Hw # 9 - 8-1, 3 & 4

8-1 $\phi = A_{40} x^4 + A_{22} x^2 y^2 + A_{04} y^4$

Eq 8.1.6, $m = n = 2$

$$(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)(n)(n-1)A_{mn} + (n+2)(n+1)n(n-1)A_{m-2,n+2} = 0$$

$$(\cancel{4})(3)(\cancel{2})(1)A_{40} + \cancel{2}(\cancel{2})(1)(\cancel{2})(1)A_{22} + (\cancel{4})(3)(\cancel{2})(1)A_{04} = 0$$

$$3A_{40} + A_{22} + 3A_{04} = 0$$

Alternate

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\frac{\partial \phi}{\partial x} = 4A_{40}x^3 + 2A_{22}xy^2$$

$$\frac{\partial^2 \phi}{\partial x^2} = 12A_{40}x^2 + 2A_{22}y^2$$

$$\frac{\partial^3 \phi}{\partial x^3} = 24A_{40}x$$

$$\frac{\partial^4 \phi}{\partial x^4} = 24A_{40}$$

$$\frac{\partial^3 \phi}{\partial x^2 \partial y} = 4A_{22}y$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 4A_{22}$$

$$\frac{\partial \phi}{\partial y} = 2A_{22}x^2y + 4A_{04}y^3$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2A_{22}x^2 + 12A_{04}y^2$$

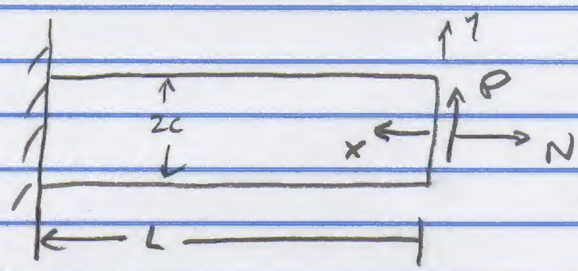
$$\frac{\partial^3 \phi}{\partial y^3} = 24A_{04}y$$

$$\frac{\partial^4 \phi}{\partial y^4} = 24A_{04}$$

$$\nabla^4 \phi = 24A_{40} + 8A_{22} + 24A_{04} = 0$$

$$3A_{40} + A_{22} + 3A_{04} = 0$$

8-3



$$\sigma_x = -\frac{3Pxy}{2c^3} + \frac{N}{2c}, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

$$e_x = \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}\left(\frac{-3Pxy}{2c^3} + \frac{N}{2c}\right)$$

$$u(x,y) = \frac{1}{E}\left(-\frac{3Px^2y}{4c^3} + \frac{N}{2c}x\right) + f(y)$$

$$e_y = \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -\frac{\nu}{E}\left(\frac{-3Pxy}{2c^3} + \frac{N}{2c}\right)$$

$$v(x,y) = -\frac{\nu}{E}\left(-\frac{3Pxy^2}{4c^3} + \frac{N}{2c}y\right) + g(x)$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1+\nu}{E}\tau_{xy} = -\frac{3(1+\nu)P}{4Ec}\left(1 - \frac{y^2}{c^2}\right)$$

$$\cancel{\frac{1}{2}}\left[\frac{1}{\cancel{E}}\left(-\frac{3Px^2}{4c^3} + \frac{3\nu P y^2}{4c^3}\right) + \frac{df}{dy} + \frac{dg}{dx}\right] = -\frac{3(1+\nu)P}{2\cancel{4}Ec}\left(1 - \frac{y^2}{c^2}\right)$$

$$-\frac{3Px^2}{4Ec^3} + \frac{dg}{dx} = -\frac{3(1+\nu)P}{2Ec}\left(1 - \frac{y^2}{c^2}\right) - \frac{3\nu P y^2}{4Ec^3} - \frac{df}{dy} = C_1$$

$$\frac{dg}{dx} = \frac{3Px^2}{4Ec^3} + C_1 \Rightarrow g(x) = \frac{Px^3}{4Ec^3} + C_1 x + C_2$$

$$\frac{df}{dy} = -\frac{3(1+\nu)P}{2Ec} + \left[\frac{\overset{6}{\cancel{3}}(1+\nu)P}{4\cancel{2}Ec^3} - \frac{3\nu P}{4Ec^3}\right]y - C_1$$

