CSE100: Design and Analysis of Algorithms
Minimum Spanning Trees and Max Flow

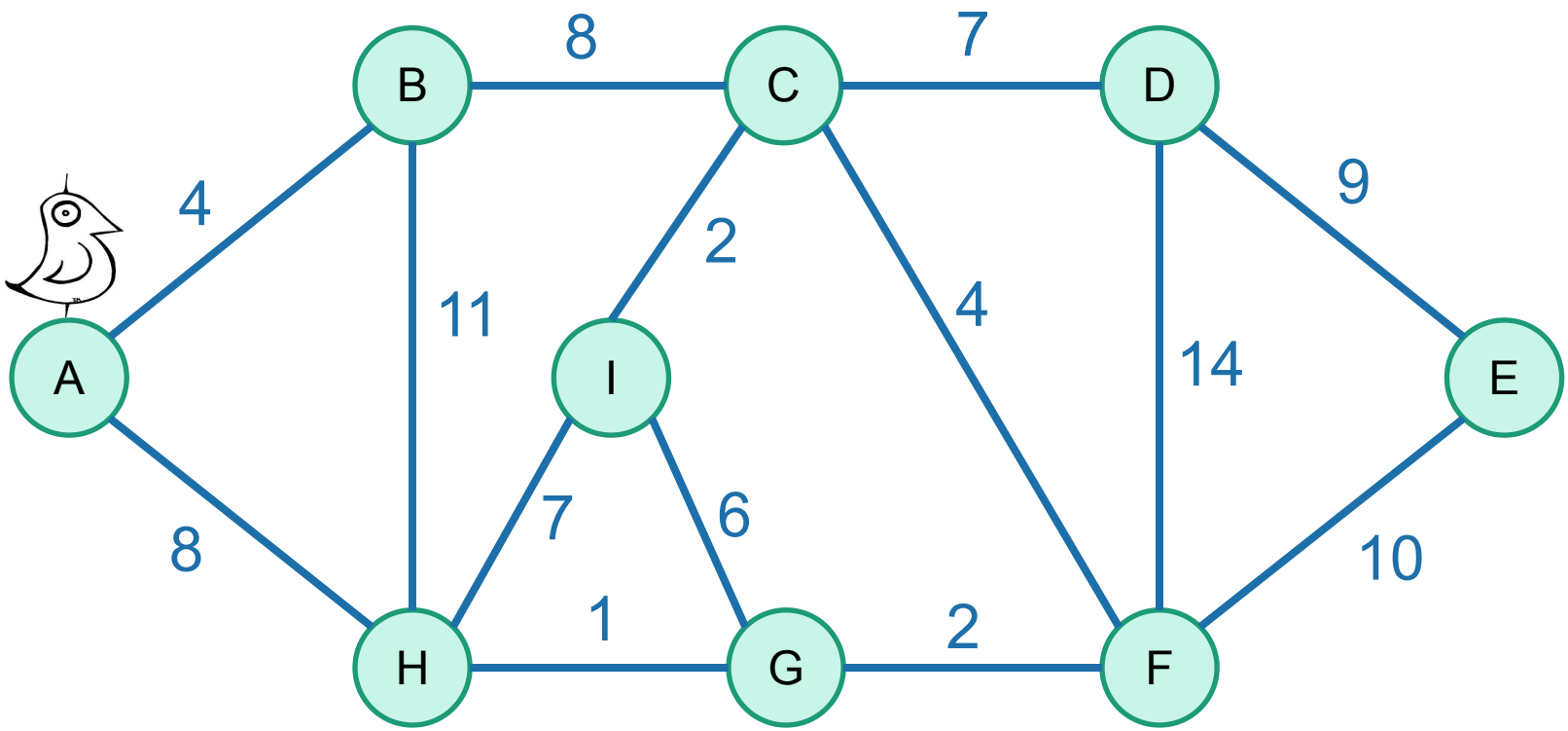
Kruskal, Max Flows, Min Cuts, and Ford-Fulkerson Algorithms

What have we learned?

- Prim's algorithm greedily grows a tree
  - smells a lot like Dijkstra's algorithm

Prim's algl:

Start growing a tree, greedily add the shortest edge we can to grow the tree.



That’s not the only greedy algorithm for MST!

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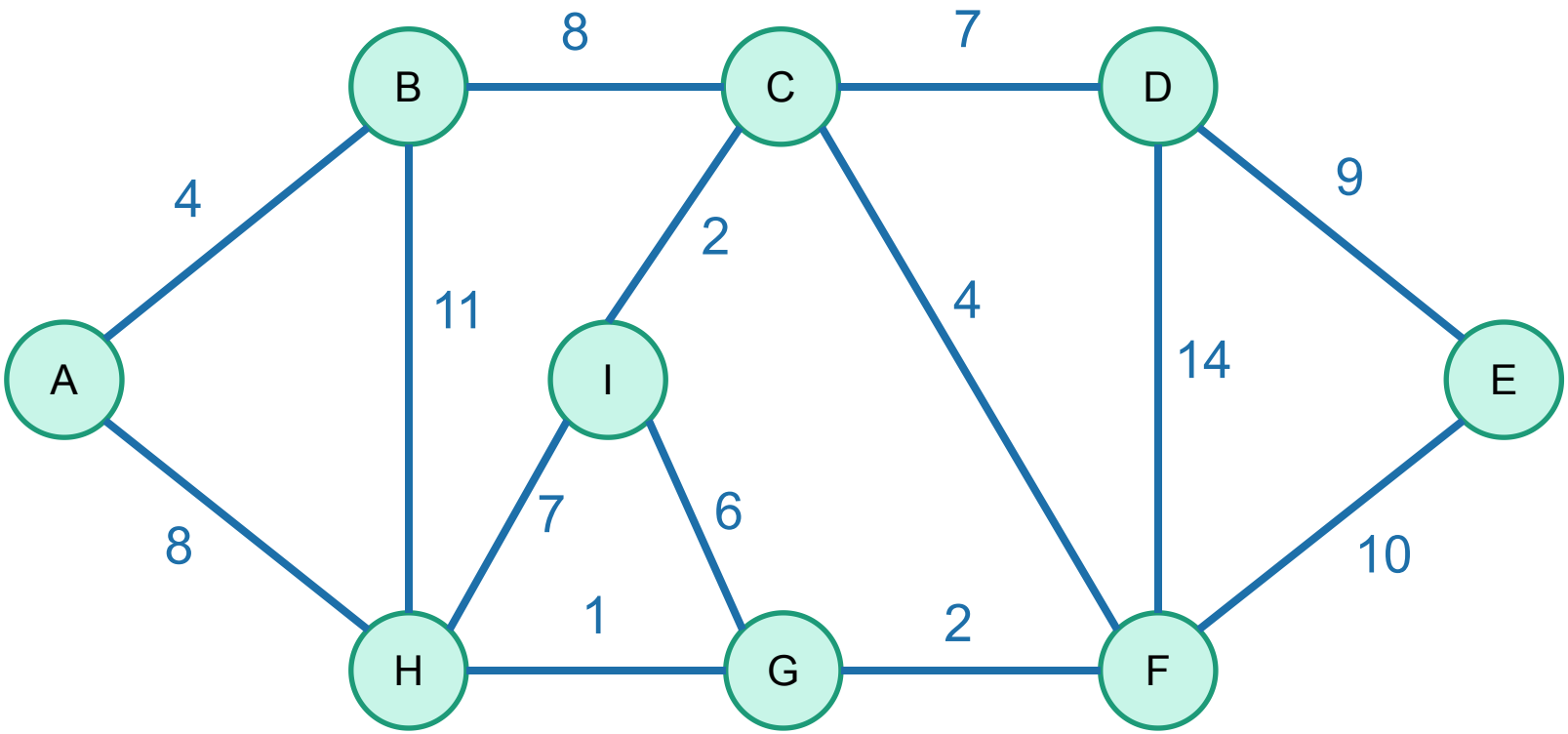
whether or not it's connected to what we have so far?



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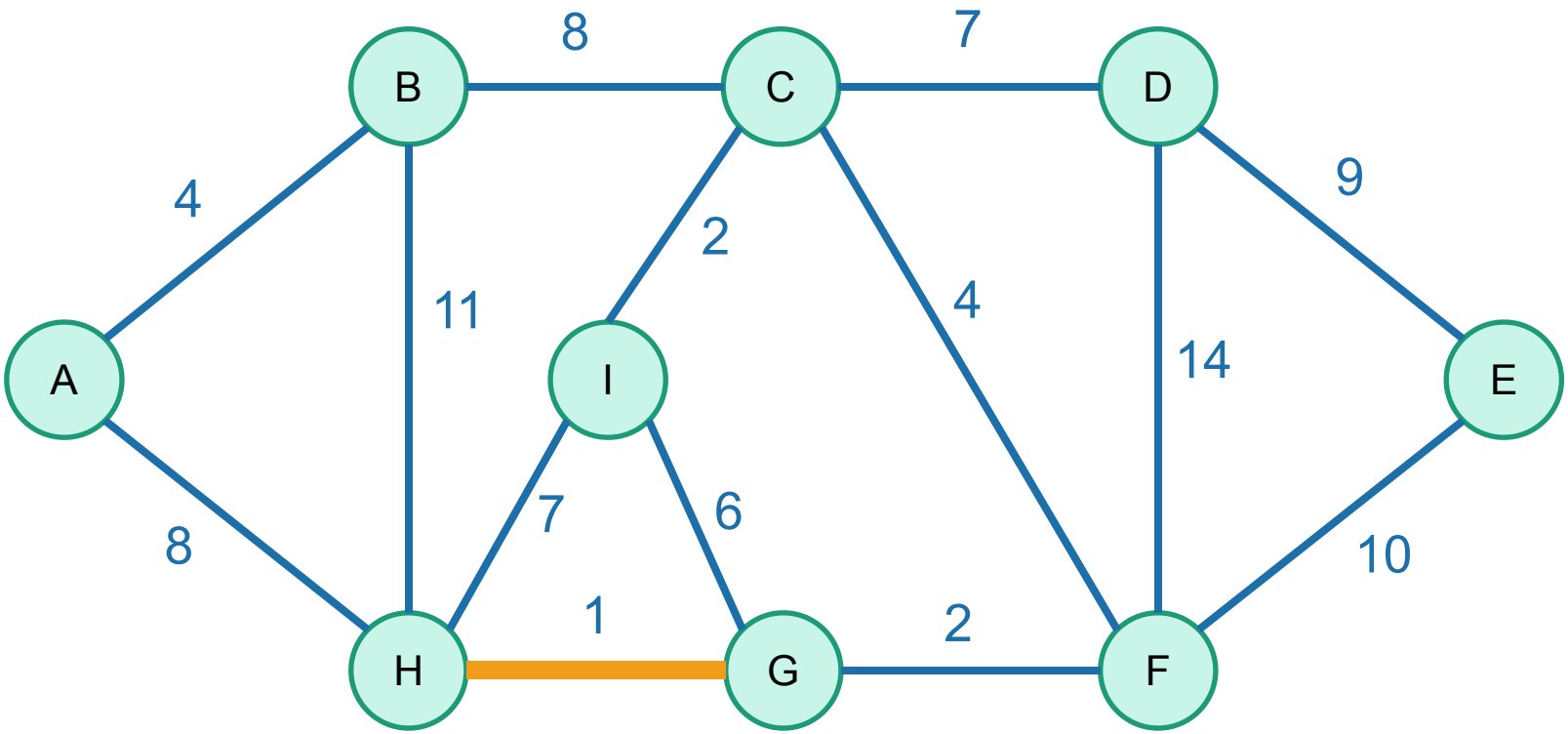

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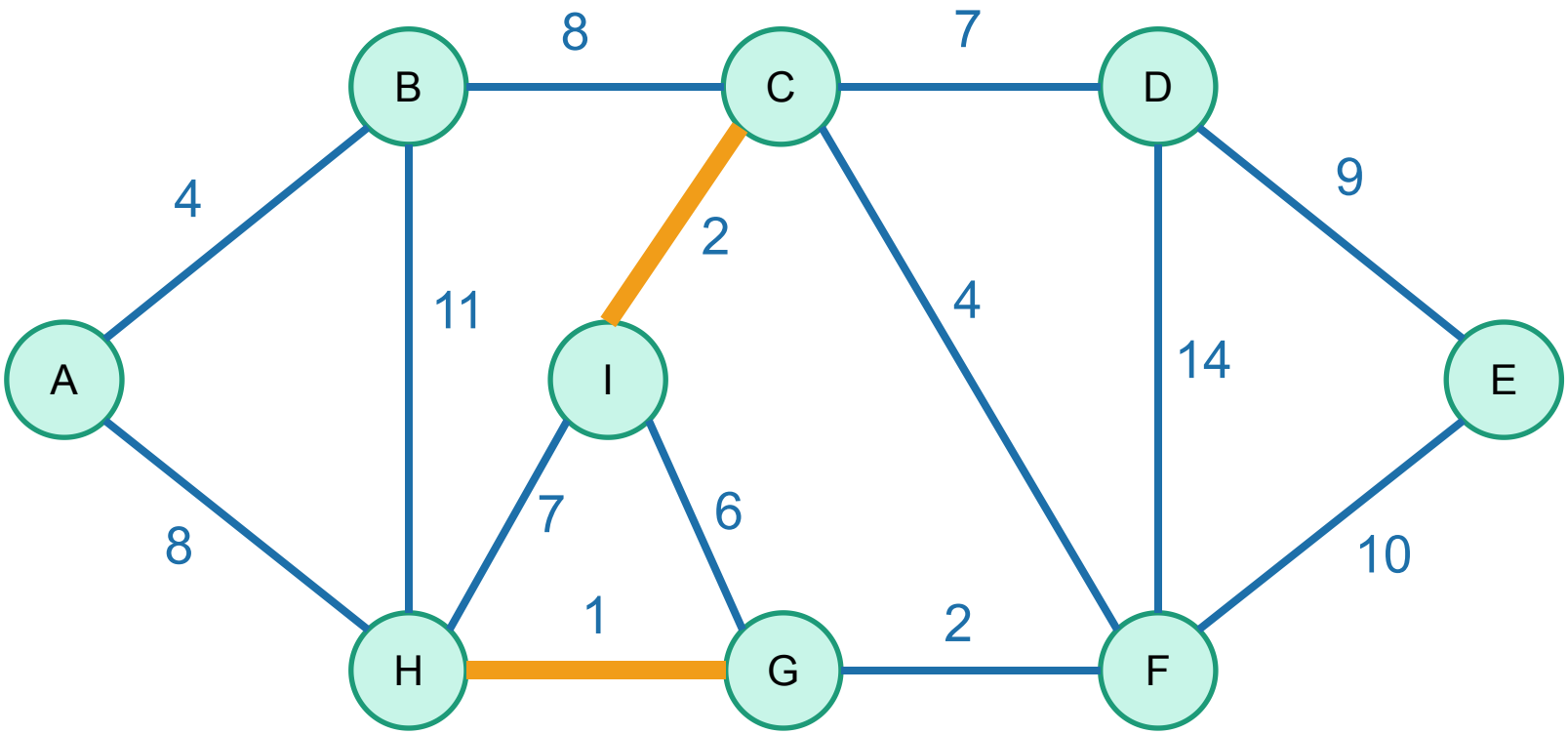

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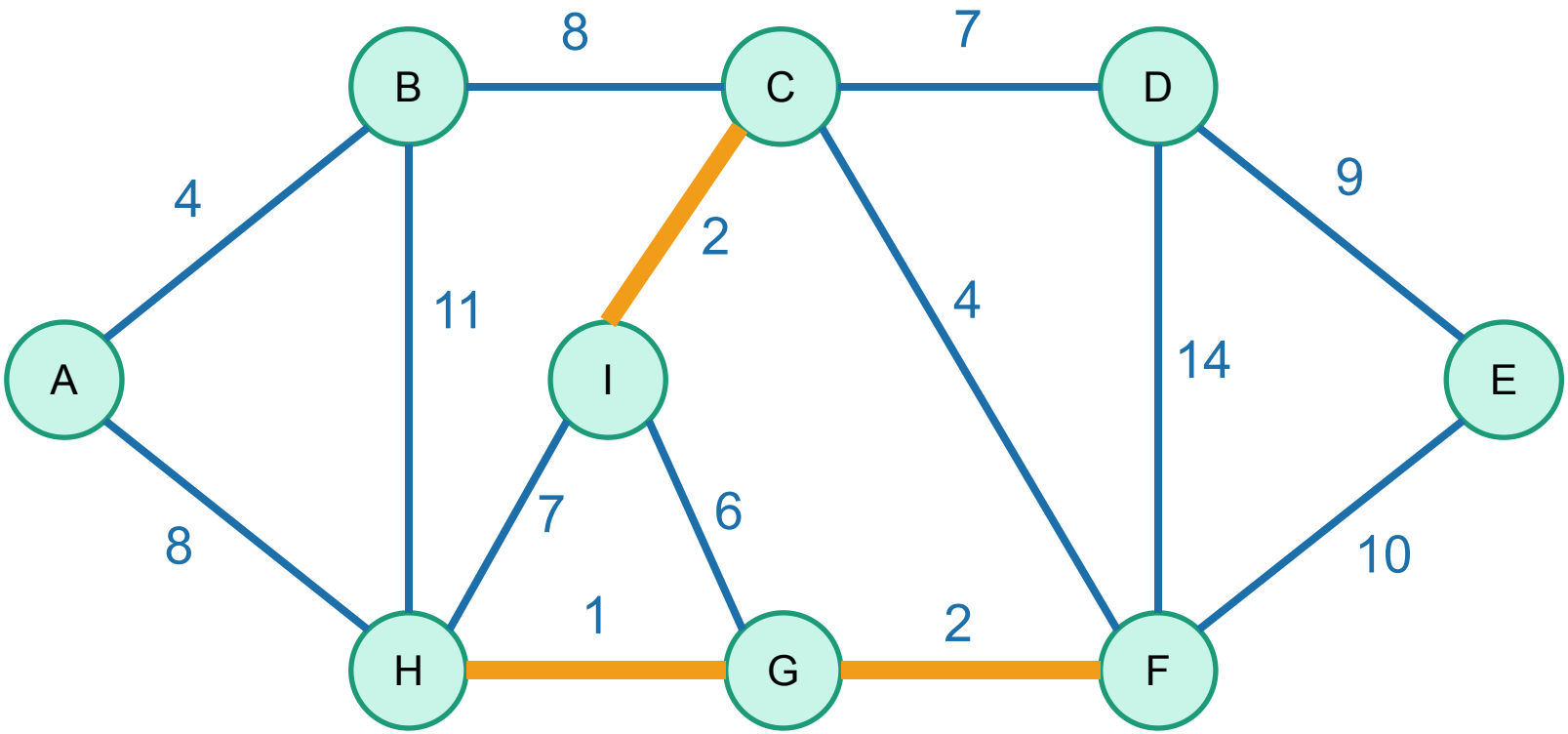

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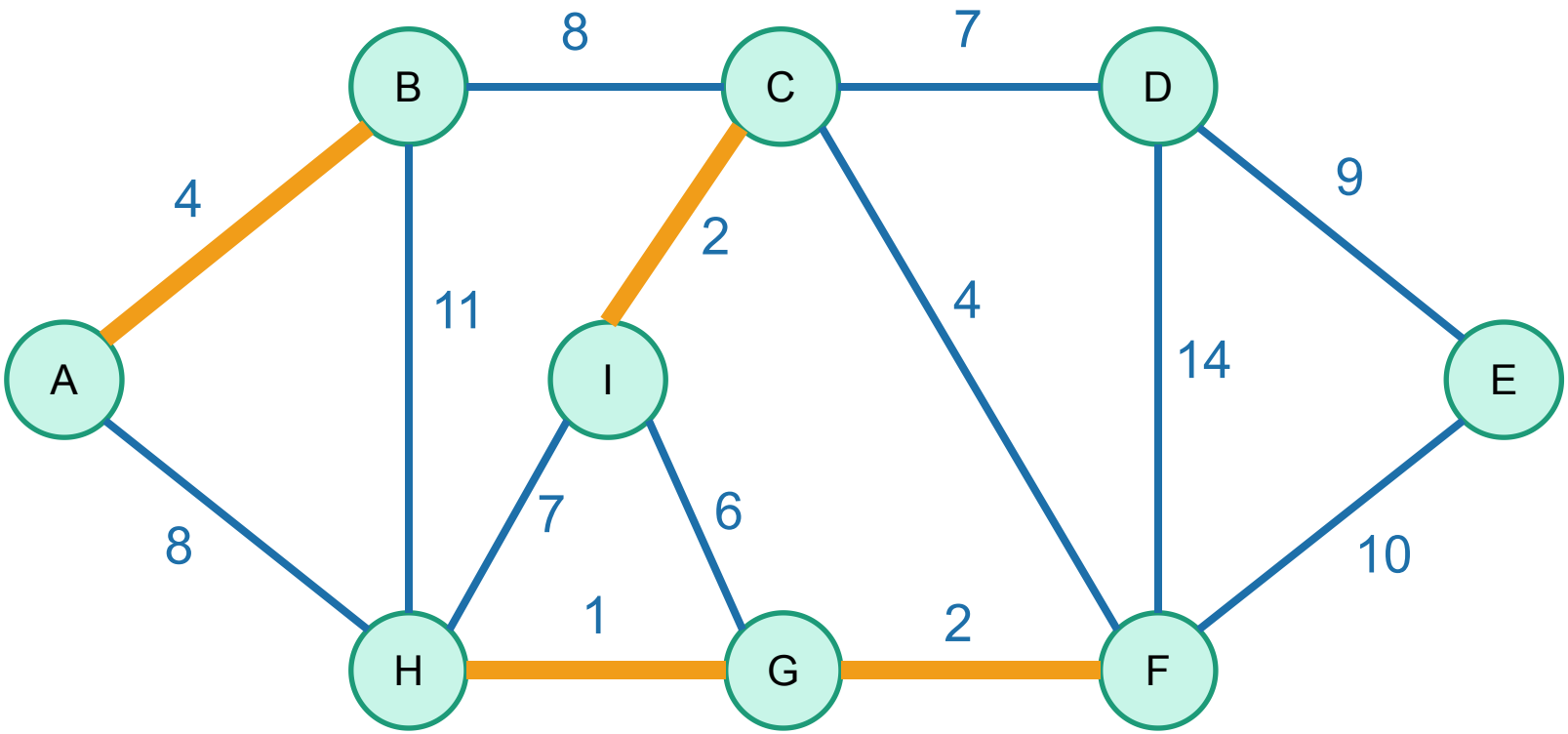

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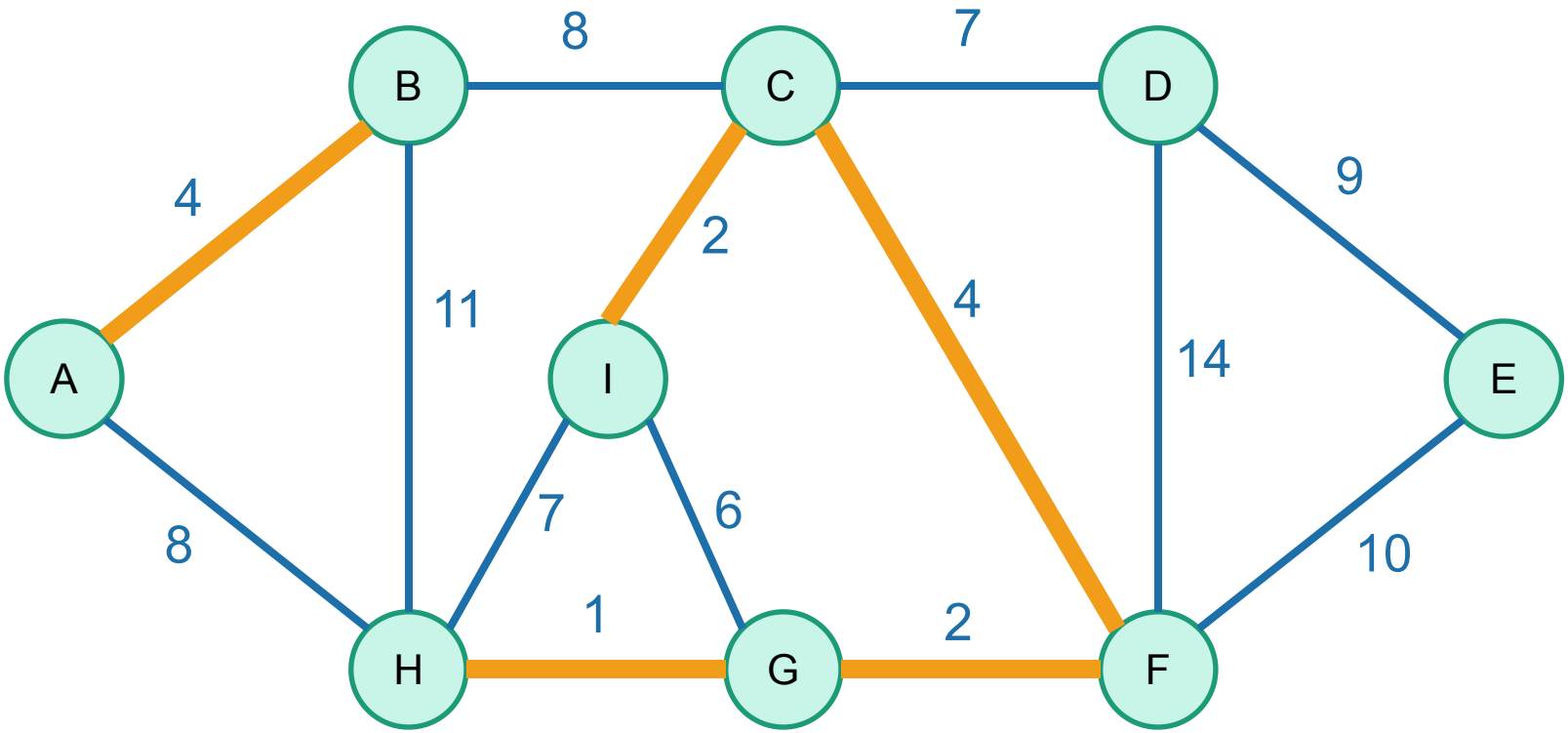

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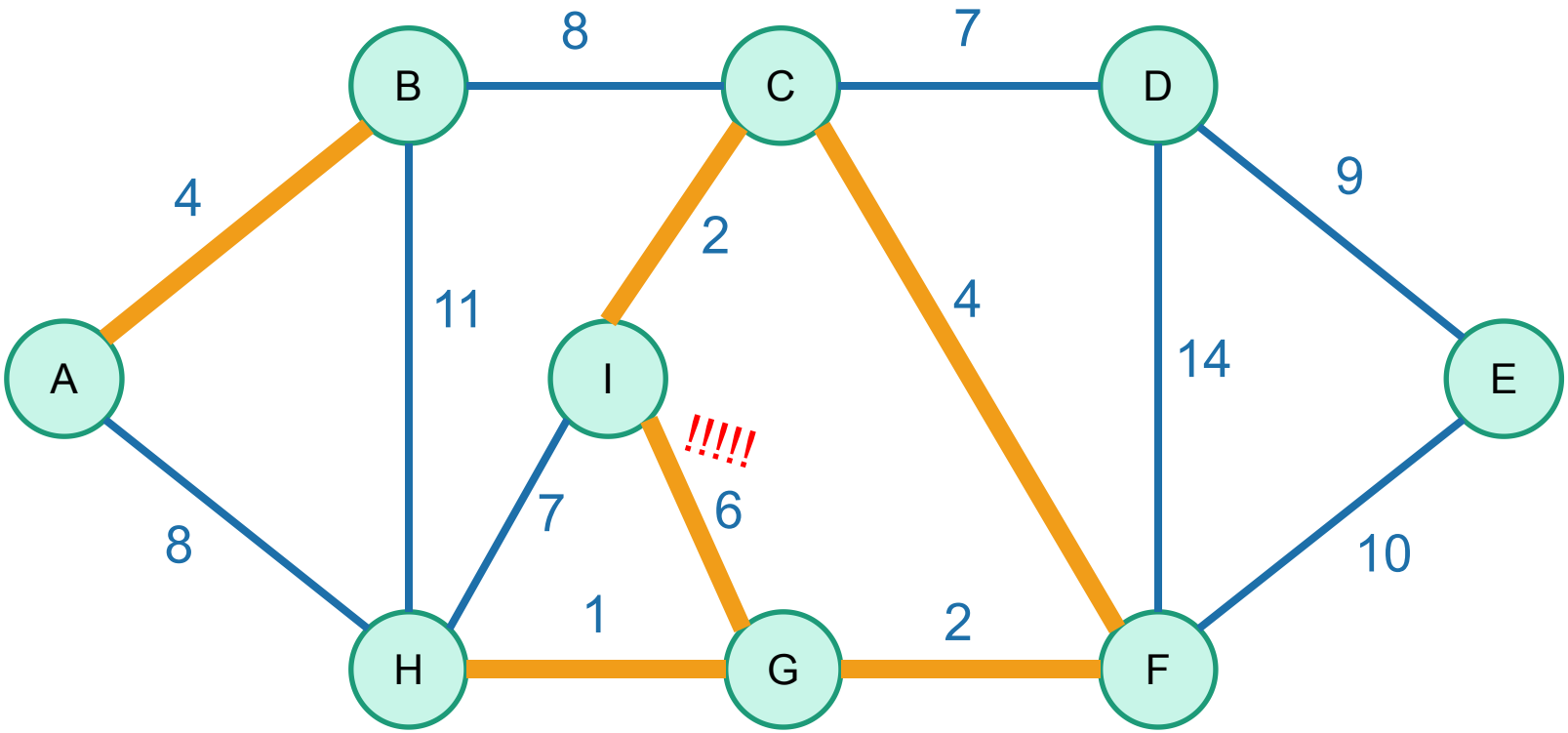

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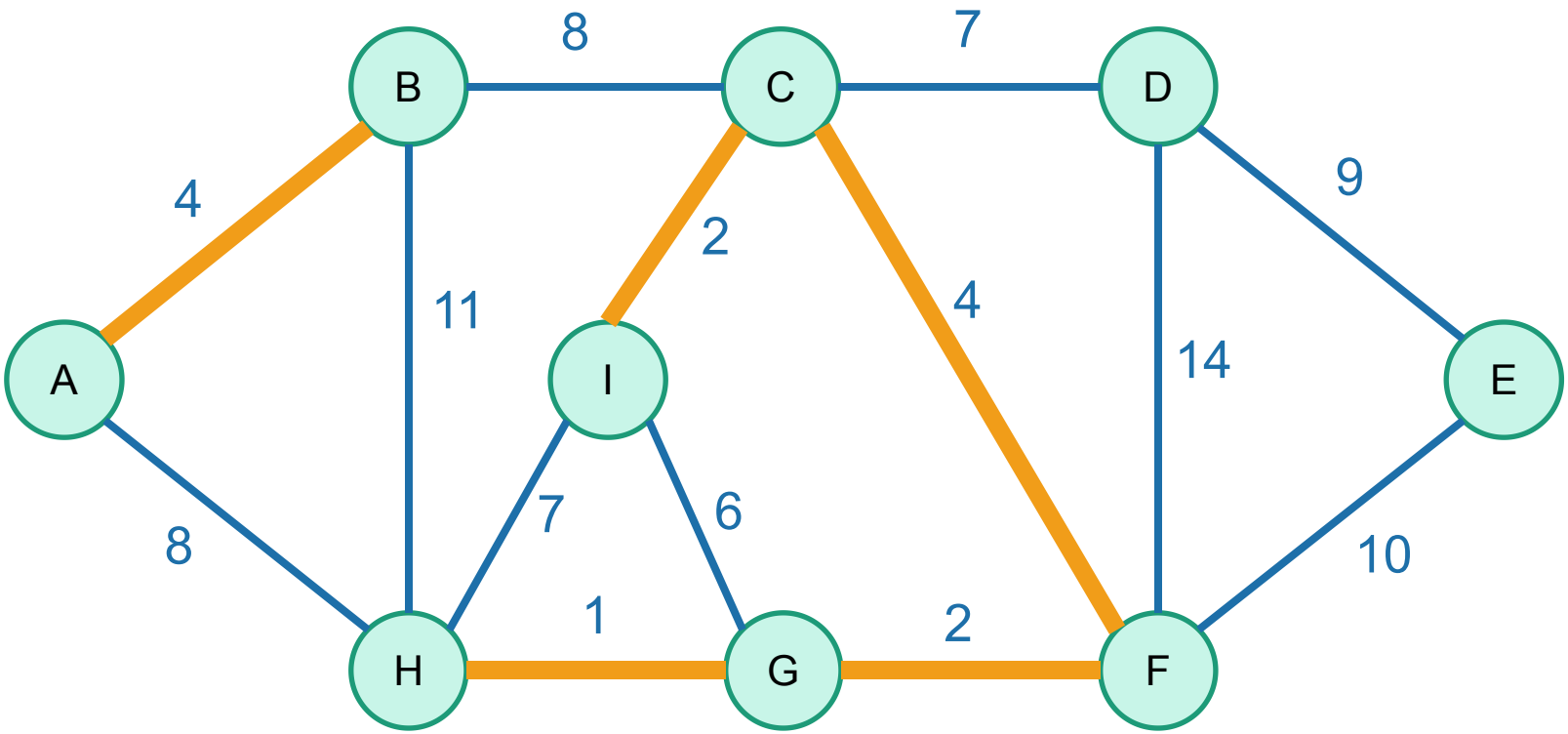

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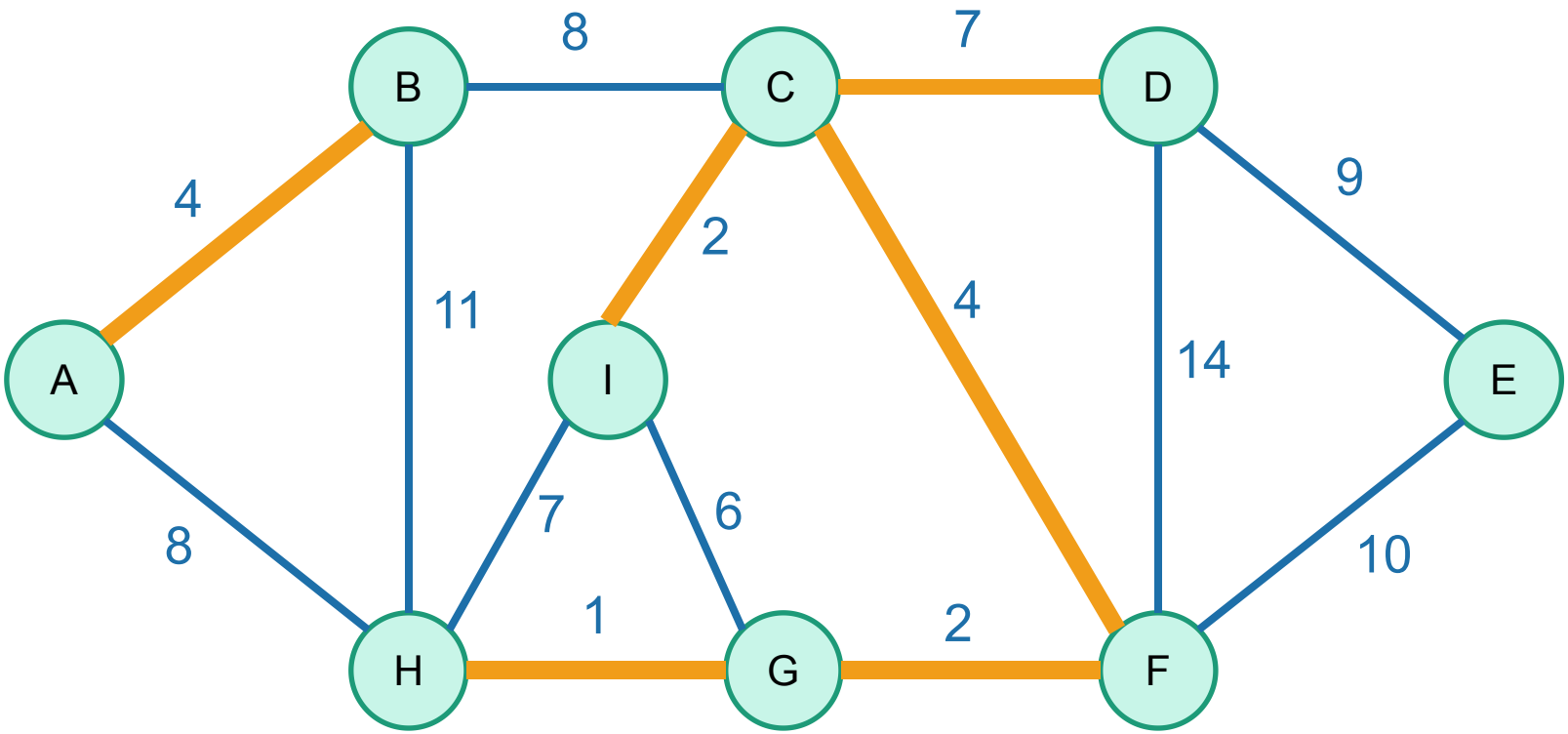

