

MATH 6370 p -ADIC HODGE THEORY

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1. MOTIVATION: COMPLEX HODGE THEORY

Cohomology is a way of measuring how many “loops” a space has. Consider the space $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Definition 1.1. A 1-cochain on \mathbb{C}^\times is a function on paths in \mathbb{C}^\times .

A 1-cochain φ is *closed* if for any continuous map f from a triangle ABC to \mathbb{C}^\times , $\phi(f(AC)) = \phi(f(AB)) + \phi(f(BC))$. It is *exact* if it is of the form

$$\psi(\text{ending point}) - \psi(\text{starting point})$$

for some function ψ on \mathbb{C}^\times .

Define

$$H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{Z}) = \{\mathbb{Z}\text{-valued closed 1-cochains}\} / \{\mathbb{Z}\text{-valued exact 1-cochains}\},$$

and define $H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{C})$ similarly.

Then $H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{Z})$ is a free abelian group of rank one, and

$$H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{C}) \cong H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

is a \mathbb{C} -vector space of dimension 1. A class in $H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{Z})$ has many representatives, but they all take on the same value on closed paths. There is a generator of $H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{Z})$ that takes any path to its winding number around the origin.

Definition 1.2. An holomorphic 1-form on \mathbb{C}^\times is an expression of the form $f(z) dz$, where $f(z)$ is an analytic function on \mathbb{C}^\times . The holomorphic functions on \mathbb{C}^\times are precisely the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$ and $|a_n| \rightarrow 0$ exponentially as $n \rightarrow \pm\infty$.

A holomorphic 1-form *exact* if it is of the form $f'(z) dz$, where $f(z)$ is a holomorphic function. (All holomorphic 1-forms are closed.)

Define

$$H_{\text{dR}}^1(\mathbb{C}^\times) = \{\text{holomorphic 1-forms}\} / \{\text{exact holomorphic 1-forms}\}$$

Then $H_{\text{dR}}^1(\mathbb{C}^\times)$ is a \mathbb{C} -vector space of dimension 1. The class of $z^{-1} dz$ is a generator. There is an isomorphism of vector spaces

$$H_{\text{dR}}^1(\mathbb{C}^\times) \xrightarrow{\sim} H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{C})$$

given by

$$(1.3) \quad f(z) dz \mapsto \left(\gamma \mapsto \int_{\gamma} f(z) dz \right).$$

For any complex manifold X , one can define the singular cohomology $H_{\text{sing}}^n(X)$ (defined using maps from simplices into X) and the de Rham cohomology $H_{\text{dR}}^n(X)$ (defined using holomorphic differentials on X). There is an isomorphism

$$H_{\text{dR}}^n(X) \cong H_{\text{sing}}^n(X, \mathbb{C})$$

given by integration.

This isomorphism is functorial: if we have a holomorphic or antiholomorphic map $\sigma: X \rightarrow Y$, then there is a commutative square

$$\begin{array}{ccc} H_{\text{dR}}^n(Y) & \xrightarrow{\sim} & H_{\text{sing}}^n(Y, \mathbb{C}) \\ \downarrow \sigma^* & & \downarrow \sigma^* \\ H_{\text{dR}}^n(X) & \xrightarrow{\sim} & H_{\text{sing}}^n(X, \mathbb{C}) \end{array}$$

If σ is holomorphic, then

$$\begin{aligned} \sigma^*(f(z) dz) &= f(\sigma(z)) d\sigma(z) \\ \sigma^*(\varphi)(\gamma) &= \varphi(\sigma(\gamma)). \end{aligned}$$

If σ is antiholomorphic, then

$$\begin{aligned} \sigma^*(f(z) dz) &= \overline{f(\sigma(z)) d\sigma(z)} \\ \sigma^*(\varphi)(\gamma) &= \overline{\varphi(\sigma(\gamma))}. \end{aligned}$$

What is the p -adic version of this story? Let K be a p -adic field. (You can assume for now that K is \mathbb{Q}_p or a finite extension, but I will make a more general definition later.) A p -adic analogue of \mathbb{C}^\times is the rigid analytic space $\mathbb{A}_K^1 \setminus \{0\}$.

