Encoding Logic Gates in Ising Hamiltonian

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Determining Field (h_i) and Interaction (J_{ij})Terms

General Outline of the Steps

- Start with any spin configuration, then start flipping individual qubits and think about how the energy of the system should *change* with those flips.
- When doing this, keep in mind the following:
    1. The lowest energy states should correspond to the states of the logic-gate of interest. So, when switching from a state that encodes one input/output set for our gate to one that does not, the energy should increase. We can think of this in terms of work/energy from classical mechanics, with this case being analogous to nonzero work being done on our system, since the total energy of the system changes.
    2. The states that correspond to our logic gate all have the same energy. This puts a restriction on how certain values (terms in the total energy equation) have to change with bit flips (i.e. do a bit flip, see which interaction terms and which field terms determine the change in energy of the system. If assumptions have been made about most of them, then the fact that the total energy should be the same can dictate the value of the final field/interaction terms we need). Following the work/energy analogy again, this would be the case where there is no work done in our system yet the relative values of internal energies change.

Concrete Example

As an example, let's look at how certain bit flips change the energy of our system in the NAND gate example that Prof. Coffey provided, where A and B are the inputs to our NAND gate and C is the output. Recall that spin UP (denoted $\uparrow$) corresponds to logical True, and spin DOWN ($\downarrow$) corresponds to logical False. Assume the state of our system is written as $|s_a s_b s_c\rangle$, where s_a, s_b, and s_c

correspond to the spins of inputs A, B, and C, respectively. Also recall that the Ising Hamiltonian takes the following form:

$$H_f = \sum_j h_j s_j + \sum_{i<j} J_{ij} s_i s_j$$

Case 1: Non-NAND to NAND Transition

Consider how the system's energy changes in the following transition:

$$|\downarrow\downarrow\downarrow\rangle \rightarrow |\downarrow\downarrow\uparrow\rangle$$

Notice first that the starting state, $|\downarrow\downarrow\downarrow\rangle$, does not correspond to one of the NAND gate configurations. Also notice that they state we end up in, $|\downarrow\downarrow\uparrow\rangle$, corresponds to A and B being False and C being True, which agrees with the NAND gate logic. Because of this, we know that the energy of our system should be lower in the second state, since the states that agree with the NAND gate logic should be the only ground-states of our system. We also need to pay attention to the spin that flipped in this transition, which was s_c. This means that, when s_c changed, the energy of our system decreased. There are two key possible reasons for this:

1. s_c flipped, which also means that its alignment with the $\vec{B}$-field term h_c flipped as well. We know the energy decreased, so it's possible that it is now anti-aligned with the component of the $\vec{B}$-field that corresponds to this qubit. However, it is also possible that this term increases the total energy, since we currently do not have an alignment assigned to the $\vec{B}$-field and there are other terms we must consider that could have decreased the total energy as well. We can only say that *magnitude* of the change caused by this term is $2 \times h_c$.

2. Before the transition, s_c was aligned with s_a and s_b; after the transition, it was not. So the interaction constants J_{ac} and J_{bc} are of interest here. We know that $J_{ac} = J_{bc}$, because A and B are equivalent as far as C is concerned. Let's assume that the interaction terms J_{ac} and J_{bc} are positive. Then, looking at the interaction terms in the Ising Hamiltonian, we see that if two spins s_i and s_j are aligned and their interaction term J_{ij} is positive, then the energy of the system includes a **positive** J_{ij} term. If they are anti-aligned, the total energy includes a **negative** J_{ij} term. Given that C went from being aligned with A and B to being anti-aligned with them, the total energy of our system must have *decreased* by $2 \times (J_{ac} + J_{bc})$.

Now we can write an equation that relates the non-zero changes in energy within our system to the overall change (which, at this point, we only know must be less than 0):

$$\pm 2 \times h_c - 2 \times (J_{ac} + J_{bc}) < 0$$

Notice that we still cannot say anything certain about the sign of the field term, despite the assumptions made about the signs of the interaction terms.

We made one assumption above, namely that the interaction terms J_{ac} and J_{bc} were positive. This was not necesary, as there are two possible terms that could decrease the energy of our system. However, making assumptions such as this narrows down the form that our problem will take; we could have said that those terms were negative instead, which would change the arguments we could make in subsequent parts of the analysis.

Case 2: NAND to NAND Transition

Now let's consider a second transition, one that goes between two states that agree with the NAND gate logic (and, thus, each correspond to a ground state of our system):

$$|\downarrow\downarrow\uparrow\rangle \rightarrow |\downarrow\uparrow\uparrow\rangle$$

This time, we flipped s_b. We know from the previous part that the first of these states agrees with the NAND logic, and we can verify that the second state, $|\downarrow\uparrow\uparrow\rangle$, does as well (A is False, B is True, so C should be True as well). As mentioned previously, that fact that both states satisfy our gate logic means that they should both correspond to the same ground state energy. We know that our field and interaction constants are not simply 0, so there must have been a tradeoff between certain energy values internal to our system. Specifically, for the transition in question:

1. s_b flipped, which means that its alignment with the relevant $\vec{B}$-field term, h_b, must have flipped as well. As in the first case, we cannot tell without further analysis whether this term increased or decreased during the transition. The magnitude of the change caused by this term is $2 \times h_b$.

2. Before the transition s_b was aligned with s_a only, and after the transition it was aligned with only s_c. Our previous assumption that J_{bc} is positive tells us that the corresponding term in the total energy equation *increased* by $2 \times J_{bc}$. However, we still do not have enough information to tell whether J_{ab} is positive or negative. However, since we already assumed J_{ac} and J_{bc} were positive, we can simply argue for consistency and say that J_{ab} is also positive. This means that corresponding term in our total energy equation *decreased* by $2 \times J_{ab}$, since s_a is spin DOWN and s_b went from spin DOWN to spin UP (which caused the sign flip).

Given all of this information, we can now characterize the relationship between the energy changes within our system as follows:

$$\pm 2 \times h_b + 2 \times (J_{bc} - J_{ab}) = 0$$

Concluding Thoughts

The process presented here is a rough outline of how we might go about assigning signs and values to the field and interaction terms for a Hamiltonian that represents a logic gate. There are steps that could be taken, such as looking at additional transitions and the corresponding changes in energy, to further simplify the analysis above. Once an adequate number of constraints have been placed on the variables, one could start assigning signs (as was done above) and concrete values to the constants, since we really just care about the relative signs and magnitudes of the h_i and J_{ij} values. Thus, there are multiple sets of values that could be assigned to these constants, as long as they obey all of the necessary constraints.