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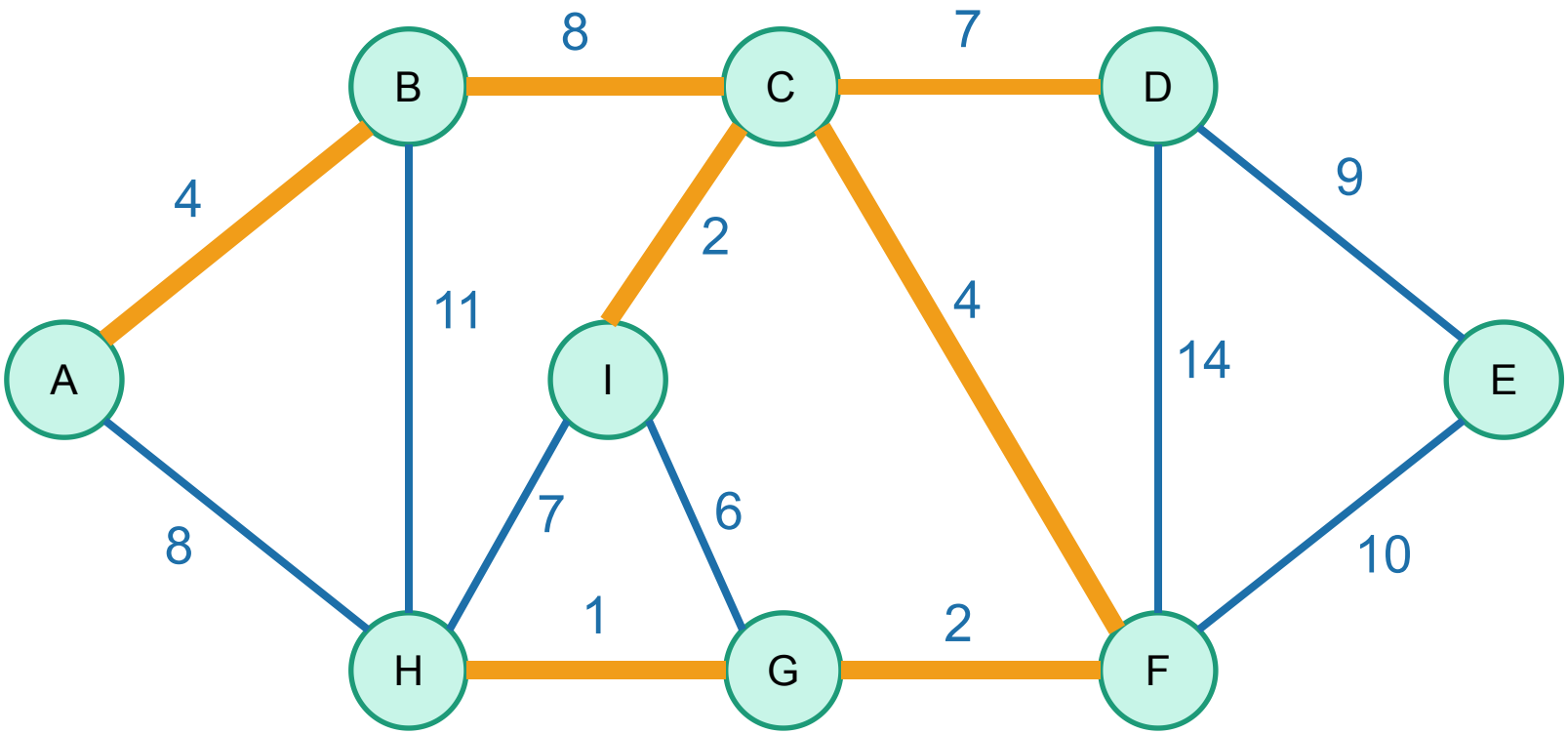

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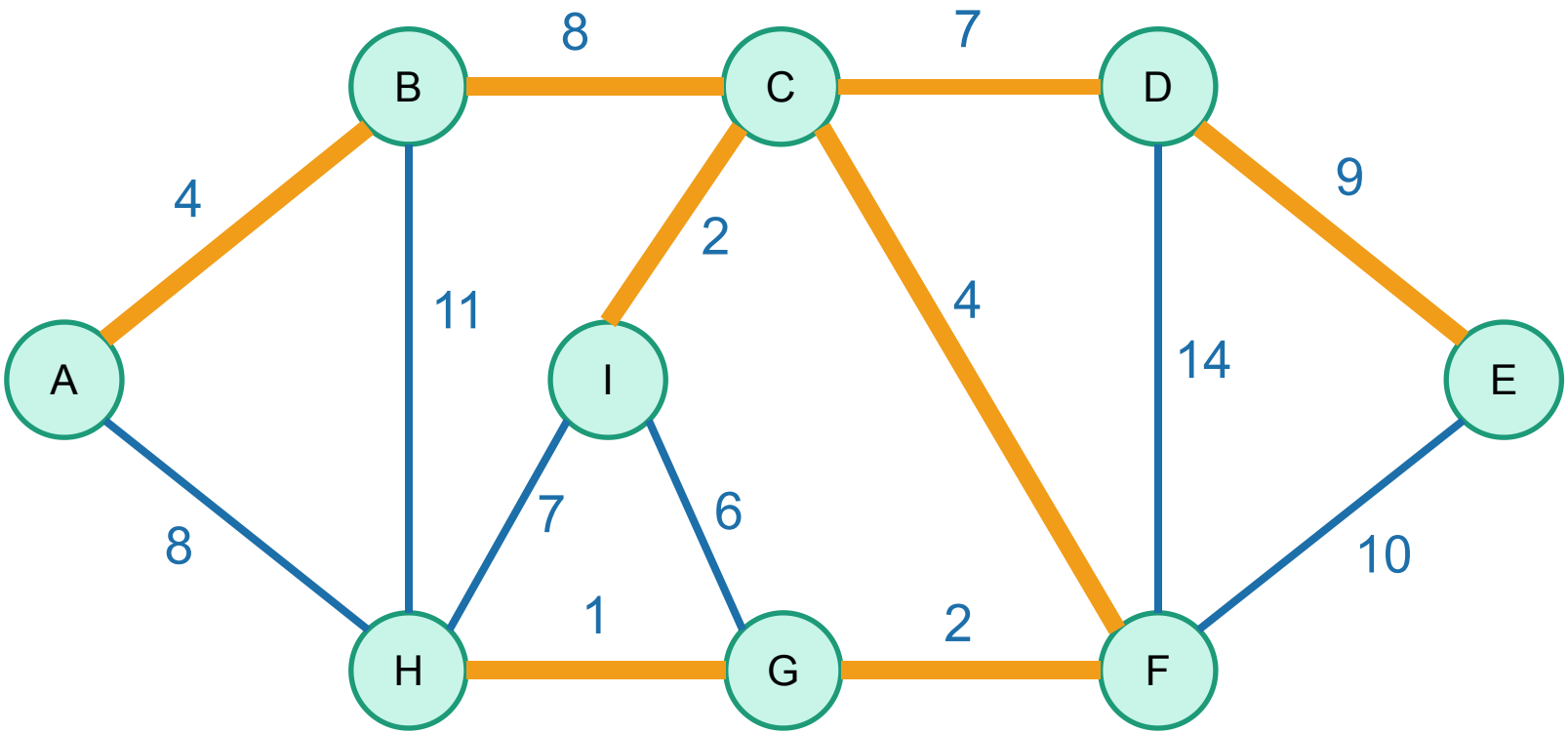


We've discovered

Kruskal's algorithm!

- **slowKruskal**(G = (V,E)):
  - Sort the edges in E by non-decreasing weight.
  - MST = {}
  - **for** e in E (in sorted order):
    - **if** adding e to MST won't cause a cycle:
      - add e to MST.
  - **return** MST

m iterations through this loop

How do we check this?

How **would** you figure out if added e would make a cycle in this algorithm?

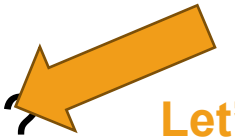
Naively, the running time is ???:

- For each of m iterations of the for loop:
  - Check if adding e would cause a cycle…

Two questions

1. Does it work?
   - That is, does it actually return a MST?

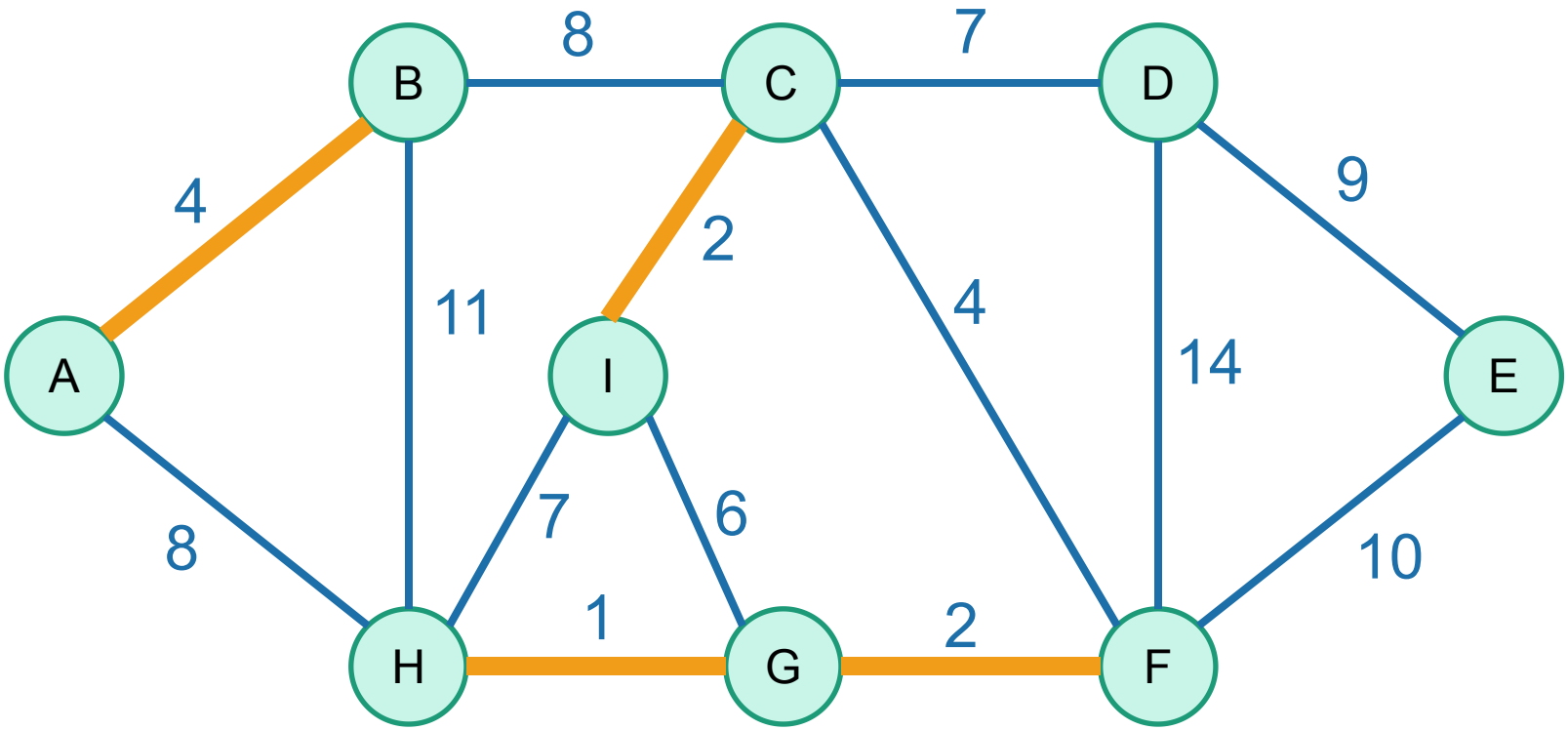
2. How do we actually implement this?
   - the pseudocode above says “slowKruskal”…



Let’s do this one first

At each step of Kruskal's, we are maintaining a forest.

A **forest** is a collection of disjoint trees



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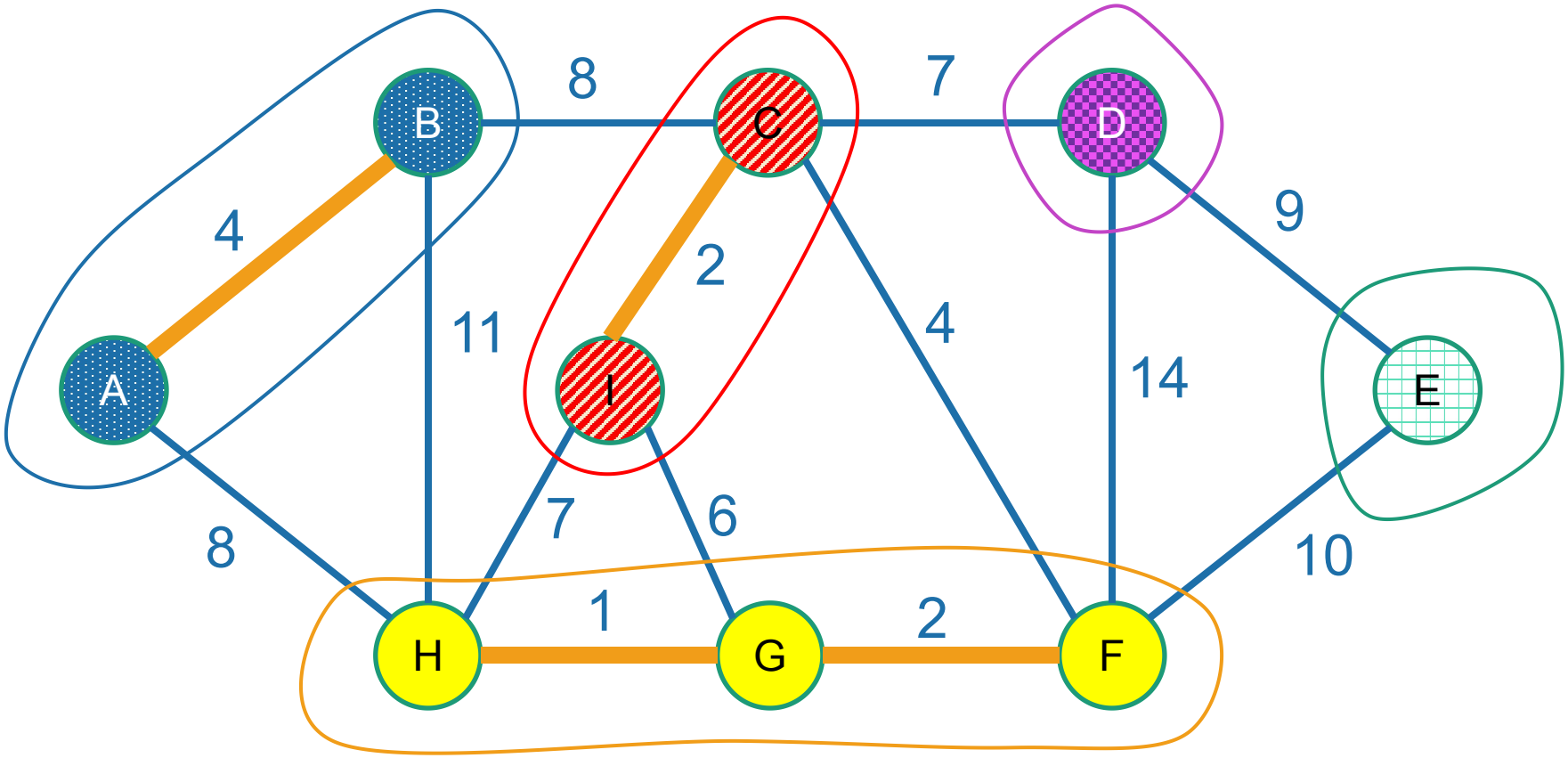

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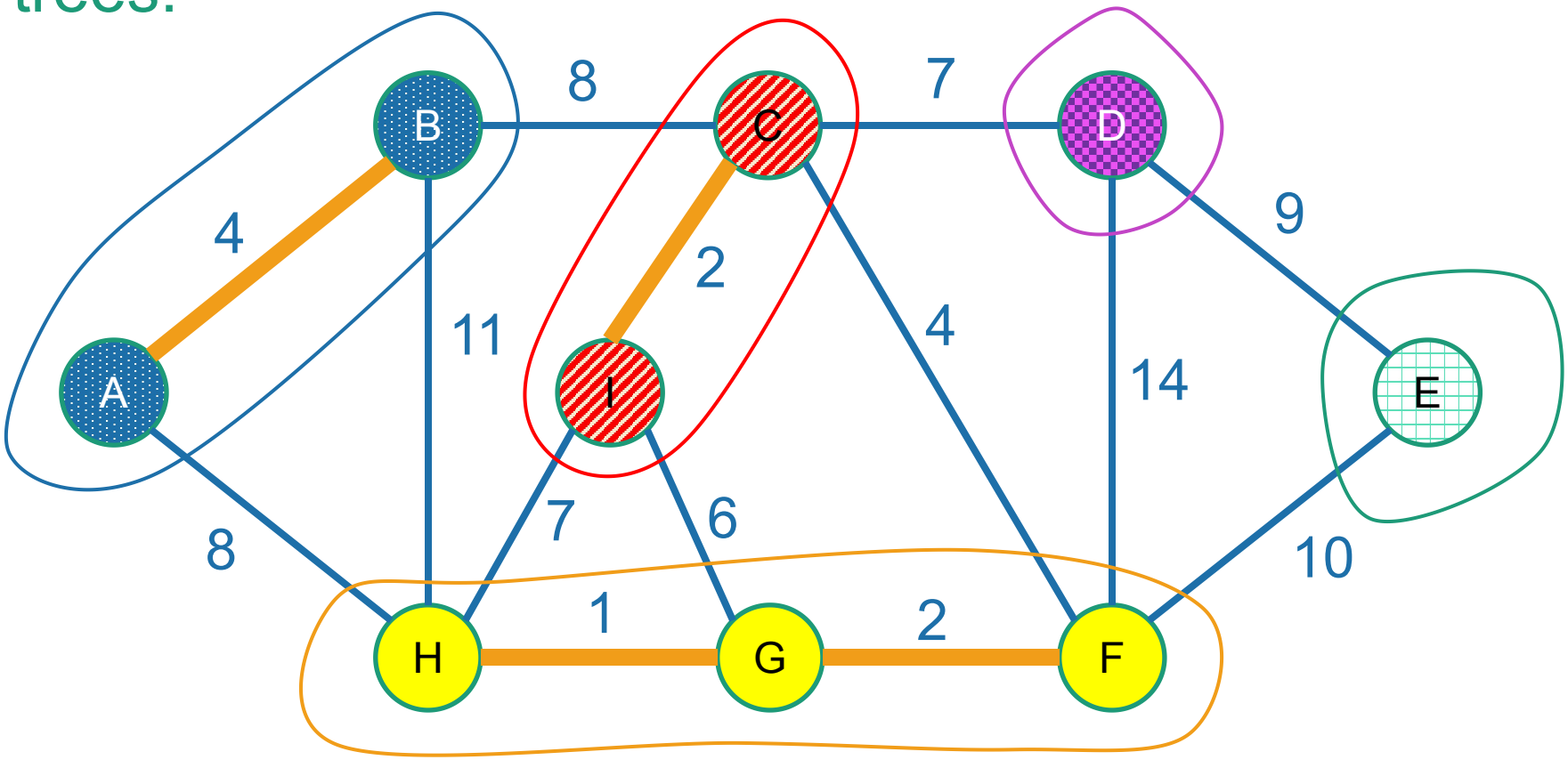

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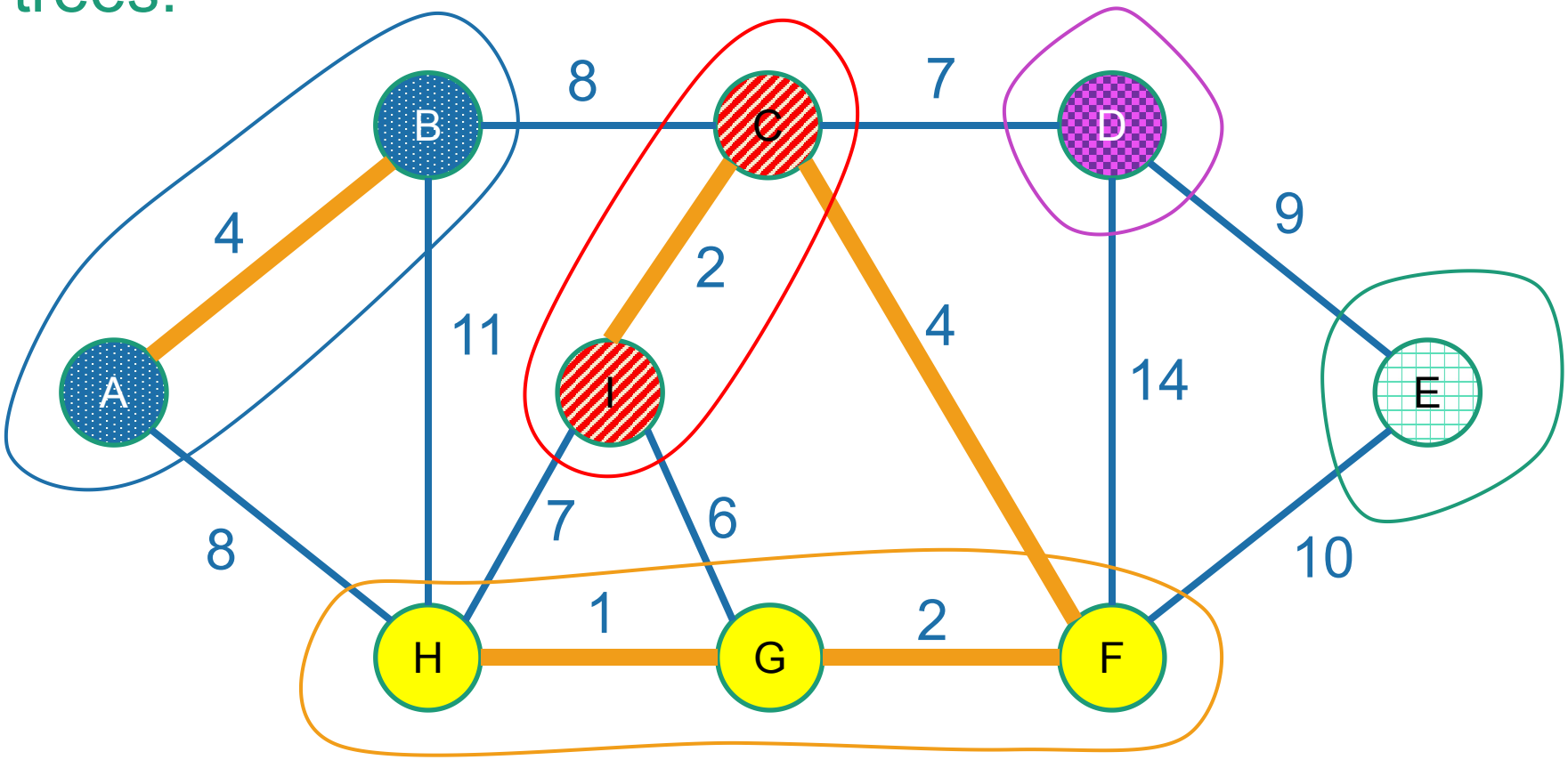

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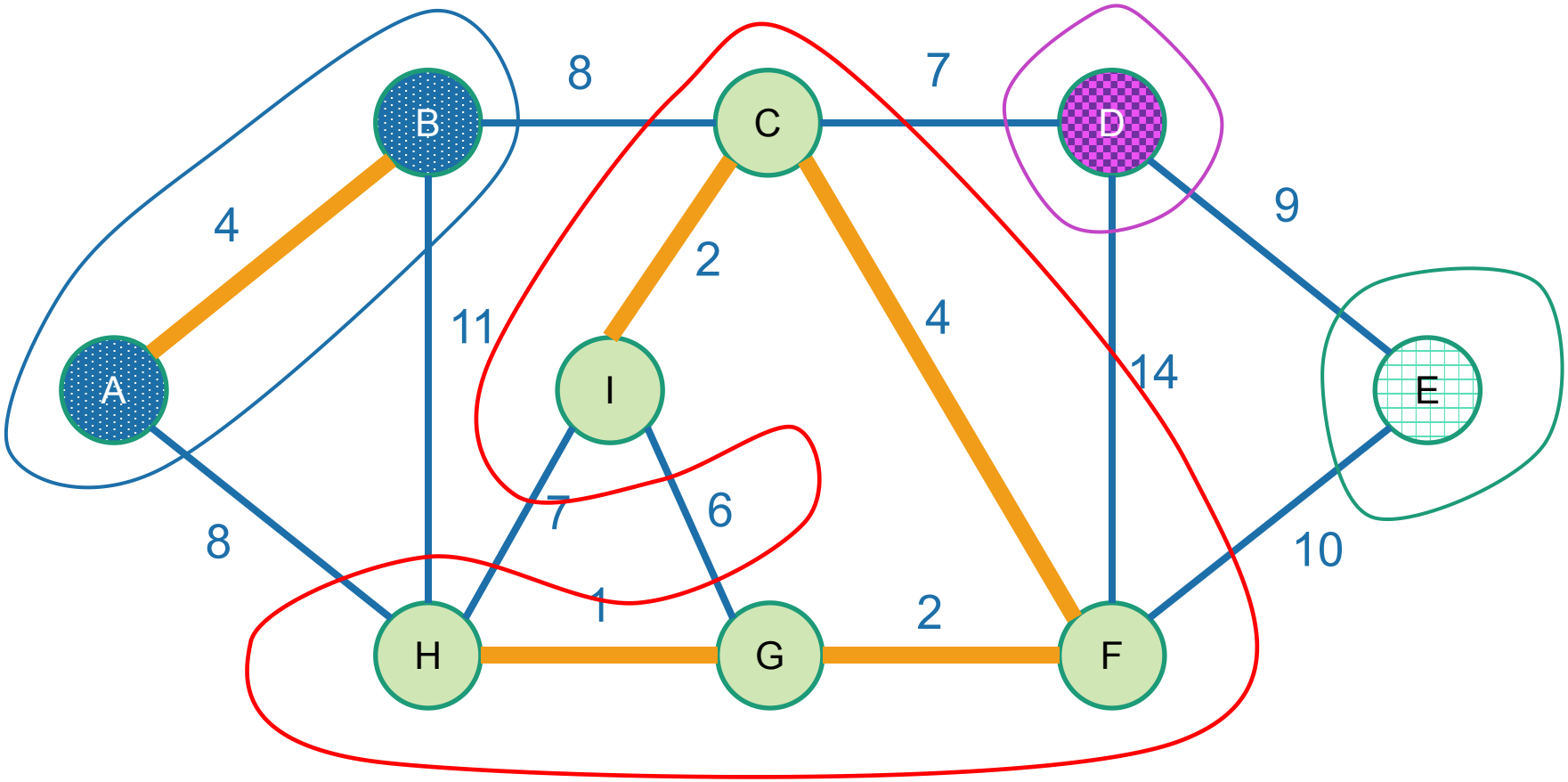


We never add an edge within a tree since that would create a cycle.

Keep the trees in a special data structure

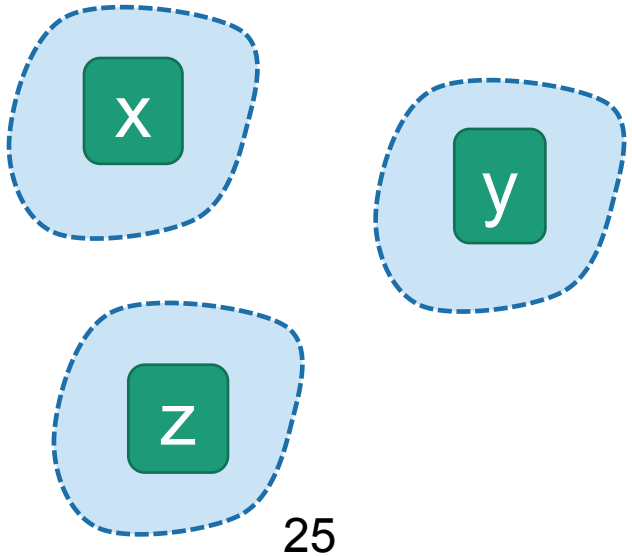
"treehouse"?

Union-find data structure
also called disjoint-set data structure

- Used for storing collections of sets
- Supports:
  - **makeSet(u):** create a set {u}
  - **find(u):** return the set that u is in
  - **union(u,v):** merge the set that u is in with the set that v is in.

makeSet(x)
makeSet(y)
makeSet(z)

union(x,y)



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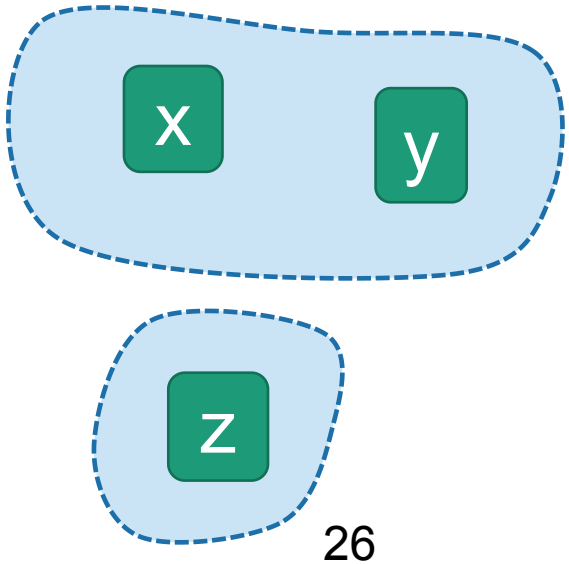

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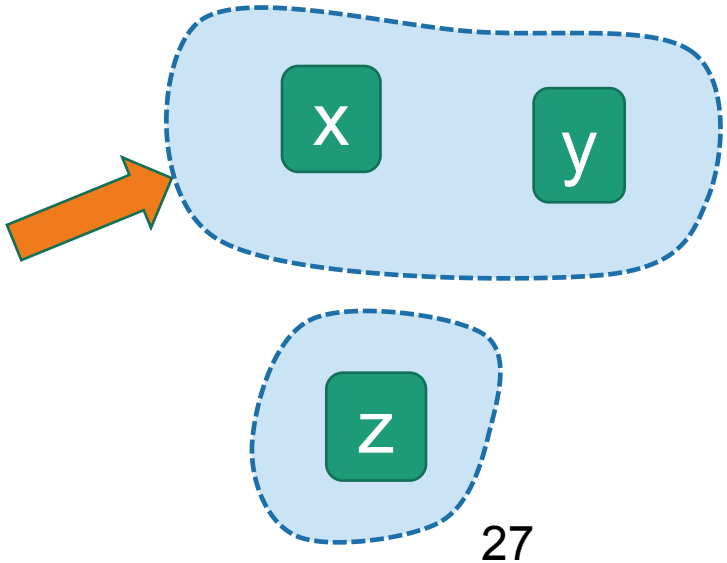
find(x)

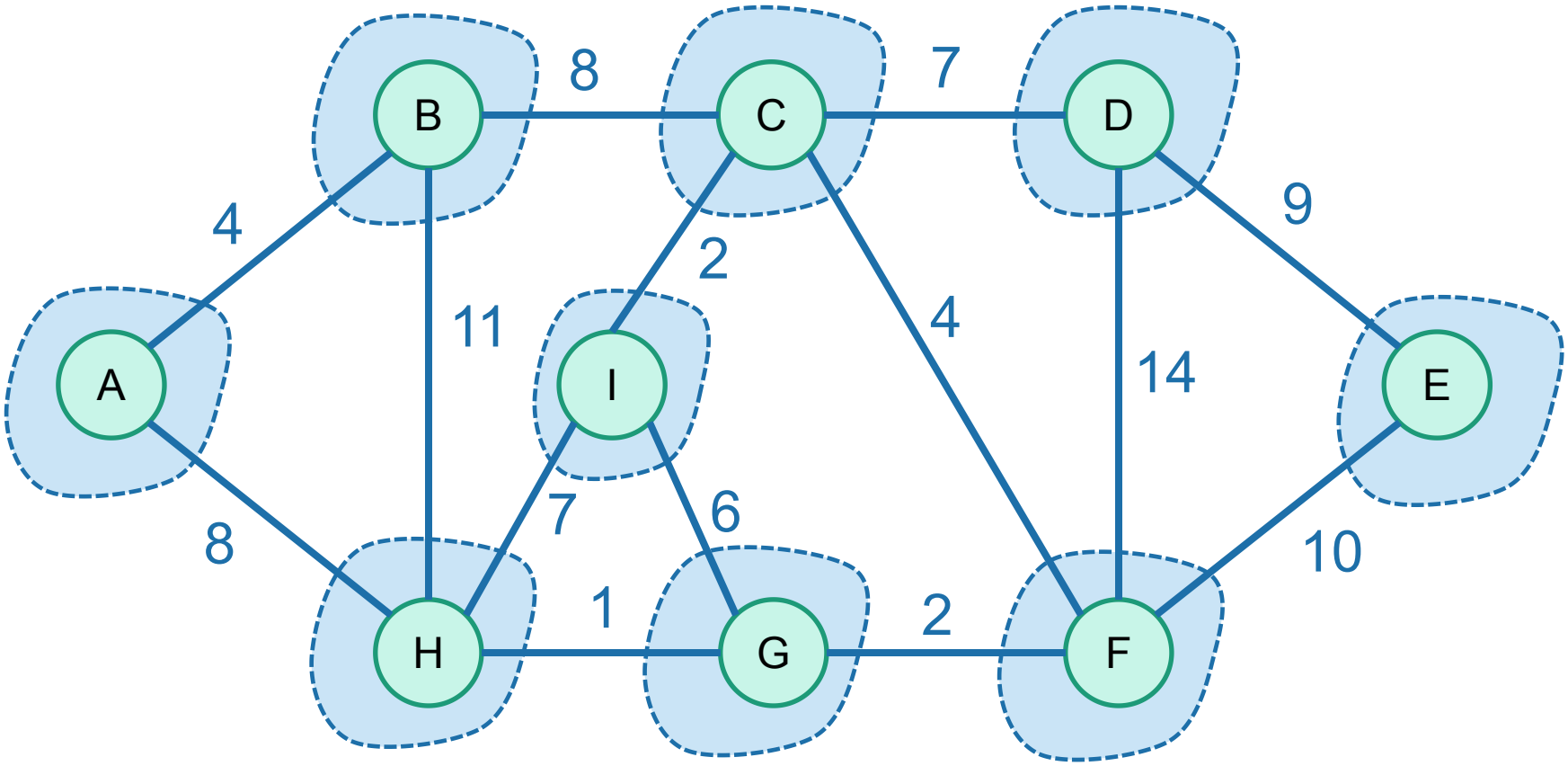


Kruskal pseudo-code

- **kruskal**(G = (V,E)):
  - Sort E by weight in non-decreasing order
  - MST = {} // initialize an empty tree
  - **for** v in V:
    - **makeSet**(v) // put each vertex in its own tree in the forest
  - **for** (u,v) in E: // go through the edges in sorted order
    - **if find**(u) != **find**(v): // if u and v are not in the same tree
      - add (u,v) to MST
      - **union**(u,v) // merge u's tree with v's tree
  - **return** MST

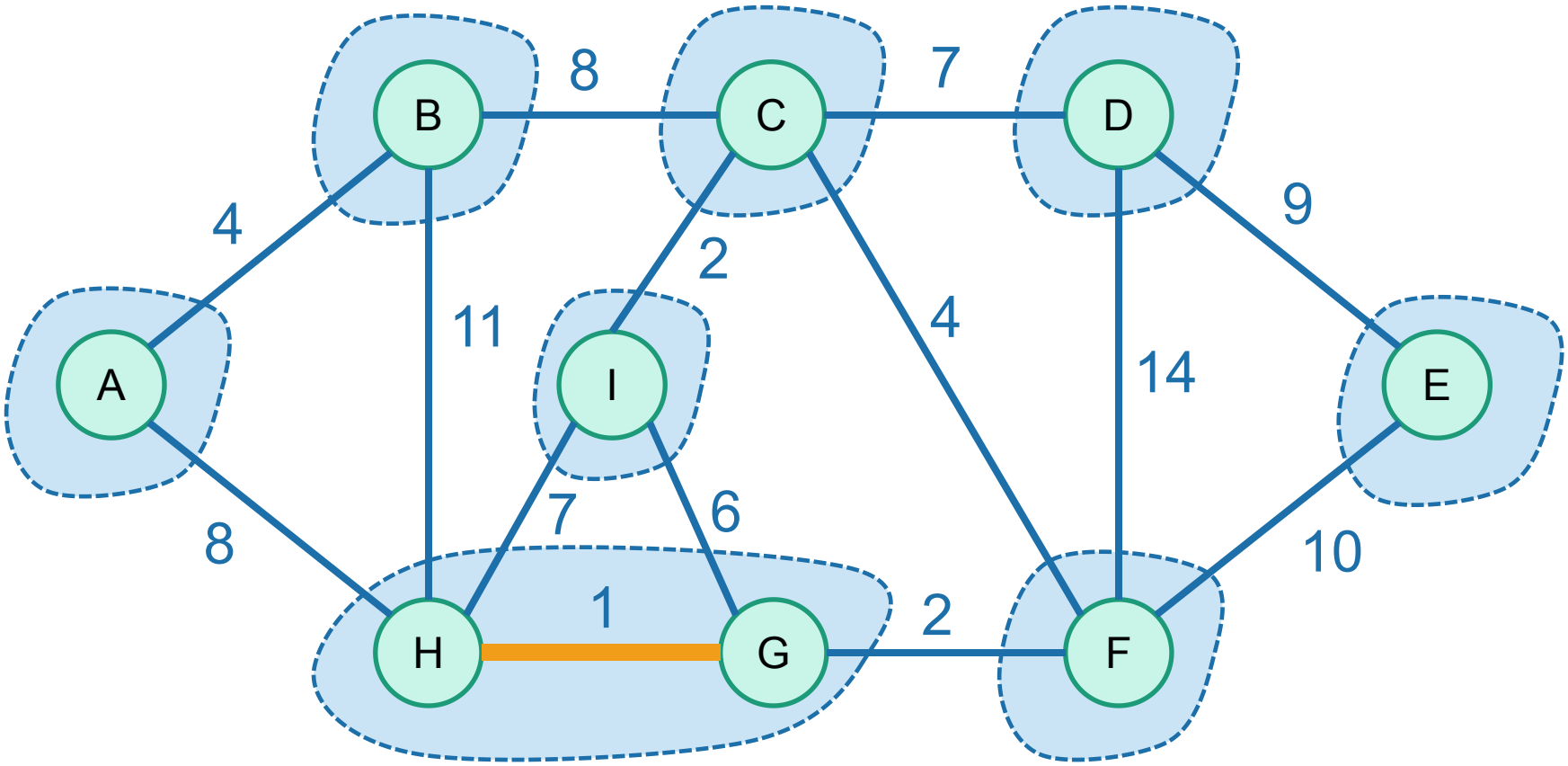
Once more…

To start, every vertex is in its own tree.



Once more…

Then start merging.



