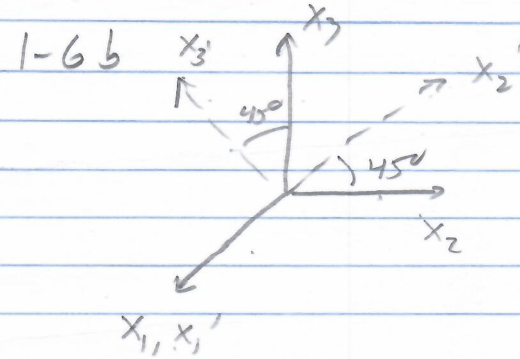
Hw 2 1-6b, 12b, 14b ; 2-1b, 2-3

1-6b



$$Q_{ij} = \cos(x_i', x_j)$$

$$Q = \begin{bmatrix} \cos 0 & \cos 90 & \cos 90 \\ \cos 90 & \cos 45 & \cos 45 \\ \cos 90 & \cos 135 & \cos 45 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$b_i' = Q_{ip} b_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix}$$

$$b_i' = \begin{Bmatrix} 2 \\ \sqrt{2} \\ 0 \end{Bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq}$$

$$= Q_{ip} a_{pq} Q_{jq}$$

$$= Q_{ip} a_{pq} Q_{qj}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3\sqrt{2} & 3\sqrt{2}/2 \\ 0 & 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 9/2 & -3/2 \\ 0 & 3/2 & -1/2 \end{bmatrix}$$

<u>1-12b</u>

$$a_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I_1 = a_{ii} = -4$$

$$I_2 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ 0 & 0 \end{vmatrix} = 3$$

$$I_3 = \det(a_{ij}) = 0$$

characteristic equation

$$-\lambda^3 - 4\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda+3)(\lambda+1) = 0$$

$$\lambda = -3, -1, 0$$

$$\lambda_1 = -3 \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} n_1^{(1)} + n_1^{(2)} = 0 \\ n_3^{(1)} = 0 \end{matrix}$$

$$\underset{\sim}{n}^{(1)} = \pm \frac{\sqrt{2}}{2} (\underset{\sim}{e}_1 - \underset{\sim}{e}_2)$$

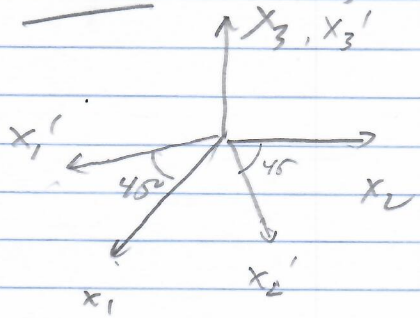
$$\lambda = -1 \qquad \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} n_1^{(2)} \\ n_2^{(2)} \\ n_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \end{matrix}$$

$$\underset{\sim}{n}^{(2)} = \pm \frac{\sqrt{2}}{2} (\underset{\sim}{e}_1 + \underset{\sim}{e}_2)$$

$$\lambda = 0 \qquad \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_3^{(3)} - \text{arbitrary} \\ n_1^{(3)} = n_2^{(3)} = 0 \end{matrix}$$

$$\underset{\sim}{n}^3 = \underset{\sim}{e}_3$$

1-12b (cont)



$$Q_{ij} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_{ij}' = [Q][a][Q]^T$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1-14b

$$\underset{\sim}{u} = x_1^2 \underset{\sim}{e}_1 + 2x_1x_2 \underset{\sim}{e}_2 + x_3^3 \underset{\sim}{e}_3$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \left(\frac{\partial}{\partial x_i} \underset{\sim}{e}_i\right) \cdot \left(u_j \underset{\sim}{e}_j\right) = \frac{\partial u_j}{\partial x_i} \delta_{ij} = \frac{\partial u_i}{\partial x_i}$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= 2x_1 + 2x_1 + 2x_3^2$$

$$= 4x_1 + 2x_3^2$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{u} = \begin{vmatrix} e_1 & e_2 & \underset{\sim}{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix} = 0 \underset{\sim}{e}_1 + 0 \underset{\sim}{e}_2 + 2x_2 \underset{\sim}{e}_3$$

$$= 2x_2 \underset{\sim}{e}_3$$

1-14b (cont.)

$$\nabla^2 \underset{\sim}{u} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \underset{\sim}{u}$$

$$= 2 \underset{\sim}{e_1} + 0 \underset{\sim}{e_2} + 6 x_3 \underset{\sim}{e_3}$$

$$= 2 \underset{\sim}{e_1} + 6 x_3 \underset{\sim}{e_3}$$

$$\underset{\sim}{\nabla} \underset{\sim}{u} = \left(\frac{\partial}{\partial x_i} \underset{\sim}{e_i}\right)\left(u_j \underset{\sim}{e_j}\right) = \frac{\partial u_j}{\partial x_i} \underset{\sim}{e_i} \underset{\sim}{e_j}$$

$$= \begin{bmatrix} \partial u_1/\partial x_1 & \partial u_2/\partial x_1 & \partial u_3/\partial x_1 \\ \partial u_1/\partial x_2 & \partial u_2/\partial x_2 & \partial u_3/\partial x_2 \\ \partial u_1/\partial x_3 & \partial u_2/\partial x_3 & \partial u_3/\partial x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 2x_2 & 0 \\ 0 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}$$

$$tr(\underset{\sim}{\nabla} \underset{\sim}{u}) = 2x_1 + 2x_1 + 3x_3^2 = 4x_1 + 3x_3^2$$

2-1b $\quad u = Ax^2, \quad v = Bxy, \quad w = Cxyz$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$= \begin{bmatrix} \partial u/\partial x & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \text{SYM} & & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 2Ax & \frac{1}{2}By & \frac{1}{2}Cyz \\ & Bx & \frac{1}{2}Cxz \\ \text{SYM} & & Cxy \end{bmatrix}$$

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \partial u_j/\partial x_i\right)$$

$$\begin{bmatrix} 0 & -1/2\,By & -1/2\,Cyz \\ 1/2\,By & 0 & -1/2\,Cxz \\ \frac{1}{2}Cyz & \frac{1}{2}Cxz & 0 \end{bmatrix}$$

2-3 $e_{11} = e_x = \frac{\partial u}{\partial x} = c_1 \Rightarrow u = c_1 x + f(y)$

$e_{22} = e_y = \frac{\partial v}{\partial y} = -c_2 \Rightarrow v = -c_2 y + g(x)$

$e_{33} = e_z = \frac{\partial w}{\partial z} = 0 \Rightarrow w = h(x, y)$

$e_{12} = e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \Rightarrow \frac{df}{dy} + \frac{dg}{dx} = 0$

$$\frac{df}{dy} = -\frac{dg}{dx} = \text{constant} = a$$

$$f = a \cdot y + d_1 \quad , \quad g = -a x + d_2$$

$$e_{23} = e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \Rightarrow \frac{\partial h}{\partial y} = 0$$

$$\Rightarrow h = f_1(x)$$

$$e_{13} = e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \Rightarrow \frac{\partial h}{\partial x} = 0$$

$$\Rightarrow h = g_1(y)$$

$$\Rightarrow h = \text{constant} = d_3$$

Hence

$$u = c_1 x + a y + d_1$$

$$v = -c_2 y - a x + d_2$$

$$w = w_0$$

Compare to eq. 2.2.9 $\Rightarrow$ $a = -\omega_z$

± rigid body rotation about z-axis

$d_1 = u_0$

$d_2 = v_0$ (rigid body translation)

$d_3 = w_0$