STAT 153 Homework 6

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1. Interpretation of Spectral Density

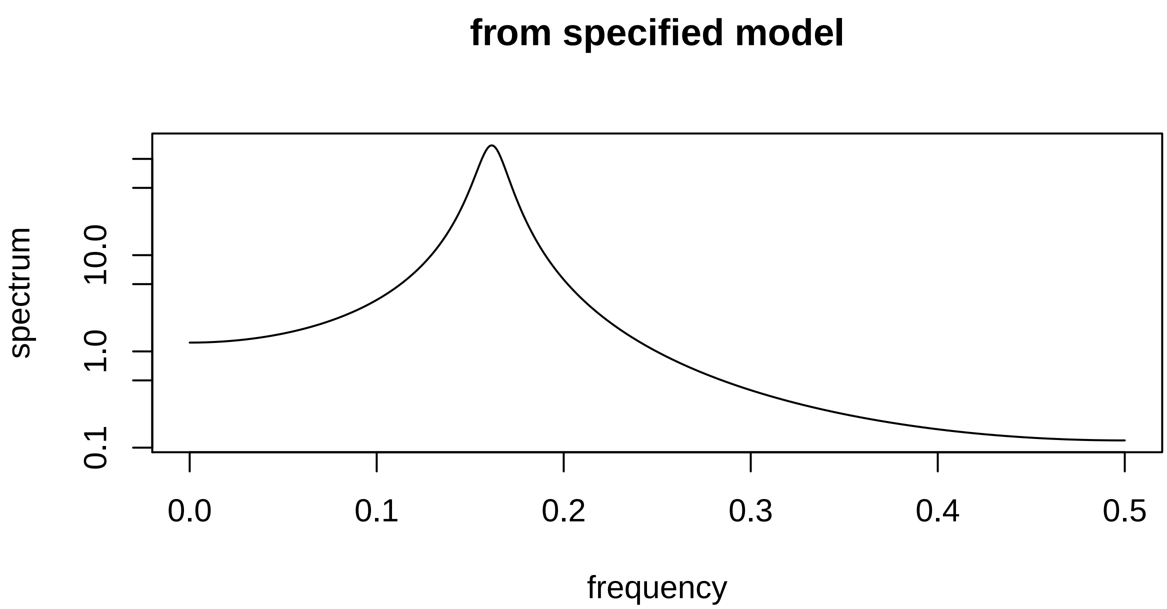
Recall that one interpretation of the spectral density is that it gives the relative contributions of sinusoids of various frequencies to the process X_t. That is, $f(\lambda)$ gives the relative strength of frequency λ.

Let Z_t be white noise with variance 1.

(a) Let $X_t = X_{t-1} - 0.9X_{t-2} + Z_t$

i) Plot its spectral density (you can use the "arma.spec" function). Which frequencies f are dominant?

```
library(TSA)
library(astsa)
plot <- arma.spec(ar = c(1, -0.9), ma = 0, var.noise = 1)
```

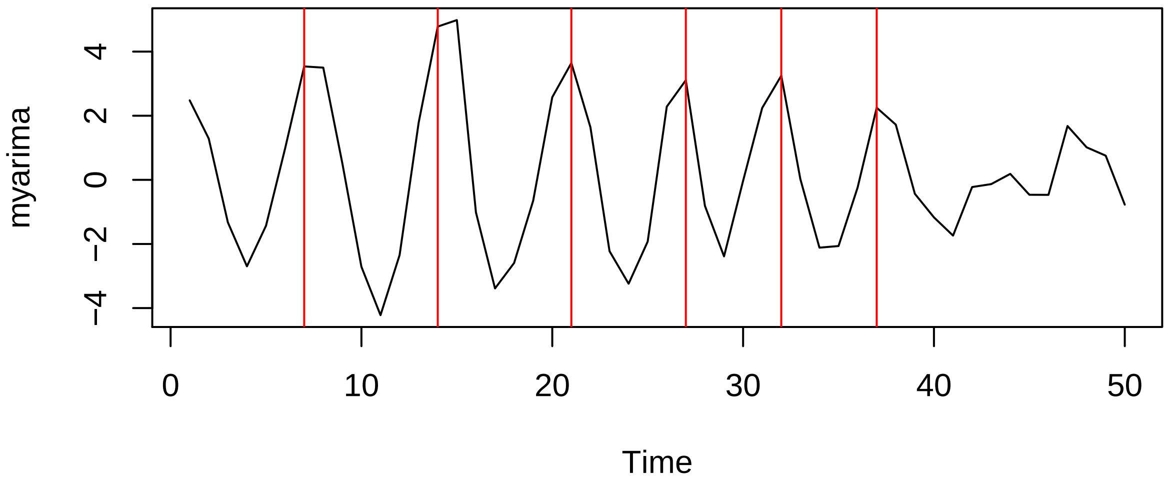



```
# dominant frequency
freq_dominant <- plot$freq[which.max(plot$spec)]
freq_dominant
```

```
## [1] 0.1613226
```

