Nominal logic for reasoning about terms with variable bindings

(Logika dziedzinowa do wnioskowania
o termach z wiązaniem zmiennych)

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Abstract

We describe logic for reasoning about terms with variable bindings.

Streszczenie

Przedstawiamy logikę dziedzinową do wnioskowania o termach z wiązaniem zmiennych.

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Introduction

1.1 Problem statement

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Chapter 2

Terms and constraints

In classical first-order logic, terms are built from variables and applications of functional symbols to other terms. In this work we expand terms with expressions closely resembling the syntax of lambda calculus, aiming to provide a flexible framework for reasoning about the lambda calculus and its derivations.

To this end we introduce an infinite set of *atoms* (denoted by lower-case letters), representing the bound variables in terms — i.e. the variables in the sense of lambda calculus. That set is disjount with the set of variables commonly found in first-order logic, which from now on we will call *variables* (and denote by uppper-case letters) as apposed to *atoms*.

Terms are given by the following grammar:

$$\begin{array}{rcll} \pi & ::= & \mathsf{id} \mid (\alpha\ \alpha)\pi & \text{(permutations)} \\ \alpha & ::= & \pi\ a & \text{(atom expressions)} \\ t & ::= & \alpha \mid \pi\ X \mid \alpha.t \mid t\ t \mid s & \text{(terms)} \end{array}$$

It is important to note that terms do not incorporate any inherent notions of computation, reduction, or binding. These expressions simply *look* like the lambda calculus but they lack the operational semantics of it. However, the intuitions associated with such expressions are not unfounded. We will observe their practical application in the sublogic of constraints that we define on top of terms to reason about notions of *freshness*, *variable binding* and *structural* order and its logical model.

Constraints are given by the following grammar:

$$\begin{array}{rcll} c & ::= & \alpha \mathbin{\#} t \mid t = t \mid t \sim t \mid t \prec t & \text{(constraints)} \end{array}$$

with following semantics:

$$
\begin{array}{rcl}
\alpha \# t & — & \text{atom } \alpha \text{ is fresh in term } t\text{, i.e. does not occur in } t \text{ as a free variable} \\
t_1 = t_2 & — & \text{terms } t_1 \text{ and } t_2 \text{ are alpha-equivalent} \\
t_1 \sim t_2 & — & \text{terms } t_1 \text{ and } t_2 \text{ possess an identical shape,} \\
 & & \text{i.e. after erasing all atoms, terms } t_1 \text{ and } t_2 \text{ would be equal} \\
t_1 \prec t_2 & — & \text{shape of term } t_1 \text{ is structurally smaller than the shape of term } t_2, \\
 & & \text{i.e. after erasing all atoms } t_1 \text{ would be equal to some subterm of } t_2
\end{array}
$$

We use metavariable Γ for finite sets of constraints.

$$
\begin{array}{lcll}
T & ::= & A \mid n \mid \$T \mid T@T \mid s & \text{(semantic terms)} \\
S & ::= & \_ \mid \_.S \mid S@S \mid s & \text{(semantic shapes)}
\end{array}
$$

$$
\begin{array}{rcl}
[\![\pi\ a]\!]_\rho & = & [\![\pi]\!]_\rho(\rho(a)) \\
[\![\pi\ X]\!]_\rho & = & [\![\pi]\!]_\rho(\rho(X)) \\
[\![\alpha.t]\!]_\rho & = & \$([\![t]\!]_\rho\uparrow)\{[\![\alpha]\!]_\rho \mapsto 0\} \\
[\![t_1\ t_2]\!]_\rho & = & [\![t_1]\!]_\rho @ [\![t_2]\!]_\rho \\
[\![s]\!]_\rho & = & s
\end{array}
$$

$$
\begin{array}{rcl}
|A| & = & \_ \\
|n| & = & \_ \\
|\$T| & = & \_.|T| \\
1|T_1@T_2| & = & |T_1|@|T_2|
\end{array}
$$

$$
\begin{array}{rll}
\rho \vDash t_1 = t_2 & \text{iff} & [\![t_1]\!]_\rho = [\![t_2]\!]_\rho \\
\rho \vDash \alpha \# t & \text{iff} & [\![\alpha]\!]_\rho \notin \mathsf{FreeAtoms}([\![t]\!]_\rho) \\
\rho \vDash t_1 \sim t_2 & \text{iff} & |[\![t_1]\!]_\rho| = |[\![t_2]\!]_\rho| \\
\rho \vDash t_1 \prec t_2 & \text{iff} & |[\![t_1]\!]_\rho| \text{ is a strict subshape of } |[\![t_2]\!]_\rho|
\end{array}
$$

