MFE Bootcamp

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1 Linear Algebra I

Motivation: As quantitative economists, we often rely on linear algebra as a tool for solving simple and complex problems. For example, linear algebra allows us to find solutions to linear systems:

$$\begin{aligned} y_1 &= \phi_1 x_{11} + \phi_2 x_{12} + \ldots + \phi_k x_{1k}, \\ &\vdots \\ y_n &= \phi_1 x_{n1} + \phi_2 x_{n2} + \ldots + \phi_k x_{nk}, \end{aligned}$$

where we are interested in the unknowns ϕ_i given series of observables x_i and y_i. An example of a system of interest is the famous Capital Asset Pricing Model:

$$r_{i,t} - r^f = \alpha_i + \beta_i (r_{m,t} - r^f),$$

where $r_{i,t}$ is the return of asset i on time t, $r_{m,t}$ is the market return on time t, and r^f is the risk-free rate of interest. In this model we try to explain the excess return of all assets by each assets sensitivity to the market (more on this later). Important things to consider when we work with these problems:

- How many solutions (if any) exists?
- If there are no closed-form solution to this system, can we still approximate it?
- How do we compute the solution?

Linear algebra can also simplify notation. Consider the simple case of computing the variance of a portfolio of two assets: We know that the total variance of two variables, in this case stock 1 and stock 2, s_1 and s_2 can be computed as

$$\text{Var}(s_1 + s_2) = Var(s_1) + Var(s_2) + 2cov(s_1, s_2).$$

Now, let ω_1 and ω_2 be the portfolio weights of stock 1 and stock 2. The portfolio variance is now:

$$\text{Var}(\omega_1 s_1 + \omega_2 s_2) = \omega_1^2 Var(s_1) + \omega_2^2 Var(s_2) + 2\omega_1\omega_2 cov(s_1, s_2).$$

For an n-asset portfolio this becomes

$$\text{Var}(\omega_1 s_1 + ... + \omega_n s_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i \omega_j Cov(s_i, s_j).$$

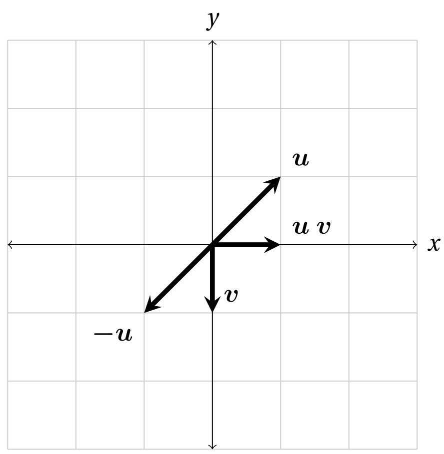
By using matrix notation we get

$$\text{Var}(\omega_1 s_1 + ... + \omega_n s_n) = \boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega},$$

where $\boldsymbol{\Sigma}$ is and $(n \times n)$ covariance matrix and $\boldsymbol{\omega}$ is a vector of weights.

Vectors

A vector in R^n is an n-tuple of numbers and represent a direction and magnitude.

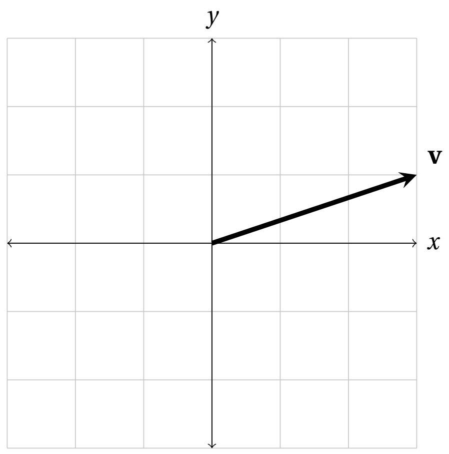


We can express the same vector in two ways:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{x}^t = [x_1, x_2, ..., x_3].$$

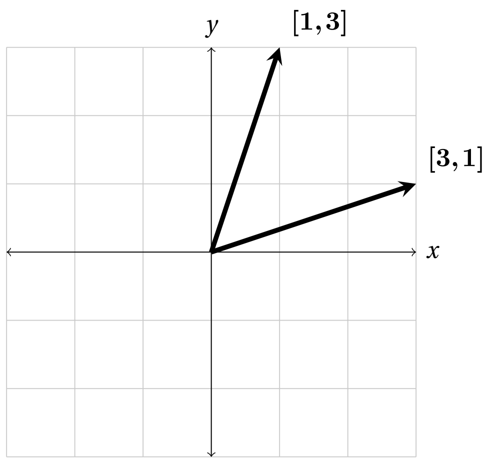
Example: In R^2 :

$$\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



The ordering matters:

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



What can we do with vectors?

