HW #7 - 6-3, 8, 16

6-3 $\quad U = \frac{1}{2}\sigma_{ij} e_{ij} \qquad (6.1.7)$

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}$$

$$U = \frac{1}{2}\left(2\mu e_{ij} + \lambda e_{kk}\delta_{ij}\right) e_{ij}$$

$$= \mu e_{ij} e_{ij} + \frac{1}{2}\lambda e_{kk} e_{ii}$$

$$U = \mu\left(e_x^2 + e_y^2 + e_z^2 + 2e_{xy}^2 + 2e_{xz}^2 + 2e_{yz}^2\right)$$

$$+ \frac{1}{2}\lambda\left(e_x + e_y + e_z\right)^2 \qquad (6.1.9)$$

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \nu/E\, \sigma_{kk}\delta_{ij}$$

$$U = \frac{1}{2}\sigma_{ij}\left(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}\right)$$

$$= \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{ii}\sigma_{kk}$$

$$= \frac{1+\nu}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\tau_{xy}^2 + 2\tau_{xz}^2 + 2\tau_{yz}^2\right)$$

$$- \frac{\nu}{2E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2 \qquad (6.1.10)$$

6-8 $\quad U_v = \frac{1}{2}\tilde{\sigma}_{ij}\tilde{e}_{ij}$

$$= \frac{1}{2}\left(\frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\frac{1}{3}e_{mm}\delta_{ij}\right)$$

$$= \frac{1}{18}\sigma_{kk} e_{mm} \cancelto{3}{\delta_{ii}}$$

$$= \frac{1}{6}\sigma_{kk} e_{mm}$$

$$= \frac{1}{6}\sigma_{kk}\left(\frac{1-2\nu}{E}\sigma_{mm}\right)$$

$$U_v = \frac{1-2\nu}{6E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2$$

$$U_d = U - U_v$$

$$= \left(\frac{1+\nu}{2E}\,\sigma_{ij}\sigma_{ij} - \frac{3\nu}{6\cancel{2}E}\,\sigma_{ii}\sigma_{kk}\right) - \left(\frac{1-2\nu}{6E}\right)\sigma_{ii}\sigma_{kk}$$

$$= \frac{1+\nu}{2E}\,\sigma_{ij}\sigma_{ij} - \frac{1+\nu}{6E}\,\sigma_{ii}\sigma_{kk}$$

$$= \frac{\overset{3}{1}}{12\cancel{4}\mu}\,\sigma_{ij}\sigma_{ij} - \frac{1}{12\mu}\,\sigma_{ii}\sigma_{kk}$$

$$= \frac{1}{12\mu}\Big[3\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\tau_{xy}^2 + 2\tau_{xz}^2 + 2\tau_{yz}^2\right) - \left(\sigma_x + \sigma_y + \sigma_z\right)^2\Big]$$

$$= \frac{1}{12\mu}\Big[\overset{2}{\cancel{3}}\sigma_x^2 + \overset{2}{\cancel{3}}\sigma_y^2 + \overset{2}{\cancel{3}}\sigma_z^2 + 6\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right) - \cancel{\sigma_x^2} - \cancel{\sigma_y^2} - \cancel{\sigma_z^2} - 2\sigma_x\sigma_y - 2\sigma_y\sigma_z - 2\sigma_x\sigma_z\Big]$$

$$= \frac{1}{12\mu}\Big[\left(\sigma_x^2 - 2\sigma_x\sigma_y + \sigma_y^2\right) + \left(\sigma_x^2 - 2\sigma_x\sigma_z + \sigma_z^2\right) + \left(\sigma_y^2 - 2\sigma_y\sigma_z + \sigma_z^2\right) + 6\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\Big]$$

$$= \frac{1}{12\mu}\Big[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_x - \sigma_z\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + 6\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\Big]$$

