Implementation and Anlysis of the Christophides Algorithm
CSCI 5454 Project

Donovan Guelde

1 Introduction

The Christofides Algorithm, first published in 1976 by Nicos Christofides, is an approximation algorithm for the Traveling Salesman Problem. The Christofides Algorithm is guaranteed to find a solution to a Travelling Salesman Problem that is within $\frac{3}{2}$ of the optimal solution length, but at greatly reduced computational cost. Before analyzing the Christofides Algorithm, let us examine the Travelling Salesman Problem.

1.1 Travelling Salesman Problem

The Travelling Salesman Problem is closely related to the Hamiltonian-cycle problem (in fact, we will see Hamiltonian cycles in the implementation of the Christofides algorithm). In the Travelling Salesman Problem, an actor, the *Salesman*, must complete a tour of n cities, visiting each city only once, in the shortest distance possible, with the stipulation that the tour begins and ends in the same city. TSP can be modeled as a complete weighted graph, $G = (V, E)$, where vertexes represent cities the *Salesman* must visit on the tour, and weights of edge $\mathrm{e}_{(u,v)} \in \mathrm{E}$ represent the cost to travel between city u and city v. Or, in other words, "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?"[1]

The Traveling Salesman Problem, in spite of its seeming simplicity, has various applications. The obvious application is in the field of logistics. Given a transportation network (road, rail, air routes, shipping routes, etc), finding the shortest path through all necessary stops reduces both time and transportation costs. Another application of the Traveling Salesman Problem is in the manufacturing of circuit boards. During the manufacturing process, printed circuit boards are drilled to accommodate the attachment of integrated circuits. The most efficient drilling pattern is a solution to the Traveling Salesman Problem, where cities represent the location of holes to be drilled, and the shortest possible path taken by the drill head is the optimal solution. Distance between drilling locations is analogous to the distance between cities. Other applications of the Travelling Salesman Problem include overhauling gas turbine engines, X-ray crystallography, computer wiring, and the order-picking problem in warehouses[2].

While the applications of a solution to the Traveling Salesman Problem are numerous, there is one major drawback. The Traveling Salesman Problem is NP-Complete.

1.2 Proof of TSP NP-Completeness

To show that TSP is NP-Complete, we must prove two things. First, that TSP is in the class NP, and second, that TSP is NP-Hard.

1.2.1 TSP is in NP

To show TSP is a member of class NP, we will consider the decision problem of TSP. That is, can we check that a given Hamiltonian cycle h of graph G has a path length l(given as part of the solution) or less? We choose the decision problem because the solution to the optimization problem, that of finding the shortest Hamiltonian path of graph G, cannot be verified in polynomial time. The Optimization problem of TSP is strictly NP-Hard.[3]

To solve the Travelling Salesman Optimization:

Given a graph G=(V,E):

1. Let start vertex s = v_0
2. Enumerate all possible permutations of the subset of V given by V'=$\{v_1, v_2, ..., v_{n-1}\}$ where n = $|V|$
3. Eliminate repetitions in permutations caused by symmetry ($\{v_1 \rightarrow v_2 \rightarrow v_3\} == \{v_3 \rightarrow v_2 \rightarrow v_1\}$)
4. Construct cycle from unique permutations. Given permutation P and start s, cycle = $\{s \rightarrow v_0 \rightarrow v_1 \rightarrow ... \rightarrow v_{n-1} \rightarrow s\}$
5. Calculate sum of distances for all cucles generated in step 4.
6. Minimum cycle distance is Travelling Salesman Optimization solution.

The number of permutations calculated in step 2 above is (n-1)! since a start node has been arbitrarily selected. Number of unique permutations remaining after step 3 is $\frac{(n-1)!}{2}$. Therefore, Travelling Salesman Optimization has runtime O((n-1)!), so the Travelling Salesman Optimization is strictly NP-Hard.

Returning to the Travelling Salesman Decision problem, let h=$\{v_1, v_2, ..., v_{(n-1)}, v_n, v_1\}$ be a Hamiltonian circuit on graph G, and $l \in \mathbb{R} > 0$ be a maximum length of the TSP solution. Taken together, (h,l) represents a potential solution to the Travelling Salesman Decision Problem on G. To verify (h,l) represents a solution to the TSP on G, one has to calculate the total distance d traveled in Hamiltonian Path h, such that d= $d_{v_1,v_2} + d_{v_2,v_3} + ... + d_{v_{n-1},v_n} + d_{v_n,v_1}$, and ensure this distance d $\leq l$. Next, one must also ensure that h is, in fact, a Hamiltonian path in G. That is, every vertex v

$\in$ V is visited once and only once in path h, and that path h begins and ends at the same vertex v_1. Verifying the solution (h,l) can be accomplished in polynomial time (O(n), in fact), making TSP a member of class NP.

