Exam 2 Solution

1. Version A

	Number of degrees of freedom per element	Number of rows	Number of columns
Plane stress 3 node (CST) triangle	**6**		
[D]		**3**	**3**
[B]		**3**	**6**
{σ}		**3**	**1**
3-D 10 node tetrahedral	**30**		
[D]		**6**	**6**
[B]		**6**	**30**
{σ}		**6**	**1**

Version B

	Number of degrees of freedom per element	Number of rows	Number of columns
Plane stress 6 node (LST) triangle	**12**		
[D]		**3**	**3**
[B]		**3**	**12**
{σ}		**3**	**1**
3-D 4 node tetrahedral	**12**		
[D]		**6**	**6**
[B]		**6**	**12**
{σ}		**6**	**1**

Version C

	Number of degrees of freedom per element	Number of rows	Number of columns
Plane stress 6 node (LST) triangle	**12**		
[D]		**3**	**3**
[B]		**3**	**12**
{σ}		**3**	**1**
3-D 8 node hexahedral (brick)	**24**		
[D]		**6**	**6**
[B]		**6**	**24**
{σ}		**6**	**1**

2.

$\alpha_1 = x_2 y_3 - y_2 x_3 = LH$

$\alpha_2 = x_3 y_1 - x_1 y_3 = 0$

$\alpha_3 = x_1 y_2 - y_2 x_1 = 0$

$\beta_1 = y_2 - y_3 = -H$

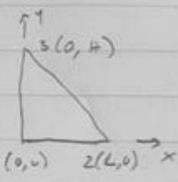
$\beta_2 = y_3 - y_1 = H$

$\beta_3 = y_1 - y_2 = 0$

$\gamma_1 = x_3 - x_2 = -L$

$\gamma_2 = x_1 - x_3 = 0$

$\gamma_3 = x_2 - x_1 = L$



$2A = LH$

Along $y = 0$

$$N_1(x,0) = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 \cancel{y}^{\,0}) = \frac{1}{LH}(LH - Hx) = 1 - x/L$$

$$N_2(x,0) = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 \cancel{y}^{\,0}) = \frac{1}{LH}(Hx) = x/L$$

$$N_3(x,0) = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 \cancel{y}^{\,0}) = \frac{1}{LH}(0) = 0$$

$$\underset{6\times1}{\{f_s\}} = \iint_S \underset{6\times2}{[N_s]^T} \underset{2\times1}{\{T\}}\, dx\, dz = t\int_0^x \begin{bmatrix} 1-x/L & 0 \\ 0 & 1-x/L \\ x/L & 0 \\ 0 & x/L \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \rho_0\left[1-\left(\frac{x}{L}\right)^2\right] \end{Bmatrix} dx$$

$$\{f_s\} = \begin{Bmatrix} 0 \\ f_{s1y} \\ 0 \\ f_{s2y} \\ 0 \\ 0 \end{Bmatrix}$$

where
$$f_{s1y} = \rho_0 t \int_0^L (1 - x/L)\left[1-\left(\frac{x}{L}\right)^2\right] dx$$
$$= \rho_0 t \int_0^L 1 - x/L - \left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 dx$$
$$= \rho_0 t \left(x + \frac{x^2}{2L} - \frac{x^3}{3L^2} + \frac{x^4}{4L^3}\right)\Big|_0^L$$
$$= \rho_0 t L\left(1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4}\right)$$
$$= \boxed{\tfrac{5}{12}\rho_0 t L}$$

$$f_{s2y} = \rho_0 t \int_0^L (x/L)\left[1-\left(\frac{x}{L}\right)^2\right] dx$$
$$= \rho_0 t \int_0^L x/L - x^3/L^3\, dx$$
$$= \rho_0 t \left(\frac{x^2}{2L} - \frac{x^4}{4L^3}\right)\Big|_0^L$$
$$= \rho_0 t L\left(\tfrac{1}{2} - \tfrac{1}{4}\right)$$
$$= \boxed{\tfrac{1}{4}\rho_0 t L}$$

Version A (L=2 mm)	$\{f_S\} = \begin{Bmatrix} 0 \\ 83.3 \\ 0 \\ 50 \\ 0 \\ 0 \end{Bmatrix}$ N
Version B (L=3 mm)	$\{f_S\} = \begin{Bmatrix} 0 \\ 125 \\ 0 \\ 75 \\ 0 \\ 0 \end{Bmatrix}$ N
Version C (L=4 mm)	$\{f_S\} = \begin{Bmatrix} 0 \\ 166.7 \\ 0 \\ 100 \\ 0 \\ 0 \end{Bmatrix}$ N

