Elegant but not totally correct — if it is deterministic it is predictable,but good summary

Visualizing Chaos Theory with Lorenz System

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PHYS304 Spring 2023, Haverford College
(Dated: June 21, 2023)

The idea of a small change in one part of a system that results in a drastic outcome in another part, akin to a butterfly flapping its wings in one location causing a hurricane thousands of miles away, is famously referred to as the Butterfly Effect. It was introduced by Edward Norton Lorenz in 1972 [1], and then has since captured the public imagination of chaos theory. Chaos theory is the concept describing how small changes in initial conditions in one state can escalate in to large differences in a later state. This project examines the chaotic behavior of the deterministic and unpredictable system, particularly Lorenz system as a strange attractor. The visualization includes the plots of Lorenz system as functions of time, and the graphical representations of Lorenz attractor in 2-dimensional phase planes and animated 3-dimensional phase space.

1. INTRODUCTION

1.1. The History of Chaos

After the discovery of the laws of motion and universal gravitation by Isaac Newton in 1600s, we thought the universe operated as if a perfect predictable machine, ruled by the laws of physics and mathematical equations. We could explain the motions of all the planets and moons, predict eclipse and the appearances of comets with pinpoint accuracy in advance for centuries [2]. In 1814, French physicist Pierre-Simon Laplace introduced a famous thought experiment. He imagined a supreme intellect referred to as Laplace's demon that knew everything about the state of the current universe such as the position of every atom and the nature of every force acting on it, thus it would be able to foresee the future of the universe, according to his statement, "if this intellect were vast enough to submit the data to analysis, then the future, just like the past, would be present before its eyes" [3].

Laplace's idea represents total determinism. With defined initial condition, the entire future is determined. However, even deterministic system can be unpredictable. Newton himself was aware that not all system is submitted to his equations, specifically the three-body problem. The problem can be summed up that if the two objects are interacting only through gravity in space, Newton's laws of motion are able to predict exactly how the two objects behave by calculating the motions of them. However, if the third body of object is added, then there is no solution to the prediction of the behavior in the system.

Later in 1889, King Oscar II of Norway and Sweden, as a part of the celebration of his sixtieth birthday, the mathematical competition was held. He offered a prize to anyone who could solve the three-body problem [4]. French mathematician Henri Poincaré won the prize without actually solving the problem. Instead, he described all the reasons why it couldn't be solved. One of the most important reasons he highlighted was how

I really enjoyed this description of the history and learned much from it — especially the more philosophical discussion of determinism

small of a change in the initial condition in the beginning of the system would lead to large differences at the end. This unpredictable deterministic system showed us a glimpse into chaos [5].

1.2. An Example of a Deterministic Chaos

One of the most famous examples of the phenomenon of chaos is the logistic map, defined by the equation,

$$x' = rx(1-x) \tag{1}$$

For a given value of the constant r, and a defined x value, the equation gives you a value of x' which then can be used again in the calculation on the right-hand side, giving another value, and so forth. This creates an iterative map on the value of x. The operation results in one of three things happens:

1. It creates a "fixed point" which is where the value settles down to a fixed number. For instance, $x = 0$ is always a fixed point of the logistic map.

2. It does not settle down to a single value, but it settles down into a periodic pattern, rotating around a set of values, such as say four values, repeating them in sequence over and over. This is called a "limit cycle".

3. It generates a seemingly random sequence of numbers that appear to have no rhyme or reason to them at all. This is referred to as "deterministic chaos" as it represents a chaotic non-periodic behavior overtime, yet the prediction of its behavior can be determined. ([6], p. 120).

In order to illustrate the behavior of the logistic map, for given values of r from 1 to 4 in steps of 0.01, we define $x = \frac{1}{2}$ as the initial condition and iterate the logistic map equation a thousand times. Starting with the value of $x = \frac{1}{2}$ from the first r, $r = 1$, then iterate the logistic map 1,000 times in a for loop for the calculation in $x^{'} = rx(1-x)$. We repeat the calculation for the value of r until $r = 4$ from a list of x values in a loop running from $r = 1$ to $r = 4$ using the append function, later increment

r from 1 to 4 in steps of 0.01. Using Matplotlib, the scatter plot (1) shows a distinctive plot that looks like a tree bent over onto its side. The plot is famously referred to as the Feigenbaum plot, named after its discoverer by Mitchell Jay Feigenbaum, or sometimes referred to as the figtree plot, a play on the fact that it looks like a tree and Feigenbaum means “figtree” in German ([6], p. 121).

