PHYS H304 Homework 6

Petra Mengistu*

(Dated: March 23, 2023)

This is the lab report write-up of the approach and methods used for the sixth problem set in PHYS H304.

1. INTRODUCTION

This is a summary of the methods and major equations used for this problem set. In this report, we explore methods of numerical integration, focusing largely on the application of Runge-Kutta-4 algorithm to solve ordinary differential equations. The major equations used are included where appropriate and a summary of the approach used to develop the code is included as well. All equations are taken from Computational Physics by Mark Newman [1].

2. METHODS

2.1. Exercise 8.4

We solve the second order differential equation as a coupled system of first order ordinary differential equations. To do so, we define the variable ω as the first order derivative of the angular displacement of the pendulum, θ. Thus, we can set up the coupled system of equations as shown in Eqns. 1 and 2.

$$\frac{d\theta}{dt} = \omega \tag{1}$$

$$\frac{d\omega}{dt} = \frac{-g}{l} \cdot \sin\theta \tag{2}$$

Then, we solve for the set of $\theta(t)$ that serves as the solutions to this system of equations using the fourth order Runge-Kutta 4 method, using the given initial conditions that the pendulum is released from an angle of 179 degrees ($\theta(0) = \frac{179}{190}\pi$ radians) and that the pendulum is initially at rest (so, $\omega = \frac{d0}{dt}_{t=0} = 0$). Using Example 8.4 from [1] as a basis, we code the fourth order Runge-Kutta algorithm to solve a system of equations. This algorithm works by approximating the value of differentiable variable from the first order derivative (slope) over a range of time divided into N slices of width, or step size, "h." We make these approximations by using the first order Taylor expansions such that the set of equations shown from Eqn. 3, 4, 5, and 6. The final estimation of the variable is determined by the sum of each "k" term weighted by a factor of $\frac{1}{6}$, as shown in Eqn. 7. Note that we can set up r as a vector so that it can simultaneously solve a system of equations for the same time range. Hence, we set the first column of r to the range of angles of the pendulum as a function of time, θ and the second column as the angular velocities, ω.

$$k1 = h\mathrm{f}(\mathrm{r}, \mathrm{t}) \tag{3}$$

$$k2 = h\mathrm{f}(\mathrm{r} + 0.5\mathrm{k1}, \mathrm{t} + 0.5\mathrm{h}) \tag{4}$$

$$k3 = h\mathrm{f}(\mathrm{r} + 0.5\mathrm{k2}, \mathrm{t} + 0.5\mathrm{h}) \tag{5}$$

$$k4 = h\mathrm{f}(\mathrm{r} + \mathrm{k3}, \mathrm{t} + \mathrm{h}) \tag{6}$$

$$r(t+h) = r(t) + \frac{1}{6}(\mathrm{k1} + \mathrm{k2} + \mathrm{k3} + \mathrm{k4}) \tag{7}$$

2.2. Exercise 8.5

With the addition of the damping term and the driving force applied to the pendulum, we set up a system of equations similar to that of Exercise 8.4, where ω remains the first order derivative of θ but we define ω as shown in Eqn. 9.

We solve this system using the fourth order Runge-Kutta method, with the initial conditions that the pendulum is initially at rest and release from an angle of zero degrees (hence $\theta = 0$ and $\omega = 0$).

$$\frac{d\theta}{dt} = \omega \tag{8}$$

$$\frac{d\omega}{dt} = \frac{-g}{l} \cdot \sin\theta + C\cos\theta\sin\Omega t \tag{9}$$

To determine the value of the driving frequency, given by Ω, at which the system achieves resonance, we use a response curve of amplitude as a function of driving frequency to determine where the systems reaches resonance. According to Smith [2], when the driving frequency approaches the natural frequency of the pendulum's swings, the system responds with an increased amplitude, and the maximum amplitude of oscillations is observed when the driving frequency is equal to the natural frequency. Hence, to determine the required Ω that

*Electronic address: pmengistu@haverford.edu

shows the system at resonance, we plot the value of the maximum amplitude of oscillation at a specific frequency for a wide range of frequencies. We estimate the resonant frequency as the frequency at which the maximum amplitude is largest, and narrow in at the frequencies near this region to determine Ω to a precision of 10^{-3}.