1.2.2 TSP is NP-Hard

Next, we must show that TSP is NP-Hard. To do so, we will show that the Hamiltonian Cycle problem, a known NP-Complete problem[3], reduces to TSP. Since the Hamiltonian Cycle problem is NP-Hard, every NP-Hard problem reduces to Hamiltonian Cycle, and if Hamiltonian Cycle reduces to TSP, so does every NP-Hard problem.

In order to reduce the Hamiltonian Cycle problem to TSP, let G=(V,E) be input for an instance of the Hamiltonian Cycle Problem. Construct weighted, complete graph G'=(V,E'), and set edge weights for all e$\in$E' such that $e_{i,j} = \begin{cases} 0 \; if \; e_{i,j} \in E \\ 1 \; if \; e_{i,j} \notin E \end{cases}$

Graph G (instance of Hamiltonian Cycle Problem) has a solution if and only if Graph G' (instance of TSP) has a cycle passing through all vertices v$\in$V with total weight 0. Consider the case where graph G does in fact have Hamiltonian Cycle h. Since every edge in h is in E, path h in graph G' has total edge weight 0.

Next, consider a tour h' in G' such that the sum of all edge weights in $h' = 0$. Since the sum of edge weights is 0, and all edges e $\in$ E' have weight 1 (not present in G) or 0 (present in G), all edges in h' must be present in graph G. It follows that since h' is a tour in G', it is also a tour in G, and is therefore a solution to the Hamiltonian Cycle problem on G. A solution to the Hamiltonian Cycle problem is a solution to TSP, and vice-versa. The Hamiltonian Cycle problem reduces to TSP. Therefore, TSP is NP-Hard. Since TSP is both a member of class NP and NP-Hard, TSP is NP-Complete.

2 The Christofides Algorithm

The Christofides Algorithm is an algorithm for approximating solutions to the Travelling Salesman Problem on complete, weighted graph G=(V,E). The Christofides Algorithm is guaranteed to find an approximate solution to the Travelling Salesman Problem that is no worse than $\frac{3}{2}$ length of the optimal solution. In order to apply the Christofides Algorithm to an instance of the Travelling Salesman Problem, graph G must satisfy the Triangle Inequality, and be symmetric. According to the Triangle inequality, given a triangle with vertices x,y,z, the distance given by $d_{x,y} + d_{y,z} \geq d_{x,z}$. If graph G is symmetric, $d_{x,y} = d_{y,x}$. The distance from vertex x to vertex y equals the distance from vertex y to vertex x. Since the solution to a TSP is a circuit, which direction the circuit is followed should not change the distance, hence the need for symmetry. We will see why adherence to the triangle inequality is required in the analysis to follow.

Nicos Christofides published the Christofides Algorithm in 1976 for the Office of Naval Research. Prior to the development of this algorithm, the best polynomial time approximation algorithm for the Travelling Salesman Problem was a 2-approximation[4]. The Christofides Algorithm improves this by 50%.

2.1 Overview of Christofides Algorithm

Nicos Christofides describes the algorithm as "An O(n^3) heuristic algorithm ... for solving n-city travelling salesman problems ... whose cost matrix satisfies the triangle condition." Utilizing the creation of a minimum spanning tree, a minimum matching on a subset of V, and the creation of a Hamiltonian circuit, the Christofides Algorithm achieves an approximate solution such that the ratio of the algorithm answer to optimal solution is strictly less than $\frac{3}{2}$. The Christofides Algorithm consists of the following steps:

Given input of graph G=(V,E), an instance of the Travelling Salesman Problem, where all e∈E have nonnegative, real cost (without self-loops), G is a complete graph($|V| * (|V| - 1)$ edges), G is symmetric, and G obeys the triangle inequality:

1. Construct a minimum spanning tree of G
2. Let U be a subset of MST obtained from (1) above such that all nodes in MST that have odd degree ∈U
3. Construct a minimum matching on U
4. Construct multigraph G' by merging MST and U minimum matching
5. Find Eulerian Circuit on G'
6. Convert Eulerian Circuit to Hamiltonian Circuit by bypassing previously visited nodes.

