Double Pendulum

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Abstract

In this project, we investigate the motion of a double pendulum. We begin with a discussion of Langrange's equations for this system, after which we apply the small angle approximation in order to find an analytic solution. To simplify this problem, we then consider the case where the masses and lengths of the two pendula are equal. Our analytic treatment of this problem ends with the general solution of this special case. We then apply numerical methods to investigate the behaviour of the double pendulum in a variety of cases. Limiting the initial angles ϕ_1 and ϕ_2 to small values, the numerical solutions match the solutions found analytically. Our simulations for large angles show behaviour which differs greatly from these analytic solutions.

1 Introduction

The double pendulum is an example of a chaotic system, meaning that even slight changes in initial conditions can lead to very different behaviors. It consists of a second pendulum hanging from the bob of the first pendulum. Despite being similar in principle to a simple pendulum, it leads to equations of motion which are significantly more complex. Due to its chaotic nature, the behaviour of a double pendulum is not always easily predictable. What this means for us is that the double pendulum is a well suited system for us to apply both analytical and numerical tools which allow us to simplify this complicated system. We will use linearization/small-angle approximation to simplify the system, and thus solve the approximate system analytically [1]. Then, we will use numerical methods and tools to help us gain some understanding of the behaviours of certain solutions. The numerical solution takes some inspiration from the strategies outlined on *The Double Pendulum* [2]. Note that both the analytical and numerical approaches used are based off of the Lagrangian formulation of Classical Mechanics, since the double pendulum is a constrained system with two degrees of freedom. Therefore, it most easily lends itself to Lagrangian analysis with two generalized coordinates.

2 Equations

We will follow the approach of the textbook [1] in analyzing the system, which is discussed in Chapter 11, Section 4. We begin with the Lagrangian

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi_1}^2 + \frac{1}{2}m_2l_2^2\dot{\phi_2}^2 + m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi_1}\dot{\phi_2} \\ -(m_1+m_2)gl_1(1-\cos\phi_1) - m_2gl_2(1-\cos\phi_2)\end{aligned} \tag{1}$$

where the potential energy, $U = (m_1+m_2)gl_1(1-\cos\phi_1) + m_2gl_2(1-\cos\phi_2)$ is chosen to simplify the equations we will use later on. Then we have the ϕ_1-equation

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi_1} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi_1}} \\ -m_2l_1l_2\dot{\phi_1}\dot{\phi_2}\sin(\phi_1-\phi_2) - (m_1+m_2)gl_1\sin\phi_1 &= (m_1+m_2)l_1^2\ddot{\phi_1} + m_2l_1l_2\cos(\phi_1-\phi_2)\ddot{\phi_2} \\ &\quad - m_2l_1l_2\dot{\phi_2}\sin(\phi_1-\phi_2)(\dot{\phi_1}-\dot{\phi_2})\end{aligned} \tag{2}$$

Similarly, we have the ϕ_2-equation

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi_2} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi_2}} \\ -m_2l_1l_2\dot{\phi_1}\dot{\phi_2}\sin(\phi_1-\phi_2) - m_2gl_2\sin\phi_2 &= -m_2l_2^2\ddot{\phi_2} + m_2l_1l_2\cos(\phi_1-\phi_2)\ddot{\phi_1} \\ &\quad - m_2l_1l_2\dot{\phi_1}\sin(\phi_1-\phi_2)(\dot{\phi_1}-\dot{\phi_2})\end{aligned} \tag{3}$$

3 Analysis

First, observe that setting $m_2 = 0$ allows us to recover the case of the single pendulum. Indeed, substituting $m_2 = 0$ into (2) yields

$$-m_1gl_1\sin\phi_1 = m_1l_1^2\ddot{\phi_1}$$

Now, the equations (2) and (3) we have obtained are analytically unsolvable [1]. By assuming ϕ_1 and ϕ_2 remain small, we can simplify our problem. However, since our equations involve their derivatives $\dot{\phi_1}$ and $\dot{\phi_2}$, we will further assume that these "velocities" remain small as well [1].
Now, for the potential energy U, we use the first two terms of the Taylor expansion of cosine, that is, $\cos\phi \approx 1 - \frac{1}{2}\phi^2$.
This gives

$$\begin{aligned}U &= (m_1+m_2)gl_1(1-\cos\phi_1) + m_2gl_2(1-\cos\phi_2) \\ &\approx (m_1+m_2)gl_1(1-(1-\frac{1}{2}\phi_1^2)) - m_2gl_2(1-(1-\frac{1}{2}\phi_2^2)) \\ &\approx \frac{1}{2}(m_1+m_2)gl_1\phi_1^2 + \frac{1}{2}m_2gl_2\phi_2^2\end{aligned} \tag{4}$$

We simplify our expression for the kinetic energy by the same process. The first and second terms, $\frac{1}{2}(m_1+m_2)l_1^2\dot{\phi_1}^2$ and $(m_1+m_2)gl_1\cos\phi_1$, do not require simplification, since they do not involve trigonometric functions. Since the third term already includes factors of $\dot{\phi}_1$ and $\dot{\phi}_2$, which are assumed to be small, we will use only the first term of the Taylor series for cosine, namely 1 [1]. This allows us to drop the factor $\cos(\phi_1-\phi_2)$. Hence we have

$$\begin{aligned} T &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi_1}^2+\frac{1}{2}m_2l_2^2\dot{\phi_2}^2+m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi_1}\dot{\phi_2} \\ &\approx \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi_1}^2+\frac{1}{2}m_2l_2^2\dot{\phi_2}^2+m_2l_1l_2\dot{\phi_1}\dot{\phi_2} \end{aligned} \tag{5}$$