3. Version A (Note: Exact = 7.7287)

n=1	$I \cong (2)(2)(\cos 0 + 4\sin^2 0) = \mathbf{4}$
n=2	$I \cong \begin{array}{l} \cos\left(\frac{-1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{-1}{\sqrt{3}}\right) + \cos\left(\frac{-1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{1}{\sqrt{3}}\right) + \\ \cos\left(\frac{1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{-1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{1}{\sqrt{3}}\right) + \end{array}$ $\cong 4\cos\left[\cos\left(\frac{1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{1}{\sqrt{3}}\right) +\right] = \mathbf{8.1181}$
n=3	$I \cong \begin{array}{l} 4\left(\frac{5}{9}\right)\left(\frac{5}{9}\right)\left[\cos\left(\sqrt{\frac{3}{5}}\right) + 4\sin^2\left(\sqrt{\frac{3}{5}}\right)\right] + \\ 2\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)\left[\cos(0) + 4\sin^2\left(\sqrt{\frac{3}{5}}\right)\right] + \\ 2\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)\left[\cos\left(\sqrt{\frac{3}{5}}\right) + 4\sin^2(0)\right] + \\ \left(\frac{8}{9}\right)\left(\frac{5}{9}\right)[\cos(0) + 4\sin^2(0)] + \end{array}$ $\cong 3.29812 + 2.92029 + 0.70588 + .79012 = \mathbf{7.7144}$

Version B (Note: Exact = 11.0946)

n=1	$I \cong (2)(2)(2\cos 0 + 4\sin^2 0) = \mathbf{8}$
n=2	$I \cong \begin{array}{c} 2\cos\left(\frac{-1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{-1}{\sqrt{3}}\right) + 2\cos\left(\frac{-1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{1}{\sqrt{3}}\right) + \\ 2\cos\left(\frac{1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{-1}{\sqrt{3}}\right) + 2\cos\left(\frac{1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{1}{\sqrt{3}}\right) + \end{array}$ $\cong 4\left[2\cos\left(\frac{1}{\sqrt{3}}\right) + 4\sin^2\left(\frac{1}{\sqrt{3}}\right)\right] = \mathbf{11.4698}$
n=3	$I \cong \begin{array}{c} 4\left(\frac{5}{9}\right)\left(\frac{5}{9}\right)\left[2\cos\left(\sqrt{\frac{3}{5}}\right) + 4\sin^2\left(\sqrt{\frac{3}{5}}\right)\right] + \\ 2\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)\left[2\cos(0) + 4\sin^2\left(\sqrt{\frac{3}{5}}\right)\right] + \\ 2\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)\left[2\cos\left(\sqrt{\frac{3}{5}}\right) + 4\sin^2(0)\right] + \\ \left(\frac{8}{9}\right)\left(\frac{5}{9}\right)\left[2\cos(0) + 4\sin^2(0)\right] + \end{array}$ $\cong 4.18050 + 1.41176 + 3.90795 + 1.58025 = \mathbf{11.0805}$

Version C (Note: Exact = 12.1853)

n=1	$I \cong (2)(2)(2\cos 0 + 5\sin^2 0) = \mathbf{8}$
n=2	$I \cong \begin{array}{c} 2\cos\left(\frac{-1}{\sqrt{3}}\right) + 5\sin^2\left(\frac{-1}{\sqrt{3}}\right) + 2\cos\left(\frac{-1}{\sqrt{3}}\right) + 5\sin^2\left(\frac{1}{\sqrt{3}}\right) + \\ 2\cos\left(\frac{1}{\sqrt{3}}\right) + 5\sin^2\left(\frac{-1}{\sqrt{3}}\right) + 2\cos\left(\frac{1}{\sqrt{3}}\right) + 5\sin^2\left(\frac{1}{\sqrt{3}}\right) + \end{array}$ $\cong 4\left[2\cos\left(\frac{1}{\sqrt{3}}\right) + 5\sin^2\left(\frac{1}{\sqrt{3}}\right)\right] = \mathbf{12.6614}$
n=3	$I \cong \begin{array}{c} 4\left(\frac{5}{9}\right)\left(\frac{5}{9}\right)\left[2\cos\left(\sqrt{\frac{3}{5}}\right) + 5\sin^2\left(\sqrt{\frac{3}{5}}\right)\right] + \\ 2\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)\left[2\cos(0) + 5\sin^2\left(\sqrt{\frac{3}{5}}\right)\right] + \\ 2\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)\left[2\cos\left(\sqrt{\frac{3}{5}}\right) + 5\sin^2(0)\right] + \\ \left(\frac{8}{9}\right)\left(\frac{5}{9}\right)\left[2\cos(0) + 5\sin^2(0)\right] + \end{array}$ $\cong 4.784448 + 4.39111 + 1.41176 + 1.58025 = 12.1676$

<u>Makeup Exam Solution</u>

1.

a) $x\left(\frac{1}{2},\frac{1}{2}\right) = 3\text{ mm}$

b) $u\left(\frac{1}{2},\frac{1}{2}\right) = 0.0169\text{ mm}$

2.

$$\{f_s\} = \begin{Bmatrix} 250\text{ N} \\ 0 \\ 0 \\ 0 \\ 150\text{ N} \\ 0 \end{Bmatrix}$$

3.

n=1: $I \cong 4.00$

n=2: $I \cong 5.0489$

n=3: $I \cong 5.4943$