Synthesizing Musical Sounds by Solving the Wave Equation for Vibrating Objects: Part I*

LEJAREN HILLER

Department of Music, State University of New York, Buffalo, N.Y. 14214

AND

PIERRE RUIZ

Bell Telephone Laboratories, Inc., Murray Hill, N.J. 07974

Solutions of differential equations describing the oscillations of vibrating objects can be used to generate musical sounds by computational means. In the present paper, we shall develop the mathematical basis for this technique as applied to the simulation of the sounds of vibrating strings. In the paper to follow, we shall discuss the computational process actually used and present typical results.

The mathematical basis of this simulation technique is the following.

1) The physical dimensions of the vibrating object and characteristics such as density and elasticity are used to set up differential equations describing its motion.

2) Boundary conditions such as the stiffness of a vibrating string and the position and rigidity of its end supports are specified.

3) Transient behavior due to friction and sound radiation is defined.

4) The mode of excitation is described mathematically. Three basic types of excitation are considered for the case of strings: plucking, striking, and bowing.

5) The resulting differential equations are converted after making certain approximations into difference equations suitable for encoding into a computer program.

INTRODUCTION: In this paper and in one to follow, we shall discuss first results in the development of a novel technique for generating musical sounds by computational means. In this method, the oscillations of a vibrating object are described by means of differential equations which are then solved numerically by means of a standard iterative method programmed for a digital computer. The values of the solutions so obtained are converted by means of digital-to-analog conversion into sound recorded on audio tape. Simultaneously, a limited number of cycles of an oscillation are graphed by means of a plotter so that a visual record of a vibration can be compared to an actual sound. The first phase of this work was limited to the study of vibrating strings, but it should be emphasized that the method is entirely general and can be correctly applied to the study of other vibrating objects; i.e., bars, plates, membranes, spheres, etc.

The present investigation was subdivided into a number of specific problems of increasing complexity which were solved individually. Subsequent to this, a more general computer program was written which incorporates routines for the solutions of all these problems taken together.

1) *The Ideal String*: This is, of course, the simplest mathematical model for a vibrating string. The analytical solution to the problem is well known. Hence it was possible to check results obtained by numerical methods against those obtained by the analytical solution. We derived a difference equation from the wave equation. This difference equation was then solved by the iterative procedure carried out by the computer in order to find discrete values of the solution.

2) *Imposition of Boundary Conditions*: An ideal string is, of course, only a crude approximation of a real

* This paper and its sequel are based on a thesis presented by Pierre Ruiz in partial fulfillment of the requirements for the degree of Master of Music at the Graduate School of the University of Illinois, 1970. This research was was sponsored by the National Science Foundation under Grant US-NSF-GK-594.

string. In order to simulate real strings more closely, we imposed three boundary conditions as follows: a) the effect of stiffness of the string itself, b) the effect of friction and sound radiation, and c) the effect of losses at the end supports.

To account for stiffness, we first obtained a new differential equation from the wave equation by adding one term representing the contribution of stiffness. We then derived the corresponding difference equation. We also defined a new parameter, the *stiffness-to-tension ratio,* since the relative contribution of stiffness and tension to the vibrations of the string can be expressed more conveniently in terms of this parameter. The expression which describes the analytical solution turned out to be quite complex and cumbersome.

We then added two more terms to the differential equation for the stiff string. One takes into account the effect of friction, the other the effect of sound radiation. Because of the greater complexity of this expression, we found that we now had to develop a new type of difference equation and a new iterative method for its solution that involves successive approximations.

Lastly, by assigning a reflection coefficient to each end support, we were able to obtain a single difference equation which accounts for losses at these end supports.

3) *Initial Conditions.* We examined three types of excitation: a) plucking, b) striking, and c) bowing. The first two modes were fairly simple to characterize analytically and two versions for each are described mathematically, the second ones being slightly more sophisticated.

The third mode of excitation, bowing, is not so simple, however. What we did was to consider the curvature of the string at the bowing point as a criterion to decide when the string is freed from the bow. Moreover, whenever the string is free, a reversal in the direction of the motion of the bowed point was used to indicate that the bow has grabbed the string again.

The first stage of this investigation involved the derivation of differential equations which represent the motions of vibrating strings subject to the imposition of the various conditions described. These equations were then transformed into difference equations for which numerical solutions could be obtained by means of standard computer programming techniques. In this first paper of a two-part series, we shall present these mathematical derivations. In the paper to follow, we shall discuss the computer program itself and present typical results.

This is a completely new approach to electronic sound synthesis insofar as the starting point is the physical description of the vibrating object. It has many interesting applications, such as the following.

1) It can be used to find out to what extent actual musical instruments can be represented by relatively simple mathematical models.

2) It can provide a composer with a new tool for synthesizing new sounds. The sounds of either real or fanciful instruments (impossible to build because of material limitations, or even instruments whose physical characteristics and geometry change in prescribed ways in time) can be obtained in this fashion.

3) It can provide assistance in the design of new instruments or in the modification of old ones. The parameters which produce the most desirable sound can then be used to construct an actual instrument.

4) It can be used to investigate playing techniques. Different ways of exciting vibrations in a musical instrument can be expressed mathematically and evaluated by means of the synthesis of the sounds resulting therefrom.

HISTORICAL

According to Cajori [1, p. 242], the motion of vibrating strings was described in terms of "wave equations" for the first time in the 18th century. In 1747 d'Alembert obtained the wave equation for the ideal string:

$$(\partial^2 y/\partial t^2) = c^2(\partial^2 y/\partial x^2) \tag{1}$$

where $c^2 = T/\rho S$, S being the cross-sectional area in cm^2, T the tension in dynes, and ρ the density in g/cm^3. He then found that the general solution of this equation had the form

$$y(x,t) = g_1(x-ct) + g_2(x+ct). \tag{2}$$

Around the same time, Bernouilli showed that the solution of the wave equation could be expressed in terms of an infinite trigonometric series. He claimed that his solution, being compounded of an infinite number of tones and overtones of all possible intensities, was a general solution of the problem. These two forms of the general solution of the wave equation were reconciled by Fourier in 1807 who proved his fundamental theorem that the trigonometric series [1, p. 270]

$$f(x) = \sum_{n=0}^{\infty} (a_n \sin nx + b_n \cos nx) \tag{3}$$

represents the function $f(x)$ for every value of x if the coefficients a_n and b_n are given by

$$\begin{aligned} a_n &= \int_{-\pi}^{\pi} f(x) \sin nx\, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx. \end{aligned} \tag{4}$$

From about 1820, the modern theory of elasticity was gradually developed and this, in turn, permitted systematic investigations of the vibrations of elastic bodies to begin. Later, Lord Rayleigh and others [1, p. 465] made extensive mathematical researches in acoustics, obtaining analytical solutions in terms of Fourier series for stiff strings, bars, and other extended objects.

