F_0 and the intensity of the sound will be discussed later in this book [see Eq. (29.6)].

Taking the simple case of normal incidence first, where the force $F_0e^{-i\omega t}$ is uniform and in phase over all the string, we use the closed form of Eq. (10.10) in Eq. (10.16) to obtain

$$y = \frac{F_0e^{-i\omega t}}{\epsilon\omega^2}\left\{\frac{\cos\left[\frac{\omega}{c}\left(x - \frac{l}{2}\right)\right]}{\cos(\omega l/2c)} - 1\right\} \tag{10.17}$$

This has resonances at every other harmonic ($\nu = \omega/2\pi = nc/2l$; $n = 1,3,5,7 \cdots$), the symmetry of the normal modes for the even harmonics precluding their excitation. The shapes exhibited by the string for different driving frequencies are shown in Fig. 21.

For the more general driving force $F_0e^{i\alpha x - i\omega t}$, it is easier to use Eq. (10.12) for $y(\xi,x,t)$, and the final result can be expressed in the series

$$y = F_0\left(\frac{4\pi l^2}{T}\right)e^{i(\frac{1}{2}\alpha l - \omega t)}\sum_{n=1}^{\infty} ne^{\frac{i\pi}{2}(n+1)}\frac{\sin[\frac{1}{2}(\alpha l - \pi n)]}{(\pi n)^2 - (\alpha l)^2}\cdot \frac{\sin(\pi n x/l)}{(\pi n)^2 - (\omega l/c)^2} \tag{10.18}$$

This series is equal to the closed form of Eq. (10.17) for $\alpha = 0$.

Transient Driving Force.—To calculate the response of the string for a transient force, we use the operational-calculus methods again. We begin by computing the response of the string to an impulsive force at $t = 0$ applied at $x = \xi$; $f(t) = \delta(x - \xi)\delta(t)$. Referring to Eq. (6.16), we see that the proper expression is given by the integral

$$y_\delta(\xi,x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Y_m(\xi,x;\omega)}{-i\omega}e^{-i\omega t}\,d\omega$$

Using Eq. (10.11), this becomes

$$y_\delta(\xi,x;t) = \frac{1}{2\pi\epsilon c}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\omega}\frac{S(\omega,x)}{\sin(\omega l/c)}\,d\omega$$

where

$$S(\omega,x) = \begin{cases} \sin\left[\left(\frac{\omega}{c}\right)(l - \xi)\right]\sin\left(\frac{\omega x}{c}\right) & (x < \xi) \\ \sin\left(\frac{\omega\xi}{c}\right)\sin\left[\left(\frac{\omega}{c}\right)(l - x)\right] & (x > \xi) \end{cases}$$

The poles of the integrand are at $\omega = (n\pi c/l)$ where n is any integer, positive or negative (these correspond to the natural frequencies). Near one of the poles for an even n, the quantity $\sin(\omega l/c)$ approaches

