implies \cos^2(x)+\sin^2(x)\\
&= \cos^2(0)+\sin^2(0)\\
&= 1+0 = 1 && \text{by Theorem D.2 page 213}
\end{aligned}$$

✏

[17] Rosenlicht (1968), page 157

Proposition D.2.

PRP
$$\sin\left(\frac{\pi}{2}\right) = 1$$

✎PROOF:

$$\begin{aligned}
\sin(\pi/2) &= \pm\sqrt{\sin^2(\pi/2) + 0} \\
&= \pm\sqrt{\sin^2(\pi/2) + \cos^2(\pi/2)} && \text{by definition of } \pi && \text{(Definition D.4 page 211)} \\
&= \pm\sqrt{1} && \text{by Theorem D.4 page 215} \\
&= \pm 1 \\
&= 1 && \text{by Lemma D.2 page 214}
\end{aligned}$$

✏

D.4 The complex exponential

Definition D.5.

DEF
The function $\mathsf{f} \in \mathbb{C}^{\mathbb{C}}$ *is the* ***exponential function*** $\exp(ix) \triangleq \mathsf{f}(x)$ *if*

1. $\frac{d^2}{dx^2}\mathsf{f} + \mathsf{f} = 0$ *(second order homogeneous differential equation) and*
2. $\mathsf{f}(0) = 1$ *(first initial condition) and*
3. $\left[\frac{d}{dx}\mathsf{f}\right](0) = i$ *(second initial condition).*

Theorem D.5 (Euler's identity). [18]

THM
$$e^{ix} = \cos(x) + i\sin(x) \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\exp(ix) &= \mathsf{f}(0)\cos(x) + \left[\frac{d}{dx}\mathsf{f}\right](0)\sin(x) && \text{by Theorem D.3 page 214} \\
&= \cos(x) + i\sin(x) && \text{by Definition D.5 page 216}
\end{aligned}$$

✏

Proposition D.3.

PRP
$$e^{-i\pi/2} = -i \quad \Big| \quad e^{i\pi/2} = i$$

✎PROOF:

$$\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem D.5 page 216)} \\
&= 0 + i && \text{by Theorem D.2 (page 213) and Proposition D.2 (page 216)} \\
e^{-i\pi/2} &= \cos(-\pi/2) + i\sin(-\pi/2) && \text{by } \textit{Euler's identity} \text{ (Theorem D.5 page 216)} \\
&= \cos(\pi/2) - i\sin(\pi/2) && \text{by Theorem D.2 page 213} \\
&= 0 - i && \text{by Theorem D.2 (page 213) and Proposition D.2 (page 216)}
\end{aligned}$$

✏

[18] Euler (1748), Bottazzini (1986), page 12

Corollary D.1.

COR
$$e^{ix} = \sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!} \qquad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{e^{ix}} &= \cos(x) + i\sin(x) && \text{by } \textit{Euler's identity} && \text{(The} \\
&= \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i \underbrace{\sum_{n\in\mathbb{W}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin(x)} && \text{by } \textit{Taylor series} && \text{(The} \\
&= \sum_{n\in\mathbb{W}} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} = \sum_{n\in\mathbb{W}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{W}} \frac{(ix)^{2n+1}}{(2n+1)!} = \sum_{n\in\mathbb{W}\cap\mathbb{Z}_e} \frac{(ix)^n}{n!} + \sum_{n\in\mathbb{W}\cap\mathbb{Z}_o} \frac{(ix)^n}{n!} = \boxed{\sum_{n\in\mathbb{W}} \frac{(ix)^n}{n!}}
\end{aligned}$$

✏

Corollary D.2 (Euler formulas). [19]

COR
$$\cos(x) = \mathbf{R_e}\left(e^{ix}\right) = \frac{e^{ix}+e^{-ix}}{2} \quad \forall x\in\mathbb{R} \qquad \sin(x) = \mathbf{I_m}\left(e^{ix}\right) = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x\in\mathbb{R}$$

✎PROOF:

$$\begin{aligned}
\boxed{\mathbf{R_e}\left(e^{ix}\right)} &\triangleq \frac{e^{ix} + \left(e^{ix}\right)^*}{2} = \frac{e^{ix}+e^{-ix}}{2} && \text{by definition of } \mathfrak{R} \\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(-x)+i\sin(-x)}{2} && \text{by } \textit{Euler's identity} && \text{(Theorem D.5 page 216)} \\
&= \frac{\cos(x)+i\sin(x)}{2} + \frac{\cos(x)-i\sin(x)}{2} && = \frac{\cos(x)}{2} + \frac{\cos(x)}{2} && = \boxed{\cos(x)} \\
\boxed{\mathbf{I_m}\left(e^{ix}\right)} &\triangleq \frac{e^{ix} - \left(e^{ix}\right)^*}{2i} = \frac{e^{ix}-e^{-ix}}{2i} && \text{by definition of } \mathfrak{I} \\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(-x)+i\sin(-x)}{2i} && \text{by } \textit{Euler's identity} && \text{(Theorem D.5 page 216)} \\
&= \frac{\cos(x)+i\sin(x)}{2i} - \frac{\cos(x)-i\sin(x)}{2i} && = \frac{i\sin(x)}{2i} + \frac{i\sin(x)}{2i} && = \boxed{\sin(x)}
\end{aligned}$$

✏

Theorem D.6. [20]

THM
$$e^{(\alpha+\beta)} = e^{\alpha}\, e^{\beta} \qquad \forall \alpha,\beta\in\mathbb{C}$$

[19] Euler (1748), Bottazzini (1986), page 12
[20] Rudin (1987) page 1

✎PROOF:

$$\begin{aligned}
e^{\alpha}\, e^{\beta} &= \left(\sum_{n\in\mathbb{W}} \frac{\alpha^n}{n!}\right)\left(\sum_{m\in\mathbb{W}} \frac{\beta^m}{m!}\right) && \text{by Corollary D.1 page 217}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \sum_{k=0}^{n} \frac{n!\,\alpha^k}{n!\,k!} \frac{\beta^{n-k}}{(n-k)!}\\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \alpha^k\, \beta^{n-k}\\
&= \sum_{n\in\mathbb{W}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \alpha^k\, \beta^{n-k}\\
&= \sum_{n\in\mathbb{W}} \frac{(\alpha+\beta)^n}{n!} && \text{by the \textit{Binomial Theorem}}\\
&= e^{\alpha+\beta} && \text{by Corollary D.1 page 217}
\end{aligned}$$

✏

D.5 Trigonometric Identities

Theorem D.7 (shift identities).

THM

$$\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= -\sin x \quad \forall x\in\mathbb{R} & \sin\left(x+\frac{\pi}{2}\right) &= \cos x \quad \forall x\in\mathbb{R}\\
\cos\left(x-\frac{\pi}{2}\right) &= \sin x \quad \forall x\in\mathbb{R} & \sin\left(x-\frac{\pi}{2}\right) &= -\cos x \quad \forall x\in\mathbb{R}
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\cos\left(x+\frac{\pi}{2}\right) &= \frac{e^{i\left(x+\frac{\pi}{2}\right)} + e^{-i\left(x+\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}\\
&= \frac{e^{ix}e^{i\frac{\pi}{2}} + e^{-ix}e^{-i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem D.6 page 217)}\\
&= \frac{e^{ix}(i) + e^{-ix}(-i)}{2} && \text{by Proposition D.3 page 216}\\
&= \frac{e^{ix} - e^{-ix}}{-2i}\\
&= -\sin x && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}\\
\cos\left(x-\frac{\pi}{2}\right) &= \frac{e^{i\left(x-\frac{\pi}{2}\right)} + e^{-i\left(x-\frac{\pi}{2}\right)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}\\
&= \frac{e^{ix}e^{-i\frac{\pi}{2}} + e^{-ix}e^{+i\frac{\pi}{2}}}{2} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem D.6 page 217)}\\
&= \frac{e^{ix}(-i) + e^{-ix}(i)}{2} && \text{by Proposition D.3 page 216}\\
&= \frac{e^{ix} - e^{-ix}}{2i}\\
&= \sin x && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}
\end{aligned}$$

$$\begin{aligned}
\sin\left(x+\frac{\pi}{2}\right) &= \cos\left(\left[x+\frac{\pi}{2}\right]-\frac{\pi}{2}\right) && \text{by previous result}\\
&= \cos(x)\\
\sin\left(x-\frac{\pi}{2}\right) &= -\cos\left(\left[x-\frac{\pi}{2}\right]+\frac{\pi}{2}\right) && \text{by previous result}\\
&= -\cos(x)
\end{aligned}$$

✏

Theorem D.8 (product identities).

THM

(A).	$\cos x \cos y$	$=$	$\tfrac{1}{2}\cos(x-y)$	$+$	$\tfrac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$
(B).	$\cos x \sin y$	$=$	$-\tfrac{1}{2}\sin(x-y)$	$+$	$\tfrac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
(C).	$\sin x \cos y$	$=$	$\tfrac{1}{2}\sin(x-y)$	$+$	$\tfrac{1}{2}\sin(x+y)$	$\forall x,y\in\mathbb{R}$
(D).	$\sin x \sin y$	$=$	$\tfrac{1}{2}\cos(x-y)$	$-$	$\tfrac{1}{2}\cos(x+y)$	$\forall x,y\in\mathbb{R}$

✎PROOF:

1. Proof for (A) using *Euler formulas* (Corollary D.2 page 217)
(algebraic method requiring *complex number system* $\mathbb{C}$):

$$\begin{aligned}
\cos x\cos y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{iy}+e^{-iy}}{2}\right) && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}\\
&= \frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(-x+y)}+e^{i(-x-y)}}{4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{4}+\frac{e^{i(x-y)}+e^{-i(x-y)}}{4}\\
&= \frac{2\cos(x+y)}{4}+\frac{2\cos(x-y)}{4} && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}\\
&= \frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)
\end{aligned}$$

2. Proof for (A) using *Volterra integral equation* (Theorem D.3 page 214)
(differential equation method requiring only *real number system* $\mathbb{R}$):

$$\begin{aligned}
&\mathsf{f}(x) \triangleq \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)\\
\implies\; &\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x) = -\tfrac{1}{2}\sin(x-y)-\tfrac{1}{2}\sin(x+y) && \text{by Theorem D.4 page 215}\\
\implies\; &\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x) = -\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y) && \text{by Theorem D.4 page 215}\\
\implies\; &\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathsf{f}(x)+\mathsf{f}(x) = 0 && \text{by \textit{additive inverse} property}\\
\implies\; &\underbrace{\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)}_{\mathsf{f}(x)} = \underbrace{[\tfrac{1}{2}\cos(0-y)+\tfrac{1}{2}\cos(0+y)]}_{\mathsf{f}''(0)}\cos(x)+\underbrace{[-\tfrac{1}{2}\sin(0-y)-\tfrac{1}{2}\sin(0+y)]}_{\mathsf{f}'(0)}\sin(x)\\
\implies\; &\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y) = \cos y\cos x+0\sin(x)\\
\implies\; &\cos x\cos y = \tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y)
\end{aligned}$$

3. Proof for (B) using *Euler formulas* (Corollary D.2 page 217):

$$\begin{aligned}
\sin x \sin y &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) && \text{by Corollary D.2 page 217}\\
&= \frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(-x+y)}+e^{i(-x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}}{-4}\\
&= \frac{e^{i(x+y)}+e^{-i(x+y)}}{-4}-\frac{e^{i(x-y)}+e^{-i(x-y)}}{-4}\\
&= \frac{2\cos(x-y)}{4}-\frac{2\cos(x+y)}{4} && \text{by Corollary D.2 page 217}\\
&= \frac{1}{2}\cos(x-y)-\frac{1}{2}\cos(x+y)
\end{aligned}$$

4. Proofs for (C) and (D) using (A) and (B):

$$\begin{aligned}
\cos x \sin y &= \cos(x)\cos\left(y-\frac{\pi}{2}\right) && \text{by } \textit{shift identities} \quad \text{(Theorem D.7 page 218)}\\
&= \frac{1}{2}\cos\left(x+y-\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(x-y+\frac{\pi}{2}\right) && \text{by (A)}\\
&= \frac{1}{2}\sin(x+y)-\frac{1}{2}\sin(x-y) && \text{by } \textit{shift identities} \quad \text{(Theorem D.7 page 218)}\\
\sin x \cos y &= \cos y \sin x\\
&= \frac{1}{2}\sin(y+x)-\frac{1}{2}\sin(y-x) && \text{by (B)}\\
&= \frac{1}{2}\sin(x+y)+\frac{1}{2}\sin(x-y) && \text{by Theorem D.2 page 213}
\end{aligned}$$

✏

Proposition D.4.

P R P						
	(A).	$\cos(\pi) = -1$	(C).	$\cos(2\pi) = 1$	(E).	$e^{i\pi} = -1$
	(B).	$\sin(\pi) = 0$	(D).	$\sin(2\pi) = 0$	(F).	$e^{i2\pi} = 0$

✎PROOF:

$$\begin{aligned}
\cos(\pi) &= -1+1+\cos(\pi)\\
&= -1+2[{}^1\!/\!_2\cos({}^\pi\!/\!_2-{}^\pi\!/\!_2)+{}^1\!/\!_2\cos({}^\pi\!/\!_2+{}^\pi\!/\!_2)] && \text{by } \cos(0)=1 \text{ result} \quad \text{(Theorem D.2 page 213)}\\
&= -1+2\cos({}^\pi\!/\!_2)\cos({}^\pi\!/\!_2) && \text{by } \textit{product identities} \quad \text{(Theorem D.8 page 219)}\\
&= -1+2(0)(0) && \text{by definition of } \pi \quad \text{(Definition D.4 page 211)}\\
&= -1\\
\sin(\pi) &= 0+\sin(\pi)\\
&= 2[-{}^1\!/\!_2\sin({}^\pi\!/\!_2-{}^\pi\!/\!_2)+{}^1\!/\!_2\sin({}^\pi\!/\!_2+{}^\pi\!/\!_2)] && \text{by } \sin(0)=0 \text{ result} \quad \text{(Theorem D.2 page 213)}\\
&= 2\cos({}^\pi\!/\!_2)\sin({}^\pi\!/\!_2) && \text{by } \textit{product identities} \quad \text{(Theorem D.8 page 219)}\\
&= 2(0)\sin({}^\pi\!/\!_2) && \text{by definition of } \pi \quad \text{(Definition D.4 page 211)}\\
&= 0\\
\cos(2\pi) &= 1+\cos(2\pi)-1\\
&= 2[{}^1\!/\!_2\cos(\pi-\pi)+{}^1\!/\!_2\cos(\pi+\pi)]-1 && \text{by } \cos(0)=1 \text{ result} \quad \text{(Theorem D.2 page 213)}\\
&= 2\cos(\pi)\cos(\pi)-1 && \text{by } \textit{product identities} \quad \text{(Theorem D.8 page 219)}\\
&= 2(-1)(-1)-1 && \text{by (A)}\\
&= 1
\end{aligned}$$

$$
\begin{aligned}
\sin(2\pi) &= 0 + \sin(2\pi) \\
&= 2[{}^1\!/\!_2\sin(\pi - \pi) + {}^1\!/\!_2\sin(\pi + \pi)] && \text{by } \sin(0) = 0 \text{ result} && \text{(Theorem D.2 page 213)} \\
&= 2\sin(\pi)\cos(\pi) && \text{by \textit{product identities}} && \text{(Theorem D.8 page 219)} \\
&= 2(0)(-1) && \text{by (A) and (B)} \\
&= 0 \\
e^{i\pi} &= \cos(\pi) + i\sin(\pi) && \text{by \textit{Euler's identity}} && \text{(Theorem D.5 page 216)} \\
&= -1 + 0 && \text{by (A) and (B)} \\
&= -1 \\
e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) && \text{by \textit{Euler's identity}} && \text{(Theorem D.5 page 216)} \\
&= 1 + 0 && \text{by (C) and (D)} \\
&= 1
\end{aligned}
$$

⇨

Theorem D.9 (double angle formulas). [21]

THM				
(A).	$\cos(x+y)$	$=$	$\cos x \cos y - \sin x \sin y$	$\forall x,y \in \mathbb{R}$
(B).	$\sin(x+y)$	$=$	$\sin x \cos y + \cos x \sin y$	$\forall x,y \in \mathbb{R}$
(C).	$\tan(x+y)$	$=$	$\dfrac{\tan x + \tan y}{1 - \tan x \tan y}$	$\forall x,y \in \mathbb{R}$

✎PROOF:

1. Proof for (A) using *product identities* (Theorem D.8 page 219).

$$
\begin{aligned}
\cos(x+y) &= \underbrace{\frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x+y)}_{\cos(x+y)} + \underbrace{\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x-y)}_{0} \\
&= \left[\frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\right] - \left[\frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\right] \\
&= \cos x \cos y - \sin x \sin y && \text{by Theorem D.8 page 219}
\end{aligned}
$$

2. Proof for (A) using *Volterra integral equation* (Theorem D.3 page 214):

$$
\begin{aligned}
\mathrm{f}(x) \triangleq \cos(x+y) &\implies \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x) = -\sin(x+y) && \text{by Theorem D.4 page 215} \\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) = -\cos(x+y) && \text{by Theorem D.4 page 215} \\
&\implies \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = 0 && \text{by \textit{additive inverse} property} \\
&\implies \cos(x+y) = \cos y \cos x - \sin y \sin x && \text{by Theorem D.3 page 214} \\
&\implies \cos(x+y) = \cos x \cos y - \sin x \sin y && \text{by \textit{commutative} property}
\end{aligned}
$$

[21] Expressions for $\cos(\alpha+\beta)$, $\sin(\alpha+\beta)$, and $\sin^2 x$ appear in works as early as Ptolemy (circa 100AD). Reference: http://en.wikipedia.org/wiki/History_of_trigonometric_functions

3. Proof for (B) and (C) using (A):

$$\begin{aligned}
\sin(x+y) &= \cos\left(x - \frac{\pi}{2} + y\right) && \text{by } \textit{shift identities} \text{ (Theorem D.7 page 218)}\\
&= \cos\left(x - \frac{\pi}{2}\right)\cos(y) - \sin\left(x - \frac{\pi}{2}\right)\sin(y) && \text{by (A)}\\
&= \sin(x)\cos(y) + \cos(x)\sin(y) && \text{by } \textit{shift identities} \text{ (Theorem D.7 page 218)}
\end{aligned}$$

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)}\\
&= \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y} && \text{by (A)}\\
&= \left(\frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y}\right)\left(\frac{\cos x\cos y}{\cos x\cos y}\right)\\
&= \frac{\frac{\sin x\cos y}{\cos x\cos y} + \frac{\cos x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y} - \frac{\sin x\sin y}{\cos x\cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x}\frac{\sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x\tan y}
\end{aligned}$$

✏

Theorem D.10 (trigonometric periodicity).

THM						
(A).	$\cos(x+M\pi) = (-1)^M\cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		(D).	$\cos(x+2M\pi) = \cos(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
(B).	$\sin(x+M\pi) = (-1)^M\sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		(E).	$\sin(x+2M\pi) = \sin(x)$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$
(C).	$e^{i(x+M\pi)} = (-1)^M e^{ix}$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$		(F).	$e^{i(x+2M\pi)} = e^{ix}$	$\forall x\in\mathbb{R},\ M\in\mathbb{Z}$

✎PROOF:

1. Proof for (A):

   (a) $M=0$ case: $\cos(x+0\pi) = \cos(x) = (-1)^0\cos(x)$

   (b) Proof for $M>0$ cases (by induction):

      i. Base case $M=1$:

$$\begin{aligned}
\cos(x+\pi) &= \cos x\cos\pi - \sin x\sin\pi && \text{by } \textit{double angle formulas} && \text{(Theorem D.9 page 221)}\\
&= \cos x(-1) - \sin x(0) && \text{by } \cos\pi = -1 \text{ result} && \text{(Proposition D.4 page 220)}\\
&= (-1)^1\cos x
\end{aligned}$$

      ii. Inductive step…Proof that M case $\implies M+1$ case:

$$\begin{aligned}
\cos(x+[M+1]\pi) &= \cos([x+\pi]+M\pi)\\
&= (-1)^M\cos(x+\pi) && \text{by induction hypothesis } (M \text{ case})\\
&= (-1)^M(-1)\cos(x) && \text{by base case (item 1(b)i page 222)}\\
&= (-1)^{M+1}\cos(x)\\
&\implies M+1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\cos(x + M\pi) &\triangleq \cos(x - N\pi) && \text{by definition of } N\\
&= \cos(x)\cos(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem D.9 page 221)}\\
&= \cos(x)\cos(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem D.2 page 213}\\
&= \cos(x)\cos(0 + N\pi) + \sin(x)\sin(0 + N\pi)\\
&= \cos(x)(-1)^N\cos(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (1b) page 222)}\\
&= (-1)^N\cos(x) && \text{by } \cos(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem D.2 page 213)}\\
&\triangleq (-1)^{-M}\cos(x) && \text{by definition of } N\\
&= (-1)^M\cos(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\cos(x + M\pi) &= \frac{e^{i(x+M\pi)} + e^{-i(x+M\pi)}}{2} && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}\\
&= e^{iM\pi}\left[\frac{e^{ix} + e^{-ix}}{2}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem D.6 page 217)}\\
&= \left(e^{i\pi}\right)^M \cos x && \text{by \textit{Euler formulas}} && \text{(Corollary D.2 page 217)}\\
&= (-1)^M \cos x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition D.4 page 220)}
\end{aligned}$$

2. Proof for (B):

(a) $M = 0$ case: $\sin(x + 0\pi) \;=\; \sin(x) \;=\; (-1)^0\sin(x)$

(b) Proof for $M > 0$ cases (by induction):

i. Base case $M = 1$:

$$\begin{aligned}
\sin(x + \pi) &= \sin x\cos\pi + \cos x\sin\pi && \text{by \textit{double angle formulas}} && \text{(Theorem D.9 page 221)}\\
&= \sin x(-1) - \cos x(0) && \text{by } \sin\pi = 0 \text{ results} && \text{(Proposition D.4 page 220)}\\
&= (-1)^1\sin x
\end{aligned}$$

ii. Inductive step…Proof that M case $\implies M + 1$ case:

$$\begin{aligned}
\sin(x + [M + 1]\pi) &= \sin([x + \pi] + M\pi)\\
&= (-1)^M\sin(x + \pi) && \text{by induction hypothesis (} M \text{ case)}\\
&= (-1)^M(-1)\sin(x) && \text{by base case (item (2(b)i) page 223)}\\
&= (-1)^{M+1}\sin(x)\\
&\implies M + 1 \text{ case}
\end{aligned}$$

(c) Proof for $M < 0$ cases: Let $N \triangleq -M \ldots \implies N > 0$.

$$\begin{aligned}
\sin(x + M\pi) &\triangleq \sin(x - N\pi) && \text{by definition of } N\\
&= \sin(x)\sin(-N\pi) - \sin(x)\sin(-N\pi) && \text{by \textit{double angle formulas}} && \text{(Theorem D.9 page 221)}\\
&= \sin(x)\sin(N\pi) + \sin(x)\sin(N\pi) && \text{by Theorem D.2 page 213}\\
&= \sin(x)\sin(0 + N\pi) + \sin(x)\sin(0 + N\pi)\\
&= \sin(x)(-1)^N\sin(0) + \sin(x)(-1)^N\sin(0) && \text{by } M \geq 0 \text{ results} && \text{(item (2b) page 223)}\\
&= (-1)^N\sin(x) && \text{by } \sin(0)=1,\ \sin(0)=0 \text{ results} && \text{(Theorem D.2 page 213)}\\
&\triangleq (-1)^{-M}\sin(x) && \text{by definition of } N\\
&= (-1)^M\sin(x)
\end{aligned}$$

(d) Proof using complex exponential:

$$\begin{aligned}
\sin(x+M\pi) &= \frac{e^{i(x+M\pi)} - e^{-i(x+M\pi)}}{2i} && \text{by } \textit{Euler formulas} && \text{(Corollary D.2 page 217)}\\
&= e^{iM\pi}\left[\frac{e^{ix}-e^{-ix}}{2i}\right] && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem D.6 page 217)}\\
&= \left(e^{i\pi}\right)^M \sin x && \text{by } \textit{Euler formulas} && \text{(Corollary D.2 page 217)}\\
&= (-1)^M \sin x && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition D.4 page 220)}
\end{aligned}$$

3. Proof for (C):

$$\begin{aligned}
e^{i(x+M\pi)} &= e^{iM\pi}e^{ix} && \text{by } e^{\alpha\beta} = e^{\alpha}e^{\beta} \text{ result} && \text{(Theorem D.6 page 217)}\\
&= \left(e^{i\pi}\right)^M\left(e^{ix}\right)\\
&= (-1)^M e^{ix} && \text{by } e^{i\pi} = -1 \text{ result} && \text{(Proposition D.4 page 220)}
\end{aligned}$$

4. Proofs for (D), (E), and (F):
$$\begin{aligned}
\cos(i(x+2M\pi)) &= (-1)^{2M}\cos(ix) &= \cos(ix) && \text{by (A)}\\
\sin(i(x+2M\pi)) &= (-1)^{2M}\sin(ix) &= \sin(ix) && \text{by (B)}\\
e^{i(x+2M\pi)} &= (-1)^{2M}e^{ix} &= e^{ix} && \text{by (C)}
\end{aligned}$$

⇨

Theorem D.11 (half-angle formulas/squared identities).

THM

(A). $\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \forall x\in\mathbb{R}$	(C). $\cos^2 x + \sin^2 x = 1 \quad \forall x\in\mathbb{R}$
(B). $\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \forall x\in\mathbb{R}$	

✎PROOF:

$$\begin{aligned}
\cos^2 x &\triangleq (\cos x)(\cos x) = \tfrac{1}{2}\cos(x-x) + \tfrac{1}{2}\cos(x+x) && \text{by } \textit{product identities} && \text{(Theorem D.8 page 219)}\\
&= \tfrac{1}{2}[1+\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem D.2 page 213)}\\
\sin^2 x &= (\sin x)(\sin x) = \tfrac{1}{2}\cos(x-x) - \tfrac{1}{2}\cos(x+x) && \text{by } \textit{product identities} && \text{(Theorem D.8 page 219)}\\
&= \tfrac{1}{2}[1-\cos(2x)] && \text{by } \cos(0)=1 \text{ result} && \text{(Theorem D.2 page 213)}\\
\cos^2 x + \sin^2 x &= \tfrac{1}{2}[1+\cos(2x)] + \tfrac{1}{2}[1-\cos(2x)] = 1 && \text{by (A) and (B)}\\
& && \text{note: see also} && \text{Theorem D.4 page 215}
\end{aligned}$$

⇨

D.6 Planar Geometry

The harmonic functions $\cos(x)$ and $\sin(x)$ are *orthogonal* to each other in the sense

$$\begin{aligned}
\langle\cos(x)\,|\,\sin(x)\rangle &= \int_{-\pi}^{+\pi}\cos(x)\sin(x)\,dx\\
&= \frac{1}{2}\int_{-\pi}^{+\pi}\sin(x-x)\,dx + \frac{1}{2}\int_{-\pi}^{+\pi}\sin(x+x)\,dx && \text{by Theorem D.8 page 219}\\
&= \frac{1}{2}\int_{-\pi}^{+\pi}\underbrace{\sin(0)}_{0}\,dx + \frac{1}{2}\int_{-\pi}^{+\pi}\sin(2x)\,dx
\end{aligned}$$

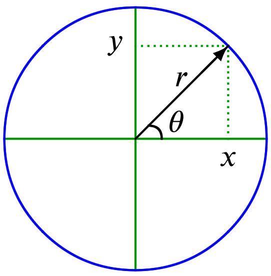
$$
\begin{aligned}
&= -\frac{1}{2}\cdot\frac{1}{2}\cos(2x)\Big|_{-\pi}^{+\pi}\cos(2x) \\
&= -\frac{1}{4}[\cos(2\pi)-\cos(-2\pi)] \\
&= 0
\end{aligned}
$$

Because $\cos(x)$ are $\sin(x)$ are orthogonal, they can be conveniently represented by the x and y axes in a plane— because perpendicular axes in a plane are also orthogonal. Vectors in the plane can be represented by linear combinations of $\cos x$ and $\sin x$. Let $\tan x$ be defined as

$$\tan x \triangleq \frac{\sin x}{\cos x}.$$

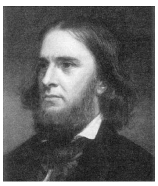
We can also define a value θ to represent the angle between such a vector and the x-axis such that

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$



$$
\begin{array}{rcl@{\qquad}rcl}
\cos\theta & \triangleq & \frac{x}{r} & \sec\theta & \triangleq & \frac{r}{x} \\
\sin\theta & \triangleq & \frac{y}{r} & \csc\theta & \triangleq & \frac{r}{y} \\
\tan\theta & \triangleq & \frac{y}{x} & \cot\theta & \triangleq & \frac{x}{y}
\end{array}
$$

D.7 The power of the exponential



❝*Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.*❞
Benjamin Peirce (1809–1880), American Harvard University mathematician after proving $e^{i\pi} = -1$ in a lecture. [22]

❝*Young man, in mathematics you don't understand things. You just get used to them.*❞
John von Neumann (1903–1957), Hungarian-American mathematician, as allegedly told to Gary Zukav by Felix T. Smith, Head of Molecular Physics at Stanford Research Institute, about a "physicist friend". [23]

[22] quote: Kasner and Newman (1940), page 104
image: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Peirce_Benjamin.html

[23] quote: Zukav (1980), page 208
image: http://en.wikipedia.org/wiki/John_von_Neumann
The quote appears in a footnote in Zukav (1980) that reads like this: Dr. Felix Smith, Head of Molecular Physics, Stanford Research Institute, once related to me the true story of a physicist friend who worked at Los Alamos after World War II. Seeking help on a difficult problem, he went to the great Hungarian mathematician, John von Neumann, who was at Los Alamos as a consultant. "Simple," said von Neumann. "This can be solved by using the method of characteristics." After the explanation the physicist said, "I'm afraid I don't understand the method of characteristics." "Young man," said von Neumann, "in mathematics you don't understand things, you just get used to them."

The following corollary presents one of the most amazing relations in all of mathematics. It shows a simple and compact relationship between the transcendental numbers π and e, the imaginary number i, and the additive and multiplicative identity elements 0 and 1. The fact that there is any relationship at all is somewhat amazing; but for there to be such an elegant one is truly one of the wonders of the world of numbers.

Corollary D.3. [24]

COR
$$e^{i\pi} + 1 = 0$$

✎PROOF:

$$\begin{aligned} e^{ix}\big|_{x=\pi} &= [\cos x + i \sin x]_{x=\pi} && \text{by Euler's identity (Theorem D.5 page 216)} \\ &= -1 + i \cdot 0 && \text{by Proposition D.4 page 220} \\ &= -1 \end{aligned}$$

✏

There are many transforms available, several of them integral transforms $[\mathbf{A}\mathsf{f}](s) \triangleq \int_t \mathsf{f}(s)\kappa(t,s)\,\mathrm{d}s$ using different kernels $\kappa(t,s)$. But of all of them, two of the most often used themselves use an exponential kernel:

① The *Laplace Transform* with kernel $\kappa(t,\ s) \triangleq e^{st}$

② The *Fourier Transform* with kernel $\kappa(t,\ \omega) \triangleq e^{i\omega t}$.

Of course, the Fourier kernel is just a special case of the Laplace kernel with $s = i\omega$ ($i\omega$ is a unit circle in s if s is depicted as a plane with real and imaginary axes). What is so special about exponential kernels? Is it just that they were discovered sooner than other kernels with other transforms? The answer in general is "no". The exponential has two properties that makes it extremely special:

- The exponential is an eigenvalue of any *linear time invariant* (LTI) operator (Theorem D.12 page 226).
- The exponential generates a *continuous point spectrum* for the *differential operator*.

Theorem D.12. [25] *Let* $\mathbf{L}$ *be an operator with kernel* $\mathsf{h}(t,\omega)$ *and*
$\check{\mathsf{h}}(s) \triangleq \langle \mathsf{h}(t,\omega) \,|\, e^{st} \rangle$ (LAPLACE TRANSFORM).

THM
$$\left\{\begin{array}{ll} 1. & \mathbf{L} \text{ is LINEAR and} \\ 2. & \mathbf{L} \text{ is TIME-INVARIANT} \end{array}\right\} \implies \left\{ \mathbf{L}e^{st} = \underbrace{\check{\mathsf{h}}^*(-s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenvector}} \right\}$$

✎PROOF:

[24] Euler (1748), Euler (1988) ⟨chapter 8?⟩, http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html

[25] Mallat (1999), page 2, ...page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf

$$
\begin{aligned}
\left[\mathbf{L}e^{st}\right](s) &= \langle e^{su} \,|\, \mathsf{h}((t;u), s)\rangle && \text{by linear hypothesis}\\
&= \langle e^{su} \,|\, \mathsf{h}((t-u), s)\rangle && \text{by time-invariance hypothesis}\\
&= \left\langle e^{s(t-v)} \,|\, \mathsf{h}(v, s)\right\rangle && \text{let } v = t-u \implies u = t-v\\
&= e^{st}\ \langle e^{-sv} \,|\, \mathsf{h}(v, s)\rangle && \text{by additivity of } \langle \triangle \,|\, \triangledown\rangle\\
&= \langle \mathsf{h}(v, s) \,|\, e^{-sv}\rangle^{*}\ e^{st} && \text{by conjugate symmetry of } \langle \triangle \,|\, \triangledown\rangle\\
&= \left\langle \mathsf{h}(v, s) \,|\, e^{(-s)v}\right\rangle^{*}\ e^{st}\\
&= \check{\mathsf{h}}^{*}(-s)\, e^{st} && \text{by definition of } \check{\mathsf{h}}(s)
\end{aligned}
$$

✎

APPENDIX E

FOURIER TRANSFORM

❝*The analytical equations . . . extend to all general phenomena. There cannot be a language more universal and more simple, more free from errors and from obscurities, . . . mathematical analysis is as extensive as nature itself; . . . it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them.*❞

Joseph Fourier (1768–1830) [1]

E.1 Definitions

This chapter deals with the *Fourier Transform* in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$, where $\mathbb{R}$ is the set of real numbers, $\mathscr{B}$ is the set of *Borel sets* on $\mathbb{R}$, μ is the standard *Borel measure* on , and

$$\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathsf{f} \in \mathbb{R}^{\mathbb{R}} | \int_{\mathbb{R}} |\mathsf{f}|^2 \, \mathsf{d}\mu < \infty \right\}.$$

Furthermore, $\langle \triangle \mid \nabla \rangle$ is the *inner product* induced by the operator $\int_{\mathbb{R}} \, \mathsf{d}\mu$ such that

$$\langle \mathsf{f} \mid \mathsf{g} \rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\mathsf{g}^*(x) \, \mathsf{d}x,$$

and $\left(\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \langle \triangle \mid \nabla \rangle \right)$ is a *Hilbert space*.

Definition E.1. *Let κ be a* FUNCTION *in* $\mathbb{C}^{\mathbb{R}^2}$.

DEF	*The function κ is the **Fourier kernel** if*	$\kappa(x,\omega) \triangleq e^{i\omega x}$	$\forall x,\omega \in \mathbb{R}$

Definition E.2. [2] *Let $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ be the space of all* LEBESGUE SQUARE-INTEGRABLE FUNCTIONS.

---

[1] quote: Fourier (1878), pages 7–8 ⟨Preliminary Discourse⟩
image: http://en.wikipedia.org/wiki/File:Fourier2.jpg, public domain

[2] Bachman et al. (2000) page 363, Chorin and Hald (2009) page 13, Loomis and Bolker (1965), page 144, Knapp (2005b) pages 374–375, Fourier (1822), Fourier (1878) page 336?

DEF: *The **Fourier Transform** operator $\tilde{\mathbf{F}}$ is defined as*

$$\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega) \triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \qquad \forall \mathsf{f}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

*This definition of the Fourier Transform is also called the **unitary Fourier Transform**.*

Remark E.1 (**Fourier transform scaling factor**). [3] If the Fourier transform operator $\tilde{\mathbf{F}}$ and inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ are defined as

$$\tilde{\mathbf{F}}\mathsf{f}(x) \triangleq \mathsf{F}(\omega) \triangleq A\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\, \mathrm{d}x \qquad \text{and} \qquad \tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}(\omega) \triangleq B\int_{\mathbb{R}} \mathsf{F}(\omega)e^{i\omega x}\, \mathrm{d}\omega$$

then A and B can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor A set equal to 1 such that $\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. In this case, the inverse Fourier transform operator $\tilde{\mathbf{F}}^{-1}$ is either defined as

- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](f) \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\, e^{i2\pi f x}\, \mathrm{d}x$ (using oscillatory frequency free variable f) or
- $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{i\omega x}\, \mathrm{d}x$ (using angular frequency free variable ω).

In short, the 2π has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi}\int\cdots$). One could argue that it is unnecessary to burden the exponential argument with the 2π factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable ω thus giving the inverse operator definition $\left[\tilde{\mathbf{F}}^{-1}\mathsf{f}(x)\right](\omega) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x$. But this causes a new problem. In this case, the Fourier operator $\tilde{\mathbf{F}}$ is not *unitary* (see Theorem E.2 page 230)—in particular, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* \neq \mathbf{I}$, where $\tilde{\mathbf{F}}^*$ is the *adjoint* of $\tilde{\mathbf{F}}$; but rather, $\tilde{\mathbf{F}}\left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right) = \left(\frac{1}{2\pi}\tilde{\mathbf{F}}^*\right)\tilde{\mathbf{F}} = \mathbf{I}$. But if we define the operators $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}^{-1}$ are inverses *and* $\tilde{\mathbf{F}}$ is *unitary*—that is, $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}$.

E.2 Operator properties

Theorem E.1 (Inverse Fourier transform). [4] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator (Definition E.2 page 229). The inverse $\tilde{\mathbf{F}}^{-1}$ of $\tilde{\mathbf{F}}$ is*

THM: $$\left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}\right](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)e^{i\omega x}\, \mathrm{d}\omega \qquad \forall \tilde{\mathsf{f}}\in \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem E.2. *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator with inverse $\tilde{\mathbf{F}}^{-1}$ and adjoint $\tilde{\mathbf{F}}^*$.*

THM: $$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$$

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{f} \mid \mathsf{g} \right\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\, e^{-i\omega x}\, \mathrm{d}x \mid \mathsf{g}(\omega)\right\rangle && \text{by definition of } \tilde{\mathbf{F}} \text{ page 229}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\, \left\langle e^{-i\omega x} \mid \mathsf{g}(\omega)\right\rangle\, \mathrm{d}x && \text{by \textit{additive property} of } \langle\triangle\mid\triangledown\rangle\\
&= \int_{\mathbb{R}} \mathsf{f}(x)\, \frac{1}{\sqrt{2\pi}}\left\langle \mathsf{g}(\omega) \mid e^{-i\omega x}\right\rangle^*\, \mathrm{d}x && \text{by \textit{conjugate symmetric property} of } \langle\triangle\mid\triangledown\rangle
\end{aligned}$$

[3] Chorin and Hald (2009) page 13, Jeffrey and Dai (2008) pages xxxi–xxxii, Knapp (2005b) pages 374–375
[4] Chorin and Hald (2009) page 13

$$= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2\pi}} \left\langle \mathsf{g}(\omega) \,|\, e^{-i\omega x} \right\rangle \right\rangle \qquad \text{by definition of } \langle \triangle \,|\, \nabla \rangle$$

$$= \left\langle \mathsf{f} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*} \mathsf{g} \right\rangle \qquad \text{by Theorem E.1 page 230}$$

The Fourier Transform operator has several nice properties:

- $\tilde{\mathbf{F}}$ is *unitary* (Corollary E.1—next corollary).
- Because $\tilde{\mathbf{F}}$ is unitary, it automatically has several other nice properties (Theorem E.3 page 231).

