Final Project: Projectile Motion

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For my final project I took a look at projectile motion in a number of different sport's balls. Starting with the equations of motion modeled for a golf ball,I worked to recreate the figures and findings from a study on the motion of cricket and golf balls as they move through the air. Once I had these equations mastered and a working code, I worked on implementing a time-dependent spin so the motion of more complicated objects could be more accurately modeled. Starting with the equations of motion for a spinning top I was able to implement those equations into my projectile motion function and create a program that models the motion with varying spin. I worked out the example of a football as it moves through the air in either a spiraling motion or an end over end tumble. My program runs and produces trajectories that have accurate shapes, but unfortunately, I do not think that they are entirely accurate given the short distance the football is traveling. Based on the numbers I was using for the spin and initial velocity, the ball should be traveling further than shown by my figures. I hypothesis that this problem is due to the lift coefficient, but after looking at other cases, I believe it is due to a lack of accurate representation of the equations of motion. I looked through any paper I could find regarding the aerodynamics of a football and was unable to find the lift coefficient. Due to this, I was decided to use the same values for the golf ball and look at the shape of the motion. Finally, I applied my equations to the case of a soccer ball and looked at some interesting cases.

Background

Projectile motion has been an area of study for various reasons. From military applications to sport balls trajectories there are a variety of ways in which objects move through the air. From a personal standpoint, I have always been interested in the physics behind projectile motion, especially when spin is involved. The paper I used to base my work off of focuses on the projectile motion of golf and cricket balls [1]. They use a set of differential equations in MATLAB to model the motion of the balls through the air. Starting with the simple equations of motion they added the forces associated with the spin and wind direction experienced by the ball. These three second order differential equations then can be broken down into pairs of first order differential equations and solved. With these six coupled equations of motion the motion of the ball can be solved for and the flight patterns of the ball can be studied as the velocity, spin, and wind vary. I was able to use the Fourth order Runge-Kutta method to solve these equations and obtain the position and velocity values in the x, y, and z directions. From here I was able to recreate their findings as well as do some expansion on their work. I looked to expand upon the effects of spin as the ball travels through the air. Putting spin on a ball as it travels is very common in sports, no matter what is the source of the balls flight (a bat, throw, shot, etc.). Given that I can get this code up and running, I will be able to apply this to a number of spins and balls in different sports.

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Magnus Effect

The Magnus force, also known as the lift force, is the force felt by the ball when it is given a spin. As the ball moves through the fluid, air, it experiences a drag force opposite to the motion. In the case of a spinning ball, there is an additional force that comes into play. This Magnus force is the explanation for how many balls are able to travel in non linear paths in sports. A curve ball in baseball or a slice in golf are examples of how players take advantage of the Magnus effect. Fig. 1 shows a ball moving to the right with a counter-clockwise spin. This is the case of 'back' or 'under' spin in laminar flow. From a visual standpoint, this ball will 'carry' as it moves through the air, typically travelling further than a ball with not spin or 'top' spin. As the ball moves through the fluid, it is visible that the spin is opposite the direction of the air velocity at the bottom of the ball and the same direction on top of the ball. This creates an area high pressure below the ball and low pressure above the ball. These areas of differing pressure lead to an upward Magnus force that pushes the ball. Looking at the flow of the fluid, because of conservation of momentum, the streamlines are deflected downwards behind the ball in a turbulent wake to account for the new upward momentum. The size of the wake left behind the ball determines the drag force pulling against the ball. A smaller wake means less drag force working against the motion of the ball.