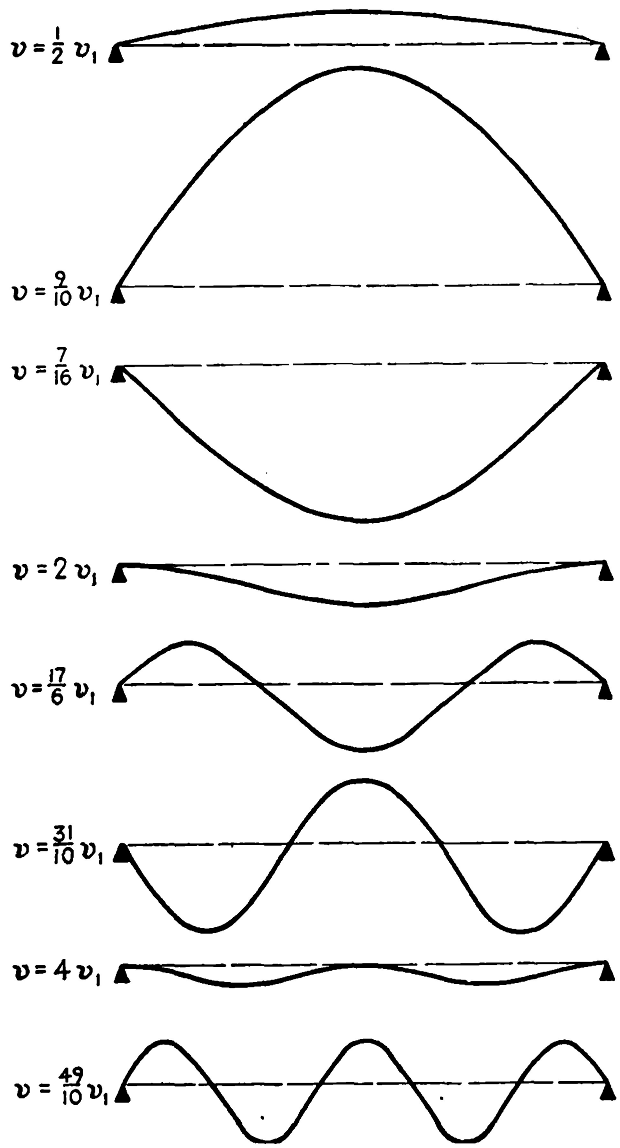

FIG. 21.—Shapes of steady-state motion of an undamped string between rigid supports, driven by a uniform force of frequency ν. The fundamental frequency of the string is ν_1. Those parts of the string above the dashed equilibrium line are in phase with the force; those below the line are 180 deg out of phase.

the value $(l/c)[\omega - (n\pi c/l)]$. In other words, for ω very near $(n\pi c/l)$ (n even) the integrand becomes

$$\frac{1}{2\pi\epsilon l}\frac{e^{-in\pi ct/l}}{(n\pi c/l)}\frac{\sin\left(n\pi - \frac{n\pi\xi}{l}\right)\sin\left(\frac{n\pi x}{l}\right)}{\omega - (n\pi c/l)} \qquad (x < \xi)$$

The residue of this expression, its limiting value when multiplied by $[\omega - (n\pi c/l)]$, as this factor approaches zero, turns out to be

$$\frac{-1}{2\pi^2 nc\epsilon}\, e^{-in\pi ct/l}\sin\left(\frac{n\pi\xi}{l}\right)\sin\left(\frac{n\pi x}{l}\right) \qquad (0 < x < l)$$

For n odd, the factor $\sin(\omega l/c)$ approaches the quantity

$$-\left(\frac{c}{l}\right)\left(\omega - \frac{n\pi c}{l}\right)$$

but the factor $\sin(n\pi - n\pi\xi/l)$ turns out to be $-\sin(n\pi\xi/l)$, so the two minus signs cancel, and the result is the same as before. The final value for the integral, $-2\pi i$ times the sum of the residues on or below the real axis of ω, gives the value for $t > 0$

$$y_\delta(\xi,x;t) = \begin{cases} 0 & (t < 0) \\ \dfrac{-1}{i\pi\epsilon c} \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n} \sin\left(\frac{\pi n\xi}{l}\right) \sin\left(\frac{\pi nx}{l}\right) e^{-in\pi ct/l} & (t > 0) \end{cases}$$

Utilizing the equation $\sin z = (1/2i)(e^{iz} - e^{-iz})$, we have

$$y_\delta(\xi,x;t) = \begin{cases} 0 & (t < 0) \\ \dfrac{2}{\pi\epsilon c} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{\pi n\xi}{l}\right) \sin\left(\frac{\pi nx}{l}\right) \sin\left(\frac{\pi nct}{l}\right) & (t > 0) \end{cases} \tag{10.19}$$

Incidentally, this equation could also have been obtained by substituting series (10.12) in the contour integral.

The series for y_δ does not converge well, any more than the series for the delta function does. However, as with the delta function, we are not interested in computing y_δ, the behavior of the string after being hit by an idealized impulsive force at a mathematical point on the string. We only intend to use the series as an easy means of computing the behavior of the string when acted on by more realistic forces, distributed along the length of the string and spread out in time.

