Warp Drive Theory: Modelling Orbital Dynamics of the Alcubierre Metric

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This report examines Einstein's theory of general relativity and Miguel Alcubierre's 1994 paper on "warp drive" as a possible method for faster-than-light (FTL) travel. Our approach involved using Python programs to analyze and model the orbits of particles in spacetime expansion-contraction geometry, which Alcubierre postulates could lead to FTL travel. To achieve this, we utilized the Alcubierre metric and Geodesic equation to calculate the force acting on particles in the relevant spacetime geometry. We then applied the fourth-order Runge-Kutta method to model particle orbits. Our program's models yielded significant results, which we discuss in the context of general relativity and the practicality of warp drive.

I. INTRODUCTION

In Special relativity, we learn that the speed of light is the maximum speed limit in the universe, and that nothing can surpass this velocity. However, while special relativity establishes that the speed of light is a universal limit, general relativity introduces the notion that nothing can travel faster than the speed of light **locally**, which means within a small region of spacetime. According to general relativity, the interconnection of space and time, along with mass creating gravity that bends the fabric of spacetime, leads to a curvature proportional to the size and energy of an object. This curvature stretches out/slow down time proportionally.

Miguel Alcubierre was a theoretical physicist that proposed an extraordinary solution to Einstein's general relativity equations: "warp drive." in his 1994 paper. This concept relies on the expansion of the universe, and naturally, expanding spacetime. By expanding spacetime behind a spacecraft and contracting it in front, Alcubierre postulated that FTL speeds of travel in space could be attained. Alcubierre provides a metric tensor in his paper that represents the specific spacetime geometry associated with this expansion and contraction.

By converting Alcubierre's metric of spacetime into four geodesic equations, which are differential equations that explain the motion of a free-falling particle influenced by the curvature of spacetime, we can gain a better understanding of the spacetime curvature described by Alcubierre. Furthermore, by utilizing the fourth-order Runge-Kutta method with a fixed step size to compute the velocities and positions of a particle over time in space through the four geodesic equations, we can graph the trajectory of a particle in Alcubierre's curved spacetime. By doing so, we can comprehend how objects in proximity to a spaceship traveling at FTL speeds are impacted by the spacetime curvature.

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II. BACKGROUND

According to Einstein's theory of General Relativity (GR), the presence of mass and energy causes spacetime to curve and warp, affecting the path of light and causing it to follow a curved trajectory when passing near massive objects. Einstein also believed that the speed of light in a vacuum remains constant, regardless of the curvature of spacetime. This concept helps us understand the phenomenon of time dilation, where the rate of passage of time is affected by the curvature of spacetime. When more massive objects stretch out the fabric of spacetime, time passes more slowly. Therefore, the constant speed of light requires the rate of time to increase proportionally with the stretched-out spacetime caused by massive objects.

In other words, the larger the mass, the stronger the gravitational field and the more spacetime is stretched out. This stretching out of spacetime causes time to slow down proportionally. General Relativity treats special relativity as a sub-theory that applies locally to small enough regions of space where the curvature of spacetime can be ignored. The three fundamental effects of special relativity — time dilation, length contraction, and synchronization — are equally important in GR, but they are modified by the presence of gravity and spacetime curvature. Unlike special relativity, GR does not prohibit faster-than-light travel or communication, but it requires that special relativity restrictions apply locally.

