CSE100: Design and Analysis of Algorithms
Dynamic Programming (wrap up) and More Dynamic Programming

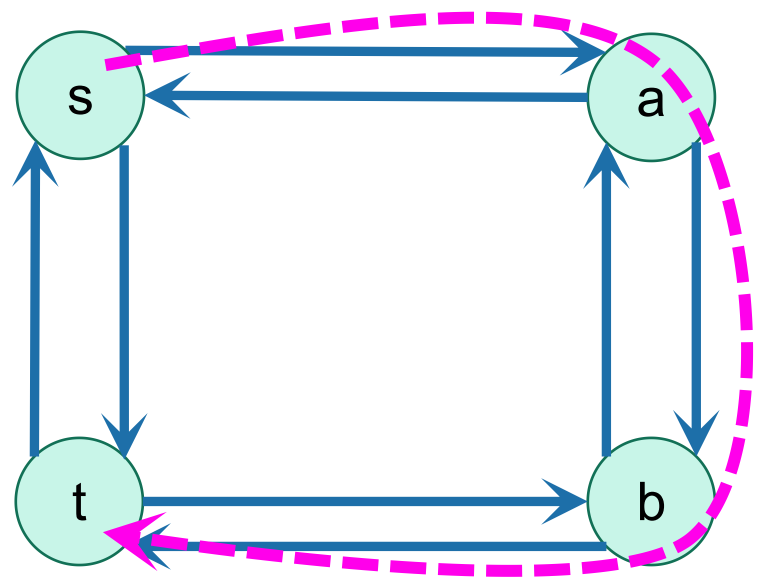
Longest Common Subsequences, Knapsack

What is ***dynamic programming***? (review)

- An algorithm design paradigm usually for solving **optimization problems**
- Elements of Dynamic programming
  - Optimal substructure: Optimal solutions to sub-problems can be used to find the optimal solution of the original problem.
  - Overlapping subproblems: The subproblems show up again and again
- Using these properties, we can design a ***dynamic programming*** algorithm:
  - Keep a table of solutions to the smaller problems.
  - Use the solutions in the table to solve bigger problems.
  - At the end we can use information we collected along the way to find the solution to the whole thing.
- Can be implemented **bottom-up** or **top-down**.

Bonus: Another Example of DP?

- Longest simple path (say all edge weights are 1):

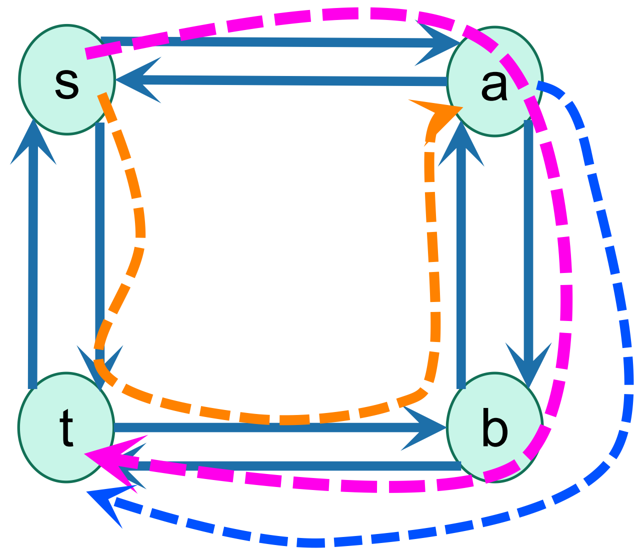


What is the longest simple path from s to t?

This doesn't give optimal sub-structure

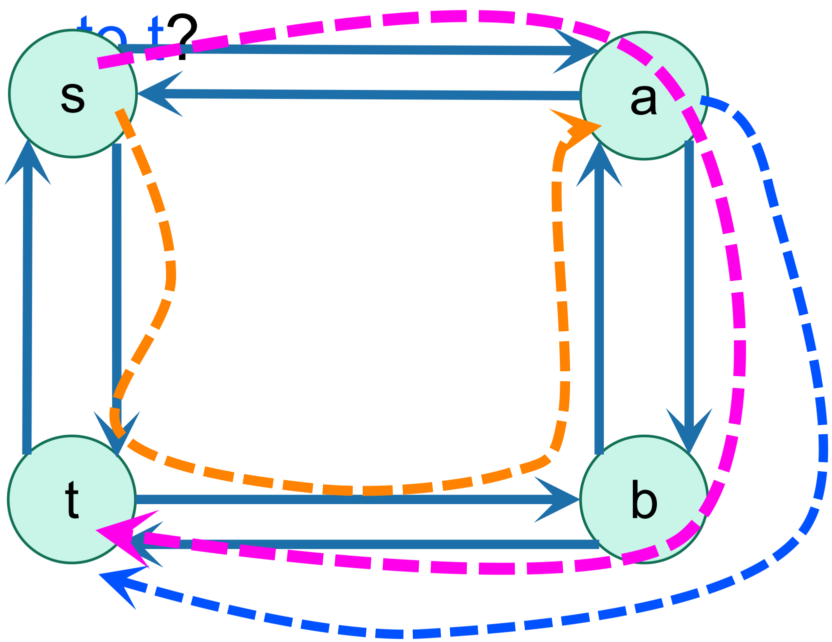
Optimal solutions to subproblems don't give us an optimal solution to the big problem. (At least if we try to do it this way).

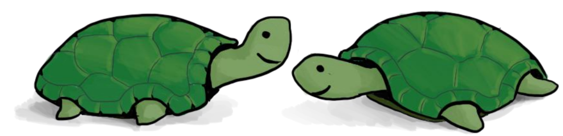
- The sub-problems we came up with aren't independent:
  - Once we've chosen the longest path from a to t
    - which uses b,
  - our longest path from s to a shouldn't be allowed to use b
    - since b was already used and that breaks the "simple-ness" of the combined path.
- Actually, the longest simple path problem is NP-complete.
  - We don't know of any polynomial-time algorithms for it, DP or otherwise!



This is an optimization problem…

- Can we use Dynamic Programming?
- Optimal Substructure?
  - Longest path from s to t = longest path from s to a + longest path from a to t?





NOPE!

