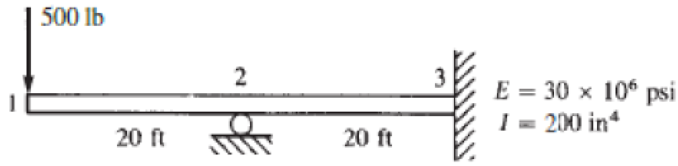
Homework #4 Solution

Problem 1

4.7



$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}$$

$E = 30 \times 10^6$, $I = 200$ in.^{4}, $L = 20$ ft $= 240$ in.

$$\{F\} = [K]\,\{d\} \;\Rightarrow\; \begin{Bmatrix} F_{1y} = -10 \\ M_1 = 0 \\ F_{2y} = ? \\ M_2 = 0 \\ F_{3y} = ? \\ M_3 = ? \end{Bmatrix} = [K]\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \text{ where } v_2 = v_3 = \phi_3 = 0$$

$$\begin{Bmatrix} -500 \\ 0 \\ 0 \end{Bmatrix} = \frac{30\times10^6(200)}{(240)^3}\begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix} \qquad (1)$$

Solving for the displacements we have

$\phi_1 = 0.0036$ rad, $\phi_2 = 0.0012$ rad, $v_1 = -0.672$ in.

Substituting in the equation $\{F\} = [K]\,\{d\}$ we have

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{30\times10^6(200)}{(240)}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}\begin{Bmatrix} \cancel{-1.344}\text{ in.} \\ \cancel{0.0072} \\ 0 \\ \cancel{0.0024} \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} -0.672 \\ 0.0036 \\ \\ 0.0012 \\ \\ \\ \end{matrix}$$

$F_{1y} = -500$, $M_1 = 0$,

$F_{2y} = 1250$ lb, $M_2 = 0$, $F_{3y} = -750$ lb, $M_3 = 60$ kip-in.

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix} \begin{Bmatrix} -1.344 \\ 0.0072 \\ 0 \\ 0.0024 \end{Bmatrix}$$

$$f_{1y} = -500 \text{ lb}$$

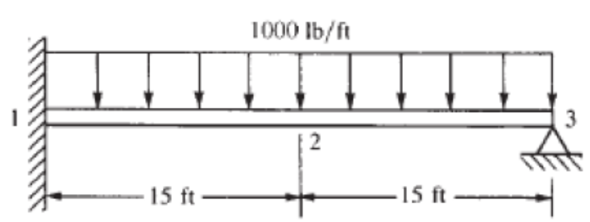
$$m_1 = 0$$
$$f_{2y} = 500 \text{ lb}$$
$$m_2 = -120{,}000 \text{ lb-in.}$$

Element 2–3

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.0024 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{aligned} f_{2y} &= 750 \text{ lb} \\ m_2 &= 120{,}000 \text{ lb-in.} \\ f_{3y} &= -750 \text{ lb} \\ m_3 &= 60{,}000 \text{ lb-in.} \end{aligned}$$

Problem 2

4.23



$E = 29 \times 10^6$ psi
$I = 200$ in.^{4}

Global stiffness matrix of the beam

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

After imposing the boundary conditions $v_1 = \phi_1 = v_3 = 0$ in $\{F\} = [K]\{d\}$

$$\begin{Bmatrix} -wL \\ 0 \\ \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} v_3 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Multiplying (1) by L and (3) by – 4 and adding them

$$-wL^2 = \frac{EI}{L^3}\,[24Lv_2 + 0\phi_2 + 6L^2\phi_3]$$

$$\frac{-wL^2}{3} = \frac{EI}{L^3}\,[-24Lv_2 - 8L^2\phi_2 - 16L^2\phi_3]$$

$$\frac{-4wL^2}{3} = \frac{EI}{L^3}\,[-8L^2\phi_2 - 10L^2\phi_3] \qquad (4)$$

Adding (2) and (4) we get

$$\frac{-4wL^2}{3} = \frac{EI}{L^3}\,[-8L^2\phi_3] \Rightarrow \boxed{\phi_3 = \frac{wL^3}{6EI}}$$