II.1. General Relativity's Fundamental Equations

II.1.1. Four-Vectors

In General Relativity, dynamic quantities and distances are expressed using four-vectors, which connect two distinct points in spacetime. They are referred to as such because they incorporate the four coordinates of spacetime: three of which are spatial and one of which is

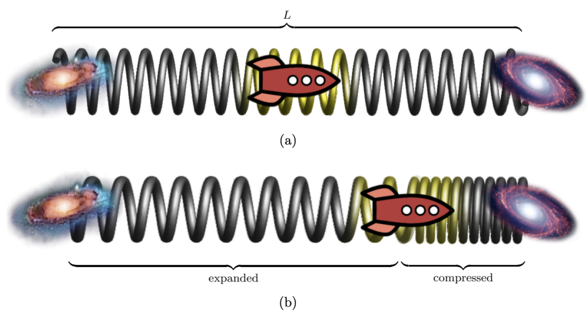

FIG. 1. This analogy provides a visual representation of the Alcubierre drive, which attempts to transport a spaceship between two galaxies by manipulating spacetime. (a) Under normal conditions, the spaceship moves through spacetime. (b) With the Alcubierre warp drive, the spring-like representation of spacetime is stretched behind the spaceship and compressed in front of it, generating a forward propulsion that appears to exceed the speed of light to an outside observer. The credit for this illustration goes to Kinach, 2017.

temporal. A standard form of the four-vector is:

$$\mathbf{a} = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = a^0 e_0 + a^1 e_1 + a^2 e_2 + a^3 e_3 \quad (1)$$

where e_0, e_1, e_2, e_3 are the basis vectors that define the coordinate axes in an inertial reference frame.

II.1.2. Metrics

The mass surrounding a particular spacetime determines its geometry, which in turn dictates the motion of a particle through that spacetime. This geometry is described by the metric, which provides the distance of the path of minimum time that a particle travels within the spacetime field. The metric is represented by a matrix called the metric tensor, which is a symmetric, rank-2 tensor that captures the local properties of spacetime, including its curvature. As a result, the metric tensor is responsible for defining the behavior of particles and light in the presence of gravitational fields. The most comprehensive equation for the metric of spacetime is:

$$ds^2 = -d\tau^2 = -g_{\mu\alpha} dx^\mu dx^\alpha \quad (2)$$

where ds^2 is the distance between two events in a given spacetime; dx^μ and dx^α are infinitesimal displacements in the α and β directions; and $g_{\mu\alpha}$ is the metric tensor of the four-dimensional spacetime [1],

$$\begin{pmatrix} g_{0,0} & g_{0,1} & g_{0,2} & g_{0,3} \\ g_{1,0} & g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,0} & g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,0} & g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix} \quad (3)$$

The four-dimensional spacetime will have some coordinate system x^0, x^1, x^2, x^3, and this coordinate system is encoded in each specific column and row in the metric tensor. As an example, imagine that the coordinate system is t, r, θ, ϕ. The first row ($\mu = 0$) and the first column ($\alpha = 0$) would encode the dimension of time, where $x^0 = t$. Therefore, for the first row of the matrix, $x^\mu = x^0 = t$, and for the first column, $x^\alpha = x^0 = t$. Similarly, the second row ($\mu = 1$) and the first column ($\alpha = 1$) would encode the dimension of radial distance, such that $x^1 = r$. Therefore, for the first row of the matrix, $x^\mu = x^1 = r$, and for the first column $x^\alpha = x^1 = r$. The entry in the first row and second column, $g_{0,1}$, would represent the intersection between the dimensions of time and radial distance, as $\mu = 0$ means $x^\mu = x^0 = t$ and $\alpha = 1$ means $x^\alpha = x^1 = r$. Moreover, $g_{0,1}$ (which would be calculated using the metric $ds^2 = g_{\mu\alpha} dx^\mu dx^\alpha$ of a specific spacetime), then the metric tensor captures the manner in which time and radial distance intersect, thereby encoding this relationship. It is clear that a symmetric matrix must be used for the metric tensor since the alternative would pose considerable difficulties.

