NAND gate example

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Let NAND(x,y)=NOT(AND(x,y)). It has the following truth table.

x	y	NAND(x,y)
F	F	T
F	T	T
T	F	T
T	T	F

We'd like to consider how to set up an Ising model that encodes this logical operation. Take

$$H_f = \sum_j h_j \sigma_j^z + \sum_{i<j} J_{ij} \sigma_i^z \sigma_j^z,$$

where h_j are the field parameters and J_{ij} the spin couplings. Choose logical 1 (true) to be represented by spin up $|\uparrow\rangle$ (or $s = +1$) and logical 0 (false) by spin down $|\downarrow\rangle$ (or $s = -1$). We'll confirm the following encoding for the NAND gate:

h_j	J_{ij}
$h_A = -1$	$J_{AB} = 1$
$h_B = -1$	$J_{AC} = 2$
$h_C = -2$	$J_{BC} = 2$

A	B	C	Energy
$\downarrow$	$\downarrow$	$\downarrow$	9
$\downarrow$	$\downarrow$	$\uparrow$	-3
$\downarrow$	$\uparrow$	$\downarrow$	1
$\downarrow$	$\uparrow$	$\uparrow$	-3
$\uparrow$	$\downarrow$	$\downarrow$	1
$\uparrow$	$\downarrow$	$\uparrow$	-3
$\uparrow$	$\uparrow$	$\downarrow$	-3
$\uparrow$	$\uparrow$	$\uparrow$	1

So we see that the ground state energy of -3 corresponds to the entries of the truth table for NAND.

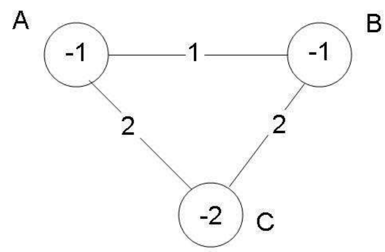

Figure 1: Graphical representation of the Hamiltonian implementing the logic of a NAND gate. Each of the vertices is a spin and the number inside is the field value on that spin; each edge is the coupling value between the respective vertices. Each ground state of this Hamiltonian enforces the logic C=NOT(AND(A,B)).

This example begs the question of how to obtain the h_j and J_{ij} parameters for a given problem. When the operation in question is a logic gate, two avenues seem to be available:
(a) Use integer programming to solve for these parameter values. This will not be computationally advantageous in general, but ok for small or specialized problems.
(b) For this special case of a logic gate (by far not the typical application of adiabatic quantum computing), one can write mathematical formulas encoding the output of the gate, and then translate that to a quadratic objective function for the Ising model.

We as a group need to elaborate and explore both of these approaches.

Consider method (a) for the NAND gate. Writing out the problem Hamiltonian in detail,

$$H_f = h_1\sigma_1^z + h_2\sigma_2^z + h_3\sigma_3^z + J_{12}\sigma_1^z\sigma_2^z + J_{13}\sigma_1^z\sigma_3^z + J_{23}\sigma_2^z\sigma_3^z.$$

We have 6 parameters in total: 3 h's and 3 J's. Suppose that we wish the ground state energy to be k. Then from the truth table encoded by spins, we have the 4 conditions

$$-h_1 - h_2 + h_3 + J_{12} - J_{13} - J_{23} = k,$$

$$-h_1 + h_2 + h_3 - J_{12} - J_{13} + J_{23} = k,$$

$$h_1 - h_2 + h_3 - J_{12} + J_{13} - J_{23} = k,$$

$$h_1 + h_2 - h_3 + J_{12} - J_{13} - J_{23} = k.$$

The other four conditions are inequalities since we demand that they yield an energy strictly greater than k. Thinking in terms of an integer programming problem, we replace the condition $> k$ with $\geq k + 1$ and get

$$-h_1 - h_2 - h_3 + J_{12} + J_{13} + J_{23} \geq k + 1,$$

$$-h_1 + h_2 - h_3 - J_{12} + J_{13} - J_{23} \geq k + 1,$$

$$h_1 - h_2 - h_3 - J_{12} - J_{13} + J_{23} \geq k + 1,$$

$$h_1 + h_2 + h_3 + J_{12} + J_{13} + J_{23} \geq k + 1.$$

If we just solve the 4 equalities, we have the reduced relations

$$h_3 = h_1 + h_2, \quad J_{12} = h_3 - k, \quad J_{13} = h_2 - k, \quad \text{and} \quad J_{23} = h_1 - k.$$

It's easily confirmed that these relations are satisfied as above when $k = -3$.

In general when there are N spins in the Ising Hamiltonian, there are N h parameters and $N(N-1)/2$ J parameters, for a total of $N(N+1)/2 = \binom{N+1}{2}$. There are 2^N conditions, so that this method is limited to small problems.