(§2.3) Linear transformations and operators

Recall the defn. of a linear transformation L.

For scalar c (usually for us $c \in \mathbb{C}$)

$L\, c|v\rangle = c L|v\rangle$

and $L(|v_1\rangle + |v_2\rangle) = L|v_1\rangle + L|v_2\rangle$

then

$$L(c_1|v_1\rangle + c_2|v_2\rangle) = c_1 L|v_1\rangle + c_2 L|v_2\rangle.$$

(L is a linear operator taking a vector space $\mathcal{H}$ to itself.) $L: \mathcal{H} \to \mathcal{H}$

The outer product of two vectors generates an operator.

Notation: $|\psi\rangle\langle\varphi|$.

E.g. in the matrix repr.

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

see sup →

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then $NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|$.

Example of Z gate $\Rightarrow$ $Z: |0\rangle \to |0\rangle$

$Z: |1\rangle \to -|1\rangle$

w/ matrix repr $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (diagonal in the comp. basis)

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

Lets quickly verify:

$Z|0\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle$ (by orthogonality)

$= |0\rangle\langle 0|0\rangle = |0\rangle$ ✓

$Z|1\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle$

$= -|1\rangle\langle 1|1\rangle = -|1\rangle$ ✓

Matrix element repr of an operator

Let T be a linear operator on Hilbert space $\mathcal{H}$.

Let $T_{n,m} = \langle b_n|T|b_m\rangle$ where $\{|b_n\rangle\}$ is an orthonormal basis for $\mathcal{H}$.

Then $T = \sum_{n,m} T_{n,m} |b_n\rangle\langle b_m|$

For an orthonormal basis $B = \{|b_n\rangle\}$, the identity operator may be written as

$$\sum_n |b_n\rangle\langle b_n| = I.$$

Called the 'resolution of the identity' (in the basis B).

Example - may skip

If T is the NOT operator,

$$\langle i | NOT | j \rangle = 0 \quad \text{if } i = j, \; i, j = 0, 1$$

$$\langle i | NOT | j \rangle = 1 \quad \text{if } i \neq j, \; i, j = 0, 1$$

possible HW problem: matrix elements for the Hadamard gate

i.e. $T_{00} = T_{11} = 0, \quad T_{01} = T_{10} = 1,$

(again) $NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Use of Dirac notation to square an operator

Example $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Then $Z^2 = (|0\rangle\langle 0| - |1\rangle\langle 1|)(|0\rangle\langle 0| - |1\rangle\langle 1|)$

$= |0\rangle\langle 0| + |1\rangle\langle 1|$

Thus $Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity (operator).

(as is clear enough from the (diagonal) matrix rep.)

Definition of adjoint operator in the matrix repr.

"Hermitian conjugate" of operator R is this

$$R^{\dagger} = (R^{*})^{T} \quad (= {R^{T}}^{*})$$

← "dagger"

↑ to avoid too many T's!

It is the operator s.t. the inner product

$$\langle \varphi | T | \psi \rangle = (\langle \psi | T^{\dagger} | \varphi \rangle)^{*}$$

← c.c.

Unitaries (unitary operators) are particularly important in QM & QC.

A unitary op. U is such that

$$U^{\dagger} = U^{-1}, \text{ the inverse op of } U.$$

Then $U^{\dagger} U = U U^{\dagger} = I$ (is an example of a normal op.)

The "observables" of QM correspond (to another important class of) operators, Hermitian (or self adjoint). These operators satisfy $H = H^{\dagger}$.

Eg?

Eg. $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is Hermitian (and unitary!)

Defn of projection operator (on a vector space $\mathcal{H}$):
a linear operator P such that $P^2 = P$.

If also $P^\dagger = P$, we have an "orthogonal projection"

Eg if $P = |1\rangle\langle 1|$, then $P^2 = P$, $P^\dagger = P$,
P is an $\perp$ projector.

Linear alg result for eigenvalues of a Hermitian operator. If $H = H^\dagger$, and $H|\psi\rangle = \lambda|\psi\rangle$,
then $\lambda \in \mathbb{R}$.

The trace of a matrix A is the sum of its diagonal elements and $tr(A) = \sum$ (eigenvalues of A), (at least for A diagonal, as for normal ops.)

could have been a HW problem!

Example - to probably skip. The (real, symmetric) matrix $m_a = \begin{bmatrix} a & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ has eigenvalues
($a \in \mathbb{R}$)
satisfying the characteristic eqn
$$x^3 - (10+a)x^2 + (10a - 14)x + a = 0$$
(I haven't found how)

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$\therefore$ the eigenvalues sum to 10 to a.

Behold: the trace of A is also 10 to a ✓

Simple 2x2 matrix case

$M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues

that add to $a+d = tr(M)$; the

char. poly. of M_2:

$$X^2 + (-1)\underbrace{(a+d)}_{tr(M_2)} X + \underbrace{ad-bc}_{det(M_2)} \equiv p_2(X)$$

to here FT. 11/22/16

Properties of the trace operation

Cyclicity: (i) $tr(ABC) = Tr(CAB) = tr(BCA)$

(ii) the trace is invariant ~~w.r.t.~~ under a similarity transformation (hence a change of bases).

By (i), if $M' \equiv S^{-1}MS$,

$tr(M') = tr(S^{-1}MS) = tr(SS^{-1}M) = tr(M)$ ✓

Coming attractions: Spectral theorem

fcns of operators

(§2.4) Spectral theorem

– Normal operators may be diagonalized

Introduce the commutator of two operators (← to explain)

$$[A, B] \equiv AB - BA.$$

A normal operator N satisfies

$$[N, N^\dagger] = 0$$ (← Hermitian conjugate)

Examples of normal ops. are the unitary & Hermitian ops (of such importance in QM & QC).

A form of the spectral theorem. (The spectrum of an op. refers to its eigenvalues.) (Here for finite dim Hilbert spaces)

For a normal operator T, there is a unitary op. (matrix) P so that

$$T = P \Delta P^\dagger$$

where Δ is a diagonal matrix.

Eg. $J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (for a spin 1 particle)