HW # 8 - 7-6, 7, 11

7-6. Eq 7.2.2

$$e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad (1)$$
$$e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad (2)$$
$$e_{xy} = \frac{1+\nu}{E}\tau_{xy} \qquad (3)$$

1st eq., solve for $\sigma_x = E e_x + \nu\sigma_y$

Substitute into 2nd equation

$$e_y = \frac{1}{E}\left[\sigma_y - \nu(E e_x + \nu\sigma_y)\right]$$

$$E e_y = (1-\nu^2)\sigma_y - \nu E e_x$$

$$\boxed{\sigma_y = \frac{E}{1-\nu^2}(e_y + \nu e_x)}$$

Substitute into 1st equation

$$\sigma_x = E e_x + \nu\left[\left(\frac{E}{1-\nu^2}\right)(e_y + \nu e_x)\right]$$

$$\sigma_x = \frac{E}{1-\nu^2}\left[(1-\cancel{\nu^2})e_x + \nu e_y + \cancel{\nu^2}e_x\right]$$

$$\boxed{\sigma_x = \frac{E}{1-\nu^2}(e_x + \nu e_y)}$$

3rd equation, solve for τ_{xy}

$$\boxed{\tau_{xy} = \frac{E}{1+\nu}e_{xy}}$$

7-7 Equilibrium (2-D)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0$$

Hooke's law (plane stress - see prob. 7-6)

$$\sigma_x = \frac{E}{1-\nu^2}(e_x + \nu e_y)$$
$$\sigma_y = \frac{E}{1-\nu^2}(e_y + \nu e_x)$$
$$\tau_{xy} = \frac{E}{1+\nu} e_{xy}$$

Equilibrium in x-direction

$$\frac{\partial}{\partial x}\left[\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\cancelto{\mu}{\frac{E}{2(1+\nu)}}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + F_x = 0$$

$$\frac{E}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 v}{\partial x \partial y}\right) + \mu\left(\frac{\partial^2 u}{\partial y^2} + \underbrace{\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}}_{=0} + \frac{\partial^2 v}{\partial x \partial y}\right) + F_x = 0$$

$$\mu \nabla^2 u + \frac{E}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 v}{\partial x \partial y}\right) + \frac{E}{2(1+\nu)}\left(-\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + F_x = 0$$

$$\mu \nabla^2 u + \frac{E}{2(1+\nu)(1-\nu)}\left[2\frac{\partial^2 u}{\partial x^2} + 2\nu\frac{\partial^2 v}{\partial x \partial y} - (1-\nu)\frac{\partial^2 u}{\partial x^2} + (1-\nu)\frac{\partial^2 v}{\partial x \partial y}\right] + F_x = 0$$

$$\mu \nabla^2 u + \frac{E}{2\cancel{(1+\nu)}(1-\nu)}\left[\cancel{(1+\nu)}\frac{\partial^2 u}{\partial x^2} + \cancel{(1+\nu)}\frac{\partial^2 v}{\partial x \partial y}\right] + F_x = 0$$

$$\boxed{\mu \nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0}$$

Equilibrium in y-direction

$$\frac{\partial}{\partial x}\left[\cancelto{\mu}{\frac{E}{2(1+\nu)}}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial y} + \nu\frac{\partial u}{\partial x}\right)\right] + F_y = 0$$

$$\mu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+\underbrace{\frac{\partial^2 v}{\partial y^2}-\frac{\partial^2 v}{\partial y^2}}_{=0}\right)+\frac{E}{(1+\nu)(1-\nu)}\left(\frac{\partial^2 v}{\partial y^2}+\nu\frac{\partial^2 u}{\partial x\partial y}\right)+F_y=0$$

$$\mu\nabla^2 v+\frac{E}{2(1+\nu)(1-\nu)}\left[(1-\nu)\frac{\partial^2 u}{\partial x\partial y}-(1-\nu)\frac{\partial^2 v}{\partial y^2}+2\frac{\partial^2 v}{\partial y^2}+2\nu\frac{\partial^2 u}{\partial x\partial y}\right]+F_y=0$$

$$\mu\nabla^2 v+\frac{E}{2\cancel{(1+\nu)}(1-\nu)}\left[\cancel{(1+\nu)}\frac{\partial^2 u}{\partial x\partial y}+\cancel{(1+\nu)}\frac{\partial^2 v}{\partial y^2}\right]+F_y=0$$

$$\boxed{\mu\nabla^2 v+\frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y=0}$$

Compatibility (2-1)

$$\frac{\partial^2 e_x}{\partial y^2}+\frac{\partial^2 e_y}{\partial x^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y}$$