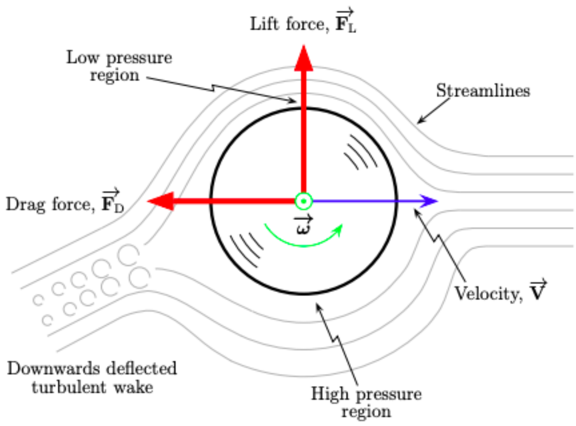

FIG. 1: Schematic of the streamline near a ball. The associated forces are shown as they affect the ball as it moves through the fluid [1].

In order to create a ball that utilizes the Magnus forces and minimizes the drag force dimples were introduced to golf balls. In order to minimize the wake, turbulence is need to keep the flow of the fluid close to the ball. A dimpled ball creates a turbulent flow at the surface of the ball and this allows the flow to 'cling' to the ball for longer and in turn creates a small wake behind the ball. This smaller wake is less of a drag force and allows the dimpled ball to travel further than a smooth ball. Fig. 2 shows the difference between a smooth and a dimpled ball. As we can see, the smooth ball has a larger wake and thus a greater drag force working against the ball's motion.

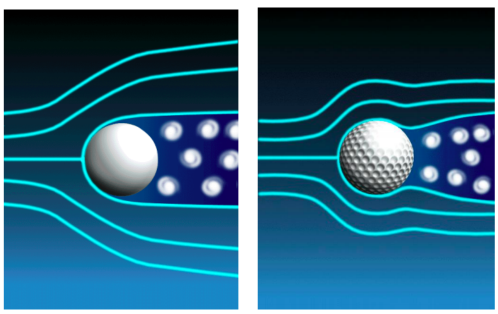

FIG. 2: Smooth vs Dimpled golf ball [2].

This paper looked at the case of constant spin (ω) values. They based this off two things: The complexity of a time dependent spin and previous research stating that a golf ball will retain 80% of its spin after 5 seconds of flight [1]. Assuming this, they worked to find the equations of force for the drag and lift and how the wind would affect them.

Equations of Motion

To get any useful information we need the equations of motion. These equations are second order differential equations that relate all of the forces to one another. The drag, lift, and gravitational forces all come together in the form of a single equation:

$$m\ddot{\vec{r}} = -\frac{1}{2}\rho A C_D|\vec{v} - \vec{W}|(\vec{v} - \vec{W}) + \frac{1}{2}\rho A C_L|\vec{v} - \vec{W}|[\frac{\omega x(\vec{v} - \vec{W})}{\omega}] + m\vec{g} \tag{1}$$

This could then be broken into its components for solving:

$$m\ddot{x} = -\frac{1}{2}\rho A|\vec{v} - W|[C_D(v_x - W_x) - C_L(\frac{\omega_y(v_z - W_z - \omega_z(v_y - W_y}{\omega})] \tag{2}$$

$$m\ddot{y} = -\frac{1}{2}\rho A|\vec{v} - W|[C_D(v_y - W_y) - C_L(\frac{\omega_z(v_x - W_x - \omega_x(v_z - W_z}{\omega})] \tag{3}$$

$$m\ddot{z} = -\frac{1}{2}\rho A|\vec{v} - W|[C_D(v_z - W_z) - C_L(\frac{\omega_x(v_y - W_y - \omega_y(v_x - W_x}{\omega})] \tag{4}$$

where:

$$|\vec{v} - W| = [(v_x - W_x)^2 + (v_y - W_y)^2 + (v_z - W_z)^2]^{1/2} \tag{5}$$

These equation only hold true given a few conditions: constant wind velocity, constant angular velocities, and constant values of the parameters ρ, A, C_L and C_D.

