Homework Assignment #5 – Solution

6.3

(c) $E = 30 \times 10^6 \qquad v = 0.25 \qquad t = 1$

Triangle coordinate definition

$i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x = 0 \quad y = 1$ $\quad$ This defines an array variable
x coordinate is the top
y coordinate is the bottom

$j = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad$ Area of triangle = $\frac{1}{2}$ base $\times$ height

$m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad A = \frac{1}{2}\,(j_x - i_x)\,(m_y - i_y)$

$\gamma_i = x_m - x_j$
$\gamma_j = x_i - x_m$
$\gamma_m = x_j - x_i$

$A = 1$

Develop stiffness matrix

$\beta_i = y_j - y_m$ $\quad \cancel{\beta_i = j_y - m_y} \quad \beta_i = -1 \quad \cancel{\gamma_i = m_x - j_x} \quad \gamma_i = -2$

$\beta_j = y_m - y_i$ $\quad \cancel{\beta_j = m_y - i_y} \quad \beta_j = 1 \quad \cancel{\gamma_j = i_x - m_x} \quad \gamma_j = 0$

$\beta_m = y_i - y_j$ $\quad \cancel{\beta_m = i_y - j_y} \quad \beta_m = 0 \quad \cancel{\gamma_m = j_x - i_x} \quad \gamma_m = 2$

$$[B_i] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{pmatrix} \qquad [B_j] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_j \end{pmatrix}$$

$$[B_m] = \frac{1}{2A}\begin{pmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{pmatrix}$$

Gradient matrix

$[B] =$ augment (B_i, B_j, B_m)

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -0.5 & 0 & 0.5 & 1 & 0 \end{pmatrix}$$

Plane stress

Constitutive matrix

$$[D] = \frac{E}{1-v^2}\begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{pmatrix}$$

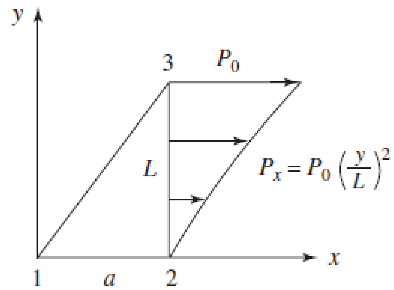
$$[D] = \begin{pmatrix} 3.2\times10^7 & 8\times10^6 & 0 \\ 8\times10^6 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{pmatrix}$$

$[k] = t\,A\,[B]^T\,[D]\,[B]$ Constant-strain triangular element stiffness matrix

$$[k] = \begin{pmatrix} 2\times10^7 & 1\times10^7 & -8\times10^6 & -6\times10^6 & -1.2\times10^7 & -4\times10^6 \\ 1\times10^7 & 3.5\times10^7 & -4\times10^6 & -3\times10^6 & -6\times10^6 & -3.2\times10^7 \\ -8\times10^6 & -4\times10^6 & 8\times10^6 & 0 & 0 & 4\times10^6 \\ -6\times10^6 & -3\times10^6 & 0 & 3\times10^6 & 6\times10^6 & 0 \\ -1.2\times10^7 & -6\times10^6 & 0 & 6\times10^6 & 1.2\times10^7 & 0 \\ -4\times10^6 & -3.2\times10^7 & 4\times10^6 & 0 & 0 & 3.2\times10^7 \end{pmatrix}$$

6.11

(b)



By Equation (6.3.11)

$$\{f_s\} = t\int_0^L \left.\begin{Bmatrix} N_1\,P_x \\ 0 \\ N_2\,P_x \\ 0 \\ N_3\,P_x \\ 0 \end{Bmatrix} dy\right|_{\substack{x=a \\ y=y}}$$