$$\frac{df}{dy} = -\frac{3(1+\nu)P}{2Ec} + \frac{(6+3\nu)P}{4Ec^3}y^2 - C_1$$

$$f(y) = \frac{(2+\nu)P}{4Ec^3}y^3 - \frac{3(1+\nu)P}{2Ec}y - C_1 y + C_3$$

$$u(0,0) = u_0 = f(0)$$

$$f(0) = C_3 = u_0$$

$$v(0,0) = v_0 = g(0)$$

$$g(0) = C_2 = v_0$$

Rotation $\omega = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$

$$\omega(0,0) = \omega_0 = \frac{1}{2}\left(\frac{df}{dy} - \frac{dg}{dx}\right)$$

$$\omega_0 = \frac{1}{2}\left[C_1 - (-C_1)\right] = C_1$$

$$u(x,y) = \frac{1}{E}\left(-\frac{3Px^2y}{4c^3} + \frac{Nx}{2c} + \frac{(2+\nu)P}{4c^3}y^3 - \frac{3(1+\nu)Py}{2Ec}\right)$$

$$-\omega_0 y + u_0$$

$$v(x,y) = \frac{1}{E}\left(\frac{3\nu Pxy^2}{4c^3} - \frac{\nu Ny}{2c} + \frac{Px^3}{4c^3}\right) + \omega_0 x + v_0$$

B.C.

$$u(L,0) = 0 = \frac{NL}{2Ec} + u_0 \Rightarrow u_0 = -\frac{NL}{2Ec}$$

$$v(L,0) = 0 = \frac{PL^3}{4Ec^3} + \omega_0 L + v_0$$

$$\frac{\partial v}{\partial x}(L,0) = 0 = \frac{3PL^2}{4Ec^3} + \omega_0 \Rightarrow \omega_0 = -\frac{3PL^2}{4Ec^3}$$

$$v_0 = -\frac{PL^3}{4Ec^2} + \frac{3PL^3}{4Ec^2} = \frac{PL^3}{2Ec^2}$$

$$u(x,y) = \frac{1}{E}\left(-\frac{3Px^2y}{3c^3} + \frac{Nx}{2c} + \frac{(2+\nu)P}{4c^3}y^3 - \frac{3(1+\nu)Py}{2Ec}\right)$$

$$+ \frac{3PL^2y}{4Ec^2} - \frac{NL}{2Ec}$$

$$v(x,y) = \frac{1}{E}\left(\frac{3\nu Pxy^2}{4c^3} - \frac{\nu Ny}{2c} + \frac{Px^3}{4c^2}\right)$$

$$-\frac{3PL^2x}{4Ec^3} + \frac{PL^3}{2Ec^2}$$

Compare to beam theory with $N=0$, $y=0$ (mid-plane)

$$v(x,0) = \frac{Px^3}{4Ec^3} - \frac{3PL^2x}{4Ec^2} + \frac{PL^3}{2Ec^2}$$

$$= \frac{P}{4Ec^3}\left(x^3 - 3L^2x + 2L^3\right)$$

(agrees with beam theory)

8-4

$$\phi(x) = C_1 x^2 + C_2 x^2 y + C_3 y^3 + C_4 y^5 + C_5 x^2 y^3$$

$$\partial\phi/\partial x = 2C_1 x + 2C_2 x y + 2C_5 x y^3$$

$$\partial^2\phi/\partial x^2 = 2C_1 + 2C_2 y + 2C_5 y^3$$

$$\partial^3\phi/\partial x^3 = \partial^4\phi/\partial x^4 = 0$$

$$\frac{\partial^3\phi}{\partial x^2 \partial y} = 2C_2 + 6C_5 y^2$$

$$\frac{\partial^4\phi}{\partial x^2 \partial y^2} = 12 C_5 y$$

$$\partial\phi/\partial y = C_2 x^2 + 3C_3 y^2 + 5C_4 y^4 + 3C_5 x^2 y^2$$

$$\frac{\partial^2\phi}{\partial y^2} = 6C_3 y + 20 C_4 y^3 + 6C_5 x^2 y$$

$$\partial^3\phi/\partial y^3 = 6C_3 + 60 C_4 y^2 + 6 C_5 x^2$$

$$\partial^4\phi/\partial y^4 = 120 C_4 y$$

$$\nabla^4\phi = 24 C_5 y + 120 C_4 y = 0$$

$$C_5 + 5C_4 = 0 \Rightarrow C_5 = -5C_4$$

Stresses

$$\sigma_x = \partial^2\phi/\partial y^2 = 6C_3 y + 20C_4 y^3 - 6 \underset{(-5C_4)}{\cancel{C_5}} x^2 y$$