6-16

$$w = \sum_{j=1}^{N} c_j \sin \frac{j\pi x}{\ell}$$

$N=2$

$$w = C_1 \sin \frac{\pi x}{\ell} + C_2 \sin \frac{2\pi x}{\ell}$$

$$\frac{dw}{dx} = \frac{C_1 \pi}{\ell} \cos \frac{\pi x}{\ell} + \frac{2C_2\pi}{\ell} \cos \frac{2\pi x}{\ell}$$

$$\frac{d^2w}{dx^2} = -\frac{C_1\pi^2}{\ell^2} \sin \frac{\pi x}{\ell} - \frac{4C_2\pi^2}{\ell^2} \sin \frac{2\pi x}{\ell}$$

$$\Pi = \int_0^\ell \left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 - q_0 w\right] dx$$

$$= \frac{EI}{2} \frac{\pi^4}{\ell^4} \int_0^\ell \left(C_1 \sin \frac{\pi x}{\ell} + 4C_2 \sin \frac{2\pi x}{\ell}\right)^2 dx$$

$$- q_0 \int_0^\ell \left(C_1 \sin \frac{\pi x}{\ell} + C_2 \sin \frac{2\pi x}{\ell}\right) dx$$

$$= \frac{EI\pi^4}{2\ell^4} \int_0^\ell \left[C_1^2 \sin^2 \frac{\pi x}{\ell} + 8C_1C_2 \sin\frac{\pi x}{\ell} \sin \frac{2\pi x}{\ell} + 16C_2^2 \sin^2 \frac{2\pi x}{\ell}\right] dx$$

$$- q_0 \int_0^\ell \left(C_1 \sin \frac{\pi x}{\ell} + C_2 \sin \frac{2\pi x}{\ell}\right) dx$$

<u>Integrals</u>

$$\int_0^{l} \sin^2\frac{\pi x}{l}\,dx = \left(\frac{x}{2} - \frac{l}{2\pi}\sin\frac{\pi x}{l}\cos\frac{\pi x}{l}\right)\Bigg|_0^{l} = l/2$$

$$\int_0^{l} \sin\frac{\pi x}{l}\sin\frac{2\pi x}{l}\,dx = \left(\frac{l}{2\pi}\sin\frac{\pi x}{l} - \frac{l}{6\pi}\sin\frac{3\pi x}{l}\right)_0^{l} = 0$$

$$\int_0^{l} \sin^2\frac{2\pi x}{l}\,dx = \left(\frac{x}{2} - \frac{l}{4\pi}\sin\frac{2\pi x}{l}\cos\frac{2\pi x}{l}\right)\Bigg|_0^{l} = l/2$$

$$\int_0^{l} \sin\frac{\pi x}{l}\,dx = -\frac{l}{\pi}\cos\frac{\pi x}{l}\Bigg|_0^{l} = \frac{2l}{\pi}$$

$$\int_0^{l} \sin\frac{2\pi x}{l}\,dx = \frac{l}{2\pi}\cos\frac{2\pi x}{l}\Bigg|_0^{l} = 0$$

$$\Pi = \frac{EI\pi^4}{2l^4}\left[c_1^2(l/2) + 8c_1c_2(0) + 16c_2^2(l/2)\right]$$
$$- q_0\left[c_1\left(\frac{2l}{\pi}\right) + c_2(0)\right]$$

$$\Pi = \frac{EI\pi^4}{2l^4}\left(\frac{c_1^2 l}{2} + 8c_2^2 l\right) - q_0\frac{2c_1 l}{\pi}$$

$$\frac{\partial\Pi}{\partial c_1} = 0 = \frac{\cancel{2}EI\pi^4 l}{2\cancel{4}l^{\cancel{4}3}}c_1 - \frac{2q_0 l}{\pi}$$

$$c_1 = \frac{4q_0 l^4}{EI\pi^5}$$

$$C_2 = \frac{\partial \Pi}{\partial C_2} = 16\, C_2\, \ell = 0$$

$$C_2 = 0$$

$$\boxed{w = \frac{4\, q_0\, \ell^4}{EI\pi^5} \sin \frac{\pi x}{\ell}}$$

at mid-span $(x = \ell/2)$

$$w(\ell/2) = \frac{4}{\pi^5} \frac{q_0 \ell^4}{EI} = 0.01307 \frac{q_0 \ell^4}{EI}$$