2.3. Exercise 8.2

To determine the cyclic behavior of rabbits and foxes within the ecosystem, we solve the ordinary system of equations given in Eqn. 10 and 11 using Runge-Kutta 4 as described in Section 2.1 that show the Lotka-Volterra equations.

$$\frac{dx}{dt} = \alpha x - \beta x y \tag{10}$$

$$\frac{dy}{dt} = \gamma x y - \delta y \tag{11}$$

To determine the behavior of the system, we take the derivative of either of the first order equations and observe the corresponding equation.

3. RESULTS AND CONCLUSIONS

For Exercise 8.4, the solution to the system of equations is given in Figure 1. While the oscillations are periodic, the figure does not quite represent a simple harmonic oscillator(SHO, due to the flatter regions at the peaks. This is the result of the large initial angle at which the pendulum is released, which causes the straying away from SHO behavior.

For Exercise 8.5a, we show the behavior of the driven oscillator in Figure 2, which shows an overlay of two frequencies a faster frequency and a slower frequency to create periodic beats observed in the nodes of groups of oscillations. According to Smith [2], the beat frequency is given by the absolute value of the difference between the two component frequencies - one of which is the driving frequency for this system. Using Eqn. 12, we determine the other frequency within the system. From Figure 2, we estimate the beat frequency to be ≈ 0.015 rad/s.

$$\omega_b = |\Omega - \omega_2| \tag{12}$$

From Figure 2, we estimate the beat frequency to be $\approx$ 0.015 rad/s. Hence, the corresponding "slow" frequency is approximately 4.985 rad/s. We note that the beat frequency is a factor of $\frac{3}{1000}$ of the driving frequency, which is linked to the nature of the relationships from which Eqn. 12 is derived. In addition, we observe that the curve of the oscillations are not smooth, there appear to be multiple peaks within each period that shows that

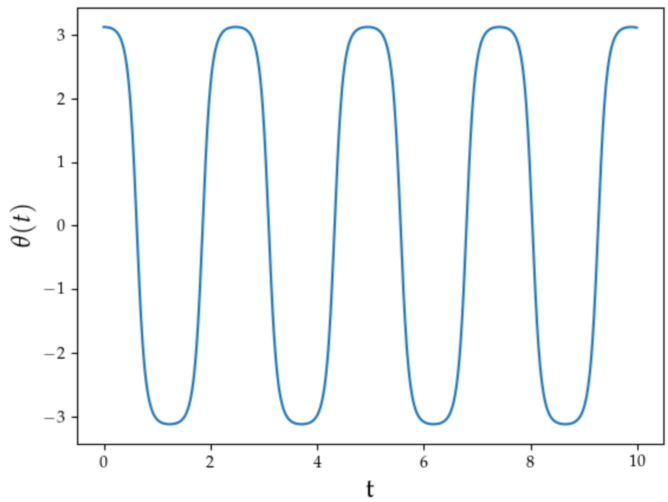

FIG. 1: A plot of the angle (in radians) of the pendulum's swing for a range of times (in seconds) at a large initial displacement of 179 degrees

the system is being driven at a frequency that differs from its natural frequency.

For part b, we plot the maximum amplitude for a variety of driving frequencies as a function of Ω shown in Figure 3. The maximum response is observed at a frequency of approximately 9.479 rad/s; therefore, we determine that this is the frequency at which resonance is achieved. Setting Ω equal to this resonant frequency, we plot $\theta(t)$ against a time range to show the behavior of the driven pendulum at resonance shown in Figure 4. We note that the amplitude of oscillation is indeed larger and that the oscillations are smooth curves as the driving frequency aligns with the natural frequency.