With these equation of motion we can see that they are all second order differential equations. In order to solve them we need to break them down into two first order differential equations. These equations will allow us to solve them in python. For example, breaking down Eq. 2 gives us these two first order differential equations:

$$\frac{dx}{dt} = v_x, \tag{6}$$

$$\frac{dv_x}{dt} = -\frac{1}{2}\frac{\rho A}{m}|\vec{v} - W|[C_D(v_x - W_x) \\ - C_L(\frac{\omega_y(v_z - W_z - \omega_z(v_y - W_y)}{\omega})] \tag{7}$$

The researchers in the paper used a MATLAB command: ode45, to solve the differential equations. I used the fourth order Runge-Kutta method to solve. To do this I needed to first define a function that contained all of the necessary constants and parameters for these 6 equations. Those 6 equations must then be added using an if/else statement. This statement is for the case when there is no spin ($\omega = 0$). Without this, the program runs into problems when we try and divide by 0.

The researchers examined the effect of pure ω_y spin ($\omega_x = \omega_z = 0$, $W_x = W_y = W_z = 0$). They modeled the cases of ω_y = -300 rad/s (under-spin) and ω_y = +300 rad/s (topspin). Under-spin or backspin is the most common spin used in golf. Almost every shot hit utilizes this spin because it allows the ball to carry farther. Topspin is rarely, if ever, used in golf. This kind of spin is used in sports like baseball and tennis. Fig. 3 is the graph they created, and Fig. 4 is my recreation.

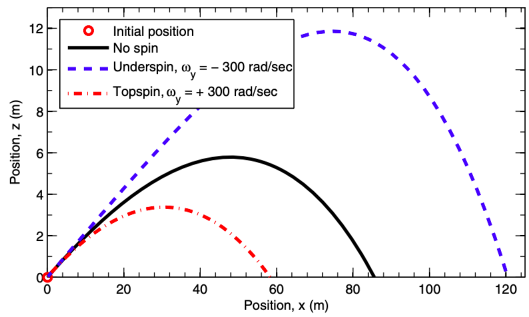

FIG. 3: The figure created by the paper. Omega values of +/- 300 rad/s.

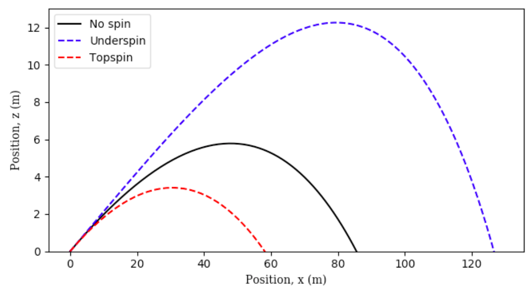

FIG. 4: My recreation of the figure above. As we can see, the values an shapes of the plot are the same. I used the same omega values for the cases of the top and under spin.

From both figures we can see the advantage of using topspin when hitting a golf ball. The ball utilizes the lift force and carries much further than the other two cases.

Along with topspin and backspin, slices and hooks are common in the game of golf. Unfortunately, they are commonly associated with a bad shot, but never the less, they happen frequently enough to be studied. A hook is defined by side spin, for example a right handed golfer hitting the ball in the left direction. This is characterized by ($\omega_x = \omega_y = 0rad/s, \omega_z = 300rad/s$). A Slice is also characterized by a side spin, but to the other direction: ($\omega_x = \omega_y = 0rad/s, \omega_z = -300rad/s$). Partial slices and hooks occur when the ball is hit with either a slight under spin or top spin along with the side spin. For the papers example, and my own, these motions are characterized by: ($\omega_x = 0, \omega_y = 212rad/s, \omega_z = 212rad/s$) for the partial hook, and ($\omega_x = 0, \omega_y = 212rad/s, \omega_z = 212rad/s$) for the partial slice. As we can see from these ω values, we can infer that the partial hook will land significantly shorter due to the top spin involved as compared to the backspin involved with the partial slice. Fig. 5 shows the results created by the paper.

