Let $K = \frac{1}{2}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z + I \otimes I)$ be an operator on two qubits.

1 Find K^2

Let $\vec{\sigma} = (\sigma_x \hat{\boldsymbol{x}} + \sigma_y \hat{\boldsymbol{y}} + \sigma_z \hat{\boldsymbol{z}})$. Note $|\vec{\sigma} \otimes \vec{\sigma}| = |\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z|$.

Then $K = \frac{1}{2}(I \otimes I + \vec{\sigma} \otimes \vec{\sigma})$,

$K^2 = \frac{1}{4}(I + (\vec{\sigma} \otimes \vec{\sigma}))^2 = \frac{1}{4}(I + 2I(\vec{\sigma} \otimes \vec{\sigma}) + (\vec{\sigma} \otimes \vec{\sigma})^2)$

By inspecting $(\vec{\sigma} \otimes \vec{\sigma})$ and $(\vec{\sigma} \otimes \vec{\sigma})^2$, a simplification is found:

$$(\vec{\sigma} \otimes \vec{\sigma}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(\vec{\sigma} \otimes \vec{\sigma})^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -4 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= 3I - 2(\vec{\sigma} \otimes \vec{\sigma})$

Thus $K^2 = \frac{1}{4}(I + 2(\vec{\sigma} \otimes \vec{\sigma}) + 3I - 2(\vec{\sigma} \otimes \vec{\sigma})) = \frac{1}{4}(4I) = I$.

2 Use result to find $exp[-i\pi K/4]$ & $exp[-i\pi K/2]$

The power series for the exponential function goes like

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$

In this case, $z = -i\pi K/4$. Making the substitution,

$e^{-i\pi K/4} = I + (-i\pi K/4) + \frac{(-i\pi K/4)^2}{2!} + \frac{(-i\pi K/4)^3}{3!} + \frac{(-i\pi K/4)^4}{4!} + \cdots$

$= I - K^2 \frac{(\pi/4)^2}{2!} + K^2 K^2 \frac{(\pi/4)^4}{4!} + \cdots + K(-i\pi/4) - iKK^2 \frac{(\pi/4)^3}{3!} + \cdots$

All factors of K^2 reduce to the identity, so these are really the $\cos(\pi/4)$ and $\sin(\pi/4)$ series with prefactors.

$= I\cos(\pi/4) + iK\sin(\pi/4) = \frac{1}{\sqrt{2}}(I + iK)$

Similarly for $e^{-i\pi K/2}$,

$e^{-i\pi K/2} = I\cos(\pi/2) + iK\sin(\pi/2) = iK$

3 Find the eigenvalues of K

Brute force:

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; |K - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\rightarrow (1-\lambda)[(1-\lambda)((-\lambda)^2 - 1)] = 0$

$\rightarrow \lambda_1 = -1, \lambda_{2,3,4} = 1$, which were expected given the form of $K2$.

4 Find a set of orthogonal eigenstates of K

We want the eigenvectors. Substituting the λ's back into $(K - \lambda I)$:

$$\lambda_1 \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \text{ or } x_1, x4 = 0, x_2 = -x_3 \rightarrow \vec{v_1} = (0, 1, -1, 0)$$

$$\lambda_{2,3,4} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ or } x_1, x4 = \text{free}, x_2 = x_3 \rightarrow \begin{array}{c} \vec{v_2} = (1,0,0,0) \\ \vec{v_3} = (0,0,0,1) \\ \vec{v_4} = (0,1,1,0) \end{array}$$

5 Express the eigenstates in Dirac notation

If we let $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$, then

$(0, 1, -1, 0) = |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle = |01\rangle - |10\rangle$,

$(1, 0, 0, 0) = |0\rangle \otimes |0\rangle = |00\rangle$,

$(0, 0, 0, 1) = |1\rangle \otimes |1\rangle = |11\rangle$,

$(0, 1, 1, 0) = |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle = |01\rangle + |10\rangle$

6 Express K in Dirac notation

Express the constituent components in Dirac notation:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle = |01\rangle + |10\rangle,$$

$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i(|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) = i(|10\rangle - |01\rangle),$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle = |00\rangle - |11\rangle$, and finally

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |00\rangle + |11\rangle.$

Then K is

$$K = \tfrac{1}{2}([|01\rangle + |10\rangle] \otimes [|01\rangle + |10\rangle] + i([|10\rangle - |01\rangle] \otimes [|10\rangle - |01\rangle])$$
$$+ [|00\rangle - |11\rangle] \otimes [|00\rangle - |11\rangle] + [|00\rangle + |11\rangle] \otimes [|00\rangle + |11\rangle])$$

$$= \tfrac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 10| - |10\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 10|$$
$$- |01\rangle\langle 01| + |00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 00| + |00\rangle\langle 11|$$
$$+ |11\rangle\langle 00| + |11\rangle\langle 11|)$$

$$= |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|$$

7 Verify that the eigenstates and K are properly expressed in Dirac notation

We expect $K\vec{v}_i = \lambda_i \vec{v}_i$:

$$K\vec{v}_1 = (|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|)(|01\rangle - |10\rangle)$$

$$= |00\rangle\langle 00|01\rangle + |01\rangle\langle 10|01\rangle + |10\rangle\langle 01|01\rangle + |11\rangle\langle 11|01\rangle - |00\rangle\langle 00|01\rangle$$
$$- |01\rangle\langle 10|10\rangle - |10\rangle\langle 01|10\rangle - |11\rangle\langle 11|10\rangle$$

Since our eigenstates comprise an orthonormal basis2, we know $\langle a|b\rangle = 0$ and $\langle a|a\rangle = 1$ for any a, b in the set. Only two terms survive from the line above:

$$K\vec{v}_1 = |10\rangle - |01\rangle = (-1)(|01\rangle - |10\rangle) = \lambda_1 \vec{v}_1$$

Similarly,

$$K\vec{v}_2 = (|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|)(|00\rangle) = (1)\,|00\rangle = \lambda_2 \vec{v}_2,$$

$$K\vec{v}_3 = (|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|)(|11\rangle) = (1)\,|11\rangle = \lambda_3 \vec{v}_3,$$

$$K\vec{v}_4 = (|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|)(|01\rangle + |10\rangle)$$

$$= |00\rangle\langle 00|01\rangle + |01\rangle\langle 10|01\rangle + |10\rangle\langle 01|01\rangle + |11\rangle\langle 11|01\rangle + |00\rangle\langle 00|01\rangle$$
$$+ |01\rangle\langle 10|10\rangle + |10\rangle\langle 01|10\rangle + |11\rangle\langle 11|10\rangle$$

$$= |10\rangle + |01\rangle = (1)(|10\rangle + |01\rangle) = \lambda_4 \vec{v}_4$$