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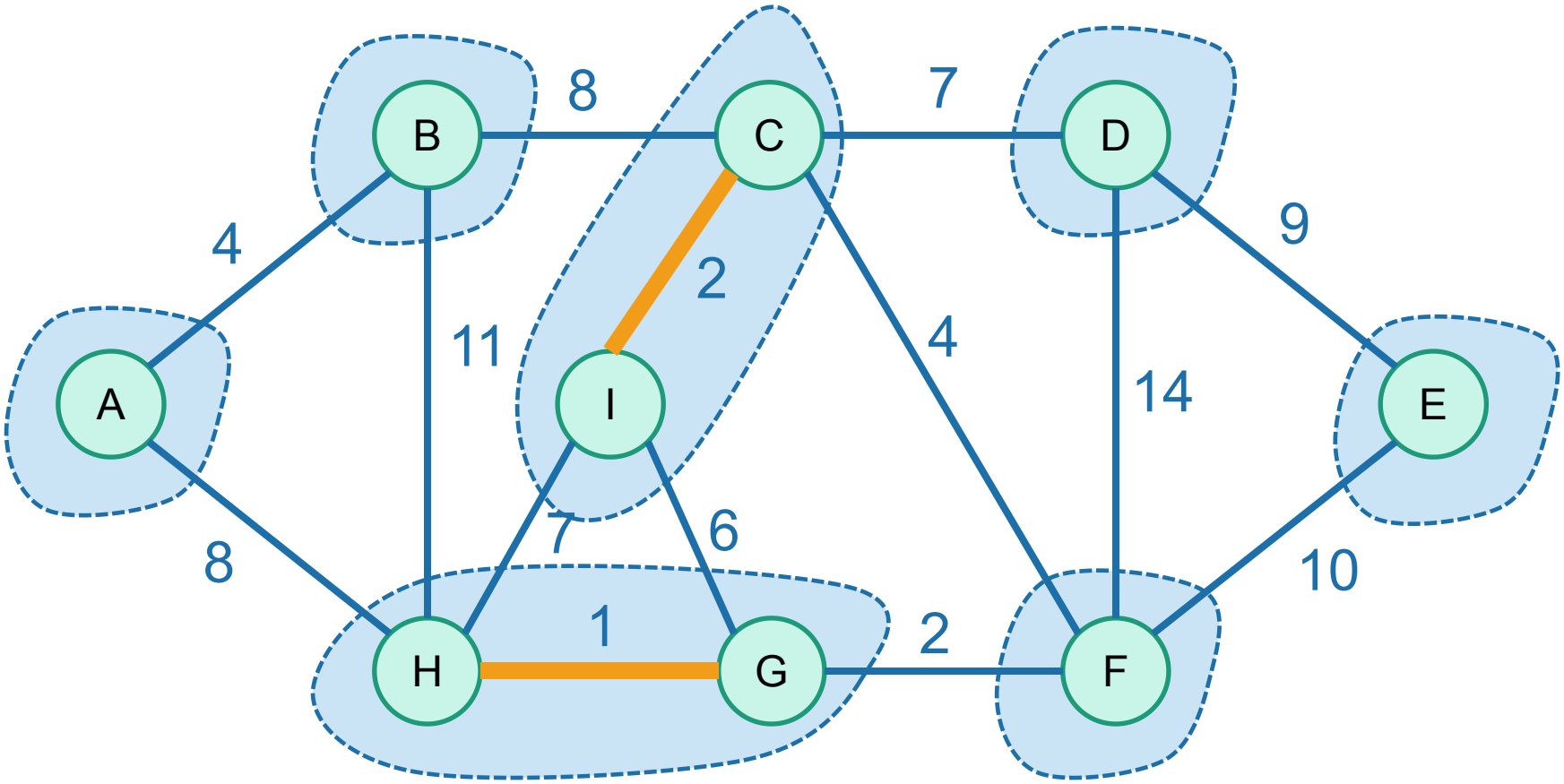

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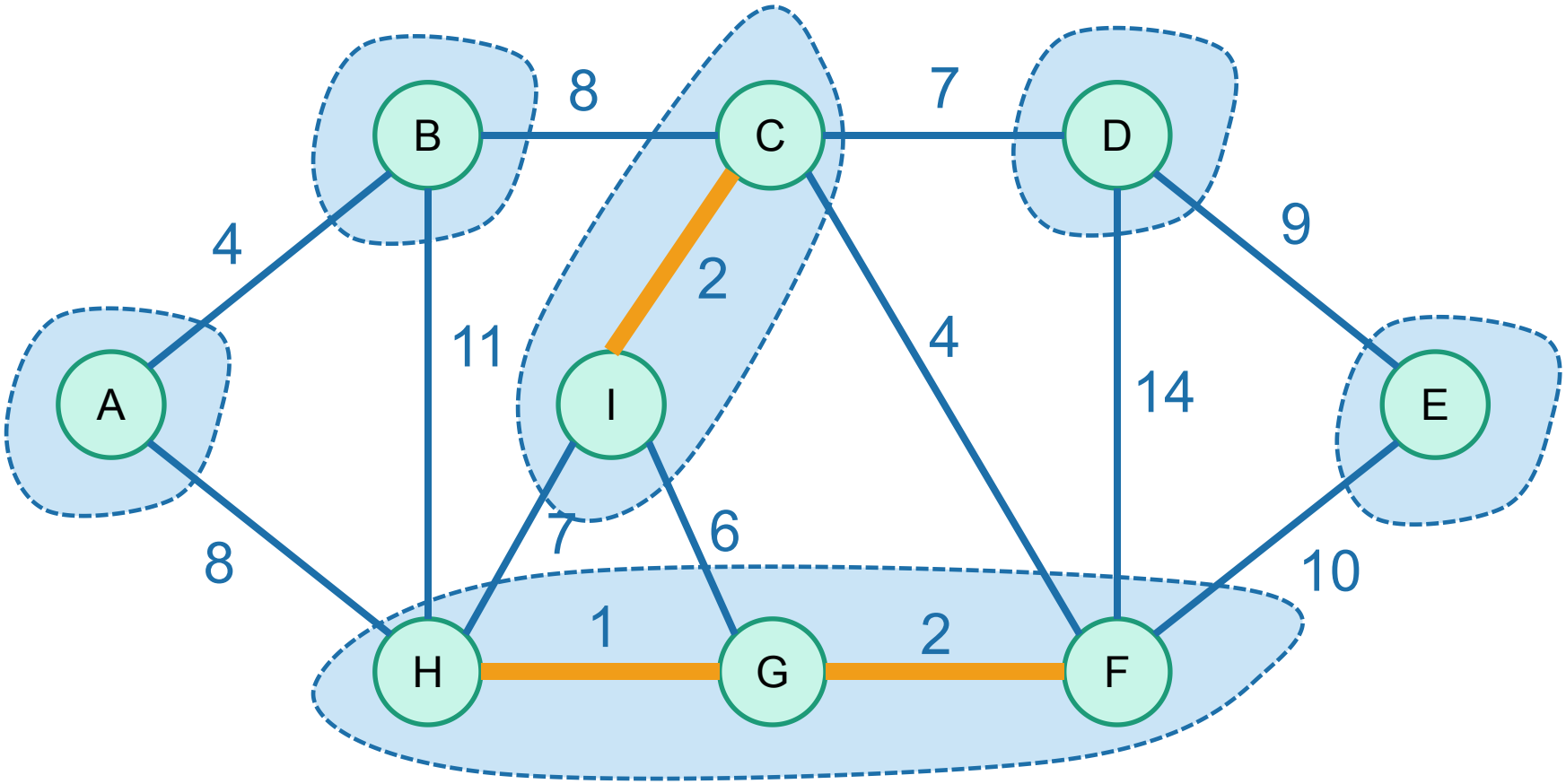

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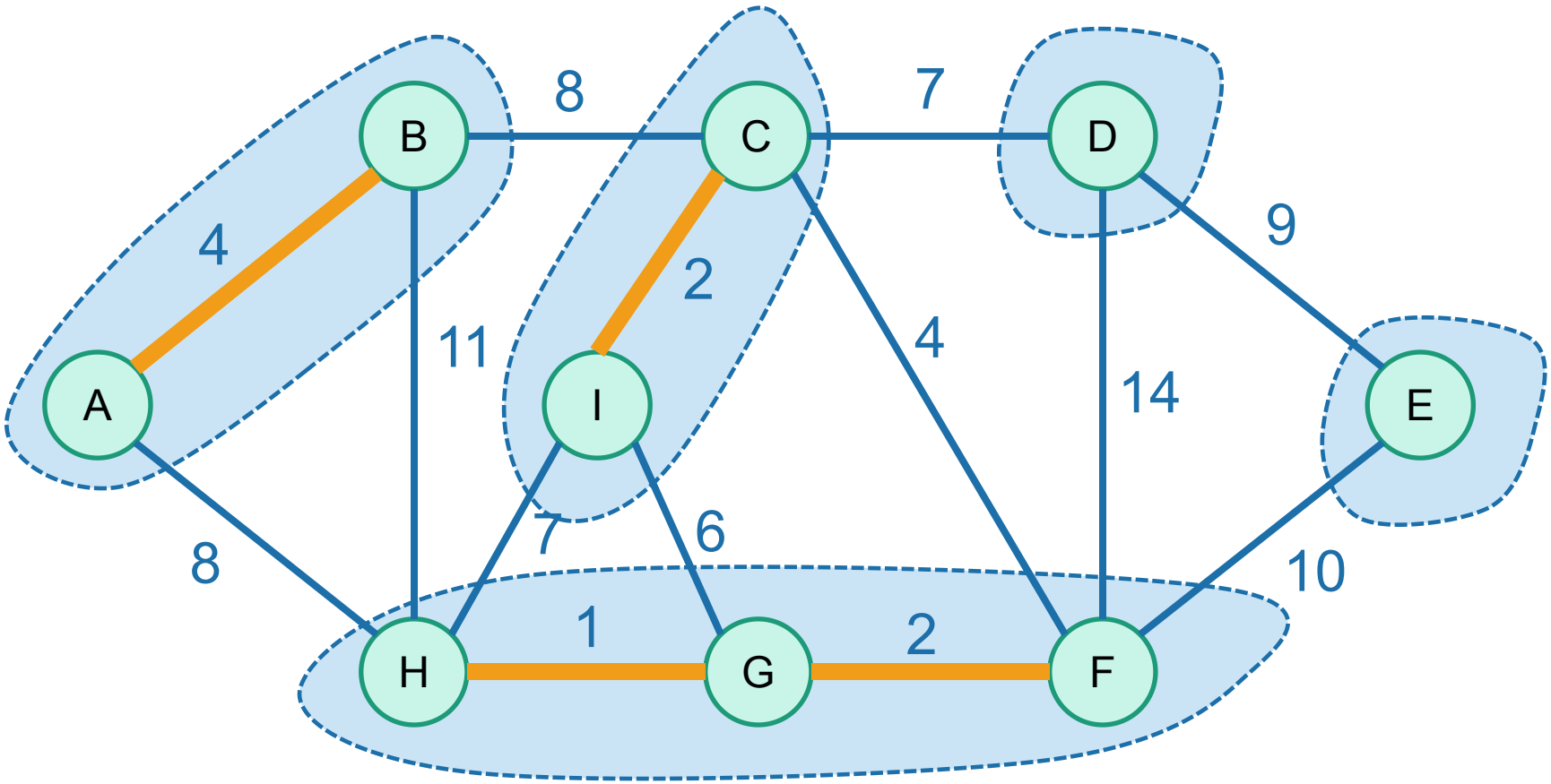

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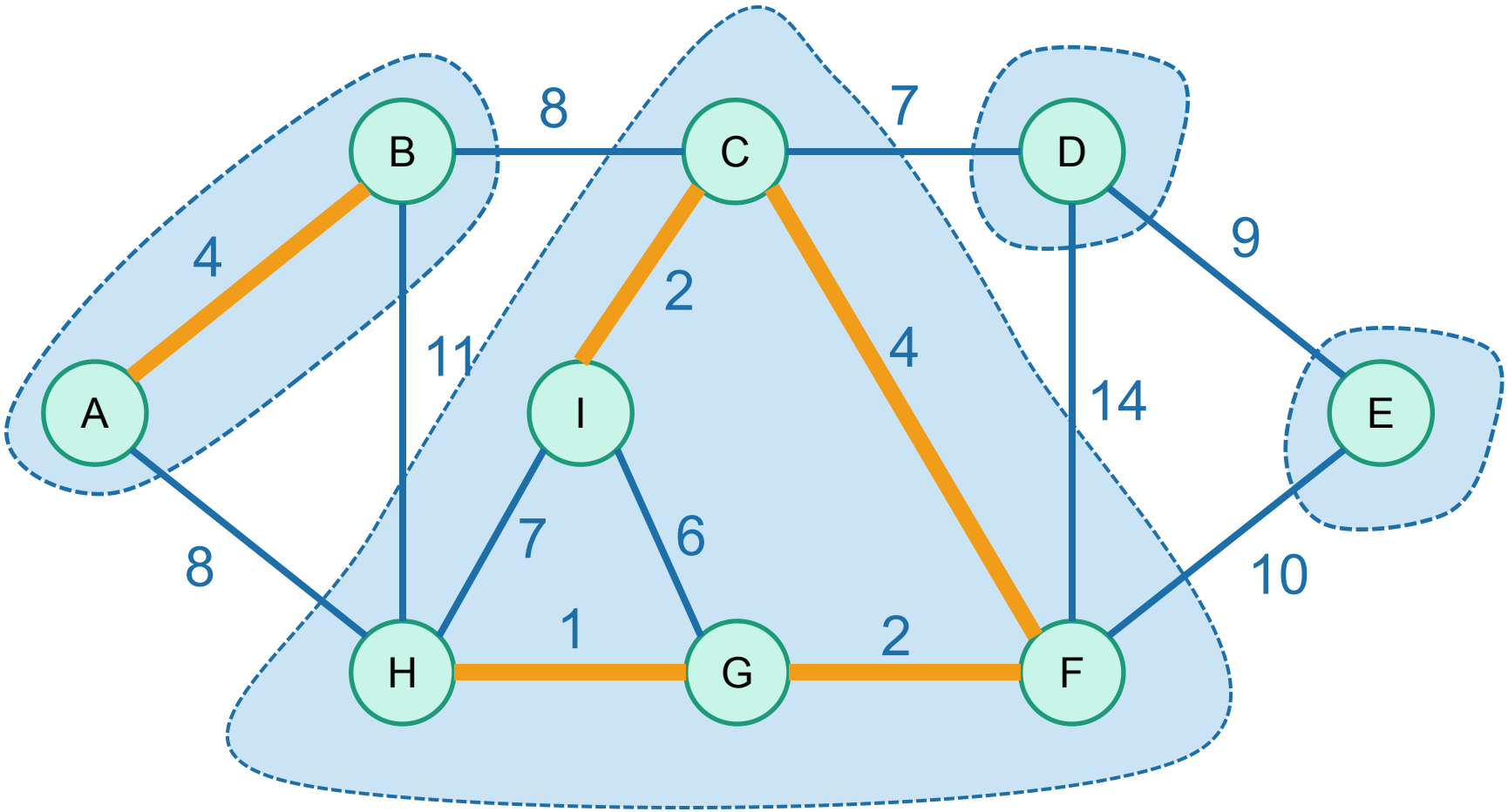

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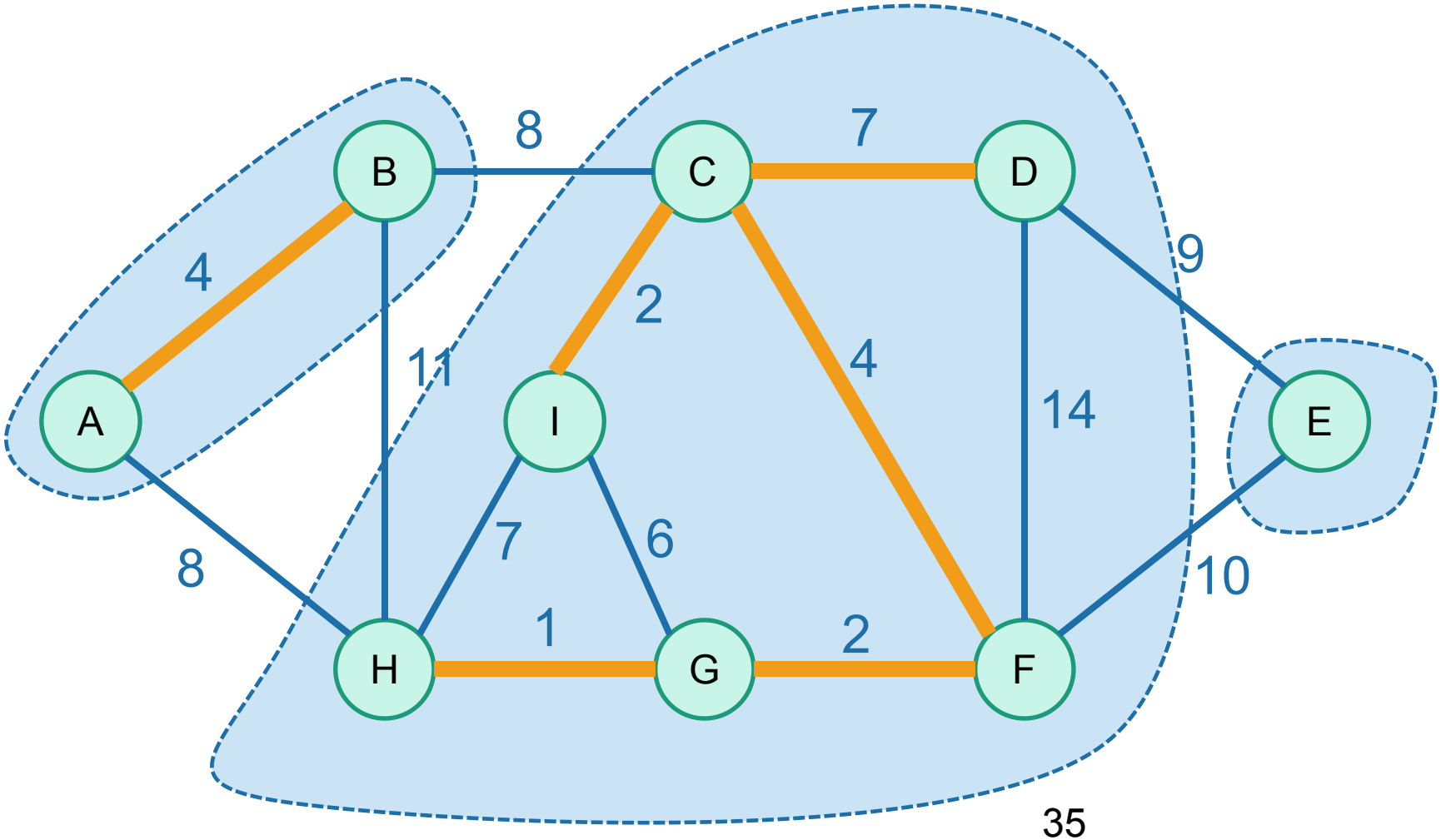

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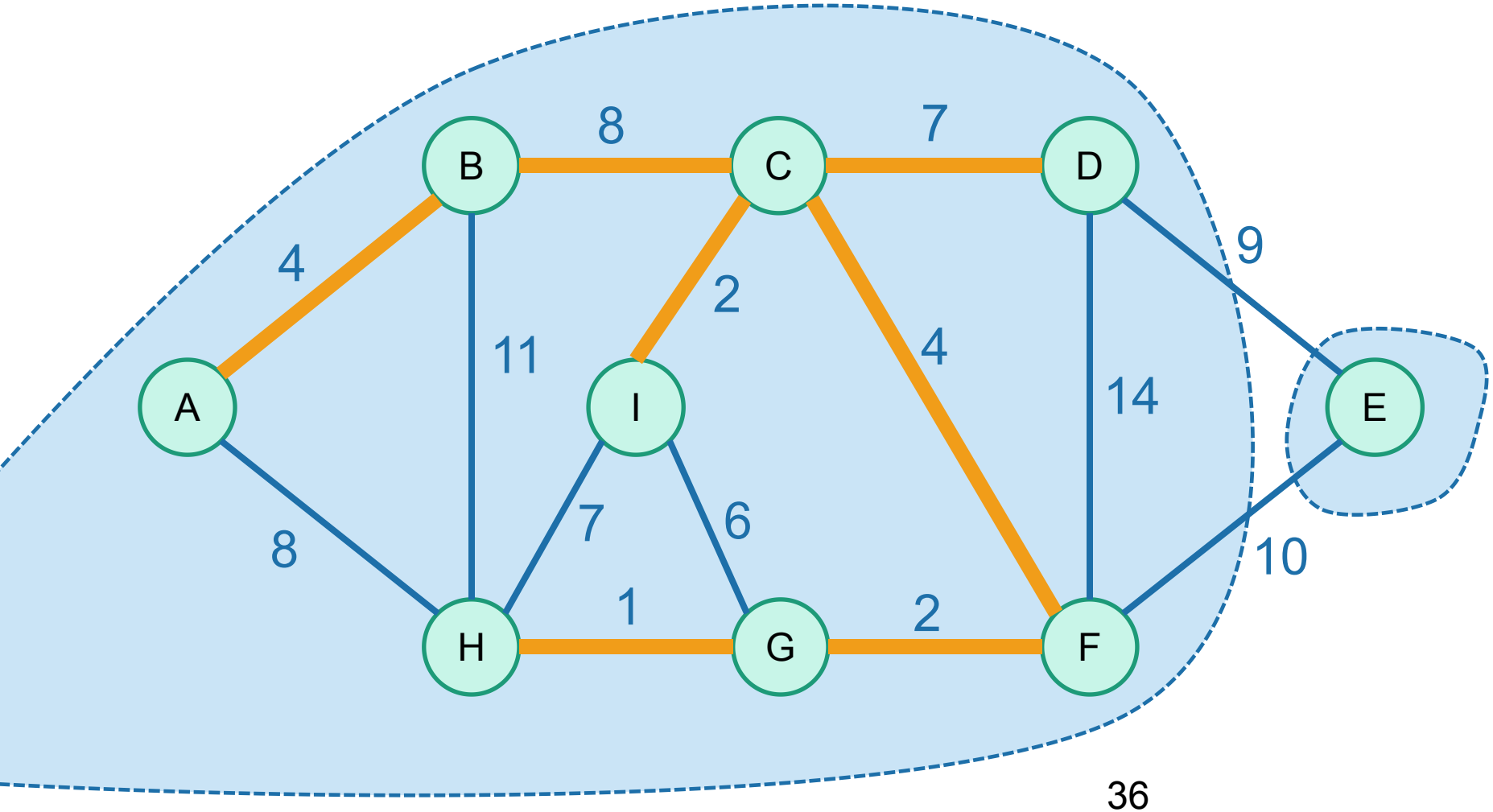
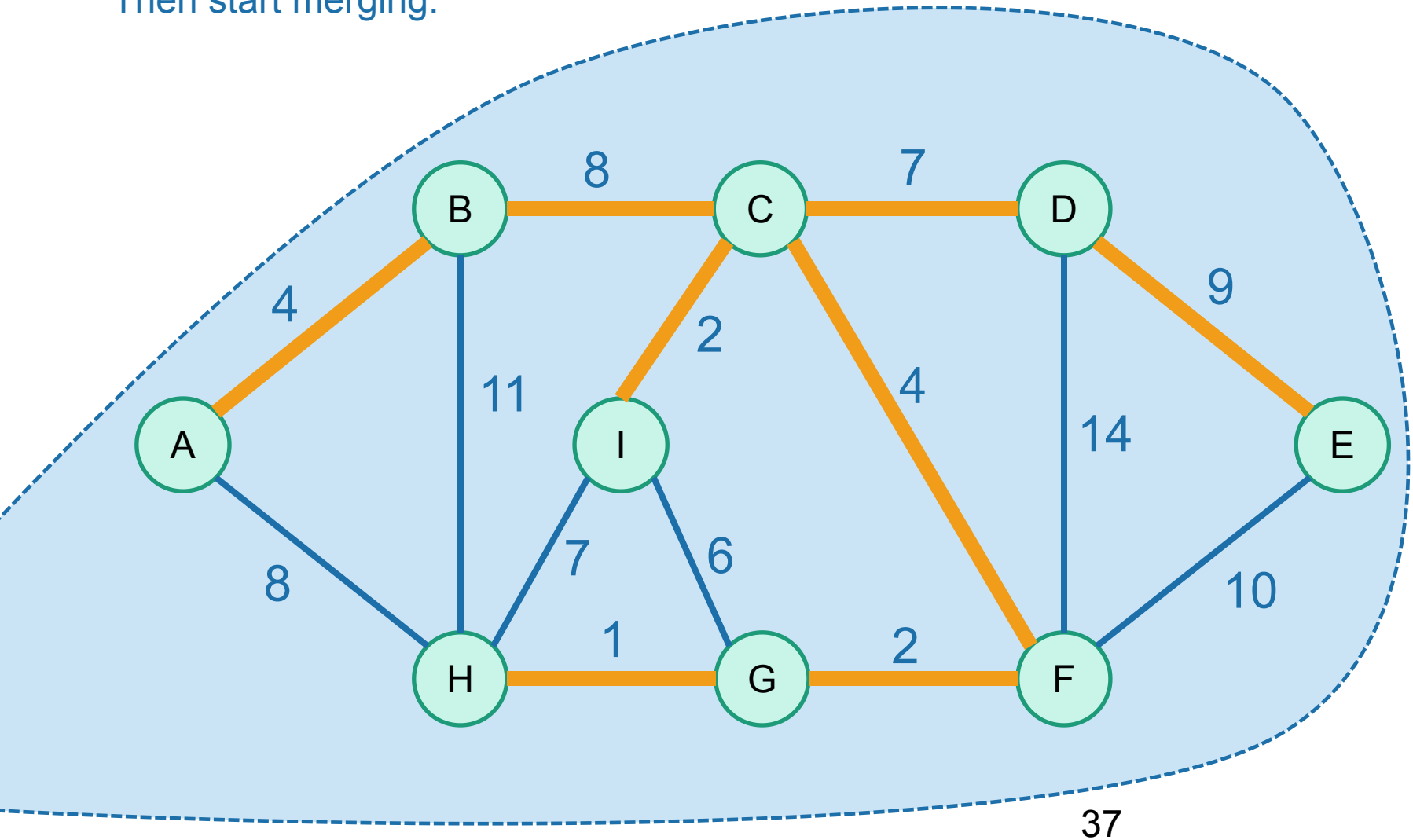


Once more…

Stop when we have one big tree!

Then start merging.



Running time

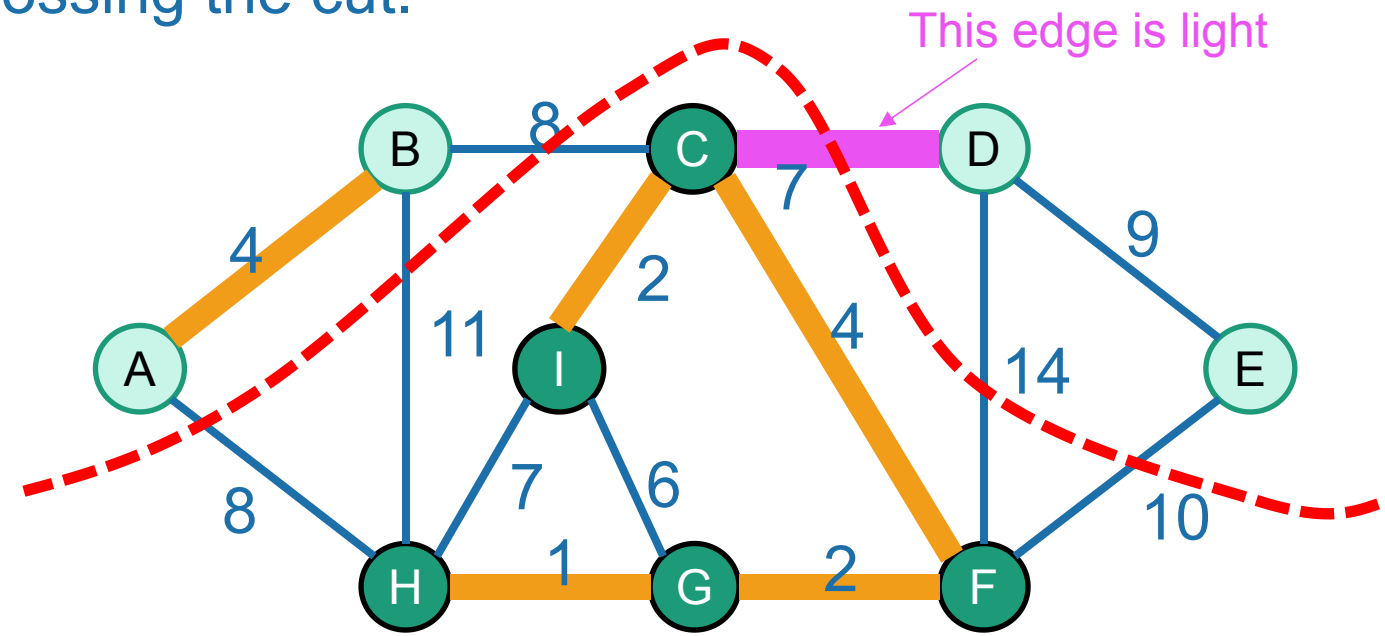
- Sorting the edges takes $O(m \log(n))$
  - In practice, if the weights are small integers we can use radixSort and take time $O(m)$
- For the rest:
  - n calls to **makeSet**
    - put each vertex in its own set
  - $2m$ calls to **find**
    - for each edge, **find** its endpoints
  - n calls to **union**
    - we will never add more than $n - 1$ edges to the tree,
    - so we will never call **union** more than $n - 1$ times.
- Total running time:
  - Worst-case $O(mlog(n))$, just like Prim with an RBtree.

In practice, each of makeSet, find, and union run in constant time*

*technically, they run in *amortized time* $O(\alpha(n))$, where $\alpha(n)$ is the *inverse Ackermann function*. $\alpha(n) \leq 4$ provided that n is smaller than the number of atoms in the universe.

Both Prim and Kruskal

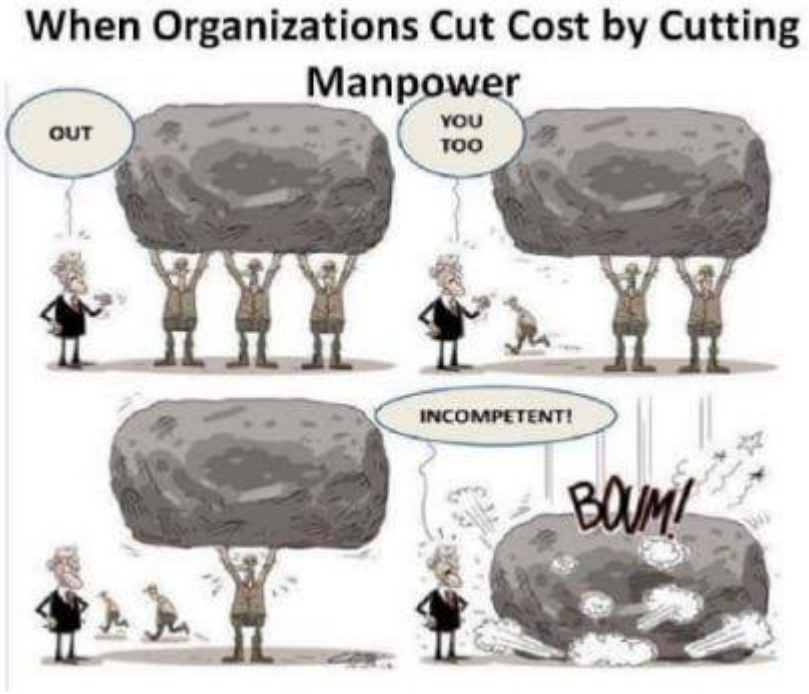
- Greedy algorithms for MST.
- Similar reasoning:
  - Optimal substructure: subgraphs generated by cuts.
  - The way to make safe choices is to choose light edges crossing the cut.



S is the set of **thick orange** edges

Today

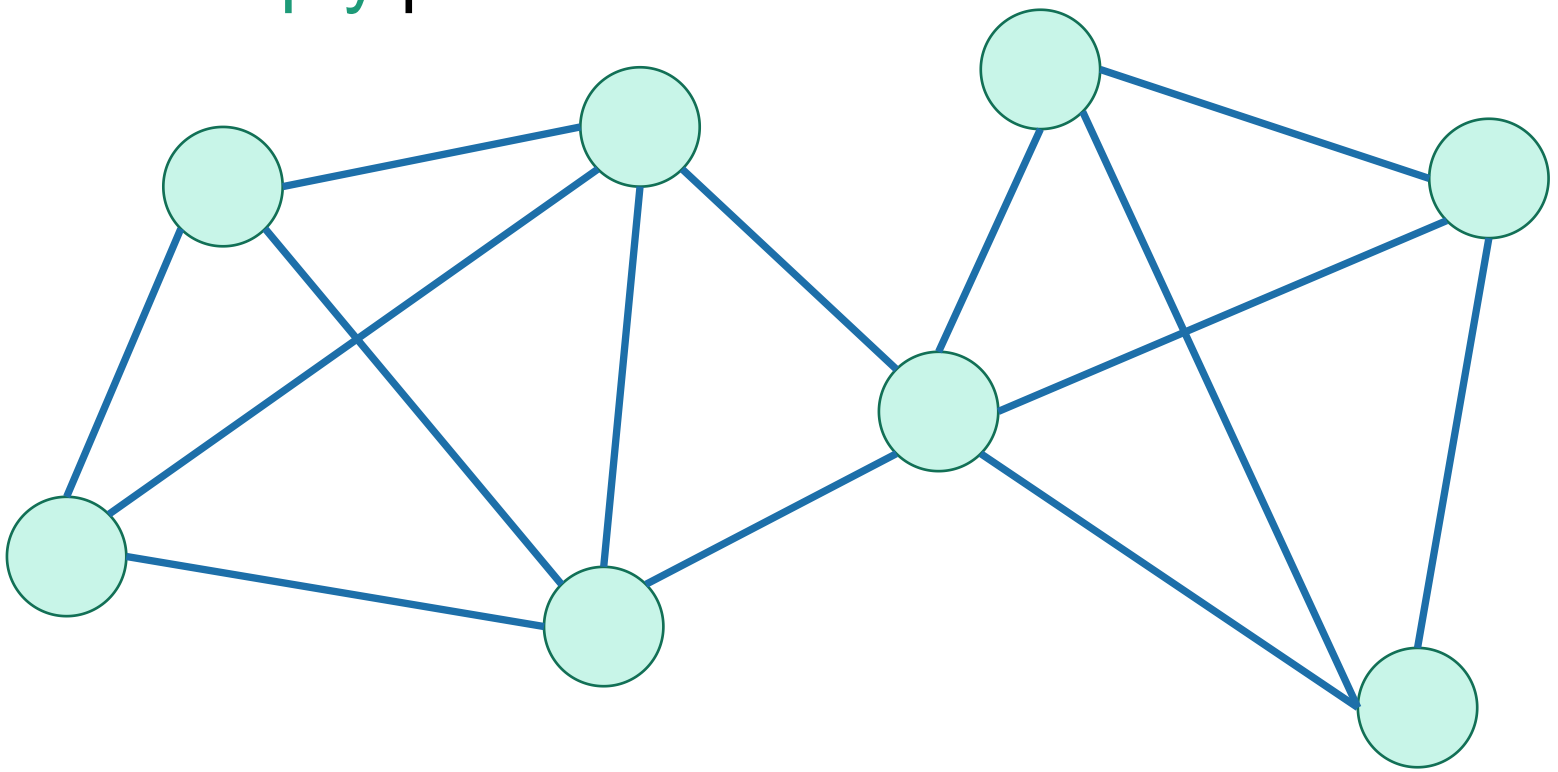
- Cuts anf Flow



Recall: cuts in graphs

*For today, all graphs are **undirected and unweighted.**

- A cut is a partition of the vertices into two nonempty parts.



