Name: ____________________

GSI's name: ____________________

Lab section: ____________________

Statistics 153 (Introduction to Time Series) Homework 4

Due on 23 October, 2018

16 October, 2018

Remember: Homework is due by 3:50pm in lecture on due date. Please staple your homework when you turn it in. For the computing exercises, be sure to attach all relevant code and plots.

Theoretical exercises:

1. Consider an invertible MA(1) model $X_t = Z_t + \theta Z_{t-1}$ for some i.i.d. white noise process $\{Z_t\}$ with variance σ^2.

   (a) Derive the explicit form of the minimum mean-square error one-step prediction

   $$\tilde{X}_{n+1} = E(X_{n+1}|X_n, X_{n-1}, X_{n-2}, \dots)$$

   for X_{n+1} based on the complete infinite past $X_n, X_{n-1}, X_{n-2}, \dots$.

   (2 Points)

   (b) Derive the mean squared error $E(\tilde{X}_{n+1} - X_{n+1})^2$.

   (1 Point)

   (c) Now consider the truncated estimate $\tilde{X}^n_{n+1}$, which equals $\tilde{X}_{n+1}$ but with unobserved data being set to zero, that is, $0 = X_0 = X_{-1} = \dots$. Show that

   $$E((X_{n+1} - \tilde{X}^n_{n+1})^2) = \sigma^2(1 + \theta^{2+2n}).$$

   (2 Points)

   (d) Comment on how well the truncated estimate $\tilde{X}^n_{n+1}$ works compared to $\tilde{X}_{n+1}$.

   (1 Point)

2. Consider an invertible MA(q) model $X_t = \theta(B)Z_t$ for some white noise $\{Z_t\}$ with variance σ^2.

   (a) Show that for any $m > q$ the best linear predictor of X_{n+m} based on $X_1, \dots, X_n$ is always zero.

   (1 Point)

(b) Now assume that the white noise $\{Z_t\}$ is also i.i.d.. Show that for any $m > q$ the best predictor (minimum mean-square error forecast) of X_{n+m} based on the full history $X_n, X_{n-1}, X_{n-2}, \ldots$ is also zero.

(1 Point)

3. Consider a causal, zero mean AR(1) model $X_t - \phi X_{t-1} = Z_t$ for some white noise $\{Z_t\}$ with variance σ^2.

   (a) Derive the general form of the best linear predictor $\tilde{X}_{n+m}$ of X_{n+m} in terms of $X_1, \ldots, X_n$.

   (1 Point)

   (b) Show that

$$E(X_{n+m} - \tilde{X}_{n+m})^2 = \sigma^2 \frac{1 - \phi^{2m}}{1 - \phi^2}$$

   (1 Point)

Computer exercises:

1. Consider the `LakeHuron` dataset in R.

   (a) Fit a linear trend function to the data and obtain residuals.

   (1 Point)

   (b) Fit an AR(1) model to the residuals using the R function *arima()*.

   (1 Point)

   (c) Assume that your fitted model coincides with the true generating model of the data. Obtain predictions for the residuals for the future $m = 30$ time points *without using the predict function in R.*

   (1 Point)

   (d) Compare your predictions with those obtained by the `predict` function in R.

   (1 Point)

   (e) Obtain predictions for the original data for the future $m = 30$ time points.

   (1 Point)

   (f) Under Gaussian noise assumption, obtain prediction intervals by using your results from the theoretical exercises.

   (1 Point)

   (g) Compare your prediction intervals with those obtained from the `predict` function.

   (1 Point)

2. Plot and describe the important characteristics of sample ACF and sample PACF for each of the following ARMA models. Use 10,000 observations in each case.

(a) $X_t = \frac{3}{5}X_{t-1} - \frac{4}{5}X_{t-2} + Z_t$

(1 Point)

(b) $X_t = Z_t + 0.8Z_{t-1} + 1.1Z_{t-2}$

(1 Point)

(c) $X_t = \frac{4}{5}X_{t-1} + Z_t + \frac{4}{5}Z_{t-1}$

(1 Point)