With strings, two problems appeared to be especially difficult to solve, namely, the problem of the struck string and the problem of the bowed string. In the case of the struck string, serious difficulties arose when attempts were made to describe mathematically the interaction between hammer and string. Kaufmann [2] gave a solution to this problem in 1895 for inelastic hammers. His assumptions were revised and the solution extended to elastic hammers by Das [3] who calculated the maximum amplitudes for different harmonics at the center of a struck string and checked his calculations against measurements made with a microscope. His analytical solution is unfortunately very complex and cumbersome to handle.

In the case of a bowed string, the main difficulty involves obtaining an adequate mathematical representation of the interaction between the bow and the string.

Helmholtz [4] concluded from his experimental observations that the motion of any point along a bowed string can be represented by a sawtooth wave with the ratio of the velocities of ascent and descent being equal to the ratio of the distances of the point in question to the ends of the string. Raman [5] later observed that other modes of vibration were also possible. These involved more than two discontinuities per period in the motion of the bowed point. More recently, Kar *et al.* [6] derived a formula for minimum bowing pressure as a function of the velocity of the bow, the bowing distance, and a coefficient of friction. This formula, valid for high velocities of bowing, was later on extended to low velocities of bowing. Bow pressures and velocities have also been measured by Saunders [7] and Bladier [8].

During the 19th century, investigators assumed that the damping effect of friction on the vibrations of strings was proportional to velocity (Helmholtz and Rayleigh, for example). In 1920, Ghosh [9] obtained experimental evidence for the existence of a damping force proportional to the square of the velocity in the case of oscillations with large amplitudes. As we shall see later, the problem of friction is a very complex one, closely related to the motion of the bridge and radiation of sound.

More recent investigations use the analogy between the wave equation for the vibrating string and the wave equation for some kind of oscillatory electrical system [10]. This treatment permits the introduction of the concept of impedance. This method has the advantage of making the expressions for the analytical solutions somewhat more compact.

It is apparent that analytical solutions, which are for the most part expressed in terms of Fourier series, become more and more impractical as conditions specifying how real vibrating objects oscillate are introduced, and that we have to turn to numerical methods if we want to go any further. The method of solving differential equations using difference equations with the aid of a computer is, of course, widely used today. We do not know, however, of any previous investigations carried out by this method to study and to simulate the sounds of various vibrating objects such as strings commonly used to produce musical tones. With computers specifically, there are currently four categories of processes for synthesizing musical sounds. These are the following.

1) *Sign Bit Extraction:* This technique is based in the fact that each type of computation in a computer has its own characteristic execution time. For example, a number can be repeatedly added to itself at a specified constant rate in the accumulator register to cause the most significant digit to change from zero to one and back again. The square wave thus generated can be amplified and recorded or heard through a loudspeaker. A typical example of this process has been described by Divilbiss [11]. The main advantage of this technique is that sound is generated in real time. Its main disadvantage is an extremely limited timbre range.

2) *Digital Simulation of Components of an Electronic Music Apparatus*: This process was developed by a group at Bell Telephone Laboratories [12]. A compiler first generates programs for a set of "instruments" (orchestra program). These instruments are then "played" by a sequencing program at the command of a sequence of note cards which contain information analogous to that given by conventional music notes. The output is then fed to a D/A converter to obtain recorded sound.

The orchestra program uses "unit generators" as building blocks in order to assemble the instruments of an "orchestra." The unit generators called "oscillators" generate digital representations of periodic signals possessing controllable frequencies and amplitudes. Amplitude modulation, frequency modulation, or both can then be imposed on these periodic signals. Since outputs of given "unit generators" can also be used to control the inputs of other "unit generators," complex sounds can be synthesized. This computational process has gone through several phases of development, the most recent of which is the "Music V" program [13].

3) *Methods Based on Fourier Analysis*: The original sound of a musical instrument is recorded on tape and changed into digital form by means of an A/D converter. A Fourier analysis is performed on the resulting data in order to find values for the amplitudes, phase angles, and attack and decay times of the first n overtones, with the amplitudes and phase angles being expressed as functions of time. The values of the parameters thus found are used to combine sinusoidal functions together and generate sound by D/A conversion.

Luce [14] investigated nonpercussive instruments of the orchestra by this technique in 1962. In 1965 Freedman [15] developed a more refined method of analysis of the samples by using certain mathematical transformations. He also produced tones simulating a clarinet, a trumpet, a saxophone, a bassoon, and a violin. Risset [16] investigated trumpet tones in 1964 and found that a) the overtones rise later than the fundamental, and b) the high-frequency overtones are stronger relative to the fundamental in loud trumpet tones than in soft ones. Risset also synthesized trumpet tones based on these data that closely matched real trumpet tones. Beauchamp [17] has developed a complete computer system for the analysis and synthesis of harmonic musical tones. One of the important aspects of his research is the investigation of the relationship between harmonic content and overall intensity level.

4) *Hybrid Machines*: The computation time, and hence the cost for synthesizing sounds by the second and third group of methods increases as the sounds become more complex. The amount of computation can be reduced by introducing analog components into the system.

This idea has been explored by Gabura and Ciamaga [18] who have used a computer to control voltage-controlled equipment. More recently, Mathews and Moore [19] have developed a computer program to compose and edit functions of time in real time. These functions are used to control the amplitude and the frequency of voltage-controlled oscillators. Features of the program include an algebra for combining functions with each other and with real time inputs to the system, such as knobs, keyboard devices, etc. A subset of the functions and a manually controllable time pointer can be displayed on an oscilloscope. Functions are stored and edited on a random access disk.

