HA1 Report

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1 Problem 1: Motion model

1.1 Is $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ a Markov chain?

No. The definition of a Markov chain requires that the probability distribution of the next state of kinematics-vector X_{n+1} depends only on the current state X_n.

In the given model, the next state X_{n+1} depends not only on the current state X_n, but also on the current driving command Z_n, where $\{Z_n\}_{n\in\mathcal{N}}$ is an independent Markov chain. Thus knowing X_n alone is insufficient to determine the distribution of X_{n+1}, which means $\{X_n\}_{n\in\mathcal{N}}$ does not satisfy the Markov property.

The noise W_{n+1} is mutually independent completely stochastic component determined at time-step $n+1$. This means we can't speak of any prior knowledge about the noise and it can be essentailly ignored for the Markov chain analysis.

1.2 Is $\{\tilde{\mathbf{X}}_n\}_{n\in\mathbb{N}}$ a Markov chain?

Yes. In this case, in order to satify the Markov property, the distribution of the next state $\tilde{X}_{n+1}$ must depend only on the current state, now defined as $\tilde{X}_n = (X_n^T, Z_n^T)$.

Again, according to the given model, to determine the next state we must know X_n, Z_n and mutually independent W_{n+1}. Thus knowing X_n and Z_n is technically enough to determine the next state's probability distribution.

Since now knowledge of both X_n and Z_n is coupled within $\tilde{X}_n$, knowing $\tilde{X}_n$ is indeed enough to determine the probability distribution of $\tilde{X}_{n+1}$.

1.3 Trajectory simulation.

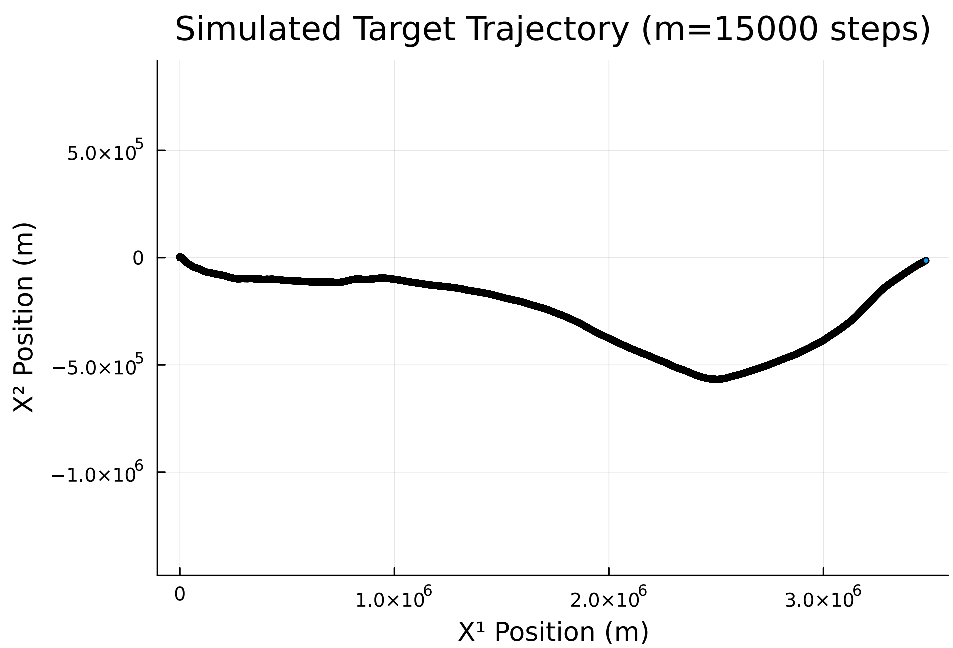

Figure 1: Simulation of a trajectory $\{(X_n^1, X_n^2)\}$ of arbitrary length $m = 15000$.

The figure above depicts a reasonable trajectory for a moving target, that is:

- path is continuous and relatively smooth,
- the direction changes seem consistent with the simulated driving commands,
- the transition matrix P makes the driving command likely to persist for several steps (0.8 probability), which explains consistently straight segments of the trajectory between major turns,
- path is not perfectly straight due to the introduced noise.

2 Problem 2: Observation model

2.1 *Convincing myself* that $\{(\tilde{\mathbf{X}}_n, \mathbf{Y}_n)\}_{n\in\mathbb{N}}$ forms a hidden Markov model.

