Homework Assignment #7 – Solution

1)

10.2 Using Equation (10.1.1 b)

For Figure (a)

(a) $$s = [x - \frac{(x_1 + x_2)}{2}]\left(\frac{2}{x_2 - x_1}\right)$$

At $A \Rightarrow x = x_A = 15$ in.

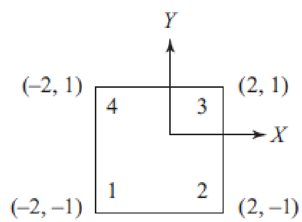
$$s = [15 - \frac{(10+20)}{2}]\left(\frac{2}{20-10}\right)$$

$$s = (15 - 15)\,\frac{2}{10} = 0$$

2)

10.11 (a)

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{pmatrix} x_1 = -z,\ x_2 = z \\ x_3 = z,\ x_4 = -z \end{pmatrix}$$



$$\frac{\partial x}{\partial s} = \frac{1}{4}(-1)(1-t)(-2) + \frac{1}{4}(1-t)(2) + \frac{1}{4}(1+t)(2) + \frac{1}{4}(-1)(1+t)(-2) = z$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}\,(-1)\,(1-s)(-2) + \frac{1}{4}\,(-1)\,(1+s)\,(2) + \frac{1}{4}\,(1+s)\,(2)$$

$$+\,\frac{1}{4}(1-s)\,(-2) = 0$$

$y_1 = 1,\ y_2 = -1$

$y_3 = 1,\ y_4 = 1$

$$\frac{\partial y}{\partial s} = \frac{1}{4}\,(-1)\,(1-t)\,(-1) + \frac{1}{4}\,(1-t)\,(-1) + \frac{1}{4}\,(1+t)\,(1)$$

$$+\,\frac{1}{4}\,(-1)\,(1+t)\,(1) = 0$$

$$\frac{\partial y}{\partial t} = \frac{1}{4}\ (-1)\,(1-s)\,(-1) + \frac{1}{4}\ (-1)\,(1+s)\,(-1) + \frac{1}{4}\ (1+s)\,(1)$$

$$+ \frac{1}{4}\ (1-s)\,(1) = 1$$

$$|\,[J]\,| = \left| \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right| = 2 - 0 = 2$$

By Equation (10.2.22)

$$|\,[J]\,| = \frac{1}{8}\ [-2\ \ 2\ \ 2\ {-2}] \times \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \begin{Bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{Bmatrix} \qquad \text{(A)}$$

Symplying by multiplying the matrices in Equation (A) yields

$|\,[J]\,| = 2$ also

and $\quad |\,[J]\,| = \dfrac{A}{4}$ as

$A = 4 \times 2 = 8$ (area of element)

$$\therefore \qquad = |\,[J]\,|\ \frac{8}{4} = 2$$

3)

10.15

(a) $\int_{-1}^{1} \cos \dfrac{s}{2}\ ds$ Use Table 10.1

$$I = \sum_{i=1}^{3} W_i \cos \frac{s_i}{2} = W_1 \cos \frac{s_1}{2} + W_2 \cos \frac{s_2}{2} + W_3 \cos \frac{s_3}{2}$$

$$= \frac{5}{9}\cos\left(\frac{0.7746}{2}\right) + \frac{8}{9}\cos(0) + \frac{5}{9}\cos\left(\frac{-0.7746}{2}\right)$$

$I = 1.918$ (Analytical $I = 1.918$)

That is

$$\int_{-1}^{1} \cos\frac{s}{2}\, ds = 2\sin\frac{s}{2}\Bigg|_{-1}^{1} = 2\sin\frac{1}{2} - 2\sin\left(-\frac{1}{2}\right)$$

$$= 4\sin\frac{1}{2} = 4(0.47)$$

$$= 1.918$$

4) 2D Gauss Quadrature Problem

1x1 ($s_1=t_1=0$, $W_1=2$):

$$f(s,t) = \cos(s)\cos(t)$$

$$I \cong \sum_{i=1}^{1}\sum_{j=1}^{1} W_i W_j f(s_i,t_j) = (2)(2)f(0,0) = 4$$

2x2 ($s_1=t_1=-.5773$, $s_2=t_2=+.5773$, $W_1=W_2=1$):

$$I \cong \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j f(s_i,t_j) = f(s_1,t_1) + f(s_1,t_2) + f(s_2,t_1) + f(s_2,t_2) = 4(0.8379)^2 = 2.809$$

3x3 ($s_1=t_1=-.7746$, $s_2=t_2=0$, $s_3=t_3=+.7746$, $W_1=W_3=5/9$, $W_2=8/9$):

$$I \cong \sum_{i=1}^{3}\sum_{j=1}^{3} W_i W_j f(s_i,t_j)$$

$$= \left(\frac{5}{9}\right)\left(\frac{5}{9}\right)f(s_1,t_1) + \left(\frac{5}{9}\right)\left(\frac{8}{9}\right)f(s_1,t_2) + \left(\frac{5}{9}\right)\left(\frac{5}{9}\right)f(s_1,t_3) + \left(\frac{8}{9}\right)\left(\frac{5}{9}\right)f(s_2,t_1)$$
$$+ \left(\frac{8}{9}\right)\left(\frac{8}{9}\right)f(s_2,t_2) + \left(\frac{8}{9}\right)\left(\frac{5}{9}\right)f(s_2,t_3) + \left(\frac{5}{9}\right)\left(\frac{5}{9}\right)f(s_3,t_1) + \left(\frac{5}{9}\right)\left(\frac{8}{9}\right)f(s_3,t_2)$$
$$+ \left(\frac{5}{9}\right)\left(\frac{5}{9}\right)f(s_3,t_3) = 2.833$$

Exact solution => I= 2.8323