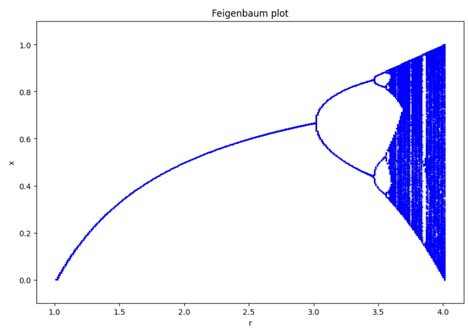

FIG. 1: Feigenbaum plot from the logistic map for given values of r from 1 to 4 in steps of 0.01 starting with $x = \frac{1}{2}$. The plot represents a fixed point at x = 0, the limit cycle period between $3 \leq r \leq 3.6$, the deterministic chaos period between $3.6 \leq r \leq 4$, and the edge of chaos at around $r = 3.6$.

According to the plot (1), for a given value of r, a fixed point on the Feigenbaum plot is at x = 0. For the period of fixed points, it lies between $1 \leq r \leq 3$. The limit cycle is a period between $3 \leq r \leq 3.6$ where the values of x does not settle down to a single value, and fluctuates between two or more values into a periodic pattern, rotating around a set of values. And, the deterministic chaos is the period between $3.6 \leq r \leq 4$ where it goes crazy and converges to a non-periodic value, generating a seemingly random sequence of numbers that appear to have no pattern. Additionally, the ”edge of chaos” where the system move from orderly behavior (fixed points or limit cycles) to chaotic behavior is at around $r = 3.6$.

1.3. The Discovery of Chaos Theory

The idea of deterministic chaos resurfaced in the mid 20^{th} century with the breakthrough by Edward Norton Lorenz, an American mathematician and meteorologist. We might have heard of the famous analogy of the Butterfly Effect. A well-known metaphor for chaotic behavior saying a butterfly flapping its wings in Brazil can set off a tornado in Texas. The idea was introduced in the talk given by Lorenz to the American Association for the Advancement of Science in 1972 [1]. The idea has since captured the public imagination of chaos theory.

Chaos theory is the concept describing how small changes in initial conditions in one state can escalate in to large differences in a later state. With an incomplete picture of the initial conditions, small errors in the approximation can have enormous consequences later in the future as Lorenz discovered in 1961 when he attempted to make a basic computer simulation of the Earth’s atmosphere. He inputted the set of 12 equations and 12 variables including temperature, pressure, humidity, etc, and had the computer printed out each time step as a row of 12 numbers, showing how they evolved over time. The breakthrough came when Lorenz wanted to redo a run but, as a shortcut, he only entered the numbers from halfway through a previous printout. Even though, the repeated run followed the previous one for a while, it diverged soon after. The result described a totally different state of the atmosphere, an entirely different weather forecast with no resemblance to the outcome from the previous run.

Lorenz first thought was that the computer had broken, soon after he realized that that the only difference was that the numbers on the printout were rounded to three decimal places while the numbers in the computer calculated six decimal places. The small difference of less than one part in a thousand was enough to create disproportionately large differences resulting in a complete different weather patterns just a short time after. Then chaos theory was born [7] [2].

2. METHOD AND RESULTS

2.1. Lorenz Equations

Based on his simulation for the discovery of chaos theory, Lorenz simplified the set of differential equations down to three equations and three variables, also known as the Lorenz system,

$$\frac{dx}{dt} = \sigma(y - x), \tag{2}$$

$$\frac{dy}{dt} = rx - y - xz, \tag{3}$$

$$\frac{dz}{dt} = xy - bz, \tag{4}$$

where σ, r, and b are constants ([6], p. 347). Even simplified, the Lorenz system behaves the same way as the previous simulation. If the initial condition is changed, the result diverges drastically. This phenomenon becomes known as sensitive dependence on initial conditions which states that with chaotic systems, it’s impossible to make firm predictions due to the unpredictable infinite decimal point of the state of the system [5] .