Hooke's law (plane stress)

$$e_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)$$
$$e_y=\frac{1}{E}(\sigma_y-\nu\sigma_x)$$
$$e_{xy}=\frac{1+\nu}{E}\tau_{xy}$$

$$\frac{1}{\cancel{E}}\left(\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2\sigma_y}{\partial y^2}\right)+\frac{1}{\cancel{E}}\left(\frac{\partial^2\sigma_y}{\partial x^2}-\nu\frac{\partial^2\sigma_x}{\partial x^2}\right)=\frac{2(1+\nu)}{\cancel{E}}\frac{\partial^2\tau_{xy}}{\partial x\partial y}$$

$$\frac{\partial^2\sigma_x}{\partial y^2}+\underbrace{\frac{\partial^2\sigma_x}{\partial x^2}-\frac{\partial^2\sigma_x}{\partial x^2}}_{=0}-\nu\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}+\underbrace{\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial^2\sigma_y}{\partial y^2}}_{=0}-\nu\frac{\partial^2\sigma_x}{\partial x^2}$$
$$=2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y}$$

$$\nabla^2\sigma_x+\nabla^2\sigma_y-(1+\nu)\frac{\partial^2\sigma_x}{\partial x^2}-(1+\nu)\frac{\partial^2\sigma_y}{\partial y^2}=2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y}$$

$$\frac{1}{1+\nu}\nabla^2(\sigma_x+\sigma_y)=2\frac{\partial^2\tau_{xy}}{\partial x\partial y}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}$$

From equilibrium

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_x=0\quad\Rightarrow\quad\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\tau_{xy}}{\partial x\partial y}=-\frac{\partial F_x}{\partial x}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0 \Rightarrow \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2} = -\frac{\partial F_y}{\partial y}$$

Hence

$$\frac{1}{1+\nu} \nabla^2 (\sigma_x + \sigma_y) = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right)$$

$$\boxed{\nabla^2 (\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right)}$$

7-11 Eq 7.1.5 (plane strain)

$$\mu \nabla^2 u + (\lambda + \mu) \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + F_x = 0$$

$$\mu \nabla^2 v + (\lambda + \mu) \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + F_y = 0$$

Table 4-1

$$\lambda + \mu = \frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{2(1+\nu)}$$

$$= \frac{\cancel{2E\nu} + (1-\cancel{2\nu})E}{2(1+\nu)(1-2\nu)}$$

$$= \frac{E}{2(1+\nu)(1-2\nu)}$$

Plane strain to plane stress (Table 7-1)

$$E \Rightarrow \frac{E(1+2\nu)}{(1+\nu)^2}, \qquad \nu \Rightarrow \frac{\nu}{1+\nu}$$

$$\frac{E}{2(1+\nu)(1-2\nu)} \Rightarrow \frac{\frac{E(1+2\nu)}{(1+\nu)^2}}{2\left(1+\frac{\nu}{1+\nu}\right)\left(1-\frac{2\nu}{1+\nu}\right)}$$

$$= \frac{E\cancel{(1+2\nu)}}{2\cancel{(1+\nu)^2}\left(\frac{\cancel{1+2\nu}}{\cancel{1+\nu}}\right)\left(\frac{1+\nu-2\nu}{\cancel{1+\nu}}\right)}$$

$$= \frac{E}{2(1-\nu)}$$

Hence, for plane stress

$$\boxed{\begin{aligned} \mu \nabla^2 u + \frac{E}{2(1-\nu)} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0 \\ \mu \nabla^2 v + \frac{E}{2(1-\nu)} \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0 \end{aligned}}$$

Eq 7.1.7

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

Plane strain to plane stress

$$\nu \Rightarrow \frac{\nu}{1+\nu}$$

$$\frac{1}{1-\nu} \Rightarrow \frac{1}{1-\frac{\nu}{1+\nu}} = \frac{1+\nu}{1+\nu-\nu} = 1+\nu$$

$$\boxed{\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)}$$

Eq 7.2.5 (plane stress)

$$\mu \nabla^2 u + \frac{E}{2(1-\nu)} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

$$\mu \nabla^2 v + \frac{E}{2(1-\nu)} \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

plane stress to plane strain (Table 7.1)

$$E \Rightarrow \frac{E}{1-\nu^2}, \quad \nu \Rightarrow \frac{\nu}{1-\nu}$$

$$\frac{E}{2(1-\nu)} \Rightarrow \frac{E/_{1-\nu^2}}{2\left(1-\frac{\nu}{1-\nu}\right)} \frac{(1-\nu)}{(1-\nu)} = \frac{E/_{1+\nu}}{2(1-\nu-\nu)}$$

$$= \frac{E}{2(1+\nu)(1-2\nu)} = \lambda + \mu \text{ (from Table 4-1)}$$

Hence, for plane strain

$$\boxed{\begin{aligned} &\mu \nabla^2 u + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x = 0 \\ &\mu \nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y = 0 \end{aligned}}$$

Eq 7.2.7 (plane stress)

$$\nabla^2(\sigma_x+\sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$

Plane stress to plane strain

$$\nu \Rightarrow \frac{\nu}{1-\nu}$$

$$1+\nu \Rightarrow 1+\frac{\nu}{1-\nu} = \frac{1-\nu+\nu}{1-\nu} = \frac{1}{1-\nu}$$

$$\boxed{\nabla^2(\sigma_x+\sigma_y) = -\left(\frac{1}{1-\nu}\right)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)}$$