We will define rigid analytic spaces later. For now, we will just define the space of analytic functions on $\mathbb{A}_K^1 \setminus \{0\}$. Motivated by the complex case, We define this space to be the set of Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n,$$

where $a_n \in K$ and the $|a_n|$'s go to zero faster than exponentially as $n \rightarrow \pm\infty$. A 1-form is an analytic function multiplied by dz . Then

$$H_{\text{dR}}^1(\mathbb{A}_K^1 \setminus \{0\}) = \{1\text{-forms}\} / \{\text{exact 1-forms}\}$$

is a 1-dimensional K -vector space generated by the class of $z^{-1} dz$.

A p -adic analogue of singular cohomology is étale cohomology. For now, we will just give a heuristic definition. Consider the map $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$. Any path in \mathbb{C}^\times that starts and ends at 1 is the image of a path in \mathbb{C} that starts at 0 and ends at $2\pi i k$, where k is the winding number of the path. So we can identify

$$H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(2\pi i \mathbb{Z}, \mathbb{Z}).$$

Unlike the complex exponential function, the p -adic exponential has a finite radius of convergence. So it is useful instead to look at the collection of maps $z \mapsto z^n$ for each integer n . A path that starts and ends at 1 and has winding number k is the image under $z \mapsto z^n$ of a path that starts at 1 and ends at $e^{2\pi i k/n}$. The collection of roots of unity $\{e^{2\pi i k/n} | n \in \mathbb{Z}_{>0}\}$ is enough to recover k .

Let μ be the set of all roots of unity of \mathbb{C}^\times . Then we can identify

$$H_{\text{sing}}^1(\mathbb{C}^\times, \mathbb{Z}) \cong \text{Hom}_{\text{cts}}(\mu, \mathbb{Q}/\mathbb{Z}).$$

Here, μ has the topology inherited from \mathbb{C}^\times , and \mathbb{Q}/\mathbb{Z} has the topology inherited from $\mathbb{R}/\mathbb{Z} \cong S^1$.

The isomorphism $\mathrm{Hom}_{\mathbb{Z}}(2\pi i\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{cts}}(\mu, \mathbb{Q}/\mathbb{Z})$ can be described as follows. Any element of the former group is multiplication by $\frac{k}{2\pi i}$ for some integer k . Its image in the latter group is the map $e^{2\pi i m/n} \mapsto mk/n$.

With this in mind, we define

$$H_{\mathrm{\acute{e}t}}^1(\mathbb{A}_{\overline{K}}^1 \setminus \{0\}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}_p}(\mu_{p^\infty}, \mathbb{Q}_p/\mathbb{Z}_p),$$

where μ_{p^∞} is the set of p -power roots of unity in \overline{K} . It is a free \mathbb{Z}_p -module of rank 1, and it has an action of $\mathrm{Gal}(\overline{K}/K)$. We will also denote this group by $\mathbb{Z}_p(-1)$.

We would like to compare $H_{\mathrm{dR}}^1(\mathbb{A}_K^1 \setminus \{0\})$ and $H_{\mathrm{\acute{e}t}}^1(\mathbb{A}_{\overline{K}}^1 \setminus \{0\}, \mathbb{Z}_p)$. More specifically, we would like to write down an $\mathrm{Gal}(\overline{K}/K)$ -equivariant isomorphism

$$H_{\mathrm{dR}}^1(\mathbb{A}_K^1 \setminus \{0\}) \otimes_K L \cong H_{\mathrm{\acute{e}t}}^1(\mathbb{A}_{\overline{K}}^1 \setminus \{0\}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} L$$

for some field L .

What should L be? The field \overline{K} has a unique multiplicative absolute value extending the one on K . We write $C = \widehat{\overline{K}}$ for the completion of \overline{K} with respect to this absolute value. The most obvious guess is that $L = C$.

However, it turns out that this guess does not work. We will show in a future lecture that there are no nonzero $\mathrm{Gal}(\overline{K}/K)$ -equivariant maps

$$H_{\mathrm{dR}}^1(\mathbb{A}_K^1 \setminus \{0\}) \otimes_K C \rightarrow H_{\mathrm{\acute{e}t}}^1(\mathbb{A}_{\overline{K}}^1 \setminus \{0\}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C.$$

Instead, we will have to define a ring B_{dR}^+ that is a completion of \overline{K} with respect to a more unusual topology. The ring B_{dR}^+ will be a discrete valuation ring with residue field C . We will take $L = B_{\mathrm{dR}} = \mathrm{Frac} B_{\mathrm{dR}}^+$.

