12.4. *Consider a system–environment interaction that leaves the system unchanged and applies the rotation gate*

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

to the environment if the system is in the state $|1\rangle$. If the system is in the state $|0\rangle$, the environment is unchanged. Let the initial state of the environment be $|\phi_E\rangle = |0_E\rangle$.

12.5. *Show that if the state of the environment is $|0_E\rangle$ and $U = \sqrt{p} I \otimes I + \sqrt{1-p} X \otimes Z$, this is the phase flip channel (12.24).*

12.6. *Consider phase damping and derive (12.51) through (12.53).*

12.7. *Verify that if the third qubit in a logical state has been flipped so that the state is $\alpha|001\rangle + \beta|110\rangle$, the algorithm described in (12.59) through (12.63) will reveal that the third qubit is flipped by setting the ancillary bits to 11.*

12.8. *What result is returned for the ancillary bits in the bit flip correcting algorithm?*

13

TOOLS OF QUANTUM INFORMATION THEORY

In this chapter we consider several important aspects of quantum information theory that are important for a thorough understanding of quantum computers. First we discuss the *no-cloning theorem*, which shows that you cannot make copies of an unknown quantum state. After recognizing this fact, we see how to measure the closeness of two states to each other. We will do this by looking at trace distance and fidelity. We can characterize the amount of entanglement in a state by looking at the concurrence, and we can determine the resources needed to create a given entangled state by calculating the entanglement of formation.

We conclude the chapter by considering how to characterize the information content in a state. This is done by calculating *entropy*.

THE NO-CLONING THEOREM

A routine task performed in information processing is making copies of data. We take it for granted that we can make as many copies as we like of something—whether it's a word processing file or a bit of music. As we have seen, the remarkable power of a quantum computer comes from the fact a qubit can exist in a superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Given this fact, can we make an exact copy of an arbitrary qubit?

It turns out the answer is no. This result, which we state below, is known as the no-cloning theorem, and it was derived by Wooters and Zurek in 1982.

Consider two pure states $|\psi\rangle$ and $|\phi\rangle$, and suppose that there exists a unitary operator U such that

$$U(|\psi\rangle \otimes |\chi\rangle) = |\psi\rangle \otimes |\psi\rangle \tag{13.1}$$

$$U(|\phi\rangle \otimes |\chi\rangle) = |\phi\rangle \otimes |\phi\rangle \tag{13.2}$$

for some target state $|\chi\rangle$. We take the inner product of the left-hand side of (13.1) with the left-hand side of (13.2) and use the fact that $U^\dagger U = I$ to get

$$(\langle\psi| \otimes \langle\chi|U^\dagger)(U|\phi\rangle \otimes |\chi\rangle) = \langle\psi|\phi\rangle\langle\chi|\chi\rangle = \langle\psi|\phi\rangle \tag{13.3}$$

However, taking the inner product of the right-hand sides of (13.1) and (13.2) gives

$$(\langle\psi|\phi\rangle)^2 \tag{13.4}$$

Equating these two results gives us the equation

$$\langle\psi|\phi\rangle = (\langle\psi|\phi\rangle)^2 \tag{13.5}$$

This equation can only be true in two cases—if $\langle\psi|\phi\rangle = 0$, in which case the states are orthogonal, or if $|\phi\rangle = |\psi\rangle$. What this result means is that there is no unitary operator U that can be used to clone arbitrary quantum states.

Here is a second proof, this time a simple proof by contradiction. In quantum mechanics we use linear operators. If U is linear, then

$$U(\alpha|\psi\rangle \otimes |\chi\rangle) = \alpha(U(|\psi\rangle \otimes |\chi\rangle)) = \alpha|\psi\rangle \otimes |\psi\rangle \tag{13.6}$$

However, we can let $|\omega\rangle = \alpha|\psi\rangle$ and then apply (13.1), giving

$$U(|\omega\rangle \otimes |\chi\rangle) = |\omega\rangle \otimes |\omega\rangle = \alpha|\psi\rangle \otimes \alpha|\psi\rangle = \alpha^2|\psi\rangle \otimes |\psi\rangle \tag{13.7}$$

Comparison of (13.6) and (13.7) gives a contradiction. Hence general cloning is not possible. So a question we may ask, given that we can't make a perfect copy of a quantum state in general, is how close is one quantum state to another? Is it possible to make imperfect copies?

TRACE DISTANCE

Because we cannot, in principle, make an exact copy of an unknown quantum state, the next question we might ask is, can we make an *approximate* copy? Before we look into the answer, let's see what tools are at our disposal that can be used to determine how similar two states are.

The first measure we will consider is the *trace distance*. Let ρ and σ be two density matrices. The trace distance $\delta(\rho, \sigma)$ is defined to be

$$\delta(\rho, \sigma) = \frac{1}{2} Tr|\rho - \sigma| \tag{13.8}$$

Note that $|\rho| = \sqrt{\rho^\dagger \rho}$. Suppose that the states ρ and σ are equally likely and that we want to do a measurement to distinguish between the two states. The average probability of success is

$$\Pr = \frac{1}{2} + \frac{1}{2}\delta(\rho, \sigma) \tag{13.9}$$

The trace distance acts like a metric on the Hilbert space. For example, the trace distance is nonnegative

$$0 \leq \delta(\rho, \sigma) \tag{13.10}$$

with equality if and only if $\rho = \sigma$. The trace distance is symmetric,

$$\delta(\rho, \sigma) = \delta(\sigma, \rho) \tag{13.11}$$

and it satisfies the triangle inequality

$$\delta(\rho, \sigma) \leq \delta(\rho, \vartheta) + \delta(\vartheta, \sigma) \tag{13.12}$$