Corollary E.1. *Let* $\mathbf{I}$ *be the identity operator and let* $\tilde{\mathbf{F}}$ *be the Fourier Transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}^{-1}$.

COR
$$\underbrace{\tilde{\mathbf{F}}\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = \mathbf{I}}_{\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}} \qquad (\tilde{\mathbf{F}} \textit{ is unitary})$$

✎PROOF: This follows directly from the fact that $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (Theorem E.2 page 230). ✏

Theorem E.3. *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator with adjoint* $\tilde{\mathbf{F}}^*$ *and inverse* $\tilde{\mathbf{F}}$. *Let* $|||\cdot|||$ *be the operator norm with respect to the vector norm* $\|\cdot\|$ *with respect to the Hilbert space* $(\mathbb{C}^{\mathbb{R}}, \langle \triangle \,|\, \nabla \rangle)$. *Let* $\mathcal{R}(\mathbf{A})$ *be the range of an operator* $\mathbf{A}$.

THM
$$\begin{array}{rclcll}
\mathcal{R}(\mathbf{F}\tau) & = & \mathcal{R}(\tilde{\mathbf{F}}^{-1}) & = & \mathcal{L}^2_{\mathbb{R}} & \\
|||\tilde{\mathbf{F}}||| & = & |||\tilde{\mathbf{F}}^{-1}||| & = & 1 & (\text{UNITARY}) \\
\langle \tilde{\mathbf{F}}\mathsf{f} \,|\, \tilde{\mathbf{F}}\mathsf{g} \rangle & = & \langle \tilde{\mathbf{F}}^{-1}\mathsf{f} \,|\, \tilde{\mathbf{F}}^{-1}\mathsf{g} \rangle & = & \langle \mathsf{f} \,|\, \mathsf{g} \rangle & (\text{PARSEVAL'S EQUATION}) \\
\|\tilde{\mathbf{F}}\mathsf{f}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f}\| & = & \|\mathsf{f}\| & (\text{PLANCHEREL'S FORMULA}) \\
\|\tilde{\mathbf{F}}\mathsf{f} - \tilde{\mathbf{F}}\mathsf{g}\| & = & \|\tilde{\mathbf{F}}^{-1}\mathsf{f} - \tilde{\mathbf{F}}^{-1}\mathsf{g}\| & = & \|\mathsf{f} - \mathsf{g}\| & (\text{ISOMETRIC})
\end{array}$$

✎PROOF: These results follow directly from the fact that $\tilde{\mathbf{F}}$ is unitary (Corollary E.1 page 231) and from the properties of unitary operators. ✏

Theorem E.4 (Shift relations). *Let* $\tilde{\mathbf{F}}$ *be the Fourier transform operator.*

THM
$$\begin{array}{rcl}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) & = & e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) \\
\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) & = & \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v)
\end{array}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}[\mathsf{f}(x-u)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x-u) e^{-i\omega x} \, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229))} \\
&= \frac{1}{\sqrt{2\pi}} \int_{v \in \mathbb{R}} \mathsf{f}(v) e^{-i\omega(u+v)} \, \mathrm{d}v && \text{where } v \triangleq x - u \implies t = u + v \\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{v \in \mathbb{R}} \mathsf{f}(v) e^{-i\omega v} \, \mathrm{d}v \\
&= e^{-i\omega u} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x} \, \mathrm{d}x && \text{by change of variable } t = v \\
&= e^{-i\omega u} \left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229))} \\
\left[\tilde{\mathbf{F}}\left(e^{ivx}\mathsf{g}(x)\right)\right](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ivx} \mathsf{g}(x) e^{-i\omega x} \, \mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229))} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{g}(x) e^{-i(\omega - v)x} \, \mathrm{d}x \\
&= \left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega - v) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229))}
\end{aligned}$$

Theorem E.5 (Complex conjugate). *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $*$ represent the complex conjugate operation on the set of complex numbers.*

THM

$$\begin{array}{rcll} \tilde{\mathbf{F}}\mathsf{f}^*(-x) &=& -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* & \forall \mathsf{f}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \\ \mathsf{f} \text{ is real} \implies \tilde{\mathsf{f}}(-\omega) &=& \left[\tilde{\mathsf{f}}(\omega)\right]^* & \forall \omega\in\mathbb{R} \qquad \text{REALITY CONDITION} \end{array}$$

✎PROOF:

$$\begin{aligned} \left[\tilde{\mathbf{F}}\mathsf{f}^*(-x)\right](\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(-x)e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229)} \\ &= \frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(u)e^{i\omega u}(-1)\,\mathrm{d}u && \text{where } u \triangleq -x \implies \mathrm{d}x = -\,\mathrm{d}u \\ &= -\left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right]^* \\ &\triangleq -\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]^* && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229)} \\ \tilde{\mathsf{f}}(-\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i(-\omega)x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229)} \\ &= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}^*(x)e^{-i\omega x}\,\mathrm{d}x\right]^* \\ &= \left[\frac{1}{\sqrt{2\pi}}\int \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]^* && \text{by } \mathsf{f} \text{ is real hypothesis} \\ &\triangleq \tilde{\mathsf{f}}^*(\omega) && \text{by definition of } \tilde{\mathbf{F}} && \text{(Definition E.2 page 229)} \end{aligned}$$

E.3 Convolution

Definition E.3. [5]

DEF

The ***convolution operation*** *is defined as*

$$\left[\mathsf{f}\star\mathsf{g}\right](x) \triangleq \mathsf{f}(x)\star\mathsf{g}(x) \triangleq \int_{u\in\mathbb{R}} \mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

Theorem E.6 (next) demonstrates that multiplication in the "time domain" is equivalent to convolution in the "frequency domain" and vice-versa.

Theorem E.6 (convolution theorem). [6] *Let $\tilde{\mathbf{F}}$ be the Fourier Transform operator and $\star$ the convolution operator.*

THM

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega)}_{\text{convolution in "time domain"}} = \underbrace{\sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{multiplication in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$$

$$\underbrace{\tilde{\mathbf{F}}\left[\mathsf{f}(x)\mathsf{g}(x)\right](\omega)}_{\text{multiplication in "time domain"}} = \underbrace{\frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)}_{\text{convolution in "frequency domain"}} \qquad \forall \mathsf{f},\mathsf{g}\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}.$$

---

[5] Bachman (1964), page 6, Bracewell (1978) page 108 ⟨Convolution theorem⟩

[6] Bracewell (1978) page 110

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\left[\mathsf{f}(x)\star\mathsf{g}(x)\right](\omega) &= \tilde{\mathbf{F}}\left[\int_{u\in\mathbb{R}}\mathsf{f}(u)\mathsf{g}(x-u)\,\mathrm{d}u\right](\omega) && \text{by definition of } \star \text{ (Definition E.3 page 232)}\\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)\left[\tilde{\mathbf{F}}\mathsf{g}(x-u)\right](\omega)\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-i\omega u}\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega)\,\mathrm{d}u && \text{by Theorem E.4 page 231}\\
&= \sqrt{2\pi}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}}\mathsf{f}(u)e^{-i\omega u}\,\mathrm{d}u\right)}_{[\tilde{\mathbf{F}}\mathsf{f}](\omega)}\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega)\\
&= \sqrt{2\pi}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition E.2 page 229)}\\
\tilde{\mathbf{F}}[\mathsf{f}(x)\mathsf{g}(x)](\omega) &= \tilde{\mathbf{F}}\left[\left(\tilde{\mathbf{F}}^{-1}\tilde{\mathbf{F}}\mathsf{f}(x)\right)\mathsf{g}(x)\right](\omega) && \text{by definition of operator inverse}\\
&= \tilde{\mathbf{F}}\left[\left(\frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)e^{ivx}\,\mathrm{d}v\right)\mathsf{g}(x)\right](\omega) && \text{by Theorem E.1 page 230}\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\left(e^{ivx}\,\mathsf{g}(x)\right)\right](\omega,v)\,\mathrm{d}v\\
&= \frac{1}{\sqrt{2\pi}}\int_{v\in\mathbb{R}}\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right](v)\left[\tilde{\mathbf{F}}\mathsf{g}(x)\right](\omega-v)\,\mathrm{d}v && \text{by Theorem E.4 page 231}\\
&= \frac{1}{\sqrt{2\pi}}\left[\tilde{\mathbf{F}}\mathsf{f}\right](\omega)\star\left[\tilde{\mathbf{F}}\mathsf{g}\right](\omega) && \text{by definition of } \star \text{ (Definition E.3 page 232)}
\end{aligned}$$

✏

E.4 Real valued functions

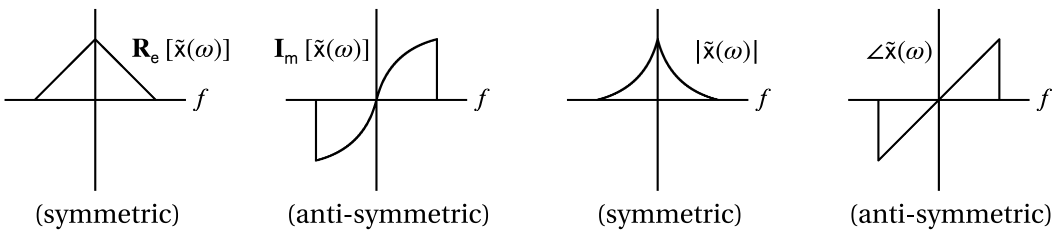


Figure E.1: Fourier transform components of real-valued signal

Theorem E.7. *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* $\tilde{\mathsf{f}}(\omega)$ *the* FOURIER TRANSFORM *of* $\mathsf{f}(x)$.

THM

$$\left\{\begin{array}{l}\mathsf{f}(x) \text{ is REAL-VALUED}\\ (\mathsf{f}\in\mathbb{R}^{\mathbb{R}})\end{array}\right\} \implies \left\{\begin{array}{rcll}
\tilde{\mathsf{f}}(\omega) &=& \tilde{\mathsf{f}}^*(-\omega) & \text{(HERMITIAN SYMMETRIC)}\\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(SYMMETRIC)}\\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right] & \text{(ANTI-SYMMETRIC)}\\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}(-\omega)| & \text{(SYMMETRIC)}\\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}(-\omega) & \text{(ANTI-SYMMETRIC).}
\end{array}\right.$$

✎PROOF:

$$\begin{array}{lclclclcl}
\tilde{\mathsf{f}}(\omega) &\triangleq& [\tilde{\mathbf{F}}\mathsf{f}(x)](\omega) &\triangleq& \left\langle \mathsf{f}(x)\,|\,e^{i\omega x}\right\rangle &=& \left\langle \mathsf{f}(x)\,|\,e^{i(-\omega)x}\right\rangle^* &\triangleq& \tilde{\mathsf{f}}^*(-\omega)\\
\mathbf{R_e}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& \mathbf{R_e}\left[\tilde{\mathsf{f}}(-\omega)\right]\\
\mathbf{I_m}\left[\tilde{\mathsf{f}}(\omega)\right] &=& \mathbf{I_m}\left[\tilde{\mathsf{f}}^*(-\omega)\right] &=& -\mathbf{I_m}\left[\tilde{\mathsf{f}}(-\omega)\right]\\
|\tilde{\mathsf{f}}(\omega)| &=& |\tilde{\mathsf{f}}^*(-\omega)| &=& |\tilde{\mathsf{f}}(-\omega)|\\
\angle\tilde{\mathsf{f}}(\omega) &=& \angle\tilde{\mathsf{f}}^*(-\omega) &=& -\angle\tilde{\mathsf{f}}(-\omega)
\end{array}$$

✏

E.5 Moment properties

Definition E.4. [7]

DEF: *The quantity* M_n *is the* n**th moment** *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$ *if*

$$\mathsf{M}_n \triangleq \int_{\mathbb{R}} x^n \mathsf{f}(x)\,\mathrm{d}x \qquad \text{for } n \in \mathbb{W}.$$

Lemma E.1. [8] *Let* M_n *be the* nTH MOMENT (Definition E.4 page 234) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition E.2 page 229) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition ?? page ??).

LEM:

$$\begin{aligned} \mathsf{M}_n &= \sqrt{2\pi}(i)^n \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \\ \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} &= \frac{1}{\sqrt{2\pi}}(-i)^n \mathsf{M}_n && \forall n\in\mathbb{W}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \end{aligned}$$

✎PROOF:

$$\begin{aligned} \sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=0} &= \sqrt{2\pi}(i)^n \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=0} && \text{by definition of } \tilde{\mathbf{F}} \text{ (Definition E.2 page 229)} \\ &= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\right] \mathrm{d}x \Big|_{\omega=0} \\ &= (i)^n \int_{\mathbb{R}} \mathsf{f}(x) \left[(-i)^n x^n e^{-i\omega x}\right] \mathrm{d}x \Big|_{\omega=0} \\ &= (-i^2)^n \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \mathsf{f}(x) x^n\,\mathrm{d}x \\ &\triangleq \mathsf{M}_n && \text{by definition of } \mathsf{M}_n \text{ (Definition E.4 page 234)} \end{aligned}$$

✏

Lemma E.2. [9] *Let* M_n *be the* nTH MOMENT (Definition E.4 page 234) *and* $\tilde{\mathsf{f}}(\omega) \triangleq [\tilde{\mathbf{F}}\mathsf{f}](\omega)$ *the* FOURIER TRANSFORM (Definition E.2 page 229) *of a function* $\mathsf{f}(x)$ *in* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition ?? page ??).

LEM:

$$\mathsf{M}_n = 0 \quad \iff \quad \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} = 0 \qquad \forall n\in\mathbb{W}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned} 0 &= \langle \mathsf{f}(x) \,|\, x^n \rangle && \text{by left hypothesis} \\ &= \sqrt{2\pi}(-i)^{-n} \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} && \text{by Lemma E.1 page 234} \\ \Longrightarrow \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\Big|_{\omega=0} &= 0 \end{aligned}$$

[7] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83

[8] Goswami and Chan (1999), pages 38–39

[9] Vidakovic (1999), pages 82–83, Mallat (1999), pages 241–242

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
0 &= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=0} && \text{by right hypothesis}\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\bigg|_{\omega=0} && \text{by definition of } \tilde{\mathsf{f}}(\omega)\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n e^{-i\omega x}\,\mathrm{d}x\bigg|_{\omega=0}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)\left[(-i)^n x^n e^{-i\omega x}\right]\,\mathrm{d}x\bigg|_{\omega=0}\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)x^n\,\mathrm{d}x\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\langle \mathsf{f}(x)\,|\,x^n\rangle && \text{by definition of } \langle\cdot\,|\,\cdot\rangle \text{ in } \boldsymbol{L}^2_{\mathbb{R}} \text{ (Definition ?? page ??)}
\end{aligned}$$

✏

Lemma E.3 (Strang-Fix condition). [10] *Let* $\mathsf{f}(x)$ *be a function in* $\boldsymbol{L}^2_{\mathbb{R}}$ *and* M_n *the* nTH MOMENT *(Definition E.4 page 234) of* $\mathsf{f}(x)$. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition F.3 page 240).*

LEM

$$\underbrace{\sum_{k\in\mathbb{Z}} \mathbf{T}^k x^n \mathsf{f}(x) = \mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "time"}} \iff \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\bigg|_{\omega=2\pi k} = \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k\mathsf{M}_n}_{\text{STRANG-FIX CONDITION in "frequency"}}$$

✎PROOF:

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} &= \sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}} \mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k && \text{by Definition E.2 page 229}\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[\int_{\mathbb{R}} x^n\mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]_{\omega=2\pi k} e^{i2\pi kx}\bar{\delta}_k\\
&= (-i)^n\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}(x-k)^n\mathsf{f}(x-k)\bar{\delta}_k && \text{by } \textit{PSF} \text{ (Theorem F.2 page 248)}\\
&= \frac{1}{\sqrt{2\pi}}(-i)^n\bar{\delta}_k\mathsf{M}_n && \text{by left hypothesis}
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}(-i)^n\mathsf{M}_n &= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\left[(-i)^n\bar{\delta}_k\mathsf{M}_n\right]e^{-i2\pi kx} && \text{by definition of } \bar{\delta}\\
&= \sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \tilde{\mathsf{f}}(\omega)\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx} && \text{by right hypothesis}\\
&= \sum_{k\in\mathbb{Z}}\left[\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\right]^n \int_{\mathbb{R}}\mathsf{f}(x)e^{-i\omega x}\,\mathrm{d}x\right]\bigg|_{\omega=2\pi k} e^{-i2\pi kx}
\end{aligned}$$

[10] Jawerth and Sweldens (1994), pages 16–17, Sweldens and Piessens (1993), page 2, Vidakovic (1999), page 83, Mallat (1999), pages 241–243, Fix and Strang (1969)

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \left[\left. \int_{\mathbb{R}} \mathsf{f}(x)(-ix)^n e^{-i\omega x}\,\mathrm{d}x \right] \right|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} \left[\left. \int_{\mathbb{R}} x^n \mathsf{f}(x) e^{-i\omega x}\,\mathrm{d}x \right] \right|_{\omega=2\pi k} e^{-i2\pi kx} \\
&= (-i)^n \sum_{k\in\mathbb{Z}} (x-k)^n \mathsf{f}(x-k) && \text{by } \textit{PSF} \quad \text{(Theorem F.2 page 248)}
\end{aligned}
$$

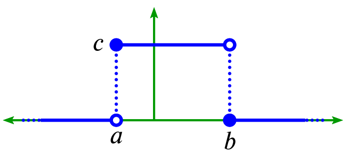
✏

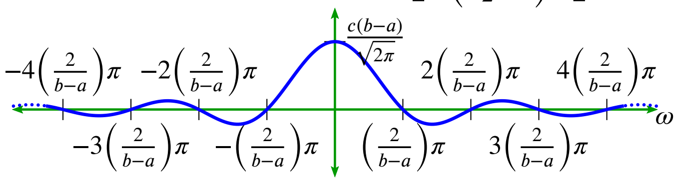
E.6 Examples

Example E.1 (rectangular pulse). Let $\tilde{\mathsf{f}}(\omega)$ be the *Fourier transform* of a function $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

EX

$$\mathsf{f}(x) = \begin{cases} c & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{\mathsf{f}}(\omega) = \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)} \right]$$





✎PROOF:

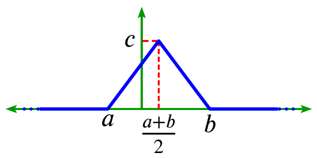
$$
\begin{aligned}
\tilde{\mathsf{f}}(\omega) &= \tilde{\mathbf{F}}[\mathsf{f}(x)](\omega) && \text{by definition of } \tilde{\mathsf{f}}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[\mathsf{f}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by } \textit{shift relation} \quad \text{(Theorem E.4 page 231)} \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{[a:b)}\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by definition of } \mathsf{f}(x) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega} \tilde{\mathbf{F}}\left[c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x)\right](\omega) && \text{by definition of } \mathbb{1} \quad \text{(Definition F.2 page 239)} \\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{\mathbb{R}} c\mathbb{1}_{\left[-\frac{b-a}{2}:\frac{b-a}{2}\right)}(x) e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition E.2 page 229)} \\
&= \frac{1}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} c e^{-i\omega x}\,\mathrm{d}x && \text{by definition of } \mathbb{1} \quad \text{(Definition F.2 page 239)} \\
&= \frac{c}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\right)\omega} \frac{1}{-i\omega} e^{-i\omega x} \Big|_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \\
&= \frac{2c}{\sqrt{2\pi}\omega} e^{-i\left(\frac{a+b}{2}\right)\omega} \left[\frac{e^{i\left(\frac{b-a}{2}\omega\right)} - e^{-i\left(\frac{b-a}{2}\omega\right)}}{2i} \right] \\
&= \frac{c(b-a)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{2}\omega\right)}{\left(\frac{b-a}{2}\omega\right)} \right] && \text{by } \textit{Euler formulas} \quad \text{(Corollary D.2 page 217)}
\end{aligned}
$$

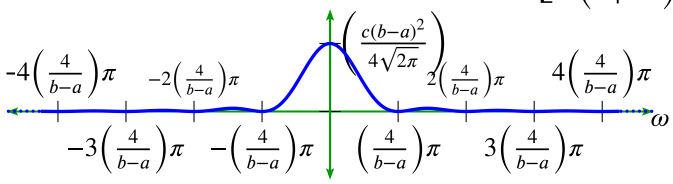
✏

Example E.2 (triangle). Let $\tilde{f}(\omega)$ be the *Fourier transform* of a function $f(x) \in L^2_{\mathbb{R}}$.

EX

$$f(x) = \begin{cases} c\left[1 - \frac{|2x-b-a|}{b-a}\right] & \text{for } x \in [a:b) \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)} \left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2$$





✎PROOF:

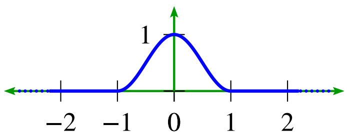
$$\begin{aligned}
\tilde{f}(\omega) &= \tilde{\mathbf{F}}[f(x)](\omega) && \text{by definition of } \tilde{f}(\omega) \\
&= e^{-i\left(\frac{a+b}{2}\right)\omega}\tilde{\mathbf{F}}\left[f\left(x - \frac{a+b}{2}\right)\right](\omega) && \text{by \textit{shift relation}} \quad \text{(Theorem E.4 page 231)} \\
&= \tilde{\mathbf{F}}\left[c\left(1 - \frac{|2x-b-a|}{b-a}\right)\mathbb{1}_{[a:b)}(x)\right](\omega) && \text{by definition of } f(x) \\
&= c\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x) \star \mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}(x)\right](\omega) \\
&= c\sqrt{2\pi}\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right] && \text{by \textit{convolution theorem}} \quad \text{(Theorem E.6 page 232)} \\
&= c\sqrt{2\pi}\left(\tilde{\mathbf{F}}\left[\mathbb{1}_{\left[\frac{a}{2}:\frac{b}{2}\right)}\right]\right)^2 \\
&= c\sqrt{2\pi}\left(\frac{\left(\frac{b}{2}-\frac{a}{2}\right)}{\sqrt{2\pi}} e^{-i\left(\frac{a+b}{4}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]\right)^2 && \text{by \textit{Rectangular pulse} ex.} \quad \text{Example E.1 page 236} \\
&= \frac{c(b-a)^2}{4\sqrt{2\pi}} e^{-i\left(\frac{a+b}{2}\omega\right)}\left[\frac{\sin\left(\frac{b-a}{4}\omega\right)}{\left(\frac{b-a}{4}\omega\right)}\right]^2
\end{aligned}$$

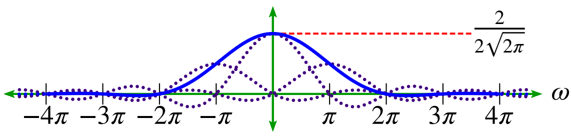
✏

Example E.3. Let a function f be defined in terms of the cosine function (Definition D.2 page 211) as follows:

EX

$$f(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{f}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\text{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\text{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\text{sinc}(\omega-\pi)}\right]$$





✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition F.2 page 239) on a set A.

$$\begin{aligned}
\tilde{f}(\omega) &\triangleq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\omega x}\,dx && \text{by definition of } \tilde{f}(\omega) \text{ (Definition E.2)} \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x)e^{-i\omega x}\,dx && \text{by definition of } f(x)
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right) e^{-i\omega x}\, \mathrm{d}x \qquad \text{by definition of } \mathbb{1} \text{ (Definition F.2)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left[\frac{e^{i\frac{\pi}{2}x} + e^{-i\frac{\pi}{2}x}}{2}\right]^2 e^{-i\omega x}\, \mathrm{d}x \qquad \text{by Corollary D.2 page 217}$$

$$= \frac{1}{4\sqrt{2\pi}} \int_{-1}^{1} \left[2 + e^{i\pi x} + e^{-i\pi x}\right] e^{-i\omega x}\, \mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}} \int_{-1}^{1} 2e^{-i\omega x} + e^{-i(\omega+\pi)x} + e^{-i(\omega-\pi)x}\, \mathrm{d}x$$

$$= \frac{1}{4\sqrt{2\pi}} \left[2\frac{e^{-i\omega x}}{-i\omega} + \frac{e^{-i(\omega+\pi)x}}{-i(\omega+\pi)} + \frac{e^{-i(\omega-\pi)x}}{-i(\omega-\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2\sqrt{2\pi}} \left[2\frac{e^{-i\omega} - e^{+i\omega}}{-2i\omega} + \frac{e^{-i(\omega+\pi)} - e^{+i(\omega+\pi)}}{-2i(\omega+\pi)} + \frac{e^{-i(\omega-\pi)} - e^{+i(\omega-\pi)}}{-2i(\omega-\pi)}\right]_{-1}^{1}$$

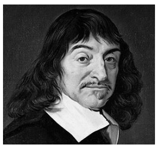
$$= \frac{1}{2\sqrt{2\pi}} \left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)}\right]$$

✎

APPENDIX F

TRANSVERSAL OPERATORS

"Je me plaisois surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons: mais je ne remarquois point encore leur vrai usage; et, pensant qu'elles ne servoient qu'aux arts mécaniques, je m'étonnois de ce que leurs fondements étant si fermes et si solides, on n'avoit rien bâti dessus de plus relevé:"



"I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them."

René Descartes, philosopher and mathematician (1596–1650) [1]

F.1 Families of Functions

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{\mathbb{R}}$ (Definition ?? page ??). The space $\boldsymbol{L}^2_{\mathbb{R}}$ is a subspace of the space $\mathbb{R}^{\mathbb{R}}$, the set of all functions with *domain* $\mathbb{R}$ (the set of real numbers) and *range* $\mathbb{R}$. The space $\mathbb{R}^{\mathbb{R}}$ is a subspace of the space $\mathbb{C}^{\mathbb{C}}$, the set of all functions with *domain* $\mathbb{C}$ (the set of complex numbers) and *range* $\mathbb{C}$. That is, $\boldsymbol{L}^2_{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}} \subseteq \mathbb{C}^{\mathbb{C}}$. In general, the notation Y^X represents the set of all functions with domain X and range Y (Definition F.1 page 239). Although this notation may seem curious, note that for finite X and finite Y, the number of functions (elements) in Y^X is $\left|Y^X\right| = |Y|^{|X|}$.

Definition F.1. *Let X and Y be sets.*

DEF *The space Y^X represents the set of all functions with* DOMAIN X *and* RANGE Y *such that*

$$Y^X \triangleq \{\mathsf{f}(x) | \mathsf{f}(x) : X \to Y\}$$

Definition F.2. [2] *Let X be a set.*

[1] quote: Descartes (1637a)
translation: Descartes (1637b) ⟨part I, paragraph 10⟩
image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg, public domain

[2] Aliprantis and Burkinshaw (1998), page 126, Hausdorff (1937), page 22, de la Vallée-Poussin (1915) page

DEF

*The **indicator function*** $\mathbb{1} \in \{0,1\}^{2^X}$ *is defined as*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \qquad \forall x\in X,\ A\in 2^X \\ 0 & \text{for } x \notin A \qquad \forall x\in X,\ A\in 2^X \end{cases}$$

The indicator function $\mathbb{1}$ *is also called the **characteristic function**.*

F.2 Definitions and algebraic properties

Much of the wavelet theory developed in this text is constructed using the **translation operator T** and the **dilation operator D** (next).

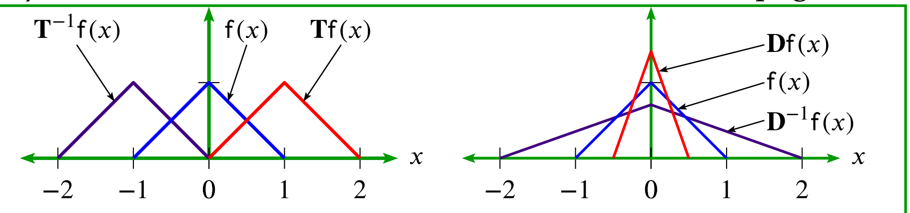
Definition F.3. [3]

DEF

$$\begin{array}{lrcll} \mathbf{T}_\tau \text{ is a \textbf{translation operator} on } \mathbb{C}^{\mathbb{C}} \text{ if} & \mathbf{T}_\tau f(x) & \triangleq & f(x-\tau) & \forall f\in\mathbb{C}^{\mathbb{C}} \quad . \\ \mathbf{D}_\alpha \text{ is a \textbf{dilation operator} on } \mathbb{C}^{\mathbb{C}} \text{ if} & \mathbf{D}_\alpha f(x) & \triangleq & f(\alpha x) & \forall f\in\mathbb{C}^{\mathbb{C}} \quad . \\ \text{Moreover, } \mathbf{T} \triangleq \mathbf{T}_1 \quad \text{and} \quad \mathbf{D} \triangleq \sqrt{2}\mathbf{D}_2 & & & & \quad . \end{array}$$

Example F.1. Let $\mathbf{T}$ and $\mathbf{D}$ be defined as in Definition F.3 (page 240).

EX



Proposition F.1. *Let* $\mathbf{T}_\tau$ *be a* TRANSLATION OPERATOR *(Definition F.3 page 240).*

PRP

$$\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n f(x) = \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n f(x+\tau) \qquad \forall f\in\mathbb{R}^{\mathbb{R}} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n f(x) \text{ is PERIODIC with period } \tau \right)$$

✎PROOF:

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \mathbf{T}_\tau^n f(x+\tau) &= \sum_{n\in\mathbb{Z}} f(x-n\tau+\tau) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition F.3 page 240)} \\ &= \sum_{m\in\mathbb{Z}} f(x-m\tau) && \text{where } m \triangleq n-1 && \Longrightarrow \quad n=m+1 \\ &= \sum_{m\in\mathbb{Z}} \mathbf{T}_\tau^m f(x) && \text{by definition of } \mathbf{T}_\tau && \text{(Definition F.3 page 240)} \end{aligned}$$

✐

In a linear space, every operator has an *inverse*. Although the inverse always exists as a *relation*, it may not exist as a *function* or as an *operator*. But in some cases the inverse of an operator is itself an operator. The inverses of the operators $\mathbf{T}$ and $\mathbf{D}$ both exist as operators, as demonstrated next.

Proposition F.2 (transversal operator inverses)**.** *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition F.3 page 240.*

PRP

$\mathbf{T}$ *has an* INVERSE $\mathbf{T}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{T}^{-1}f(x) = f(x+1) \qquad \forall f\in\mathbb{C}^{\mathbb{C}} \qquad \text{(\textbf{translation operator inverse})}.$$

$\mathbf{D}$ *has an* INVERSE $\mathbf{D}^{-1}$ *in* $\mathbb{C}^{\mathbb{C}}$ *expressed by the relation*

$$\mathbf{D}^{-1}f(x) = \frac{\sqrt{2}}{2} f\left(\frac{1}{2}x\right) \qquad \forall f\in\mathbb{C}^{\mathbb{C}} \qquad \text{(\textbf{dilation operator inverse})}.$$

[3] Walnut (2002) pages 79–80 ⟨Definition 3.39⟩, Christensen (2003) pages 41–42, Wojtaszczyk (1997) page 18 ⟨Definitions 2.3,2.4⟩, Kammler (2008) page A-21, Bachman et al. (2000) page 473, Packer (2004) page 260, Benedetto and Zayed (2004) page , Heil (2011) page 250 ⟨Notation 9.4⟩, Casazza and Lammers (1998) page 74, Goodman et al. (1993a), page 639, Heil and Walnut (1989) page 633 ⟨Definition 1.3.1⟩, Dai and Lu (1996), page 81, Dai and Larson (1998) page 2

✎PROOF:

1. Proof that $\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T}\mathsf{f}(x) &= \mathbf{T}^{-1}\mathsf{f}(x-1) && \text{by defintion of } \mathbf{T} && \text{(Definition F.3 page 240)}\\
&= \mathsf{f}([x+1]-1)\\
&= \mathsf{f}(x)\\
&= \mathsf{f}([x-1]+1)\\
&= \mathbf{T}\mathsf{f}(x+1) && \text{by defintion of } \mathbf{T} && \text{(Definition F.3 page 240)}\\
&= \mathbf{T}\mathbf{T}^{-1}\mathsf{f}(x)\\
\implies \mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} = \mathbf{T}\mathbf{T}^{-1}
\end{aligned}$$

2. Proof that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$:

$$\begin{aligned}
\mathbf{D}^{-1}\mathbf{D}\mathsf{f}(x) &= \mathbf{D}^{-1}\sqrt{2}\mathsf{f}(2x) && \text{by defintion of } \mathbf{D} && \text{(Definition F.3 page 240)}\\
&= \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\mathsf{f}\left(2\left[\frac{1}{2}x\right]\right)\\
&= \mathsf{f}(x)\\
&= \sqrt{2}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}[2x]\right)\right]\\
&= \mathbf{D}\left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)\right] && \text{by defintion of } \mathbf{D} && \text{(Definition F.3 page 240)}\\
&= \mathbf{D}\mathbf{D}^{-1}\mathsf{f}(x)\\
\implies \mathbf{D}^{-1}\mathbf{D} &= \mathbf{I} = \mathbf{D}\mathbf{D}^{-1}
\end{aligned}$$

✏

Proposition F.3. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as defined in Definition F.3 page 240.*
Let $\mathbf{D}^0 = \mathbf{T}^0 \triangleq \mathbf{I}$ *be the* IDENTITY OPERATOR.

PRP
$$\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) = 2^{j/2}\mathsf{f}\left(2^jx-n\right) \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

F.3 Linear space properties

Proposition F.4. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition F.3 page 240.*

PRP
$$\mathbf{D}^j\mathbf{T}^n[\mathsf{f}\mathsf{g}] = 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}\right] \qquad \forall j,n\in\mathbb{Z}, \mathsf{f}\in\mathbb{C}^{\mathbb{C}}$$

✎PROOF:

$$\begin{aligned}
\mathbf{D}^j\mathbf{T}^n[\mathsf{f}(x)\mathsf{g}(x)] &= 2^{j/2}\mathsf{f}\left(2^jx-n\right)\mathsf{g}\left(2^jx-n\right) && \text{by Proposition F.3 page 241}\\
&= 2^{-j/2}\left[2^{j/2}\mathsf{f}\left(2^jx-n\right)\right]\left[2^{j/2}\mathsf{g}\left(2^jx-n\right)\right]\\
&= 2^{-j/2}\left[\mathbf{D}^j\mathbf{T}^n\mathsf{f}(x)\right]\left[\mathbf{D}^j\mathbf{T}^n\mathsf{g}(x)\right] && \text{by Proposition F.3 page 241}
\end{aligned}$$

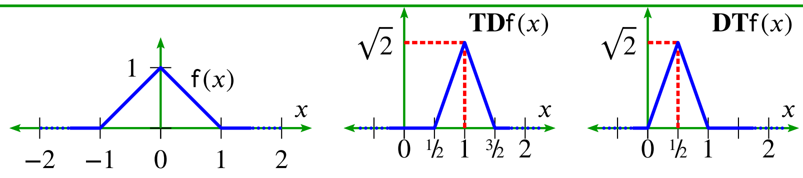
✏

In general the operators $\mathbf{T}$ and $\mathbf{D}$ are *noncommutative* ($\mathbf{TD} \neq \mathbf{DT}$), as demonstrated by Counterexample F.1 (next) and Proposition F.5 (page 242).

Counterexample F.1.

CNT

As illustrated to the right, it is **not** always true that
$\mathbf{TD} = \mathbf{DT}$:



Proposition F.5 (commutator relation). [4] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition F.3 page 240.*

PRP

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j && \forall j,n\in\mathbb{Z} \\ \mathbf{T}^n\mathbf{D}^j &= \mathbf{D}^j\mathbf{T}^{2^j n} && \forall n,j\in\mathbb{Z} \end{aligned}$$

✎PROOF:

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^{2^j n}\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - 2^j n) && \text{by Proposition F.4 page 241} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j[x-n]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition ?? page ??)} \\ &= \mathbf{T}^n 2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition F.3 page 240)} \\ &= \mathbf{T}^n\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition F.3 page 240)} \end{aligned}$$

$$\begin{aligned} \mathbf{D}^j\mathbf{T}^n\mathsf{f}(x) &= 2^{j/2}\,\mathsf{f}(2^j x - n) && \text{by Proposition F.4 page 241} \\ &= 2^{j/2}\,\mathsf{f}\left(2^j\left[x-2^{-j/2}n\right]\right) && \text{by } \textit{distributivity} \text{ of the field } (\mathbb{R},+,\cdot,0,1) && \text{(Definition ?? page ??)} \\ &= \mathbf{T}^{2^{-j/2}n}2^{j/2}\,\mathsf{f}\left(2^j x\right) && \text{by definition of } \mathbf{T} && \text{(Definition F.3 page 240)} \\ &= \mathbf{T}^{2^{-j/2}n}\mathbf{D}^j\mathsf{f}(x) && \text{by definition of } \mathbf{D} && \text{(Definition F.3 page 240)} \end{aligned}$$

✏

F.4 Inner product space properties

In an inner product space, every operator has an *adjoint* and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator $\mathbf{U}$ coincide, then $\mathbf{U}$ is said to be *unitary*. And in this case, $\mathbf{U}$ has several nice properties (see Proposition F.9 and Theorem F.1 page 245). Proposition F.6 (next) gives the adjoints of $\mathbf{D}$ and $\mathbf{T}$, and Proposition F.7 (page 243) demonstrates that both $\mathbf{D}$ and $\mathbf{T}$ are unitary. Other examples of unitary operators include the *Fourier Transform operator* $\tilde{\mathbf{F}}$ and the *rotation matrix operator*.

Proposition F.6. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition F.3 page 240) with* ADJOINT $\mathbf{T}^*$ *and* $\mathbf{D}$ *the* DILATION OPERATOR *with* ADJOINT $\mathbf{D}^*$.