The p -adic analogues of complex manifolds are called rigid analytic spaces. If you are not familiar with rigid analytic spaces, you can just think about algebraic varieties over a p -adic field—there is an analytification functor that turns any such variety into a rigid analytic space. Given a rigid analytic space X over a p -adic field K , one can define étale cohomology groups

$$H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbb{Z}_p)$$

and de Rham cohomology groups

$$H_{\mathrm{dR}}^i(X).$$

Theorem 1.4 (Scholze, [Sch13]). *If X is proper and smooth, then there is a Galois equivariant isomorphism*

$$H_{\mathrm{dR}}^i(X) \otimes_K B_{\mathrm{dR}} \cong H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

Here, B_{dR} is the fraction field of B_{dR}^+ .

If X comes from an algebraic variety, then this isomorphism was previously proved by Tsuji [Tsu99] and Faltings [Fal02]. There is also a version of the comparison theorem for certain non-proper varieties, including $\mathbb{A}_K^1 \setminus \{0\}$, due to Li–Pan [LP19].

In p -adic Hodge theory, we study cohomology theories in p -adic geometry and the relations between them. Because $H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbb{Z}_p)$ has a $\mathrm{Gal}(\overline{K}/K)$ -action, and $\mathrm{Gal}(\overline{K}/K)$ is much more interesting than $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, the study of Galois representations is an important component of the theory.

2. INFINITE GALOIS THEORY

Definition 2.1. Let L/K be extension of fields. Let $\text{Aut}(L/K)$ denote the group of automorphisms of L fixing each element of K .

Give $\text{Aut}(L/K)$ the weakest topology such that the stabilizer of any finite subset of L is open.

Example 2.2. The group $\text{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ has two elements. The nontrivial element sends $\sqrt{2} \mapsto -\sqrt{2}$.

Example 2.3. The group $\text{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ is trivial. If ω is a nontrivial cube root of unity, then $\text{Aut}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q})$ permutes $\{\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}\}$, and this permutation action induces an isomorphism $\text{Aut}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) \cong S_3$.

Example 2.4. Let p be a prime number. For each positive integer n , there is a field \mathbb{F}_{p^n} with p^n elements. It is unique up to isomorphism. We have $\text{Aut}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$, where the Frobenius automorphism $x \mapsto x^p$ corresponds to the element $1 \in \mathbb{Z}/n\mathbb{Z}$. Then $\text{Aut}(\mathbb{F}_p/\mathbb{F}_p) = \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$.

Example 2.5. For each positive integer n , there is a field $\mathbb{Q}(\mu_{p^n})$ obtained by adjoining all p -power roots of unity to \mathbb{Q} . There is an isomorphism

$$\begin{aligned} \text{Aut}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) &\cong (\mathbb{Z}/p^n\mathbb{Z})^\times \\ (\zeta \mapsto \zeta^m) &\leftrightarrow m. \end{aligned}$$

Let

$$\mathbb{Q}(\mu_{p^\infty}) = \varinjlim_n \mathbb{Q}(\mu_{p^n}).$$

Then

$$\text{Aut}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p^\times.$$

Similarly,

$$\text{Aut}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times.$$

Example 2.6. Let K be a field. Then $\text{Aut}(K(t)/K) = \text{PGL}_2(K)$, with the discrete topology. (We leave the proof as an exercise to the reader.)

Lemma 2.7. *If L/K is finite, then $\text{Aut}(L/K)$ has the discrete topology.*

Proof. Choose a K -vector space basis for L . An automorphism of L fixing this basis must be the identity. \square

Lemma 2.8. *If L/K is algebraic, then the map*

$$\text{Aut}(L/K) \rightarrow \varprojlim_{K'} \text{Aut}(K'/K)$$

is an isomorphism of topological groups, where K' runs over $\text{Aut}(L/K)$ -stable finite extensions of K .

Proof. Since L/K is algebraic, any $\alpha \in L$ has finite orbit under $\text{Aut}(L/K)$. So the field obtained by adjoining the orbit of α to K is a finite $\text{Aut}(L/K)$ -stable extension of K . Specifying compatible automorphisms of each K' is equivalent to specifying an automorphism of L . \square

Definition 2.9. A topological space is *profinite* if it is the inverse limit of a collection of finite sets having the discrete topology.

Lemma 2.10 ([Sta, Tag 08ZY]). *A topological space is profinite if and only if it is totally disconnected and compact.*

If H is a group acting on a field K , we denote by K^H the subfield of K fixed by H .