If $\rho = |\psi\rangle\langle\psi|$ is a pure state, then $\delta(\rho, \sigma)$ is given by

$$\delta(\rho, \sigma) = \sqrt{1 - \langle\psi|\sigma|\psi\rangle} \tag{13.13}$$

If ρ and σ commute, meaning $[\rho, \sigma] = 0$, and they are both diagonal with respect to some basis $\{|u_i\rangle\}$ such that the eigenvalues of ρ are r_i and the eigenvalues of σ are s_i, then

$$\delta(\rho, \sigma) = \frac{1}{2} Tr \left| \sum_i (r_i - s_i)|u_i\rangle\langle u_i| \right| \tag{13.14}$$

Example 13.1

Compute the trace distance between

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

and each of

$$\sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|, \quad \pi = \frac{1}{8}|0\rangle\langle 0| + \frac{7}{8}|1\rangle\langle 1|$$

Solution

Looking at each of these states. Intuitively you would expect that ρ and σ are closer together than ρ and π, since π is more weighted toward $|1\rangle$. First we have

$$\rho - \sigma = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| - \left(\frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|\right)$$
$$= \frac{1}{12}|0\rangle\langle 0| - \frac{1}{12}|1\rangle\langle 1|$$

Let's recall a couple of facts about the trace. It's linear, meaning $Tr(\alpha A + \beta B) = \alpha Tr(A) + \beta Tr(B)$. Second, the trace turns outer products into inner products, meaning $Tr(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle$. So we can write

$$\delta(\rho, \sigma) = \frac{1}{2}Tr|\rho - \sigma|$$
$$= \frac{1}{2}Tr\left|\frac{1}{12}|0\rangle\langle 0| - \frac{1}{12}|1\rangle\langle 1|\right|$$
$$= \frac{1}{2}\left(\frac{1}{12}\right)(Tr(|0\rangle\langle 0|) + Tr(|1\rangle\langle 1|)) = \frac{1}{2}\left(\frac{1}{12}\right)(\langle 0|0\rangle + \langle 1|1\rangle)$$
$$= \frac{1}{2}\left(\frac{1}{12}\right)(2) = \frac{1}{12}$$

Now let's see what the trace distance $\delta(\rho, \pi)$ is. We have

$$\rho - \pi = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| - \left(\frac{1}{8}|0\rangle\langle 0| + \frac{7}{8}|1\rangle\langle 1|\right)$$
$$= \frac{5}{8}(|0\rangle\langle 0| - |1\rangle\langle 1|)$$

We find that

$$\delta(\rho, \pi) = \frac{1}{2}Tr|\rho - \pi|$$
$$= \frac{1}{2}Tr\left|\frac{5}{8}(|0\rangle\langle 0| - |1\rangle\langle 1|)\right|$$
$$= \frac{1}{2}\left(\frac{5}{8}\right)(Tr(|0\rangle\langle 0|) + Tr(|1\rangle\langle 1|)) = \frac{1}{2}\left(\frac{5}{8}\right)(\langle 0|0\rangle + \langle 1|1\rangle)$$
$$= \frac{1}{2}\left(\frac{5}{8}\right)(2) = \frac{5}{8}$$

As expected, since ρ and σ are more heavily weighted toward $|0\rangle$, we have $\delta(\rho, \pi)\rangle\delta(\rho, \sigma)$. This tells us the states ρ and σ are more similar than the states ρ and π are.

You Try It

Write down the matrix representations of $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$, and $\pi = \frac{1}{8}|0\rangle\langle 0| + \frac{7}{8}|1\rangle\langle 1|$, and calculate (13.8). Verify the result obtained in Example 13.1.

A simple way to calculate the trace distance is to use the eigenvalues of the matrix $\rho - \sigma$. If we denote the eigenvalues by λ_i, then the trace distance is

$$\delta(\rho, \sigma) = \frac{1}{2}\sum_i |\lambda_i| = \frac{1}{2}\sum_i \sqrt{\lambda_i^* \lambda_i} \tag{13.15}$$

If we know the Bloch vectors of each density matrix, then we can calculate the trace distance easily. Suppose that $\vec{r}$ is the Bloch vector of ρ and $\vec{s}$ is the Bloch vector of σ. Then the trace distance $\delta(\rho, \sigma)$ can be calculated as

$$\delta(\rho, \sigma) = \frac{1}{2}|\vec{r} - \vec{s}| \tag{13.16}$$

Example 13.2

Find the trace distance between the states

$$\rho = \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \frac{2}{5} & \frac{-i}{8} \\ \frac{i}{8} & \frac{3}{5} \end{pmatrix}$$

Solution

Let's do it using (13.8) first. We have

$$\rho - \sigma = \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix} - \begin{pmatrix} \frac{2}{5} & \frac{-i}{8} \\ \frac{i}{8} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{9}{40} & \frac{i3}{8} \\ \frac{-i3}{8} & \frac{-9}{40} \end{pmatrix}$$

Now $(\rho - \sigma)^\dagger = \rho - \sigma$, so

$$(\rho - \sigma)^\dagger \rho - \sigma = \begin{pmatrix} \frac{9}{40} & \frac{i3}{8} \\ \frac{-i3}{8} & \frac{-9}{40} \end{pmatrix} \begin{pmatrix} \frac{9}{40} & \frac{i3}{8} \\ \frac{-i3}{8} & \frac{-9}{40} \end{pmatrix} = \begin{pmatrix} \frac{153}{800} & 0 \\ 0 & \frac{153}{800} \end{pmatrix}$$