PRP

$$\begin{aligned} \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(TRANSLATION OPERATOR ADJOINT)} \\ \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\,\mathsf{f}\left(\frac{1}{2}x\right) && \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} && \text{(DILATION OPERATOR ADJOINT)} \end{aligned}$$

✎PROOF:

[4] Christensen (2003) page 42 ⟨equation (2.9)⟩, Dai and Larson (1998) page 21, Goodman et al. (1993a), page 641, Goodman et al. (1993b), page 110

1. Proof that $\mathbf{T}^*\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$
\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{T}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{T}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{T}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of adjoint } \mathbf{T}^* \\
&= \langle \mathsf{g}(u-1) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{T} \quad \text{(Definition F.3 page 240)} \\
&= \langle \mathsf{g}(x) \,|\, \mathsf{f}(x+1)\rangle && \text{where } x \triangleq u-1 \implies u = x+1 \\
\implies \mathbf{T}^*\mathsf{f}(x) &= \mathsf{f}(x+1)
\end{aligned}
$$

2. Proof that $\mathbf{D}^*\mathsf{f}(x) = \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$
\begin{aligned}
\langle \mathsf{g}(x) \,|\, \mathbf{D}^*\mathsf{f}(x)\rangle &= \langle \mathsf{g}(u) \,|\, \mathbf{D}^*\mathsf{f}(u)\rangle && \text{by change of variable } x \to u \\
&= \langle \mathbf{D}\mathsf{g}(u) \,|\, \mathsf{f}(u)\rangle && \text{by definition of } \mathbf{D}^* \\
&= \left\langle \sqrt{2}\mathsf{g}(2u) \,|\, \mathsf{f}(u)\right\rangle && \text{by definition of } \mathbf{D} \quad \text{(Definition F.3 page 240)} \\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{g}(2u)\mathsf{f}^*(u)\,\mathrm{d}u && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
&= \int_{x\in\mathbb{R}} \mathsf{g}(x)\left[\sqrt{2}\mathsf{f}\left(\frac{x}{2}\right)\frac{1}{2}\right]^* \mathrm{d}x && \text{where } x = 2u \\
&= \left\langle \mathsf{g}(x) \,|\, \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)\right\rangle && \text{by definition of } \langle \triangle \,|\, \nabla\rangle \\
\implies \mathbf{D}^*\mathsf{f}(x) &= \frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{x}{2}\right)
\end{aligned}
$$

✏

Proposition F.7. [5] *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition F.3 (page 240). Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition F.2 (page 240).*

PRP
$\mathbf{T}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $(\mathbf{T}^{-1} = \mathbf{T}^*$ *in* $L^2_{\mathbb{R}})$.
$\mathbf{D}$ *is* UNITARY *in* $L^2_{\mathbb{R}}$ $(\mathbf{D}^{-1} = \mathbf{D}^*$ *in* $L^2_{\mathbb{R}})$.

✎PROOF:

$$
\begin{aligned}
\mathbf{T}^{-1} &= \mathbf{T}^* && \text{by Proposition F.2 page 240 and Proposition F.6 page 242} \\
&\implies \mathbf{T} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D}^{-1} &= \mathbf{D}^* && \text{by Proposition F.2 page 240 and Proposition F.6 page 242} \\
&\implies \mathbf{D} \text{ is } \textit{unitary} && \text{by the definition of } \textit{unitary} \text{ operators}
\end{aligned}
$$

✏

F.5 Normed linear space properties

Proposition F.8. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR *(Definition F.3 page 240).*

PRP
$$
\left\{\begin{array}{lll} (1). & \mathbf{D}\mathsf{f}(x) = \sqrt{2}\mathsf{f}(x) & \textit{and} \\ (2). & \mathsf{f}(x) \textit{ is } \text{CONTINUOUS} & \end{array}\right\} \iff \{\mathsf{f}(x) \textit{ is a } \text{CONSTANT}\} \qquad \forall \mathsf{f}\in L^2_{\mathbb{R}}
$$

✎PROOF:

[5] Christensen (2003) page 41 ⟨Lemma 2.5.1⟩, Wojtaszczyk (1997) page 18 ⟨Lemma 2.5⟩

1. Proof that (1) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\mathbf{D}\mathsf{f}(x) &\triangleq \sqrt{2}\mathsf{f}(2x) && \text{by definition of } \mathbf{D} && \text{(Definition F.3 page 240)}\\
&= \sqrt{2}\mathsf{f}(x) && \text{by } \textit{constant} \text{ hypothesis}
\end{aligned}$$

2. Proof that (2) $\Longleftarrow$ *constant* property:

$$\begin{aligned}
\|\mathsf{f}(x)-\mathsf{f}(x+h)\| &= \|\mathsf{f}(x)-\mathsf{f}(x)\| && \text{by } \textit{constant} \text{ hypothesis}\\
&= \|0\|\\
&= 0 && \text{by } \textit{nondegenerate} \text{ property of } \|\cdot\|\\
&\le \varepsilon\\
&\implies \forall h>0,\ \exists\varepsilon \quad \text{such that} \quad \|\mathsf{f}(x)-\mathsf{f}(x+h)\| < \varepsilon\\
&\overset{\text{def}}{\iff} \mathsf{f}(x) \text{ is } \textit{continuous}
\end{aligned}$$

3. Proof that (1,2) $\Longrightarrow$ *constant* property:

   (a) Suppose there exists $x, y \in \mathbb{R}$ such that $\mathsf{f}(x) \neq \mathsf{f}(y)$.

   (b) Let $(\!(x_n)\!)_{n\in\mathbb{N}}$ be a sequence with limit x and $(\!(y_n)\!)_{n\in\mathbb{N}}$ a sequence with limit y

   (c) Then

$$\begin{aligned}
0 &< \|\mathsf{f}(x)-\mathsf{f}(y)\| && \text{by assumption in item (3a) page 244}\\
&= \lim_{n\to\infty}\left\|\mathsf{f}(x_n)-\mathsf{f}(y_n)\right\| && \text{by (2) and definition of } (\!(x_n)\!) \text{ and } (\!(y_n)\!) \text{ in item (3b) page 244}\\
&= \lim_{n\to\infty}\left\|\mathsf{f}(2^m x_n)-\mathsf{f}(2^\ell y_n)\right\| \quad \forall m,\ell\in\mathbb{Z} && \text{by (1)}\\
&= 0
\end{aligned}$$

   (d) But this is a *contradiction*, so $\mathsf{f}(x) = \mathsf{f}(y)$ for all $x, y \in \mathbb{R}$, and $\mathsf{f}(x)$ is *constant*.

✏

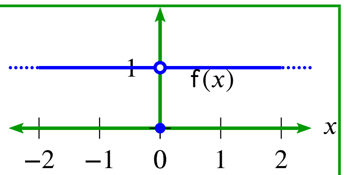
Remark F.1.

REM In Proposition F.8 page 243, it is not possible to remove the *continuous* constraint outright, as demonstrated by the next two counterexamples.

Counterexample F.2. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 0 & \text{for } x = 0\\ 1 & \text{otherwise.}\end{cases}$

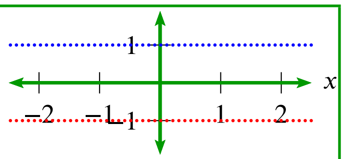
Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Counterexample F.3. Let $\mathsf{f}(x)$ be a function in $\mathbb{R}^{\mathbb{R}}$.
Let $\mathbb{Q}$ be the set of *rational numbers* and $\mathbb{R}\backslash\mathbb{Q}$ the set of *irrational numbers*.

CNT Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } x \in \mathbb{Q}\\ -1 & \text{for } x \in \mathbb{R}\backslash\mathbb{Q}.\end{cases}$

Then $\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) = \sqrt{2}\mathsf{f}(x)$, but $\mathsf{f}(x)$ is *not constant*.



Proposition F.9 (Operator norm). *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition F.3 page 240. Let* $\mathbf{T}^{-1}$ *and* $\mathbf{D}^{-1}$ *be as in Proposition F.2 page 240. Let* $\mathbf{T}^*$ *and* $\mathbf{D}^*$ *be as in Proposition F.6 page 242. Let* $\|\cdot\|$ *and* $\langle \triangle \mid \nabla \rangle$ *be as in Definition* **??** *page* **??**. *Let* $|\!|\!|\cdot|\!|\!|$ *be the operator norm induced by* $\|\cdot\|$.

PRP $$|\!|\!|\mathbf{T}|\!|\!| = |\!|\!|\mathbf{D}|\!|\!| = |\!|\!|\mathbf{T}^*|\!|\!| = |\!|\!|\mathbf{D}^*|\!|\!| = |\!|\!|\mathbf{T}^{-1}|\!|\!| = |\!|\!|\mathbf{D}^{-1}|\!|\!| = 1$$

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* and from properties of unitary operators. ✏

Theorem F.1. *Let $\mathbf{T}$ and $\mathbf{D}$ be as in Definition F.3 page 240. Let $\mathbf{T}^{-1}$ and $\mathbf{D}^{-1}$ be as in Proposition F.2 page 240. Let $\|\cdot\|$ and $\langle \triangle \mid \nabla \rangle$ be as in Definition ?? page ??.*

THM

1.	$\|\mathbf{T}f\|$	$=\|\mathbf{D}f\|$	$=\|f\|$	$\forall f \in L^2_{\mathbb{R}}$	(ISOMETRIC IN LENGTH)
2.	$\|\mathbf{T}f - \mathbf{T}g\|$	$=\|\mathbf{D}f - \mathbf{D}g\|$	$=\|f-g\|$	$\forall f,g \in L^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
3.	$\left\|\mathbf{T}^{-1}f - \mathbf{T}^{-1}g\right\|$	$=\left\|\mathbf{D}^{-1}f - \mathbf{D}^{-1}g\right\|$	$=\|f-g\|$	$\forall f,g \in L^2_{\mathbb{R}}$	(ISOMETRIC IN DISTANCE)
4.	$\langle \mathbf{T}f \mid \mathbf{T}g\rangle$	$=\langle \mathbf{D}f \mid \mathbf{D}g\rangle$	$=\langle f \mid g\rangle$	$\forall f,g \in L^2_{\mathbb{R}}$	(SURJECTIVE)
5.	$\left\langle \mathbf{T}^{-1}f \mid \mathbf{T}^{-1}g\right\rangle$	$=\left\langle \mathbf{D}^{-1}f \mid \mathbf{D}^{-1}g\right\rangle$	$=\langle f \mid g\rangle$	$\forall f,g \in L^2_{\mathbb{R}}$	(SURJECTIVE)

✎PROOF: These results follow directly from the fact that $\mathbf{T}$ and $\mathbf{D}$ are *unitary* (Proposition F.7 page 243) and from properties of unitary operators. ✏

Proposition F.10. *Let $\mathbf{T}$ be as in Definition F.3 page 240. Let $\mathbf{A}^*$ be the* ADJOINT *of an operator $\mathbf{A}$.*

PRP

$$\left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) = \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* \qquad \left(\text{The operator } \left[\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right] \text{ is SELF-ADJOINT}\right)$$

✎PROOF:

$$\begin{aligned}
\left\langle \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) f(x) \,\middle|\, g(x)\right\rangle &= \left\langle \sum_{n\in\mathbb{Z}} f(x-n) \,\middle|\, g(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition F.3 page 240)}\\
&= \left\langle \sum_{n\in\mathbb{Z}} f(x+n) \,\middle|\, g(x)\right\rangle && \text{by } \textit{commutative} \text{ property} && \text{(Definition ?? page ??)}\\
&= \sum_{n\in\mathbb{Z}} \langle f(x+n) \mid g(x)\rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \sum_{n\in\mathbb{Z}} \langle f(u) \mid g(u-n)\rangle && \text{where } u \triangleq x+n\\
&= \left\langle f(u) \,\middle|\, \sum_{n\in\mathbb{Z}} g(u-n)\right\rangle && \text{by } \textit{additive} \text{ property of } \langle \triangle \mid \nabla \rangle\\
&= \left\langle f(x) \,\middle|\, \sum_{n\in\mathbb{Z}} g(x-n)\right\rangle && \text{by change of variable: } u \to x\\
&= \left\langle f(x) \,\middle|\, \sum_{n\in\mathbb{Z}} \mathbf{T}^n g(x)\right\rangle && \text{by definition of } \mathbf{T} && \text{(Definition F.3 page 240)}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) &= \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right)^* && \text{by definition of } \textit{adjoint}\\
\iff \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\right) & \text{ is } \textit{self-adjoint} && \text{by definition of } \textit{self-adjoint}
\end{aligned}$$

✏

F.6 Fourier transform properties

Proposition F.11. *Let $\mathbf{T}$ and $\mathbf{D}$ be as in Definition F.3 page 240. Let $\mathbf{B}$ be the* TWO-SIDED LAPLACE TRANSFORM *defined as $[\mathbf{B}f](s) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-sx}\, dx$.*

PRP

1. $\mathbf{BT}^n = e^{-sn}\mathbf{B}$ $\quad\forall n\in\mathbb{Z}$
2. $\mathbf{BD}^j = \mathbf{D}^{-j}\mathbf{B}$ $\quad\forall j\in\mathbb{Z}$
3. $\mathbf{DB} = \mathbf{BD}^{-1}$ $\quad\forall n\in\mathbb{Z}$
4. $\mathbf{BD}^{-1}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} = \mathbf{D}$ $\quad\forall n\in\mathbb{Z}$ $\quad$ ($\mathbf{D}^{-1}$ *is* SIMILAR *to* $\mathbf{D}$)
5. $\mathbf{DBD} = \mathbf{D}^{-1}\mathbf{BD}^{-1} = \mathbf{B}$ $\quad\forall n\in\mathbb{Z}$

✎PROOF:

$$
\begin{aligned}
\mathbf{BT}^n\mathsf{f}(x) &= \mathbf{B}\mathsf{f}(x-n) && \text{by definition of } \mathbf{T} && \text{(Definition F.3 page 240)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x-n)e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s(u+n)}\,\mathrm{d}u && \text{where } u \triangleq x-n\\
&= e^{-sn}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right]\\
&= e^{-sn}\,\mathbf{B}\mathsf{f}(x) && \text{by definition of } \mathbf{B}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{BD}^j\mathsf{f}(x) &= \mathbf{B}\left[2^{j/2}\,\mathsf{f}\left(2^jx\right)\right] && \text{by definition of } \mathbf{D} && \text{(Definition F.3 page 240)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}\left(2^jx\right)\right]e^{-sx}\,\mathrm{d}x && \text{by definition of } \mathbf{B}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[2^{j/2}\,\mathsf{f}(u)\right]e^{-s2^{-j}}2^{-j}\,\mathrm{d}u && \text{let } u \triangleq 2^jx \implies x = 2^{-j}u\\
&= \frac{\sqrt{2}}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-s2^{-j}u}\,\mathrm{d}u\\
&= \mathbf{D}^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(u)e^{-su}\,\mathrm{d}u\right] && \text{by Proposition F.6 page 242 and} && \text{Proposition F.7 page 243}\\
&= \mathbf{D}^{-j}\,\mathbf{B}\,\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{DB}\,\mathsf{f}(x) &= \mathbf{D}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-sx}\,\mathrm{d}x\right] && \text{by definition of } \mathbf{B}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}(x)e^{-2sx}\,\mathrm{d}x && \text{by definition of } \mathbf{D} && \text{(Definition F.3 page 240)}\\
&= \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathsf{f}\left(\frac{u}{2}\right)e^{-su}\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies x = \tfrac{1}{2}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\frac{\sqrt{2}}{2}\mathsf{f}\left(\frac{u}{2}\right)\right]e^{-su}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\mathbf{D}^{-1}\mathsf{f}\right](u)\,e^{-su}\,\mathrm{d}u && \text{by Proposition F.6 page 242 and} && \text{Proposition F.7 page 243}\\
&= \mathbf{BD}^{-1}\mathsf{f}(x) && \text{by definition of } \mathbf{B}\\
\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{BD} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse}\\
\mathbf{BD}^{-1}\mathbf{B}^{-1} &= \mathbf{DBB}^{-1} && \text{by previous result}\\
&= \mathbf{D} && \text{by definition of operator inverse}\\
\mathbf{DBD} &= \mathbf{DD}^{-1}\mathbf{B} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse}\\
\mathbf{D}^{-1}\mathbf{BD}^{-1} &= \mathbf{D}^{-1}\mathbf{DB} && \text{by previous result}\\
&= \mathbf{B} && \text{by definition of operator inverse}
\end{aligned}
$$

Corollary F.1. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition F.3 page 240. Let* $\tilde{f}(\omega) \triangleq \tilde{\mathbf{F}}f(x)$ *be the* FOURIER TRANSFORM *(Definition E.2 page 229) of some function* $f \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition ?? page ??).*

COR

$$\begin{array}{llcll}
1. & \tilde{\mathbf{F}}\mathbf{T}^n & = & e^{-i\omega n}\tilde{\mathbf{F}} & \\
2. & \tilde{\mathbf{F}}\mathbf{D}^j & = & \mathbf{D}^{-j}\tilde{\mathbf{F}} & \\
3. & \mathbf{D}\tilde{\mathbf{F}} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1} & \\
4. & \mathbf{D} & = & \tilde{\mathbf{F}}\mathbf{D}^{-1}\tilde{\mathbf{F}}^{-1} & = \tilde{\mathbf{F}}^{-1}\mathbf{D}^{-1}\tilde{\mathbf{F}} \\
5. & \tilde{\mathbf{F}} & = & \mathbf{D}\tilde{\mathbf{F}}\mathbf{D} & = \mathbf{D}^{-1}\tilde{\mathbf{F}}\mathbf{D}^{-1}
\end{array}$$

✎PROOF: These results follow directly from Proposition F.11 page 245 with $\tilde{\mathbf{F}} = \mathbf{B}|_{s=i\omega}$. ✏

Proposition F.12. *Let* $\mathbf{T}$ *and* $\mathbf{D}$ *be as in Definition F.3 page 240. Let* $\tilde{f}(\omega) \triangleq \tilde{\mathbf{F}}f(x)$ *be the* FOURIER TRANSFORM *(Definition E.2 page 229) of some function* $f \in \boldsymbol{L}^2_{\mathbb{R}}$ *(Definition ?? page ??).*

PRP

$$\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n f(x) = \frac{1}{2^{j/2}}e^{-i\frac{\omega}{2^j}n}\tilde{f}\left(\frac{\omega}{2^j}\right)$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\mathbf{D}^j\mathbf{T}^n f(x) &= \mathbf{D}^{-j}\tilde{\mathbf{F}}\mathbf{T}^n f(x) && \text{by Corollary F.1 page 247 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{\mathbf{F}}f(x) && \text{by Corollary F.1 page 247 (3)}\\
&= \mathbf{D}^{-j}e^{-i\omega n}\tilde{f}(\omega) && \\
&= 2^{-j/2}e^{-i2^{-j}\omega n}\tilde{f}\left(2^{-j}\omega\right) && \text{by Proposition F.2 page 240}
\end{aligned}$$

✏

Proposition F.13. *Let* $\mathbf{T}$ *be the translation operator (Definition F.3 page 240). Let* $\tilde{f}(\omega) \triangleq \tilde{\mathbf{F}}f(x)$ *be the* FOURIER TRANSFORM *(Definition E.2 page 229) of a function* $f \in \boldsymbol{L}^2_{\mathbb{R}}$*. Let* $\breve{a}(\omega)$ *be the* DTFT *(Definition ?? page ??) of a sequence* $(a_n)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2_{\mathbb{R}}$ *(Definition ?? page ??).*

PRP

$$\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) = \breve{a}(\omega)\tilde{\phi}(\omega) \qquad \forall (a_n)\in\boldsymbol{\ell}^2_{\mathbb{R}}, \phi(x)\in\boldsymbol{L}^2_{\mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\tilde{\mathbf{F}}\sum_{n\in\mathbb{Z}} a_n\mathbf{T}^n\phi(x) &= \sum_{n\in\mathbb{Z}} a_n\tilde{\mathbf{F}}\mathbf{T}^n\phi(x) && \\
&= \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\tilde{\mathbf{F}}\phi(x) && \text{by Corollary F.1 page 247}\\
&= \left[\sum_{n\in\mathbb{Z}} a_n e^{-i\omega n}\right]\tilde{\phi}(\omega) && \text{by definition of } \tilde{\phi}(\omega)\\
&= \breve{a}(\omega)\tilde{\phi}(\omega) && \text{by definition of } \textit{DTFT} \text{ (Definition ?? page ??)}
\end{aligned}$$

✏

Definition F.4. *Let* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *be the* SPACE OF LEBESGUE SQUARE-INTEGRABLE FUNCTIONS *(Definition ?? page ??). Let* $\boldsymbol{\ell}^2_{\mathbb{R}}$ *be the* SPACE OF ALL ABSOLUTELY SQUARE SUMMABLE SEQUENCES OVER $\mathbb{R}$ *(Definition ?? page ??).*

DEF

$\mathbf{S}$ *is the **sampling operator** in* $\boldsymbol{\ell}^2_{\mathbb{R}}{}^{\boldsymbol{L}^2_{\mathbb{R}}}$ *if* $\quad [\mathbf{S}f(x)](n) \triangleq f\left(\frac{2\pi}{\tau}n\right) \qquad \forall f\in\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}, \tau\in\mathbb{R}^+$

Theorem F.2 (Poisson Summation Formula—PSF). [6] *Let* $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition E.2 page 229) *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\mathbf{S}$ *be the* SAMPLING OPERATOR (Definition F.4 page 247).

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_\tau \mathsf{f}(x) = \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)}_{\text{summation in "time"}} = \underbrace{\sqrt{\frac{2\pi}{\tau}}\, \hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}[\mathsf{f}(x)]}_{\text{operator notation}} = \underbrace{\frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx}}_{\text{summation in "frequency"}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(x) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)$ then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:

   Note that $\mathsf{h}(x)$ is *periodic* with period τ. Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and thus $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of PSF (this theorem—Theorem F.2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) &= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau) && \text{by (1) lemma page 248} \\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{\tau}}\int_0^\tau \left(\sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)\right)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right]}_{\hat{\mathbf{F}}[\sum_{n\in\mathbb{Z}}\mathsf{f}(x+n\tau)]} && \text{by definition of } \hat{\mathbf{F}} \quad \text{(Definition ?? page ??)} \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_0^\tau \mathsf{f}(x+n\tau)e^{-i\frac{2\pi}{\tau}kx}\,\mathrm{d}x\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}k(u-n\tau)}\,\mathrm{d}u\right] && \text{where } u \triangleq x+n\tau \implies x = u-n\tau \\
&= \hat{\mathbf{F}}^{-1}\left[\frac{1}{\sqrt{\tau}}\sum_{n\in\mathbb{Z}}\cancelto{1}{e^{i2\pi kn}}\int_{u=n\tau}^{u=(n+1)\tau} \mathsf{f}(u)e^{-i\frac{2\pi}{\tau}ku}\,\mathrm{d}u\right] \\
&= \sqrt{\frac{2\pi}{\tau}}\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{u\in\mathbb{R}} \mathsf{f}(u)e^{-i\left(\frac{2\pi}{\tau}k\right)u}\,\mathrm{d}u\right]}_{[\tilde{\mathbf{F}}\mathsf{f}]\left(\frac{2\pi}{\tau}k\right)} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem ?? page ??)} \\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\left[\left[\tilde{\mathbf{F}}\mathsf{f}(x)\right]\left(\frac{2\pi}{\tau}k\right)\right] && \text{by definition of } \tilde{\mathbf{F}} \quad \text{(Definition E.2 page 229)} \\
&= \sqrt{\frac{2\pi}{\tau}}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}\mathsf{f} && \text{by definition of } \mathbf{S} \quad \text{(Definition F.4 page 247)} \\
&= \frac{\sqrt{2\pi}}{\tau}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\frac{2\pi}{\tau}n\right)e^{i\frac{2\pi}{\tau}nx} && \text{by evaluation of } \hat{\mathbf{F}}^{-1} \quad \text{(Theorem ?? page ??)}
\end{aligned}$$

✏

Theorem F.3 (Inverse Poisson Summation Formula—IPSF). [7]
Let $\tilde{\mathsf{f}}(\omega)$ *be the* FOURIER TRANSFORM (Definition E.2 page 229) *of a function* $\mathsf{f}(x) \in \boldsymbol{L}^2_{\mathbb{R}}$.

[6] Andrews et al. (2001), page 624, Knapp (2005b) page 389, Lasser (1996), page 254, Rudin (1987), pages 194–195, Folland (1992), page 337

[7] Gauss (1900), page 88

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n_{2\pi/\tau}\tilde{\mathsf{f}}(\omega) \triangleq \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega-\frac{2\pi}{\tau}n\right)}_{\text{summation in ``frequency''}} \quad = \quad \underbrace{\frac{\tau}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \mathsf{f}(n\tau)e^{-i\omega n\tau}}_{\text{summation in ``time''}}$$

✎PROOF:

1. lemma: If $\mathsf{h}(\omega) \triangleq \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)$, then $\mathsf{h} \equiv \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$. Proof:
   Note that $\mathsf{h}(\omega)$ is periodic with period $2\pi/T$:
   $$\mathsf{h}\left(\omega+\frac{2\pi}{\tau}\right) \triangleq \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}+\frac{2\pi}{\tau}n\right) = \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+(n+1)\frac{2\pi}{\tau}\right) = \sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right) \triangleq \mathsf{h}(\omega)$$
   Because h is periodic, it is in the domain of $\hat{\mathbf{F}}$ and is equivalent to $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\mathsf{h}$.

2. Proof of IPSF (this theorem—Theorem F.3):

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right) \\
&= \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right) && \text{by (1) lemma page 249} \\
&= \hat{\mathbf{F}}^{-1}\underbrace{\left[\sqrt{\frac{\tau}{2\pi}}\int_0^{\frac{2\pi}{\tau}}\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)e^{-i\omega\frac{2\pi}{2\pi/\tau}k}\,\mathsf{d}\omega\right]}_{\hat{\mathbf{F}}\left[\sum_{n\in\mathbb{Z}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)\right]} && \text{by definition of }\hat{\mathbf{F}} && \text{(Definition ?? page ??)} \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{\tau}}\tilde{\mathsf{f}}\left(\omega+\frac{2\pi}{\tau}n\right)e^{-i\omega Tk}\,\mathsf{d}\omega\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\int_{u=\frac{2\pi}{\tau}n}^{u=\frac{2\pi}{\tau}(n+1)}\tilde{\mathsf{f}}(u)e^{-i\left(u-\frac{2\pi}{\tau}n\right)Tk}\,\mathsf{d}u\right] && \text{where } u \triangleq \omega+\frac{2\pi}{\tau}n \implies && \omega = u-\frac{2\pi}{\tau}n \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\sum_{n\in\mathbb{Z}}\cancel{e^{i2\pi nk}}^{\,1}\int_{\frac{2\pi}{\tau}n}^{\frac{2\pi}{\tau}(n+1)}\tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathsf{d}u\right] \\
&= \hat{\mathbf{F}}^{-1}\left[\sqrt{\frac{\tau}{2\pi}}\int_{\mathbb{R}}\tilde{\mathsf{f}}(u)e^{-iu\tau k}\,\mathsf{d}u\right] \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\tilde{\mathsf{f}}(u)e^{iu(-\tau k)}\,\mathsf{d}u\right]}_{[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\left[[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{f}}](-k\tau)\right] && \text{by value of }\tilde{\mathbf{F}}^{-1} && \text{(Theorem E.1 page 230)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\tilde{\mathbf{F}}^{-1}\,\tilde{\mathsf{f}} && \text{by definition of }\mathbf{S} && \text{(Definition F.4 page 247)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathbf{S}\mathsf{f}(x) && \text{by definition of }\tilde{\mathbf{F}} && \text{(Definition E.2 page 229)} \\
&= \sqrt{\tau}\,\hat{\mathbf{F}}^{-1}\mathsf{f}(-k\tau) && \text{by definition of }\mathbf{S} && \text{(Definition F.4 page 247)} \\
&= \sqrt{\tau}\frac{1}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}}\mathsf{f}(-k\tau)e^{i2\pi\frac{1}{\frac{2\pi}{\tau}}k\omega} && \text{by definition of }\hat{\mathbf{F}}^{-1} && \text{(Theorem ?? page ??)} \\
&= \frac{\tau}{\sqrt{\frac{2\pi}{\tau}}}\sum_{k\in\mathbb{Z}}\mathsf{f}(-k\tau)e^{ik\tau\omega} && \text{by definition of }\hat{\mathbf{F}}^{-1} && \text{(Theorem ?? page ??)} \\
&= \frac{\tau}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\mathsf{f}(m\tau)e^{-i\omega m\tau} && \text{let } m \triangleq -k
\end{aligned}$$

Remark F.2. The left hand side of the *Poisson Summation Formula* (Theorem F.2 page 248) is very similar to the *Zak Transform* $\mathbf{Z}$: [8]

$$(\mathbf{Z}\mathsf{f})(t,\omega) \triangleq \sum_{n\in\mathbb{Z}} \mathsf{f}(x+n\tau)e^{i2\pi n\omega}$$

Remark F.3. A generalization of the *Poisson Summation Formula* (Theorem F.2 page 248) is the **Selberg Trace Formula**.[9]

F.7 Examples

Example F.2 (linear functions). [10] Let $\mathbf{T}$ be the *translation operator* (Definition F.3 page 240). Let $\mathcal{L}(\mathbb{C},\mathbb{C})$ be the set of all *linear* functions in $\boldsymbol{L}^2_{\mathbb{R}}$.

EX
1. $\{x,\ \mathbf{T}x\}$ is a *basis* for $\mathcal{L}(\mathbb{C},\mathbb{C})$ and
2. $\mathsf{f}(x) = \mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x$ $\quad\forall \mathsf{f} \in \mathcal{L}(\mathbb{C},\mathbb{C})$

✎PROOF: By left hypothesis, f is *linear*; so let $\mathsf{f}(x) \triangleq ax+b$

$$\begin{aligned}
\mathsf{f}(1)x - \mathsf{f}(0)\mathbf{T}x &= \mathsf{f}(1)x - \mathsf{f}(0)(x-1) && \text{by Definition F.3 page 240}\\
&= (ax+b)|_{x=1}\, x - (ax+b)|_{x=0}\,(x-1) && \text{by left hypothesis and definition of f}\\
&= (a+b)x - b(x-1)\\
&= ax + bx - bx + b\\
&= ax+b\\
&= \mathsf{f}(x) && \text{by left hypothesis and definition of f}
\end{aligned}$$

Example F.3 (Cardinal Series). Let $\mathbf{T}$ be the *translation operator* (Definition F.3 page 240). The *Paley-Wiener* class of functions $\boldsymbol{PW}^2_\sigma$ are those functions which are "*bandlimited*" with respect to their Fourier transform. The cardinal series forms an orthogonal basis for such a space. The *Fourier coefficients* for a projection of a function f onto the Cardinal series basis elements is particularly simple—these coefficients are samples of $\mathsf{f}(x)$ taken at regular intervals. In fact, one could represent the coefficients using inner product notation with the *Dirac delta distribution* δ as follows:

$$\langle \mathsf{f}(x) \,|\, \mathbf{T}^n\delta(x)\rangle \triangleq \int_{\mathbb{R}} \mathsf{f}(x)\delta(x-n)\,\mathsf{d}x \triangleq \mathsf{f}(n)$$

EX
1. $\left\{ \mathbf{T}^n \frac{\sin(\pi x)}{\pi x} \middle|\, n\in\mathbb{N} \right\}$ is a *basis* for $\boldsymbol{PW}^2_\sigma$ and
2. $\mathsf{f}(x) = \underbrace{\sum_{n=1}^{\infty} \mathsf{f}(n)\mathbf{T}^n \frac{\sin(\pi x)}{\pi x}}_{\textit{Cardinal series}}$ $\quad\forall \mathsf{f} \in \boldsymbol{PW}^2_\sigma,\ \sigma \le \frac{1}{2}$

Example F.4 (Fourier Series).

EX
1. $\left\{ \mathbf{D}_n e^{ix} \middle|\, n\in\mathbb{Z} \right\}$ is a *basis* for $\boldsymbol{L}(0:2\pi)$ and
2. $\mathsf{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{D}_n e^{ix}$ $\quad\forall x\in(0:2\pi), \mathsf{f}\in\boldsymbol{L}(0:2\pi)$ where
3. $\alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathsf{f}(x)\mathbf{D}_n e^{-ix}\,\mathsf{d}x$ $\quad\forall \mathsf{f}\in\boldsymbol{L}(0:2\pi)$

[8] Janssen (1988), page 24, Zayed (1996), page 482
[9] Lax (2002), page 349, Selberg (1956), Terras (1999)
[10] Higgins (1996) page 2

✎PROOF: See Theorem ?? page ??. ✏

Example F.5 (Fourier Transform). [11]

EX

1. $\left\{\mathbf{D}_\omega e^{ix} | \omega\in\mathbb{R}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} \tilde{\mathsf{f}}(\omega)\mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\tilde{\mathsf{f}}(\omega) \triangleq \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}_\omega e^{-ix}\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example F.6 (Gabor Transform). [12]

EX

1. $\left\{\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{ix}\right)\Big|\, \tau,\omega\in\mathbb{R}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \displaystyle\int_{\mathbb{R}} \mathsf{G}(\tau,\omega)\,\mathbf{D}_x e^{i\omega}\, \mathrm{d}\omega \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\mathsf{G}(\tau,\omega) \triangleq \displaystyle\int_{\mathbb{R}} \mathsf{f}(x)\left(\mathbf{T}_\tau e^{-\pi x^2}\right)\left(\mathbf{D}_\omega e^{-ix}\right)\, \mathrm{d}x \quad \forall x\in\mathbb{R}, \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

Example F.7 (wavelets). Let $\psi(x)$ be a *wavelet*.

EX

1. $\left\{\mathbf{D}^k\mathbf{T}^n\psi(x)\Big|\, k,n\in\mathbb{Z}\right\}$ is a *basis* for $\boldsymbol{L}^2_{\mathbb{R}}$ and
2. $\mathsf{f}(x) = \displaystyle\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \alpha_{k,n}\mathbf{D}^k\mathbf{T}^n\psi(x) \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$ where
3. $\alpha_n \triangleq \displaystyle\int_{\mathbb{R}} \mathsf{f}(x)\mathbf{D}^k\mathbf{T}^n\psi^*(x)\, \mathrm{d}x \quad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$

[11] cross reference: Definition E.2 page 229

[12] Gabor (1946), Qian and Chen (1996) ⟨Chapter 3⟩, Forster and Massopust (2009) page 32 ⟨Definition 1.69⟩

APPENDIX G

OPERATORS ON LINEAR SPACES

❝*And I am not afraid to say that there is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients.... we need still another analysis which is distinctly geometrical or linear, and which will express situation directly as algebra expresses magnitude directly.*❞

Gottfried Leibniz (1646–1716), German mathematician, in a September 8, 1679 letter to Christian Huygens. [1]

G.1 Operators on linear spaces

G.1.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition G.1. [2] *Let* $(\mathbb{F}, +, \cdot, 0, 1)$ *be a* FIELD. *Let* X *be a set, let* $+$ *be an* OPERATOR *(Definition G.2 page 254) in* X^{X^2}, *and let* $\otimes$ *be an operator in* $X^{\mathbb{F}\times X}$.

[1] quote: Leibniz (1679) pages 248–249
image: http://en.wikipedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg, public domain

[2] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991), page 41, Halmos (1948), pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b), pages 119–120, Banach (1922) pages 134–135

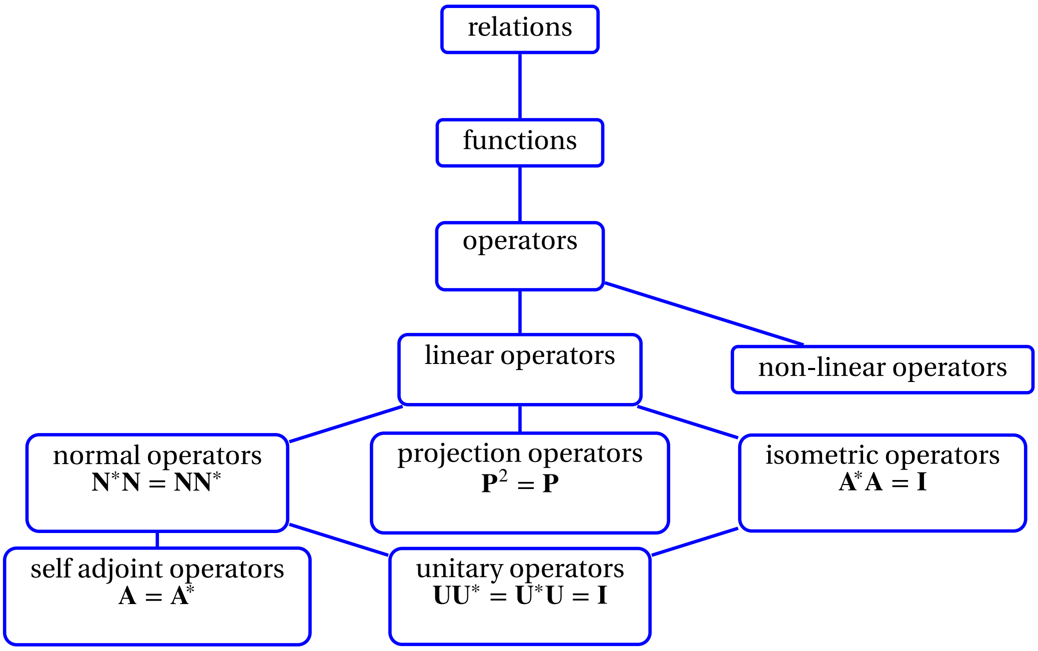


Figure G.1: Some operator types

DEF

The structure $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *is a* ***linear space*** *over* $(\mathbb{F}, +, \cdot, 0, 1)$ *if*

1.	$\exists \mathbb{0} \in X$ such that	$x + \mathbb{0}$	$=$	x	$\forall x \in X$	(+ IDENTITY)	*
2.	$\exists y \in X$ such that	$x + y$	$=$	$\mathbb{0}$	$\forall x \in X$	(+ INVERSE)	
3.		$(x + y) + z$	$=$	$x + (y + z)$	$\forall x,y,z \in X$	(+ *is* ASSOCIATIVE)	
4.		$x + y$	$=$	$y + x$	$\forall x,y \in X$	(+ *is* COMMUTATIVE)	
5.		$1 \cdot x$	$=$	x	$\forall x \in X$	($\cdot$ IDENTITY)	
6.		$\alpha \cdot (\beta \cdot x)$	$=$	$(\alpha \cdot \beta) \cdot x$	$\forall \alpha,\beta \in S$ *and* $x \in X$	($\cdot$ ASSOCIATES *with* $\cdot$)	
7.		$\alpha \cdot (x + y)$	$=$	$(\alpha \cdot x) + (\alpha \cdot y)$	$\forall \alpha \in S$ *and* $x,y \in X$	($\cdot$ DISTRIBUTES *over* +)	
8.		$(\alpha + \beta) \cdot x$	$=$	$(\alpha \cdot x) + (\beta \cdot x)$	$\forall \alpha,\beta \in S$ *and* $x \in X$	($\cdot$ PSEUDO-DISTRIBUTES *over* +)	

The set X *is called the* ***underlying set***. *The elements of* X *are called* ***vectors***. *The elements of* $\mathbb{F}$ *are called* ***scalars***. *A linear space is also called a* ***vector space***. *If* $\mathbb{F} \triangleq \mathbb{R}$, *then* $\boldsymbol{\Omega}$ *is a* ***real linear space***. *If* $\mathbb{F} \triangleq \mathbb{C}$, *then* $\boldsymbol{\Omega}$ *is a* ***complex linear space***.

