Synthesizing Musical Sounds by Solving the Wave Equation for Vibrating Objects: Part II*

LEJAREN HILLER

Department of Music, State University of New York, Buffalo, N.Y. 14214

AND

PIERRE RUIZ

Bell Telephone Laboratories, Inc., Murray Hill, N.J. 07974

In Part I of this paper, derivations from the wave equation of difference equations describing the oscillations of vibrating objects were presented. The vibrations of ideal and nonideal strings set into motion by different types of excitation were considered in particular.

In Part II, the following is discussed.

1) The resulting difference equations are solved by means of a standard iterative procedure with the aid of a computer.

2) Discrete values of the solution corresponding to the motion of a selected point of the object are written on a file of digital magnetic tape.

3) The numerical values written on this tape are converted into analog signals by means of D/A conversion and recorded on audio tape.

4) A few cycles of the solution are also plotted by means of a microfilm plotter in order to compare the visual appearance of the vibration with the sound on tape.

The simulation was successful in reproducing the main features of string tones. It was found that losses play a very important role in defining the timbres of these particular tones. More generally, however, this research illustrates how musical sounds of all sorts might be generated with a digital computer by utilizing a conceptual approach that heretofore has not been attempted.

INTRODUCTION: In Part I of this paper [1], we presented the mathematical basis for the writing of a computer program that permits the simulation of the sounds of musical instruments, either real or fanciful. The idea is that we use the wave equation for vibrating objects to define the oscillations of the system under investigation. Thus far we have been concerned especially with the vibrations of strings, but we have also emphasized that this is only a particular application of a general technique that can be used for investigating the behavior of all types of vibrating objects.

The mathematical basis of this simulation technique can be summarized as follows.

1) The physical dimensions of the vibrating object and characteristics such as density and elasticity are used to set up differential equations describing its motions.

2) Boundary conditions such as the stiffness of a vibrating string and the rigidity of its end supports are specified.

3) Transient behavior that depends upon friction and sound radiation is defined.

4) Five modes of excitation are described mathematically, namely. a) simple model of a plucked string, b) refined model of a plucked string (transient 1), c) simple model of a struck string, d) refined model of a struck string (transient 2), e) bowed string.

In order to make the differential equations amendable to encoding into a computer program, it was also necessary to convert these equations into difference equations that can, in turn, be solved numerically. In the case of the ideal string, this was a relatively simple task, but once the more complex conditions described were imposed, various approximations and limitations had also to be dealt with in order to obtain useful expressions. Each one of these was tested in turn by means of writing computer programs for typical numerical solutions.

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Subsequently, a more general computer program for sound synthesis was written that would permit its user to impose at will the various constraints and initial conditions thus far investigated. This program is concerned particularly with the solution of Eq. (23) in Part I which is reproduced here for convenient reference

$$[1+b_1\Delta t+(2b_3/\Delta t)]y(i,j+1)=2(1-3a)y(i,j)+4a[y(i+1,j)+y(i-1,j)]-a[y(i+2,j)+y(i-2,j)]-[1-b_1\Delta t-(2b_3/\Delta t)]y(i,j-1)+(b_3/\Delta t)[y(i,j+2)-y(i,j-2)]. \quad (1)$$

As we noted, the most successful method for solving this equation is based upon the following considerations. Suppose that $y(i,j)$ has been computed for all i up to $j = j_1-1$ and that we have an estimate for all the $y(i,j)$; then we proceed as follows.

1) Obtain an estimate for all the y (i, j_1+1) by deleting the term $(b_3/\Delta t)[y(i, j+2)-y(i, j-2)]$ in Eq. (1) and using the resulting expression.

2) Recompute $y(i, j_1)$ using Eq. (1).

3) Repeat steps 1) and 2).

It was also pointed out that this method will not work unless $b_3/\Delta t<1$, and that the smaller $b_3/\Delta t$ is, the more accurate the method becomes.

In implementing the programming to be described hereafter, the authors carried out its first stages at the University of Illinois when both of them were still affiliated with that institution. Plots of a few cycles of the typical solutions to these problems were obtained with the aid of a CALCOMP plotter. This work was done on an IBM-7094 computer. At the same time, we had hoped to convert the results into recorded sound by means of a sound-generating system attached to the ILLIAC II computer. For various technical and administrative reasons, this proved not to be feasible, so the project was completed at a later date at Bell Telephone Laboratories on a Honeywell DDP-224 computer system to which a D/A converter system is attached. The tape generated by the DDP-224 was also used to make a second set of graphs plotted with the aid of a Stromberg–Carlson 4020 microfilm plotter coupled to a GE-645 computer. It is this revised programming that is described here.

THE COMPUTER PROGRAM

The computer program for sound synthesis was written to generate sounds that depend on any one of six types of boundary conditions. The first five of these correspond to the five modes of excitation described: plucking, striking, bowing, transient 1, and transient 2. The sixth type defines a fanciful string with variable stiffness.

Six different tasks can be executed by the program: 1) reading the data and providing the initialization corresponding to one of the boundary conditions, 2) printing the first two consecutive shapes of the string, 3) choosing the point on the string to be used for sound synthesis, 4) synthesizing and writing samples on a digital tape, 5) playing back this output through a D/A converter, and 6) displaying samples on the scope.

The main program has an "option control" section to which the program returns after each task has been performed. The user selects each option by typing an appropriate two-digit name.

The sampling rate of the simulation is computed by the formula

$$\text{sampling rate} = 2fn, \quad (2)$$

where f is the fundamental frequency and n is the number of string subdivisions. This latter quantity is computed in turn by means of the formula $n = 10500/f+1$. This ensures a sampling rate slightly greater than 20 kHz. The output is later desampled to exactly 20 kHz by linear extrapolation.

A subroutine carries out the actual iteration of the wave equation. Samples are synthesized in groups of three. The roles of these three arrays are permuted cyclically to avoid being forced either to use very large arrays or to spend computing time transferring data.

The three samples thus computed then go through the desampling procedure and are scaled to convert them into 12-bit numbers which are stored in yet another array with samples that are out of range being clipped. As soon as the last array is filled, a system subroutine transfers the data to magnetic tape. This process recycles until we write an end of file.

