MCE 466 – Summary of Finite Element Equations

Spring, 1-D Truss, 2-D Truss and Beam Elements

| Element | Force, {f} | Displacement, {d} | Element Stiffness Matrix, [k] | Strain-displacement matrix, [B] | Stress-strain matrix, [D] | Stress, {σ} |
|--------------|--|--|---|---|------------------------------|---|
| Spring | $\begin{cases} f_{1x} \\ f_{2x} \end{cases}$ | $\begin{cases} u_1 \\ u_2 \end{cases}$ | $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ | - | ı | - |
| 1-D Truss | $\begin{cases} f_{1x} \\ f_{2x} \end{cases}$ | $\begin{cases} u_1 \\ u_2 \end{cases}$ | $\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$ | [E] | $E\left[-\frac{1}{L} \frac{1}{L}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ |
| 2-D Truss | $\begin{cases} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{cases}$ | $\begin{cases} u_1 \\ v_1 \\ u_2 \\ v_2 \end{cases}$ | $\frac{AE}{L} \begin{bmatrix} c^{2} & cs & -c^{2} & -cs \\ cs & s^{2} & -cs & -s^{2} \\ -c^{2} & -cs & c^{2} & cs \\ -cs & -s^{2} & cs & s^{2} \end{bmatrix}$ | $\frac{1}{L} \begin{bmatrix} -c & -s & c & s \end{bmatrix}$ | [E] | $\frac{E}{L}\begin{bmatrix} -c & -s & c & s \end{bmatrix} \begin{cases} u_1 \\ v_1 \\ u_2 \\ v_2 \end{cases}$ |

Definition of an element stiffness matrix:

$$\{f\} = [k]\{d\} \tag{2.1.1}$$

Definition of global or total stiffness matrix for a structure:

$$\{F\} = [K]\{d\} \tag{2.1.2}$$

Displacement function assumed for linear spring element:

$$u = a_1 + a_2 x (2.2.2)$$

Shape functions for linear spring element:

$$N_1 = 1 - x/L$$
 $N_2 = x/L$ (2.2.10)

Basic matrix equation relating nodal forces to nodal displacement for spring element:

$$\begin{cases}
 f_{1x} \\
 f_{2x}
 \end{cases} = \begin{bmatrix}
 k & -k \\
 -k & k
\end{bmatrix} \begin{Bmatrix} u_1 \\ u_2
 \end{cases}
 (2.2.17)$$

Stiffness matrix for linear spring element:

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{2.2.18}$$

Global equations for a spring assemblage:

$$[F] = [K]{d} (2.2.20)$$

Total potential energy:

$$\pi_p = U + \Omega \tag{2.6.1}$$

For a system of springs:

$$U = \frac{1}{2} \{d\}^{T} [K] \{d\}$$
 (2.6.20)

Truss/Bar element equations

Displacement function assumed for two-noded bar element:

$$u = a_1 + a_2 x (3.1.1)$$

Shape functions for bar:

$$N_1 = 1 \quad \frac{x}{L} \qquad N_2 = \frac{x}{L}$$
 (3.1.4)

Stiffness matrix for bar:

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3.1.14}$$

Transformation matrix relating vectors in the plane in two different coordinate systems:

$$[T] = \begin{bmatrix} C & S \\ S & C \end{bmatrix} \tag{3.3.18}$$

Global stiffness matrix for bar arbitrarily oriented in the plane:

$$[k] = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ S^2 & -CS & -S^2 \\ C^2 & CS \\ \text{Symmetry} & S^2 \end{bmatrix}$$
(3.4.23)

Axial stress in a bar:

$$\{\sigma\} = [C']\{d\}$$
 (3.5.6)

where

$$[C'] = \frac{E}{L}[\neg C \quad \neg S \quad C \quad S] \tag{3.5.8}$$

Total potential energy for bar:

$$\pi_p = \frac{AL}{2} \{d\}^T \{B\}^T [D]^T [B] \{d\} \quad \{d\}^T \{f\}$$
 (3.10.19)

where

$$\{f\} = \{P\} + \iint_{S_{s}} [N_{S}]^{T} \{T_{x}\} ds + \iiint_{V} [N]^{T} \{X_{b}\} dV$$

Quadratic form of bar strain energy:

$$U = \frac{1}{2} \{d\}^{T} [k] \{d\} = \frac{1}{2} [u_{1} u_{2}] \frac{AE}{L} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{1} \\ u_{2} \end{Bmatrix} = \frac{AE}{2L} [u_{1}^{2} \quad 2u_{1}u_{2} + u_{2}^{2}]$$
(3.10.28b)

Beam element equations

Displacement function assumed for beam transverse displacement:

$$v(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4 (4.1.2)$$

Shape functions for beam element:

$$N_{1} = \frac{1}{L^{3}}(2x^{3} - 3x^{2}L + L^{3}) \qquad N_{2} = \frac{1}{L^{3}}(x^{3}L - 2x^{2}L^{2} + xL^{3})$$

$$N_{3} = \frac{1}{L^{3}}(-2x^{3} + 3x^{2}L) \qquad N_{4} = \frac{1}{L^{3}}(x^{3}L - x^{2}L^{2})$$

$$(4.1.7)$$

Beam bending stress or flexure formula:

$$\sigma_{x} = \frac{-My}{I} \tag{4.1.10b}$$

Stiffness matrix for beam element:

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$
(4.1.14)

Work due to distributed loading:

$$W_{\text{distributed}} = \int_0^L w(x)v(x) \ dx \tag{4.4.1}$$

Work due to discrete nodal forces:

$$W_{\text{discrete}} = m_1 \phi_1 + m_2 \phi_2 + f_{1y} v_1 + f_{2y} v_2 \tag{4.4.2}$$

General formulation for beam with distributed loading:

$$\{F\} = [K]\{d\} - \{F_0\} \tag{4.4.8}$$

Work-equivalent replacement matrix for beam with uniform load:

$$\{F_0\} = \begin{cases} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{cases}$$
(4.4.10)

 f_{2y}

Table D–1 Single element equivalent joint forces f_0 for different types of loads

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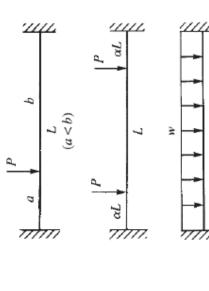
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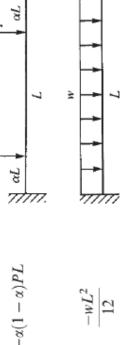


 $\alpha(1-\alpha)PL$

 $\frac{wL^2}{12}$

 $\frac{wL^2}{30}$

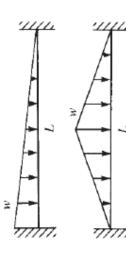
 $\frac{5wL^2}{96}$



 $\frac{-wL^2}{12}$

 $\frac{-wL}{2}$

4



$$\frac{-wL}{2}$$

$$\frac{-wL}{2}$$

$$\frac{-wL}{20}$$

$$\frac{-wL}{20}$$

$$\frac{w}{20}$$

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