For the most general type of force $f(\xi,t)$, a function of time and of position, the response of the string is

$$\begin{aligned} y &= \int_{-\infty}^{\infty} d\tau \int_0^l d\xi\, f(\xi,\tau) y_\delta(\xi,x,t-\tau) \\ &= \frac{2}{\pi\epsilon c} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \int_{-\infty}^{\infty} \sin\left[\frac{\pi nc}{l}(t-\tau)\right] d\tau \cdot \right. \\ &\qquad \left. \cdot \int_0^l \sin\left(\frac{\pi n\xi}{l}\right) f(\xi,\tau)\, d\xi \right\} \sin\left(\frac{\pi nx}{l}\right) \end{aligned} \tag{10.20}$$

This series *does* converge for reasonable forms of $f(\xi,t)$.

The Piano String.—A reasonable approximation to the force of a piano hammer on a string is

$$F(\xi,t) = \begin{cases} 0 & (-\tfrac{1}{2}\sigma > t > \tfrac{1}{2}\sigma;\ x_0 - \tfrac{1}{2}\zeta > \xi > x_0 + \tfrac{1}{2}\zeta) \\ A \cos\left[\left(\dfrac{\pi}{\zeta}\right)(\xi - x_0)\right] \cos\left(\dfrac{\pi t}{\sigma}\right) & \\ & (-\tfrac{1}{2}\sigma < t < \tfrac{1}{2}\sigma;\ x_0 - \tfrac{1}{2}\zeta < \xi < x_0 + \tfrac{1}{2}\zeta) \end{cases} \tag{10.21}$$

The quantity σ is the time duration of the application of force, and ζ is the length of the portion of string acted on by the force. The force starts acting at $t = -\frac{1}{2}\sigma$, rises to maximum at $t = 0$, and goes again to zero at $t = \frac{1}{2}\sigma$. The distribution of the force along the string is also like the positive half of the cosine curve, with the center of force, where it is greatest, falling at $\xi = x_0$.

Substituting this into Eq. (10.20) and carrying out the usual accompaniment of trigonometry and algebra, we can work out the expressions for the shape of the string. The integration over ξ is from $x_0 - \frac{1}{2}\zeta$ to $x_0 + \frac{1}{2}\zeta$, but the integration over τ is a bit more elusive. When t is less than $-\frac{1}{2}\sigma$, $y_\delta(\xi,x;\,t - \tau)$ is zero for all values of τ for which $F(\xi,\tau)$ differs from zero, so the integral is zero, as it should be (since the string has not been hit yet). For $t > \frac{1}{2}\sigma$ the range of integration over τ is from $-\frac{1}{2}\sigma$ to $+\frac{1}{2}\sigma$; but for $-\frac{1}{2}\sigma < t < \frac{1}{2}\sigma$ the only range over which Fy_δ is not zero is between $\tau = -\frac{1}{2}\sigma$, where F goes to zero, and $\tau = t$, where $y_\delta(\xi,x;\,t - \tau)$ goes to zero. Consequently, we have three expressions for the resulting shape $y(x,t)$

For $t < -\frac{1}{2}\sigma$, $y(x,t) = 0$

For $-\frac{1}{2}\sigma < t < \frac{1}{2}\sigma$,

$$y(x,t) = \frac{4A\sigma\zeta}{\pi^3\epsilon c} \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{\cos(n\pi\zeta/2l)}{1 - (n\zeta/l)^2}\right] \left[\frac{1}{1 - (nc\sigma/l)^2}\right] \cdot \tag{10.22}$$
$$\cdot \sin\left(\frac{\pi n x_0}{l}\right) \sin\left(\frac{\pi n x}{l}\right) \left\{\sin\left[\left(\frac{\pi n c}{l}\right)\left(t + \frac{1}{2}\sigma\right)\right] - \left(\frac{nc\sigma}{l}\right) \cos\left(\frac{\pi t}{\sigma}\right)\right\}$$

For $t > \frac{1}{2}\sigma$,

$$y(x,t) = \frac{8A\sigma\zeta}{\pi^3\epsilon c} \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{\cos(n\pi\zeta/2l)}{1 - (n\zeta/l)^2}\right] \left[\frac{\cos(n\pi c\sigma/2l)}{1 - (nc\sigma/l)^2}\right] \cdot$$
$$\cdot \sin\left(\frac{\pi n x_0}{l}\right) \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi n c t}{l}\right)$$

This formula appears quite formidable, but it can be computed if need be. Certainly it is not very difficult to obtain from it the relative magnitude of the various harmonics in the free vibration after the end of the blow ($t > \frac{1}{2}\sigma$).