The subroutines used to iterate Eq. (1) by the three-step process outlined carries out the process by a set of

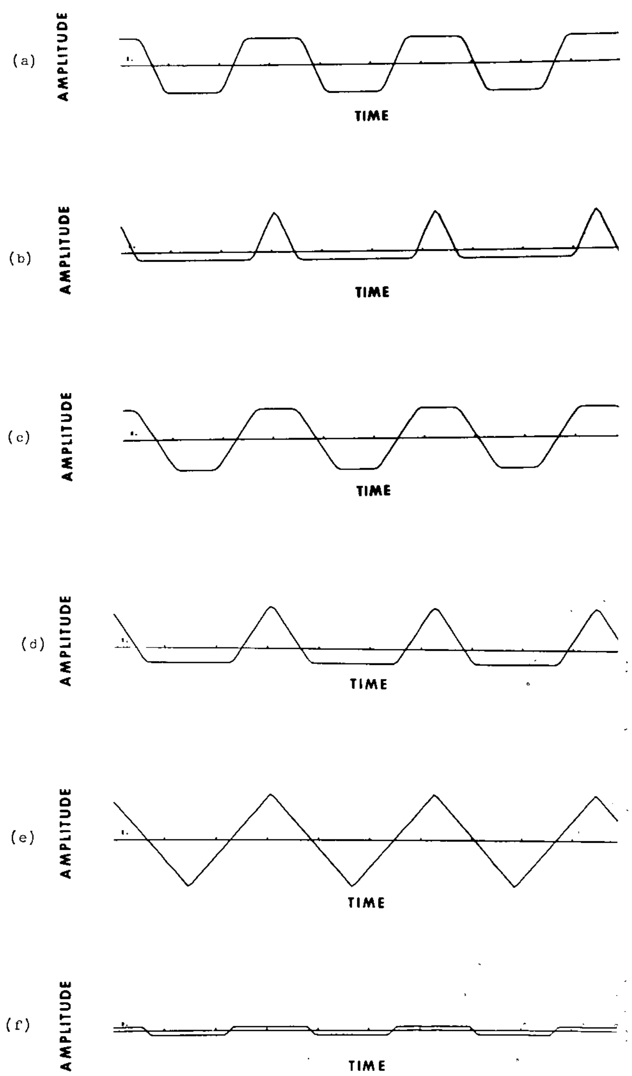

Fig. 1. Ideal plucked string tuned to 500 Hz.
a. $x_p = 15.0$, $x_0 = 50.0$, b. $x_p = 15.0$, $x_0 = 15.0$,
c. $x_p = 25.0$, $x_0 = 50.0$, d. $x_p = 25.0$, $x_0 = 25.0$,
e. $x_p = 50.0$, $x_0 = 50.0$, f. $x_p = 50.0$, $x_0 = 5.0$.

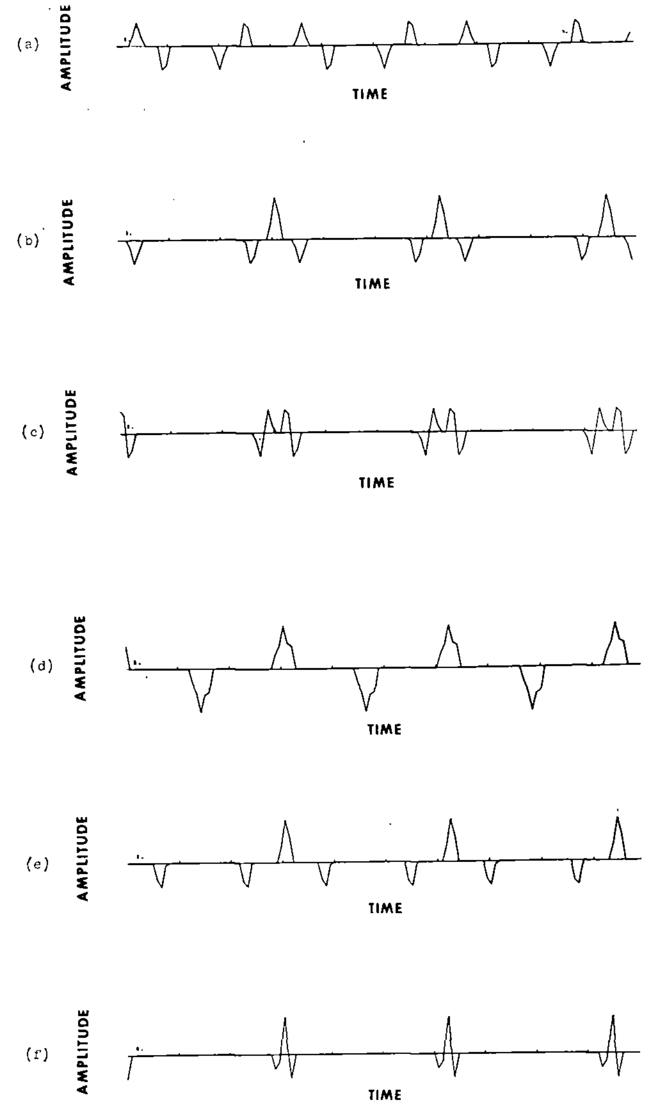

Fig. 2. Ideal struck string tuned to 500 Hz.
a. $x_p = 15.0$, $x_0 = 50.0$, b. $x_p = 15.0$, $x_0 = 15.0$, c. $x_p = 15.0$, $x_0 = 5.0$, d. $x_p = 50.0$, $x_0 = 50.0$, e. $x_p = 25.0$, $x_0 = 25.0$, f. $x_p = 5.0$, $x_0 = 5.0$.

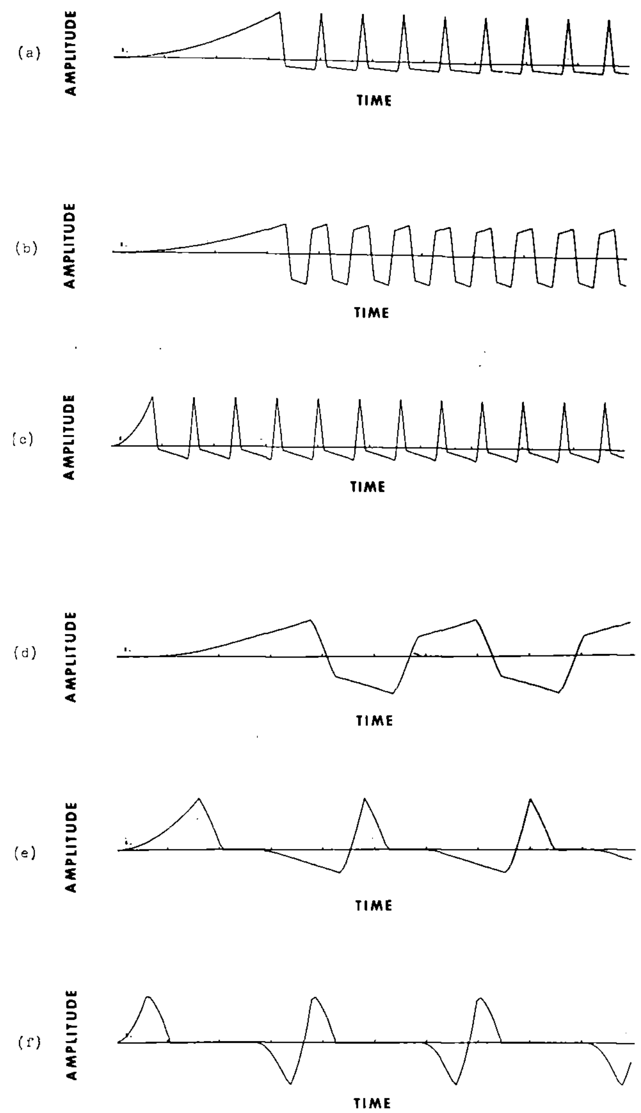

Fig. 3. Effect of transient 1 on a string tuned to 500 Hz.
a. $x_0 = 15.0$, $TRANS = 4.0$, b. $x_0 = 50.0$, $TRANS = 4.0$, c. $x_0 = 15.0$, $TRANS = 1.0$, d. $x_0 = 50.0$, $TRANS = 1.0$, e. $x_0 = 15.0$, $TRANS = 0.5$, f. $x_0 = 50.0$, $TRANS = 0.2$.

nine instructions that is the heart of the synthesis program. The rest of this subroutine is made up of sequences which take care of particular cases such as losses at the end supports and the fanciful stiff string.