Substituting in (2)

$$0 = \frac{EI}{L^3}\left[8L^2\phi_2 + \frac{2L^2wL^3}{6EI}\right] \Rightarrow \boxed{\phi_2 = \frac{-wL^3}{24EI}}$$

Substituting in (1)

$$-wL = \frac{EI}{L^3}\left[24v_2 + \frac{6LwL^3}{6EI}\right] \Rightarrow -wL - wL = 24\,\frac{EI}{L^3}\,v_2$$

$$\Rightarrow \boxed{v_2 = \frac{-wL^4}{12EI}}$$

$$\Rightarrow \phi_3 = \frac{(\frac{1000}{12})(15\times 12)^3}{6(29\times 10^6)200} = 0.01396 \text{ rad}$$

$$\Rightarrow \phi_3 = \frac{(\frac{1000}{12})(15\times 12)^3}{6(29\times 10^6)200} = 0.01396 \text{ rad}$$

$$\phi_2 = \frac{-(\frac{1000}{12})(15\times 12)^3}{24(29\times 10^6)200} = -0.003491 \text{ rad}$$

$$v_2 = \frac{-1}{12}\,\frac{(\frac{1000}{12})(15\times 12)^4}{(29\times 10^6)\times 200} = -1.2569 \text{ in.} \downarrow$$

Substituting back in the global equation

$\{F\} = [K]\,\{d\} - \{F_0\}$ we can find the reactions

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1.2569 \\ -0.003491 \\ 0 \\ 0.01396 \end{Bmatrix}$$

$$- \begin{Bmatrix} \frac{-wL}{2} = -7500 \text{ lb} \\ \frac{-wL^2}{12} = -225{,}000 \text{ lb} \cdot \text{in.} \\ -wL = -15{,}000 \text{ lb} \\ 0 \\ \frac{-wL}{2} = -7500 \text{ lb} \\ \frac{wL^2}{12} = 225{,}000 \text{ lb} \cdot \text{in.} \end{Bmatrix}$$

$F_{1y} = 18750$ lbs, $M_1 = 1350$ K·in.

$F_{2y} = 0$, $M_2 = 0$

$F_{3y} = 11250$ lb, $M_3 = 0$

Element 1–2

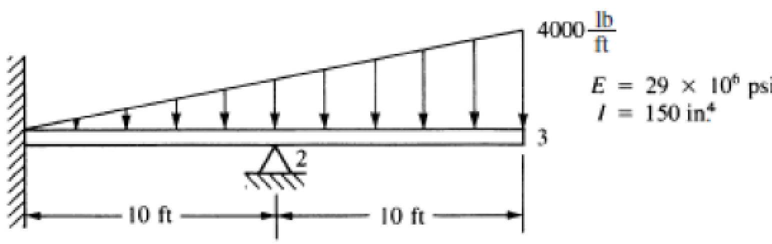
$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1.2569 \\ -0.003491 \end{Bmatrix} - \begin{Bmatrix} -7500 \\ -225000 \\ -7500 \\ 225000 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1y} = 18750 \text{ lb} \\ m_1 = 1350 \text{ k} \cdot \text{in.} \\ f_{2y} = -3750 \text{ lb} \\ m_2 = 675 \text{ k} \cdot \text{in.} \end{matrix}$$

