# Calculus 3 Study Guide

## Coordinate Systems

### Polar Coordinates

**Note:**
Polar coordinates describe a point in 2D space using two values:

- $r$: the distance from the origin
- $\theta$: the angle from the positive x-axis

This system is ideal for problems with circular or rotational symmetry, such as spirals or radial fields.

**Polar to Cartesian**

- $x = r\cos\theta$
- $y = r\sin\theta$

**Cartesian to Polar**

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \frac{x}{y} \rightarrow \theta = \tan^{-1}\left(\frac{x}{y}\right)$

#### Converting Coordinates

**Example 1:** Convert the Cartesian point $(3, 3)$ into Polar coordiantes.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

**Answer:** $\left(3\sqrt{2}, \frac{\pi}{4}\right)$

**Example 2:** Convert the Polar point $\left(2, \frac{\pi}{3}\right)$ to Cartesian coordiantes.

$$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

**Answer:** $\left(1, \sqrt{3}\right)$

### Converting Equations

**Example 1:** Convert the equation $r^2 = 4$ and $\theta = \frac{\pi}{4}$ to Cartesian coordiantes.

**Step 1:** Polar to Cartesian conversions:

$$x = r\cos(\theta), y = r\sin(\theta)$$

**Step 2:** Substitute the gien values into these formulas:

$$x = 2\cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$y = 2\sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

**Answer:**

$$x = \sqrt{2}, y = \sqrt{2}$$

So the equation in Cartesian coordinates would be: $x^2 + y^2 = 4$

## Cylindrical Coordinates

**Note:**
Polar coordinates describe a point in 3D space using three values:

- $r$: The distance from the origin in the xy-plane
- $\theta$: The angle from the positive x-axis in the xy-plane (same as polar coordiantes)
- $z$: The height above (or below) the xy-plane

This system is useful for objects with circular symmetry around the z-axis, like cylinders and spirals.

**Cylindrical to Cartesian**

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

**Cartesian to Cylindrical**

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \frac{x}{y} \to \theta = \tan^{-1}\left(\frac{x}{y}\right)$
- $z = z$

## Converting Coordinates

**Example 1:** Convert the Cartesian point $(3, 3, 4)$ into Cylindrical coordiantes.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = 4$$

**Answer:** $\left(3\sqrt{2}, \frac{\pi}{4}, 4\right)$

**Example 2:** Convert the Cylindrical point $\left(2, \frac{\pi}{6}, 5\right)$ into Cartesian coordiantes.

$$x = r\cos\theta = 2\cos\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = r\sin\theta = 2\sin\left(\frac{\pi}{6}\right) = 2 \cdot \frac{1}{2} = 1$$

$$z = 5$$

**Answer:** $\left(\sqrt{3}, 1, 5\right)$

## Converting Equations

**Example 1:** Convert the equation $r = 3$ to Cartesian coordinates.

**Step 1:** Cylindrical to Cartesian conversion:

$$r = \sqrt{x^2 + y^2}$$

**Step 2:** Substitute the given value:

$$\sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9$$

**Answer:**
$x^2 + y^2 = 9$

**Example 2:** Convert the equation $r^2 + (z-2)^2 = 4$ to Cartesian coordinates.

**Step 1:** Use the cylindrical to Cartesian identity:

$$r^2 = x^2 + y^2$$

**Step 2:** Substitute into the original equation:

$$x^2 + y^2 + (z-2)^2 = 4$$

**Answer:**
$x^2 + y^2 + (z-2)^2 = 4$

**Example 3:** Convert the cylindrical equation $r^2 + (z-2)^2 = 4$ into spherical coordinates.

**Step 1:** Recall the cylindrical-to-spherical conversions:

$$r = \rho \sin\phi, \quad z = \rho\cos\phi$$

**Step 2:** Substitute into the equation:

$$(\rho\sin\phi)^2 + (\rho\cos\phi - 2)^2 = 4$$

**Step 3:** Simplify the expression:

$$\rho^2\sin^2\phi + (\rho\cos\phi - 2)^2 = 4$$

**Answer:**
The equation in spherical coordinates is: $\rho^2\sin^2\phi + (\rho\cos\phi - 2)^2 = 4$

## Spherical Coordinates

**Note:**
Spherical coordinates describe a point in 3D space using three values:

- $\rho$: the ditance from the origin
- $\theta$: the angle from the positive x-axis in the xy-plane (same as polar coordiantes)
- $\phi$: the angle from the postive z-axis down to the point

This system is useful for problems with radial symmetry, like spheres and cones.

**Spherical to Cartesian**

- $x = \rho\sin\phi\cos\theta$
- $y = \rho\sin\phi\sin\theta$
- $z = \rho\cos\phi$

**Cartesian to Spherical**

- $\rho = \sqrt{x^2+y^2+z^2}$
- $\tan\theta = \frac{x}{y} \to \theta = \tan^{-1}\left(\frac{x}{y}\right)$
- $\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

**Example 1:** Convert the Cartesian point $(2, 2, 1)$ into Spherical coordiantes.

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$
$\theta = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$
$\phi = \arccos\left(\frac{z}{\rho}\right) = arccos\left(\frac{1}{3}\right)$
**Answer:** $\left(3, \frac{\pi}{4}, \arccos\left(\frac{1}{3}\right)\right)$

**Example 2:** Convert the Spherical point $\left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$

$x = \rho \sin\phi \cos\theta = 4 \cdot \sin\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \sqrt{2}$
$y = \rho \sin\phi \sin\theta = 4 \cdot \sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}$
$z = \rho \cos\phi = 4 \cdot \cos\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$
**Answer:** $\left(\sqrt{2}, \sqrt{6}, 2\sqrt{2}\right)$

## Converting Equations

**Example 1:** Convert the equation $\rho = 5$ (spherical) to Cartesian coordinates.

**Step 1:** Spherical to Cartesian conversion:
$\rho = \sqrt{x^2 + y^2 + z^2}$

**Step 2:** Substitute the given value:
$\sqrt{x^2 + y^2 + z^2} = 5 \Rightarrow x^2 + y^2 + z^2 = 25$

**Answer:**
$x^2 + y^2 + z^2 = 25$