PHYS 304 Final Project
Thermal Diffusion in Earth's Crust

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The heat equation is a fundamental one-dimensional physical equation used to model the flow of heat in thermal systems. Using the *forward-time centered-space method* of partial differential equation solving, the heat equation is solved and plotted in order to display how heat diffusion in the Earth's crust is impacted by seasonal temperature variation. Experiments are then performed to examine how these temperature profiles are affected by different crustal compositions and global mean surface temperatures.

1. CONTEXT AND MOTIVATIONS

The diffusion equation 1, also known as the heat equation, is a partial differential equation used to one-dimensionally model the diffusion of gases and liquids and the flow of heat in thermal systems.

$$\frac{\delta T}{\delta t} = D\frac{\delta^2 T}{\delta x^2} \tag{1}$$

Here, T is temperature, t is time, and x is position. D is a constant known as the thermal diffusivity, or heat diffusion constant. Thermal diffusivity is dependent on the properties of the material that the heat is flowing through. It is defined as the thermal conductivity, k, divided by the density, ρ, and the specific heat capacity, C_p, of the specific material 2.

$$D = \frac{k}{\rho C_p} \tag{2}$$

One interesting application of the heat equation is by utilizing it to model the diffusion of heat in Earth's crust. While the boundaries of an isolated system can be held constant, the boundary conditions of the heat equation vary in time due to seasonal changes and variations in Earth's surface temperature. The variation in mean daily temperature at an arbitrary point on the Earth's surface is given by the following equation 3, where A is the mean surface temperature, B the temperature at the deepest point, t is time, and τ is 365, the number of days in a year.

$$T_0(t) = A + B\sin\frac{2\pi t}{\tau} \tag{3}$$

The initial goal of this project is to create a program that will solve the heat equation and produce a model of the crust's temperature gradient up to a depth of 20 meters. The program will run for a simulated 9 years in order to allow any patterns to settle, and will then plot four curves, each representing a period of three months to show seasonal differences. This project will examine the effects of changing two parameters of this model. First, how different crustal material, and therefore different thermal diffusivity values, impact this model. The crust is made up of a variety of different materials, how would these profiles change if they were localized in an area of the crust primarily made of granite, or sandstone? Second, how do these profiles change with different average surface temperatures? On an anthropological timescale, Earth's average temperature has not changed drastically, however, considering ice ages and warming events, there have been periods in Earth history with dramatically different surface temperatures.

2. COMPUTATIONAL METHODS

2.1. Partial Differential Equation Solving

The method by which the heat equation is solved is called the *forward-time centered-space method* (FTCS) of solving partial differential equations. Because the heat equation lacks temporal boundary conditions, and instead presents us with only an initial condition, methods of solving partial differential equations that involve relaxation cannot be used. Rather, we must use a *forward integration method.*

Starting with the heat equation in the following form 4, where ϕ is $\phi(x,t)$, a variable dependent on both position and time,

$$\frac{\delta\phi}{\delta t} = D\frac{\delta^2\phi}{\delta x^2} \tag{4}$$

We first begin by dividing the spatial dimension into a grid of points, with an arbitrary, even spacing of a. The derivative term on the right-hand side of equation 4 can now be rewritten as the following 5,

$$\frac{\delta^2\phi}{\delta x^2} = \frac{\phi(x+a,t) + \phi(x-a,t) - 2\phi(x,t)}{a^2} \tag{5}$$

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This utilizes a known approximation technique for the second derivative involving central differences, referenced in an earlier chapter of Newman's textbook (6),

$$\begin{aligned} f''(x) &\simeq \frac{f'(x+h/2) - f'(x-h/2)}{h} \\ &= \frac{[f(x+h) - f(x)]/h - [f(x) - f(x-h]/h}{h} \\ &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \end{aligned} \tag{6}$$

Returning to the heat equation, it can now be written as the following 7,

$$\frac{d\phi}{dt} = \frac{D}{a^2}[\phi(x+a,t) + \phi(x-a,t) - 2\phi(x,t)] \tag{7}$$

Now, we have produced a set of ordinary differential equations that can be solved with one of the various methods we have learned. This program utilizes Euler's method. Applying Euler's method to the previous equation 7, yields 8

$$\begin{aligned} \phi(x,t+h) = \phi(x,t) + h\frac{D}{a^2}[\phi(x+a,t) \\ + \phi(x-a,t) - 2\phi(x,t)] \end{aligned} \tag{8}$$

Since we can calculate the value of ϕ at every point x and some time t, this equation calculates the value at every time t+h, where h is small. Doing this at every grid point will derive the solution to the differential equation.