Using these approximations, which we will denote T_0 and U_0, we now have a new Lagrangian $\mathcal{L}_0$ given by

$$\begin{aligned} \mathcal{L}_0 &= T_0-U_0 \\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi_1}^2+\frac{1}{2}m_2l_2^2\dot{\phi_2}^2+m_2l_1l_2\dot{\phi_1}\dot{\phi_2} \\ &\quad -\frac{1}{2}(m_1+m_2)gl_1\phi_1^2-\frac{1}{2}m_2gl_2\phi_2^2 \end{aligned} \tag{6}$$

The resulting Lagrange's equations will be simpler. For our ϕ_1-equation, instead of our previous equation (2), we have

$$\begin{aligned} \frac{\partial\mathcal{L}_0}{\partial\phi_1} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}_0}{\partial\dot{\phi_1}} \\ -(m_1+m_2)gl_1\phi_1 &= \frac{\mathrm{d}}{\mathrm{d}t}((m_1+m_2)l_1^2\dot{\phi_1}+m_2l_1l_2\dot{\phi_2}) \\ -(m_1+m_2)gl_1\phi_1 &= (m_1+m_2)l_1^2\ddot{\phi_1}+m_2l_1l_2\ddot{\phi_2} \end{aligned} \tag{7}$$

Similarly, instead of (3), our new ϕ_2-equation is

$$\begin{aligned} \frac{\partial\mathcal{L}_0}{\partial\phi_2} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}_0}{\partial\dot{\phi_2}} \\ -m_2gl_2\phi_2 &= \frac{\mathrm{d}}{\mathrm{d}t}(m_2l_1l_2\dot{\phi_1}+m_2l_2^2\dot{\phi_2}) \\ -m_2gl_2\phi_2 &= m_2l_1l_2\ddot{\phi_1}+m_2l_2^2\ddot{\phi_2} \end{aligned} \tag{8}$$

In order to solve the system given by (7) and (8), we can reformulate it as the matrix equation

$$\begin{pmatrix} (m_1+m_2)l_1^2 & m_2l_1l_2 \\ m_2l_1l_2 & m_2l_2^2 \end{pmatrix}\begin{pmatrix}\ddot{\phi_1}\\ \ddot{\phi_2}\end{pmatrix} = -\begin{pmatrix} (m_1+m_2)gl_1 & 0 \\ 0 & m_2gl_2 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{9}$$

which we can write as

$$\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{K}\boldsymbol{\phi} \tag{10}$$

where

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

and $\mathbf{M}$ and $\mathbf{K}$ are the matrices on the left and right side of (9), respectively. While the entries of $\mathbf{M}$ do not have units of mass, and likewise the entries of $\mathbf{K}$ are not spring constants, their roles in the equations justify our choice in naming them in this way [1]. Recall that a mass oscillating on a horizontal spring satisfies equation $m\ddot{x} = -kx$.

In order to solve this system, we will consider the case where the masses and lengths are equal, that is $m_1 = m_2 = m$ and $l_1 = l_2 = l$ for some m and l. Substituting this into our expressions for $\mathbf{M}$ and $\mathbf{K}$ yields

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix} \\ &= ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathbf{K} &= \begin{pmatrix} 2mgl & 0 \\ 0 & mgl \end{pmatrix} \\ &= ml^2 \begin{pmatrix} 2\frac{g}{l} & 0 \\ 0 & \frac{g}{l} \end{pmatrix} \end{aligned}$$

If we recall that a simple pendulum of length l (and any mass) has frequency $\omega = \sqrt{\frac{g}{l}}$ (for small amplitudes), we can rewrite $\mathbf{K}$ as

$$\mathbf{K} = ml^2 \begin{pmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{pmatrix} \tag{12}$$

Now, the matrix equation (10) we are to solve is similar in form to the equation $x'' = -x$, which has sinusoidal solutions $x = \cos t$ and $x = \sin t$. Thus, we will look for solutions of the form $\mathbf{z} = \mathbf{a}e^{i\omega t}$ with ω real and a constant vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, and consider only the real part, giving our solutions $\boldsymbol{\phi}$ [1]. Suppose we have

$$\mathbf{M}\ddot{\mathbf{z}} = -\mathbf{K}\mathbf{z}$$

Then

$$\begin{aligned} \mathbf{M}\frac{d^2}{dt^2}(\mathbf{a}e^{i\omega t}) &= -\mathbf{K}\mathbf{a}e^{i\omega t} \\ -w^2 e^{i\omega t}\mathbf{M}\mathbf{a} &= -e^{i\omega t}\mathbf{K}\mathbf{a} \end{aligned}$$

We can divide by $e^{i\omega t}$ and rearrange to give

$$(\mathbf{K} - w^2\mathbf{M})\mathbf{a} = 0 \tag{13}$$

Now, if $(\mathbf{K} - w^2\mathbf{M})$ is invertible, then $\mathbf{a}$ must be zero. This corresponds to the stable solution $\phi_1 = \phi_2 = 0$ for all time. Otherwise, our matrix $(\mathbf{K} - w^2\mathbf{M})$ is not invertible, hence its determinant must be 0. We have

$$\begin{aligned}(\mathbf{K} - w^2\mathbf{M}) &= ml^2 \begin{pmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{pmatrix} - \omega^2 ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= ml^2 \begin{pmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{pmatrix} \end{aligned} \tag{14}$$