(ii). Simulate the process for 50 time steps. If there is periodic behavior, compare the observed period with $\frac{1}{f}$, the inverse of the dominant frequencies from (i).

```
set.seed(1117)
# simulate the process
myarima <- arima.sim(n = 50, list(ar = c(1, -0.9)), sd = 1)
plot(myarima)
abline(v = 7, col = "red"); abline(v = 14, col = "red")
abline(v = 21, col = "red"); abline(v = 27, col = "red")
abline(v = 32, col = "red"); abline(v = 37, col = "red")
```



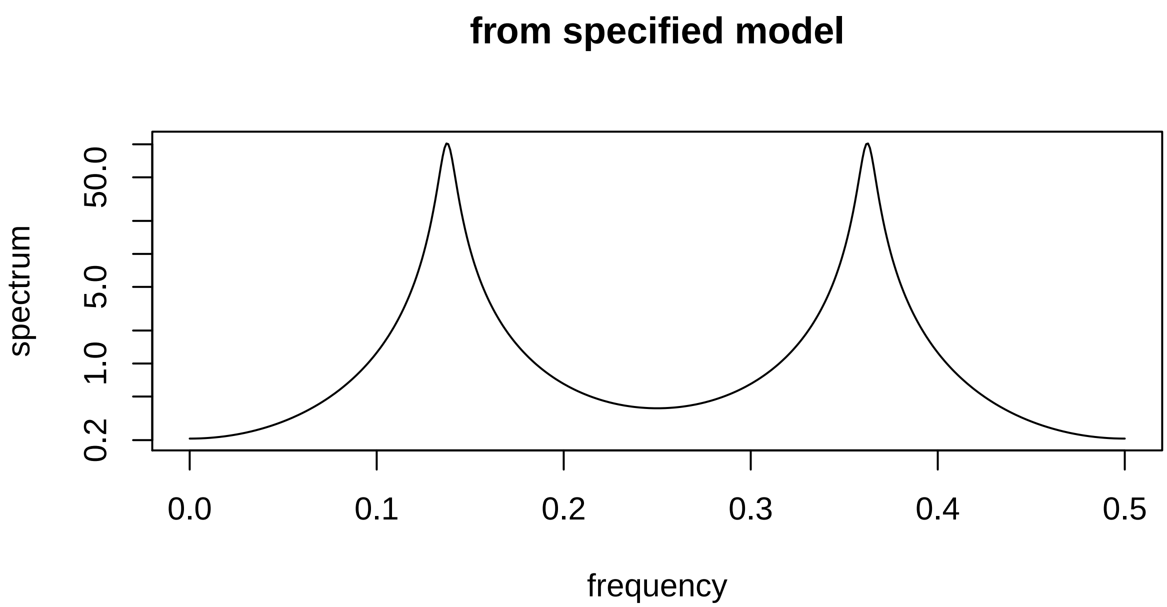

```r
# 1 / dominant frequency
1 / freq_dominant
```

```
## [1] 6.198758
```

The plot of the simulated process shows it has period 5, 6 or 7, and $1/f \approx 6.2$.

b. Let $X'_t = -0.3X'_{t-2} - 0.9X'_{t-4} + Z_t$. Repeat the same procedure as above for this process.

```r
plot <- arma.spec(ar = c(0, -0.3, 0, -0.9), ma = 0, var.noise = 1)
```