THE IDEAL STRING

The ideal string, the simplest mathematical model of a vibrating string, is defined by the following restrictions.

1) The string vibrates in one plane only.
2) Every point moves in a straight line perpendicular to the line joining the end supports of the string; i.e., the string does not stretch and does not vibrate longitudinally.
3) The tension changes negligibly.
4) The string is perfectly flexible.
5) The weight of the string is very small compared to the tension.
6) The string is rigidly supported at both ends.
7) The string has a uniform linear density.
8) The amplitude of the oscillations is small compared to its length.
9) The effect of the surrounding medium is negligible.

A two-dimensional Cartesian coordinate system is conventionally used to describe the vibrations of this string with the string being assumed to lie along the x axis and with one end at the origin. The displacement y of a particular point is a function of both x and time t.

As we already saw, the ideal string can be represented by the wave equation (1), derivations of which are easily found in the literature (see, for example, [20, p. 80 ff.]). We also noted that there exist many solutions of Eq. (1). We felt that it was important that we start our research by solving this classic problem, not only so that we could check our work against known results, but also because we were interested in generating the sound of even this simple vibrating system. We anticipated that this sound would not at all resemble a real vibrating string; however, this is the system with which we wished our catalog of sounds to commence.

Let us return to Eq. (2). Since g_1 and g_2 are the functions $x-ct$ and $x+ct$, respectively, they satisfy the equations

$$g_1(x+\Delta x, t+\Delta t) = g_1(x,t) \tag{5a}$$
$$g_2(x+\Delta x, t) = g_2(x, t+\Delta t) \tag{5b}$$

for $x = c\Delta t$. Because of these relations, g_1 and g_2 can be thought of as waves traveling in the positive and negative directions, respectively. The terms, g_1 and g_2 are related to the initial position $f_1(x) = y(x,0)$ and to the initial velocity of the string $f_2(x) = (\partial y/\partial x)(x,0)$ by

$$g_1(x) = \tfrac{1}{2}[f_1(x) - (1/c)f_2(x)] \tag{5c}$$
$$g_2(x) = \tfrac{1}{2}[f_1(x) + (1/c)f_2(x)] \tag{5d}$$

for $0 \leq x \leq L$.

Numerical methods for solving the wave equation are well known. The method we have used is based on a difference equation that is itself derived from Eqs. (5a) and (5b). Specifically, let us consider $n+1$ equally spaced points on the string, $x_r, \ldots, x_n$. The distance between adjacent points is $\Delta x = L/n$. Since x_0 and x_n are the end points of the string, $x_0 = 0$ and $x_n = L$. Now let $\Delta t = x/L$. Δt is the time taken by the waves g_1 and g_2 to travel the distance Δx in the positive and negative directions, respectively.

Let $y(i,j)$ be the position of x_i at the instant $j\Delta t$. Let $y_1(i,j)$ and $y_2(i,j)$ be the values of g_1 and g_2, respectively, corresponding to $x=x_i$ and to $t=j\Delta t$. Application of Eqs. (5a) and (5b) yields the following equations:

$$\begin{aligned} y_1(i,j+1) &= y_1(i-1,j) \\ y_2(i,j+1) &= y_2(i+1,j) \\ y_1(i,j-1) &= y_1(i+1,j) \\ y_2(i,j-1) &= y_2(i-1,j). \end{aligned} \tag{6}$$

By adding these equations together and reordering terms in the right-hand side we obtain

$$\begin{aligned} &y_1(i,j+1) + y_2(i,j+1) + y_1(i,j-1) + y_2(i,j-1) = \\ &y_1(i-1,j) + y_2(i-1,j) + y_1(i+1,j) + y_2(i+1,j) \end{aligned} \tag{7}$$

so that

$$y(i,j+1) = y(i+1,j) + y(i-1,j) - y(i,j-1). \tag{8}$$

This is an exact difference equation for an ideal string and is, in fact, the only exact difference equation we have used in this work. We can make the following observations concerning its behavior.

1) Because of the presence of the terms $y(i+1,j)$ and $y(i-1,j)$, Eq. (8) is valid only for $1 \leq i \leq n$. But since x_0 and x_n are the end points of the string, we have

$$y(0,j) = y(n+1,j) = 0, \quad \text{for any } j. \tag{9}$$

2) If $y(i,j)$ and $y(i,j-1)$ are known for all values of i in the range $1 \leq i \leq n-1$, then $y(i,j+1)$ can be computed for all these values of i. In other words, if the positions of all the points of the string at two successive instants differing by Δt are known, then we can compute the positions of all the points of the string at the next instant.

3) This implies that if $y(i,0)$ and $y(i,1)$ are known for $1 \leq i \leq n-1$, we can compute the positions of all the points of the string at any future time by using Eq. (8).

4) The solution obtained from the iteration of Eq. (8) is exact regardless of the value of n (n can be as small as 2 in which case only the center point of the string is computed).

5) n and Δt are also related by the expression

$$\Delta t = \Delta x/c = L/nc = 1/(2nf). \tag{10}$$

For given values of L and c, Δt decreases as n increases; in effect, closer points in time require closer points in space. For a given Δt, n increases as the fundamental frequency f of the string decreases.

