

HW 4

$$1) \quad \sigma_{ij} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

$$a) \quad \underline{n} = \frac{1}{2} \underline{e}_1 - \frac{1}{2} \underline{e}_2 + \frac{1}{\sqrt{2}} \underline{e}_3$$

$$\text{Note } |\underline{n}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}} = 1$$

$$\underline{T}^n = \underline{n} \cdot \underline{\sigma} = (n_i \underline{e}_i) \cdot (\sigma_{jkl} \underline{e}_j \underline{e}_k)$$

$$= n_j \sigma_{jkl} \underline{e}_k$$

$$= (n_1 \sigma_{11} + n_2 \sigma_{21} + n_3 \sigma_{31}) \underline{e}_1 +$$

$$(n_1 \sigma_{12} + n_2 \sigma_{22} + n_3 \sigma_{32}) \underline{e}_2 +$$

$$(n_1 \sigma_{13} + n_2 \sigma_{23} + n_3 \sigma_{33}) \underline{e}_3$$

$$= \left[ \left(\frac{1}{2}\right)(1) + \left(-\frac{1}{2}\right)(0) + \left(\frac{1}{\sqrt{2}}\right)(-4) \right] \underline{e}_1 +$$

$$\left[ \left(\frac{1}{2}\right)(0) + \left(-\frac{1}{2}\right)(3) + \left(\frac{1}{\sqrt{2}}\right)(0) \right] \underline{e}_2 +$$

$$\left[ \left(\frac{1}{2}\right)(-4) + \left(-\frac{1}{2}\right)(0) + \left(\frac{1}{\sqrt{2}}\right)5 \right] \underline{e}_3$$

$$\underline{T}^n = \left( \frac{1-4\sqrt{2}}{2} \right) \underline{e}_1 - \left( \frac{3}{2} \right) \underline{e}_2 + \left( \frac{-4+5\sqrt{2}}{2} \right) \underline{e}_3$$

$$b) \quad |\underline{T}^n| = \sqrt{\left( \frac{1-4\sqrt{2}}{2} \right)^2 + \left( \frac{3}{2} \right)^2 + \left( \frac{-4+5\sqrt{2}}{2} \right)^2}$$

$$= \sqrt{27-12\sqrt{2}}$$

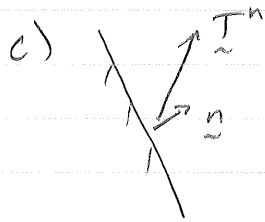
$$|\underline{T}^n| = 3.167$$

## Matlab code:

```
% a
disp('1a)')
sigma=[1 0 -4; 0 3 0; -4 0 5]
n=[1/2; -1/2; 1/sqrt(2)]
Tn=sigma*n
check_Tn=[(1-4*sqrt(2))/2; -3/2; (-4+5*sqrt(2))/2]
% b
disp('1b)')
magnitude_Tn=norm(Tn)
check_magnitude_Tn=sqrt(27-12*sqrt(2))
% c
disp('1c)')
N=norm(Tn'*n)
check_N=7/2-2*sqrt(2)
% d
disp('1d)')
S=norm(cross(Tn,n))
check_S=sqrt(27/4+2*sqrt(2))
check_mag_Tn_2=sqrt(N^2+S^2)
```

## Command window output:

```
1a)
sigma =
     1     0    -4
     0     3     0
    -4     0     5
n =
    0.5000
   -0.5000
    0.7071
Tn =
   -2.3284
   -1.5000
    1.5355
check_Tn =
   -2.3284
   -1.5000
    1.5355
1b)
magnitude_Tn =
    3.1669
check_magnitude_Tn =
    3.1669
1c)
N =
    0.6716
check_N =
    0.6716
1d)
S =
    3.0949
check_S =
    3.0949
check_mag_Tn_2 =
    3.1669
```



$$N = \underline{T}^n \cdot \underline{n}$$

$$= \left(1 - \frac{4\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{-4+5\sqrt{2}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= 7/2 - 2\sqrt{2}$$

$$= 0.6716$$

d)

$$S = \sqrt{|\underline{T}^n|^2 - N^2}$$

$$= \sqrt{(27 - 12\sqrt{2}) - (7 - 2\sqrt{2})^2}$$

$$= \sqrt{27/4 - 2\sqrt{2}}$$

$$= 3.095$$

(see attached Matlab)