2.2 Christofides Algorithm Correctness

Since the Christofides Algorithm is an approximation of the Travelling Salesman Problem, we will prove correctness of the $\frac{3}{2}$ ratio of $\frac{ALG}{OPT}$.

Theorem: Given optimal cost of TSP solution C_o, cost of Christofides solution = $C_c \leq \frac{3}{2}C_o$.

Lemma 1: If MST is a minimum spanning tree of Graph G, an instance of the Travelling Salesman Problem, then the sum of all edge costs in MST = C_{MST} < C_o.

Proof of Lemma 1: Optimal solution to the Travelling Salesman Problem is a circuit. Removing any edge e from a circuit results in tree T. We can see this since a circuit is a path that visits every vertex exactly once and returns to the starting vertex. Since the path forms a closed loop and every vertex is visited only once, removing any edge results in a connected graph with no cycles, hence, a tree.

$C_{MST} \leq C_T$ and

$C_T = C_o - C_{arbitrary\ edge\ e}$ so

$C_{MST} < C_o$.

Lemma 2: Given a graph G=(V,E) where $|V|$ is even, the optimal solution of Travelling Salesman Problem H = $\{v_1, v_2, ..., v_i, v_1\}$, and cost of optimal solution C_o, the sum of all edge weights in a minimal matching M_{min} on all v∈V = $C_{M_{min}} \leq \frac{1}{2}C_o$.

Proof of Lemma 2: Since $|V|$ is even, $|H|$ is also even, and we can divide the edges in H into two subsets, M_1 and M_2. $M_1 = \{e_{v_1,v_2}, e_{v_3,v_4}, ..., e_{v_{i-1},v_i}\}$ and $M_2 = \{e_{v_2,v_3}, e_{v_4,v_5}, ..., e_{v_i,v_1}\}$. We can say, without loss of generality, that $C_{M_1} \leq C_{M_2}$. Therefore:

$C_o = C_{M_1} + C_{M_2}$

$C_o \geq 2 * C_{M_1}$

$\frac{1}{2}C_o \geq C_{M_1}$

$C_{M_{min}} \leq C_{M_1} \leq \frac{1}{2}C_o$

Proof of Theorem:

1. For a graph G=(V,E), $C_{MST} < C_o$ by Lemma 1
2. Let U be a subset of V where all nodes in MST that have odd degree ∈U
3. Because MST is a connected graph with no cycles, number of edges in MST = (n-1). Sum of degree of nodes in MST = 2(n-1), which is even. Therefore,

$|U|$ is even.

4. Cost of traversing all edges in a minimum matching on U = $C_U \leq \frac{1}{2}$ C_o by Lemma 2.

5. Let G' be the multigraph obtained by merging MST and U. Construct a Eulerian Circuit on G'. We know G' contains a Eulerian Circuit because any nodes with odd degree in MST received exactly one added edge from the merge with U. Therefore, all nodes in G' have even degree, and a Eulerian Circuit exists.

6. As the Salesman follows the Eulerian Circuit on G', cost of traversing edge (u,v) = cost of MST($e_{u,v}$) or cost of U($e_{u,v}$). If the Eulerian Circuit visits a previously visited node, bypass that node and travel to the next unvisited node in the Eulerian Circuit. Because of the Triangle Inequality, the cost of shortcutting $\leq$ cost of following Eulerian Circuit.

7. Continue in this manner until Salesman returns to start node. Upon completion of the circuit, every edge followed by the salesman between node u and v has cost $C_{u,v} = C_{MST_{e u,v}}$ or $C_{U_{e u,v}}$.

8. Therefore, $C_c \leq C_{MST} + C_U < \frac{3}{2}$ C_o.

2.3 Christofides Algorithm Space and Time Usage

2.3.1 Christofides Space Usage

Space Usage of Christofides Algorithm is $O(n^2)$

Data Structure	Space Usage
complete graph G=(V,E)	$n+n^2$
MST of G	n + (n-1)
U, a subset of V	n
G', a union of MST and Min. Matching on U	n + 2n
H, a Eulerian Circuit on G'	2n+1
Array A, to store algorithm path	n+1

The space usage of Christofides Algorithm is dominated by the complete graph G. Assuming that a list of nodes is kept for reference and edge weights are stored in an association matrix, this matrix would be the dominating factor in space usage.

Therefore, space usage is O(n^2).

MST, U, G', and H all use O(n) space, but these are temporary data structures created and destroyed after use. The association matrix of G, however, is required in all phases of the algorithm.

2.3.2 Christofides Runtime

Runtime of Christofides Algorithm is O(n^3).

