HW # 10 - 8-15, 26 and 48

8-15 Given $\phi = a_4 \theta$

Eq 7.6.7

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = 0$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -a_4 \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -a_4\left(\frac{-1}{r^2}\right) = a_4/r^2$$

Boundary condition at $r = r_2$, $\tau(r_2) = \tau = a_4/r_2^2$

$$a_4 = \tau r_2^2$$

Displacements

$$e_r = \frac{\partial u_r}{\partial r} = 0 \Rightarrow u_r = \text{constant}$$

$$u_r(r = r_1) = 0 \Rightarrow u_r(r) = 0$$

$$e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) = 0 \Rightarrow u_\theta = u_\theta(r)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\cancel{\frac{\partial u_r}{\partial \theta}}^{\,0} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \frac{1+\nu}{E}\tau_{r\theta}$$

$$\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \frac{2(1+\nu)}{E}\frac{a_4}{r^2}$$

Note

$$\frac{d}{dr}\left(\frac{u_\theta}{r}\right) = \frac{r\frac{du_\theta}{r} - u_\theta}{r^2} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = \frac{2(1+\nu)}{E}\frac{a_4}{r^2}$$

Integrating

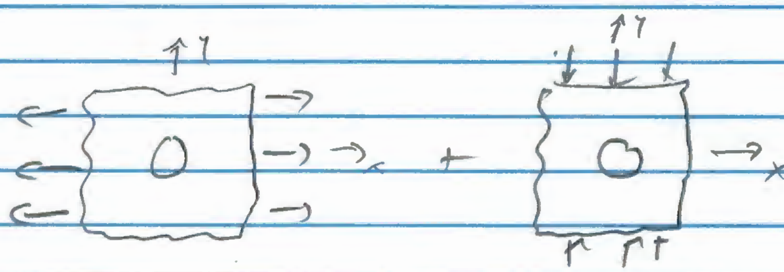
$$u_\theta = -\frac{(1+\nu)\overset{\tau r_2^2}{a_4}}{E}\frac{1}{r} + C r$$

$$u_\theta(r=r_1) = 0 = -\frac{(1+\nu)\tau r_2^2}{E}\left(\frac{1}{r_1}\right) + C r_1$$

$$C = -\frac{(1+\nu)\tau r_2^2}{E}\,\frac{1}{r_1^2}$$

$$\boxed{u_\theta = \frac{(1+\nu)\tau r_2^2}{E}\left(r/r_1^2 - 1/r\right)}$$

8-26



① ②

① From 8.4.15

$$\sigma_r^{(1)} = \frac{T}{2}\left(1-a^2/r^2\right) + \frac{T}{2}\left(1 + 3a^4/r^4 - 4a^2/r^2\right)\cos 2\theta$$

$$\sigma_\theta^{(1)} = \frac{T}{2}\left(1+a^2/r^2\right) - \frac{T}{2}\left(1 + 3a^4/r^4\right)\cos 2\theta$$

$$\tau_{r\theta}^{(1)} = -\frac{T}{2}\left(1 - 3a^4/r^4 + 2a^2/r^2\right)\sin 2\theta$$

② replace $\theta$ with $\theta + \pi/2$, $T$ with $-T$

Note $\cos\left[2(\theta+\pi/2)\right] = \cos(2\theta+\pi)$

$$\cos(2\theta+\pi) = \cos 2\theta\,\overset{-1}{\cancel{\cos\pi}} - \sin 2\theta\,\overset{0}{\cancel{\sin\pi}} = -\cos 2\theta$$

$$\sin(2\theta+\pi) = \sin 2\theta\,\overset{-1}{\cancel{\cos\pi}} + \cos 2\theta\,\overset{0}{\cancel{\sin\pi}} = -\sin 2\theta$$

$$\sigma_r^{(2)} = -\frac{T}{2}\left(1-a^2/r^2\right) + \frac{T}{2}\left(1 + 3a^4/r^4 - 4a^2/r^2\right)\cos 2\theta$$

$$\sigma_\theta^{(2)} = -\frac{T}{2}\left(1+a^2/r^2\right) - \frac{T}{2}\left(1 + 3a^4/r^4\right)\cos 2\theta$$

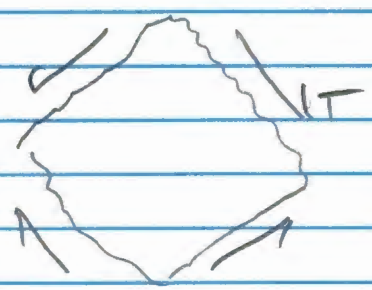
$$\tau_{r\theta} = -\frac{T}{2}\left(1 + \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

Superpose ① + ②

$$\boxed{\begin{aligned} \sigma_r &= T\left(1 + 3a^4/r^4 - 4a^2/r^2\right)\cos 2\theta \\ \sigma_\theta &= -T\left(1 + 3a^4/r^4\right)\cos 2\theta \\ \tau_{r\theta} &= -T\left(1 + 3a^4/r^4 + 2a^2/r^2\right)\sin 2\theta \end{aligned}}$$

