Vanishing theorems for Shimura varieties at unipotent level

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I am going to talk about some results related to the (p-adic) Langlands correspondence. Roughly, the Langlands correspondence relates two different kinds of objects: automorphic representations and Galois representations.

$$\text{automorphic representations} \iff \text{Galois representations}$$

Here is a classic example. Let $\mathcal{H}$ be the upper half complex plane $\operatorname{Im} z > 0$. Consider the function $f\colon \mathcal{H} \to \mathbb{C}$ defined by

$$f(z) := \eta(z)^2\eta(11z)^2 = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2, \quad q = e^{2\pi i z}\,.$$

One can show that is a modular form of weight 2 and level $\Gamma_0(11)$:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z)$$

for all $a,b,c,d \in \mathbb{Z}$ satisfying $ad-bc=1$, $c \equiv 0 \pmod{11}$. If $f = \sum_{n=1}^{\infty} a_n q^n$, then define the associated L-function by

$$L(f,s) := \sum_{n=1}^{\infty} a_n n^{-s}\,.$$

$$L(f,s) = 1 - 2\cdot 2^{-s} - 3^{-s} + 2\cdot 4^{-s} + 5^{-s} + 2\cdot 6^{-s} - 2\cdot 7^{-s} + \cdots$$

Now consider the elliptic curve E defined by the equation

$$y^2 + y = x^3 - x^2 - 10x - 20\,.$$

The curve E also has an associated L-function. For ℓ prime, let $a_\ell := \ell+1-\#E(\mathbb{F}_\ell)$.

$$L(E,s) := (1-a_{11}11^{-s})^{-1}\prod_{\ell\neq 11}(1-a_\ell\ell^{-s}+\ell^{1-2s})^{-1}$$

Then one can compute

$$L(E,s) = 1 - 2\cdot 2^{-s} - 3^{-s} + 2\cdot 4^{-s} + 5^{-s} + 2\cdot 6^{-s} - 2\cdot 7^{-s} + \cdots = L(f,s)\,.$$

How does this relate to the Langlands correspondence?

- The modular form f is a vector in an automorphic representation Π of GL_2.
- For any prime p, the Tate module $T_p(E)$ is a Galois representation.

The Tate module is defined as follows. Since E has a group law, $E(\overline{\mathbb{Q}})$ is an abelian group, and it carries an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For any prime p,

$$T_p(E) := \varprojlim_n E(\overline{\mathbb{Q}})[p^n]$$

is a free $\mathbb{Z}_p$-module of rank 2. So for each p, it determines a continuous representation

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p)\,.$$

So how do we generalize this story? Generalizing the Galois side is fairly straightforward. Let F be a number field, let n be a positive integer, and let p be a prime. A Galois representation is a continuous representation

$$\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)\,.$$

In the classical Langlands correspondence:

- Automorphic representations have $\mathbb{C}$ coefficients
- Galois representations are required to be "geometric" (ramified at finitely many places, de Rham at p).

The p-adic Langlands correspondence interpolates the classical correspondence p-adically. So, on the Galois side, we drop the de Rham condition, and on the automorphic side, we need to find a p-adic replacement for automorphic forms.

One reason to be interested in the p-adic Langlands correspondence is that a lot of recent progress in the classical Langlands correspondence uses p-adic interpolation (construction of Galois representations by Harris–Lan–Taylor–Thorne and Scholze, potential modularity by Allen–Calegari–Caraiani–Gee–Helm–Le Hung–Newton–Scholze–Taylor–Thorne and Boxer–Calegari–Gee–Pilloni).

To see how to find a p-adic replacement for automorphic forms, let's go back to the modular form example. Let

$$\Gamma_0(11) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1, 11 \mid c \right\}.$$

Let $Y := \Gamma_0(11)\backslash\mathcal{H}$. Consider the cohomology group $H^1(Y, \mathbb{C})$. We can think of elements of this group as functions on the space of loops in Y. The differential $f(z)\,dz$ is $\Gamma_0(11)$-invariant, so we can define an element $\phi \in H^1(Y, \mathbb{C})$ by $\phi(\gamma) = \int_\gamma f(z)\,dz$. We think of ϕ as a cohomological replacement for f and $H^1(Y, \mathbb{Z})$ as a p-adic replacement for modular forms of weight 2.