Now,

$$\begin{aligned} |\mathbf{K} - w^2\mathbf{M}| &= 0 \\ (ml^2)^2(2(\omega_0^2 - \omega^2)^2 - \omega^2) &= 0 \\ \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 &= 0 \\ (\omega^2 - (2+\sqrt{2})\omega_0^2)(\omega^2 - (2-\sqrt{2})\omega_0^2) &= 0 \end{aligned} \tag{15}$$

Giving two similar solutions

$$\omega_1^2 = (2+\sqrt{2})\omega_0^2, \qquad \omega_2^2 = (2-\sqrt{2})\omega_0^2 \tag{16}$$

Since $m, l \neq 0$.
Now, all that remains is to substitute these solutions into our equation (13) to find the corresponding values of $\mathbf{a}$. First,

$$\begin{aligned} (\mathbf{K} - w_1^2\mathbf{M})\mathbf{a} &= 0 \\ ml^2 \begin{pmatrix} -(2+2\sqrt{2})\omega_0^2 & -(2+\sqrt{2})\omega_0^2 \\ -(2+\sqrt{2})\omega_0^2 & -(1+\sqrt{2})\omega_0^2 \end{pmatrix} \mathbf{a} &= 0 \end{aligned}$$

which we can factor as

$$-ml^2\omega_0^2(1+\sqrt{2}) \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \mathbf{a} = 0$$

Thus

$$\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

which is equivalent to

$$\begin{pmatrix} 0 & 0 \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \tag{17}$$

Hence we have $a_2 = -\sqrt{2}a_1$, and thus $\{\left(\begin{smallmatrix}1\\-\sqrt{2}\end{smallmatrix}\right)\}$ is a basis for the eigenspace corresponding to ω_1.
Note that the expression for $\mathbf{z}$ and hence $\boldsymbol{\phi}$ involve ω rather than ω^2, but for real ω, we have $Re(e^{\pm i\omega t}) = \cos(\omega t)$. Thus, we have found the solutions

$$\begin{aligned}\boldsymbol{\phi} &= Re(\mathbf{a}e^{i\omega_1 t})\\ &= A_1 \begin{pmatrix}1\\-\sqrt{2}\end{pmatrix}\cos(\omega_1 t - \alpha_1)\end{aligned} \tag{18}$$

where $\omega_1 = \sqrt{2+\sqrt{2}}\omega_0$, A_1 is a constant, and α_1 is a some phase shift [1]. This solution is known as the first normal mode. Notice that in this case we have $\phi_2 = -\sqrt{2}\phi_1$, hence the two pendulum have opposite phase, and the lower pendulum oscillates with an amplitude larger by a factor of $\sqrt{2}$ [1].

Now we will follow the same steps for ω_2 to find the second normal mode. We have

$$(\mathbf{K} - w_2^2\mathbf{M})\mathbf{a} = 0$$
$$ml^2 \begin{pmatrix}(-2+2\sqrt{2})\omega_0^2 & (-2+\sqrt{2})\omega_0^2\\ (-2+\sqrt{2})\omega_0^2 & (-1+\sqrt{2})\omega_0^2\end{pmatrix}\mathbf{a} = 0$$

That is,

$$-ml^2\omega_0^2(-1+\sqrt{2})\begin{pmatrix}2 & -\sqrt{2}\\ -\sqrt{2} & 1\end{pmatrix}\mathbf{a} = 0$$
$$\begin{pmatrix}2 & -\sqrt{2}\\ -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = 0$$

Equivalently,

$$\begin{pmatrix}0 & 0\\ -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = 0 \tag{19}$$

Thus $a_2 = \sqrt{2}a_1$, and thus $\{\left(\begin{smallmatrix}1\\\sqrt{2}\end{smallmatrix}\right)\}$ is a basis for the eigenspace corresponding to ω_2. Then, just as above, we have

$$\begin{aligned}\phi &= Re(\mathbf{a}e^{i\omega_2 t})\\ &= A_2 \begin{pmatrix}1\\\sqrt{2}\end{pmatrix}\cos(\omega_2 t - \alpha_2)\end{aligned} \tag{20}$$

where $\omega_2 = \sqrt{2-\sqrt{2}}\omega_0$, A_2 is a constant, and again α_2 is some phase shift. In this second normal mode, we have $\phi_2 = \sqrt{2}\phi_1$, hence there is no phase difference. As in the first normal mode, the lower pendulum oscillates with an amplitude larger by a factor of $\sqrt{2}$.

We can take any linear combination of these two normal modes, hence the general solution is

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = c_1 A_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_1 t - \alpha_1) + c_2 A_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_2 t - \alpha_2)$$

$$= c_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_1 t - \alpha_1) + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_2 t - \alpha_2) \tag{21}$$

where we have dropped A_1 and A_2 since c_1 and c_2 are arbitrary constants.