Definition G.2. [3]

DEF

A function $\mathbf{A}$ *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *is an* ***operator*** *in* $\boldsymbol{Y}^{\boldsymbol{X}}$ *if*
$\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both* LINEAR SPACES (Definition G.1 page 253).

Two operators $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ are **equal** if $\mathbf{A}x = \mathbf{B}x$ for all $x \in \boldsymbol{X}$. The inverse relation of an operator $\mathbf{A}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ always exists as a *relation* in $2^{\boldsymbol{X}\boldsymbol{Y}}$, but may not always be a *function* (may not always be an operator) in $\boldsymbol{Y}^{\boldsymbol{X}}$.

The operator $\mathbf{I} \in \boldsymbol{X}^{\boldsymbol{X}}$ is the *identity* operator if $\mathbf{I}x = \mathbf{I}$ for all $x \in \boldsymbol{X}$.

Definition G.3. [4] *Let* $\boldsymbol{X}^{\boldsymbol{X}}$ *be the set of all operators with from a* LINEAR SPACE $\boldsymbol{X}$ *to* $\boldsymbol{X}$. *Let* $\mathbf{I}$ *be an operator in* $\boldsymbol{X}^{\boldsymbol{X}}$. *Let* $①(\boldsymbol{X})$ *be the* IDENTITY ELEMENT *in* $\boldsymbol{X}^{\boldsymbol{X}}$.

DEF

$\mathbf{I}$ *is the* ***identity operator*** *in* $\boldsymbol{X}^{\boldsymbol{X}}$ *if* $\qquad \mathbf{I} = ①(\boldsymbol{X})$.

[3] Heil (2011) page 42
[4] Michel and Herget (1993) page 411

G.1.2 Linear operators

Definition G.4. [5] *Let* $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *and* $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be linear spaces.*

DEF

An operator $\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *is* ***linear*** *if*

$$\begin{array}{rrcll} 1. & \mathbf{L}(\boldsymbol{x}+\boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x}+\mathbf{L}\boldsymbol{y} & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X} \quad \text{(ADDITIVE)} \quad \text{and} \\ 2. & \mathbf{L}(\alpha\boldsymbol{x}) & = & \alpha\mathbf{L}\boldsymbol{x} & \forall \boldsymbol{x} \in \boldsymbol{X}, \quad \forall \alpha \in \mathbb{F} \quad \text{(HOMOGENEOUS).} \end{array}$$

The set of all linear operators from $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *is denoted* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *such that*

$$\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \{\mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}} | \mathbf{L} \text{ is linear}\}.$$

Theorem G.1. [6] *Let* $\mathbf{L}$ *be an operator from a linear space* $\boldsymbol{X}$ *to a linear space* $\boldsymbol{Y}$*, both over a field* $\mathbb{F}$*.*

THM

$$\{\mathbf{L} \text{ is LINEAR}\} \implies \left\{\begin{array}{rrcll} 1. & \mathbf{L}\mathbb{0} & = & \mathbb{0} & \text{and} \\ 2. & \mathbf{L}(-\boldsymbol{x}) & = & -(\mathbf{L}\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \boldsymbol{X} & \text{and} \\ 3. & \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) & = & \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} \quad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X} & \text{and} \\ 4. & \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) & = & \sum_{n=1}^{N} \alpha_n (\mathbf{L}\boldsymbol{x}_n) \quad \boldsymbol{x}_n \in \boldsymbol{X}, \alpha_n \in \mathbb{F} & \end{array}\right\}$$

✎PROOF:

1. Proof that $\mathbf{L}\mathbb{0} = \mathbb{0}$:

$$\begin{aligned} \mathbf{L}\mathbb{0} &= \mathbf{L}(0 \cdot \mathbb{0}) && \text{by additive identity property} \\ &= 0 \cdot (\mathbf{L}\mathbb{0}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 255)} \\ &= \mathbb{0} && \text{by } \textit{additive identity} \text{ property} \end{aligned}$$

2. Proof that $\mathbf{L}(-\boldsymbol{x}) = -(\mathbf{L}\boldsymbol{x})$:

$$\begin{aligned} \mathbf{L}(-\boldsymbol{x}) &= \mathbf{L}(-1 \cdot \boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \\ &= -1 \cdot (\mathbf{L}\boldsymbol{x}) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 255)} \\ &= -(\mathbf{L}\boldsymbol{x}) && \text{by } \textit{additive inverse} \text{ property} \end{aligned}$$

3. Proof that $\mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) = \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}$:

$$\begin{aligned} \mathbf{L}(\boldsymbol{x}-\boldsymbol{y}) &= \mathbf{L}(\boldsymbol{x}+(-\boldsymbol{y})) && \text{by } \textit{additive inverse} \text{ property} \\ &= \mathbf{L}(\boldsymbol{x})+\mathbf{L}(-\boldsymbol{y}) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 255)} \\ &= \mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y} && \text{by item (2)} \end{aligned}$$

4. Proof that $\mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) = \sum_{n=1}^{N} \alpha_n (\mathbf{L}\boldsymbol{x}_n)$:

   (a) Proof for $N = 1$:

$$\begin{aligned} \mathbf{L}\left(\sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n\right) &= \mathbf{L}(\alpha_1 \boldsymbol{x}_1) && \text{by } N = 1 \text{ hypothesis} \\ &= \alpha_1 (\mathbf{L}\boldsymbol{x}_1) && \text{by } \textit{homogeneous} \text{ property of } \mathbf{L} && \text{(Definition G.4 page 255)} \end{aligned}$$

[5] Kubrusly (2001) page 55, Aliprantis and Burkinshaw (1998) page 224, Hilbert et al. (1927) page 6, Stone (1932) page 33

[6] Berberian (1961) page 79 ⟨Theorem IV.1.1⟩

(b) Proof that N case $\implies$ $N+1$ case:

$$\begin{aligned}
\mathbf{L}\left(\sum_{n=1}^{N+1}\alpha_n\boldsymbol{x}_n\right) &= \mathbf{L}\left(\alpha_{N+1}\boldsymbol{x}_{N+1}+\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right)\\
&= \mathbf{L}(\alpha_{N+1}\boldsymbol{x}_{N+1})+\mathbf{L}\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{x}_n\right) && \text{by } \textit{linearity} \text{ property of } \mathbf{L} \quad \text{(Definition G.4 page 255)}\\
&= \alpha_{N+1}\mathbf{L}(\boldsymbol{x}_{N+1})+\sum_{n=1}^{N}\mathbf{L}(\alpha_n\boldsymbol{x}_n) && \text{by left } N+1 \text{ hypothesis}\\
&= \sum_{n=1}^{N+1}\mathbf{L}(\alpha_n\boldsymbol{x}_n)
\end{aligned}$$

✏

Theorem G.2. [7] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of all linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM			
$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	*is a linear space*		*(space of linear transforms)*
$\mathcal{N}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{X}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	
$\mathcal{I}(\mathbf{L})$	*is a linear subspace of $\boldsymbol{Y}$*	$\forall \mathbf{L}\in\boldsymbol{Y}^{\boldsymbol{X}}$	

✎PROOF:

1. Proof that $\mathcal{N}(\mathbf{L})$ is a linear subspace of $\boldsymbol{X}$:
   (a) $\mathbb{0}\in\mathcal{N}(\mathbf{L}) \implies \mathcal{N}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{N}(\mathbf{L})\triangleq\{\boldsymbol{x}\in\boldsymbol{X}|\mathbf{L}\boldsymbol{x}=\mathbb{0}\}\subseteq\boldsymbol{X}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{N}(\mathbf{L}) \implies \mathbb{0}=\mathbf{L}(\boldsymbol{x}+\boldsymbol{y})=\mathbf{L}(\boldsymbol{y}+\boldsymbol{x}) \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\boldsymbol{X} \implies \mathbb{0}=\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\alpha\mathbf{L}\boldsymbol{x} \implies \mathbb{0}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{N}(\mathbf{L})$
2. Proof that $\mathcal{I}(\mathbf{L})$ is a linear subspace of $\boldsymbol{Y}$:
   (a) $\mathbb{0}\in\mathcal{I}(\mathbf{L}) \implies \mathcal{I}(\mathbf{L})\neq\varnothing$
   (b) $\mathcal{I}(\mathbf{L})\triangleq\{\boldsymbol{y}\in\boldsymbol{Y}|\exists\boldsymbol{x}\in\boldsymbol{X}\ \text{ such that }\ \boldsymbol{y}=\mathbf{L}\boldsymbol{x}\}\subseteq\boldsymbol{Y}$
   (c) $\boldsymbol{x}+\boldsymbol{y}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{v}\in\boldsymbol{X}\ \text{ such that }\ \mathbf{L}\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x} \implies \boldsymbol{y}+\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$
   (d) $\alpha\in\mathbb{F},\ \boldsymbol{x}\in\mathcal{I}(\mathbf{L}) \implies \exists\boldsymbol{x}\in\boldsymbol{X}\ \text{ such that }\ \boldsymbol{y}=\mathbf{L}\boldsymbol{x} \implies \alpha\boldsymbol{y}=\alpha\mathbf{L}\boldsymbol{x}=\mathbf{L}(\alpha\boldsymbol{x}) \implies \alpha\boldsymbol{x}\in\mathcal{I}(\mathbf{L})$

✏

Example G.1. [8] Let $C([a:b],\mathbb{R})$ be the set of all *continuous* functions from the closed real interval $[a:b]$ to $\mathbb{R}$.

EX
$C([a:b],\mathbb{R})$ is a linear space.

Theorem G.3. [9] *Let $\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ be the set of linear operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of a linear operator $\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$.*

THM		
$\mathbf{L}\boldsymbol{x}=\mathbf{L}\boldsymbol{y}$	$\iff$	$\boldsymbol{x}-\boldsymbol{y}\in\mathcal{N}(\mathbf{L})$
$\mathbf{L}$ *is* INJECTIVE	$\iff$	$\mathcal{N}(\mathbf{L})=\{\mathbb{0}\}$

[7] Michel and Herget (1993) pages 98–104, Berberian (1961) pages 80–85 ⟨Theorem IV.1.4 and Theorem IV.3.1⟩
[8] Eidelman et al. (2004) page 3
[9] Berberian (1961) page 88 ⟨Theorem IV.1.4⟩

✎PROOF:

1. Proof that $\mathbf{L}x = \mathbf{L}y \implies x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}(x-y) &= \mathbf{L}x - \mathbf{L}y && \text{by Theorem G.1 page 255}\\
&= \mathbb{0} && \text{by left hypothesis}\\
&\implies x - y \in \mathcal{N}(\mathbf{L}) && \text{by definition of } \textit{null space}
\end{aligned}$$

2. Proof that $\mathbf{L}x = \mathbf{L}y \impliedby x - y \in \mathcal{N}(\mathbf{L})$:

$$\begin{aligned}
\mathbf{L}y &= \mathbf{L}y + \mathbb{0} && \text{by definition of linear space (Definition G.1 page 253)}\\
&= \mathbf{L}y + \mathbf{L}(x-y) && \text{by right hypothesis}\\
&= \mathbf{L}y + (\mathbf{L}x - \mathbf{L}y) && \text{by Theorem G.1 page 255}\\
&= (\mathbf{L}y - \mathbf{L}y) + \mathbf{L}x && \text{by \textit{associative} and \textit{commutative} properties (Definition G.1 page 253)}\\
&= \mathbf{L}x
\end{aligned}$$

3. Proof that $\mathbf{L}$ is *injective* $\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}$:

$$\begin{aligned}
\mathbf{L} \text{ is \textit{injective}} &\iff \{(\mathbf{L}x = \mathbf{L}y \iff x = y) \quad \forall x, y \in X\}\\
&\iff \left\{\left[\mathbf{L}x - \mathbf{L}y = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \left\{\left[\mathbf{L}(x-y) = \mathbb{0} \iff (x-y) = \mathbb{0}\right] \quad \forall x, y \in X\right\}\\
&\iff \mathcal{N}(\mathbf{L}) = \{\mathbb{0}\}
\end{aligned}$$

✏

Theorem G.4. [10] *Let W, X, Y, and Z be linear spaces over a field $\mathbb{F}$.*

THM

1.	$\mathbf{L}(\mathbf{M}\mathbf{N})$	$=$	$(\mathbf{L}\mathbf{M})\mathbf{N}$	$\forall \mathbf{L}\in\mathcal{L}(Z,W), \mathbf{M}\in\mathcal{L}(Y,Z), \mathbf{N}\in\mathcal{L}(X,Y)$	(ASSOCIATIVE)
2.	$\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})$	$=$	$(\mathbf{L}\mathbf{M}) \dotplus (\mathbf{L}\mathbf{N})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(X,Y), \mathbf{N}\in\mathcal{L}(X,Y)$	(LEFT DISTRIBUTIVE)
3.	$(\mathbf{L} \dotplus \mathbf{M})\mathbf{N}$	$=$	$(\mathbf{L}\mathbf{N}) \dotplus (\mathbf{M}\mathbf{N})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(Y,Z), \mathbf{N}\in\mathcal{L}(X,Y)$	(RIGHT DISTRIBUTIVE)
4.	$\alpha(\mathbf{L}\mathbf{M})$	$=$	$(\alpha\mathbf{L})\mathbf{M} = \mathbf{L}(\alpha\mathbf{M})$	$\forall \mathbf{L}\in\mathcal{L}(Y,Z), \mathbf{M}\in\mathcal{L}(X,Y), \alpha\in\mathbb{F}$	(HOMOGENEOUS)

✎PROOF:

1. Proof that $\mathbf{L}(\mathbf{M}\mathbf{N}) = (\mathbf{L}\mathbf{M})\mathbf{N}$: Follows directly from property of *associative* operators.

2. Proof that $\mathbf{L}(\mathbf{M} \dotplus \mathbf{N}) = (\mathbf{L}\mathbf{M}) \dotplus (\mathbf{L}\mathbf{N})$:

$$\begin{aligned}
[\mathbf{L}(\mathbf{M} \dotplus \mathbf{N})]x &= \mathbf{L}[(\mathbf{M} \dotplus \mathbf{N})x]\\
&= \mathbf{L}[(\mathbf{M}x) \dotplus (\mathbf{N}x)]\\
&= [\mathbf{L}(\mathbf{M}x)] \dotplus [\mathbf{L}(\mathbf{N}x)] && \text{by \textit{additive} property Definition G.4 page 255}\\
&= [(\mathbf{L}\mathbf{M})x] \dotplus [(\mathbf{L}\mathbf{N})x]
\end{aligned}$$

3. Proof that $(\mathbf{L} \dotplus \mathbf{M})\mathbf{N} = (\mathbf{L}\mathbf{N}) \dotplus (\mathbf{M}\mathbf{N})$: Follows directly from property of *associative* operators.

4. Proof that $\alpha(\mathbf{L}\mathbf{M}) = (\alpha\mathbf{L})\mathbf{M}$: Follows directly from *associative* property of linear operators.

5. Proof that $\alpha(\mathbf{L}\mathbf{M}) = \mathbf{L}(\alpha\mathbf{M})$:

$$\begin{aligned}
[\alpha(\mathbf{L}\mathbf{M})]x &= \alpha[(\mathbf{L}\mathbf{M})x]\\
&= \mathbf{L}[\alpha(\mathbf{M}x)] && \text{by \textit{homogeneous} property Definition G.4 page 255}\\
&= \mathbf{L}[(\alpha\mathbf{M})x]\\
&= [\mathbf{L}(\alpha\mathbf{M})]x
\end{aligned}$$

[10] Berberian (1961) page 88 ⟨Theorem IV.5.1⟩

✏

Theorem G.5 (Fundamental theorem of linear equations). *Michel and Herget (1993) page 99 Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ (Definition ?? page ??).*

THM

$$\dim \mathcal{I}(\mathbf{L}) + \dim \mathcal{N}(\mathbf{L}) = \dim \boldsymbol{X} \qquad \forall \mathbf{L} \in \boldsymbol{Y}^{\boldsymbol{X}}$$

✎PROOF: Let $\{\psi_k | k = 1, 2, \ldots, p\}$ be a basis for $\boldsymbol{X}$ constructed such that $\{\psi_{p-n+1}, \psi_{p-n+2}, \ldots, \psi_p\}$ is a basis for $\mathcal{N}(\mathbf{L})$.

Let $p \triangleq \dim \boldsymbol{X}$.
Let $n \triangleq \dim \mathcal{N}(\mathbf{L})$.

$$\begin{aligned}
\dim \mathcal{I}(\mathbf{L}) &= \dim \{\boldsymbol{y} \in \boldsymbol{Y} | \exists \boldsymbol{x} \in \boldsymbol{X} \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L}\boldsymbol{x}\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \mathbf{L} \sum_{k=1}^{p} \alpha_k \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \sum_{k=1}^{n} \alpha_k \mathbf{L} \psi_k \right\} \\
&= \dim \left\{ \boldsymbol{y} \in \boldsymbol{Y} | \exists (\alpha_1, \alpha_2, \ldots, \alpha_p) \quad \text{such that} \quad \boldsymbol{y} = \sum_{k=1}^{p-n} \alpha_k \mathbf{L} \psi_k + \mathbb{0} \right\} \\
&= p - n \\
&= \dim \boldsymbol{X} - \dim \mathcal{N}(\mathbf{L})
\end{aligned}$$

Note: This "proof" may be missing some necessary detail. ✏

G.2 Operators on Normed linear spaces

G.2.1 Operator norm

Definition G.5. [11] *Let $\boldsymbol{V} = (X, \mathbb{F}, \dotplus, \cdot)$ be a linear space and $\mathbb{F}$ be a field with absolute value function $|\cdot| \in \mathbb{R}^{\mathbb{F}}$.*

DEF

*A **norm** is any functional $\|\cdot\|$ in $\mathbb{R}^X$ that satisfies*

1.	$\|\boldsymbol{x}\|$	$\geq$	0	$\forall \boldsymbol{x} \in X$	(STRICTLY POSITIVE)	*and*
2.	$\|\boldsymbol{x}\|$	$=$	$0 \iff \boldsymbol{x} = \mathbb{0}$	$\forall \boldsymbol{x} \in X$	(NONDEGENERATE)	*and*
3.	$\|a\boldsymbol{x}\|$	$=$	$\lvert a\rvert \, \|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X, a \in \mathbb{C}$	(HOMOGENEOUS)	*and*
4.	$\|\boldsymbol{x} + \boldsymbol{y}\|$	$\leq$	$\|\boldsymbol{x}\| + \|\boldsymbol{y}\|$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	(SUBADDITIVE/*triangle inquality*).	

*A **normed linear space** is the pair $(\boldsymbol{V}, \|\cdot\|)$.*

[11] Aliprantis and Burkinshaw (1998) pages 217–218, Banach (1932a) page 53, Banach (1932b) page 33, Banach (1922) page 135

Definition G.6. [12] *Let $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of linear operators over normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.* [13]

DEF

*The **operator norm** $|||\cdot|||$ is defined as*

$$|||\mathbf{A}||| \triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} \qquad \forall \mathbf{A} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$$

*The pair $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is the **normed space of linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$.*

Proposition G.1 (next) shows that the functional defined in Definition G.6 (previous) is a *norm* (Definition G.5 page 258).

Proposition G.1. [14] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the normed space of linear operators over the normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

PRP

*The functional $|||\cdot|||$ is a **norm** on $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$. In particular,*

1.	$|||\mathbf{A}||| \geq 0$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(NON-NEGATIVE)	and
2.	$|||\mathbf{A}||| = 0 \iff \mathbf{A} \mathrel{\circeq} \mathbb{0}$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(NONDEGENERATE)	and
3.	$|||\alpha\mathbf{A}||| = |\alpha|\ |||\mathbf{A}|||$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), \alpha\in\mathbb{F}$	(HOMOGENEOUS)	and
4.	$\left|\left|\left|\mathbf{A} \mathrel{\mathring{+}} \mathbf{B}\right|\right|\right| \leq |||\mathbf{A}||| + |||\mathbf{B}|||$	$\forall \mathbf{A}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$	(SUBADDITIVE).	

*Moreover, $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ is a **normed linear space**.*

✎PROOF:

1. Proof that $|||\mathbf{A}||| > 0$ for $\mathbf{A} \neq \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 259)} \\ &> 0 \end{aligned}$$

2. Proof that $|||\mathbf{A}||| = 0$ for $\mathbf{A} \mathrel{\circeq} \mathbb{0}$:

$$\begin{aligned} |||\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 259)} \\ &= \sup_{x\in\boldsymbol{X}} \{\|\mathbb{0}x\| \mid \|x\| \leq 1\} \\ &= 0 \end{aligned}$$

3. Proof that $|||\alpha\mathbf{A}||| = |\alpha|\ |||\mathbf{A}|||$:

$$\begin{aligned} |||\alpha\mathbf{A}||| &\triangleq \sup_{x\in\boldsymbol{X}} \{\|\alpha\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 259)} \\ &= \sup_{x\in\boldsymbol{X}} \{|\alpha|\ \|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 259)} \\ &= |\alpha| \sup_{x\in\boldsymbol{X}} \{\|\mathbf{A}x\| \mid \|x\| \leq 1\} && \text{by definition of sup} \\ &= |\alpha|\ \ |||\mathbf{A}||| && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 259)} \end{aligned}$$

[12] Rudin (1991) page 92, Aliprantis and Burkinshaw (1998) page 225
[13] The operator norm notation $|||\cdot|||$ is introduced (as a Matrix norm) in Horn and Johnson (1990), page 290
[14] Rudin (1991) page 93

4. Proof that $\lVert\lvert \mathbf{A} \mathring{+} \mathbf{B} \rvert\rVert \le \lVert\lvert\mathbf{A}\rvert\rVert + \lVert\lvert\mathbf{B}\rvert\rVert$:

$$\begin{aligned}
\lVert\lvert \mathbf{A} \mathring{+} \mathbf{B} \rvert\rVert &\triangleq \sup_{\boldsymbol{x}\in\boldsymbol{X}} \left\{ \left\| (\mathbf{A} \mathring{+} \mathbf{B})\boldsymbol{x} \right\| \mid \|\boldsymbol{x}\| \le 1 \right\} && \text{by definition of } \lVert\lvert\cdot\rvert\rVert \text{ (Definition G.6 page 259)} \\
&= \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x}\| + \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} && \text{by definition of } \lVert\lvert\cdot\rvert\rVert \text{ (Definition G.6 page 259)} \\
&\le \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{ \|\mathbf{A}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} + \sup_{\boldsymbol{x}\in\boldsymbol{X}} \{ \|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \\
&\triangleq \lVert\lvert\mathbf{A}\rvert\rVert + \lVert\lvert\mathbf{B}\rvert\rVert && \text{by definition of } \lVert\lvert\cdot\rvert\rVert \text{ (Definition G.6 page 259)}
\end{aligned}$$

✏

Lemma G.1. *Let* $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), \lVert\lvert\cdot\rvert\rVert)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *and* $\boldsymbol{Y} \triangleq \left(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$.

LEM	$\lVert\lvert\mathbf{L}\rvert\rVert = \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} \qquad \forall \boldsymbol{x} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$

✎PROOF: [15]

1. Proof that $\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \ge \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \}$:

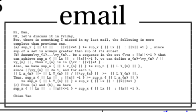
$$\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \ge \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} \qquad \text{because } A \subseteq B \implies \sup A \le \sup B$$

2. Let the subset $Y \subsetneq X$ be defined as

$$Y \triangleq \left\{ \boldsymbol{y} \in X \mid \begin{array}{ll} 1. & \|\mathbf{L}\boldsymbol{y}\| = \sup_{\boldsymbol{x}\in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \text{ and} \\ 2. & 0 < \|\boldsymbol{y}\| \le 1 \end{array} \right\}$$

3. Proof that $\sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} \le \sup_{\boldsymbol{x}} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \}$:

$$\begin{aligned}
\sup_{\boldsymbol{x}\in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1 \} &= \|\mathbf{L}\boldsymbol{y}\| && \text{by definition of set } Y \\
&= \frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \|\mathbf{L}\boldsymbol{y}\| \\
&= \|\boldsymbol{y}\| \left\| \frac{1}{\|\boldsymbol{y}\|} \mathbf{L}\boldsymbol{y} \right\| && \text{by homogeneous property (page 258)} \\
&= \|\boldsymbol{y}\| \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| && \text{by homogeneous property (page 255)} \\
&\le \|\boldsymbol{y}\| \sup_{\boldsymbol{y}\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \right\} && \text{by definition of supremum} \\
&= \|\boldsymbol{y}\| \sup_{\boldsymbol{y}\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \mid \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \text{ for all } \boldsymbol{y} \in Y \\
&\le \sup_{\boldsymbol{y}\in Y} \left\{ \left\| \mathbf{L} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \mid \left\| \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| = 1 \right\} && \text{because } 0 < \|\boldsymbol{y}\| \le 1 \\
&\le \sup_{\boldsymbol{x}\in X} \{ \|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} && \text{because } \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in X \quad \forall \boldsymbol{y} \in Y
\end{aligned}$$

[15] 
Many many thanks to former NCTU Ph.D. student Chien Yao (Chinese: 姚建; PinYin: Yáo Jiàn) for his brilliant help with this proof. (If you are viewing this text as a pdf file, zoom in on the figure to the left to see text from Chien Yao's 2007 April 16 email.)

4. By (1) and (3),

$$\sup_{x\in X}\{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} = \sup_{x\in X}\{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1\}$$

✏

Proposition G.2. [16] *Let* $\mathbf{I}$ *be the identity operator in the normed space of linear operators* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{X}), |||\cdot|||)$.

PRP: $|||\mathbf{I}||| = 1$

✎PROOF:

$$\begin{aligned}
|||\mathbf{I}||| &\triangleq \sup\{\|\mathbf{I}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by definition of } |||\cdot||| \text{ (Definition G.6 page 259)}\\
&= \sup\{\|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} && \text{by definition of } \mathbf{I} \text{ (Definition G.3 page 254)}\\
&= 1
\end{aligned}$$

✏

Theorem G.6. [17] *Let* $(\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), |||\cdot|||)$ *be the normed space of linear operators over normed linear spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

THM:

$$\begin{aligned}
\|\mathbf{L}\boldsymbol{x}\| &\le |||\mathbf{L}|||\ \|\boldsymbol{x}\| && \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}),\ \boldsymbol{x}\in\boldsymbol{X}\\
|||\mathbf{K}\mathbf{L}||| &\le |||\mathbf{K}|||\ |||\mathbf{L}||| && \forall \mathbf{K},\mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})
\end{aligned}$$

✎PROOF:

1. Proof that $\|\mathbf{L}\boldsymbol{x}\| \le |||\mathbf{L}|||\ \|\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{L}\boldsymbol{x}\| &= \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\|\mathbf{L}\boldsymbol{x}\|\\
&= \|\boldsymbol{x}\|\left\|\frac{1}{\|\boldsymbol{x}\|}\mathbf{L}\boldsymbol{x}\right\| && \text{by property of norms}\\
&= \|\boldsymbol{x}\|\left\|\mathbf{L}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\| && \text{by property of linear operators}\\
&\triangleq \|\boldsymbol{x}\|\ \|\mathbf{L}\boldsymbol{y}\| && \text{where } y \triangleq \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\\
&\le \|\boldsymbol{x}\| \sup_{y}\|\mathbf{L}\boldsymbol{y}\| && \text{by definition of supremum}\\
&= \|\boldsymbol{x}\| \sup_{y}\{\|\mathbf{L}\boldsymbol{y}\| \mid \|\boldsymbol{y}\| = 1\} && \text{because } \|\boldsymbol{y}\| = \left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\| = \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|} = 1\\
&\triangleq \|\boldsymbol{x}\|\ \|\mathbf{L}\| && \text{by definition of operator norm}
\end{aligned}$$

[16] Michel and Herget (1993) page 410
[17] Rudin (1991) page 103, Aliprantis and Burkinshaw (1998) page 225

2. Proof that $|||\mathbf{KL}||| \le |||\mathbf{K}|||\ |||\mathbf{L}|||$:

$$
\begin{aligned}
|||\mathbf{KL}||| &\triangleq \sup_{x\in X}\{\|(\mathbf{KL})x\| \mid \|x\| \le 1\} && \text{by Definition G.6 page 259 } (|||\cdot|||)\\
&= \sup_{x\in X}\{\|\mathbf{K}(\mathbf{L}x)\| \mid \|x\| \le 1\}\\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ \|\mathbf{L}x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&\le \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ \|x\| \mid \|x\| \le 1\} && \text{by 1.}\\
&= \sup_{x\in X}\{|||\mathbf{K}|||\ |||\mathbf{L}|||\ 1 \mid \|x\| \le 1\} && \text{by definition of sup}\\
&= |||\mathbf{K}|||\ |||\mathbf{L}||| && \text{by definition of sup}
\end{aligned}
$$

✏

G.2.2 Bounded linear operators

Definition G.7. [18] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be a normed space of linear operators.*

DEF

*An operator $\mathbf{B}$ is **bounded** if $|||\mathbf{B}||| < \infty$.*
*The quantity $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ is the set of all **bounded linear operators** on $(\boldsymbol{X}, \boldsymbol{Y})$ such that*
$$\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y}) \triangleq \{\mathbf{L} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}) \mid |||\mathbf{L}||| < \infty\}.$$

Theorem G.7. [19] *Let $(\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y}), |||\cdot|||)$ be the set of linear operators over normed linear spaces $\boldsymbol{X} \triangleq (X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ and $\boldsymbol{Y} \triangleq (Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.*

THM

The following conditions are all EQUIVALENT*:*

1. $\mathbf{L}$ *is continuous at* A SINGLE POINT $x_0 \in X$ $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
2. $\mathbf{L}$ *is* CONTINUOUS *(at every point $x \in X$)* $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
3. $|||\mathbf{L}||| < \infty$ *($\mathbf{L}$ is* BOUNDED*)* $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}) \quad \Longleftrightarrow$
4. $\exists M \in \mathbb{R}$ such that $\|\mathbf{L}x\| \le M\,\|x\|$ $\quad \forall \mathbf{L}\in\mathcal{L}(\boldsymbol{X},\boldsymbol{Y}), x\in X$

✎PROOF:

1. Proof that $1 \implies 2$:

$$
\begin{aligned}
\epsilon &> \|\mathbf{L}x - \mathbf{L}x_0\| && \text{by hypothesis 1}\\
&= \|\mathbf{L}(x - x_0)\| && \text{by linearity (Definition G.4 page 255)}\\
&= \|\mathbf{L}(x + y - x_0 - y)\|\\
&= \|\mathbf{L}(x + y) - \mathbf{L}(x_0 + y)\| && \text{by linearity (Definition G.4 page 255)}\\
&\implies \mathbf{L} \text{ is continuous at point } x + y\\
&\implies \mathbf{L} \text{ is continuous at every point in } X && \text{(hypothesis 2)}
\end{aligned}
$$

2. Proof that $2 \implies 1$: obvious:

[18] Rudin (1991) pages 92–93
[19] Aliprantis and Burkinshaw (1998) page 227

3. Proof that $4 \implies 2$:[20]

$$\begin{aligned}
|||\mathbf{L}\boldsymbol{x}||| \le M\,\|\boldsymbol{x}\| &\implies |||\mathbf{L}(\boldsymbol{x}-\boldsymbol{y})||| \le M\,\|\boldsymbol{x}-\boldsymbol{y}\| && \text{by hypothesis 4}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le M\,\|\boldsymbol{x}-\boldsymbol{y}\| && \text{by linearity of } \mathbf{L} \text{ (Definition G.4 page 255)}\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } M\,\|\boldsymbol{x}-\boldsymbol{y}\| < \epsilon\\
&\implies |||\mathbf{L}\boldsymbol{x}-\mathbf{L}\boldsymbol{y}||| \le \epsilon \text{ whenever } \|\boldsymbol{x}-\boldsymbol{y}\| < \frac{\epsilon}{M} && \text{(hypothesis 2)}
\end{aligned}$$

4. Proof that $3 \implies 4$:

$$\begin{aligned}
\|\mathbf{L}\boldsymbol{x}\| &\le \underbrace{|||\mathbf{L}|||}_{M}\ \|\boldsymbol{x}\| && \text{by Theorem G.6 page 261}\\
&= M\ \|\boldsymbol{x}\| && \text{where } M \triangleq |||\mathbf{L}||| < \infty \text{ (by hypothesis 1)}
\end{aligned}$$

5. Proof that $1 \implies 3$:[21]

$$\begin{aligned}
|||\mathbf{L}||| = \infty &\implies \{\|\mathbf{L}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\} = \infty\\
&\implies \exists\,(\boldsymbol{x}_n) \quad \text{such that} \quad \|\boldsymbol{x}_n\| = 1 \text{ and } |||\mathbf{L}||| = \left\{\|\mathbf{L}\boldsymbol{x}_n\| \mid \|\boldsymbol{x}_n\| \le 1\right\} = \infty\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \infty = |||\mathbf{L}||| = \|\mathbf{L}\boldsymbol{x}_n\|\\
&\implies \|\boldsymbol{x}_n\| = 1 \text{ and } \|\mathbf{L}\boldsymbol{x}_n\| \ge n\\
&\implies \frac{1}{n}\|\boldsymbol{x}_n\| = \frac{1}{n} \text{ and } \frac{1}{n}\|\mathbf{L}\boldsymbol{x}_n\| \ge 1\\
&\implies \left\|\frac{\boldsymbol{x}_n}{n}\right\| = \frac{1}{n} \text{ and } \left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \lim_{n\to\infty}\left\|\frac{\boldsymbol{x}_n}{n}\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|\mathbf{L}\frac{\boldsymbol{x}_n}{n}\right\| \ge 1\\
&\implies \mathbf{L} \text{ is not continuous at } 0
\end{aligned}$$

But by hypothesis, $\mathbf{L}$ *is* continuous. So the statement $|||\mathbf{L}||| = \infty$ must be *false* and thus $|||\mathbf{L}||| < \infty$ ($\mathbf{L}$ is *bounded*).

✏

G.2.3 Adjoints on normed linear spaces

Definition G.8. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\boldsymbol{X}^*$ be the* TOPOLOGICAL DUAL SPACE *of $\boldsymbol{X}$.*

DEF: $\mathbf{B}^*$ *is the **adjoint** of an operator $\mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ if*

$$\mathsf{f}(\mathbf{B}\boldsymbol{x}) = \left[\mathbf{B}^*\mathsf{f}\right](\boldsymbol{x}) \qquad \forall \mathsf{f} \in \boldsymbol{X}^*,\ \boldsymbol{x} \in \boldsymbol{X}$$

Theorem G.8. [22] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on* NORMED LINEAR SPACES *$\boldsymbol{X}$ and $\boldsymbol{Y}$.*

THM:

$$\begin{aligned}
(\mathbf{A} \mathbin{\dot{+}} \mathbf{B})^* &= \mathbf{A}^* \mathbin{\dot{+}} \mathbf{B}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\lambda\mathbf{A})^* &= \lambda\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})\\
(\mathbf{A}\mathbf{B})^* &= \mathbf{B}^*\mathbf{A}^* && \forall \mathbf{A},\mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{Y})
\end{aligned}$$

[20] Bollobás (1999), page 29
[21] Aliprantis and Burkinshaw (1998), page 227
[22] Bollobás (1999), page 156

✎PROOF:

$$\begin{aligned}
\left[\mathbf{A} \mathring{+} \mathbf{B}\right]^* \mathrm{f}(\boldsymbol{x}) &= \mathrm{f}\left(\left[\mathbf{A} \mathring{+} \mathbf{B}\right]\boldsymbol{x}\right) && \text{by definition of adjoint} && \text{(Definition G.8 page 263)}\\
&= \mathrm{f}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}) && \text{by definition of linear operators} && \text{(Definition G.4 page 255)}\\
&= \mathrm{f}(\mathbf{A}\boldsymbol{x}) + \mathrm{f}(\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{linear functional}\\
&= \mathbf{A}^*\mathrm{f}(\boldsymbol{x}) + \mathbf{B}^*\mathrm{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 263)}\\
&= \left[\mathbf{A}^* + \mathbf{B}^*\right]\mathrm{f}(\boldsymbol{x}) && \text{by definition of } \textit{linear functional}
\end{aligned}$$

$$\begin{aligned}
[\lambda\mathbf{A}]^*\mathrm{f}(\boldsymbol{x}) &= \mathrm{f}([\lambda\mathbf{A}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 263)}\\
&= \lambda\mathrm{f}(\mathbf{A}\boldsymbol{x}) && \text{by definition of } \textit{linear functional}\\
&= \left[\lambda\mathbf{A}^*\right]\mathrm{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 263)}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{A}\mathbf{B}]^*\mathrm{f}(\boldsymbol{x}) &= \mathrm{f}([\mathbf{A}\mathbf{B}]\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 263)}\\
&= \mathrm{f}(\mathbf{A}[\mathbf{B}\boldsymbol{x}]) && \text{by definition of } \textit{linear operators} && \text{(Definition G.4 page 255)}\\
&= \left[\mathbf{A}^*\mathrm{f}\right](\mathbf{B}\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 263)}\\
&= \mathbf{B}^*\left[\mathbf{A}^*\mathrm{f}\right](\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 263)}\\
&= \left[\mathbf{B}^*\mathbf{A}^*\right]\mathrm{f}(\boldsymbol{x}) && \text{by definition of } \textit{adjoint} && \text{(Definition G.8 page 263)}
\end{aligned}$$

✎

Theorem G.9. [23] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{B}^*$ be the adjoint of an operator $\mathbf{B}$.*

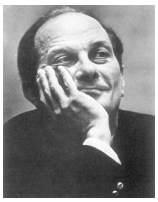
THM
$$|||\mathbf{B}||| = |||\mathbf{B}^*||| \qquad \forall \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$$

✎PROOF:

$$\begin{aligned}
|||\mathbf{B}||| &\triangleq \sup\left\{\|\mathbf{B}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| \le 1\right\} && \text{by Definition G.6 page 259}\\
&\stackrel{?}{=} \sup\left\{|\mathrm{g}(\mathbf{B}\boldsymbol{x}; \boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\}\\
&= \sup\left\{|\mathrm{f}(\boldsymbol{x}; \mathbf{B}^*\boldsymbol{y}^*)| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\}\\
&\triangleq \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \|\boldsymbol{x}\| \le 1,\ \left\|\boldsymbol{y}^*\right\| \le 1\right\}\\
&= \sup\left\{\left\|\mathbf{B}^*\boldsymbol{y}^*\right\| \mid \left\|\boldsymbol{y}^*\right\| \le 1\right\}\\
&\triangleq |||\mathbf{B}^*||| && \text{by Definition G.6 page 259}
\end{aligned}$$

✎

G.2.4 More properties



❝*Beginning with the third year of studies, most of my mathematical work was really started in conversations with Mazur and Banach. And according to Banach some of my own contributions were characterized by a certain "strangeness" in the formulation of problems and in the outline of possible proofs. As he told me once some years later, he was surprised how often these "strange" approaches really worked.*❞

Stanislaus M. Ulam (1909–1984), Polish mathematician [24]

[23] Rudin (1991) page 98

Theorem G.10 (Mazur-Ulam theorem). [25] *Let $\phi \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ be a function on normed linear spaces $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ and $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$. Let $\mathbf{I} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$ be the identity operator on $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.*

T H M

$$\left.\begin{array}{ll} 1. & \underbrace{\phi^{-1}\phi = \phi\phi^{-1} = \mathbf{I}}_{\text{bijective}} \quad \text{and} \\ 2. & \underbrace{\|\phi\boldsymbol{x} - \phi\boldsymbol{y}\|_{\boldsymbol{Y}} = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{X}} \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}}_{\text{isometric}} \end{array}\right\} \implies \underbrace{\phi([1-\lambda]\boldsymbol{x} + \lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x} + \lambda\phi\boldsymbol{y} \forall \lambda \in \mathbb{R}}_{\text{affine}}$$

✎PROOF: Proof not yet complete.