We write $\rho \vDash \Gamma$ iff for all $c \in \Gamma$ we have $\rho \vDash c$. We write $\Gamma \vDash c$ iff for every ρ such that $\rho \vDash \Gamma$ we have $\rho \vDash c$.

With this model in mind we will se that there exists a decidibile algorithm for determining whether C1,...,Cn —= C0, i.e. a deterministic way of checking if constraints c1, ..., cn imply c0. We present such algorithm in the next chapter.

Chapter 3

Constraint solver

Bird's eye view: Solver breaks down constraints (on both sides of the turnstile) to irreducible components that are solved easily.

At the core of our work lies the Solver — the algorithm of resolving the constraints. Given a list of assumptions $c_1, \ldots, c_n$ it checks whether given goal c_0 holds. In other words it is an algorithm that verifies whether, for every possible substitution of closed terms (in terms of variables, not atoms) for variables in $c_0, c_1, \ldots, c_n$ such that the constraints $c_1, \ldots, c_n$ are satisfied, c_0 is also satisfied.

For convenience and effectiveness of implementation, the Solver works with constraints a little different constraints (although not more expressive) than those occuring in formulas and kinds, main difference being use of *shapes* instead of terms for shape constraints. Solver contraints and shapes are given by the following grammar:

$$\begin{array}{lcll} \mathcal{C} & ::= & \alpha \,\#\, t \mid t = t \mid S \sim S \mid S \prec S & \text{(solver constraints)} \\ S & ::= & \_ \mid X \mid \_.S \mid S\ S \mid s & \text{(shapes)} \end{array}$$

Solver erases atoms from terms in shape contstraints, effectively transforming them from *constraints* to *solver constraints*.

We add another environment Δ to distinguish between the potentially-reducible assumptions in Γ. For convenience we will write $a \neq \alpha$ instead of $a \,\#\, \alpha$ as it gives good intuition of atom freshness implying inequality and for $\alpha = \pi a$ we will write $\alpha \,\#\, t$ meaning $a \,\#\, \pi^{-1} t$. Irreducible constraints are:

$$\begin{array}{rcl} a_1 \neq a_2 & — & \text{atoms } a_1 \text{ and } a_2 \text{ are different} \\ a \,\#\, X & — & \text{atom } a \text{ is fresh in variable } X \\ X_1 \sim X_2 & — & \text{variables } X_1 \text{ and } X_2 \text{ posses the same shape} \\ X \sim t & — & \text{variable } X \text{ has a shape of term } t \\ t \prec X & — & \text{term } t \text{ strictly subshapes variable } X \end{array}$$

After all the constraints are reduced to such simple constraints we reduce the goal-constraint and repeat the reduction procedure on new assumptions and goal. We either arrive on a contradictory environment or all the assumptions and goal itself are reduced to irreducible constraints which is as simple as checking if the goal occurs on the left side of the turnstile.

$$\frac{\mathcal{C} \in \Delta}{\Gamma; \Delta \vDash \mathcal{C}}$$

Decidability of atom equality plays an important role in the reduce procedure:

$$\frac{\Gamma; \Delta \vDash a = \pi^{-1}\alpha}{\Gamma; \Delta \vDash \pi a = \alpha} \qquad \frac{\begin{array}{c} a \neq \alpha_1, a \neq \alpha_2, \Gamma; \Delta \vDash a = \alpha \\ a = \alpha_1, a \neq \alpha_2, \Gamma; \Delta \vDash \alpha_2 = \alpha \qquad a = \alpha_2, \Gamma; \Delta \vDash \alpha_1 = \alpha \end{array}}{\Gamma; \Delta \vDash a = (\alpha_1\ \alpha_2)\alpha}$$