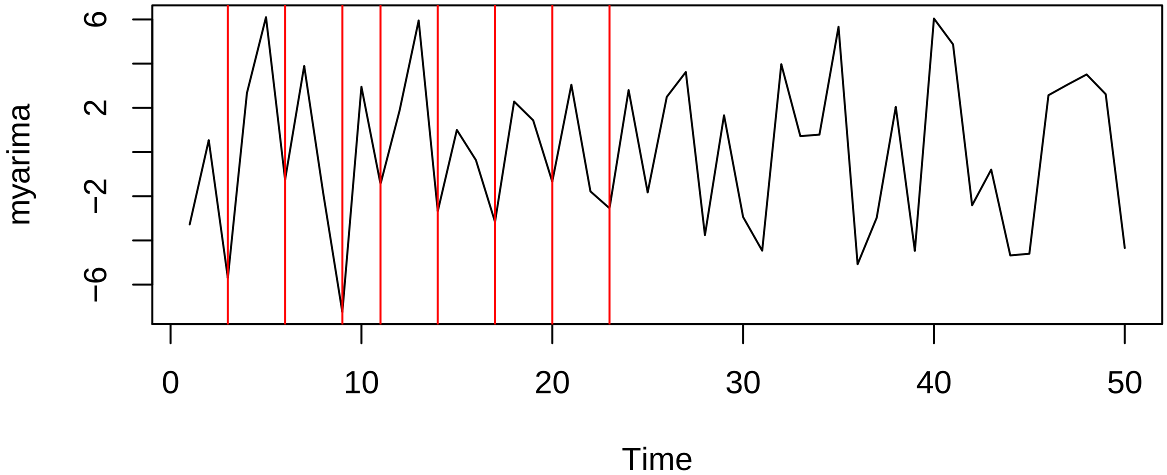



```r
# dominant frequency
freq_dominant <- plot$freq[which.max(plot$spec)]
freq_dominant
```

```
## [1] 0.3627255
```

```r
set.seed(1117)
# simulate the process
myarima <- arima.sim(n = 50, list(ar = c(0, -0.3, 0, -0.9)), sd = 1)
plot(myarima)
abline(v = 3, col = "red"); abline(v = 6, col = "red")
```

```r
abline(v = 9, col = "red"); abline(v = 11, col = "red")
abline(v = 14, col = "red"); abline(v = 17, col = "red")
abline(v = 20, col = "red"); abline(v = 23, col = "red")
```



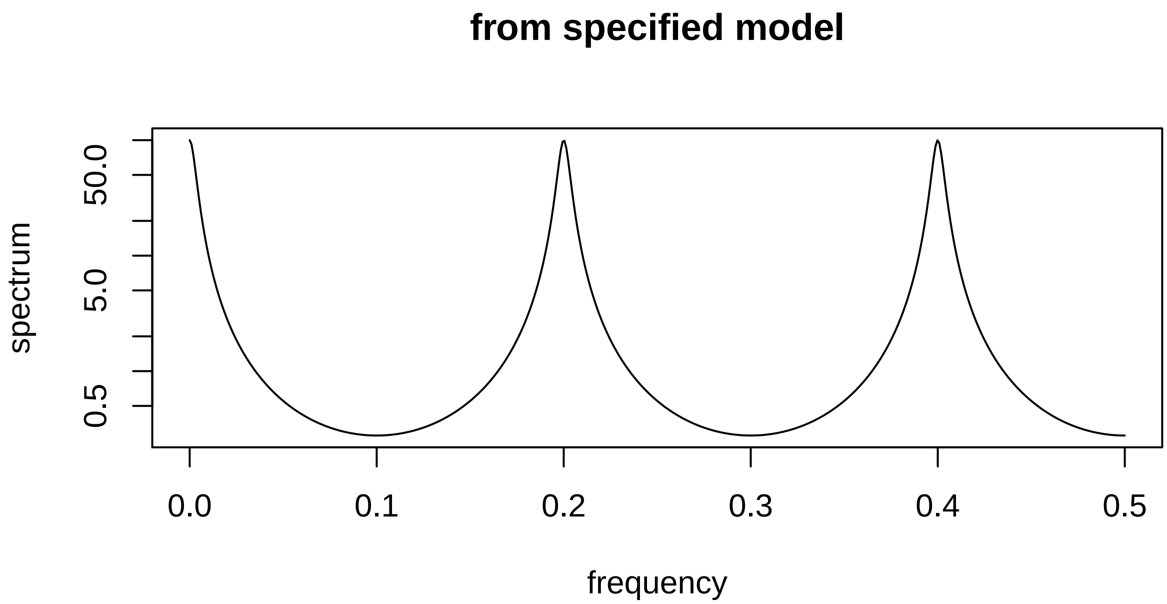

```r
# 1 / dominant frequency
1 / freq_dominant
```

```
## [1] 2.756906
```