Next we find

$$|\rho - \sigma| = \sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)} = \sqrt{\begin{pmatrix} \frac{153}{800} & 0 \\ 0 & \frac{153}{800} \end{pmatrix}} = \frac{1}{20} \begin{pmatrix} 3\sqrt{\frac{17}{2}} & 0 \\ 0 & 3\sqrt{\frac{17}{2}} \end{pmatrix}$$

Hence

$$\delta(\rho, \sigma) = \frac{1}{2} \left(\frac{1}{20} \right) (2) \left(3\sqrt{\frac{17}{2}} \right) \approx 0.437$$

The Bloch vector for ρ was found in Example 5.12:

$$S_x = Tr(X\rho) = Tr\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix} \right] = Tr \begin{pmatrix} \frac{-i}{4} & \frac{3}{8} \\ \frac{5}{8} & \frac{i}{4} \end{pmatrix} = 0$$

$$S_y = Tr(Y\rho) = Tr\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix} \right] = Tr \begin{pmatrix} \frac{-1}{4} & \frac{-i3}{8} \\ \frac{i5}{8} & \frac{-1}{4} \end{pmatrix} = \frac{-1}{2}$$

$$S_z = Tr(Z\rho) = Tr\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix} \right] = Tr \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{i}{4} & \frac{-3}{8} \end{pmatrix} = \frac{1}{4}$$

$$\vec{r} = -\frac{1}{2}\hat{y} + \frac{1}{4}\hat{z}$$

The Bloch vector for σ, which was calculated as Exercise 5.10, is

$$\vec{s} = \frac{1}{4}\hat{y} - \frac{1}{5}\hat{z}$$

Therefore

$$\vec{r} - \vec{s} = -\frac{3}{4}\hat{y} + \frac{9}{20}\hat{z}$$

The magnitude of this vector is

$$|\vec{r} - \vec{s}| = \sqrt{\left(-\frac{3}{4}\right)^2 + \left(\frac{9}{20}\right)^2} = \frac{\sqrt{306}}{20}$$

Hence

$$\delta(\rho, \sigma) = \frac{1}{2}|\vec{r} - \vec{s}| = \frac{1}{2}\frac{\sqrt{306}}{20} \approx 0.437$$

Example 13.3

A system is in the pure state

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

Find the trace distance between ρ and $\sigma = |\psi\rangle\langle\psi|$, where

$$|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$$

Solution

Both density matrices need to be written with respect to the same basis. Let's start by rewriting ρ in terms of the computational basis. In Example 5.5 we found that

$$\begin{aligned} \rho &= \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-| \\ &= \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)((|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)) \\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \end{aligned}$$

The matrix representation of this density operator is

$$\rho = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Now for

$$|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$$

we found in Example 5.4 that

$$\sigma = |\psi\rangle\langle\psi| = \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)\left(\frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|\right)$$
$$= \frac{1}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|$$

The matrix representation is

$$\sigma = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

The matrix $\rho - \sigma$ is given by

$$\rho - \sigma = \frac{1}{20}\begin{pmatrix} 6 & -3 \\ -3 & -6 \end{pmatrix}$$

This matrix has two eigenvalues, namely

$$\lambda_1 = -\frac{3}{4\sqrt{5}}, \quad \lambda_2 = \frac{3}{4\sqrt{5}}$$

Using (13.15), we find the trace distance to be

$$\delta(\rho, \sigma) = \frac{1}{2}\sum_i |\lambda_i| = \frac{1}{2}\left(\left|-\frac{3}{4\sqrt{5}}\right| + \left|\frac{3}{4\sqrt{5}}\right|\right) = \frac{3}{4\sqrt{5}}$$

FIDELITY

Another measure that can be used to determine how close one state is to another is based on the notion of the amount of statistical overlap between two distributions, called the *fidelity*. Once again, let ρ and σ be two density operators. Then the fidelity is given by

$$F(\rho, \sigma) = Tr\left(\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right) \tag{13.17}$$

In short, fidelity is a concept that comes from the inner product of two quantum states. Let $|\psi\rangle$and $|\phi\rangle$ be two states. The inner product $|\langle\phi|\psi\rangle|^2$ gives the probability of finding the system in the state $|\phi\rangle$ if it is known to be in the state $|\psi\rangle$, and vice versa. Hence this is a kind of measure of how similar the two states are or how much overlap there is between them. Suppose that they are pure states and with density operators $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$. Since these are pure states, $\rho^2 = \rho$, $\sigma^2 = \sigma$, and hence $\rho = \sqrt{\rho}, \sigma = \sqrt{\sigma}$. Then

$$\begin{aligned} F(\rho, \sigma) = Tr\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right) &= Tr\sqrt{(|\psi\rangle\langle\psi|)(|\phi\rangle\langle\phi|)(|\psi\rangle\langle\psi|)} \\ &= Tr\sqrt{(|\langle\phi|\psi\rangle|^2)(|\psi\rangle\langle\psi|)} = |\langle\phi|\psi\rangle|\sqrt{\langle\psi|\psi\rangle} = |\langle\phi|\psi\rangle| \end{aligned} \tag{13.18}$$

