HW 1 – 1-1b, 2(a or b), 5 Solution

1-1b

$$a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \qquad b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 2 + 2 + 5 \quad (\text{scalar})$$

$$\begin{aligned} a_{ij}a_{ij} &= a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + \\ &\quad a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + \\ &\quad a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33} \\ &= 1 + 4 + 0 + 0 + 4 + 1 + 0 + 16 + 4 \\ &= 30 \quad (\text{scalar}) \end{aligned}$$

$$a_{ij}a_{jk} = \begin{bmatrix} (a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{31}) & (a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32}) & (\quad) \\ (\quad) & (\quad) & (\quad) \\ (\quad) & (\quad) & (\quad) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix} \quad (\text{matrix})$$

$$a_{ij}b_j = \begin{cases} a_{11}b_1 + a_{12}b_2 + a_{13}b_3, & i = 1 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3, & i = 2 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3, & i = 3 \end{cases}$$

$$= \underset{3\times 3}{[a]}\underset{3\times 1}{\{b\}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 3 \\ 6 \end{Bmatrix} \quad (\text{vector}$$

$$a_{ij}\, b_i\, b_j = a_{ij}\, b_j\, b_i = c_i\, b_i$$

where $c_i = a_{ij}\, b_j = \begin{Bmatrix} 4 \\ 3 \\ 6 \end{Bmatrix}$ (from above)

$$c_i b_i = c_1 b_1 + c_2 b_2 + c_3 b_3$$
$$= (4)(2) + (3)(1) + (6)(1) = 17 \quad \text{(scalar)}$$

$$b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{(matrix}$$

$$b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$$
$$= 4 + 1 + 1$$
$$= 6 \quad \text{(scalar)}$$

1-2a

$$[a] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$a_{(ij)} = \tfrac{1}{2}(a_{ij} + a_{ji})$$
$$= \tfrac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 4 & 3/2 \\ 1/2 & 3/2 & 1 \end{bmatrix} \quad \text{(symmetric)}$$

$$a_{[ij]} = \tfrac{1}{2}(a_{ij} - a_{ji}) = \begin{bmatrix} 0 & 1/2 & 1/2 \\ -1/2 & 0 & 1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix} \quad \text{(anti-symmetric)}$$

$$a_{(ij)} + a_{[ij]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = a_{ij} \quad \checkmark$$

1-2b

$$[a] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$$a_{(ij)} = \frac{1}{2}(a_{ij} + a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 5/2 \\ 0 & 5/2 & 2 \end{bmatrix}$$

$$a_{[ij]} = \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -3/2 \\ 0 & 3/2 & 0 \end{bmatrix}$$

$$a_{(ij)} + a_{[ij]} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = a_{ij} \checkmark$$

1-5 $\det(a_{ij}) = e_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}$

Sum on i & j, retaining non-zero terms

$$\det(a_{ij}) = \overset{+1}{\cancel{e_{123}}}\, a_{11} a_{22} a_{33} + \overset{+1}{\cancel{e_{231}}}\, a_{12} a_{23} a_{31} + \overset{+1}{\cancel{e_{312}}}\, a_{13} a_{21} a_{32} + \overset{-1}{\cancel{e_{321}}}\, a_{13} a_{22} a_{31} + \overset{-1}{\cancel{e_{132}}}\, a_{11} a_{23} a_{32} + \overset{-1}{\cancel{e_{213}}}\, a_{12} a_{21} a_{33}$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \checkmark$$