HW7 Write-up

Nina Martinez Diers*
Bryn Mawr College Department of Physics
(Dated: March 29, 2024)

This week, I wrote algorithms using a fourth-order Runge-Kutta method to describe the oscillation of a simple pendulum and the interdependence of predator/prey populations.

1. NEWMAN 8.4 OSCILLATIONS OF PENDULA AND 8.5

1.1. Introduction

Classically, a simple pendulum has the following equation of motion:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} sin\theta \tag{1}$$

We can rewrite the equation of motion of the pendulum, Eq. 1 as two first-order equations:

$$\frac{d\theta}{dt} = \omega \tag{2}$$

$$\frac{d\omega}{dt} = -\frac{g}{l} sin\theta \tag{3}$$

This makes it possible to implement first-order computational methods, such as the fourth-order Runge-Kutta method implemented in this problem, to solve the second-order equation of motion. When the pendulum is also subject to a driving force, the equation of motion becomes:

$$\frac{d\omega}{dt} = -\frac{g}{l} sin\theta + C cos\theta \, sin\Omega t \tag{4}$$

Selecting for different values for the driving frequency Ω causes the oscillations of the pendulum to be resonant or chaotic.

1.2. Experiment

First, I wrote an algorithm implementing Eqs. 2 and 3 in an algorithm using a fourth-order Runge-Kutta method to calculate the angle of a simple pendulum hanging from a string with length $l = 10$cm under the influence of the earth's gravitational field $g = 9.81$m/s^{-2} as a function $\theta(t)$. The initial angle of the pendulum is 179º, and it is released from rest. The angle was calculated over $t = 0$s to 30s. These data were saved as a list of values of the angle of the string from vertical in both degrees and radians. The values of the angle in degrees were used to plot the angle as a function of time, shown in Fig. 2. The

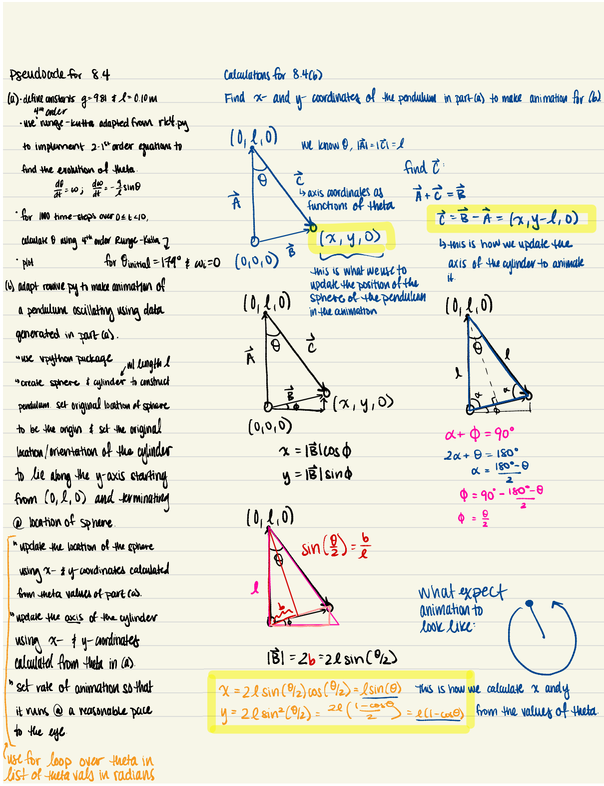


FIG. 1: Pseudocode and calculations for Newman 8.4

values of the angle in radians were used to create an animation of the pendulum. For each time-point, the value of theta was used to calculate the x- and y-coordinates of the ball of the pendulum, as well as the axis of the string of the pendulum to update the position of the objects in time in the pendulum. The Cartesian coordinates used to update the position of the sphere were given by:

$$x = l \, sin\theta \tag{5}$$

$$y = l \, (1 - cos\theta) \tag{6}$$

$$z = 0 \tag{7}$$

and the vector that described the the axis of the string of the pendulum was given by $\vec{C} = \langle x, y - l, 0 \rangle$. The pseudocode and calculations for the algorithms that generated the values for the angle as a function of time and the animation of the pendulum is given by Fig. 1.