A version of this program written in Fortran is given elsewhere [2]. It should be noted, however, that in order to increase the speed of the computations, this program has also been written in DAP-2, the assembly language of the DDP-224 computer, and has been recently written also in Compass, the CDC-6400 assembly language.

ANALYSIS OF THE RESULTS

For present purposes, we shall let x_p be the point of excitation and x_0 be the point at which the solution of the wave equation was computed for sound synthesis. We shall further assume that one end support is located at the origin and that the string is 100 units long.

A. Graphs from the SC-4020 Plotter

Fig. 1 shows six graphs of the behavior of an ideal plucked string tuned to 500 Hz. For this frequency, 22 string subdivisions are required. The graphs represent the first 125 points (6.25 ms at a sampling rate of 20 kHz) of the solution of the wave equation for the six cases listed. These graphs illustrate the following well-known facts about plucked strings.

Upon release, the slope discontinuity at x_p gives rise to two slope discontinuities traveling in opposite directions. These discontinuities change sign and reverse direction upon hitting an end support. As a result, a graph of any x_0 (except x_p) will show four slope discontinuities per period (Fig. 1a, c, and f). Since the two slope discontinuities must coincide once per period at x_p, a graph of x_p will show only three slope discontinuities per period (Fig. 1b and d). If the string is plucked at the center, the two discontinuities always pass through x_0 at the same time. In that case, the graph of x_0 shows only two slope discontinuities per period (Fig. 1e).

By examining separately the cases $x_0 > x_p$ and $x_0 < x_p$, we see that the first four slope discontinuities occur at $(x_p - x_0)/200f$, $(x_p + x_0)/200f$, $[200 - (x_0 + x_p)]/$

$200f$, and $[200+(x_p-x_0)]/200f$, respectively, where f is the frequency.

Fig. 2 shows six graphs of the behavior of an ideal struck string. The string is tuned to 500 Hz and is struck with a hammer of width W. Again, the graphs show the first 125 points of the solution of the wave equation for six particular cases.

When the hammer strikes the string, it produces two pulses traveling in opposite directions. These two pulses behave exactly as do the two slope discontinuities of the plucked string. Fig. 2a and c show four pulses per period, Fig. 2b, e, and f three pulses, and Fig. 2d two pulses. But in this case the two traveling pulses overlap partially when they go through the observed point.

Because of the finite number of string subdivisions (22 in these examples), the pulses are of trapezoidal shape except for $W = 0$, where the trapezoid reduces to a triangle. The width of the base of the trapezoid is equal to the hammer width plus two string subdivisions. The graphs in Fig. 2 show waveforms made of pulses each lasting only a small fraction of the vibration cycle. This fraction is equal to the hammer width divided by the string length. Although the graphs show periodic vibrations, the 2, 3, or 4 pulses which make up each cycle are not exactly alike due to the desampling discussed earlier. This error decreases as hammer width increases.

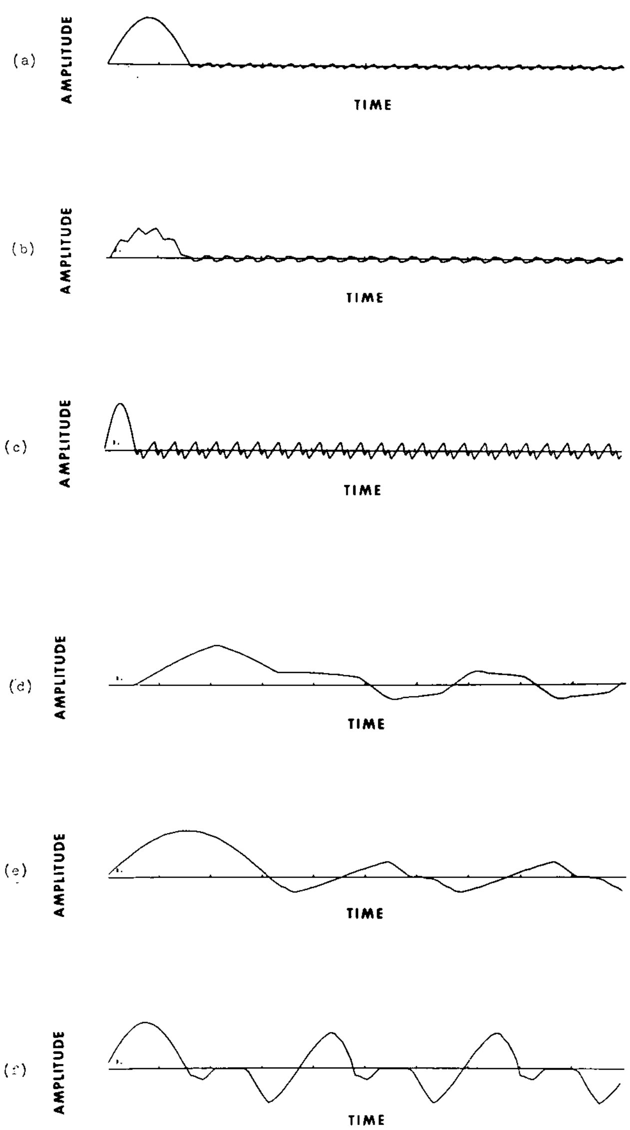

Fig. 4. Effect of transient 2 on a string tuned to 500 Hz. **a.** $x_0 = 15.0$, $TRANS = 4.0$, **b.** $x_0 = 50.0$, $TRANS = 4.0$, **c.** $x_0 = 15.0$, $TRANS = 1.5$, **d.** $x_0 = 50.0$, $TRANS = 1.5$, **e.** $x_0 = 15.0$, $TRANS = 1.0$, **f.** $x_0 = 15.0$, $TRANS = 0.5$.