Operation	Runtime
Construct MST of G	O(n^2)
generate U	O(n)
Find Minimum Matching on U	O(n^3)
Merge MST, Minimum Matching	O(n)
Generate Eulerian Circuit	O(n)
Generate Hamiltonian Circuit	O(n)

Discussion:

Construction of MST takes O(n^2) time when using a common algorithm such as Primms and an n x n association matrix. This step could be faster, but would require a different data structure, such as a Fibonacci Heap and adjacency list. Because the space requirements include an association matrix, O(n^2) reflects this choice.

Generating U, a subset of V, is done in linear time. Simply checking the degree of each node in the MST is all that is required.

Minimum matching on U is O(n^3). [6]

Merging MST and Minimum Matching take O(n) time. There are n nodes and n-1 edges in MST, and a maximum of n nodes and $\frac{n}{2}$ edges in minimum matching.

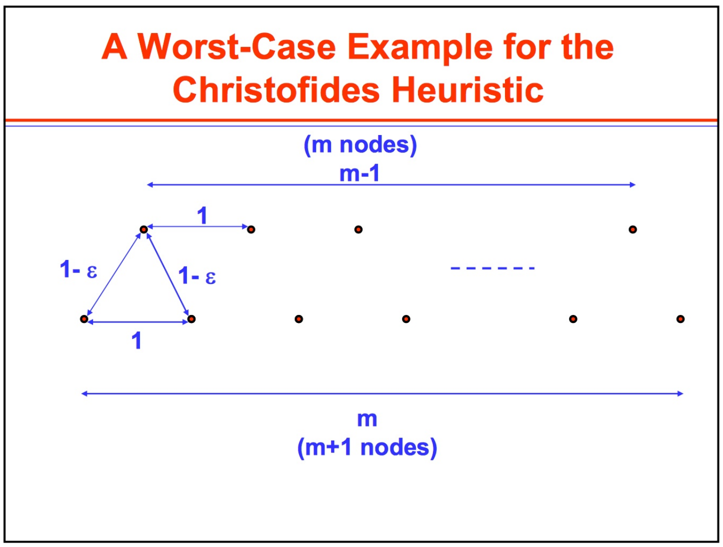
Eulerian circuit passes through each edge exactly once. There is a maximum of 2n edges in the graph at this step, so constructing the Eulerian Circuit is O(n).

Hamiltonian circuit is generated by shortcutting the Eulerian circuit, so it is also O(n).

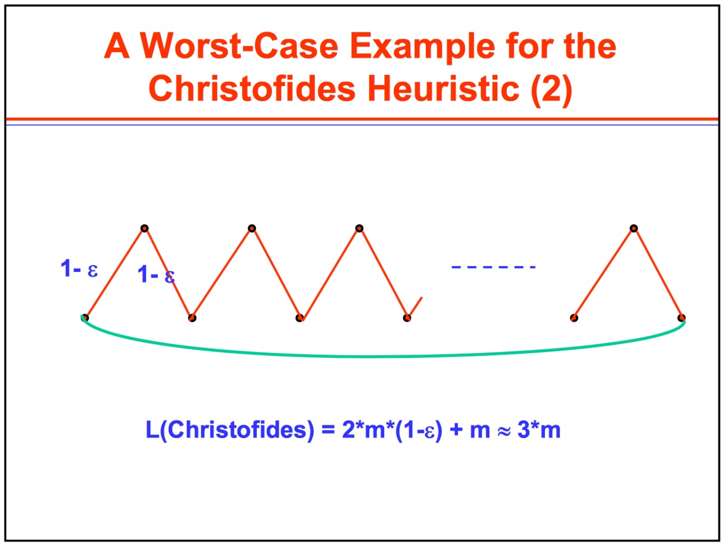
2.4 Worst-, Average-, and Best-Case Inputs