For Exercise 8.2, we obtain the set of solutions plotted in Figure 5. Choosing to differentiate Eqn. 10, we obtain Eqn. 13 and simplify the resulting equations. From this equation, it would appear as though damping is occuring in the system. However, substituting Eqn. 10 11 where appropriate shows the relationship that is similar to that of a simple harmonic oscillator of the form $\frac{d^2x}{dt^2} = -\omega_0 x^2$. However, looking at the resultant expression in Eqn. 14, there is an additional term of order "x" which contributes to the non-harmonic oscillatory behavior shown in Figure 5.

$$\frac{d^2x}{dt^2} = \alpha\frac{dx}{dt} - \beta y\frac{dx}{dt} - \beta x\frac{dy}{dt} \tag{13}$$

$$\frac{d^2x}{dt^2} = -x(-\alpha^2 + \alpha\beta y + \beta^2 y^2 + \beta\gamma y) - x^2\beta y(\alpha + \gamma) \tag{14}$$

In terms of the physical predator-prey interactions, we can interpret the differential equations as rates of population growth for rabbits and foxes. The first term in Eqn.

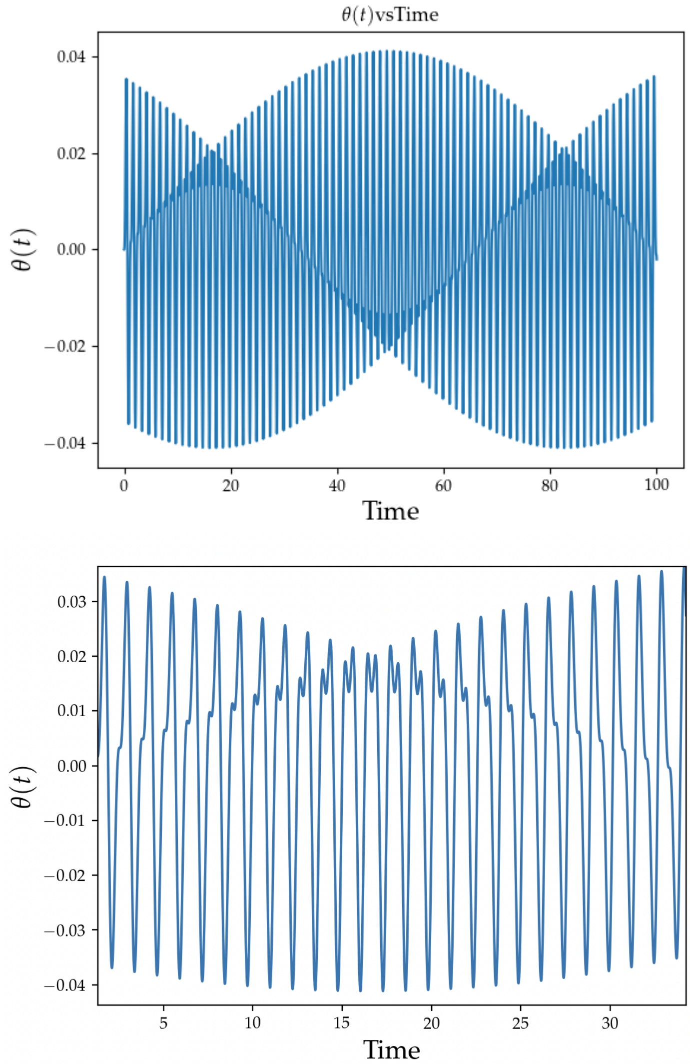

FIG. 2: A plot of the angle (in radians) of the pendulum's swing for a range of times (in seconds) at the given driving frequency of $\Omega = 5rad/s$. Lower panel shows a few cycles of oscillation that show the incongruence of the driving frequency with the natural frequency of the system.

