Introduction to Quantum Computing Homework Set 1 Solutions

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Abstract

Key words and phrases

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A1. For the Turing machine model described in class that used the unary representation of numbers: (a) How many steps of the program are needed to find $1 + 2 = 3$? (b) How many steps of the program are needed to find $3 + 4 = 7$? (c) How many steps of the program do you expect are required to find $n + m$ for positive integers n and m?

(a) 8 (b) 16 (c) The number of steps is linear in n and m and seems to be given by $3n + m + 3$. The minimum number of steps will be the minimum of that expression and $3m + n + 3$.

A2. Construct AND and OR gates from NAND and FANOUT.

Recall that (classically) we may copy expressions using FANOUT. We first consider the logical expression (a NAND b) NAND (a NAND b) using a truth table:

a	b	a NAND b	(a NAND b) NAND (a NAND b)	a AND b
0	0	1	0	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	1

The last two columns of the table agree and therefore the AND gate has been reproduced.

Similarly, we consider the logical expression (a NAND a) NAND (b NAND b) using a truth table:

a	b	a NAND a	b NAND b	(a NAND a) NAND (b NAND b)	a OR b
0	0	1	1	0	0
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	1	1

The last two columns of the table agree and therefore the OR gate has been reproduced.

We can also perform a confirmation algebraically. E.g.,

(a NAND b) NAND (a NAND b)=NOT((a NAND b) AND (a NAND b)) =1-(a NAND b)^2-1- (a NAND b)=1-(1-ab)=ab= a AND b,

and

(a NAND a) NAND (b NAND b)=(1-a AND a) NAND (1-b AND b) =(1-a^2) NAND (1-b^2)=(1-a) NAND (1-b)=NOT((1-a) AND (1-b)) =1-(1-a)(1-b)=a+b-ab=a+b-ab=a OR b.

A3. Let σ_x and σ_y denote the usual Pauli spin matrices. For *each* of these matrices: (i) Find each of the projection operators $P_\pm$ corresponding to their eigenvalues. (ii) Verify that $P_\pm^2 = P_\pm$ and $P_+ P_- = 0$. (iii) Resolve the identity matrix in terms of P_+ and P_-. (iv) Write the spectral decomposition of each in terms of P_+ and P_-.

See separate file for discussion of this problem.

A4. See separate file.

A5. Consider n unitary matrices V_i, $1 \leq i \leq n$. Let $W = V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_n$. Show that W is unitary.

First, if $n = 2$, $W_2 = V_1 \otimes V_2$, $W_2^\dagger = (V_1 \otimes V_2)^\dagger = V_1^\dagger \otimes V_2^\dagger$ (by properties of $\otimes$ under $*$ and T). Then

$$W_2 W_2^\dagger = (V_1 \otimes V_2)(V_1^\dagger \otimes V_2^\dagger)$$

$$= V_1 V_1^\dagger \otimes V_2 V_2^\dagger = I \otimes I = I.$$

This is also the basis for an inductive proof in the case of n factors. If we assume

$W_n = \otimes_{i=1}^{n} V_i$ is unitary, with each V_i unitary, then

$$W_{n+1} = \otimes_{i=1}^{n+1} V_i = \otimes_{i=1}^{n} V_i \otimes V_{n+1} = W_n \otimes V_{n+1},$$

is also unitary by the above.

A6. Let T be a linear transformation on C^n such that $T|\psi\rangle = T|\phi\rangle = 0$ with $|\psi\rangle \neq |\phi\rangle$. Show that T is not norm-preserving.

Consider the effect of T on the vector $|\psi\rangle - |\phi\rangle$. Because $|\psi\rangle \neq |\phi\rangle$ we have $|||\psi\rangle - |\phi\rangle|| \neq 0$. On the other hand, because T is linear, we must have that $||T(|\psi\rangle - |\phi\rangle)|| = |||0, \ldots, 0\rangle - |0, \ldots, 0\rangle|| = 0$. Hence T is not norm preserving.

A7. Show that $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ form an orthonormal basis for $\mathbb{C}^2$. Construct the projectors P_+ and P_- corresponding to these basis vectors.

First, $|+\rangle$ and $|-\rangle$ have unit norm and are orthogonal: $\langle +|+\rangle = \langle -|-\rangle = 1$ and $\langle +|-\rangle = 0$.

The projection operators are given by

$$P_+ = |+\rangle\langle +| = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$$

$$= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$P_- = |-\rangle\langle -| = \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)$$

$$= \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$