THE STIFF STRING

Among ordinary musical instruments, the effect of stiffness is most evident in piano strings (especially on upright pianos which have shorter strings). It is responsible for the "stretching of octaves" employed in piano tuning whereby if all the C's are tuned by octaves starting with middle C, the C's above middle C become increasingly sharp and those below middle C increasingly flat [21]. The stiff string is actually a case intermediate between an ideal string and a rigid bar. The differential equation which describes its behavior is

$$(\partial^2 y/\partial t^2) = (T/\rho S)(\partial^2 y/\partial x^2) - (ER^2/\rho)(\partial^4 y/\partial x^4) \tag{11}$$

where E is Young's modulus of elasticity in dyn/cm^2, R is the radius of gyration of a cross section in cm, also $=$ I/S with I being the moment of inertia in cm^4 of a cross section about a plane going through the center of the string perpendicular to the direction of the vibration, and the other terms are the same as defined previously. A normalized form of this equation can be obtained by making the substitution $t' = ct$, where $c^2 = (T/\rho S)$:

$$(\partial^2 y/\partial t'^2) = (\partial^2 y/\partial x^2) - (ESR^2/T)(\partial^4 y/\partial x^4). \tag{12}$$

Eq. (11) has many solutions, but a unique solution requires two additional boundary conditions. We shall consider the particular case of a stiff string clamped at both ends, i.e., the case where $\partial y/\partial x = 0$ for $x = 0$ and $x = L$. The solution of this problem has been given by Morse [20, pp. 166-188]. The fundamental frequency of the motion is

$$f = 2\pi\mu_2[(\mu_2{}^2+2\beta^2)ER^2/\rho]^{1/2} \quad (13)$$

where $\beta^2 = T/8\pi^2ESR^2$ and μ_2 is the smallest solution of

$$\tan(\pi\mu/2) = -(1+1/\epsilon^2\mu^2)^{1/2} \tanh[(\pi/2)(\mu^2+1/\epsilon^2)^{1/2}]. \quad (14)$$

In Eq. (14), $\epsilon = (\pi/L)(ESR^2/T)^{1/2}$. The quantity ϵ, defined by this ratio, will be called the *stiffness-to-tension ratio.*

A difference equation for the stiff string is best derived by substituting difference quotients for the partial derivatives in Eq. (12). Then, as in the case of the ideal string, if we also let $\Delta t = \Delta x/c$, and in addition, we define

$$a = ESR^2/T\Delta x^2 = (L\epsilon/\pi\Delta x)^2 = (n\epsilon/\pi)^2, \quad (15)$$

by substituting difference quotients in Eq. (12), we obtain the expression

$$\begin{aligned}[y(i,j+1) - 2y(i,j) + y(i,j-1)]/c^2\Delta t^2 = \\ [y(i+1,j) - 2y(i,j) + y(i-1,j)]/\Delta x^2 \\ -(ESR^2/T)[y(i+2,j) - 4y(i+1,j) \\ +6y(i,j) - 4y(i-1,j) + y(i-2,j)]/\Delta x^4. \quad (16)\end{aligned}$$

If we now multiply both sides of Eq. (16) by Δx^2 and further simplify, we arrive at the recursion equation

$$\begin{aligned}y(i,j+1) = \\ -6ay(i,j) + (1+4a)[y(i+1,j) + y(i-1,j)] \\ -a[y(i+2,j) + y(i-2,j)] - y(i,j-1). \quad (17)\end{aligned}$$

Since x_0 and x_n are the end points of the string just as with the ideal string, $y(0,j) = y(n+1,j) = 0$ for any j. In addition, if we assume that the string is clamped at its end points, we have the additional boundary condition that $y(1,j) = y(n,j) = 0$ for any j. A set of values for $y(i,0)$ and $y(i,j)$ for $2 \leqq i \leqq n\text{-}1$ now determines a solution uniquely. However, the stiffness parameter a in the difference equation is proportional to the square of the number of string subdivisions. This means that Eq. (17) will yield correct solutions for small values of ϵ only. Moreover, the greater the number of string divisions used, the smaller a must be. Consequently, we shall limit ourselves to strings with a stiffness-to-tension ratio and a fundamental frequency such that $a < 0.25$.

Finally, for small values of ϵ we note that the solution of Eq. (14) is

$$\mu \approx 1.0 + 2/\pi. \quad (18)$$

In Table I we give the solution of Eq. (14) for various values of ϵ.

Table I. Solution of $\tan(\pi\mu/2) = (1 + 1/\epsilon^2\mu^2)^{1/2} \tanh[(\pi/2)(\mu^2 + 1/\epsilon^2)^{1/2}]$ for different values of ϵ

ϵ	μ	ϵ	μ	ϵ	μ
0	1.0000	1.6	1.4779	4.1	1.5012
0.01	1.0064	1.7	1.4810	4.2	1.5014
0.02	1.0129	1.8	1.4835	4.3	1.5016
0.03	1.0194	1.9	1.4857	4.4	1.5018
0.04	1.0261	2.0	1.4876	4.5	1.5020
0.05	1.0328	2.1	1.4892	4.6	1.5021
0.06	1.0396	2.2	1.4906	4.7	1.5023
0.07	1.0464	2.3	1.4919	4.8	1.5024
0.08	1.0533	2.4	1.4930	4.9	1.5025
0.09	1.0603	2.5	1.4939	5.0	1.5026
0.1	1.0673	2.6	1.4948	6	1.5036
0.2	1.1392	2.7	1.4956	7	1.5041
0.3	1.2092	2.8	1.4963	8	1.5045
0.4	1.2710	2.9	1.4969	9	1.5047
0.5	1.3210	3.0	1.4974	10	1.5049
0.6	1.3595	3.1	1.4980	11	1.5050
0.7	1.3887	3.2	1.4984	12	1.5051
0.8	1.4107	3.3	1.4988	13	1.5052
0.9	1.4274	3.4	1.4992	14	1.5052
1.0	1.4403	3.5	1.4996	15	1.5053
1.1	1.4504	3.6	1.4999	20	1.5054
1.2	1.4584	3.7	1.5002	30	1.5055
1.3	1.4648	3.8	1.5005	40	1.5056
1.4	1.4701	3.9	1.5007	50	1.5056
1.5	1.4744	4.0	1.5010	60	1.5056

THE EFFECT OF FRICTION

Friction, present in every vibrating string, not only dampens free oscillations, but also slightly changes the allowed frequencies. One resistive force results from interaction with the surrounding medium. A second resistive force arises from the internal friction of stiff strings. When such a string vibrates, it dissipates heat as a result of the nonelastic deformations to which it is subjected. However, since the strings used in many musical instruments can be assumed to be almost perfectly elastic for small oscillations, the effect of internal friction will not be considered at this time.

The medium acts in three different ways on the string: 1) it dissipates energy from the string in the form of heat because of its viscosity, 2) it draws energy from the string when outgoing sound waves are propagated (this is the radiation resistance of the medium), and 3) it adds an effective mass per unit length. The result is that most vibrations decay more or less exponentially with a damping rate usually expressed in terms of a time constant τ. This is defined as the time it takes for the amplitude to decrease by a factor of $e = 2.718$, which in turn is equivalent to a drop of 10 log 2.718 = 4.34 dB.

