Domain-specific logic for terms with variable binding

(Logika dziedzinowa do wnioskowania
o termach z wiązaniem zmiennych)

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Abstract

In this work, we address a fundamental distinction between manual and computer-based proof systems, emphasizing the challenge of maintaining precision and transparency in handling variable binding. The common practice of making unspoken assumptions in pen-and-paper proofs, particularly the use of imprecise notion of „sufficiently fresh names”, introduces potential pitfalls when translating to formal and rigorous proof systems.

Nominal Logic, as introduced by Andrew M. Pitts, emerges as a promising solution to bridge this gap, offering a first-order theory of names and binding. This approach allows for the definition of essential concepts, including alpha-equivalence, freshness, and variable binding, solely in terms of name swapping rather than classical renaming.

Building upon Pitts' work, we introduce a specialized variant of Nominal Logic, where we define constraints — precise descriptors of syntactical properties — and use them to reason about terms with variable binding. We introduce The Solver, an algorithm of automated constraint resolution, which forms the logical core of the constraints sublogic, and acts as a middle ground between human and computer provers. Layered on top of the constraints, we define a higher order logic, with constraints embedded into propositional formulas and relations.

Alongside this logic, we establish a proof system and a proof assistant implemented in OCaml and inspired by HOL theorem provers. The integration of these components forms a cohesive framework for precise articulation of and reasoning about complex syntactic properties. To demonstrate its potential for programming language reasoning, we conduct proofs of classical properties of simply typed lambda calculus using this framework.

Streszczenie

Przedstawiamy logikę dziedzinową do wnioskowania o termach z wiązaniem zmiennych.

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Chapter 1

Introduction

One of the fundamental distinctions between conducting proofs manually with pen and paper and using a computer lies in the flexibility and liberties one can take in the first case. Human provers and reviewers often agree upon unexplained or unproven assumptions and may skip some unimportant boilerplate. Computers, on the other hand, are less forgiving and demand transparency and justification down to the smallest details.

A common assumption we commonly make when writing pen-and-paper proofs pertains to working with abstract syntax trees, where we assume that the variables we choose are fresh enough or that substitutions avoid issues like variable capture. For instance, when dealing with lambda calculus, we often construct inductive proofs over the structure of expression, where in the case for an abstracion we will implicitly only show the case where the variable bound in that abstraction is *sufficiently fresh.* Addressing the general case could introduce unnecessary complexities unrelated to the theorem at hand. Justifiably, we skip over this detail — however, the induction principle obliges us to prove the case for arbitrary variable names.

Addressing this gap in formal reasoning requires careful considerations to come up with a resolution. Fortunately, there exist some solutions to that problem — and one particular approach, coined *nominal logic* and introduced by Andrew M. Pitts[2] is of most interest to this work.

1.1 Nominal approach

Pitts' work introduces *nominal logic*, a first-order theory of names, swapping, and freshness, that amongst other novelties, introduces the precise mathematical definition describing the concept of "sufficiently fresh names", which, as Pitts argues, bridges the gap between formal mathematical reasoning and the informal practices mentioned earlier.

Andrew M. Pitts, *"Nominal logic, a first order theory of names and binding"*[2]:

Names of what? Names of entities that may be subject to binding by some of the syntactical constructions under consideration. In Nominal Logic these sorts of names, the ones that may be bound and hence that may be subjected to swapping without changing the validity of predicates involving them, will be called atoms.

Pitts chose to found his theory around the notion of swapping names as opposed to the classical renaming. In the author's previous work[6], written together with Murdoch J. Gabbay, it was shown that a theory based on this operation allows for all necessary concepts, including alpha-equivalence, freshness, and variable-binding, to be defined solely in terms of swapping pairs of names.

Additionaly, swapping has one other useful logical properties — it is involutive (i.e. a swap gets nullified by applying the same swap again, while substitutions cannot always be reversed), which, as Pitts argues, means that *equivariant* predicates (i.e. those whose validity is invariant under atom-swapping) have excellent logical properties. This class of equivariant predicates includes equality, alpha-equivalence and is closed under standard logical connectives, universal and existential quantification, and formation of least and greatest fixpoint.

As an example of Nominal Logic at work, consider the abstract syntax tree of untyped lambda calculus, given by the grammar below, where a ranges over an infinite set of names — or rather *atoms*.

$$t ::= a \mid \lambda a.t \mid t\ t \qquad \text{(lambda terms)}$$

Figure 1.1: Terms of untyped lambda calculus

The definition of swapping atoms a and b in some tree t, written $(a\ b)\ t$, is rather straightforward. It naturally follows the tree structure, touching only the affected

$$\begin{array}{rcl} (a\ b)(\lambda c.t) & := & \lambda((a\ b)c).((a\ b)t) \\ (a\ b)(t_1\ t_2) & := & ((a\ b)t_1)\ ((a\ b)t_2) \end{array} \qquad (a\ b)c := \begin{cases} a & \text{if } c = b \\ b & \text{if } c = a \\ c & \text{otherwise} \end{cases}$$

Figure 1.2: Swapping procedure

atoms, and doesn't need to distinct betwen free and bound names (like substitutions do), but simply changes them all the same exact way.

$$\frac{a \neq b}{a \mathrel{\#} b} \qquad \frac{a \mathrel{\#} t_1 \quad a \mathrel{\#} t_2}{a \mathrel{\#} t_1\ t_2} \qquad \frac{}{a \mathrel{\#} \lambda a.t} \qquad \frac{a \mathrel{\#} t}{a \mathrel{\#} \lambda b.t}$$

Figure 1.3: Freshness relation

Relation of *freshness* of atom a in tree t, written $a \mathrel{\#} t$, is similarly simple to define.[1] Note that it only assumes the comparability of atoms and is an *equivariant* relation, which can be shown by simplest induction.

$$\frac{}{a =_\alpha a} \qquad \frac{t_1 =_\alpha t_1' \quad t_2 =_\alpha t_2'}{t_1\ t_2 =_\alpha t_1'\ t_2'} \qquad \frac{(a\ b)t =_\alpha (a'\ b)t' \quad b \mathrel{\#} t \quad b \mathrel{\#} t'}{\lambda a.t =_\alpha \lambda a'.t'}$$

Figure 1.4: Alpha-equivalence relation

With *swapping* and *freshness* already established, we define the alpha-equivalence of terms, written $t_1 =_\alpha t_2$. We built this definition of alpha-equivalence using only induction, swapping, and freshness then, as Pitts argues, it is equivariant as well.

Andrew M. Pitts, *"Nominal logic, a first order theory of names and binding"*[2]:

The fundamental assumption underlying Nominal Logic is that *the only predicates we ever deal with* (when describing properties of syntax) *are equivariant ones, in the sense that their validity is invariant under swapping* (i.e., transposing, or interchanging) *names.*

1.2 Motivation and contributions

Nominal logic opens avenues for expressing alpha-equivalence, freshness and other fundamental syntactic properties with elegance. Formalizing theories within such system calls for a robust framework, ideally accompanied by a proof assistant. To achieve the bigger goal of abstracting away the mundane hanlding of these properties, which are so obvious to the human eye, yet non-trivial from the point of view of rigorous computer accuracy, we strive for automatic deductive process.

We categorize the fundamental properties of terms with variable binding, such as alpha equivalence and freshness, as *constraints.* As a middle ground between human and computer provers, we introduce *the Solver*, an algorithm designed to automatically resolve new constraints based on the pre-established ones. It serves as the logical core of the constraints sublogic, that together with the embedding

[1]Pitts defines it as a not being a member of the *support set* of $t$but for our purposes, the simple inductive definition will suffice.

of constraints into propositional formulas constructs a higher-order logic capable of seamlessly expressing these properties. This approach we've taken, liberates users from the painstaking task of manually proving the seemingly trivial but crucial details, through automated resolution of constraints, while ensuring the completeness and correctness of written proofs.

For the user interface, we have developed a proof checker and proof assistant, tying all the parts together in a cohesive framework. The proof assistant draws inspiration from the HOL family of theorem provers, initially introduced by Michael J. C. Gordon[7]. Similar to HOL, it utilizes the OCaml programming language as the interface to writing proofs and encoding theorems. While currently somewhat low-level, with further automation efforts, it should achieve intuitiveness and user-friendliness akin to other, more mature and powerful proof assistants.

1.3 Related work

Of course, there's other works that focus on reasoning about syntactical properties of binders, as they are essential in formalizing properites of programming languages.

- **Higher-Order Abstract Syntax** (HOAS) introduced by Frank Pfenning and Conal Elliott[9] is a uniform and generic representation of terms, formulas, programs, and other syntactic objects used in formal reasoning systems that focus on substitution and unification under the presence of binders. Authors utilize the binding construct of the implementation language to represent the binding in the language being formalized.

- **Beluga** is a programming framework designed for reasoning about formal systems. Based on the LF logical framework, it encodes HOAS approach using dependent types and provides support for reasoning with context and contextual objects. It's developed at the Complogic group at McGill University, led by Professor Brigitte Pientka[11].

- **Twelf** is a framework used to specify, implement, and prove properties of deductive systems and logics. It's based on the LF logical framework, and uses Elf constraint logic programming langauge. The principal authors of Twelf are Frank Pfenning, and Carsten Schürmann[10]. Multiple reasearch projects were developed using Twelf, including a type safety proof for Standard ML[8].

- **Parametric Higher-Order Abstract Syntax** (PHOAS) improves on the idea of HOAS, by utilizing dependently-typed abstract syntax trees to formalize it in general-purpose type theories, like Coq's Calculus of Inductive Constructions. Introduced by Adam Chlipala[4], it has been used to develop certified, executable program transformations over several formalizations of statically-typed functional programming languages.

- **Locally Nameless Representation** is an approach to representation of syntax with variable binders, introduced by Arthur Charguéraud[3]. It represents the bound variables through de Bruijn indices, while retaining names of the free variables, achieving strong induction principles. Utilizing the Coq library TLC developed by Charguéraud, the approach has successfully formalized diverse type systems and semantics.

- **Autosubst**[12] is a Coq library that automates some crucial parts of formalizing syntactic theories with variable binders, developed by Steven Schäfer, Tobias Tebbi, and Gert Smolka. Authors employ de Bruijn representation of terms with additional binding annotations to automatically derive the substitution operation and proofs of substitution lemmas. They introduce an automation tactic that solves equations involving terms and substitutions, based on their work on the decision procedure of equational theory of an extension of the sigma-calculus by Abadi et al[1].

Chapter 2

Terms and constraints

To properly describe our framework and constraints sublogic, we must start with the simplest elements: *names*, *terms*, and *constraints*.

$$
\begin{array}{rcll}
\pi & ::= & \mathsf{id} \mid (\alpha\ \alpha)\pi & \text{(permutations)} \\
\alpha & ::= & \pi\ a & \text{(atom expressions)} \\
t & ::= & \alpha \mid \pi\ X \mid \alpha.t \mid t\ t \mid f & \text{(terms)} \\
c & ::= & \alpha \# t \mid t = t \mid t \sim t \mid t \prec t \mid \text{symbol}\ t & \text{(constraints)} \\
s & ::= & \_ \mid X \mid \_.s \mid s\ s \mid f & \text{(shapes)}
\end{array}
$$

Figure 2.1: Syntax of constraint sublogic

The names are drawn from an infinite set of *atoms* (represented by lowercase letters) and correspond to the bound variables in terms, analogous to the variables in the lambda calculus. This set is disjoint from the set of variables commonly used in first-order logic, which we will refer to as *variables* (denoted by uppercase letters).

The terms are constructed to mimic the structure of abstract syntax trees of the lambda calculus, extending it with notion of permutations (of atoms) and functional symbols, denoted by metavariable f, that are drawn from yet another set disjoint with atoms and variables.

Construction $\alpha.t$ represents a *binder* — informally, we think of it as binding the occurences of α in t, similarly to a lambda abstraction — yet it *isn't* a binder, but a simple syntactical construction glueing together an atom with another term. The semantics of binding will apply only after we interpret this syntactical term in the model.

Also note that we do not restrict this construction to the form of $a.t$, but allow permuted atoms to appear under binders. Additionaly, when dealing with atom expressions with identity permutation $\mathsf{id}\ a$ we will skip the permutation and simply write a, and sometimes call such atom expressions *pure*. The same rules apply to permuted variables.

$$\begin{array}{rcl}
\pi\ (\pi'\ a) & := & (\pi \mathbin{+\!\!+} \pi')\ a \\
\pi\ (\pi'\ X) & := & (\pi \mathbin{+\!\!+} \pi')\ X \\
\pi\ (\alpha.t) & := & (\pi\ \alpha).(\pi\ t) \\
\pi\ (t_1\ t_2) & := & (\pi\ t_1)\ (\pi\ t_2) \\
\pi\ f & := & f
\end{array}
\qquad
\begin{array}{rcl}
|\pi\ a| & := & \_ \\
|\pi\ X| & := & X \\
|\alpha.t| & := & \_.|t| \\
|t_1\ t_2| & := & |t_1|\ |t_2| \\
|f| & := & f
\end{array}$$

$$\begin{array}{rcl}
\downarrow \mathsf{id}\ a & := & a \\
\downarrow (\pi \mathbin{+\!\!+} (\alpha_1\ \alpha_2))\ a & := & \downarrow \pi\ a' \\
\text{where }\ a' & := & \downarrow (\alpha_1\ \alpha_2) a
\end{array}
\qquad
\begin{array}{rcl}
\downarrow (\alpha_1\ \alpha_2)\ a & := & \begin{cases} a_2 & \text{if } a = a_1 \\ a_1 & \text{if } a = a_2 \\ a & \text{otherwise} \end{cases} \\
\text{where }\ a_1 & := & \downarrow \alpha_1 \\
\text{and }\ a_2 & := & \downarrow \alpha_2
\end{array}$$

Figure 2.2: Operations on atoms

Although permutations only appear next to atoms and variables, we define the operation of applying a permutation to a whole term. We also define the shapes and how to compute the shape of term, which we will use to resolve contraints about shape. Permutations only affect atoms, they are stored at the variables too, for when the variable is substituted for a term, they would be used to permute the substituted term and present the ordering of applying swaps in that permutation.

Additionally, we define the operation of normalizing atom expressions, denoted as $\downarrow \alpha$, which essentially "computes" or "applies" the permutation on an atom. Technically, within our framework, this operation is never explicitly utilized; instead, permuted atoms are handled through the Solver rules. However, we include its definition here to offer the reader insight into interpreting atom expressions.

The constraints are precise descriptions of syntactical properties, describing the relationship between their arguments — atoms and terms. It's crucial to emphasize that these terms and constraints function solely as data structures do not incorporate notions of binding or reduction by themselves. These properties can only appear after we interpret constraints within the logical model, which allows us to then reason about concepts such as *freshness*, *variable binding*, and *structural* order.

2.1 Model

To build the mathematical model of terms and constraints, we introduce *semantic terms* and *semantic shapes* that will inhabit it. We will use metavariable A for

$\alpha \# t$	Atom α is fresh in term t, meaning it does not occur in t as a free variable.
$t_1 = t_2$	Terms t_1 and t_2 are alpha-equivalent.
$t_1 \sim t_2$	Terms t_1 and t_2 possess an identical shape, i.e., after erasing all atoms, terms t_1 and t_2 would be equal.
$t_1 \prec t_2$	The shape of term t_1 is structurally smaller than the shape of term t_2, i.e., after erasing all atoms, t_1 would be equal to some subterm of t_2.
symbol t	term t is equal to some functional symbol.

