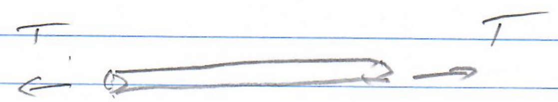
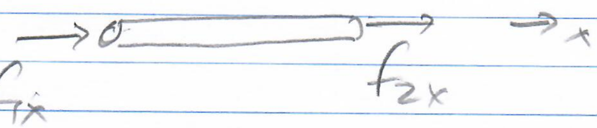
Step 4 - Element Stiffness matrix (or element equations)

$$T = \sigma A$$

$$T = A E \left(\frac{u_2 - u_1}{L} \right)$$





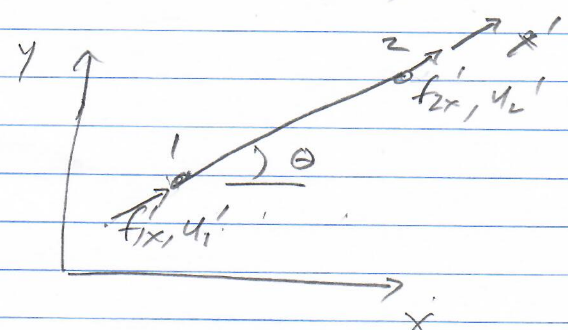
$$f_{1x} = -T = -\left(\frac{AE}{L}\right)(u_2 - u_1)$$

$$f_{2x} = T = \left(\frac{AE}{L}\right)(u_2 - u_1)$$

Matrix form

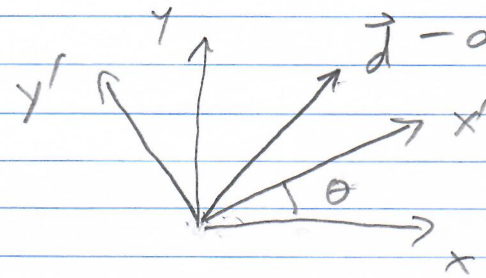
$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \left(\frac{AE}{L}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

2-D (Plane) Truss



$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \underbrace{\begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}}_{[k]} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

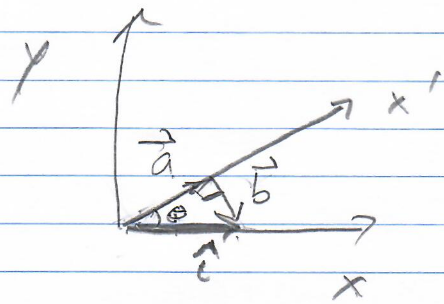
2-D vector transformat.



$\vec{d}$ – arbitrary vector

$$\vec{d} = u\,\hat{i} + v\,\hat{j}$$

$$\vec{d} = u'\,\hat{i}' + v'\,\hat{j}'$$



$$\vec{a} + \vec{b} = \hat{i}$$

$$|\vec{a}| = |\hat{i}|\cos\theta = \cos\theta$$

$$|\vec{b}| = -|\hat{i}|\sin\theta = -\sin\theta$$

$$\vec{a} = \cos\theta\,\hat{i}'$$

$$\vec{b} = -\sin\theta\,\hat{j}'$$

$$\hat{i} = \cos\theta\,\hat{i}' - \sin\theta\,\hat{j}'$$

Similarly

$$\hat{j} = \sin\theta\,\hat{i}' + \cos\theta\,\hat{j}'$$

$$\vec{d} = u(\cos\theta\,\hat{i}' - \sin\theta)\hat{j} + v(\sin\theta\,\hat{i}' + \cos\theta\,\hat{j}')$$

$$\vec{d} = \underbrace{(u\cos\theta + v\sin\theta)}_{u'}\,\hat{i}' + \underbrace{(-u\sin\theta + v\cos\theta)}_{v'}\,\hat{j}'$$