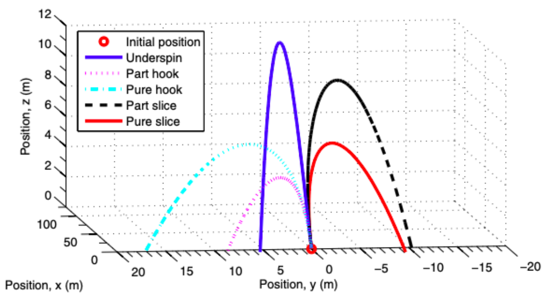

FIG. 5: 3D plot created in the paper. This shows the different effects of a hand full of spins.

Fig. 6 shows what I was able to recreate. To do this, I used the same functions and Runge-Kutta methods. I needed to create mutliple motion functions with the varying ω values and plot their respective x,y, and z coordinates along a sing MatLib 3D plot. We can see that the pure hook and slice are just mirror images of each other, while the partial hook and slice are very much different. Thinking about these two shots from a golfers perspective, taking the x direction as the direction they are aiming, while there is a distance advantage of hitting a slice, the ball also travels further off course in the y direction. With the partial hook, there isn't as great a distance covers in the x direction, but the shot is not as far off in the y direction.

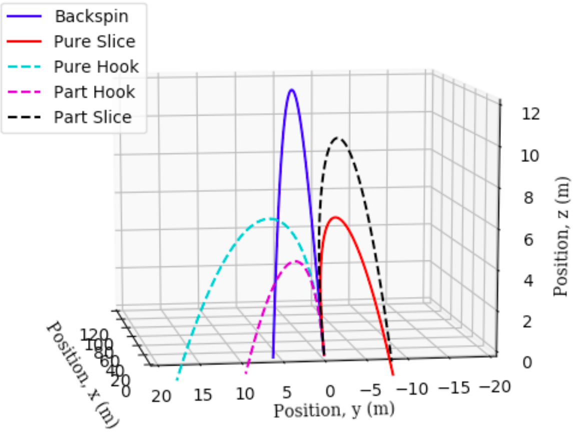

FIG. 6: My recreation of the figure above. As we can see, the values an shapes of the plot are the same. I used the same omega values for the cases of each type of spin.

This code was challenging to get working, but once I had it running it was a lot of fun to play with the values and see different paths of the balls flight. Taking this project to the next level, Dan suggested that I investigate a varying spin.

Adding a Time-dependent Spin

Once I felt the I had the code for the projectile motion function properly, I decided to change the spin from a constant value, to a time-dependent value in an attempt to model a football. Dan suggested trying to model the motion of a football tumbling. This presents a number of challenges given the shape of the football. To do this Dan suggested that I take the case of a spinning top and work from there. The first place I looked was the *Classical Mechanics* book[3]. This showed me the basics behind the motion of the spinning top and why it could work for a way to implement the kind of spin we were looking for. From there, I found an example online that derived the equations of motion for this case [4]. They used a similar approach as the book, in that they started by looking at the Lagrange equation of the system. The Lagrange of the spinning top is given below:

$$L = \frac{1}{2}I_1(\dot{\theta^2} + \dot{\phi}^2\sin(\theta)) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos(\theta))^2 - mgl\cos(\theta) \tag{8}$$

The three equations of motion can be found. Those three equations are listed below:

$$\ddot{\theta} = \frac{\dot{\phi}^2\sin(\theta)\cos(\theta)[I_1 - I_3] - I_3\dot{\phi}\dot{\psi}\sin(\theta) + mgl\sin(\theta)}{I_1} \tag{9}$$

$$\ddot{\phi} = \frac{2[I_1 - I_3]\dot{\phi}\dot{\theta}\sin(\theta)\cos(\theta) - I_3\ddot{\psi}\cos(\theta) + I_3\dot{\psi}\dot{\theta}\sin(\theta)}{I_1\sin^2(\theta) + I_3\cos^2(\theta)} \tag{10}$$

$$\ddot{\psi} = \dot{\phi}\dot{\theta}\sin(\theta) - \ddot{\phi}\cos(\theta) \tag{11}$$

As was done for the equations of motion for the golf ball, these second order differential equations need to be broken down into pairs of first order differential equations. Fig 5 shows a diagram of the top and it the angles and values of interest. For a football spinning end over end, we would want to look at the case of a spin in the θ direction, a spin about the y axis.