2.2. Lorenz System as Functions Over Time

Given initial conditions for the case of $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$, we can solve the the Lorenz system by creating the array of values of (x, y, z) = (0, 1, 0) as initial conditions in the range from t = 0 to t = 50. Then we can use the given values to solve the differential equations using the fourth order Runge-Kutta method which are calculated in a loop of the initial conditions. After solving the Lorenz equations, using Matplotlib, we can make the plots of x, y, and z as a function of time, resulted in 2, 3, and 4 accordingly ([6], p. 347).

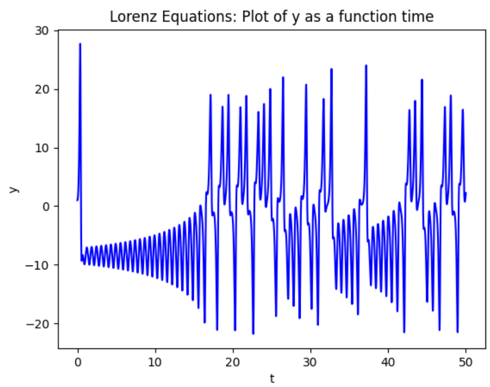

FIG. 2: The plot of Lorenz system for y as a function of time over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

Nice discussion of non-periodicity and phase planes

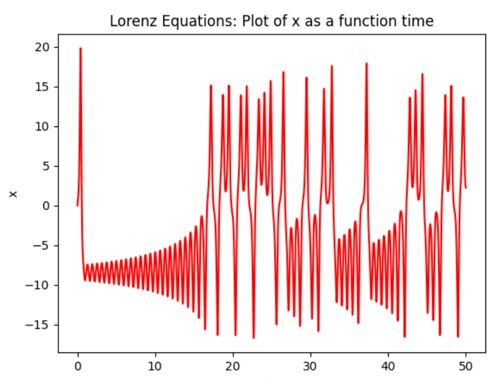

FIG. 3: The plot of Lorenz system for x as a function of time over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

In the plots showing the Lorenz system for y (2) and x (3) as functions of time, we can see that the two plots share relatively similar behavior of the system over a period of time. They both begin with a large swing starting at the fixed point of 0 then quickly jump to peak and drop

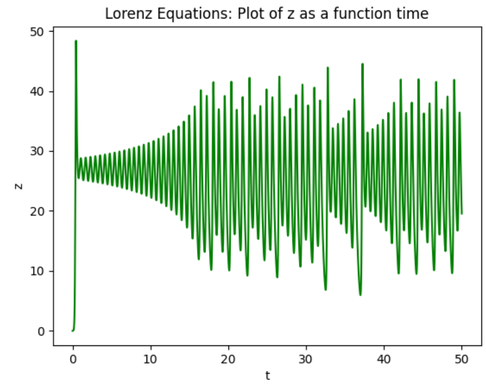

FIG. 4: The plot of Lorenz system for z as a function of time over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

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down immediately to around -10 on both y and x axis, then oscillate with a stabilizing and slowly increasing period up until around t = 16. After that point, the graphs oscillate into a non-periodic behavior showing the unpredictable nature of the motion of the system. One more notice is that the overall size of the oscillation of y as a function of time is greater than x as a function of time as the peaks of 2 are reaching around 30 on the y axis, while the peaks of 3 are reaching only around 20 on the x axis.

For the case of z as a function of time (4), it begins in the same manner to y (2) and x (3) as functions of time, starting at the fixed point of 0 with a large swing. However, its first peak of the swing nearly reach 50 on the z axis and instantly drop down to around 26. Then it oscillates with a stabilizing and slowly increasing period up until around t = 16. After that point, the graph oscillates into a non-periodic behavior similar to the other two plots. Also, 4 seems to be "less crazy" and more deterministic over the chaotic period compared to 2, and 3. These differences in the behaviors between the three plots as functions of time will direct the graphical representations of Lorenz system in the following sections.

2.3. Graphical Representations of Lorenz System as Strange Attractor

2.3.1. The Phase Planes of Lorenz Attractor

Given the same initial conditions as in section 2.2, we can also plot the phase planes of Lorenz system. With the defined constants of $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$, this specific Lorenz system is an example of a "strange attractor". An attractor is a set of points in the space, invariant under the dynamics, which attracts all the neighboring states asymptotically in the course of dynamic evolution.

This behavior is known as "the basin of attraction" [8].