Definition 2.11. We say that L/K is *Galois* if it is algebraic and $L^{\text{Aut}(L/K)} = K$. If L/K is Galois, then we will also denote $\text{Aut}(L/K)$ by $\text{Gal}(L/K)$.

Example 2.12. Of the extensions mentioned in Examples 2.2–2.5, $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$, $\overline{\mathbb{F}}_p/\mathbb{F}_p$, $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}$, $\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p$ are Galois. The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not Galois since the fixed field of the automorphism group is $\mathbb{Q}(\sqrt[3]{2})$. The extension $K(t)/K$ is not Galois since it is not algebraic.

There is a more concrete characterization of Galois extensions in terms of splitting fields.

Definition 2.13. Let K be a field. A polynomial $f(x) \in K[x]$ *factors completely* if it can be written in the form $f(x) = c \prod_{i=1}^n (x - x_i)$ with $n \in \mathbb{Z}_{\geq 0}$, $c, x_1, \dots, x_n \in K$ and $c \neq 0$.

Definition 2.14. Let L/K be an algebraic extension, and let $P \subset K[x] \setminus \{0\}$. We say that L is a *splitting field* for P if every element of P factors completely over L , and no proper subfield of L has this property.

Lemma 2.15. *Every subset of $K[x] \setminus \{0\}$ admits a splitting field. It is unique up to isomorphism.*

Proof. First, suppose P consists of a single element $f(x)$. Then we can construct a splitting field inductively as follows. Letting $K_0 = K$, and for $i \geq 0$, let $f_i(x)$ be an irreducible factor of $f(x)$ of degree > 1 over $K_i[x]$, and let $K_{i+1} = K[x]/f_i(x)$. Eventually, $f(x)$ factors completely in some K_n , and this K_n is a splitting field for $\{f(x)\}$.

If L is any splitting field of $\{f(x)\}$, we can construct an isomorphism $K_n \xrightarrow{\sim} L$ as follows. For each i , we construct an embedding $K_{i+1} \hookrightarrow L$ by sending the generator of K_{i+1} to some root of the polynomial $f_i(x)$ in L (using the map $K_i \hookrightarrow L$ to consider $f_i(x)$ as an element of $L[x]$). Since f does not factor completely over any subfield of L , $K_n \rightarrow L$ must be surjective, hence an isomorphism.

To prove the lemma for arbitrary P , we use Zorn's lemma. □

Definition 2.16. A polynomial $f(x) \in K[x]$ is *separable* if $f(x)$ and $f'(x)$ generate the unit ideal.

Lemma 2.17. *Suppose the polynomial $f(x) \in K[x]$ factors completely. The factors are distinct if and only if f is separable.*

Proof. Suppose $f(x)$ is divisible by $(x - \alpha)^2$ for some $\alpha \in K$. Then $f'(x)$ is divisible by $x - \alpha$. So $(f(x), f'(x)) \subset (x - \alpha)$.

Conversely, suppose $f(x) = \prod_{i=1}^n (x - \alpha_i)$ has no repeated factors. By the Chinese remainder theorem, $f(x)$ and $f'(x)$ generate the unit ideal if and only if $f'(\alpha_i) \neq 0$ for all i . In fact, we have

$$f'(\alpha_i) = \prod_{j \neq i} (\alpha_i - \alpha_j) \neq 0.$$

□

Lemma 2.18. *An extension L/K is Galois iff it is the splitting field of a set of separable polynomials.*

Proof. Suppose L/K is Galois. Let $\alpha \in L$. Then α is a zero of some polynomial over K . Any element of the $\text{Gal}(L/K)$ -orbit of α is also a zero of this polynomial. So the orbit is finite. Let $\{\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n\}$ be the orbit. Then

$$\prod_{i=1}^n (x - \alpha_i)$$

is a polynomial that is $\text{Gal}(L/K)$ -invariant. Since L/K is Galois, the polynomial has coefficients in K . Then L is a splitting field for the set of all polynomials that can be constructed in this way.

Conversely, suppose L/K is the splitting field of a set of separable polynomials over K . WLOG we may assume that they are irreducible over K . If $\alpha, \alpha' \in L$ are two roots of the same irreducible polynomial, then the construction of Lemma 2.15 produces an automorphism of L fixing K and sending α to α' . \square

Lemma 2.19. *Let L/K be a Galois extension. If K' is a subfield of L containing K , then L/K' is Galois, and $\text{Gal}(L/K')$ is closed in $\text{Gal}(L/K)$.*

Proof. By Lemma 2.18, L is a splitting field for some set of polynomials over K . Then L is a splitting field for the same set of polynomials over K' , so L/K' is Galois.