Using Eq. 4, the algorithm for calculating $\theta(t)$ can be applied to the case of the driven pendulum, as outlined by the pseudocode in Fig. 3. The initial angle was 0º and

*Electronic address: nmartinezd@brynmawr.edu

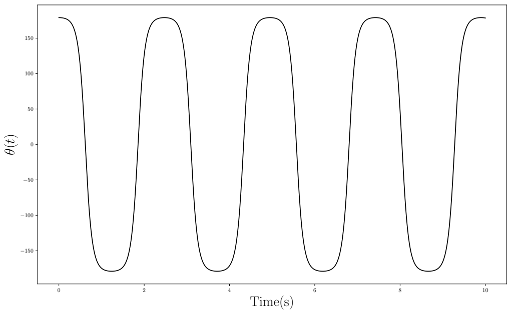
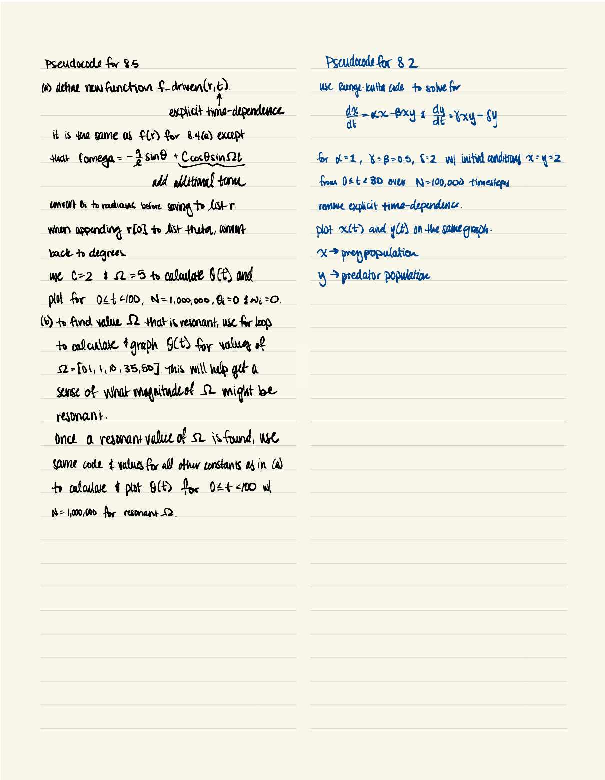


FIG. 2: Oscillation of a simple vertical pendulum under influence of gravity with a string length of 10cm.

it was also released from rest. The angle was calculated over $t = 0$s to 100s.



Pseudocode for 8.5

(a) define new function f_driven(r,t)

explicit time-dependence

it is the same as f(r) for 8.4(a) except

that fomega = $-\frac{g}{\ell}\sin\theta + \underbrace{C\cos\theta\sin\Omega t}_{\text{add additional term}}$

convert θ_i to radians before saving to list r.

when appending r[0] to list theta, convert

back to degrees.

use $C=2$ & $\Omega=5$ to calculate $\theta(t)$ and

plot for $0 \le t < 100$, $N = 1{,}000{,}000$, $\theta_i = 0$ & $\omega_i = 0$.

(b) to find value Ω that is resonant, use for loop

to calculate & graph $\theta(t)$ for values of

$\Omega = [0.1, 1, 10, 35, 50]$. This will help get a

sense of what magnitude of Ω might be

resonant.

Once a resonant value of Ω is found, use

same code & values for all other constants as in (a)

to calculate & plot $\theta(t)$ for $0 \le t < 100$ w/

$N = 1{,}000{,}000$ for resonant Ω.

Pseudocode for 8.2

use Runge-Kutta code to solve for

$\frac{dx}{dt} = \alpha x - \beta xy$ & $\frac{dy}{dt} = \gamma xy - \delta y$

for $\alpha = 1$, $\gamma = \beta = 0.5$, $\delta = 2$ w/ initial conditions $x = y = 2$

from $0 \le t < 30$ over $N = 100{,}000$ timesteps

remove explicit time-dependence.

plot $x(t)$ and $y(t)$ on the same graph.

$x \to$ prey population

$y \to$ predator population

FIG. 3: Pseudocode for Newman 8.5 and Newman 8.2

1.3. Results

When the driving frequency is $\Omega = 5\text{s}^{-1}$ with a driving amplitude of $C = 2\text{s}^{-1}$, the driven pendulum exhibits somewhat chaotic behavior as shown in Fig. 4. The maximum amplitude of the oscillations is very small, at 2.5^{o}, but there is a high degree of disturbances in the oscillation that introduce a periodic irregularity in the oscillation.

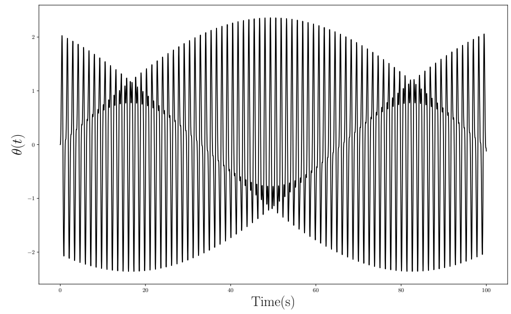


FIG. 4: Oscillation of a driven pendulum with the driving force amplitude of 2s^{-1} frequency of 5s^{-1}.