$$\vec{d} = u'\,\hat{i}' + v'\,\hat{j}'$$

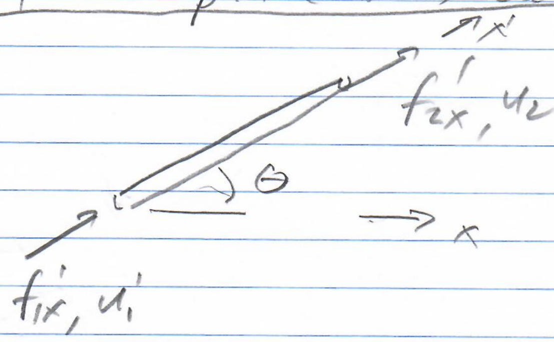
$$\{d'\} = \begin{Bmatrix} u' \\ v' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$c = \cos\theta, \quad s = \sin\theta$$

$$\{d'\} = \begin{Bmatrix} u' \\ v' \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$[T]$ transformation matrix

Apply to plane truss element



$$\begin{Bmatrix} f_{1x}' \\ f_{2x}' \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

$$\underset{2\times1}{\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}} = \underset{2\times4}{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}} \underset{4\times1}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}} = \underset{2\times4}{[T^*]} \underset{4\times1}{\{d\}}$$

$$\begin{Bmatrix} f_{1x}' \\ f_{2x}' \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \underset{2\times4}{[T^*]} \underset{4\times1}{\{f\}}$$

$$\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \left(\frac{AE}{L}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

expand $\underset{2\times4}{[T^*]}$ to $\underset{4\times4}{[T]}$

$$\begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{Bmatrix} = \underset{4\times4}{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}} \underset{4\times1}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}} = [T]\{d\}$$

$$\begin{Bmatrix} f_{1x}' \\ f_{1y}' \\ f_{2x}' \\ f_{2y}' \end{Bmatrix} = [T] \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = [T]\{f\}$$

$$\begin{Bmatrix} f_{1x}' \\ f_{1y}' \\ f_{2x}' \\ f_{2y}' \end{Bmatrix} = \frac{AE}{L} \underset{4\times4}{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{Bmatrix}$$

$$\underset{4\times4}{[T]} \underset{4\times1}{\{f\}} = \underbrace{\frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underset{4\times4}{[k']}} \underset{4\times4}{[T]} \underset{4\times1}{\{d\}}$$

(4 equation

Need $[T]^{-1}$

It can be shown

$$[T]^{-1} = [T]^T$$

$$\underset{4\times1}{\{f\}} = \underset{4\times4}{[T]^T} \underset{4\times4}{[k']} \underset{4\times4}{[T]} \underset{4\times1}{\{d\}}$$

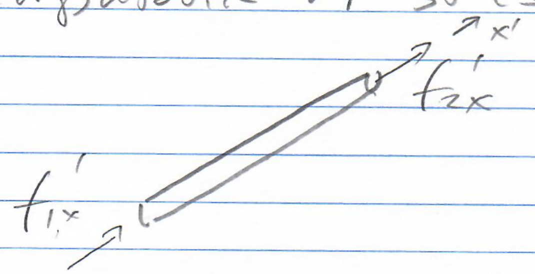
$$\{f\} = \underbrace{[T]^T[k'][T]}\{d\}$$

$$\{f\} = \underset{4\times4}{[k]}\{d\}$$

element stiffness matrix in global coordinate

$$[k] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ & s^2 & -cs & -s^2 \\ \text{SYM} & & c^2 & cs \\ & & & s^2 \end{bmatrix}$$

Computation of stress



Tension

$f'_{2x} > 0$

Compression

$f'_{2x} < 0$

Since $\sigma > 0$ - tension

$\sigma < 0$ - compression

$$\sigma = \frac{f'_{2x}}{A}$$

$$f'_{2x} = \frac{AE}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

$$\sigma = \frac{f'_{2x}}{A} = \frac{E}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \underset{2\times4}{\begin{bmatrix} T^* \end{bmatrix}} \underset{4\times1}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}}$$

$$\{d'\} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\sigma = \frac{E}{L} \underset{1\times2}{\begin{bmatrix} -1 & 1 \end{bmatrix}} \underset{2\times4}{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}}_{\{d\}\ 4\times1}$$

$$\boxed{\sigma = \frac{E}{L} \begin{bmatrix} -c & -s & c & s \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}}$$