2.4.1 Worst-Case Input

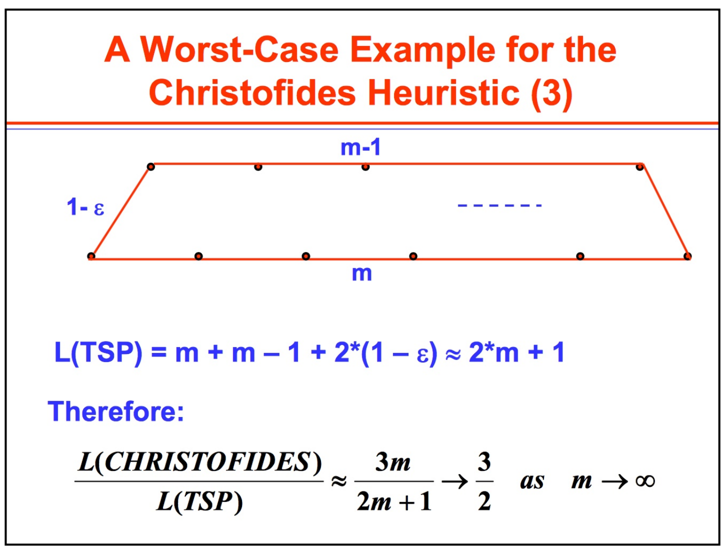
Recall that $C_c = C_{MST} + C_M$, where $C_{MST} < C_o$ and $C_M \leq \frac{1}{2}\, C_o$. To force worst-case performance of the Christofides Algorithm, we can imagine a graph G such that the optimal TSP solution is one that avoids the edges in the MST, and 'saves cost' via the Triangle inequality by bypassing edges in the MST, yet still passes through all nodes in G. One such example is a graph of odd-numbered nodes arranged in two offset rows such that one row contains m nodes, and the other contains m+1 nodes. Distance between adjacent nodes in the same row is 1, and distance between nearest-neighbors between rows is some distance less than 1, say $(1\text{-}\epsilon) = d^*$.



The MST of this graph will be a zig-zag path through the nodes, alternating between rows. Only two nodes will have odd degree, the start and end of the path found by the MST (first and last node of the (m+1)-node row). Christofides Algorithm will simply follow the path found in MST, and connect the first and last node. $C_c = 2m(d^*)+m$.



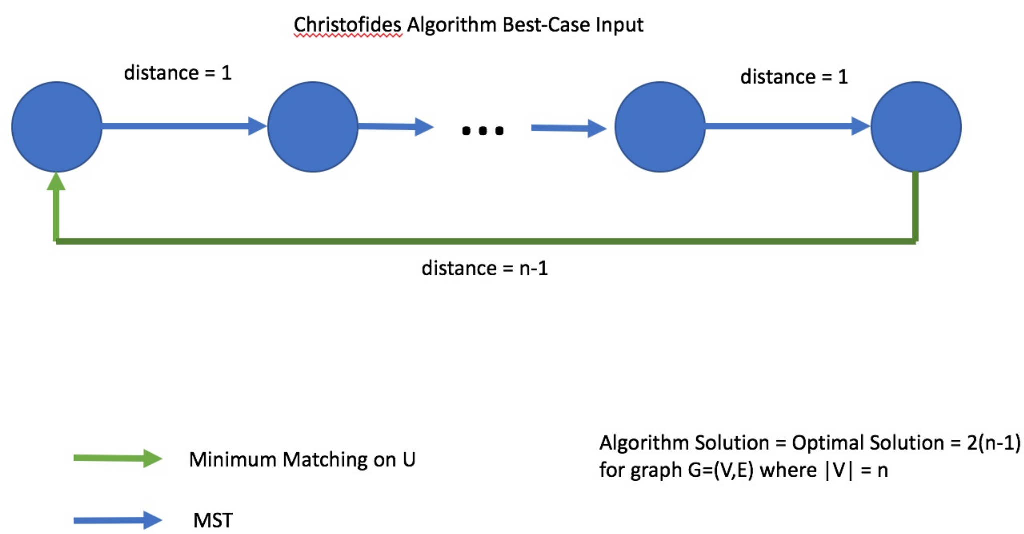
The optimal solution in this case is to start at one end of either row, connect all nodes in the row linearly, connect to the nearest end of the next row, connect all nodes in that row linearly, and return to the starting node. C_o=m+m-1+2(d*)



In this case, the approximation ratio is $\frac{2m(d*)+m}{2m+2d*-1} = \frac{3m+2d*}{2m+2d*-1}$. As $n \to \infty$, $\frac{3m+2d*}{2m+2d*-1}$ $\to \frac{3}{2}$
* example and images from [7].

2.4.2 Best-Case Input

A best-case input for the Christofides Algorithm is one in which the optimal solution follows the edges of the MST on G to the fullest extent possible. A trivial example of such a graph is a graph G with n nodes arranged linearly. Distance between adjacent nodes d* =1. Distance between first and last nodes = (n-1). The MST of G will be a linear path through all nodes. Every edge has length 1 for a total length n-1. Christofides Algorithm will follow this MST through all nodes, then connect the two odd-degree nodes, first and last. This will be identical to the optimal solution.