Figure 2.3: Informal semantics of constraints

$$\begin{array}{lcll} T & ::= & A \mid n \mid \$T \mid T@T \mid f & \text{(semantic terms)} \\ S & ::= & \_ \mid \$S \mid S@S \mid f & \text{(semantic shapes)} \end{array}$$

Figure 2.4: Semantic representation of terms and shapes

semantic names drawn from an infinite set of names, representing the free variables. Binders in semantic terms are achieved by De Bruijn indices[5] and consequently the bound names are represented by natural numbers, denoted by n, and the binding construction has no explicit argument, denoted by \$.

$$\begin{array}{rcl rcl} [\![\pi\ a]\!]_\rho & := & [\![\pi]\!]_\rho(\rho(a)) & |A| & := & \_ \\ [\![\pi\ X]\!]_\rho & := & [\![\pi]\!]_\rho(\rho(X)) & |n| & := & \_ \\ [\![\alpha.t]\!]_\rho & := & \$([\![t]\!]_\rho\!\uparrow)\{[\![\alpha]\!]_\rho \mapsto 0\} & |\$T| & := & \$|T| \\ [\![t_1\ t_2]\!]_\rho & := & [\![t_1]\!]_\rho@[\![t_2]\!]_\rho & |T_1@T_2| & := & |T_1|@|T_2| \\ [\![f]\!]_\rho & := & f & |f| & := & f \end{array}$$

Figure 2.5: Interpretation of terms and shapes in the model

The term interpretation function, denoted $[\![\cdot]\!]_\rho$, maps syntactic terms to semantic terms, utilizing the standard shifting of De Bruijn indices (denoted by $\uparrow$). It is parametrized by function ρ that maps atoms and variables to semantic shapes.
The shape interpretation function, denoted $|\cdot|$, maps semantic terms to semantic shapes by erasing names.

With above machinery, establish the relation $\rho \vDash c$ that interprets the constraints in our model, using some mapping ρ. Note that the freshness can be expressed through membership check of FreeAtoms set, which is trivial to compute as a consequence of using of De Bruijn indices. Note that it's possible for terms of form $a.X$ and $b.Y$ to be equal in this model.

We will use metavariable Γ to represent finite sets of constraints, and write $\rho \vDash \Gamma$

$$\begin{array}{lll}
\rho \vDash t_1 = t_2 & \text{iff} & [\![t_1]\!]_\rho = [\![t_2]\!]_\rho \\
\rho \vDash \alpha \mathrel{\#} t & \text{iff} & [\![\alpha]\!]_\rho \notin \mathsf{FreeAtoms}([\![t]\!]_\rho) \\
\rho \vDash t_1 \sim t_2 & \text{iff} & |[\![t_1]\!]_\rho| = |[\![t_2]\!]_\rho| \\
\rho \vDash t_1 \prec t_2 & \text{iff} & |[\![t_1]\!]_\rho| \text{ is a strict subshape of } |[\![t_2]\!]_\rho|
\end{array}$$

Figure 2.6: Constraint interpretation in the model

if for all $c \in \Gamma$, we have $\rho \vDash c$, as well as write $\Gamma \vDash c$ if for every ρ such that $\rho \vDash \Gamma$, we have $\rho \vDash c$. In the next chapter, we present the deterministic *Solver* algorithm that emulates this model by syntatically verifying statements of form $\Gamma \vDash c$.

Chapter 3

Constraint solver

At the heart of our work lies the Solver, an algorithm designed to resolve constraints. For any assumed constraints $c_1, \ldots, c_n$, and goal constraint c_0, the Solver determines whether judgment $c_1, \ldots, c_n \vDash c_0$ holds. Meaning that for every possible substitution of variables into closed terms in constraints $c_0, c_1, \ldots, c_n$, such that $c_1, \ldots, c_n$ are satisfied, would also satisfy c_0.

$$\mathcal{C} \quad ::= \quad \alpha \,\#\, t \mid t = t \mid s \sim s \mid s \prec s \mid \text{symbol } t \qquad \text{(solver constraints)}$$

Figure 3.1: Solver's internal representation of constraints

For the sake of convenience and implementation efficiency, the Solver operates on its own internal representation of constraints, that slightly differs from constraints described in the previous section. It erases atoms in terms under shape constraints, effectively transforming them into *shapes*. Additionaly, we add some syntax sugar for convenience of notation.

$$\begin{array}{rcl@{\qquad}rcl} a \neq \alpha & := & a \,\#\, \alpha & \mathsf{id}^{-1} & := & \mathsf{id} \\ (\pi\ a) \,\#\, t & := & a \,\#\, \pi^{-1}\ t & ((\alpha_1\ \alpha_2)\pi)^{-1} & := & \pi^{-1} +\!\!\!+ (\alpha_1\ \alpha_2) \end{array}$$

Figure 3.2: Constraints' syntax sugaring

A high level perspective of the Solver is that it works on judgments of form $\Gamma; \Delta \vdash \mathcal{C}$, veryfying whether a given goal-constrint $\mathcal{C}$ holds in environments of assumed constraints (kept in Γ and Δ) through dissecting constraints on both sides of the turnstile into irreducible components that are straightforward to handle.
Environment Γ keeps the yet unprocessed assumptions, while another environment Δ keeps track of already analysed and irreducible assumptions. These assumptions usually flow from the former to the latter, but if we analyse a constraint that that affects other assumptions in Δ, they may flow back to Γ to be further disected by

$a_1 \neq a_2$	Atoms a_1 and a_2 are different.
$a \# X$	Atom a is Fresh in variable X.
$X_1 \sim X_2$	Variables X_1 and X_2 posses the same shape.
$X \sim t$	Variable X has a shape of term t.
$t \prec X$	Term t strictly subshapes variable X.
symbol X	Variable X is some functional symbol.

Figure 3.3: Irreducible constraints

the Solver.

After all assumptions in Γ are reduced to irreducible constraints, we break down the goal-constraint $\mathcal{C}$ and repeat the reduction procedure on new assumptions and goal.

$$\frac{}{\Gamma; \lightning \vdash \mathcal{C}} \qquad \frac{\mathcal{C} \text{ is trivial}}{\Gamma; \Delta \vdash \mathcal{C}} \qquad \frac{\mathcal{C} \in \Delta}{\Gamma; \Delta \vdash \mathcal{C}}$$

Figure 3.4: Base cases of the Solver's judgement

This recursive procedure may stop at a contradictory environment $\lightning$, that short-cuircuts the procedure, or at a state in which all the assumptions and goal itself are reduced to irreducible components, which is then as simple as checking if the goal is trivial or if it occurs on the left side of the turnstile.

3.1 Goal-reducing rules

$$\frac{}{\varnothing; \Delta \vdash a = a} \qquad \frac{}{\varnothing; \Delta \vdash X = X} \qquad \frac{}{\varnothing; \Delta \vdash f = f}$$

$$\frac{\varnothing; \Delta \vdash t_1 = t_2 \qquad \varnothing; \Delta \vdash t_1' = t_2'}{\varnothing; \Delta \vdash t_1 t_1' = t_2 t_2'}$$

$$\frac{\varnothing; \Delta \vdash \alpha_1 \# \alpha_2.t_2 \qquad \varnothing; \Delta \vdash t_1 = (\alpha_1\ \alpha_2)t_2}{\varnothing; \Delta \vdash \alpha_1.t_1 = \alpha_2.t_2}$$

Figure 3.5: Equality-reduction rules

Checking equality of terms is rather straightforward and follows from the term structure if no permutations are involved. Only the case for abstraction terms is more involved: the left side's argument must be fresh in the whole right side's term (which informally means that either arguments are the same or the left's argument doesn't occur at all in the right's body) and that left body must be equal to the right body with if its argument was swapped for the left one.

TODO: write why we show goal-reducing first while assumption-reducing are actually done beforehand.

To compare a *pure* atom a with permuted one, we employ the decidability of atom equality to reduce the right hand-side's permutation by applying it's outermost swap $(\alpha_1\ \alpha_2)$ on the left side's atom. There's three possible cases:

1. a is different from both α_1 and α_2, so the swap doesn't change the goal,
2. a is equal to α_1 but different from α_2, so the swap substitutes it for α_2,
3. a is equal to α_2, so the swap substitutes it for α_1.

Notice that it is impossible for any two of these assumption to be valid at the same time — the contradictory branches will resolve through absurd environment.

$$\frac{\begin{array}{c} a \neq \alpha_1, a \neq \alpha_2; \Delta \vdash a = \alpha \\ a = \alpha_1, a \neq \alpha_2; \Delta \vdash \alpha_2 = \alpha \\ a = \alpha_2; \Delta \vdash \alpha_1 = \alpha \end{array}}{\varnothing; \Delta \vdash a = (\alpha_1\ \alpha_2)\alpha}$$

$$\frac{\varnothing; \Delta \vdash a = \pi^{-1}\alpha}{\varnothing; \Delta \vdash \pi a = \alpha} \qquad \frac{\varnothing; \Delta \vdash X_1 = \pi_1^{-1}\pi_2 X_2}{\varnothing; \Delta \vdash \pi_1 X_1 = \pi_2 X_2}$$

$$\frac{\varnothing; \Delta \vdash \pi \text{ idempotent on } X}{\varnothing; \Delta \vdash X = \pi X} \qquad \frac{\forall a \in \pi.\ \varnothing; \Delta \vdash a = \pi a\ \vee\ \varnothing; \Delta \vdash a \# X}{\varnothing; \Delta \vdash \pi \text{ idempotent on } X}$$

Figure 3.6: Permutation-reduction rules

If the left-hand side's term is permuted we move the permutation to the right-hand side by inverting it. There's also special check for variables equal to their permuteded selves — the only way they could be equal is that if that permutation is idempotent on them.

$$\frac{a_1 \neq a_2 \in \Delta}{\varnothing; \Delta \vdash a_1 \# a_2} \qquad \frac{a \# X \in \Delta}{\varnothing; \Delta \vdash a \# X} \qquad \frac{}{\varnothing; \Delta \vdash a \# f}$$

$$\frac{a \neq \alpha; \Delta \vdash a \# t}{\varnothing; \Delta \vdash a \# \alpha.t} \qquad \frac{\varnothing; \Delta \vdash a \# t_1 \qquad \varnothing; \Delta \vdash a \# t_2}{\varnothing; \Delta \vdash a \# t_1 t_2}$$

Figure 3.7: Freshness-reduction rules

Freshness follows the term structure and breaks down into assumption check or trivial case. Unlike to how we defined freshness in abstraction in the introduction, we do not have two rules that differencing on whether $a = \alpha$. If they are indeed equal, then the assumption of inequality will immediately result in contradiction of environment, but if it wasn't yet established then we continue the solver procedure with an additional assumption.

$$\frac{}{\varnothing;\Delta \vdash \_ \sim \_} \qquad \frac{}{\varnothing;\Delta \vdash f \sim f}$$

$$\frac{X_1 \sim X_2 \in \Delta}{\varnothing;\Delta \vdash X_1 \sim X_2} \qquad \frac{X \sim s' \in \Delta \quad \varnothing;\Delta \vdash s' \sim s}{\varnothing;\Delta \vdash X \sim s}$$

$$\frac{\varnothing;\Delta \vdash s_1 \sim s_2}{\varnothing;\Delta \vdash \_.s_1 \sim \_.s_2} \qquad \frac{\varnothing;\Delta \vdash s_1 \sim s_2 \quad \varnothing;\Delta \vdash s_1' \sim s_2'}{\varnothing;\Delta \vdash s_1 s_1' \sim s_2 s_2'}$$

Figure 3.8: Shape rules

Shape equality is naturally structural. All atoms are considered to have the same shape. Symbols have the same shape if they are equal. Variables can share shape and be have their shape stored by Δ, which enables transitivity.

$$\frac{\varnothing;\Delta \vdash s_1 \sim s_2 \quad s_2 \prec X \in \Delta}{\varnothing;\Delta \vdash s_1 \prec X} \qquad \frac{\varnothing;\Delta \vdash s_1 \prec s_2 \quad s_2 \prec X \in \Delta}{\varnothing;\Delta \vdash s_1 \prec X}$$

Figure 3.9: Subshape rules

Solving subshape recurses through right-hand side shape's structure to find a shape-equal sub-shape. Otherwise, we use assunmptions in environment Δ, to identify all shapes that given variable subshapes, enabling transitivity.

$$\frac{}{\varnothing;\Delta \vdash \text{symbol } f} \qquad \frac{\text{symbol } X \in \Delta}{\varnothing;\Delta \vdash \text{symbol } X} \qquad \frac{\text{symbol } X \in \Delta}{\varnothing;\Delta \vdash a \,\#\, X}$$

Figure 3.10: Symbol rules

Symbol constraints are really simple to check, either the term is already a symbol, or it is a variable that we already assumed to be a symbol.

3.2 Assumptions-reducing rules

But before the Solver can reduce the goal-constraint, it must first reduce all assumptions in the Γ environment. We will now present the rules for reducing the constraints on the left side of the turnstile, which are mostly anologous to the goal reducing rules.

$$\dfrac{X = \pi^{-1}t, \Gamma; \Delta \vdash \mathcal{C}}{\pi X = t, \Gamma; \Delta \vdash \mathcal{C}} \qquad \dfrac{a = \pi^{-1}\alpha, \Gamma; \Delta \vdash \mathcal{C}}{\pi a = \alpha, \Gamma; \Delta \vdash \mathcal{C}} \qquad \textsc{Perm Reduce}$$

$$\dfrac{\pi \text{ idempotent on } X, \Gamma; \Delta \vdash \mathcal{C}}{X = \pi X, \Gamma; \Delta \vdash \mathcal{C}} \qquad \textsc{Perm Idempotent}$$

$$\dfrac{\varnothing \vdash \text{ idempotent on } X \qquad \Gamma; \Delta \vdash \mathcal{C}}{\pi \text{ idempotent on } X, \Gamma; \Delta \vdash \mathcal{C}} \qquad \textsc{Perm Shortcircuit}$$

$$\dfrac{(\forall a \in \pi.\ \Gamma; \Delta \vdash a = \pi a \ \vee\ \Gamma; \Delta \vdash a \mathrel{\#} X), \Gamma; \Delta \vdash \mathcal{C}}{\pi \text{ idempotent on } X, \Gamma; \Delta \vdash \mathcal{C}} \qquad \textsc{Perm Explode}$$

Figure 3.11: Permutation-reducing rules

For variables equal to some term and atoms equal to some atom expressions, we first deal with permutation by inverting it and moving it to the right-hand side. Then we consider the special case where a variable is equal to itself when permuted. While the assumption of the permutation being idempotent might appear to multiply the number of assumptions exponentially based on the number of atoms in the given permutation, it's worth noting that this number is unlikely to be very high, as permutations rarely consist of more than a few swaps.

In practice, the solver implementation will initially check whether the permutation is idempotent with an empty set of assumptions. Only if this initial check fails, will it proceed to examine the permutation atom by atom. Otherwise both equality and freshness assumptions follow from the term structure.

TODO: describe the meta-assumption $\forall a \in \pi \ldots$

$$\dfrac{\alpha_1 \mathrel{\#} \alpha_2.t_2\ ,\ t_1 = (\alpha_1\ \alpha_2)t_2\ ,\ \Gamma; \Delta \vdash \mathcal{C}}{\alpha_1.t_1 = \alpha_2.t_2, \Gamma; \Delta \vdash \mathcal{C}} \qquad \dfrac{\begin{array}{c} a = \alpha,\ \Gamma; \Delta \vdash \mathcal{C} \\ a \neq \alpha,\ a \mathrel{\#} t,\ \Gamma; \Delta \vdash \mathcal{C} \end{array}}{a \mathrel{\#} \alpha.t, \Gamma; \Delta \vdash \mathcal{C}}$$

Figure 3.12: Binding assumption rules

Again, the binding term constructor is of most interest to use: equality behaves the same as on the goal side, we simply split up the assumption into two assumptions the same way we would split the goal. For freshness of an atom in an abstraction, we consider two cases: either the atom is equal to the argument, or different from the argument but fresh in the body. In constrast to the goal-reducing rules where we would be satisifed with just one branch successing, here we expect both possibilities to be satisfiable.

$$\frac{\Gamma\{X \mapsto t\}; \Delta\{X \mapsto t\} \vdash \mathcal{C}\{X \mapsto t\}}{X = t, \Gamma; \Delta \vdash \mathcal{C}}\ \textsc{Subst Term}$$

$$\frac{\Gamma\{a_1 \mapsto a_2\}; \Delta\{a_1 \mapsto a_2\} \vdash \mathcal{C}\{a_1 \mapsto a_2\}}{a_1 = a_2, \Gamma; \Delta \vdash \mathcal{C}}\ \textsc{Subst Atom}$$

Figure 3.13: Substitution rules

In the end, all assumptions reach the irreducible components that are handled through the special environment Δ enviroment. Equality assumptions reduce to substitution of the name for the expression, and while substitution over the environment Γ and goal $\mathcal{C}$ is indeed a simple substitution, substituting in Δ environment is a more involved process that can can arrive at a contradiction or extract assumptions from Δ back into Γ.

$$\frac{\Gamma; \{a_1 \neq a_2\} \cup \Delta \vdash \mathcal{C}}{a_1 \neq a_2,\ \Gamma; \Delta \vdash \mathcal{C}} \qquad \frac{\Gamma; \{a \# X\} \cup \Delta \vdash \mathcal{C}}{a \# X,\ \Gamma; \Delta \vdash \mathcal{C}}$$

$$\frac{\Gamma; \{X_1 \sim X_2\} \cup \Delta \vDash \mathcal{C}}{X_1 \sim X_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{\Gamma; \{X \sim s\} \cup \Delta \vDash \mathcal{C}}{X \sim s,\ \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{\Gamma; \{t \prec X\} \cup \Delta \vDash \mathcal{C}}{t \prec X, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{\Gamma; \{\text{symbol } X\} \cup \Delta \vDash \mathcal{C}}{\text{symbol } X, \Gamma; \Delta \vDash \mathcal{C}}$$

Figure 3.14: Moving irreducible assumptions inside Δ

Otherwise assumption are simply moved to the environment of irreducible constraints via procedure that we describe in the next section.