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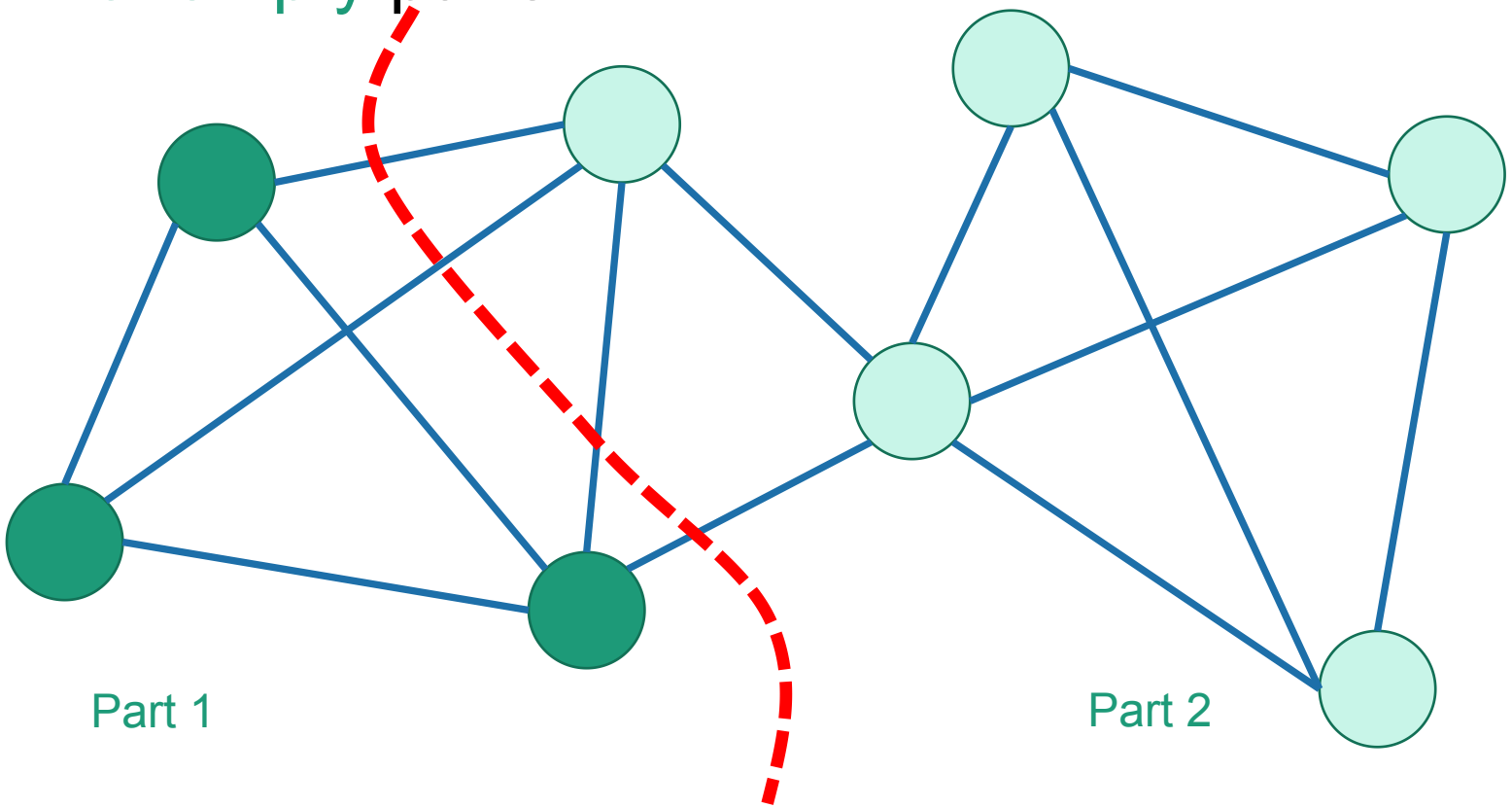
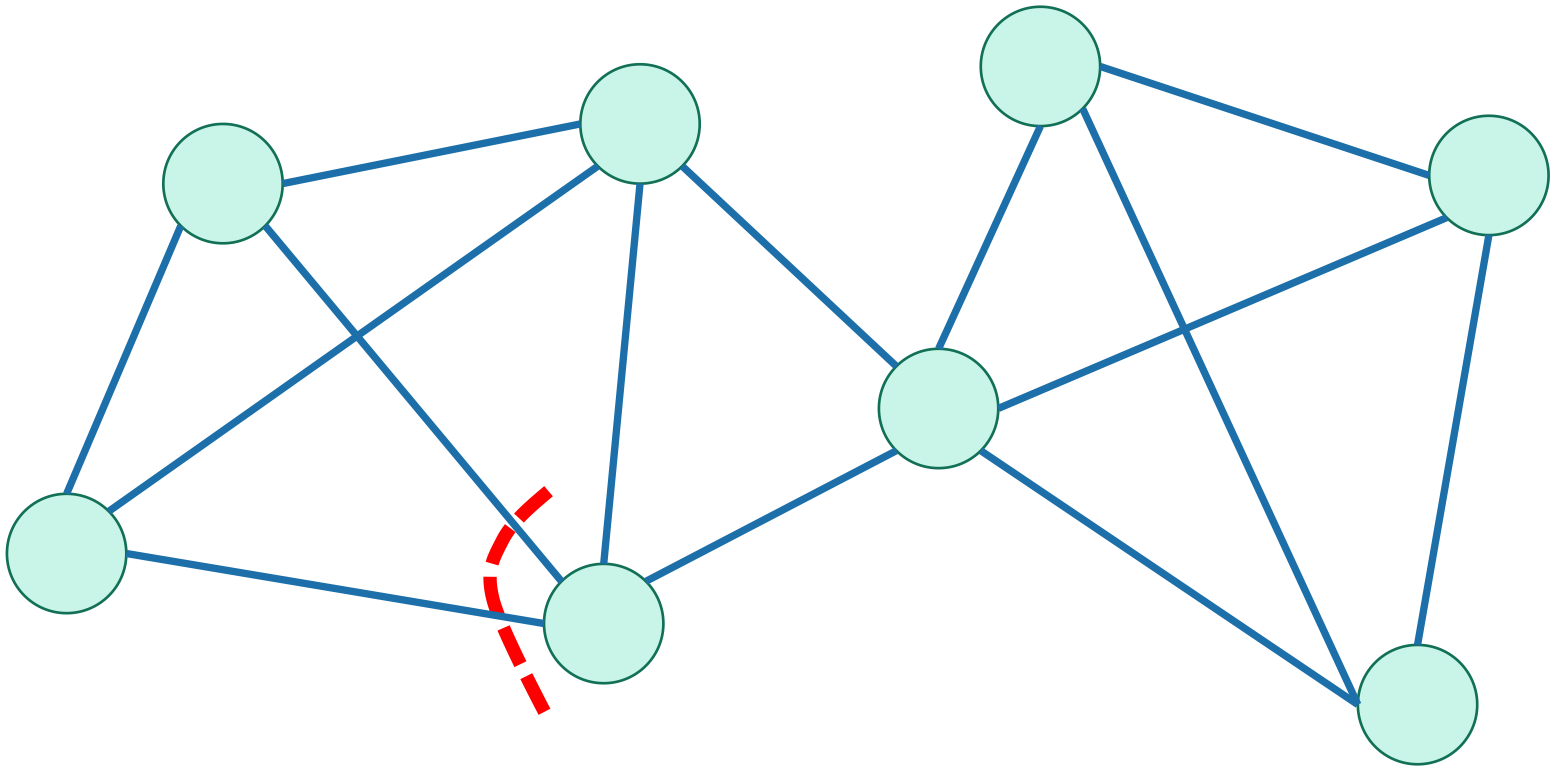


This is not a cut



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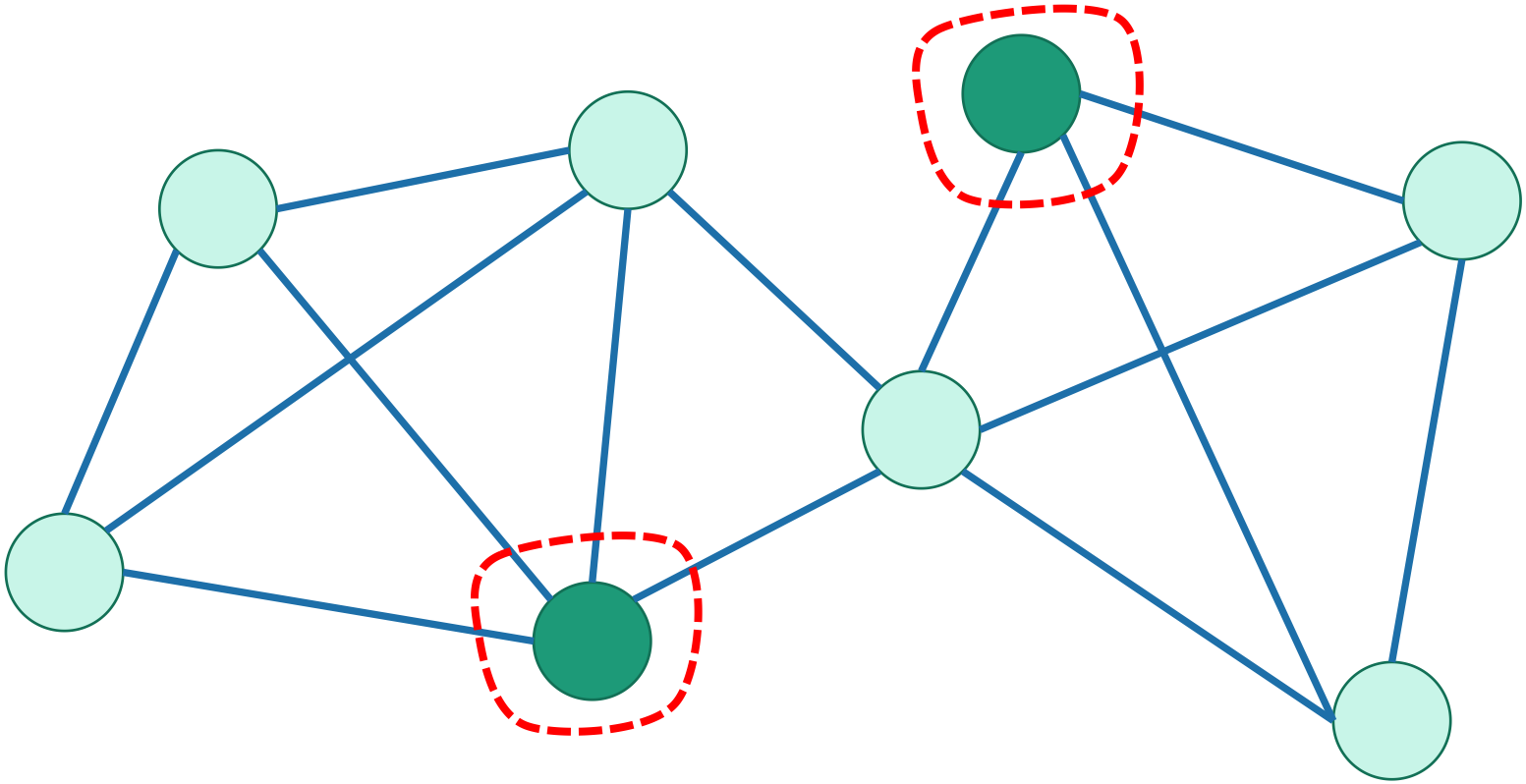
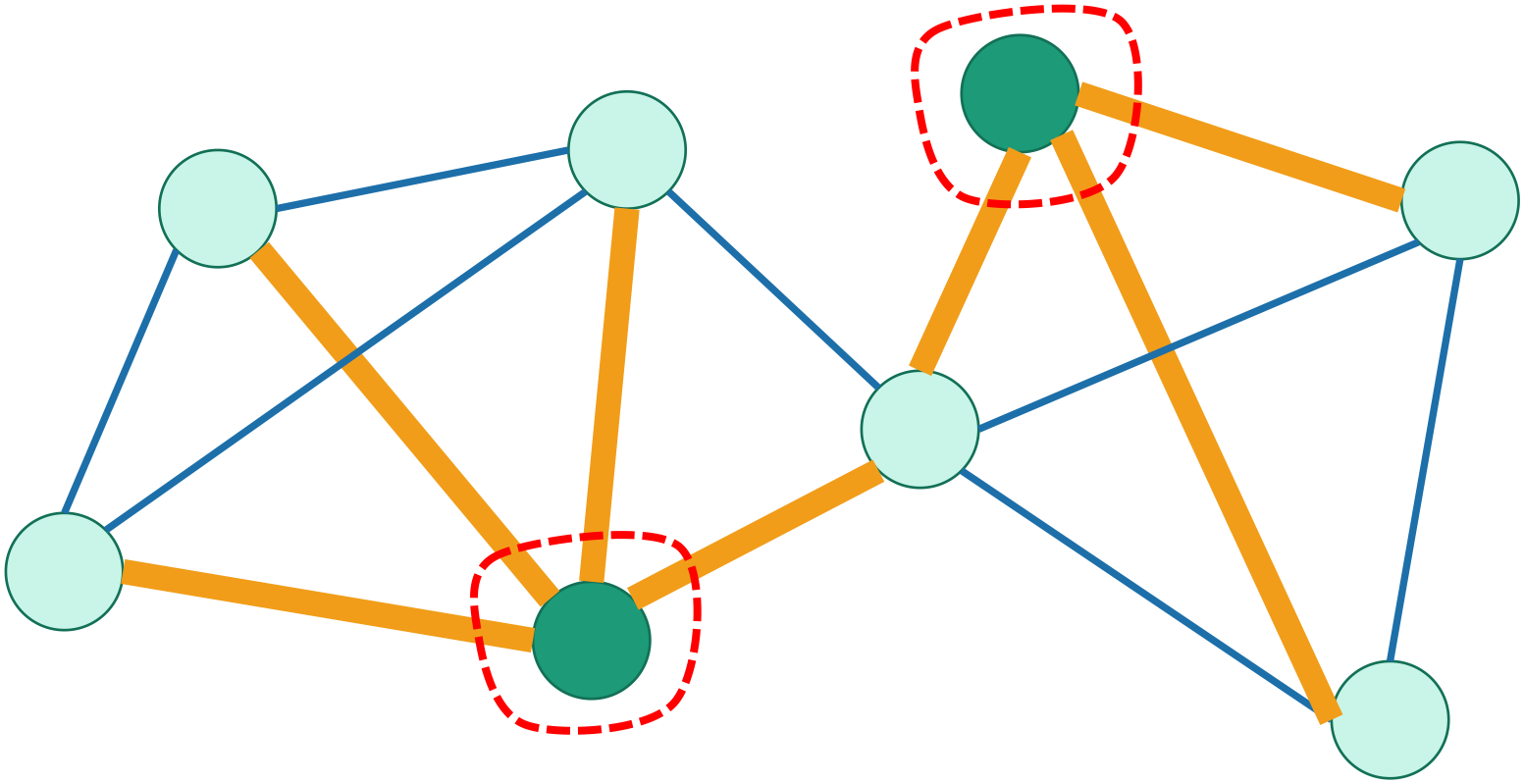


This is a cut

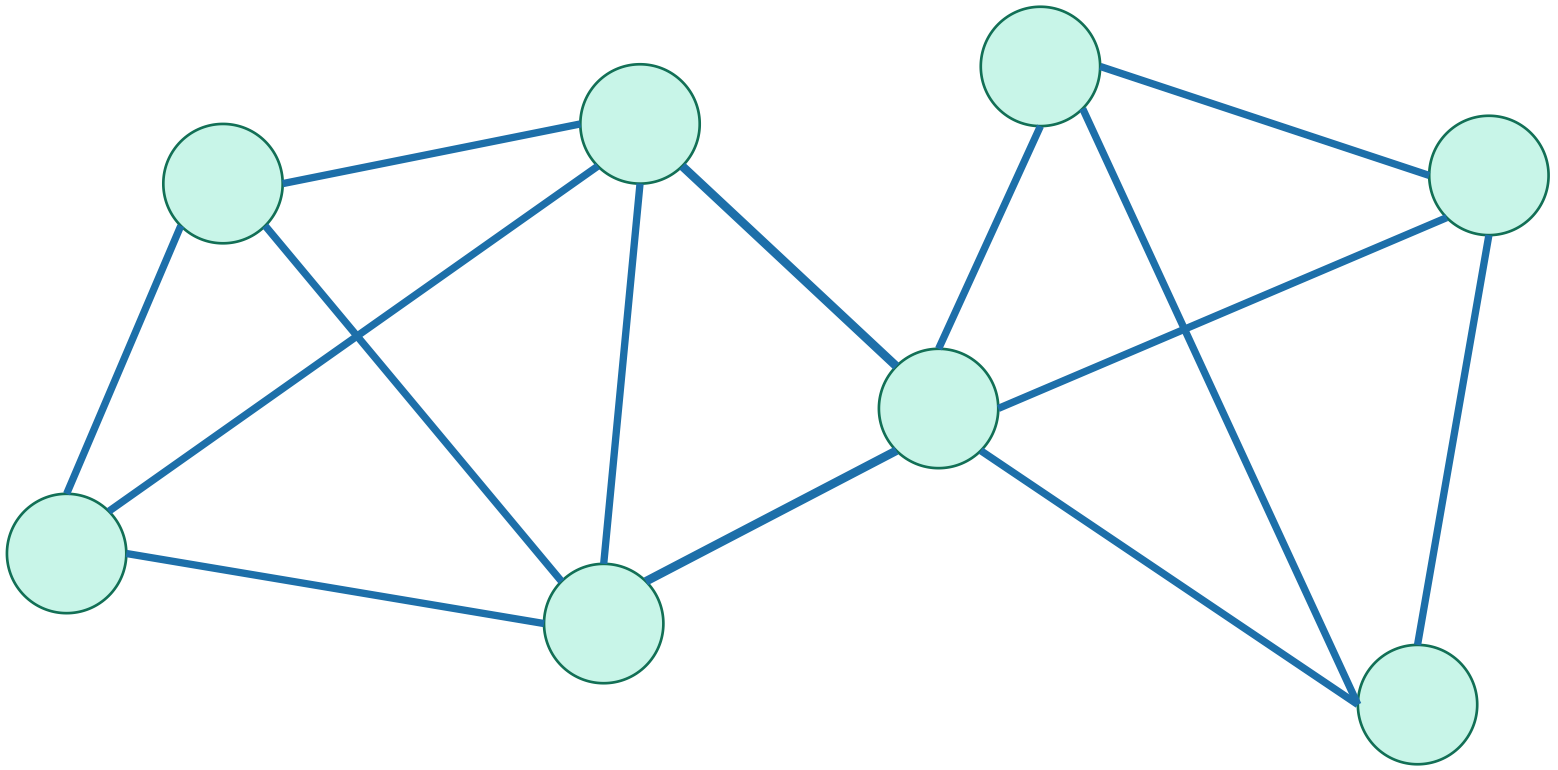
These edges **cross the cut.**

- They go from one part to the other



A (global) minimum cut

is a cut that has the fewest edges possible crossing it.



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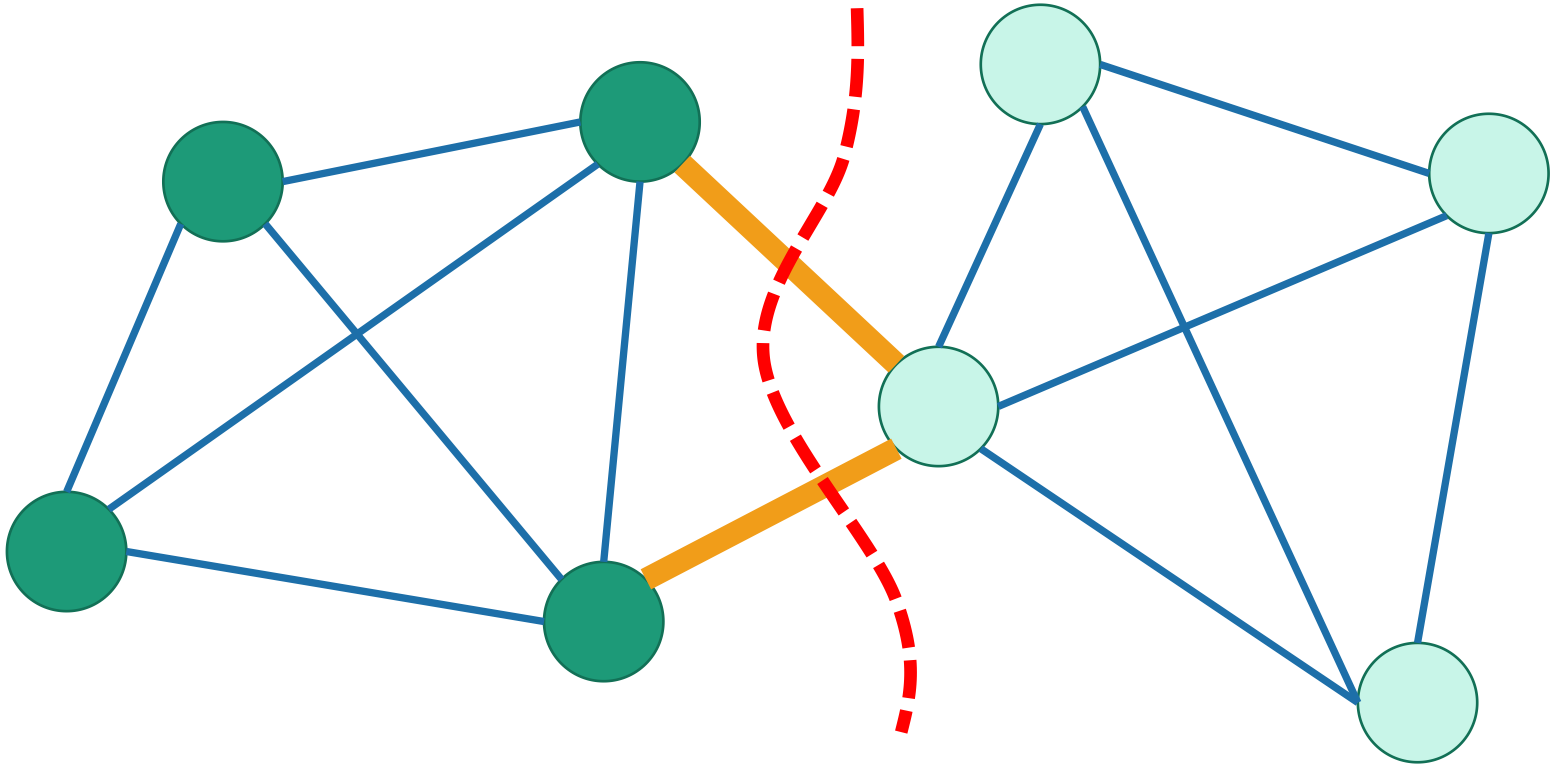
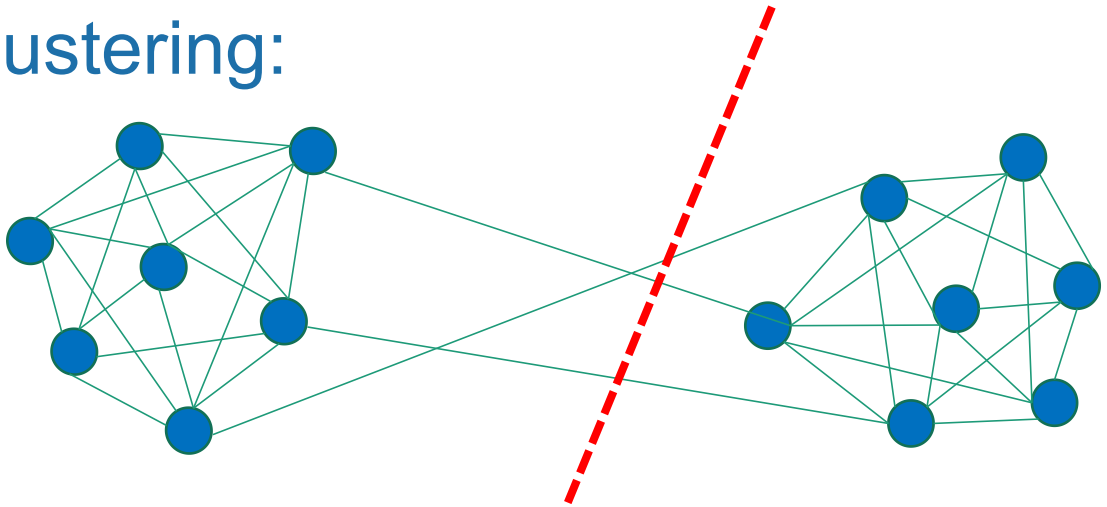
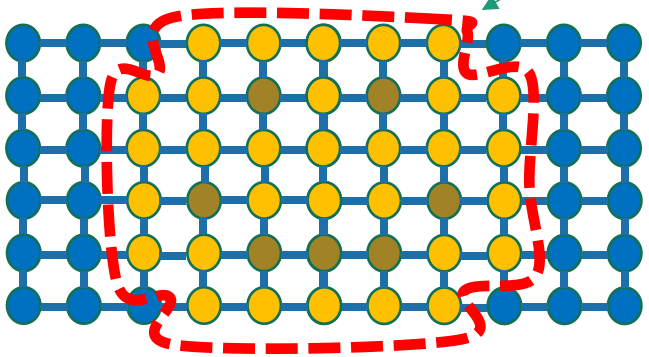


Why might we care about global minimum cuts?

- Clustering:



- Image Segmentation



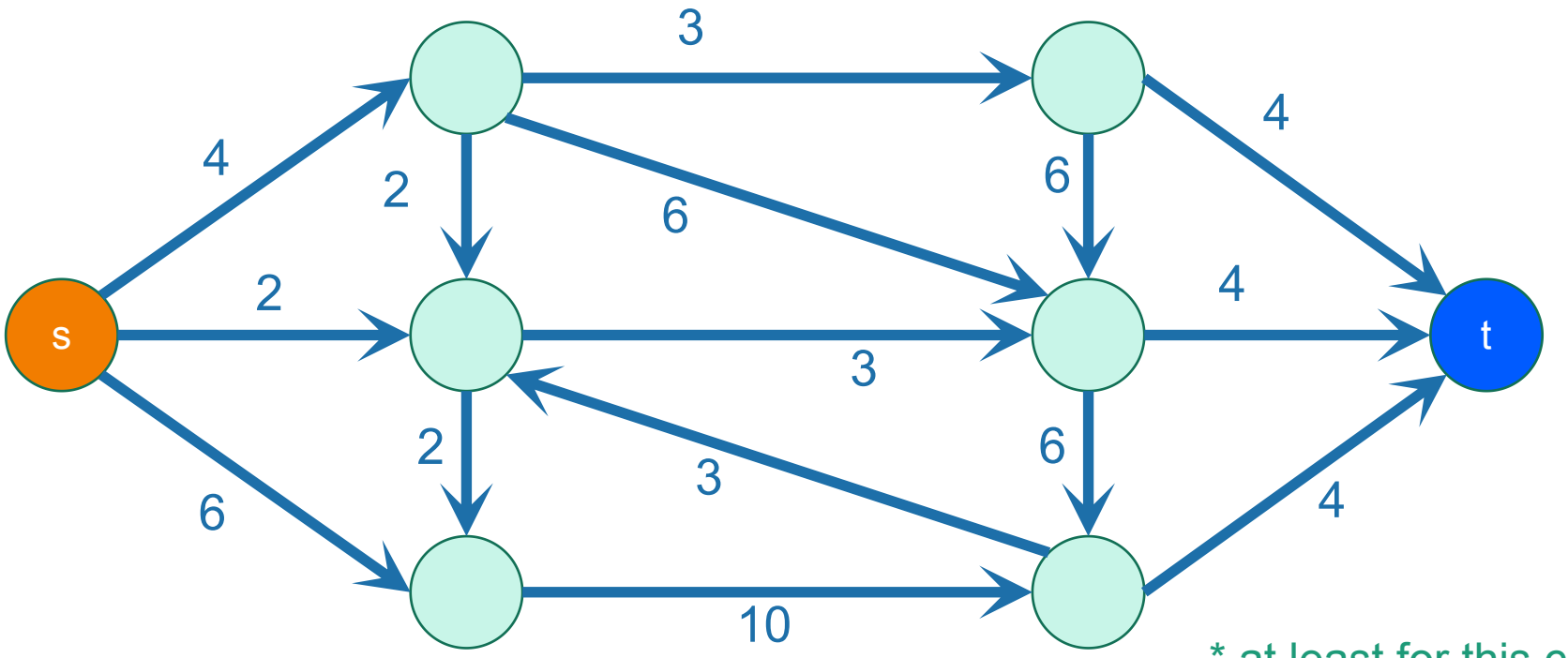
*For the rest of today edges aren't weighted; but the algorithm can be adapted to deal with edge weights.

Karger's algorithm (Please Read it. Not Covered Today)

- Finds **global minimum cuts** in undirected graphs
- Randomized algorithm
  - But a different sort of randomized algorithm than Quicksort!
- Karger's algorithm **might be wrong**.
  - While QuickSort, which just might be slow.
- Why would we want an algorithm that might be wrong?
  - **With high probability it won't be wrong.**
  - Maybe the stakes are low and the cost of a deterministic algorithm is high.

s-t cut

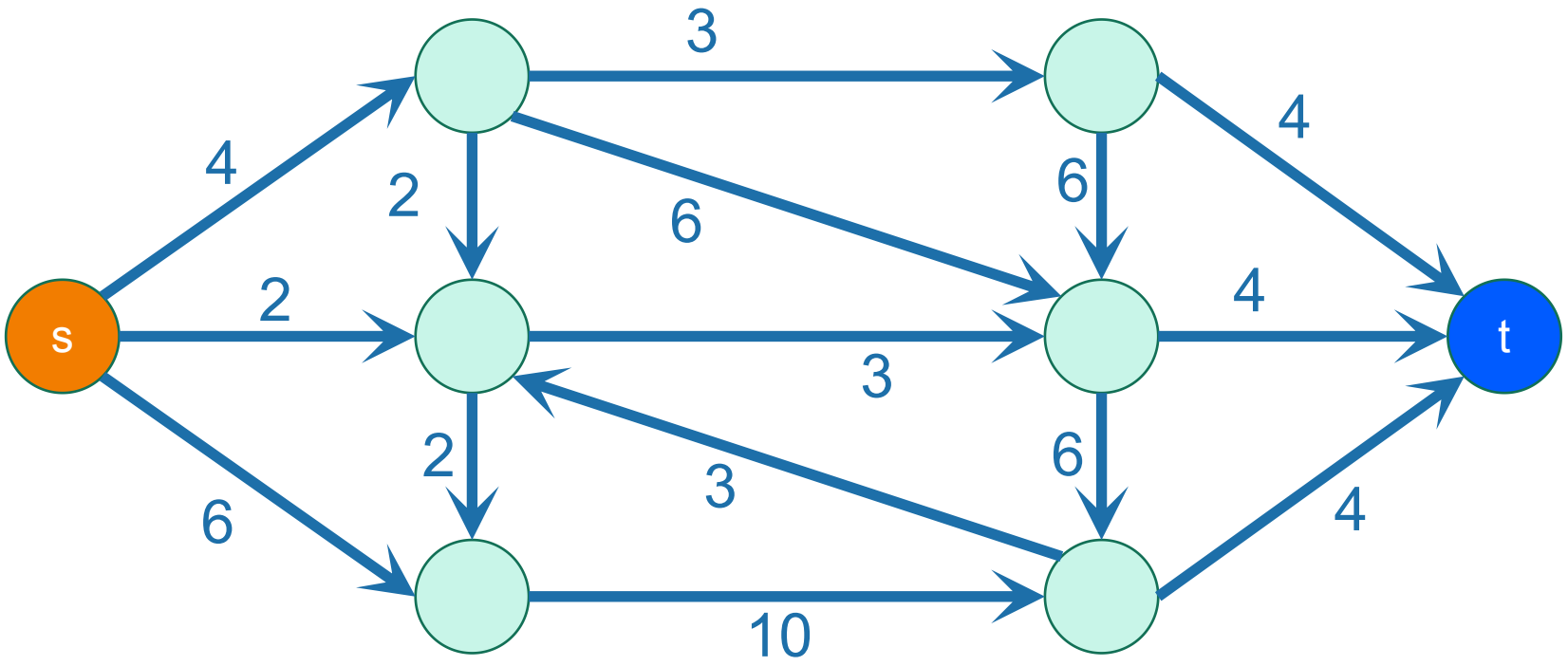
- Graphs are directed and edges have “capacities” (weights)
- We have a special “source” vertex s and “sink” vertex t.
  - s has only outgoing edges*
  - t has only incoming edges*



* at least for this class

An **s-t cut**

is a cut which separates s from t



An **s-t cut**

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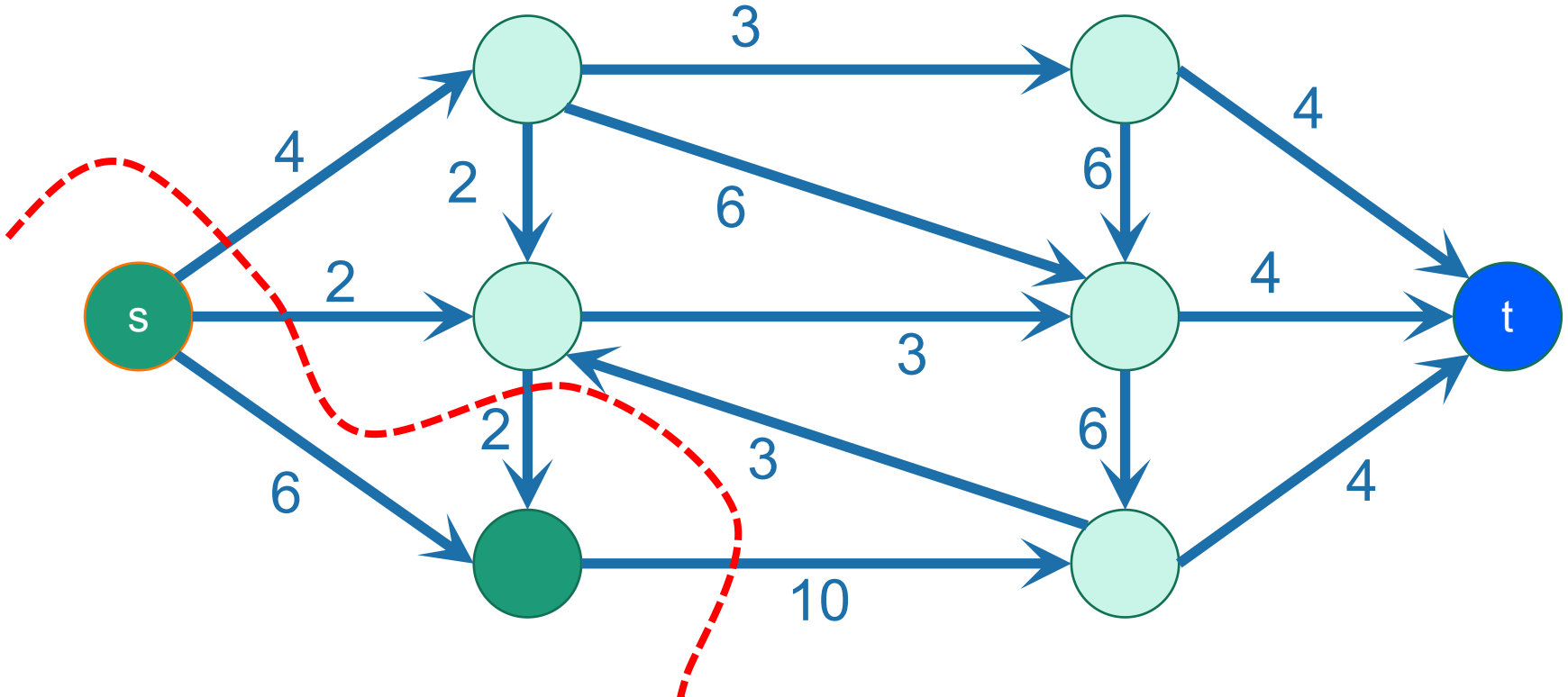
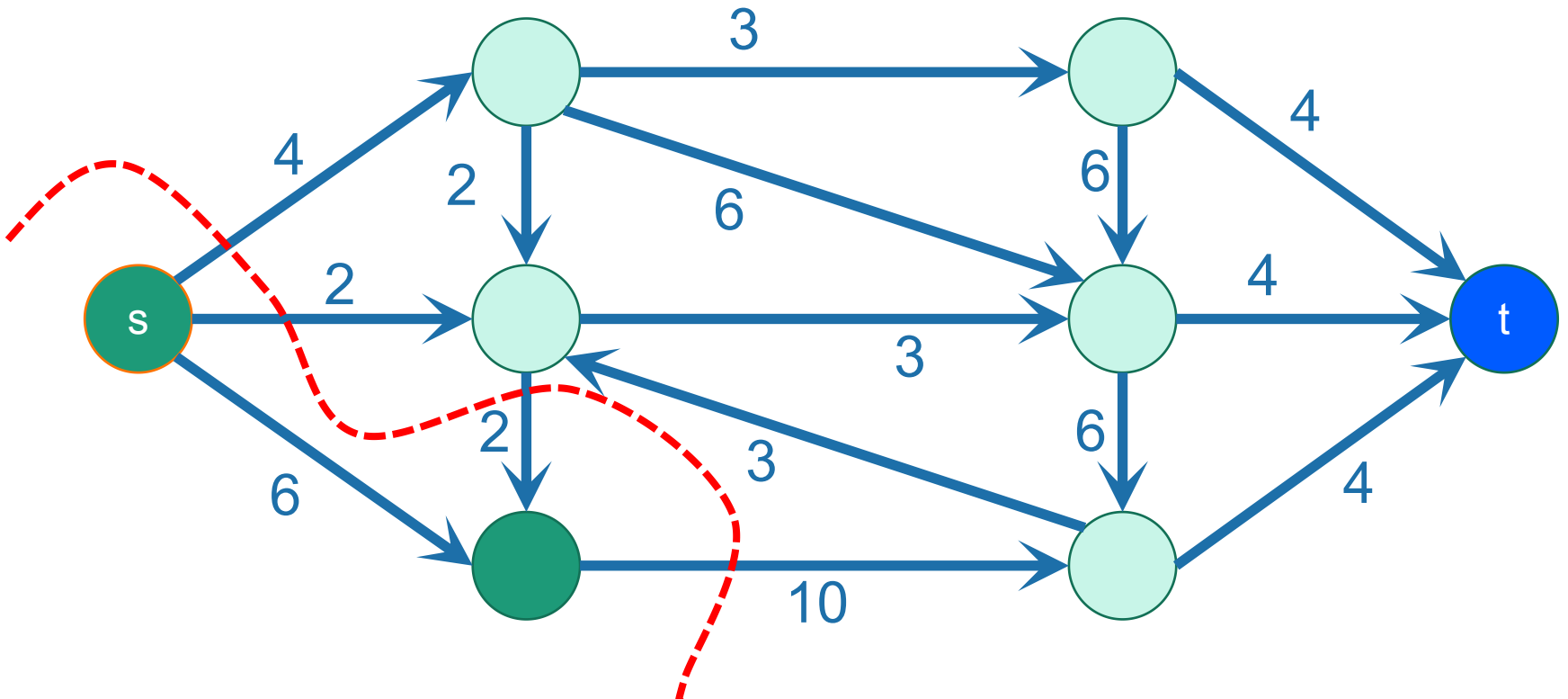


An **s-t cut**

is a cut which separates s from t

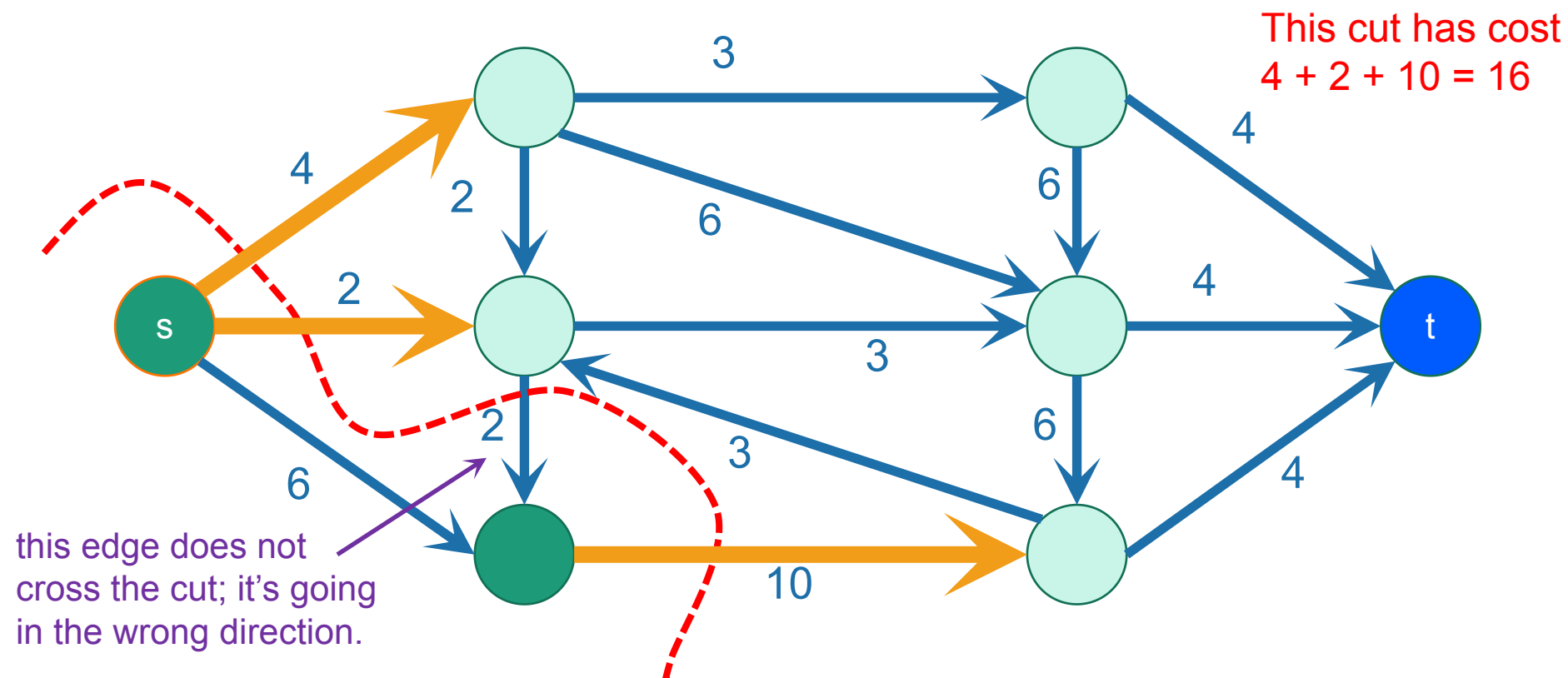
- An edge crosses the cut if it goes from s's side to t's side.