Setup:

- Hidden State: $\tilde{\mathbf{X}}_n = (\mathbf{X}_n^T, Z_n^T)^T$. Let $\mathbf{pos}_n = (X_n^1, X_n^2)^T$ be the position component of $\mathbf{X}_n$.
- Observation: $\mathbf{Y}_n = (Y_n^1, \ldots, Y_n^s)^T$ $(s = 6)$.
- Observation Model: $Y_n^\ell = \nu - 10\eta \log_{10}(\|\mathbf{pos}_n - \boldsymbol{\pi}_\ell\|) + V_n^\ell$, with $V_n^\ell \sim \mathcal{N}(0, \zeta^2)$ i.i.d. noise $(\nu = 90, \eta = 3, \zeta = 1.5)$, independent of $\{\tilde{\mathbf{X}}_k\}$.

Checks:

1. Verifying whether our hidden state is a Markov Chain:
   - The hidden state process is $\{\tilde{\mathbf{X}}_n\}_{n\in\mathbb{N}}$.
   - From Problem 1, we established that this process satisfies the Markov property: the distribution of $\tilde{\mathbf{X}}_n$ depends only on $\tilde{\mathbf{X}}_{n-1}$.
   - This fulfils the first requirement for a hidden Markov model
2. Verifying conditional independence of our observations:

- The observation Y_n^ℓ depends on the current position $\mathbf{pos}_n$ (determined by $\tilde{\mathbf{X}}_n$) and the current independent noise V_n^ℓ.
- Since the noise vector $\mathbf{V}_n = (V_n^1, \dots, V_n^s)^T$ is independent of the state process and its components are independent, the observation $\mathbf{Y}_n$ is conditionally independent of past states and observations given the current state $\tilde{\mathbf{X}}_n$.
- This fulfills the second requirement, i.e.: $p(\mathbf{y}_n|\tilde{\mathbf{x}}_0, \dots, \tilde{\mathbf{x}}_n, \mathbf{y}_0, \dots, \mathbf{y}_{n-1}) = p(\mathbf{y}_n|\tilde{\mathbf{x}}_n)$.
- Therefore, $\{(\tilde{\mathbf{X}}_n, \mathbf{Y}_n)\}$ forms a hidden Markov model.

2.2 Transition density $p(\mathbf{y}_n \mid \tilde{\mathbf{x}}_n)$.

- The searched transition density is the conditional probability density of $\mathbf{Y}_n$ given $\tilde{\mathbf{X}}_n = \tilde{\mathbf{x}}_n$.
- Let $\mathbf{pos}_n = (x_n^1, x_n^4)^T$ be the position extracted from $\tilde{\mathbf{x}}_n$.
- Let's define the conditional mean

$$\mu^\ell(\tilde{\mathbf{x}}_n) = \nu - 10\eta \log_{10}(\|\mathbf{pos}_n - \boldsymbol{\pi}_\ell\|)$$

- The conditional distribution is

$$Y_n^\ell|\tilde{\mathbf{x}}_n \sim \mathcal{N}(\mu^\ell(\tilde{\mathbf{x}}_n), \zeta^2)$$

- Due to noise independence, the components of $\mathbf{Y}_n$ are conditionally independent given $\tilde{\mathbf{x}}_n$.
- The joint density is the product of individual Gaussian PDFs:

$$p(\mathbf{y}_n \mid \tilde{\mathbf{x}}_n) = \prod_{\ell=1}^{s} p(y_n^\ell|\tilde{\mathbf{x}}_n) = \prod_{\ell=1}^{s} \left[\frac{1}{\sqrt{2\pi}\zeta} \exp\left(-\frac{(y_n^\ell - \mu^\ell(\tilde{\mathbf{x}}_n))^2}{2\zeta^2} \right) \right]$$

- Combining terms gives the multivariate Gaussian density:

$$p(\mathbf{y}_n \mid \tilde{\mathbf{x}}_n) = \frac{1}{(2\pi)^{s/2}\zeta^s} \exp\left(-\frac{1}{2\zeta^2} \sum_{\ell=1}^{s} (y_n^\ell - \mu^\ell(\tilde{\mathbf{x}}_n))^2 \right)$$

- Substituting known constants ($s = 6$, $\zeta = 1.5$, $\nu = 90$, $\eta = 3$) we obtain the resulting density:

$$p(\mathbf{y}_n \mid \tilde{\mathbf{x}}_n) = \frac{1}{(2\pi)^3(1.5)^6} \exp\left(-\frac{1}{2(1.5)^2} \sum_{\ell=1}^{6} \left(y_n^\ell - \left(90 - 30 \log_{10}(\|(x_n^1, x_n^4)^T - \boldsymbol{\pi}_\ell\|) \right) \right)^2 \right)$$