Element 2–3

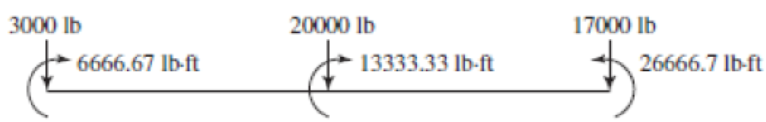
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -1.2569 \\ -0.003491 \\ 0 \\ 0.01396 \end{Bmatrix} - \begin{Bmatrix} -7500 \\ -225000 \\ -7500 \\ 225000 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2y} = 3750 \text{ lb} \\ m_2 = -675 \text{ k} \cdot \text{in.} \\ f_{3y} = 11250 \text{ lb} \\ m_3 = 0 \end{matrix}$$

<u>Problem 3</u>

4.24



$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6 & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}$$



After applying the boundary conditions

$$v_1 = \phi_1 = v_2 = 0 \text{ in } \{F_0\} = [K]\ \{d\}$$

$$\begin{Bmatrix} -13333.33 \text{ ft}\cdot\text{lb} \\ -17000 \text{ lb} \\ 26666.67 \text{ ft}\cdot\text{lb} \end{Bmatrix} = 30208.33 \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12 & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Rewriting equations (1) (2) and (3) we get

$$-0.441379 = 8L^2\phi_2 - 6Lv_3 + 2L^2\phi_3 \qquad (1)$$

$$-0.562759 = -6L\phi_2 + 12v_3 - 6L\phi_3 \qquad (2)$$

$$0.882759 = 2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3 \qquad (3)$$

Adding (1) to $-4 \times (3)$ we get

$$-0.441379 = 8L^2\phi_2 - 6Lv_3 + 2L^2\phi_3$$

$$-3.53103 = -8L^2\phi_2 + 24Lv_3 - 16L^2\phi_3$$

$$-3.9724 = 18Lv_3 - 14L^2\phi_3 \qquad (4)$$

Adding $L \times (2)$ to $3 \times (3)$ we get (where $L = 10$ ft)

$$-5.62759 = -6L^2\phi_2 + 12Lv_3 - 6L^2\phi_3$$

$$2.64827 = 6L^2\phi_2 - 18Lv_3 + 12L^2\phi_3$$

$$-297931 = -6Lv_3 + 6L^2\phi_3 \qquad (5)$$

Adding Equation (4) to $3 \times (5)$ we have

$$-12.91034 = 4L^2\phi_3 \;\Rightarrow\; \boxed{\phi_3 = -3.22758 \times 10^{-2} \text{ rad}}$$

Substituting in (4)

$$\Rightarrow -3.9724 = 180v_3 - 1400\,(-3.22758 \times 10^{-2})$$

$$\Rightarrow \boxed{v_3 = -2.73103 \times 10^{-1} \text{ ft} = -3.27724 \text{ in.}}$$

Substituting in (1)

$$\Rightarrow -0.441379 = 8L^2\phi_2 - 6L\,(-2.73103 \times 10^{-1}) + 2L^2\,(-3.22758)$$

$$\Rightarrow \boxed{\phi_2 = -1.29655 \times 10^{-2} \text{ rad}}$$

$$F_{1y}^{(e)} = \frac{6EI}{L^2}\phi_2 = \frac{6(29 \times 10^6)(150 \text{ in.}^4)}{(120)\,\text{in.}^2}\,(-1.29655 \times 10^{-2}) = -23500 \text{ lb}$$

$$M_1^{(e)} = \frac{2EI}{L}\phi_2 = \frac{2(29 \times 10^6)(150 \text{ in.}^4)}{120 \times 12''}\,(-1.29655 \times 10^{-2}) = 78333 \text{ lb}\cdot\text{ft}$$

$$F_{2y}^{(e)} = \frac{-12EI}{L^3}v_3 + \frac{6EI}{L^2}\phi_3$$

$$= \frac{-12(29 \times 10^6)(150)}{120 \times 120 \times 120}\,(-3.27724) + \frac{6(29 \times 10^6)}{(120)^2} \times 150$$