2.2. The Program

I have produced two separate versions of the program, one where the user inputs values needed to calculate the thermal diffusivity of a certain material, and one where these values are set, and the user inputs the global mean surface temperature value. However, these aspects are just simple input functions and don't require much discussion.

This program was originally adapted from *heat.py*, a program by Mark Newman from the textbook *Computational Physics* [1], though significant changes have been made. Due to the precision desired in this situation, the program took a long time to run. The biggest change is that the iteration is now done in a separate function, rather than in a loop with the plotting commands. Calculations for the first nine years of the simulation are done and then values for the tenth year, the year that will be plotted, are divided into the four quarters and calculated separately.

A way in which the program could have been made faster is by implementing the Crank-Nicolson method of solving partial differential equations. I did, in fact, make efforts to implement this method, but I was ultimately unsuccessful.

3. RESULTS

3.1. Baseline

To gain a baseline understanding of what the crustal heat diffusion profile looks like, we first run the program using values supplied by problem 9.4 in *Computational Physics* by Mark Newman [1]. This problem also supplied the equation for the variation of mean daily temperature at a given point on the Earth's surface 3. The textbook provides the following constants:

$$\begin{aligned} \tau &= 365 \text{ days} \\ A &= 10°\text{C} \\ B &= 12°\text{C} \\ D &= 0.1\ m^2/\text{day} \end{aligned}$$

Where, again, A is the average surface temperature, B is the temperature at the deepest point (20 meters), τ is the length of a year, and D is the thermal diffusivity. Thermal diffusivity does vary with crustal material, as this project will later examine, but for simplicity the crust is assigned a homogeneous value. Furthermore, the problem states that at a depth of 20 meters, almost all annual temperature variation is irrelevant, and this area can be assigned a constant value of 11°C.

For this project, I decided upon a grid size of 100 points and a time step of 0.0001 seconds. The resulting profile is shown in figure 1. The figure displays the temperature profiles of the crust as a function of depth and time, with each line representing the profile of a different season. In accordance with reality, the surface temperatures of summer and winter start at the two extremes of very hot and very cold, but as we travel deeper, the temperatures converge. The same is true of spring and fall, however they both begin at around the same surface temperature. The figure shows the ultimate convergence of temperatures at approximately the depth of 80 meters, but we can clearly see the beginnings of convergence as shallow as 20 meters.

3.2. Crustal Composition

The first aspect explored is how different crustal compositions affect these temperature profiles. Despite the value of D = 0.1 m^2/day being a good approximation of the crust's thermal diffusivity, it ignores the fundamentally non-homogeneous nature of the Earth's crust, which is especially prevalent at shallow depths like the ones dealt with here. This section will discuss four common crust materials, creating profiles assuming a bulk crustal composition of each one.

The four materials analyzed here are some of the most common in Earth's upper crust, and it is not at all unreasonable to assume that large swaths of these materials, several kilometers wide and meters deep, exist on Earth. The materials are granite, shale, sandstone and

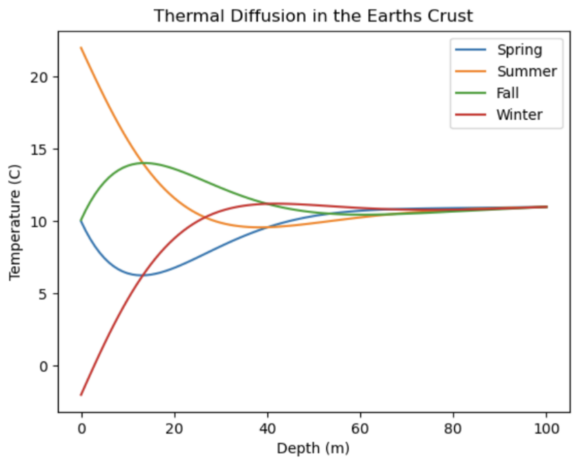

FIG. 1: Temperature profiles of the Earth's crust as a function of depth and time. Each line represents a different season.

limestone. Basalt, an abundant crustal rock, was deliberately excluded as its properties vary so greatly that it would be impossible to assign any singular value to all basalts.