1. Let ψ be the *reflection* of $\boldsymbol{z}$ in $\boldsymbol{X}$ such that $\psi\boldsymbol{x} = 2\boldsymbol{z} - \boldsymbol{x}$
   (a) $\|\psi\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{z}\|$
2. Let $\lambda \triangleq \sup_{\mathrm{g}} \{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\}$
3. Proof that $\mathrm{g} \in W \implies \mathrm{g}^{-1} \in W$:

   Let $\hat{\boldsymbol{x}} \triangleq \mathrm{g}^{-1}\boldsymbol{x}$ and $\hat{\boldsymbol{y}} \triangleq \mathrm{g}^{-1}\boldsymbol{y}$.

$$\begin{aligned} \left\|\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}^{-1}\boldsymbol{y}\right\| &= \|\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\mathrm{g}\hat{\boldsymbol{x}} - \mathrm{g}\hat{\boldsymbol{y}}\| && \text{by left hypothesis} \\ &= \left\|\mathrm{g}\mathrm{g}^{-1}\boldsymbol{x} - \mathrm{g}\mathrm{g}^{-1}\boldsymbol{y}\right\| && \text{by definition of } \hat{\boldsymbol{x}} \text{ and } \hat{\boldsymbol{y}} \\ &= \|\boldsymbol{x} - \boldsymbol{y}\| && \text{by definition of } \mathrm{g}^{-1} \end{aligned}$$

4. Proof that $\mathrm{g}\boldsymbol{z} = \boldsymbol{z}$:

$$\begin{aligned} 2\lambda &= 2\sup\{\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\|\} && \text{by definition of } \lambda \text{ item (2)} \\ &\le 2\|\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\| && \text{by definition of sup} \\ &= \|2\boldsymbol{z} - 2\mathrm{g}\boldsymbol{z}\| \\ &= \|\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}\boldsymbol{z}\| && \text{by definition of } \psi \text{ item (1)} \\ &= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \mathrm{g}^{-1}\mathrm{g}\boldsymbol{z}\right\| && \text{by item (3)} \\ &= \left\|\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| && \text{by definition of } \mathrm{g}^{-1} \\ &= \left\|\psi\mathrm{g}^{-1}\psi\mathrm{g}\boldsymbol{z} - \boldsymbol{z}\right\| \\ &= \left\|\mathrm{g}^{*}\boldsymbol{z} - \boldsymbol{z}\right\| \\ &\le \lambda && \text{by definition of } \lambda \text{ item (2)} \\ &\implies 2\lambda \le \lambda \\ &\implies \lambda = 0 \\ &\implies \mathrm{g}\boldsymbol{z} = \boldsymbol{z} \end{aligned}$$

5. Proof that $\phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) = \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y}$:

$$\begin{aligned} \phi\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right) &= \\ &= \frac{1}{2}\phi\boldsymbol{x} + \frac{1}{2}\phi\boldsymbol{y} \end{aligned}$$

[24] quote: Ulam (1991), page 33
image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html

[25] Oikhberg and Rosenthal (2007), page 598, Väisälä (2003), page 634, Giles (2000), page 11, Dunford and Schwartz (1957), page 91, Mazur and Ulam (1932)

6. Proof that $\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) = [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}$:

$$\begin{aligned}\phi([1-\lambda]\boldsymbol{x}+\lambda\boldsymbol{y}) &= \\ &= [1-\lambda]\phi\boldsymbol{x}+\lambda\phi\boldsymbol{y}\end{aligned}$$

✏

Theorem G.11 (Neumann Expansion Theorem). [26] *Let* $\mathbf{A} \in \boldsymbol{X}^{\boldsymbol{X}}$ *be an operator on a linear space* $\boldsymbol{X}$. *Let* $\mathbf{A}^0 \triangleq \mathbf{I}$.

THM

$$\left.\begin{array}{ll} 1. & \mathbf{A} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X}) \quad (\mathbf{A}\text{ is bounded}) \\ 2. & |||\mathbf{A}||| < 1 \end{array}\right\} \implies \left\{\begin{array}{llcl} 1. & (\mathbf{I}-\mathbf{A})^{-1} & & \text{exists} \\ 2. & |||(\mathbf{I}-\mathbf{A})^{-1}||| & \le & \dfrac{1}{1-|||\mathbf{A}|||} \\ 3. & (\mathbf{I}-\mathbf{A})^{-1} & = & \sum_{n=0}^{\infty}\mathbf{A}^n \\ & \multicolumn{3}{l}{\text{with uniform convergence}} \end{array}\right.$$

G.3 Operators on Inner product spaces

G.3.1 General Results

Definition G.9. [27] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ *be a linear space.*

DEF

A function $\langle \triangle \mid \nabla \rangle \in \mathbb{F}^{X\times X}$ *is an* ***inner product*** *on* $\boldsymbol{\Omega}$ *if*

1.	$\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle \ge 0$		$\forall x \in X$	(non-negative)	and
2.	$\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 \iff \boldsymbol{x} = \mathbb{0}$		$\forall x \in X$	(nondegenerate)	and
3.	$\langle \alpha\boldsymbol{x} \mid \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$		$\forall x,y \in X,\ \forall \alpha\in\mathbb{C}$	(homogeneous)	and
4.	$\langle \boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{u}\rangle = \langle \boldsymbol{x} \mid \boldsymbol{u}\rangle + \langle \boldsymbol{y} \mid \boldsymbol{u}\rangle$		$\forall x,y,u \in X$	(additive)	and
5.	$\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle = \langle \boldsymbol{y} \mid \boldsymbol{x}\rangle^*$		$\forall x,y \in X$	(conjugate symmetric).	

An inner product is also called a ***scalar product***.
An ***inner product space*** *is the pair* $(\boldsymbol{\Omega}, \langle \triangle \mid \nabla \rangle)$.

Theorem G.12. [28] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X},\boldsymbol{X})$ *be* BOUNDED LINEAR OPERATORS *on an inner product space* $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \langle \triangle \mid \nabla \rangle\right)$.

THM

$$\begin{array}{lclcl} \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle = 0 & \forall x\in X & \iff & \mathbf{B}\boldsymbol{x} = \mathbb{0} & \forall x \in X \\ \langle \mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle = \langle \mathbf{B}\boldsymbol{x} \mid \boldsymbol{x}\rangle & \forall x\in X & \iff & \mathbf{A} = \mathbf{B} & \end{array}$$

✎PROOF:

[26] Michel and Herget (1993) page 415
[27] Haaser and Sullivan (1991), page 277, Aliprantis and Burkinshaw (1998) page 276, Peano (1888b) page 72
[28] Rudin (1991) page 310 ⟨Theorem 12.7, Corollary⟩

1. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \implies \mathbf{B}x = \mathbb{0}$:

$$\begin{aligned}
0 &= \langle \mathbf{B}(x + \mathbf{B}x) \mid (x + \mathbf{B}x)\rangle + i\,\langle \mathbf{B}(x + i\mathbf{B}x) \mid (x + i\mathbf{B}x)\rangle && \text{by left hypothesis}\\
&= \left\{\left\langle \mathbf{B}x + \mathbf{B}^2x) \mid x + \mathbf{B}x\right\rangle\right\} + i\left\{\left\langle \mathbf{B}x + i\mathbf{B}^2x) \mid x + i\mathbf{B}x\right\rangle\right\} && \text{by Definition G.4 page 255}\\
&= \left\{\langle \mathbf{B}x \mid x\rangle + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + \left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\} && \text{by Definition G.9 page 266}\\
&\quad + i\left\{\langle \mathbf{B}x \mid x\rangle - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2\left\langle \mathbf{B}^2x \mid \mathbf{B}x\right\rangle\right\}\\
&= \left\{0 + \langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle + 0\right\} + i\left\{0 - i\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle + i\left\langle \mathbf{B}^2x \mid x\right\rangle - i^2 0\right\} && \text{by left hypothesis}\\
&= \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle + \left\langle \mathbf{B}^2x \mid x\right\rangle\right\} + \left\{\langle \mathbf{B}x \mid \mathbf{B}x\rangle - \left\langle \mathbf{B}^2x \mid x\right\rangle\right\}\\
&= 2\,\langle \mathbf{B}x \mid \mathbf{B}x\rangle\\
&= 2\,\|\mathbf{B}x\|^2\\
&\implies \mathbf{B}x = \mathbb{0} && \text{by Definition G.5 page 258}
\end{aligned}$$

2. Proof that $\langle \mathbf{B}x \mid x\rangle = 0 \impliedby \mathbf{B}x = \mathbb{0}$: by property of inner products.

3. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \implies \mathbf{A} \triangleq \mathbf{B}$:

$$\begin{aligned}
0 &= \langle \mathbf{A}x \mid x\rangle - \langle \mathbf{B}x \mid x\rangle && \text{by left hypothesis}\\
&= \langle \mathbf{A}x - \mathbf{B}x \mid x\rangle && \text{by \textit{additivity} property of } \langle \triangle \mid \nabla\rangle \text{ (Definition G.9 page 266)}\\
&= \langle (\mathbf{A} - \mathbf{B})x \mid x\rangle && \text{by definition of operator addition}\\
\implies (\mathbf{A} - \mathbf{B})x &= \mathbb{0} && \text{by item 1}\\
\implies \mathbf{A} &= \mathbf{B} && \text{by definition of operator subtraction}
\end{aligned}$$

4. Proof that $\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \impliedby \mathbf{A} \triangleq \mathbf{B}$:

$$\langle \mathbf{A}x \mid x\rangle = \langle \mathbf{B}x \mid x\rangle \qquad \text{by } \mathbf{A} \triangleq \mathbf{B} \text{ hypothesis}$$

✏

G.3.2 Operator adjoint

A fundamental concept of operators on inner product spaces is the *operator adjoint* (Proposition G.3 page 267). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem G.13 page 268).
- Both support decomposition into "real" and "imaginary" parts (Theorem **??** page **??**).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *null space* of an operator (Theorem G.14 page 269).

Proposition G.3. [29] *Let $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *(Definition G.7 page 262) on a* HILBERT SPACE $\boldsymbol{H}$*.*

PRP *An operator $\mathbf{B}^*$ is the **adjoint** of $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ if*
$$\langle \mathbf{B}x \mid y\rangle = \langle x \mid \mathbf{B}^*y\rangle \qquad \forall x, y \in \boldsymbol{H}.$$

✎PROOF:

[29] Michel and Herget (1993) page 220, Rudin (1991) page 311, Giles (2000), page 182, von Neumann (1929) page 49, Stone (1932) page 41

1. For fixed y, $\mathsf{f}(\boldsymbol{x}) \triangleq \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle$ is a *functional* in $\mathbb{F}^{\boldsymbol{X}}$.

2. $\mathbf{B}^*$ is the *adjoint* of $\mathbf{B}$ because

$$\begin{aligned}
\langle \mathbf{B}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle &\triangleq \mathsf{f}(\mathbf{B}\boldsymbol{x}) \\
&\triangleq \mathbf{B}^*\mathsf{f}(\boldsymbol{x}) && \text{by definition of } \textit{operator adjoint} && \text{(Definition G.8 page 263)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{B}^*\boldsymbol{y} \rangle
\end{aligned}$$

✏

Example G.2.

EX	In matrix algebra ("linear algebra")	
	The inner product operation $\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle$ is represented by	$\boldsymbol{y}^H \boldsymbol{x}$.
	The linear operator is represented as a matrix	$\boldsymbol{A}$.
	The operation of $\boldsymbol{A}$ on a vector $\boldsymbol{x}$ is represented as	$\boldsymbol{A}\boldsymbol{x}$.
	The adjoint of matrix $\boldsymbol{A}$ is the Hermitian matrix	$\boldsymbol{A}^H$.

✎PROOF:

$$\langle \boldsymbol{A}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H \boldsymbol{A}\boldsymbol{x} = [(\boldsymbol{A}\boldsymbol{x})^H \boldsymbol{y}]^H = [\boldsymbol{x}^H \boldsymbol{A}^H \boldsymbol{y}]^H = (\boldsymbol{A}^H \boldsymbol{y})^H \boldsymbol{x} \triangleq \left\langle \boldsymbol{x} \,|\, \boldsymbol{A}^H \boldsymbol{y} \right\rangle$$

✏

Structures that satisfy the four conditions of the next theorem are known as **-algebras* ("*star-algebras*" (Definition ?? page ??). Other structures which are *-algebras include the *field of complex numbers* $\mathbb{C}$ and any *ring of complex square* $n \times n$ *matrices*.[30]

Theorem G.13 (operator star-algebra). [31] *Let* $\boldsymbol{H}$ *be a* HILBERT SPACE *with operators* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *and with adjoints* $\mathbf{A}^*, \mathbf{B}^* \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$. *Let* $\bar{\alpha}$ *be the complex conjugate of some* $\alpha \in \mathbb{C}$.

THM *The pair* $(\boldsymbol{H}, *)$ *is a* $*$-ALGEBRA (STAR-ALGEBRA). *In particular,*

1.	$(\mathbf{A} \dotplus \mathbf{B})^*$	$=$	$\mathbf{A}^* + \mathbf{B}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(DISTRIBUTIVE)	*and*
2.	$(\alpha\mathbf{A})^*$	$=$	$\bar{\alpha}\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(CONJUGATE LINEAR)	*and*
3.	$(\mathbf{A}\mathbf{B})^*$	$=$	$\mathbf{B}^*\mathbf{A}^*$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(ANTIAUTOMORPHIC)	*and*
4.	$\mathbf{A}^{**}$	$=$	$\mathbf{A}$	$\forall \mathbf{A},\mathbf{B} \in \boldsymbol{H}$	(INVOLUTARY)	

✎PROOF:

$$\begin{aligned}
\left\langle \boldsymbol{x} \,|\, (\mathbf{A} \dotplus \mathbf{B})^*\boldsymbol{y} \right\rangle &= \left\langle (\mathbf{A} \dotplus \mathbf{B})\boldsymbol{x} \,|\, \boldsymbol{y} \right\rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)} \\
&= \langle \mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle + \langle \mathbf{B}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition G.9 page 266)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A}^*\boldsymbol{y} \rangle + \langle \boldsymbol{x} \,|\, \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of operator addition} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A}^*\boldsymbol{y} + \mathbf{B}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition G.9 page 266)} \\
&= \langle \boldsymbol{x} \,|\, (\mathbf{A}^* + \mathbf{B}^*)\boldsymbol{y} \rangle && \text{by definition of operator addition}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, (\alpha\mathbf{A})^*\boldsymbol{y} \rangle &= \langle (\alpha\mathbf{A})\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)} \\
&= \langle \alpha(\mathbf{A}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle && \text{by definition of scalar multiplication} \\
&= \alpha \langle \mathbf{A}\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition G.9 page 266)} \\
&= \alpha \langle \boldsymbol{x} \,|\, \mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)} \\
&= \langle \boldsymbol{x} \,|\, \alpha^*\mathbf{A}^*\boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition G.9 page 266)}
\end{aligned}$$

[30] Sakai (1998) page 1
[31] Halmos (1998), pages 39–40, Rudin (1991) page 311

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, (\mathbf{AB})^* \boldsymbol{y} \rangle &= \langle (\mathbf{AB})\boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)} \\
&= \langle \mathbf{A}(\mathbf{B}\boldsymbol{x}) \,|\, \boldsymbol{y} \rangle && \text{by definition of operator multiplication} \\
&= \langle (\mathbf{B}\boldsymbol{x}) \,|\, \mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{B}^*\mathbf{A}^* \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)}
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{A}^{**} \boldsymbol{y} \rangle &= \langle \mathbf{A}^* \boldsymbol{x} \,|\, \boldsymbol{y} \rangle && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)} \\
&= \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle^* && \text{by definition of inner product} && \text{(Definition G.9 page 266)} \\
&= \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle^* && \text{by definition of adjoint} && \text{(Proposition G.3 page 267)} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{A} \boldsymbol{y} \rangle && \text{by definition of inner product} && \text{(Definition G.9 page 266)}
\end{aligned}$$

✎

Theorem G.14. [32] *Let $\boldsymbol{Y}^{\boldsymbol{X}}$ be the set of all operators from a linear space $\boldsymbol{X}$ to a linear space $\boldsymbol{Y}$. Let $\mathcal{N}(\mathbf{L})$ be the* NULL SPACE *of an operator $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ and $\mathcal{I}(\mathbf{L})$ the* IMAGE SET *of $\mathbf{L}$ in $\boldsymbol{Y}^{\boldsymbol{X}}$.*

THM

$$\mathcal{N}(\mathbf{A}) = \mathcal{I}(\mathbf{A}^*)^\perp$$
$$\mathcal{N}(\mathbf{A}^*) = \mathcal{I}(\mathbf{A})^\perp$$

✎PROOF:

$$\begin{aligned}
\mathcal{I}(\mathbf{A}^*)^\perp &= \left\{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}^*) \right\} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \mathbf{A}^* \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} \\
&= \{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A} \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathbf{A}^* && \text{(Proposition G.3 page 267)} \\
&= \{ y \in \boldsymbol{H} \,|\, \mathbf{A} \boldsymbol{y} = 0 \} \\
&= \mathcal{N}(\mathbf{A}) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\mathbf{A})^\perp &= \{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \boldsymbol{u} \rangle = 0 \quad \forall u \in \mathcal{I}(\mathbf{A}) \} \\
&= \{ y \in \boldsymbol{H} \,|\, \langle \boldsymbol{y} \,|\, \mathbf{A} \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \} && \text{by definition of } \mathcal{I} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \langle \mathbf{A}^* \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{H} \right\} && \text{by definition of } \mathbf{A}^* && \text{(Proposition G.3 page 267)} \\
&= \left\{ y \in \boldsymbol{H} \,|\, \mathbf{A}^* \boldsymbol{y} = 0 \right\} \\
&= \mathcal{N}(\mathbf{A}^*) && \text{by definition of } \mathcal{N}(\mathbf{A})
\end{aligned}$$

✎

G.4 Special Classes of Operators

G.4.1 Projection operators

Definition G.10. [33] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF

$\mathbf{P}$ *is a **projection** operator if* $\quad \mathbf{P}^2 = \mathbf{P}$.

[32] Rudin (1991) page 312
[33] Rudin (1991) page 133 ⟨5.15 Projections⟩, Kubrusly (2001) page 70, Bachman and Narici (1966) page 6, Halmos (1958) page 73 ⟨§41. Projections⟩

Theorem G.15. [34] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ with* NULL SPACE $\mathcal{N}(\mathbf{P})$ *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM
$$\left.\begin{array}{llcll}
1. & \mathbf{P}^2 & = & \mathbf{P} & (\mathbf{P} \text{ is a projection operator}) \quad \text{and}\\
2. & \boldsymbol{\Omega} & = & \boldsymbol{X} \dotplus \boldsymbol{Y} & (\boldsymbol{Y} \text{ compliments } \boldsymbol{X} \text{ in } \boldsymbol{\Omega}) \quad \text{and}\\
3. & \mathbf{P}\boldsymbol{\Omega} & = & \boldsymbol{X} & (\mathbf{P} \text{ projects onto } \boldsymbol{X})
\end{array}\right\} \implies \left\{\begin{array}{llcll}
1. & \mathcal{I}(\mathbf{P}) & = & \boldsymbol{X} & \text{and}\\
2. & \mathcal{N}(\mathbf{P}) & = & \boldsymbol{Y} & \text{and}\\
3. & \boldsymbol{\Omega} & = & \mathcal{I}(\mathbf{P}) \dotplus \mathcal{N}(\mathbf{P}) &
\end{array}\right.$$

✎PROOF:

$$\begin{aligned}
\mathcal{I}(\mathbf{P}) &= \mathbf{P}\boldsymbol{\Omega}\\
&= \mathbf{P}(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2)\\
&= \mathbf{P}\boldsymbol{\Omega}_1 + \mathbf{P}\boldsymbol{\Omega}_2\\
&= \boldsymbol{\Omega}_1 + \{\mathbb{0}\}\\
&= \boldsymbol{\Omega}_1
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(\mathbf{P}) &= \{\boldsymbol{x} \in \boldsymbol{\Omega} | \mathbf{P}\boldsymbol{x} = \mathbb{0}\}\\
&= \left\{\boldsymbol{x} \in (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2) | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\}\\
&= \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_1 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\} + \left\{\boldsymbol{x} \in \boldsymbol{\Omega}_2 | \mathbf{P}\boldsymbol{x} = \mathbb{0}\right\}\\
&= \{\mathbb{0}\} + \boldsymbol{\Omega}_2\\
&= \boldsymbol{\Omega}_2
\end{aligned}$$

✏

Theorem G.16. [35] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{P}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

THM
$$\underbrace{\mathbf{P}^2 = \mathbf{P}}_{\mathbf{P} \text{ is a projection operator}} \iff \underbrace{(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})}_{(\mathbf{I} - \mathbf{P}) \text{ is a projection operator}}$$

✎PROOF:

Proof that $\mathbf{P}^2 = \mathbf{P} \implies (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned}
(\mathbf{I} - \mathbf{P})^2 &= (\mathbf{I} - \mathbf{P})(\mathbf{I} - \mathbf{P})\\
&= \mathbf{I}(\mathbf{I} - \mathbf{P}) + (-\mathbf{P})(\mathbf{I} - \mathbf{P})\\
&= \mathbf{I} - \mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2\\
&= \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} && \text{by left hypothesis}\\
&= \mathbf{I} - \mathbf{P}
\end{aligned}$$

Proof that $\mathbf{P}^2 = \mathbf{P} \impliedby (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P})$:

$$\begin{aligned}
\mathbf{P}^2 &= \underbrace{\mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P}^2}_{(\mathbf{I} - \mathbf{P})^2} - (\mathbf{I} - \mathbf{P} - \mathbf{P})\\
&= (\mathbf{I} - \mathbf{P})^2 - (\mathbf{I} - \mathbf{P} - \mathbf{P})\\
&= (\mathbf{I} - \mathbf{P}) - (\mathbf{I} - \mathbf{P} - \mathbf{P}) && \text{by right hypothesis}\\
&= \mathbf{P}
\end{aligned}$$

✏

[34] Michel and Herget (1993) pages 120–121
[35] Michel and Herget (1993) page 121

Theorem G.17. [36] *Let $\boldsymbol{H}$ be a* HILBERT SPACE *and* $\mathbf{P}$ *an operator in* $\boldsymbol{H}^{\boldsymbol{H}}$ *with adjoint* $\mathbf{P}^*$, NULL SPACE $\mathcal{N}(\mathbf{P})$, *and* IMAGE SET $\mathcal{I}(\mathbf{P})$.

THM

If $\mathbf{P}$ *is a* PROJECTION OPERATOR, *then the following are equivalent:*

1.	$\mathbf{P}^* = \mathbf{P}$	($\mathbf{P}$ *is* SELF-ADJOINT)	$\iff$
2.	$\mathbf{P}^*\mathbf{P} = \mathbf{P}\mathbf{P}^*$	($\mathbf{P}$ *is* NORMAL)	$\iff$
3.	$\mathcal{I}(\mathbf{P}) = \mathcal{N}(\mathbf{P})^\perp$		$\iff$
4.	$\langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x} \rangle = \|\mathbf{P}\boldsymbol{x}\|^2 \quad \forall x \in X$		

✎PROOF: This proof is incomplete at this time.

Proof that (1) $\implies$ (2):

$$\begin{aligned} \mathbf{P}^*\mathbf{P} &= \mathbf{P}^{**}\mathbf{P}^* && \text{by (1)} \\ &= \mathbf{P}\mathbf{P}^* && \text{by Theorem G.13 page 268} \end{aligned}$$

Proof that (1) $\implies$ (3):

$$\begin{aligned} \mathcal{I}(\mathbf{P}) &= \mathcal{N}(\mathbf{P}^*)^\perp && \text{by Theorem G.14 page 269} \\ &= \mathcal{N}(\mathbf{P})^\perp && \text{by (1)} \end{aligned}$$

Proof that (3) $\implies$ (4):

Proof that (4) $\implies$ (1):

✏

G.4.2 Self Adjoint Operators

Definition G.11. [37] *Let* $\mathbf{B} \in \mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ *be a* BOUNDED *operator with adjoint* $\mathbf{B}^*$ *on a* HILBERT SPACE $\boldsymbol{H}$.

DEF

The operator $\mathbf{B}$ *is said to be* ***self-adjoint*** *or* ***hermitian*** *if* $\mathbf{B} \stackrel{\circ}{=} \mathbf{B}^*$.

Example G.3 (Autocorrelation operator). Let $\mathsf{x}(t)$ be a random process with autocorrelation
$\mathsf{R}_{\mathsf{xx}}(t,u) \triangleq \underbrace{\mathsf{E}[\mathsf{x}(t)\mathsf{x}^*(u)]}_{\text{expectation}}$.

Let an autocorrelation operator $\mathbf{R}$ be defined as $[\mathbf{R}\mathsf{f}](t) \triangleq \int_{\mathbb{R}} \underbrace{\mathsf{R}_{\mathsf{xx}}(t,u)}_{\text{kernel}} \mathsf{f}(u)\, \mathsf{d}u$.

EX

$\mathbf{R} = \mathbf{R}^*$ (The auto-correlation operator $\mathbf{R}$ is *self-adjoint*)

Theorem G.18. [38] *Let* $\mathbf{S} : \boldsymbol{H} \to \boldsymbol{H}$ *be an operator over a* HILBERT SPACE $\boldsymbol{H}$ *with eigenvalues* $\{\lambda_n\}$ *and eigenfunctions* $\{\psi_n\}$ *such that* $\mathbf{S}\psi_n = \lambda_n\psi_n$ *and let* $\|\mathbf{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$.

THM

$$\left\{ \underbrace{\mathbf{S} = \mathbf{S}^*}_{\mathbf{S} \text{ is self adjoint}} \right\} \implies \left\{ \begin{array}{lll} 1. & \langle \mathbf{S}\boldsymbol{x} \mid \boldsymbol{x} \rangle \in \mathbb{R} & \text{(the hermitian quadratic form of } \mathbf{S} \text{ is REAL-VALUED)} \\ 2. & \lambda_n \in \mathbb{R} & \text{(eigenvalues of } \mathbf{S} \text{ are REAL-VALUED)} \\ 3. & \lambda_n \neq \lambda_m \implies \langle \psi_n \mid \psi_m \rangle = 0 & \text{(eigenvectors are ORTHOGONAL)} \end{array} \right\}$$

[36] Rudin (1991) page 314

[37] Historical works regarding self-adjoint operators: von Neumann (1929), page 49, "linearer Operator R selbstadjungiert oder Hermitesch", Stone (1932), page 50 ⟨"self-adjoint transformations"⟩

[38] Lax (2002), pages 315–316, Keener (1988), pages 114–119, Bachman and Narici (1966) page 24 ⟨Theorem 2.1⟩,
Bertero and Boccacci (1998) page 225 ⟨§"9.2 SVD of a matrix …If all eigenvectors are normalized…"⟩

✎PROOF:

1. Proof that $\mathbf{S} = \mathbf{S}^* \implies \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \in \mathbb{R}$:

$$
\begin{aligned}
\langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle &= \langle \mathbf{S}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle && \text{by left hypothesis} \\
&= \langle \boldsymbol{x} \,|\, \mathbf{S}\boldsymbol{x} \rangle^* && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition G.9 page 266}
\end{aligned}
$$

2. Proof that $\mathbf{S} = \mathbf{S}^* \implies \lambda_n \in \mathbb{R}$:

$$
\begin{aligned}
\lambda_n \|\psi_n\|^2 &= \lambda_n \langle \psi_n \,|\, \psi_n \rangle && \text{by definition} \\
&= \langle \lambda_n \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition G.9 page 266} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_n \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_n \psi_n \rangle && \text{by definition of eigenpairs} \\
&= \lambda_n^* \langle \psi_n \,|\, \psi_n \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition G.9 page 266} \\
&= \lambda_n^* \|\psi_n\|^2 && \text{by definition}
\end{aligned}
$$

3. Proof that $\mathbf{S} = \mathbf{S}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$
\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition G.9 page 266} \\
&= \langle \mathbf{S}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{S}\psi_m \rangle && \text{by left hypothesis} \\
&= \langle \psi_n \,|\, \lambda_m \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \lambda_m^* \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \triangledown \rangle \text{ Definition G.9 page 266} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{because } \lambda_m \text{ is real}
\end{aligned}
$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

G.4.3 Normal Operators

Definition G.12. [39] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{N}^*$ be the adjoint of an operator $\mathbf{N} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

DEF

$\mathbf{N}$ ***is normal*** *if* $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*$.

Theorem G.19. [40] *Let $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE *$\boldsymbol{H}$. Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H}, \boldsymbol{H})$.*

THM

$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N} \text{ is normal}} \iff \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\| \qquad \forall \boldsymbol{x} \in \boldsymbol{H}$$

[39] Rudin (1991) page 312, Michel and Herget (1993) page 431, Dieudonné (1969), page 167, Frobenius (1878), Frobenius (1968), page 391

[40] Rudin (1991) pages 312–313

✎PROOF:

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\|\mathbf{N}\boldsymbol{x}\|^2 &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}^*\mathbf{N}\boldsymbol{x}\rangle && \text{by Proposition G.3 page 267 (definition of } \mathbf{N}^*)\\
&= \langle \boldsymbol{x} \,|\, \mathbf{N}\mathbf{N}^*\boldsymbol{x}\rangle && \text{by left hypothesis (}\mathbf{N}\text{ is normal)}\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by Proposition G.3 page 267 (definition of } \mathbf{N}^*)\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by definition}
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \impliedby \|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|$:

$$\begin{aligned}
\langle \mathbf{N}^*\mathbf{N}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}^{**}\boldsymbol{x}\rangle && \text{by Proposition G.3 page 267 (definition of } \mathbf{N}^*)\\
&= \langle \mathbf{N}\boldsymbol{x} \,|\, \mathbf{N}\boldsymbol{x}\rangle && \text{by Theorem G.13 page 268 (property of adjoint)}\\
&= \|\mathbf{N}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\mathbf{N}^*\boldsymbol{x}\|^2 && \text{by right hypothesis } (\|\mathbf{N}^*\boldsymbol{x}\| = \|\mathbf{N}\boldsymbol{x}\|)\\
&= \langle \mathbf{N}^*\boldsymbol{x} \,|\, \mathbf{N}^*\boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{N}\mathbf{N}^*\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by Proposition G.3 page 267 (definition of } \mathbf{N}^*)
\end{aligned}$$

✏

Theorem G.20. [41] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}} \implies \underbrace{\mathcal{N}(\mathbf{N}^*) = \mathcal{N}(\mathbf{N})}_{\mathbf{N}\text{ and }\mathbf{N}^*\text{ have the same null space}}$$

✎PROOF:

$$\begin{aligned}
\mathcal{N}(\mathbf{N}^*) &= \{\boldsymbol{x} | \mathbf{N}^*\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} && \text{(definition of } \mathcal{N})\\
&= \{\boldsymbol{x} | \, \|\mathbf{N}^*\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} && \text{by definition of } \|\cdot\| \text{ (Definition G.5 page 258)}\\
&= \{\boldsymbol{x} | \, \|\mathbf{N}\boldsymbol{x}\| = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} &&\\
&= \{\boldsymbol{x} | \mathbf{N}\boldsymbol{x} = 0 \quad \forall \boldsymbol{x} \in \boldsymbol{X}\} && \text{by definition of } \|\cdot\| \text{ (Definition G.5 page 258)}\\
&= \mathcal{N}(\mathbf{N}) && \text{(definition of } \mathcal{N})
\end{aligned}$$

✏

Theorem G.21. [42] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{N}(\mathbf{N})$ be the* NULL SPACE *of an operator $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ and $\mathcal{I}(\mathbf{N})$ the* IMAGE SET *of $\mathbf{N}$ in $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$.*

THM
$$\left\{\underbrace{\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^*}_{\mathbf{N}\text{ is normal}}\right\} \implies \left\{\underbrace{\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m\rangle = 0}_{\text{eigenfunctions associated with distinct eigenvalues are orthogonal}}\right\}$$

✎PROOF: The proof in (1) is flawed. This implies that (2) is also flawed. [Rudin] claims both to be true.(Rudin, 1991)313

[41] 🕮 Rudin (1991) pages 312–313
[42] 🕮 Rudin (1991) pages 312–313

1. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies \mathbf{N}^*\psi = \lambda^*\psi$:

$$\begin{aligned}
&\mathbf{N}\psi = \lambda\psi \\
&\Longleftarrow \\
&\quad 0 = \mathcal{N}(\mathbf{N} - \lambda\mathbf{I}) \\
&\quad\;\; = \mathcal{N}([\mathbf{N} - \lambda\mathbf{I}]^*) && \text{by } \mathcal{N}(\mathbf{N}) = \mathcal{N}(\mathbf{N}^*) \\
&\quad\;\; = \mathcal{N}(\mathbf{N}^* - [\lambda\mathbf{I}]^*) && \text{by Theorem G.13 page 268} \\
&\quad\;\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}^*) && \text{by Theorem G.13 page 268} \\
&\quad\;\; = \mathcal{N}(\mathbf{N}^* - \lambda^*\mathbf{I}) \\
&\Longrightarrow \\
&(\mathbf{N}^* - \lambda^*\mathbf{I})\psi = 0 \\
&\iff \mathbf{N}^*\psi = \lambda^*\psi
\end{aligned}$$

2. Proof that $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* \implies [\lambda_n \neq \lambda_m \implies \langle \psi_n \,|\, \psi_m \rangle = 0]$:

$$\begin{aligned}
\lambda_n \langle \psi_n \,|\, \psi_m \rangle &= \langle \lambda_n\psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 266} \\
&= \langle \mathbf{N}\psi_n \,|\, \psi_m \rangle && \text{by definition of eigenpairs} \\
&= \langle \psi_n \,|\, \mathbf{N}^*\psi_m \rangle && \text{by Proposition G.3 page 267 (definition of adjoint)} \\
&= \langle \psi_n \,|\, \lambda_m^*\psi_m \rangle && \text{by (4.)} \\
&= \lambda_m \langle \psi_n \,|\, \psi_m \rangle && \text{by definition of } \langle \triangle \,|\, \nabla \rangle \text{ Definition G.9 page 266}
\end{aligned}$$

This implies for $\lambda_n \neq \lambda_m \neq 0$, $\langle \psi_n \,|\, \psi_m \rangle = 0$.

✏

G.4.4 Isometric operators

An operator on a pair of normed linear spaces is *isometric* (next definition) if it is an *isometry*.

Definition G.13. *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES *(Definition G.5 page 258).*

DEF *An operator* $\mathbf{M} \in \mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$ *is **isometric** if*
$$\|\mathbf{M}x\| = \|x\| \qquad \forall x \in X.$$

Theorem G.22. [43] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *and* $(Y, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ *be* NORMED LINEAR SPACES. *Let* $\mathbf{M}$ *be a linear operator in* $\mathcal{L}(\boldsymbol{X}, \boldsymbol{Y})$.

THM
$$\underbrace{\|\mathbf{M}x\| = \|x\| \quad \forall x \in X}_{\text{isometric in length}} \iff \underbrace{\|\mathbf{M}x - \mathbf{M}y\| = \|x - y\| \quad \forall x, y \in X}_{\text{isometric in distance}}$$

✎PROOF:

1. Proof that $\|\mathbf{M}x\| = \|x\| \implies \|\mathbf{M}x - \mathbf{M}y\| = \|x - y\|$:

$$\begin{aligned}
\|\mathbf{M}x - \mathbf{M}y\| &= \|\mathbf{M}(x - y)\| && \text{by definition of linear operators (Definition G.4 page 255)} \\
&= \|\mathbf{M}u\| && \text{let } u \triangleq x - y \\
&= \|x - y\| && \text{by left hypothesis}
\end{aligned}$$

[43] Kubrusly (2001) page 239 ⟨Proposition 4.37⟩, Berberian (1961) page 27 ⟨Theorem IV.7.5⟩

2. Proof that $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \iff \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$:

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \|\mathbf{M}(\boldsymbol{x} - \mathbb{0})\| \\
&= \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\mathbb{0})\| && \text{by definition of linear operators (Definition G.4 page 255)} \\
&= \|\boldsymbol{x} - \mathbb{0}\| && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\|
\end{aligned}$$

✏

Isometric operators have already been defined (Definition G.13 page 274) in the more general normed linear spaces, while Theorem G.22 (page 274) demonstrated that in a normed linear space $\boldsymbol{X}$, $\|\mathbf{M}\boldsymbol{x}\| = \|\boldsymbol{x}\| \iff \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$. Here in the more specialized inner product spaces, Theorem G.23 (next) demonstrates two additional equivalent properties.