For $\theta = \pm 45$, $\cos 2\theta = 0$, $\sin 2\theta = \pm 1$

Hence, as $r \to \infty$, the state of stress on planes oriented at $\pm 45$ are under pure shear, $T$




8-48. From 8.4.80

$$\sigma_r = -\frac{3+\nu}{8}\rho\omega^2 r^2 + \frac{C_1}{2} + C_2/r^2$$

$$\sigma_\theta = \frac{1+3\nu}{8}\rho\omega^2 r^2 + C_1/2 - C_2/r^2$$

$$\sigma_r(a) = 0 = -\frac{3+\nu}{8}\rho\omega^2 a^2 + C_1/2 + C_2/a^2 \qquad (1)$$

$$\sigma_r(b) = 0 = -\frac{3+\nu}{8}\rho\omega^2 b^2 + C_1/2 + C_2/b^2 \qquad (2)$$

(1)-(2)

$$-\frac{3+\nu}{8}\rho\omega^2\cancel{(a^2-b^2)} = C_2\left(\cancel{1/a^2 - 1/b^2}\right)\left(\frac{b^2-a^2}{a^2b^2}\right)$$

$$C_2 = -\left(\frac{3+\nu}{8}\right)\rho\omega^2 a^2 b^2$$

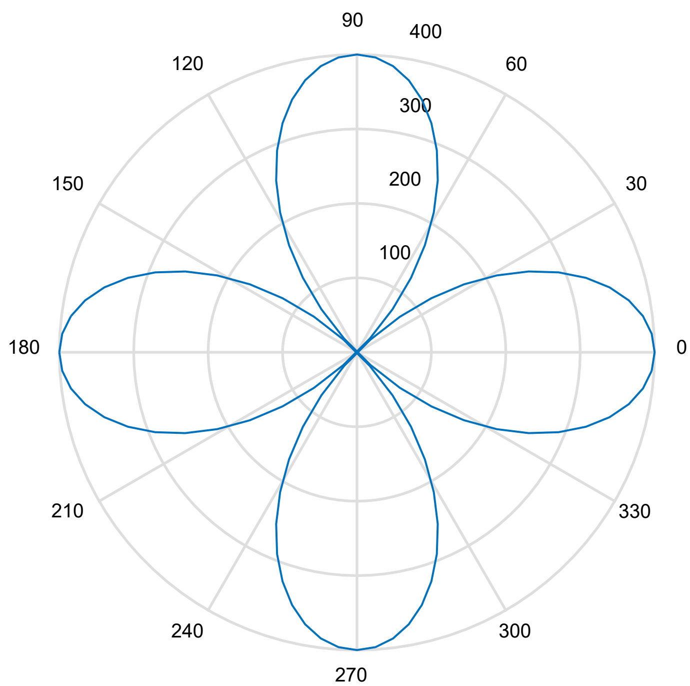
8-26 - Polar plot

$$\sigma_\theta\ (r = a) = \ -4\ T\ \cos(2\theta)$$

Matlab code

```
theta=linspace(0,2*pi,101);
% at r=a:
T=100;
sig_theta=-4*T*cos(2*theta);
polar(theta,sig_theta)
```

Fogure window output



(1) $$-\frac{3+\nu}{8}\rho w^2 a^2 + \frac{c_1}{2} - \frac{3+\nu}{8}\rho w^2 b^2 = 0$$

$$c_1 = 2\left(\frac{3+\nu}{8}\right)\rho w^2(a^2+b^2)$$

$$\sigma_r(r) = -\left(\frac{3+\nu}{8}\right)\rho w^2 r^2 + \left(\frac{3+\nu}{8}\right)\rho^2 w^2(a^2+b^2)$$
$$-\left(\frac{3+\nu}{8}\right)\rho w^2 a^2 b^2/r^2$$
$$= -\frac{3+\nu}{8}\rho w^2\left(r^2 + a^2b^2/r^2 - a^2 - b^2\right)$$

$$\sigma_\theta(r) = -\frac{1+3\nu}{8}\rho w^2 r^2 + \frac{3+\nu}{8}\rho w^2(a^2+b^2)$$
$$+\left(\frac{3+\nu}{8}\right)\rho w^2 a^2 b^2/r^2$$

$$\sigma_\theta(r)\Big|_{max} = \sigma_\theta(a)$$

$$\sigma_\theta\Big|_{max} = -\frac{1+3\nu}{8}\rho w^2 a^2 + \frac{3+\nu}{8}\rho w^2(a^2+b^2)$$
$$+\frac{3+\nu}{8}\rho w^2 b^2$$

for $b \gg a$

$$\sigma_\theta\Big|_{max} = \frac{3+\nu}{4}\rho w^2 b^2$$

Ex 8-11, $b \gg a$

$$\sigma_{max} = \sigma_r(0) = \sigma_\theta(0) = \frac{3+\nu}{8}\rho w^2 b^2$$

(½ that of annular disk)