By Lemma 2.10, $\text{Gal}(L/K')$ and $\text{Gal}(L/K)$ are compact Hausdorff spaces. So $\text{Gal}(L/K')$ must be closed in $\text{Gal}(L/K)$. \square

Lemma 2.20. *If L is a field and $H \subset \text{Aut } L$ is a finite subgroup, then the map $H \rightarrow \text{Gal}(L/L^H)$ is an isomorphism.*

Proof. The map is injective, so it suffices to prove that $|H| \geq |\text{Gal}(L/L^H)|$. From the construction of Lemma 2.15, we see that $|\text{Gal}(L/L^H)| = [L : L^H]$. We will show that $[L : L^H] \leq |H|$.

Let $H = \{\sigma_1 = 1, \sigma_2, \dots, \sigma_n\}$. Let $\alpha_1, \dots, \alpha_{n+1} \in L$. The system

$$(2.21) \quad \sum_{j=1}^{n+1} \sigma_i(\alpha_j) X_j = 0$$

has $n+1$ variables and n equations, so it has a nonzero solution. Among all solutions, choose a solution (c_1, \dots, c_{n+1}) with the fewest nonzero elements. After reordering the α_j and multiplying by a scalar, we may assume $c_1 \neq 0$ and $c_1 \in F$. For any i ,

$$(c_1 - \sigma(c_1), \dots, c_{n+1} - \sigma(c_{n+1}))$$

is a solution to (2.21) with fewer nonzero terms, so it must be zero. So the c_j 's are all in F , and $\alpha_1, \dots, \alpha_{n+1}$ are linearly dependent over F . Therefore, $[L : L^H] \leq |H|$, as desired. \square

Theorem 2.22 (Fundamental theorem of infinite Galois theory). *There is a bijection between closed subgroups H of $\text{Gal}(L/K)$ and subfields K' of L containing K , given by*

$$\begin{aligned} H &\mapsto L^H \\ K' &\mapsto \text{Gal}(L/K'). \end{aligned}$$

Proof. By Lemma 2.19, for any subfield K' of L containing K , $L^{\text{Gal}(L/K')} = L$.

Conversely, suppose H is a closed subgroup of $\text{Gal}(L/K)$, and suppose $\sigma \in \text{Gal}(L/K) \setminus H$. Since H is closed, we can find some finite Galois extension K'' of K such that the action of σ on K'' does not agree with the action of any element of H . By Lemma 2.20, σ cannot fix $(K'')^H$. So it cannot fix L^H . Therefore, $H \rightarrow \text{Gal}(L/L^H)$ is an isomorphism. \square

Definition 2.23. A *separable closure* of a field K is a splitting field for the set of all separable polynomials in $K[x]$.

We denote a separable closure of K by K^{sep} . We will sometimes write G_K for $\text{Gal}(K^{\text{sep}}/K)$. If K has characteristic zero, then a separable closure is the same thing as an algebraic closure.

3. ELLIPTIC CURVES

In p -adic Hodge theory, we consider étale cohomology groups $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Z}_p)$. In general, it is difficult to describe these groups explicitly. In some situations, we can be more explicit. One of these is the case where X is an elliptic curve.

Definition 3.1. Let K be a field. An elliptic curve over K is pair (E, O) , where E is a complete smooth geometrically irreducible curve of genus 1 over K and $O \in E(K)$.

Sometimes, we will abuse notation and call E an elliptic curve.

If K has characteristic $\neq 2$, then any elliptic curve is isomorphic to one of the form

$$y^2 = x^3 + ax^2 + bx + c,$$

where $x^3 + ax^2 + bx + c$ is separable. When E is written in this form, we usually take O to be the point at ∞ .

The curve E has a group structure, meaning that there are morphisms

$$+: E \times E \rightarrow E$$

$$-: E \rightarrow E$$

$$O: \text{Spec } k \rightarrow E$$

(with O being the point chosen above), satisfying the usual group axioms.

The group structure can be described as follows. Given two points P_1, P_2 on E , there is exactly one other point Q where the line through P_1, P_2 intersects E . (If $P_1 = P_2$, we use the tangent line through P_1 .) Define $P_1 + P_2$ to be the reflection of Q about the x -axis.