Let us first assume that a vibrating string is subjected to a damping force proportional to the velocity at each point and at each instant in time in order to account for energy lost in the form of heat. This damping force attenuates all the overtones by the same amount. Consequently, for heat dissipation alone, Eq. (11) becomes

$$\begin{aligned}(\partial^2y/\partial t^2) = (T/\rho S)(\partial^2y/\partial x^2) - (ER^2/\rho)(\partial^4y/\partial x^4) \\ -b_1(\partial y/\partial t) \quad (19)\end{aligned}$$

where $-b_1(\partial y/\partial t)$ is the damping term. For sound radiation we now need a second term that discriminates among frequencies. To obtain it, let us choose the simplest damping term that discriminates among frequencies, namely, a term proportional to $-\partial^3y/\partial t^3$, which represents a damping force proportional to ω^2, and add it to Eq. (19):

$$\begin{aligned}(\partial^2y/\partial t^2) = (T/\rho S)(\partial^2y/\partial x^2) - (ER^2/\rho)(\partial^4y/\partial x^4) \\ -2b_1(\partial y/\partial t) + 2b_3(\partial^3y/\partial t^3). \quad (20)\end{aligned}$$

In Eq. (20), b_1 accounts for heat dissipation, b_3 accounts for sound radiation, and the 2's are inserted for convenience.

The normalized form of this equation is obtained by again making the substitution $t' = ct$:

$$(\partial^2 y/\partial t'^2) = (\partial^2 y/\partial x^2) - a\Delta x^2 (\partial^2 y/\partial x^4) \\ - (2b_1/c)(\partial y/\partial t') + 2b_3 c (\partial^3 y/\partial t'^3). \quad (21)$$

Our next task is the derivation of a difference equation for this wave equation. By substituting difference quotients for the derivatives in Eq. (21) we get:

$$[y(i,j+1) - 2y(i,j) + y(i,j-1)]/\Delta t'^2 = \\ [y(i+1,j) - 2y(i,j) + y(i-1,j)]/\Delta x^2 \\ - a\Delta x^2 [y(i+2,j) - 4y(i+1,j) + 6y(i,j) \\ - 4y(i-1,j) + y(i-2,j)]/\Delta x^4 \\ - (2b_1/c)[y(i,j+1) - y(i,j-1)]/2\Delta t' \\ + 2b_3 c [y(i,j+2) - 2y(i,j+1)]/2\Delta t'^3 \\ + [2y(i,j-1) - y(i,j-2)]/2\Delta t'^3. \quad (22)$$

Then, if we multiply both sides by $\Delta t'^2$, use the equality $\Delta t' = c\Delta t$, and simplify, we get

$$[1 + b_1\Delta t + (2b_3/\Delta t)]\, y(i,j+1) = \\ 2(1-3a)\, y(i,j) + 4a[y(i+1,j) + y(i-1,j)] \\ -a[y(i+2,j) + y(i-2,j)] - [1 - b_1\Delta t \\ - (2b_3/\Delta t)]\, y(i,j-1) + (b_3/\Delta t)[y(i,j+2) \\ - y(i,j-2)]. \quad (23)$$

The most successful method for solving this equation is the following. Suppose that $y(i,j)$ has been computed for all i up to $j = j_1 - 1$ and that we have an estimate for all the $y(i,j)$, then we proceed as follows.

1) Obtain an estimate for all the $y(i,j_1+1)$ by deleting the term $(b_3/\Delta t)[y(i,j+2) - y(i,j-2)]$ in Eq. (23) and using the resulting expression.

2) Recompute $y(i,j_1)$ using Eq. (23).

3) Repeat steps 1) and 2).

This method will not work unless $b_3/\Delta t < 1$, and the smaller $b_3/\Delta t$ is, the more accurate the method becomes. Eq. (23) is perhaps the most important equation we have thus far derived because its solution forms the central computation loop of the computer program to be discussed in Part II of this paper. It is also possible to obtain an analytical solution to Eq. (20) and in fact we have done so [22]. For our present purposes, the solution so obtained also provided expressions for evaluating the damping constants b_1 and b_3. These expressions are

$$n_1 = (b_1 + b_3\omega^2)/(1 + 9b_3^2\omega^2) \\ n_2 = (-3b_3^2\omega^2)/(1 + 9b_3^2\omega^2). \quad (24)$$

The terms n_1 and n_2 are small integration constants, $\omega = 2\pi f$, and b_1 and b_2 are assumed to be small in order to eliminate other terms in the general solution.

EFFECT OF THE END SUPPORTS

On stringed instruments, one end of the vibrating string is supported by a bridge that is in contact with some sort of resonance box. On some instruments like the guitar, this bridge is firmly attached to the resonance box. On others, like those of the violin family, the bridge is held in its position only by the tension of the string. Most of the energy of the string is transmitted to the resonance box via this bridge. The other end of the vibrating portion of the string is sometimes supported by a second bridge as in the piano. For instruments of the violin family, on the other hand, it is the point of contact between the string and the fingerboard for an "open string" if not the point of contact with the player's finger. The greater absorbing properties of flesh explain in part the difference in tone between open and stopped strings.

The way in which energy is absorbed by the bridge depends on the bridge itself as well as on the resonance box. A complete analysis would have to consider five vibrating systems coupled together: the string, the bridge, the resonance box, the air inside the resonance box, and the air outside the resonance box. A complete analysis of such a system would be quite overwhelming at this point. Instead, let us approach the problem by using a highly simplified model of the end supports.