Theorem G.23. [44] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$ be the space of* BOUNDED LINEAR OPERATORS *on a normed linear space $\boldsymbol{X} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$. Let $\mathbf{N}$ be a bounded linear operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle}$.*

THM *The following conditions are all **equivalent**:*

1.	$\mathbf{M}^*\mathbf{M}$	$=$	$\mathbf{I}$			$\iff$
2.	$\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y}\rangle$	$=$	$\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	*($\mathbf{M}$ is surjective)*	$\iff$
3.	$\|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|$	$=$	$\|x - y\|$	$\forall \boldsymbol{x}, \boldsymbol{y} \in X$	*(isometric in distance)*	$\iff$
4.	$\|\mathbf{M}\boldsymbol{x}\|$	$=$	$\|\boldsymbol{x}\|$	$\forall \boldsymbol{x} \in X$	*(isometric in length)*	

✎PROOF:

1. Proof that (1) $\implies$ (2):

$$\begin{aligned}
\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y}\rangle &= \langle \boldsymbol{x} \mid \mathbf{M}^*\mathbf{M}\boldsymbol{y}\rangle && \text{by Proposition G.3 page 267 (definition of adjoint)} \\
&= \langle \boldsymbol{x} \mid \mathbf{I}\boldsymbol{y}\rangle && \text{by (1)} \\
&= \langle \boldsymbol{x} \mid \boldsymbol{y}\rangle && \text{by Definition G.3 page 254 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\implies$ (4):

$$\begin{aligned}
\|\mathbf{M}\boldsymbol{x}\| &= \sqrt{\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{x}\rangle} && \text{by definition of } \|\cdot\| \\
&= \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x}\rangle} && \text{by right hypothesis} \\
&= \|\boldsymbol{x}\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

3. Proof that (2) $\impliedby$ (4):

$$\begin{aligned}
4\langle \mathbf{M}\boldsymbol{x} \mid \mathbf{M}\boldsymbol{y}\rangle &= \|\mathbf{M}\boldsymbol{x} + \mathbf{M}\boldsymbol{y}\|^2 - \|\mathbf{M}\boldsymbol{x} - \mathbf{M}\boldsymbol{y}\|^2 + i\,\|\mathbf{M}\boldsymbol{x} + i\mathbf{M}\boldsymbol{y}\|^2 - i\,\|\mathbf{M}\boldsymbol{x} - i\mathbf{M}\boldsymbol{y}\|^2 && \text{by polarization id.} \\
&= \|\mathbf{M}(\boldsymbol{x} + \boldsymbol{y})\|^2 - \|\mathbf{M}(\boldsymbol{x} - \boldsymbol{y})\|^2 + i\,\|\mathbf{M}(\boldsymbol{x} + i\boldsymbol{y})\|^2 - i\,\|\mathbf{M}(\boldsymbol{x} - i\boldsymbol{y})\|^2 && \text{by Definition G.4} \\
&= \|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 + i\,\|\boldsymbol{x} + i\boldsymbol{y}\|^2 - i\,\|\boldsymbol{x} - i\boldsymbol{y}\|^2 && \text{by left hypothesis}
\end{aligned}$$

4. Proof that (3) $\iff$ (4): by Theorem G.22 page 274

[44] Michel and Herget (1993) page 432 ⟨Theorem 7.5.8⟩, Kubrusly (2001) page 391 ⟨Proposition 5.72⟩

5. Proof that (4) $\implies$ (1):

$$\begin{aligned}
\langle \mathbf{M}^*\mathbf{M}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle &= \langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}^{**}\boldsymbol{x}\rangle && \text{by Proposition G.3 page 267 (definition of adjoint)}\\
&= \langle \mathbf{M}\boldsymbol{x} \,|\, \mathbf{M}\boldsymbol{x}\rangle && \text{by Theorem G.13 page 268 (property of adjoint)}\\
&= \|\mathbf{M}\boldsymbol{x}\|^2 && \text{by definition}\\
&= \|\boldsymbol{x}\|^2 && \text{by left hypothesis with } y=0\\
&= \langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by definition}\\
&= \langle \mathbf{I}\boldsymbol{x} \,|\, \boldsymbol{x}\rangle && \text{by Definition G.3 page 254 (definition of } \mathbf{I})\\
\implies \quad \mathbf{M}^*\mathbf{M} &= \mathbf{I} && \forall \boldsymbol{x} \in X
\end{aligned}$$

✏

Theorem G.24. [45] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{M}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\mathbf{I}$ the identity operator in $\mathcal{L}(\boldsymbol{X}, \boldsymbol{X})$. Let Λ be the set of eigenvalues of $\mathbf{M}$. Let $\|\boldsymbol{x}\| \triangleq \sqrt{\langle \boldsymbol{x} \,|\, \boldsymbol{x}\rangle}$.*

THM

$$\underbrace{\mathbf{M}^*\mathbf{M} = \mathbf{I}}_{\mathbf{M} \text{ is isometric}} \implies \left\{ \begin{array}{rcll} |||\mathbf{M}||| &=& 1 & \text{(UNIT LENGTH)} \quad \text{and} \\ |\lambda| &=& 1 \quad \forall \lambda \in \Lambda & \end{array} \right.$$

✎PROOF:

1. Proof that $\mathbf{M}^*\mathbf{M} = \mathbf{I} \implies |||\mathbf{M}||| = 1$:

$$\begin{aligned}
|||\mathbf{M}||| &= \sup_{\boldsymbol{x}\in X} \{ \|\mathbf{M}\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} && \text{by Definition G.6 page 259}\\
&= \sup_{\boldsymbol{x}\in X} \{ \|\boldsymbol{x}\| \mid \|\boldsymbol{x}\| = 1 \} && \text{by Theorem G.23 page 275}\\
&= \sup_{\boldsymbol{x}\in X} \{1\}\\
&= 1
\end{aligned}$$

2. Proof that $|\lambda| = 1$: Let $(\boldsymbol{x}, \lambda)$ be an eigenvector-eigenvalue pair.

$$\begin{aligned}
1 &= \frac{1}{\|\boldsymbol{x}\|} \|\boldsymbol{x}\|\\
&= \frac{1}{\|\boldsymbol{x}\|} \|\mathbf{M}\boldsymbol{x}\| && \text{by Theorem G.23 page 275}\\
&= \frac{1}{\|\boldsymbol{x}\|} \|\lambda\boldsymbol{x}\| && \text{by definition of } \lambda\\
&= \frac{1}{\|\boldsymbol{x}\|} |\lambda| \; \|\boldsymbol{x}\| && \text{by homogeneous property of } \|\cdot\|\\
&= |\lambda|
\end{aligned}$$

✏

Example G.4 (One sided shift operator). [46] Let $\boldsymbol{X}$ be the set of all sequences with range $\mathbb{W}$ $(0, 1, 2, \ldots)$ and shift operators defined as

1. $\mathbf{S}_r \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \triangleq \left(\!\left(0, x_0, x_1, x_2, \ldots \right)\!\right)$ (right shift operator)
2. $\mathbf{S}_l \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \triangleq \left(\!\left(x_1, x_2, x_3, \ldots \right)\!\right)$ (left shift operator)

EX

1. $\mathbf{S}_r$ is an isometric operator.
2. $\mathbf{S}_r^* = \mathbf{S}_l$

[45] Michel and Herget (1993) page 432
[46] Michel and Herget (1993) page 441

✎PROOF:

1. Proof that $\mathbf{S}_r^* = \mathbf{S}_l$:

$$\begin{aligned}
\langle \mathbf{S}_r \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \,|\, \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \rangle &= \langle \left(\!\left(0, x_0, x_1, x_2, \ldots \right)\!\right) \,|\, \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \rangle \\
&= \sum_{n=1}^{\infty} x_{n-1}\; y_n^* \\
&= \sum_{n=0}^{\infty} x_n\; y_{n+1}^* \\
&= \sum_{n=0}^{\infty} x_n\; y_{n+1}^* \\
&= \langle \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \,|\, \left(\!\left(y_1, y_2, y_3, \ldots \right)\!\right) \rangle \\
&= \left\langle \left(\!\left(x_0, x_1, x_2, \ldots \right)\!\right) \,\middle|\, \underbrace{\mathbf{S}_l}_{\mathbf{S}_r^*} \left(\!\left(y_0, y_1, y_2, \ldots \right)\!\right) \right\rangle
\end{aligned}$$

2. Proof that $\mathbf{S}_r$ is isometric ($\mathbf{S}_r^* \mathbf{S}_r = \mathbf{I}$):

$$\begin{aligned}
\mathbf{S}_r^* \mathbf{S}_r &= \mathbf{S}_l \mathbf{S}_r && \text{by 1.} \\
&= \mathbf{I}
\end{aligned}$$

✑

G.4.5 Unitary operators

Definition G.14. [47] *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ be a bounded linear operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$, and $\boldsymbol{I}$ the identity operator in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{X})$.*

DEF: *The operator* $\mathbf{U}$ *is* ***unitary*** *if* $\mathbf{U}^*\mathbf{U} \;=\; \mathbf{U}\mathbf{U}^* = \mathbf{I}$.

Proposition G.4. *Let $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of* BOUNDED LINEAR OPERATORS *on normed linear spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathbf{U}$ and $\mathbf{V}$ be* BOUNDED LINEAR OPERATORS *in $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$.*

PRP: $\left.\begin{array}{l}\mathbf{U} \text{ is UNITARY } \quad \text{and} \\ \mathbf{V} \text{ is UNITARY}\end{array}\right\} \implies (\mathbf{UV}) \text{ is UNITARY.}$

✎PROOF:

$$\begin{aligned}
(\mathbf{UV})(\mathbf{UV})^* &= (\mathbf{UV})\left(\mathbf{V}^*\mathbf{U}^*\right) && \text{by Theorem G.8 page 263} \\
&= \mathbf{U}\left(\mathbf{V}\mathbf{V}^*\right)\mathbf{U}^* && \text{by associative property} \\
&= \mathbf{U}\mathbf{I}\mathbf{U}^* && \text{by definition of unitary operators—Definition G.14 page 277} \\
&= \mathbf{I} && \text{by definition of unitary operators—Definition G.14 page 277}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{UV})^*(\mathbf{UV}) &= \left(\mathbf{V}^*\mathbf{U}^*\right)(\mathbf{UV}) && \text{by Theorem G.8 page 263} \\
&= \mathbf{V}^*\left(\mathbf{U}^*\mathbf{U}\right)\mathbf{V} && \text{by associative property} \\
&= \mathbf{V}^*\mathbf{I}\mathbf{V} && \text{by definition of unitary operators—Definition G.14 page 277} \\
&= \mathbf{I} && \text{by definition of unitary operators—Definition G.14 page 277}
\end{aligned}$$

[47] Rudin (1991) page 312, Michel and Herget (1993) page 431, Autonne (1901) page 209, Autonne (1902), Schur (1909), Steen (1973)

Theorem G.25. [48] *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let $\mathcal{I}(\mathbf{U})$ be the* IMAGE SET *of* $\mathbf{U}$.

THM

If $\mathbf{U}$ *is a* ***bounded linear operator*** $(\mathbf{U} \in \mathcal{B}(\boldsymbol{H},\boldsymbol{H}))$, *then the following conditions are* ***equivalent***:

1. $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ (UNITARY) $\iff$
2. $\langle \mathbf{U}\boldsymbol{x} \,|\, \mathbf{U}\boldsymbol{y}\rangle = \langle \mathbf{U}^*\boldsymbol{x} \,|\, \mathbf{U}^*\boldsymbol{y}\rangle = \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle$ *and* $\mathcal{I}(\mathbf{U}) = X$ (SURJECTIVE) $\iff$
3. $\|\mathbf{U}\boldsymbol{x} - \mathbf{U}\boldsymbol{y}\| = \|\mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN DISTANCE) $\iff$
4. $\|\mathbf{U}\boldsymbol{x}\| = \|\boldsymbol{x}\|$ *and* $\mathcal{I}(\mathbf{U}) = X$ (ISOMETRIC IN LENGTH)

✎PROOF:

1. Proof that (1) $\implies$ (2):
   (a) $\langle \mathbf{U}\boldsymbol{x} \,|\, \mathbf{U}\boldsymbol{y}\rangle = \langle \mathbf{U}^*\boldsymbol{x} \,|\, \mathbf{U}^*\boldsymbol{y}\rangle = \langle \boldsymbol{x} \,|\, \boldsymbol{y}\rangle$ by Theorem G.23 (page 275).
   (b) Proof that $\mathcal{I}(\mathbf{U}) = X$:

$$\begin{aligned}
X &\supseteq \mathcal{I}(\mathbf{U}) && \text{because } \mathbf{U} \in X^X \\
&\supseteq \mathcal{I}(\mathbf{U}\mathbf{U}^*) \\
&= \mathcal{I}(\mathbf{I}) && \text{by left hypothesis } (\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}) \\
&= X && \text{by Definition G.3 page 254 (definition of } \mathbf{I})
\end{aligned}$$

2. Proof that (2) $\iff$ (3) $\iff$ (4): by Theorem G.23 page 275.
3. Proof that (3) $\implies$ (1):
   (a) Proof that $\|\mathbf{U}\boldsymbol{x} - \mathbf{U}\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \implies \mathbf{U}^*\mathbf{U} = \mathbf{I}$: by Theorem G.23 page 275
   (b) Proof that $\|\mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \implies \mathbf{U}\mathbf{U}^* = \mathbf{I}$:

$$\begin{aligned}
\|\mathbf{U}^*\boldsymbol{x} - \mathbf{U}^*\boldsymbol{y}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \implies \quad & \mathbf{U}^{**}\mathbf{U}^* = \mathbf{I} && \text{by Theorem G.23 page 275} \\
& \mathbf{U}\mathbf{U}^* = \mathbf{I} && \text{by Theorem G.13 page 268}
\end{aligned}$$

Theorem G.26. *Let $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$ be the space of* BOUNDED LINEAR OPERATORS *on a* HILBERT SPACE $\boldsymbol{H}$. *Let* $\mathbf{U}$ *be a bounded linear operator in* $\mathcal{B}(\boldsymbol{H},\boldsymbol{H})$, $\mathcal{N}(\mathbf{U})$ *the* NULL SPACE *of* $\mathbf{U}$, *and* $\mathcal{I}(\mathbf{U})$ *the* IMAGE SET *of* $\mathbf{U}$.

THM

$$\underbrace{\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}}_{\mathbf{U} \text{ is unitary}} \implies \left\{\begin{array}{lllll}
\mathbf{U}^{-1} & = & \mathbf{U}^* & & \text{and} \\
\mathcal{I}(\mathbf{U}) & = & \mathcal{I}(\mathbf{U}^*) & = X & \text{and} \\
\mathcal{N}(\mathbf{U}) & = & \mathcal{N}(\mathbf{U}^*) & = \{\mathbb{0}\} & \text{and} \\
|||\mathbf{U}||| & = & |||\mathbf{U}^*||| & = 1 & \text{(UNIT LENGTH)}
\end{array}\right\}$$

✎PROOF:

1. Note that $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all both *isometric* and *normal*:

$$\begin{array}{lllllll}
\mathbf{U}^*\mathbf{U} & & & = & \mathbf{I} & \implies & \mathbf{U} \text{ is } \textit{isometric} \\
\mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & \mathbf{I} & \implies & \mathbf{U}^* \text{ is } \textit{isometric} \\
\mathbf{U}^{-1} & = & \mathbf{U}^* & & & \implies & \mathbf{U}^{-1} \text{ is } \textit{isometric} \\
\\
\mathbf{U}^*\mathbf{U} & = & \mathbf{U}\mathbf{U}^* & = & & \implies & \mathbf{U} \text{ is } \textit{normal} \\
\mathbf{U}\mathbf{U}^* & = & \mathbf{U}^*\mathbf{U} & = & & \implies & \mathbf{U}^* \text{ is } \textit{normal} \\
\mathbf{U}^{-1} & = & \mathbf{U}^* & & & \implies & \mathbf{U}^{-1} \text{ is } \textit{normal}
\end{array}$$

[48] Rudin (1991) pages 313–314 ⟨Theorem 12.13⟩, Knapp (2005a) page 45 ⟨Proposition 2.6⟩

2. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U}^*) = \boldsymbol{H}$: by Theorem G.25 page 278.

3. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{U}^*) = \mathcal{N}(\mathbf{U}^{-1})$:

$$\begin{aligned}
\mathcal{N}(\mathbf{U}^*) &= \mathcal{N}(\mathbf{U}) && \text{because } \mathbf{U} \text{ and } \mathbf{U}^* \text{ are both } \textit{normal} \text{ and by Theorem G.21 page 273} \\
&= \mathcal{I}(\mathbf{U})^\perp && \text{by Theorem G.14 page 269} \\
&= X^\perp && \text{by above result} \\
&= \{\mathbb{0}\}
\end{aligned}$$

4. Proof that $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \implies \|\mathbf{U}^{-1}\| = \|\mathbf{U}^*\| = \|\mathbf{U}\| = 1$:
Because $\mathbf{U}$, $\mathbf{U}^*$, and $\mathbf{U}^{-1}$ are all isometric and by Theorem G.24 page 276.

✏

Example G.5. Examples of *Fredholm integral operators* include

EX								
1.	**Fourier Transform**	$[\tilde{\mathbf{F}}\mathsf{x}](f)$	$=$	$\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-i2\pi ft}\,\mathrm{d}t$	$\kappa(t,f)$	$=$	$e^{-i2\pi ft}$
2.	**Inverse Fourier Transform**	$[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}](t)$	$=$	$\int_{f\in\mathbb{R}} \tilde{\mathsf{x}}(f)e^{i2\pi ft}\,\mathrm{d}f$	$\kappa(f,t)$	$=$	$e^{i2\pi ft}$
3.	**Laplace operator**	$[\mathbf{L}\mathsf{x}](s)$	$=$	$\int_{t\in\mathbb{R}} \mathsf{x}(t)e^{-st}\,\mathrm{d}t$	$\kappa(t,s)$	$=$	e^{-st}

Example G.6 (Translation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{T}\mathsf{f}(x) \triangleq \mathsf{f}(x-1) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(translation operator)}$$

EX					
1.	$\mathbf{T}^{-1}\mathsf{f}(x)$	$=$	$\mathsf{f}(x+1)$	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(inverse translation operator)
2.	$\mathbf{T}^*$	$=$	$\mathbf{T}^{-1}$		($\mathbf{T}$ is invertible)
3.	$\mathbf{T}^*\mathbf{T}$	$=$	$\mathbf{T}\mathbf{T}^* = \mathbf{I}$		($\mathbf{T}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{T}^{-1}\mathsf{f}(x) = \mathsf{f}(x+1)$:

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{T} &= \mathbf{I} \\
\mathbf{T}\mathbf{T}^{-1} &= \mathbf{I}
\end{aligned}$$

2. Proof that $\mathbf{T}$ is unitary:

$$\begin{aligned}
\langle \mathbf{T}\mathsf{f}(x) \,|\, \mathsf{g}(x) \rangle &= \langle \mathsf{f}(x-1) \,|\, \mathsf{g}(x) \rangle && \text{by definition of } \mathbf{T} \\
&= \int_x \mathsf{f}(x-1)\mathsf{g}^*(x)\,\mathrm{d}x \\
&= \int_x \mathsf{f}(x)\mathsf{g}^*(x+1)\,\mathrm{d}x \\
&= \langle \mathsf{f}(x) \,|\, \mathsf{g}(x+1) \rangle \\
&= \left\langle \mathsf{f}(x) \,\middle|\, \underbrace{\mathbf{T}^{-1}}_{\mathbf{T}^*}\mathsf{g}(x) \right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example G.7 (Dilation operator). Let $\boldsymbol{X} = \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{T} \in \boldsymbol{X}^{\boldsymbol{X}}$ be defined as

$$\mathbf{D}\mathsf{f}(x) \triangleq \sqrt{2}\mathsf{f}(2x) \qquad \forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}} \qquad \text{(dilation operator)}$$

EX					
1.	$\mathbf{D}^{-1}\mathsf{f}(x)$	$=$	$\frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$	$\forall \mathsf{f}\in\boldsymbol{L}^2_{\mathbb{R}}$	(inverse dilation operator)
2.	$\mathbf{D}^*$	$=$	$\mathbf{D}^{-1}$		($\mathbf{D}$ is invertible)
3.	$\mathbf{D}^*\mathbf{D}$	$=$	$\mathbf{D}\mathbf{D}^* = \mathbf{I}$		($\mathbf{D}$ is unitary)

✎PROOF:

1. Proof that $\mathbf{D}^{-1}\mathsf{f}(x) = \frac{1}{\sqrt{2}}\mathsf{f}\left(\frac{1}{2}x\right)$:

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$$
$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$$

2. Proof that $\mathbf{D}$ is unitary:

$$\begin{aligned}
\langle \mathbf{D}\mathsf{f}(x) \,|\, \mathsf{g}(x)\rangle &= \left\langle \sqrt{2}\mathsf{f}(2x) \,|\, \mathsf{g}(x)\right\rangle && \text{by definition of } \mathbf{D}\\
&= \int_x \sqrt{2}\mathsf{f}(2x)\mathsf{g}^*(x)\,\mathrm{d}x\\
&= \int_{u\in\mathbb{R}} \sqrt{2}\mathsf{f}(u)\mathsf{g}^*\left(\frac{1}{2}u\right)\frac{1}{2}\,\mathrm{d}u && \text{let } u \triangleq 2x \implies \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u\\
&= \int_{u\in\mathbb{R}} \mathsf{f}(u)\left[\frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}u\right)\right]^*\,\mathrm{d}u\\
&= \left\langle \mathsf{f}(x) \,|\, \frac{1}{\sqrt{2}}\mathsf{g}\left(\frac{1}{2}x\right)\right\rangle\\
&= \left\langle \mathsf{f}(x) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}\mathsf{g}(x)\right\rangle && \text{by 1.}
\end{aligned}$$

✏

Example G.8 (Delay operator). Let $\boldsymbol{X}$ be the set of all sequences and $\mathbf{D} \in \boldsymbol{X}^{\boldsymbol{X}}$ be a delay operator.

EX The delay operator $\mathbf{D}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq (\!(x_{n-1})\!)_{n\in\mathbb{Z}}$ is unitary.

✎PROOF: The inverse $\mathbf{D}^{-1}$ of the delay operator $\mathbf{D}$ is

$$\mathbf{D}^{-1}\,(\!(x_n)\!)_{n\in\mathbb{Z}} \triangleq (\!(x_{n+1})\!)_{n\in\mathbb{Z}}.$$

$$\begin{aligned}
\langle \mathbf{D}\,(\!(x_n)\!) \,|\, (\!(y_n)\!)\rangle &= \langle (\!(x_{n-1})\!) \,|\, (\!(y_n)\!)\rangle && \text{by definition of } \mathbf{D}\\
&= \sum_n \boldsymbol{x}_{n-1}\,\boldsymbol{y}_n^*\\
&= \sum_n \boldsymbol{x}_n\,\boldsymbol{y}_{n+1}^*\\
&= \left\langle (\!(x_n)\!) \,|\, (\!(y_{n+1})\!)\right\rangle\\
&= \left\langle (\!(x_n)\!) \,|\, \underbrace{\mathbf{D}^{-1}}_{\mathbf{D}^*}(\!(y_n)\!)\right\rangle
\end{aligned}$$

Therefore, $\mathbf{D}^* = \mathbf{D}^{-1}$. This implies that $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}$ which implies that $\mathbf{D}$ is unitary. ✏

Example G.9 (Fourier transform). Let $\tilde{\mathbf{F}}$ be the *Fourier Transform* and $\tilde{\mathbf{F}}^{-1}$ the *inverse Fourier Transform* operator

$$[\tilde{\mathbf{F}}\mathsf{x}](f) \triangleq \int_t \mathsf{x}(t)\underbrace{e^{-i2\pi ft}}_{\kappa(t,f)}\,\mathrm{d}t \qquad \left[\tilde{\mathbf{F}}^{-1}\tilde{\mathsf{x}}\right](t) \triangleq \int_f \tilde{\mathsf{x}}(f)\underbrace{e^{i2\pi ft}}_{\kappa^*(t,f)}\,\mathrm{d}f.$$

EX $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$ (the Fourier Transform operator $\tilde{\mathbf{F}}$ is unitary)

✎PROOF:

$$\begin{aligned}
\left\langle \tilde{\mathbf{F}}\mathsf{x} \,|\, \tilde{\mathsf{y}} \right\rangle &= \left\langle \int_t \mathsf{x}(t) e^{-i2\pi f t} \,\mathrm{d}t \,|\, \tilde{\mathsf{y}}(f) \right\rangle \\
&= \int_t \mathsf{x}(t) \left\langle e^{-i2\pi f t} \,|\, \tilde{\mathsf{y}}(f) \right\rangle \,\mathrm{d}t \\
&= \int_t \mathsf{x}(t) \int_f e^{-i2\pi f t} \tilde{\mathsf{y}}^*(f) \,\mathrm{d}f \,\mathrm{d}t \\
&= \int_t \mathsf{x}(t) \left[\int_f e^{i2\pi f t} \tilde{\mathsf{y}}(f) \,\mathrm{d}f \right]^* \mathrm{d}t \\
&= \left\langle \mathsf{x}(t) \,|\, \int_f \tilde{\mathsf{y}}(f) e^{i2\pi f t} \,\mathrm{d}f \right\rangle \\
&= \left\langle \mathsf{x} \,|\, \underbrace{\tilde{\mathbf{F}}^{-1}}_{\tilde{\mathbf{F}}^*} \tilde{\mathsf{y}} \right\rangle
\end{aligned}$$

This implies that $\tilde{\mathbf{F}}$ is unitary ($\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-1}$). ✎

Example G.10 (Rotation matrix). [49] Let the rotation matrix $\mathbf{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\mathbf{R}_\theta \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

EX

1. $\mathbf{R}^{-1}{}_\theta = \mathbf{R}_{-\theta}$
2. $\mathbf{R}^*{}_\theta = \mathbf{R}^{-1}{}_\theta$ ($\mathbf{R}$ is unitary)

✎PROOF:

$$\begin{aligned}
\mathbf{R}^* &= \mathbf{R}^H \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^H && \text{by definition of } \mathbf{R} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} && \text{by definition of Hermetian transpose operator } H \\
&= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} && \text{by Theorem D.2 page 213} \\
&= \mathbf{R}_{-\theta} && \text{by definition of } \mathbf{R} \\
&= \mathbf{R}^{-1} && \text{by 1.}
\end{aligned}$$

✎

G.5 Operator order

Definition G.15. [50] *Let* $\mathbf{P} \in \boldsymbol{Y}^{\boldsymbol{X}}$ *be an operator.*

DEF

$\mathbf{P}$ *is **positive** if* $\langle \mathbf{P}\boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \;\forall \boldsymbol{x} \in \boldsymbol{X}$.
This condition is denoted $\mathbf{P} \geq 0$.

[49] Noble and Daniel (1988), page 311
[50] Michel and Herget (1993) page 429 ⟨Definition 7.4.12⟩

Theorem G.27. [51]

THM

$$\underbrace{\mathbf{P} \geq 0 \text{ and } \mathbf{Q} \geq 0}_{\mathbf{P} \text{ and } \mathbf{Q} \text{ are both positive}} \implies \left\{ \begin{array}{lllll} (\mathbf{P}+\mathbf{Q}) & \geq & 0 & & ((\mathbf{P}+\mathbf{Q}) \text{ is positive}) \\ \mathbf{A}^*\mathbf{P}\mathbf{A} & \geq & 0 & \forall \mathbf{A}\in\mathcal{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{P}\mathbf{A} \text{ is positive}) \\ \mathbf{A}^*\mathbf{A} & \geq & 0 & \forall \mathbf{A}\in\mathcal{B}(\boldsymbol{X},\boldsymbol{X}) & (\mathbf{A}^*\mathbf{A} \text{ is positive}) \end{array}\right.$$

✎PROOF:

$$\begin{aligned}
\langle (\mathbf{P}+\mathbf{Q})\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x}\rangle + \langle \mathbf{Q}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by additive property of } \langle \triangle \mid \nabla\rangle \text{ (Definition G.9 page 266)} \\
&\geq \langle \mathbf{P}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by left hypothesis} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{A}^*\mathbf{P}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{P}\mathbf{A}\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x}\rangle && \text{by definition of adjoint (Proposition G.3 page 267)} \\
&= \langle \mathbf{P}\boldsymbol{y} \mid \boldsymbol{y}\rangle && \text{where } \boldsymbol{y} \triangleq \mathbf{A}\boldsymbol{x} \\
&\geq 0 && \text{by left hypothesis} \\
\langle \mathbf{I}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition of } \mathbf{I} \text{ (Definition G.3 page 254)} \\
&\geq 0 && \text{by non-negative property of } \langle \triangle \mid \nabla\rangle \text{ (Definition G.9 page 266)} \\
&\implies \mathbf{I} \text{ is positive} \\
\langle \mathbf{A}^*\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle &= \langle \mathbf{A}^*\mathbf{I}\mathbf{A}\boldsymbol{x} \mid \boldsymbol{x}\rangle && \text{by definition of } \mathbf{I} \text{ (Definition G.3 page 254)} \\
&\geq 0 && \text{by two previous results}
\end{aligned}$$

✏

Definition G.16. [52] *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ *be* BOUNDED *operators.*

DEF

$\mathbf{A} \geq \mathbf{B}$ *("*$\mathbf{A}$ *is greater than or equal to* $\mathbf{B}$*") if*
$\mathbf{A} - \mathbf{B} \geq 0$ *("*$(\mathbf{A}-\mathbf{B})$ *is positive")*

[51] Michel and Herget (1993) page 429
[52] Michel and Herget (1993) page 429

APPENDIX H

PARTITION OF UNITY

H.1 Definition and motivation

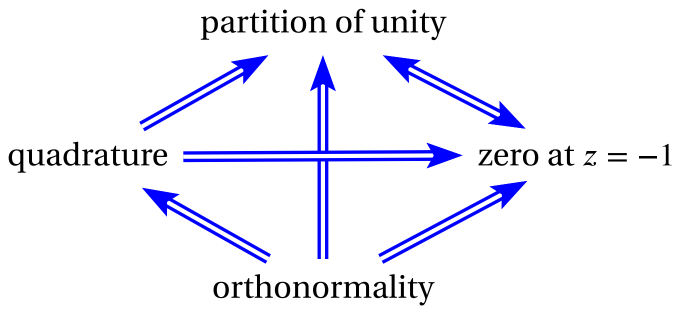


Figure H.1: Implications of scaling function properties

A very common property of scaling functions (Definition ?? page ??) is the *partition of unity* property (Definition H.1 page 284). The partition of unity is a kind of generalization of *orthonormality*; that is, *all* orthonormal scaling functions form a partition of unity. But the partition of unity property is not just a consequence of orthonormality, but also a generalization of orthonormality, in that if you remove the orthonormality constraint, the partition of unity is still a reasonable constraint in and of itself.

There are two reasons why the partition of unity property is a reasonable constraint on its own:

- Without a partition of unity, it is difficult to represent a function as simple as a constant.[1]
- For a multiresolution system $\left(L^2_{\mathbb{R}}, \left(\!\left(V_j\right)\!\right), \phi, \left(\!\left(h_n\right)\!\right)\right)$, the partition of unity property is equivalent to $\sum_{n\in\mathbb{Z}}(-1)^n h_n = 0$ (Theorem ?? page ??). As viewed from the perspective of discrete time signal processing (Appendix ?? page ??), this implies that the scaling coefficients form a "*lowpass filter*"; lowpass filters provide a kind of "coarse approximation" of a function. And that is what the scaling function is "supposed" to do—to provide a coarse approximation at some resolution or "scale" (Definition ?? page ??).

[1] Jawerth and Sweldens (1994) page 8

Definition H.1. [2]

DEF

A function $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ *forms a* ***partition of unity*** *if*

$$\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) = 1 \qquad \forall x \in \mathbb{R}.$$

H.2 Results

Theorem H.1. [3] *Let* $\left(\boldsymbol{L}^2_{\mathbb{R}}, (\!(V_j)\!), \phi, (\!(h_n)\!)\right)$ *be a multiresolution system* (Definition ?? page ??). *Let* $\tilde{\mathbf{F}}\mathsf{f}(\omega)$ *be the* FOURIER TRANSFORM (Definition E.2 page 229) *of a function* $\mathsf{f} \in \boldsymbol{L}^2_{\mathbb{R}}$. *Let* $\bar{\delta}_n$ *be the* KRONECKER DELTA FUNCTION.

THM

$$\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f} = c}_{\text{PARTITION OF UNITY in "time"}} \iff \underbrace{\left[\tilde{\mathbf{F}}\mathsf{f}\right](2\pi n) = \bar{\delta}_n}_{\text{PARTITION OF UNITY in "frequency"}}$$

✎PROOF: Let $\mathbb{Z}_e$ be the set of even integers and $\mathbb{Z}_o$ the set of odd integers.

1. Proof for ($\Longrightarrow$) case:

$$\begin{aligned}
c &= \sum_{m\in\mathbb{Z}} \mathbf{T}^m \mathsf{f}(x) && \text{by left hypothesis} \\
&= \sum_{m\in\mathbb{Z}} \mathsf{f}(x-m) && \text{by definition of } \mathbf{T} \quad \text{(Definition F.3 page 240)} \\
&= \sqrt{2\pi} \sum_{m\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi m) e^{i2\pi m x} && \text{by } \textit{PSF} \quad \text{(Theorem F.2 page 248)} \\
&= \underbrace{\sqrt{2\pi}\,\tilde{\mathsf{f}}(2\pi n) e^{i2\pi n x}}_{\text{real and constant for } n=0} + \underbrace{\sqrt{2\pi} \sum_{m\in\mathbb{Z}\setminus n} \tilde{\mathsf{f}}(2\pi m) e^{i2\pi m x}}_{\text{complex and non-constant}} \\
\Longrightarrow \sqrt{2\pi}\tilde{\mathsf{f}}(2\pi n) &= c\bar{\delta}_n && \text{because } c \text{ is real and constant for all } x
\end{aligned}$$

2. Proof for ($\Longleftarrow$) case:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) && \text{by definition of } \mathbf{T} \quad \text{(Definition F.3 page 240)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \tilde{\mathsf{f}}(2\pi n) e^{-i2\pi n x} && \text{by } \textit{PSF} \quad \text{(Theorem F.2 page 248)} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \frac{c}{\sqrt{2\pi}} \bar{\delta}_n e^{-i2\pi n x} && \text{by right hypothesis} \\
&= \sqrt{2\pi} \frac{c}{\sqrt{2\pi}} e^{-i2\pi 0 x} && \text{by definition of } \bar{\delta}_n \quad \text{(Definition ?? page ??)} \\
&= c
\end{aligned}$$

✏

[2] Kelley (1955) page 171, Munkres (2000) page 225, Jänich (1984) page 116, Willard (1970), page 152 ⟨item 20C⟩, Willard (2004) page 152 ⟨item 20C⟩

[3] Jawerth and Sweldens (1994) page 8

Corollary H.1.

COR
$$\left\{\begin{array}{l}\exists \mathsf{g} \in \boldsymbol{L}^2_{\mathbb{R}} \textit{ such that} \\ \mathsf{f}(x) = \mathbb{1}_{[-1:1)}(x) \star \mathsf{g}(x)\end{array}\right\} \implies \left\{\begin{array}{l}\mathsf{f}(x) \textit{ generates} \\ \textit{a } \text{PARTITION OF UNITY}\end{array}\right\}$$

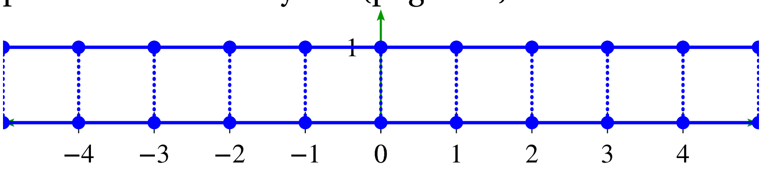
✎PROOF:

$$\begin{aligned}
\mathsf{f}(x) = \mathbb{1}_{[0:1)}(x) \star \mathsf{g}(x) &\implies \tilde{\mathsf{f}}(\omega) = \tilde{\mathbf{F}}\left[\mathbb{1}_{[-1:1)}\right](\omega)\tilde{\mathsf{g}}(\omega) && \text{by \textit{convolution theorem}} && \text{(Theorem E.6 page 232)}\\
&\iff \tilde{\mathsf{f}}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{\sin(\omega)}{\omega}\tilde{\mathsf{g}}(\omega) && \text{by \textit{rectangular pulse} ex.} && \text{(Example E.1 page 236)}\\
&\implies \tilde{\mathsf{f}}(2\pi n) = 0 \\
&\iff \mathsf{f}(x) \text{ generates a \textit{partition of unity}} && \text{by Theorem H.1 page 284}
\end{aligned}$$

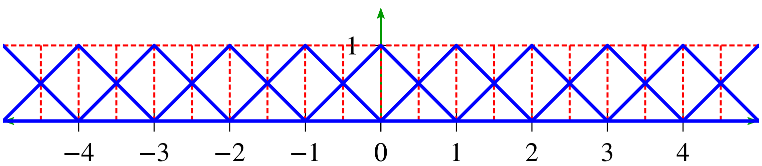
✏

H.3 Examples

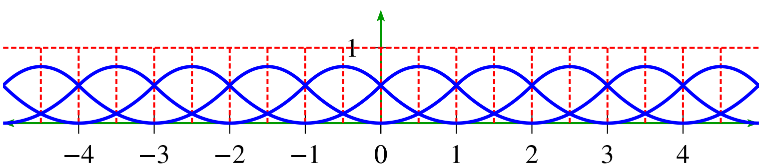
Example H.1. All *B-splines* (Definition 13.3 page 150) form a partition of unity (Theorem ?? page ??). All B-splines of order $n = 1$ or greater can be generated by convolution with a *pulse* function, similar to that specified in Corollary H.1 (page 285) and as illustrated below:



(Example ?? page ??)



(Example ?? page ??)

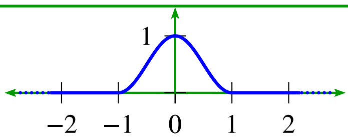


(Example ?? page ??)

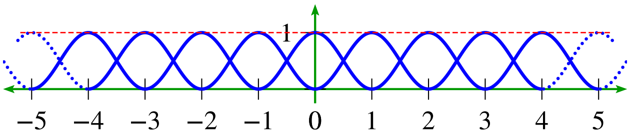
Example H.2. Let a function f be defined in terms of the cosine function (Definition D.2 page 211) as follows:

EX

$$\mathsf{f}(x) \triangleq \begin{cases} \cos^2\left(\frac{\pi}{2}x\right) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

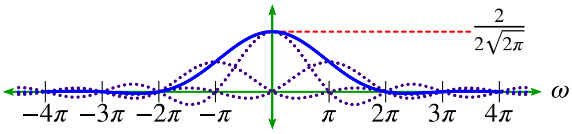


Then f induces a *partition of unity*:



Note that $\tilde{\mathsf{f}}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\underbrace{\frac{2\sin\omega}{\omega}}_{2\,\mathrm{sinc}(\omega)} + \underbrace{\frac{\sin(\omega-\pi)}{(\omega-\pi)}}_{\mathrm{sinc}(\omega-\pi)} + \underbrace{\frac{\sin(\omega+\pi)}{(\omega+\pi)}}_{\mathrm{sinc}(\omega+\pi)}\right]$

and so $\tilde{\mathsf{f}}(2\pi n) = \frac{1}{\sqrt{2\pi}}\bar{\delta}_n$:



✎PROOF: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition F.2 page 239) on a set A.