3 Problem 3: Mobility tracking using SMC methods (SIS)

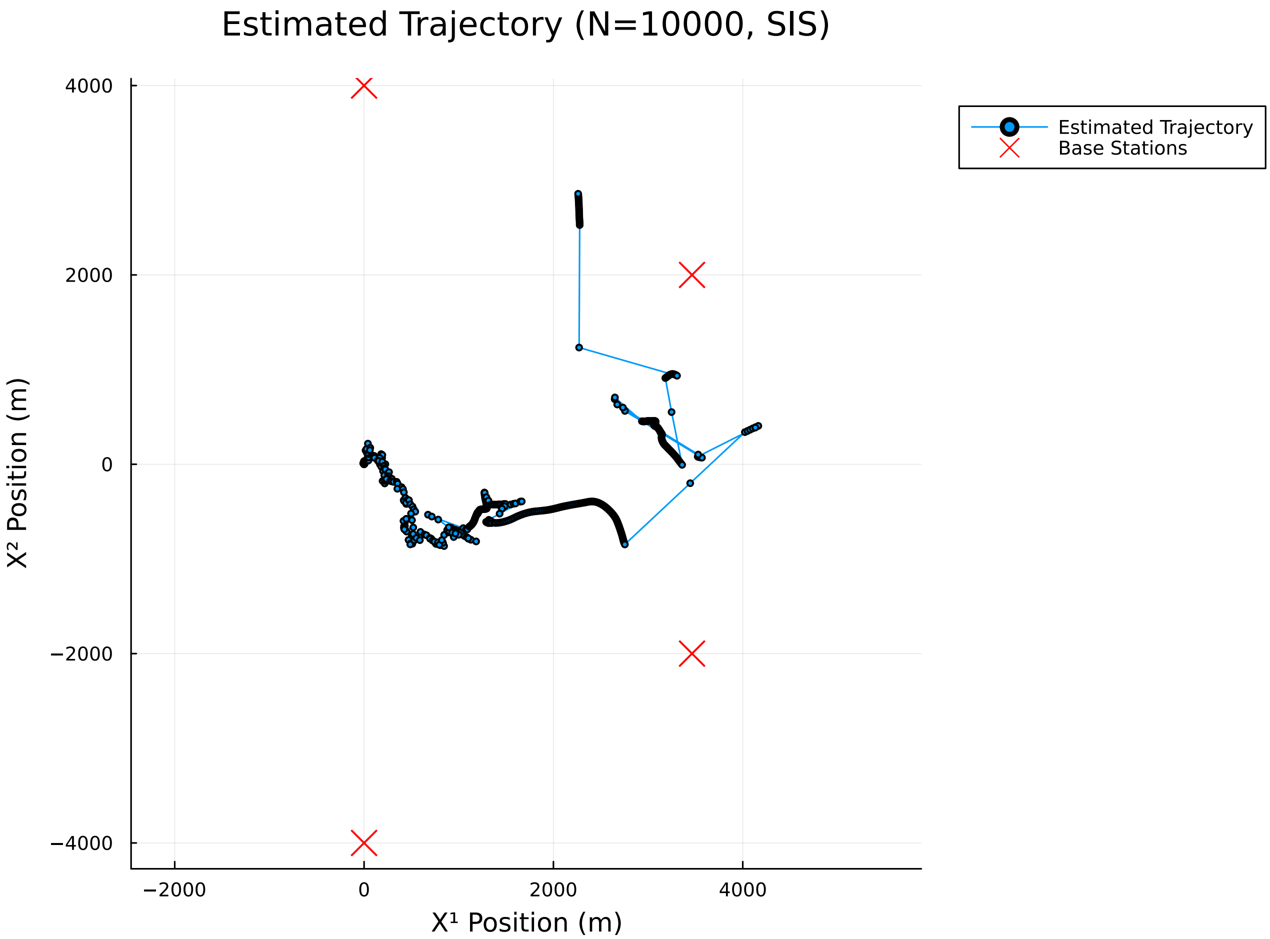

Figure 2: Estimates of $\{(\tau_n^1, \tau_n^2)\}_{n=0}^{m=500}$ and the locations of the basis stations (two fell outside of ROI) for SIS.