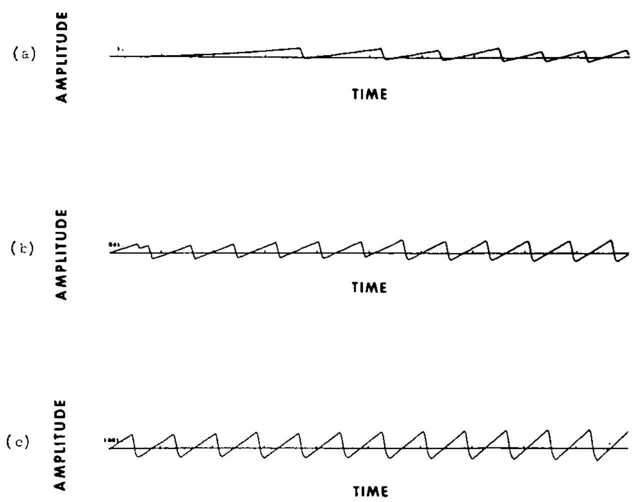

Fig. 5. Bowed string tuned to 500 Hz where $x_p = 10.0$, $x_0 = 10.0$, $PRESS = 0.4$, $TRANS = 50.0$, and $\tau_2 = 0.5$.

The effect of transient 1 is shown in the six graphs of Fig. 3. The string is again tuned to 500 Hz. The graphs of Fig. 3a, b, and c show 500 points per line, those of Fig. 3d, e, and f, 125 points per line. The amplitude of the transient (the maximum displacement of x_p) is 1.0, $x_p = 15.0$, and $TRANS$ is the duration of the transient in vibration cycles.

Each graph begins with a parabolic arc. This arc represents the time during which x_0 is forced to move with constant acceleration.

Transient 1 and the simple model for the plucked string give very similar results for $TRANS \geqq 1$. This can be seen by comparing Fig. 3a and c with Fig. 1b and by comparing Fig. 3b and d with Fig. 1a.

For small values of $TRANS$ (transients lasting less than one vibration cycle), transient 1 has the same effect as striking the string. The vibrations consist of pulses as shown in Fig. 3e and f.

When a stringed instrument is played by plucking the strings, the player's finger remains in contact with the string at least 0.1 second. This time is much longer than a vibration cycle for ordinary musical notes; consequently, the vibration patterns shown in Fig. 3e and f cannot be obtained by plucking the string manually.

The effect of transient 2 is illustrated by the six graphs shown in Fig. 4. The string is still tuned to 500 Hz. The graphs of Fig. 4a, b, and c show 1000 points per line, those of Fig. 4d, e, and f, 125 points per line. The amplitude of the transient is 1.0 and $x_p = 15.0$.

The sinusoidal arc which appears at the beginning of the graph for x_p (Fig. 4a, c, e, and f) represents the time during which the hammer is in contact with the string. For long strike times, very little energy is transferred to the string. This is illustrated by the first three graphs. The optimum strike time for maximum energy transfer for $x_p = 15.0$, is $TRANS = 0.85$. This strike time falls between the strike time used for Fig. 4e ($TRANS = 1.0$) and the strike time used in Fig. 4f ($TRANS = 0.5$).

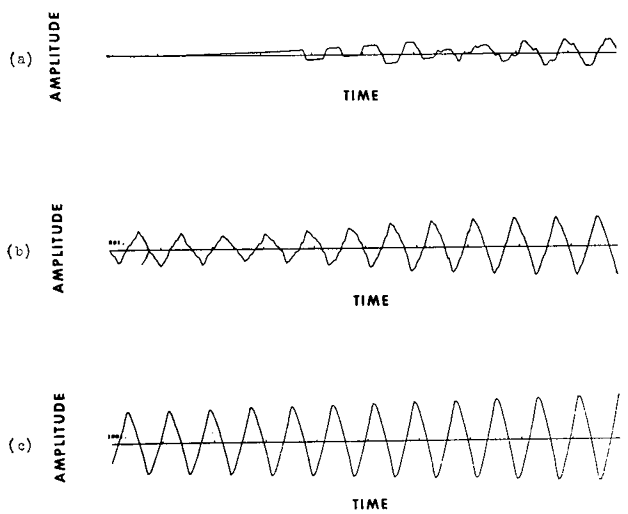

Fig. 6. Bowed string tuned to 500 Hz where $x_p = 10.0$, $x_0 = 50.0$, $PRESS = 0.4$, $TRANS = 50.0$, and $\tau_2 = 0.5$.

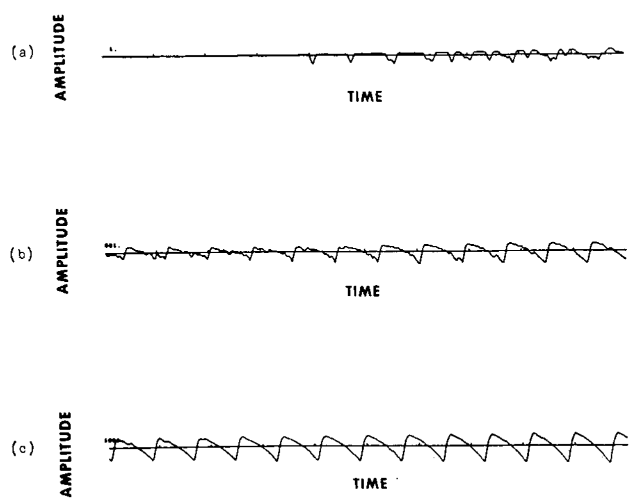

Fig. 7. Bowed string tuned to 500 Hz where $x_p = 10.0$, $x_0 = 95.0$, $PRESS = 0.4$, $TRANS = 50.0$, and $\tau_2 = 0.5$.

For short striking times, transient 2 and the simple model for the struck string give very similar results. This can be seen by comparing Fig. 4f with Fig. 2b.

The oscillation of a bowed string is illustrated in Figs. 5 through 10. Each group shows 500 points per line. The string is tuned to 500 Hz, and the maximum displacement of the bowed point (H) is set to 0.5. *PRESS* is the measure of the bow pressure defined by the equation

$$P = PRESS \times (h/\Delta x)n/m(m-n) \qquad (3)$$

TRANS is the number of vibration cycles during which the bow acquires its final velocity, and τ_2 is the time constant for damping due to sound radiation.

