NAND gate example

Mark W. Coffey
Department of Physics
Colorado School of Mines
Golden, CO 80401

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Let NAND(x,y)=NOT(AND(x,y)). It has the following truth table.

x	y	NAND(x,y)
F	F	T
F	T	T
T	F	T
T	T	F

We'd like to consider how to set up an Ising model that encodes this logical operation. Take

$$H_f = \sum_j h_j \sigma_j^z + \sum_{i<j} J_{ij} \sigma_i^z \sigma_j^z,$$

where h_j are the field parameters and J_{ij} the spin couplings. Choose logical 1 (true) to be represented by spin up $|\uparrow\rangle$ (or $s = +1$) and logical 0 (false) by spin down $|\downarrow\rangle$ (or $s = -1$). We'll confirm the following encoding for the NAND gate:

h_j	J_{ij}
$h_A = -1$	$J_{AB} = 1$
$h_B = -1$	$J_{AC} = 2$
$h_C = -2$	$J_{BC} = 2$

A	B	C	Energy
$\downarrow$	$\downarrow$	$\downarrow$	9
$\downarrow$	$\downarrow$	$\uparrow$	-3
$\downarrow$	$\uparrow$	$\downarrow$	1
$\downarrow$	$\uparrow$	$\uparrow$	-3
$\uparrow$	$\downarrow$	$\downarrow$	1
$\uparrow$	$\downarrow$	$\uparrow$	-3
$\uparrow$	$\uparrow$	$\downarrow$	-3
$\uparrow$	$\uparrow$	$\uparrow$	1

So we see that the ground state energy of -3 corresponds to the entries of the truth table for NAND.

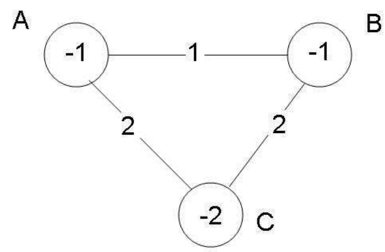

Figure 1: Graphical representation of the Hamiltonian implementing the logic of a NAND gate. Each of the vertices is a spin and the number inside is the field value on that spin; each edge is the coupling value between the respective vertices. Each ground state of this Hamiltonian enforces the logic C=NOT(AND(A,B)).

This example begs the question of how to obtain the h_j and J_{ij} parameters for a given problem. When the operation in question is a logic gate, two avenues seem to be available:
(a) Use integer programming to solve for these parameter values. This will not be computationally advantageous in general, but ok for small or specialized problems.
(b) For this special case of a logic gate (by far not the typical application of adiabatic quantum computing), one can write mathematical formulas encoding the output of the gate, and then translate that to a quadratic objective function for the Ising model.

We as a group need to elaborate and explore both of these approaches.

Consider method (a) for the NAND gate. Writing out the problem Hamiltonian in detail,

$$H_f = h_1\sigma_1^z + h_2\sigma_2^z + h_3\sigma_3^z + J_{12}\sigma_1^z\sigma_2^z + J_{13}\sigma_1^z\sigma_3^z + J_{23}\sigma_2^z\sigma_3^z.$$

We have 6 parameters in total: 3 h's and 3 J's. Suppose that we wish the ground state energy to be k. Then from the truth table encoded by spins, we have the 4 conditions

$$-h_1 - h_2 + h_3 + J_{12} - J_{13} - J_{23} = k,$$

$$-h_1 + h_2 + h_3 - J_{12} - J_{13} + J_{23} = k,$$

$$h_1 - h_2 + h_3 - J_{12} + J_{13} - J_{23} = k,$$

$$h_1 + h_2 - h_3 + J_{12} - J_{13} - J_{23} = k.$$

The other four conditions are inequalities since we demand that they yield an energy strictly greater than k. Thinking in terms of an integer programming problem, we replace the condition $> k$ with $\geq k+1$ and get

$$-h_1 - h_2 - h_3 + J_{12} + J_{13} + J_{23} \geq k+1,$$

$$-h_1 + h_2 - h_3 - J_{12} + J_{13} - J_{23} \geq k+1,$$

$$h_1 - h_2 - h_3 - J_{12} - J_{13} + J_{23} \geq k+1,$$

$$h_1 + h_2 + h_3 + J_{12} + J_{13} + J_{23} \geq k+1.$$

If we just solve the 4 equalities, we have the reduced relations

$$h_3 = h_1 + h_2, \quad J_{12} = h_3 - k, \quad J_{13} = h_2 - k, \quad \text{and} \quad J_{23} = h_1 - k.$$

It's easily confirmed that these relations are satisfied as above when $k = -3$.

In general when there are N spins in the Ising Hamiltonian, there are N h parameters and $N(N-1)/2$ J parameters, for a total of $N(N+1)/2 = \binom{N+1}{2}$. There are 2^N conditions, so that this method is limited to small problems.

CNOT gate

Let c denote the control qubit and t the target qubit. Then we recall the truth table for this gate in the following form (corresponding to addition modulo 2).

c	t	c	$c \oplus t$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

Motivated by Pudenz and Lidar (Table 1), we consider the following penalty function:

$$2x_i x_j - 2(x_i + x_j)x_k - 4(x_i + x_j)x_a + 4x_k x_a + x_i + x_j + x_k + 4x_a. \tag{1}$$

The ancilla qubit a is used to avoid a cubic term so as to be able to develop an Ising model (with terms only linear and quadratic in the spin variables). We may note that the penalty function (1) is symmetric in the Boolean inputs x_i and x_j. x_k is the output such that (1) is minimized.

x_i	x_j	$x_a = x_i x_j$	(1)	x_k to minimize (1)
0	0	0	x_k	0
0	1	0	$-x_k + 1$	1
1	0	0	$-x_k + 1$	1
1	1	1	x_k	0

For the corresponding Ising model, the linear terms in (1) yield the field (h_j) coefficients and the other terms the spin-spin couplings J_{ij}.
Possible exercises for the reader. (i) Explicitly write the h_j and J_{ij} coefficients for the Ising model 4-qubit Hamiltonian. (ii) Generate a Mathematica notebook to illustrate AQC with this Hamiltonian. For Boolean variable $x_i \in \{0, 1\}$ and spins $s_i \in \{-1, 1\}$, there is the mapping $s_i = 2x_i - 1$.