1. Proof that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f} = 1$ (time domain proof):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\cos^2(x)\mathbb{1}_{[-1:1]}(x) && \text{by definition of } \mathsf{f}(x) \\
&= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\cos^2(x)\mathbb{1}_{[-1:1)}(x) && \text{because } \cos^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 1 \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n) && \text{by definition of } \mathbf{T} \text{ (Definition F.3 page 240)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}_o} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n)}_{\text{odd part}} + \underbrace{\sum_{n\in\mathbb{Z}_e} \cos^2\left(\frac{\pi}{2}(x-n)\right)\mathbb{1}_{[-1:1)}(x-n)}_{\text{even part}} \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-2n)\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-2n-1)\right)\mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x - n\pi\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x - n\pi - \frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \sum_{n\in\mathbb{Z}} (-1)^{2n}\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} (-1)^{2n}\cos^2\left(\frac{\pi}{2}x - \frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \sum_{n\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n) + \sum_{n\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x-2n-1) && \text{by Theorem D.11 page 224} \\
&= \cos^2\left(\frac{\pi}{2}x\right)\sum_{n\in\mathbb{Z}} \mathbb{1}_{[-1:1)}(x-2n) + \sin^2\left(\frac{\pi}{2}x\right)\sum_{n\in\mathbb{Z}} \mathbb{1}_{[-1:1)}(x-2n-1) \\
&= \cos^2\left(\frac{\pi}{2}x\right)\cdot 1 + \sin^2\left(\frac{\pi}{2}x\right)\cdot 1 \\
&= 1 && \text{by } \textit{square identity} \text{ (Theorem D.11 page 224)}
\end{aligned}$$

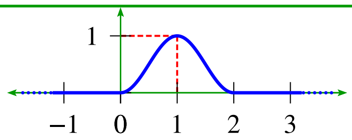
2. Proof that $\tilde{\mathsf{f}}(\omega) = \cdots$: by Example E.3 page 237

✏

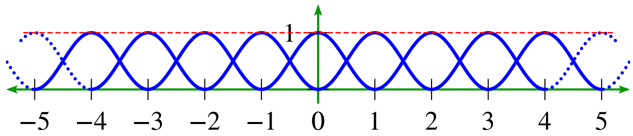
Example H.3. Let a function f be defined in terms of the sine function (Definition D.3 page 211) as follows:

C N T

$$\mathsf{f}(x) \triangleq \begin{cases} \sin^2\left(\frac{\pi}{2}x\right) & \text{for } x \in [0:2] \\ 0 & \text{otherwise} \end{cases}$$



Then $\int_{\mathbb{R}} \mathsf{f}(x)\,\mathrm{d}x = 1$ and f induces a *partition of unity*



✎PROOF:

1. Proof that $\int_{\mathbb{R}} \mathsf{f}(x)\,\mathrm{d}x = 1$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathsf{f}(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x)\,\mathrm{d}x && \text{by definition of } \mathsf{f}(x) \\
&= \int_0^2 \sin^2\left(\frac{\pi}{2}x\right)\mathrm{d}x && \text{by definition of } \mathbb{1}_{A(x)} \text{ (Definition F.2 page 239)} \\
&= \int_0^2 \frac{1}{2}[1-\cos(\pi x)]\,\mathrm{d}x && \text{by Theorem D.11 page 224} \\
&= \frac{1}{2}\left[x - \frac{1}{\pi}\sin(\pi x)\right]_0^2 \\
&= \frac{1}{2}[2-0-0-0] \\
&= 1
\end{aligned}$$

2. Proof that $\mathsf{f}(x)$ forms a *partition of unity*:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2]}(x) && \text{by definition of } \mathsf{f}(x) \\
&= \sum_{n\in\mathbb{Z}} \mathbf{T}^n\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{because } \sin^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 2 \\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^{m-1}\sin^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[0:2)}(x) && \text{where } m \triangleq n+1 \implies n = m-1 \\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m+1)\right)\mathbb{1}_{[0:2)}(x-m+1) && \text{by definition of } \mathbf{T} \text{ (Definition F.3 page 240)} \\
&= \sum_{m\in\mathbb{Z}} \sin^2\left(\frac{\pi}{2}(x-m)+\frac{\pi}{2}\right)\mathbb{1}_{[-1:1)}(x-m) \\
&= \sum_{m\in\mathbb{Z}} \cos^2\left(\frac{\pi}{2}(x-m)\right)\mathbb{1}_{[-1:1)}(x-m) && \text{by Theorem D.11 page 224} \\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1)}(x) && \text{by definition of } \mathbf{T} \text{ (Definition F.3 page 240)} \\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m\cos^2\left(\frac{\pi}{2}x\right)\mathbb{1}_{[-1:1]}(x) && \text{because } \cos^2\left(\frac{\pi}{2}x\right) = 0 \text{ when } x = 1 \\
&= 1 && \text{by Example H.2 page 285}
\end{aligned}$$

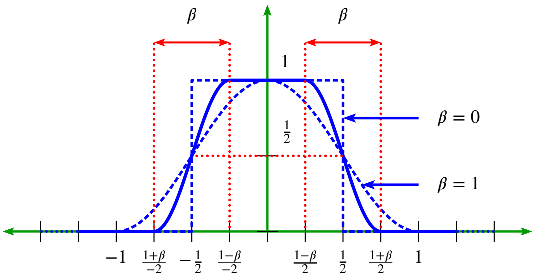
✏

Example H.4 (raised cosine). [4] Let a function f be defined in terms of the cosine function (Definition D.2 page 211) as follows:

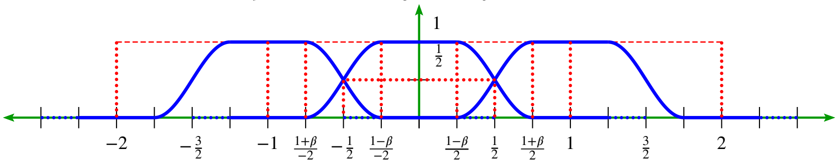
[4] Proakis (2001) pages 560–561

EX

Let $\mathsf{f}(x) \triangleq \begin{cases} 1 & \text{for } 0 \le |x| < \frac{1-\beta}{2} \\ \frac{1}{2}\left\{1 + \cos\left[\frac{\pi}{\beta}\left(|x| - \frac{1-\beta}{2}\right)\right]\right\} & \text{for } \frac{1-\beta}{2} \le |x| < \frac{1+\beta}{2} \\ 0 & \text{otherwise} \end{cases}$



Then f induces a *partition of unity*:



✎PROOF:

1. definition: Let $\mathbb{1}_A(x)$ be the *set indicator function* (Definition F.2 page 239) on a set A.
   Let $A \triangleq \left[\frac{1+\beta}{-2} : \frac{1-\beta}{-2}\right)$, $\quad B \triangleq \left[\frac{1-\beta}{-2} : \frac{1-\beta}{2}\right)$, and $\quad C \triangleq \left[\frac{1-\beta}{2} : \frac{1+\beta}{2}\right)$

2. lemma: $\mathbb{1}_A(x-1) = \mathbb{1}_C(x)$. Proof:

$$\begin{aligned}
\mathbb{1}_A(x-1) &\triangleq \begin{cases} 1 & \text{if } -\frac{1+\beta}{2} \le x - 1 < -\frac{1-\beta}{2} \\ 0 & \text{otherwise} \end{cases} && \text{by definition of } \mathbb{1} \text{ (Definition F.2 page 239) and } A \text{ ((2) lemma page 288)} \\
&= \begin{cases} 1 & \text{if } 1 - \frac{1+\beta}{2} \le x < 1 - \frac{1-\beta}{2} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \frac{1-\beta}{2} \le x < \frac{1+\beta}{2} \\ 0 & \text{otherwise} \end{cases} \\
&\triangleq \mathbb{1}_C(x) && \text{by definition of } \mathbb{1} \text{ (Definition F.2 page 239) and } C \text{ ((2) lemma page 288)}
\end{aligned}$$

3. lemma: $-1 + \frac{1-\beta}{2} = -\beta - \frac{1-\beta}{2}$. Proof:

$$-1 + \frac{1-\beta}{2} = \frac{-2+1-\beta}{2} \quad = \frac{-1-\beta}{2} = (-\beta+\beta) - \left(\frac{1+\beta}{2}\right) \quad = -\beta + \frac{2\beta-1-\beta}{2} = -\beta - \frac{1-\beta}{2}$$

4. Proof that $\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f} = 1$:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathsf{f}(x) &= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n) && \text{by Definition F.3} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_A(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by definition 1 page 288} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) \\
&\quad + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n-1)\mathbb{1}_A(x-n-1) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by Proposition F.1} \\
&= \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n-1)\mathbb{1}_C(x-n) + \sum_{n\in\mathbb{Z}} \mathsf{f}(x-n)\mathbb{1}_B(x-n) && \text{by (2) lemma page 288}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n|-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad+\frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|x-n-1|-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) && \text{by definition of } \mathsf{f}(x) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left((x-n)-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad+\frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(-(x-n-1)-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) && \text{by def. of } \mathbb{1}_C(x) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad+\frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-1+\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad+\frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\beta-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) && \text{by (3) lemma page 288} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad+\frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}-\frac{\pi\beta}{\beta}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1+\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n) \\
&\quad+\frac{1}{2}\sum_{n\in\mathbb{Z}}\left\{1-\cos\left[\frac{\pi}{\beta}\left(x-n-\frac{1-\beta}{2}\right)\right]\right\}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\mathbb{1}_C(x-n)+\frac{1}{2}\sum_{n\in\mathbb{Z}}\mathbb{1}_C(x-n)+\sum_{n\in\mathbb{Z}}\mathbb{1}_B(x-n) \\
&= \sum_{n\in\mathbb{Z}}\mathbb{1}_{B\cup C}(x-n) \\
&= 1
\end{aligned}
$$

✏

APPENDIX I

MATRIX CALCULUS

Optimization problems often require finding the value of some parameter which results in some measure reaching a minimum or maximum value. Often this optimal parameter value can be found by solving the single equation generated by the partial derivative of the measure with respect to the parameter. When there are several parameters, optimization often requires several simultaneous equations generated by the partial derivatives of the measure with respect to each parameter. The need for several partial derivatives and several simultaneous equations leads to a natual union of two branches of mathematics— partial differential equations and linear algebra. In general, we would like to not only be able to take the partial derivative of a scalar with respect to another scalar, but to be able to take the partial derivative of a vector with respect to another vector. This generalization is the problem addressed in this section. Other references are also available.[1]

I.1 First derivative of a vector with respect to a vector

Definition I.1.

DEF

$\boldsymbol{x}$ is a vector with the following properties:

1. $\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ *(n element column vector)*
2. $\dfrac{\partial}{\partial x_k} x_j = \bar{\delta}_{kj}$ *($(x_1, x_2, \ldots, x_n)$ are mutually independent)*

Definition I.2 (Jacobian matrix). [2] *The **gradient of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$**, as well as the **gradient of $\boldsymbol{y}^T$ with respect to $\boldsymbol{x}$**, is defined as*

[1] Graham (1981) ⟨Chapter 4⟩, Haykin (2001) ⟨Appendix B⟩, Moon and Stirling (2000) ⟨Appendix E⟩, Scharf (1991), pages 274–276, Trees (2002) ⟨Section A.7⟩, Felippa (1999)

[2] Graham (1981), page 52, Graham (2018), page 529780486824178§"4.2 The Derivatives of Vectors", Scharf (1991), page 274, Trees (2002), page 1398, Anderson (1984) page 13 ⟨§"2.2.5 Transformation of Variables"⟩, Anderson (1958), page 11 ⟨§"2.2.5 Transformation of Variables"⟩

DEF

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \triangleq \frac{\partial \boldsymbol{y}^T}{\partial \boldsymbol{x}} \triangleq \underbrace{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}}_{n \times m \text{ matrix}} \qquad \forall \boldsymbol{y} \in \mathbb{C}^m$$

Remark I.1. Depending on whether $\boldsymbol{x}$ and $\boldsymbol{y}$ are scalars or vectors, $\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}$ takes on the following forms:[3]

	y scalar	y vector
x scalar	$\frac{\partial y}{\partial x}$	$\frac{\partial \boldsymbol{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} & \cdots & \frac{\partial y_m}{\partial x} \end{bmatrix}$
x vector	$\frac{\partial y}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}$	$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \triangleq \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$

Lemma I.1. *Let $\boldsymbol{x} \in \mathbb{R}^n$ be a vector. Then*

LEM

$$\frac{\partial}{\partial x_k} x_i x_j = \bar{\delta}_{ik} x_j + \bar{\delta}_{jk} x_i = \begin{cases} 2x_k & \text{for } i = j = k \\ x_j & \text{for } i = k \text{ and } j \neq k \\ x_i & \text{for } i \neq k \text{ and } j = k \\ 0 & \text{otherwise} \end{cases}$$

Lemma I.2.

LEM

$$\left(\boldsymbol{x}^H \mathbf{A} \boldsymbol{x}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i^* x_j \qquad \forall \quad \begin{array}{lll} \mathbf{A} \in (\mathbb{C}^n \times \mathbb{C}^n) & (n \times n \text{ array}) & \text{and} \\ \boldsymbol{x} \in \mathbb{C}^n & (n \text{ element column vector}) & \end{array}$$

✎PROOF:

$$\begin{aligned}
\boldsymbol{x}^H \mathbf{A} \boldsymbol{x} &\triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^* \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} && \text{by definitions of } \mathbf{A} \text{ and } \boldsymbol{x} \\
&= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^* \sum_{i=1}^{n} x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \\
&= \sum_{i=1}^{n} x_i \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^* \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ji} x_j^*
\end{aligned}$$

[3] For the generalization of the partial derivative of a matrix with respect to a matrix, see Graham (1981) ⟨chapter 6⟩. Graham uses *kronecker products* to handle the additional dimensions(?)

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i^* x_j$$

✎

Lemma I.3.

LEM

$$\frac{\partial}{\partial \boldsymbol{x}}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] = \mathsf{a}(\boldsymbol{x})\left[\frac{\partial}{\partial \boldsymbol{x}}\mathsf{b}(\boldsymbol{x})\right] + \left[\frac{\partial}{\partial \boldsymbol{x}}\mathsf{a}(\boldsymbol{x})\right]\mathsf{b}(\boldsymbol{x}) \qquad \underbrace{\forall \mathsf{a}, \mathsf{b} : \mathbb{R}^n \to \mathbb{R}}_{\mathsf{a}(\boldsymbol{x}),\,\mathsf{b}(\boldsymbol{x}) \textit{ are functions from a vector } \boldsymbol{x} \textit{ to a scalar in } \mathbb{R}}$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] &= \begin{bmatrix} \frac{\partial}{\partial x_1}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] \\ \frac{\partial}{\partial x_2}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] \\ \vdots \\ \frac{\partial}{\partial x_n}[\mathsf{a}(\boldsymbol{x})\,\mathsf{b}(\boldsymbol{x})] \end{bmatrix} && \text{by definition of } \frac{\partial}{\partial \boldsymbol{x}} && \text{(Definition I.2 page 291)} \\
&= \begin{bmatrix} \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_1} + \mathsf{b}(\boldsymbol{x})\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_1} \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_2} + \mathsf{b}(\boldsymbol{x})\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_2} \\ \vdots \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_n} + \mathsf{b}(\boldsymbol{x})\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_n} \end{bmatrix} && \text{by } \textit{linearity} \text{ of } \frac{\partial}{\partial \boldsymbol{x}} \\
&= \begin{bmatrix} \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_1} \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_2} \\ \\ \mathsf{a}(\boldsymbol{x})\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial x_n} \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_1}\mathsf{b}(\boldsymbol{x}) \\ \frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_2}\mathsf{b}(\boldsymbol{x}) \\ \vdots \\ \frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial x_n}\mathsf{b}(\boldsymbol{x}) \end{bmatrix} && \text{by } \textit{linearity} \text{ of } \textit{vector addition} \\
&= \mathsf{a}(\boldsymbol{x})\left[\frac{\partial \mathsf{b}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right] + \left[\frac{\partial \mathsf{a}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right]\mathsf{b}(\boldsymbol{x})
\end{aligned}$$

✎

Theorem I.1. [4]

LEM

$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x} = \mathbf{I} \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x} &= \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial x_n} & \frac{\partial x_2}{\partial x_n} & \cdots & \frac{\partial x_n}{\partial x_n} \end{bmatrix} && \text{by Definition I.2 page 291} \\
&= \begin{bmatrix} \bar{\delta}_{11} & \bar{\delta}_{21} & \cdots & \bar{\delta}_{n1} \\ \bar{\delta}_{12} & \bar{\delta}_{22} & \cdots & \bar{\delta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\delta}_{1n} & \bar{\delta}_{2n} & \cdots & \bar{\delta}_{nn} \end{bmatrix} && \text{by Definition I.1 page 291 (mutual independence property)}
\end{aligned}$$

[4] Scharf (1991), page 274, Trees (2002), page 1398

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \text{by definition of kronecker delta function } \bar{\delta}$$

$$= \mathbf{I} \qquad \text{by definition of identity operator } \mathbf{I}$$

✏

Theorem I.2.

THM
$$\frac{\partial}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x}) = \mathbf{A}^T + \sum_{i=1}^{n} \left(\frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi} \end{bmatrix} \right) x_i \qquad \forall \boldsymbol{x} \in \mathbb{R}^n,\ \mathbf{A} \in \mathbb{C}^m \times \mathbb{C}^n$$

✎PROOF: Let

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{A} \triangleq \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\frac{\partial}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{x}} \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) \qquad \text{by definition of } A \text{ and } \boldsymbol{x}$$

$$= \frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} \sum_{i=1}^{n} a_{1i} x_i \\ \sum_{i=1}^{n} a_{2i} x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} x_i \end{bmatrix} \qquad \text{by matrix multiplication}$$

$$= \frac{\partial}{\partial \boldsymbol{x}} \sum_{i=1}^{n} \begin{bmatrix} a_{1i} x_i \\ a_{2i} x_i \\ \vdots \\ a_{mi} x_i \end{bmatrix}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} x_i \\ a_{2i} x_i \\ \vdots \\ a_{mi} x_i \end{bmatrix}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial a_{1i} x_i}{\partial x_1} & \frac{\partial a_{2i} x_i}{\partial x_1} & \cdots & \frac{\partial a_{mi} x_i}{\partial x_1} \\ \frac{\partial a_{1i} x_i}{\partial x_2} & \frac{\partial a_{2i} x_i}{\partial x_2} & \cdots & \frac{\partial a_{mi} x_i}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_{1i} x_i}{\partial x_n} & \frac{\partial a_{2i} x_i}{\partial x_n} & \cdots & \frac{\partial a_{mi} x_i}{\partial x_n} \end{bmatrix} \qquad \text{by Definition I.2 page 291}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} a_{1i} \frac{\partial x_i}{\partial x_1} + \frac{\partial a_{1i}}{\partial x_1} x_i & a_{2i} \frac{\partial x_i}{\partial x_1} + \frac{\partial a_{2i}}{\partial x_1} x_i & \cdots & a_{mi} \frac{\partial x_i}{\partial x_1} + \frac{\partial a_{mi}}{\partial x_1} x_i \\ a_{1i} \frac{\partial x_i}{\partial x_2} + \frac{\partial a_{1i}}{\partial x_2} x_i & a_{2i} \frac{\partial x_i}{\partial x_2} + \frac{\partial a_{2i}}{\partial x_2} x_i & \cdots & a_{mi} \frac{\partial x_i}{\partial x_2} + \frac{\partial a_{mi}}{\partial x_2} x_i \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i} \frac{\partial x_i}{\partial x_n} + \frac{\partial a_{1i}}{\partial x_n} x_i & a_{2i} \frac{\partial x_i}{\partial x_n} + \frac{\partial a_{2i}}{\partial x_n} x_i & \cdots & a_{mi} \frac{\partial x_i}{\partial x_n} + \frac{\partial a_{mi}}{\partial x_n} x_i \end{bmatrix} \qquad \text{by Lemma I.3 page 293}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} a_{1i}\frac{\partial x_i}{\partial x_1} & a_{2i}\frac{\partial x_i}{\partial x_1} & \cdots & a_{mi}\frac{\partial x_i}{\partial x_1} \\ a_{1i}\frac{\partial x_i}{\partial x_2} & a_{2i}\frac{\partial x_i}{\partial x_2} & \cdots & a_{mi}\frac{\partial x_i}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i}\frac{\partial x_i}{\partial x_n} & a_{2i}\frac{\partial x_i}{\partial x_n} & \cdots & a_{mi}\frac{\partial x_i}{\partial x_n} \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial a_{1i}}{\partial x_1}x_i & \frac{\partial a_{2i}}{\partial x_1}x_i & \cdots & \frac{\partial a_{mi}}{\partial x_1}x_i \\ \frac{\partial a_{1i}}{\partial x_2}x_i & \frac{\partial a_{2i}}{\partial x_2}x_i & \cdots & \frac{\partial a_{mi}}{\partial x_2}x_i \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_{1i}}{\partial x_n}x_i & \frac{\partial a_{2i}}{\partial x_n}x_i & \cdots & \frac{\partial a_{mi}}{\partial x_n}x_i \end{bmatrix}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} a_{1i}\bar{\delta}_{i1} & a_{2i}\bar{\delta}_{i1} & \cdots & a_{mi}\bar{\delta}_{i1} \\ a_{1i}\bar{\delta}_{i2} & a_{2i}\bar{\delta}_{i2} & \cdots & a_{mi}\bar{\delta}_{i2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i}\bar{\delta}_{in} & a_{2i}\bar{\delta}_{in} & \cdots & a_{mi}\bar{\delta}_{in} \end{bmatrix} + \sum_{i=1}^{n} \left(\frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi} \end{bmatrix} \right) x_i \qquad \text{by Lemma I.1}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} + \sum_{i=1}^{n} \left(\frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi} \end{bmatrix} \right) x_i \qquad \text{by definition of } \bar{\delta}$$

$$= \mathbf{A}^T + \sum_{i=1}^{n} \left(\frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi} \end{bmatrix} \right) x_i$$

✏

Theorem I.3 (Affine equations). [5]

THM

$\mathbf{A}$ *and* $\mathbf{B}$ *are independent of* $\boldsymbol{x}$ $\implies \begin{cases} \frac{\partial}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x}) = \mathbf{A}^T & \forall \boldsymbol{x} \in \mathbb{R}^n,\ \mathbf{A} \in \mathbb{C}^m \times \mathbb{C}^n \\ \frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T\mathbf{B}) = \mathbf{B} & \forall \boldsymbol{x} \in \mathbb{R}^n,\ \mathbf{B} \in \mathbb{C}^n \times \mathbb{C}^m \end{cases}$

✎PROOF: Let $\mathbf{B} \triangleq \mathbf{A}^T$.

1. Proof that $\frac{\partial}{\partial x}(A\boldsymbol{x}) = \mathbf{A}^T$:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x}) &= \mathbf{A}^T + \sum_{i=1}^{n} \left(\frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi} \end{bmatrix} \right) x_i && \text{by Theorem I.2 page 294} \\
&= \mathbf{A}^T + \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial}{\partial \boldsymbol{x}} a_{1i} & \frac{\partial}{\partial \boldsymbol{x}} a_{2i} & \cdots & \frac{\partial}{\partial \boldsymbol{x}} a_{mi} \end{bmatrix} x_i \\
&= \mathbf{A}^T + \underbrace{\sum_{i=1}^{n} \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} x_i}_{0} && \text{by left hypothesis} \\
&= \mathbf{A}^T
\end{aligned}$$

2. Proof that $\frac{\partial}{\partial x}\left(\boldsymbol{x}^T\mathbf{B}\right) = \mathbf{B}$:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T\mathbf{B}\right) &= \frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T\mathbf{A}^T\right) && \text{by definition of } \mathbf{B} \\
&= \frac{\partial}{\partial \boldsymbol{x}}\left[(\mathbf{A}\boldsymbol{x})^T\right] \\
&= \frac{\partial}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x}) && \text{by Definition I.2 page 291} \\
&= \mathbf{A}^T && \text{by Theorem I.3 page 295} \\
&= \mathbf{B} && \text{by definition of } \mathbf{B}
\end{aligned}$$

✏

[5] Graham (1981), page 54, Graham (2018), page 549780486824178§"4.2 The Derivatives of Vectors"

Theorem I.4 (Product rule). [6] *Let $\mathbf{y}$ and $\mathbf{z}$ be functions of $\mathbf{x}$ and*

THM
$$\frac{\partial}{\partial \mathbf{x}}\mathbf{z}^T\mathbf{y} = \frac{\partial \mathbf{z}}{\partial \mathbf{x}}\mathbf{y} + \frac{\partial \mathbf{y}}{\partial \mathbf{x}}\mathbf{z} \qquad \forall \mathbf{x}\in\mathbb{R}^n,\ \mathbf{y}\in\mathbb{R}^m,\ \mathbf{z}\in\mathbb{R}^m$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x}}\mathbf{z}^T\mathbf{y} &= \frac{\partial}{\partial \mathbf{x}}\sum_{k=1}^m z_k y_k \\
&= \sum_{k=1}^m \frac{\partial}{\partial \mathbf{x}} z_k y_k \\
&= \sum_{k=1}^m \frac{\partial z_k}{\partial \mathbf{x}} y_k + \sum_{k=1}^m \frac{\partial y_k}{\partial \mathbf{x}} z_k && \text{by Lemma I.3 page 293} \\
&= \begin{bmatrix}
\frac{\partial z_1}{\partial x_1}y_1 & + & \frac{\partial z_2}{\partial x_1}y_2 & + & \cdots & + & \frac{\partial z_n}{\partial x_1}y_n \\
\frac{\partial z_1}{\partial x_1}y_1 & + & \frac{\partial z_2}{\partial x_1}y_2 & + & \cdots & + & \frac{\partial z_n}{\partial x_1}y_n \\
\vdots & & & & \ddots & & \vdots \\
\frac{\partial z_1}{\partial x_1}y_1 & + & \frac{\partial z_2}{\partial x_1}y_2 & + & \cdots & + & \frac{\partial z_n}{\partial x_1}y_n
\end{bmatrix} + \begin{bmatrix}
\frac{\partial y_1}{\partial x_1}z_1 & + & \frac{\partial y_2}{\partial x_1}z_2 & + & \cdots & + & \frac{\partial y_n}{\partial x_1}z_n \\
\frac{\partial y_1}{\partial x_1}z_1 & + & \frac{\partial y_2}{\partial x_1}z_2 & + & \cdots & + & \frac{\partial y_n}{\partial x_1}z_n \\
\vdots & & & & \ddots & & \vdots \\
\frac{\partial y_1}{\partial x_1}z_1 & + & \frac{\partial y_2}{\partial x_1}z_2 & + & \cdots & + & \frac{\partial y_n}{\partial x_1}z_n
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_1} \\
\frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_1} \\
\vdots & & \ddots & \vdots \\
\frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_1}
\end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\
\vdots & & \ddots & \vdots \\
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1}
\end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \frac{\partial \mathbf{z}}{\partial \mathbf{x}}\mathbf{y} + \frac{\partial \mathbf{y}}{\partial \mathbf{x}}\mathbf{z}
\end{aligned}$$

✏

Theorem I.5.

THM
$$\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^T A\mathbf{x}\right) = \mathbf{A}\mathbf{x} + \mathbf{A}^T\mathbf{x} + \left[\sum_{i=1}^n \left(\frac{\partial}{\partial \mathbf{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni}\end{bmatrix}\right)x_i\right]\mathbf{x} \qquad \forall \mathbf{x}\in\mathbb{R}^n, \mathbf{A}\in\mathbb{R}^n\times\mathbb{R}^n$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^T\mathbf{A}\mathbf{x}\right) &= \left[\frac{\partial}{\partial \mathbf{x}}\mathbf{x}\right]\mathbf{A}\mathbf{x} + \left[\frac{\partial}{\partial \mathbf{x}}\mathbf{A}\mathbf{x}\right]\mathbf{x} && \text{by Theorem I.4 page 296} \\
&= \mathbf{I}\mathbf{A}\mathbf{x} + \left[\mathbf{A}^T + \sum_{i=1}^n \left(\frac{\partial}{\partial \mathbf{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni}\end{bmatrix}\right)x_i\right]\mathbf{x} && \text{by Theorem I.1 and Theorem I.2} \\
&= \mathbf{A}\mathbf{x} + \mathbf{A}^T\mathbf{x} + \left[\sum_{i=1}^n \left(\frac{\partial}{\partial \mathbf{x}}\begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni}\end{bmatrix}\right)x_i\right]\mathbf{x} && \text{by definition of identity operator } \mathbf{I}
\end{aligned}$$

✏

Theorem I.6 (Quadratic form). [7]

THM
$$\mathbf{A} \text{ is independent of } \mathbf{x} \quad\Longrightarrow\quad \frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^T A\mathbf{x}\right) = A\mathbf{x} + A^T\mathbf{x} \qquad \forall \mathbf{x}\in\mathbb{R}^n, \mathbf{A}\in\mathbb{R}^n\times\mathbb{R}^n$$

[6] Scharf (1991), page 274, Trees (2002), page 1398
[7] Graham (1981), page 54

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T \mathbf{A}\boldsymbol{x}\right) &= \left[\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}\right]\mathbf{A}\boldsymbol{x} + \left[\frac{\partial}{\partial \boldsymbol{x}}\mathbf{A}\boldsymbol{x}\right]\boldsymbol{x} && \text{by Theorem I.4 page 296}\\
&= \mathbf{I}\mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x} && \text{by Theorem I.1 page 293 and Theorem I.3 page 295}
\end{aligned}$$

✏

Corollary I.1. [8]

COR
$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T\boldsymbol{x}) = 2\boldsymbol{x} \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

✎PROOF:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T\boldsymbol{x}) &= \frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T\mathbf{I}\boldsymbol{x}) && \text{by property of identity operator } I\\
&= \mathbf{I}\boldsymbol{x} + \mathbf{I}^T\boldsymbol{x} && \text{by previous result 3.}\\
&= \boldsymbol{x} + \boldsymbol{x} && \text{by property of identity operator } I\\
&= 2\boldsymbol{x}
\end{aligned}$$

✏

Theorem I.7 (Chain rule). [9] *Let* $\boldsymbol{z}$ *be a function of* $\boldsymbol{y}$ *and* $\boldsymbol{y}$ *a function of* $\boldsymbol{x}$ *and*

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} \qquad \boldsymbol{y} \triangleq \begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_m\end{bmatrix} \qquad \boldsymbol{z} \triangleq \begin{bmatrix} z_1\\ z_2\\ \vdots\\ z_k\end{bmatrix}$$

THM
$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{z} = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{y}}$$

✎PROOF:

$$\begin{aligned}
\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}} &= \begin{bmatrix}
\frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} & \cdots & \frac{\partial z_k}{\partial x_1}\\
\frac{\partial z_1}{\partial x_2} & \frac{\partial z_2}{\partial x_2} & \cdots & \frac{\partial z_k}{\partial x_2}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial z_1}{\partial x_n} & \frac{\partial z_2}{\partial x_n} & \cdots & \frac{\partial z_k}{\partial x_n}
\end{bmatrix}\\
&= \begin{bmatrix}
\sum_{j=0}^{m}\frac{\partial z_1}{\partial y_j}\frac{\partial y_j}{\partial x_1} & \sum_{j=0}^{m}\frac{\partial z_2}{\partial y_j}\frac{\partial y_j}{\partial x_1} & \cdots & \sum_{j=0}^{m}\frac{\partial z_k}{\partial y_j}\frac{\partial y_j}{\partial x_1}\\
\sum_{j=0}^{m}\frac{\partial z_1}{\partial y_j}\frac{\partial y_j}{\partial x_2} & \sum_{j=0}^{m}\frac{\partial z_2}{\partial y_j}\frac{\partial y_j}{\partial x_2} & \cdots & \sum_{j=0}^{m}\frac{\partial z_k}{\partial y_j}\frac{\partial y_j}{\partial x_2}\\
\vdots & \vdots & \ddots & \vdots\\
\sum_{j=0}^{m}\frac{\partial z_1}{\partial y_j}\frac{\partial y_j}{\partial x_n} & \sum_{j=0}^{m}\frac{\partial z_2}{\partial y_j}\frac{\partial y_j}{\partial x_n} & \cdots & \sum_{j=0}^{m}\frac{\partial z_k}{\partial y_j}\frac{\partial y_j}{\partial x_n}
\end{bmatrix}\\
&= \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1}\\
\frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial z_1}{\partial y_1} & \frac{\partial z_2}{\partial y_1} & \cdots & \frac{\partial z_k}{\partial y_1}\\
\frac{\partial z_1}{\partial y_2} & \frac{\partial z_2}{\partial y_2} & \cdots & \frac{\partial z_k}{\partial y_2}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial z_1}{\partial y_m} & \frac{\partial z_2}{\partial y_m} & \cdots & \frac{\partial z_k}{\partial y_m}
\end{bmatrix}\\
&= \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{y}}
\end{aligned}$$

[8] Graham (1981), page 54
[9] Graham (1981), pages 54–55

✏

I.2 First derivative of a matrix with respect to a scalar

Definition I.3. *Let* $x \in \mathbb{R}$, $\{y_{jk} \in \mathbb{C} | j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n\}$ *and*

$$Y = \underbrace{\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}}_{m \times n \text{ matrix}}$$

The ***derivative of*** Y ***with respect to*** x *is*

DEF
$$\frac{\mathrm{d}Y}{\mathrm{d}x} \triangleq \underbrace{\begin{bmatrix} \frac{\mathrm{d}y_{11}}{\mathrm{d}x} & \frac{\mathrm{d}y_{12}}{\mathrm{d}x} & \cdots & \frac{\mathrm{d}y_{1n}}{\mathrm{d}x} \\ \frac{\mathrm{d}y_{21}}{\mathrm{d}x} & \frac{\mathrm{d}y_{22}}{\mathrm{d}x} & \cdots & \frac{\mathrm{d}y_{2n}}{\mathrm{d}x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}y_{m1}}{\mathrm{d}x} & \frac{\mathrm{d}y_{m2}}{\mathrm{d}x} & \cdots & \frac{\mathrm{d}y_{mn}}{\mathrm{d}x} \end{bmatrix}}_{m \times n \text{ matrix}}$$

Theorem I.8. [10] *Let* $x \in \mathbb{R}$, $\{y_{jp} \in \mathbb{C} | j = 1, 2, \ldots, m;\ p = 1, 2, \ldots, n\}$, $\{w_{jp} \in \mathbb{C} | j = 1, 2, \ldots, n;\ p = 1, 2, \ldots, k\}$, *and*

$$Y = \underbrace{\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}}_{m \times n \text{ matrix}} \qquad W = \underbrace{\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{p1} & w_{p2} & \cdots & w_{pk} \end{bmatrix}}_{p \times k \text{ matrix}}$$

THM
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y + W\right) &= \frac{\mathrm{d}}{\mathrm{d}x}Y + \frac{\mathrm{d}}{\mathrm{d}x}W && (\text{for } p = m,\ k = n) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(YW\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}Y\right)W + Y\left(\frac{\mathrm{d}}{\mathrm{d}x}W\right) && (\text{for } p = n) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^T\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}Y\right)^T && \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^{-1}\right) &= -Y^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x}Y\right)Y^{-1} && (\text{for } m = n \text{ and } Y \text{ invertible})
\end{aligned}$$

✎PROOF:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(Y + W\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{bmatrix}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} y_{11} + w_{11} & y_{12} + w_{12} & \cdots & y_{1n} + w_{1n} \\ y_{21} + w_{21} & y_{22} + w_{22} & \cdots & y_{2n} + w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} + w_{m1} & y_{m2} + w_{m2} & \cdots & y_{mn} + w_{mn} \end{bmatrix}
\end{aligned}$$

[10] Gradshteyn and Ryzhik (1980), pages 1106–1107

$$
\begin{aligned}
&= \begin{bmatrix} (y_{11}+w_{11})' & (y_{12}+w_{12})' & \cdots & (y_{1n}+w_{1n})' \\ (y_{21}+w_{21})' & (y_{22}+w_{22})' & \cdots & (y_{2n}+w_{2n})' \\ \vdots & \vdots & \ddots & \vdots \\ (y_{m1}+w_{m1})' & (y_{m2}+w_{m2})' & \cdots & (y_{mn}+w_{mn})' \end{bmatrix} \\
&= \begin{bmatrix} y'_{11}+w'_{11} & y'_{12}+w'_{12} & \cdots & y'_{1n}+w'_{1n} \\ y'_{21}+w'_{21} & y'_{22}+w'_{22} & \cdots & y'_{2n}+w'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{m1}+w'_{m1} & y'_{m2}+w'_{m2} & \cdots & y'_{mn}+w'_{mn} \end{bmatrix} \\
&= \begin{bmatrix} y'_{11} & y'_{12} & \cdots & y'_{1n} \\ y'_{21} & y'_{22} & \cdots & y'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{m1} & y'_{m2} & \cdots & y'_{mn} \end{bmatrix} + \begin{bmatrix} w'_{11} & w'_{12} & \cdots & w'_{1n} \\ w'_{21} & w'_{22} & \cdots & w'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w'_{m1} & w'_{m2} & \cdots & w'_{mn} \end{bmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} + \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{bmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} Y + \frac{\mathrm{d}}{\mathrm{d}x} W
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(YW\right) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nk} \end{bmatrix} \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} \sum_{j=1}^{n} y_{1j}w_{j1} & \sum_{j=1}^{n} y_{1j}w_{j2} & \cdots & \sum_{j=1}^{n} y_{1j}w_{jk} \\ \sum_{j=1}^{n} y_{2j}w_{j1} & \sum_{j=1}^{n} y_{2j}w_{j2} & \cdots & \sum_{j=1}^{n} y_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} y_{mj}w_{j1} & \sum_{j=1}^{n} y_{mj}w_{j2} & \cdots & \sum_{j=1}^{n} y_{mj}w_{jk} \end{bmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{n} \begin{bmatrix} y_{1j}w_{j1} & y_{1j}w_{j2} & \cdots & y_{1j}w_{jk} \\ y_{2j}w_{j1} & y_{2j}w_{j2} & \cdots & y_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y_{mj}w_{j1} & y_{mj}w_{j2} & \cdots & y_{mj}w_{jk} \end{bmatrix} \\
&= \sum_{j=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} y_{1j}w_{j1} & y_{1j}w_{j2} & \cdots & y_{1j}w_{jk} \\ y_{2j}w_{j1} & y_{2j}w_{j2} & \cdots & y_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y_{mj}w_{j1} & y_{mj}w_{j2} & \cdots & y_{mj}w_{jk} \end{bmatrix} \\
&= \sum_{j=1}^{n} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x}(y_{1j}w_{j1}) & \frac{\mathrm{d}}{\mathrm{d}x}(y_{1j}w_{j2}) & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}(y_{1j}w_{jk}) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y_{2j}w_{j1}) & \frac{\mathrm{d}}{\mathrm{d}x}(y_{2j}w_{j2}) & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}(y_{2j}w_{jk}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}x}(y_{mj}w_{j1}) & \frac{\mathrm{d}}{\mathrm{d}x}(y_{mj}w_{j2}) & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}(y_{mj}w_{jk}) \end{bmatrix} \\
&= \sum_{j=1}^{n} \begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ y'_{1j}w_{j1}+y_{1j}w'_{j1} & y'_{1j}w_{j2}+y_{1j}w'_{j2} & \cdots & y'_{1j}w_{jk}+y_{1j}w'_{jk} \\ y'_{2j}w_{j1}+y_{2j}w'_{j1} & y'_{2j}w_{j2}+y_{2j}w'_{j2} & \cdots & y'_{2j}w_{jk}+y_{2j}w'_{jk} \\ y'_{mj}w_{j1}+y_{mj}w'_{j1} & y'_{mj}w_{j2}+y_{mj}w'_{j2} & \cdots & y'_{mj}w_{jk}+y_{mj}w'_{jk} \end{bmatrix}
\end{aligned}
$$

$$= \sum_{j=1}^{n} \left(\begin{bmatrix} y'_{1j}w_{j1} & y'_{1j}w_{j2} & \cdots & y'_{1j}w_{jk} \\ y'_{2j}w_{j1} & y'_{2j}w_{j2} & \cdots & y'_{2j}w_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{mj}w_{j1} & y'_{mj}w_{j2} & \cdots & y'_{mj}w_{jk} \end{bmatrix} + \begin{bmatrix} y_{1j}w'_{j1} & y_{1j}w'_{j2} & \cdots & y_{1j}w'_{jk} \\ y_{2j}w'_{j1} & y_{2j}w'_{j2} & \cdots & y_{2j}w'_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ y_{mj}w'_{j1} & y_{mj}w'_{j2} & \cdots & y_{mj}w'_{jk} \end{bmatrix} \right)$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x} Y\right) W + Y \left(\frac{\mathrm{d}}{\mathrm{d}x} W\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^T\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}^T \right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ y_{12} & y_{22} & \cdots & y_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} y'_{11} & y'_{21} & \cdots & y'_{n1} \\ y'_{12} & y'_{22} & \cdots & y'_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{1n} & y'_{2n} & \cdots & y'_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} y'_{11} & y'_{12} & \cdots & y'_{1n} \\ y'_{21} & y'_{22} & \cdots & y'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{m1} & y'_{m2} & \cdots & y'_{mn} \end{bmatrix}^T$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} \right)^T$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Y^{-1}\right) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\operatorname{adj} Y}{|Y|}$$

$$\vdots$$

no proof at this time

$$\vdots$$

$$= -Y^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x} Y\right) Y^{-1}$$

✏

I.3 Second derivative of a scalar with respect to a vector

Definition I.4. [11] *Let*

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

[11] Lieb and Loss (2001), page 240, Horn and Johnson (1990), page 167

*The **Hessian matrix** of a scalar y with respect to the vector $\boldsymbol{x}$ is*

DEF
$$\frac{\partial^2 y}{\partial \boldsymbol{x}^2} = \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial y}{\partial \boldsymbol{x}}\right) = \frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1}\frac{\partial y}{\partial x_1} & \frac{\partial}{\partial x_1}\frac{\partial y}{\partial x_2} & \cdots & \frac{\partial}{\partial x_1}\frac{\partial y}{\partial x_n} \\ \frac{\partial}{\partial x_2}\frac{\partial y}{\partial x_1} & \frac{\partial}{\partial x_2}\frac{\partial y}{\partial x_2} & \cdots & \frac{\partial}{\partial x_2}\frac{\partial y}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_n}\frac{\partial y}{\partial x_1} & \frac{\partial}{\partial x_n}\frac{\partial y}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n}\frac{\partial y}{\partial x_n} \end{bmatrix}}_{n\times n \text{ matrix}}$$

I.4 Multiple derivatives of a vector with respect to a scalar

Definition I.5. *Let*

$$\boldsymbol{y} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

The derivative of a vector $\boldsymbol{y}$ with respect to the scalar x is

DEF
$$\begin{bmatrix} \boldsymbol{y} \\ \frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{y} \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{y} \\ \vdots \\ \frac{\mathrm{d}^n}{\mathrm{d}x^n}\boldsymbol{y} \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_m \\ \frac{\mathrm{d}}{\mathrm{d}x}y_1 & \frac{\mathrm{d}}{\mathrm{d}x}y_2 & \cdots & \frac{\mathrm{d}}{\mathrm{d}x}y_m \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}y_1 & \frac{\mathrm{d}^2}{\mathrm{d}x^2}y_2 & \cdots & \frac{\mathrm{d}^2}{\mathrm{d}x^2}y_m \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}^n}{\mathrm{d}x^n}y_1 & \frac{\mathrm{d}^n}{\mathrm{d}x^n}y_2 & \cdots & \frac{\mathrm{d}^n}{\mathrm{d}x^n}y_m \end{bmatrix}}_{(n+1)\times m \text{ matrix}}$$

APPENDIX J

TRANSLATION SPACES

J.1 Translation

J.1.1 Definitions

Definition J.1. *Let X be a set and $\mathbf{I}$ be the identity operator on X.*

DEF

$\mathbf{T}_x$ *is a **translation operator** on X if*

1. $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$ $\quad \forall A \in 2^X$ $\quad$ (IDENTITY) $\quad$ *and*
2. $\mathbf{T}_x \mathbf{T}_y = \mathbf{T}_y \mathbf{T}_x$ $\quad \forall x, y \in X$ $\quad$ (COMMUTATIVE) $\quad$ *and*
3. $\mathbf{T}_x \bigcup_{i \in I} A_i = \bigcup_{i \in I} \mathbf{T}_x A_i$ $\quad \forall A, Y \in 2^X, x \in X$ $\quad$ (DISTRIBUTIVE *over* $\cup$) $\quad$ *and*
4. $\bigcup_{b \in B} \mathbf{T}_b A = \bigcup_{a \in A} \mathbf{T}_a B$ $\quad \forall A, B \in 2^X$ $\quad$ *and*
5. $\mathbf{T}_x (A \cap B) = \left(\mathbf{T}_x A\right) \cap \left(\mathbf{T}_x B\right)$ $\quad \forall A, B \in 2^X, x \in X$ $\quad$ *and*
6. $\mathbf{T}_x \left(A^c\right) = \mathsf{c}\left(\mathbf{T}_x A\right)$ $\quad \forall A, B \in 2^X, x \in X.$

*The pair $(X, \mathbf{T})$ is a **translation space** on X.*

Definition J.2. [1] *Let X be a set on which is defined the translation opertor $\mathbf{T}_x$. **Minkowski addition** $\oplus$ and **Minkowski subtraction** $\ominus$ is defined as follows:*

DEF

$$A \oplus B = \bigcup_{b \in B} \mathbf{T}_b A \qquad \forall\, A, B \in 2^X \qquad \text{(MINKOWSKI ADDITION)}$$

$$A \ominus B = \bigcap_{b \in B} \mathbf{T}_b A \qquad \forall\, A, B \in 2^X \qquad \text{(MINKOWSKI SUBTRACTION)}$$

Theorem J.1 (next) shows a relationship between Minkowski addition and Minkowski subtraction.