To calculate the thermal diffusivities of these materials, we need their thermal conductivity (k), density (ρ), and specific heat (C_p). These values are then plugged in to an altered version of equation 2, which converts the value to units of square meters per day, 9.

$$D = 86400\frac{k}{\rho C_p} \tag{9}$$

The values of these constants and the calculated thermal diffusivities are shown in table I. The density values of these materials were found through encyclopedic websites, like Wikipedia. Other values were found through scholarly articles and will be cited. For any values that are ranges, the average value was used. The value of D is calculated inside of the program itself through the use of input commands for the three necessary constants. This allows the easy plotting of other materials as long as their constants are known. For these runs, the temperature values were not changed from those in the baseline experiment.

The first material analyzed is granite, shown in figure 2. We see a small, but still noticeable difference from the baseline 1, which makes sense. As shown in the table I, granite has a thermal diffusivity of 0.1353 m^2/day, which is similar to the average value of 0.1 m^2/day presented by Newman.

Skipping past shale for a moment, we see a similar pattern with sandstone 3 and limestone 4, as their thermal diffusivity values are even closer to the average value presented by Newman

Finally, we see some strong deviation from the norm in the form of shale 5. With the smallest thermal conductivity and largest specific heat out of the selected materials,

Material	Thermal Conductivity (W/mK)	Density (kg/m3)	Specific Heat (J/kgK)	Thermal Diffusivity (m2/day)
Granite	3.1[2]	2750	720[2]	0.1353
Shale	1.05-1.45[3]	2060-2670	871[4]	0.0524
Sandstone	2.25-2.47[4]	2420	787-811[5]	0.1055
Limestone	2.39-2.78[4]	2630	751-780[4]	0.111

TABLE I: The thermal conductivity, density, specific heat, and calculated thermal diffusivity of common crust materials.

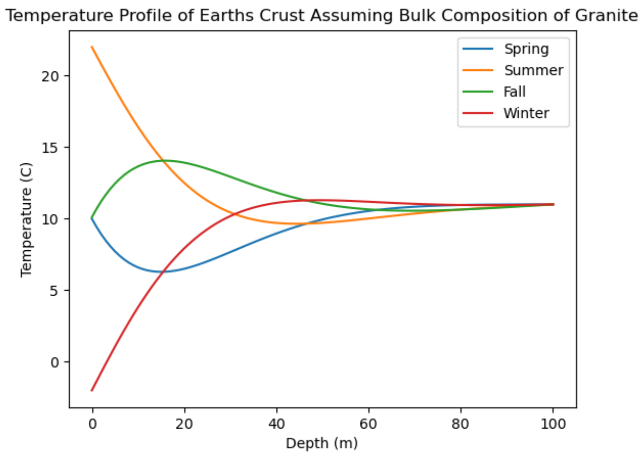

FIG. 2: Temperature profiles of the Earth's crust as a function of depth and time assuming a bulk crustal composition of granite. Each line represents a different season.

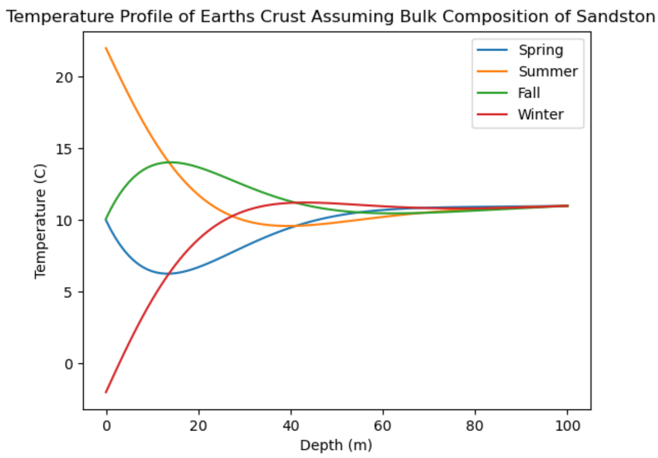

FIG. 3: Temperature profiles of the Earth's crust as a function of depth and time assuming a bulk crustal composition of sandstone. Each line represents a different season.