From (13.18), a few general properties of fidelity can be seen. The first is that fidelity is a number that ranges between 0 and 1,

$$0 \le F(\rho, \sigma) \le 1 \tag{13.19}$$

with unity if the states ρ and σ are the same state and 0 if there is no overlap whatsoever. We can also see from (13.18) that the fidelity of two pure states is symmetric. In fact this is true in general, that is,

$$F(\rho, \sigma) = F(\sigma, \rho) \tag{13.20}$$

The fidelity is futher invariant under unitary operations, that is,

$$F(U\rho U^{\dagger}, U\sigma U^{\dagger}) = F(\rho, \sigma) \tag{13.21}$$

If ρ and σ commute and are hence diagonal in the same basis, which we denote by $|u_i\rangle$, then we can write the fidelity in terms of the eigenvalues of ρ and σ. Suppose that $\rho = \sum_i r_i |u_i\rangle\langle u_i|$ and $\sigma = \sum_i s_i |u_i\rangle\langle u_i|$, then

$$F(\rho, \sigma) = \sum_i \sqrt{r_i s_i} \tag{13.22}$$

Example 13.4

Compute the fidelity between

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

and each of

$$\sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|, \quad \pi = \frac{1}{8}|0\rangle\langle 0| + \frac{7}{8}|1\rangle\langle 1|$$

Can the fidelity be calculated using (13.18) or (13.22)? Compare with Example 13.1.

Solution

First we compute

$$\rho^2 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{9}{16} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$$

ince $Tr(\rho^2) = 10/16 < 1$, ρ is not a pure state. Similar calculations show that $Tr(\sigma^2) =$ /9 < 1 and $Tr(\pi^2) = 50/64 < 1$, so σ and π are also mixed states and (13.18) does not pply. However, since all three density operators are diagonal in the computational basis, we an use (13.22). Notice that

$$\rho\sigma = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{12} \end{pmatrix}$$

$$\sigma\rho = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{12} \end{pmatrix}$$

herefore $[\rho, \sigma] = 0$. Using (13.22), we find that the fidelity is

$$F(\rho, \sigma) = \sum_i \sqrt{r_i s_i} = \sqrt{\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)} + \sqrt{\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{12}} = \frac{1+\sqrt{6}}{\sqrt{12}} = 0.996$$

Since the fidelity is close to 1, this tells us that the two states are very similar—they have lot of overlap. In Example 13.1 we found that the trace distance between the states was 1/12, small number indicating that there is not much "distance" between the two states. Hence ates that are similar have a high fidelity and a small trace distance.

For the other state we find that

$$F(\rho, \pi) = \sum_i \sqrt{r_i s_i} = \sqrt{\left(\frac{3}{4}\right)\left(\frac{1}{8}\right)} + \sqrt{\left(\frac{1}{4}\right)\left(\frac{7}{8}\right)} = \sqrt{\frac{3}{32}} + \sqrt{\frac{7}{32}} = \frac{\sqrt{3}+\sqrt{7}}{\sqrt{32}} = 0.774$$

he smaller fidelity indicates there is not nearly as much overlap between these two states as ere is in the first case. In Example 13.1 we found that the trace distance was 5/8. So a state at is not as similar results in a smaller fidelity and a larger trace distance.

Fidelity can be looked at as a transition probability. In other words, the probability that ρ evolves into σ is

$$\Pr(\rho \to \sigma) = (F(\rho, \sigma))^2 \tag{13.23}$$

Example 13.5

What is the probability that ρ evolves into each of the states σ and π in the previous example?

Solution

The probability that ρ evolves into σ is

$$\Pr(\rho \to \sigma) = (F(\rho, \sigma))^2 = (0.996)^2 = 0.992$$

The probability that ρ evolves into π is

$$\Pr(\rho \to \pi) = (F(\rho, \pi))^2 = (0.774)^2 = 0.599$$

The *Bures distance function* is a distance measure between quantum states that makes use of the fidelity. It is

$$d_B^2(\rho, \sigma) = 2(1 - F(\rho, \sigma)) \tag{13.24}$$

The *Bures distance function* or *modified Bures metric* is a distance measure between quantum states that makes use of the fidelity. It is

$$d_B^2(\rho, \sigma) = 2(1 - F(\rho, \sigma)) \tag{13.25}$$

The *Bures metric* is given by

$$d_B(\rho, \sigma) = 2 - 2\sqrt{F(\rho, \sigma)} \tag{13.26}$$

Example 13.6

Consider the states in Example 13.4 and show that the Bures distance between ρ and π is much larger than the Bures distance between ρ and σ.

Solution

In the first case we find that

$$d_B^2(\rho, \sigma) = 2(1 - F(\rho, \sigma)) = 2(1 - 0.996) = 0.008$$

For the other two states we have

$$d_B^2(\rho, \pi) = 2(1 - F(\rho, \pi)) = 2(1 - 0.774) = 0.452$$

$d_B^2(\rho, \pi) \gg d_B^2(\rho, \sigma)$ once again indicating that ρ and σ are far more similar than ρ and π.

In practical cases of interest we often want to find the *minimum* fidelity for a given channel. This is because we do not know the quantum state $|\psi\rangle$, so finding the minimum fidelity gives us a worst-case analysis of a given quantum channel.

Example 13.7

On a certain quantum channel there is a probability $p = 1/9$ that there is a bit flip error. What is the minimum fidelity of the bit flip channel in this case? Assume that system starts in some pure state $\rho = |\psi\rangle\langle\psi|$.