Then the point at infinity is the identity, and the inverse of any point is its reflection about the x -axis.

To see that the group operation is associative, we consider line bundles on E . Let $dx + ey = f$ be the equation of the line through P_1, P_2 , and let g be the x -coordinate of the third intersection of this line with the elliptic curve. The rational function $\frac{dx+ey-f}{x-g}$ has zeros at P_1, P_2 and poles at $P_1 + P_2$ and O . It determines an isomorphism of line bundles

$$\mathcal{O}([P_1] + [P_2] - [O]) \cong \mathcal{O}([P_1 + P_2]).$$

Given a third point P_3 , we have

$$\mathcal{O}([P_1] + [P_2] + [P_3] - 2[O]) \cong \mathcal{O}([(P_1 + P_2) + P_3]) \cong \mathcal{O}([P_1 + (P_2 + P_3)])$$

Since E does not have genus 0, it cannot have a rational function with a single zero and pole, so we must have $(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$.

The above proof is somewhat sketchy. A more rigorous treatment uses the Picard functor. There is a contravariant functor $\text{Pic}: \text{Scheme} \rightarrow \text{Ab}$ that sends a scheme X to the group of isomorphism classes of line bundles on X , with the group operation being tensor product. For any map of schemes $X \rightarrow S$, there is a contravariant functor $\text{Pic}_{X/S}: \text{Scheme}/S \rightarrow \text{Ab}$ that sends a scheme T over S to $\text{Pic}(X \times_S T)/\text{Pic}(T)$. For any curve X over S , there is a natural transformation $X \mapsto \text{Pic}_{X/S}$. If X is projective, then any invertible sheaf on X has a well-defined degree, so we can write

$$\text{Pic}_{X/S} = \bigsqcup_{d \in \mathbb{Z}} \text{Pic}_{X/S}^d.$$

If X is an elliptic curve, then one can show that $X \mapsto \text{Pic}_{X/S}^1$ is an isomorphism. The point O also determines an isomorphism $\text{Pic}_{X/S}^1 \cong \text{Pic}_{X/S}^0$. Since $\text{Pic}_{X/S}$ has a group structure, E does as well. For more details, see [KM85, Theorem 2.1.2].

Remark 3.2. If X is a singular curve defined by a Weierstrass equation $y^2 = x^3 + ax^2 + bx + c$, then the nonsingular locus of X still has a group structure. (If a line passes through a singular point of X , the intersection multiplicity is at least 2. So if a line intersects the curve at two nonsingular points, then the third intersection point must also be nonsingular.)

For example, the additive group $\mathbb{G}_a = \text{Spec } K[t]$ is isomorphic to the nonsingular locus of the cuspidal cubic $y^2 = x^3$ via the map $t \mapsto (t^{-3}, t^{-2})$. We leave it as an exercise to the reader to check that this map is a homomorphism of groups.

Remark 3.3. Although the focus of today's lecture will be elliptic curves over fields, one can define an elliptic curve over an arbitrary scheme S . It is a pair (E, O) , where E is a smooth proper curve over S with geometrically connected fibers of genus 1, and $O: S \rightarrow E$ is a section of $E \rightarrow S$.

If E is a smooth degree 3 curve in \mathbb{P}_S^2 equipped with a section $O: S \rightarrow E$, then (E, O) is an elliptic curve. Even if E is not smooth, the nonsingular locus still has a group structure with identity O .

4. FORMAL GROUPS (GUEST LECTURE BY PETAR BAKIĆ)

Notes may be added later.

5. ELLIPTIC CURVES, CONTINUED

Definition 5.1. Let (E, O) , (E', O') be elliptic curves over K . A morphism $(E, O) \rightarrow (E', O')$ is a morphism $E \rightarrow E'$ sending O to O' .

Lemma 5.2. Any morphism $\phi: (E, O) \rightarrow (E', O')$ of elliptic curves is a group homomorphism.

Proof. If ϕ is constant, then it is a group homomorphism. Otherwise, ϕ is a finite locally free morphism, so there is an induced homomorphism $\text{Pic}_{E/K}^0 \rightarrow \text{Pic}_{E'/K}^0$. Since we can identify E, E' with $\text{Pic}_{E/K}^0$, $\text{Pic}_{E'/K}^0$, respectively, ϕ must also be a group homomorphism. \square

We will write $\text{Hom}(E, E')$ for the set of morphisms $(E, O) \rightarrow (E', O')$, and $\text{End}(E)$ for $\text{Hom}(E, E')$. Lemma 5.2 implies that $\text{End}(E)$ is a (not necessarily commutative) ring.