4 Numerical Methods

To begin our Numerical Analysis of the double-pendulum, we need to simplify our system of DEs. Currently, we have a system of two 2nd-order differential equations, but to apply a numerical integrator we want to express the system as four 1st-order differential equations. First, since we are limiting our scope to a double pendulum with bobs of equal length and mass ($m_1 = m_2 = m$, $l_1 = l_2 = l$), we can make simplifications to obtain the following DEs:

$$-2\frac{g}{l}\sin\phi_1 = 2\ddot{\phi}_1 + \ddot{\phi}_2\cos(\phi_1 - \phi_2) + \dot{\phi}_2^{\,2}\sin(\phi_1 - \phi_2)$$

$$-\frac{g}{l}\sin\phi_2 = \ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1 - \phi_2) - \dot{\phi}_1^{\,2}\sin(\phi_1 - \phi_2)$$

From here, it will be useful for us to work with the DEs in matrix form $A\ddot{\vec{\phi}} = \vec{f}$, where $\vec{\phi} = (\phi_1, \phi_2)^T$.

$$\begin{bmatrix} 2 & cos(\phi_1 - \phi_2) \\ cos(\phi_1 - \phi_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} = \begin{bmatrix} -2\frac{g}{l}\sin\phi_1 - \dot{\phi}_2^{\,2}\sin(\phi_1 - \phi_2) \\ -\frac{g}{l}\sin\phi_2 + \dot{\phi}_1^{\,2}\sin(\phi_1 - \phi_2) \end{bmatrix}$$

In order to cast this into a system of 1st-order DEs, first we need to use the fact that $\ddot{\vec{\phi}} = A^{-1}\vec{f}$. Note that we can find $A^{-1}\vec{f}$ in our numerical implementation, rather than carry out a very long and messy algebraic computation. Now if we define the state vector $\vec{y} = (\phi, \dot{\phi})^T$, then we finally have a form which we can work with numerically:

$$\dot{\vec{y}} = \begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ A^{-1}\vec{f} \end{bmatrix}$$

From here, a numerical integrator was applied using the RK45 method 7, and this concludes the discussion of how numerical methods were implemented to solve the equations of motion.

Next, we will discuss the particular numerical solutions analyzed. Firstly, notice that our equations of motion are independent of m, so the only parameter values we need to consider are g and l. We have $g = 9.81 m \cdot s^{-1}$ as We will consider the double pendulum subject to Earth's gravitational acceleration. For simplicity, we will consider $l = 1$, since this is really just an arbitrary scale factor. Now we have established the parameters being considered, next the table below will provide the initial conditions which we will examine:

IVPs considered			
$\phi_{1,0}$	$\phi_{2,0}$	$\dot{\phi}_{1,0}$	$\dot{\phi}_{2,0}$
0.1	0.141	0	0
0.1	-0.141	0	0
1	1.41	0	0
1	-1.41	0	0
0	0.2	0	0

4.1 1st Normal Mode

The first IVP which we found numerical solutions to corresponds to the 1st normal mode solutions which we found analytically using small angle approximation. In this case the behaviour of the exact solution numerically does not differ from the approximated solution found analytically because the angle remains sufficiently small (< 0.26 radians). We can observe from Fig. 1 that both bobs swing perfectly in phase, while ϕ_2 has a larger amplitude by $\sqrt{2}$. We can see from Fig. 2 that the phase space trajectories for both bobs are neat closed and elliptical paths.

4.2 2nd Normal Mode

In the next IVP examined numerically, we have the 2nd normal mode. Once again, the angles ϕ_1 and ϕ_2 remain sufficiently small throughout, so we observe the neat behaviour predicted by the small angle approximation. In this case the bobs start out of phase and from Fig. 3 we can see that they continue to oscillate perfectly out of phase throughout. From Fig. 4, the phase portraits are once again simple elliptical trajectories, however phase plane portraits do not encode how the particles travel along these trajectories. In the previous IVP, the bobs travel along the ellipses in phase. Meanwhile in this IVP the bobs travel out of phase along the ellipses.

4.3 1st Normal Mode, Large Angle

Next, We will examine what happens if the double pendulum oscillates at approximately the 1st normal mode, but at angles greater than where small angle approximation applies. As seen in Fig. 5, the bobs appear close to oscillating in phase, however there are many imperfect perturbations in the system. Looking at the phase space portraits in Fig. 6, we see the chaotic behaviour of the double pendulum once the angles become large. The trajectories vaguely resemble the simple ellipses from before, however they are noticeably messier. Despite the fact that small angle approximation no longer applies perfectly, it is still useful for giving us a rough idea of how the system behaves in this scenario.

4.4 2nd Normal Mode, Large Angle

When we increase the angles for the 2nd Normal Mode past where small angle approximation applies, something very similar happens as for the 1st mode. Fig. 7 suggests that although the bobs are not oscillating perfectly out of phase nor precisely sinusoidal, the behaviour is similar to the 2nd normal mode with some extra perturbations. Similarly in the phase space portraits in Fig. 8, the trajectories are no longer perfectly elliptical, however they suggest a similar behaviour but with an added element of chaos.