Incidentally, this example is a good one to show the power of the operational-calculus methods. The algebraic gymnastics necessary to obtain Eq. (10.22) from the combination of Eqs. (10.20) and (10.21) are not particularly easy for one mathematically muscle-bound. Nevertheless, the calculations are only laborious, not subtle. On the other hand, to obtain the final formula for $y(x,t)$ by any other method would involve still more labor, and more mathematical subtleties than we care to include in this volume. Our applications of operational calculus in Chap. II may have seemed rather like using a sledge hammer to drive a tack. We see now, however, that the problem of the simple string already provides a spike worthy of the sledge.

To be honest, the resulting series for $y(x,t)$ is not too good an approximation for the actual motion of an actual piano string when it is struck, partly because the actual piano string is not a perfect string but has stiffness. We shall indicate how to correct for this in the next chapter.

The Effect of Friction.—In the foregoing analysis we have neglected friction, although it is present in every vibrating string. To complete our discussion we should show, as with the simple oscillator, that the effect of friction is to damp out the free vibrations and to change slightly the allowed frequencies. To show that this is so is not difficult by the use of operational calculus, although it would be difficult by any other method.

The difficulty lies with the nature of the frictional term. The resistive force per unit length opposing the string's motion is due to the medium surrounding the string, the medium gaining the energy that the string loses. Part of the energy goes into heating the medium, the amount depending on the *viscosity* of the medium; and part goes into outgoing sound waves in the medium, the amount depending on the *radiation resistance* of the medium. The medium also adds an effective mass per unit length, which may not be negligible if the medium is a liquid. The important point, however, the one that is responsible for our difficulties, is that the effective resistance due to the medium (and also its added effective mass) *depends on the frequency* of the string's motion.

The equation of motion for the string when friction is included is

$$\epsilon \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - R(\omega) \frac{\partial y}{\partial t}$$

where R is the effective frictional resistance per unit length of string. To find the "normal modes" of the string involves a sort of circular process, since we cannot solve for the natural frequencies until we know the value of R, and we cannot obtain the value of R unless we know the frequency of motion. In this case it is actually easier to start with steady-state driven motion, for then the frequency is known and the value of R is definite.

The equation for the string acted on by a simple harmonic force of frequency $(\omega/2\pi)$ exerted on the point $x = \xi$ is

$$\frac{\partial^2 y}{\partial t^2} + 2k(\omega)\frac{\partial y}{\partial t} - c^2\frac{\partial^2 y}{\partial x^2} = \frac{F(\xi)}{\epsilon}\,\delta(x-\xi)e^{-i\omega t} \qquad c^2 = (T/\epsilon), \quad k(\omega) = R(\omega)/2\epsilon \tag{10.23}$$

We are assuming that the added mass due to the reaction of the medium is negligible compared with the weight of the string. The more general case, where we must assume that ϵ is also a function of ω, will be discussed later.

As has been done previously, we shall assume that the steady-state motion of the string can be expressed in terms of a Fourier series

$$y = \sum a_n \sin\left(\frac{\pi n x}{l}\right) e^{-i\omega t}$$

Substituting this in Eq. (10.23), multiplying both sides by $\sin(\pi m x/l)$ and integrating over x from 0 to l, gives an equation for a_m, from which one eventually obtains a series for y

$$y = -\frac{2F(\xi)}{l\epsilon}\sum_{n=1}^{\infty}\frac{\sin(\pi n\xi/l)\sin(\pi n x/l)}{(\omega + ik - w_n)(\omega + ik + w_n)}\,e^{-i\omega t} \tag{10.24}$$

where

$$w_n(\omega) = \left[\left(\frac{\pi n c}{l}\right)^2 - k^2\right]^{\frac{1}{2}}$$

We note that both w_n and k are functions of ω, the driving frequency.