TODO: Example how the solver works? Here or someplace earlier?

3.3 Irreducible constraints

Environment Δ that containts all the irreducible assumptions is given by a sextuple $(\texttt{neq\_atoms}_\Delta, \texttt{fresh}_\Delta, \texttt{var\_shape}_\Delta, \texttt{shape}_\Delta, \texttt{subshape}_\Delta, \texttt{symbols}\Delta)$.

`neq_atoms`	Set of pairs of atoms that are known to be different.
`fresh`	Set of pairs of atom and variable, indicating that the atom is *fresh* in the variable.
`var_shape`	Mapping from variables to shape-representative variables. All variables mapped to the same representative are considered to inhabit the same shape.
`shape`	Mapping from shape-representative variables to the actual shape it must inhabit.
`subshape`	Set of pairs of shape-representative variables and shapes that subshape the variable.
`symbols`	Set of shape-representative variables that are known to be some unknown functional symbols.

Figure 3.15: Description of environment Δ

```
$X_\Delta$ :=
  | if $Y$ ← var_shape$_\Delta$ $X$ then $Y_\Delta$
  | otherwise $X$

$|X|_\Delta$ :=
  | if $Y$ ← var_shape$_\Delta$ $X$ then $|Y|_\Delta$
  | if $s$ ← shape$_\Delta$ $X$ then $s$
  | otherwise $X$
```

$$\begin{aligned}
|\_|_\Delta &:= \_ \\
|\_.s|_\Delta &:= \_.|s|_\Delta \\
|s_1 s_2|_\Delta &:= |s_1|_\Delta |s_2|_\Delta \\
|f|_\Delta &:= f \\
|t|_\Delta &:= ||t||_\Delta
\end{aligned}$$

Figure 3.16: Shape interpretation in Δ

We can now establish a method to compute the shape-representative variable and outline the procedure for reconstructing the shape within the environment Δ, denoted $|s|_\Delta$. With such environment structure, verifying whether a constraint is included in Δ can be accomplished straightforwardly.

$$
\begin{array}{rcl}
(a_1 \neq a_2) \in \Delta & := & (a_1 \neq a_2) \in \texttt{neq\_atoms}_\Delta \\
(a \# X) \in \Delta & := & X \in \texttt{fresh}_\Delta(a) \\
(X_1 \sim X_2) \in \Delta & := & |X_1|_\Delta = |X_2|_\Delta \\
(X \sim s) \in \Delta & := & s = \texttt{shape}_\Delta(X_\Delta) \\
(s \prec X) \in \Delta & := & s \in \texttt{subshape}_\Delta(X_\Delta)
\end{array}
$$

Figure 3.17: Shape and assumptions interpretation in Δ

Additionally, we establish rules for a special occurs check procedure, which safeguards against handling circular references, and does so while considering all occurences in the assumptions of Δ. This is needed because of the shape assumptions we introduced, we must go with the occurence check through the "shape-similar" variables and shapes.

$$
\frac{X_\Delta \text{ occurs syntactically in } |s|_\Delta}{\Delta \vdash X \text{ occurs in } s}
$$

$$
\frac{\begin{array}{c} X' \text{ occurs syntactically in } |s|_\Delta \\ (s' \prec X') \in \Delta \qquad \Delta \vdash X \text{ occurs in } s' \end{array}}{\Delta \vdash X \text{ occurs in } s}
$$

Figure 3.18: Occurs check rules

Incorporating atom constraints into Δ proceeds as follows: freshness of an atom in a in a variables is simply acknowledged in the `fresh` mapping. Inequality of two atoms simply adds to the set `neq_atoms`, unless invoked with identical atoms, in which case we report a contradiction.

```
{a # X} ∪ Δ :=
  Δ  |> fresh += (a # X)

{a ≠ a'} ∪ Δ :=
  | if a = a' then ↯
  | otherwise Δ |> neq_atoms += (a ≠ a')
```

Figure 3.19: Adding constraints to Δ

We are using OCaml's pipelining notation of `x |> f1 |> ... |> fn` for `fn (... (f1 x))` and treat expressions like `fresh += x` as functions, meaning `fun Δ -> {` `Δ with fresh = x :: Δ.fresh` } and alike.

```
$\{X \sim X'\} \cup \Delta :=$
  | if $X_\Delta = X'_\Delta$ then $\Delta$
  | if $|X|_\Delta = |X'|_\Delta$ then $\Delta$
  | if $X_\Delta$ occurs in $|X'|_\Delta$ then $\lightning$
  | if $X'_\Delta$ occurs in $|X|_\Delta$ then $\lightning$
  | otherwise $\Delta$ |> symbols        $\{X_\Delta \rightsquigarrow X'_\Delta\}$
                      |> subshape       $\{X_\Delta \rightsquigarrow X'_\Delta\}$
                      |> transfer_shape $\{X_\Delta \rightsquigarrow X'_\Delta\}$
                      |> var_shape += $(X_\Delta \mapsto X'_\Delta)$
                      |> shape     -= $X_\Delta$
                      |> subshape  -= $X_\Delta$

$\{X \sim s\} \cup \Delta :=$
  | if $X_\Delta$ occurs in $|s|_\Delta$ then $\lightning$
  | otherwise $\Delta$ |> symbols  $\{X_\Delta \rightsquigarrow |s|_\Delta\}$
                      |> subshape $\{X_\Delta \rightsquigarrow |s|_\Delta\}$
                      |> shape    $\{X_\Delta \rightsquigarrow |s|_\Delta\}$
```

Figure 3.20: Adding constraints to Δ

To set variable shape, we first make sure to perform occurs check on the proposed shape and then substitute the shape-variable in all affected fields. To meld together two shape-variables, we first check whether they have already been merged. If they have, we return contradiction. Next, we conduct an occurs check to ensure that merging them won't create a circular reference. If this check fails, we again report a contradiction. Finally, we merge all the information pertaining to X into X' and remove any traces of X from within Δ environment.

To maintain a high-level description, we delegate the detailed implementation aspects to auxiliary functions responsible for substituting shape-variables within the given field of Δ.

```
$\Delta\ \{X \mapsto t\}$ :=
  $\Delta$ |> fresh -= $X$
    |> assumptions += $(X \sim |t|_\Delta)$
    |> assumptions += $\bigcup_{(a \,\#\, X) \in \Delta} (a \,\#\, t)$

$\Delta\ \{a \mapsto a'\}$ :=
  $\Delta$ |> fresh -= $a$
    |> fresh += $(a' \,\#\, \mathtt{fresh}_\Delta a)$
    |> clear neq_atoms
    |> assumptions += $\bigcup_{(a_1 \neq a_2) \in \Delta} (a_1\{a \mapsto a'\} \neq a_2\{a \mapsto a'\})$
```

Figure 3.21: Substitution in Δ

Finally, we demonstrate how the substitution of variables and atoms is accomplished, thereby concluding the description of the *Solver* and its environment. Note that we are using the meta-field of `assumptions` to indicate that some of the assumptions in Δ are no longer "simple" and escape from Δ back to Γ to be broken up by the *Solver*.

```
symbols $\{X \rightsquigarrow s\}$ $\Delta$ :=
  | if $X_\Delta \notin$ symbols$_\Delta$ then $\Delta$
  | otherwise $\Delta$ |> symbols -= $X$
                       |> assumptions += $(\text{symbol } s)$

shape $\{X \rightsquigarrow s\}$ $\Delta$ :=
  | if $s' \leftarrow$ shape$_\Delta$ $X$ then $\Delta$ |> assumptions += $(s \sim s')$
  | otherwise $\Delta$ |> shapes += $(X \mapsto s)$

subshape $\{X \rightsquigarrow s\}$ $\Delta$ :=
  $\Delta$ |> assumptions += (subshapes$_\Delta X \prec s$)

transfer_shape $\{X \rightsquigarrow X'\}$ $\Delta$ :=
  | if $s \leftarrow$ shape$_\Delta$ $X$ then $\Delta$ |> shape $\{X' \rightsquigarrow s\}$
  | otherwise $\Delta$
```

Figure 3.22: Auxiliary functions in Δ

Now the curious reader should feel obliged to ask themselves an important question: does that procedure always stop?

To address this question, we define the state of the Solver as a triple $(\Gamma, \Delta, \mathcal{C})$. Upon analyzing the Solver rules, it becomes evident that each rule consistently leads to a lesser state by reducing it through one or more of the following actions:

1. Decreasing the number of distinct variables in Γ, Δ, and $\mathcal{C}$, or maintaining the same number while:
2. Decreasing the depth of $\mathcal{C}$, or preserving the current depth while:
3. Reducing assumptions with a given depth in either Γ or Δ into assumptions with lower depth, or maintaining the number and depth of assumptions, while:
4. Eliminating an assumption from Γ and introducing an assumption of the same depth into Δ.

In the following chapters, we will write $\Gamma \vDash c$ but mean $\Gamma; \varnothing \vdash \mathcal{C}$, as by the construction of $\vdash$ we consider it equivalent to $\vDash$ defined in the model.

Chapter 4

Higher Order Logic

By constructing the Solver, we have built the sound logical system designed for handling and resolving constraints. We will now extend its utility and accesibility by introducing a higher-order logic layered atop the sublogic of constraints. This logical framework includes all essential elements necessary for formalizing theories, such as traditional connectives and quantifiers, functions and relations, and special formulae that have constraints embedded inside them, providing a versatile platform for expressing and reasoning about syntactic properties.

The Solver's decidable procedure allows us to us to integrate it with the very core of our logic to ehance it's capabilites by reasoning about constraints. In the subsequent chapter we will see how the Solver lets us treat constraints as propositions, ensures that constraints guarding formulas are satisfied, and enables us to express safe recursive predicates through fixpoint operator.

Next, our focus will shift towards constructing a dedicated proof system for this higher-order logic, including a proof assistant. Binding together all these components creates a cohesive framework for precise articulation and reasoning of complex syntactic properties.

4.1 Kinds

To organize the different types of formulas within this logic, we introduce the concept of *kinds*. The kind checker ensures that the formulas under consideration are coherent, given the multiple ways atoms, terms, binders, and constraints may appear within them. This step is essential for maintaining the logical integrity and meaningful interpretation of the formulas.

$$\kappa \quad ::= \quad \star \mid \kappa \to \kappa \mid \forall_A a.\, \kappa \mid \forall_T X.\, \kappa \mid [c]\kappa \qquad \text{(kinds)}$$

Figure 4.1: Kinds grammar

Notice that as constraints occur in kinds, we cannot simply give functions from atoms some kind $Atom \to \kappa$, but we must know *which* atom is bound there, to substitute for it in it's kind κ — the same way we substitute that atom for an atom expression in the function body when applying it to the formula. The *guarded kind* $[c]\kappa$ is most importantly used in kinding of the fixpoint formulas, which we will explain in later sections.

$\varphi :: \star$	φ is a propositional formula.
$\varphi :: \kappa_1 \to \kappa_2$	φ is a function that takes a formula of kind κ_1, and produces a formula of kind κ_2.
$\varphi :: \forall_A a.\, \kappa$	φ is a function that takes an atom expression, binds it to a, and produces a formula of kind κ.
$\varphi :: \forall_T X.\, \kappa$	φ is a function that takes a term, binds it to X, and produces a formula of kind κ.
$\varphi :: [c]\kappa$	φ is a formula of kind κ as long as c is satisfied.

Figure 4.2: Kinds semantics

4.2 Subkinding

We relax kinding rules are through the *subkinding* relation. Function kind is con-

$$\frac{}{\Gamma \vdash \kappa <: \kappa}\ \textsc{Subkind Refl} \qquad \frac{\Gamma \vdash \kappa_1 <: \kappa_2 \quad \Gamma \vdash \kappa_2 <: \kappa_3}{\Gamma \vdash \kappa_1 <: \kappa_3}\ \textsc{Subkind Trans}$$

$$\frac{\Gamma \vdash \kappa_1 <: \kappa_2}{\Gamma \vdash \forall_A a.\, \kappa_1 <: \forall_A a.\, \kappa_2}\ \textsc{Subkind ForallAtom}$$

$$\frac{\Gamma \vdash \kappa_1 <: \kappa_2}{\Gamma \vdash \forall_T X.\, \kappa_1 <: \forall_T X.\, \kappa_2}\ \textsc{Subkind ForallTerm}$$

$$\frac{\Gamma \vdash \kappa_1' <: \kappa_1 \quad \Gamma \vdash \kappa_2 <: \kappa_2'}{\Gamma \vdash \kappa_1 \to \kappa_2 <: \kappa_1' \to \kappa_2'}\ \textsc{Subkind Function}$$

$$\frac{\Gamma \vDash c}{\Gamma \vdash [c]\kappa <: \kappa}\ \textsc{Subkind Reduce} \qquad \frac{\Gamma, c \vdash \kappa_1 <: \kappa_2}{\Gamma \vdash \kappa_1 <: [c]\kappa_2}\ \textsc{Subkind Guard}$$

Figure 4.3: Subkinding Rules

travariant to the subkinding relation on the left argument: Universally quantified kinds only subkind if they are quantified over the same name. Constraints from the left side that are solved through $\vDash$ relation can be dropped, and constraints from the right-hand side can be moved inside of the enviroment.

$$\frac{\Gamma \vdash \kappa_1 <: \kappa_2}{\Gamma \vdash [c]\kappa_1 <: [c]\kappa_2}$$

Note that there is no structural subkinding rule for guarded kinds like the one above, but such a rule can be derived from SubkindReduce, SubkindGuard, transitivity, and weakening.