In general relativity, the concept of metrics is fundamental as it arises from the observation that different observers in relative motion perceive time and distance differently. However, an external observer outside these frames of reference can universally agree on the time elapsed and distance between two events. This is what the metric captures by relating the measurements in each reference frame that are affected by time dilation and length contraction. Thus, the metric allows for the transformation between different reference frames and underscores the interconnectedness of space and time, which is why spacetime is a commonly used term. Additionally, the metric of a mass is influenced by other nearby masses in an inextricable way. In the case of a particle and a planet, the particle is too small to influence the geometry of the planet's spacetime, and thus only the planet's metric is relevant. The Alcubierre Drive provides a specific metric for a spacecraft to determine how it shapes the geometry of the surrounding spacetime fabric.

II.1.3. Geodesic Equations

When a particle is influenced solely by the curvature of spacetime, it is considered a free particle moving along geodesic lines, which are free from external forces. The curvature of spacetime is responsible for gravitational interaction between objects, and this interaction is not considered a force itself. In other words, the geodesic equation serves as the GR equivalent of $F = ma$!

When travelling on geodesic lines, a particle's motion obeys the geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} \quad (4)$$

where $\Gamma^\mu_{\rho\sigma}$, called the Christoffel symbols, are constructed from the metric of that spacetime and its first derivatives. The Christoffel symbol is computed based on particular elements of the metric tensor, and produces a vector that traces the direction of angular movement in the curved spacetime described by the metric tensor. Given the metric tensor $g_{\mu\alpha}$ for a four-dimensional spacetime,

$$\begin{pmatrix} g_{0,0} & g_{0,1} & g_{0,2} & g_{0,3} \\ g_{1,0} & g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,0} & g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,0} & g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix}$$

The Christoffel symbols are calculated with respect to any two desired dimensions, ρ and α, of the four-dimensional spacetime:

$$\Gamma^{\mu}_{\rho\sigma} = \frac{g^{\mu\alpha}}{2}\left(\frac{\partial g_{\alpha\rho}}{\partial x_{\sigma}} + \frac{\partial g_{\alpha\sigma}}{\partial x_{\rho}} - \frac{\partial g_{\rho\sigma}}{\partial x_{\alpha}}\right) \tag{5}$$

where μ gives the row number and α gives the column number. $g^{\mu\alpha}$ is the inverse metric — an inverse matrix that, when multiplied by the metric tensor, returns the identity matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

II.2. Alcubierre Drive

Based on our current understanding of the spacetime fabric in our local universe, it is impossible to make a round trip between two points separated by a proper spatial distance Dfaster than the speed of light c,as measured by an observer stationed at the initial point. This can be mathematically expressed as a minimum travel time of $\frac{2D}{c}$according to the simple relationship of distance=(rate)×(time). However, Alcubierre proposed that it is possible to achieve faster-than-light (FTL) travel within the framework of general relativity, as measured by an observer in an outside reference frame. It is worth noting that this theory disregards non-trivial topologies, such as wormholes, that disrupt flat spacetime and make the theory more complex.

To illustrate this idea, consider two observers at rest in the fabric of spacetime. As the universe expands, new space is created between the observers, even though they remain stationary in their own reference frames. According to the standard model of cosmology, some parts of the universe are moving away from us at FTL speeds, which means that the distance between the two observers may increase at a rate faster than the speed of light. By defining the relative speed of separation as the rate of change of proper spatial distance over proper time, we can calculate a value that exceeds the speed of light. However, this does not mean that the observers are actually moving faster than light; rather, the FTL speed of separation arises from the expansion of spacetime itself.

Alcubierre's theory proposes a concept similar to the idea of utilizing the expanding universe to propel a spacecraft to faster-than-light speeds through space. The theory relies on the principle of rapidly creating new space at the back of the moving volume to simulate the expansion of the universe, while simultaneously annihilating existing space at the front of the moving volume. Alcubierre also provides a metric that describes the geometry of spacetime contraction-expansion, which would push the spaceship forward at FTL speeds, assuming it is moving along the x-axis of a Cartesian coordinate system.