These graphs illustrate the stabilization process which takes place before the string reaches a steady mode of vibration. The sequence of events can be described as follows. a) x_p moves along with the bow until the curvature of the string at x_p exceeds the critical value *PRESS*. As in the case of the plucked string, two slope discontinuities traveling in opposite directions are produced upon release. One of them is reflected at the near end (the bridge in the case of a violin) and, upon coming back through x_p, causes x_p to reverse its direction of motion thus causing the hairs of the bow to grab the string again. b) When the other slope discontinuity comes back after reflection at the far end, x_p is still moving along with the bow. In most cases, this discontinuity is not strong enough to cause the release of the string and is reflected back toward the far end by the hairs of the bow. Figs. 5 and 8 illustrate the case where such a reflection takes place. Here the time difference between the first two successive releases is greater than the period of the stable vibrations. On the other hand, Fig. 9 illustrates the case where the discontinuity causes the release of the string. Here the time difference between the first two successive releases is exactly one period. In this example, vibrations of steady frequency are obtained immediately. c) The first few releases of the string might or might not coincide with the passing of a slope discontinuity through x_p. Eventually there will be slope discontinuities which will pass through x_p at "the right time" and cause the release of the string. The slope discontinuities which are not strong enough to cause release are reflected back and forth and are gradually damped out. d) The stable mode reached in Figs. 5 through 9 is essentially the mode described by Helmholtz

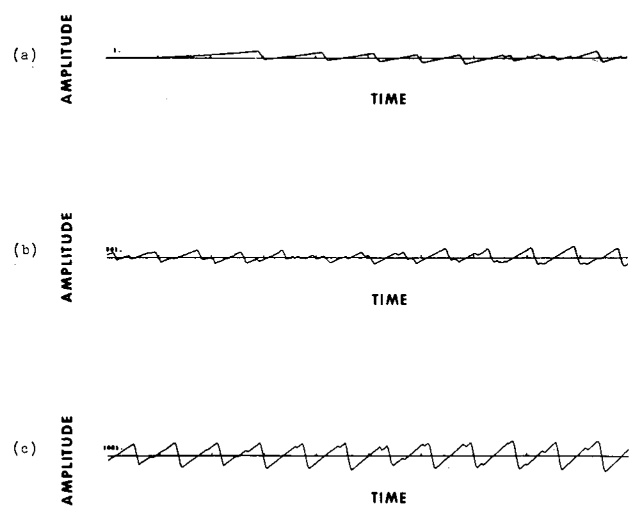

Fig. 8. Bowed string tuned to 500 Hz where $x_p = 15.0$, $x_0 = 15.0$, $PRESS = 0.3$, $TRANS = 50.0$, and $\tau_2 = 2.0$.

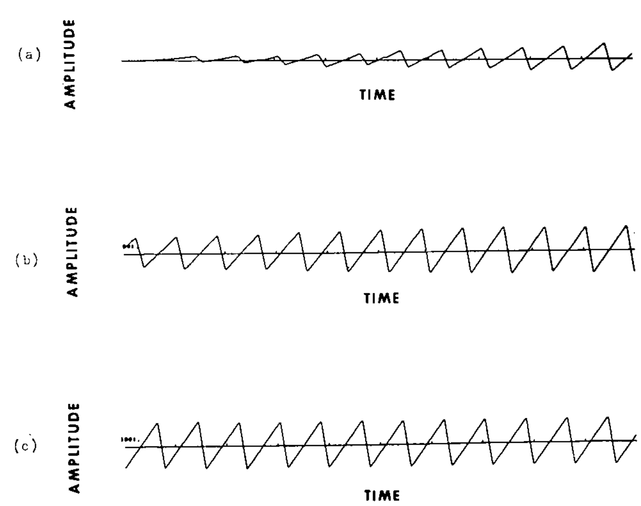

Fig. 9. Bowed string tuned to 500 Hz where $x_p = 20.0$, $x_0 = 20.0$, $PRESS = 0.2$, $TRANS = 20.0$, and $\tau_2 = 1.0$.

[3]. The vibration pattern of any point x_0 along the string is a sawtooth wave. The duration of the ascending part is $(100-x_0)/100f$. The string itself is always of triangular shape, the vertex of the triangle moving along two symmetrical parabolic arcs.

A comparison of the duration of the ascending part of the sawtooth wave measured from the graphs with the value predicted by Helmholtz is the following:

Fig. 5	$D1 = 0.812$	$D2 = 20/22 = 0.909$
Fig. 6	$D1 = 0.500$	$D2 = 11/22 = 0.500$
Fig. 7	$D1 = 0.288$	$D2 = 2/22 = 0.091$
Fig. 8	$D1 = 0.848$	$D2 = 19/22 = 0.864$
Fig. 9	$D1 = 0.781$	$D2 = 18/22 = 0.818.$

$D1$ is measured from the graphs and is given as a fraction of the total period. $D2$ is the value predicted by Helmholtz. In computing $D2$, x_0 was quantized to multiples of a string subdivision (in this instance to fractions if 22).

We see that the value predicted by Helmholtz is slightly higher than the measured value in all instances for which $x_0<50.0$. This discrepancy results from the damping which slows down the motion of x_p when the string is free. This effect of damping can be observed in Fig. 5, and corresponds to the curved shape of the descending portion of the cycle. The best match between $D1$ and $D2$ for points off center is shown in Fig. 8, where damping is smallest.

Another effect of damping which can be observed in all instances is the raising of the dc value of the vibration pattern for $x_0<50.0$ and lowering it for $x_0>50.0$.

The examples shown in Figs. 9 and 10 have identical parameters except for the value of *PRESS*. In Fig. 9, the value of *PRESS* is lower and the string settles into a vibration pattern with a frequency $= 2f$. Of the three kinds of damping (damping due to friction, sound radiation, and reflections at the end supports) only the second kind, which attenuates higher frequencies more severely, was successful in stabilizing the string in Helmholtz's simple mode of vibration. From this we conclude that the greater attenuation of higher frequencies is crucial during the stabilization period of the bowed string.

All three kinds of damping are illustrated for a

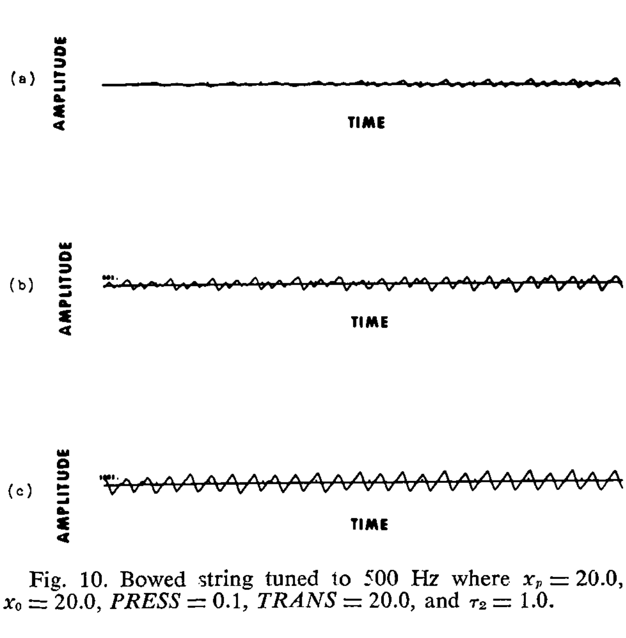

Fig. 10. Bowed string tuned to 500 Hz where $x_p = 20.0$, $x_0 = 20.0$, $PRESS = 0.1$, $TRANS = 20.0$, and $\tau_2 = 1.0$.