4.5 Linear Combination of Normal Modes

Finally, we will examine a more general case of the double pendulum for small angles. If we keep the angles ϕ_1 and ϕ_2 sufficiently small, in general the solutions will be a linear combination of the two normal modes. As such, instead of oscillating perfectly in phase or perfectly out of phase, the bobs will oscillate with a mix of both. We can see from Fig. 9 that the bobs oscillate more erratically as energy is exchanged from one bob to another. In the phase space portrait, Fig. 10, we see that the phase space trajectories of the bobs trade between many different circular patterns, which ultimately add together into the simple ellipses we had from before. This is a result of the linear combinations of the two normal modes. Perhaps the plot which best illustrates the behaviour in this scenario is when we plot ϕ_1 against ϕ_2 in Fig. 11. The curve parameterized in this plot is known as a Lissajous Curve, and it indicative of this solution being a superposition of the two normal modes.

4.6 Numerical Methods Wrap up

This concludes our discussion of Numerical Solutions of the Double Pendulum. More general solutions of the Double Pendulum at large angles can demonstrate very erratic and chaotic behaviours. However, the cases which we have focused on have helped us to gain insight and understanding of the system and its behaviour.

5 Conclusions

In conclusion, we made many simplifying assumptions around the double pendulum system in order to help us gain insight and understanding of the physical system. We made assumptions of that the bobs were of equal mass and length, and then linearized the equations of motion by using small angle approximations. After making these assumptions we were able to find closed form analytical solutions to our approximate system. These solutions told us that for small angles, there are two special solutions where the bobs of the pendulum oscillate either exactly in phase or exactly out of phase. These special solutions are known as normal modes. In general, the bobs will oscillate as a superposition (linear combination) of the two normal modes. Armed with this knowledge, we studied numerical solutions of the non-linear system (no small-angle approximation). We found that even with the non-linear terms added in, when the angles were sufficiently small enough the behaviour of the non-linear double pendulum agree nicely with the predictions made with the small angle approximation. The numerical solutions indicated that the pendulums can oscillate at the two normal modes, and in general oscillate with a superposition of modes. The numerical solutions also showed that at slightly larger angles where small-angle approximation cannot be made, there were similar behaviours as the small angle case but with extra chaos introduced by the non-linear terms. However note that once the angles become much larger the behaviour of the system becomes far too unpredictably chaotic.

References

[1] Taylor, J. R. *Classical mechanics*. University Science Books, 2005.

[2] O'Reilly, O. M. (n.d.). *The Double Pendulum*. Rotations. https://rotations.berkeley.edu/the-double-pendulum/

6 Appendix A: Figures

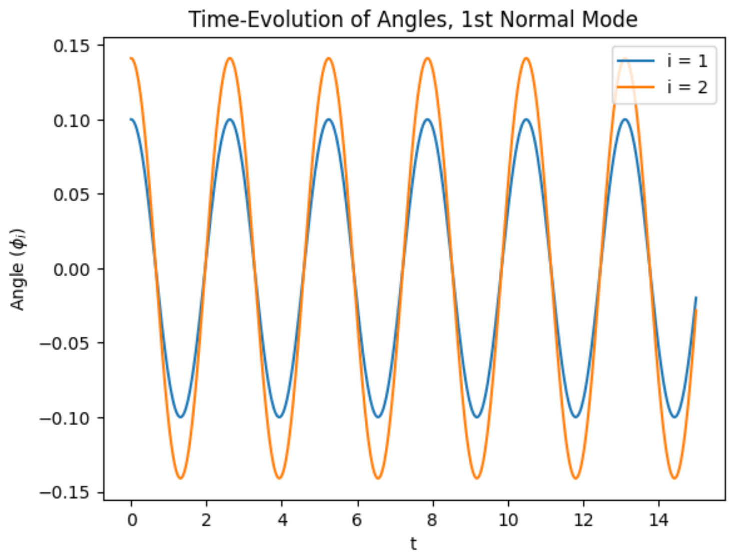

Figure 1: This figure shows the time evolution of the angles of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at the 1st normal mode

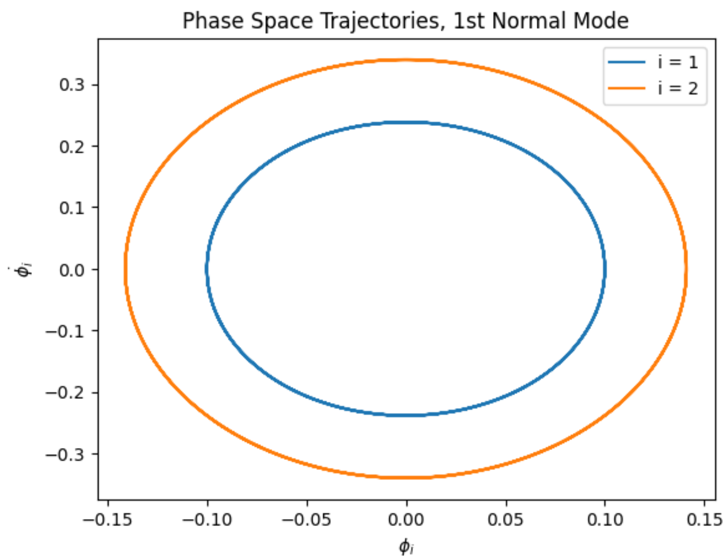

Figure 2: This figure shows the phase plane portraits of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at the 1st normal mode

