Calculus 3 Study Guide

Coordinate Systems

Polar Coordinates

Note:
Polar coordinates describe a point in 2D space using two values:

- r: the distance from the origin
- θ: the angle from the positive x-axis

This system is ideal for problems with circular or rotational symmetry, such as spirals or radial fields.

Polar to Cartesian

- $x = r\cos\theta$
- $y = r\sin\theta$

Cartesian to Polar

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Converting Coordinates

Example 1: Convert the Cartesian point $(3,3)$ into Polar coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Answer: $\left(3\sqrt{2}, \frac{\pi}{4}\right)$

Example 2: Convert the Polar point $\left(2, \frac{\pi}{3}\right)$ to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Answer: $\left(1, \sqrt{3}\right)$

Converting Equations

Example 1: Convert the equation $r^2 = 4$ and $\theta = \frac{\pi}{4}$ to Cartesian coordinates.

Step 1: Polar to Cartesian conversions:

$$x = r\cos(\theta), y = r\sin(\theta)$$

Step 2: Substitute the given values into these formulas:

$$x = 2\cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$y = 2\sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

Answer:

$$x = \sqrt{2}, y = \sqrt{2}$$

So the equation in Cartesian coordinates would be: $x^2 + y^2 = 4$

Cylindrical Coordinates

Note:
Polar coordinates describe a point in 3D space using three values:

- r: The distance from the origin in the xy-plane
- θ: The angle from the positive x-axis in the xy-plane (same as polar coordinates)
- z: The height above (or below) the xy-plane

This system is useful for objects with circular symmetry around the z-axis, like cylinders and spirals.

Cylindrical to Cartesian

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

Cartesian to Cylindrical

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \frac{y}{x} \to \theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Converting Coordinates

Example 1: Convert the Cartesian point $(3, 3, 4)$ into Cylindrical coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = 4$$

Answer: $\left(3\sqrt{2}, \frac{\pi}{4}, 4\right)$

Example 2: Convert the Cylindrical point $\left(2, \frac{\pi}{6}, 5\right)$ into Cartesian coordinates.

$$x = r\cos\theta = 2\cos\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = r\sin\theta = 2\sin\left(\frac{\pi}{6}\right) = 2 \cdot \frac{1}{2} = 1$$

$$z = 5$$

Answer: $\left(\sqrt{3}, 1, 5\right)$

Converting Equations

Example 1: Convert the equation $r = 3$ to Cartesian coordinates.

Step 1: Cylindrical to Cartesian conversion:

$$r = \sqrt{x^2 + y^2}$$

Step 2: Substitute the given value:

$$\sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9$$

Answer:
$x^2 + y^2 = 9$

Example 2: Convert the equation $r^2 + (z-2)^2 = 4$ to Cartesian coordinates.

Step 1: Use the cylindrical to Cartesian identity:

$$r^2 = x^2 + y^2$$

Step 2: Substitute into the original equation:

$$x^2 + y^2 + (z-2)^2 = 4$$

Answer:

$x^2 + y^2 + (z-2)^2 = 4$

Example 3: Convert the cylindrical equation $r^2 + (z-2)^2 = 4$ into spherical coordinates.

Step 1: Recall the cylindrical-to-spherical conversions:

$$r = \rho \sin\phi, \quad z = \rho\cos\phi$$

Step 2: Substitute into the equation:

$$(\rho \sin\phi)^2 + (\rho\cos\phi - 2)^2 = 4$$

Step 3: Simplify the expression:

$$\rho^2 \sin^2\phi + (\rho\cos\phi - 2)^2 = 4$$

Answer:
The equation in spherical coordinates is: $\rho^2 \sin^2\phi + (\rho\cos\phi - 2)^2 = 4$

Spherical Coordinates

Note:
Spherical coordinates describe a point in 3D space using three values:

- ρ: the distance from the origin
- θ: the angle from the positive x-axis in the xy-plane (same as polar coordinates)
- ϕ: the angle from the positive z-axis down to the point

This system is useful for problems with radial symmetry, like spheres and cones.

Spherical to Cartesian

- $x = \rho \sin\phi \cos\theta$
- $y = \rho \sin\phi \sin\theta$
- $z = \rho\cos\phi$

Cartesian to Spherical

- $\rho = \sqrt{x^2 + y^2 + z^2}$
- $\tan\theta = \frac{y}{x} \to \theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

Example 1: Convert the Cartesian point $(2, 2, 1)$ into Spherical coordinates.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{1}{3}\right)$$

Answer: $\left(3, \frac{\pi}{4}, \arccos\left(\frac{1}{3}\right)\right)$

Example 2: Convert the Spherical point $\left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$

$$x = \rho \sin\phi \cos\theta = 4 \cdot \sin\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \sqrt{2}$$

$$y = \rho \sin\phi \sin\theta = 4 \cdot \sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}$$

$$z = \rho \cos\phi = 4 \cdot \cos\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$$

Answer: $\left(\sqrt{2}, \sqrt{6}, 2\sqrt{2}\right)$

Converting Equations

Example 1: Convert the equation $\rho = 5$ to Cartesian coordinates.

Step 1: Spherical to Cartesian conversion:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Step 2: Substitute the given value:

$$\sqrt{x^2 + y^2 + z^2} = 5 \Rightarrow x^2 + y^2 + z^2 = 25$$

Answer:
$x^2 + y^2 + z^2 = 25$

Example 2: Convert the spherical equation $\rho = 2\sin\phi$ into both Cartesian and cylindrical coordinates.

Step 1: Use the identity for $\rho \sin\phi = r$, where $r = \sqrt{x^2 + y^2}$
Multiply both sides by $\sin\phi$:

$$\rho \sin\phi = 2\sin^2\phi$$

Step 2: Use substitution $\rho \sin\phi = r \Rightarrow r = 2\sin^2\phi$
Now convert $\sin^2\phi$ in terms of z and ρ using:

$$z = \rho\cos\phi \Rightarrow \cos\phi = \frac{z}{\rho}, \quad \sin^2\phi = 1 - \cos^2\phi = 1 - \frac{z^2}{\rho^2}$$

Step 3: Substitute back:

$$r = 2\left(1 - \frac{z^2}{\rho^2}\right) \quad \text{and since} \quad \rho^2 = r^2 + z^2$$

Substituting ρ^2 and simplifying:

$$r = 2\left(1 - \frac{z^2}{r^2 + z^2}\right) = 2\left(\frac{r^2}{r^2 + z^2}\right)$$

$$\Rightarrow r(r^2 + z^2) = 2r^2 \Rightarrow rz^2 = r^2 \Rightarrow z^2 = r \quad \text{(in cylindrical)}$$

Answer (Cylindrical): $z^2 = r$

Step 4: Convert to Cartesian:
Use $r^2 = x^2 + y^2$

$$z^2 = \sqrt{x^2 + y^2}$$

Answer (Cartesian): $z^2 = \sqrt{x^2 + y^2}$

Tangent Planes