

MCE 466 – Summary of Finite Element Equations

Spring, 1-D Truss, 2-D Truss and Beam Elements

Element	Force, $\{f\}$	Displacement, $\{d\}$	Element Stiffness Matrix, [k]	Strain-displacement matrix, [B]	Stress-strain matrix, [D]	Stress, $\{\sigma\}$
Spring	$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}$	$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$	$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$	-	-	-
1-D Truss	$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}$	$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$	$\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$	[E]	$E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$
2-D Truss	$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$	$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$	$\frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$	$\frac{1}{L} \begin{bmatrix} -c & -s & c & s \end{bmatrix}$	[E]	$\frac{E}{L} \begin{bmatrix} -c & -s & c & s \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$

Definition of an element stiffness matrix:

$$\{f\} = [k]\{d\} \quad (2.1.1)$$

Definition of global or total stiffness matrix for a structure:

$$\{F\} = [K]\{d\} \quad (2.1.2)$$

Displacement function assumed for linear spring element:

$$u = a_1 + a_2x \quad (2.2.2)$$

Shape functions for linear spring element:

$$N_1 = 1 - x/L \quad N_2 = x/L \quad (2.2.10)$$

Basic matrix equation relating nodal forces to nodal displacement for spring element:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (2.2.17)$$

Stiffness matrix for linear spring element:

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad (2.2.18)$$

Global equations for a spring assemblage:

$$[F] = [K]\{d\} \quad (2.2.20)$$

Total potential energy:

$$\pi_p = U + \Omega \quad (2.6.1)$$

For a system of springs:

$$U = \frac{1}{2} \{d\}^T [K] \{d\} \quad (2.6.20)$$

Displacement function assumed for two-noded bar element:

$$u = a_1 + a_2x \quad (3.1.1)$$

Shape functions for bar:

$$N_1 = 1 - \frac{x}{L} \quad N_2 = \frac{x}{L} \quad (3.1.4)$$

Stiffness matrix for bar:

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (3.1.14)$$

Transformation matrix relating vectors in the plane in two different coordinate systems:

$$[T] = \begin{bmatrix} C & S \\ S & C \end{bmatrix} \quad (3.3.18)$$

Global stiffness matrix for bar arbitrarily oriented in the plane:

$$[k] = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{Symmetry} & & & S^2 \end{bmatrix} \quad (3.4.23)$$

Axial stress in a bar:

$$\{\sigma\} = [C']\{d\} \quad (3.5.6)$$

where

$$[C'] = \frac{E}{L} \begin{bmatrix} -C & -S & C & S \end{bmatrix} \quad (3.5.8)$$

Total potential energy for bar:

$$\pi_p = \frac{AL}{2} \{d\}^T \{B\}^T [D]^T [B] \{d\} - \{d\}^T \{f\} \quad (3.10.19)$$

where

$$\{f\} = \{P\} + \int \int_{S_1} [N_S]^T \{T_x\} ds + \int \int \int_V [N]^T \{X_b\} dV$$

Quadratic form of bar strain energy:

$$U = \frac{1}{2} \{d\}^T [k] \{d\} = \frac{1}{2} [u_1 \ u_2] \frac{AE}{L} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{AE}{2L} [u_1^2 \quad 2u_1u_2 + u_2^2] \quad (3.10.28b)$$

Beam element equations

Displacement function assumed for beam transverse displacement:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4 \quad (4.1.2)$$

Shape functions for beam element:

$$\begin{aligned} N_1 &= \frac{1}{L^3} (2x^3 - 3x^2L + L^3) & N_2 &= \frac{1}{L^3} (x^3L - 2x^2L^2 + xL^3) \\ N_3 &= \frac{1}{L^3} (-2x^3 + 3x^2L) & N_4 &= \frac{1}{L^3} (x^3L - x^2L^2) \end{aligned} \quad (4.1.7)$$

Beam bending stress or flexure formula:

$$\sigma_x = \frac{-My}{I} \quad (4.1.10b)$$

Stiffness matrix for beam element:

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (4.1.14)$$

Work due to distributed loading:

$$W_{\text{distributed}} = \int_0^L w(x)v(x) dx \quad (4.4.1)$$

Work due to discrete nodal forces:

$$W_{\text{discrete}} = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2 \quad (4.4.2)$$

General formulation for beam with distributed loading:

$$\{F\} = [K]\{d\} - \{F_0\} \quad (4.4.8)$$

Work-equivalent replacement matrix for beam with uniform load:

$$\{F_0\} = \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} \quad (4.4.10)$$



Table D-1 Single element equivalent joint forces f_0 for different types of loads

	f_{1y}	m_1	Loading case	f_{2y}	m_2
1.	$-\frac{P}{2}$	$-\frac{PL}{8}$		$-\frac{P}{2}$	$\frac{PL}{8}$
2.	$-\frac{Pb^2(L+2a)}{L^3}$	$-\frac{Pab^2}{L^2}$		$-\frac{Pa^2(L+2b)}{L^3}$	$\frac{Pa^2b}{L^2}$
3.	$-P$	$-\alpha(1-\alpha)PL$		$-P$	$\alpha(1-\alpha)PL$
4.	$-\frac{wL}{2}$	$-\frac{wL^2}{12}$		$-\frac{wL}{2}$	$\frac{wL^2}{12}$
5.	$-\frac{7wL}{20}$	$-\frac{wL^2}{20}$		$-\frac{3wL}{20}$	$\frac{wL^2}{30}$
6.	$-\frac{wL}{4}$	$-\frac{5wL^2}{96}$		$-\frac{wL}{4}$	$\frac{5wL^2}{96}$