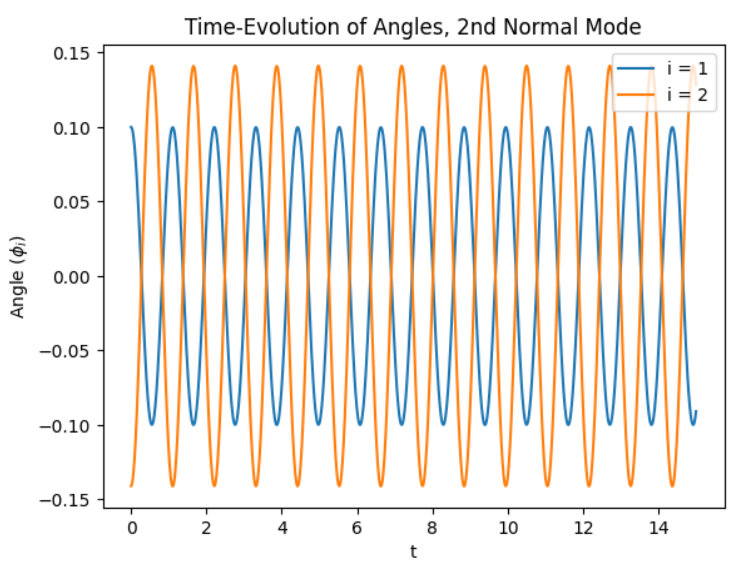

Figure 3: This figure shows the time evolution of the angles of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at the 2nd normal mode

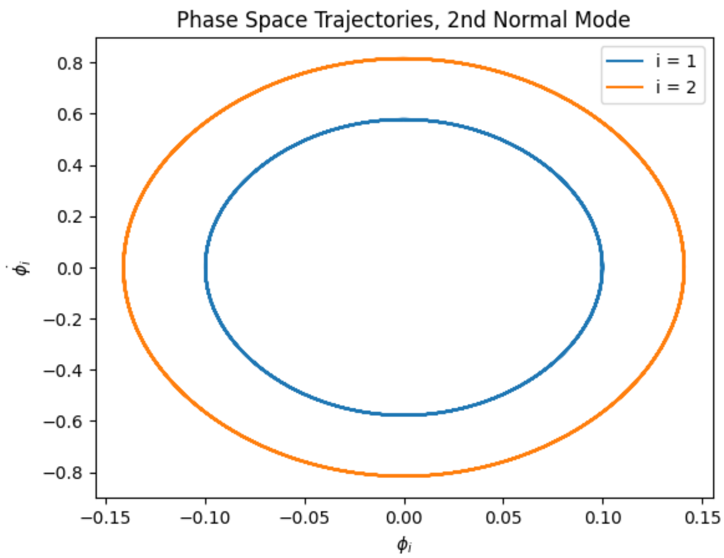

Figure 4: This figure shows the phase plane portraits of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at the 2nd normal mode

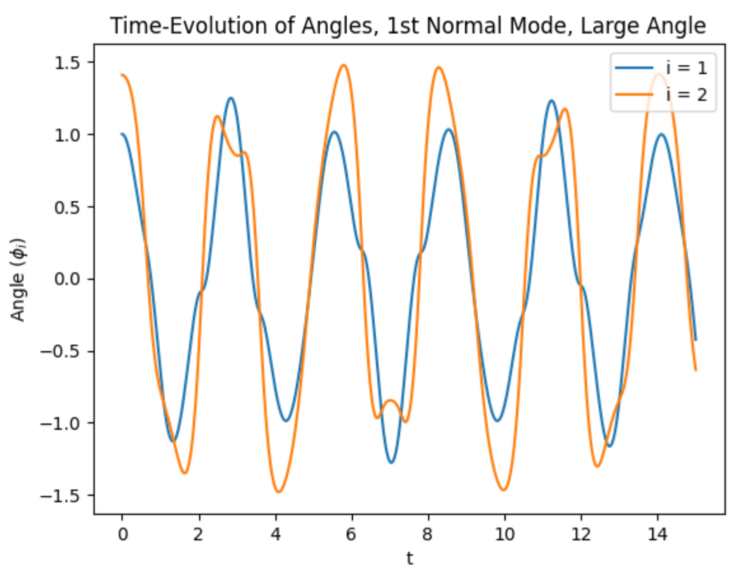

Figure 5: This figure shows the time evolution of the angles of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at almost the 1st normal mode, but since the angles are larger there are added perturbations

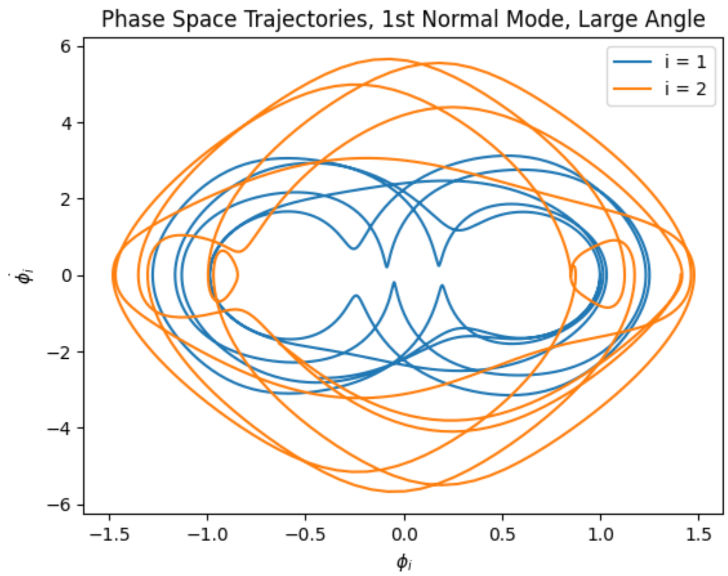

Figure 6: This figure shows the phase plane portraits of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at the 1st normal mode plus extra perturbations due to the large angle