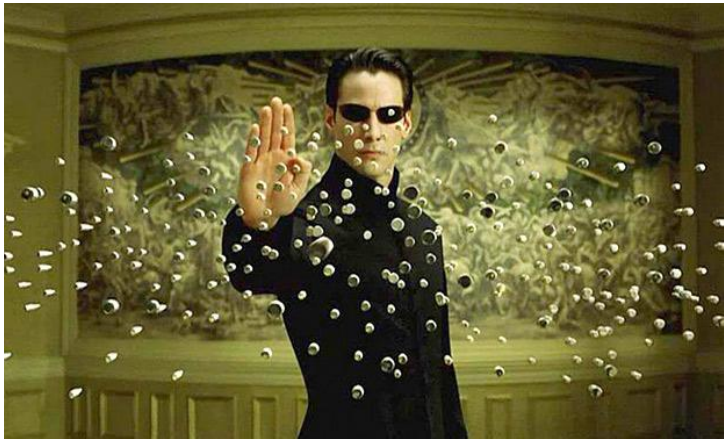
Recap

- Two shortest-path algorithms:
  - Bellman-Ford for single-source shortest path
  - Floyd-Warshall for all-pairs shortest path
- ***Dynamic programming!***
  - This is a fancy name for:
    - Break up an optimization problem into smaller problems
      - The optimal solutions to the sub-problems should be sub-solutions to the original problem.
    - Build the optimal solution iteratively by filling in a table of sub-solutions.
      - Take advantage of overlapping sub-problems!

Rest of Today

- More examples of ***dynamic programming***!

We will stop bullets with our action-packed coding skills, and also maybe find longest common subsequences.



Remember…

Dynamic Programming!

- Not coding in an action movie

Tom Cruise programs dynamically in Mission Impossible

Last Lecture

Dynamic Programming!

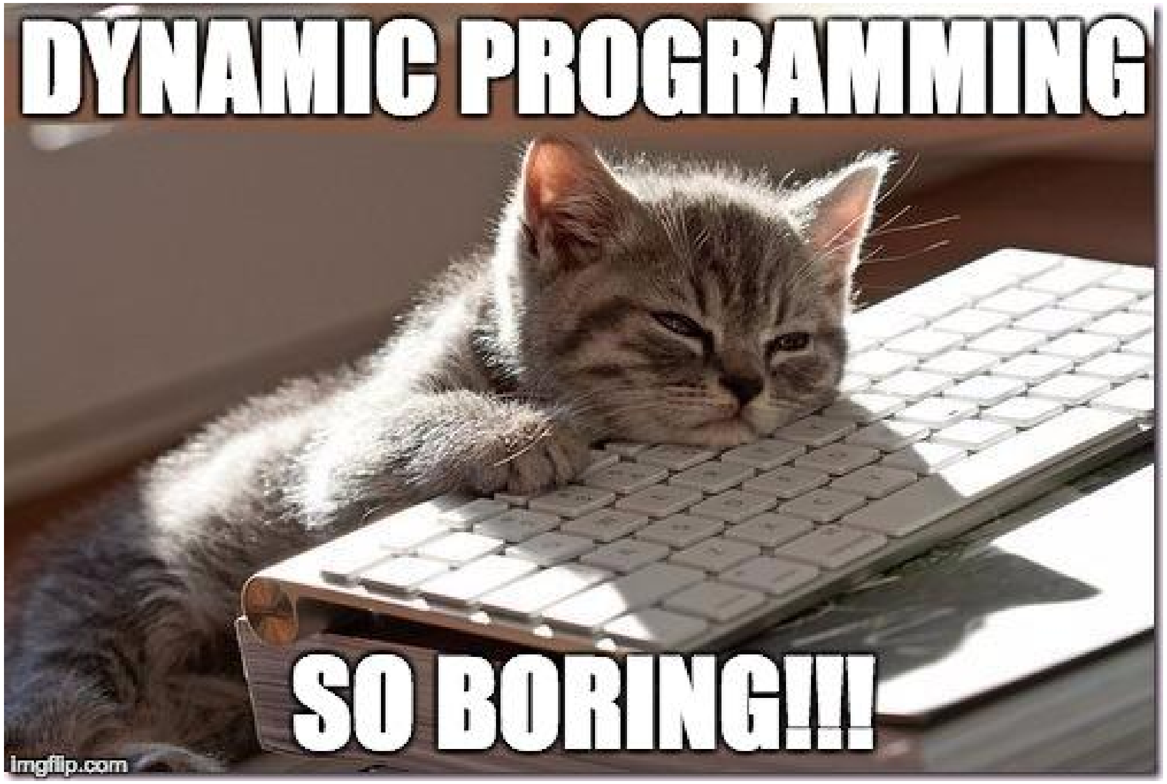
- Dynamic programming is an algorithm design paradigm.
- Basic idea:
  - Identify **optimal sub-structure**
    - Optimum to the big problem is built out of optima of small sub-problems
  - Take advantage of **overlapping sub-problems**
    - Only solve each sub-problem once, then use it again and again
  - Keep track of the solutions to sub-problems in a table as you build to the final solution.

Today

- Examples of dynamic programming:
  1. Longest common subsequence
  2. Knapsack problem
     - Two versions!
  3. Independent sets in trees
     - If we have time…
     - (If not the slides will be there as a reference)

The remaining goal of today's lecture

- For you to get really bored of dynamic programming



Longest Common Subsequence

- How similar are these two species?

DNA:
AGCCCTAAGGGCTACCTAGCTT

DNA:
GACAGCCTACAAGCGTTAGCTTG

Longest Common Subsequence

- How similar are these two species?

DNA:
AGCCCTAAGGGCTACCTAGCTT

DNA:
GACAGCCTACAAGCGTTAGCTTG

- Pretty similar, their DNA has a long common subsequence:

AGCCTAAGCTTAGCTT

Longest Common Subsequence

- Subsequence:
  - BDFH is a **subsequence** of ABCDEFGH
- If X and Y are sequences, a **common subsequence** is a sequence which is a subsequence of both.
  - BDFH is a **common subsequence** of ABCDEFGH and of ABDFGHI
- A **longest common subsequence**…
  - …is a common subsequence that is longest.
  - The **longest common subsequence** of ABCDEFGH and ABDFGHI is ABDFGH.