The plot of the simulated process shows it has period 2 or 3, and $1/f \approx 2.76$

c. Let $X''_t = 0.9X''_{t-5} + Z_t$. Repeat the same procedure as above for this process.

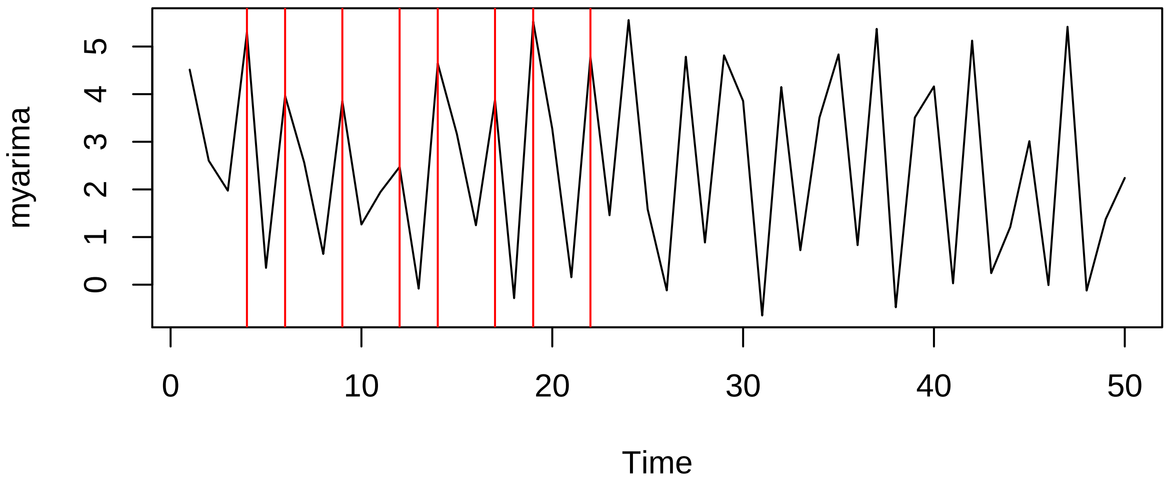
```r
plot <- arma.spec(ar = c(0, 0, 0, 0, 0.9), var.noise = 1)
```




```r
# dominant frequency
freq_dominant <- plot$freq[which.max(plot$spec[-1])]
freq_dominant
```

```
## [1] 0.3987976
```

```r
set.seed(1117)
# simulate the process
myarima <- arima.sim(n = 50, list(ar = c(0, 0, 0, 0, 0.9)), sd = 1)
plot(myarima)
abline(v = 4, col = "red"); abline(v = 6, col = "red")
abline(v = 9, col = "red"); abline(v = 12, col = "red")
abline(v = 14, col = "red"); abline(v = 17, col = "red")
abline(v = 19, col = "red"); abline(v = 22, col = "red")
```




```r
# 1 / dominant frequency
1 / freq_dominant
```

```
## [1] 2.507538
```

The plot of the simulated process shows it has period 2 or 3, and $1/f \approx 2.51$

2. Spectral Density of AR Processes

Let Z_t be a white noise process with variance 1. Consider the AR(3) process:

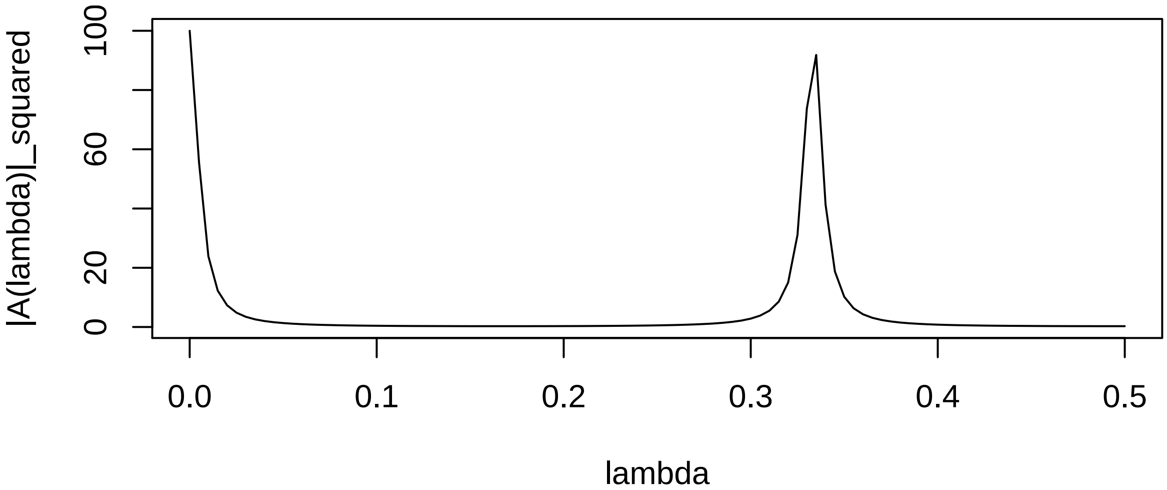
$$(1 - 0.9B^3)X_t = Z_t$$

a. Compute the transfer and power transfer functions associated with the AR polynomial $(1 - 0.9B^3)$. Also, compute the spectral density $f_X(\lambda)$.

   $a_0 = 1, a_3 = -0.9$, and $a_j = 0$ for all other j.
   $A(\lambda) = \sum_j a_j \exp(-2\pi i j\lambda) = 1 - 0.9\exp(-6\pi i\lambda) = 1 - 0.9\cos(6\pi\lambda) + 0.9i\sin(6\pi\lambda)$
   $|A(\lambda)|^2 = (1 - 0.9\cos(6\pi\lambda))^2 + 0.81\sin^2(6\pi\lambda) = 1 + 0.81 - 1.8\cos(6\pi\lambda) = 1.81 - 1.8\cos(6\pi\lambda)$
   $f_X(\lambda) = \frac{|A_\theta(\lambda)|^2}{|A_\phi(\lambda)|^2} \cdot \sigma_Z^2 = \frac{1}{1.81 - 1.8\cos(6\pi\lambda)}$

b. Plot the spectral density $f_X(\lambda)$. Do you think X_t will oscillate? If so, what period?

```r
spectral_density <- function(x) {1 / (1.81 - 1.8 * cos(6 * pi * x))}
curve(spectral_density, from = 0, to = 1/2, xlab = "lambda",
      ylab = "|A(lambda)|_squared")
```




```r
x <- seq(from = 0.3, to = 0.4, length.out = 1000)
# dominant frequency
freq_dominant <- x[which.max(spectral_density(x))]
freq_dominant
```

```
## [1] 0.3333333
```

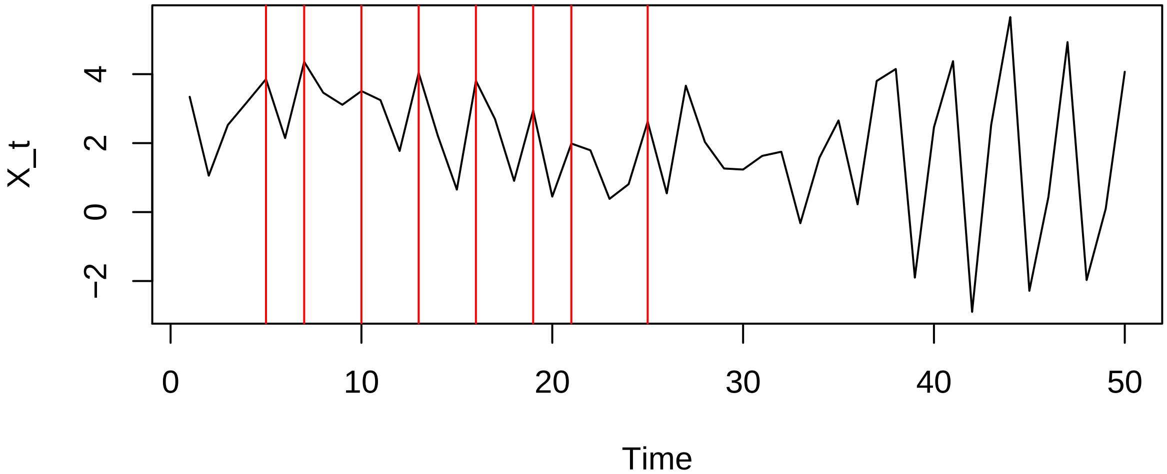
```r
# period
1 / freq_dominant
```

```
## [1] 3
```