4.3 Formulas

Formulas include standard connectives (of kind $\star$):

$$\varphi \quad ::= \quad \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \Longrightarrow \varphi \mid \ldots \quad \text{(formulas)}$$

Quantification over atoms and terms (on formulas of kind $\star$):

$$\varphi \quad ::= \quad \ldots \mid \forall_A a.\, \varphi \mid \forall_T X.\, \varphi \mid \forall_\kappa P. \varphi \mid \exists_A a.\, \varphi \mid \exists_T X.\, \varphi \mid \exists_\kappa P. \varphi \mid \ldots \quad \text{(formulas)}$$

Propositional variables, functions and applications:

$$\varphi \quad ::= \quad \ldots \mid P \mid \lambda_A a.\, \varphi \mid \lambda_T X.\, \varphi \mid \lambda P :: \kappa.\, \varphi \mid \varphi\ \alpha \mid \varphi\ t \mid \varphi\ \varphi \mid \ldots \quad \text{(formulas)}$$

Constraints and guards:

$$\varphi \quad ::= \quad \ldots \mid c \mid [c] \wedge \varphi \mid [c] \Longrightarrow \varphi \mid \ldots \text{(formulas)}$$

$$\frac{(P :: \kappa) \in \Sigma}{\Gamma; \Sigma \vdash P :: \kappa}\ \textsc{Kind Var} \qquad \frac{}{\Gamma; \Sigma \vdash c :: \star}\ \textsc{Kind Constr}$$

$$\frac{\Gamma, c; \Sigma \vdash \varphi :: \star}{\Gamma; \Sigma \vdash [c] \wedge \varphi :: \star}\ \textsc{Kind ConstrAnd} \qquad \frac{\Gamma, c; \Sigma \vdash \varphi :: \star}{\Gamma; \Sigma \vdash [c] \Longrightarrow \varphi :: \star}\ \textsc{Kind ConstrImp}$$

$$\frac{\Gamma; \Sigma \vdash \varphi :: \kappa}{\Gamma; \Sigma \vdash \lambda_A a.\, \varphi :: \forall_A a.\, \kappa}\ \textsc{Kind FunAtom} \qquad \frac{\Gamma; \Sigma \vdash \varphi :: \forall_A a.\, \kappa}{\Gamma; \Sigma \vdash \varphi\ \alpha :: \kappa\{a \mapsto \alpha\}}\ \textsc{Kind AppAtom}$$

$$\frac{\Gamma; \Sigma \vdash \varphi :: \kappa}{\Gamma; \Sigma \vdash \lambda_T X.\, \varphi :: \forall_T X.\, \kappa}\ \textsc{Kind FunTerm} \qquad \frac{\Gamma; \Sigma \vdash \varphi :: \forall_T X.\, \kappa}{\Gamma; \Sigma \vdash \varphi\ t :: \kappa\{X \mapsto t\}}\ \textsc{Kind FunAtom}$$

$$\frac{\Gamma; \Sigma, P :: \kappa_1 \vdash \varphi :: \kappa_2}{\Gamma; \Sigma \vdash \lambda P :: \kappa_1.\, \varphi :: \kappa_1 \to \kappa_2}\ \textsc{Kind FunForm} \qquad \frac{\begin{array}{c}\Gamma; \Sigma \vdash \varphi_1 :: \kappa' \to \kappa \\ \Gamma; \Sigma \vdash \varphi_2 :: \kappa'\end{array}}{\Gamma; \Sigma \vdash \varphi_1\ \varphi_2 :: \kappa}\ \textsc{Kind AppForm}$$

Figure 4.4: Selected kinding rules

Naturally, constraints can act as propositions, as we can reason about their validity, and thus they are of kind $\star$. Constructions $[c] \Longrightarrow \varphi$ and $[c] \wedge \varphi$ are called *guards* and make assumptions about the environment in which one shall interpret the guarded formula. The former states that the formula φ holds if the constraint c is valid, analogously to a propositional implication. The latter additionaly requires that c already holds. We will see how guards interact with kinding rules after we define the fixpoint operator.

The binding constructs in functions and quantifiers follow the classical binder properties: we have the flexibility to perform alpha renaming on the bound names, and we can substitute the bound name with an expression within the body. This differs from the abstraction term $a.t$, which does not function as a true binder. Instead the binding term is simply a piece of data — an atom followed by a term — lacking any inherent mathematical properties typically associated with binders.

4.4 Fixpoint

We finish the definition of formulas with the *greatest fixpoint operator* that allows us to write recursive predicates over terms:

$$\varphi \quad ::= \quad \ldots \mid \text{fix } P(X) :: \kappa = \varphi \quad \text{(formulas)}$$

$$\frac{\Gamma; \Sigma, (P :: \forall_T Y.\, [Y \prec X]\kappa\{X \mapsto Y\}) \vdash \varphi :: \kappa}{\Gamma; \Sigma \vdash (\text{fix } P(X) :: \kappa = \varphi) :: \forall_T X.\, \kappa} \ \textsc{Kind Fixpoint}$$

$$(\text{fix } P(X) :: \kappa = \varphi)\ t \ \equiv\ \varphi\{X \mapsto t\}\{P \mapsto (\text{fix } P(X) :: \kappa = \varphi)\} \quad \textsc{Fixpoint Unwrap}$$

Figure 4.5: Fixpoint kinding rule

By the kinding rules, the fixpoint can only be recursively applied on structurally smaller terms, which is expressed through the kinding $(P :: \forall_T Y.\, [Y \prec X]\ \kappa\{X \mapsto Y\})$. To evaluate a fixpoint function applied to a term, simply substitute the bound variable with the given term and replace recursive calls inside the fixpoint's body with the fixpoint itself. This way we enable the kind-checker to verify the soundness of fixpoint formulas and enforce usage of special guard formulas resembling implications and conjunctions. Because the applied term is finite and we always recurse on structurally smaller terms, the final formula after all substitutions must also be finite and safe —— thanks to the semantics of constraints and kinds.

To familiarize the reader with the fixpoint formulas, we present how Peano arithmetic can be modeled in our logic. Given symbols 0 and S for natural number construction, one can write a predicate $(Nat\ N)$ that a term N models some natural number, and $(PlusEq\ N\ M\ K)$ that two terms N and M added together are equal to K.

Notice how the constraint $(N = S\ M)$ guards the recursive call to Nat, ensuring

$$\text{fix } \mathit{Nat}(N) :: \star = (N = 0) \vee (\exists_T M.\, [N = S\ M] \wedge (\mathit{Nat}\ M))$$

$$\begin{aligned}&\text{fix } \mathit{PlusEq}(N) :: \forall_T M.\, \forall_T K.\, \star = \lambda_T M.\, \lambda_T K.\\ &\quad ([N = 0] \wedge (M = K)) \vee\\ &\qquad (\exists_T N', K'.\, [N = S\ N'] \wedge [K = S\ K'] \wedge (\mathit{PlusEq}\ N'\ M\ K'))\end{aligned}$$

Figure 4.6: Peano arithmetic predicates expressed with fixpoint

that constraint $(M \prec N)$ will be satisfied during kind checking of $(\mathit{Nat}\ M)$ in the kind derivation of the whole formula $(\mathit{Nat} :: \forall_T N.\, \star)$, analogously in PlusEq. This is exactly the reason of introducing kinds — to allow us to use recursive predicates in a safe and sound fashion.

See more interesting examples of fixpoints usage in the chapter on STLC.

4.5 Natural deduction

$$\frac{\varphi \in \Theta}{\Gamma; \Theta \vdash \varphi}\ \textsc{Assumption} \qquad \frac{\Gamma; \Theta \vdash \bot}{\Gamma; \Theta \vdash \varphi}\ \textsc{ExFalso}$$

$$\frac{\Gamma \vDash c}{\Gamma; \Theta \vdash c}\ \textsc{ConstrI} \qquad \frac{\Gamma \vDash \bot_c}{\Gamma; \Theta \vdash \varphi}\ \textsc{ConstrE}$$

$$\frac{\Gamma; \Theta \vdash \varphi_1}{\Gamma; \Theta \vdash \varphi_1 \lor \varphi_2}\ \textsc{OrI1} \qquad \frac{\Gamma; \Theta \vdash \varphi_2}{\Gamma; \Theta \vdash \varphi_1 \lor \varphi_2}\ \textsc{OrI2}$$

$$\frac{\begin{array}{c}\Gamma; \Theta \vdash \varphi_1 \lor \varphi_2 \\ \Gamma; \Theta, \varphi_1 \vdash \psi \qquad \Gamma; \Theta, \varphi_2 \vdash \psi\end{array}}{\Gamma; \Theta \vdash \psi}\ \textsc{OrE}$$

Figure 4.7: Selected rules of natural deduction

Finally, we come to the definition of proof-theoretic rules of natural deduction. Starting with inference rules for assumption, we have an analogous rules for between the worlds of propositional logic and constraint sublogic. And while the $\vdash$ relation we define is purely syntactic, we can still use semantic $\vDash$ because of its decidability and equivalence to our description from the chapter about the Solver.

Again, for *ex falso*, we define an analogous proof constructor for dealing with a contradictory constraint environment. Note that there are many constraints that can be used as $\bot_c$, i.e. constraints that are always false, and the solver will only *prove* them if we supply it with contradictory assumptions.

$$\frac{\Gamma; \Theta, \varphi_1 \vdash \varphi_2}{\Gamma; \Theta \vdash \varphi_1 \Longrightarrow \varphi_2}\ \textsc{ImpI} \qquad \frac{\Gamma_1; \Theta_1 \vdash \varphi_1 \quad \Gamma_2; \Theta_2 \vdash \varphi_1 \Longrightarrow \varphi_2}{\Gamma_1 \cup \Gamma_2; \Theta_2 \cup \Theta_2 \vdash \varphi_2}\ \textsc{ImpE}$$

$$\frac{\Gamma, c; \Theta \vdash \varphi}{\Gamma; \Theta \vdash [c] \Longrightarrow \varphi}\ \textsc{ConstrImpI} \qquad \frac{\Gamma_1; \Theta_1 \vdash c \quad \Gamma_2; \Theta_2 \vdash [c] \Longrightarrow \varphi}{\Gamma_1 \cup \Gamma_2; \Theta_2 \cup \Theta_2 \vdash \varphi}\ \textsc{ConstrImpE}$$

$$\frac{\Gamma_1; \Theta_1 \vdash \varphi_1 \quad \Gamma_2; \Theta_2 \vdash \varphi_2}{\Gamma_1 \cup \Gamma_2; \Theta_2 \cup \Theta_2 \vdash \varphi_1 \wedge \varphi_2}\ \textsc{AndI}$$

$$\frac{\Gamma; \Theta \vdash \varphi_1 \wedge \varphi_2}{\Gamma; \Theta \vdash \varphi_1}\ \textsc{AndE1} \qquad \frac{\Gamma; \Theta \vdash \varphi_1 \wedge \varphi_2}{\Gamma; \Theta \vdash \varphi_2}\ \textsc{AndE2}$$

$$\frac{\Gamma_1; \Theta_1 \vdash c \quad \Gamma_2; \Theta_2 \vdash \varphi}{\Gamma_1 \cup \Gamma_2; \Theta_2 \cup \Theta_2 \vdash [c] \wedge \varphi}\ \textsc{ConstrAndI}$$

$$\frac{\Gamma; \Theta \vdash [c] \wedge \varphi}{\Gamma; \Theta \vdash c}\ \textsc{ConstrAndE1} \qquad \frac{\Gamma; \Theta \vdash [c] \wedge \varphi \quad \Gamma; \Theta \vdash \varphi : \star}{\Gamma; \Theta \vdash \varphi}\ \textsc{ConstrAndE2}$$

Figure 4.8: Natural deduction for guard formulas

We present constraint guard rules alongside implication and conjunction to direct the reader to the similarities between them. Despite these similiartiy, in the rule for eliminating constraint conjunction guard (ConstrAndE2), we restrict the guarded formulas to be of kind $\star$. If one wants to eliminate the guard to use the inner formula, one can only do so with formulas that *make sense* (or rather: pass kind checking) on their own, without the constraint c guarding them, as opposed to the kinding rule where c was added to the kinding environment.

$$\frac{a \notin \mathrm{FV}(\Gamma;\Theta) \qquad \Gamma;\Theta \vdash \varphi}{\Gamma;\Theta \vdash \forall_A a.\, \varphi}\ \textsc{Forall AtomI} \qquad \frac{\Gamma;\Theta \vdash \forall_A a.\, \varphi}{\Gamma;\Theta \vdash \varphi\{a \mapsto \alpha\}}\ \textsc{Forall AtomE}$$

$$\frac{X \notin \mathrm{FV}(\Gamma;\Theta) \qquad \Gamma;\Theta \vdash \varphi}{\Gamma;\Theta \vdash \forall_T X.\, \varphi}\ \textsc{Forall TermI} \qquad \frac{\Gamma;\Theta \vdash \forall_T X.\, \varphi}{\Gamma;\Theta \vdash \varphi\{X \mapsto t\}}\ \textsc{Forall TermE}$$

$$\frac{\begin{array}{c}\Gamma;\Theta \vdash P :: \kappa \\ P \notin \mathrm{FV}(\Gamma;\Theta) \qquad \Gamma;\Theta \vdash \varphi\end{array}}{\Gamma;\Theta \vdash \forall_\kappa P.\varphi}\ \textsc{Forall FormI} \qquad \frac{\begin{array}{c}\Gamma;\Theta \vdash \psi :: \kappa \\ \Gamma;\Theta \vdash \forall_\kappa P.\varphi\end{array}}{\Gamma;\Theta \vdash \varphi\{P \mapsto \psi\}}\ \textsc{Forall FormE}$$

$$\frac{\Gamma;\Theta \vdash \varphi\{a \mapsto a'\}}{\Gamma;\Theta \vdash \exists_A a.\, \varphi}\ \textsc{Exists AtomI} \qquad \frac{\begin{array}{c}\Gamma_1;\Theta_1 \vdash \exists_A a.\, \varphi \\ \Gamma_2;\Theta_2, \varphi\{a \mapsto a'\} \vdash \psi \\ a' \notin \mathrm{FV}(\Gamma_1 \cup \Gamma_2;\Theta_2 \cup \Theta_2)\end{array}}{\Gamma_1 \cup \Gamma_2;\Theta_2 \cup \Theta_2 \vdash \psi}\ \textsc{Exists AtomE}$$

$$\frac{\Gamma;\Theta \vdash \varphi\{X \mapsto X'\}}{\Gamma;\Theta \vdash \exists_T X.\, \varphi}\ \textsc{Exists TermI} \qquad \frac{\begin{array}{c}\Gamma_1;\Theta_1 \vdash \exists_T X.\, \varphi \\ \Gamma_2;\Theta_2, \varphi\{X \mapsto X'\} \vdash \psi \\ X' \notin \mathrm{FV}(\Gamma_1 \cup \Gamma_2;\Theta_2 \cup \Theta_2)\end{array}}{\Gamma_1 \cup \Gamma_2;\Theta_2 \cup \Theta_2 \vdash \psi}\ \textsc{Exists TermE}$$

$$\frac{\begin{array}{c}\Gamma;\Theta \vdash P' :: \kappa \\ \Gamma;\Theta \vdash \varphi\{P \mapsto P'\}\end{array}}{\Gamma;\Theta \vdash \exists_\kappa P.\varphi}\ \textsc{Exists FormI} \qquad \frac{\begin{array}{l}\Gamma_1;\Theta_1 \vdash \exists_\kappa P.\varphi \\ \Gamma_2;\Theta_2, \varphi\{P \mapsto P'\}, P' :: \kappa' \vdash \psi \\ P' \notin \mathrm{FV}(\Gamma_1 \cup \Gamma_2;\Theta_2 \cup \Theta_2) \\ \Gamma_1 \cup \Gamma_2;\Theta_2 \cup \Theta_2 \vdash \kappa <: \kappa'\end{array}}{\Gamma_1 \cup \Gamma_2;\Theta_2 \cup \Theta_2 \vdash \psi}\ \textsc{Exists FormE}$$

Figure 4.9: Natural deduction for quantifiers

Inference rules for quantifiers are rather straightforward, with the only novelty being that we differtiate between atom and term quantification, and restrict the quantified name to be *fresh* in the environment — it should never occur in any of the assumptions.

The only axioms of our logic are strictly related to constraints:

1. We can deterministically compare any two atoms,
2. There always exists a fresh atom,
3. We can always deduce the structure of a term.

$$\frac{}{\vdash \forall_A\, a,\, a'.\, (a = a') \vee (a \neq a')}\ \textsc{Axiom Compare}$$

$$\frac{}{\vdash \forall_T X.\ \exists_A a.\, (a \mathrel{\#} X)}\ \textsc{Axiom Fresh}$$

$$\frac{}{\begin{array}{c}\vdash \forall_T X.\, (\exists_A a.\ X = a) \vee (\exists_A a.\ \exists_T X'.\ X = a.X') \\ \vee\, (\exists_T X_1,\, X_2.\ X = a.X') \vee (\mathit{symbol}\, X)\end{array}}\ \textsc{Axiom Inversion}$$

Figure 4.10: Axioms

$$\frac{\Gamma \vDash a = \alpha \qquad \Gamma; \Theta \vdash \varphi}{\Gamma\{a \mapsto \alpha\}; \Theta\{a \mapsto \alpha\} \vdash \varphi\{a \mapsto \alpha\}}\ \textsc{Sub Atom}$$

$$\frac{\Gamma \vDash X = t \qquad \Gamma; \Theta \vdash \varphi}{\Gamma\{X \mapsto t\}; \Theta\{X \mapsto t\} \vdash \varphi\{X \mapsto t\}}\ \textsc{Sub Term}$$

$$\frac{\Gamma; \Theta \vdash \psi \qquad \Gamma; \Theta \vdash \psi \equiv \varphi}{\Gamma; \Theta \vdash \varphi}\ \textsc{Equiv}$$

$$\frac{\Gamma; \Theta, (\forall_T X'.\, [X' \prec X] \Longrightarrow \varphi(X')) \vdash \varphi(X)}{\Gamma; \Theta \vdash \forall_T X.\, \varphi(X)}\ \textsc{Induction}$$

Figure 4.11: Flexibility rules

To make the framework more flexible we introduce a way for using equivalent formulas: And a way to substitute atoms for atomic expression and variables for terms, if the solver can prove their equality: Finally, we define induction over term structure, and thanks to the constraints sublogic we can easily define the notion of *smaller terms* needed for the inductive hypothesis.