$$ds^2 = -dt^2 + (dx - v_s(t)f(r_s)dt)^2 + dy^2 + dz^2. \tag{6}$$

It is evident that the spacecraft is following the x-axis in the Alcubierre metric equation provided, as the term dx is the only one that evolves over time. Hence, the equation implies that the spacecraft is moving along the x-axis.

$$f(r_s) = \frac{tanh(\sigma(r_s + R)) - tanh(\sigma(r_s - R))}{2tanh(\sigma R)} \tag{7}$$

The warp drive contains a flat spacetime area called the warp bubble that encloses the object in the warp drive. The warp bubble has a radius R, and the spaceship's trajectory in the x-direction is given by an arbitrary function $x_s(t)$. The velocity of the spaceship in the warp bubble, as it moves in the x-direction, is obtained by differentiating $x_s(t)$ with respect to t, and is denoted as $v_s(t)$.

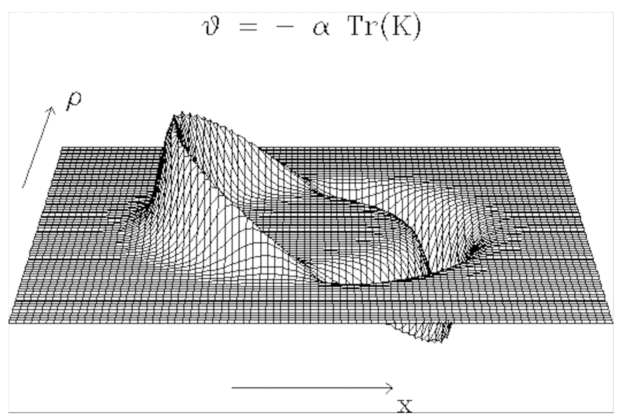

FIG. 2. Key figure from Alcubierre's original paper, showing the curvature of space in the region of the travelling warp.

In his 2017 thesis titled "Light Propagation in a Warp Drive Spacetime," Michael Peter Kinach demonstrated the computation of the Alcubierre metric tensor and its conversion into a matrix [2]. While we will not delve into the specifics of these calculations, as our project mainly centers around the simulation of particle movement in this distorted spacetime, we will provide the outcomes of Kinach's analysis - the Alcubierre metric represented in matrix form.

$$\begin{pmatrix} v_s(t)^2 f(r_s)^2 - 1 & -v_s(t)f(r_s) & 0 & 0 \\ -v_s(t)f(r_s) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

It is worth noting that the Alcubierre metric is independent of the mass or presence of any free particle affected by the spacetime curvature induced by a spaceship traveling at warp speed. This is analogous to the classical gravitational force equation, $ma = \frac{GMm}{r^2}$, in which the orbit of a mass is solely determined by the central mass, as the orbiting mass cancels out. Similarly, in the context of general relativity, the mass of the spaceship is predetermined and encoded in the metric, meaning that it does not affect the spacetime curvature.

III. SIMULATING THE MOTION OF OBJECTS IN SPACETIME METRICS AND THE ALCUBIERRE WARP DRIVE

III.1. Analytic Approach

To simulate the motion of particles in the curved spacetime of specific metrics, the typical analytical method to generate geodesic equations is to utilize the metric tensor matrix to compute the two Christoffel symbols concerning the desired dimensions, ρ and α in the four-dimensional spacetime. Subsequently, the Christoffel symbol of the desired dimensions is utilized to derive the Geodesic equation for motion along the axes of each dimension.

As an illustration, in the Alcubierre metric, we assumed the spaceship's movement to be in a straight line along the x-axis. Consequently, to investigate how a particle behaves in the presence of the curved spacetime, it is necessary to examine its motion in the y and z directions. This can be accomplished by plotting the particle's orbit in the xy and xz planes, and thus want $\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}$. The associated dimensions we use to calculate the Christoffel symbol would be $\rho = y$ and $\alpha = z$.