An **s-t cut**

is a cut which separates s from t

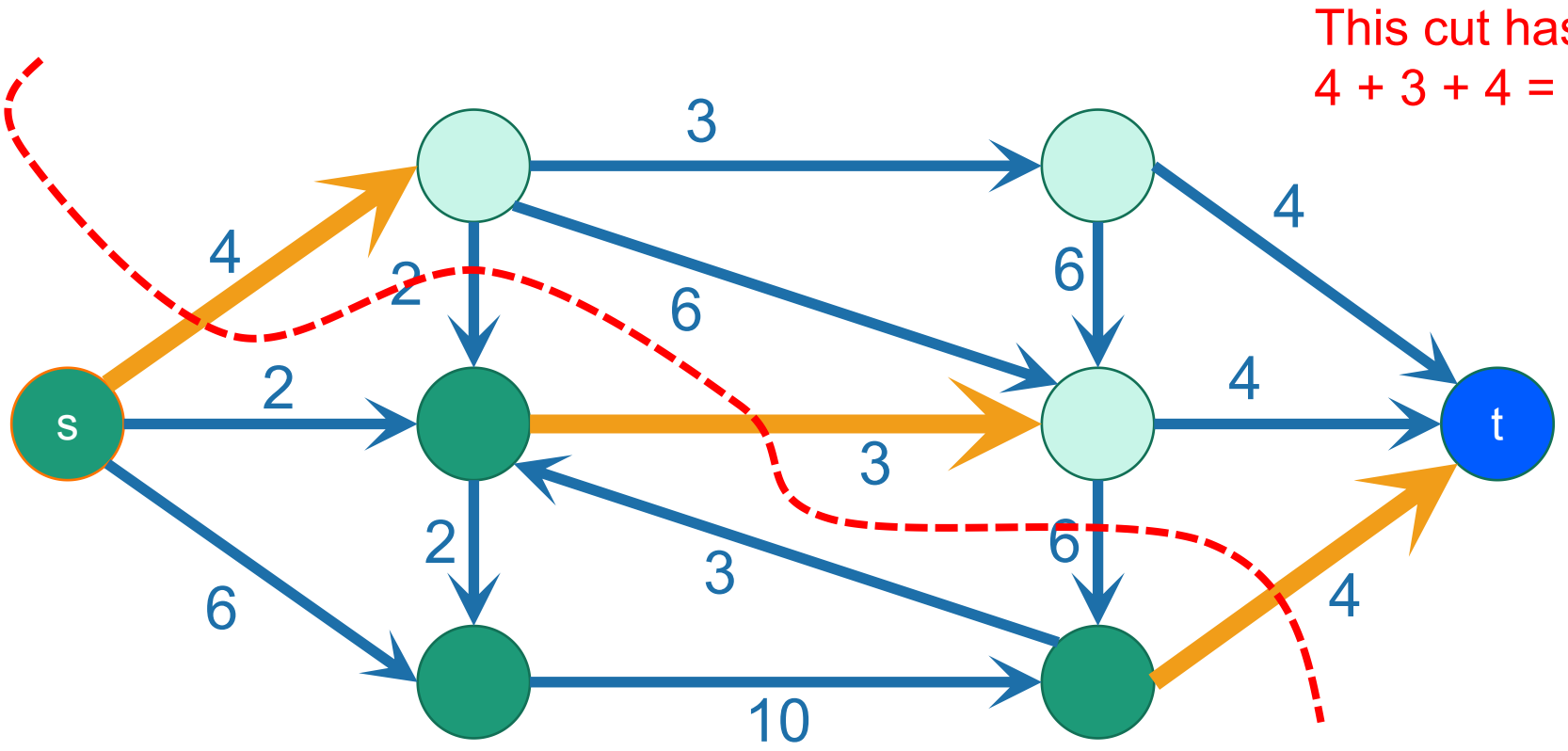
- An edge crosses the cut if it goes from s's side to t's side.
- The cost (or capacity) of a cut is the sum of the capacities of the edges that cross the cut



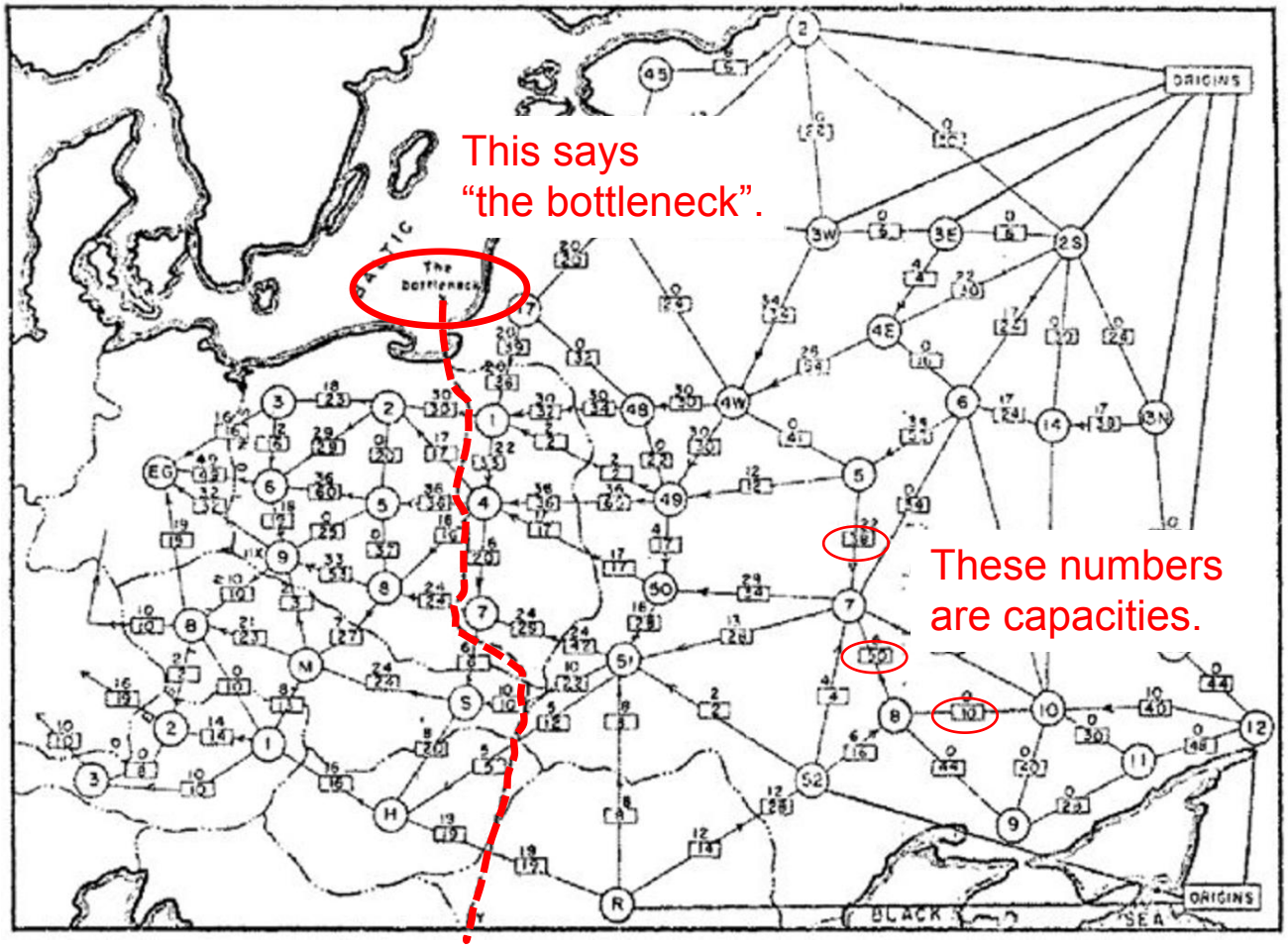
A minimum **s-t cut**

is a cut which separates s from t with minimum capacity.

- Question: how do we find a minimum s-t cut?



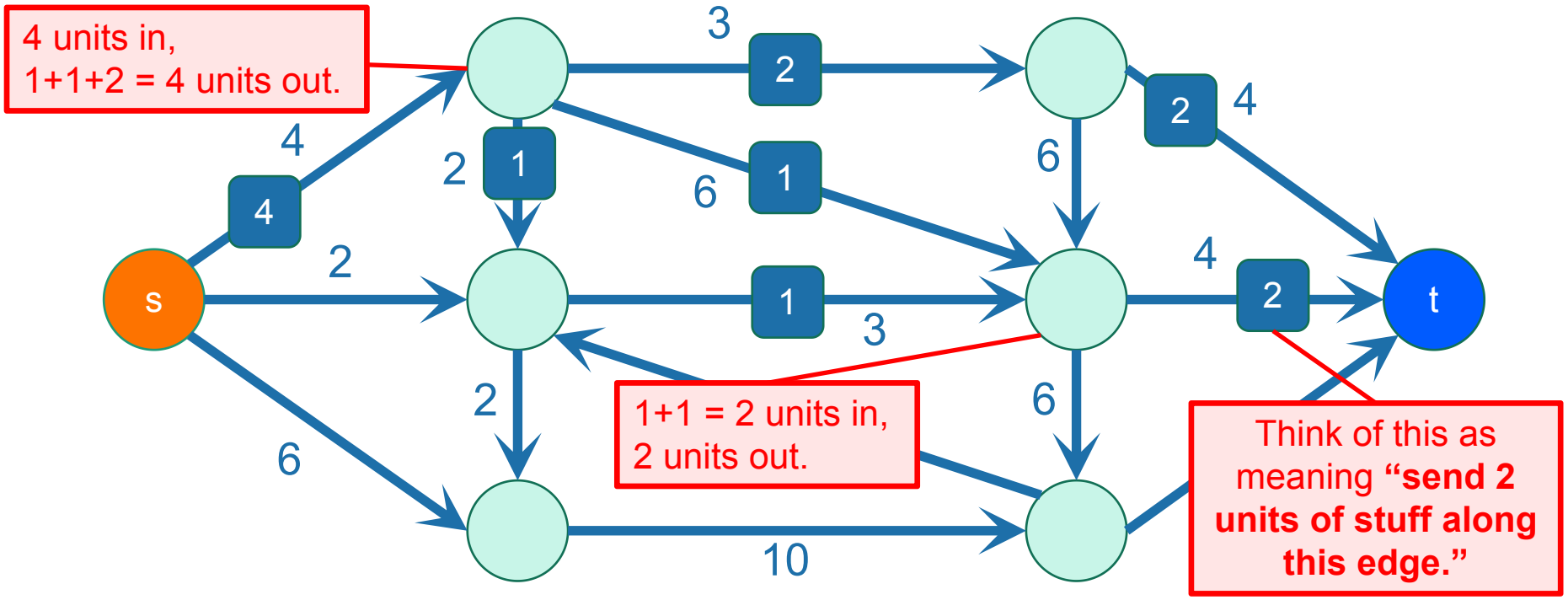
Example where this comes up



- 1955 map of rail networks from the Soviet Union to Eastern Europe.
  - Declassified in 1999.
  - 44 vertices, 105 edges
- The US wanted to cut off routes from suppliers in Russia to Eastern Europe as efficiently as possible
- In 1955, Ford and Fulkerson at the RAND corporation gave an algorithm which finds the optimal s-t cut.

Flows

- In addition to a capacity, each edge has a **flow**
  - (unmarked edges in the picture have flow 0)
- The flow on an edge must be less than its capacity.
- At each vertex, the incoming flows must equal outgoing flows.

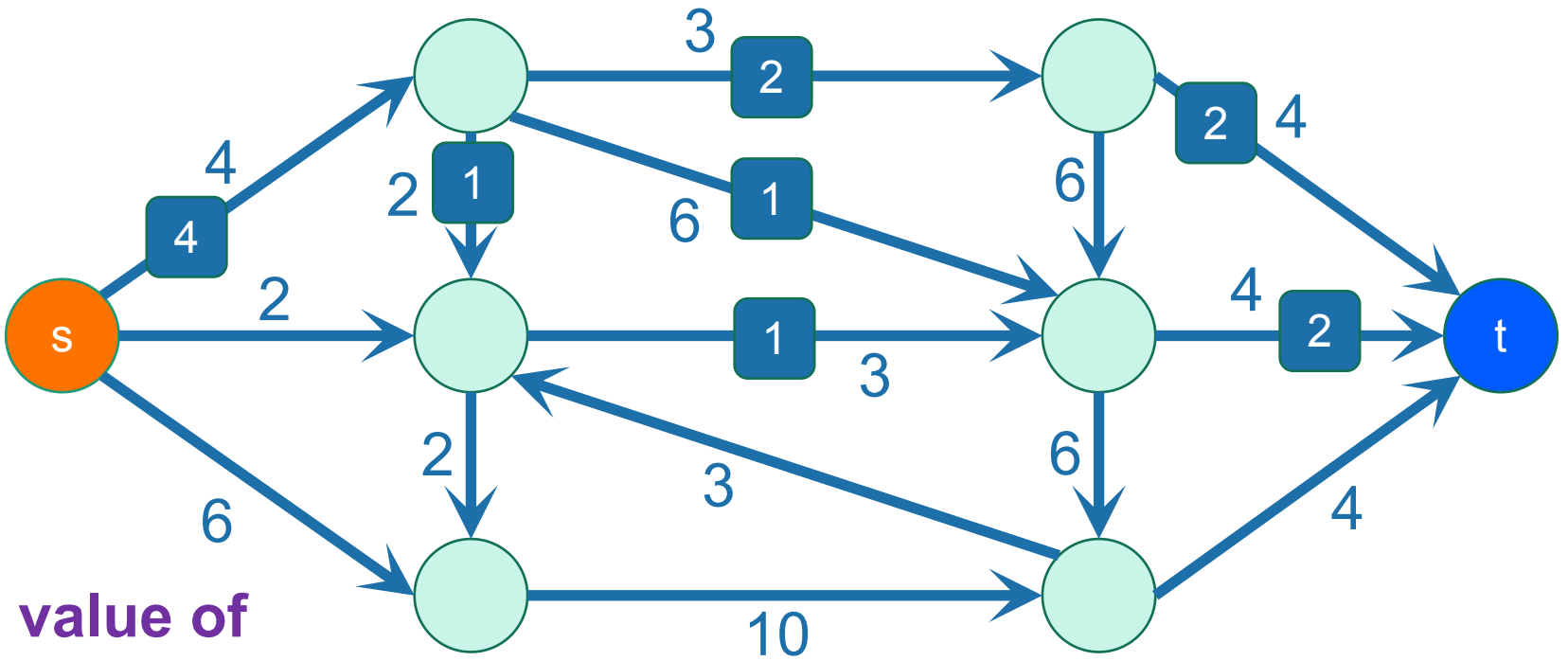


Flows

- The value of a flow is:
  - The amount of stuff coming out of s
  - The amount of stuff flowing into t
  - These are the same!

Because of conservation of flows at vertices,

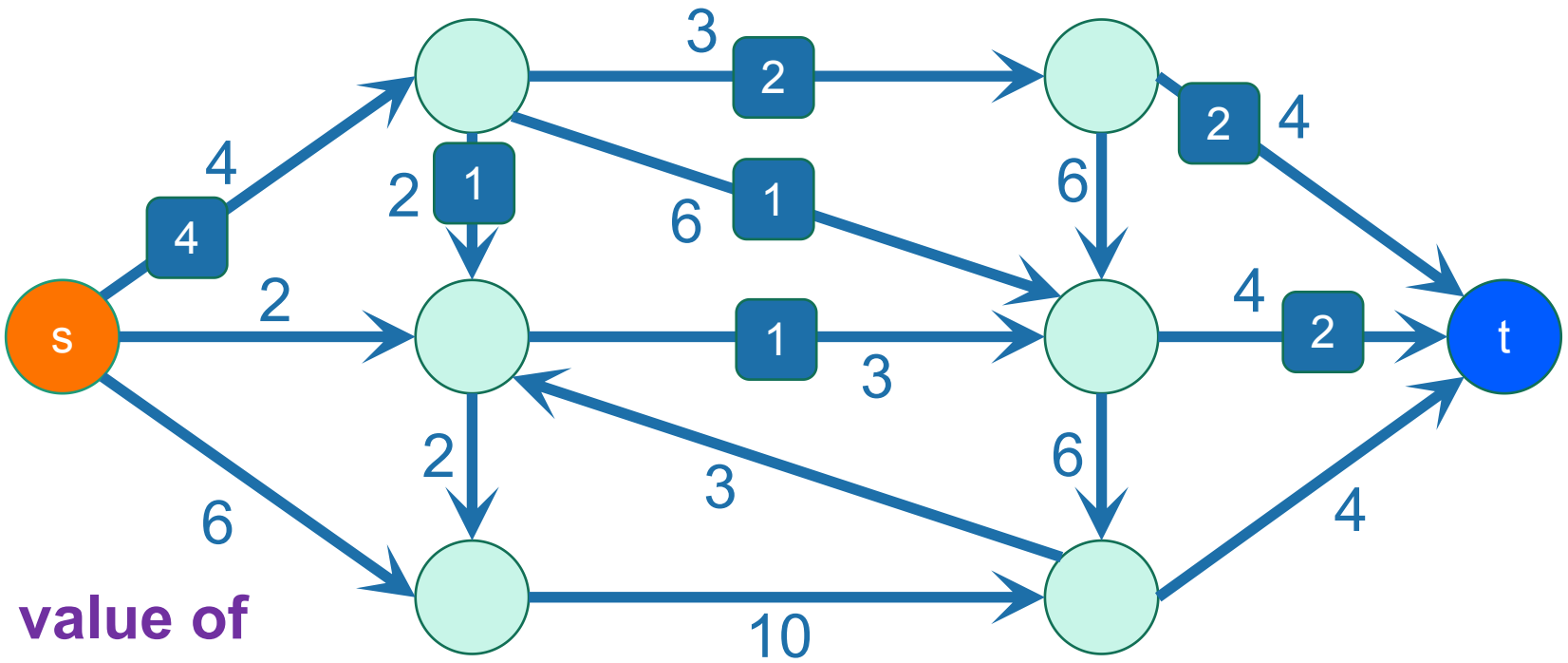
stuff you put in
=
stuff you take out.



The value of this flow is 4.

A maximum flow
is a flow of maximum value.

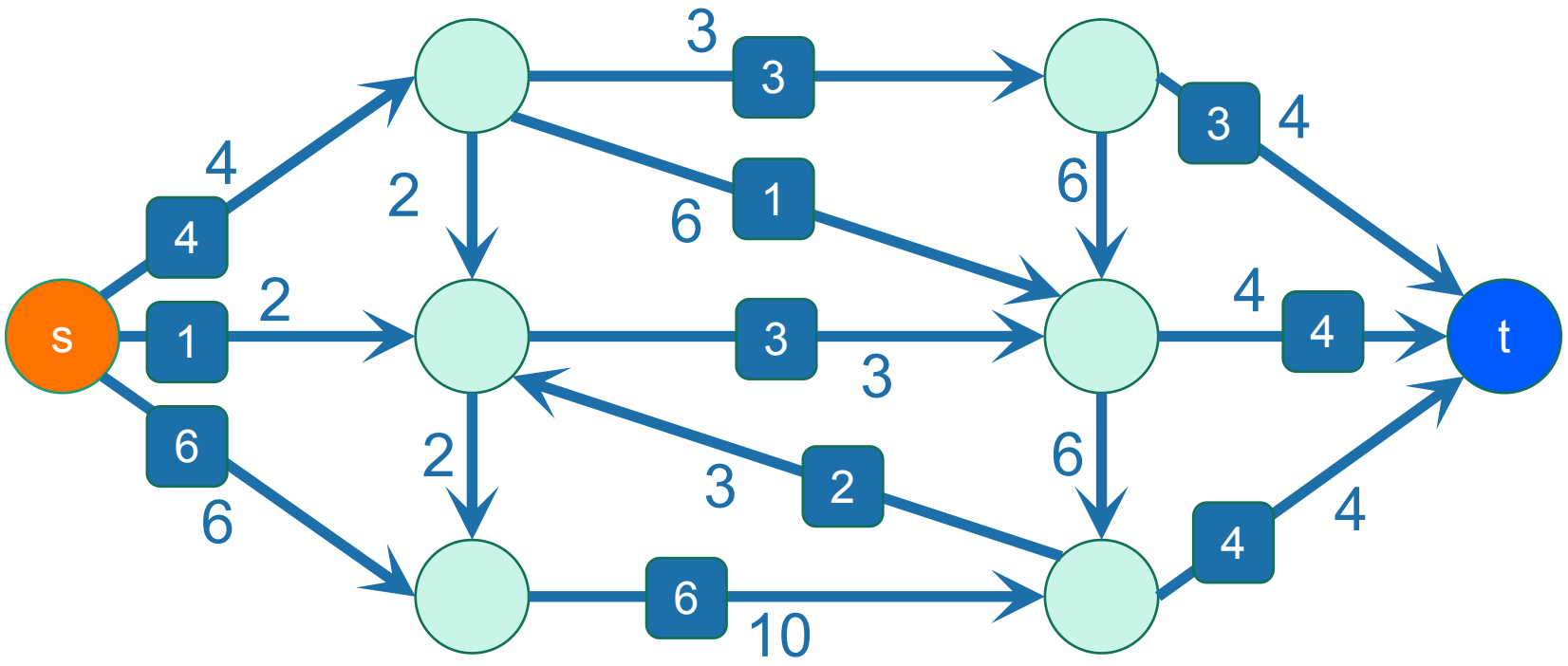
- This example flow is pretty wasteful, I'm not utilizing the capacities very well.



The value of this flow is 4.

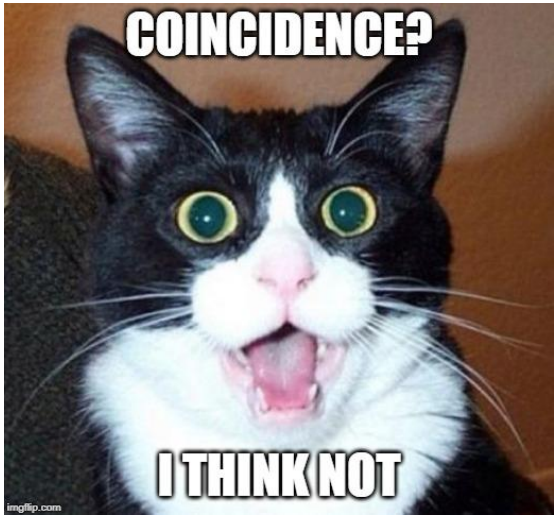
A maximum flow
is a flow of maximum value.

- This one is maximal.
  - It has value 11.



A maximum flow
is a flow of maximum value.

That's the same as the minimum cut in this graph!



- This one is maximal.
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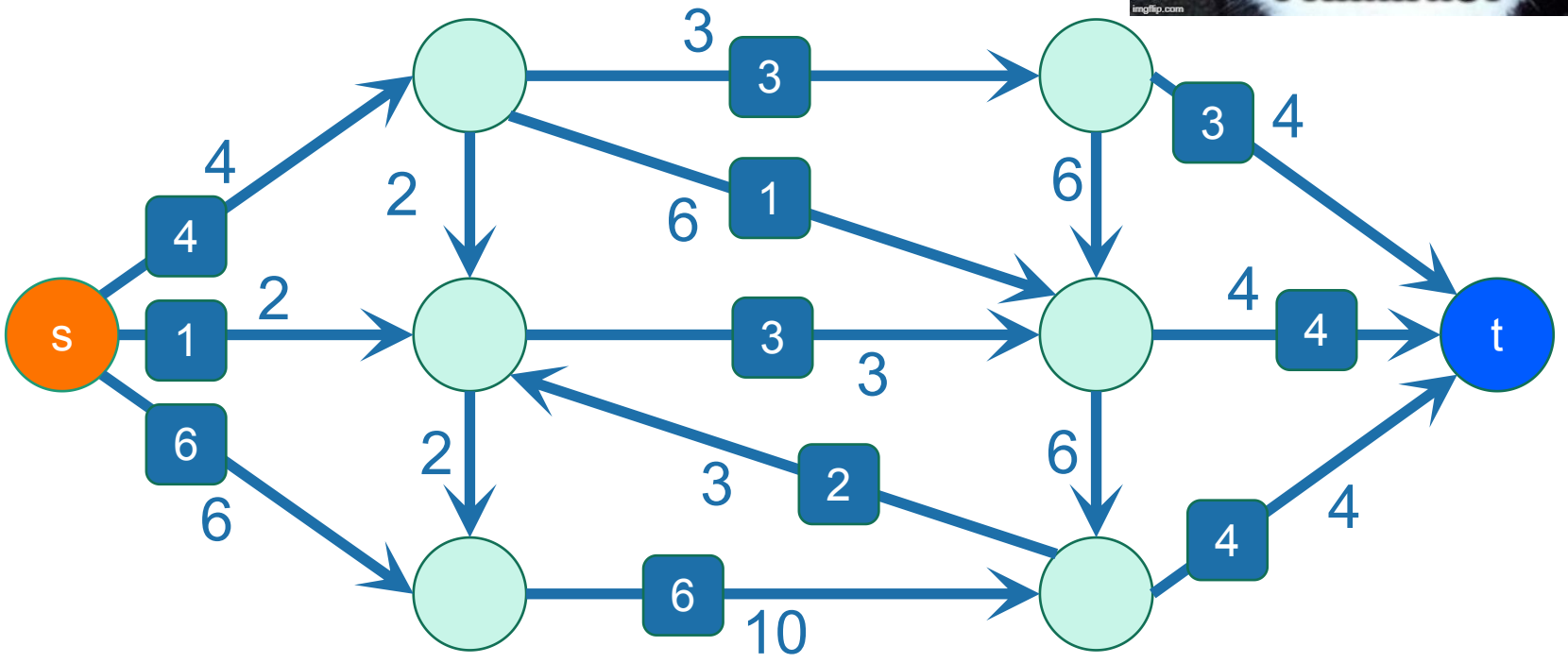
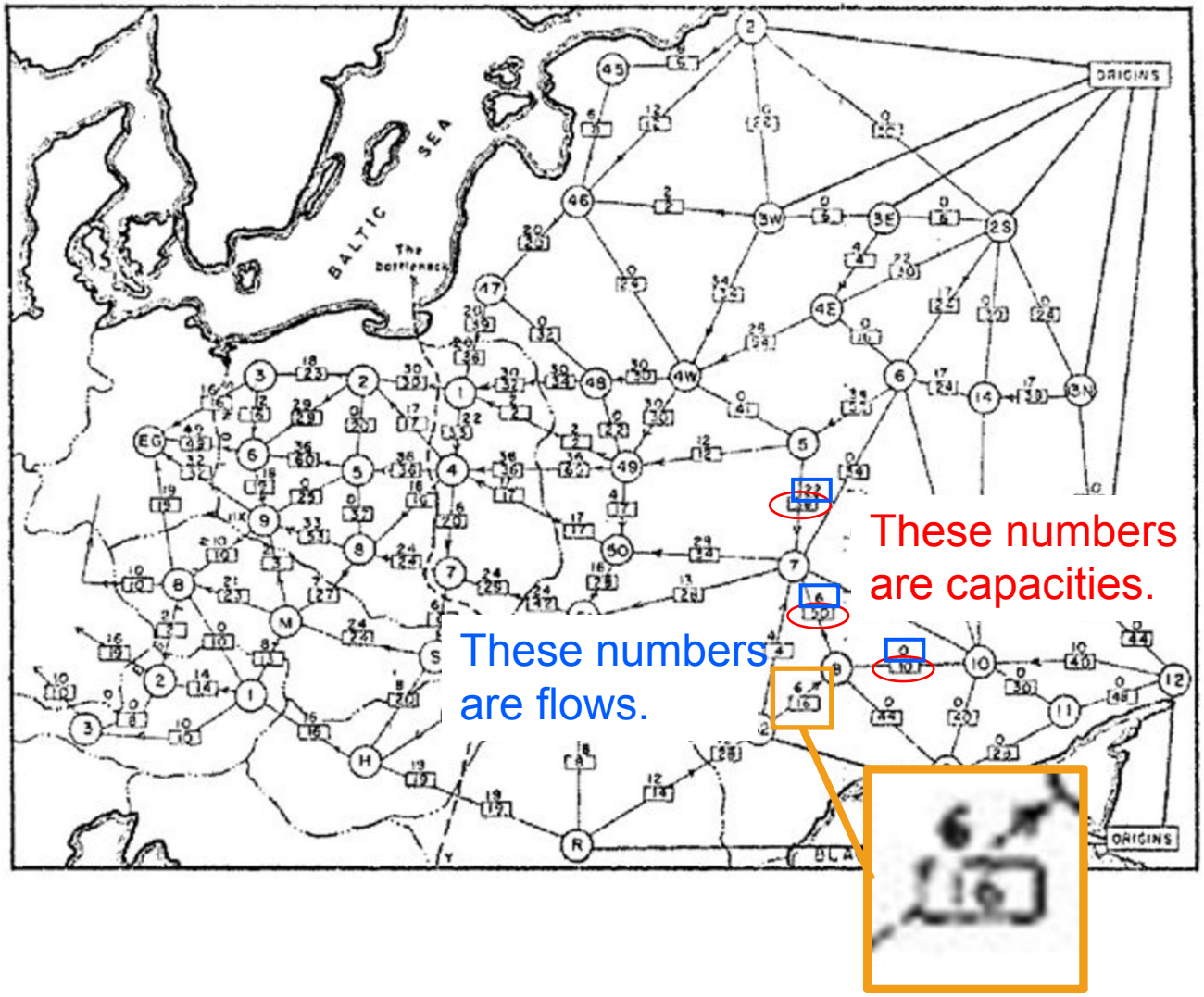


Example



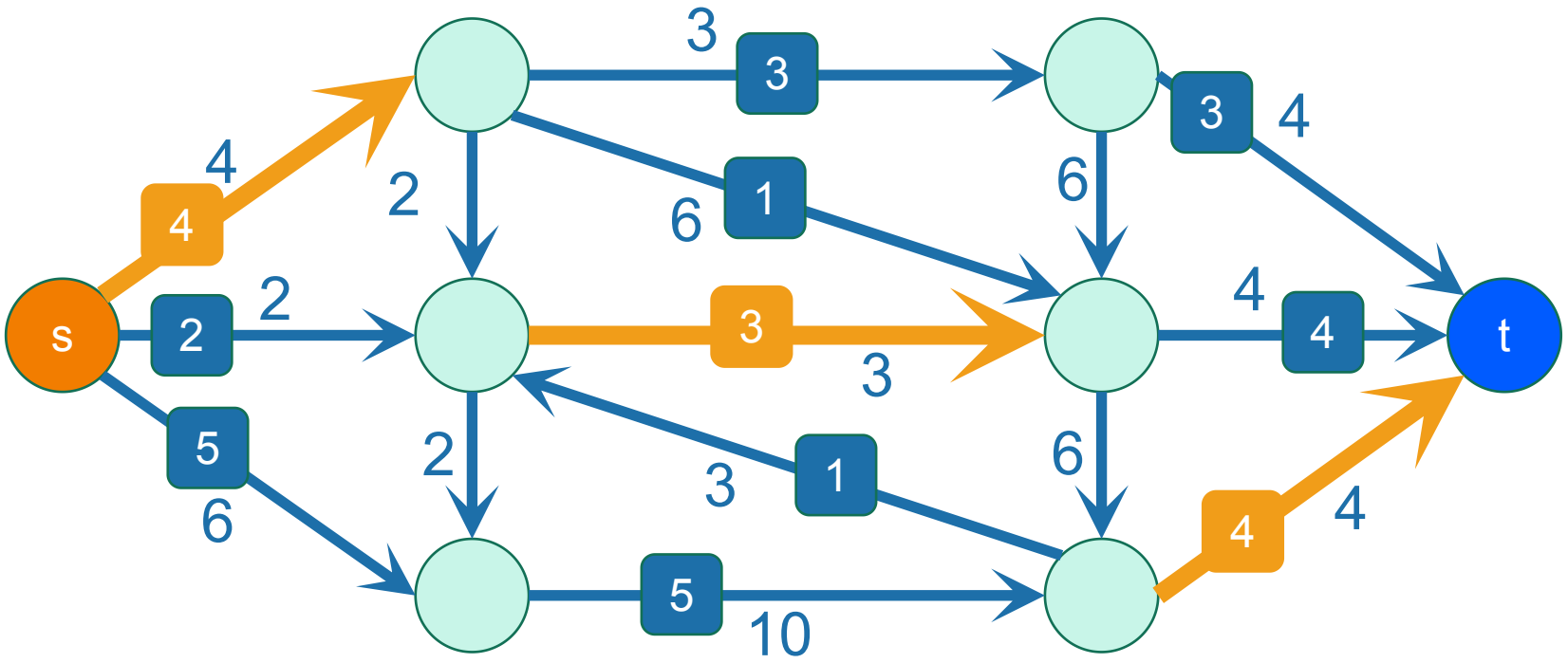
- 1955 map of rail networks from the Soviet Union to Eastern Europe.
  - Declassified in 1999.
  - 44 vertices, 105 edges
- The Soviet Union wants to route supplies from suppliers in Russia to Eastern Europe as efficiently as possible

Theorem

Max-flow min-cut theorem

The value of a max flow from s to t
is equal to
the cost of a min s-t cut.

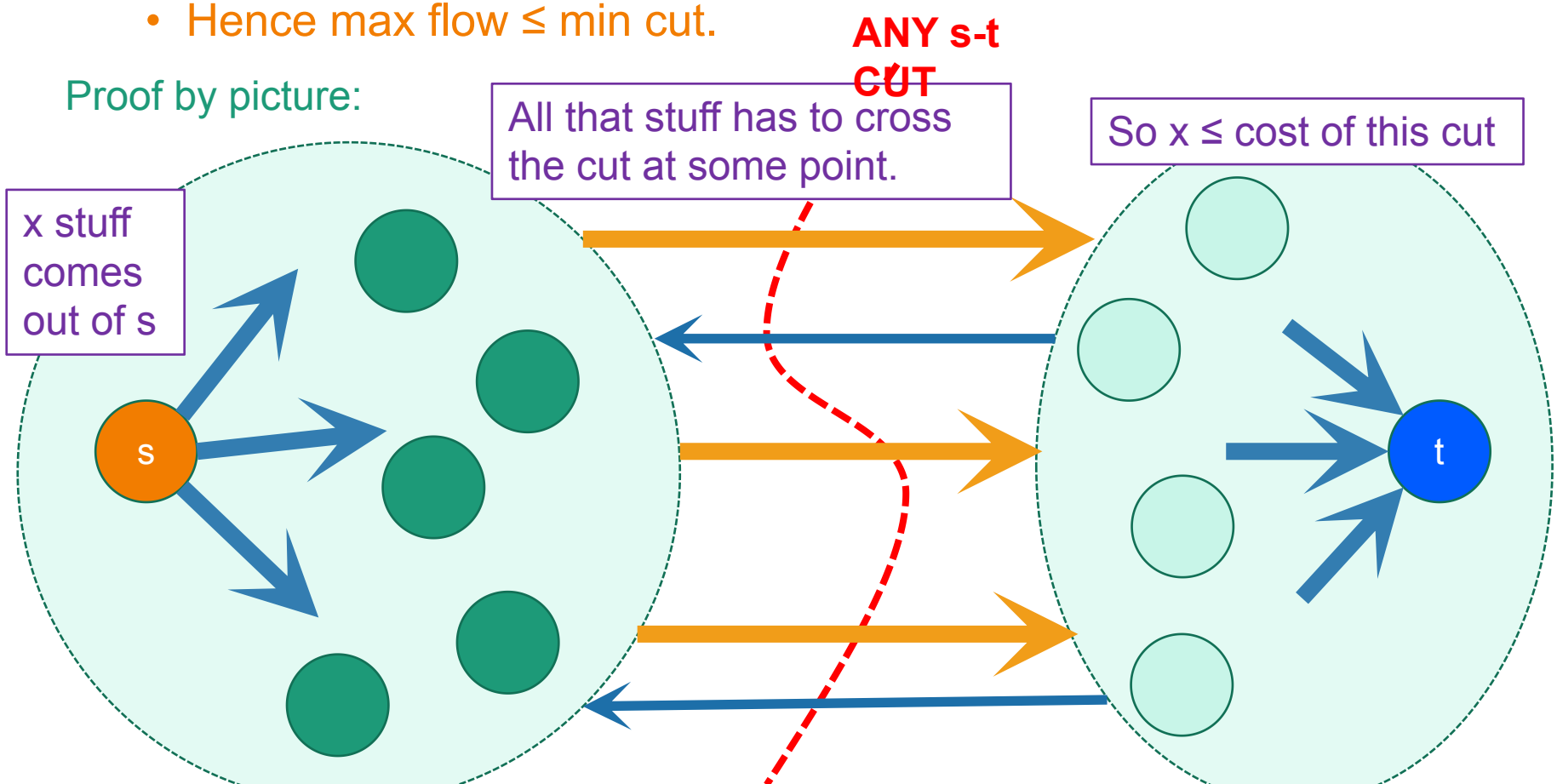
Intuition: in a max flow, the min cut better fill up, and this is the bottleneck.