$$\frac{\Gamma; \Delta \vDash \pi \text{ idempotent on } X}{\Gamma; \Delta \vDash X = \pi X} \qquad \frac{\Gamma; \Delta \vDash X_1 = \pi_1^{-1}\pi_2 X_2}{\Gamma; \Delta \vDash \pi_1 X_1 = \pi_2 X_2}$$

$$\frac{\Gamma; \Delta \vDash \alpha_1 \# t_2 \qquad \Gamma; \Delta \vDash t_1 = (\alpha_1\ \alpha_2)t_2}{\Gamma; \Delta \vDash \alpha_1.t_1 = \alpha_2.t_2} \qquad \frac{\Gamma; \Delta \vDash t_1 = t_2 \qquad \Gamma; \Delta \vDash t_1' = t_2'}{\Gamma; \Delta \vDash t_1 t_1' = t_2 t_2'}$$

$$\frac{}{\Gamma; \Delta \vDash a = a} \qquad \frac{}{\Gamma; \Delta \vDash X = X} \qquad \frac{}{\Gamma; \Delta \vDash s = s}$$

$$\frac{\forall a \in \pi.\, \Gamma; \Delta \vDash a = \pi a \ \vee\ \Gamma; \Delta \vDash a \# X}{\Gamma; \Delta \vDash \pi \text{ idempotent on } X}$$

$$\frac{a_1 \neq a_2 \in \Delta}{\Gamma; \Delta \vDash a_1 \# a_2} \qquad \frac{\begin{array}{c} a \neq \alpha_1, a \neq \alpha_2, \Gamma; \Delta \vDash a \# \alpha \\ a = \alpha_1, a \neq \alpha_2, \Gamma; \Delta \vDash \alpha_1 \# \alpha \qquad a = \alpha_2, \Gamma; \Delta \vDash \alpha_2 \# \alpha \end{array}}{\Gamma; \Delta \vDash a \# (\alpha_1\ \alpha_2)\alpha}$$

$$\frac{a \# X \in \Delta}{\Gamma; \Delta \vDash a \# X} \qquad \frac{\begin{array}{c} a \neq \alpha_1, a \neq \alpha_2, \Gamma; \Delta \vDash a \# \pi X \\ a = \alpha_1, a \neq \alpha_2, \Gamma; \Delta \vDash \alpha_1 \# \pi X \qquad a = \alpha_2, \Gamma; \Delta \vDash \alpha_2 \# \pi X \end{array}}{\Gamma; \Delta \vDash a \# (\alpha_1\ \alpha_2)\pi X}$$

$$\frac{a \neq \alpha, \Gamma; \Delta \vDash a \# t}{\Gamma; \Delta \vDash a \# \alpha.t} \qquad \frac{\Gamma; \Delta \vDash a \# t_1 \qquad \Gamma; \Delta \vDash a \# t_2}{\Gamma; \Delta \vDash a \# t_1 t_2} \qquad \frac{}{\Gamma; \Delta \vDash a \# s}$$

$$\frac{X_1 \sim X_2 \in \Delta}{\Gamma; \Delta \vDash X_1 \sim X_2} \qquad \frac{X \sim S' \in \Delta \qquad \Gamma; \Delta \vDash S' \sim S}{\Gamma; \Delta \vDash X \sim S}$$

$$\frac{\Gamma; \Delta \vDash S_1 \sim S_2}{\Gamma; \Delta \vDash \_.S_1 \sim \_.S_2} \qquad \frac{\Gamma; \Delta \vDash S_1 \sim S_2 \qquad \Gamma; \Delta \vDash S_1' \sim S_2'}{\Gamma; \Delta \vDash S_1 S_1' \sim S_2 S_2'} \qquad \frac{}{\Gamma; \Delta \vDash s \sim s}$$