An n-vector, $\mathbf{u}$, can be multiplied by a number c. This is called scalar multiplication and yields a new vector n-vector, $c\mathbf{u}$.

Example: (n=3) Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $c = 3$. Then,

$$3\mathbf{u} = 3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \times 2 \\ 3 \times (-1) \\ 3 \times 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}.$$

We can also add and subtract to vectors $\mathbf{u}$, $\mathbf{v}$:

$$\mathbf{u}' + \mathbf{v}' = (u_1 + v_1, u_2 + v_2, +...+, u_n + v_n),$$
$$\mathbf{u}' - \mathbf{v}' = (u_1 - v_1, u_2 - v_2, -...-, u_n - v_n).$$

Definition 1.1: Vector Space

Formally, a vector space over $\mathbb{R}$ is a tuple $(V, +, \cdot)$ where V is a set, +: $V \times V \to V$ and $\cdot$: $\mathbb{R} \times V \to V$ such that

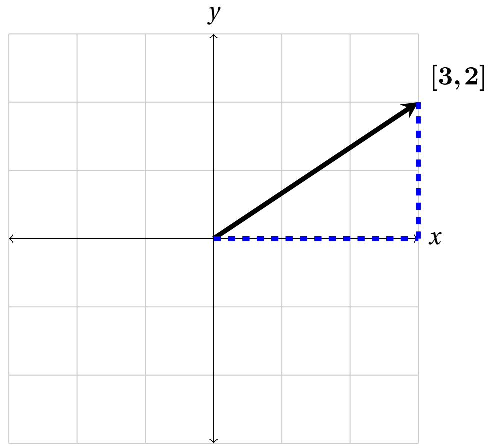
1. $x + y = y + x$ for all $x, y \in V$,
2. $(x + y) + z = x + (y + z)$ for all $x, y \in V$,
3. there exist $0 \in V$ such that $x + 0$=x for all $x \in V$,
4. for every $x \in V$, there exist $-x \in V$ such that $x + (-x) = 0$,
5. $1 \cdot x = x$ for all $x \in V$,
6. $(ab) \cdot x = a \cdot (b \cdot x)$ for all $a, b \in \mathbb{R}$ and $x \in V$,
7. $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in \mathbb{R}$ and $x, y \in V$.
8. $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in \mathbb{R}$ and $x \in V$

Example: Let $n \in \mathbb{N}$. Let
$$V = \mathbb{R}^n = \{(x_1, ..., x_n) | x_1, ..., x_n \in \mathbb{R}\}.$$
Given $(x_1, ..., x_n), (y_1, ..., y_n) \in V$ and $a \in \mathbb{R}$, define $(x_1, ..., x_n) + (y_1, ..., y_n) \in V$ and $a \cdot (x_1, ..., x_n) \in V$ by
$$(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$$
and
$$a \cdot (x_1, ..., x_n) = (ax_1, ..., ax_n).$$
Then $(V, +, \cdot)$ is a vector space.

1.1 Exercise

1. Let S be a nonempty set. Let $V = \{f : S \to \mathbb{R}\}$. Given $f, g \in V$ and $a \in \mathbb{R}$, define $f + g \in V$ and $a \times f \in V$ by
$$(f + g)(s) = f(s) \quad \text{for } s \in S,$$
and
$$(a \cdot f)(s) = af(s) \quad \text{for } s \in S.$$
Show that $(V, +, \cdot)$ is a vector space.

1.2 Length of a vector, inner product, and the Cauchy-Schwarz inequality



Lenght of a vector: This one is easy, we have a vector of length 2 and we remember the Pythagorean theorem:

$$c^2 = b^2 + a^2.$$

For a vector **x** of length n:

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

This is called the **Euclidean norm**. We can verify it for the example above where $\mathbf{x}' = (3,2) : \|\mathbf{x}\| = \sqrt{3^2 + 2^2}$, which is equal to what we learned using the Pythagorean theorem. Also, for any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x} - \mathbf{y}\|$ measures the distance between **x** and **y**.

