MCE 571 - Exam 1 Solution Fall 2021

1. $\mathbf{u} = x_1 x_2^3 \mathbf{e}_1 + 4 x_2 x_3^2 \mathbf{e}_2 + x_1^2 \mathbf{e}_3$

a) $\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x_i} \mathbf{e}_i \cdot u_j \mathbf{e}_j = \frac{\partial u_j}{\partial x_i} \delta_{ij} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = x_2^3 + 4x_3^2$ (scalar)

b) $\nabla \times \mathbf{u} = \frac{\partial}{\partial x_i} \mathbf{e}_i \times \frac{\partial}{\partial x_j} \mathbf{e}_j = \frac{\partial u_j}{\partial x_i} e_{ijk} \mathbf{e}_k = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\mathbf{e}_2$
$+ \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{e}_3$

$= -8 x_2 x_3 \mathbf{e}_1 - 2 x_1 \mathbf{e}_2 - 3 x_1 x_2^2 \mathbf{e}_3$ (vector)

c) $\nabla \cdot \nabla \mathbf{u} = \frac{\partial}{\partial x_i} \mathbf{e}_i \cdot \frac{\partial}{\partial x_j} \mathbf{e}_j (u_k \mathbf{e}_k) = \frac{\partial^2}{\partial x_i \partial x_i} u_k \mathbf{e}_k$

$= \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right)\mathbf{e}_1 + \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right)\mathbf{e}_2 + \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right)\mathbf{e}_3$

$= 6 x_1 x_2 \mathbf{e}_1 + 8 x_2 \mathbf{e}_2 + 2 \mathbf{e}_3$ (vector)

d) $\nabla \mathbf{u} = \frac{\partial}{\partial x_i} \mathbf{e}_i u_j \mathbf{e}_j = \frac{\partial u_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j = \begin{bmatrix} x_2^3 & 0 & 2x_1 \\ 3x_1 x_2^2 & 4x_3^2 & 0 \\ 0 & 8x_2 x_3 & 0 \end{bmatrix}$ (matrix)

alternate

$\nabla \mathbf{u} = u_{i,j} \mathbf{e}_i \mathbf{e}_j = \begin{bmatrix} x_2^3 & 3x_1 x_2^2 & 0 \\ 0 & 4x_3^2 & 8x_2 x_3 \\ 2x_1 & 0 & 0 \end{bmatrix}$

2. $\sigma_{ij} = \begin{bmatrix} 6 & -3 & 2 \\ -3 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$, $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 \Rightarrow \mathbf{n} = \frac{1}{\sqrt{3}}\mathbf{e}_1 + \frac{1}{\sqrt{3}}\mathbf{e}_2 - \frac{1}{\sqrt{3}}\mathbf{e}_3$

a) $\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

b) $\hat{\sigma}_{ij} = \sigma_{ij} - \tilde{\sigma}_{ij} = \begin{bmatrix} 2 & -3 & 2 \\ -3 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

c) $\hat{\sigma}_{ij}\hat{\sigma}_{ij} = 4 + 9 + 4 + 9 + 4 + 4 = \boxed{34}$

d) $\tau_{oct} = \frac{1}{3}(2I_1^2 - 6I_2)^{1/2}$

$I_1 = \sigma_{kk} = 12$, $I_2 = \begin{vmatrix} 6 & -3 \\ -3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 3 + 8 + 20 = 31$

$\tau_{oct} = \frac{1}{3}[288 - 186]^{1/2} = \boxed{3.367}$

e) $\sigma_{vm} = \frac{3}{\sqrt{2}}\tau_{oct} = \boxed{7.141}$

2. with $\underset{\sim}{n} = \underset{\sim}{e}_1 + \underset{\sim}{e}_2 - \underset{\sim}{e}_3$ (Forgot to normalize)

f) $\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \begin{bmatrix} 6 & -3 & 2 \\ -3 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ -2 \end{Bmatrix}$

g) $N = \underset{\sim}{T}^n \cdot \underset{\sim}{n} = (1)(1) + (1)(-1) + (-1)(-2) = 2$

h) $|\underset{\sim}{T}^n| = \sqrt{1+1+4} = \sqrt{6}$

$S = \sqrt{|T^n|^2 - N^2} = \sqrt{6-4} = \sqrt{2}$

$$E = -3k_2 \left(1 - \cancel{\frac{k_2 k_2}{k_1}}\right)^{-1}$$

$$= -3k_2 \left(\frac{k_1}{k_1 - 2k_2}\right)$$

$$= \frac{-3k_1 k_2}{k_1 - 2k_1}$$

$$= \frac{3k_1 k_2}{2k_2 - k_1}$$