The figure above shows a path that, following the logic from previous task, doesn't look reasonable. The subsequent coordinates shouldn't just jump so far away from each other as they do here, what we can see from the markers placement.

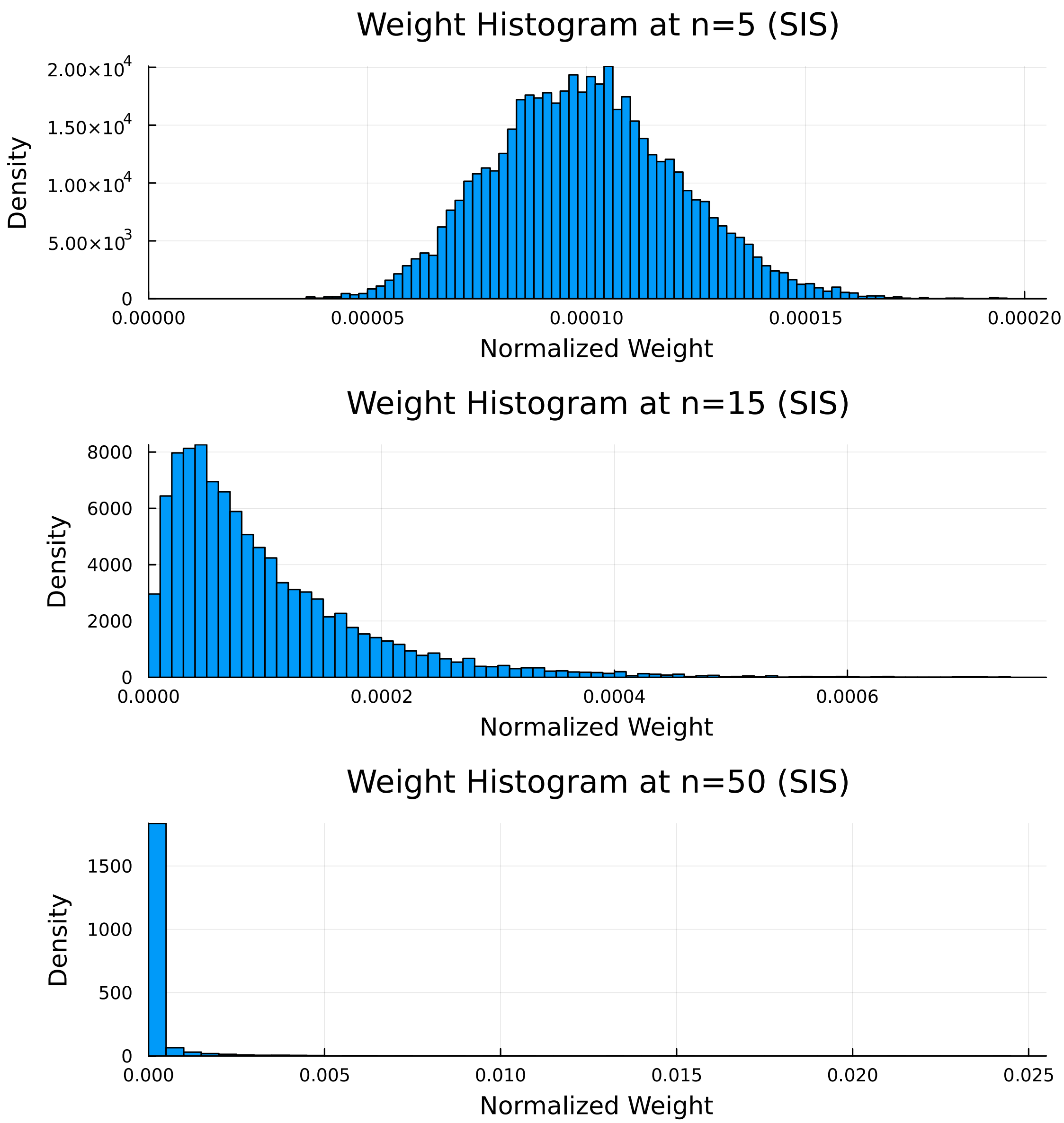

Figure 3: Histograms of the importance weights for SIS.

The histograms illustrate the concept of weight degeneracy, which is expected for SIS without resampling. We obtain efficient sample sizes around $n = 5$, which is very low. With growing n, a stronger and stronger degeneracy is observed. At $n = 50$ only around 230/10000 particles remain any useful, and it quickly goes to 1 with further growing n (up to 500). This strongly motivates the need for resampling.