Proof of Min-Cut Max-Flow Thm

- **Lemma1:**
  - For ANY s-t flow and ANY s-t cut, the value of the flow is at most the cost of the cut.
  - Hence max flow ≤ min cut.

Proof by picture:



Proof of Min-Cut Max-Flow Thm

- **Lemma1:**
  - For ANY s-t flow and ANY s-t cut, the value of the flow is at most the cost of the cut.
  - Hence max flow ≤ min cut.

  - That was proof-by-picture.
  - See the notes for the proof-by-proof.
    - You are **not** responsible for the proof-by-proof on the final.

Proof of Min-Cut Max-Flow Thm

- **Lemma1:**
  - For ANY s-t flow and ANY s-t cut, the value of the flow is at most the cost of the cut.
  - Hence max flow ≤ min cut.

- **The theorem is stronger:**
  - max flow = min cut
  - Proof by algorithm
    - up next!

Ford-Fulkerson algorithm

- Usually we state the algorithm first and then prove that it works.
- Today we're going to just start with the proof, and this will inspire the algorithm.

Outline of algorithm:

- Start with zero flow
- We will maintain a "residual graph" G_f
- A path from s to t in G_f will give us a way to improve our flow
- We will continue until there are no s-t paths left.

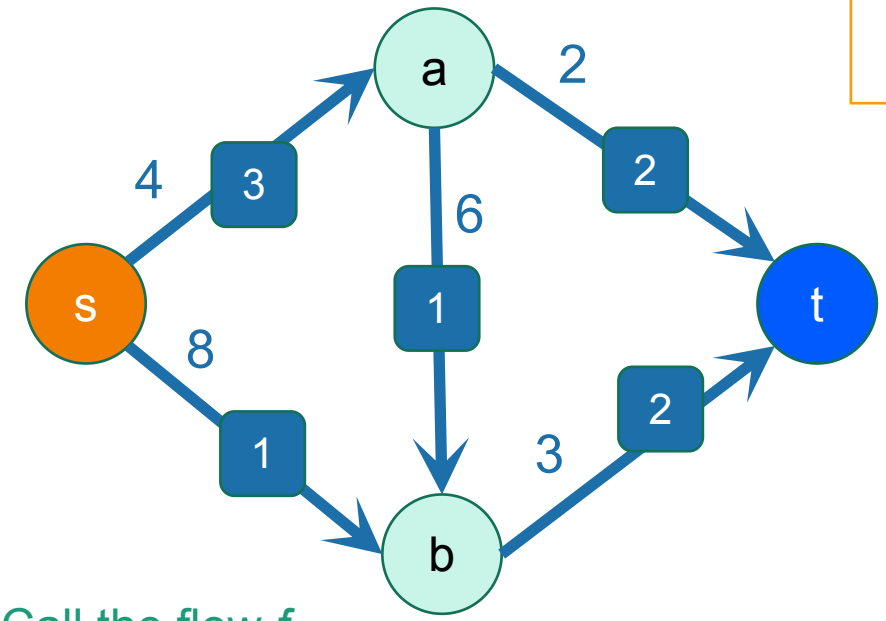
Assume for today that we don't have edges like this, although it's not necessary.

Tool: Residual networks

Say we have a flow

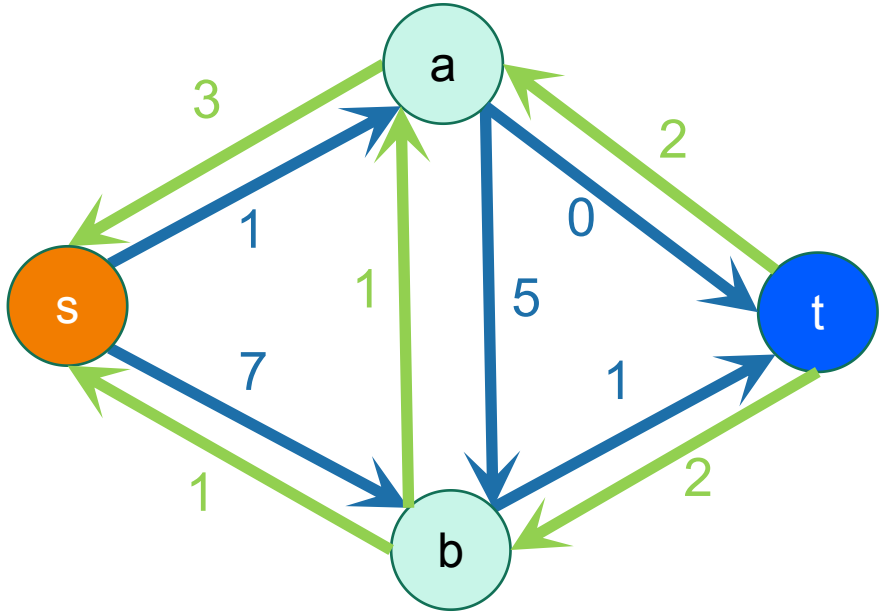
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & if\ (u,v) \in E \\ f(v,u) & if\ (v,u) \in E \\ 0 & else \end{cases}$$

- $f(u,v)$ is the flow on edge (u,v).
- $c(u,v)$ is the capacity on edge (u,v).



Call the flow *f*
Call the graph *G*

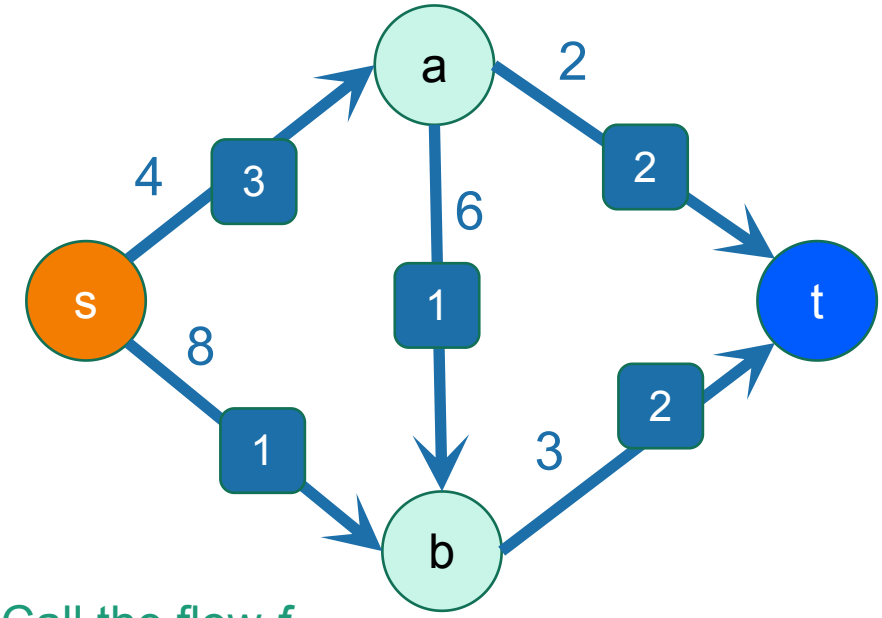
Create a new **residual network** from this flow:



Call the graph G_f

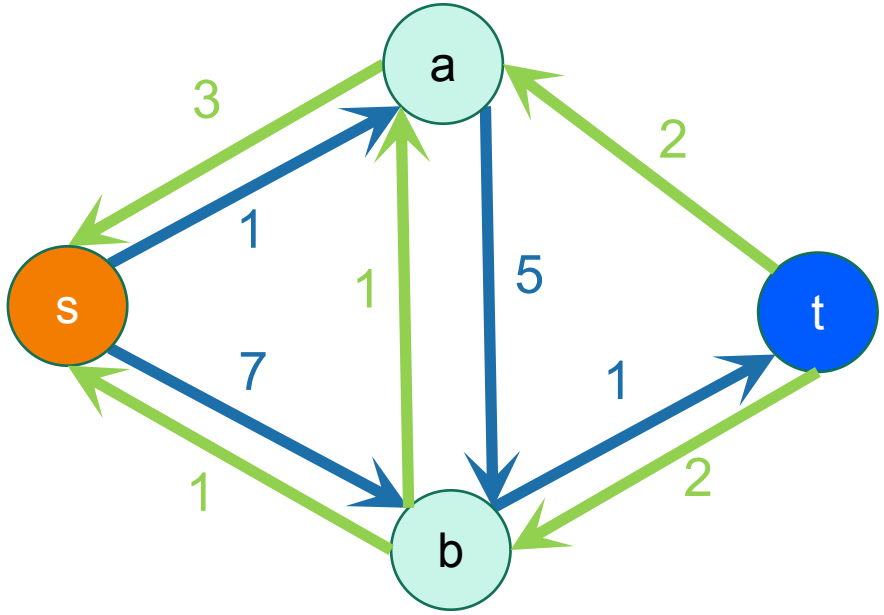
Tool: Residual networks

Say we have a flow



Call the flow *f*
Call the graph *G*

Forward edges are the amount that's left.
Backwards edges are the amount that's been used.



Call the graph G_f

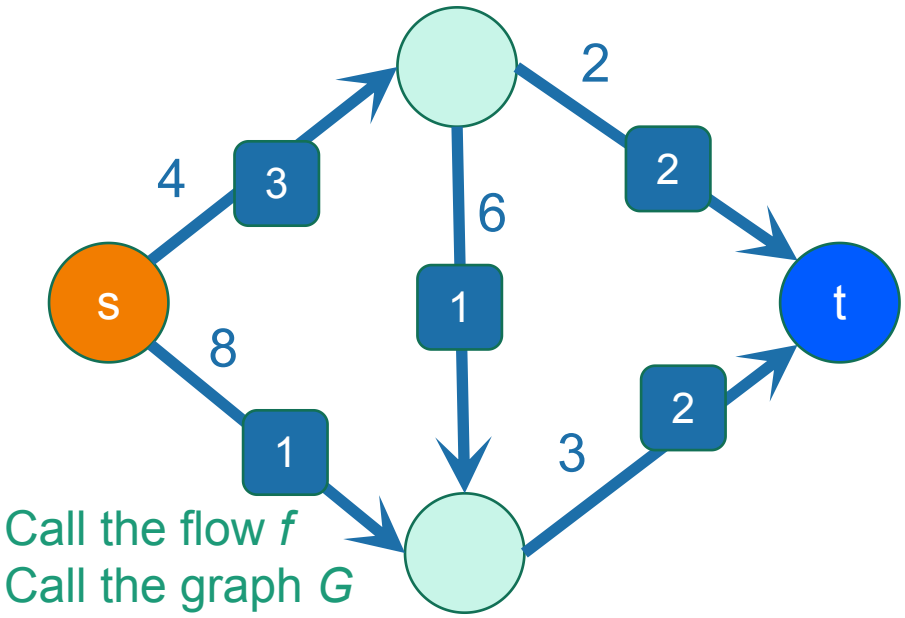
Create a new **residual network** from this flow:

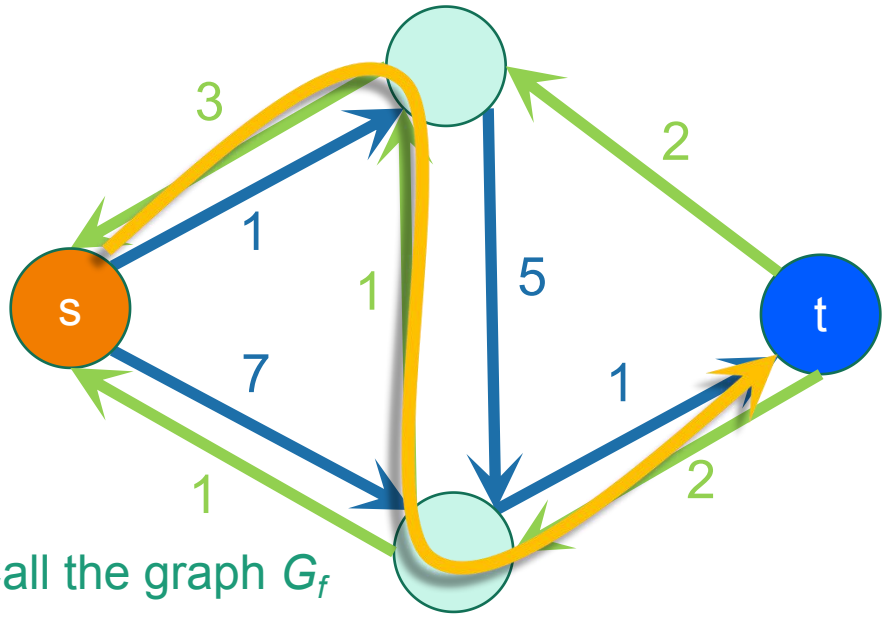
Why do we care about residual networks?

Lemma:

- t is not reachable from s in G_f f is a max flow.

Example: **t is reachable from s in this example, so not a max flow.**



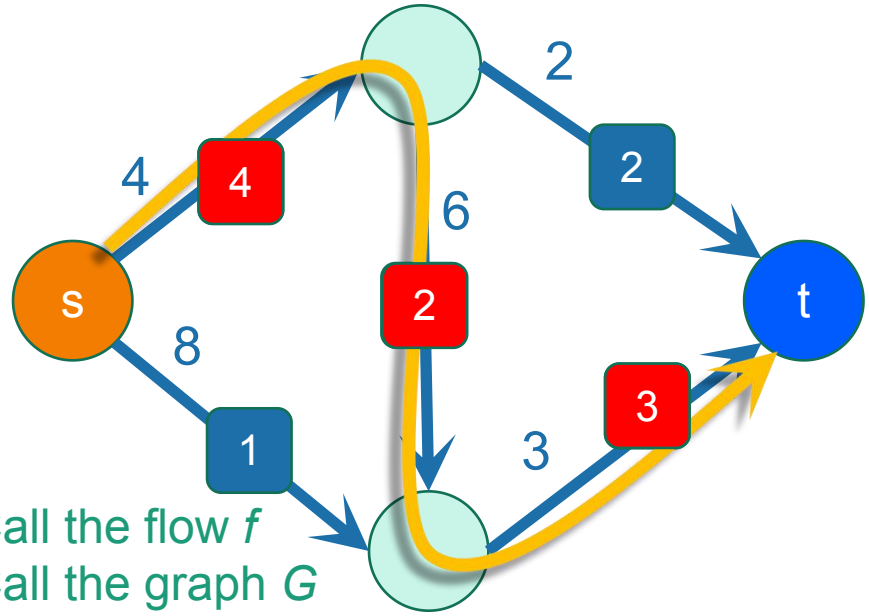


Why do we care about residual networks?

Lemma:

- t is not reachable from s in G_f f is a max flow.

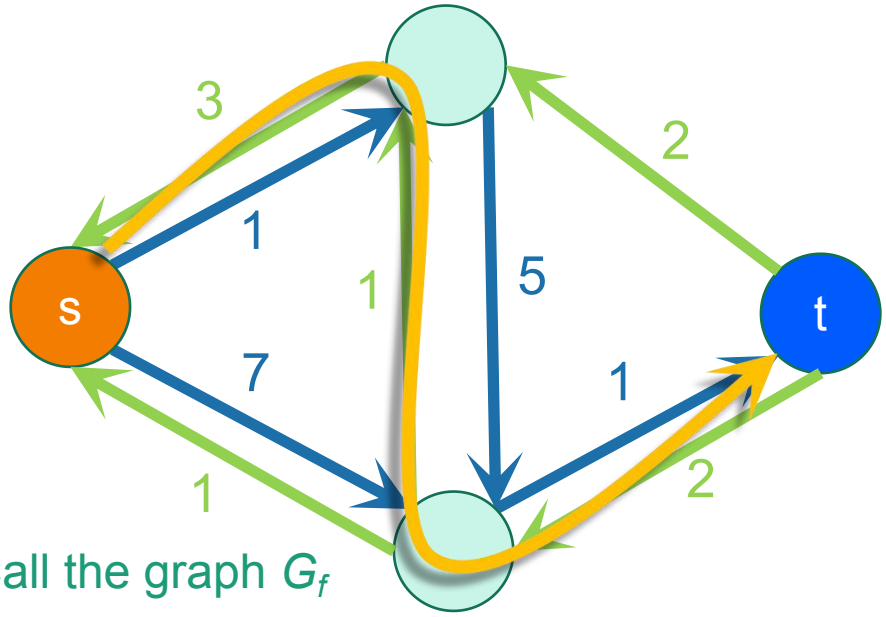
To see that this flow is not maximal, notice that we can improve it by sending one more unit of stuff along this path:



Call the flow f
Call the graph G

Example: **s is reachable from t in this example, so not a max flow.**

Now update the residual graph…



Call the graph G_f

Why do we care about residual networks?

Lemma:

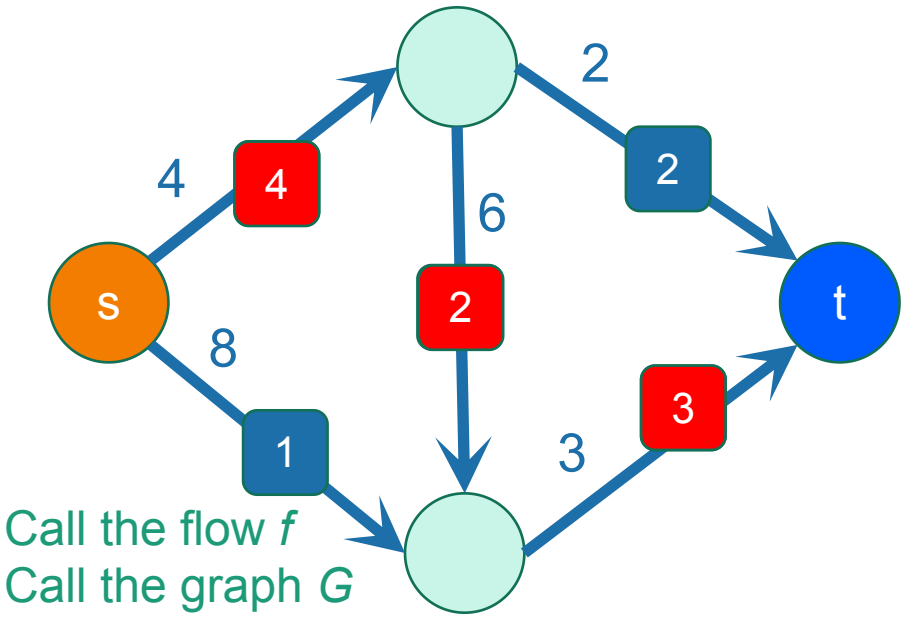
- t is not reachable from s in G_f f is a max flow.

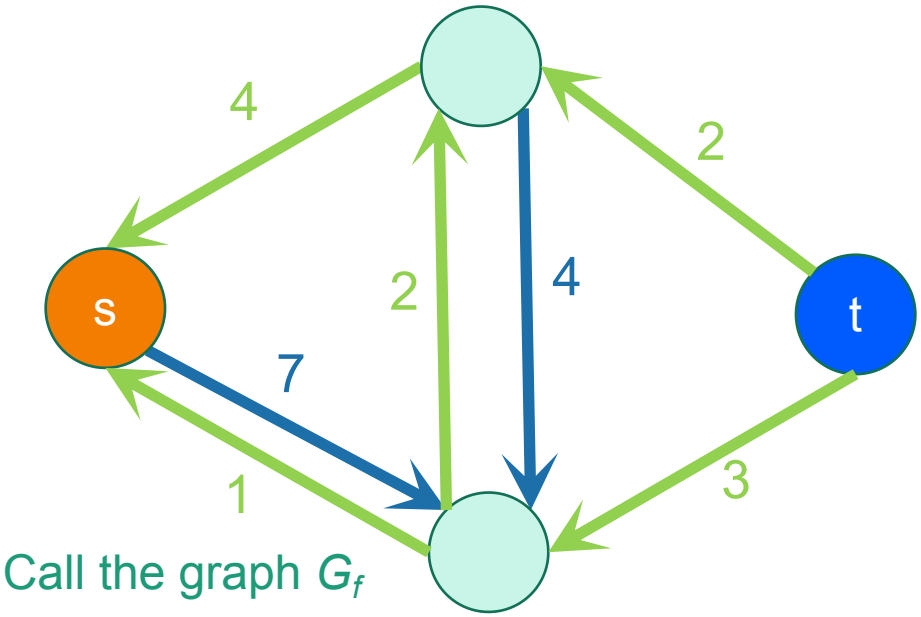
Example:

Now we get this residual graph:

Now we can't reach t from s.

So the lemma says that f is a max flow.





Let’s prove the Lemma

- t is not reachable from s in G_f **<->** *f* is a max flow.

Lemma:

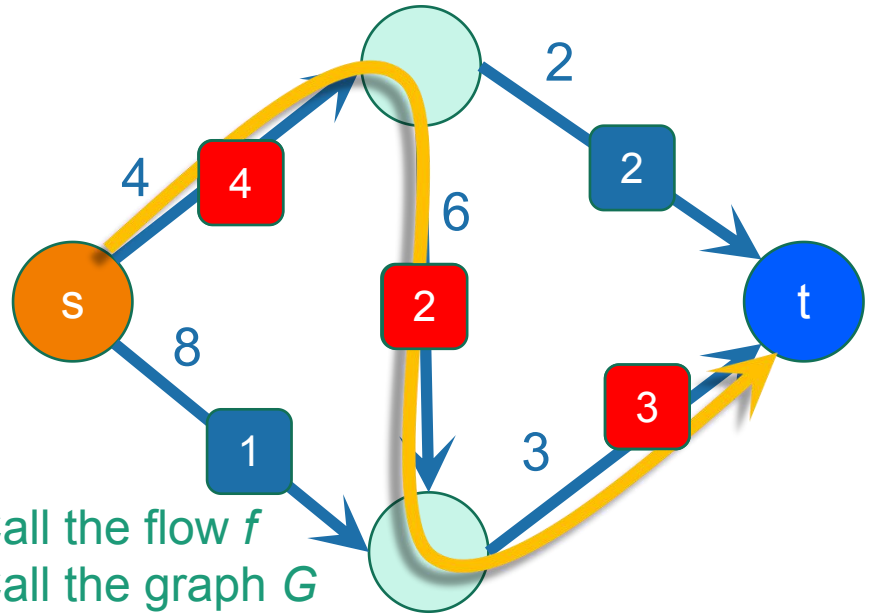
first this direction

We will prove the contrapositive

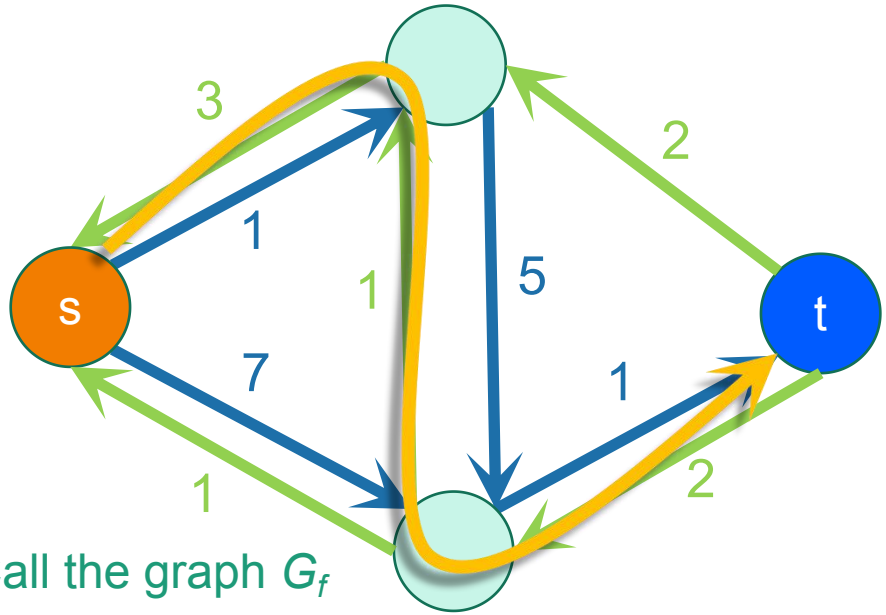
t is not reachable from s in G_f ***f*** is a max flow.

- Suppose there is a path from s to t in G_f.
  - This is called augmenting path.
- Claim: if there is an augmenting path, we can increase the flow along the path.
- So do that and update flow.
- This results in a bigger flow
  - So we can't have started with a max flow.

We will come back to this in a second.



Call the flow *f*
Call the graph *G*

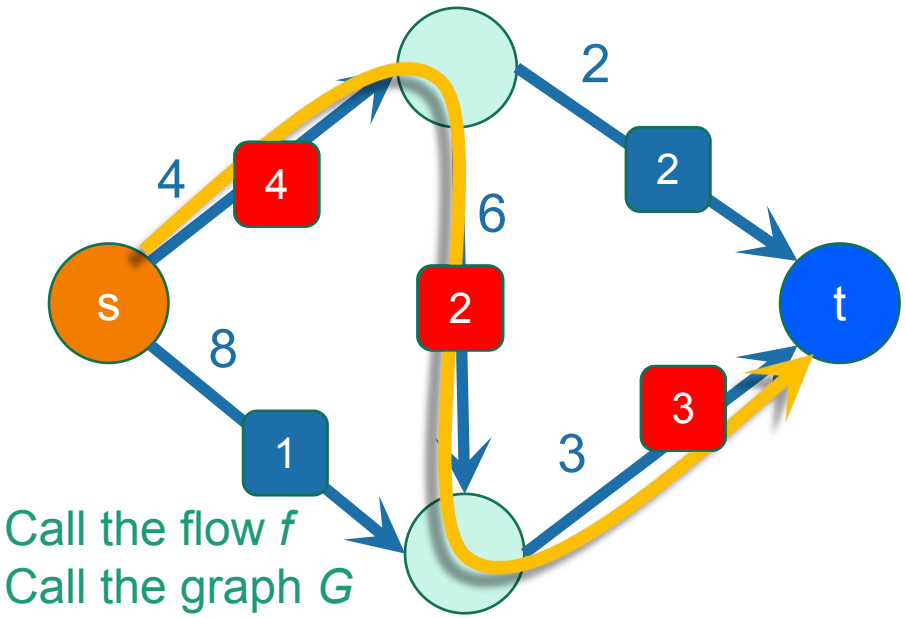


Call the graph G_f

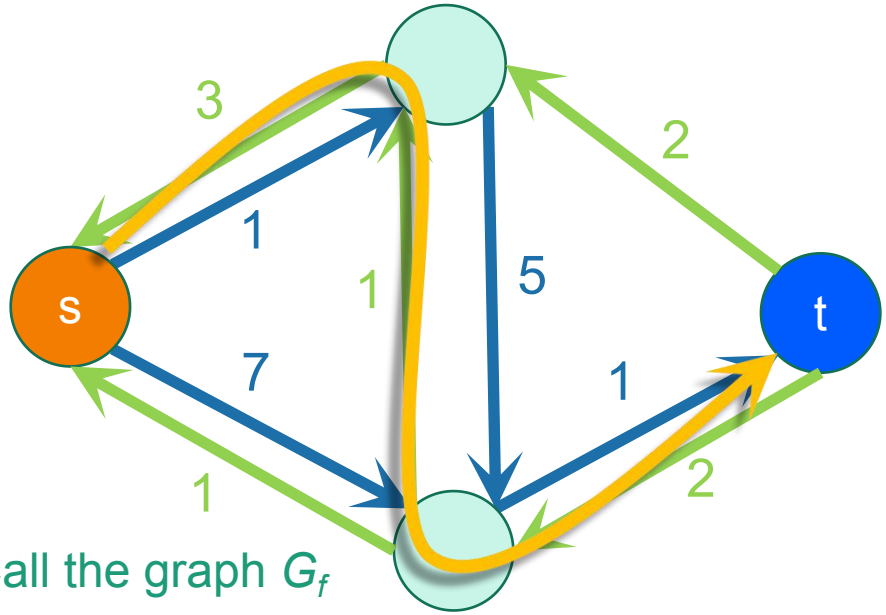
Claim:

if there is an augmenting path, we can increase the flow along the path.

- In the situation we just saw, this is pretty obvious.



Call the flow *f*
Call the graph *G*



Call the graph G_f

- Every edge on the path was below capacity, so just increase the flow on all the edges.

Claim:

if there is an augmenting path, we can increase the flow along the path.

Check that this always makes a bigger flow!

Ollie the over-achieving ostrich

- increaseFlow(path P in G_f , flow f):
  - x = min weight on any edge in P
  - **for** (u,v) in P:
    - **if** (u,v) in E, $f'(u,v) \leftarrow f(u,v) + x.$
    - **if** (v,u) in E, $f'(u,v) \leftarrow f(u,v) - x.$
  - **return** *f'*

That proves the **claim**

If there is an augmenting path, we can increase the flow along that path

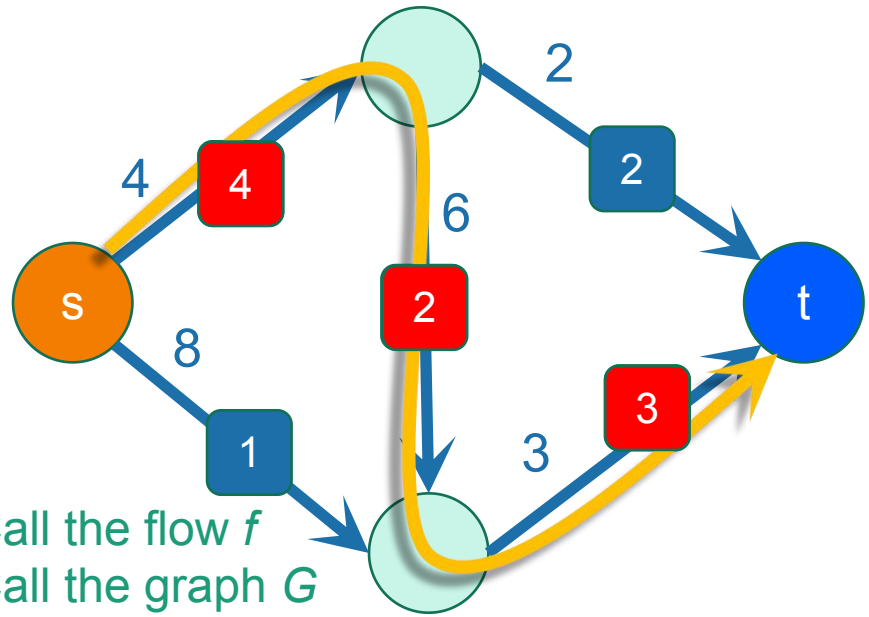
Lemma:

first this direction <-

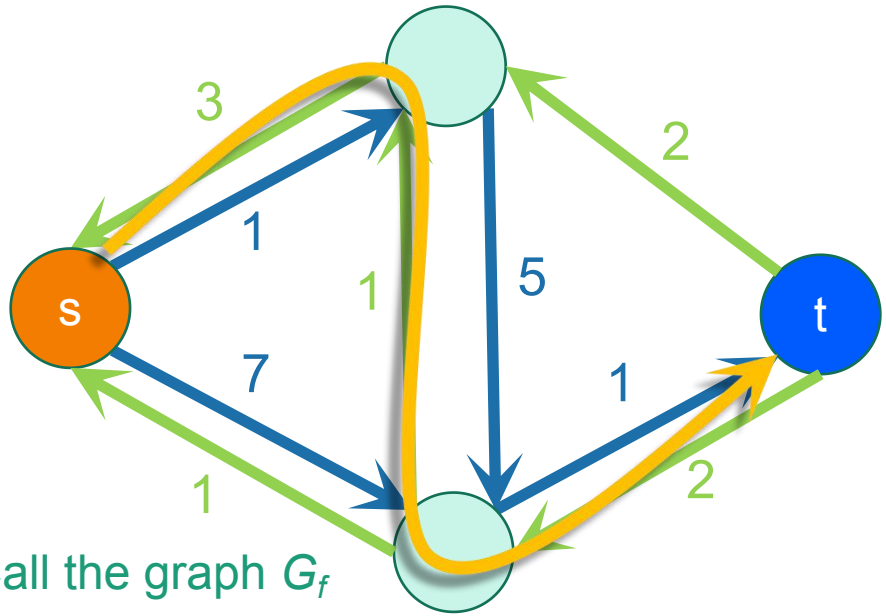
We will prove the contrapositive

t is not reachable from s in G_f <- f is a max flow.

- Suppose there is a path from s to t in G_f.
  - This is called augmenting path.
- Claim: if there is an augmenting path, we can increase the flow along the path.
- So do that and update flow.
- This results in a bigger flow
  - So we can't have started with a max flow.



Call the flow f
Call the graph G



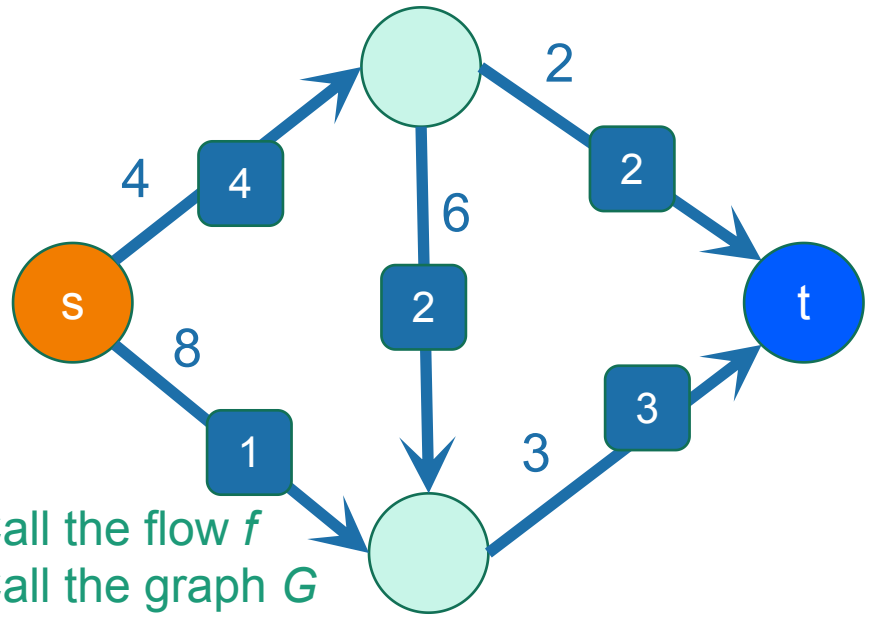
Call the graph G_f

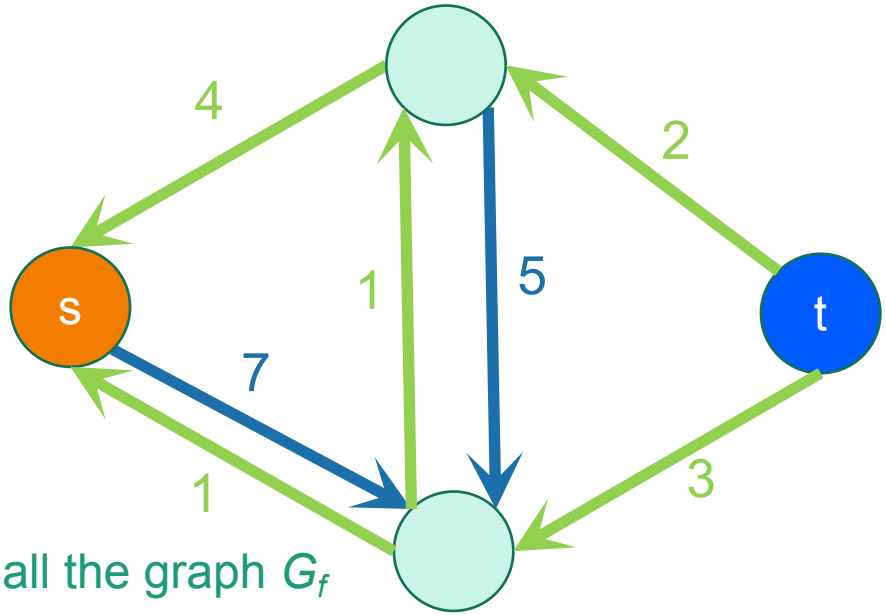
Lemma: now this direction ->

t is not reachable from s in G_f -> ***f*** is a max flow.

- Suppose there is no path from s to t in G_f.
- Consider the cut given by:

  {things reachable from s} , **{things not reachable from s}**





Lemma:

now this direction ->

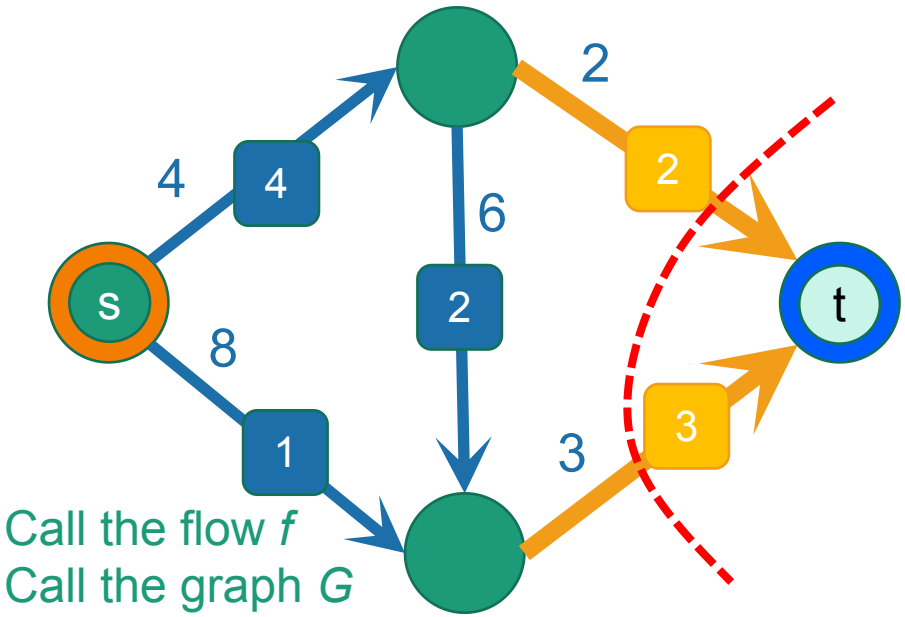
t is not reachable from s in G_f -> *f* is a max flow.