$$= 6C_3 y + C_4 (20y^3 - 30x^2 y)$$

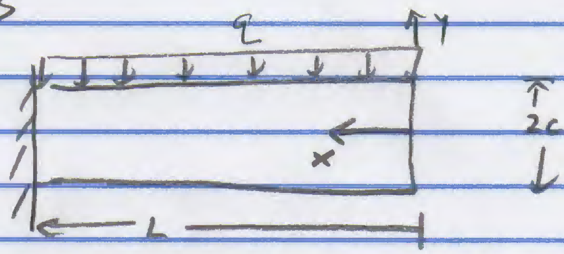
$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 2C_1 + 2C_2 y + 2\overset{-5C_4}{\cancel{C_5}}\, y^3$$

$$= 2C_1 + 2C_2 y - 10\, C_4\, y^3$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} = -2C_2 x - 6\overset{-5C_4}{\cancel{C_5}}\, x y^2$$

$$= -2\, C_2 x + 30\, C_4\, x y^2$$

B.C.'s



$$\tau_{xy}(x, c) = 0 = -\cancel{2}C_2\cancel{x} + 3\overset{15}{\cancel{0}}\, C_4 \cancel{x}\, c^2$$

$$C_2 = 15\, C_4\, c^2 \qquad (1)$$

$$\sigma_y(x, c) = -q = 2C_1 + 2C_2 c - \overset{2}{\cancel{10}}\, \overset{(C_2/15c^2)}{\cancel{C_4}}\, \overset{3}{\cancel{c^2}}\, c$$

$$-q = 2C_1 + \tfrac{4}{3}\, C_2\, c \qquad (2)$$

$$\tau_{xy}(x, -c) = 0 = -2\, C_2\, x + 3\, C_4\, x c^2 \Rightarrow C_2 = 15 C_4 c^2$$

$$\sigma_y(x, -c) = 0 = \cancel{2}C_1 - \cancel{2}C_2 c + \tfrac{1}{3}\cancel{10}\, \overset{C_2/15c^2}{\cancel{C_4}}\, c^{\cancel{3}}$$

$$C_1 = C_2 c\left(1 - \tfrac{1}{3}\right) = \frac{2C_2 c}{3}$$

$$C_1 = \frac{2c}{\cancel{3}}\left(\overset{5}{\cancel{15}}\, C_4 c^2\right) = 10\, C_4\, c^3 \qquad (3)$$

At $x = 0$

$$\int_{-c}^{c} \sigma_x(0,y)\,dy = 0 = \int_{-c}^{c} 6C_3 y + C_4(20y^3)\,dy$$

$$= \left[\frac{C_3 y^2}{3} + 5C_4 y^4\right]_{-c}^{c}$$

$$= 0 \quad ✓$$

$$\int_{-c}^{c} \tau_{xy}(0,y)\,dy = 0 = \int_{-c}^{c} (0) = 0 \quad ✓$$

$$\int_{-c}^{c} \sigma_x(0,y)\,y\,dy = 0 = \int_{-c}^{c} 6C_3 y^2 + 20C_4 y^4\,dy$$