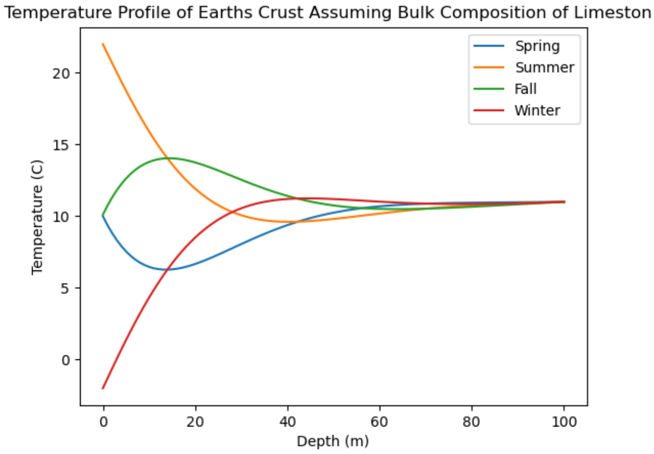

FIG. 4: Temperature profiles of the Earth's crust as a function of depth and time assuming a bulk crustal composition of limestone. Each line represents a different season.

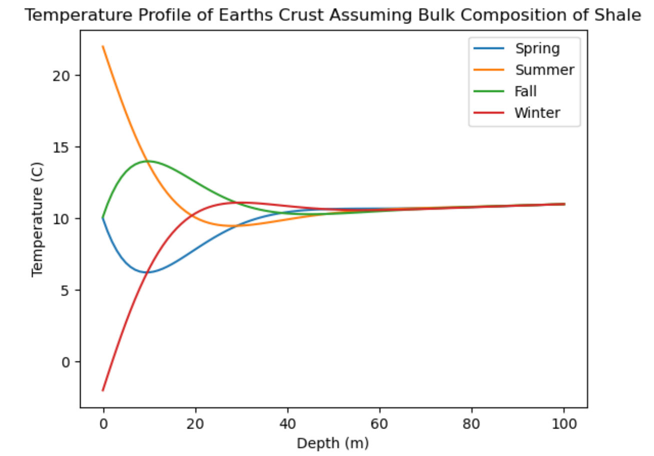

FIG. 5: Temperature profiles of the Earth's crust as a function of depth and time assuming a bulk crustal composition of shale. Each line represents a different season.

shale has by far the lowest thermal diffusivity constant. The temperature profiles of the seasons converge much quicker than in the other materials.

If you take the average of the thermal diffusivity of these materials, you get a value of 0.1010 m^2/day. Knowing that these are some of the most common materials in the Earth's crust, we can confidently say that Newman's proposed value of 0.1 m^2/day is a good estimation of the crust's bulk thermal diffusivity. However, looking at these materials independently shows that they can have a significant difference and that in order to create proper, realistic temperature profiles of the crust in specific locations, it is necessary to know its composition. Furthermore, all of these profiles agree with the widely accepted fact that seasonal variations in temperature become irrelevant to crustal heating at very shallow depths, less than a single kilometer

It is imperative to note that this model is only representative of one of the many sources of heating the Earth receives. These profiles demonstrate the effects of surface temperature only, which is controlled by solar radiation and the passage of the seasons. The model does not account for any heat produced by radioactive decay within the crust [6]. It does account somewhat for the Earth's geothermal gradient [7], the rate of temperature change of the Earth's interior with respect to depth, by setting the temperature at the deepest point higher than the surface. Geophysicists estimate the geothermal gradient in the crust to be approximately 25°C per kilometer of depth [6], however, this model only deals with temperature variations at depths of 20 meters.

3.3. Surface Temperature

This section will investigate how these profiles are affected by drastically different average surface temperatures. In terms of modern human history, there has not been much variation in average surface temperature. However, on geological timescales, Earth's temperature has varied greatly due to a number of different causes. Two well known periods of extreme climate will be analyzed: the Paleocene-Eocene thermal maximum (PETM) and the Neoproterozoic snowball Earth.

The Paleocene-Eocene thermal maximum was a period in Earth's history approximately 56 million years ago where Earth experienced a sudden increase in surface temperature. Paleogeologists disagree on the exact figure, but it is generally accepted that the Earth experienced a 5-9°C increase in global average surface temperature. The cause of this event is also unknown, but is most likely attributed to a rise in greenhouse gases [8]. The global mean surface temperature for this period has been calculated through the use of multiple models as approximately 31.6°C [9]. For these models we assume baseline conditions for thermal diffusivity, and while the value of A in equation 3 changes, the value of B remains at 12°C, as the surface temperature should not affect the temperature at depths of 20 meters.