- Suppose there is no path from s to t in G_f.
- Consider the cut given by:

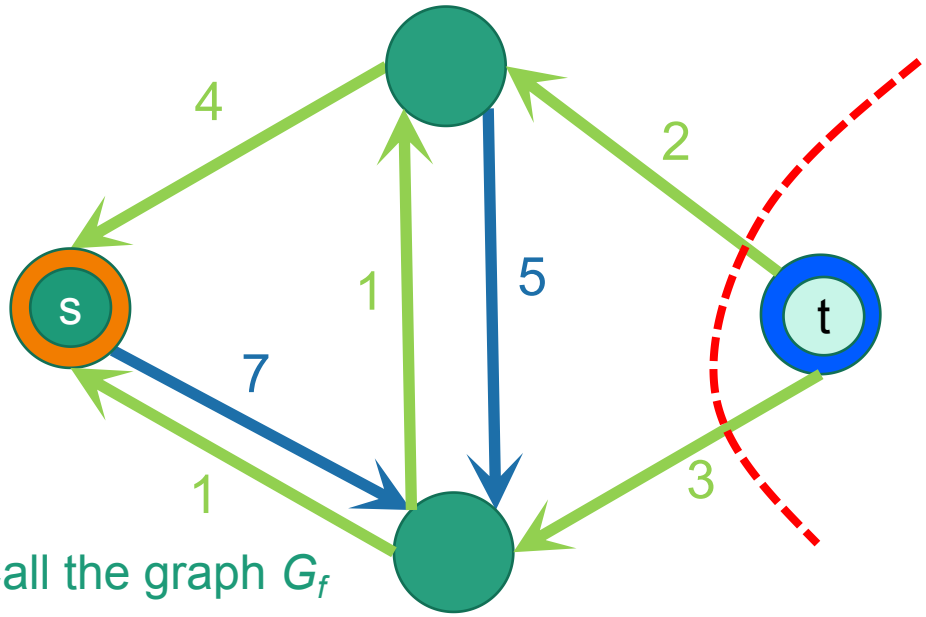
  t lives here

  {things reachable from s} , **{things not reachable from s}**
- The flow from s to t is **equal** to the cost of this cut
  - Similar to proof-by-picture we saw before:
    - All the stuff has to cross the cut
    - The edges in the cut are full because they don't exist in G_f
- **thus**: this flow value = cost of this cut = cost of min cut = max flow

Lemma 1



Call the flow *f*
Call the graph *G*



Call the graph G_f

Lemma: now this direction ->

t is not reachable from s in G_f -> *f* is a max flow.

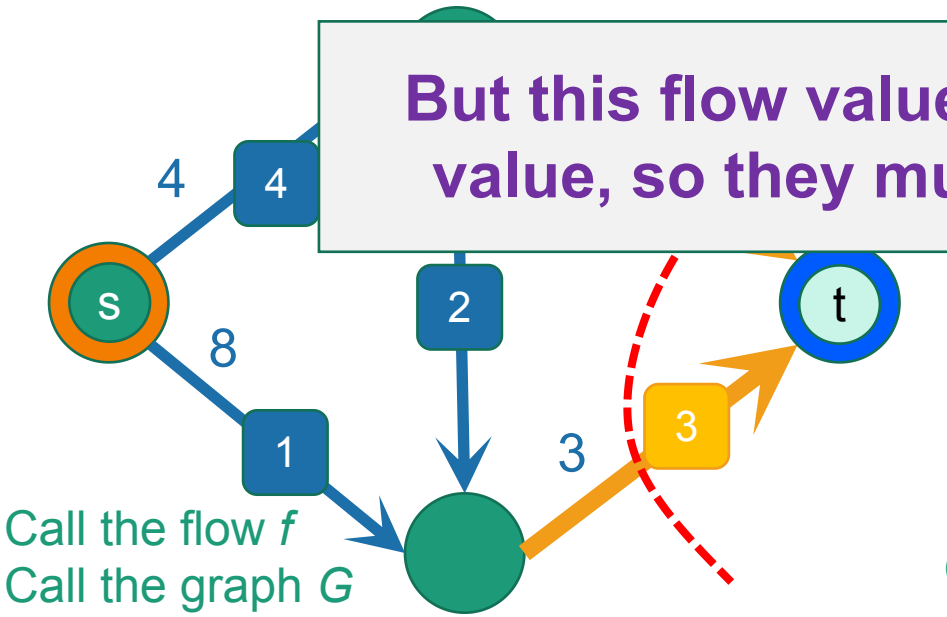
- Suppose there is no path from s to t in G_f.
- Consider the cut given by:

  t lives here

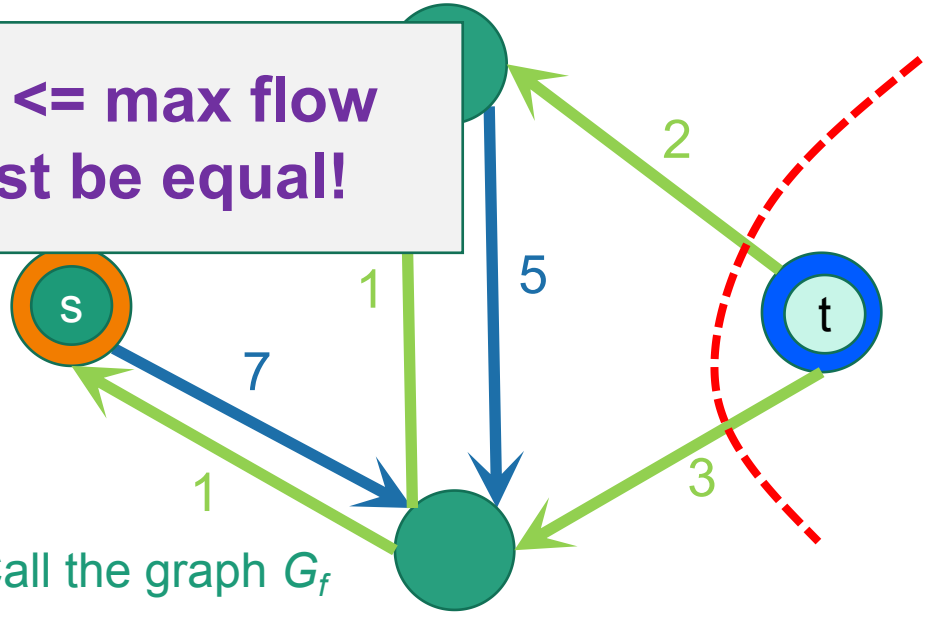
  {things reachable from s} , **{things not reachable from s}**
- The flow from s to t is **equal** to the cost of this cut
  - Similar to proof-by-picture we saw before:
    - All the stuff has to cross the cut
    - The edges in the cut are full because they don't exist in G_f
- **thus**: this flow value = cost of this cut = cost of min cut = max flow

Lemma 1

But this flow value <= max flow value, so they must be equal!



Call the flow *f*
Call the graph *G*



Call the graph G_f

We've proved:

- t is not reachable from s in G_f **<->** f is a max flow

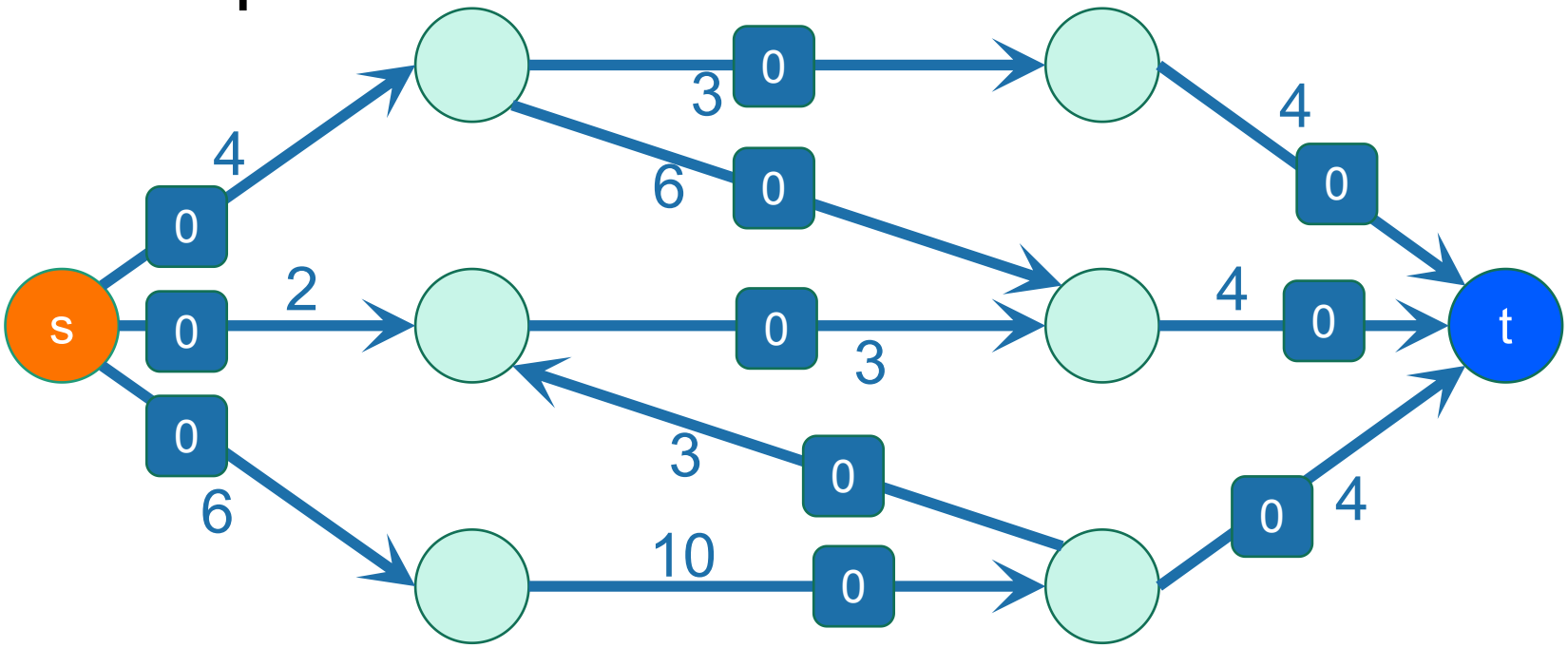
- This inspires an **algorithm**:
- Ford-Fulkerson(G):
  - f <-all zero flow.
  - G_f **<-** G
  - **while** t is reachable from s in G_f
    - Find a path P from s to t in G_f // eg, use BFS
    - f **<- increaseFlow**(P, f)
    - update G_f
  - return f

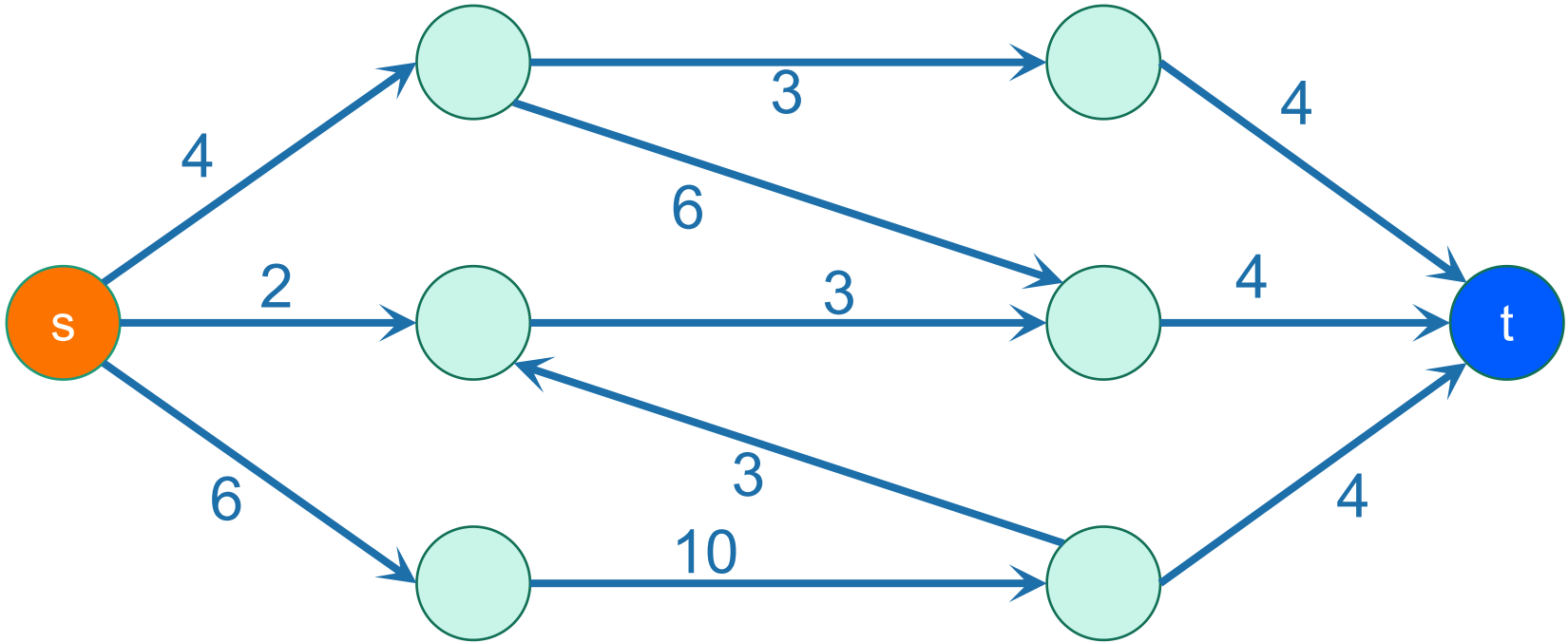
How do we choose which paths to use?

- The analysis we did still works no matter how we choose the paths.
  - That is, the algorithm will be **correct** if it terminates.
- **However, the algorithm may not be efficient!!!**
  - May take a long time to terminate
  - (Or may actually never terminate?)

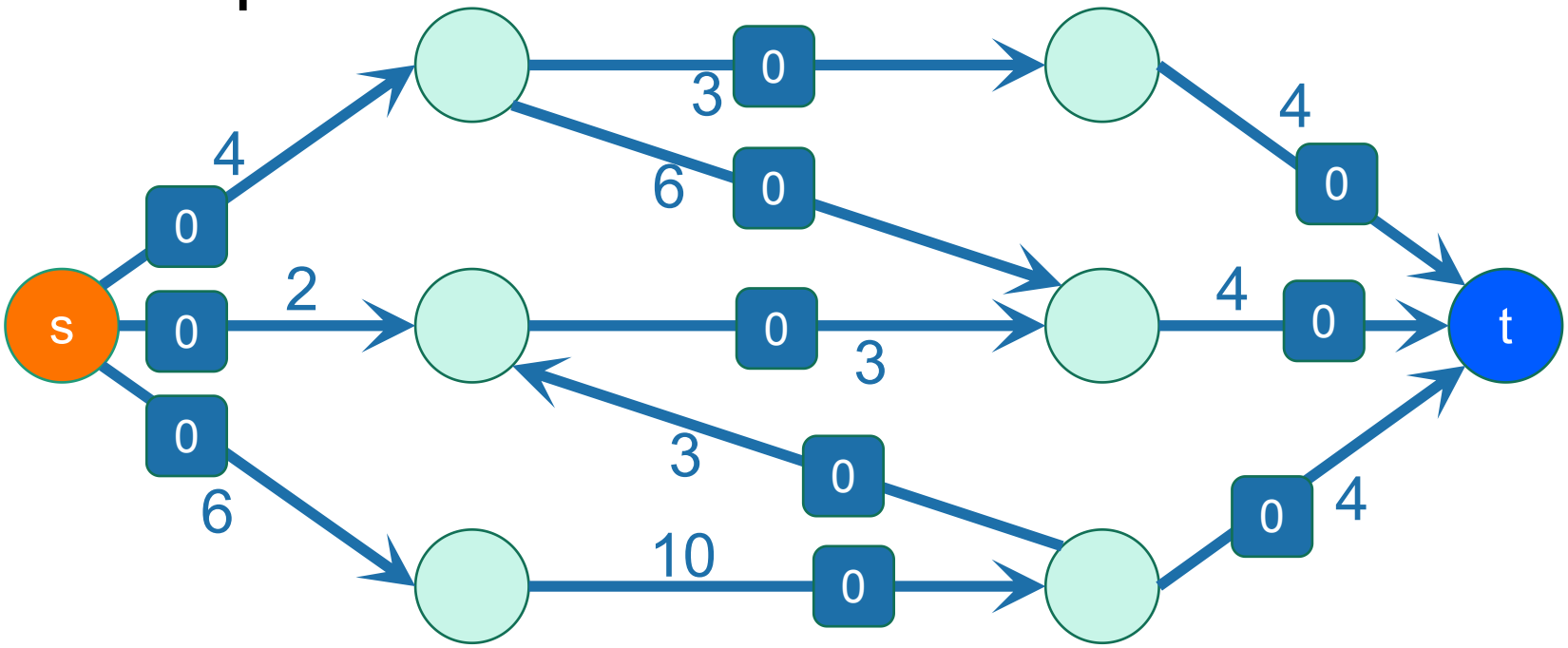
- We need to be careful with our path selection to make sure the algorithm terminates quickly.
  - Using BFS leads to the **Edmonds-Karp algorithm.** *(today)*
  - There's another way in the notes. *(optional)*

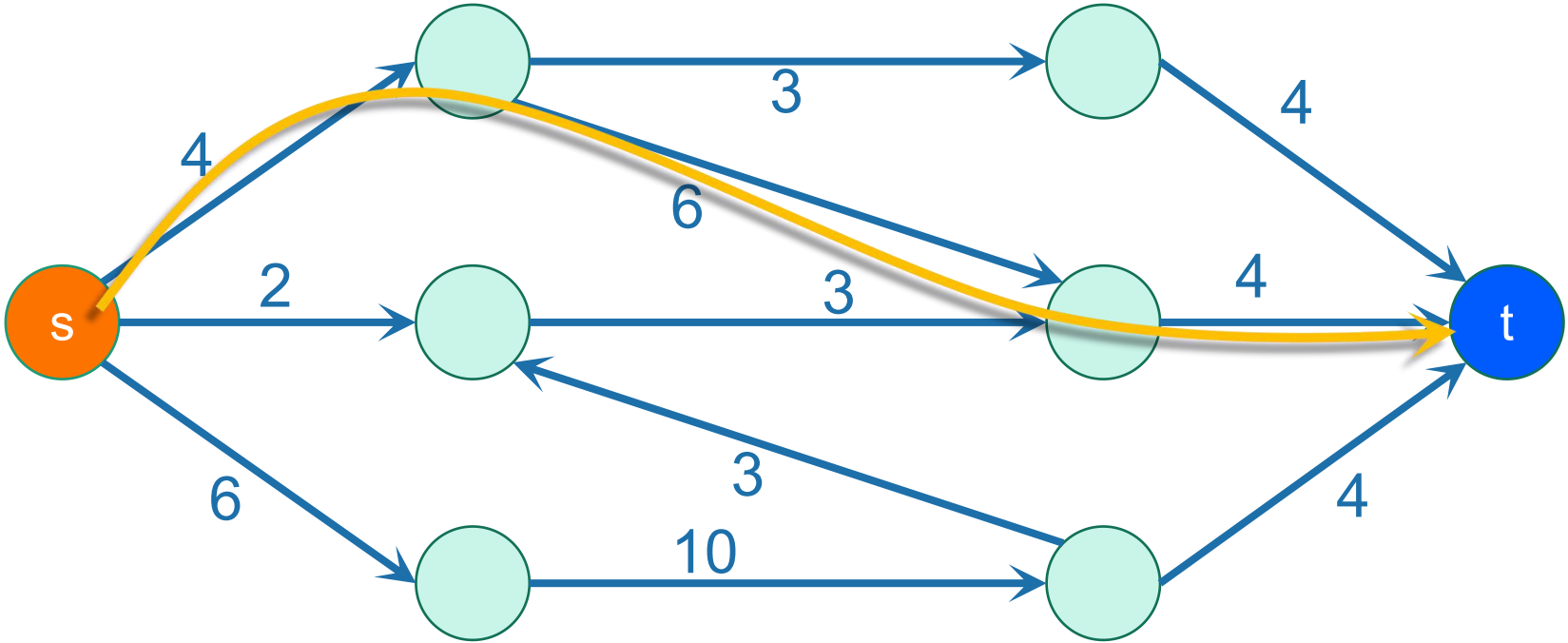
Example of Ford-Fulkerson



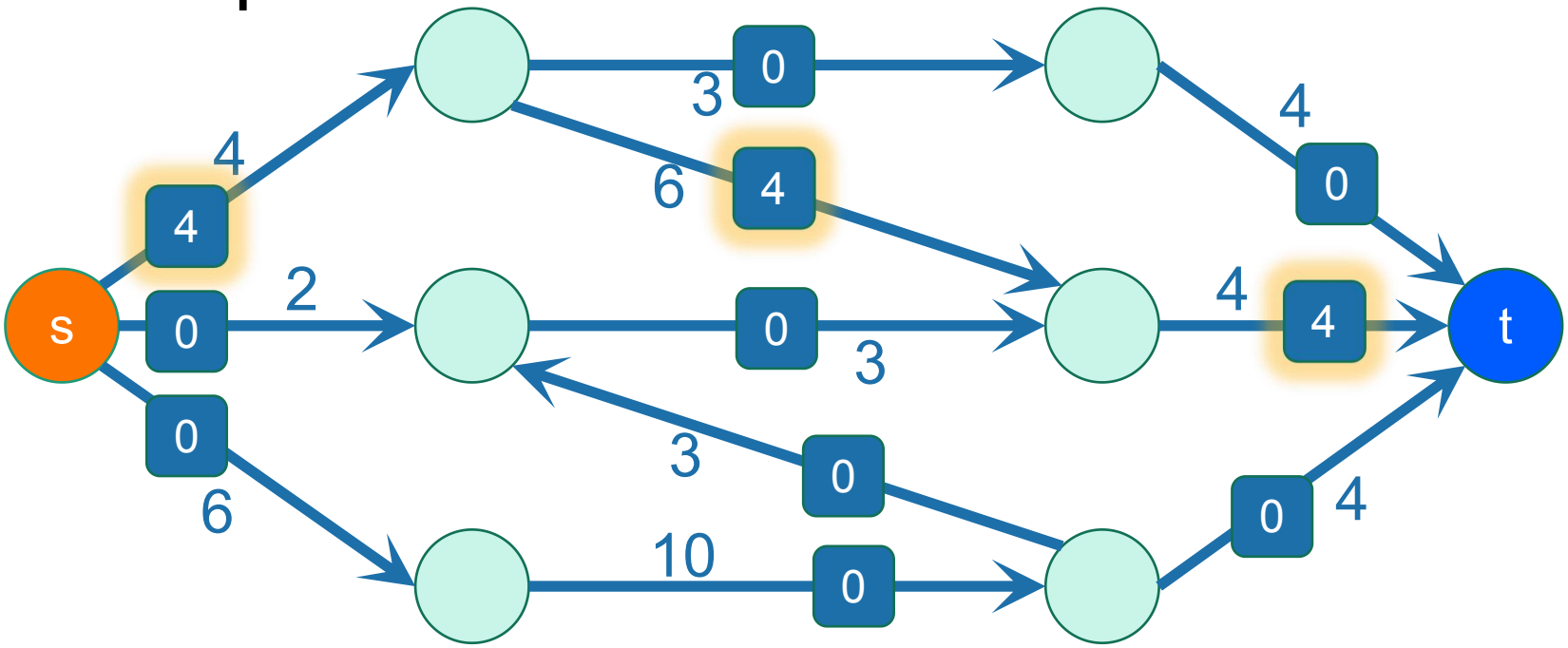


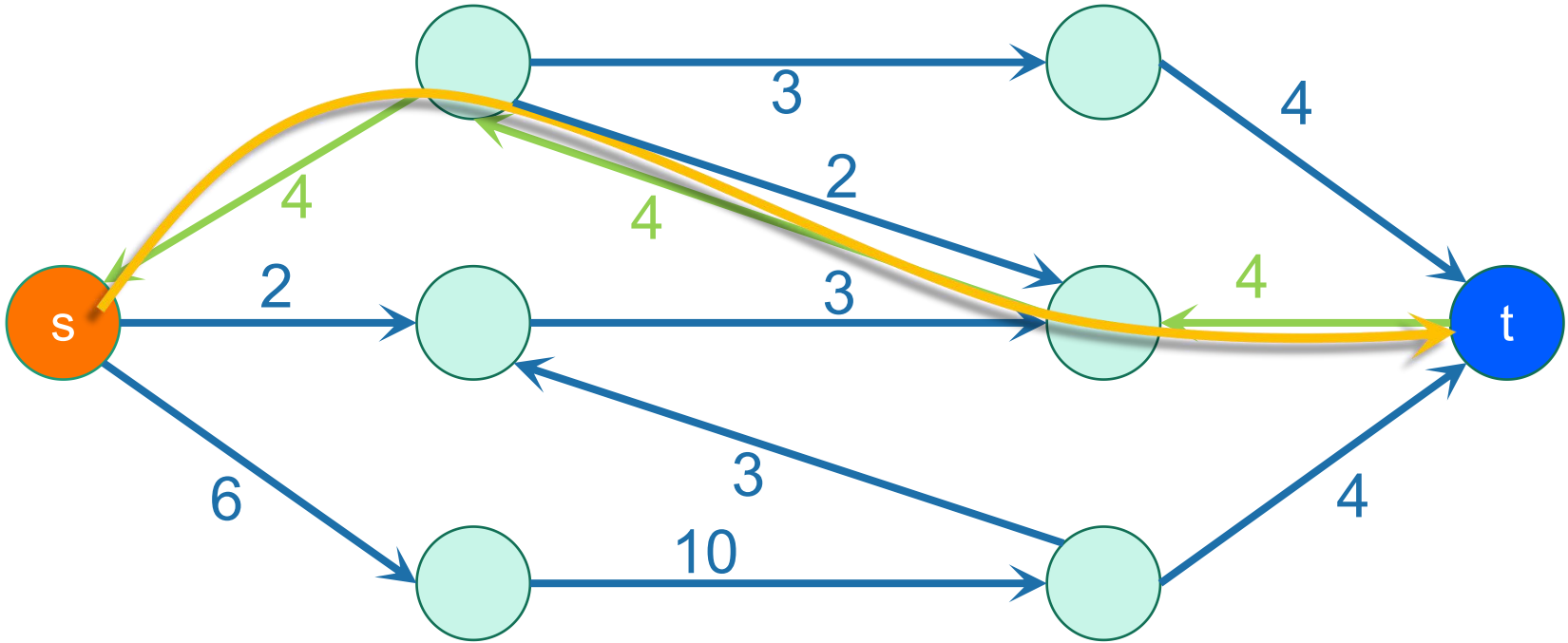
Example of Ford-Fulkerson





Example of Ford-Fulkerson





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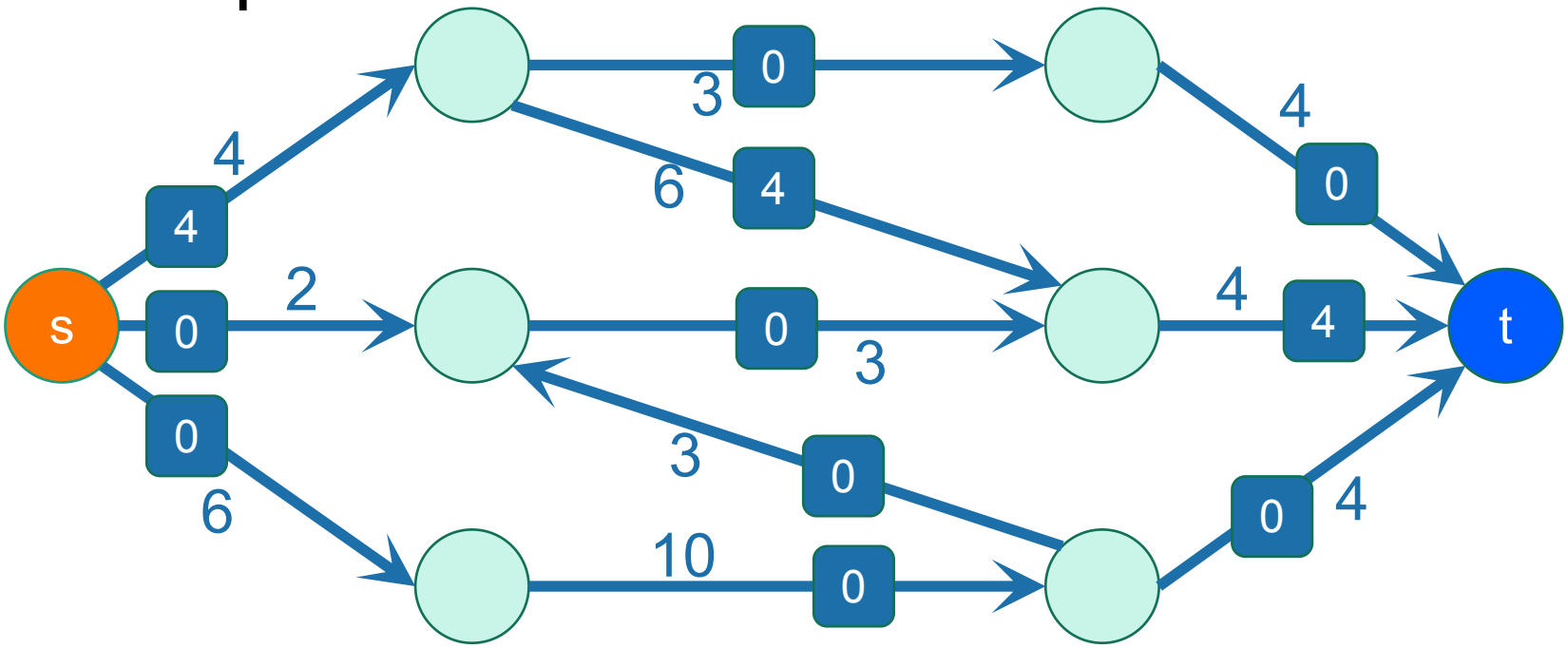

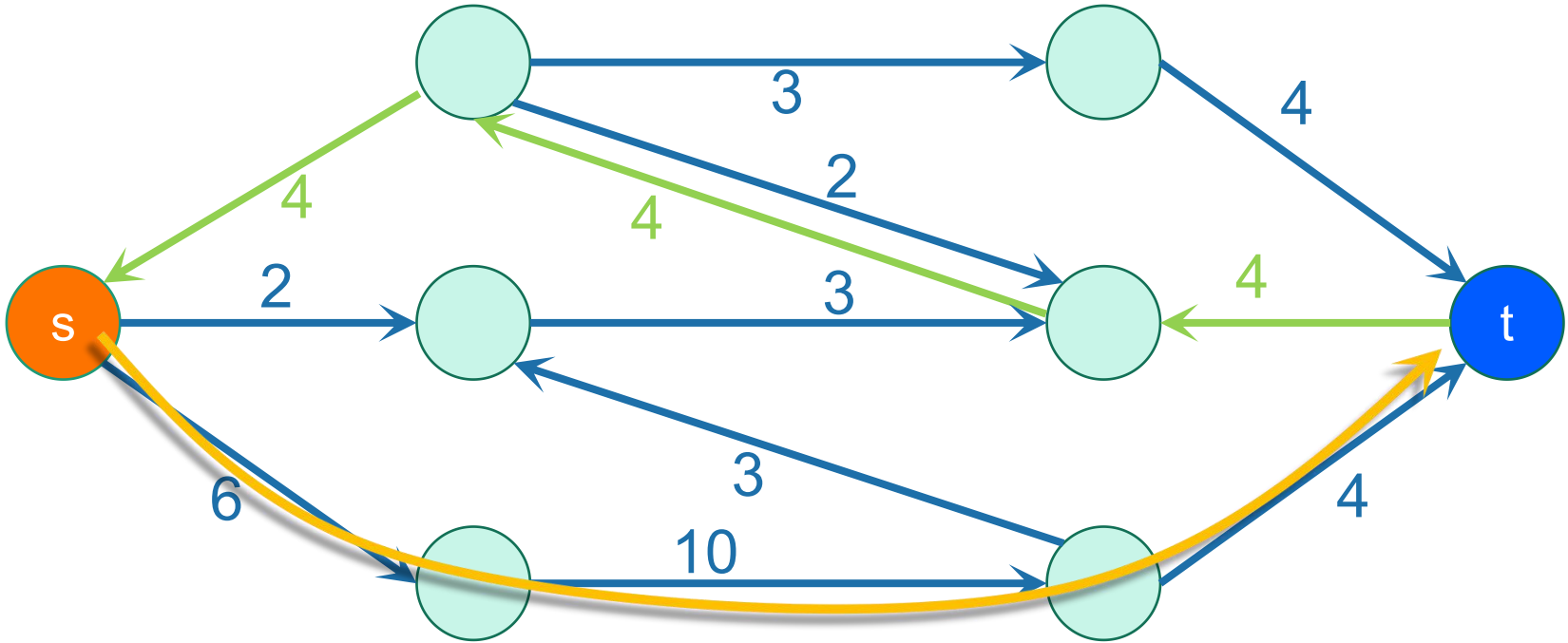

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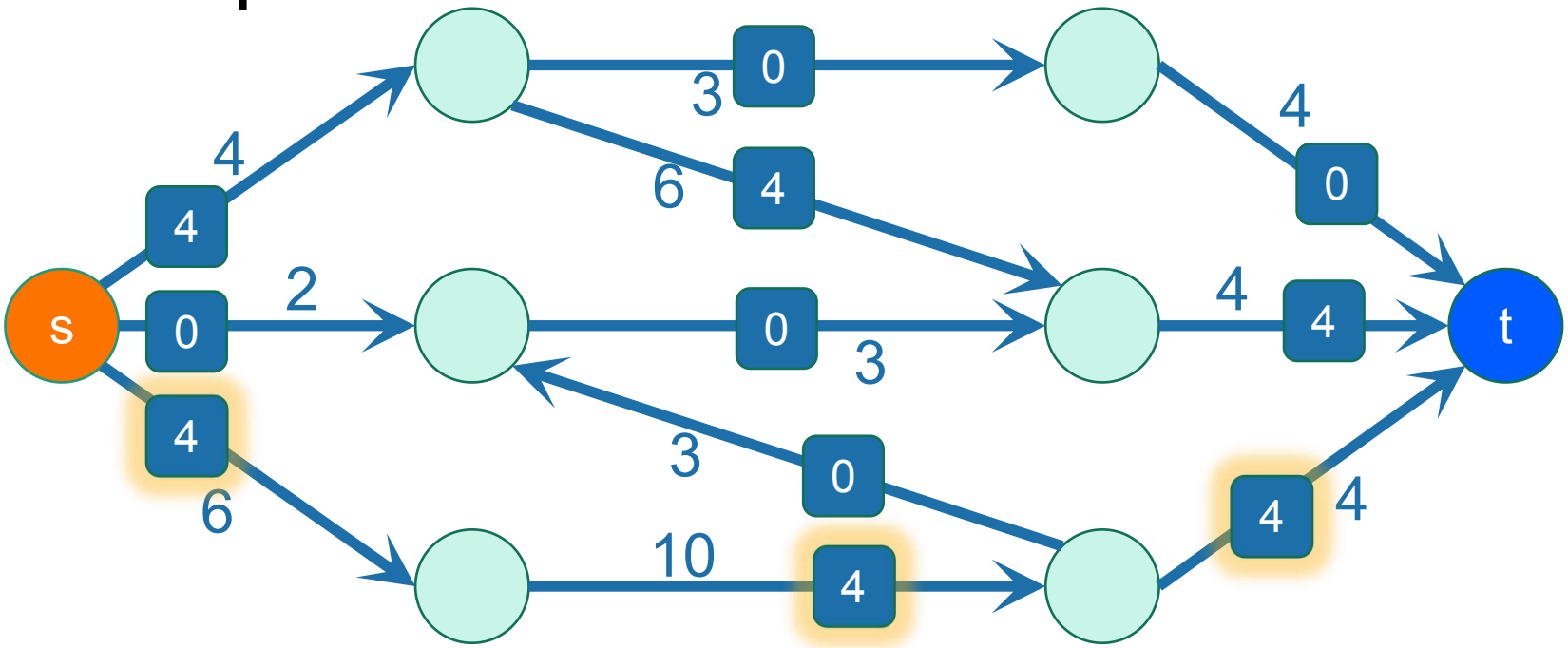