$$\frac{S_2 \prec X \in \Delta \qquad \Gamma; \Delta \vDash S_2 \sim X}{\Gamma; \Delta \vDash S_1 \prec X} \qquad \frac{S_2 \prec X \in \Delta \qquad \Gamma; \Delta \vDash S_2 \prec X}{\Gamma; \Delta \vDash S_1 \prec X}$$

$$\frac{\Gamma; \Delta \vDash S_1 \sim S_2}{\Gamma; \Delta \vDash S_1 \prec \_.S_2} \qquad \frac{\Gamma; \Delta \vDash S_1 \prec S_2}{\Gamma; \Delta \vDash S_1 \prec \_.S_2}$$

$$\frac{\Gamma;\Delta\vDash S_1\sim S_2}{\Gamma;\Delta\vDash S_1\prec S_2S_2'}\qquad\frac{\Gamma;\Delta\vDash S_1\sim S_2'}{\Gamma;\Delta\vDash S_1\prec S_2S_2'}\qquad\frac{\Gamma;\Delta\vDash S_1\prec S_2}{\Gamma;\Delta\vDash S_1\prec S_2S_2'}\qquad\frac{\Gamma;\Delta\vDash S_1\prec S_2'}{\Gamma;\Delta\vDash S_1\prec S_2S_2'}$$

$$\frac{a_1\neq a_2\in\Delta}{a_1=a_2,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{\Gamma\{a_1\mapsto a_2\};\Delta\{a_1\mapsto a_2\}\vDash\mathcal{C}\{a_1\mapsto a_2\}}{a_1=a_2,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{\begin{array}{c}a\neq\alpha_1,a\neq\alpha_2,a=\alpha,\Gamma;\Delta\vDash\mathcal{C}\\ a=\alpha_1,a\neq\alpha_2,\alpha_2=\alpha,\Gamma;\Delta\vDash\mathcal{C}\qquad a=\alpha_2,\alpha_1=\alpha,\Gamma;\Delta\vDash\mathcal{C}\end{array}}{a=(\alpha_1\ \alpha_1)\alpha,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{a=\pi^{-1}\alpha,\Gamma;\Delta\vDash\mathcal{C}}{\pi a=\alpha,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{}{a=t_1t_2,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{}{a=\alpha.t,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{}{a=s,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{\vDash\ \text{idempotent on}\ X}{X=\pi X,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{\pi\ \text{idempotent on}\ X,\Gamma;\Delta\vDash\mathcal{C}}{X=\pi X,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{\Gamma\{X\mapsto t\};\Delta\{X\mapsto t\}\vDash\mathcal{C}\{X\mapsto t\}}{X=t,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{X=\pi^{-1}t,\Gamma;\Delta\vDash\mathcal{C}}{\pi X=t,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{\alpha_1\#\alpha_2.t_2,\ t_1=(\alpha_1\ \alpha_2)t_2,\ \Gamma;\Delta\vDash\mathcal{C}}{\alpha_1.t_1=\alpha_2.t_2,\Gamma;\Delta\vDash\mathcal{C}}\qquad\text{Other term constructors trivial}$$

$$\frac{t_1=t_2,\ t_1'=t_2',\ \Gamma;\Delta\vDash\mathcal{C}}{t_1t_1'=t_2t_2',\Gamma;\Delta\vDash\mathcal{C}}\qquad\text{Other term constructors trivial}$$

$$\frac{s_1\neq s_2}{s_1=s_2,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{}{s=s,\Gamma;\Delta\vDash\mathcal{C}}\qquad\text{Other term constructors trivial}$$