In order to account for losses that may occur when reflections take place at the end supports, let us assign a reflection coefficient to each of the supports so defined that at any instant the magnitude of the reflected wave at either end support is equal to the incident wave multiplied by the respective reflection coefficient. The motion of the string and the supports (which are no longer fixed) can therefore be described by the four following equations, where $i = 1,n$:

$$g_1(i+1,j) = g_1(i,j-1) \quad (25a)$$
$$g_2(i,j) = g_2(i+1,j-1) \quad (25b)$$
$$g_1(1,j) = \alpha_1 g_2(1,j) \quad (25c)$$
$$g_2(n+1,j) = \alpha_2 g_2(n+1,j). \quad (25d)$$

Eq. (25a) and (25b) again say that g_1 and g_2 are waves traveling in opposite directions. Eqs. (25c) and (25d) in which α_1 and α_2 are the reflection coefficients of the end supports, and which account for losses at the supports, supersede previously employed boundary conditions. If the supports are lossfree, α_1 and $\alpha_2 = 1$; in general, however, they will be numbers between 0 and 1. By definition, we can now write

$$y(2,j+1) = g_1(2,j+1) + g_2(2,j+1). \quad (26)$$

By making use of Eqs. (25a) and (25b), we get

$$y(2,j+1) = g_1(1,j) + g_2(1,j+2). \quad (27)$$

Then by using Eq. (25c) we get

$$y(2,j+1) = -\alpha_1 g_2(1,j) + g_2(1,j+2). \quad (28)$$

By definition, we also have

$$y(1,j) = g_1(1,j) + g_2(1,j). \quad (29)$$

By substitution from Eq. (25c) we get

$$y(1,j) = g_2(1,j)(1 - \alpha_1). \quad (30)$$

Combining this result and Eq. (28) and rearranging, we get

$$y(1,j+2) = (1-\alpha_1)\, y(2,j+1) + \alpha_1 y(1,j). \quad (31a)$$

This is the difference equation for the support located at the origin. By symmetry the difference equation for the other support must be

$$y(n+1,j+2) = (1-\alpha_2)\, y(n,j+1) + \alpha_2 y(n+1,j). \quad (31b)$$

These last two equations together with Eq. (23) can be iterated to obtain the motion of the string and constitute the most general set of difference equations that will be used to simulate a vibrating string.

INITIAL CONDITIONS

Finally, we must determine how the string is excited; i.e., we need to know not only the shape of the string at $t=0$, namely, $y(x,0)$ as a function of t but also the velocities of all the points of the string at $t=0$, namely, $y'(x,0)$ as a function of x. In musical practice, strings are most often set into vibration by plucking, striking, or bowing. With plucking and striking, the energy is provided just at the beginning and is then absorbed by the surrounding medium. In bowing, however, energy is provided continuously in order to obtain a sustained tone.

We shall define two models for the plucked string, a simple one labeled "plucked string" and a more sophisticated one labeled "transient 1." We shall also define two models for the struck string, a simple one labeled "struck string" and a more sophisticated one labeled "transient 2." Finally, we shall define the bowed string, a much more complex problem

1) *The Plucked String*: Let us assume that a point on the string is displaced from its equilibrium position and is released from rest. The triangular shape of the string at time $t = 0$ can be represented by the following equations:

$$\begin{aligned} y &= (y_p/x_p)\,x, && 0 < x \leq x_p \\ y &= y_p(1-x)/(1-x_p), && x_p < x \leq L \end{aligned} \tag{32}$$

where x_p and y_p represent the abscissa and ordinate, respectively, of the plucked point. Since the string is released from rest, we have the additional boundary condition that $(\partial y/\partial t) = 0$ and $0 \leq x \leq L$ at $t = 0$. These boundary conditions can be transformed so that we can use them together with the wave equation in its difference equation form. In other words, the initial state thus described can be sampled to yield discrete values for the $y(i,0)$. Since the string is released from rest, we have from Eqs. (5c) and (5d)

$$g_1(x) = g_2(x) = \tfrac{1}{2} f_1(x) \tag{33}$$

where $f_1(x)$ is expressed by the $y(i,0)$. By the time the string assumes the shape given by the $y(i,1)$ g_1 will have moved in the positive direction by Δx (a string subdivision) and g_2 in the negative direction by the same amount. Therefore, $y(i,1)$ can be computed by adding g_1 and g_2 together to yield the following result:

$$\begin{aligned} y(i,1) &= y(i,0), && i = 1, \cdots, n-1, i \neq k, k+1 \\ y(i,1) &= \tfrac{1}{2}[y(i-1,0) + y(i+1,0)], && i = k, k+1 \end{aligned} \tag{34}$$

where $y(k,0)$ and $y(k+1,0)$ are the points where the initial string shape has discontinuities in slope.

Although Eqs. (34) were derived for an ideal string, they will be assumed to be an adequate approximation. However, if a large number of subdivisions is used, the vertex of the triangle will have to be rounded to avoid the slope discontinuity at x_p. This is especially important in the case of stiff strings where discontinuities in slope might cause excessive bending and make the calculation very inaccurate.

2) *Transient 1*: When real instruments are played, the point at which a string is plucked is in contact with a player's finger for a certain amount of time. During this time the plucked point moves away from its equilibrium position so that when the string is finally released, all of its points will be moving with nonzero velocity. For our purposes, we shall assume that the plucked point moves with constant acceleration during the time it is in contact with the player's finger. This is also a somewhat ideal model since the total force acting in the y direction on the plucked point is assumed to be constant during the time of contact with the player's finger. This would be approximately true only if this time of contact is short and the force exerted by the player is large, so that the effect of the changing restoring force due to the tension of the string is negligible.

If the plucked point moves in the positive y direction with a constant acceleration a for t_1 seconds at the end of which its displacement from equilibrium is y_p, we have

$$y_p = at^2/2, \qquad 0 \leq t \leq t_1. \tag{35}$$

The displacement of the plucked point, $y(p,j)$, is obtained by sampling this continuous function. The first position to be computed after the string is free must be obtained from the expression

$$y(p, N_p+1) = \tfrac{1}{2}[y(p-1,N_p) + y(p+1,N_p)] \tag{36}$$

where N_p is the number of sampling intervals corresponding to t_1, and not taken from the wave equation. This can be seen by the reasoning which led to the derivation of Eqs. (34). For $j > N_p + 1$, $y(p,j)$ can be obtained from the wave equation.