The equivalence relation ($\varphi_1 \equiv \varphi_2$) is a bit complicated due to subkinding, existence of formulas with fixpoints, functions, applications, and presence of an environment with variable mapping. Nonetheless, it's simply that — *an equivalence relation* — and it behaves as expected. We will only highlight the interesting parts.

$$\begin{aligned}
\texttt{compute}\ \Sigma\ n\ P \ &\leadsto\ \texttt{compute}\ \Sigma\ n\ \varphi \\
\text{when}\ &\ \ \Sigma(P) = \varphi \\[1em]
\texttt{compute}\ \Sigma\ n\ (\varphi\,\alpha) \ &\leadsto\ \texttt{compute}\ \Sigma\ (n'-1)\ \varphi'\{a \mapsto \alpha\} \\
\text{when}\ &\ \ \texttt{compute}\ \Sigma\ n\ \varphi \leadsto^{*} (n', \lambda_A a.\, \varphi') \\[1em]
\texttt{compute}\ \Sigma\ n\ (\varphi\, t) \ &\leadsto\ \texttt{compute}\ \Sigma\ (n'-1)\ \varphi'\{X \mapsto t\} \\
\text{when}\ &\ \ \texttt{compute}\ \Sigma\ n\ \varphi \leadsto^{*} (n', \lambda_T X.\, \varphi') \\[1em]
\texttt{compute}\ \Sigma\ n\ (\varphi\, t) \ &\leadsto\ \texttt{compute}\ \Sigma\{P \mapsto \phi'\}\ (n'-1)\ \varphi'\{X \mapsto t\} \\
\text{when}\ &\ \ \texttt{compute}\ \Sigma\ n\ \varphi \leadsto^{*} (n', \text{fix}\ P(X) :: \kappa = \varphi') \\[1em]
\texttt{compute}\ \Sigma\ n\ (\varphi_1\, \varphi_2) \ &\leadsto\ \texttt{compute}\ \Sigma\ (n_2-1)\ \psi_1\{P \mapsto \psi_2\} \\
\text{when}\ &\ \ \texttt{compute}\ \Sigma\ n\ \varphi_1 \leadsto^{*} (n_1, \lambda P :: \kappa.\, \psi_1) \\
\text{and}\ &\ \ \texttt{compute}\ \Sigma\ n_1\ \varphi_2 \leadsto^{*} (n_2, \psi_2)
\end{aligned}$$

Figure 4.12: Computing weak head normal form

Equivalence checking procedure starts by computing weak head normal form (WHNF). Because of the fixpoint formulas unfolding indefinetely, we restrict that computation up to some *depth* denoted by n.

If we have a WHNF computed or if we've reached the limit of computation (when $n \leqslant 0$) then we try to progress with equivelnce by recursing on the structure of formulas:

$$\frac{\Gamma; \Sigma \vdash \varphi_1 \equiv \varphi_2 \qquad \Gamma; \Sigma \vdash \psi_1 \equiv \psi_2}{\Gamma; \Sigma \vdash \varphi_1 \Longrightarrow \psi_1 \equiv \varphi_2 \Longrightarrow \psi_2}$$

$$\frac{\Gamma \vdash c_1 \equiv c_2 \qquad \Gamma; \Sigma \vdash \varphi_1 \equiv \varphi_2}{\Gamma; \Sigma \vdash [c_1] \wedge \varphi_1 \equiv [c_2] \wedge \varphi_2} \qquad \frac{\Gamma \vDash a_1 = a_2 \qquad \Gamma \vDash t_1 = t_2}{\Gamma \vdash (a_1 \# t_1) \equiv (a_2 \# t_2)}$$

$$\frac{\begin{array}{c} X \notin \mathrm{FV}(\Gamma; \Sigma) \\ \Gamma; \Sigma \vdash \varphi_1[X_1 \mapsto X] \equiv \varphi_2[X_2 \mapsto X] \end{array}}{\Gamma; \Sigma \vdash \lambda_T X_1.\, \varphi_1 \equiv \lambda_T X_2.\, \varphi_2} \qquad \frac{\Gamma \vDash t_1 = t_2 \qquad \Gamma; \Sigma \vdash \varphi_1 \equiv \varphi_2}{\Gamma; \Sigma \vdash \varphi_1\ t_1 \equiv \varphi_2\ t_2}$$

$$\frac{\begin{array}{c} \kappa_1 <: \kappa_2 \qquad P \notin \mathrm{FV}(\Gamma; \Sigma) \qquad X \notin \mathrm{FV}(\Gamma; \Sigma) \\ \Gamma; \Sigma \vdash \varphi_1[P_1 \mapsto P, X_1 \mapsto X] \equiv \varphi_2[P_2 \mapsto P, X_2 \mapsto X] \end{array}}{\Gamma; \Sigma \vdash \text{fix } P_1(X_1) :: \kappa_1 = \varphi_1 \equiv \text{fix } P_2(X_2) :: \kappa_2 = \varphi_2}$$

Figure 4.13: Selected equivalence rules

Note that we allow *different terms* in equivalent formulas as long as constraints-enviroment Γ ensures their equality is provable. For functions, we simply substitute the arguments of both left and right side to the same, fresh name.

Quantifiers are handled the same way as function above — as they all are a form of bind. To handle formulas with constraints we introduce *constraint equivalence* relation, which does nothing more than use the Solver to check that the constructors of constraint are the same and that arguments are equal to each other in the Solver's sense, analogusly as with terms above.

Chapter 5

Implementation

All the concepts discussed in previous chapters are acompanied by our code implementation in OCaml. Atoms and variables are represented internally by integers (yet remain disjoint sets) — and their string *names* are kept within the environment and stored in binders themselves (quantifiers and functions). Along with terms, constraints, kinds, and formulas, they're defined in `Types` module, mirroring their previously described grammars. The only difference is that we allow conjunction and disjunction to be used with more than two arguments, with the added feature of arguments being labeled by string names. This naming approach lets the user to easily select desired branches while composing proofs or to give meaningful names within the definition of properties.

The *Solver* inhabits its own dedicated `Solver` module along with `SolverEnv` responsible for implementing the specialized environment Δ handling the irreducible assumptions. Analogously, the `KindChecker` and `KindCheckerEnv` modules serve similar roles. The natural deduction from previous chapter is distributed over modules `Proof`, `ProofEnv`, `ProofEquiv`, and is a direct implementation of the described rules.

```
(* Module: Proof *)

(* ----------- *)
(*  Γ; f ⊢ f   *)
val assumption : 'a env -> formula -> proof

(*    Γ; Θ, f1 ⊢ f2   *)
(* ------------------- *)
(*  Γ; Θ ⊢ f1  ⟹  f2  *)
val imp_i : formula -> proof -> proof

(*  Γ1; Θ1 ⊢ f1  ⟹  f2    Γ2; Θ2 ⊢ f2  *)
(* ------------------------------------- *)
(*        Γ1 ∪ Γ2; Θ1 ∪ Θ2 ⊢ f2         *)
val imp_e : proof -> proof -> proof

(*  Γ; Θ ⊢ ⊥  *)
(* ----------- *)
(*  Γ; Θ ⊢ f  *)
val bot_e : formula -> proof -> proof

(*   Γ ⊨ c    *)
(* ---------- *)
(*  Γ; Θ ⊢ c  *)
val constr_i : proof_env -> constr -> proof

...

TODO: this is kinda interesting, that we have deduction rules
    impolemented as functions (or smart constructors), that's nice
```

Module `Proof` provides methods for constructing forward proofs, i.e. those in which more complex conclusions are built from simpler, already proven facts. Smart constructors of `proof` data type ensure correctness by implementing the descripted natural deduction and serve as the logical core for writing proofs.

Human provers, working within intuitionistic logic, generally prefer to conduct proofs not in this *bottom-up* fasion, but through simplifying the goal to be proven until we reach the trivial matters. To accomodate for that, we included the *top-down* proof structure in `incproof` data type. As such proofs have incomplete parts by nature, they must have *holes*, and live within some *proof context*, as defined in modules `IncProof`, which serves the role of being a convenient facade for writing proofs, while responsibility of keeping proofs correct is delegated to the `Proof` module.

5.1 Proof assistant

To facilitate user interaction with our framework, we provide a practical *proof assistant*. While simple, it is also powerful and easy to use. The interface defined in modules `Prover`, `ProverInternals`, and `Tactics` provides multiple *tactics* (functions that manipulate *prover state*) and ways to combine them.

```
type prover_state = S_Unfinished of (goal * proof_context)
                  | S_Finished of proof

type tactic = prover_state → prover_state
```

Figure 5.1: Basic interface of the Prover.

The unfinished leaves in the incomplete proof trees are represented by the empty proof constructor, denoted $\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi$, defined by the environments Γ (of assumed constraints), Θ (of propositional assumptions), Σ (introduced names and their kinds), implicit context, and the goal formula φ. We will skip the context in this description and ask the reader to assume that proper handling of multiple goals is achieved automatically.

$$
\begin{array}{rcl}
\texttt{proof}\ (\Gamma; \Theta; \Sigma)\ \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash \bullet :: \varphi \\
\\
\texttt{intro} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: [c] \Longrightarrow \varphi & \rightsquigarrow & \Gamma, c; \Theta; \Sigma \vdash \bullet :: \varphi \\
\\
\texttt{intro' x} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \psi \Longrightarrow \varphi & \rightsquigarrow & \Gamma; \Theta, \texttt{x} :: \psi; \Sigma \vdash \bullet :: \varphi \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \forall_A a.\, \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma, \texttt{x} :: a \vdash \bullet :: \varphi \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \forall_T X.\, \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma, \texttt{x} :: X \vdash \bullet :: \varphi \\
\\
\texttt{apply}\ (\psi \Longrightarrow \varphi) & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash \bullet :: \psi \\
 & \text{and} & \Gamma; \Theta; \Sigma \vdash \bullet :: \psi \Longrightarrow \varphi \\
\\
\texttt{apply\_assm H} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash \varphi \\
 & \text{when} & (\texttt{H} :: \varphi) \in \Theta \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash \bullet :: \psi \\
 & \text{when} & (\texttt{H} :: \psi \Longrightarrow \varphi) \in \Theta \\
\\
\texttt{solve} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: c & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash c \\
 & \text{when} & \Gamma \vDash c
\end{array}
$$

Figure 5.2: Basic tactics

Some typical tactics include introduction of names and assumptions into environment, using those assumptions to progress proofs and transforming goals by using implications. We can complete the proof by matching the goal with assumption by `apply` (which can be made automatically via tactical `assumption`) or by calling the solver with constraint-assumptions through `solve`. Technical detail is that all formulas in Θ that are actually constraints will also be included in Solver assumptions.

$$\begin{array}{rcl}
\texttt{apply\_thm}\ \mathcal{T} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash \bullet :: \psi \\
& \text{where} & \mathcal{T} \text{ is a proof of } \psi \Longrightarrow \varphi \\
& & \\
\texttt{apply\_assm\_spec H [e; a]} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi(\mathsf{e}, \mathsf{a}) & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash \bullet :: \psi(\mathsf{e}, \mathsf{a}) \\
& \text{when} & (\mathsf{H} :: \forall_T X.\, \forall_A a.\, \psi(X, a) \Longrightarrow \varphi(X, a)) \in \Theta \\
& & \\
\texttt{apply\_in\_assm H1 H2} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \psi & \rightsquigarrow & \Gamma; \Theta; \Sigma' \vdash \bullet :: \psi \\
& \text{when} & \Sigma = \{\mathsf{H1} :: \psi_2 \Longrightarrow \psi_1, \mathsf{H2} :: \psi_2\} \cup \Sigma'' \\
& \text{and} & \Sigma' = \{\mathsf{H1} :: \psi_1, \mathsf{H2} :: \psi_2\} \cup \Sigma''
\end{array}$$

Figure 5.3: More ways to use `apply` tactic

External theorems can be applied via tactic `apply_thm` similarly how an assumption is applied. Universal assumptions are specialized by `apply_assm_spec`, as well as theorems by `apply_thm_spec`. Note that propositions can be applied not only on the goal, but also on other assumptions via `apply_in_assm` tactic.

$$\begin{array}{rcl}
\texttt{add\_assm}\ \psi & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma; \Theta; \Sigma \vdash \bullet :: \psi \\
& \text{and} & \Gamma; \psi, \Theta; \Sigma \vdash \bullet :: \varphi \\
& & \\
\texttt{add\_assm\_thm}\ \mathcal{T} & & \\
\Gamma; \Theta; \Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma; \psi, \Theta; \Sigma \vdash \bullet :: \varphi \\
& \text{where} & \mathcal{T} \text{ is a proof of } \psi
\end{array}$$

Figure 5.4: Adding assumptions

One can also add introduce assumptions to the environment by `add_assm`, together with a new goal (of proving that assumption) or add external theorem via `add_assm_thm`, which can already be specialized if needed via `add_assm_thm_spec`, analogously to `apply_thm_spec`.

$$\begin{array}{rcl}
\texttt{destr\_goal} & & \\
\Gamma;\Theta;\Sigma \vdash \bullet :: [c] \wedge \varphi & \rightsquigarrow & \Gamma;\Theta;\Sigma \vdash \bullet :: c \\
& \text{and} & \Gamma, c;\Theta;\Sigma \vdash \bullet :: \varphi \\
\Gamma;\Theta;\Sigma \vdash \bullet :: \varphi_1 \wedge \varphi_2 & \rightsquigarrow & \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi_1 \\
& \text{and} & \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi_2 \\
\\
\texttt{left} & \equiv & \texttt{case l} \\
\Gamma;\Theta;\Sigma \vdash \bullet :: (\texttt{l:}\ \varphi_1) \vee (\texttt{r:}\ \varphi_2) & \rightsquigarrow & \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi_1 \\
\texttt{right} & \equiv & \texttt{case r} \\
\Gamma;\Theta;\Sigma \vdash \bullet :: (\texttt{l:}\ \varphi_1) \vee (\texttt{r:}\ \varphi_2) & \rightsquigarrow & \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi_2
\end{array}$$

Figure 5.5: Tactis that disect the goal

To progress the conjunction proofs we provide tactics `destr_goal` and. Disjunction can be handled by `left` and `right` tactic, or `(destr_goal n)` for choosing the `n`-th disjunct. For convenience and clarity, the `case` tactic allows us to focus on the chosen disjunct by it's name.

$$\begin{array}{rcl}
\texttt{destr\_assm H} & & \\
\Gamma;\Theta \cup \{\texttt{H} :: [c] \wedge \varphi\};\Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma \cup \{c\};\Theta \cup \{\texttt{H} :: \varphi\};\Sigma \vdash \bullet :: \varphi \\
\Gamma;\Theta \cup \{\texttt{H} :: \varphi_1 \wedge \varphi_2\};\Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma;\Theta \cup \{\texttt{H\_1} :: \varphi_1, \texttt{H\_2} :: \varphi_2\};\Sigma \vdash \bullet :: \varphi \\
\Gamma;\Theta \cup \{\texttt{H} :: \varphi_1 \vee \varphi_2\};\Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma;\Theta \cup \{\texttt{H} :: \varphi_1\};\Sigma \vdash \bullet :: \varphi \\
& \text{and} & \Gamma;\Theta \cup \{\texttt{H} :: \varphi_2\};\Sigma \vdash \bullet :: \varphi \\
\\
\texttt{destr\_assm' H x} & & \\
\Gamma;\Theta \cup \{\texttt{H} :: \exists_A a.\, \varphi\};\Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma;\Theta \cup \{\texttt{H} :: \varphi\{a \mapsto \texttt{x}\}\};\Sigma' \vdash \bullet :: \varphi \\
& \text{where} & \Sigma' = \Sigma \cup \{\texttt{x} :: A\} \\
\Gamma;\Theta \cup \{\texttt{H} :: \exists_T X.\, \varphi\};\Sigma \vdash \bullet :: \varphi & \rightsquigarrow & \Gamma;\Theta \cup \{\texttt{H} :: \varphi\{X \mapsto \texttt{x}\}\};\Sigma' \vdash \bullet :: \varphi \\
& \text{where} & \Sigma' = \Sigma \cup \{\texttt{x} :: T\} \\
& \text{when} & \texttt{x} \notin \mathrm{FV}(\Gamma;\Theta;\Sigma)
\end{array}$$

Figure 5.6: Tactics that disect assumptions

Naturally, we can also case-analyse assumptions by `destr_assm`. Note that the user provides `destr_assm'` with a string *name* that will be bound to the existential variable, but the binding is done "behind the scenes" and actually any string can be given, as an unique internal identifier is generated.