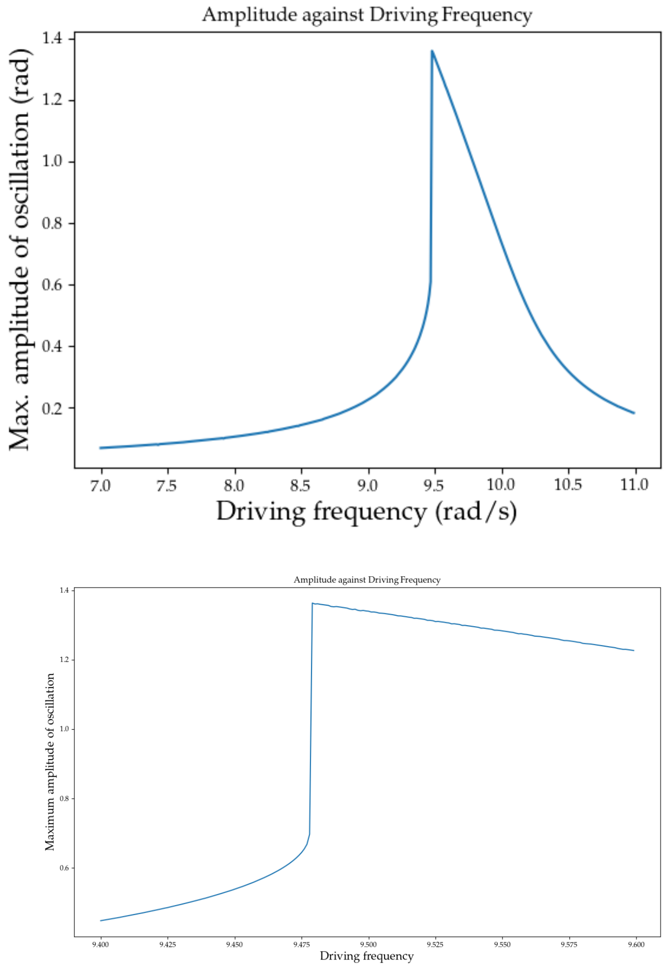

FIG. 3: A plot of the maximum amplitude as a function of $\Omega =$. Upper panel shows the broad range of responses for widely spaced frequencies while lower panel shows the zoomed in portion of a narrow range of frequencies to determine the resonant frequency to 3 decimal places of accuracy.

10, αx represents the number of rabbits born/added to the system. The second term, βxy, shows the number of rabbits killed by the foxes when the two species encounter each other. The total number of rabbits in the ecosystem per unit time is thus the difference between the amount of rabbits produced and those that are killed. In contrast, when rabbits and fox interact, the death of a rabbit serves as a source of nourishment to the foxes and increases the number of foxes in the ecosystem. The second term in Eqn. 11 refers to the number of foxes that die due to natural causes. Hence, $\frac{dy}{dt}$, the rate of foxes present per unit time is the difference between the number of foxes being "added" to the system and those lost from the system. As the number of rabbits increases sharply, the number of foxes declines. When the number of rabbits is at its sharpest decline, the number of foxes reach its peak, as they have acquired their source of food. However, as the rabbits to consume grow low, the number of foxes also begin to decline, which results in an increase in the number of rabbits again, leading a cyclical relationship.

4. SURVEY QUESTION

The most interesting problem was 8.2. The homework took me about 7 hours to complete. I learnt how to apply the coding we have been learning to practical concepts in Physics, particularly in solving ODEs. I think the problem set length and difficulty were just right.