Lemma 5.3. Let E be an elliptic curve.

- (1) $\text{End } E$ has no zero divisors.
- (2) For any nonzero integer n , the multiplication by n map $E \rightarrow E$ is not zero.

Proof. For the first item, observe that any nonzero element of $\text{End } E$ is surjective, and the composition of two surjections is a surjection.

For the second item, see [Sil09, Proposition III.4.2(a)]. \square

Since the Picard functor is contravariant, any homomorphism $\phi: E \rightarrow E'$ also induces a homomorphism $\hat{\phi}: \text{Pic}_{E'/K}^0 \rightarrow \text{Pic}_{E/K}^0$, or equivalently, a homomorphism $\hat{\phi}: E' \rightarrow E$.

Lemma 5.4.

- (1) For any $\phi \in \text{Hom}(E, E')$, $\hat{\phi}\phi$ is multiplication by $\deg \phi$.
- (2) For any $\phi \in \text{Hom}(E, E')$, $\psi \in \text{Hom}(E', E'')$, $\widehat{\psi\phi} = \hat{\phi}\hat{\psi}$.
- (3) For any $\phi, \psi \in \text{Hom}(E, E')$, $\widehat{\psi + \phi} = \hat{\psi} + \hat{\phi}$.
- (4) For any integer n , the image of n in $\text{Hom}(E, E)$ is self dual.
- (5) For any $\phi \in \text{Hom}(E, E')$, $\deg \hat{\phi} = \deg \phi$.
- (6) $\hat{\hat{\phi}} = \phi$

Proof. See [Sil09, Theorem 6.1 and 6.2]. \square

Corollary 5.5. The degree map $\deg: \text{Hom}(E, E') \rightarrow \mathbb{Z}$ is a positive definite quadratic form.

Corollary 5.6. For any elliptic curve E , the multiplication by N map has degree N^2 .

For any positive integer N , let

$$E(K^{\text{sep}})[N] = \{P \in E(K^{\text{sep}}) | NP = 0\}.$$

Corollary 5.7.

If the characteristic of K does not divide N , then $E(K^{\text{sep}})[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$.

Note that $\text{Gal}(K^{\text{sep}}/K)$ acts on $E(K^{\text{sep}})[N]$.

For any prime p , define the Tate module

$$T_p(E) = \varprojlim_n E(K^{\text{sep}})[p^n].$$

If the characteristic of K is different from p , then $T_p(E)$ is a free \mathbb{Z}_p -module of rank 2

Theorem 5.8. *Suppose the characteristic of K is different from p . Then the natural map*

$$\mathrm{Hom}(E, E') \otimes \mathbb{Z}_p \rightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(E), T_p(E'))$$

is injective.

Proof. See [Sil09, Theorem III.7.4]. □

Corollary 5.9. $\mathrm{Hom}(E, E')$ is a free \mathbb{Z} -module of rank at most 4.

Corollary 5.10. $\mathrm{End}(E) \otimes \mathbb{Q}$ is isomorphic to one of the following:

- (1) \mathbb{Q} ;
- (2) An imaginary quadratic extension of \mathbb{Q} ;
- (3) A quaternion algebra over \mathbb{Q} , ramified at $p = \mathrm{char} K$ and ∞ , and at no other places.

A quaternion algebra over \mathbb{Q} is a division algebra D with center \mathbb{Q} satisfying $[D : \mathbb{Q}] = 4$. By “ramified at p and ∞ ”, we mean that $D \otimes \mathbb{Q}_p$ and $D \otimes \mathbb{R}$ are division algebras, while $D \otimes \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell)$ is the ring of 2×2 matrices for all $\ell \neq p$.

Remark 5.11. If L is an extension of K , then $\mathrm{End}(E_L)$ can be larger than $\mathrm{End}(E)$. For example, if E is the elliptic curve $y^2 = x^3 - x$ over \mathbb{Q} , then $\mathrm{End} E = \mathbb{Z}$, but $\mathrm{End} E_{\mathbb{Q}(i)} = \mathbb{Z}[i]$, where i acts by $(x, y) \mapsto (-x, iy)$.