$$0 = \left[2C_3 y^3 + 4C_4 y^5\right]_{-c}^{c}$$

$$0 = \cancel{4}C_3\cancel{c^3} + \cancel{8}^{\,2}C_4 c^{\cancel{5}\,2}$$

$$C_3 = -2C_4 c^2 \quad (4)$$

Eq. 2

$$-q = 2C_1 + \tfrac{4}{3}C_2 c$$

$$= C_4\left(20c^3 + \cancel{\tfrac{4}{3}\,15}^{\,20}c^3\right)$$

$$C_4 = -q/40c^3$$

$$C_1 = -q/4,\quad C_2 = -\frac{3q}{8c},\quad C_3 = q/20c$$

Stresses

$$\sigma_x = 6C_3 y + C_4(20y^3 - 30x^2y)$$

$$= \frac{3qy}{10c} - \frac{q}{40c^3}(20y^3 - 30x^2y)$$

$$= \frac{3qy}{10c} - \frac{q}{4c^3}(2y^3 - 3x^2y)$$

$$\sigma_y = 2C_1 + 2C_2 y - 10C_4 y^3$$

$$= -\frac{q}{2} - \frac{3qy}{4c} + \frac{qy^3}{4c^3}$$

$$\tau_{xy} = -2C_2 x + 30C_4 xy^2$$

$$= \frac{3q}{4c}x - \frac{3q}{4c^3}xy^2$$

at $x = L$

$$\int_{-c}^{c} \sigma_x(L,y)\,dy = \int_{-c}^{c}\left[\frac{3qy}{10c} - \frac{q}{4c^3}(2y^3 - 3x^2y)\right]dy$$

$$= 0 \checkmark$$

$$\int_{-c}^{c} \tau_{xy}(L,y)\,dy = \int_{-c}^{c} \frac{3qL}{4c} - \frac{3qLy^2}{4c^3}\,dy$$

$$= \frac{3qL}{4\cancel{c}}\left[2\cancel{c} - \frac{(2\cancel{c^3})}{3\cancel{c^2}}\right]^{4/3}$$

$$= qL \checkmark$$

$$\int_{-c}^{c} \sigma_x(L,y)\,y\,dy = \int_{-c}^{c} \frac{3qy^2}{10c} - \frac{q}{4c^3}\left(2y^4 - 3L^2y^2\right)dy$$

$$= \frac{3q}{10c}\left(\frac{2c^3}{3}\right) - \frac{q}{4c^3}\left(\frac{2c^5}{5} - L^2(2c^3)\right)$$

$$= \frac{qc^2}{10} - \frac{qc^2}{10} + \frac{qL^2}{2}$$

$$= qL^2/2 \quad \checkmark$$

9 - 10

8-1, 3 & 4

8-1.

$$\phi = A_{40}x^4 + A_{22}x^2y^2 + A_{04}y^4$$
$$\nabla^4\phi = 24A_{40} + 8A_{22} + 24A_{04} = 0 \Rightarrow$$
$$3A_{40} + A_{22} + 3A_{04} = 0$$

8-3.

$$\sigma_x = -\frac{3Pxy}{2c^3} + \frac{N}{2c}\ ,\ \sigma_y = 0\ ,\ \tau_{xy} = -\frac{3P}{4c}(1-\frac{y^2}{c^2})$$

$$\frac{\partial u}{\partial x} = e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}(-\frac{3Pxy}{2c^3} + \frac{N}{2c}) \Rightarrow u = \frac{1}{E}(-\frac{3Px^2y}{4c^3} + \frac{N}{2c}x) + f(y)$$

$$\frac{\partial v}{\partial y} = e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -\frac{\nu}{E}(-\frac{3Pxy}{2c^3} + \frac{N}{2c}) \Rightarrow v = -\frac{\nu}{E}(-\frac{3Pxy^2}{4c^3} + \frac{N}{2c}y) + g(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2e_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = -\frac{3(1+\nu)P}{2Ec}(1-\frac{y^2}{c^2})$$

$$\Rightarrow -\frac{3Px^2}{4Ec^3} + f'(y) + \frac{3P\nu y^2}{4Ec^3} + g'(x) = -\frac{3(1+\nu)P}{2Ec}(1-\frac{y^2}{c^2})$$

Rearranging and separating the variables $\Rightarrow$

$$-\frac{3Px^2}{4Ec^3} + g'(x) = -\frac{3(1+\nu)P}{2Ec}(1-\frac{y^2}{c^2}) - f'(y) - \frac{3P\nu y^2}{4Ec^3} = \text{constant} = \omega_o$$

$$\therefore\ g(x) = \frac{Px^3}{4Ec^3} + \omega_o x + v_o\ ,\ f(y) = -\frac{P\nu y^3}{4Ec^3} - \frac{3(1+\nu)P}{2Ec}(y - \frac{y^3}{3c^2}) - \omega_o y + u_o$$

$$u = -\frac{3Px^2y}{4Ec^3} + \frac{N}{2Ec}x - \frac{P\nu y^3}{4Ec^3} - \frac{3(1+\nu)P}{2cE}(y - \frac{y^3}{3c^2}) - \omega_o y + u_o$$

$$v = \frac{3P\nu xy^2}{4Ec^3} - \frac{N\nu}{2Ec}y + \frac{Px^3}{4Ec^3} + \omega_o x + v_o$$

