MCE 466 – Summary of Finite Element Equations

Spring, 1-D Truss, 2-D Truss and Beam Elements

Element	Force, {*f*}	Displacement, {*d*}	Element Stiffness Matrix, [k]	Strain-displacement matrix, [B]	Stress-strain matrix, [D]	Stress, {σ}
Spring	$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}$	$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$	$\begin{bmatrix} \mathrm{k} & -\mathrm{k} \\ -\mathrm{k} & \mathrm{k} \end{bmatrix}$	-	-	-
1-D Truss	$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}$	$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$	$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$	$[E]$	$E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$
2-D Truss	$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$	$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$	$\frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$	$\frac{1}{L}\begin{bmatrix} -c & -s & c & s \end{bmatrix}$	$[E]$	$\frac{E}{L}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$

Spring element equations

Definition of an element stiffness matrix:

$$\{f\} = [k]\{d\} \tag{2.1.1}$$

Definition of global or total stiffness matrix for a structure:

$$\{F\} = [K]\{d\} \tag{2.1.2}$$

Displacement function assumed for linear spring element:

$$u = a_1 + a_2 x \tag{2.2.2}$$

Shape functions for linear spring element:

$$N_1 = 1 - x/L \qquad N_2 = x/L \tag{2.2.10}$$

Basic matrix equation relating nodal forces to nodal displacement for spring element:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{2.2.17}$$

Stiffness matrix for linear spring element:

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{2.2.18}$$

Global equations for a spring assemblage:

$$[F] = [K]\{d\} \tag{2.2.20}$$

Total potential energy:

$$\pi_p = U + \Omega \tag{2.6.1}$$

For a system of springs:

$$U = \frac{1}{2}\{d\}^T[K]\{d\} \tag{2.6.20}$$

Truss/Bar element equations

Displacement function assumed for two-noded bar element:

$$u = a_1 + a_2 x \tag{3.1.1}$$

Shape functions for bar:

$$N_1 = 1 \quad \frac{x}{L} \qquad N_2 = \frac{x}{L} \tag{3.1.4}$$

Stiffness matrix for bar:

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & \quad 1 \\ \quad 1 & 1 \end{bmatrix} \tag{3.1.14}$$

Transformation matrix relating vectors in the plane in two different coordinate systems:

$$[T] = \begin{bmatrix} C & S \\ \quad S & C \end{bmatrix} \tag{3.3.18}$$

Global stiffness matrix for bar arbitrarily oriented in the plane:

$$[k] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & \quad C^2 & \quad CS \\ & S^2 & \quad CS & \quad S^2 \\ & & C^2 & CS \\ \text{Symmetry} & & & S^2 \end{bmatrix} \tag{3.4.23}$$

Axial stress in a bar:

$$\{\sigma\} = [C']\{d\} \tag{3.5.6}$$

where

$$[C'] = \frac{E}{L}[\quad C \quad\quad S \quad C \quad S] \tag{3.5.8}$$

Total potential energy for bar:

$$\pi_p = \frac{AL}{2}\{d\}^T\{B\}^T[D]^T[B]\{d\} \quad\quad \{d\}^T\{f\} \tag{3.10.19}$$

where

$$\{f\} = \{P\} + \iint_{S_1} [N_S]^T\{T_x\}ds + \iiint_V [N]^T\{X_b\}dV$$

Quadratic form of bar strain energy:

$$U = \frac{1}{2}\{d\}^T[k]\{d\} = \frac{1}{2}[u_1\, u_2]\frac{AE}{L}\begin{bmatrix} 1 & \quad 1 \\ \quad 1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{AE}{2L}[u_1^2 \quad\quad 2u_1u_2 + u_2^2] \tag{3.10.28b}$$

Beam element equations

Displacement function assumed for beam transverse displacement:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4 \tag{4.1.2}$$