Finally we can alter goals through generalization, by finding a witness, by induction on terms, and through reduction to absurd. We also provide tactics for using the axioms of our logic, described in previous chapter.

`ex_falso`

$$\Gamma;\Theta;\Sigma \vdash \bullet :: \varphi \quad \leadsto \quad \Gamma;\Theta;\Sigma \vdash \bullet :: \bot$$

`discriminate`

$$\begin{array}{rcl} \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi & \leadsto & \Gamma;\Theta;\Sigma \vdash \varphi \\ & \text{when} & \Gamma \vDash \bot_c \end{array}$$

`exists e`

$$\begin{array}{rcl} \Gamma;\Theta;\Sigma \vdash \bullet :: \exists_A a.\, \varphi & \leadsto & \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi\{a \mapsto \mathtt{e}\} \\ \Gamma;\Theta;\Sigma \vdash \bullet :: \exists_T X.\, \varphi & \leadsto & \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi\{X \mapsto \mathtt{e}\} \end{array}$$

`generalize x`

$$\begin{array}{rcl} \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi & \leadsto & \Gamma;\Theta;\Sigma' \vdash \bullet :: \forall_T \mathsf{x}.\, \varphi \\ & \text{when} & \Sigma = \Sigma' \cup \{\mathsf{x}\} \text{ and } \mathsf{x} \notin \mathrm{FV}(\Gamma) \end{array}$$

`by_induction x IH`

$$\begin{array}{rcl} \Gamma;\Theta;\Sigma \vdash \bullet :: (\forall_T X.\, \varphi(X)) & \leadsto & \Gamma;\Theta \cup \Theta';\Sigma \cup \{\mathsf{x} :: T\} \vdash \bullet :: \varphi(X) \\ & \text{where} & \Theta' = \{\mathtt{IH} :: \forall_T \mathsf{x}.\, [\mathsf{x} \prec X] \Longrightarrow \varphi(\mathsf{x})\} \end{array}$$

`compare_atoms a b`

$$\Gamma;\Theta;\Sigma \vdash \bullet :: \varphi \quad \leadsto \quad \Gamma;\Theta;\Sigma \vdash \bullet :: (\mathsf{a} = \mathsf{b} \vee \mathsf{a} \neq \mathsf{b}) \Longrightarrow \varphi$$

`get_fresh_atom a e`

$$\begin{array}{rcl} \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi & \leadsto & \Gamma \cup \{\mathsf{a} \# \mathsf{e}\};\Theta;\Sigma \cup \{\mathsf{a} :: A\} \vdash \bullet :: \varphi \\ & \text{when} & \mathsf{a} \notin \mathrm{FV}(\Gamma;\Theta;\Sigma) \end{array}$$

`inverse_term e`

$$\begin{array}{rcl} \Gamma;\Theta;\Sigma \vdash \bullet :: \varphi & \leadsto & \Gamma;\Theta;\Sigma \vdash \bullet :: (\exists_A \mathsf{a}.\ \ \mathsf{e} = \mathsf{a}) \Longrightarrow \varphi \\ & \text{and} & \Gamma;\Theta;\Sigma \vdash \bullet :: \\ & & (\exists_A \mathsf{a}.\, \exists_T \mathtt{e'}.\ \ \mathsf{e} = \mathtt{a.e'}) \Longrightarrow \varphi \\ & \text{and} & \Gamma;\Theta;\Sigma \vdash \bullet :: (\exists_T \mathtt{e1}, \mathtt{e2}.\ \ \mathsf{e} = \mathtt{e1\ e2}) \Longrightarrow \varphi \\ & \text{and} & \Gamma;\Theta;\Sigma \vdash \bullet :: (\text{symbol } \mathsf{e}) \Longrightarrow \varphi \end{array}$$

Figure 5.7: More tactics and axioms

Proofs are written as OCaml programs, but can be similiarly easy to read as the ones written with dedicated domain-specific languages, as provide the users with some helper functions and tacticals.

operator `(|>)`	Applies a tactic on the prover state.
operator `(%>)`	Combines two tactics together.
`subst`	Substitutes atoms for atom expressions and variables for terms in goal and environmentas long as Solver proves their equality.
`compute`	Computes WHNF of the current goal.
`try_tactic`	Tries applying a tactic and returns unchanged state if it fails.
`repeat`	Applies given tactic mutliple times (until failure).
`assumption`	Finds the appropriate assumption to `apply`.
`trivial`	Tries applying some simple tactics to progress the proof.
`qed`	Turns prover state of a finished proof into a forwards proof. Correctness of proof transformations is guaranteed through the usage of `proof` smart constructors that implement the natural deduction rules.

Figure 5.8: More tactics and operators.

Naturally, we also provide a pretty-printer, created using the `EasyFormat` library, along with a parser developed using the `Angstrom` parser combinator library, designed to handle terms, constraints, kinds, and formulas.

See how predicates such as Nat and $PlusEq$ can be expressed using the programmer-friendly syntax:

```
(* define symbols used in arithmetical theorems *)
let arith_symbols = symbols ["0"; "S"]

let nat_predicate = (* Nat n *)
 "fix Nat(n) : * =
   zero: (n = 0)
   ∨
   succ: (∃ m :term. [n = S m] ∧ Nat m)"

let plus_eq_relation = (* PlusEq n m k *)
  "fix PlusEq(n) : ∀ m k : term. * = fun m k : term →
    zero: ([n = 0] ∧ [m = k])
    ∨
    succ: (∃ n' k' :term. [n = S n'] ∧ [k = S k'] ∧ PlusEq n' m k')"
```

And a theorem that 1 is a natural number, together with a short proof:

```
let nat_1_thm = arith_thm "Nat {S 0}"

let nat_1 =
  proof' nat_1_thm (* goal: Nat {S 0} *)
  |> case "succ"   (* goal: ∃ m :term. [S 0 = S m] ∧ Nat m *)
  |> exists "0"    (* goal: [S 0 = S 0] ∧ Nat 0 *)
  |> solve         (* goal: Nat 0 *)
  |> case "zero"   (* goal: 0 = 0 *)
  |> solve         (* finished *)
  |> qed
```

Another example theorem could be the symmetry of addition:

```
let plus_symm_thm = arith_thm
  "∀ x y z :term. (IsNum x) ⟹ (IsNum y) ⟹
    (PlusEq x y z) ⟹ (PlusEq y x z)"
```

The proof of which is included in the `examples` subdirectory of the project, together with the case study from the next chapter.

Chapter 6

Case study: Progress and Preservation of STLC

The ultimate goal of our work is to create a logic for dealing with variable binding, and there's no better way to put it to work than to prove some things about lambda calculus.

We will take a look at simply typed lambda calculus and examine proofs of its two major properties of *type soundness*: *progress* and *preservation*. But before we delve into the proofs, let's first establish the needed predicates:

```
(* define symbols used in lambda calculus theorems *)
let lambda_symbols = ["lam"; "app"; "base"; "arrow"; "nil"; "cons"]

let term_predicate = (* Term e *)
  "fix Term(e): * =
     var: (∃ a :atom. [e = a])
     ∨
     lam: (∃ a :atom.∃ e' :term. [e = lam (a.e')] ∧ (Term e'))
     ∨
     app: (∃ e1 e2 :term. [e = app e1 e2] ∧ (Term e1) ∧ (Term e2))"

let type_predicate = (* Type t *)
  "fix Type(t): * =
     base: (t = base)
     ∨
     arrow: (∃ t1 t2 :term. [t = arrow t1 t2] ∧ (Type t1) ∧ (Type t2))"
```

Then we define the standard relations of typing through env, typing through the term structure and substitution. As we do not consider functions, relation `(Sub e a v e')` is used to mean that `e` with `a` substituted for `v` is equal to `e'`.

```
let inenv_relation = (* InEnv env a t *)
  "fix InEnv(env): ∀ a :atom. ∀ t :term. * = fun (a :atom) (t :term) →
     current: (∃ env': term. [env = cons a t env'])
     ∨
     next: (∃ b :atom. ∃ s env': term.
             [env = cons b s env'] ∧ [a ≠ b] ∧ (InEnv env' a t))"

let typing_relation = (* Typing e env t *)
  "fix Typing(e): ∀ env t :term. * = fun env t :term →
     var: (∃ a :atom. [e = a] ∧ (InEnv env a t))
     ∨
     lam: (∃ a :atom.∃ e' t1 t2 :term.
            [e = lam (a.e')] ∧ [t = arrow t1 t2]
              ∧ (Type t1) ∧ (Typing e' {cons a t1 env} t2))
     ∨
     app: (∃ e1 e2 t2 :term.
            [e = app e1 e2]
              ∧ (Typing e1 env {arrow t2 t}) ∧ (Typing e2 env t2))"

 let sub_relation = (* Sub e a v e' *)
   "fix Sub(e): ∀ a :atom. ∀ v e':term.* = fun (a :atom) (v e' :term) →
      var_same: ([e = a] ∧ [e' = v])
      ∨
      var_diff: (∃ b :atom. [e = b] ∧ [e' = b] ∧ [a ≠ b])
      ∨
      lam: (∃ b :atom. ∃ e_b e_b' :term.
             [e = lam (b.e_b)] ∧ [e' = lam (b.e_b')] ∧
             [b # v] ∧ [a ≠ b] ∧ (Sub e_b a v e_b') )
      ∨
      app: (∃ e1 e2 e1' e2' :term.
             [e = app e1 e2] ∧ [e' = app e1' e2']
               ∧ (Sub e1 a v e1') ∧ (Sub e2 a v e2') )"
```

Notice that in the definition of `Sub`, in the case for abstracion we only consider the case where the substituted name is different than the abstraction's argument (`a` $\neq$ `b`). If we wanted to substitute `a` for `v` in term `a.e`, then we could swap the argument's name for a different atom b that is fresh in `e`, as then know that `a.e` $=$ `b.(a b)e` and can substitute in that term. In the end, as `b` was fresh in `e`, then `a` must be fresh in (`a b`)`e`, so either way we arrive at identity — but have one less case to consider while writing proofs.

To state the theorem of *progress*, we will naturally need the predicate that a term is *progressive*:

```
let value_predicate = (* Value v *)
  "fun e :term →
     var: (∃ a :atom. [e = a])
     ∨
     lam: (∃ a :atom. ∃ e' : term. [e = lam (a.e')] ∧ (Term e'))"

let steps_relation = (* Steps e e' *)
  "fix Steps(e): ∀ e' :term.* = fun e' :term →
     app_l: (∃ e1 e1' e2 :term. [e = app e1 e2]
              ∧ [e' = app e1' e2] ∧ (Steps e1 e1') )
     ∨
     app_r: (∃ v e2 e2' :term. [e = app v e2]
              ∧ [e' = app v e2'] ∧ (Value v) ∧ (Steps e2 e2') )
     ∨
     app: (∃ a :atom.∃ e_a v :term. [e = app (lam (a.e_a)) v]
            ∧ (Value v) ∧ (Sub e_a a v e') )"

let progressive_predicate = (* Progressive e *)
  "fun e:term →
     value: (Value e)
     ∨
     steps: (∃ e' :term. Steps e e')"

(* lambda_thm parses the theorem in an env that includes lambda_symbols
    and all lambda predicates and relations *)
let progress_thm = lambda_thm
  "∀ e t :term. (Typing e nil t)  ⟹  (Progressive e)"
```

We will also require a lemma about *canonical forms*, which states that all values in the empty environment are of *arrow* type and can be *inversed* into an abstraction term (since we did not consider any true base types like `Bool` or `Int`).

```
let canonical_form_thm = lambda_thm
  "∀ v :term. (Value v)  ⟹
   ∀ t :term. (Typing v nil t)  ⟹
     (∃ a :atom. ∃ e :term. [v = lam (a.e)] ∧ (Term e))"
```

As well as some boilerplate lemmas and relations:

```
let empty_contradiction_thm = lambda_thm
  "∀ a :atom. ∀ t :term. (InEnv nil a t)  ⟹  false"

let typing_terms_thm = lambda_thm
  "∀ e env t : term. (Typing e env t)  ⟹  (Term e)"

let subst_exists_thm = lambda_thm
  "∀ a :atom.
   ∀ v :term. (Value v)  ⟹
   ∀ e :term. (Term e)  ⟹
     ∃ e' :term. (Sub e a v e')"

let env_inclusion_relation = (* EnvInclusion e1 *)
  "fun env1 env2 : term →
     ∀ a : atom. ∀ t : term. (InEnv env1 a t)  ⟹  (InEnv env2 a t)"
```

Lets begin with the proof of *canonical forms*:

```
let canonical_form =
  proof' canonical_form_thm
  |> intros ["v"; "t"; "Hv"; "Ht"]
(* Proof state:
[ ]
[ Ht : Typing v nil t ;
  Hv : Value v
]
⊢ ∃ a :atom. ∃ e :term. [v = lam (a.e)] ∧ Term e
*)
```

The proof will follow from case analysis of `Typing` relation, so let's *destruct* assumption `Ht` and consider the first case, where `v` is some variable `a`. This case is impossible in empty environment, so we named the assumption `contra` and show it through the tactic `ex_falso`.

```
  |> destruct_assm "Ht"
  |> intros' ["contra"; "a"; ""]
     %> ex_falso
(* Proof state:
[ v = a ]
[ Hv : Value v ;
  contra : InEnv nil a t
]
⊢ ⊥
*)
     %> apply_thm_spec empty_contradiction ["a"; "t"]
        (* InEnv nil a t  ⟹  ⊥ *)
     %> apply_assm "contra"
```

Next case is the only sensible one: that `v` is some `lam (a.e)` of type `arrow t1 t2`.

```
  |> intros' ["Hlam"; "a"; "e"; "t1"; "t2"; ""; ""; ""]
     %> exists' ["a"; "e"]
     %> solve
(* Proof state:
[ v = lam (a.e) ; t = arrow t1 t2]
[ Hlam : Type t1 ∧ Typing e {cons a t1 nil} t2 ;
  ...
]
⊢ Term e
*)
```

Now, obviously every term that *types* is indee a proper *term*, so we simply use the `typing_terms` lemma and we're done here.

```
     %> apply_thm_spec typing_terms ["e"; "cons a t1 nil"; "t2"]
        (* Typing e {cons a t1 nil} t2  ⟹  Term e *)
     %> assumption
```

Final case is that `e` is an application, but then it can't be a value, so we analyse the `Hv` assumption, arriving at contradiction in either case:

```
  |> intros' ["contra"; "e1"; "e2"; "t2"; ""]
     %> ex_falso
     %> destruct_assm "Hv"
(* Proof state:
[ v = app e1 e2 ]
[ contra : Typing e1 nil {arrow t2 t} ∧ Typing e2 nil t2 ]
⊢ (∃ a : atom. v = a)  ⟹  ⊥
*)
     %> intros' ["contra_var"; "a"]
     %> discriminate
(* Proof state:
[ v = app e1 e2 ]
[ contra : Typing e1 nil {arrow t2 t} ∧ Typing e2 nil t2 ]
⊢ (∃ a : atom. ∃ e' : term. v = lam (a.e))  ⟹  ⊥
*)
     %> intros' ["contra_lam"; "a"; "e"; ""] %> discriminate
     %> discriminate
  |> qed
```