Solution

The bit flip channel was described in (12.22). Recall that there is a probability p that nothing happens to the qubit, while there is a probability $1 - p$ that there is a bit flip error. The quantum operation is

$$\rho' = \Phi(\rho) = p\rho + (1 - p)X\rho X$$

The fidelity between this state and $\rho = |\psi\rangle\langle\psi|$ is given by

$$F(\rho, \rho') = F(\rho', \rho) = Tr(\sqrt{\sqrt{\rho'}\rho\sqrt{\rho'}}) = tr\sqrt{\sqrt{\rho'}(|\psi\rangle\langle\psi|)\sqrt{\rho'}}$$
$$= \sqrt{\langle\psi|\sqrt{\rho'}\sqrt{\rho'}|\psi\rangle} = \sqrt{\langle\psi|\rho'|\psi\rangle}$$

Using (12.22) simplifies the fidelity to

$$F(\rho', \rho) = \sqrt{\langle\psi|(p\rho + (1 - p)X\rho X)|\psi\rangle} = \sqrt{\langle\psi|(p|\psi\rangle\langle\psi| + (1 - p)X|\psi\rangle\langle\psi|X)|\psi\rangle}$$
$$= \sqrt{p + (1 - p)\langle\psi|X|\psi\rangle\langle\psi|X|\psi\rangle}$$
$$= \sqrt{p + (1 - p)\langle\psi|X|\psi\rangle^2}$$

Now we want to find the state where F is a minimum, which gives the worst-case scenario. Since it is always the case that $0 \leq p \leq 1$, it follows that$1 - p \geq 0$ and F will assume the smallest value when $(1 - p)\langle\psi|X|\psi\rangle^2 = 0$. So we need to find the state for which $\langle\psi|X|\psi\rangle = 0$. Notice that if

$$|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

Then

$$\langle\psi|X|\psi\rangle = \left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}}\right) X \left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right) = \left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}}\right)\left(\frac{|1\rangle + i|0\rangle}{\sqrt{2}}\right)$$
$$= \frac{i\langle 0 \mid 0\rangle - i\langle 1 \mid 1\rangle}{2} = 0$$

Let's verify that $|\psi\rangle$ is a pure state. We find that

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$
$$\Rightarrow \rho^2 = \frac{1}{4}\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix}$$

So we have $Tr(\rho^2) = \frac{1}{4}(2 + 2) = 1$, and this is a pure state. So the minimum fidelity occurs when $\langle\psi|X|\psi\rangle = 0$, in which case

$$F(\rho, \rho') = \sqrt{p}$$

For the case where $p = 1/9$ the minimum fidelity is $F_{\min} = \sqrt{1/9} \approx 0.33$.

ENTANGLEMENT OF FORMATION AND CONCURRENCE

Here we return to the examination of entangled states of two qubits. Two questions we can ask are how much entanglement does a state have, and second, what is the cost of creating a given entangled state? One way to characterize entanglement is by calculating the *concurrence*. To characterize the resources required to create a given entangled state, we can calculate the *entanglement of formation*.

First let's consider the concurrence. Basically this is just the amount of overlap between a state $|\psi\rangle$ and a state $|\tilde{\psi}\rangle$:

$$C(\psi) = |\langle\psi|\tilde{\psi}\rangle| \tag{13.27}$$

where $|\tilde{\psi}\rangle = Y \otimes Y|\psi^*\rangle$ and ψ^* is the complex conjugate of the state. The concurrence can also be calculated using the density operator by considering the quantity given by $\rho\ (Y \otimes Y)\rho^\dagger(Y \otimes Y)$.

Example 13.8

Concurrence can be a measure of entanglement. Consider the product state

$$|\psi\rangle = |0\rangle \otimes |1\rangle$$

and show that the concurrence is zero.

Solution

We have

$$|\tilde{\psi}\rangle = Y \otimes Y|\psi\rangle = Y|0\rangle \otimes Y|1\rangle = -i|1\rangle \otimes i|0\rangle = |1\rangle \otimes |0\rangle$$

So we have

$$\langle\tilde{\psi}|\psi\rangle = (\langle 1| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) = \langle 1|0\rangle\langle 0|1\rangle = 0$$

So the concurrence is zero. We can also see the concurrence vanishes by using the matrix representations of the operators. First, we have

$$Y \otimes Y = \begin{pmatrix} 0 & -iY \\ iY & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The density operator for this state is

$$\rho = |01\rangle\langle 01| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\begin{aligned}
&\rho(Y \otimes Y)\rho^{\dagger}(Y \otimes Y) \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

We can find the concurrence by looking at the eigenvalues of the resulting matrix. The eigenvalues of this matrix all vanish, hence the concurrence is zero.

A second way to define concurrence is to look at the eigenvalues of the matrix

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}} \tag{13.28}$$

which are denoted by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The concurrence is

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{13.29}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. In the next two examples we consider the concurrence of entangled states.

Example 13.9

Find the concurrence of

$$|S\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Solution

Again, note that

$$Y \otimes Y = \begin{pmatrix} 0 & -iY \\ iY & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The density operator is

$$\begin{aligned}
\rho = |S\rangle\langle S| &= \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right) \\
&= \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)
\end{aligned}$$

The matrix representation is

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we have

$$\begin{aligned}
&\rho(Y \otimes Y)\rho^{\dagger}(Y \otimes Y) \\
&= \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

The eigenvalues of this matrix are

$$\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 0$$

Using (13.29), we find the concurrence to be

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} = \max\{0, 1\} = 1$$

Example 13.10

Find the concurrence of

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Solution

The density operator in this case is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Therefore

$$\rho(Y \otimes Y)\rho^\dagger(Y \otimes Y)$$
$$= \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues are $\{1, 0, 0, 0\}$, and hence the concurrence is 1.