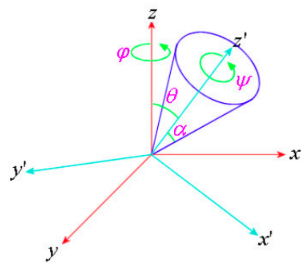

FIG. 7: .

Before I jumped into the complexity of combining the two equations, I worked out the spinning top example alone. Using similar methods as before, I created a function that took in 6 inputs related to the three angles and their associated ω values. This function contained the three second order differential equations broken down into pairs of first order differential equations. I used a fourth order Runge-Kutta to solve and create a plot of the motion of the top. Once I had the spinning top working, in added the three pairs of differential equations to the simple projectile motion function. The number of inputs doubled from 6 to 12, as did the outputs. The fourth order Runge-Kutta also needed a few minor adjustments but overall was very similar to before. The variables were adjusted accordingly and the entire system was set to measure the motion of a projectile with the variable spin addition.

The first case I wanted to investigate was the spin of a football going end over end. Given the shape, I figured that a football tumbling end over end will be greatly affected by the presents of wind. I looked at both the case of topspin, as well as backspin. Before I could do this, I needed to research and find the appropriate moments of inertia, drag coefficients, and other measurements for the football. The moments of inertia were found to be: $I_1 = 0.002829$ kg m^2 $I_3 = 0.001982$ kg m^2[5]. To calculate the coefficient of drag I averaged the coefficient of drags found by a study that looked at the aerodynamics of footballs [6]. Fig 8 shows an image from their study.

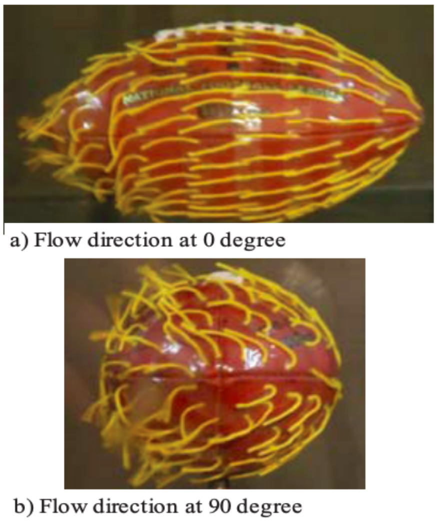

FIG. 8: This figure shows the experimental set up used to help calculate the drag force of the football at these two orientations [6].

Looking at the drag created by a football in the orientations shown below, they calculated a coefficient of drag of 0.2 for the case of the ball pointing in the direction of the flow (0 degree of rotation) and 0.78 when the ball is rotated 90 degrees. These are the two most extreme cases of the drag coefficients, so I figured averaging the two would provide a usable drag coefficient for my calculations. For the launch angles and velocity I looked to find studies that modeled punts and kicks of a football and found that an appropriate value for these conditions is a launch angle around 50 degrees and 25 m/s. [7]

I then plotted the case of an end over end spin in the presence of a head on wind ($W_x = -5$ m/s) as well as the case of no wind. Fig. 8 shows the results of putting topspin on a football rotating end over end.

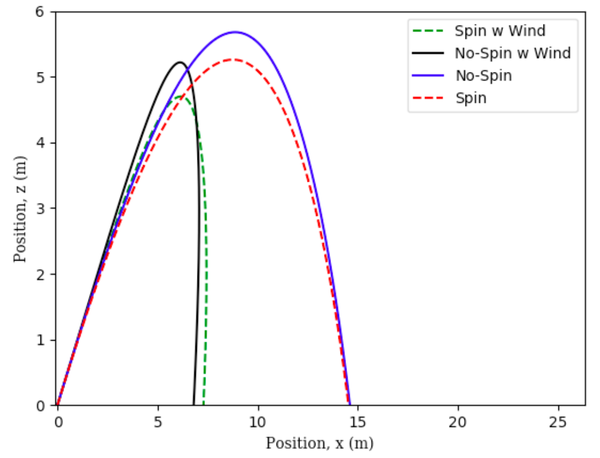

FIG. 9: Plot of 4 separate trajectories. A tumbling football ($\omega_y = 60 rad/s$) in the presence of wind and without wind. A football with no spin, both in the presence of wind and no wind.