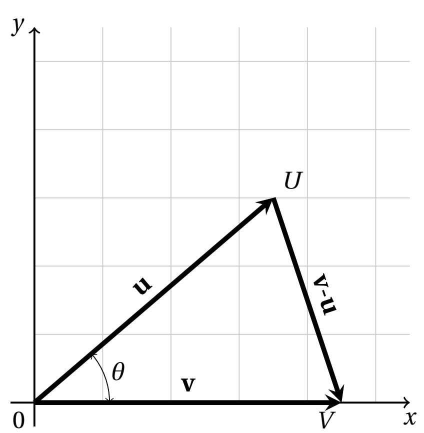
Inner product of two vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ is defined by $\mathbf{v} \cdot \mathbf{u} = \sum_{i=1}^{n} x_i y_i$. For example, let $\boldsymbol{\omega}$ be a vector of portfolio weights and **p** be a price vector for every asset in the portfolio. The value of the portfolio is then given by $\boldsymbol{\omega} \cdot \mathbf{p} = \sum_{i=1}^{n} \omega_i p_i$.

Cauchy-Schwarz inequality:

$$\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$$

In fact,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot cos(\theta)$$



According to Pythagora's theorem, the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is a right angle $(= 90°)$ if and only if $(OU)^2 + (OB)^2 = (VU)^2$, or $\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 = \|\mathbf{v} - \mathbf{u}\|^2$. This implies that $\theta = 90°$ if and only if

$$\mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} = (\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v}.$$

Because of symmetry, $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$, so we need $2\mathbf{u}\cdot\mathbf{v} = 0$ for the equality to hold, and so $\mathbf{u}\cdot\mathbf{v} = 0$. When the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $90°$, we say that they are **orthogonal.** If two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal we write $\mathbf{u} \perp \mathbf{v}$. Here we proved that two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are orthogonal if and only if their inner product is 0. For vectors in $\mathbb{R}^n$, we define orthogonality between $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathbf{u} \perp \mathbf{v} \iff \mathbf{u}\cdot\mathbf{v} = 0.$$

Example: Suppose we have n observations of a commodity's price and quantity demanded $(p_1, d_1), (p_2, d_2), ..., (p_n,$
Define the means as

$$\overline{\mathbf{p}} = \frac{1}{n}\sum_{i=1}^{n} p_i, \quad \overline{\mathbf{d}} = \frac{1}{n}\sum_{i=1}^{n} d_i,$$

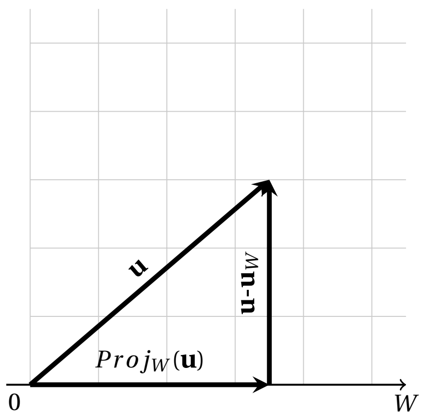
and

$$\mathbf{v} = (p_1 - \overline{\mathbf{p}}, p_2 - \overline{\mathbf{p}}, ..., p_n - \overline{\mathbf{p}}), \quad \mathbf{u} = (d_1 - \overline{\mathbf{d}}, d_2 - \overline{\mathbf{d}}, ..., d_n - \overline{\mathbf{d}}).$$

Then the **correlation coefficient** ρ can be computed as

$$\rho = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\cdot\|\mathbf{u}\|} = cos(\theta).$$

Projection



$\mathbf{u}_W$ is such that

1. $\mathbf{u}_W$ is parallell with W: There exist a scalar c such that $\mathbf{u}_W = cW$,
2. $\mathbf{u}_w \perp (\mathbf{u} - \mathbf{u}_W)$

Now we can find c, i.e. find $\mathbf{u}_w$. 2. means that

$$\begin{aligned} \mathbf{u}_W\cdot(\mathbf{u}-\mathbf{u}_W) &= 0, \\ \mathbf{u}_W\cdot\mathbf{u} - \mathbf{u}_W\cdot\mathbf{u}_W &= 0, \quad \text{(Distributive law (VS 7))} \\ \mathbf{u}_W\cdot\mathbf{u} &= \|\mathbf{u}_W\|^2. \quad (*) \end{aligned}$$