Theorem 1 (Eichler–Shimura)**.** *For any congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, the set of L-functions arising from weight 2 modular forms of level Γ is the same as the set of L-functions arising from $H^1(\Gamma\backslash\mathcal{H}, \mathbb{C})$.*

Definition 2 (Emerton)**.**

$$\widetilde{H}^1(\Gamma) := \varprojlim_m \varinjlim_n H^i((\Gamma \cap \Gamma(p^n))\backslash\mathcal{H}, \mathbb{Z}/p^m\mathbb{Z})$$

Here

$$\Gamma(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p^n} \right\}.$$

It turns out that modular forms of *all* weights, not just weight 2, show up in the completed cohomology.

The spaces $\Gamma\backslash\mathcal{H}$ are examples of *locally symmetric spaces*, i.e. they are quotients of symmetric spaces by arithmetic subgroups. A slightly more general example is $\Gamma\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ for $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ a congruence subgroup. (Note that $\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ since $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ by linear fractional transformations and the stabilizer of the point i is $\mathrm{SO}_2(\mathbb{R})$).

For an arbitrary connected reductive group G over $\mathbb{Q}$, one can define locally symmetric spaces

$$Y_{K^pK_p} := G(\mathbb{Q})\backslash G(\mathbb{A})/A_\infty K_\infty K^p K_p\,.$$

Definition 3 (Emerton)**.**

$$\tilde{H}^i(K^p) := \varprojlim_m \varinjlim_{K_p} H^i(Y_{K^pK_p}, \mathbb{Z}/p^m\mathbb{Z})\,.$$

where the limit runs over compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. Define $\tilde{H}^i_c$ similarly.

We still have only a basic understanding of completed cohomology. In particular, we still need to determine in which degrees completed cohomology can appear.

Conjecture 4 (Calegari–Emerton)**.** *Let*

$$\ell_0 := \operatorname{rk} G - \operatorname{rk} A_\infty K_\infty$$
$$q_0 := \frac{1}{2}(\dim Y_K - \ell_0)$$

Then $\tilde{H}^i = \tilde{H}^i_c = 0$ *for* $i > q_0$*.*

Theorem 5 (Scholze)**.** *Suppose the* Y_K *have the structure of Shimura varieties of Hodge type. Then* $\tilde{H}^i_c = 0$ *for* $i > q_0$*.*

Theorem 6 (Caraiani-G-Johansson)**.** *Suppose the* Y_K *have the structure of Shimura varieties of Hodge type, and* $G_{\mathbb{Q}_p}$ *is split. Let* N *be a compact unipotent subgroup of* $G(\mathbb{Q}_p)$*. Then*

$$\varprojlim_m \varinjlim_{K_p \supset N} H^i_c(Y_{K^pK_p}, \mathbb{Z}/p^m\mathbb{Z}) = 0$$

for $i > q_0$*.*

Theorem 7 (Scholze)**.** *Let* F *be a totally real or CM number field. Let* $Y^M_{K_M}$ *be a locally symmetric space for* $M = \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GL}_n$*. Let* $\mathbb{T}$ *be the image of the Hecke algebra in* $\operatorname{End} H^*(Y^M_{K_M}, \mathbb{Z}_p)$*, and let* $\mathfrak{m} \subseteq \mathbb{T}$ *be a maximal ideal. There exists a Galois representation*

$$\bar{\rho}_{\mathfrak{m}} \colon \operatorname{Gal}(\bar{F}/F) \to \operatorname{GL}_n(\bar{\mathbb{F}}_p)$$

so that the Satake parameters of $\mathfrak{m}$ *match the Frobenius eigenvalues of* $\bar{\rho}_{\mathfrak{m}}$*. If* $\bar{\rho}_{\mathfrak{m}}$ *is irreducible, this extends to*

$$\rho_m \colon \operatorname{Gal}(\bar{F}/F) \to \operatorname{GL}_n(\mathbb{T}_{\mathfrak{m}}/I)$$

for some nilpotent ideal I *of* $\mathbb{T}_{\mathfrak{m}}$*.*

Theorem 8 (Newton–Thorne)**.** *In the above theorem, one can take* $I^4 = 0$*.*

Theorem 9 (CGHJMRS)**.** *In the above theorem, if* F *is totally split above* p*, one can take* $I = 0$*.*