From the figure we can see some interesting behavior. Looking at the case of no wind we see that the footballs both travel about the same distance with and without topspin. The football with topspin doesn't reach the same height as the ball with no spin. Looking at the balls in the presence of wind we see that the ball with the topspin travels slightly farther than that of the ball with no spin. The ball with no spin remains in the air longer and thus the wind has more time to affect the forward motion of the ball. The variation in the spin is shown below in Fig 10:

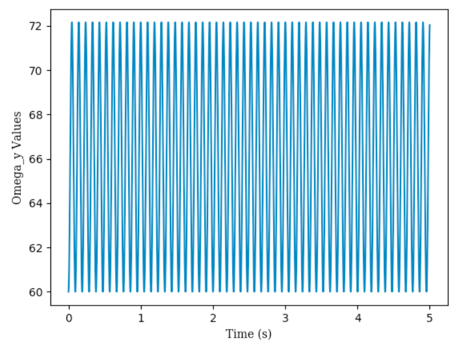

FIG. 10: The variation in the omega values as time progresses.

I also plotted the same trajectories, but with backspin ($\omega_y = -60 rad/s$) Fig 11. In the case of no wind, the balls one again travel the same distance, but as expected

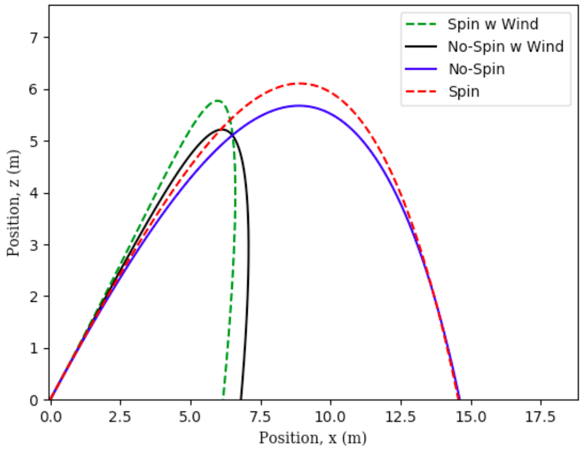

FIG. 11: Plot of 4 separate trajectories. A tumbling football ($\omega_y = -60 rad/s$) in the presence of wind and without wind. A football with no spin, both in the presence of wind and no wind.

the ball with the backspin reaches a higher altitude. For the two balls in the presence of wind, we see that the ball with the backspin hangs in the air longer, and this the wind has a great effect on its forward movement. As before, the variation in the spin is shown below in Fig 12:

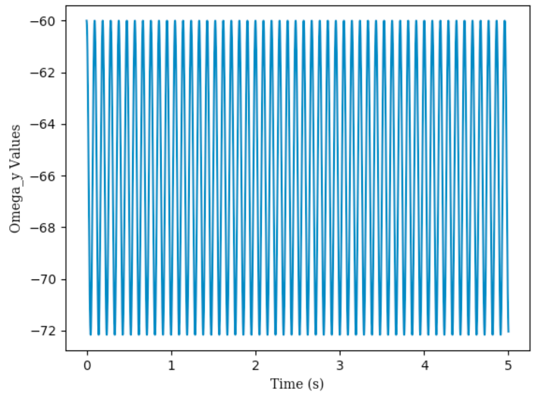

FIG. 12: The variation in the omega values as time progresses.