In the next section we will see that we can write the *Shannon entropy* as

$$h(p) = -p \log_2 p - (1 - p)\log_2(1 - p) \tag{13.30}$$

The *entanglement of formation* is defined in terms of the concurrence as

$$E(\rho) = h\left(\frac{1 + \sqrt{1 - C(\rho)^2}}{2}\right) \tag{13.31}$$

This is a mathematical characterization of the resources required to create an entangled state.

Example 13.11

Find the entanglement of formation for the Werner state

$$\rho = \frac{5}{6}|\phi^+\rangle\langle\phi^+| + \frac{1}{24}I_4 = \begin{pmatrix} \frac{11}{24} & 0 & 0 & \frac{5}{12} \\ 0 & \frac{1}{24} & 0 & 0 \\ 0 & 0 & \frac{1}{24} & 0 \\ \frac{5}{12} & 0 & 0 & \frac{11}{24} \end{pmatrix}$$

For $\rho\ (Y \otimes Y)\rho^\dagger(Y \otimes Y)$.

Solution

First we have

$$\rho(Y \otimes Y)\rho^\dagger(Y \otimes Y) \Rightarrow \rho^2 = \begin{pmatrix} \frac{221}{576} & 0 & 0 & \frac{55}{144} \\ 0 & \frac{1}{576} & 0 & 0 \\ 0 & 0 & \frac{1}{576} & 0 \\ \frac{55}{144} & 0 & 0 & \frac{221}{576} \end{pmatrix}$$

The eigenvalues of this matrix are

$$\lambda_i = \left\{\frac{49}{64}, \frac{1}{576}, \frac{1}{576}, \frac{1}{576}\right\}$$

From (13.29) the concurrence is

$$C(\rho) = 0.76$$

From (13.31) the entanglement of formation is

$$E(\rho) = -\frac{1+\sqrt{1-C(\rho)^2}}{2}\log_2\frac{1+\sqrt{1-C(\rho)^2}}{2} - \frac{1-\sqrt{1-C(\rho)^2}}{2}\log_2\frac{1-\sqrt{1-C(\rho)^2}}{2}$$
$$= 0.67$$

You Try It

Show that the entanglement of formation for the case considered in Example 13.9 is 1 and the entanglement of formation for the case in Example 13.10 is also 1.

INFORMATION CONTENT AND ENTROPY

Entropy is a way to quantify the information content in a signal. Specifically, suppose that there exists a random variable X. Entropy tells us the amount of ignorance we have about the random variable X prior to measurement. Or put another way, entropy provides the answer on how much information we will gain when we measure X.

We define entropy by using the probabilities that each possible outcome occurs. Suppose that p_j is the probability of the j th outcome where there are n total possible outcomes. The Shannon entropy H is given by

$$H = -\sum_{j=1}^{n} p_j \log_2 p_j \tag{13.32}$$

An example of an entropy function is shown in Figure 13.1, where we have a plot of the *binary entropy function* $H_2(x) = -x\log x - (1-x)\log(1-x)$. In the plot it is clear that entropy is a *concave* function. This means that for $0 \leq \lambda \leq 1$,

$$\lambda H(p) + (1-\lambda)H(q) \leq H(\lambda p + (1-\lambda)q) \tag{13.33}$$

The maximum entropy occurs for the case where we have the least amount of knowledge. In the case of discrete probabilities p_j, we have the least amount of knowledge about the outcome of a measurement when each of the possible outcomes is equally likely. That is, with n possible outcomes, each of the probabilities is given by

$$p_j = \frac{1}{n}$$

A simple example is the binary entropy function. There are two possible outcomes. We first suppose that both are equally likely so that $x = 1/2$. Then we have

$$-x\log x - (1-x)\log(1-x) = 1$$

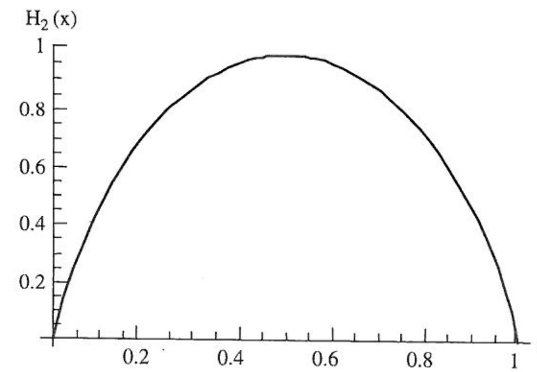

Figure 13.1 The binary entropy function

(note we are using base 2 logarithms). Now suppose that one outcome is substantially more likely. For example, if $x = 0.2$, the probability of finding the alternative is $1 - x = 0.8$. This is a situation where we have more knowledge about the state prior to measurement because one alternative is far more likely than the other. In this case

$$-x\log x - (1-x)\log(1-x) = 0.72$$

If $x = 0.05$, meaning that there is a 95% chance of seeing the alternative, then

$$-x\log x - (1-x)\log(1-x) = 0.29$$

So, if we are completely uncertain as to what outcome will occur—meaning all possible outcomes are equally likely—then with two possible outcomes

$$H(x) = 1$$

This case represents maximum entropy. For all other cases

$$H(x) < 1$$

The general rule is that the larger the entropy, the more ignorance you have about the outcome.