2.4.3 Average-Case Input

For most real-world graphs, the optimal solution will make use of some shortcuts that the algorithm will not, but not to the extent that we see in the worst-case scenario, above. Results were obtained experimentally via random graph generation, and will be discussed in a following section.

3 Experiment

3.1 Algorithm Implementation

The Christofides Algorithm was implemented in Python, using networkX, numpy, and itertools libraries for implementation of helper functions. Major steps of implementation were:

1. Randomly generate graph. Using a specified number of nodes n, n random points were generated in a 1000 x 1000 unit 'space'. Euclidean distances were then calculated between all nodes and stored in a numpy matrix for fast recall/operations.

2. A graph object was generated using networkX functionality. NetworkX was chosen for its useful graph functionality, such as finding MST and calculating maximum-weighted matchings. This was easy to convert to a minimum-weight match.

3. Using MST functionality of networkX, as described above, the MST of G was generated.

4. Nodes of odd degree in MST were found.

5. Minimum Matching was constructed via networkX Maximum Matching functionality. To use networkX maximum matching functionality to make a minimum matching, I simply subtracted edge weights from an arbitrarily large number. Smallest edge weights became largest, and vice-versa. When constructing the graph based on this matching, I re-inserted the original edge weights.

6. Merging was done via networkX add_edges() method, and a Eulerian Circuit was generated via networkX eulerian_circuit() method.

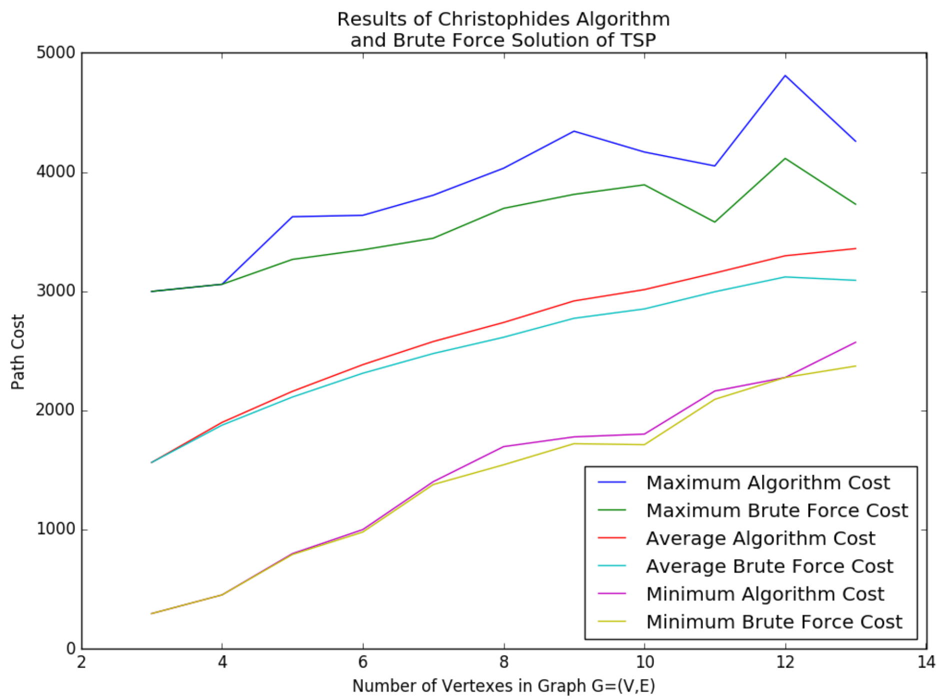
7. Converting the Eulerian Circuit to Hamiltonian Circuit was trivial. All that is required is to keep a list of visited nodes. When a visited node is encountered a second time, simply continue along the Eulerian Circuit until a previously unvisited node is encountered. Add nodes to a solution array in the order that they are visited.

8. To calculate C_c, simply add up the edge costs to reflect the edges that are traversed, as reflected by the solution array.

To compare algorithm results against optimal results, I simply brute-forced the same graphs that were randomly generated for the algorithm implementation. Using itertools.permutations(), I generated all permutations of numbers of the set {1,2,...,n-1}. 0 was reserved as the start/finish node. To avoid checking duplicate/reversed paths, any iteration where element[0] < element[n] of the permutations was ignored. Once permutations were generated and duplicates eliminated, distances were summed up to reflect the path generated by itertools. Time and path cost were recorded for all brute-force and algorithm iterations.

