MCE 466 – Summary of Finite Element Equations (Exam 2)

2-D elements

Stress vector for two-dimensional stress state:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{6.1.1}$$

Principal stresses for two-dimensional stress state:

$$\begin{aligned} \sigma_1 &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\max} \\ \sigma_2 &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\min} \end{aligned} \tag{6.1.2}$$

Principal angle:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{6.1.3}$$

Strain-displacement equations for two-dimensional stress state:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{6.1.4}$$

Strain vector for two-dimensional stress state:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{6.1.5}$$

Stress–strain relationship for two-dimensional stress state:

$$\{\sigma\} = [D]\{\varepsilon\} \tag{6.1.7}$$

Stress–strain or constitutive matrix for plane stress condition:

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1 - \nu}{2} \end{bmatrix} \tag{6.1.8}$$

Stress–strain matrix for plane strain condition:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{6.1.10}$$

Displacement functions for three-noded triangular element:

$$\begin{aligned} u(x,y) &= a_1 + a_2x + a_3y \\ v(x,y) &= a_4 + a_5x + a_6y \end{aligned} \tag{6.2.2}$$

Shape functions for three-noded triangular element:

$$\begin{aligned} N_i &= \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y) \\ N_j &= \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y) \\ N_m &= \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y) \end{aligned} \tag{6.2.18}$$

where

$$\begin{aligned} &\alpha_i = x_j y_m - y_j x_m \qquad &&\alpha_j = y_i x_m - x_i y_m \qquad &&\alpha_m = x_i y_j - y_i x_j \\ &\beta_i = y_j - y_m &&\beta_j = y_m - y_i &&\beta_m = y_i - y_j \\ &\gamma_i = x_m - x_j &&\gamma_j = x_i - x_m &&\gamma_m = x_j - x_i \end{aligned} \tag{6.2.10}$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \tag{6.2.8}$$

Shape function matrix for three-noded triangular element:

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \tag{6.2.22}$$

Strain-displacement equations in matrix form:

$$\{\varepsilon\} = [[B_i]\ [B_j]\ [B_m]] \begin{Bmatrix} \{d_i\} \\ \{d_j\} \\ \{d_m\} \end{Bmatrix} \tag{6.2.31}$$

where the gradient matrix is

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix} \tag{6.2.32}$$

$$[B] = [[B_i]\ [B_j]\ [B_m]] \tag{6.2.34}$$

Stress–strain relationship as function of displacement matrix:

$$\{\sigma\} = [D][B]\{d\} \tag{6.2.36}$$

CST element equations:

$$\{f\} = [k][d]$$

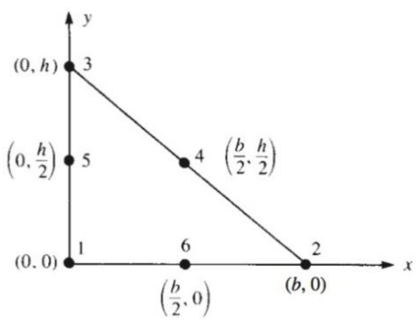
where

$$\{f\} = \iiint_V [N]^T \{X\}\, dV + \{P\} + \iint_S [N_S]^T \{T_S\}\, dS$$

$$[k] = tA[B]^T[D][B]$$

$$\{\boldsymbol{d}\} = \begin{Bmatrix} \boldsymbol{u_1} \\ \boldsymbol{v_1} \\ \boldsymbol{u_2} \\ \boldsymbol{v_2} \\ \boldsymbol{u_3} \\ \boldsymbol{v_3} \end{Bmatrix}$$

LST element equations:



$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2} \qquad N_2 = \frac{-x}{b} + \frac{2x^2}{b^2}$$

$$N_3 = \frac{-y}{h} + \frac{2y^2}{h^2} \qquad N_4 = \frac{4xy}{bh} \qquad N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2}$$

$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$

$$\{\varepsilon\} = [B]\{d\}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix}$$

$$\beta_1 = -3h + \frac{4hx}{b} + 4y \qquad \beta_2 = -h + \frac{4hx}{b} \qquad \beta_3 = 0$$

$$\beta_4 = 4y \qquad \beta_5 = -4y \qquad \beta_6 = 4h - \frac{8hx}{b} - 4y$$

$$\gamma_1 = -3b + 4x + \frac{4by}{h} \qquad \gamma_2 = 0 \qquad \gamma_3 = -b + \frac{4by}{h}$$

$$\gamma_4 = 4x \qquad \gamma_5 = 4b - 4x - \frac{8by}{h} \qquad \gamma_6 = -4x$$

$$[k] = t\iint_A [B]^T[D][B]\, dx\, dy$$

12x12 12x3 3x3 3x12

Axisymmetric element equations:

Displacement functions for axisymmetric triangle element:

$$u(r,z) = a_1 + a_2 r + a_3 z$$
$$w(r,z) = a_4 + a_5 r + a_6 z \tag{9.1.3}$$

Shape functions for axisymmetric triangle element:

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i r + \gamma_i z)$$
$$N_j = \frac{1}{2A}(\alpha_j + \beta_j r + \gamma_j z) \tag{9.1.12}$$
$$N_m = \frac{1}{2A}(\alpha_m + \beta_m r + \gamma_m z)$$

Gradient matrix:

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \dfrac{\alpha_i}{r} + \beta_i + \dfrac{\gamma_i z}{r} & 0 \\ \gamma_i & \beta_i \end{bmatrix} \tag{9.1.19}$$

and

$$[B] = [[B_i] \ \ [B_j] \ \ [B_m]] \tag{9.1.21}$$

Strain–displacement equations in matrix form:

$$\{\varepsilon\} = [B]\{d\} \tag{9.1.20}$$

Stress–displacement equations in matrix form:

$$\{\sigma\} = [D][B]\{d\} \tag{9.1.22}$$

Element stiffness matrix:

$$[k] = 2\pi \iint_A [B]^T [D][B] r \, dr \, dz \tag{9.1.24}$$

(All pertain to axisymmetric element).

Strain-displacement relationships for axisymmetric behavior:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{9.1.1e}$$

Stress–strain relationships for isotropic material:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \tag{9.1.2}$$

Chapter 10 equations

Natural coordinates related to global for a two-noded bar element:

$$x = a_1 + a_2 s \tag{10.1.2}$$

Shape functions in natural coordinate s for two-noded bar:

$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2} \tag{10.1.5}$$

Displacement function for two-noded bar:

$$\{u\} = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{10.1.6}$$

Gradient matrix for two-noded bar:

$$[B] = \left[-\frac{1}{L}\ \frac{1}{L}\right] \tag{10.1.11}$$

Determinant of Jacobian matrix for bar:

$$|[J]| = \frac{dx}{ds} = \frac{L}{2} \tag{10.1.14}$$

Stiffness matrix for two-noded bar:

$$[k] = \frac{L}{2}\int_{-1}^{1}[B]^T E[B]A\, ds \tag{10.1.15}$$

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{10.1.16}$$

Body force matrix for two-noded bar:

$$\{f_b\} = \frac{ALX_b}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{10.1.20}$$

Surface force matrix for two-noded bar:

$$\{f_s\} = \{T_x\}\frac{L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{10.1.24}$$

Relation between global and natural coordinates for quadrilateral element:

$$\begin{aligned} x &= a_1 + a_2 s + a_3 t + a_4 st \\ y &= a_5 + a_6 s + a_7 t + a_8 st \end{aligned} \tag{10.2.2}$$

and

$$\begin{aligned} x = \frac{1}{4}[&(1-s)(1-t)x_1 + (1+s)(1-t)x_2 \\ &+ (1+s)(1+t)x_3 + (1-s)(1+t)x_4] \\ y = \frac{1}{4}[&(1-s)(1-t)y_1 + (1+s)(1-t)y_2 \\ &+ (1+s)(1+t)y_3 + (1-s)(1+t)y_4] \end{aligned} \tag{10.2.3}$$

Shape functions for four-noded quadrilateral element expressed in natural coordinates:

$$N_1 = \frac{(1-s)(1-t)}{4} \quad N_2 = \frac{(1+s)(1-t)}{4}$$
$$N_3 = \frac{(1+s)(1+t)}{4} \quad N_4 = \frac{(1-s)(1+t)}{4} \tag{10.2.5}$$

Strain-displacement equations in natural coordinates:

$$\{\varepsilon\} = [D'][N]\{d\} \tag{10.2.15}$$

Determinant of Jacobian matrix for four-noded quadrilateral element:

$$|[J]| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} \tag{10.2.22}$$

$$\{X_c\}^T = [x_1 \; x_2 \; x_3 \; x_4]$$

where

$$\{Y_c\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{10.2.24}$$

Stiffness matrix for four-noded quadrilateral expressed in natural coordinates:

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T[D][B]h\,|[J]|\,ds\,dt \tag{10.2.27}$$

In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1}\left[\sum_i W_i f(s_i,t)\right] dt$$
$$= \sum_j W_j\left[\sum_i W_i f(s_i,t_j)\right] = \sum_i \sum_j W_i W_j f(s_i,t_j) \tag{10.3.13}$$

Table 10–2 Table for Gauss points for integration from minus one to one, $\int_{-1}^{1} y(x)\,dx = \sum_{i=1}^{n} W_i y_i$

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962 = 1/\sqrt{3}$	1.000
3	$x_1, x_3 = \pm 0.77459666924148 = \sqrt{3/5}$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549