X_t will oscillate with period 3.

c. Simulate X_t for 50 time steps. Is the simulation consistent with your answer to (b)?

```r
set.seed(1117)
myarima <- arima.sim(n = 50, list(ar = c(0, 0, 0.9)), sd = 1)
plot(myarima, ylab = "X_t")
abline(v = 5, col = "red"); abline(v = 7, col = "red")
abline(v = 10, col = "red"); abline(v = 13, col = "red")
abline(v = 16, col = "red"); abline(v = 19, col = "red")
abline(v = 21, col = "red"); abline(v = 25, col = "red")
```



The plot of the simulated process shows that it has periods 2, 3, 4, or 5, but dominant period is 3. Thus, the simulation consistent with the answer to (b).

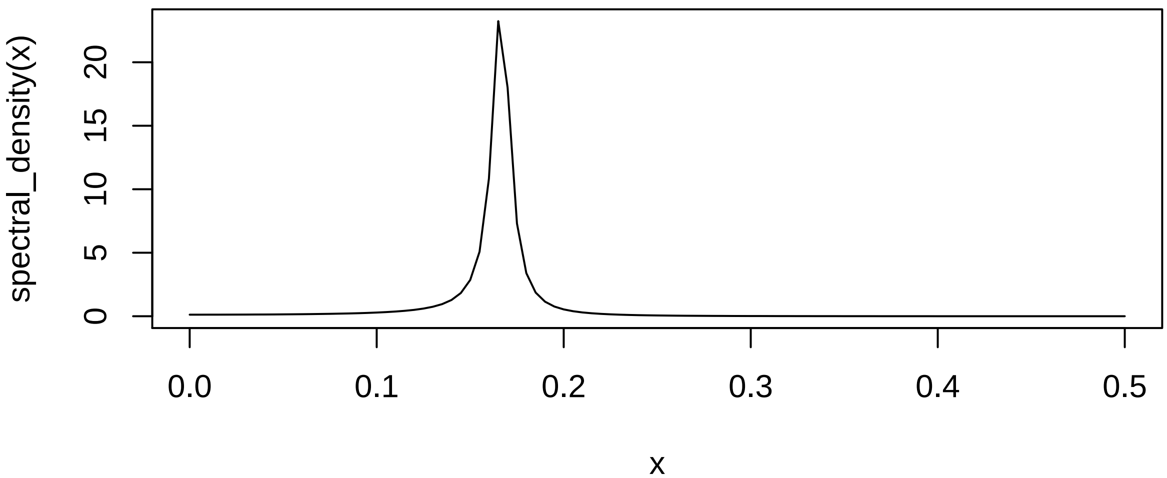
d. Consider the linear filter with weights $a_{-1} = a_0 = a_1 = \frac{1}{3}$; $a_j = 0$ otherwise. Let Y_t be the time series obtained by applying this filter to X_t. Compute the transfer function, power transfer function, and spectral density $f_Y(\lambda)$.