4 Problem 4: Mobility tracking using SMC methods (SISR)

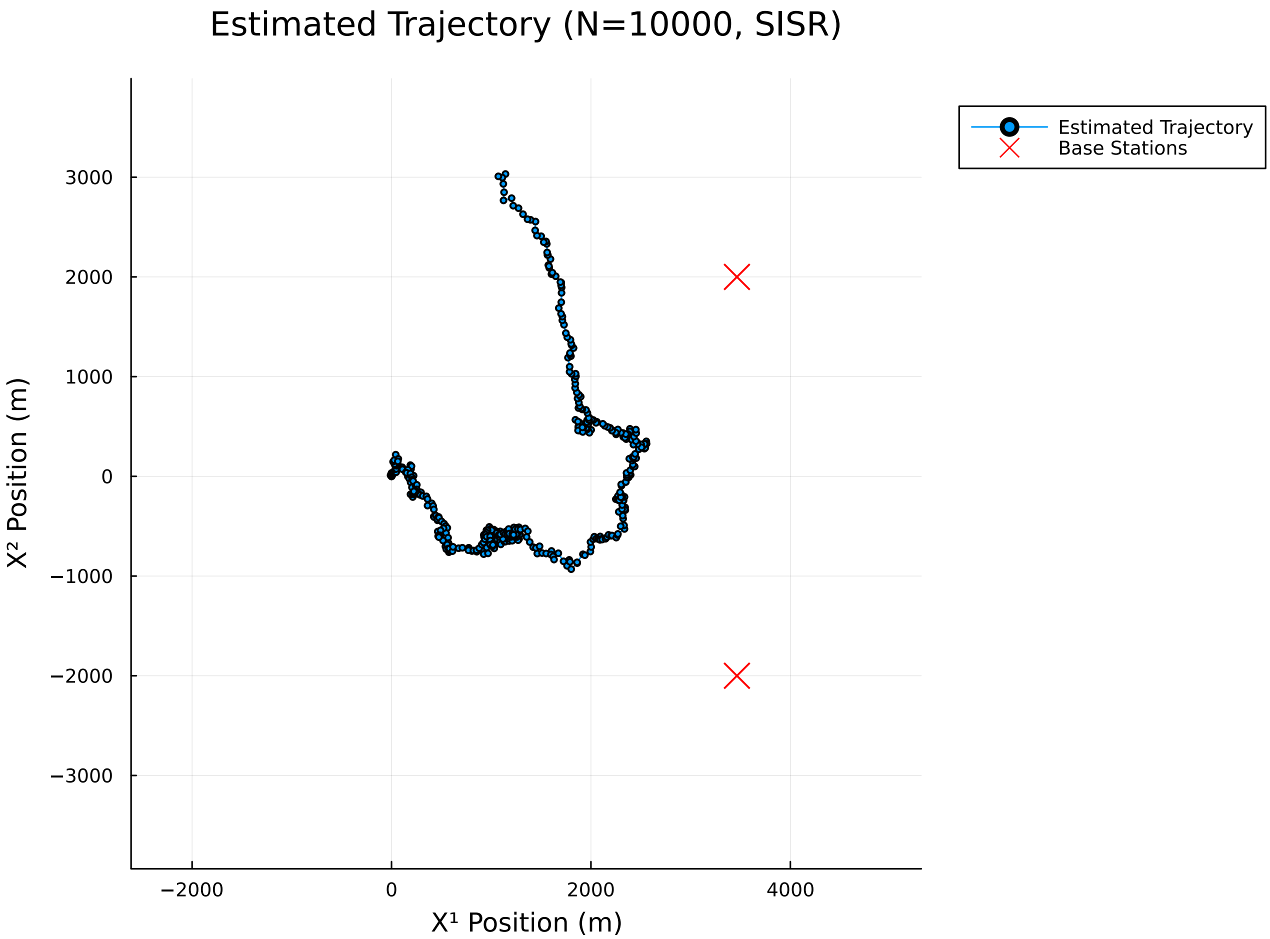

Figure 4: Estimates of $\{(\tau_n^1, \tau_n^2)\}_{n=0}^{m=500}$ and the locations of the local basis stations for SISR.

Compared to the path obtained in the previous problem, this looks way better. Now each subsequent marker is placed closely to the previous one.

If we were to plot the histograms this time as well, we would see that even for large n, all the way up to $n = 500$ there are no signs of weight degeneracy and the distributions resemble normal, much like the first histogram from Figure 3.