Characteristic Impedances and Admittances.—An interesting alternative method of writing this equation is in terms of transfer admittances. The ratio between the string's velocity $v = -i\omega y$ and the driving force $F(\xi)e^{-i\omega t}$ is

$$\left.\begin{aligned} Y_m(\xi,x;\omega) &= \sum_{n=1}^{\infty}\left[\frac{1}{Z_m(\xi,x;\omega,n)}\right. \\ Z_m(\xi,x;\omega;n) &= \left[-i\omega\left(\frac{l\epsilon}{2}\right) + \frac{lR}{2}\right. \\ &\qquad \left. - \frac{1}{i\omega}\left(\frac{\pi^2 n^2 T}{2l}\right)\right]\csc\left(\frac{\pi n\xi}{l}\right)\csc\left(\frac{\pi n x}{l}\right) \end{aligned}\right\} \tag{10.25}$$

The input admittance is obtained by setting $x = \xi$. Considering v to be analogous to a current and F analogous to a voltage, the reaction of the string (at $x = \xi$) is analogous to that of an electric circuit of an infinite number of parallel branches, the nth branch consisting of an inductance $(l\epsilon/2)\csc^2(\pi n\xi/l)$, a resistance of $(lR/2)\csc^2(\pi n\xi/l)$, and a capacitance of $(2l/\pi^2 n^2 T)\sin^2(\pi n\xi/l)$, all three in series.

The response of the string to a unit impulsive force concentrated at $x = \xi$ is obtained by computing the contour integral

$$y_\delta(\xi,x;t) = -\frac{1}{\pi l\epsilon}\int_{-\infty}^{\infty}\left\{\sum{}' \frac{\sin(\pi n\xi/l)\sin(\pi nx/l)}{(\omega + ik - w_n)(\omega + ik + w_n)} e^{-i\omega t}\right\} d\omega$$

One pole for the nth term occurs when $\omega + ik(\omega) - w_n(\omega)$ is zero. This may be difficult to solve algebraically if $k(\omega)$ is a complicated function of ω. However, it can usually be solved graphically or by successive approximations. We can write the solution symbolically as

$$\left.\begin{aligned} \omega &= \omega_n - ik_n;\quad k_n = k(\omega_n - ik_n) \\ \omega_n &= w_n(\omega_n - ik_n) = [(\pi nc/l)^2 - k_n^2]^{\frac{1}{2}} \end{aligned}\right\} \tag{10.26}$$

It turns out that the other pole of the nth term is at $\omega = -\omega_n - ik_n$.

Taking residues at all the poles we finally obtain

$$y_\delta(\xi,x;t) = \begin{cases} 0 & (t < 0) \\ \dfrac{2}{l\epsilon}\displaystyle\sum_{n=1}^{\infty} \frac{e^{-k_n t}}{\omega_n}\sin\left(\frac{\pi n\xi}{l}\right)\sin\left(\frac{\pi nx}{l}\right)\sin(\omega_n t) & (t > 0) \end{cases} \tag{10.27}$$

which is to be compared with Eq. (10.19). This expression can be used in Eq. (10.20) to obtain the response of the string, with friction, to any transient force.

The expression for y_δ gives the free vibrations of a string when started with an impulsive blow. It shows that the effect of friction is to introduce a damping term $e^{-k_n t}$ into each of the component vibrations. The frequencies of free vibration $(\omega_n/2\pi)$ do not greatly differ from the harmonics for the undamped motion, $(nc/2l)$, if the frictional constant k_n is small. In a good many cases k_n increases as n increases, so that the higher harmonics damp out more rapidly than the lower. In such a case the sound emitted by the string will be harsh just after the start of the motion, owing to the initial intensity of the higher harmonics, becoming "smoother" as the motion damps out.

The amount of energy radiated by a string directly into the air is quite small compared with that which can be radiated by a sounding board attached to the string supports, as may be determined by com-