$A(\lambda) = \sum_j a_j \exp(-2\pi i j\lambda) = \frac{1}{3}\exp(2\pi i\lambda) + \frac{1}{3}\exp(0) + \frac{1}{3}\exp(-2\pi i\lambda) = \frac{1}{3} + \frac{1}{3}\cos(2\pi\lambda)$
$|A(\lambda)|^2 = (\frac{1}{3} + \frac{1}{3}\cos(2\pi\lambda))^2 = \frac{1}{9} + \frac{2}{9}\cos(2\pi\lambda) + \frac{1}{9}\cos^2(2\pi\lambda)$
(Since $\frac{1}{3} + \frac{1}{3}\cos(2\pi\lambda) \geq 0$ for $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$)

$$f_Y(\lambda) = f_X(\lambda)|A(\lambda)|^2 = \frac{\frac{1}{9} + \frac{2}{9}\cos(2\pi\lambda) + \frac{1}{9}\cos^2(2\pi\lambda)}{1.81 - 1.8\cos(6\pi\lambda)}$$
$$= \frac{1 + 2\cos(2\pi\lambda) + \cos^2(2\pi\lambda)}{16.29 - 16.2\cos(6\pi\lambda)}$$

e. Plot the spectral density $f_Y(\lambda)$. Do you think Y_t will oscillate? If so, what period?

```
spectral_density <- function(x) {
  (1 + 2 * cos(2 * pi * x) + cos(2 * pi * x)^2) / (16.29 + 16.2 * cos(6 * pi * x))
}
curve(spectral_density, from = 0, to = 1/2)
```




```
x <- seq(from = 0.1, to = 0.2, length.out = 1000)
# dominant frequency
freq_dominant <- x[which.max(spectral_density(x))]
freq_dominant
```

```
## [1] 0.1665666
```

```
# period
1 / freq_dominant
```

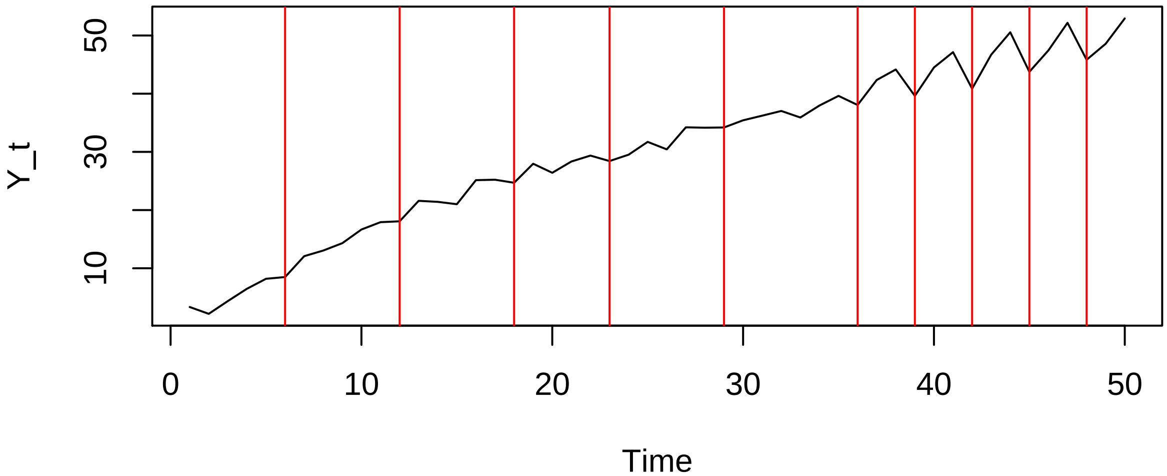
```
## [1] 6.003606
```

Y_t will oscillate with period 3.

f. Simulate Y_t by applying the filter from (d) to your simulated X_t from (c). Is the simulation consistent with your answer to (d)?

```
plot(filter(myarima, filter = c(1/3, 1/3, 1/3), method = "recursive"),
     ylab = "Y_t")
```

```r
abline(v = 6, col = "red"); abline(v = 12, col = "red")
abline(v = 18, col = "red"); abline(v = 23, col = "red")
abline(v = 29, col = "red"); abline(v = 36, col = "red")
abline(v = 39, col = "red"); abline(v = 42, col = "red")
abline(v = 45, col = "red"); abline(v = 48, col = "red")
```



Domiant period is 3, so the simulation is not consistent with the answer to (d).