Strange attractors are unique from other phase space attractors for that the position of the system on the attractor is undetermined with fractal structure presenting an infinite geometric shapes with endless patterns driven by recursion. Two states or sets of points that are near each other at one time will be arbitrarily far apart at later times as the motion of the system never repeats. The only restriction is that the state of system remain on the attractor [9].

The phase planes of Lorenz system as a strange attractor, or "Lorenz attractor" displays the "snapshot" of the 3-dimensional representation of the attractor, representing in the 2-dimensional plane, illustrating the system from different perspectives. Plotting using Matplotlib, the phase planes of strange attractor are as follow: x-z plane (5), x-y plane (6), and y-z plane (7), resulting in the butterfly-shape graphs which the Butterfly Effect was named after.

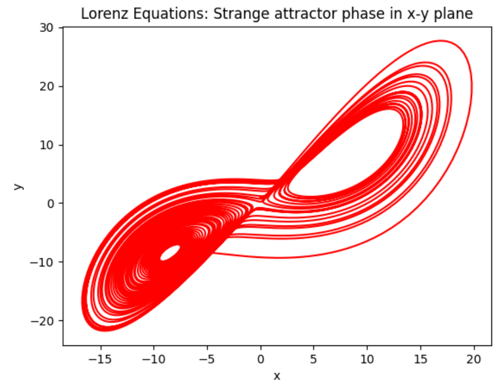

FIG. 6: The plot of Lorenz attractor in x-y phase plane over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

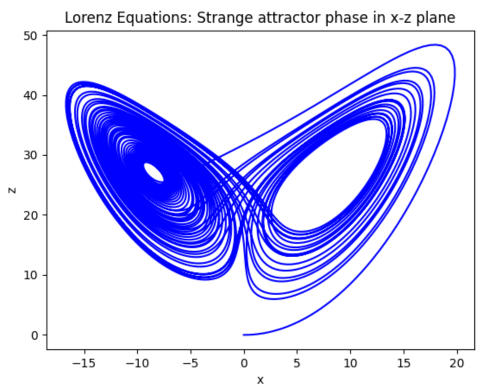

FIG. 5: The plot of Lorenz attractor in x-z phase plane over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

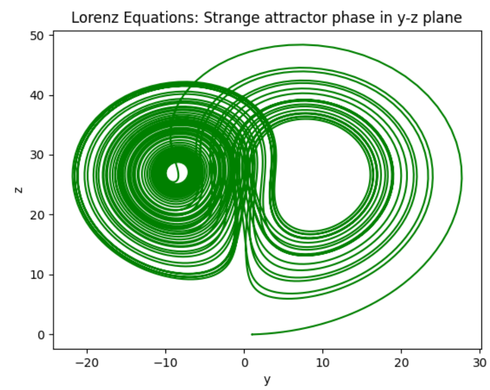

FIG. 7: The plot of Lorenz attractor in y-z phase plane over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

According to the plots of the phase planes of Lorenz attractor in 5, 6, 7, we can see the properties of strange attractor shown in the Lorenz attractor. No point in the planes is ever visited more than once by the same trajectory and no two trajectories intersect [10]. We can see this behavior in action in phase space of the Lorenz attractor in the next section.

2.3.2. The 3-Dimensional Animated Visualization of Strange Attractor

Via the use of Matplotlib Animation, we can reveal features of Lorenz attractor in phase space as a 3-dimensional graphical representation in which better describes the state of the system at various points in time.

We define the phase space by setting the sets of values in x, y, and z axes, creating points and lines in the space for the animation. Then we can set up these sets of lines and points to submit to Lorenz system over the defined parameters, resulted in the animated Lorenz attractor in 3-dimensional display. The plots 8, 9, 10, and 11 show some of the animation frames grabbed over time from various angles and viewpoints.

The animation further demonstrates and confirms that Lorenz attractor behave as a strange attractor, for the reason that the motion or the state of the system never repeats as no point in the space is repeated by the same trajectory and no two trajectories intersect, thus the structure is in fact fractal. Meaning that, as the system evolves over time, a single trajectory will visit an infinite number of points in the space, and the space will have an infinite number of trajectories, resulting in chaos.

Non linearity is important but not mentioned — similarly no surfaces of section to show the strange attractor phenomenon (ie grabbing once every n periods and plotting that, instead of just showing the phase planes). I was not able to get your animations to work, but the screen grabs in the report help me understand what you did.