3.2 Experiment Results

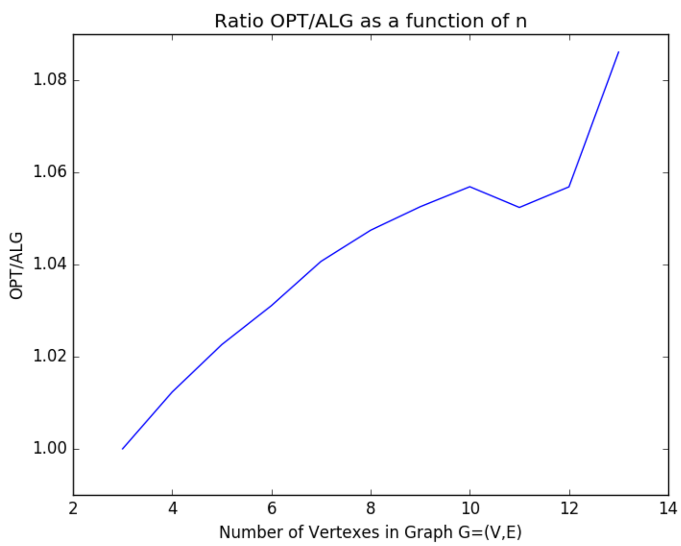
Data was collected over multiple iterations for n=[3,4,5,6,7,8,9,10,11,12,13]. Due to runtime constraints, number of iterations was decreased as n increased. At n=3 to 10, 1000 iterations were performed. At n=11 and 12, 100 iterations were performed, and at n=13, 50 iterations. Beyond n=13, runtime was prohibitive. Results are as follows:



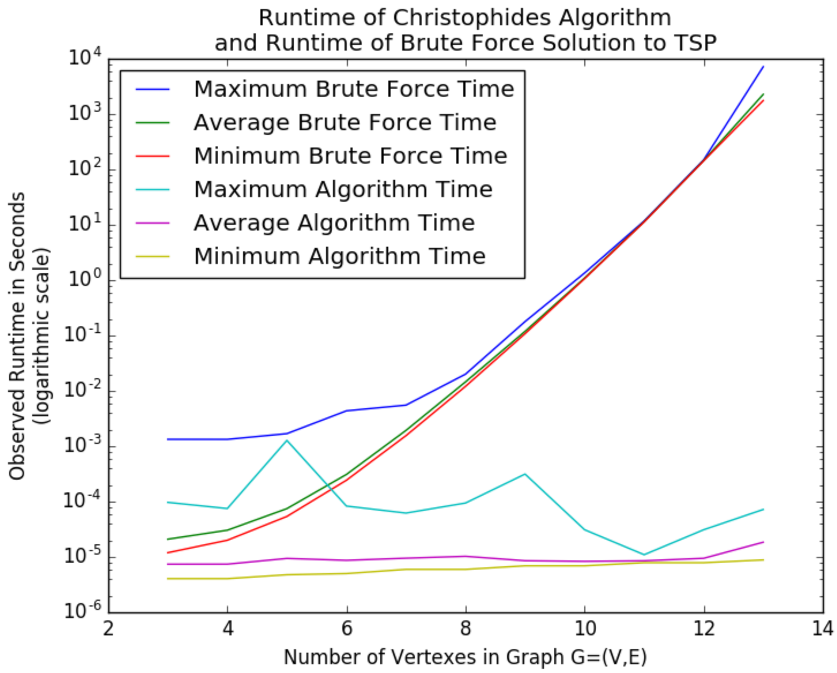
To measure accuracy of the Christofides Algorithm, we can compare average values of C_c and C_o for all n values for which simulations were run.

C_c	C_o	$\frac{ALG}{OPT}$
1564.29889711	1564.29889711	1.0
1900.19945718	1877.04997719	1.0123329055
2160.71515929	2112.91559425	1.02262256248
2384.53768121	2312.67646758	1.03107274824
2578.42763296	2477.64256672	1.04067780704
2738.23113853	2614.22379746	1.04743562552
2919.01722133	2773.3136069	1.05253773467
3013.94297714	2851.72091876	1.05688567114
3152.81751083	2995.88517515	1.05238262701
3297.50312162	3120.09718795	1.05685910501
3357.8141498	3091.62924492	1.0860985855

From the above table, it appears that, on a graph of randomly-generated locations, Christofides Algorithm is well below the worst-case ratio of $\frac{3}{2}$. However, there does appear to be a gradual increase in $\frac{OPT}{ALG}$ as n increases.



