Homework Problems 1-1a, 2a, 5, and 6a

1-1. For the given matrix/vector pairs, compute the following quantities: a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_ib_j, b_ib_i. For each case, point out whether the result is a scalar, vector, or matrix. Note that $a_{ij}b_j$ is actually the matrix product $[\boldsymbol{a}]\{\boldsymbol{b}\}$, while $a_{ij}a_{jk}$ is the product $[\boldsymbol{a}][\boldsymbol{a}]$.

$$\text{(a) } a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

1-2. Use the decomposition result (1.2.10) to express a_{ij} from Exercise 1-1 in terms of the sum of symmetric and antisymmetric matrices. Verify that $a_{(ij)}$ and $a_{[ij]}$ satisfy the conditions given in the last paragraph of Section 1.2.

Last paragraph of Section 1.2:

A useful identity may be written as

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) = a_{(ij)} + a_{[ij]} \tag{1.2.10}$$

The first term $a_{(ij)} = 1/2(a_{ij} + a_{ji})$ is symmetric, while the second term $a_{[ij]} = 1/2(a_{ij} - a_{ji})$ is antisymmetric, and thus an arbitrary symbol a_{ij} can be expressed as the sum of symmetric and antisymmetric pieces. Note that if a_{ij} is symmetric, it has only six independent components. On the other hand, if a_{ij} is antisymmetric, its diagonal terms a_{ii} (no sum on i) must be zero, and it has only three independent components. Note that since $a_{[ij]}$ has only three independent components, it can be related to a quantity with a single index, for example, a_i (see Exercise 1-15).

1-5. Formally expand the expression (1.3.4) for the determinant and justify that either index notation form yields a result that matches the traditional form for $\det[a_{ij}]$.

Equation 1.3.3 and 1.3.4

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{otherwise} \end{cases} \tag{1.3.3}$$

$$\det[a_{ij}] = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{ijk}a_{i1}a_{j2}a_{k3} \tag{1.3.4}$$

1-6. Determine the components of the vector b_i and matrix a_{ij} given in Exercise 1-1 in a new coordinate system found through a rotation of 45° ($\pi/4$ radians) about the x_1-axis. The rotation direction follows the positive sense presented in Example 1-2.

Example 1-2

EXAMPLE 1-2: Transformation Examples

The components of a first- and second-order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \; a_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

Determine the components of each tensor in a new coordinate system found through a rotation of 60° ($\pi/6$ radians) about the x_3-axis. Choose a counterclockwise rotation when viewing down the negative x_3-axis (see Figure 1-2).

The original and primed coordinate systems shown in Figure 1-2 establish the angles between the various axes. The solution starts by determining the rotation matrix for this case:

$$Q_{ij} = \begin{bmatrix} \cos 60^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 150^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

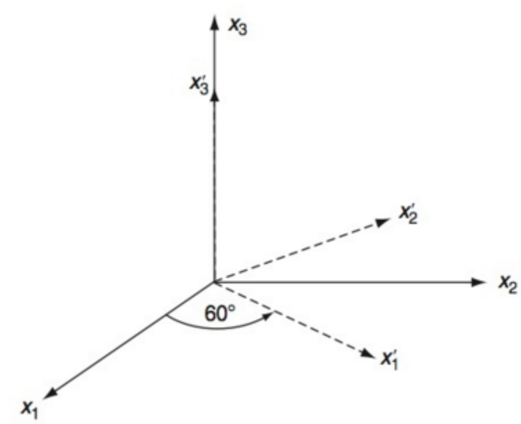

FIGURE 1-2 *Coordinate transformation.*

The transformation for the vector quantity follows from equation $(1.5.1)_2$:

$$a_i' = Q_{ij} a_j = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 + 2\sqrt{3} \\ 2 - \sqrt{3}/2 \\ 2 \end{bmatrix}$$

and the second-order tensor (matrix) transforms according to $(1.5.1)_3$:

$$
\begin{aligned}
a'_{ij} = Q_{ip}Q_{jq}a_{pq} &= \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} 7/4 & \sqrt{3}/4 & 3/2+\sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1-3\sqrt{3}/2 \\ 3/2+\sqrt{3} & 1-3\sqrt{3}/2 & 4 \end{bmatrix}
\end{aligned}
$$

where $[\ ]^T$ indicates transpose (defined in Section 1.7). Although simple transformations can be worked out by hand, for more general cases it is more convenient to use a computational scheme to evaluate the necessary matrix multiplications required in the transformation laws (1.5.1). MATLAB software is ideally suited to carry out such calculations, and an example program to evaluate the transformation of second-order tensors is given in Example C-1 in Appendix C.