A final case I looked at with the football was the case of a spiral with a crosswind. I plotted three scenarios, one with no spin and a 10 m/s crosswind, one with a spin that modeled a spiral ($\omega_x = 60 rad/s$) with a cross wind in the same direction of the spin (+wind), and a spiral with a crosswind opposite the direction of the spin (-wind). Fig 13 shows the resulting plots.

This plot gives some cool results. We can see that when the wind is in the direction of the spin, the ball travels farther than when there is no spin with the same wind. I believe that this is due to the increase in rotation speed that then increases the lift force and allows the ball to carry for longer. In the case of the spin with an opposite

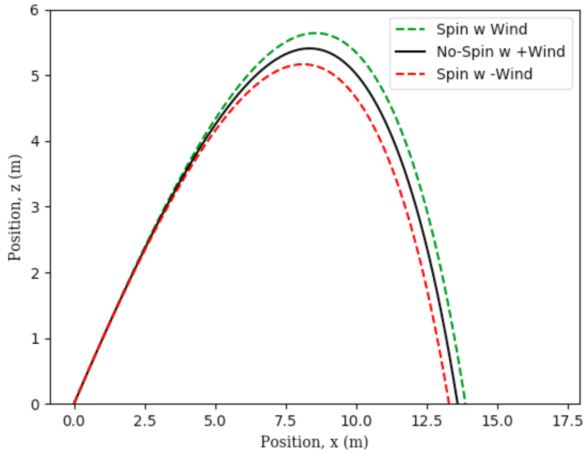

FIG. 13: The plot of three trajectories. Two balls with spin ($\omega_x = 60 rad/s$) and one without. A crosswind in all three cases, with a wind in the opposite direction of the spin for one case.

crosswind we can see that it travels a shorter distance than both other trajectories. I believe that this is due to the fact that the wind and spin are fighting each other and causing a decrease in the lift force.

Soccer Ball

Another case that I wanted to look at was the motion of a soccer ball. Soccer players utilize spin in most of their shots and have the ability to "bend" the ball around defenders and into the goal. After some looking around, I was able to find that NASA had found the coefficients of lift and drag for the soccer ball ($C_L = .25$ and $C_D = .25$) [8][9]. The area, radius, etc. [10] were all adjusted and the following plots were created:

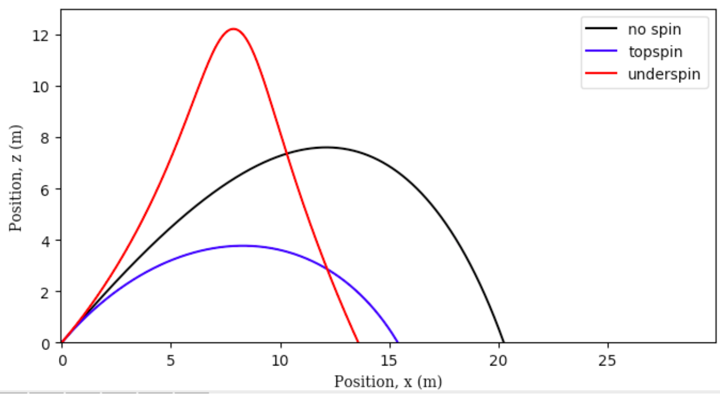

FIG. 14: The plot of three trajectories. A soccer ball with no spin, topspin, and under-spin. We see typical behavior for the cases of the ball with no spin and topspin. For the under-spin, once the ball reaches its peak, we see some interesting motion as it falls down.