5 Problem 5: SMC-based model calibration

We approximate the intractable likelihood function using a particle filter (SISR) and then perform a grid search over possible values of ζ to find the one that maximizes this approximate likelihood, yielding the approximate MLE $\hat{\zeta}_m \approx 2.2$. Then we plot the positions expected under this MLE below. Again, the obtained trajectory looks reasonable.

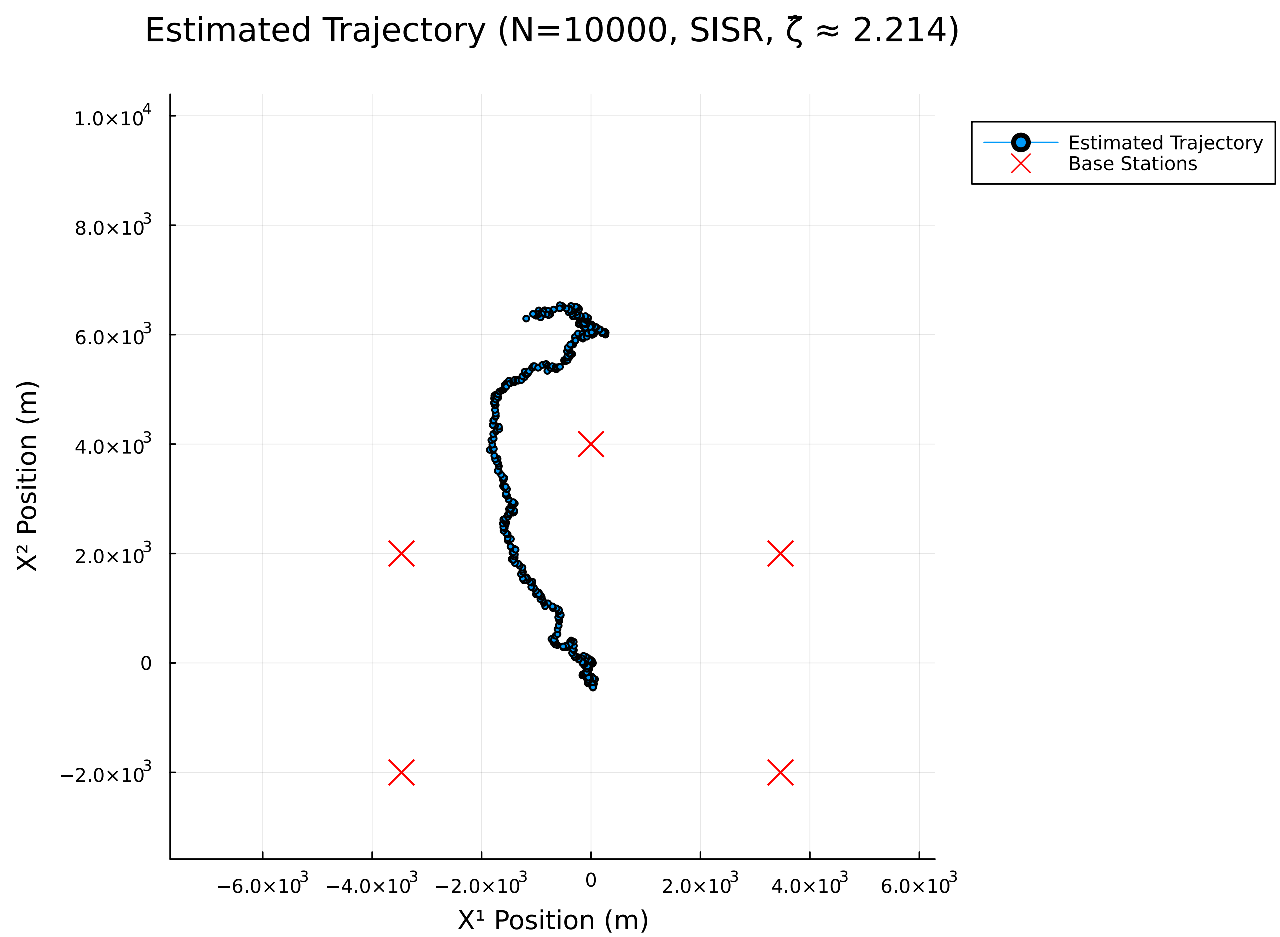

Figure 5: Estimates of $\{(\tau_n^1, \tau_n^2)\}_{n=0}^{m=500}$ with $\hat{\zeta}_m = 2.5$ and the locations of the local basis stations.