2.  $[\sigma_{ij}] = \begin{bmatrix} 5 & -3 & 8 \\ -3 & 7 & 1 \\ 8 & 1 & 9 \end{bmatrix}$

a)  $\begin{vmatrix} 5-\sigma & -3 & 8 \\ -3 & 7-\sigma & 1 \\ 8 & 1 & 9-\sigma \end{vmatrix} = -\sigma^3 + 21\sigma^2 - 64\sigma - 267 = 0$

use Matlab command

$$\text{roots}([-1 \ 21 \ -64 \ -267])$$

yields

$$\sigma_1 = 15.389, \quad \sigma_2 = 7.828, \quad \sigma_3 = -2.217$$

b) using Matlab command  $[V, sig] = \text{eig}(S_{sym})$

$$\underline{\tilde{n}}^{(1)} = -0.7689 \underline{e}_1 - 0.3095 \underline{e}_2 + 0.5703 \underline{e}_3$$

$$\underline{\tilde{n}}^{(2)} = 0.1611 \underline{e}_1 - 0.9415 \underline{e}_2 - 0.2961 \underline{e}_3$$

$$\underline{\tilde{n}}^{(3)} = 0.6286 \underline{e}_1 - 0.1335 \underline{e}_2 + 0.7663 \underline{e}_3$$

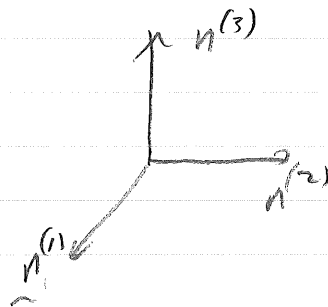
Show

$$\underline{\tilde{n}}^{(1)} \cdot \underline{\tilde{n}}^{(2)} = \underline{\tilde{n}}^{(2)} \cdot \underline{\tilde{n}}^{(3)} = \underline{\tilde{n}}^{(3)} \cdot \underline{\tilde{n}}^{(1)} = 0$$

(orthogonal)

Show

$$\underline{\tilde{n}}^{(1)} \times \underline{\tilde{n}}^{(2)} = \underline{\tilde{n}}^{(3)}, \quad \underline{\tilde{n}}^{(2)} \times \underline{\tilde{n}}^{(3)} = \underline{\tilde{n}}^{(1)}, \quad \underline{\tilde{n}}^{(3)} \times \underline{\tilde{n}}^{(1)} = \underline{\tilde{n}}^{(2)}$$



(right-handed)

$$c) \quad T_{oct} = \frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$

$$= 7.211$$

$$\sigma_m = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) = 7$$

Note  $\sigma_m = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) = 7$  since

$\sigma_{kk}$  is invariant

$$\begin{aligned}
 \sigma_{\text{von Mises}} &= \frac{1}{\sqrt{2}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \\
 &= \frac{3}{\sqrt{2}} \tau_{\text{oct}} \\
 &= 15.297
 \end{aligned}$$

d) Spherical stress

$$\tilde{\sigma}_{ij} = \frac{1}{3} \sigma_{kk} \delta_{ij} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Deviatoric stress

$$\hat{\sigma}_{ij} = \sigma_{ij} - \tilde{\sigma}_{ij} = \begin{bmatrix} -2 & -3 & 8 \\ -3 & 0 & 1 \\ 8 & 1 & 2 \end{bmatrix}$$

e) Stress tensor invariants

$$I_1 = \sigma_{ii} = 21$$

$$I_2 = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji})$$

$$= \frac{1}{2} \left[ (15+7+9)(15+7+9) - (25+9+64+9+49+1+64+1+81) \right]$$

$$= 69$$

Alternate

$$I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 = 69$$

$$I_3 = \begin{vmatrix} 5 & -3 & 8 \\ -3 & 7 & 1 \\ 8 & 1 & 9 \end{vmatrix} = -267$$

Note

$$\frac{1}{3} (2I_1^2 - 6I_2)^{1/2} = 7.211 = \tau_{oct}$$

Alternate

$$I_3 = \sigma_1 \sigma_2 \sigma_3 = -267$$

stress deviator invariants

$$J_1 = \hat{\sigma}_{ii} = 0$$

$$J_2 = \frac{1}{2} (\hat{\sigma}_{11}^2 + \hat{\sigma}_{22}^2 + \hat{\sigma}_{33}^2 - \hat{\sigma}_{11}\hat{\sigma}_{22} - \hat{\sigma}_{11}\hat{\sigma}_{33} - \hat{\sigma}_{22}\hat{\sigma}_{33})$$

$$= \frac{1}{2} (4 + 4 + 18 + 2 + 128)$$

$$= -78$$

$$J_3 = \begin{vmatrix} -2 & -3 & 8 \\ -3 & 0 & 1 \\ 8 & 1 & 2 \end{vmatrix} = -64$$

$$f) \underline{T}^n = \underline{n} \cdot \underline{\sigma} = \left[ \left(\frac{1}{\sqrt{3}}\right)(5) + \left(\frac{1}{\sqrt{5}}\right)(-3) - \left(\frac{1}{\sqrt{3}}\right)(8) \right] \underline{e}_1 +$$

