Topology

DEFINITION

(X, T)

a) $\phi \in T$

b) $X \in T$

c) $U_1 \dots U_n \in T \Rightarrow \bigcap_{i=1}^{n} U_i \in T$

(Closed under finite intersections)

d) $U_i \in T \quad i \in I$ (any indexing set)

then $\boxed{\bigcup_{i \in I} U_i \in T}$

("closed under arbitrary unions")

Elements of X are called __POINTS__

Elements of T " " __OPEN SETS__.

e.g. Euclidean topology

"the motivating example"

Take $X = \mathbb{R}^n$

(usual Euclidean distance: $d(x,y) = \sqrt{x^2+r^2}$ ...)

Given $x \in X$ $r > 0$ set

$$B(x,r) = \{ y \in \mathbb{R}^n \mid \|x-y\| < r \}$$

open ball centred at x of radius r.

A set $U \subseteq \mathbb{R}^n$ is open if given any $x \in U$ there exists $\varepsilon > 0$ s.t.

$$B(x, \varepsilon) \subseteq U$$

Call this the Euclidean topology or the usual topology.

any x has an open ball.

If in doubt, this is the topology being used.

- If $n=1$, then write $\mathbb{R}$ (instead of $\mathbb{R}^1$)

Other examples. ~ <u>Extreme cases.</u>

- Maximise or minimise the ~~of~~ number of open sets.

1) DISCRETE TOPOLOGY

Given X (a set), we take as our open sets <u>all</u> <u>subsets of X</u> (ie. $P(X)$)

$\Rightarrow$ Clearly $\phi \in P(X)$ and $X \in P(X)$.

(a) and (b) are satisfied.

$$\bigcap_{i=1}^{n} G_i \subseteq P(X) \quad \text{finite intersection}$$

$$\bigcup_{i \in I} u_i \subseteq P(X) \quad \text{'infinite' unions.}$$

2) INDISCRETE TOPOLOGY.

(X, T) with $T = \{\phi, X\}$ only.

the only open sets are ϕ and X.

Can show this satisfies

(a), (b), (c), (d).

$$\phi \cap X = \phi \in T$$

since.

$$\phi \cup X = X \in T$$

NOTE (c) says if $U_1, \ldots U_n$ are open sets then $\bigcap_{i=1}^{n} U_i \in T$ [finite]

(d) says if $U_1, U_2, \ldots$

then $\bigcup U_i \in T$ arbitrary Union.

- Discrete / indiscrete are opposite extremes.
- Euclidean is 'somewhere between the extremes'.

The PARTICULAR POINT Topology.

let X be a set. and fix $x_0 \in X$.

The PPT based at $x_0 \in X$ has as open sets ϕ and anything that contains x_0

let's check ...

$\emptyset \in T \Rightarrow$ (a) satisfied.

$x_0 \in X \Rightarrow X \in T \Rightarrow$ (b) satisfied.

(c) Finite intersections:

let $u_1 \ldots u_n$ be open.

$\Rightarrow$ if $u_i = \emptyset$ for some i

then $\bigcap_{i=1}^{n} u_i = \emptyset \in T$

if $u_i \neq \emptyset$ then

$x_0 \in \bigcap_{i=1}^{n} u_i \Rightarrow \bigcap_{i=1}^{n} u_i$ is open.

$\Rightarrow$ (c) is satisfied.

(d) Arbitrary unions.

if u_i $(i \in I)$ be open.

if $u_i = \emptyset$ then $\bigcup u_i = \emptyset \in T$

if $u_i \neq \emptyset$ atleast once then

$\exists$ u_i s.t. $x_0 \in u_i \Rightarrow \bigcup u_i$ is open.

$(u_i \subseteq \bigcup u_i) \Rightarrow$ (d)

Note: this <u>doesn't</u> look much like the Euclidean topology.

$\mathbb{R}$ and subsets containing 0.

open	NOT OPEN
$\mathbb{Q}$	$\mathbb{Z}^+$
$[-1, +1]$	$(0,1)$
$\{-6, 0, 11, 3\sqrt{91}\}$	$\{-3, 6, \sqrt{23}, 10^{10^{10}}\}$

So V. DIFFERENT from 'usual' topology.

EX// Consider $\mathbb{R}$ and let τ be the collection of subsets that are either the empty set or their complement is finite. Show this is a top. space.

cofinite topology.