Initially, we computed a set of Christoffel symbols manually and later on utilized a Mathematica notebook to determine the Christoffel symbols and subsequently use them to derive the geodesic equations. This notebook was a component of a comprehensive compilation provided by Hartle along with his book "Gravity: An Introduction to Einstein's General Relativity"[3] and was adapted from the "Curvature and the Einstein equation" notebook written by Leonard Parker. The notebook guided us through a sequence of procedures to derive the geodesic equations.

1. Set dimensions of spacetime, n, equal to 4
2. Define the coordinate system as a list stored in the variable *coord*
3. Define the metric tensor as stored in the variable *metric*, inputting the covariant components of the metric tensor by editing the relevant input line in the notebook
4. Calculate the inverse metric by using **inversemetric = Simplify[Inverse[*metric*]]**
5. Calculate the Christoffel symbols, also known as the affine connection, using **Simplify[Table[(1/2)*Sum[(*inversemetric*[[i, s]])* (D[*metric*[[s, j]], *coord*[[k]]] + D[*metric*[[s, k]], *coord*[[j]]] - D[*metric*[[j, k]], *coord*[[s]]]), s, 1, n], i, 1, n, j, 1, n, k, 1, n]]**
6. Calculate the geodesic equations using **Simplify[Table[-Sum[*affine*[[i, j, k]] u[j] u[k], j, 1, n, k, 1, n], i, 1, n]]**

After inputting the coordinates and the entries of the metric tensor, the process was really just running each cell in the Mathematica notebook.

We followed a process in which we utilized the Mathematica notebook to model each chosen metric, inputting the coordinates and metric tensor entries and running all subsequent cells to generate four geodesic equations of motion (one for each dimension of spacetime) for each spacetime curvature. Initially, we analyzed the Alcubierre metric, followed by other metrics that were relatively simpler, such as flat spacetime. After obtaining the geodesic equations, we created a user-defined Python function for each metric and orbital path to apply the fourth-order Runge-Kutta method and plot the outcomes.

III.2. Computational Approach: Fourth-Order Runge-Kutta Method

To simulate the particle orbits in each spacetime using Runge-Kutta, we began with four geodesic equations for each spatial dimension. The earlier section details how these equations were computed in Mathematica.,

$$\frac{dv_t}{d\tau}, \frac{dv_x}{d\tau}, \frac{dv_y}{d\tau}, \frac{dv_z}{d\tau},$$

We started by understanding that the acceleration in each dimension is described as the derivative of velocity in any spatial dimension with respect to time. To solve the geodesic first-order differential equations using the fourth-order Runge-Kutta method with a fixed step size, we used a modified version of Mark Newman's *odesim.py* program that solves simultaneous first-order differential equations. Our modifications included redefining the necessary constants, starting with the arbitrary parameter R, which determines the boundary of the warp drive at a distance R from its center. We experimented with multiple random values of R to observe the orbit's behavior. We also defined the parameter $\sigma = 1$, which determines the sharpness of the warp bubble's boundary. The initial x, y, and z values were set to zero, but different initial velocities for x, y, and z were tested to examine how a particle with varying velocities relative to the warp bubble would behave near a spaceship moving in the warp drive. Finally, the last constant, rs, was defined as a function of t that always returns zero since $r_s = 0$ corresponds to the center of the warp drive.

Next, we defined individual functions for each of the metrics we aimed to model. These functions receive an r-vector and an array of time points as arguments, and output the derivatives of position and velocity in the desired spatial dimensions, which are used in Runge-Kutta calculations. We initialized empty arrays to store positions and velocities at each time step, and also created two arrays to hold the initial conditions, one for each dimension. The initial position coordinate and velocity in each direction are stored in the first and second entries of their respective arrays.

To perform calculations for the fourth-order Runge-Kutta method, we utilized a *for loop* to iterate through each time point and obtain the results from a user-defined function. The loop added the current position and velocity values to empty arrays at each time step. We determined the step size through trial-and-error, adjusting values and rerunning the code to observe the expected orbital behavior. Ultimately, we visualized the plotted array of x-values versus y-values using Matplotlib.