We sometimes want to find these

- Applications in bioinformatics

- The unix command diff
- Merging in version control
    - svn, git, etc…

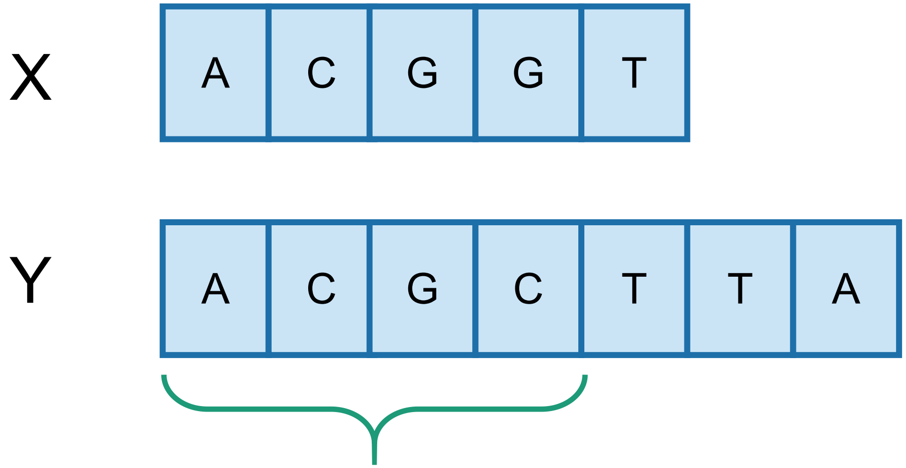
```
5:55pm acerpa@ubuntu:~>[1261]cat file1
A
B
C
D
E
F
G
H
5:55pm acerpa@ubuntu:~>[1262]cat file2
A
B
D
F
G
H
I
5:55pm acerpa@ubuntu:~>[1263]diff file1 file2
3d2
< C
5d3
< E
8a7
> I
5:55pm acerpa@ubuntu:~>[1264]
```

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
- **Step 5:** If needed, code this up like a reasonable person.

Step 1: Optimal substructure

Prefixes:



Notation: denote this prefix **ACGC** by Y_4

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let $C[i, j] = \text{length\_of\_LCS}(X_i, Y_j)$

Examples: $C[2,3] = 2$
$C[4,4] = 3$

Optimal substructure ctd.

- Subproblem:
  - finding LCS's of prefixes of X and Y.

- Why is this a good choice?
  - As we will see, there's some relationship between LCS's of prefixes and LCS's of the whole things.
  - These subproblems overlap a lot.

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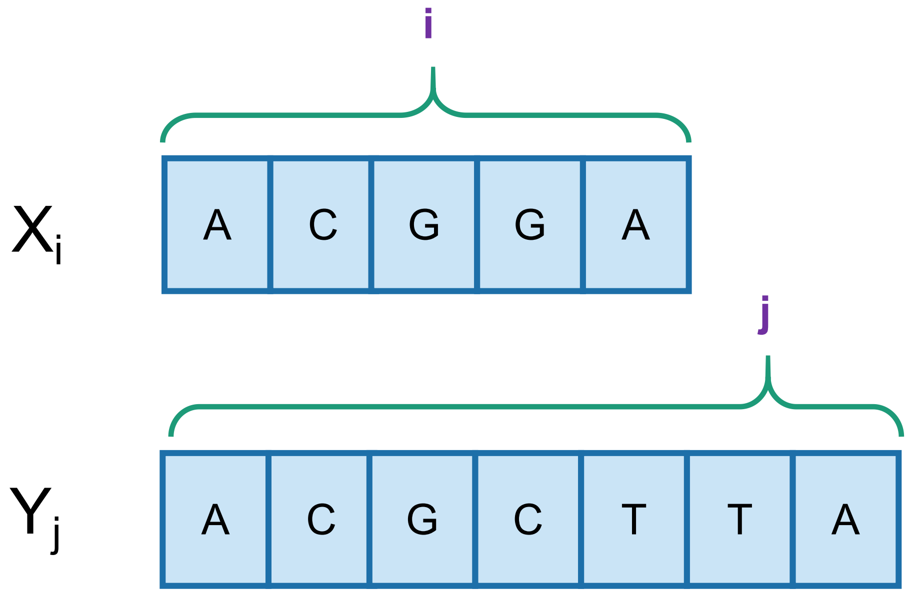
Goal

- Write C[i,j] in terms of the solutions to smaller sub-problems

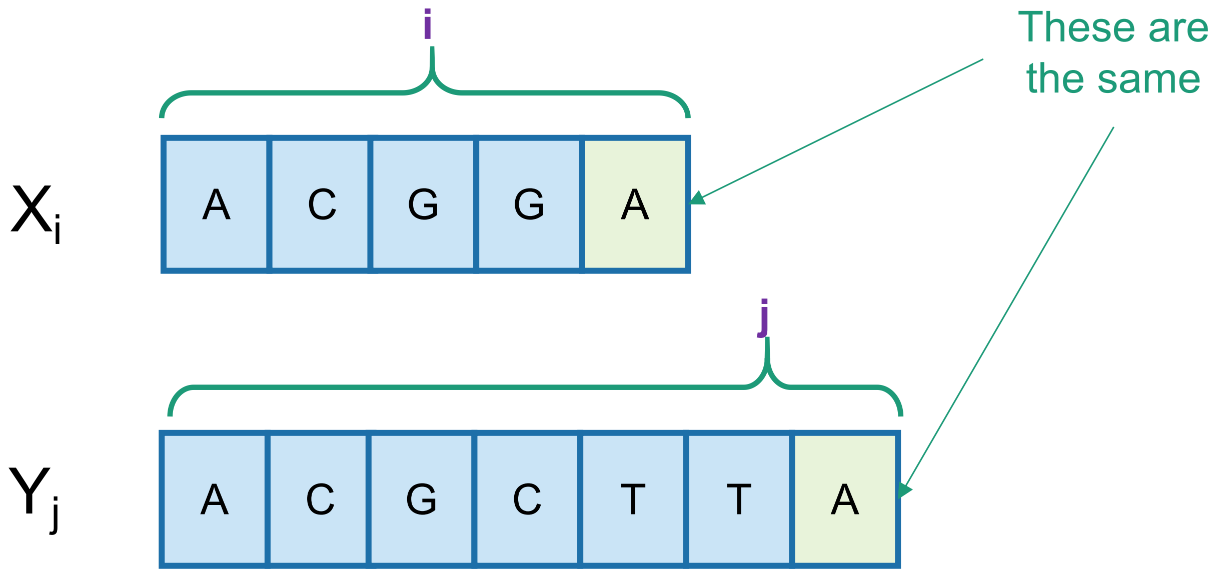


$C[i, j] = \text{length\_of\_LCS}(X_i, Y_j)$

Two cases

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i, j] = length_of_LCS(X_i, Y_j)

