

Calculus 3 Study Guide

Coordinate Systems

Polar Coordinates

Note:

Polar coordinates describe a point in 2D space using two values:

- r : the distance from the origin
- θ : the angle from the positive x-axis

This system is ideal for problems with circular or rotational symmetry, such as spirals or radial fields.

Polar to Cartesian

- $x = r \cos \theta$
- $y = r \sin \theta$

Cartesian to Polar

- $r = \sqrt{x^2 + y^2}$
- $\tan \theta = \frac{x}{y} \rightarrow \theta = \tan^{-1} \left(\frac{x}{y} \right)$

Converting Coordinates

Example 1: Convert the Cartesian point $(3, 3)$ into Polar coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1} \left(\frac{x}{y} \right) = \tan^{-1} \left(\frac{3}{3} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Answer: $\left(3\sqrt{2}, \frac{\pi}{4} \right)$

Example 2: Convert the Polar point $\left(2, \frac{\pi}{3} \right)$ to Cartesian coordinates.

$$x = r \cos \theta = 2 \cos \left(\frac{\pi}{3} \right) = 2 \cdot \frac{1}{2} = 1$$

$$y = r \sin \theta = 2 \sin \left(\frac{\pi}{3} \right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Answer: $\left(1, \sqrt{3} \right)$

Converting Equations

Example 1: Convert the equation $r^2 = 4$ and $\theta = \frac{\pi}{4}$ to Cartesian coordinates.

Step 1: Polar to Cartesian conversions:

$$x = r \cos(\theta), y = r \sin(\theta)$$

Step 2: Substitute the given values into these formulas:

$$x = 2 \cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$y = 2 \sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

Answer:

$$x = \sqrt{2}, y = \sqrt{2}$$

So the equation in Cartesian coordinates would be: $x^2 + y^2 = 4$

Cylindrical Coordinates

Note:

Polar coordinates describe a point in 3D space using three values:

- r : The distance from the origin in the xy-plane
- θ : The angle from the positive x-axis in the xy-plane (same as polar coordinates)
- z : The height above (or below) the xy-plane

This system is useful for objects with circular symmetry around the z-axis, like cylinders and spirals.

Cylindrical to Cartesian

- $x = r \cos \theta$
- $y = r \sin \theta$
- $z = z$

Cartesian to Cylindrical

- $r = \sqrt{x^2 + y^2}$
- $\tan \theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Converting Coordinates

Example 1: Convert the Cartesian point $(3, 3, 4)$ into Cylindrical coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1} \left(\frac{x}{y} \right) = \tan^{-1} \left(\frac{3}{3} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = 4$$

Answer: $\left(3\sqrt{2}, \frac{\pi}{4}, 4 \right)$

Example 2: Convert the Cylindrical point $\left(2, \frac{\pi}{6}, 5 \right)$ into Cartesian coordinates.

$$x = r \cos \theta = 2 \cos \left(\frac{\pi}{6} \right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = r \sin \theta = 2 \sin \left(\frac{\pi}{6} \right) = 2 \cdot \frac{1}{2} = 1$$

$$z = 5$$

Answer: $\left(\sqrt{3}, 1, 5 \right)$

Converting Equations

Example 1: Convert the equation $r = 3$ to Cartesian coordinates.

Step 1: Cylindrical to Cartesian conversion:

$$r = \sqrt{x^2 + y^2}$$

Step 2: Substitute the given value:

$$\sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9$$

Answer:

$$x^2 + y^2 = 9$$

Example 2: Convert the equation $r^2 + (z - 2)^2 = 4$ to Cartesian coordinates.

Step 1: Use the cylindrical to Cartesian identity:

$$r^2 = x^2 + y^2$$

Step 2: Substitute into the original equation:

$$x^2 + y^2 + (z - 2)^2 = 4$$

Answer:

$$x^2 + y^2 + (z - 2)^2 = 4$$

Example 3: Convert the cylindrical equation $r^2 + (z - 2)^2 = 4$ into spherical coordinates.

Step 1: Recall the cylindrical-to-spherical conversions:

$$r = \rho \sin \phi, \quad z = \rho \cos \phi$$

Step 2: Substitute into the equation:

$$(\rho \sin \phi)^2 + (\rho \cos \phi - 2)^2 = 4$$

Step 3: Simplify the expression:

$$\rho^2 \sin^2 \phi + (\rho \cos \phi - 2)^2 = 4$$

Answer:

The equation in spherical coordinates is: $\rho^2 \sin^2 \phi + (\rho \cos \phi - 2)^2 = 4$

Spherical Coordinates

Note:

Spherical coordinates describe a point in 3D space using three values:

- ρ : the distance from the origin
- θ : the angle from the positive x-axis in the xy-plane (same as polar coordinates)
- ϕ : the angle from the positive z-axis down to the point

This system is useful for problems with radial symmetry, like spheres and cones.

Spherical to Cartesian

- $x = \rho \sin \phi \cos \theta$
- $y = \rho \sin \phi \sin \theta$

- $z = \rho \cos \phi$

Cartesian to Spherical

- $\rho = \sqrt{x^2 + y^2 + z^2}$
- $\tan \theta = \frac{x}{y} \rightarrow \theta = \tan^{-1} \left(\frac{x}{y} \right)$
- $\phi = \arccos \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$

Example 1: Convert the Cartesian point $(2, 2, 1)$ into Spherical coordinates.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$\theta = \tan^{-1} \left(\frac{x}{y} \right) = \tan^{-1} \left(\frac{2}{2} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\phi = \arccos \left(\frac{z}{\rho} \right) = \arccos \left(\frac{1}{3} \right)$$

Answer: $\left(3, \frac{\pi}{4}, \arccos \left(\frac{1}{3} \right) \right)$

Example 2: Convert the Spherical point $\left(4, \frac{\pi}{3}, \frac{\pi}{4} \right)$

$$x = \rho \sin \phi \cos \theta = 4 \cdot \sin \left(\frac{\pi}{4} \right) \cdot \cos \left(\frac{\pi}{3} \right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \sqrt{2}$$

$$y = \rho \sin \phi \sin \theta = 4 \cdot \sin \left(\frac{\pi}{4} \right) \cdot \sin \left(\frac{\pi}{3} \right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}$$

$$z = \rho \cos \phi = 4 \cdot \cos \left(\frac{\pi}{4} \right) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$$

Answer: $\left(\sqrt{2}, \sqrt{6}, 2\sqrt{2} \right)$

Converting Equations

Example 1: Convert the equation $\rho = 5$ (spherical) to Cartesian coordinates.

Step 1: Spherical to Cartesian conversion:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Step 2: Substitute the given value:

$$\sqrt{x^2 + y^2 + z^2} = 5 \Rightarrow x^2 + y^2 + z^2 = 25$$

Answer:

$$x^2 + y^2 + z^2 = 25$$