Now $\quad N_1 = 0,\ N_2 = \dfrac{L_x - ay}{2A},\ N_3 = \dfrac{ay}{2A}$

$$\{f_s\} = t\int_0^L \left.\begin{Bmatrix} 0 \\ 0 \\ \left(\frac{Lx-ay}{2A}\right) P_0\left(\frac{y}{L}\right)^2 \\ 0 \\ \frac{ay}{2A}\left(\frac{y}{L}\right)^2 P_0 \\ 0 \end{Bmatrix} dy\right|_{\substack{x=a \\ y=y}}$$

Simplifying and integrating

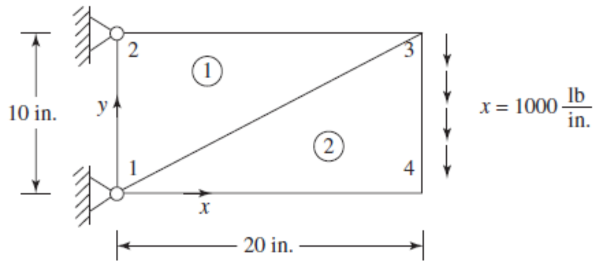
$$\{f_s\} = \frac{P_0\,t}{2\left(\frac{1}{2}aL\right)}\begin{Bmatrix} 0 \\ 0 \\ \left.\left(\frac{Lay^3}{3L^2} - \frac{ay^4}{4L^2}\right)\right|_0^L \\ 0 \\ \left.\frac{ay^4}{4L^2}\right|_0^L \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{P_0 tL}{12} \\ 0 \\ \frac{P_0 tL}{4} \\ 0 \end{Bmatrix}$$

or

$$f_{2x} = \frac{P_0\, tL}{12}$$

$$f_{3x} = \frac{P_0\, tL}{4}$$

6.13



Refer to Section 6.5 for $[K]$

Since $\quad u_1 = v_1 = 0,\ u_2 = v_2 = 0$

$$\begin{Bmatrix} 0 \\ -5{,}000 \\ 0 \\ -5{,}000 \end{Bmatrix} = \frac{75000 \times 5}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 \\ & 87 & 12 & -80 \\ & & 48 & -26 \\ \text{Symmetry} & & & 87 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Solving

$$u_3 = 0.50 \times 10^{-3} \text{ in.} \qquad v_3 = -0.275 \times 10^{-2} \text{ in.}$$
$$u_4 = -0.609 \times 10^{-3} \text{ in.} \qquad v_4 = -0.293 \times 10^{-2} \text{ in.}$$

Using element (1)

By Equation (6.2.36), $\{\sigma\} = [D]\,[B]\,\{d\}$

$$\{\sigma\} = \frac{30 \times 10^6}{(0.91)(200)} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$

$$\times \begin{Bmatrix} 0 \\ 0 \\ 0.5\times10^{-3} \\ -0.275\times10^{-2} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 824 \\ 247 \\ -1586 \end{Bmatrix} \text{ psi}$$

$$\sigma_{1,2} = \frac{824+247}{2} \pm \sqrt{\left(\frac{824-247}{2}\right)^2 + (-1586)^2}$$

$$\sigma_1 = 2149 \text{ psi} \qquad \sigma_2 = -\,1077 \text{ psi}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{-2\times1586}{824-247}\right)$$

$$\theta_p = -\,40^\circ$$

Using element (2)

$$\{\sigma\} = \frac{30\times10^6}{0.91(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix}$$

$$\times \begin{Bmatrix} 0 \\ 0 \\ -0.609\times10^{-3} \\ -0.293\times10^{-2} \\ 0.5\times10^{-3} \\ -0.275\times10^{-2} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -825 \\ 292 \\ -411 \end{Bmatrix} \text{ psi}$$

$$\sigma_{1,2} = \frac{-825 + 292}{2} \pm \sqrt{\left(\frac{-825 - 292}{2}\right)^2 + (-411)^2}$$

$$\sigma_1 = 426 \text{ psi} \qquad \sigma_2 = -960 \text{ psi}$$

$$\theta_p = \frac{1}{2} \tan^{-1}\left(\frac{-2 \times 411}{-825 - 292}\right)$$

$$\theta_p = 18.15^\circ$$