Case 1: $X_i[i] = Y_j[j]$

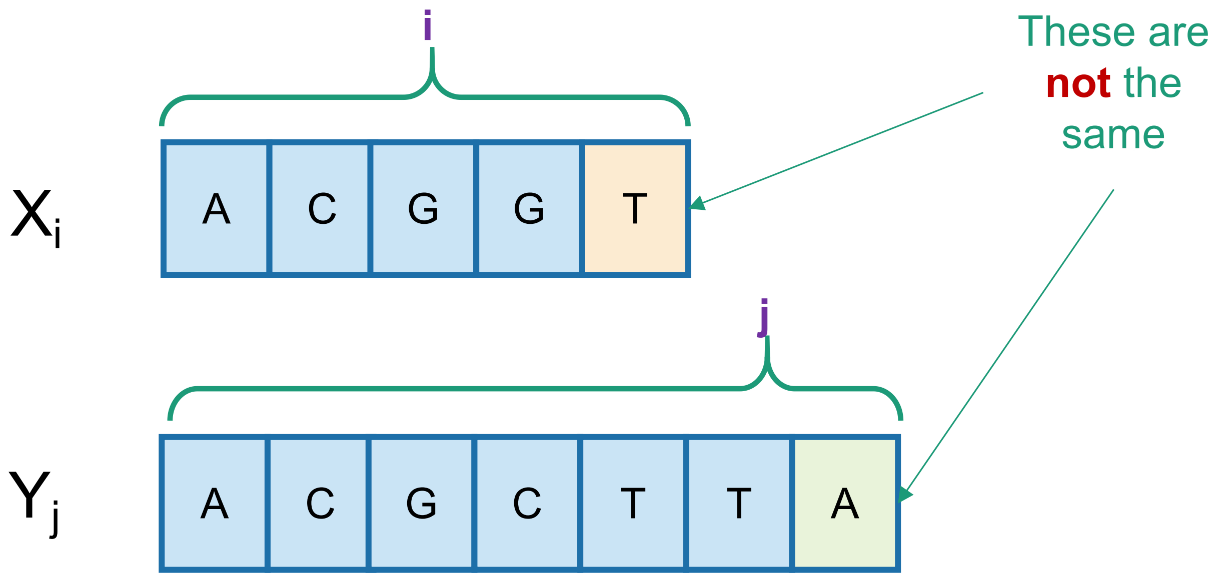


- Then C[i, j] = 1 + C[i-1, j-1].
  - because LCS(X_i, Y_j) = LCS(X_{i-1}, Y_{j-1}) followed by $\boxed{A}$

Two cases

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i, j] = length_of_LCS(X_i, Y_j)

Case 2: $X_i[i] \neq Y_j[j]$



- Then C[i, j] = max{C[i-1, j], C[i, j-1]}.
  - either LCS(X_i, Y_j) = LCS(X_{i-1}, Y_j) and $\boxed{T}$ is not involved,
  - or LCS(X_i, Y_j) = LCS(X_i, Y_{j-1}) and $\boxed{A}$ is not involved,
  - (maybe both are not involved, that's covered by the "or").

Recursive formulation
of the optimal solution

X_0	

Y_j	A	C	G	C	T	T	A

Case 0

- $C[i, j] =$

$$\begin{cases} 0 & \text{if } \ i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } Xi[i] = Yj[j] \ \text{ and } i, j > 0 \\ \max\{C[i, j-1], \ C[i-1, j]\} & \text{if } Xi[i] \neq Yj[j] \ \text{ and } i, j > 0 \end{cases}$$

Case 1

X_i	A	C	G	G	A

Y_j	A	C	G	C	T	T	A

Case 2

X_i	A	C	G	G	T

Y_j	A	C	G	C	T	T	A

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LCS DP

- **LCS**(X, Y):
  - C[i,0] = C[0,j] = 0 for all i = 0,…,m, j=0,…n.
  - **For** i = 1,…,m and j = 1,…,n:
    - **If** $X_i[i] = Y_j[j]$:
      - C[i,j] = C[i-1,j-1] + 1
    - **Else:**
      - C[i,j] = max{ C[i,j-1], C[i-1,j] }
  - Return C[m,n]

Running time: O(nm)

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } Xi[i] = Yj[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1],\ C[i-1, j]\} & \text{if } Xi[i] \neq Yj[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X i[i] = Y j[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X i[i] \neq Y j[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

So the LCS of X and Y has length 3.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X i[i] = Y j[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X i[i] \neq Y j[j] \text{ and } i, j > 0 \end{cases}$$

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Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X: | A | C | G | G | A |

Y: | A | C | T | G |

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X: A C G G A

Y: A C T G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.

That 3 must have come from the 3 above it.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X: | A | C | G | G | A |

Y: | A | C | T | G |

		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

This 3 came from that 2 – we found a match!

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

	Y		A	C	T	G
		0	0	0	0	0
X	A	0	1	1	1	1
	C	0	1	2	2	2
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

That 2 may as well have come from this other 2.

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X | A | C | G | G | A

Y | A | C | T | G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

C	G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1],\ C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	**A**	**C**	**T**	**G**	

Y

		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

X

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

A C G

This is the LCS!

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i,j-1],\ C[i-1,j] \} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Finding an LCS

- See CLRS for pseudocode
- Takes time O(mn) to fill the table
- Takes time O(n + m) on top of that to recover the LCS
  - We walk up and left in an n-by-m array
  - We can only do that for n + m steps.
- Altogether, we can find LCS(X,Y) in time O(mn).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 5:** If needed, code this up like a reasonable person.

This pseudocode actually isn't so bad

- If we are only interested in the length of the LCS we can do a bit better on space:
  - Since we go across the table one-row-at-a-time, we can only keep two rows if we want.
- If we want to recover the LCS, we need to keep the whole table.

- Can we do better than O(mn) time?
  - A bit better.
    - By a log factor or so.
  - But doing much better (polynomially better) is an open problem!
    - If you can do it let us know :D

What have we learned?

- We can find LCS(X,Y) in time O(nm)
  - if |X|=m, |Y|=n
- We went through the steps of coming up with a dynamic programming algorithm.
  - We kept a 2-dimensional table, breaking down the problem by decrementing the length of X and Y.