3) *Struck String*: Let us assume that at time $t = 0$ the string has just been struck and the struck portion of the string, which lies between $x = L_1$ and $x = L_2$, has acquired its maximum displacement y_0. Let us assume further the striking time is so short that portions of the string which were not struck are still at rest at time $t = 0$. Mathematically,

$$\begin{aligned} y(x,0) &= 0, && 0 \leq x \leq L_1, L_2 \leq x \leq L \\ y(x,0) &= y_0, && L_1 < x < L_2 \\ y'(x,0) &= 0, && 0 \leq x \leq L. \end{aligned} \tag{37}$$

In order to find corresponding values for the $y(i,0)$ and the $y(i,1)$, let us assume that the struck portion of the string is represented by the points $x_j, x_{j+1}, \ldots, x_k$. The reasoning leading to Eqs. (34) yields in this case

$$\begin{aligned} y(i,1) &= 0, && i = 1, \cdots, j-2,\ i = k+2, \cdots, n-1 \\ y(i,1) &= y_0/2, && i = j-1, j, k, k+1 \\ y(i,1) &= y_0, && i = j+1, \cdots, k-1. \end{aligned} \tag{38}$$

4) *Transient 2*: To improve the above model, let us assume this time that the string is struck by a hammer of mass M at a point a cm from one end. Let V_0 be the velocity of the hammer before it strikes the string. The hammer will be in contact with the string T_1 seconds. During this time, the only force acting on the hammer is the restoring force of the string. Let us assume additionally that the string has a triangular shape.[1] Then, if the displacement y is small compared to the length L, we can write for the restoring force

$$F = T[y/a + y/(L-a)] = yLt/a(L-a). \tag{39}$$

The equation for the hammer is therefore

$$M(d^2y/dt^2) + M\omega^2 y = 0 \tag{40}$$

where $\omega^2 = LT/a(L-a)M$. Since the boundary conditions

[1] This is actually an oversimplification. See Kock [10].

are $y(0) = 0$ and $y'(0) = V_0$, the solution of the differential equation is

$$y = (V_0/\omega) \sin \omega t. \tag{41}$$

Since the restoring force is proportional to the displacement y, the hammer will be in contact with the string until y reaches 0, *i.e.*, until $T_1 = \pi/\omega$. Then, if A is the maximum displacement of the struck point, we obtain $A = V_0/\omega$.

If we wish maximum energy transfer, T_1 should be set equal to $(1-a/L)/f$, and the string should be struck at a point such that

$$(4/\pi)(1-a/L)^2 \cot(\pi a/L) = M/m \tag{42}$$

where m is the mass of the string and M the mass of the hammer.

If p is the coordinate of the struck point, we have from Eq. (41)

$$y(p,j) = (V_0/\omega) \sin(\omega \cdot \Delta t \cdot j), \qquad j = 0, N_p \tag{43}$$

where N_p is defined as before and Eq. (36) is applied.

5) *Bowed String*: Let us assume the following. a) The bow is initially at rest. Its velocity increases linearly with time until its steady-state velocity has been reached. From then on, the velocity of the bow remains constant. b) The bowing pressure is constant. c) The portion of the string which is bowed is so short that it can be assumed to be a point. d) At any given instant of time, the string is in one of two states: the bowed point moves along with the bow without slipping or the string slips away from the bow with the effect of friction being negligible so that in effect the string is free.

The motion of all the points of this string other than the bowing point is accounted for by the wave equation. If the string is in mode 2, the wave equation also applies to the bowing point. However, if the string is in mode 1, the bowed point is constrained to move at the velocity of the bow. Moreover, if we can also assume that the effect of the initial acceleration of the bow is negligible, then the sum of all the forces acting on the bowed point must be zero. The frictional force must be equal and opposite to the restoring force due to tension. Eventually, this restoring force, equal to $T(\partial^2 y/\partial x^2)$, will become larger than the maximum frictional force which can be provided by the bow, whereupon the string will shift to mode 2. In other words, if the string is in mode 1, it will go into mode 2 as soon as $|\partial^2 y/\partial x^2| > p$ for a certain positive constant p which depends on the bowing pressure. Conversely, if the string is free (mode 2), it will be caught by the bow (mode 1) as soon as the bowed point starts moving in the same direction as the bow. Let us now establish some numerical relations between p, the velocity of the bow V, the bowed point x_m, the maximum displacement of the bowed point h, the string length L, and the frequency f.

Assuming Helmholtz's result, i.e., that the string is in mode 1 during $(L-x)/L$ of the vibration cycle, and assuming that the bowed point is displaced by $2h$ during that time, we can write:

$$V = (2hfL)/(L-x_m) = (2hfn)/(n-m). \tag{44}$$

In order to get an estimate for p, let us assume again that the string is vibrating as described by Helmholtz. When the bowed point reaches its maximum displacement h, the value of the difference quotient for $(\partial^2 y/\partial x^2)$ at this point is given by

$$\begin{aligned} P_1 &= h/(L-x_m) + h/x_m \\ &= hL/x_m(L-x_m) = (h/\Delta x)n/m(m-n). \end{aligned} \tag{45}$$

Since the string is just about to go into mode 2, we must have $P < P_1$. We will therefore let

$$P = \text{PRESS} \times (h/\Delta x) n/m(m-n) \tag{46}$$

where *PRESS* is a number between 0 and 1 to be determined experimentally.

When the boundary conditions for the bowed point are implemented together with the wave equation, we must remember to use Eq. (36), where in this case p corresponds to the bowed point and N_p to the time of release, for the first computation of the point each time the string goes into mode 2.

It is also necessary to use the general version of the wave equation (23), since it was found experimentally that severe damping of higher frequencies is essential for the establishment of Helmholtz's mode of vibration.

REFERENCES

[1] F. Cajori, *A History of Mathematics* (Macmillan, New York, 1920).

[2] W. Kaufmann, "Uber die Bewegungen geschlagener Saiten," *Ann. Phys.*, vol. 54, p. 675 (1895).

[3] P. Das, "Investigations on the Acoustics of the Pianoforte," *Proc. Assoc. Sci.*, vol. 8, p. 107 (1923); "On the Impact of an Elastic Hammer on a Pianoforte String," *Ind. J. Phys.*, vol. 10, p. 75 (1926).

[4] H. L. F. Helmholtz, *On the Sensations of Tone*, 2nd English ed., A. Ellis, Ed., based on 4th German ed., 1877 (Dover Publ., New York, 1954), p. 83.