Theorem J.1 (de Morgan relations)**.** [2] *Let $(X, +)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\mathsf{c}(A \oplus B) = A^c \ominus B \qquad \forall\, A, B \in 2^X$$

$$\mathsf{c}(A \ominus B) = A^c \oplus B \qquad \forall\, A, B \in 2^X$$

[1] Matheron (1975) page 17, Lay (1982) page 7

[2] Pitas and Venetsanopoulos (1991) page 159

✎PROOF:

$$\begin{aligned}
\mathsf{c}(A \oplus B) &= \mathsf{c}\left(\bigcup_{b\in B} \mathbf{T}_b A\right) && \text{by Definition J.2 page 303}\\
&= \bigcap_{b\in B} \mathsf{c}(\mathbf{T}_b A) && \text{by Demorgan relation page 303}\\
&= \bigcap_{b\in B} \mathbf{T}_b(A^{\mathsf{c}}) && \text{by Definition J.1 page 303}\\
&= A^{\mathsf{c}} \ominus B && \text{by Theorem J.2 page 306}
\end{aligned}$$

$$\begin{aligned}
\mathsf{c}(A \ominus B) &= \mathsf{c}\left(\bigcap_{b\in B} \mathbf{T}_b A\right) && \text{by Definition J.2 page 303}\\
&= \bigcup_{b\in B} \mathsf{c}(\mathbf{T}_b A) && \text{by Demorgan relation page 303}\\
&= \bigcup_{b\in B} \mathbf{T}_b(A^{\mathsf{c}}) && \text{by Definition J.1 page 303}\\
&= A^{\mathsf{c}} \oplus B && \text{by Theorem J.2 page 306}
\end{aligned}$$

✏

J.1.2 Examples

Example J.1 (Translation on groups). [3] Let $\oplus$ be the Minkowski addition operator defined in terms of the *translation operator* $\mathbf{T}$. Let $(X, +)$ be a *group*.

EX

$$\left\{ \mathbf{T}_x A \triangleq \{a + x | a \in A\} \quad \forall A \in 2^X \right\} \implies \left\{ \begin{array}{ll} \mathbf{T}_x \text{ is a translation operator} & \text{and} \\ A \oplus B = \{a + b | a \in A \text{ and } b \in B\} & \forall A, B \in 2^X \end{array} \right\}$$

✎PROOF:

1. Proof that $\exists 0 \in X$ such that $\mathbf{T}_0 = \mathbf{I}$:

$$\begin{aligned}
\mathbf{T}_0 A &= \{a + 0 | a \in A\} && \text{by definition of } \mathbf{T}_x\\
&= \{a | a \in A\} && \text{by additive identity property of groups}\\
&= A
\end{aligned}$$

2. Proof that $\mathbf{T}_x\mathbf{T}_y = \mathbf{T}_y\mathbf{T}_x$:

$$\begin{aligned}
\mathbf{T}_x\mathbf{T}_y A &= \mathbf{T}_x \{a + y | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a + y + x | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \{a + x + y | a \in A\} && \text{by commutative property of groups}\\
&= \mathbf{T}_y \{a + x | a \in A\} && \text{by definition of } \mathbf{T}_y\\
&= \mathbf{T}_y\mathbf{T}_x \{a | a \in A\} && \text{by definition of } \mathbf{T}_x
\end{aligned}$$

[3] Matheron (1975) pages 16–17, Pitas and Venetsanopoulos (1991) page 159, Lay (1982) page 7

3. Proof that $\mathbf{T}_x \bigcup_{i\in I} A_i = \bigcup_{i\in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcup_i A_i &= \left\{ y+x | y \in \bigcup_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y+x | \bigvee_i y \in A_i \right\} \\
&= \bigcup_i \left\{ y+x | y \in A_i \right\} \\
&= \bigcup_i \mathbf{T}_x \left\{ y | y \in A_i \right\} \\
&= \bigcup_i \mathbf{T}_x A
\end{aligned}$$

4. Proof that $\bigcup_{b\in B} \mathbf{T}_b A = \bigcup_{a\in A} \mathbf{T}_a B$:

$$\begin{aligned}
\bigcup_{b\in B} \mathbf{T}_b A &= \bigcup_{b\in B} \{a+b | a \in A\} && \text{by definition of } \mathbf{T}_x \\
&= \left\{ a+b | a \in A \text{ and } b \in B \right\} \\
&= \left\{ b+a | b \in B \text{ and } a \in A \right\} \\
&= \bigcup_{a\in A} \{b+a | b \in B\} \\
&= \bigcup_{a\in A} \mathbf{T}_a B
\end{aligned}$$

5. Proof that $\mathbf{T}_x \bigcap_{i\in I} A_i = \bigcap_{i\in I} \mathbf{T}_x A_i$:

$$\begin{aligned}
\mathbf{T}_x \bigcap_i A_i &= \left\{ y+x | y \in \bigcap_i A_i \right\} && \text{by definition of } \mathbf{T}_y \\
&= \left\{ y+x | \bigwedge_i y \in A_i \right\} \\
&= \bigcap_i \left\{ y+x | y \in A_i \right\} \\
&= \bigcap_i \mathbf{T}_x \left\{ y | y \in A_i \right\} \\
&= \bigcap_i \mathbf{T}_x A
\end{aligned}$$

6. Proof that $\mathbf{T}_x \left(A^{\mathsf{c}}\right) = \mathsf{c}\left(\mathbf{T}_x A\right)$:

$$\begin{aligned}
\mathbf{T}_x \mathsf{c} A &= \mathbf{T}_x \left\{ a | a \in A^{\mathsf{c}} \right\} \\
&= \left\{ a+x | a \in A^{\mathsf{c}} \right\} \\
&= \{a+x | a \notin A\} \\
&= \{a+x | \neg(a \in A)\} \\
&= \mathsf{c}\{a+x | a \in A\} \\
&= \mathsf{c}\mathbf{T}_x A
\end{aligned}$$

$$\begin{aligned}
A \oplus B &= \bigcup_{b\in B} \mathbf{T}_b A && \text{by Definition J.2 page 303} \\
&= \left\{ a+b | a \in A \text{ and } b \in B \right\} && \text{by Definition J.1 page 303}
\end{aligned}$$

✏

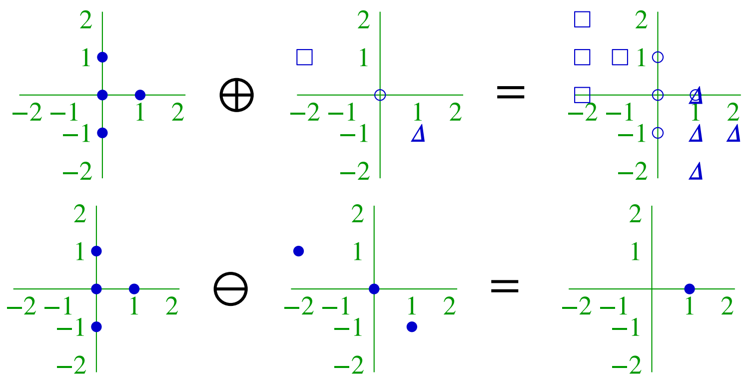


Figure J.1: Illustration for Example J.2 (page 306)

Example J.2. Let

$$A \triangleq \{(0,0),\ (0,1),\ (0,-1),\ (1,1)\}$$
$$B \triangleq \{(0,0),\ (-2,1),\ \ (1,-1)\}$$

Then

$$A \oplus B = \{(0,0),\ (0,1),\ (0,-1),\ (1,1),\ (-2,1),\ (-2,2),\ (-2,0),\ (-1,2),\ (1,-1),\ (1,-2),\ (2,0)\}$$
$$A \ominus B = \{(1,0)\}$$

These relationships are illustrated in Figure J.1 (page 306).

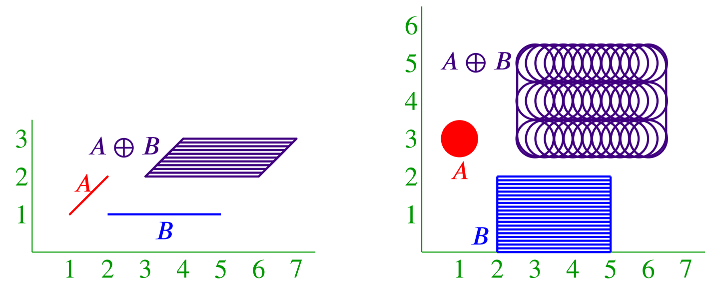


Figure J.2: Illustration for Example J.3 page 306

Example J.3. [4] Two more examples are illustrated in Figure J.2 (page 306).

J.1.3 Additive properties

Theorem J.2. [5] *Let $(X,+)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

THM	$A \oplus \{0\}$	$=$	A	$\forall A \subseteq X$	
	$A \oplus B$	$=$	$B \oplus A$	$\forall A,B \subseteq X$	(COMMUTATIVE)
	$A \oplus (B \oplus C)$	$=$	$(A \oplus B) \oplus C$	$\forall A,B,C \subseteq X$	(ASSOCIATIVE)
	$\mathbf{T}_x(A \oplus B)$	$=$	$\left(\mathbf{T}_x A\right) \oplus B$	$\forall A,B \subseteq X,\ x \in X$	(TRANSLATION INVARIANT)

[4] Lay (1982) page 7
[5] Pitas and Venetsanopoulos (1991) pages 163–164

✎PROOF:

$$
\begin{aligned}
A \oplus \{0\} &= A \oplus B|_{B=\{0\}} \\
&= \left.\bigcup_{b \in B} \mathbf{T}_b A\right|_{B=\{0\}} && \text{by Definition J.2 page 303} \\
&= \mathbf{T}_0 A \\
&= A && \text{by Definition J.1 page 303}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus B &= \bigcup_{b \in B} \mathbf{T}_b A && \text{by Definition J.2 page 303} \\
&= \bigcup_{a \in A} \mathbf{T}_a B && \text{by Definition J.1 page 303} \\
&= B \oplus A && \text{by Definition J.2 page 303}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus (B \oplus C) &= \bigcup_{y \in B \oplus C} \mathbf{T}_y A && \text{by Definition J.2 page 303} \\
&= \bigcup_{a \in A} \mathbf{T}_a (B \oplus C) && \text{by Definition J.1 page 303} \\
&= \bigcup_{a \in A} \mathbf{T}_a \left(\bigcup_{c \in C} \mathbf{T}_c B \right) && \text{by Definition J.2 page 303} \\
&= \bigcup_{a \in A} \left(\bigcup_{c \in C} \mathbf{T}_a \mathbf{T}_c B \right) && \text{by Definition J.1 page 303} \\
&= \bigcup_{a \in A} \left(\bigcup_{c \in C} \mathbf{T}_c \mathbf{T}_a B \right) && \text{by Definition J.1 page 303} \\
&= \bigcup_{c \in C} \mathbf{T}_c \left(\bigcup_{a \in A} \mathbf{T}_a B \right) && \text{by Definition J.1 page 303} \\
&= \bigcup_{c \in C} \mathbf{T}_c \left(\bigcup_{b \in B} \mathbf{T}_b A \right) && \text{by Definition J.1 page 303} \\
&= \bigcup_{c \in C} \mathbf{T}_c (A \oplus B) && \text{by Definition J.2 page 303} \\
&= (A \oplus B) \oplus C && \text{by Definition J.2 page 303}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T}_x (A \oplus B) &= \mathbf{T}_x \bigcup_{b \in B} \mathbf{T}_b A && \text{by Definition J.2 page 303} \\
&= \bigcup_{b \in B} \mathbf{T}_x \mathbf{T}_b A && \text{by Definition J.1 page 303} \\
&= \bigcup_{b \in B} \mathbf{T}_b \mathbf{T}_x A && \text{by Definition J.1 page 303} \\
&= \left(\mathbf{T}_x A \right) \oplus B && \text{by Definition J.2 page 303}
\end{aligned}
$$

✏

Theorem J.3. [6] *Let $(X, +)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$.*

[6] Pitas and Venetsanopoulos (1991) page 163

THM

$$\begin{array}{rcll}
A \oplus (B \cup C) &=& (A \oplus B) \cup (A \oplus C) & \forall A,B,C \subseteq X \quad (\oplus \text{ is LEFT DISTRIBUTIVE over } \cup)\\
(A \cup B) \oplus C &=& (A \oplus C) \cup (B \oplus C) & \forall A,B,C \subseteq X \quad (\oplus \text{ RIGHT DISTRIBUTIVE over } \cup)\\
A \oplus (B \cap C) &\subseteq& (A \oplus B) \cap (A \oplus C) & \forall A,B,C \subseteq X\\
(A \cap B) \oplus C &\subseteq& (A \oplus C) \cap (B \oplus C) & \forall A,B,C \subseteq X
\end{array}$$

✎PROOF:

$$\begin{aligned}
(A \cup B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c (A \cup B) && \text{by Definition J.2 page 303}\\
&= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cup (\mathbf{T}_c B) \right] && \text{by Definition J.1 page 303}\\
&= \left(\bigcup_{c \in C} \mathbf{T}_c A \right) \cup \left(\bigcup_{c \in C} \mathbf{T}_c B \right)\\
&= (A \oplus C) \cup (B \oplus C) && \text{by Definition J.2 page 303}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \cup C) &= (B \cup C) \oplus A && \text{by Theorem J.2 page 306}\\
&= (B \oplus A) \cup (C \oplus A) && \text{by previous result}\\
&= (A \oplus B) \cup (A \oplus C) && \text{by Theorem J.2 page 306}
\end{aligned}$$

$$\begin{aligned}
(A \cap B) \oplus C &= \bigcup_{c \in C} \mathbf{T}_c (A \cap B) && \text{by Theorem J.2 page 306}\\
&= \bigcup_{c \in C} \left[(\mathbf{T}_c A) \cap (\mathbf{T}_c B) \right] && \text{by Definition J.1 page 303}\\
&\subseteq \left(\bigcup_{c \in C} \mathbf{T}_c A \right) \cap \left(\bigcup_{c \in C} \mathbf{T}_c B \right) && \text{by minimax inequality}\\
&= (A \oplus C) \cap (B \oplus C) && \text{by Theorem J.2 page 306}
\end{aligned}$$

$$\begin{aligned}
A \oplus (B \cap C) &= (B \cap C) \oplus A && \text{by Theorem J.2 page 306}\\
&\subseteq (B \oplus A) \cap (C \oplus A) && \text{by previous result}\\
&= (A \oplus B) \cap (A \oplus C) && \text{by Theorem J.2 page 306}
\end{aligned}$$

✏

J.1.4 Subtractive properties

Theorem J.4. [7] *Let $(X, +)$ be a group with with Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM

$$\begin{array}{rrcll}
 & A \ominus \{0\} &=& A & \forall A \subseteq X\\
 & A \ominus B &=& B^c \ominus A^c & \forall A,B \subseteq X\\
 & \mathbf{T}_x (A \ominus B) &=& (\mathbf{T}_x A) \ominus B & \forall A,B \subseteq X,\ x \in X \quad (\text{TRANSLATION INVARIANT})\\
A \subseteq B \implies & A \ominus C &\subseteq& B \ominus C & \forall A,B,C \subseteq X \quad (\text{INCREASING})
\end{array}$$

[7] Pitas and Venetsanopoulos (1991) pages 164–165

✎PROOF:

$$\begin{aligned}
A \ominus \{0\} &= \mathsf{c}(A^\mathsf{c} \oplus \{0\}) && \text{by Theorem J.1 page 303}\\
&= \mathsf{c}(A^\mathsf{c}) && \text{by Theorem J.2 page 306}\\
&= A
\end{aligned}$$

$$\begin{aligned}
A \ominus B &= \mathsf{cc}(A \ominus B)\\
&= \mathsf{c}(A^\mathsf{c} \oplus B) && \text{by Theorem J.1 page 303}\\
&= \mathsf{c}(B \oplus A^\mathsf{c}) && \text{by Theorem J.2 page 306}\\
&= B^\mathsf{c} \ominus A^\mathsf{c} && \text{by Theorem J.1 page 303}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_x(A \ominus B) &= \mathbf{T}_x\mathsf{c}\left(A^\mathsf{c} \oplus B\right) && \text{by Theorem J.1 page 303}\\
&= \mathsf{c}\mathbf{T}_x\left(A^\mathsf{c} \oplus B\right) && \text{by Definition J.1 page 303}\\
&= \mathsf{c}\left(\mathbf{T}_x A^\mathsf{c} \oplus B\right) && \text{by Theorem J.2 page 306}\\
&= \mathsf{c}\left(\mathsf{c}\mathbf{T}_x A \oplus B\right) && \text{by Definition J.1 page 303}\\
&= \mathbf{T}_x A \ominus B && \text{by Theorem J.1 page 303}
\end{aligned}$$

$$\begin{aligned}
A \ominus C &= \bigcap_{c \in C} A_c && \text{by Theorem J.2 page 306}\\
&\subseteq \bigcap_{c \in C} B_c && \text{by } A \subseteq B \text{ hypothesis}\\
&= B \ominus C && \text{by Definition J.2 page 303}
\end{aligned}$$

✏

Theorem J.5. [8] *Let $(X,+)$ be a group with with Minokowski subtraction operator* $\ominus : X^2 \to X$.

THM	$A \ominus (B \cup C)$ $=$ $(A \ominus B) \cap (A \ominus C)$	$\forall A,B,C \subseteq X$	($\ominus$ LEFT DISTRIBUTIVE *over* $\cup$)
	$(A \cap B) \ominus C$ $=$ $(A \ominus C) \cap (B \ominus C)$	$\forall A,B,C \subseteq X$	($\ominus$ RIGHT DISTRIBUTIVE *over* $\cap$)
	$(A \cup B) \ominus C$ $\supseteq$ $(A \ominus C) \cup (B \ominus C)$	$\forall A,B,C \subseteq X$	
	$A \ominus (B \cap C)$ $\supseteq$ $(A \ominus B) \cup (A \ominus C)$	$\forall A,B,C \subseteq X$	

✎PROOF:

$$\begin{aligned}
A \ominus (B \cup C) &= \mathsf{cc}\Big[\, A \ominus (B \cup C) \,\Big]\\
&= \mathsf{c}\Big[\, A^\mathsf{c} \oplus (B \cup C) \,\Big] && \text{by Theorem J.1 page 303}\\
&= \mathsf{c}\Big[\, (A^\mathsf{c} \oplus B) \cup (A^\mathsf{c} \oplus C) \,\Big] && \text{by Theorem J.3 page 307}\\
&= \left[\mathsf{c}\left(A^\mathsf{c} \oplus B\right)\right] \cap \left[\mathsf{c}\left(A^\mathsf{c} \oplus C\right)\right] && \text{by Demorgan relation page 303}\\
&= (A \ominus B) \cap (A \ominus C) && \text{by Theorem J.1 page 303}
\end{aligned}$$

$$\begin{aligned}
(A \cap B) \ominus C &= \mathsf{c}[(A \cap B) \ominus C]\\
&= \mathsf{c}[\mathsf{c}(A \cap B) \oplus C] && \text{by Theorem J.1 page 303}\\
&= \mathsf{c}\left[(A^\mathsf{c} \cup B^\mathsf{c}) \oplus C\right]\\
&= \mathsf{c}\left[(A^\mathsf{c} \oplus C) \cup (B^\mathsf{c} \oplus C)\right] && \text{by Theorem J.3 page 307}\\
&= \mathsf{c}(A^\mathsf{c} \oplus C) \cap \mathsf{c}(B^\mathsf{c} \oplus C)\\
&= (A \ominus C) \cap (B \ominus C) && \text{by Theorem J.1 page 303}
\end{aligned}$$

[8] Pitas and Venetsanopoulos (1991) page 165

$$
\begin{aligned}
A \ominus (B \cap C) &= \mathsf{cc}\left[\, A \ominus (B \cap C) \,\right] \\
&= \mathsf{c}\left[\, A^{\mathsf{c}} \oplus (B \cap C) \,\right] && \text{by Theorem J.1 page 303} \\
&\supseteq \mathsf{c}\left[(A^{\mathsf{c}} \oplus B) \cap (A^{\mathsf{c}} \oplus C)\right] && \text{by Theorem J.3 page 307} \\
&= \left[\mathsf{c}(A^{\mathsf{c}} \oplus B)\right] \cup \left[\mathsf{c}(A^{\mathsf{c}} \oplus C)\right] && \text{by Demorgan relation page 303} \\
&= (A \ominus B) \cup (A \ominus C) && \text{by Theorem J.1 page 303}
\end{aligned}
$$

$$
\begin{aligned}
(A \cup B) \ominus C &= \mathsf{cc}[(A \cup B) \ominus C] \\
&= \mathsf{c}[\mathsf{c}(A \cup B) \oplus C] && \text{by Theorem J.1 page 303} \\
&= \mathsf{c}\left[\left(A^{\mathsf{c}} \cap B^{\mathsf{c}}\right) \oplus C\right] && \text{by Demorgan relation page 303} \\
&\supseteq \mathsf{c}\left[\left(A^{\mathsf{c}} \oplus C\right) \cap \left(B^{\mathsf{c}} \oplus C\right)\right] \\
&= \mathsf{c}\left(A^{\mathsf{c}} \oplus C\right) \cup \mathsf{c}\left(B^{\mathsf{c}} \oplus C\right) \\
&= (A \ominus C) \cup (B \ominus C) && \text{by Theorem J.1 page 303}
\end{aligned}
$$

⇰

Theorem J.6. [9] *Let $(X, +)$ be a group with with Minokowski addition operator $\oplus : X^2 \to X$ and Minokowski subtraction operator $\ominus : X^2 \to X$.*

THM				
	$A \ominus (B \oplus C)$	$=$	$(A \ominus B) \ominus C$	$\forall A,B,C \subseteq X$
	$A \oplus (B \ominus C)$	$\subseteq$	$(A \oplus B) \ominus C$	$\forall A,B,C \subseteq X$

✎PROOF:

$$
\begin{aligned}
A \ominus (B \oplus C) &= \mathsf{cc}\left[\, A \ominus (B \oplus C) \,\right] \\
&= \mathsf{c}\left[\, A^{\mathsf{c}} \oplus (B \oplus C) \,\right] && \text{by Theorem J.1 page 303} \\
&= \mathsf{c}\left[\, (A^{\mathsf{c}} \oplus B) \oplus C \,\right] && \text{by Theorem J.2 page 306} \\
&= \mathsf{c}(A^{\mathsf{c}} \oplus B) \ominus C && \text{by Theorem J.1 page 303} \\
&= (A \ominus B) \ominus C && \text{by Theorem J.1 page 303}
\end{aligned}
$$

$$
\begin{aligned}
A \oplus (B \ominus C) &= A \oplus \left(\bigcap_{c \in C} \mathbf{T}_c B \right) && \text{by Definition J.2 page 303} \\
&= \left(\bigcap_{c \in C} \mathbf{T}_c B \right) \oplus A && \text{by Theorem J.2 page 306} \\
&= \bigcup_{a \in A} \mathbf{T}_a \left(\bigcap_{c \in C} \mathbf{T}_c B \right) && \text{by Definition J.2 page 303} \\
&= \bigcup_{a \in A} \bigcap_{c \in C} \mathbf{T}_a \mathbf{T}_c B && \text{by Definition J.1 page 303} \\
&\subseteq \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_a \mathbf{T}_c B && \text{by minimax inequality} \\
&= \bigcap_{c \in C} \bigcup_{a \in A} \mathbf{T}_c \mathbf{T}_a B && \text{by Definition J.1 page 303}
\end{aligned}
$$

[9] Pitas and Venetsanopoulos (1991) page 166

$$
\begin{aligned}
&= \bigcap_{c\in C} \mathbf{T}_c\left(\bigcup_{a\in A} \mathbf{T}_a B\right) && \text{by Definition J.1 page 303}\\
&= \bigcap_{c\in C} \mathbf{T}_c(B \oplus A) && \text{by Definition J.2 page 303}\\
&= (B \oplus A) \ominus C && \text{by Definition J.2 page 303}\\
&= (A \oplus B) \ominus C && \text{by Theorem J.2 page 306}
\end{aligned}
$$

J.2 Operations

Definition J.3. [10] *Let $(X,+)$ be a group.*

DEF: *The **symmetric set** of A is the set* $\check{A} \triangleq -A \qquad \forall A\subseteq X$

Definition J.4. [11] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and D^s be the symmetric set of set D.*

DEF:
*The **dilation** of A by D is the operation* $A \oplus \check{D} \qquad \forall A,D\subseteq X.$
*The **erosion** of A by E is the operation* $A \ominus \check{E} \qquad \forall A,E\subseteq X.$

Definition J.5. [12] *Let $(X,+)$ be a group with Minokowski addition operator $\oplus : X^2 \to X$, Minokowski subtraction operator $\ominus : X^2 \to X$, and B^s be the symmetric set of a set B.*

DEF:
*The **opening** of A with respect to B is the set* $A_B \triangleq \underbrace{\underbrace{(A \ominus \check{B})}_{\text{erosion}} \oplus B}_{\text{dilation}} \qquad \forall A,B\subseteq X.$

*The **closing** of A with respect to B is the set* $A^B \triangleq \underbrace{\underbrace{(A \oplus \check{B})}_{\text{dilation}} \ominus B}_{\text{erosion}} \qquad \forall A,B\subseteq X.$

Theorem J.7. [13] *Let $(X,+)$ be a group with A_B representing the opening of a set A with respect to a set B and A^B representing the closing of a set A with respect to a set B.*

THM:
(complement of the opening)→ $\mathsf{c}(A_B) = (A^{\mathsf{c}})^B$ *←(closing of the complement)* $\forall A,B\subseteq X$
(complement of the closing)→ $\mathsf{c}(A^B) = (A^{\mathsf{c}})_B$ *←(opening of the complement)* $\forall A,B\subseteq X$

✎PROOF:

$$
\begin{aligned}
\mathsf{c}(A_B) &= \mathsf{c}\Big[\,(A \ominus \check{B}) \oplus B\,\Big] && \text{by Definition J.5 page 311}\\
&= \mathsf{c}(A \ominus \check{B}) \ominus B && \text{by Theorem J.1 page 303}\\
&= \mathsf{c}(A \ominus \check{B}) \ominus B && \text{by Theorem J.1 page 303}\\
&= (A^{\mathsf{c}} \oplus \check{B}) \ominus B && \text{by Theorem J.1 page 303}\\
&= (A^{\mathsf{c}})^B && \text{by Definition J.5 page 311}
\end{aligned}
$$

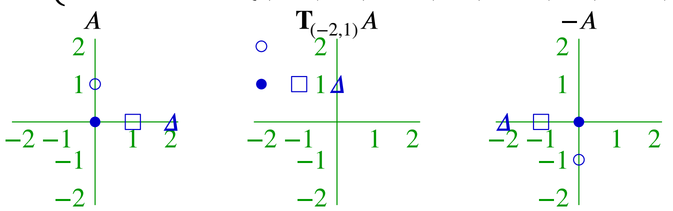
[10] Matheron (1975) page 17
[11] Pitas and Venetsanopoulos (1991) page 161
[12] Serra (1982) page 50
[13] Serra (1982) page 51

$$
\begin{aligned}
\mathsf{c}(A^B) &= \mathsf{c}\left[(A \oplus \check{B}) \ominus B \right] && \text{by Definition J.5 page 311} \\
&= \mathsf{c}(A \oplus \check{B}) \oplus B && \text{by Theorem J.1 page 303} \\
&= \mathsf{c}(A \oplus \check{B}) \oplus B && \text{by Theorem J.1 page 303} \\
&= (A^{\mathsf{c}} \ominus \check{B}) \oplus B && \text{by Theorem J.1 page 303} \\
&= (A^{\mathsf{c}})_B && \text{by Definition J.5 page 311}
\end{aligned}
$$

Example J.4.

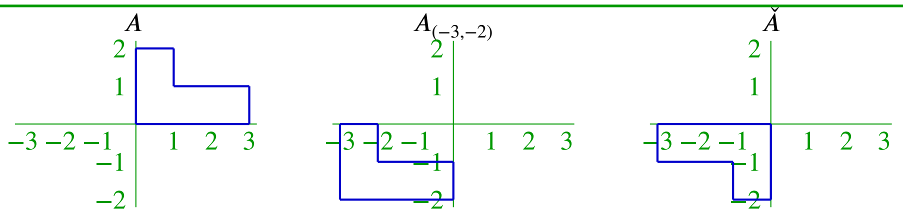
EX

$$A \triangleq \{(0,0),\ (1,0),\ (2,0),\ (0,1)\} \implies \left\{ \begin{array}{lcl} A_{(-2,1)} &=& \{(-2,1),\ (-1,1),\ (0,1),\ (-2,2)\} \quad \text{and} \\ \check{A} &=& \{(0,0),\ (-1,0),\ (-2,0),\ (0,-1)\} \end{array} \right\}$$



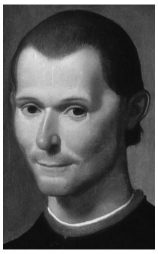
Example J.5. An example similar to Example J.4 (page 312) but using solid shapes is illustrated next:

EX

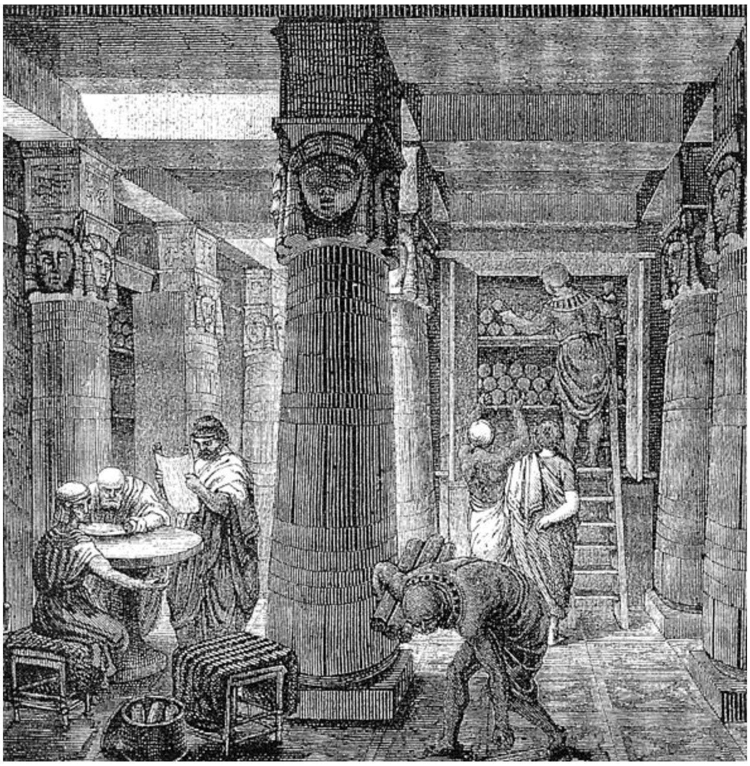


Back Matter

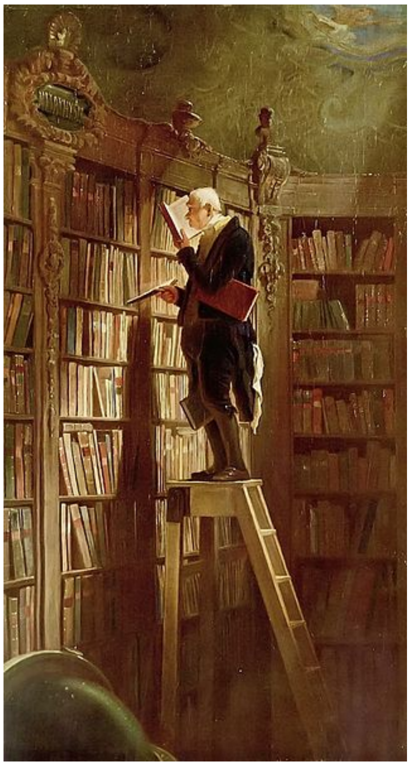
❝*It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils.*❞
Niels Henrik Abel (1802–1829), Norwegian mathematician [14]



❝*When evening comes, I return home and go to my study. On the threshold I strip naked, taking off my muddy, sweaty workaday clothes, and put on the robes of court and palace, and in this graver dress I enter the courts of the ancients and am welcomed by them, and there I taste the food that alone is mine, and for which I was born. And there I make bold to speak to them and ask the motives of their actions, and they, in their humanity reply to me. And for the space of four hours I forget the world, remember no vexation, fear poverty no more, tremble no more at death; I pass indeed into their world.*❞
Niccolò Machiavelli (1469–1527), Italian political philosopher, in a 1513 letter to friend Francesco Vettori. [15]



ancient library of Alexandria



The Book Worm by *Carl Spitzweg*, circa 1850

[16]

❝*To sit alone in the lamplight with a book spread out before you, and hold intimate converse with men of unseen generations—such is a pleasure beyond compare.*❞
Yoshida Kenko (Urabe Kaneyoshi) (1283? – 1350?), Japanese author and Buddhist monk [17]

[14] quote: Simmons (2007), page 187.
image: http://en.wikipedia.org/wiki/Image:Niels_Henrik_Abel.jpg, public domain

[15] quote: Machiavelli (1961), page 139?.
image: http://commons.wikimedia.org/wiki/File:Santi_di_Tito_-_Niccolo_Machiavelli%27s_portrait_headcrop.jpg, public domain

[16] http://en.wikipedia.org/wiki/File:Ancientlibraryalex.jpg, public domain http://en.wikipedia.org/wiki/File:Carl_Spitzweg_021.jpg,

[17] quote: Kenko (circa 1330)
image: http://en.wikipedia.org/wiki/Yoshida_Kenko

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