When the driving frequency is $\Omega = 10\text{s}^{-1}$ with a driving amplitude of $C = 2\text{s}^{-1}$, the driven pendulum exhibits resonant behavior as shown in Fig. 5. The amplitude of the pendulum gradually increases to 45^{o}, then gradually decreases back to 0^{o} degrees with a period of roughly 20s.

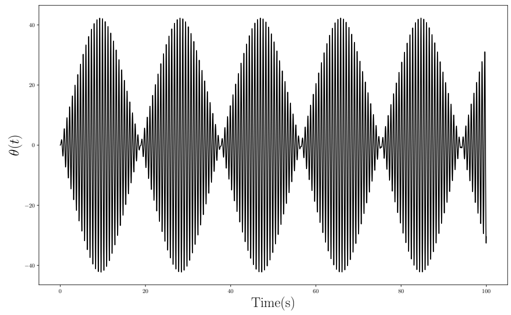


FIG. 5: Oscillation of a driven pendulum with the driving force amplitude of 2s^{-1} frequency of 10s^{-1}. The driving frequency is resonant with the natural frequency of the pendulum.

1.4. Conclusions

The driven pendulum is resonant at when the driving frequency is $\Omega = 10\text{s}^{-1}$ with a driving amplitude of $C = 2\text{s}^{-1}$. With no driving or damping force, a simple pendulum released from rest at 179^{o} oscillates regularly with an amplitude equal to the initial angle.

2. NEWMAN 8.2 LOTKA-VOLTERRA EQUATIONS

2.1. Introduction

The Lotka-Volterra equations,

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{8}$$

$$\frac{dy}{dt} = \gamma xy - \delta y \tag{9}$$

form a simplistic model that can be used to model patterns of predator and prey populations.[1] The units of time of the equations is not defined in the problem. The proportionality constant α represents the reproduction rate of the population of prey. The proportionality constant β represents the rate that the prey are eaten by predators. The proportionality constant γ represents the reproduction rate of the population of predators, which is also dependent on the population of prey due to availability of the predator's food source. The proportionality constant δ represents the death rate of the predators. Assumptions that the model makes are that the prey do not have additional causes of death besides the specific species of predators and it also ignores other food sources for the predators.

2.2. Experiment

Using values of $\alpha = 1$, $\beta = \gamma = 0.5$ and $\delta = 2$, we generate a model of predator and prey populations over $0 \leq t < 30$) when the starting population for each species is equal and proportional to 2 ($x = y = 2$). As outlined in Fig. 3, 4th order Runge-Kutta method is implemented to solve the differential equations 8 and 9. $N = 100,000$ time-steps are used to evaluate the population size. When double the number of time-steps are used, the resulting graph appears identical, indicating that the errors associated with using too large of a time-step the model are not impacting the fidelity of the results.

2.3. Results

Figure 6 shows that there is a phase difference between the oscillation of the prey population and predator population, which indicates the interactions between the populations. When the predator population drops, the prey population can rise which then allows the predator population to rise again until a critical point is reached causing the behavior to switch again.

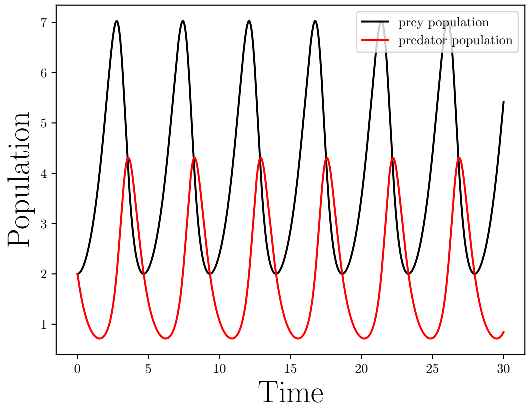


FIG. 6: Predator-prey interactions for identical starting populations.

2.4. Conclusions

With no other external factors, the populations of predators and prey can stably oscillate in time with the population of prey never exceeding a proportionality factor of 7 and never dipping below a factor of 2, and the population of predators not exceeding a proportionality factor of 4.3 and not dropping below a factor of 0.5. The populations of predators and prey are identical when the population of predators is at its maximum and when the population of prey is at its minimum.

[1] M. Newman, *Computational Physics* (2013), revised and expanded ed., ISBN 978-1-4801-4551-1.

Appendix A: Statement of Collaborators

I did not collaborate with anyone writing the code for this assignment. The algorithms written were based off 4th order runge-kutta code provided by Mark Newman[1] and Dan.

Appendix B: Survey Question

The most interesting problem was the pendulum problem. I enjoyed learning how to do make the animation and I thought it was fun to get some chaotic graphs from the driven pendulum. In total, I spent about 2 hours coding, 2 hours coding for the extra-credit and 6 hours doing the write-up. I thought that the length of the problem set (including time spent on the extra-credit) was perfect.