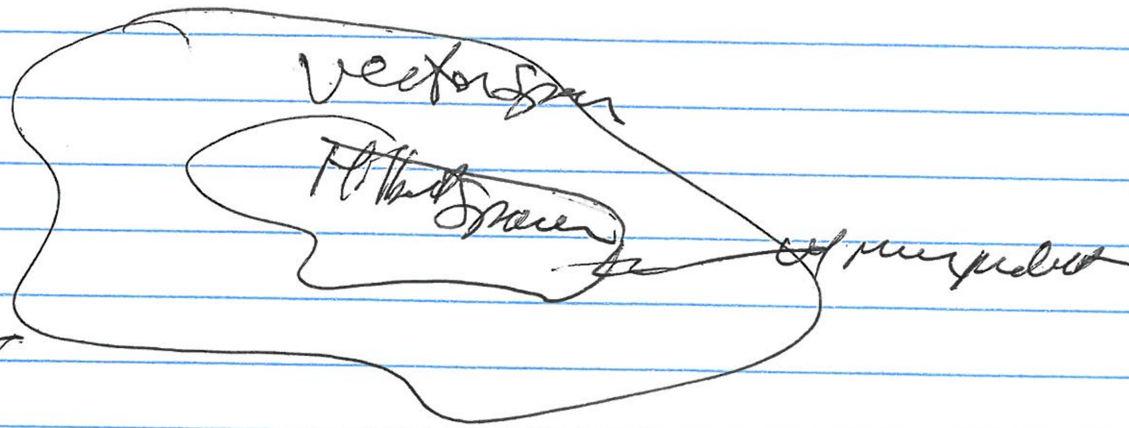
Ch. 2 linear algebra & Dirac notation

QM (quantum mechanics) deals w/ vectors (states) in "Hilbert space".

A Hilbert space is a complete, normed, vector space.
norm
finite/boundal from inner product



Dirac notation of physics
– a first take/cut of how to transcribe:

vector $\vec{a}$: so-called ket $|a\rangle$

transposed conjugate vector $\vec{a}^{*T}$ (c.c.) $\langle a|$ "bra"

inner product "bra-ket" $\langle a|b\rangle$

No big deal for us since we deal w/ (some degree of freedom-based) finite dimensional Hilbert spaces.

Basis vectors

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

For a 2-dim. Hilbert space. allow superposition of

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right].$$

Higher dimensional basis vectors may be generated as tensor products

$$\text{eg } |00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

In general, for 2^n-dimensional of basis vectors $|00\cdots0\rangle, |00\cdots01\rangle, \ldots |11\cdots1\rangle$

the matrix repr are:

$|00\cdots 0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (of length 2^n) $|00\cdots 01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$|11\cdots 10\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$, $|11\cdots 11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

*

Example let $i = \sqrt{-1}$. In $2^n = 4$ dimen the vector

$\frac{1}{\sqrt{3}}\left(\sqrt{2}\, i\, |01\rangle + |11\rangle\right) =$

$= \frac{1}{\sqrt{3}}\left(\sqrt{2}\, i\, |0\rangle + |1\rangle\right) \otimes |1\rangle$

has the matrix repr

$\frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ \sqrt{2}\, i \\ 0 \\ 1 \end{bmatrix}.$

*(This basis is referred to as the computational bases) (all vec same, have vector only 1 nonzero entry each)

Dual ~~vectors~~ inner product over the complex ($\mathbb{C}$)

W4

The dual of vector $|x\rangle$ is $\langle x|$, key pt the matrix's complex-conjugate transposed (Hermitian conjugate),

E.g. $|x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$, then $\langle x| = \frac{1}{\sqrt{2}}(-i, 1)$.

In Dirac notation an inner product like $\langle \vec{v}, \vec{w} \rangle$ is written $\langle v|w\rangle$.

This way, for vectors over the complex, the norm squared is ≥ 0 as needed,

or $\|\vec{v}\|^2$ (math notation)

$$|v|^2 = \langle v|v\rangle \geq 0.$$

E.g. w/ x as above,

$$\langle x|x\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(-i, 1)\cdot\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{2}{2} = 1$$

(as normalized as we need in QM)

more generally if $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ then

$$\langle v|w\rangle = (v_1^* \; v_2^* \cdots v_n^*)\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} v_i^* w_i$$

(Mathematicians!) no comma

wrt the computational basis for dim $2^2 = 4$ (or for 2 qubits), e.g.

$\langle 00|00\rangle = 1 = \langle 01|01\rangle = \langle 10|10\rangle = \langle 11|11\rangle$.

& e.g. $\langle 01|10\rangle = 0 = \langle 11|01\rangle = \ldots$

- we have an orthonormal basis such that if b_n, b_m are one of the 4 basis vectors then

$$\langle b_n|b_m\rangle = \delta_{nm}, \text{ the Kronecker delta}$$

symbol $\ni$ $\delta_{nm} = \begin{cases} 1 & \text{when } n=m \\ 0 & \text{when } n \neq m. \end{cases}$

(Ex. 2.26, p. 26 for the Hadamard basis 'given away' part of a HW problem HW 1).

dual vectors generally:

If $|\psi\rangle = \sum_i \alpha_i |\phi_i\rangle$, $\alpha_i \in \mathbb{C}$,
then
$\langle\psi| = \sum_i \alpha_i^* \langle\phi_i|$.

note

Further, if the $|\phi_i\rangle$ are an orthonormal basis, then

$\langle\psi|\psi\rangle = \|\psi\|^2 = \sum_i |\alpha_i|^2 = \sum_i \alpha_i \alpha_i^*$

(p. 27, line 2 of text has a typo)