Shape functions for beam element:

$$\begin{aligned} N_1 &= \frac{1}{L^3}(2x^3 - 3x^2L + L^3) \qquad & N_2 &= \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3) \\ N_3 &= \frac{1}{L^3}(-2x^3 + 3x^2L) \qquad & N_4 &= \frac{1}{L^3}(x^3L - x^2L^2) \end{aligned} \tag{4.1.7}$$

Beam bending stress or flexure formula:

$$\sigma_x = \frac{-My}{I} \tag{4.1.10b}$$

Stiffness matrix for beam element:

$$[k] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{4.1.14}$$

Work due to distributed loading:

$$W_{\text{distributed}} = \int_0^L w(x)v(x)\ dx \tag{4.4.1}$$

Work due to discrete nodal forces:

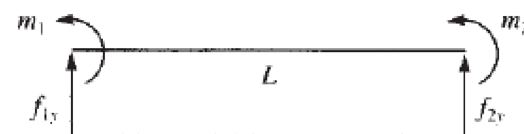
$$W_{\text{discrete}} = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2 \tag{4.4.2}$$

General formulation for beam with distributed loading:

$$\{F\} = [K]\{d\} - \{F_0\} \tag{4.4.8}$$

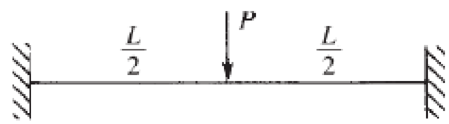
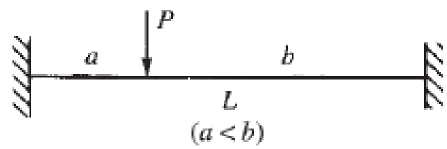
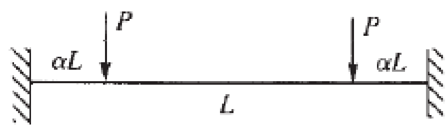
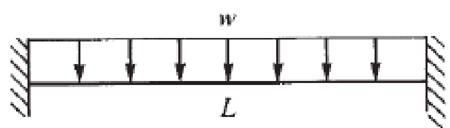
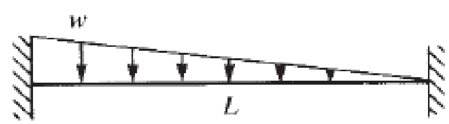
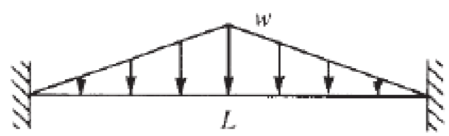
Work-equivalent replacement matrix for beam with uniform load:

$$\{F_0\} = \begin{Bmatrix} \dfrac{-wL}{2} \\ \dfrac{-wL^2}{12} \\ \dfrac{-wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} \tag{4.4.10}$$



Positive nodal force conventions

Table D–1 Single element equivalent joint forces f_0 for different types of loads

	f_{1y}	m_1	Loading case	f_{2y}	m_2
1.	$\frac{-P}{2}$	$\frac{-PL}{8}$		$\frac{-P}{2}$	$\frac{PL}{8}$
2.	$\frac{-Pb^2(L+2a)}{L^3}$	$\frac{-Pab^2}{L^2}$		$\frac{-Pa^2(L+2b)}{L^3}$	$\frac{Pa^2b}{L^2}$
3.	$-P$	$-\alpha(1-\alpha)PL$		$-P$	$\alpha(1-\alpha)PL$
4.	$\frac{-wL}{2}$	$\frac{-wL^2}{12}$		$\frac{-wL}{2}$	$\frac{wL^2}{12}$
5.	$\frac{-7wL}{20}$	$\frac{-wL^2}{20}$		$\frac{-3wL}{20}$	$\frac{wL^2}{30}$
6.	$\frac{-wL}{4}$	$\frac{-5wL^2}{96}$		$\frac{-wL}{4}$	$\frac{5wL^2}{96}$