$$\frac{(\forall a\in\pi.\ \Gamma;\Delta\vDash a=\pi a\ \vee\ \Gamma;\Delta\vDash a\#X),\Gamma;\Delta\vDash\mathcal{C}}{\pi\ \text{idempotent on}\ X,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{}{a\neq a,\Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{\Gamma;a_1\neq a_2\cup\Delta\vDash\mathcal{C}}{a_1\neq a_2,\ \Gamma;\Delta\vDash\mathcal{C}}\qquad\frac{\Gamma;a\#X\cup\Delta\vDash\mathcal{C}}{a\#X,\ \Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{\begin{array}{c}a\neq\alpha_1,a\neq\alpha_2,a\#\alpha,\Gamma;\Delta\vDash\mathcal{C}\\ a=\alpha_1,a\neq\alpha_2,\alpha_2\#\alpha,\Gamma;\Delta\vDash\mathcal{C}\qquad a=\alpha_2,\alpha_1\#\alpha,\Gamma;\Delta\vDash\mathcal{C}\end{array}}{a\#(\alpha_1\ \alpha_1)\alpha,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{\begin{array}{c}a\neq\alpha_1,a\neq\alpha_2,a\#X,\Gamma;\Delta\vDash\mathcal{C}\\ a=\alpha_1,a\neq\alpha_2,\alpha_2\#X,\Gamma;\Delta\vDash\mathcal{C}\qquad a=\alpha_2,\alpha_1\#X,\Gamma;\Delta\vDash\mathcal{C}\end{array}}{a\#(\alpha_1\ \alpha_1)X,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{a\#\alpha,\ \Gamma;\Delta\vDash\mathcal{C}\qquad a\#\alpha,\ a\#t\cup\Gamma;\Delta\vDash\mathcal{C}}{a\#\alpha.t,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{a\#t_1,\Gamma;\Delta\vDash\mathcal{C}\qquad a\#t_2,\Gamma;\Delta\vDash\mathcal{C}}{a\#t_1t_2,\Gamma;\Delta\vDash\mathcal{C}}$$

$$\frac{\Gamma; \Delta \vDash \mathcal{C}}{a \# s, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{\Gamma; X_1 \sim X_2 \cup \Delta \vDash \mathcal{C}}{X_1 \sim X_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{\Gamma; X \sim S \cup \Delta \vDash \mathcal{C}}{X \sim S,\ \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{\Gamma; \Delta \vDash \mathcal{C}}{a_1 \sim a_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \text{Other term constructors trivial}$$

$$\frac{t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C}}{\_.t_1 \sim \_.t_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \text{Other term constructors trivial}$$

$$\frac{t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C} \qquad t_1' \sim t_2', \Gamma; \Delta \vDash \mathcal{C}}{t_1 t_1' \sim t_2 t_2', \Gamma; \Delta \vDash \mathcal{C}} \qquad \text{Other term constructors trivial}$$

$$\frac{s_1 \neq s_2}{s_1 \sim s_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{}{s \sim s, \Gamma; \Delta \vDash \mathcal{C}} \qquad \text{Other term constructors trivial}$$

$$\frac{\Gamma; t \prec X \cup \Delta \vDash \mathcal{C}}{t \prec X, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C} \qquad t_1 \prec t_2, \Gamma; \Delta \vDash \mathcal{C}}{t_1 \prec \_.t_2, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C} \qquad t_1 \sim t_2', \Gamma; \Delta \vDash \mathcal{C} \qquad t_1 \prec t_2, \Gamma; \Delta \vDash \mathcal{C} \qquad t_1 \prec t_2', \Gamma; \Delta \vDash \mathcal{C}}{t_1 \prec t_2 t_2', \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{}{t \prec \alpha; \Delta \vDash \mathcal{C}} \qquad \frac{}{t \prec s; \Delta \vDash \mathcal{C}}$$

TODO: explain what is $\mathcal{C} \cup \Delta$

Define state of the solver by triple $(\Gamma, \Delta, \mathcal{C}_0)$ and such ordering of the states:

1. Number of distinct variables in Γ, Δ,$\mathcal{C}_0$.
2. Depth of $\mathcal{C}_0$.
3. Number of assumptions of given depth in Γ and Δ.
4. Number of assumptions of given depth in Γ.

Then by analysing each rule we can see the reductions always arrive in a smaller state.

3.1 Implementation

TODO: Description of the special operations over environment Δ and occurs check

Chapter 4

Higher Order Logic

4.1 Kinds

4.2 Subkinding

4.3 Formulas

4.4 Fixpoint

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Chapter 5

Proof theory

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Chapter 6

Proof assistant

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Chapter 7

Example: Progress and Preservation of STLC

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Chapter 8

Conclusion and future work

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Bibliography

[1] ...