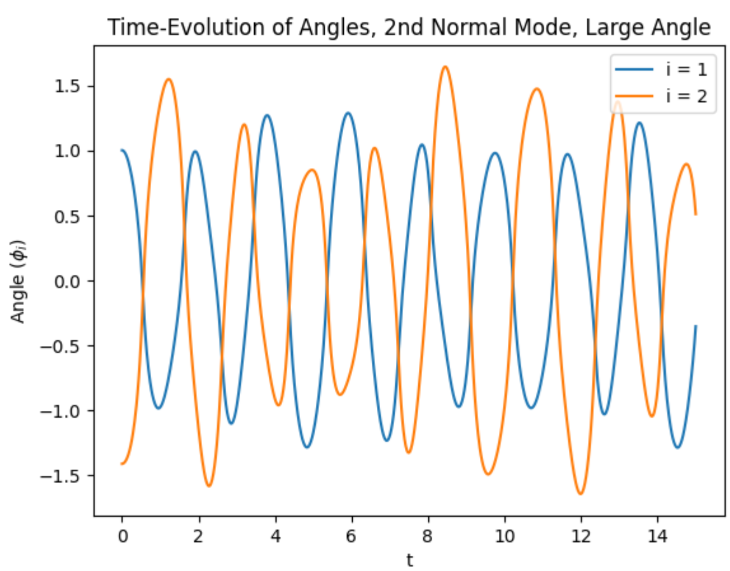

Figure 7: This figure shows the time evolution of the angles of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at almost the 2nd normal mode, but since the angles are larger there are added perturbations

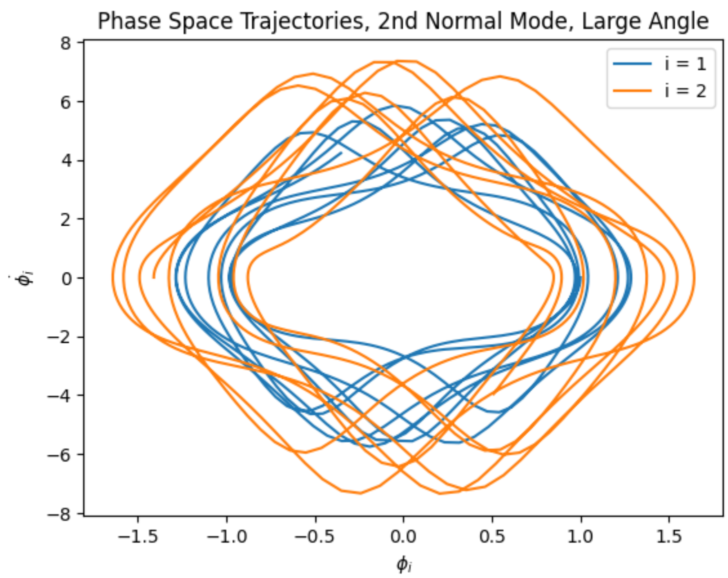

Figure 8: This figure shows the phase plane portraits of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate at the 2nd normal mode plus extra perturbations due to the large angle

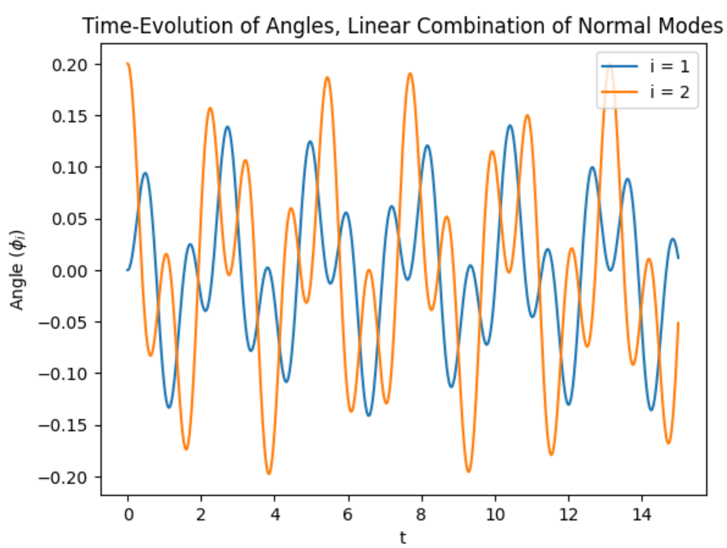

Figure 9: This figure shows the time evolution of the angles of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate with a linear combination of modes