Now suppose that there are two random variables, X and Y. If the probability that we find result $X = x_i$ and $Y = y_i$ is $p(x, y)$, then the joint Shannon entropy is

$$H(X,Y) = -\sum_{x,y} p(x,y)\log(p(x,y)) \tag{13.34}$$

The following inequality called subadditivity is obeyed in general:

$$H(X,Y) \leq H(X) + H(Y) \tag{13.35}$$

Equality holds in (13.35) if the distributions X and Y are independent. The *conditional entropy* for X given Y is

$$H(X|Y) = H(X,Y) - H(Y) \tag{13.36}$$

To determine the amount of entropy in a quantum state, we seek an analogue to the Shannon entropy. This is done by using density operators instead of elements of the probability distribution as in (13.32). The entropy of a quantum state with density operator ρ is called the *Von Neumann entropy* and is given by

$$S(\rho) = -Tr(\rho \log_2 \rho) \tag{13.37}$$

The *relative Von Neumann entropy* of states ρ and σ is

$$S(\rho\|\sigma) = Tr(\rho \log \rho) - Tr(\rho \log \sigma) \tag{13.38}$$

Note that $S(\rho\|\sigma) \geq 0$ with equality if and only if $\rho = \sigma$.

Suppose that the eigenvalues of the density operator ρ are given by λ_i. We can write the Von Neumann entropy in terms of the eigenvalues as

$$S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i \tag{13.39}$$

Example 13.12

The maximally mixed state for a qubit is

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

What is the Von Neumann entropy for this state?

Solution

The eigenvalues are

$$\{\lambda_1, \lambda_2\} = \left\{\frac{1}{2}, \frac{1}{2}\right\}$$

Using (13.39), we find the entropy to be

$$S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i = -\frac{1}{2}\log_2\left(\frac{1}{2}\right) - \frac{1}{2}\log_2\left(\frac{1}{2}\right) = -\log_2\left(\frac{1}{2}\right) = \log_2 2 = 1$$

In general, if a quantum system is in a Hilbert space of dimension n, the completely mixed state has entropy

$$\log_2 n \tag{13.40}$$

Example 13.13

Find the entropy of the two states

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \frac{9}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}$$

Solution

Let's look at the two states. We might guess that ρ has higher entropy because we have a little less information about the possible outcomes. Using the eigenvalues of ρ together with (13.39), we see that the entropy is

$$S(\rho) = -\frac{3}{4}\log_2\left(\frac{3}{4}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right) = 0.81$$

For σ we find that

$$S(\sigma) = -\frac{9}{10}\log_2\left(\frac{9}{10}\right) - \frac{1}{10}\log_2\left(\frac{1}{10}\right) = 0.47$$

Our intuition is confirmed, we have more knowledge about the state σ before a measurement is made because its far more certain that the outcome is $|0\rangle$.

Example 13.14

Find the entropy of the state

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Solution

The eigenvalues of this matrix are

$$\lambda_{1,2} = \left\{\frac{3}{4}, \frac{1}{4}\right\}$$

Is this a pure state or a mixed state? We met this state in Example 5.5 where we found that

$$\rho^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{16} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{16} \end{pmatrix}$$

$$\Rightarrow Tr(\rho^2) = \frac{5}{16} + \frac{5}{16} = \frac{10}{16} = \frac{5}{8}$$

Hence this is a mixed state. The entropy is the same as the matrix in the previous example, even though it looks very different:

$$S(\rho) = -\frac{3}{4}\log_2\left(\frac{3}{4}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right) = 0.81$$

While a completely mixed state has the highest entropy, a pure state has the smallest possible entropy.

Example 13.15

Find the entropy of the state

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Solution

First let's write down the density operator. We find that

$$\rho = |\psi\rangle\langle\psi| = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right)$$
$$= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

The matrix representation of this density operator is thus

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues of this matrix are

$$\lambda_{1,2} = \{1, 0\}$$

To calculate the entropy, we use the fact that

$$\lim_{x\to 0} x \log x = 0$$

So we only need to consider $\lambda = 1$, and we find that

$$S(\rho) = -\log_2(1) = 0$$

This state is a pure state and it has zero entropy. We know with certainty what the state is prior to measurement—or put another way—there is no ignorance as to what the state of the system is.

In n dimensions, the entropy of a quantum state obeys the following inequality:

$$\log_2 n \geq S(\rho) \geq 0 \tag{13.41}$$

We have seen examples of both extrema.The completely mixed state—which is characterized by equally probable outcomes $p_i = 1/n$, has entropy given by $\log_2 n$— The pure state with zero entropy is the lower bound in (13.41). The Von Neumann entropy is invariant under a change of basis as we show in the next example.

Example 13.16

Let $\rho = \frac{3}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$. Show that the entropy of the state is invariant under a change of basis.

Solution

The matrix representation of this state is

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Note that this matrix is written in the $\{|\pm\rangle\}$ basis. The eigenvalues are $\lambda_{1,2} = \left\{\frac{3}{4}, \frac{1}{4}\right\}$ and we have already seen that the entropy in this case is

$$S(\rho) = -\frac{3}{4}\log_2\left(\frac{3}{4}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right) = 0.81$$

What happens if we write the state in the computational basis? In that case the matrix representation is given by

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

We met this matrix earlier—in Example 13.14. It has the same eigenvalues, and we find that once again,

$$S(\rho) = -\frac{3}{4}\log_2\left(\frac{3}{4}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right) = 0.81$$

Now let's consider states formed by tensor products of qubits. If a composite state is separable, that is, a product state of the form $\rho \otimes \sigma$, entropy is additive in the sense that

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma) \tag{13.42}$$

In general, entropy is *subadditive*. Let ρ_A and ρ_B be the reduced density matrices of a composite system ρ. The subadditivity inequality states that

$$S(\rho) \leq S(\rho_A) + S(\rho_B) \tag{13.43}$$

The result (13.43) indicates that in order to have the most information about an entangled system, you need to consider the system as a whole— that is, $S(\rho)$ is smaller than the entropies of the reduced density matrices, because ignorance decreases when considering the system as a whole. Alice and Bob, in possession of the reduced density matrices ρ_A and ρ_B, have more ignorance about the state when considering their parts of the system alone.