Now we can proceed with the proof of *progress*, a simple induction over *Typing* derivation:

```
let progress =
  proof' progress_thm
  |> by_induction "e0" "IH" %> intro
(* Proof state:
[ ]
[ IH : ∀ e0 : term. [e0 ≺ e]  ⟹  ∀ t'1 : term.
        (Typing e0 nil t'1)  ⟹  Progressive e0 ]
⊢ (Typing e nil t)  ⟹  Progressive e
*)
```

To analyze all the possible branches of the `Typing` predicate, we simply use `intro'` tactic to destruct the assumption into multiple branches.

```
  |> intro'
```

First one is that `e` is a variable — which again contradicts with empty enviroment:

```
  |> intros' ["contra"; "a"; ""]
     %> ex_falso
(* Proof state:
[ e = a ]
[
  contra : InEnv nil a t ;
  ...
]
⊢ ⊥
*)
     %> apply_thm_spec empty_contradiction ["a"; "t"]
     %> assumption
```

Next, `e` is a lambda abstraction — so a value.

```
  |> intros' ["Hlam"; "a"; "e_a"; "t1"; "t2"; ""] %> case "value"
(* Proof state:
[ e = lam (a.e_a) ; t = arrow t1 t2 ]
[
  Hlam : Typing e_a {cons a t1 nil} t2 ∧ Type t1 ;
  ...
]
⊢ Value e
*)
     %> case "lam"
     %> case "lam"
     %> exists' ["a"; "e_a"]
```

```
    %> solve
```

Then `e` must be an application and thus must be reducing by taking steps, so we apply inductive hypothesis on its sub-expressions `e1` and `e2` and examine the possible cases.

```
  |> intros' ["Happ"; "e1"; "e2"; "t2"; ""; ""] %> case "steps"
  |> add_assm_parse "He1" "Progressive e1"
    %> apply_assm_spec "IH" ["e1"; "arrow t2 t"] %> solve
  |> add_assm_parse "He2" "Progressive e2"
    %> apply_assm_spec "IH" ["e2"; "t2"] %> solve
  |> subst "e" "app e1 e2"
(* Proof state:
[ e = app e1 e2 ]
[
  Happ1 : Typing e1 nil {arrow t2 t} ;
  Happ2 : Typing e2 nil t2 ;
  He1 : Progressive e1 ;
  He2 : Progressive e2 ;
]
⊢ ∃ e' : term. Steps {app e1 e2} e'
*)
```

First we consider the case of both `e1` and `e2` being a value. From `canonical_form` theorem we know then `e1` must be an abstraction — we just need to ensure the Prover that all preconditions are met.

```
  |> destruct_assm "He1" %> intros ["Hv1"]
    %> destruct_assm "He2" %> intros ["Hv2"]  (* Value e1, Value e2 *)
    %> add_assm_thm_spec "He1lam"
         canonical_form ["e1"; "arrow t2 t"]
(* Proof state:
[ e = app e1 e2 ]
[
  He1lam : (Value e1)  ⟹  (Typing e1 nil {arrow t2 t})
         ⟹  ∃ a : atom. ∃ e'1 : term. [e1 = lam (a.e'1)] ∧ Term e'1 ;
  Hv1 : Value e1 ;
  Hv2 : Value e2 ;
  ...
]
⊢ ∃ e' : term. Steps {app e1 e2} e'
*)
    %> apply_in_assm "He1lam" "Hv1"
    %> apply_in_assm "He1lam" "Happ_1"
    %> destruct_assm' "He1lam" ["a"; "e_a"; ""]
    %> subst "e1" "lam (a.e_a)"
(* Proof state:
[ e = app e1 e2 ; e1 = lam (a.e_a) ]
[
  He1lam : Term e_a ;
  ...
]
⊢ ∃ e' : term. Steps {app (lam (a.e_a)) e2} e'
*)
```

Then we need to find the `e'` that `app e1 e2` reduces to, and now that we know `e1` is an abstraction, then we can use beta-reduction rule and find the term of abstracion body `e_a` with argument `a` substituted with `e2`. Again, we ensure the Prover that preconditions are met and destruct on the final assumption to extract the term that we searched for: `e_a'`.

```
    %> add_assm_thm_spec "He_a"
         subst_exists ["a"; "e2"; "e_a"]
(* Proof state:
[ ... ]
[
  He_a : (Value e2) ⟹ (Term e_a) ⟹ ∃ e' : term. Sub e_a a e2 e' ;
  ...
]
⊢ ∃ e' : term. Steps e e'
*)
    %> apply_in_assm "He_a" "Hv2"
    %> apply_in_assm "He_a" "He1lam"
    %> destruct_assm' "He_a" ["e_a'"]
    %> exists "e_a'"
(* Proof state:
[ ... ]
[
  He_a : Sub e_a a e2 e_a' ;
  ...
]
⊢ Steps {app (lam (a.e_a)) e2} e_a'
*)
    %> case "app" %> exists' ["a"; "e_a"; "e2"] %> solve
(* Proof state:
[ ... ]
[ ... ]
⊢ Value e2 ∧ Sub e_a a e2 e_a'
*)
    %> destruct_goal %> apply_assm "Hv2" %> apply_assm "He_a"
```

Now what's left is to examine straightforward cases where either `e1` or `e2` steps.

```
  |> intros' ["Hs2"; "e2'"] (* Value e1, Steps e2 e2' *)
     %> exists "app e1 e2'"
(* Proof state:
[ ... ]
[
  Hv1 : Value e1 ;
  Hs2 : Steps e2 e2' ;
  ...
]
⊢ Steps {app e1 e2} {app e1 e2'}
*)
    %> case "app_r"
    %> exists' ["e1"; "e2"; "e2'"]
    %> repeat solve
(* Proof state:
[ ... ]
[ ... ]
⊢ Value e1 ∧ Steps e2 e2’
*)
    %> destruct_goal
    %> apply_assm "Hv1"
    %> apply_assm "Hs2"
  |> intros’ ["Hs1"; "e1’"] (* Steps e1 *)
(* Proof state:
[ ... ]
[
  Hs1 : Steps e1 e1’ ;
  ...
]
⊢ Steps {app e1 e2} {app e1’ e2}
*)
    %> exists "app e1’ e2"
    %> case "app_l"
    %> exists’ ["e1"; "e1’"; "e2"]
    %> repeat solve
    %> apply_assm "Hs1"
  |> apply_assm "Happ_2" %> apply_assm "Happ_1"
  |> qed
```

Now, to prove *Preservation*, we will need some more lemmas:

1. Substitution lemma: if term `e` has a type `t` in enviroment `{cons a ta env}`, then we can substitute `a` for any value `v` of type `ta` in `e` without breaking the typing.

```
let sub_lemma_thm = lambda_thm
  "∀ e env t :term.
   ∀ a : atom. ∀ ta :term.
   ∀ v e’ :term.
     (Typing v env ta)  ⟹
     (Typing e {cons a ta env} t)  ⟹
     (Sub e a v e’)  ⟹
       (Typing e’ env t)"
```

2. Weakening lemma: for any enviroment `env1`, we can use larger enviroment `env2` without breaking the typing.

```
let weakening_lemma_thm = lambda_thm
  "∀ e env1 t env2 : term.
     (Typing e env1 t)  ⟹
     (EnvInclusion env1 env2)  ⟹
       (Typing e env2 t)"
```

3. Lambda abstraction typing inversion: If term `lam (a.e)` has a type `{arrow t1 t2}` in environment `env`, then it must be that the body `e` has a type `t2` in enviroment extended with the argument `{cons a t1 env}`.

```
let lambda_typing_inversion_thm = lambda_thm
  "∀ a :atom. ∀ e env t1 t2 :term.
     (Typing {lam (a.e)} env {arrow t1 t2})  ⟹
       (Typing e {cons a t1 env} t2)"
```

To maintain reader engagement and prevent excessive technicality, we will omit here the proofs of rather obvious lemmas 2 and 3 and instead focus on the more important lemma 1:

```
let sub_lemma =
  proof' sub_lemma_thm
  |> by_induction "e0" "IH"
     %> repeat intro %> intros ["Hv"; "He"; "Hsub"]
(* Proof state:
[ ]
[
  He : Typing e {cons a ta env} t ;
  Hsub : Sub e a v e' ;
  Hv : Typing v env ta ;
  IH : ∀ e0 : term. [e0 ≺ e]  ⟹
        ∀ env'1 t'1 : term. ∀ a'1 : atom. ∀ ta'1 v'1 e''1 : term.
          Typing v'1 env'1 ta'1  ⟹
          Typing e0 {cons a'1 ta'1 env'1} t'1  ⟹
          Sub e0 a'1 v'1 e''1  ⟹
            Typing e''1 env'1 t'1
]
⊢ Typing e' env t
*)
%> destruct_assm "He"
```

First case is that `e` is some variable `b`, with first subcases that it is equal to `a` and substitutes to `v`:

```
  |> intros' ["Hb"; "b"; ""]
     %> destruct_assm "Hsub"
     %> ( intros' ["Heq"; ""; ""]
(* Proof state:
[ e = a ; e' = v ; e = b ]
[
  Hb : InEnv {cons a ta env} b t ;
  Hv : Typing v env ta ;
  ...
]
⊢ Typing e' env t
*)
```

Now because in the goal `e'` has type `t`, but in assumption `Hv` it has `ta`, then we again case-analyse the assumption `Hb` and get that either `t = ta` or arrive at contradiction:

```
        %> destruct_assm "Hb"
        %> ( intros' ["Heq"; "env'"; ""] (* t = ta *)
             %> apply_assm "Hv" )
        %> ( intros' ["Hdiff"; "b'"; "t'"; "env'"; ""; ""] (* a ≠ b *)
              %> discriminate )
```

Second subcase is that `b` is be different than `a` and thus is not be affected by the subistution. We will again case-analyse `Hb` assumption to extract additional facts.

```
    %> ( intros' ["Hdiff"; "b'"; ""; ""; ""] (* a ≠ b *)
       %> destruct_assm "Hb"
       %> ( intros' ["Heq"; "env'"; ""] (* a = b *)
            %> discriminate )
       %> ( intros' ["Hdiff"; "a'"; "ta'"; "env'"; ""; ""]
(* Proof state:
[  e = b ; e' = b ; a ≠ b ; ... ]
[
  Hdiff : InEnv env' b t ;
  ...
]
⊢ Typing e' env t
*)
              %> case "var"
              %> exists "b"
              %> solve
              %> assumption )
```

Second case is that `e` is some abstraction `lam (b.e_b)`. Because of the way we defined subsitution, abstraction argument must be different than the substituted variable and not occur in the substitutee value —which is made possible by swapping atoms while maintaining alpha-equality. Consequence of that is when we destruct `Hsub` we get that `e = lam (c.e_c)` and `e' = lam (c.e_c')` —while `b.e_b` and `c.e_c` are equal, `b` and `c` don't have to be. Abstracting the mundane details to auxiliary lemmas allows us to present the derivation in a simple chain of applications and assumptions:

```
  |> intros' ["Hlam"; "b"; "e_b"; "t1"; "t2"; ""; ""; ""]
     %> destruct_assm "Hsub"
     %> intros' ["Hsub"; "c"; "e_c"; "e_c'"; ""; ""; ""; ""]
     %> case "lam"
     %> exists' ["c"; "e_c'"; "t1"; "t2"]
     %> repeat solve
(* Proof state:
[ e = lam (b.e_b) ; e = lam (c.e_c) ; e' = lam (c.e_c') ;
  a ≠ c ; c # v ; t = arrow t1 t2 ]
[
  Hsub : Sub e_c a v e_c' ;
  Hlam_1 : Type t1 ;
  Hlam_2 : Typing e_b {cons b t1 (cons a ta env)} t2 ;
  Hv : Typing v env ta ;
  ...
]
⊢ Type t1 ∧ Typing e_c' {cons c t1 env} t2
*)
     %> destruct_goal
```

```
    %> assumption
    %> apply_assm_spec
       "IH" ["e_c"; "cons c t1 env"; "t2"; "a"; "ta"; "v"; "e_c'"]
    (* [e_c ≺ e]  ⟹  Typing v {cons c t1 env} ta  ⟹
       Typing e_c {cons a ta (cons c t1 env)} t2  ⟹
         Sub e_c a v e_c'  ⟹  Typing e_c' {cons c t1 env} t2  *)
    %> solve
    %> ( apply_thm_spec
         cons_fresh_typing ["v"; "env"; "ta"; "c"; "t1"]
         (* [c # v]  ⟹  Typing v env ta  ⟹
              Typing v {cons c t1 env} ta *)
       %> solve
       %> apply_assm "Hv" )
    %> ( apply_thm_spec
        typing_env_shuffle ["e_c"; "env"; "t2"; "c"; "t1"; "a"; "ta"]
        (* [c ≠ a]  ⟹
             Typing e_c {cons c t1 (cons a ta env)} t2  ⟹
               Typing e_c {cons a ta (cons c t1 env)} t2 *)
       %> solve
       %> apply_thm_spec swap_lambda_typing
           ["b"; "e_b"; "c"; "e_c"; "cons a ta env"; "t1"; "t2"]
           (* [b.e_b = c.e_c]  ⟹
                Typing e_b {cons b t1 (cons a ta env)} t2  ⟹
                  Typing e_c {cons c t1 (cons a ta env)} t2 *)
       %> solve
       %> apply_assm "Hlam_2" )
    %> apply_assm "Hsub"
```

Finally, we consider the case that `e` is an application `e1 e2`, which goes straightly from inductive hypothesis, so we omit this part here.

```
  |> intros' ["Happ"; "e1"; "e2"; "t2"; ""; ""]
    %> intros' ["Hsub"; "_e1"; "_e2"; "e1'"; "e2'"; ""; ""; ""]
    %> case "app"
    %> exists' ["e1'"; "e2'"; "t2"]
    %> solve
(* Proof state:
[ e = app e1 e2 ; e' = app e1' e2']
[
  Happ_1 : Typing e1 {cons a ta env} {arrow t2 t} ;
  Happ_2 : Typing e2 {cons a ta env} t2 ;
  Hsub_1 : Sub e1 a v e1' ;
  Hsub_2 : Sub e2 a v e2' ;
  ...
]
⊢ Typing e1' env {arrow t2 t} ∧ Typing e2' env t2
*)
    ...
  |> qed
```

Now that we've shown the `sub_lemma`, we can go on with the final proof of *preservation*. The proof goes through induction on term `e` the case analysis on assumption `Steps e e'`.

```
let preservation = proof’ preservation_thm
  |> by_induction "e0" "IH"
  |> intro %> intro %> intro %> intros ["Htyp"; "Hstep"]
(* Proof state:
[ ]
[
  Hstep : Steps e e’ ;
  Htyp : Typing e env t ;
  IH : ∀ e0 : term. [e0 ≺ e]
          ⟹ ∀ e’1 env’1 t’1 : term. (Typing e0 env’1 t’1)
            ⟹ (Steps e0 e’1)
              ⟹ Typing e’1 env’1 t’1
]
⊢ Typing e’ env t
*)
  |> destruct_assm "Hstep"
```

First two cases are rather simple: `e` is `app e1 e2` and either `e1` or `e2` take a step.

```
  |> intros’ ["He1"; "e1"; "e1’"; "e2"; ""; ""]
     %> case "app"
     %> exists’ ["e1’"; "e2"; "t2"]
     %> solve
(* Proof state:
[ e = app e1 e2 ; e’ = app e1’ e2 ]
[
  Happ_2 : Typing e2 env t2 ;
  Happ_1 : Typing e1 env {arrow t2 t} ;
  He1 : Steps e1 e1 ;
  ...
]
⊢ Typing e1’ env {arrow t2 t} ∧ Typing e2 env t2
*)
     %> destruct_goal
       %> (apply_assm_spec "IH" ["e1"; "e1’"; "env"; "arrow t2 t"]
            (* [e1 ≺ e] ⟹
                 Typing e1 env {arrow t2 t} ⟹
                   Steps e1 e1’ ⟹
                     Typing e1’ env {arrow t2 t} *)
            %> solve
            %> apply_assm "Happ_1"
            %> apply_assm "He1" )
       %> apply_assm "Happ_2"
  |> intros’ ["He2"; "v1"; "e2"; "e2’"; ""; ""; ""]
     %> case "app"
     %> exists’ ["v1"; "e2’"; "t2"]
     %> solve
(* Proof state:
[ e = app e1 e2 ; e’ = app e1’ e2 ]
[
  He2 : Value v1 ∧ Steps e2 e2’ ;
  ...
]
⊢ Typing e1 env {arrow t2 t} ∧ Typing e2’ env t2
*)
     %> destruct_goal
       %> apply_assm "Happ_1"
       %> ( apply_assm_spec "IH" ["e2"; "e2’"; "env"; "t2"]
            (* [e2 ≺ e] ⟹
                 Typing e2 env t2 ⟹
                   Steps e2 e2’ ⟹
                     Typing e2’ env t2 *)
            %> solve
            %> apply_assm "Happ_2"
            %> apply_assm "He2_2" )
```

The next, final case is where we will need the established lemmas: application `app e1 e2` beta-reduces into some term `e’` and we use the `sub_lemma` to show that `e’` still types.

```
  |> intros’ ["Hbeta"; "a"; "e_a"; "v"; ""; ""]
(* Proof state:
[ e = app (lam (a.e_a)) v ]
[
  Happ_2 : Typing v env t2 ;
  Happ_1 : Typing (lam (a.e_a)) env {arrow t2 t} ;
  Hbeta_1 : Value v ;
  Hbeta_2 : Sub e_a a v e’ ;
  ...
]
⊢ Typing e’ env t
*)
    %> apply_thm_spec
         sub_lemma ["e_a"; "env"; "t"; "a"; "t2"; "v"; "e’"]
    (* Typing v env t2 ⟹
         Typing e_a {cons a t2 env} t ⟹
           Sub e_a a v e’ ⟹
             Typing e’ env t *)
    %> apply_assm "Happ_2"
    %> ( apply_thm_spec
           lambda_typing_inversion ["a"; "e_a"; "env"; "t2"; "t"]
           (* Typing {lam (a.e_a)} env {arrow t2 t}
               ⟹ Typing e_a {cons a t2 env} t *)
         %> apply_assm "Happ_1" )
    %> apply_assm "Hbeta_2"
  |> qed
```

And that’s it.