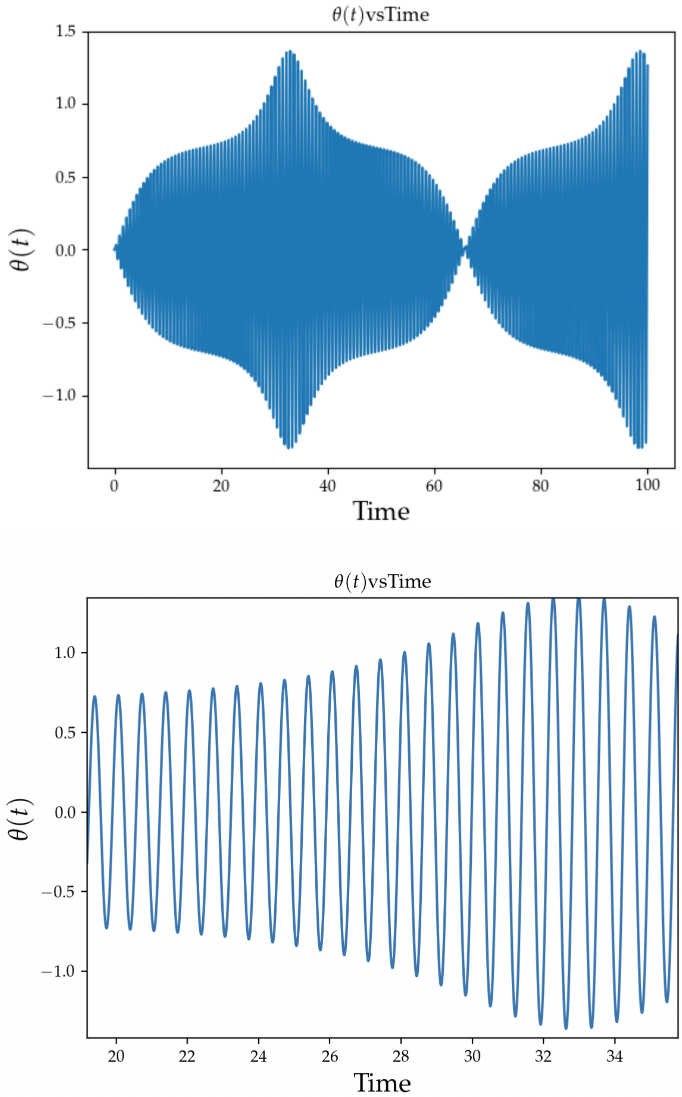

FIG. 4: Upper panel: a plot of the angle (in radians) of the pendulum's swing for a range of times (in seconds) at the determined driving frequency of $\Omega = 9.479 rad/s$ that achieves resonance. Lower panel shows a few cycles of oscillation that show the driving frequency matches the natural frequency of the system due to the smoothness of oscillations.

[1] M. Newman, *Computational Physics* ([Createspace], 2012), URL http://www-personal.umich.edu/~mejn/cp/index.html.

[2] W. F. Smith, *Waves and Oscillations: A Prelude to Quantum Mechanics* ([Oxford University Press], 2010).

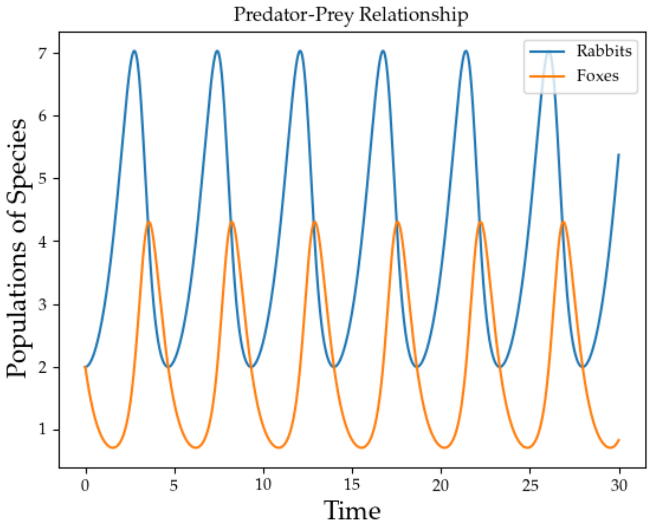

FIG. 5: A plot of the amount of foxes and rabbits (overplotted in the blue and orange lines respectively) as a function of time. The amount foxes and rabbits are interdependent, as shown by the oscillatory behavior.