Example 2: Knapsack Problem

- We have n items with weights and values:

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- And we have a knapsack:
  - it can only carry so much weight:

Capacity: 10

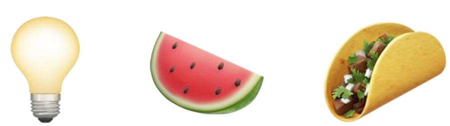
Capacity: 10

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?



Total weight: 9
Total value: 35

Some notation

Item:					
Weight:	w_1	w_2	w_3	…	w_n
Value:	v_1	v_2	v_3		v_n

Capacity: W

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.
  - K[x] = value you can fit in a knapsack of capacity x

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks

Optimal substructure

item i

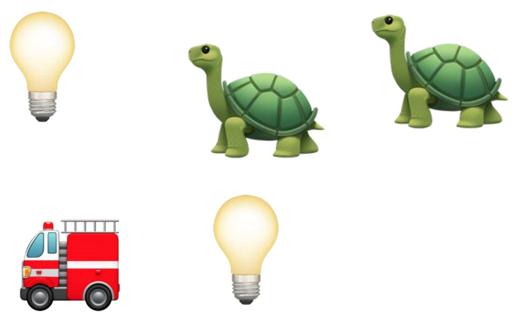
- Suppose this is an optimal solution for capacity x:

Say that the optimal solution contains at least one copy of some item labelled i.

Weight w_i
Value v_i

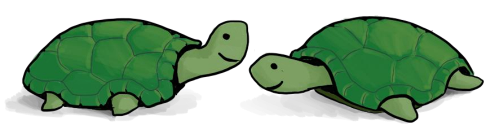
Capacity x
Value V

- Then this is optimal for capacity x - w_i:



Capacity x – w_i
Value V - v_i

Why?

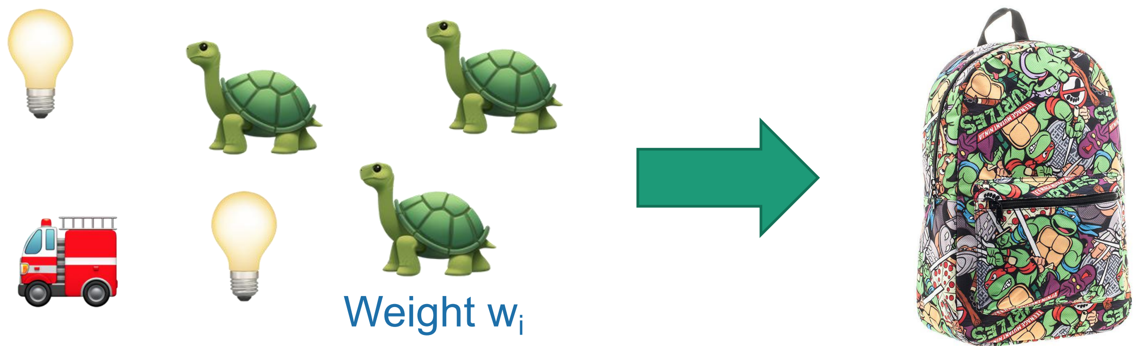


Optimal substructure

item i

- Suppose this is an optimal solution for capacity x:

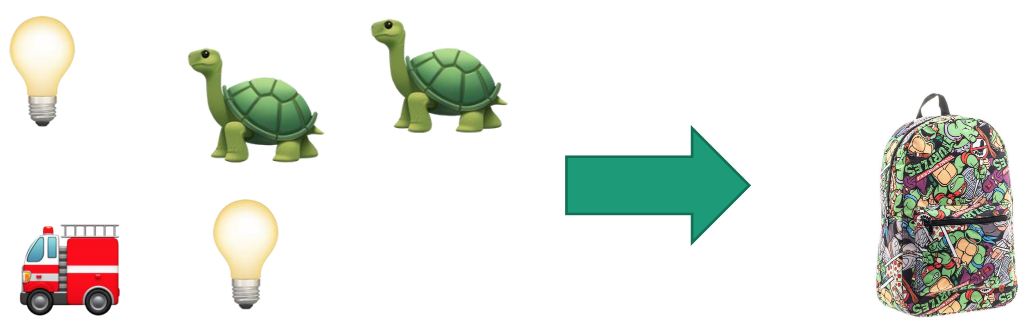
Say that the optimal solution contains at least one copy of item i.



Weight w_i
Value v_i

Capacity x
Value V

- Then this is optimal for capacity x - w_i:



Capacity x – w_i
Value V - v_i

If I could do better than the second solution, then adding a turtle to that improvement would improve the first solution.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x] be the optimal value for capacity x.

$K[x] = \max_i \{$  $+$  $\}$

The maximum is over all i for which $w_i \leq x$

Optimal way to fill the smaller knapsack

The value of item i.

$$K[x] = \max_i \{ K[x - w_i] + v_i \}$$

- (And K[x] = 0 if the maximum is empty).
  - That is, if there are no i so that $w_i \leq x$

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - **for** x = 1, …, W:
    - $K[x] = 0$
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\ K[x],\ K[x - w_i] + v_i\ \}$
  - **return** K[W]

Running time: O(nW)

K[x] = $\max_i$ {  +  }

= $\max_i$ { $K[x - w_i] + v_i$ }

Why does this work?

Because our recursive relationship makes sense.

Can we do better?

- Writing down W takes log(W) bits.
- Writing down all n weights takes at most nlog(W) bits.
- Input size: nlog(W).
  - Maybe we could have an algorithm that runs in time O(nlog(W)) instead of O(nW)?
  - Or even O($n^{1000000} \log^{1000000}(W)$)?