For plotting orbits in the Alcubierre spacetime metric, we defined the spaceship's motion to be linear along the x-axis. In order to observe the behavior of particles in the y and z directions, we plotted orbital behavior in the xy plane and xz plane. This required specifying the two spatial dimensions to plot and creating arrays to store position and velocity values for each dimension, as calculated by the Runge-Kutta method at each time step through the *for loop.* Thus, to plot orbits in the Alcubierre metric, we repeated the above process twice: the first time with x and y as spatial dimensions, and the second time with x and z.

Finally, using the generated x, y, and z values for a path usinf the fourth-order Runge-Kutta methods stored in "$x - values$, $y - values$ and $z - values$ lists. Then using the "$mpl - toolkits.mplot3d$" library we called the function "ax.scatter()" to create a scatter plot in 3D, where we used the x, y, and z values lists as the x, y, and z coordinates for each point in the scatter plot.

IV. RESULTS

IV.1. Orbits in the Alcubierre Metric

Once we input the Alcubierre metric tensor in the Mathematica notebook and define the coordinate system for the spacetime, we obtained the geodesic equations for each of the four spacetime dimensions.

$$\begin{gathered}
\frac{dv_t}{d\tau} = 0, \\
\frac{dv_x}{d\tau} = \frac{tanh(\sigma(r_s+R)) - tanh(\sigma(r_s-R))}{2tanh(\sigma R)} v_t^2 \frac{d^2x}{dt^2}, \\
\frac{dv_y}{d\tau} = 0, \\
\frac{dv_z}{d\tau} = 0,
\end{gathered} \tag{9}$$

The Geodesic differential equation is non-zero only in the x-direction, while all others are zero. This suggests that a particle close to the spaceship undergoing warp drive would experience no acceleration in the y or z-directions, but would still accelerate forward in the x-direction with the spaceship's movement. Furthermore, since we assumed the particle to be initially at rest, we anticipated that the xy and xz plane plots would depict an orbit that resembles an exponential increase. Presented below are the plots of the orbit in the xy and xz planes, which were generated using the Alcubierre metric to model the spacetime curvature.

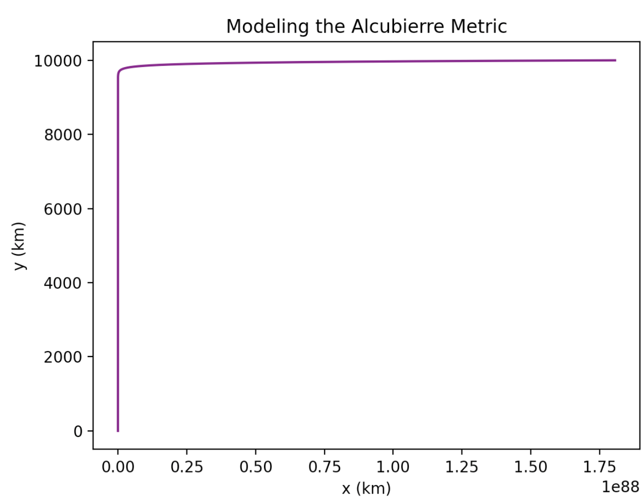

FIG. 3. Graph showing the orbit in the xy plane in the spacetime curvature if the Alcubierre metric

