Problem 1

1. Let $\{W_t\}_{t\geq 0}$, be a Brownian motion. Show that the process $Y(t) = W^2(t) - t$ is a martingale.

2. Consider the process $Z_t = \int_0^t W_u du\,,\, t > 0$

   b) Show that $E[Z_T|\mathcal{F}_t] = Z_t + W_t(T-t)$ for any $t < T$.

   b) Prove that the process $M_t = Z_t - tW_t$ is a martingale.

3. Let $\{W_t\}_{t\geq 0}$ be a Brownian motion and let $\{\mathcal{F}_t\}_{t\geq 0}$, be a filtration associated with this Brownian motion. Show that the process $Y(t) = e^{-t2}e^{W_t}$ is a martingale.

4. The process $X(t) = |W(t)|$ is called *Brownian motion reflected at the origin*. Show that $E[X(t)] = \sqrt{2t/\pi}$ and $Var(X(t)) = t(1-\frac{2}{\pi})$.

5. Consider a symmetric random walk $M_3 = \sum_{j=1}^3 X_j = \begin{cases} 1, \text{probability } p \\ -1, \text{probability } (1-p) \end{cases}$

   a) Compute the the variance $Var(M_3)$ without assuming $p = \frac{1}{2}$ and the quadratic variation $[M, M]_3$.

   b) Show the variance and the quadratic variation are equal for $p = \frac{1}{2}$.

6. Brownian bridge. The process $X_t = W_t - \frac{t}{T}W_T$ is called the Brownian bridge fixed at 0 and T.

   a) Compute $E[X_t]$ and $Var(X_t)$.

   b) Let $Y_t = X_t^2$. Show that $Y_0 = Y_T = 0$ and find $E[Y_t]$ and $Var(Y_t)$.

7. Let W_t be a Brownian motion wrt. a filtration $\mathcal{F}_t$.

   a) Evaluate $Cov(W_t^2, W_s^2)$, for $0 < s \leq t$.

   b) Evaluate $E[W_t^2|\mathcal{F}_s]$ for $0 < s \leq t$.

8. Show that $E[e^{W_s+W_u}] = e^{\frac{u+s}{2}}e^{min\{u,s\}}$.

9. Let $s \leq t$. Find the covariance of the two Brownian bridge processes $X_t = W_t - \frac{t}{T}W_T$ and $Y_s = W_s - \frac{s}{T}W_T$.

Problem 2 - Riemann-Stieltjes Integral

1. Solve the integral $\int_0^t s dW(s)$.
2. Solve the integral $\int_0^t dW(s)$.
3. Specify distribution of $\int_0^1 s dW(s)$, and calculate its expectation and variance.
4. Specify distribution of $\int_0^1 (1-s) dW(s)$, and calculate its expectation and variance.
5. Specify distribution of $\int_0^1 dW(s)$, and calculate its expectation and variance.
6. A standard Ornstein-Uhlenbeck process is given by

$$X_c(t) = e^{ct} \int_0^t e^{-cs} dW(s) \qquad t \geq 0 \text{ and } X_c(0) = 0$$

   a) Use integration by parts to find an equivalent expression, $X_c(t)$.
   b) Find the distribution of $X_c(t)$.

7. Prove the integration by parts formula for Riemann-Stieltjes integrals.
8. Show that $X_t = \int_0^t (2t-s) dW_s$ and $Y_t = \int_0^t (3t-4s) dW_s$ are Gaussian processes with mean 0 and variance $\frac{7}{3}t^3$.

Problem 3 - Itô Calculus Problems

1. Find the following differentials
   a) $d(e^{t+W_t^2})$.
   b) $d(tW_t^2)$.
   c) $d(\sqrt{W_t})$.

2. Given the dynamics $dX_t = \alpha X_t dt + \sigma X_t dW_t$ and the constant r, derive the following functions of X and t.
   a) $Y_t = e^{r(T-t)}X_t$.
   b) $Y_t = X_t^{-1}$.
   c) $Y_t = e^{X_t}$.

3. Find the increment $W_t^2 - W_0^2$.

4. Suppose $\{W_t\}_{t\geq 0}$ is a Brownian motion. Determine the stochastic differential equation satisfied by
   $X_t = e^{\mu t+\sigma W_t}$.

5. Given $dSt = \alpha S_t dt + \sigma S_t dW_t$ find an expression of $f(t, S) = \ln(S)$. Assume that μ and σ are constant.

6. Find the differential of a discounted stock price when we assume that the stock price is modelled by a Geometric Brownian motion, i.e. $dS_t = \alpha S_t dt + \sigma S_t dW_t$.

7. Find the dynamics of the process $Y(t, S_1, S_2) = \frac{S_1}{S_2}$ when

8. Find the dynamics of the process $Y_t = \frac{S_t P_t}{B_t}$ given the dynamics

$$
\begin{aligned}
dS_t &= \mu_S S_t dt + \sigma_S S_t dW_t^S \\
dP_t &= \mu_P P_t dt + \sigma_P P_t dW_t^P \\
dB_t &= rBtdt \\
dW_t^S dW_t^P &= \alpha dt
\end{aligned}
$$

9. Let $S_t = S_0 e^{(\alpha-\frac{1}{2}\sigma^2)t+\sigma W_t}$ be a geometric Brownian motion.
   a) Derive the stochastic differential equation satisfied by S_t.

b) Let p be a positive constant. Compute $d(S_t^p)$ (i.e. the differential to the power p).

10. Consider the dynamics of two Itô processes :

$$dX_t = \alpha_X X_t dt + \sigma_X X_t dW_t^X$$
$$dY_t = \alpha_Y Y_t dt + \sigma_Y Y_t dW_t^Y$$

Where Brownian motions are W_t^X, W_t^Y are correlated with $[W_t^X, W_t^X] = \rho_{XY} t$. Derive the stochastic differential equations for the following processes:

a) $U_t = X_t Y_t$.

b) $V_t = X_t / Y_t$.