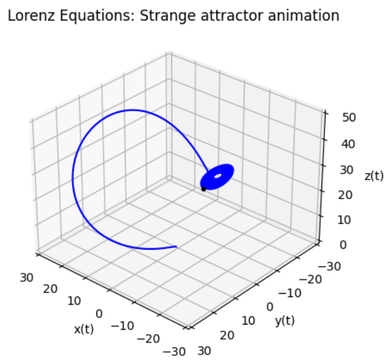

FIG. 8: A frame of an animated Lorenz attractor in 3-dimensional over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

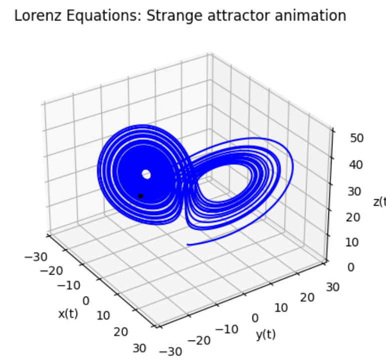

FIG. 10: A frame of an animated Lorenz attractor in 3-dimensional over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

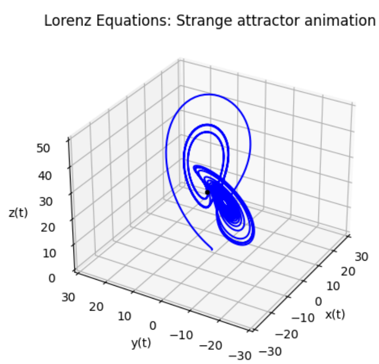

FIG. 9: A frame of an animated Lorenz attractor in 3-dimensional over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

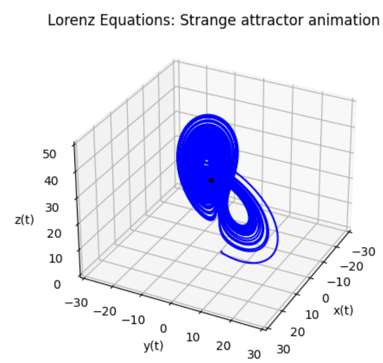

FIG. 11: A frame of an animated Lorenz attractor in 3-dimensional over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0).

3. CONCLUSION AND DISCUSSION

According to Lorenz equations in section 2.1, for the case of $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$ over the period of t = 0 to t = 50 with initial conditions of (x, y, z) = (0, 1, 0), we solve the set of differential equations using the fourth order Runge-Kutta method.

Using Matplotlib, we plotted Lorenz system as x, y, and z as a function of time: 2, 3, and 4, results in a pattern of the starting behavior across the three axes with a large swing begins at the fixed point of 0 then quickly jump to peak and drop down immediately, then oscillate with a stabilizing and slowly increasing period up until around t = 16. After that point, the graphs oscillate into a non-periodic behavior showing the unpredictable nature of the motion of the system.

The butterfly-shape plots of the phase planes of Lorenz attractor in x-z plane, x-y plane, and y-z plane: (5), (6), and (7), represents Lorenz system as a strange attractor. The phase planes display the 2-dimensional plane of the representation of the attractor, illustrating the system from different perspectives. According to the phase planes, no point in the planes is repeated by the same trajectory and no two trajectories intersect which is the chaotic behavior of a strange attractor.

In order to further visualize the behavior of Lorenz attractor, we animate the 3-dimensional graphical representation of the system via Matplotlib Animation. The animation demonstrates and confirms that Lorenz attractor behaves as a strange attractor, for the reason that the motion or the state of the system never repeats. A single trajectory will visit an infinite number of points in the space, and the space will have an infinite number of trajectories as the system evolves over time, creating fractal structure.

In conclusion, Lorenz attractor shows that Lorenz system is at the same time both deterministic and unpredictable for that, with known initial conditions, the result is determined, yet the tiniest of change in the conditions would diverge the result drastically into a totally different state.

4. ACKNOWLEDGEMENT

I would like to express my sincere gratitude to Prof. Daniel Grin for all the support throughout the semester. Thank you very much for being so understanding and caring, helping me going through difficult times. I could not have undertaken this semester without you.

I would also like extend my acknowledgement to Margaret Hamilton and Ellen Fetter who were the two mathematicians working with Edward Lorenz on the innovative computing which required for modern chaos theory, however they were not credited as often as Lorenz [11].

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