The resulting profile 6 is a familiar shape, but distorted. Note the change in the scale of the vertical axis. The temperature profiles converge at a similar depth to those seen previously, but do not stabilize completely until much deeper. Interestingly, here, the temperature appears to be *decreasing* as we move deeper into the crust, which seems to contradict the known geothermal gradient. However, it is important to once again note that this is not a complete model of all of Earth's heat sources.

The Neoproterozoic snowball Earth was a period of global cooling that took place approximately 650 million years ago. The "snowball Earth" phenomenon is a hypothesis that proposes that during this cold period, Earth's surface was entirely, or nearly entirely, frozen.

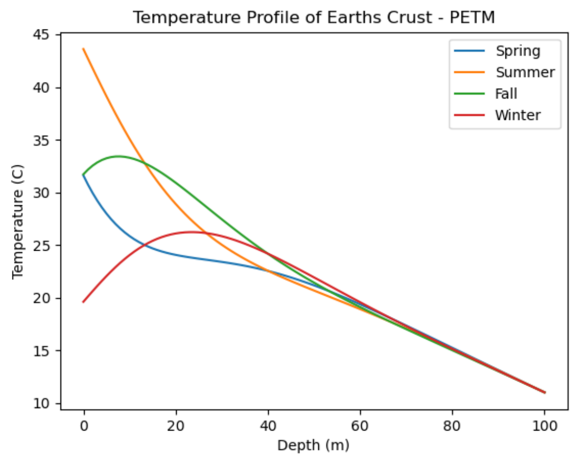

FIG. 6: Temperature profiles of the Earth's crust during the Paleocene-Eocene thermal maximum as a function of depth and time. Each line represents a different season.

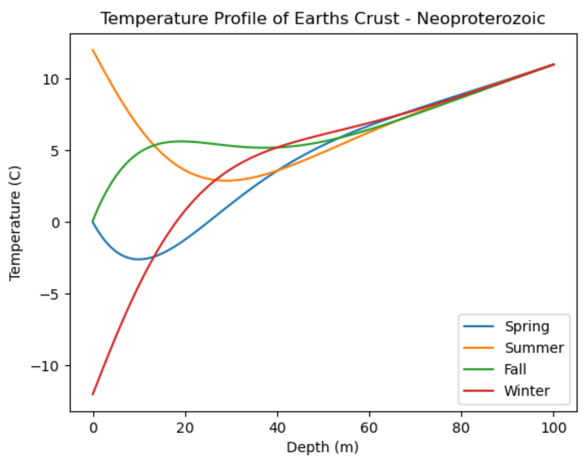

FIG. 7: Temperature profiles of the Earth's crust during the Neoproterozoic snowball Earth as a function of depth and time. Each line represents a different season.

Determining a global mean surface temperature for this period is difficult as, regardless of its snowball status, Earth was covered in a lot of ice, which would reflect incoming solar radiation back into the atmosphere. Furthermore, exact greenhouse gas content is unknown, Nevertheless, estimates have been made, and the value used here will be 0°C [10].

The resulting profiles 7 display similarities to those of the PETM, though mirrored. This aligns with expectations. In the interest of fully exploring the prospect of the snowball Earth hypothesis, an experiment was run where the crust was assumed to have a bulk composition of ice. However, the thermal diffusivity of ice is 0.10709 m^2/day, extremely similar to the baseline value, and the profiles produced were insignificantly different from those seen in figure 7.

4. DISCUSSION

While some interesting results have been produced here, it is important to recognize that a lot of this relies on assumptions. This is particularly true of the surface temperature experiments. Because so little is concretely known about the geophysics of these periods in Earth history, we must make assumptions. The equation that defines the variability of mean daily surface temperature, equation 3, may not necessarily hold for time periods with such drastically different climates. Furthermore, it cannot be said for certain that the temperature at a depth of 20 meters was the same then as it is now.

These analyses only scratch the surface of the topic of heat diffusion in Earth's crust. Further research, experimentation, and code implementation can lead to a more thorough understanding of the crust's geothermal properties.

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