$$\left[ \left(\frac{1}{\sqrt{3}}\right)(-3) + \left(\frac{1}{\sqrt{5}}\right)(7) - \left(\frac{1}{\sqrt{3}}\right)(1) \right] \underline{e}_2 +$$

$$\left[ \left(\frac{1}{\sqrt{3}}\right)(8) + \left(\frac{1}{\sqrt{5}}\right)(1) - \left(\frac{1}{\sqrt{3}}\right)(9) \right] \underline{e}_3$$

$$= \left(\frac{-6}{\sqrt{3}}\right) \underline{e}_1 + \left(\frac{3}{\sqrt{3}}\right) \underline{e}_2 + 0 \underline{e}_3$$

$$g) N = \underline{T}^n \cdot \underline{n} = \left(\frac{-6}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{3}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{5}}\right) = -1$$

## Matlab code:

```

format compact; clear all; clc
% a
disp('2a')
sigma=[5 -3 8; -3 7 1; 8 1 9]
[V,sig]=eig(sigma)
check_eigs=roots([-1 21 -69 -267])
% b
disp('2b')
eigv1=V(:,1)
eigv2=V(:,2)
eigv3=-V(:,3)
check_1=eigv1'*eigv2
check_2=eigv2'*eigv3
check_3=eigv3'*eigv1
check_4=cross(eigv1,eigv2)
check_5=cross(eigv2,eigv3)
check_6=cross(eigv3,eigv1)
% c
disp('2c')
sig_1=sig(3,3)
sig_2=sig(2,2)
sig_3=sig(1,1)
tauoct=(1/3)*sqrt((sig_1-sig_2)^2+(sig_2-sig_3)^2+(sig_3-sig_1)^2)
sigm=trace(sigma)/3
sigm_2=trace(sig)/3
sig_vm=(1/sqrt(2))*sqrt((sig_1-sig_2)^2+(sig_2-sig_3)^2+(sig_3-sig_1)^2)
% d
disp('2d')
sig_sph=diag([sigm sigm sigm])
sig_dev=sigma-sig_sph
% e
disp('2e')
I1=trace(sig)
I2=sig(1,1)*sig(2,2)+sig(2,2)*sig(3,3)+sig(3,3)*sig(1,1)...
    -sig(1,2)^2-sig(2,3)^2-sig(1,3)^2
I3=det(sig)
J1=trace(sig_dev)
J2=sig_dev(1,1)*sig_dev(2,2)+sig_dev(2,2)*sig_dev(3,3)+ ...
    sig_dev(3,3)*sig_dev(1,1)-sig_dev(1,2)^2-sig_dev(2,3)^2-...
    sig_dev(1,3)^2
J3=det(sig_dev)
% f
disp('2f')
n=(1/sqrt(3))*[1; 1; -1]
Tn=sigma*n
check_Tn=[-6/sqrt(3); 3/sqrt(3); 0]
% g
disp('2g')
N=norm(Tn'*n)
S=norm(cross(Tn,n))

```

## Command window output:

2a)

sigma =

5	-3	8
-3	7	1
8	1	9

V =

-0.7609	0.1611	-0.6286
-0.3095	-0.9415	0.1335
0.5703	-0.2961	-0.7662

sig =

-2.2165	0	0
0	7.8278	0
0	0	15.3887

check\_eigs =

15.3887
7.8278
-2.2165

2b)

eigv1 =

-0.7609
-0.3095
0.5703

eigv2 =

0.1611
-0.9415
-0.2961

eigv3 =

0.6286
-0.1335
0.7662

check\_1 =

2.7756e-016

check\_2 =

0

check\_3 =

5.5511e-017

check\_4 =

0.6286
-0.1335
0.7662

check\_5 =

-0.7609
-0.3095
0.5703

check\_6 =

0.1611
-0.9415
-0.2961



2c)  
 sig\_1 =  
     15.3887  
 sig\_2 =  
     7.8278  
 sig\_3 =  
     -2.2165  
 tauoct =  
     7.2111  
 sigm =  
     7  
 sigm\_2 =  
     7  
 sig\_vm =  
     15.2971

2d)  
 sig\_sph =  
     7      0      0  
     0      7      0  
     0      0      7  
 sig\_dev =  
     -2     -3      8  
     -3      0      1  
     8      1      2

2e)  
 I1 =  
     21  
 I2 =  
     69.0000  
 I3 =  
     -267.0000  
 J1 =  
     0  
 J2 =  
     -78  
 J3 =  
     -64

2f)  
 n =  
     0.5774  
     0.5774  
     -0.5774

Tn =  
     -3.4641  
     1.7321  
     0  
 check\_Tn =  
     -3.4641  
     1.7321  
     0

2g)  
 N =  
     1.0000  
 S =  
     3.7417