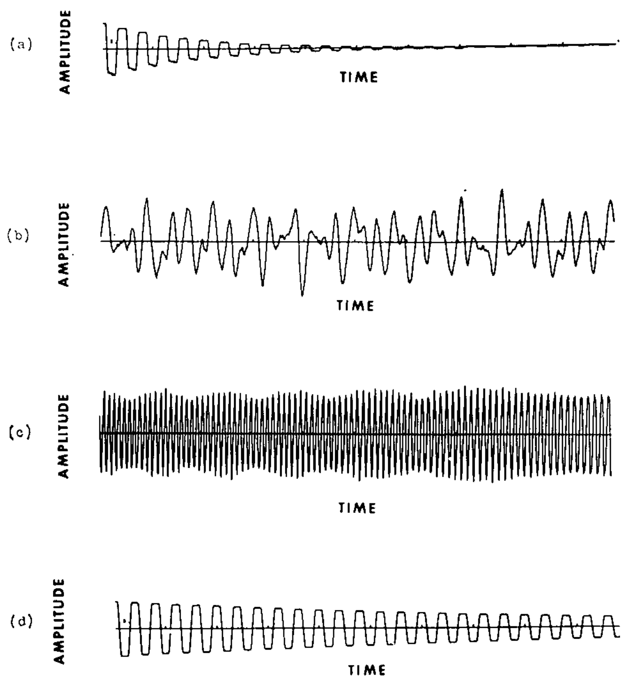

Fig. 11. Effect of damping upon plucked string. **a.** Effect of friction on a string tuned to 500 Hz where $x_p = 15.0$, $x_0 = 50.0$, and $\tau_1 = 0.05$, **b.** Effect of sound radiation on a string tuned to 750 Hz where $x_p = 15.0$, $x_0 = 50.0$, and $\tau_2 = 0.3$. **c.** Combined effect of damping due to sound radiation and of stiffness on a string tuned to 500 Hz where $x_p = x_0 = 50.0$, $\tau_2 = 5.0$, and $\epsilon_1 = 0.04$, **d.** Effect of end supports on a string tuned to 500 Hz where $x_p = 10.0$, $x_0 = 50.0$, and $a_1 = 0.08$.

plucked string in Fig. 11 where the graphs show 1000 points per line. The effect of friction is shown in Fig. 11a. The string is tuned to 500 Hz, $x_p = 15.0$, $x_0 = 50.0$, and $\tau_1 = 0.05$. This kind of damping produces a simple exponential decay which attenuates all frequencies at the same rate. At the end of 0.05 second (1000 points), the amplitude of the oscillations has decreased by 63.6%. This agrees very closely with the theoretical value of $1 - e^{-1} = 63.2\%$.

The effect of sound radiation is shown in Fig. 11b. The string is tuned to 750 Hz, $x_p = 15.0$, $x_0 = 50.0$, and $\tau = 0.3$. Fig. 11b shows that higher frequencies are damped faster. At the end of 0.05 second, the amplitude of the oscillations has decreased by 83.5%. Theoretically, the first harmonic should have decayed by $1 - e^{-(0.05/0.3)} = 15.6\%$, the second harmonic by $1 - e^{-4(0.05/0.3)} = 48.6\%$, the third harmonic by $1 - e^{-9(0.5/0.3)} = 77.7\%$, etc. A Fourier analysis remains to be done in order to obtain values for the rates of decay of the different harmonics.

Fig. 11c illustrates the effect of sound radiation combined with the effect of stiffness. The frequency due to tension alone is $f_t = 500$ Hz, $x_p = x_0 = 50.0$, $\tau_2 = 5.0$, and $\epsilon_1 = 0.04$. Stiffness increases the fundamental frequency (in this case to 660 Hz) and introduces inharmonic partials. The presence of inharmonic partials accounts for the progressive distortion of the waveform which can be observed in Fig. 11c. The amount of damping in this example is small.

The effect of the end supports is shown in Fig. 11d. The string is tuned to 500 Hz, $x_p = 10.0$, $x_0 = 50.0$, and $a_1 = 0.8$. This kind of damping is quite similar to the damping due to friction, producing a simple exponential decay and attenuating all the overtones at the same

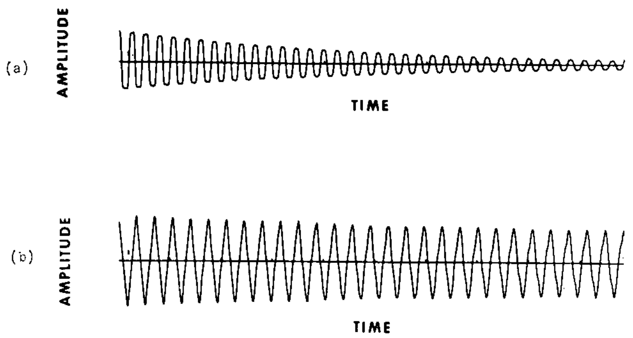

Fig. 12. Two fanciful strings. **a.** Tuned to 500 Hz and with negative stiffness. **b.** Tuned to 1500 Hz and with variable stiffness.

rate. The amplitude of the oscillations decreases by $1-\alpha_1$ (in this case by 0.2) every period. The time constant of this decay is given by $1/f \log_e(1/\alpha_1)$ and is therefore 0.0088 second.

Finally, a fanciful stiff string is illustrated by means of the two examples shown in Fig. 12 (1000 points per line). The first example is a string with negative stiffness ($\epsilon_1 = -0.07$) excited by Transient 2, $f_t = 500$ Hz, $x_p = x_0 = 15.0$, $TRANS = 0.5$, $AMPL = 1.0$ (amplitude of transient). Negative stiffness lowers the fundamental frequency and introduces inharmonic partials. The last example illustrates a fanciful string with variable stiffness. The string is plucked ($x_p = x_0 = 50.0$), $f_t = 1500$ Hz, and the stiffness-to-tension ratio varies linearly between $\epsilon_1 = +0.19$ and $\epsilon_2 = -0.19$ in 150 seconds. The main effect of this transition is to decrease continuously the fundamental frequency.

B. Synthesized Sounds Recorded on Audio Tape

A list of the parameters used to synthesize these sounds is given in Table I (pp. 550-551). These parameters correspond to a sampling rate of 20 kHz. Recordings were obtained by playing back the digital tape through the D/A converter at various sampling rates. In every case, the output of the D/A converter was run through a low-pass filter having a cutoff frequency equal to half the sampling frequency. The sounds fall into eight groups. The clock frequency (sampling rate) is given for each example. The perceived pitch depends upon the sampling rate as explained earlier.