Remark 5.12. The action of G_K on $T_p(E)$ commutes with endomorphisms of E . From the classification of Theorem 5.10, we deduce:

- If $\mathrm{End}(E) \otimes \mathbb{Q}$ is an imaginary quadratic extension F of \mathbb{Q} , then the map $G_K \rightarrow \mathrm{End} T_p(E)$ factors through $(\mathbb{Z}_p \otimes \mathcal{O}_F)^\times$.
- If $\mathrm{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra, then G_K acts by scalars on $T_p(E)$.

Remark 5.13. Tate modules are closely related to étale cohomology. If E is an elliptic curve over a field K , then

$$H_{\mathrm{ét}}^i(E_{\overline{K}}, \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{(with trivial } G_K\text{-action), } i = 0 \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p(E), \mathbb{Z}_p), & i = 1 \\ \mathbb{Z}_p(-1), & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

For each N and each extension L of K , there is a Weil pairing

$$e_N : E(L)[N] \times E(L)[N] \rightarrow \mu_N(L).$$

It can be defined as follows. Let $P, Q \in E(L)[N]$. Then

$$\sum_{m=0}^{N-1} ([P + mQ] - [mQ])$$

is a principal divisor. Let g be a function with this divisor. Let $T_Q g$ denote the translation of g by Q . Then $T_Q g$ and g have the same divisor, so $T_Q g = \omega g$ for some constant ω . Since T_Q^N is the identity, ω is an N th root of unity. We define

$$e_N(P, Q) = \omega.$$

Theorem 5.14. *The Weil pairing has the following properties:*

(1) *It is bilinear:*

$$e_N(P + Q, R) = e_N(P, R) + e_N(Q, R), \quad e_N(P, Q + R) = e_N(P, Q) + e_N(P, R)$$

(2) *It is alternating:*

$$e_N(P, P) = 1$$

(3) *If N does not divide the characteristic of K , then it is a perfect pairing.*

(4) *It is $\text{Aut}(L/K)$ -invariant: for $\sigma \in \text{Aut}(L/K)$,*

$$e_N(P^\sigma, Q^\sigma) = e_N(P, Q)^\sigma$$

(5) *The Weil pairings for various N are compatible: if $P \in E(L)[MN], Q \in E(L)[M]$,*

$$e_{MN}(P, Q) = e_M(NP, Q).$$

If the characteristic of K is not p , then we can take inverse limits to get a perfect pairing

$$T_p(E) \times T_p(E) \rightarrow \mathbb{Z}_p(1).$$

(Recall that $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}(K^{\text{sep}})$, where $\mu_{p^n}(K^{\text{sep}})$ is the group of p^n th roots of unity of K^{sep} .) We are particularly interested in elliptic curves over p -adic fields. We still need to define what these are.

Definition 5.15. A *nonarchimedean field* is a field K that is complete with respect to a norm $|\cdot|: K \rightarrow \mathbb{R}^{\geq 0}$ such that:

$$|xy| = |x||y| \text{ and } |x + y| \leq \max(|x|, |y|) \text{ for all } x, y \in K$$

$$|0| = 0, \quad |1| = 1$$

$$0 < |x| < 1 \text{ for some } x \in K$$

We will write

$$\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$$

$$\mathfrak{m}_K = \{x \in K \mid |x| < 1\}.$$

We say that K is *discretely valued* if the image of K^\times under $|\cdot|$ is a discrete subset of $\mathbb{R}^{>0}$. Equivalently,

$$\sup_{x \in \mathfrak{m}_K} |x| < 1.$$

Example 5.16. Let p be a prime. We can define a norm $|\cdot|$ on \mathbb{Q} by $|p^m \frac{r}{s}| = p^{-m}$ for all integers r, s not divisible by p . Then \mathbb{Q}_p is defined to be the completion of \mathbb{Q} with respect to this norm. It is a nonarchimedean field.

The norm on \mathbb{Q}_p extends uniquely to any algebraic extension of \mathbb{Q}_p . Any finite extension of \mathbb{Q}_p is nonarchimedean, as is the completion of any infinite algebraic extension of \mathbb{Q}_p .

Definition 5.17. A field K is *perfect* if every irreducible polynomial in $K[x]$ is separable.

Proposition 5.18. *Every field of characteristic 0 is perfect. A field of characteristic $p > 0$ is perfect iff every element is a p th power.*

Definition 5.19. Let p be a prime. A *p -adic field* is a discretely valued nonarchimedean field K of characteristic zero, such that its residue field $\mathcal{O}_K/\mathfrak{m}_K$ is perfect of characteristic p .

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