Calculus 3 Study Guide

Coordinate Systems

Polar Coordinates

Note:
Polar coordinates describe a point in 2D space using two values:

- r: the distance from the origin
- θ: the angle from the positive x-axis

This system is ideal for problems with circular or rotational symmetry, such as spirals or radial fields.

Polar to Cartesian

- $x = r\cos\theta$
- $y = r\sin\theta$

Cartesian to Polar

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \frac{x}{y} \to \theta = \tan^{-1}\left(\frac{x}{y}\right)$

Converting Coordinates

Example 1: Convert the Cartesian point $(3, 3)$ into Polar coordiantes.

$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$
$\theta = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$
Answer: $\left(3\sqrt{2}, \frac{\pi}{4}\right)$

Example 2: Convert the Polar point $\left(2, \frac{\pi}{3}\right)$ to Cartesian coordiantes.

$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$
$y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
Answer: $\left(1, \sqrt{3}\right)$

Converting Equations

Example 1: Convert the equation $r^2 = 4$ and $\theta = \frac{\pi}{4}$ to Cartesian coordiantes.

Step 1: Polar to Cartesian conversions:
$x = r\cos(\theta), y = r\sin(\theta)$

Step 2: Substitute the gien values into these formulas:
$x = 2\cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$
$y = 2\sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$

Answer:
$x = \sqrt{2}, y = \sqrt{2}$

So the equation in Cartesian coordinates would be: $x^2 + y^2 = 4$

Cylindrical Coordinates

Note:
Polar coordinates describe a point in 3D space using three values:

- r: The distance from the origin in the xy-plane
- θ: The angle from the positive x-axis in the xy-plane (same as polar coordiantes)
- z: The height above (or below) the xy-plane

This system is useful for objects with circular symmetry around the z-axis, like cylinders and spirals.

Cylindrical to Cartesian

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

Cartesian to Cylindrical

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \frac{x}{y} \rightarrow \theta = \tan^{-1}\left(\frac{x}{y}\right)$
- $z = z$

Example 1: Convert the Cartesian point $(3, 3, 4)$ into Cylindrical coordiantes.

$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$
$\theta = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$
$z = 4$
Answer: $\left(3\sqrt{2}, \frac{\pi}{4}, 4\right)$

Example 2: Convert the Cylindrical point $\left(2, \frac{\pi}{6}, 5\right)$ into Cartesian coordiantes.

$x = r\cos\theta = 2\cos\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
$y = r\sin\theta = 2\sin\left(\frac{\pi}{6}\right) = 2 \cdot \frac{1}{2} = 1$
$z = 5$
Answer: $\left(\sqrt{3}, 1, 5\right)$

Spherical Coordinates

Note:
Spherical coordinates describe a point in 3D space using three values:

- ρ: the ditance from the origin
- θ: the angle from the positive x-axis in the xy-plane (same as polar coordiantes)
- ϕ: the angle from the postive z-axis down to the point

This system is useful for problems with radial symmetry, like spheres and cones.

Spherical to Cartesian

- $x = \rho \sin\phi \cos\theta$
- $y = \rho \sin\phi \sin\theta$
- $z = \rho \cos\phi$

Cartesian to Spherical

- $\rho = \sqrt{x^2 + y^2 + z^2}$
- $\tan\theta = \frac{x}{y} \to \theta = \tan^{-1}\left(\frac{x}{y}\right)$
- $\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

Example 1: Convert the Cartesian point $(2, 2, 1)$ into Spherical coordiantes.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$
$$\theta = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$
$$\phi = \arccos\left(\frac{z}{\rho}\right) = arccos\left(\frac{1}{3}\right)$$

Answer: $\left(3, \frac{\pi}{4}, \arccos\left(\frac{1}{3}\right)\right)$

Example 2: Convert the Spherical point $\left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$

$$x = \rho \sin\phi \cos\theta = 4 \cdot \sin\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \sqrt{2}$$
$$y = \rho \sin\phi \sin\theta = 4 \cdot \sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}$$
$$z = \rho \cos\phi = 4 \cdot \cos\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$$

Answer: $\left(\sqrt{2}, \sqrt{6}, 2\sqrt{2}\right)$