Group I: Ideal Plucked String

The six examples contained in this group are steady tones with an "electronic" quality which does not resemble the sound of a stringed instrument. Obviously, the ideal string is a very poor approximation to a real string. However, we do notice that the tone quality depends on both the point of excitation. x_p, and the point used for sound synthesis x_0. The sound quality becomes richer as x_p or x_0 or both come closer to one end of the string. The purest tone corresponds to $x_p = x_0 = 50.0$. By looking at the graphs of Figs. 1, we confirm that richness of tone correlates with richness in higher overtones.

Group II. Ideal Struck String

Like the sounds in Group I, the sounds in this group fail to evoke the sound of a stringed instrument. They have a nasal quality which increases as x_p or x_0 or both come closer to one end of the string.

Group III: Plucked String with Damping Due to Friction

The examples in this group show that simple exponential damping contributes greatly to the simulation of string tones. They evoke the sound obtained by plucking a stopped string of a violin. The simulation was good when values of τ_1 (the time constant for damping due to friction) are smaller than 0.2 second. The examples having $\tau_1 = 0.01$ second sound like a violin string stopped in a high position—a very dry sound.

Group IV: Struck String with Damping Due to Friction

The examples which were recorded at 20 and 10 kHz have a harpsichord-like quality for $x_p = 25.0$, 15.0, 15.0, and 10.0. The corresponding examples recorded at 5 kHz begin to evoke piano tones. The richness and "warmth" of real piano tones, however, is not yet here. The nasal quality of the examples corresponding to $x_p = 5.0$ is rather interesting and pleasing. The examples corresponding to $x_p = 33.3$ and 50.0 lack richness.

Group V: Transient 1

These sounds also have simple exponential damping. Their quality is very similar to those of the preceding group since short transients, transient 1, and striking the string produce similar effects. However, they sound less nasal. Those recorded at 20 kHz have a guitar-like quality, especially because of the attack. Those recorded at 10 kHz sound in between a harpsichord and a piano.

Group VI: Transient 2

These sounds illustrate both the effect of stiffness and the effect of damping due to sound radiation. This kind of damping attenuates higher frequencies more rapidly and gives more warmth to these guitar-like sounds. The decay is a bit too fast for the sounds recorded at 20 kHz but seems about right for those recorded at 10 kHz. Stiffness raises the fundamental frequency and introduces inharmonicity. This inharmonicity causes slow fluctations in amplitude which can be perceived in these examples as a slight vibrato. This vibrato is most obvious in the examples recorded at 20 kHz. It is totally absent in the one example without stiffness.

Group VII: Bowed String

Eight examples were synthesized for sampling rates of both 10 and 20 kHz. The first five examples have a distinct bowed string quality. The somewhat scratchy quality of the attack of bowed string sounds is rendered quite well by example 1 and even better by example 5. These two examples are identical, except for the choice of x_0. In the case of Example 1, x_0 is the bowed point itself; in the case of example 5, it is a point closer to the end which corresponds to the bridge for a violin string. Examples 6 and 7 are identical to example 1, except for the choice of x_0 (50.0 and 70.0, respectively); however, they do not sound as good. Their tone quality is much too pure for bowed string sounds. This seems to indicate that a point close to the bridge is the best choice for x_0 for simulating the sound of a bowed string. Example 8 was recorded to illustrate the sudden change from one mode of vibration to another. The first mode of vibration is illustrated in Fig. 10. The second mode is Helmholtz's simple mode. In general, the bowed string sounds have a somewhat harsh quality which can be diminished by low-pass filtering.

Group VIII: Fanciful Stiff Strings

This last group of examples gives an idea of what would happen if the physical and geometrical characteristics of strings are allowed to change in unusual ways.

The first example is a string with negative stiffness excited by transient 2 with an amplitude *AMPL*. The result is a chime-like sound.

The second example is a plucked string the stiffness of which changes continuously from a negative value to zero during the 0.1-second attack and then remains constant. The resulting sound belongs to the realm of "computer sounds" and does not evoke any concrete object.

The third example is a struck string the stiffness of which changes rapidly from a negative value to a positive value: the recording at 20 kHz sounds like a whip.

The fourth example is like the third backward (except for the damping).

The fifth example is like the third, except that the frequency is higher (1500 Hz) and the change in stiffness takes place over a long period of time (10 seconds at a sampling rate of 20 kHz). The result is a complex tone slowly gliding down in pitch accompanied by a second tone. This tone glides up and down several times. It results from foldover of the overtones above half the frequency of the sampling rate [4].

The sounds described are recorded on the sound sheet bound in at the end of the Journal. Its contents are given in Table I.

CONCLUDING REMARKS

The synthesis program written in assembly language runs at about $32\,000/f$ times real time for a string without stiffness and without frequency-dependent losses. Stiffness increases the computation time to about $45\,000/f$. For $f = 500$ Hz, this is 64 and 90 times real time, respectively. Frequency-dependent damping (damping due to sound radiation) doubles the computation time to $64\,000/f$ and $90\,000/f$ times real time, respectively. This is rather slow for any application requiring a large production of sound. Savings in computation time could be achieved by accounting for losses only at the end supports of the string. The simple procedure which exists now to account for losses at the end supports would have to be extended to frequency-dependent damping.

The procedure used to iterate the wave equation yields correct results in the case of the ideal string and for strings with simple exponential damping (damping due to friction and damping due to reflections at the end supports). However, the accuracy of the computations for stiff strings is limited to small stiffness-to-tension ratios. Frequency measurements were performed on sounds generated for various values of ϵ. The measured frequency was always higher than the predicted value. For $\epsilon < 0.03$, the difference is less than 6% at 500 Hz. It is higher for lower frequencies and lower for higher frequencies. In order to find out how critical this limitation is for the simulation of real strings, measurements would have to be made in order to obtain stiffness-to-tension ratios for various types of strings under realistic tensions. Also, the extension of this investigation of strings with considerable stiffness and to bars would require the transformation of the difference equation for the stiff string into a form which can be iterated for all values of ϵ.

In spite of these limitations, the synthesis program was successful in reproducing the main features of the sounds produced by most string instruments. From this we conclude that the behavior of the string itself (or strings) is the prime determining factor of the perceivable characteristics of the sounds produced by string instruments.

As far as sound production is concerned, damping seems to be, paradoxically enough, the prime factor. It is so basic that without it, no resemblance to a string tone can be obtained. The investigation of the bowed string showed that the type of damping is particularly crucial for that mode of excitation.

The way in which the string is excited comes next in importance and the physical and geometrical characteristics of the string come last (they contribute to the sound quality only when they deviate significantly from those of an ideal string as in the case of a stiff string).

As far as the behavior of strings is concerned, the first extension of this investigation would require more information about damping. This information can only be obtained by a study of the resonant box and of the bridge (or bridges) which couple the string to the resonant box. Such a study should reveal how the damping of the string at different frequencies is a function of the physical and geometrical characteristics of the bridge (or bridges) and of the resonant box.