In order to complete the solution, we must choose additional boundary conditions to properly constrain the cantilever beam; thus choose

$$\frac{\partial v(L,0)}{\partial x} = 0 \Rightarrow \frac{3PL^2}{4Ec^3} + \omega_o = 0 \Rightarrow \omega_o = -\frac{3PL^2}{4Ec^3}$$

$$u(L,0) = 0 \Rightarrow u_o = \frac{N}{2Ec}L$$

$$v(L,0) = 0 \Rightarrow v_o = -\frac{PL^3}{4Ec^3} - \omega_o L = \frac{PL^3}{2Ec^3}$$

Note that we cannot ensure pointwise conditions such as $u(L,y) = 0$ and $v(L,y) = 0$ with our approximate St. Venant type solution

For the case $N = 0$

$$v(x,0) = \frac{Px^3}{4Ec^3} - \frac{3PL^2}{4Ec^3}x + \frac{PL^3}{2Ec^3} = \frac{P}{4Ec^3}(x^3 - 3L^2x + 2L^3)$$

From Strength of Materials $v(x) = \dfrac{P}{6EI}(x^3 - 3L^2x + 2L^3) = \dfrac{P}{4Ec^3}(x^3 - 3L^2x + 2L^3)$

Therefore the two displacement solutions are the same!

8-4.

$$\phi = C_1x^2 + C_2x^2y + C_3y^3 + C_4y^5 + C_5x^2y^3$$

$$\nabla^4\phi = 0 \Rightarrow 120C_4y + 24C_5y = 0 \Rightarrow C_5 = -5C_4$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 6C_3y + C_4(20y^3 - 30x^2y),\ \sigma_y = \frac{\partial^2\phi}{\partial x^2} = 2C_1 + 2C_2y - 10C_4y^3$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -2C_2x + 30C_4xy^2$$

Boundary Conditions :

$$\tau_{xy}(x,c) = 0 \Rightarrow C_2 = 15C_4c^2$$

$$\sigma_y(x,c) = -q \Rightarrow C_1 + C_2c - 5C_4c^3 = -q/2$$

$$\tau_{xy}(x,-c) = 0 \Rightarrow C_2 = 15C_4c^2 \text{ (same)}$$

$$\sigma_y(x,-c) = 0 \Rightarrow C_1 - C_2c + 5C_4c^3 = 0$$

Conditions at free end $x = 0$:

$$\int_{-c}^{c}\sigma_x(0,y)dy = 0 \Rightarrow \int_{-c}^{c}(6C_3y + 20C_4y^3)dy = 0 \text{, satisfied}$$

$$\int_{-c}^{c}\tau_{xy}(0,y)dy = 0 \Rightarrow 0 = 0 \text{ ,satisfied}$$

$$\int_{-c}^{c}\sigma_x(0,y)ydy = 0 \Rightarrow \int_{-c}^{c}(6C_3y^2 + 20C_4y^4)dy = 4C_3c^3 + 8C_4c^5 = 0$$

Solving for the four constants $\Rightarrow$ $C_1 = -q/4,\ C_2 = -3q/8c, C_3 = q/20c, C_4 = -q/40c^3$

$$\therefore \sigma_x = \frac{3}{10}qy - \frac{q}{4c^3}(y^3 - 3x^2y),\ \sigma_y = -\frac{q}{2} - \frac{qy}{4c}\left(3 - \frac{y^2}{c^2}\right),\ \tau_{xy} = \frac{3qx}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

Check remaining conditions at fixed end $x = L$:

$$\int_{-c}^{c}\sigma_x(L,y)dy = 0 \Rightarrow \int_{-c}^{c}\left(\frac{3}{10}qy - \frac{q}{4c^3}(y^3 - 3L^2y)\right)dy = 0 \text{, satisfied}$$

$$\int_{-c}^{c}\tau_{xy}(L,y)dy = qL \Rightarrow \frac{3qL}{4c}\int_{-c}^{c}\left(1 - \frac{y^2}{c^2}\right)dy = qL \text{, satisfied}$$

$$\int_{-c}^{c}\sigma_x(L,y)ydy = qL^2/2 \Rightarrow \int_{-c}^{c}\left(\frac{3}{10}qy - \frac{q}{4c^3}(y^3 - 3L^2y)\right)ydy = qL^2/2 \text{, satisfied}$$