3. DFT and Convolution
For data $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{n-1}$, we define their convolution z_t as:

$$z_t = \sum_{k=0}^{n-1} x_{t-k} y_k$$

where $x_{-m} = x_{n-m}$ for negative values in the sum above. Show that the j^{th} coefficient of DFT of z_t is $b_j^Z = b_j^X b_j^Y$. That is, DFT coefficients of the convolution of x_t and y_t is equal to the product of their DFT coefficients. (This is analogous to the spectral density result from Shumway Problem 4.8)

$$\begin{aligned}
b_j^Z &= \sum_{t=0}^{n-1} z_t \exp(\frac{-2\pi ijt}{n}) = \sum_{t=0}^{n-1}\sum_{k=0}^{n-1} x_{t-k} y_k \exp(\frac{-2\pi ijt}{n}) \\
&= \sum_{k=0}^{n-1} y_k \sum_{t=0}^{n-1} x_{t-k} \exp(\frac{-2\pi ijt}{n}) = \sum_{k=0}^{n-1} y_k \sum_{s=-k}^{n-k-1} x_s \exp(\frac{-2\pi ij(s+k)}{n}) \\
&= \sum_{k=0}^{n-1} y_k \exp(\frac{-2\pi ijk}{n}) \sum_{s=-k}^{n-k-1} x_s \exp(\frac{-2\pi ijs}{n})
\end{aligned}$$

Since $x_{-k} = x_{n-k}, \cdots, x_{-1} = x_{n-1}$ and
$\exp(\frac{-2\pi ij(-k)}{n}) = \exp(\frac{2\pi ijk}{n}) \exp(-2\pi ijk) = \exp(\frac{2\pi ijk}{n}) \exp(\frac{-2\pi ijnk}{n}) = \exp(\frac{-2\pi ij(n-k)}{n})$
$\implies \sum_{s=-k}^{n-k-1} x_s \exp(\frac{-2\pi ijs}{n}) = \sum_{s=0}^{n-1} x_s \exp(\frac{-2\pi ijs}{n})$
$\implies b_j^Z = b_j^X b_j^Y$

4. DFT and Periodic Data
Suppose the data x_t is h-cyclic for some positive integer h: $x_{t+h} = x_t$ for all integers t. Now, suppose the DFT of the first cycle $x_0, \ldots, x_{h-1}$ is $\beta_0, \ldots, \beta_{h-1}$.
For $n = kh$, show that the DFT of $x_0, \ldots, x_{n-1}$ is $b_j = b_{mk} = k\beta_m$ for multiples of k and $b_j = 0$ otherwise.

$$
\begin{aligned}
b_j &= \sum_{t=0}^{n-1} x_t \exp(\frac{-2\pi ijt}{n}) = \sum_{t=0}^{kh-1} x_t \exp(\frac{-2\pi ijt}{kh}) \\
&= \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi ijt}{kh}) + \sum_{t=h}^{2h-1} x_t \exp(\frac{-2\pi ijt}{kh}) + \cdots + \sum_{t=(k-1)h}^{kh-1} x_t \exp(\frac{-2\pi ijt}{kh}) \\
&= \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi ijt}{kh}) + \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi ij(t+h)}{kh}) + \cdots + \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi ij(t+(k-1)h)}{kh}) \\
&= \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi ijt}{kh}) \left(1 + \exp(\frac{-2\pi ij}{k}) + \cdots + \exp(\frac{-2\pi ij(k-1)}{k})\right)
\end{aligned}
$$

For $j = mk$

$$
\begin{aligned}
b_j = b_{mk} &= \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi imt}{h}) \left(1 + \exp(-2\pi im) + \cdots + \exp(-2\pi im(k-1))\right) \\
&= k \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi imt}{h}) = k\beta_m
\end{aligned}
$$

For $j \neq mk$

$$
\begin{aligned}
b_j &= \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi ijt}{kh}) \left(1 + \exp(\frac{-2\pi ij}{k}) + \cdots + \exp(\frac{-2\pi ij(k-1)}{k})\right) \\
&= \sum_{t=0}^{h-1} x_t \exp(\frac{-2\pi ijt}{kh}) \left(\frac{1 - \exp(-2\pi ij)}{1 - \exp(\frac{-2\pi ij}{k})}\right) = 0
\end{aligned}
$$