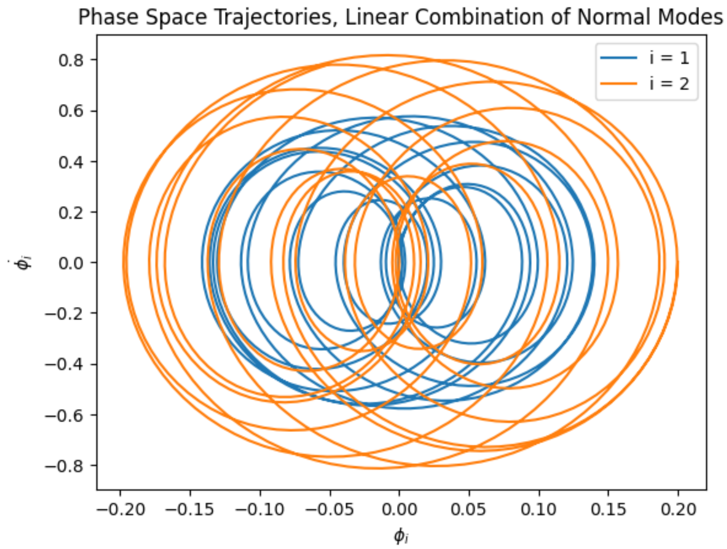

Figure 10: This figure shows the phase plane portraits of bob 1 (blue) and bob 2 (orange), with initial conditions such that the bobs oscillate with a linear combination of modes

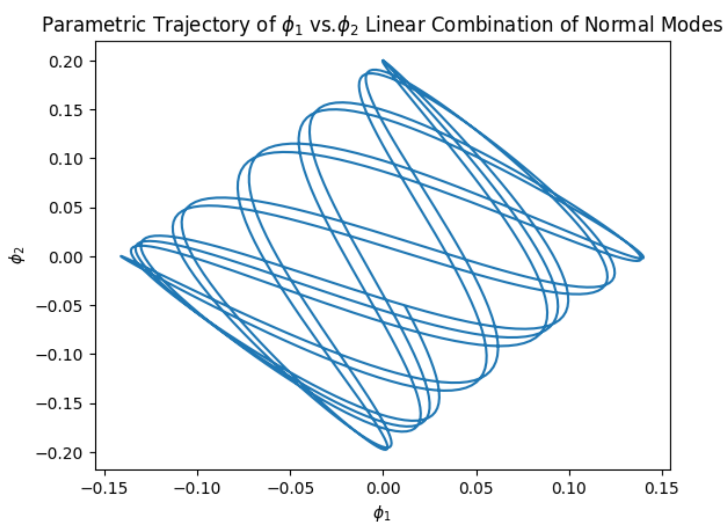

Figure 11: This figure shows the Lissajous Curve parameterized by plotting ϕ_1 vs. ϕ_2

7 Appendix B: Code

```
%matplotlib inline

import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint

def equations_of_motion(x, t):
  # constants:
  g = 9.81
  l = 1
  w = g/l

  #functions:
  phi_1 = x[0]
  phi_2 = x[1]
  phi_1_dot = x[2]
  phi_2_dot = x[3]

  # Matrix Stuff:
  M = np.array([[2, np.cos(phi_1-phi_2)],[np.cos(phi_1-phi_2), 1]])
  M_inverse = np.linalg.inv(M)
  f = np.array([-2*w*np.sin(phi_1)-phi_2_dot**2*np.sin(phi_1-phi_2),
      -w*np.sin(phi_2)+phi_1_dot**2*np.sin(phi_1-phi_2)])
  phi_ddot_vec = np.matmul(M_inverse, f)

  #DEs:
  d_phi_1_dt = phi_1_dot
  d_phi_2_dt = phi_2_dot
  d_phi_1_dot_dt = phi_ddot_vec[0]
  d_phi_2_dot_dt = phi_ddot_vec[1]

  return [d_phi_1_dt, d_phi_2_dt, d_phi_1_dot_dt, d_phi_2_dot_dt]

# initial_conditions:

#1st Normal Mode:
#x0 = [0.1, 0.141, 0, 0]

#2nd Normal Mode:
#x0 = [0.1, -0.141, 0, 0]

#Large Angle 1st Mode:
#x0 = [1, 1.41, 0, 0]

#Large Angle 2nd Mode:
```

```
#x0 = [1, -1.41, 0, 0]

#Combination of Modes:
#x0 = [0, 0.2, 0, 0]

#General Large Angle:
x0 = [0.7, 0.7, 0, 0]

#time
t = np.linspace(0, 15, 1000)
x = odeint(equations_of_motion, x0, t)

#Solutions:
phi_1 = x[:, 0]
phi_2 = x[:, 1]
phi_1_dot = x[:, 2]
phi_2_dot = x[:, 3]

descriptor = 'Linear Combination of Normal Modes'

plt.title('Time-Evolution of Angles, '+ descriptor )
plt.xlabel('t')
plt.ylabel('Angle ' + r'($\phi_i$)')
plt.plot(t,phi_1, label ='i = 1')
plt.plot(t, phi_2, label = 'i = 2')
plt.legend(loc = 'upper right')
plt.show()


plt.title('Parametric Trajectory of '+ r'$\phi_1$'+ ' vs.' + r'$\phi_2$'
    +' ' + descriptor)
plt.xlabel(r'$\phi_1$')
plt.ylabel(r'$\phi_2$')
plt.plot(phi_1, phi_2)
plt.show()

plt.title('Phase Space Trajectories, ' + descriptor)
plt.xlabel(r'$\phi_i$')
plt.ylabel(r'$\dot{\phi_i}$')
plt.plot(phi_1, phi_1_dot, label = 'i = 1')
plt.plot(phi_2, phi_2_dot, label = 'i = 2')
plt.legend(loc = 'upper right')
plt.show()
```