$$\times\,(-3.22758 \times 10^{-1}) = 40500 \text{ lb}$$

$$M_2^{(e)} = \frac{8L^2EI}{L^3}\phi_2 - \frac{6LEI}{L^3}v_2 + \frac{2L^2EI}{L^3}\phi_3 = -13333.33 \text{ ft}\cdot\text{lb}$$

$$F_{3y}^{(e)} = \frac{-6LEI}{L^3}\phi_2 + \frac{12LEI}{L^3}v_3 - \frac{6LEI}{L^3}\phi_3 = -17000 \text{ lb}$$

$$M_3^{(e)} = \frac{2L^2EI}{L^3}\phi_2 - \frac{6LEI}{L^3}v_3 + \frac{4L^2EI}{L^3}\phi_3 = 26{,}666.67 \text{ ft}\cdot\text{lb}$$

Global forces $\{F\} = \{F^{(e)}\} - \{F_0\}$

$F_{1y} = -23500 + 3000 = -20500$ lb

$M_1 = -78333.33 + 6666.67 = -71{,}666.67$ ft $\cdot$ lb

$F_{2y} = 40{,}500 + 20{,}000 = 60500$ lb

$M_2 = -13333.33 + 13333.33 = 0$

$F_{3y} = -17000 + 17000 = 0$

$M_3 = 26666.67 - 26666.67 = 0$

Element 1–2	Element 2–3
$f_{1y} = -20500$ lb	$f_{2y} = -30{,}000$ lb
$m_1 = -71666.67$ ft $\cdot$ lb	$m_2 = 2000$ kip $\cdot$ in.
$f_{2y} = 30{,}500$ lb	$f_{3y} = 0$
$m_2 = -2000$ kip $\cdot$ in.	$m_3 = 0$

Problem 4

a)

4a) $$v(x) = -\frac{w_0 l^4}{EI}\left[\frac{1}{16 l^2}x^2 - \frac{5}{48 l^3}x^3 + \frac{1}{24 l^4}x^4\right]$$

$$\frac{dv}{dx} = -\frac{w_0 l^4}{EI}\left[\frac{1}{8 l^2}x - \frac{5}{16 l^3}x^2 + \frac{1}{6 l^4}x^3\right]$$

$$\frac{d^2v}{dx^2} = -\frac{w_0 l^4}{EI}\left[\frac{1}{8 l^2} - \frac{5}{8 l^3}x + \frac{1}{2 l^4}x^2\right]$$

$$\frac{d^3v}{dx^3} = -\frac{w_0 l^4}{EI}\left[-\frac{5}{8 l^3} + \frac{1}{l^4}x\right]$$

$$\frac{d^4v}{dx^4} = -\frac{w_0 \cancel{l^4}}{EI}\left(\frac{1}{\cancel{l^4}}\right)$$

$$\boxed{w_0 = -EI\frac{d^4v}{dx^4} = w(x)}$$

b)