- Open problem!
  - (But probably the answer is **no**…otherwise P = NP)

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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Let's write a bottom-up DP algorithm

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  - **return** K[W]

$K[x] = \max_i \{$  $+$  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

Let’s write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\ K[x],\ K[x - w_i] + v_i\ \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[$x - w_i$] ∪ { item i }
  - **return** ITEMS[W]

$K[x] = \max_i \{$  +  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

Example

	0	1	2	3	4
K	0				
ITEMS					

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
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      - **if** $w_i \leq x$:
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        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\ K[x],\ K[x - w_i] + v_i\ \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

	0	1	2	3	4
K	0	1			
ITEMS					

ITEMS[1] = ITEMS[0] 

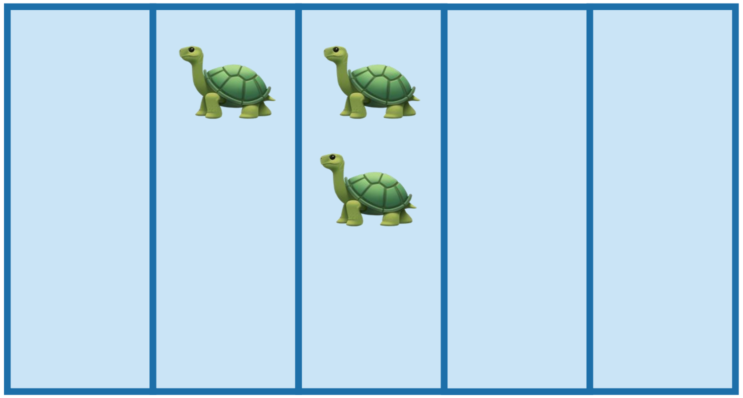
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	2		

ITEMS



ITEMS[2] = ITEMS[1] 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\ K[x],\ K[x - w_i] + v_i\ \}$
        - If K[x] was updated:
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  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
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      - **if** $w_i \leq x$:
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        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

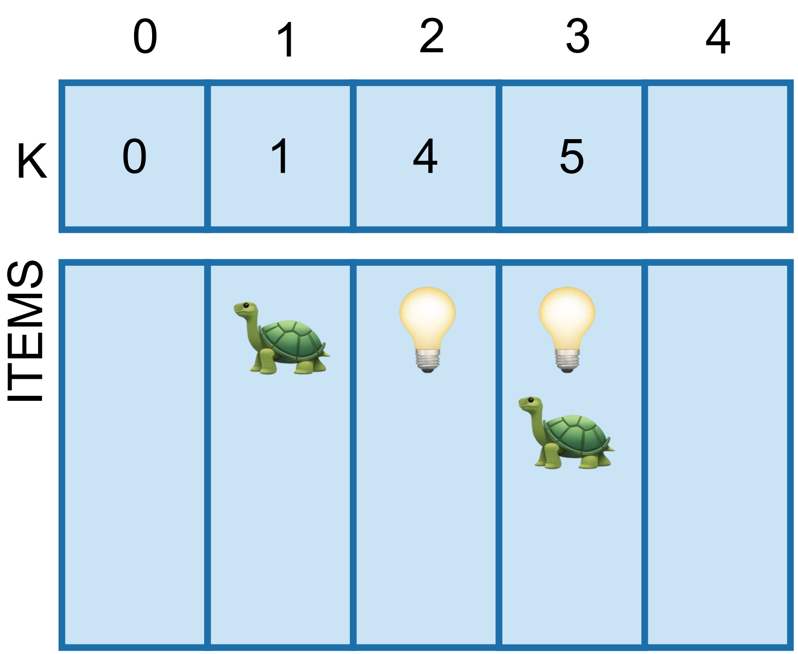
	0	1	2	3	4
K	0	1	4		
ITEMS		🐢	💡		

ITEMS[2] = ITEMS[0] 💡

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



- UnboundedKnapsack(W, n, weights, values):
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      - **if** $w_i \leq x$:
        - $K[x] = \max\{\ K[x],\ K[x - w_i] + v_i\ \}$
      - If K[x] was updated:
        - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item: 

Weight:	1	2	3
Value:	1	4	6

ITEMS[3] = ITEMS[2] 

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	
ITEMS					

ITEMS[3] = ITEMS[0] 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\ K[x],\ K[x - w_i] + v_i\ \}$
      - If K[x] was updated:
        - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	7
ITEMS					 

ITEMS[4] = ITEMS[3] 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	8
ITEMS					 

ITEMS[4] = ITEMS[2] 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\ K[x],\ K[x - w_i] + v_i\ \}$
      - If K[x] was updated:
        - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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(Pass)

What have we learned?

- We can solve unbounded knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.
- We again went through the steps to create DP solution:
  - We kept a one-dimensional table, creating smaller problems by making the knapsack smaller.

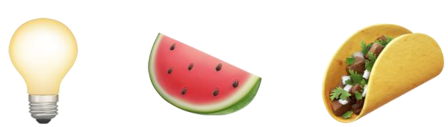
Capacity: 10

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?



Total weight: 9
Total value: 35

Recipe for applying Dynamic Programming

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Optimal substructure: try 1

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks

This won’t quite work…

- We are only allowed **one copy of each item**.
- The sub-problem needs to “know” what items we’ve used and what we haven’t.

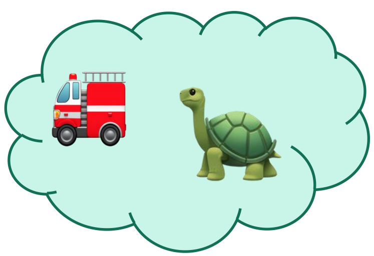
Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

We’ll still increase the size of the knapsacks.

Then more items

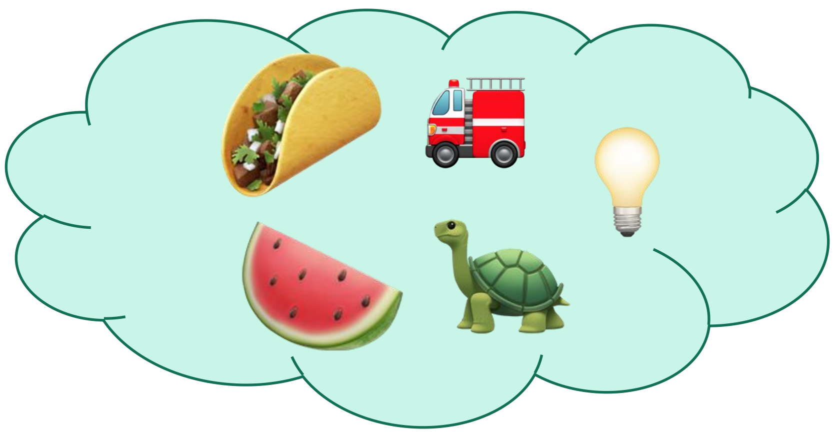


(We’ll keep a two-dimensional table).