The second major step in extending this investigation is closely related to the preceding one: a simulation of the resonant box itself. Two approaches seem possible. One consists in setting up differential equations for the box and solving them for the boundary conditions imposed by the motion of the string. This approach seems overwhelming at present in terms of complexity and computing time. The other approach, which seems more feasible, consists first of designing a digital filter with a transfer function approximating the transfer function of the resonant box and second, running the signal corresponding to a point of the string close to the bridge through this filter. There would be a correspondence between the control parameters of the filter and the geometrical and physical characteristics of the resonant box. Fanciful instruments could be stimulated by varying these control parameters in time.

Transients 1 and 2 are only two examples of many conceivable functions for exciting the string. Any other excitation function can be implemented by the synthesis program in exactly the same way. This opens up possibilities for the design of new stringed instruments. If certain excitation functions are found to produce desirable sounds, mechanical devices can be designed to excite the string in the manner described by these functions.

As far as bowing is concerned, this investigation is only the beginning. The effect of bow positions, pressures, and speeds on the bowing process should be studied further. The torsional effect of the bow on the string should also be taken into account. The results of such a study could then be checked against the theoretical work reviewed in Part I.

At present we are extending this work at the State University of New York in Buffalo to the study of the vibrations of rigid bars and other objects that resemble percussion instruments. We believe that this should prove to be a particularly interesting field, since the behavior of percussion instruments has been less well defined acoustically than any of the other major groups of instruments.

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REFERENCES

[1] L. Hiller and P. Ruiz, "Synthesizing Musical Sounds by Solving the Wave Equation for Vibrating Objects: Part I," *J. Audio Eng. Soc.*, vol. 19, pp. 462-470 (June 1971).

[2] P. M. Ruiz, " Technique for Simulating the Vibrations of Strings with a Digital Computer," *M. Music* dissertation, University of Illinois, Urbana, pp. 123-130 (1970).

[3] H. L. F. Helmholtz, *On the Sensations of Tone*, 2nd English Ed., A. Ellis, Ed., based on 4th German Ed., 1877 (Dover Publ., New York, 1954) p. 387.

[4] M. V. Mathews (with the collaboration of J. E. Miller, F. R. Moore, J. R. Pierce, and J. C. Risset), *The Technology of Computer Music* (M.I.T. Press, Cambridge, Mass., 1969), pp. 8-11.

Table I. List of sounds contained on sound sheet.

		x_p	x_0	
I	Ideal plucked string	15.0	50.0	
	$f = 500.0$	15.0	15.0	
	time = 3.0	25.0	50.0	
	clock = 15 kHz	25.0	25.0	
		50.0	50.0	
		50.0	5.0	
		x_p	x_0	Width
II	Ideal struck string	15.0	50.0	7.0
	$f = 500.0$	15.0	15.0	7.0
	time = 3.0	15.0	5.0	7.0
	clock = 15 kHz	50.0	50.0	10.0
		25.0	50.0	5.0
		5.0	50.0	0.0
		τ_1	Time	
III	Plucked string with damping due to friction	0.5	2.5	
		0.2	1.5	
	$f = 500.0$	0.1	1.0	
	$x_p = x_0 = 10.0$	0.05	0.7	
	a) clock = 20 kHz	0.01	0.3	
	b) clock = 10 kHz	All 5 examples		
	c) clock = 5 kHz	Last 3 examples only		
		x_p	Width	
IV	Struck string with damping due to friction	50.0	7.0	
		33.3	7.0	
	$f = 500.0$	25.0	7.0	
	$x_p = x_0$	15.0	7.0	
	time = 2.0	10.0	7.0	
	$\tau_1 = 0.5$	5.0	3.0	
	a) clock = 20 kHz			
	b) clock = 10 kHz	All 6 examples		
	c) clock = 5 kHz	Last 3 examples only		

Table I (continued)

		TRANS				
V	Transient 1	0.5				
	$f = 500.0$	0.2				
	$x_p = x_0 = 15.0$					
	$\tau_1 = 0.7$					
	time = 1.5					
	AMPL = 1.0					
	a) clock = 20 kHz					
	b) clock = 10 kHz	Both examples				
		x_p	ϵ_1	*TRANS*		
VI	Transient 2	10.0	0.02	0.1		
	$f = 500.0$	10.0	0.02	0.2		
	$x_p = x_0$	10.0	0.02	0.3		
	$\tau_2 = 5.0$	10.0	0.02	0.4		
	time = 10.0	15.0	0.03	0.5		
	AMPL = 1.0	15.0	0.03	0.75		
	a) clock = 20 kHz	15.0	0.0	0.75		
	b) clock = 10 kHz	All 7 examples				
		x_p	x_0	τ_2	*PRESS*	*TRANS*
VII	Bowed string	10.0	10.0	0.5	0.4	50.0
	$f = 500.0$	10.0	10.0	0.5	0.5	20.0
	time = 2.0	15.0	15.0	2.0	0.3	20.0
	$H = 0.5$	20.0	20.0	1.0	0.2	20.0
	a) clock = 10 kHz	10.0	5.0	0.5	0.4	50.0
		10.0	50.0	0.5	0.4	50.0
		10.0	75.0	0.5	0.4	50.0
		20.0	20.0	1.0	0.1	20.0
	b) clock = 20 kHz	All 8 examples				
				Fanciful stiff strings		
VIII	1) Transient 2	f	ϵ_1	τ_2	*AMPL*	*TRANS*
	$x_0 = x_p = 15.0$	500.0	—0.07	30.0	1.0	0.5
	time = 3.0					
	a) clock = 10 kHz					
	b) clock = 20 kHz					
	2) Plucked string	f	ϵ_1	ϵ_2	*TRANS*	T_1
	$x_0 = x_p = 50.0$	500.0	—0.07	0.0	50.0 or 0.1 second	0.8
	time = 1.0					
	a) clock = 10 kHz					
	b) clock = 20 kHz					
	3) Struck string	f	ϵ_1	ϵ_2	*TRANS*	T_1
	$x_0 = x_p = 5.0$	500.0	0.07	—0.07	200.0 or 0.4 second	0.1
	$W = 3.0$					
	time = 0.4					
	a) clock = 5 kHz					
	b) clock = 10 kHz					
	c) clock = 20 kHz					
	4) Struck string	f	ϵ_1	ϵ_2	*TRANS*	T_1
	$x_0 = x_p = 5.0$	500.0	—0.07	0.07	200.0 or 0.4 second	0.1
	$W = 3.0$					
	time = 0.4					
	a) clock = 5 kHz					
	b) clock = 10 kHz					
	c) clock = 20 kHz					
	5) Struck string	f	ϵ_1	ϵ_2	*TRANS*	
	$x_0 = x_p = 50.0$	1500.0	0.19	—0.19	15000.0 or 10.0 seconds	
	$W = 3.0$					
	time = 10.0					
	a) clock = 20 kHz					
	b) clock = 5 kHz	No damping				