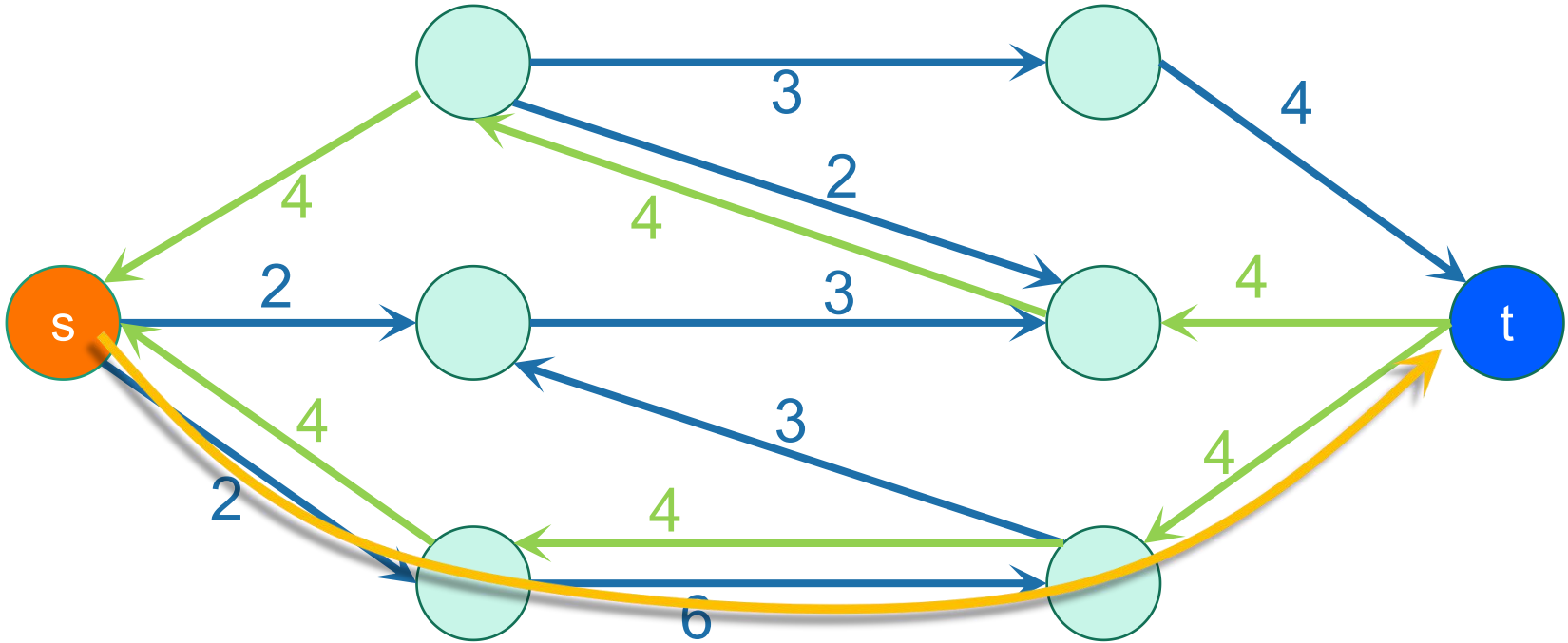

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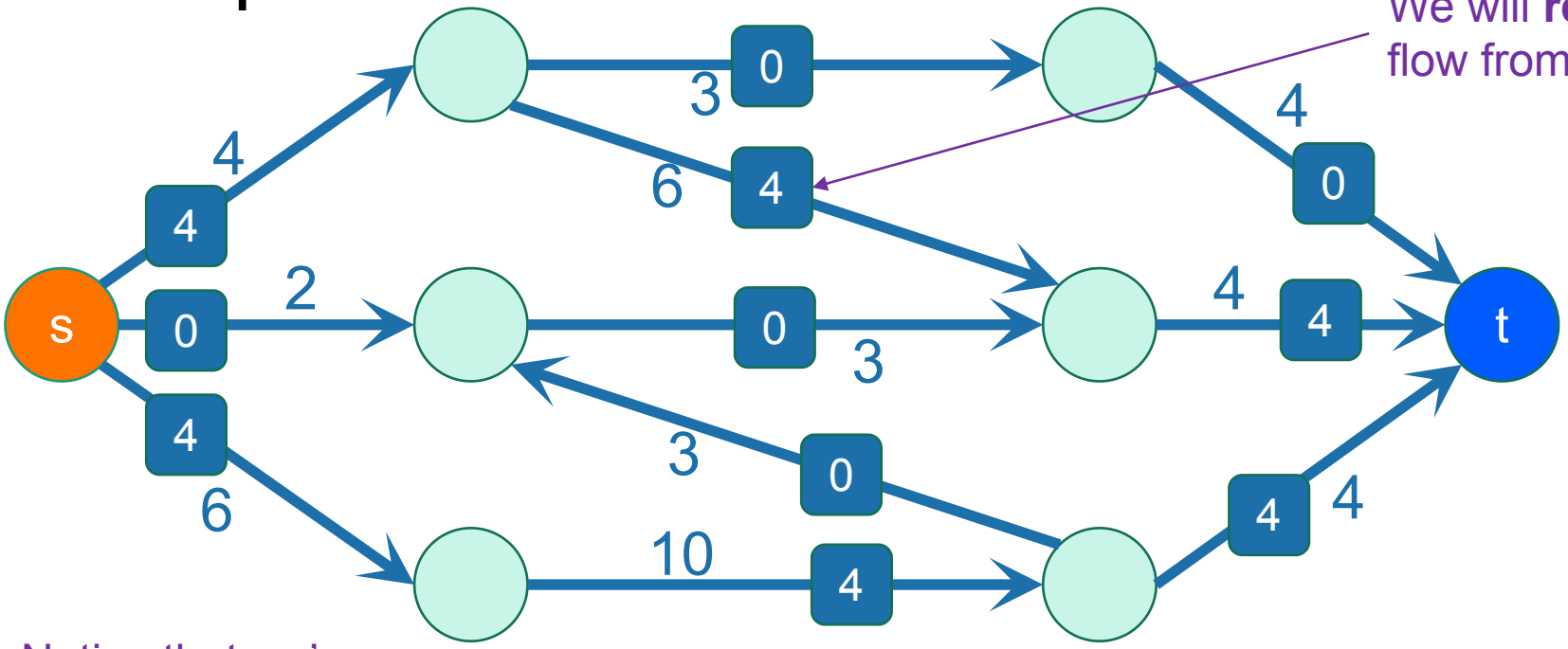
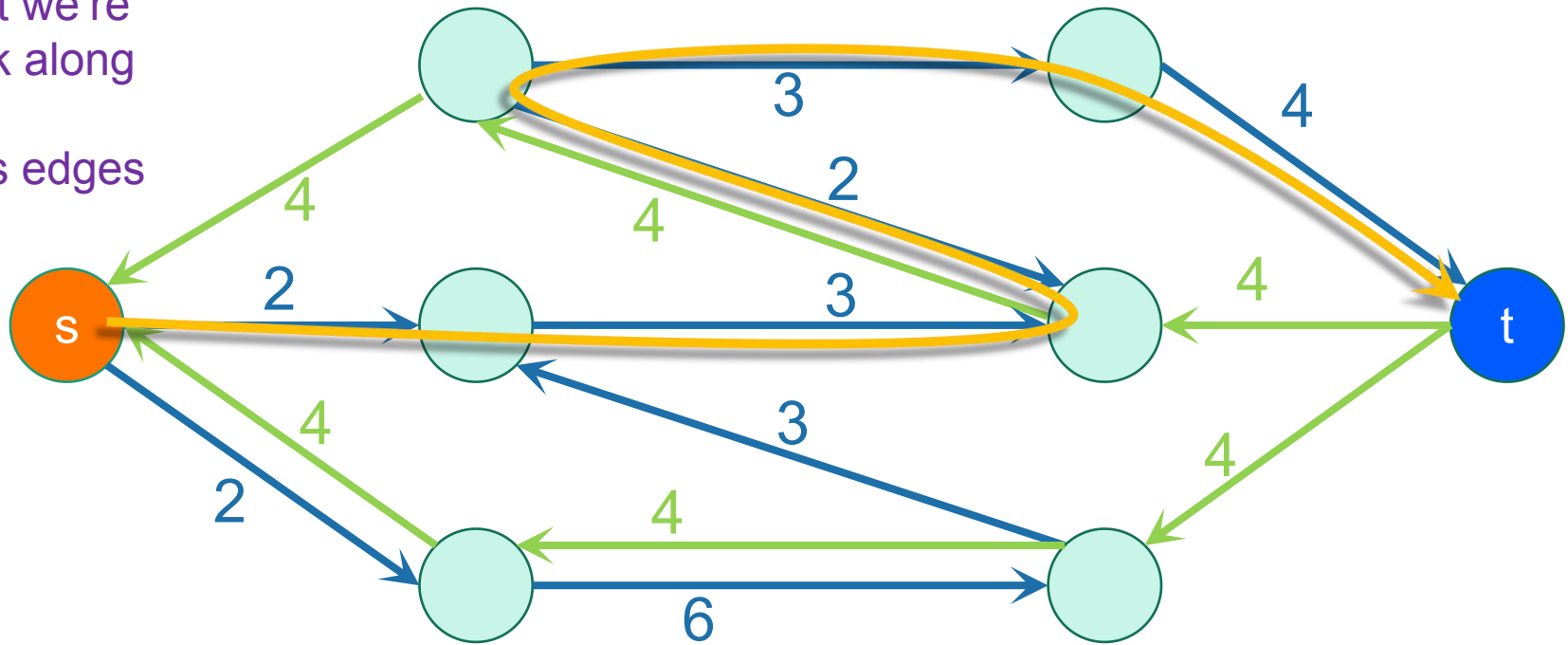


Notice that we're going back along one of the backwards edges we added.



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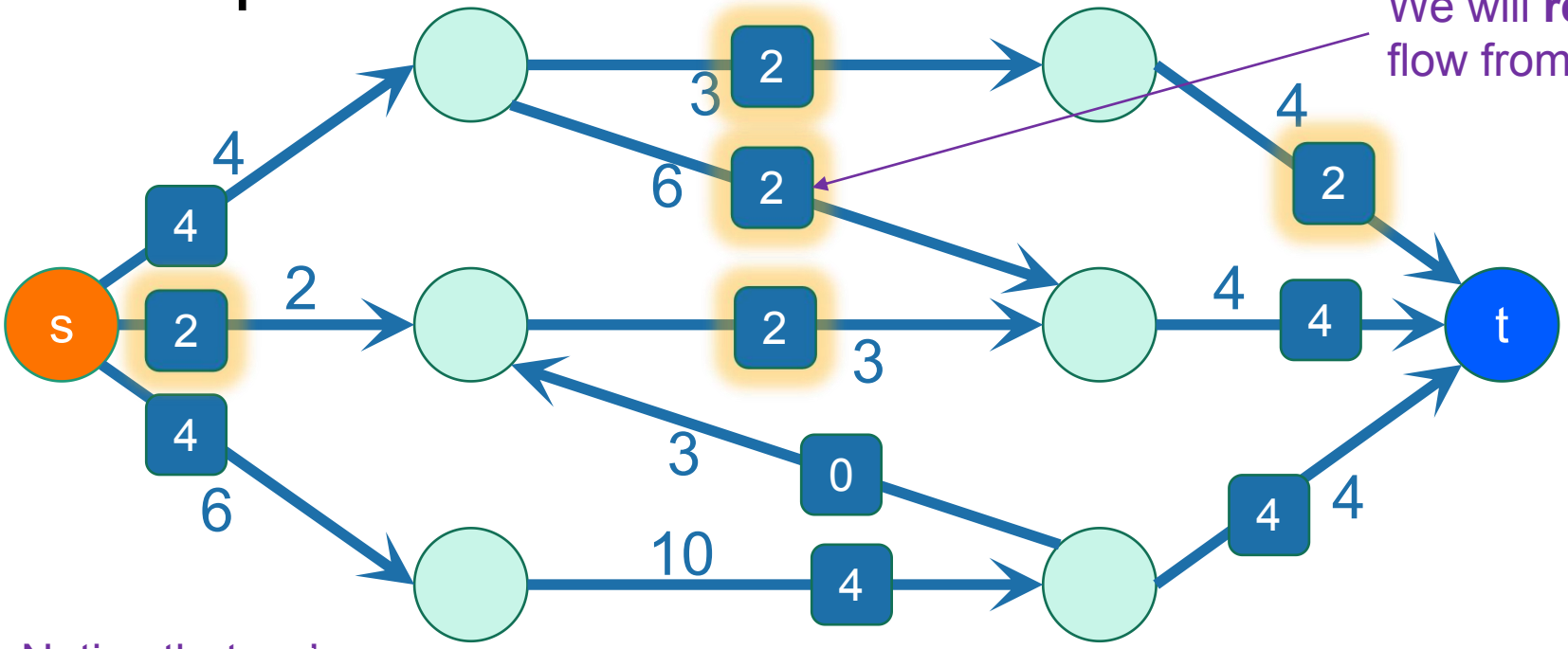

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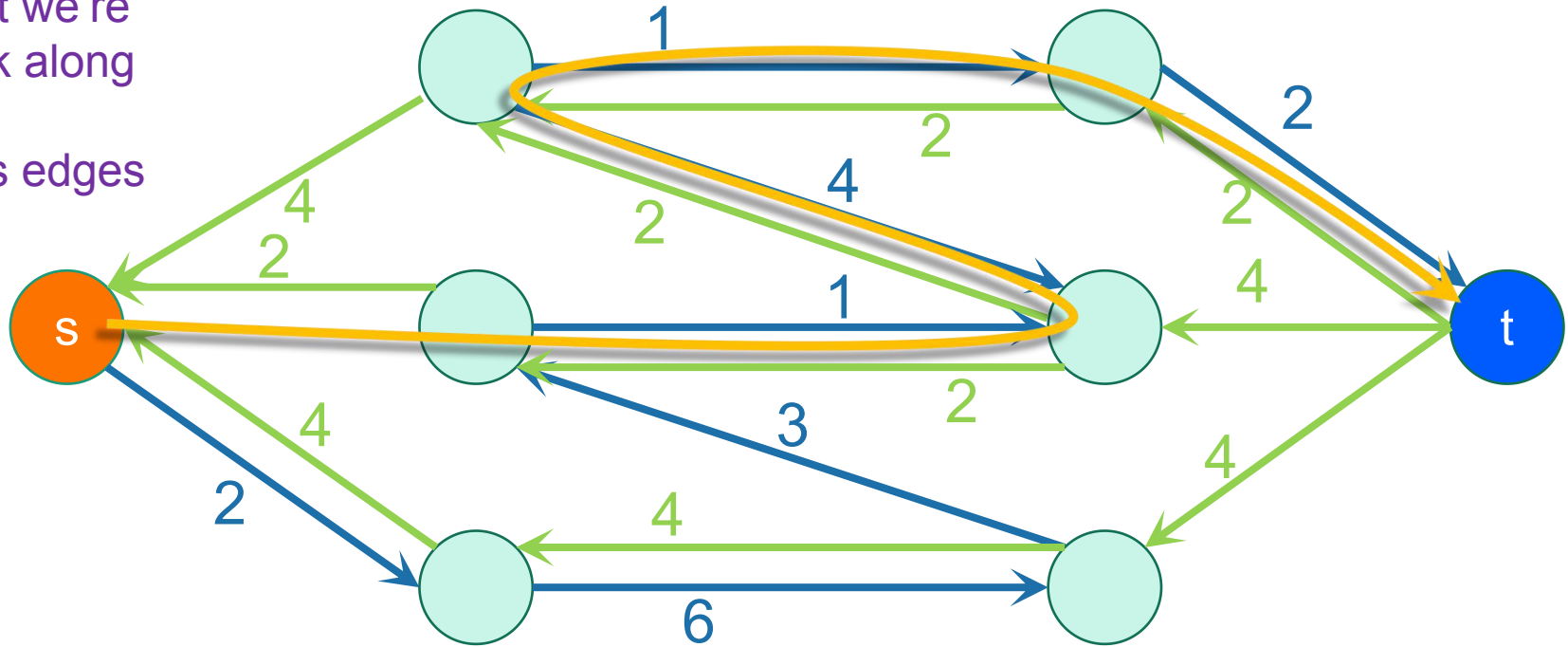
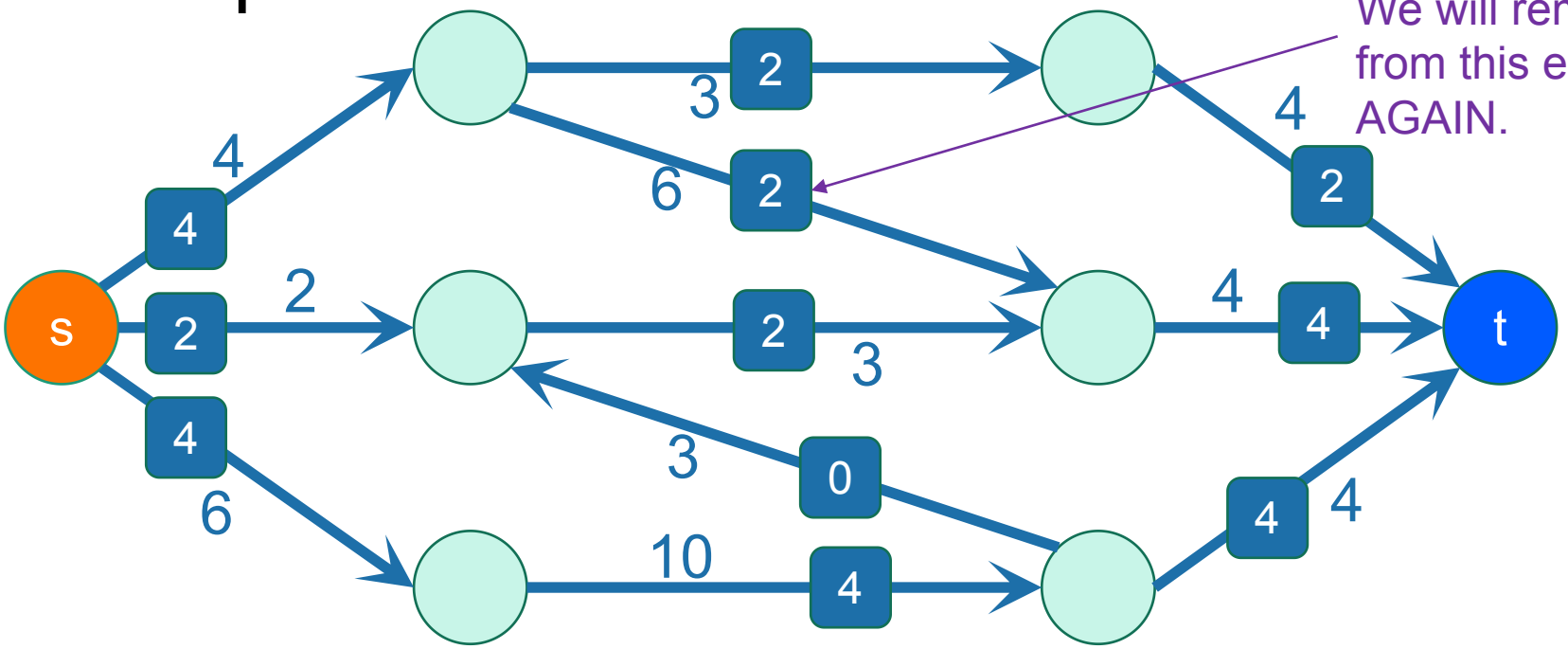
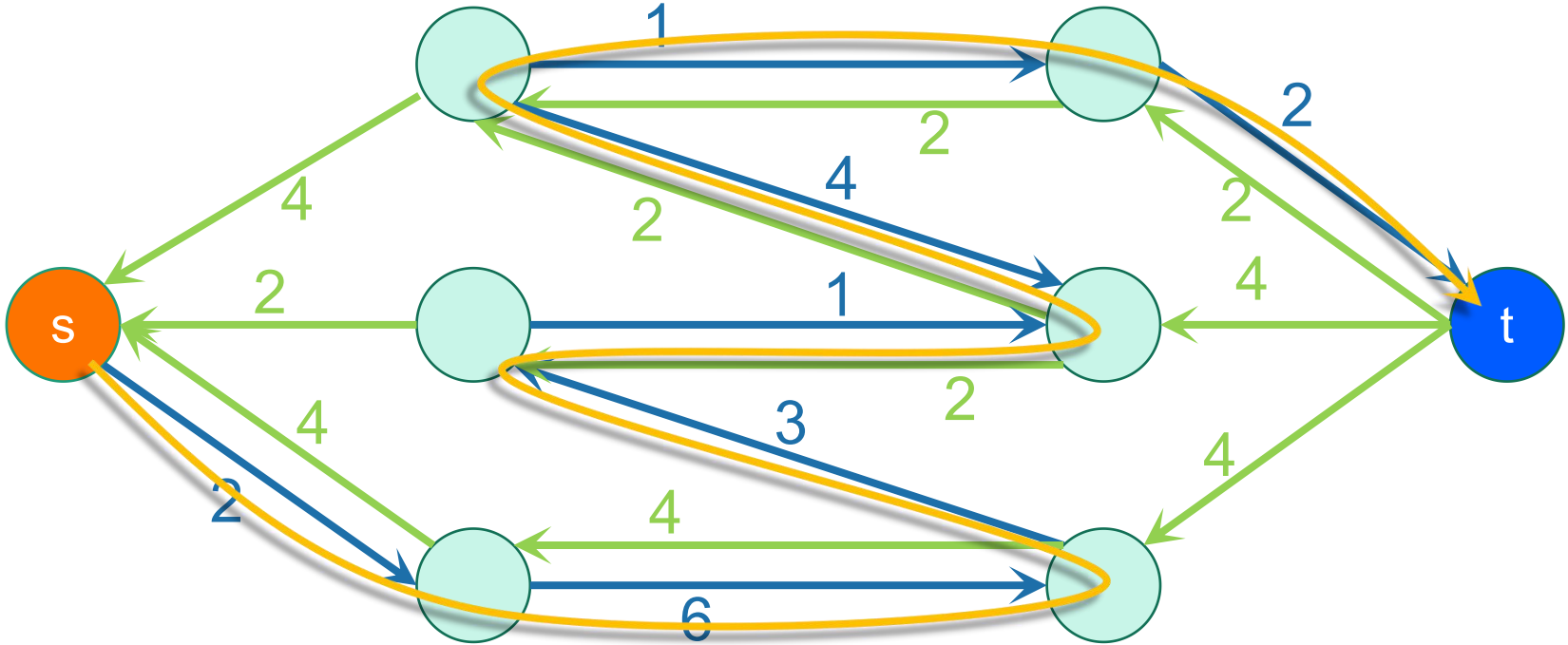


Example of Ford-Fulkerson





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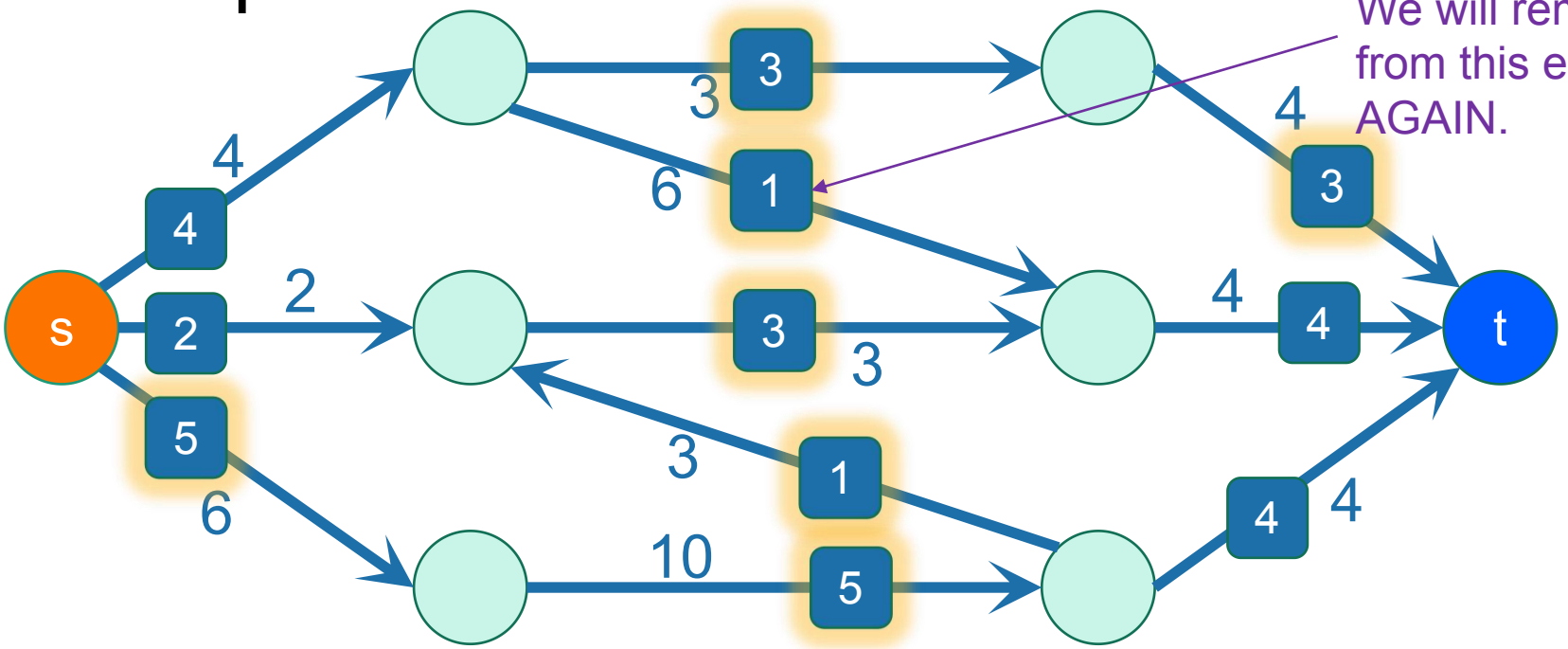

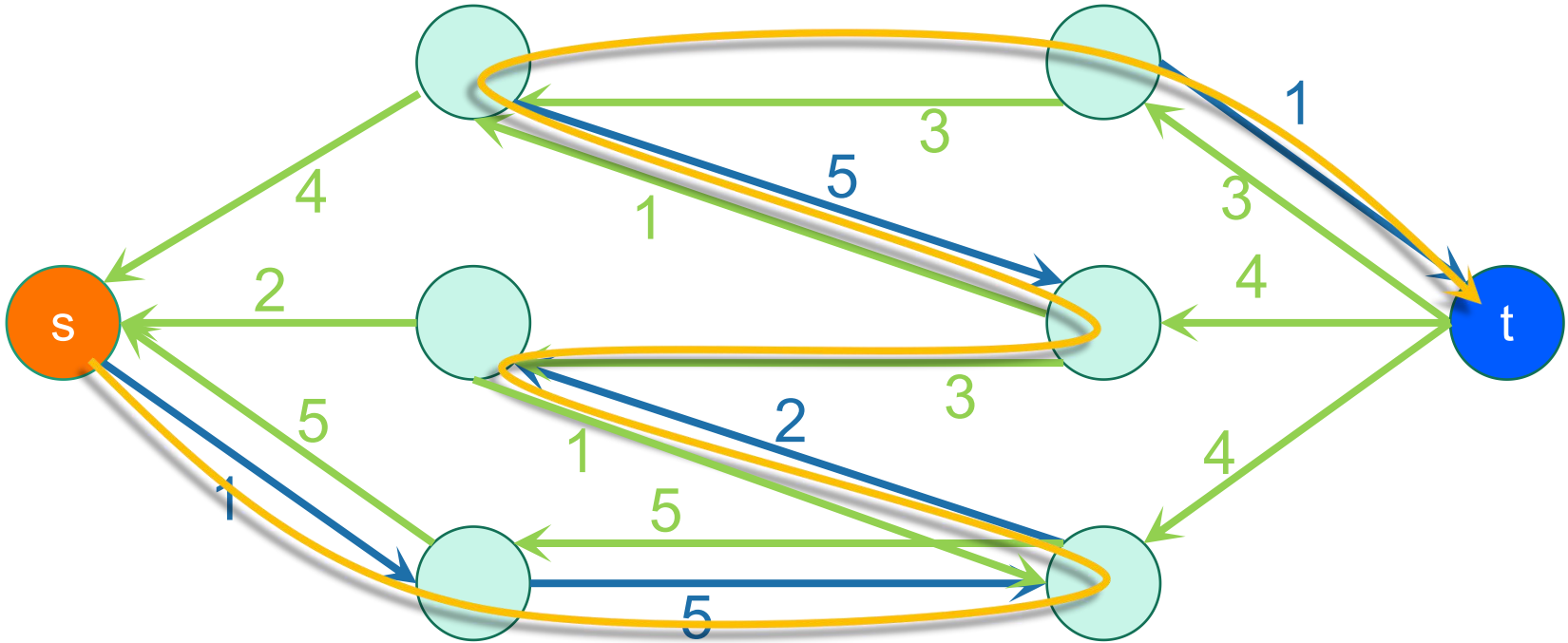

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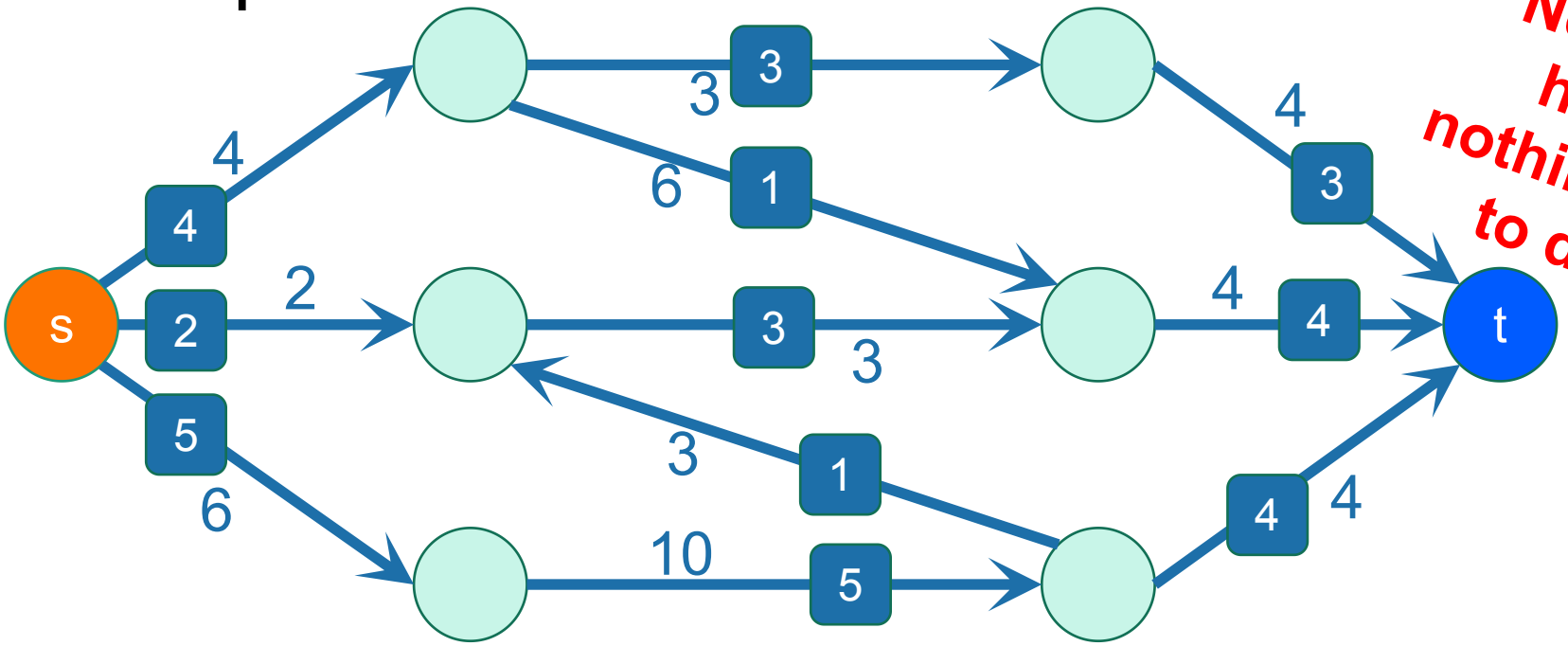
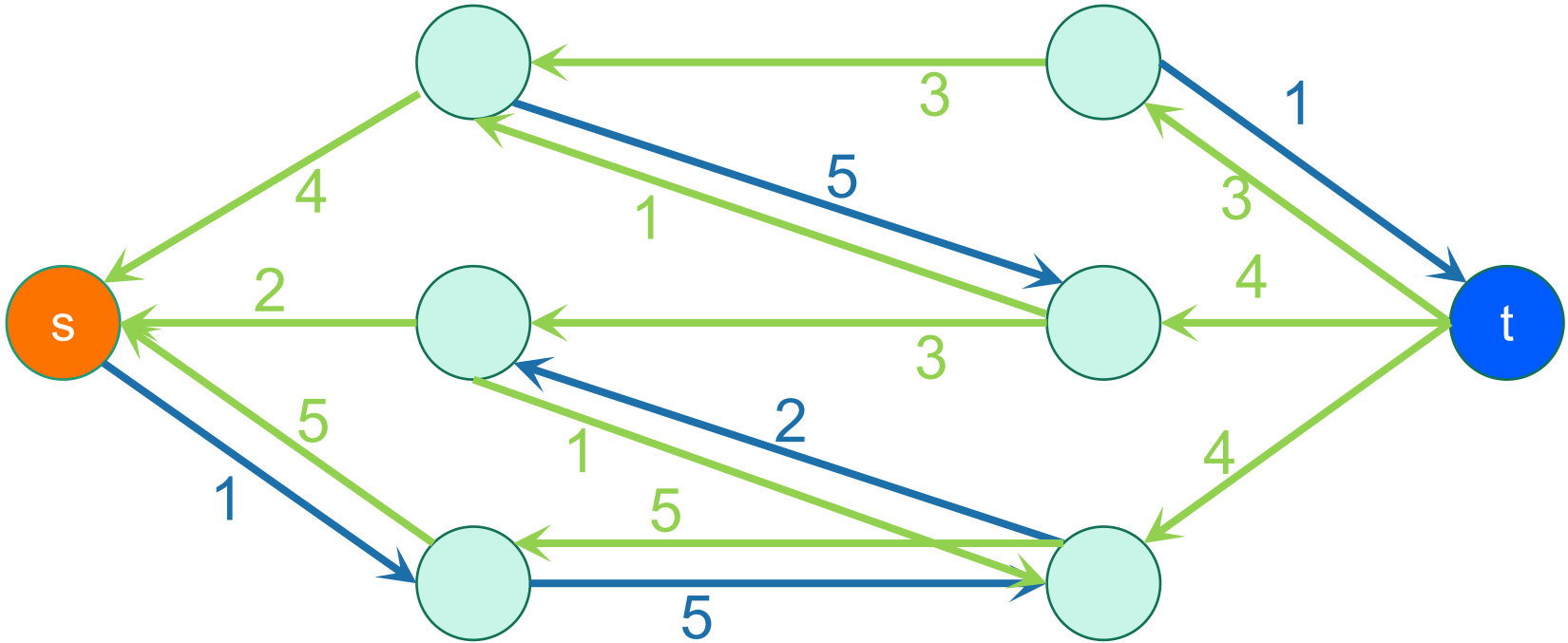


Now we have nothing left to do!



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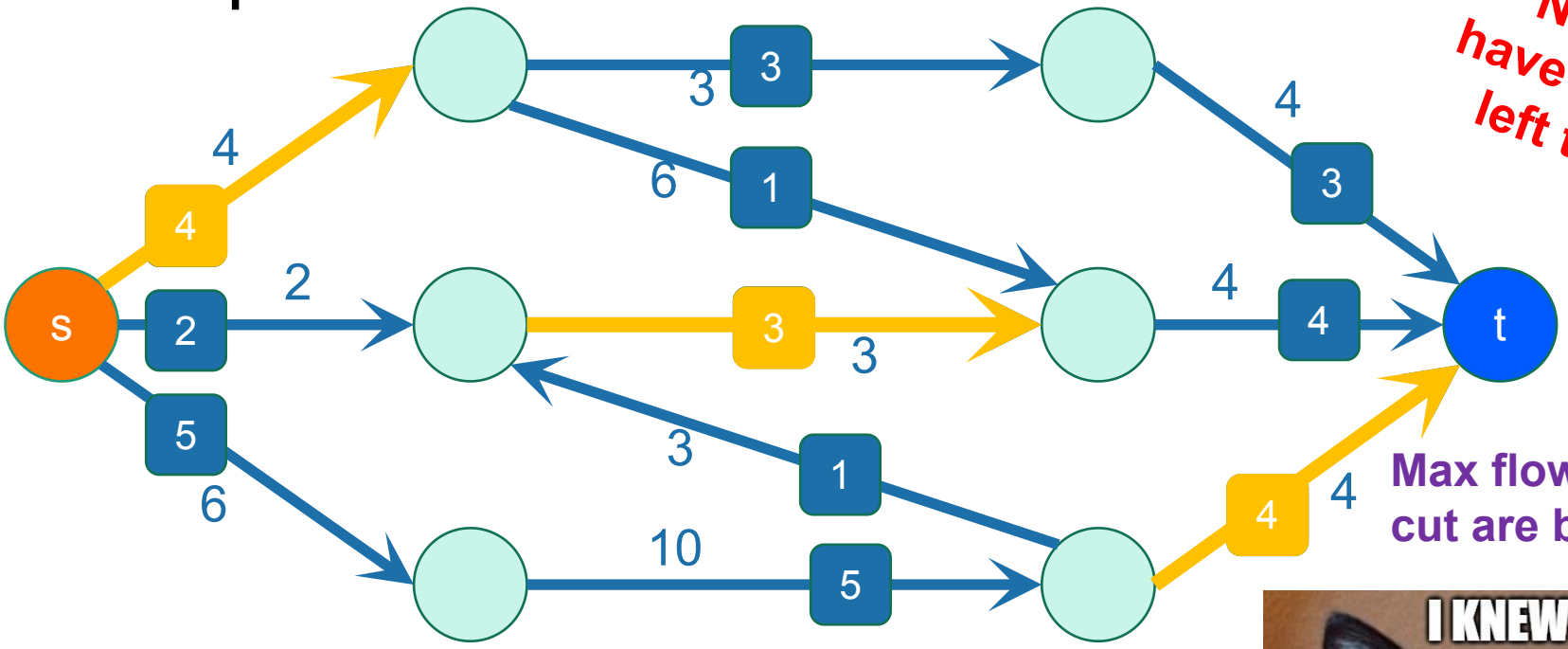
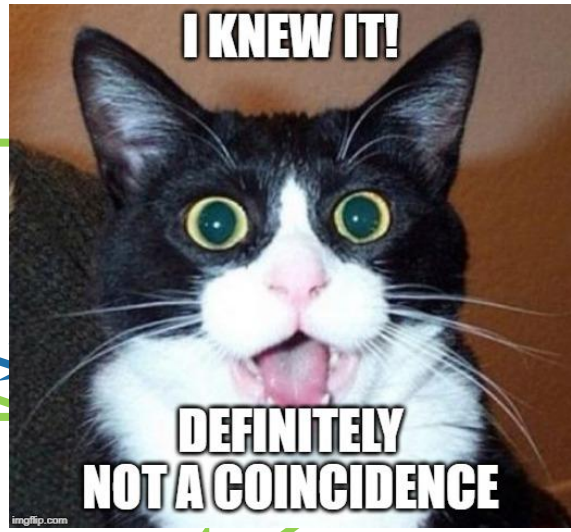


Now we have nothing left to do!

Max flow and min cut are both 11.



There's no path from s to t, and here's the cut to prove it.

What have we learned?

- Max s-t flow is equal to min s-t cut!
  - The USSR and the USA were trying to solve the same problem…
- The Ford-Fulkerson algorithm can find the min-cut/max-flow.
  - Repeatedly improve your flow along an augmenting path.
- **How long does this take???**