Then yet more items

Our sub-problems:

- Indexed by x and j



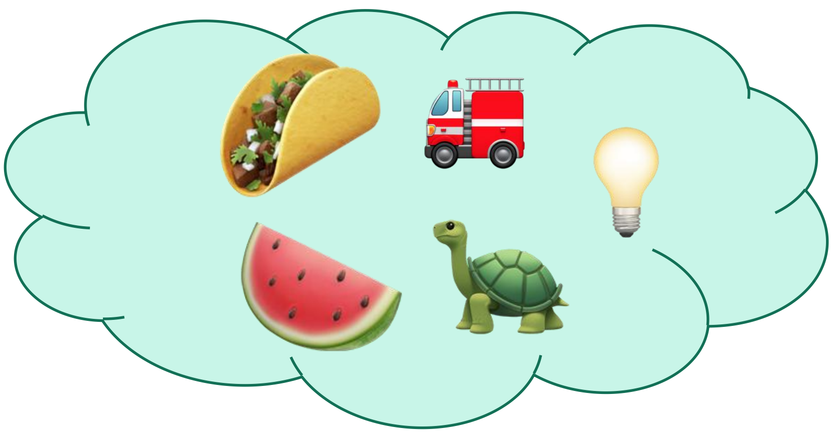
First j items

Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Relationship between sub-problems

- Want to write K[x,j] in terms of smaller sub-problems.



First j items

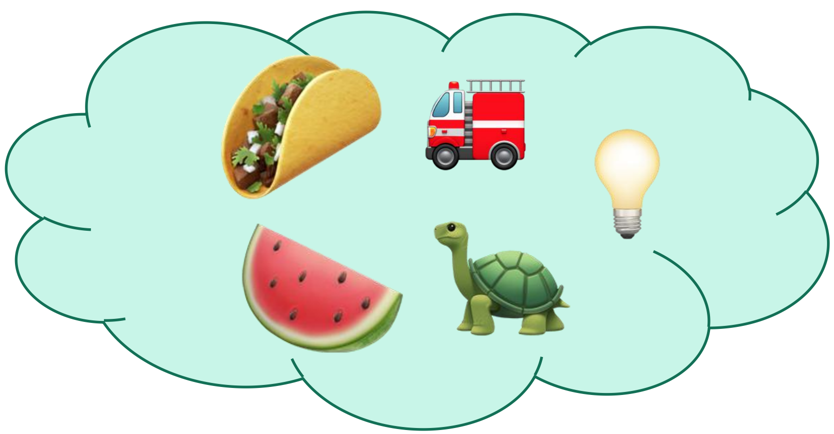
Capacity x

K[x,j] = optimal solution for a knapsack of size x
using only the first j items.

Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.
- **Case 2**: Optimal solution for j items does use item j.



First j items

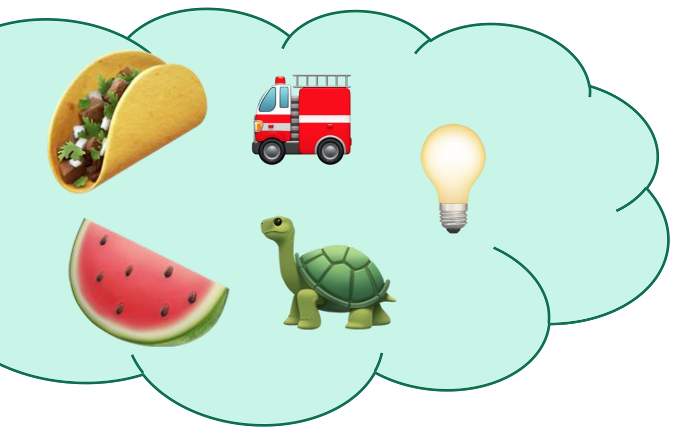
Capacity x

K[x,j] = optimal solution for a knapsack of size x
using only the first j items.

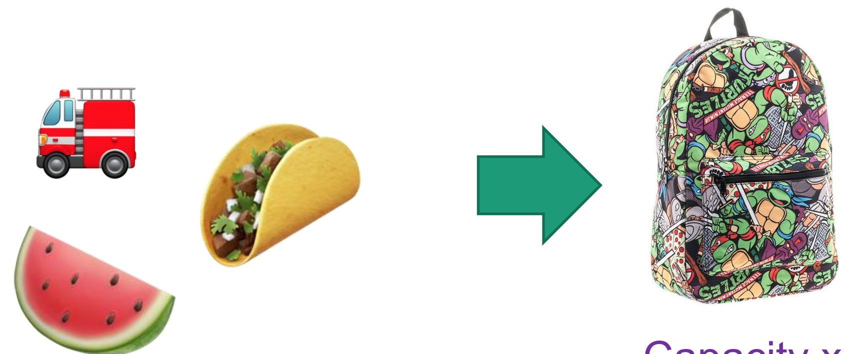
Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.

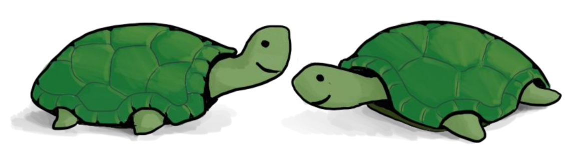


First j items



Capacity x
Value V
Use only the first j items

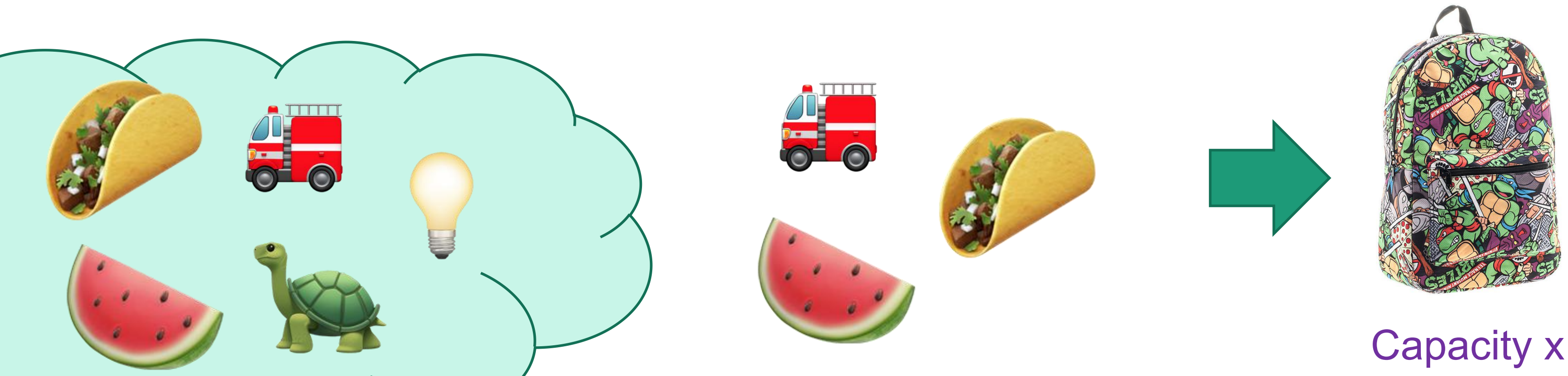
What lower-indexed
problem should we solve to
solve this problem?



Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.



Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items:

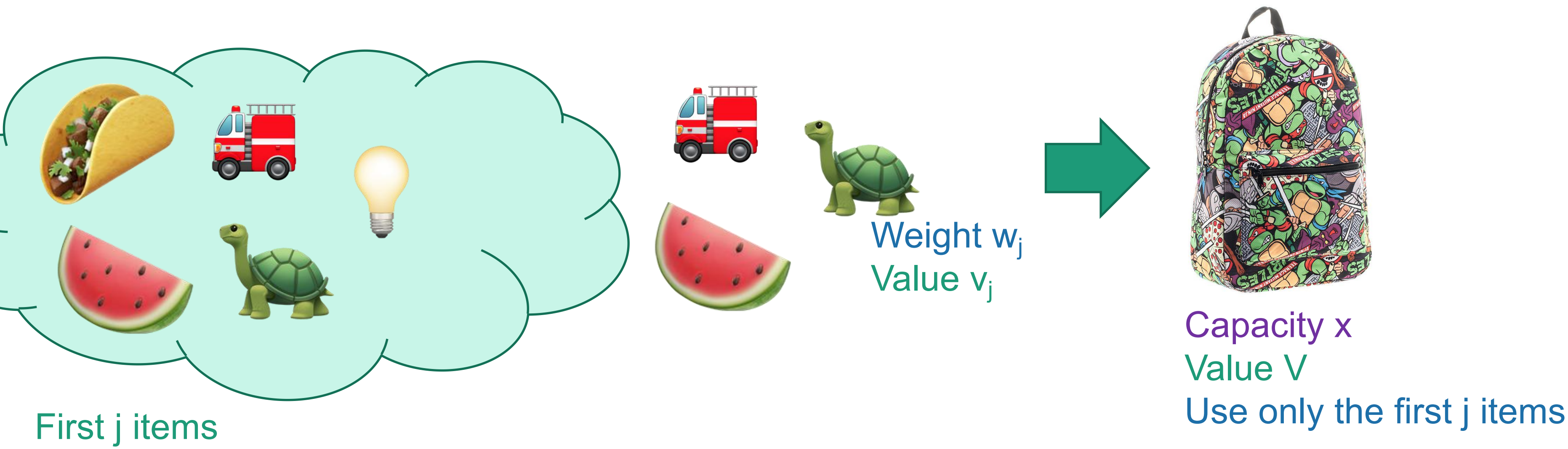
First j items

First j-1 items

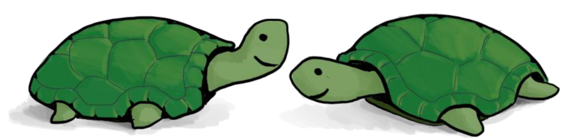
Capacity x
Value V
Use only the **first j-1 items**.

Two cases

- **Case 2**: Optimal solution for j items uses item j.



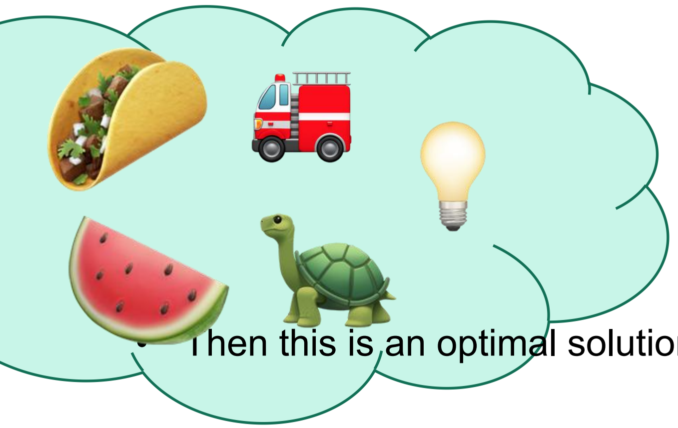
What lower-indexed problem should we solve to solve this problem?



Two cases

item j

- **Case 2**: Optimal solution for j items uses item j.

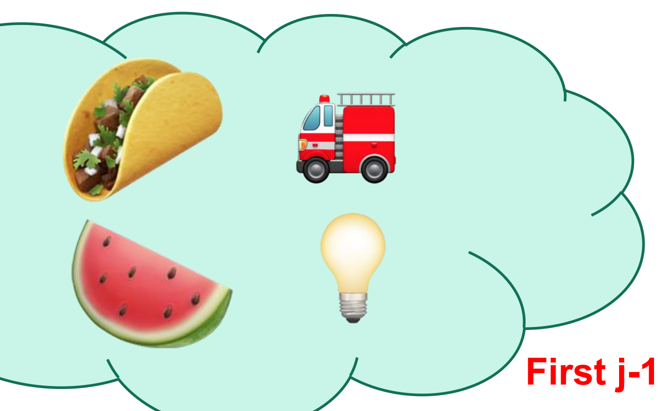


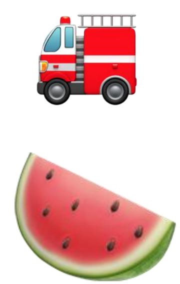
Weight w_j
Value v_j

- Then this is an optimal solution for j-1 items and a smaller knapsack:

Capacity x
Value V
Use only the first j items

First j items





Capacity $x - w_j$
Value $V - v_j$
Use only the **first j-1 items**.

First j-1 items

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- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x,j] be the optimal value for:
  - capacity x,
  - with j items.

$$K[x,j] = \max\{ \underbrace{K[x, j-1]}_{\text{Case 1}} , \underbrace{K[x - w_j, j-1] + v_j}_{\text{Case 2}} \}$$

- (And K[x,0] = 0 and K[0,j] = 0).