(0.4% above exact)

Example 6-2

$$w(\ell/2) = \frac{1}{96} \frac{q_0 \ell^4}{EI} = 0.01042 \frac{q_0 \ell^4}{EI}$$

(20% below exact)

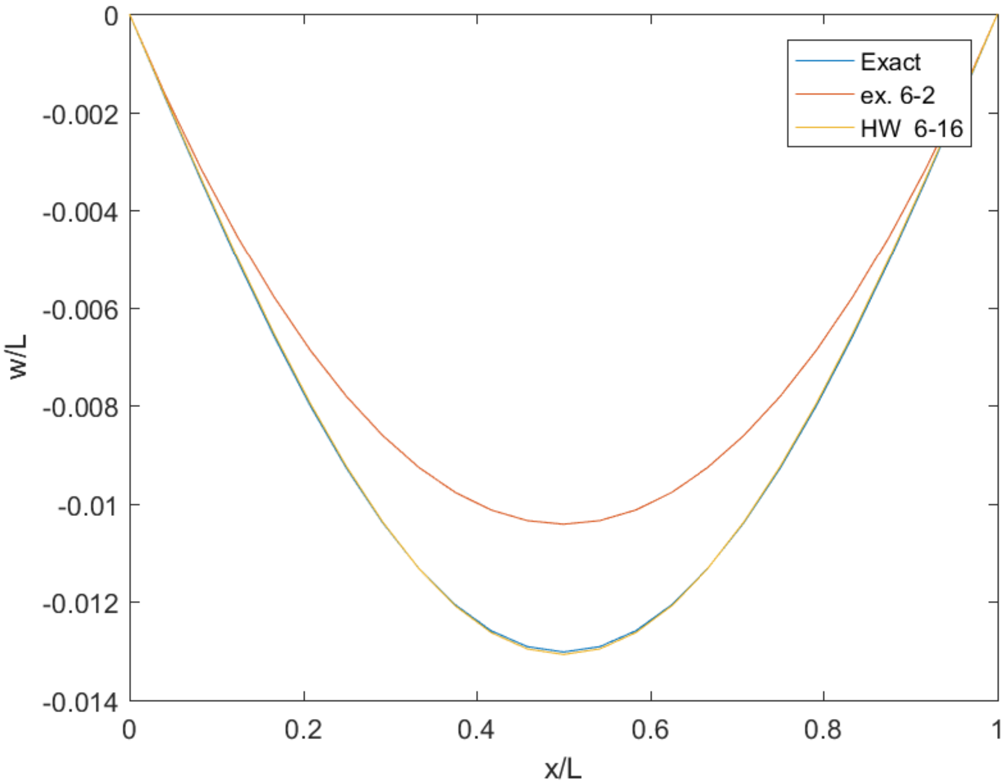
Exact

$$w(\ell/2) = \frac{q_0 (\ell/2)}{24EI} \left(\ell^3 + (\ell/2)^3 - \frac{2\ell^3}{4} \right)$$

$$= \frac{q_0 \ell^4}{48EI} \left(\frac{\overset{5}{\cancel{8 + 1 - 4}}}{8} \right)$$

$$= \frac{5}{384} \frac{q_0 \ell^4}{EI}$$

$$= 0.01302 \frac{q_0 \ell^4}{EI}$$



0
-0.002
-0.004
-0.006
-0.008
-0.01
-0.012
-0.014
w/L
0
0.2
0.4
0.6
0.8
1
x/L
Exact
ex. 6-2
HW 6-16