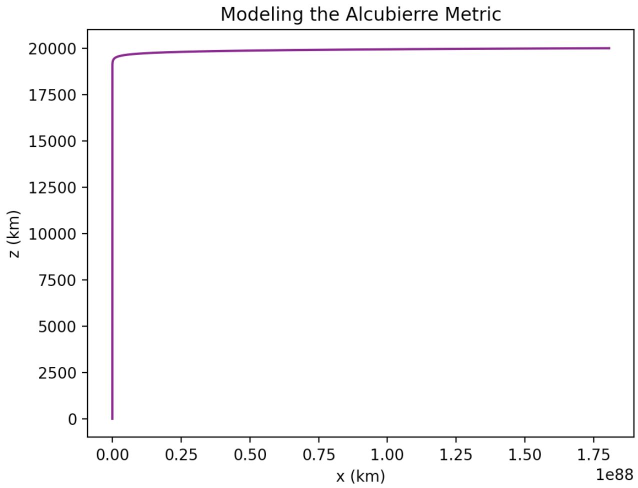

FIG. 4. Graph showing the orbit in the xz plane in the spacetime curvature if the Alcubierre metric

Below is our 3D plot of orbit in the spacetime curvature of the Alcubierre metric.

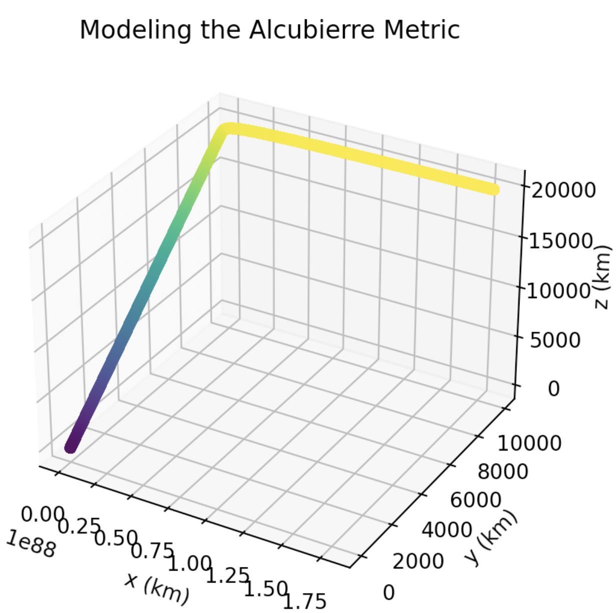

FIG. 5. Graph showing the orbit in the 3D plane of the Alcubierre metric

V. CONCLUSION AND DISCUSSION

In conclusion, this report explores the possibility to model the Alcubierre metric, which describes the curvature of spacetime in a region around a hypothetical ”warp drive.” The warp drive is a theoretical spacecraft propulsion system that would allow faster-than-light travel by contracting spacetime in front of the spacecraft and expanding it behind. The Alcubierre metric describes the shape of the ”warp bubble” that would be created by the spacecraft. The approach taken involve using mathematica to calculate the Christoffel symbols, that in turn are used to find the geodesic equations necessary to calculate the path of a test particle moving in the gravitational field described by the Alcubierre metric.

To do this, we use Python and the fourth-order Range-Kutta method to solve the geodesic equation and generate a series of x, y and z coordinates that describe the path of the particle. We then used that to plot the xy and xz planes, as well as a 3D plot to simulate the motion of the test particle in the spacetime curvature described by the Alcubierre metric. Through this study, we have gained a deeper understanding of the theory behind the Alcubierre metric, as well as the numerical and computational elements used to model the motion of a particle in a gravitational field described by the Alcubierre metric. However, FTL travel is still not physically possible, and there is still much work to do to explore this topic.

V.1. Citizenship assessment

a) Contributed to their agreed upon code component of the project: Yes
b) Responded to communications in a timely manner and attended mutually agreed-upon meeting times: Yes
c) Contributed to preparation of their agreed-upon component of the oral presentation: Yes

[1] M. Alcubierre, “The warp drive: hyper-fast travel within general relativity,” (1994).
[2] M. P. Kinach, *Light Propagation in a Warp Drive Spacetime*, Ph.D. thesis, St. Francis Xavier University (2017).
[3] J. Hartle, *Gravity: An Introduction to Einstein’s General Relativity* (Pearson, 2002).