Similar examination of observed runtime of the algorithm and brute-force methods gives the following results:



Observed runtime for the brute-force method increases rapidly as n increases. Expected runtime of the brute-force method of solving the Travelling Salesman Problem was determined to be O((n-1)!), and the data of observed runtime seems to confirm that. Observed runtime for the Christofides Algorithm appears to be mostly constant, with perhaps a very slight upward trend. Expected runtime for Christofides Algorithm was determined to be $O(n^3)$, but at such small values of n, it may be that runtime is being dominated by constant-time operations, such as overhead involved in function calls, etc. As all coding was done in Python, with numpy used for matrix and numerical operations, this seems likely. Operations implemented via numpy can be expected to complete much quicker than those implemented in pure Python, since numpy is implemented in C.

To determine if the observed runtime matches expected, the ratio of $\frac{Brute-force\ Runtime}{Christofides\ Runtime}$ can be determined. Expected ratio is $\frac{(n-1)!}{n^3}$.

n	Brute-force Runtime	Algorithm Runtime	$\frac{Brute-Force}{Algorithm}$	$\frac{Expected\ brute\ force}{expected\ algorithm}$	$\frac{observed\ ratio}{expected\ ratio}$
3	2.09114551544e-05	7.39550590515e-06	2.82758954189	0.222222222222	12.7241529385
4	3.0389547348e-05	7.40838050842e-06	4.10205001126	0.375	10.93880003
5	7.45985507965e-05	9.39774513245e-06	7.93792018672	0.96	8.26866686117
6	0.000308976411819	8.68225097656e-06	35.5871320299	3.33333333333	10.676139609
7	0.00192969703674	9.49168205261e-06	203.304011454	14.693877551	13.8359674462
8	0.0145330505371	1.02622509003e-05	1416.16597356	78.75	17.9830599817
9	0.118852747202	8.53967666626e-06	13917.7104528	497.777777778	27.9596861776
10	1.0934151926	8.33058357239e-06	131253.132881	3628.8	36.1698448195
11	11.388480289	8.51631164551e-06	1337254.99524	29990.0826446	44.5899069731
12	144.883050308	9.45329666138e-06	15326193.1259	277200.0	55.2892969908
13	2271.94656908	1.84488296509e-05	123148547.202	2834328.99408	43.4489247575

*numbers represent time in seconds for one iteration

Since the observed ratio of $\frac{brute-force\ runtime}{algorithm\ runtime}$ is so much larger than the expected ratio, the observed ratio was divided by the expected ratio to see if there was a consistent relationship between the two, and that does, in fact, appear to be the case. Trends in observed runtime appear to be in line with expected.

4 Conclusions

In this paper, we have examined the Christofides Algorithm in detail. After examining the steps of the algorithm and proving the worst-case bound of $\frac{3}{2}$ of the optimal solution, experimental results were presented. These results confirm the worst-case bound, and seem to indicate that average performance is closer to optimal than the worst case limit. Observed runtime also appears to match expected runtime, $O(n^3)$. Comparison against a simple brute-force algorithm seems to confirm the expected runtime.

5 Sources

[1] "Travelling salesman problem," in Wikipedia, Wikimedia Foundation, 2016. [Online]. Available: https://en.wikipedia.org/wiki/Travelling_salesman_problem. Accessed: Dec. 8, 2016.

[2] . Matai, S. Singh and M. Lal Mittal (2010). Traveling Salesman Problem: an Overview of Applications, Formulations, and Solution Approaches, Traveling Salesman Problem, Theory and Applications, Prof. Donald Davendra (Ed.), InTech, DOI: 10.5772/12909. Available from: http://www.intechopen.com/books/traveling-salesman-problem-theory-and-applications/traveling-salesman-problem-an-overview-of-applications-formulations-and-solution-approaches

[3] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to Algorithms, 3rd ed. Cambridge, MA: The MIT Press, 2009.

[4] N. Christofides, Worst-case analysis of a new heuristic for the travelling salesman problem, Report 388, Graduate School of Industrial Administration, CMU, 1976.

[5] N. Christofides, The shortest hamiltonian chain of a graph, SIAMJ. on Appl. Math., 19, 1970, p. 689

[6] C. H. Papadimitriou and K. Steiglitz, Combinatorial optimization: Algorithms and complexity. New York, NY, United States: Dover Publications, 2000.

[7] . [Online]. Available: https://ocw.mit.edu/courses/civil-and-environmental-engineering/1-203j-logistical-and-transportation-planning-methods-fall-2006/lecture-notes/lec15.pdf. Accessed: Dec. 9, 2016.