b) $\underset{}{v(x)} = \underset{1\times 4}{[N]}\underset{4\times 1}{\{d\}}$

<u>element 1</u>

$$v(x) = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

<u>element 2</u>

$$v(x) = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

where $v_1 = \phi_1 = v_3 = 0$

$v_2 = -2.5138$ in, $\phi_2 = -.0069877$, $\phi_3 = .02793$

Matlab script:

```
function hw4
clear; clc; close all;
% HW 4, problem 3c
%
E=29e6;
I=200;
L=15*12;
% FEA solution
%   element 1
x=linspace(0,L,101);
d=[0;0;-2.5138;-.0069827];
for i=1:101
    xp(i)=x(i);
    N=interp(x(i),L);
    v(i)=N*d;
end
plot(xp,v,'b')
hold on
%   element 2
d=[-2.5138;-.0069827; 0; .02793];
for i=1:101
    xp(i)=x(i)+L;
    N=interp(x(i),L);
    v(i)=N*d;
end
plot(xp,v,'g')
% exact
l=30*12;
w0=2000/12;
xe=linspace(0,l,101);
ve=-((w0*l^4)/(E*I))*((1/16)*(xe/l).^2-
(5/48)*(xe/l).^3+(1/24)*(xe/l).^4);
plot(xe,ve,'r')
legend('element 1','element 2','exact')
xlabel('x')
ylabel('v(x)')
title('Problem 4')
%
function N=interp(x,L)
%
N1=(2*x.^3-3*x.^2*L+L^3)/L^3;
N2=(x.^3*L-2*x.^2*L^2+x*L^3)/L^3;
N3=(-2*x.^3+3*x.^2*L)/L^3;
N4=(x.^3*L-x.^2*L^2)/L^3;
N=[N1, N2, N3, N4];
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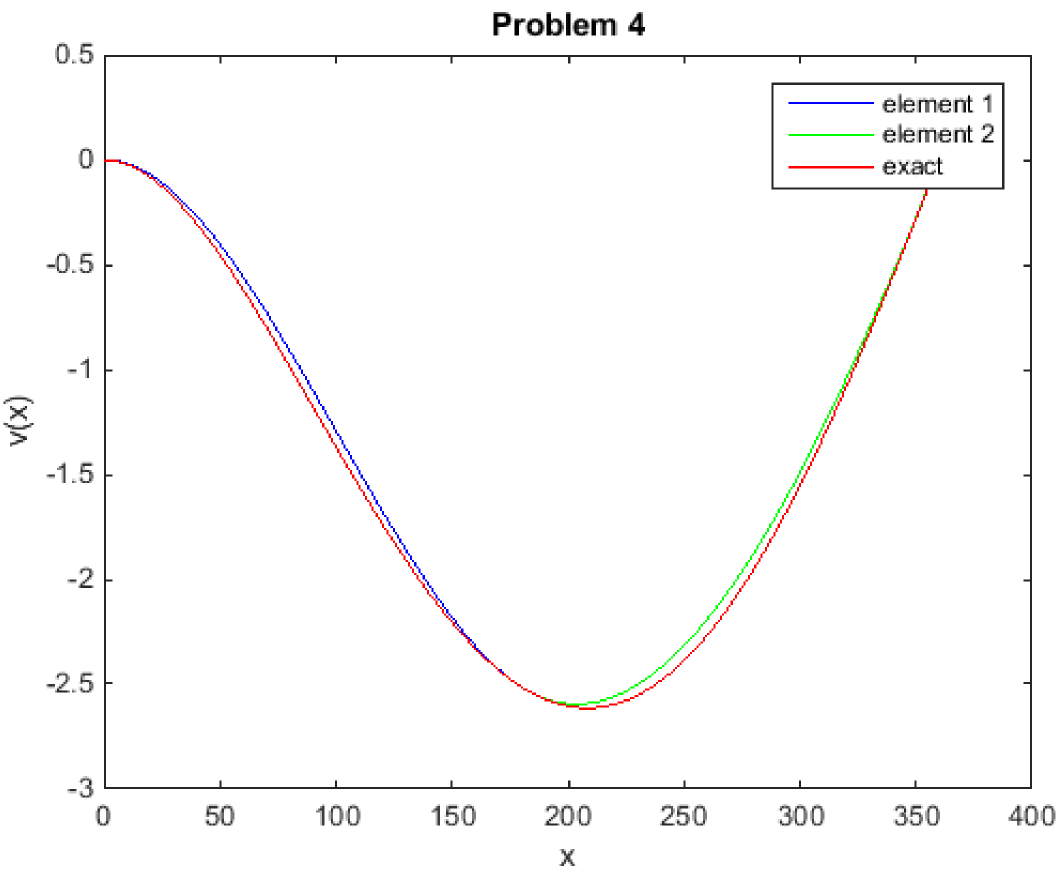
Problem 4
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
50
100
150
200
250
300
350
400
x
v(x)
element 1
element 2
exact