

# Calculus 3 Study Guide

## Coordinate Systems

### Polar Coordinates

**Note:**

Polar coordinates describe a point in 2D space using two values:

- $r$ : the distance from the origin
- $\theta$ : the angle from the positive x-axis

This system is ideal for problems with circular or rotational symmetry, such as spirals or radial fields.

**Polar to Cartesian**

- $x = r \cos \theta$
- $y = r \sin \theta$

**Cartesian to Polar**

- $r = \sqrt{x^2 + y^2}$
- $\tan \theta = \frac{x}{y} \rightarrow \theta = \tan^{-1} \left( \frac{x}{y} \right)$

**Converting Coordinates**

**Example 1:** Convert the Cartesian point  $(3, 3)$  into Polar coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1} \left( \frac{x}{y} \right) = \tan^{-1} \left( \frac{3}{3} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

**Answer:**  $\left( 3\sqrt{2}, \frac{\pi}{4} \right)$

**Example 2:** Convert the Polar point  $\left( 2, \frac{\pi}{3} \right)$  to Cartesian coordinates.

$$x = r \cos \theta = 2 \cos \left( \frac{\pi}{3} \right) = 2 \cdot \frac{1}{2} = 1$$

$$y = r \sin \theta = 2 \sin \left( \frac{\pi}{3} \right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

**Answer:**  $\left( 1, \sqrt{3} \right)$

## Converting Equations

**Example 1:** Convert the equation  $r^2 = 4$  and  $\theta = \frac{\pi}{4}$  to Cartesian coordinates.

**Step 1:** Polar to Cartesian conversions:

$$x = r \cos(\theta), y = r \sin(\theta)$$

**Step 2:** Substitute the given values into these formulas:

$$x = 2 \cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$y = 2 \sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

**Answer:**

$$x = \sqrt{2}, y = \sqrt{2}$$

So the equation in Cartesian coordinates would be:  $x^2 + y^2 = 4$

## Cylindrical Coordinates

### Note:

Polar coordinates describe a point in 3D space using three values:

- $r$ : The distance from the origin in the xy-plane
- $\theta$ : The angle from the positive x-axis in the xy-plane (same as polar coordinates)
- $z$ : The height above (or below) the xy-plane

This system is useful for objects with circular symmetry around the z-axis, like cylinders and spirals.

### Cylindrical to Cartesian

- $x = r \cos \theta$
- $y = r \sin \theta$
- $z = z$

### Cartesian to Cylindrical

- $r = \sqrt{x^2 + y^2}$
- $\tan \theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

## Converting Coordinates

**Example 1:** Convert the Cartesian point  $(3,3,4)$  into Cylindrical coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = 4$$

**Answer:**  $(3\sqrt{2}, \frac{\pi}{4}, 4)$

**Example 2:** Convert the Cylindrical point  $(2, \frac{\pi}{6}, 5)$  into Cartesian coordinates.

$$x = r \cos \theta = 2 \cos\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = r \sin \theta = 2 \sin\left(\frac{\pi}{6}\right) = 2 \cdot \frac{1}{2} = 1$$

$$z = 5$$

**Answer:**  $(\sqrt{3}, 1, 5)$

## Converting Equations

**Example 1:** Convert the equation  $r = 3$  to Cartesian coordinates.

**Step 1:** Cylindrical to Cartesian conversion:

$$r = \sqrt{x^2 + y^2}$$

**Step 2:** Substitute the given value:

$$\sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9$$

**Answer:**

$$x^2 + y^2 = 9$$

**Example 2:** Convert the equation  $r^2 + (z - 2)^2 = 4$  to Cartesian coordinates.

**Step 1:** Use the cylindrical to Cartesian identity:

$$r^2 = x^2 + y^2$$

**Step 2:** Substitute into the original equation:

$$x^2 + y^2 + (z - 2)^2 = 4$$

**Answer:**

$$x^2 + y^2 + (z - 2)^2 = 4$$

**Example 3:** Convert the cylindrical equation  $r^2 + (z - 2)^2 = 4$  into spherical coordinates.

**Step 1:** Recall the cylindrical-to-spherical conversions:

$$r = \rho \sin \phi, \quad z = \rho \cos \phi$$

**Step 2:** Substitute into the equation:

$$(\rho \sin \phi)^2 + (\rho \cos \phi - 2)^2 = 4$$

**Step 3:** Simplify the expression:

$$\rho^2 \sin^2 \phi + (\rho \cos \phi - 2)^2 = 4$$

**Answer:**

The equation in spherical coordinates is:  $\rho^2 \sin^2 \phi + (\rho \cos \phi - 2)^2 = 4$

## Spherical Coordinates

**Note:**

Spherical coordinates describe a point in 3D space using three values:

- $\rho$ : the distance from the origin
- $\theta$ : the angle from the positive x-axis in the xy-plane (same as polar coordinates)
- $\phi$ : the angle from the positive z-axis down to the point

This system is useful for problems with radial symmetry, like spheres and cones.

### Spherical to Cartesian

- $x = \rho \sin \phi \cos \theta$
- $y = \rho \sin \phi \sin \theta$
- $z = \rho \cos \phi$

### Cartesian to Spherical

- $\rho = \sqrt{x^2 + y^2 + z^2}$
- $\tan \theta = \frac{x}{y} \rightarrow \theta = \tan^{-1} \left( \frac{x}{y} \right)$
- $\phi = \arccos \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$

**Example 1:** Convert the Cartesian point  $(2, 2, 1)$  into Spherical coordinates.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$\theta = \tan^{-1} \left( \frac{x}{y} \right) = \tan^{-1} \left( \frac{2}{2} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\phi = \arccos \left( \frac{z}{\rho} \right) = \arccos \left( \frac{1}{3} \right)$$

**Answer:**  $\left( 3, \frac{\pi}{4}, \arccos \left( \frac{1}{3} \right) \right)$

**Example 2:** Convert the Spherical point  $(4, \frac{\pi}{3}, \frac{\pi}{4})$

$$x = \rho \sin \phi \cos \theta = 4 \cdot \sin \left( \frac{\pi}{4} \right) \cdot \cos \left( \frac{\pi}{3} \right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \sqrt{2}$$

$$y = \rho \sin \phi \sin \theta = 4 \cdot \sin \left( \frac{\pi}{4} \right) \cdot \sin \left( \frac{\pi}{3} \right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}$$

$$z = \rho \cos \phi = 4 \cdot \cos \left( \frac{\pi}{4} \right) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$$

**Answer:**  $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$

### Converting Equations

**Example 1:** Convert the equation  $\rho = 5$  (spherical) to Cartesian coordinates.

**Step 1:** Spherical to Cartesian conversion:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

**Step 2:** Substitute the given value:

$$\sqrt{x^2 + y^2 + z^2} = 5 \Rightarrow x^2 + y^2 + z^2 = 25$$

**Answer:**

$$x^2 + y^2 + z^2 = 25$$