Example 13.17

Alice and Bob each share one member of an EPR pair that is in the Bell state:

$$|\beta_{10}\rangle = \frac{|0_A\rangle|0_B\rangle - |1_A\rangle|1_B\rangle}{\sqrt{2}}$$

Determine the entropy of the entire system, and the entropy as seen by Alice and Bob alone.

Solution

We showed in (5.15) that the density operator for this state is

$$\begin{aligned}\rho &= |\beta_{10}\rangle\langle\beta_{10}| \\ &= \left(\frac{|0_A\rangle|0_B\rangle - |1_A\rangle|1_B\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0_A|\langle 0_B| - \langle 1_A|\langle 1_B|}{\sqrt{2}}\right) \\ &= \frac{|0_A\rangle|0_B\rangle\langle 0_A|\langle 0_B| - |0_A\rangle|0_B\rangle\langle 1_A|\langle 1_B| - |1_A\rangle|1_B\rangle\langle 0_A|\langle 0_B| + |1_A\rangle|1_B\rangle\langle 1_A|\langle 1_B|}{2}\end{aligned}$$

In matrix form we have

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of this matrix are

$$\lambda_{1,2,3,4} = \{1, 0, 0, 0\}$$

So we quickly see that the entropy is $S(\rho) = -\log 1 = 0$. To get the states as seen by Alice and Bob individually, we compute the partial trace of this density operator. For example, in Chapter 5 we found that the reduced density matrix for Bob is

$$\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is the completely mixed state with entropy given by

$$S_B(\rho_B) = -\log_2\left(\frac{1}{2}\right) = 1$$

We find a similar result for Alice, and clearly (13.43) is satisfied.

EXERCISES

13.1. *Verify that $\delta(\rho, \sigma) = 3/4\sqrt{5}$ for the density matrices in Example 13.3 by calculating $\frac{1}{2}Tr\sqrt{(\rho - \sigma)^{\dagger}(\rho - \sigma)}$ with the matrices written in the $\{|\pm\rangle\}$ basis.*

13.2. *Let $\rho = \sum_i r_i|u_i\rangle\langle u_i|$ and $\sigma = \sum_i s_i|u_i\rangle\langle u_i|$. Prove (13.22).*

13.3. *Compute the trace distance and fidelity between*

$$\rho = \frac{4}{7}|0\rangle\langle 0| + \frac{3}{7}|1\rangle\langle 1|$$

and each of

$$\sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|, \quad \pi = \frac{3}{5}|0\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1|$$

13.4. *On a certain quantum channel there is a probability $p = 1/11$ that there is a phase flip error. What pure state leads to the minimum fidelity in this case? What is the minimum fidelity of the bit flip channel in this case? Assume that the system starts in some pure state $\rho = |\psi\rangle\langle\psi|$.*

13.5. *Determine the concurrence for the Werner state where*

$$\rho = \begin{pmatrix} \frac{7}{16} & 0 & 0 & \frac{3}{8} \\ 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ \frac{3}{8} & 0 & 0 & \frac{7}{16} \end{pmatrix}$$

13.6. *Find the entanglement of formation for the state in Exercise 13.5.*

13.7. *Find the entropy of the state*

$$\rho = \begin{pmatrix} \frac{5}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$$

13.8. *Find the entropy of the state*

$$|\psi\rangle = \frac{2}{3}|0\rangle + \frac{\sqrt{5}}{3}|1\rangle$$

13.9. *Consider the product state $|A\rangle|B\rangle$ used in Example 5.11 where*

$$|A\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}, \quad |B\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

The density matrix is

$$\rho = \begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{18}} & \frac{i}{3} & \frac{i}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} & \frac{1}{6} & \frac{i}{\sqrt{18}} & \frac{i}{6} \\ \frac{-i}{3} & \frac{-i}{\sqrt{18}} & \frac{1}{3} & \frac{1}{\sqrt{18}} \\ \frac{-i}{\sqrt{18}} & \frac{-i}{6} & \frac{1}{\sqrt{18}} & \frac{1}{6} \end{pmatrix}$$

(A) Show that the entropy of the density operator for $|A\rangle|B\rangle$ is zero.
(B) Find the entropy for the density operator seen only by Alice, ρ_A.

13.10. *Consider the density operators for the state*

$$|\psi\rangle = \sqrt{\frac{3}{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle$$

and the state

$$|\phi\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

Which state has a higher entropy?

14

ADIABATIC QUANTUM COMPUTATION

Adiabatic quantum computation is an alternative approach to quantum computation based on the time evolution of a quantum system. Before describing adiabatic processes, let's quickly review the dynamics of a quantum system. The time evolution of a quantum state $|\psi(t)\rangle$ is described by the Schrödinger equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle \tag{14.1}$$

where H is the Hamiltonian operator. This operator is the total energy of the system and can be expressed in terms of kinetic and potential energies

$$H = \frac{p^2}{2m} + V = -\frac{\hbar^2}{2m}\nabla^2 + V \tag{14.2}$$

where ∇^2 is the *Laplacian* operator and V is the potential energy function. In one dimension (14.2) becomes

$$H = \frac{p^2}{2m} + V = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \tag{14.3}$$

Quantum Computing Explained, by David McMahon