MCE 571 - Exam 1 Solution — Fall 2021

1. $\underset{\sim}{u} = x_1 x_2^3 \underset{\sim}{e}_1 + 4 x_2 x_3^2 \underset{\sim}{e}_2 + x_1^2 \underset{\sim}{e}_3$

a) $\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \frac{\partial}{\partial x_i} \underset{\sim}{e}_i \cdot u_j \underset{\sim}{e}_j = \frac{\partial u_j}{\partial x_i} \delta_{ij} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = x_2^3 + 4x_3^2$ (scalar)

b) $\underset{\sim}{\nabla} \times \underset{\sim}{u} = \frac{\partial}{\partial x_i} \underset{\sim}{e}_i \times \frac{\partial}{\partial x_j} \underset{\sim}{e}_j = \frac{\partial u_j}{\partial x_i} e_{ijk} \underset{\sim}{e}_k = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\underset{\sim}{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\underset{\sim}{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\underset{\sim}{e}_3$

$= -8 x_2 x_3 \underset{\sim}{e}_1 - 2x_1 \underset{\sim}{e}_2 - 3 x_1 x_2^2 \underset{\sim}{e}_3$ (vector)

c) $\underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} = \frac{\partial}{\partial x_i} \underset{\sim}{e}_i \cdot \frac{\partial}{\partial x_j} \underset{\sim}{e}_j (u_k \underset{\sim}{e}_k) = \frac{\partial^2}{\partial x_i \partial x_i} u_k \underset{\sim}{e}_k$

$= \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right)\underset{\sim}{e}_1 + \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right)\underset{\sim}{e}_2 + \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right)\underset{\sim}{e}_3$

$= 6 x_1 x_2 \underset{\sim}{e}_1 + 8 x_2 \underset{\sim}{e}_2 + 2 \underset{\sim}{e}_3$ (vector)

d) $\underset{\sim}{\nabla} \underset{\sim}{u} = \frac{\partial}{\partial x_i} \underset{\sim}{e}_i \, u_j \underset{\sim}{e}_j = \frac{\partial u_j}{\partial x_i} \underset{\sim}{e}_i \underset{\sim}{e}_j = \begin{bmatrix} x_2^3 & 0 & 2x_1 \\ 3x_1x_2^2 & 4x_3^2 & 0 \\ 0 & 8x_2x_3 & 0 \end{bmatrix}$ (matrix)

alternate

$\underset{\sim}{\nabla} \underset{\sim}{u} = u_{i,j} \underset{\sim}{e}_i \underset{\sim}{e}_j = \begin{bmatrix} x_2^3 & 3x_1x_2^2 & 0 \\ 0 & 4x_3^2 & 8x_2x_3 \\ 2x_1 & 0 & 0 \end{bmatrix}$

2. $\sigma_{ij} = \begin{bmatrix} 6 & -3 & 2 \\ -3 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$, $\underset{\sim}{a} = \underset{\sim}{e}_1 + \underset{\sim}{e}_2 - \underset{\sim}{e}_3 \Rightarrow \underset{\sim}{n} = \frac{1}{\sqrt{3}} \underset{\sim}{e}_1 + \frac{1}{\sqrt{3}} \underset{\sim}{e}_2 - \frac{1}{\sqrt{3}} \underset{\sim}{e}_3$

a) $\tilde{\sigma}_{ij} = \frac{1}{3} \sigma_{kk} \delta_{ij} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

b) $\hat{\sigma}_{ij} = \sigma_{ij} - \tilde{\sigma}_{ij} = \begin{bmatrix} 2 & -3 & 2 \\ -3 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

c) $\hat{\sigma}_{ij} \hat{\sigma}_{ij} = 4 + 9 + 4 + 9 + 4 + 4 = \boxed{34}$

d) $\tau_{oct} = \frac{1}{3} (2 I_1^2 - 6 I_2)^{1/2}$

$I_1 = \sigma_{kk} = 12$, $I_2 = \begin{vmatrix} 6 & -3 \\ -3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 3 + 8 + 20 = 31$

$\tau_{oct} = \frac{1}{3} [288 - 186]^{1/2} = \boxed{3.367}$

e) $\sigma_{vm} = \frac{3}{\sqrt{2}} \tau_{oct} = \boxed{7.141}$

f) $\underset{\sim}{T}^{n} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \begin{bmatrix} 6 & -3 & 2 \\ -3 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -2/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} .5774 \\ -.5774 \\ -1.1547 \end{Bmatrix}$

g) $N = \underset{\sim}{T}^{n} \cdot \underset{\sim}{n} = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{-1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{-2}{\sqrt{3}}\right)\left(\frac{-1}{\sqrt{3}}\right) = 2/3 = 0.667$

h) $S = \sqrt{|\underset{\wedge}{T}^{n}|^2 + N^2} = \sqrt{\left(\underbrace{\tfrac{1}{3} + \cancel{\tfrac{1}{3}}}_{2} + \tfrac{4}{3}\right) - \tfrac{4}{9}} = \sqrt{\tfrac{14}{9}} = \frac{\sqrt{14}}{3} = 1.247$

3. $\sigma_{ij} = \begin{bmatrix} 2T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$

$e_x = \frac{1+\nu}{E}(2T) - \frac{\nu}{E}(3T) = \left(\frac{2-\nu}{E}\right)T$

$e_y = \frac{1+\nu}{E}(T) - \frac{\nu}{E}(3T) = \left(\frac{1-2\nu}{E}\right)T$

$\left.\begin{aligned} K_1 &= T/e_x = \frac{E}{2-\nu} \\ K_2 &= T/e_y = \frac{E}{1-2\nu} \end{aligned}\right\}$ $E = (2-\nu)K_1 = (1-2\nu)K_2$

$\nu(2K_2 - K_1) = K_2 - 2K_1$

$$\boxed{\nu = \frac{K_2 - 2K_1}{2K_2 - K_1}}$$

$E = (2-\nu)K_1 = \frac{4K_2 - \cancel{2K_1} - K_2 + \cancel{2K_1}}{2K_2 - K_1} K_1$

$$\boxed{E = \frac{3K_1K_2}{2K_2 - K_1}}$$