[5] C. V. Raman, "On the Mechanical Theory of Bowed Strings and of Musical Instruments of the Violin Family, with Experimental Verification of the Results," *Proc. Ind. Assoc. Sci.*, vol. 15, p. 3 (1918).

[6] K. C. Kar, N. K. Datta, and S. K. Ghosh, "Investigations on the Bowed String with an Electrically Driven Bow," *Ind. J. Theor. Phys.*, p. 423 (1951); K. C. Kar and C. Dutta, "Extension of Kar Formula for Minimum Bowing Pressure," *ibid.*, p. 31 (1957); C. Dutta, "On the Verification of the Extended Kar Formula for Minimum Bowing Pressure," *ibid.*, p. 105 (1957); C. Dutta, "On the Work Done During Bowing," *ibid.*, p. 5 (1959).

[7] F. A. Saunders, "The Mechanical Action of Violins," *J. Acoust. Soc. Am.*, vol. 9, p. 81 (1937); F. A. Saunders, "Recent Work on Violins," *ibid.*, vol. 25, p. 491 (1953).

[8] B. Bladier, "Sur les phénoménes transitoires des cordes vibrantes," *Acustica*, vol. 14, p. 331 (1964).

[9] R. N. Ghosh, "On the Forced Oscillations of Strings Under Damping Proportional to the Square of the Velocity," *Ind. J. Phys.*, no. 10, p. 67 (1920).

[10] W. E. Kock, "The Vibrating String Considered as an Electrical Transmission Line," *J. Acoust. Soc. Am.*, vol. 8, p. 227 (1937).

[11] J. L. Divilbiss, "Real Time Generation of Music with a Digital Computer," *J. Music Theory*, vol. 8, p. 99 (1964).

[12] M. V. Mathews, "An Acoustic Compiler for Music and Psychological Stimuli," *Bell Syst. Tech. J.*, vol. 40, p. 677 (1961); M. V. Mathews, J. R. Pierce, and N. Guttman, "Musical Sounds from Digital Computers," *Gravesaner Blätter*, no. 23/24, p. 119 (1962).

[13] M. V. Mathews (with the collaboration of J. E. Miller, F. R. Moore, J. R. Pierce, and J. C. Risset), *The Technology of Computer Music* (M.I.T. Press, Cambridge, Mass., 1969).

[14] D. A. Luce, "Physical Correlates of Non-Percussive Musical Instrument Tones," Ph.D. dissertation, M.I.T., Cambridge, Mass., 1963.

[15] M. D. Freedman, "A Technique for the Analysis of Musical Instrument Tones," Ph.D. dissertation, University of Illinois, Urbana, (1965); also Biological Computer Lab., Elec. Eng. Res., University of Illinois, Urbana, Tech. Rep. 6 (Oct. 1965).

[16] J. C. Risset, "Computer Study of Trumpet Tones," Bell Telephone Lab., Int. Rep. (1966).

[17] J. W. Beauchamp, "A Computer System for Time-Variant Harmonic Analysis and Synthesis of Musical Tones," in *Music by Computers*, H. von Foerster and J. W. Beauchamp, Eds. (Wiley, New York, 1969), pp. 19-62.

[18] J. Gabura and C. Ciamaga, "Digital Computer Control of Sound Generating Apparatus for the Production of Electronic Music," *Electron. Music Rev.*, vol. 1, p. 56 (1967).

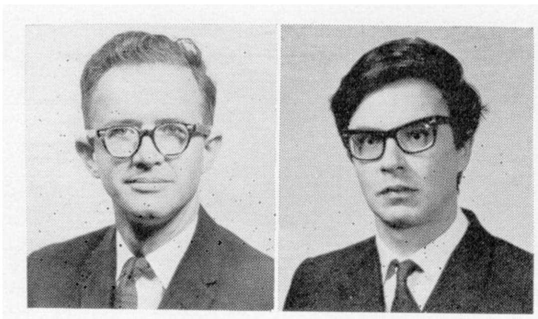
[19] M. V. Mathews and F. R. Moore, "GROOVE—A Program to Compose, Store, and Edit Functions of Time," *Communications of the Association for Computing Machinery*, vol. 13, p. 715 (1970).

[20] P. M. Morse, *Vibrations and Sound*, 2nd ed. (McGraw-Hill, New York, 1948).

[21] R. Young, "Inharmonicity of Plain Wire Piano Strings," *J. Acoust. Soc. Am.*, vol. 24, p. 269 (1952); E. D. Blackham, "The Physics of the Piano," *Sci. Am.*, vol. 213, no. 10, p. 88 (1965); H. Fletcher, E. D. Blackham, and R. Stratton, "Quality of Piano Tones," *J. Acoust. Soc. Am.*, vol. 34, p. 749 (1962).

[22] P. M. Ruiz, "A Technique for Simulating the Vibrations of Strings with a Digital Computer," M. Music dissertation, University of Illinois, Urbana, pp. 43-47 (1970).

THE AUTHORS



L. HILLER R. RUIZ

Lejaren Hiller has been Frederick Slee Professor of Composition at the State University of New York at Buffalo since 1968. He is at the same time Co-Director with Lukas Foss of the Center for the Creative and Performing Arts in Buffalo. At present, he is continuing research on the application of electronic technology to the study of musical acoustical phenomena under a continuation of the National Science grant that made the present work possible.

Dr. Hiller is the author of some 50 articles in science and music and the composer of some 46 scores in all media, electronic and computer-generated as well as conventional music. Prior to his present post, Dr. Hiller was Professor of Music and Director of the Experimental Music Studio at the University of Illinois, Urbana.

•

Pierre Ruiz received the B.S. degree in electrical engineering in 1961, the M.A. degree in mathematics in 1962, and the M. Music degree in 1970, all from the University of Illinois, Urbana.

Since 1968 he has been engaged in speech research with computers at Bell Telephone Laboratories, Inc., Murray Hill, N.J. He also adapted the "Music V" program to a DDP-224 computer and composed several musical works with this program. One was used in a score for Robert Wilson's play "Deafman Glance," at the Brooklyn Academy of Music in 1970.