Chapter 7

Conclusion

In summary, we've introduced and demonstrated a specialized variant of Nominal Logic, designed for reasoning about variable binding through the utilization of constraints solving. We've also successfully implemented this logic in OCaml, complemented by essential tools, including a proof assistant.

Through the proofs of classical properties of simply typed lambda calculus we have validated the logic's suitablility for reasoning about programming languages. However, the true potential of this framework is expected to shine when applied to specific theorems reliant on the notions of variable binding.

We must also acknowledge that our framework is still in its infancy, requiring substantial refinement to ensure a user-friendly experience, as the awkardness and low-level nature of the current tooling obscures the benefits of underlying constraint-based sublogic. Consequently, it cannot be directly compared to other theorem-proving frameworks like Coq or Twelf.

Nonetheless, we are confident that with enough refinement, our framework can prove to be a valuable resource for the specific use cases and remain enthusiastic about the framework's potential to contribute to the field of formal methods and reasoning based on Nominal Logic.

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Appendices

Appendix A

Solver rules

Goal-reducing equality rules:

$$\frac{}{\Gamma;\Delta \vDash a = a} \qquad \frac{}{\Gamma;\Delta \vDash X = X} \qquad \frac{}{\Gamma;\Delta \vDash f = f}$$

$$\frac{\Gamma;\Delta \vDash t_1 = t_2 \qquad \Gamma;\Delta \vDash t_1' = t_2'}{\Gamma;\Delta \vDash t_1 t_1' = t_2 t_2'}$$

$$\frac{\begin{array}{c}\Gamma;\Delta \vDash \alpha_1 \# \alpha_2.t_2 \\ \Gamma;\Delta \vDash t_1 = (\alpha_1\ \alpha_2)t_2\end{array}}{\Gamma;\Delta \vDash \alpha_1.t_1 = \alpha_2.t_2} \qquad \frac{\begin{array}{c}a \neq \alpha_1, a \neq \alpha_2, \Gamma;\Delta \vDash a = \alpha \\ a = \alpha_1, a \neq \alpha_2, \Gamma;\Delta \vDash \alpha_2 = \alpha \\ a = \alpha_2, \Gamma;\Delta \vDash \alpha_1 = \alpha\end{array}}{\Gamma;\Delta \vDash a = (\alpha_1\ \alpha_2)\alpha}$$

$$\frac{\Gamma;\Delta \vDash a = \pi^{-1}\alpha}{\Gamma;\Delta \vDash \pi a = \alpha} \qquad \frac{\Gamma;\Delta \vDash X_1 = \pi_1^{-1}\pi_2 X_2}{\Gamma;\Delta \vDash \pi_1 X_1 = \pi_2 X_2}$$

$$\frac{\Gamma;\Delta \vDash \pi \text{ idempotent on } X}{\Gamma;\Delta \vDash X = \pi X} \qquad \frac{\forall a \in \pi.\ \Gamma;\Delta \vDash a = \pi a \ \vee\ \Gamma;\Delta \vDash a \# X}{\Gamma;\Delta \vDash \pi \text{ idempotent on } X}$$

Goal-reducing freshness rules:

$$\frac{a_1 \neq a_2 \in \Delta}{\Gamma;\Delta \vDash a_1 \# a_2} \qquad \frac{a \# X \in \Delta}{\Gamma;\Delta \vDash a \# X} \qquad \frac{}{\Gamma;\Delta \vDash a \# f}$$

$$\frac{a \neq \alpha, \Gamma;\Delta \vDash a \# t}{\Gamma;\Delta \vDash a \# \alpha.t} \qquad \frac{\Gamma;\Delta \vDash a \# t_1 \qquad \Gamma;\Delta \vDash a \# t_2}{\Gamma;\Delta \vDash a \# t_1 t_2}$$

$$\frac{\begin{array}{c}a \neq \alpha_1, a \neq \alpha_2, \Gamma;\Delta \vDash a \# \alpha \\ a = \alpha_1, a \neq \alpha_2, \Gamma;\Delta \vDash \alpha_1 \# \alpha \\ a = \alpha_2, \Gamma;\Delta \vDash \alpha_2 \# \alpha\end{array}}{\Gamma;\Delta \vDash a \# (\alpha_1\ \alpha_2)\alpha} \qquad \frac{\begin{array}{c}a \neq \alpha_1, a \neq \alpha_2, \Gamma;\Delta \vDash a \# \pi X \\ a = \alpha_1, a \neq \alpha_2, \Gamma;\Delta \vDash \alpha_1 \# \pi X \\ a = \alpha_2, \Gamma;\Delta \vDash \alpha_2 \# \pi X\end{array}}{\Gamma;\Delta \vDash a \# (\alpha_1\ \alpha_2)\pi X}$$

Goal reducing shape rules:

$$\frac{}{\Gamma;\Delta \vDash \_ \sim \_} \qquad \frac{}{\Gamma;\Delta \vDash f \sim f}$$

$$\frac{X_1 \sim X_2 \in \Delta}{\Gamma;\Delta \vDash X_1 \sim X_2} \qquad \frac{X \sim s' \in \Delta \quad \Gamma;\Delta \vDash s' \sim s}{\Gamma;\Delta \vDash X \sim s}$$

$$\frac{\Gamma;\Delta \vDash s_1 \sim s_2}{\Gamma;\Delta \vDash \_.s_1 \sim \_.s_2} \qquad \frac{\Gamma;\Delta \vDash s_1 \sim s_2 \quad \Gamma;\Delta \vDash s_1' \sim s_2'}{\Gamma;\Delta \vDash s_1 s_1' \sim s_2 s_2'}$$

Goal-reducing subshape rules:

$$\frac{\Gamma;\Delta \vDash s_1 \sim s_2}{\Gamma;\Delta \vDash s_1 \prec \_.s_2} \qquad \frac{\Gamma;\Delta \vDash s_1 \prec s_2}{\Gamma;\Delta \vDash s_1 \prec \_.s_2}$$

$$\frac{\Gamma;\Delta \vDash s_1 \sim s_2}{\Gamma;\Delta \vDash s_1 \prec s_2 s_2'} \qquad \frac{\Gamma;\Delta \vDash s_1 \sim s_2'}{\Gamma;\Delta \vDash s_1 \prec s_2 s_2'} \qquad \frac{\Gamma;\Delta \vDash s_1 \prec s_2}{\Gamma;\Delta \vDash s_1 \prec s_2 s_2'} \qquad \frac{\Gamma;\Delta \vDash s_1 \prec s_2'}{\Gamma;\Delta \vDash s_1 \prec s_2 s_2'}$$

$$\frac{s_2 \prec X \in \Delta \quad \Gamma;\Delta \vDash s_2 \sim X}{\Gamma;\Delta \vDash s_1 \prec X} \qquad \frac{s_2 \prec X \in \Delta \quad \Gamma;\Delta \vDash s_2 \prec X}{\Gamma;\Delta \vDash s_1 \prec X}$$

Assumption-reducing equality rules:

$$\frac{X = \pi^{-1}t, \Gamma;\Delta \vDash \mathcal{C}}{\pi X = t, \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{\pi \text{ idempotent on } X, \Gamma;\Delta \vDash \mathcal{C}}{X = \pi X, \Gamma;\Delta \vDash \mathcal{C}} \qquad \frac{\begin{matrix}\vDash \text{ idempotent on } X \\ \Gamma;\Delta \vDash \mathcal{C}\end{matrix}}{\pi \text{ idempotent on } X, \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{(\forall a \in \pi.\ \Gamma;\Delta \vDash a = \pi a \ \vee \ \Gamma;\Delta \vDash a \mathrel{\#} X), \Gamma;\Delta \vDash \mathcal{C}}{\pi \text{ idempotent on } X, \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{\Gamma\{X \mapsto t\};\Delta\{X \mapsto t\} \vDash \mathcal{C}\{X \mapsto t\}}{X = t, \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{\Gamma\{a_1 \mapsto a_2\};\Delta\{a_1 \mapsto a_2\} \vDash \mathcal{C}\{a_1 \mapsto a_2\}}{a_1 = a_2, \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{a = \pi^{-1}\alpha, \Gamma;\Delta \vDash \mathcal{C}}{\pi a = \alpha, \Gamma;\Delta \vDash \mathcal{C}} \qquad \frac{\begin{matrix} a \neq \alpha_1, a \neq \alpha_2, a = \alpha, \Gamma;\Delta \vDash \mathcal{C} \\ a = \alpha_1, a \neq \alpha_2, \alpha_2 = \alpha, \Gamma;\Delta \vDash \mathcal{C} \\ a = \alpha_2, \alpha_1 = \alpha, \Gamma;\Delta \vDash \mathcal{C}\end{matrix}}{a = (\alpha_1\ \alpha_1)\alpha, \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{}{a = t_1 t_2, \Gamma;\Delta \vDash \mathcal{C}} \qquad \frac{}{a = \alpha.t, \Gamma;\Delta \vDash \mathcal{C}} \qquad \frac{}{a = f, \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{\alpha_1 \mathrel{\#} \alpha_2.t_2,\ t_1 = (\alpha_1\ \alpha_2)t_2,\ \Gamma;\Delta \vDash \mathcal{C}}{\alpha_1.t_1 = \alpha_2.t_2, \Gamma;\Delta \vDash \mathcal{C}} \quad \text{Other term constructors trivial}$$

$$\frac{t_1 = t_2,\ t_1' = t_2',\ \Gamma;\Delta \vDash \mathcal{C}}{t_1 t_1' = t_2 t_2', \Gamma;\Delta \vDash \mathcal{C}} \quad \text{Other term constructors trivial}$$

$$\frac{f_1 \neq f_2}{f_1 = f_2, \Gamma;\Delta \vDash \mathcal{C}} \qquad \frac{\Gamma;\Delta \vDash \mathcal{C}}{f = f, \Gamma;\Delta \vDash \mathcal{C}} \quad \text{Other term constructors trivial}$$

Assumption-reducing freshness rules:

$$\frac{\Gamma;\{a_1 \neq a_2\} \cup \Delta \vDash \mathcal{C}}{a_1 \neq a_2,\ \Gamma;\Delta \vDash \mathcal{C}} \qquad \frac{\Gamma;\{a \mathrel{\#} X\} \cup \Delta \vDash \mathcal{C}}{a \mathrel{\#} X,\ \Gamma;\Delta \vDash \mathcal{C}}$$

$$\frac{\begin{array}{c} a \neq \alpha_1, a \neq \alpha_2, a \mathrel{\#} \alpha, \Gamma; \Delta \vDash \mathcal{C} \\ a = \alpha_1, a \neq \alpha_2, \alpha_2 \mathrel{\#} \alpha, \Gamma; \Delta \vDash \mathcal{C} \\ a = \alpha_2, \alpha_1 \mathrel{\#} \alpha, \Gamma; \Delta \vDash \mathcal{C} \end{array}}{a \mathrel{\#} (\alpha_1\ \alpha_1)\alpha, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{\begin{array}{c} a \neq \alpha_1, a \neq \alpha_2, a \mathrel{\#} \pi X, \Gamma; \Delta \vDash \mathcal{C} \\ a = \alpha_1, a \neq \alpha_2, \alpha_2 \mathrel{\#} \pi X, \Gamma; \Delta \vDash \mathcal{C} \\ a = \alpha_2, \alpha_1 \mathrel{\#} \pi X, \Gamma; \Delta \vDash \mathcal{C} \end{array}}{a \mathrel{\#} (\alpha_1\ \alpha_1)\pi X, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{\begin{array}{c} a \mathrel{\#} \alpha,\ \Gamma; \Delta \vDash \mathcal{C} \\ a \mathrel{\#} \alpha,\ a \mathrel{\#} t,\ \Gamma; \Delta \vDash \mathcal{C} \end{array}}{a \mathrel{\#} \alpha.t, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{a \mathrel{\#} t_1, \Gamma; \Delta \vDash \mathcal{C} \qquad a \mathrel{\#} t_2, \Gamma; \Delta \vDash \mathcal{C}}{a \mathrel{\#} t_1 t_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{\Gamma; \Delta \vDash \mathcal{C}}{a \mathrel{\#} f, \Gamma; \Delta \vDash \mathcal{C}}$$

Assumption-reducing shape rules:

$$\frac{\Gamma; \{X_1 \sim X_2\} \cup \Delta \vDash \mathcal{C}}{X_1 \sim X_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{\Gamma; \{X \sim s\} \cup \Delta \vDash \mathcal{C}}{X \sim s,\ \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{\Gamma; \Delta \vDash \mathcal{C}}{a_1 \sim a_2, \Gamma; \Delta \vDash \mathcal{C}} \quad \text{Other term constructors trivial}$$

$$\frac{t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C}}{\_.t_1 \sim \_.t_2, \Gamma; \Delta \vDash \mathcal{C}} \quad \text{Other term constructors trivial}$$

$$\frac{t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C} \qquad t_1' \sim t_2', \Gamma; \Delta \vDash \mathcal{C}}{t_1 t_1' \sim t_2 t_2', \Gamma; \Delta \vDash \mathcal{C}} \quad \text{Other term constructors trivial}$$

$$\frac{f_1 \neq f_2}{f_1 \sim f_2, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{}{f \sim f, \Gamma; \Delta \vDash \mathcal{C}} \quad \text{Other term constructors trivial}$$

Assumption-reducing subshape rules:

$$\frac{\Gamma; \{t \prec X\} \cup \Delta \vDash \mathcal{C}}{t \prec X, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C} \qquad t_1 \prec t_2, \Gamma; \Delta \vDash \mathcal{C}}{t_1 \prec \_.t_2, \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{\begin{array}{cc} t_1 \sim t_2, \Gamma; \Delta \vDash \mathcal{C} & t_1 \sim t_2', \Gamma; \Delta \vDash \mathcal{C} \\ t_1 \prec t_2, \Gamma; \Delta \vDash \mathcal{C} & t_1 \prec t_2', \Gamma; \Delta \vDash \mathcal{C} \end{array}}{t_1 \prec t_2 t_2', \Gamma; \Delta \vDash \mathcal{C}}$$

$$\frac{}{t \prec \alpha, \Gamma; \Delta \vDash \mathcal{C}} \qquad \frac{}{t \prec f, \Gamma; \Delta \vDash \mathcal{C}}$$