Fig 14 gives some interesting results for the case of under-spin. As the ball reaches its peak, the concavity of the trajectory is mirrored on the descent. We also notice that unlike the other projectiles, the soccer ball does not travel the furthest distance with under-spin. Unfortunalty, it does appear that I am still running into some problems with the distance the ball is traveling. After changing the lift coefficent I figured that it would fix that problem, but there must be something else wrong with the code that is causing this problem. Looking at the second half of the under-spin case led me to want to look at the example of dropping a soccer ball of a large height and giving it an initial backspin. I remember seeing this trick being done and how the ball carries a great distance away from the place it is let go. Below is a quick example of what that looks like:

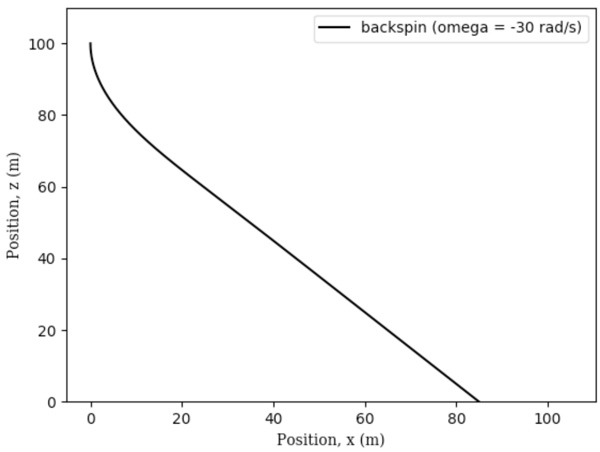

FIG. 15: This plot shows how the Magnus effect works to carry the ball in the x direction even when there is no initial x velocity.

This is a simple experiment that shows the Magnus effect. The ball if pushed in the x direction even though there is not forward intital velocity.

Conclusions

Working on this project was great learning tool and fun way to encompass some of the things I have learned over the course of this semester. Looking at the program and results from the golf ball example I am confident that this accurately models the motion of this type of ball. Even without a time dependent spin, the ball will travel in a very similar motion as shown in the figures produced above. When it comes to the football, I feel that the motion of the football and the shapes of the plots is somewhat accurate, but upon further analysis the distances simply are not. In just about every case, the ball should be traveling further than what I am getting in my results. I thought the lift coefficient was the problem, but after adjusting that in the soccer ball example I do not think it can be the only reason I am getting strange values. Due to this error, I can not say that my programs accurately model the spin and trajectory of the football or the soccer ball. I would assume that there is something inherently different about the equations of motion for these larger projectiles that is not being included in the equations of motion modeled by the golf ball. If, these equations had worked, I would have tried to model a more complex system, such as a satellite falling out of orbit, or something of that nature. Unfortunately, I was not able to move that far.

In the future, I would keep researching online and look for more information on trajectories and other research on spin to see if there is a more accurate way of looking at an object such as a football. If I were able to find this information, the code could be adjusted and more accurate representations could be created to model the trajectories.

Program names

Fig 4 = hartman-final-golf-1.py
Fig 6 = hartman-final-golf-3D-plot.py
Fig 9/Fig 10 = hartman-final-football-2.py
Fig 11/Fig 12 = hartman-final-football-3.py
Fig 13 = hartman-final-football-1.py
Fig 14 = hartman-final-soccer-2.py
Fig 15 = hartman-final-soccer-3.py

Acknowledgements

I would like to thank Professor Grin for his constant help and support throughout the entirety of this project. I know that even through we were given the circumstances, he made sure that he made time to talk to each of us about or project and was there to help when needed. Thank you Dan for a great semester!

[1] G. Robinson and I. Robinson, "The motion of an arbitrarily rotating spherical projectile and its application to ball games," *Physica Scripta*, vol. 88, p. 018101, July 2013.
[2] B. Burglund and R. Street, "Golf Ball Flight Dynamics," p. 19, May 2011.
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the Related Mechanical Systems," *Symmetry, Integrability and Geometry: Methods and Applications*, Apr. 2012. arXiv: 1204.1123.

[5] C. Horn and H. Fearn, "On the flight of the american football," p. 22.

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[7] *The Physics of Kicking a Football*.

[8] "Lift of a Soccer Ball."

[9] "Drag on a Soccer Ball."

[10] "Ball (association football)," Apr. 2020. Page Version ID: 953797114.