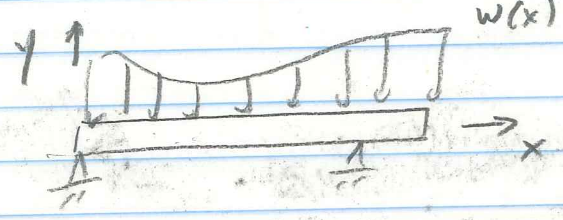
ch4a - video notes

Chapter 4 - Beam elements



V - transverse displacement (y-direction)
E - Young's modulus
I - moment of inertia of cross-section

Bending theory

$$M(x) = \underset{\uparrow \text{ flexural stiffness}}{EI} \; \underset{\uparrow \text{ curvature}}{\kappa} \approx \frac{d^2 v}{dx^2}$$

Shear force

$$V(x) = \frac{dM}{dx}$$

Distributed load

$$w(x) = -\frac{dV}{dx} = -\frac{d^2 M}{dx^2}$$

$$w(x) = -\frac{d^2}{d^2x}\left(EI \frac{d^2 v}{dx^2}\right)$$

If EI = constant

$$w(x) = -EI \frac{d^4 v}{dx^4}$$

If no distributed loads are applied (only concentrated forces & moment)

$$\boxed{\frac{d^4 v}{dx^4} = 0}$$

Solving

$$V(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

ch 4b - video notes

$$v(x) = [N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)] \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

where

$$N_1 = \tfrac{1}{L^3}(2x^3 - 3x^2L + L^3)$$

$$N_2 = \ldots$$

$$N_3 = \ldots$$

$$N_4 = \ldots$$

(see text / powerpoint)

$$f_{1y} = V = EI \left.\frac{d^3v}{dx^3}\right|_{x=0}$$

$$m_1 = -m = -EI \left.\frac{d^2v}{dx^2}\right|_{x=0}$$

$$f_{2y} = -V = -EI \left.\frac{d^3v}{dx^3}\right|_{x=L}$$

$$m_2 = m = EI \left.\frac{d^2v}{dx^2}\right|_{x=L}$$

Ex.
$$\frac{dN_1}{dx} = \frac{1}{L^3}(6x^2 - 6xL)$$

$$\frac{d^2N_1}{dx^2} = \frac{1}{L^3}(12x - 6L)$$

$$\frac{d^3N_1}{dx^3} = \frac{1}{L^3}(12)$$

Similarly

$$\frac{d^3N_2}{dx^3} = \tfrac{6}{L^2}$$

$$\frac{d^3N_3}{dx^3} = -\tfrac{12}{L^3}$$

$$\frac{d^3N_4}{dx^3} = \tfrac{6}{L^2}$$

$$\Rightarrow f_{1y} = \frac{EI}{L^3}[12 \quad 6L \quad -12 \quad 6L] \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{SYM} & & & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

↑ beam element equations

Ch4c - Video Notes

Potential energy - Beam element

Strain

$$v(x) = [N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)] \underbrace{\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}}_{\substack{\{d\} \\ 4\times1}}$$

$$\{\epsilon_x\} = -y \frac{d^2v}{dx^2}$$

$$\frac{d^2N_1}{dx^2} = \frac{12x - 6L}{L^3}, \quad \text{etc.}$$

$$\underset{1\times1}{\{\epsilon_x\}} = -y \underbrace{\left[\frac{12x-6L}{L^3} \quad \frac{6xL-4L^2}{L^3} \quad -\frac{12x+6L}{L^3} \quad \frac{6xL-2L^2}{L^3} \right]}_{\substack{[B(x)] \\ 1\times4}} \underbrace{\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}}_{\substack{\{d\} \\ 4\times1}}$$

$$\sigma_x = E \epsilon_x$$

E ← Young's Modulus

$$\underset{1\times1}{\{\sigma_x\}} = \underset{1\times1}{[D]} \underset{1\times1}{\{\epsilon_x\}}$$

$$\{\sigma_x\} = -y [D][B]\{d\}$$

$$\frac{1}{2}\sigma_x \epsilon_x = \frac{1}{2} \underset{1\times1}{\{\sigma_x\}^T} \underset{1\times1}{\{\epsilon_x\}}$$

$$\frac{1}{2}\sigma_x \epsilon_x = -\frac{1}{2} y^2 \{d\}^T [B]^T [D] [B] \{d\}$$

$[D]$ ← always symmetric

$$\int_X \iint_A \frac{1}{2}\sigma_x \epsilon_x \, dA \, dx$$

$$= \int_x \iint_A \frac{1}{2} y^2 \{d\}^T [B]^T \overset{E}{\cancel{[D]}} [B] \{d\} \, dA \, dx$$

$$= \int_{x=0}^{L} \frac{E}{2} \underbrace{\left[\iint_A y^2 \, dA \right]}_{\text{moment of inertia, } I} \{d\}^T [B]^T [B] \{d\} \, dx$$

$$= \frac{EI}{2} \int_0^L \{d\}^T [B]^T [B] \{d\} \, dx$$

$$= \frac{EI}{2} \{d\}^T \underbrace{\left[\int_0^L [B]^T [B] \, dx \right]}_{[K]} \{d\}$$

$$\int_0^L \underset{4\times 1}{\begin{bmatrix} \frac{12x-6L}{L^3} \\ (\quad) \\ (\quad) \\ (\quad) \end{bmatrix}} \underset{1\times 4}{\begin{bmatrix} (\quad)(\quad)(\quad)(\quad) \end{bmatrix}} \, dx$$

$$\int_0^L \underset{4\times 4}{\begin{bmatrix} \quad \end{bmatrix}} \, dx = \ldots$$

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ \text{SYM} & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}$$