EX// $X = \{ f : [0,1] \to \mathbb{R} \}$

$U \subseteq X$ is open if either

1) $U = \emptyset$

OR 2) for each $f_i \in U$ $\exists$ a finite set F and $\varepsilon > 0$ s.t. $\{ f : [0,1] \to \mathbb{R} \mid$ 2,

$|f(t) - f_i(t)| \leq \varepsilon \quad \forall t \in F\} \subset \mathcal{U}$

Show $(X, \mathcal{T})$ is top. space.

Non-examples. Let $X = \mathbb{R}$

(1) All subsets of $\mathbb{R}$ containing 0.

(b, c, d) satisfied.

BUT $\emptyset$ not an open ~~set~~ set.

$\Rightarrow$ (a) NOT satisfied.

$$\boxed{\text{FAILS TO BE A TOP SPACE.}}$$

(2) All subsets of $\mathbb{R}$ that don't contain zero.

(a), (c), (d) satisfied.

BUT (b) X is not an open set (contains zero).

(3) All Subsets of $\mathbb{R}$ containing at least one of 0 or 1. or is $\emptyset$.

a, b, d satisfied.

BUT (c) NOT

$[-1, \frac{1}{2}]$ $\downarrow$ contains zero

$[\frac{1}{2}, 2]$ $\downarrow$ contains $\underline{1}$.

$$\left[-1, \frac{1}{2}\right] \cap \left[\frac{1}{2}, 2\right] = \left\{\frac{1}{2}\right\}.$$

and $\left\{\frac{1}{2}\right\} \notin T$.

$\rightarrow$ does not contain 0, or $\underline{1}$ and is not $\emptyset$.

Missed an example here.

i.e. must show all 4 axioms are satisfied.

let (X, T) be a top. space.

and $Y \subseteq X$ then the subspace topology $(Y, \mathcal{U}')$

$V \in \mathcal{U}'$ if $V = Y \cap U$ $U, \in T$

e.g. $S^n \subseteq \mathbb{R}^{n+1}$
↳ (n-sphere)

$$S^n = \{ \underbrace{(x_1, \dots x_{n+1})}_{\in \mathbb{R}^{n+1}} \mid x_1^2 + x_2^2 + \dots x_{n+1}^2 = 1 \}$$

[S^1 is a circle 'embedded' in $\mathbb{R}^2$.]

S^n can be given the subspace topology

$B^n \subseteq \mathbb{R}^{n+1}$

$$B^n = \{ (x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 \dots \leq 1 \}$$

ie all points inside
== n-ball

CONTINUOUS FUNCTIONS.

DEF. Let X, Y be top. spaces.

$f: X \to Y$ is CONTINUOUS if

$$\boxed{f^{-1}(U) \subseteq X \text{ is open whenever } U \subseteq Y \text{ is open.}}$$

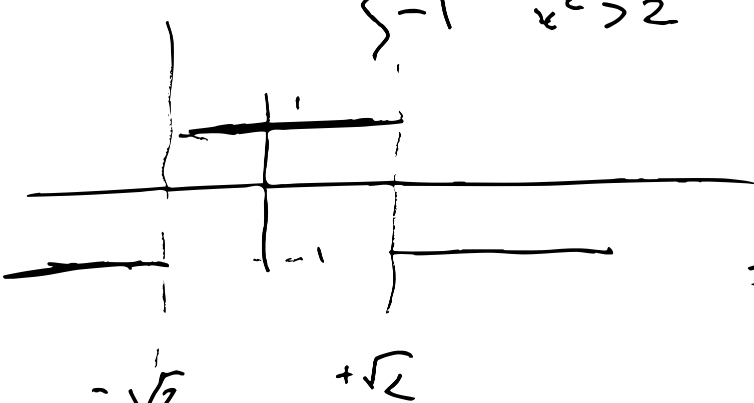
"inverse images of open sets are open"

i.e. the notion of continuous depends on the topologies of the domain and target spaces.

e.g. $\mathbb{Q} \subseteq \mathbb{R}$ $\quad f: \mathbb{Q} \to \mathbb{Q}$

$$f(x) = \begin{cases} 1 & x^2 < 2 \\ -1 & x^2 > 2 \end{cases}$$

$\sqrt{2} \notin \mathbb{Q}$



$\Rightarrow$ continuous

ex $X \xrightarrow{f} Y \xrightarrow{g} Z$

f continuous g continuous.

$f \circ g$ continuous

$\underbrace{f^{-1}(g^{-1}(U))}_{\text{open. (since } f \text{ is continuous)}}$ $g^{-1}(U)$ open (g is continuous) $U \in Z$ open.

$\Rightarrow$ $\boxed{f \circ g \text{ is continuous.}}$